
MAGNANIMITER CRUCEM SUSTINE
GEORGE KENYON
of Peel Esqr

Short, but yet Plain

ELEMENTS
OF
GEOMETRY.

SHEWING

How by a Brief and Easie Method, most of what is Necessary and Useful in EUCLID, ARCHIMEDES, APOLLONIUS, and other Excellent *Geometricians*, both Ancient and Modern, may be Understood.

Written in *French*
By *F.* IGNAT. GASTON PARDIES.

And render'd into *English*,
By *JOHN HARRIS*, D. D.
And Secretary to the *Royal Society*.

The SIXTH EDITION.

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TO

My Worthy Friend
CHARLES COX *Eſq*;

Member of Parliament for the Burgh of *Southwark*.

Dear SIR,

A*MONG the many Obligations You have conferred on me, I account it not the leaſt, that you have given me a* Riſe *to revive my* Mathematical Studies ; *in which, as I have formerly ſpent ſome Time, ſo I know of no more* uſeful Way *of employing my leiſure Hours.*

And indeed, Sir, the Diverſion *and* Advantage *I have lately reaped from them, hath (by the Divine Bleſſing) both ſupported me under, and in a good Meaſure carried me through ſuch Preſſures and Difficulties, as I once almoſt deſpaired of ſurmounting.*

The Mathematick Lecture *which You at firſt ſet up* gratis *in your* Burgh, *and which out of an uncommon Generoſity, You did afterwards remove into the City of* London, *is a demonſtrative Proof both of your ſincere Endeavour to promote the Good of your Country, and alſo of your Capacity to do it the beſt Way. And as I have already, in a good Degree, ſo I hope to ſee ſuch Effects from ſo noble a Deſign, as will render your Name juſtly honourable to Poſterity, as well as this preſent Age. Sir, You know* your Self *and* Me *too well to take this for Flattery. 'Tis what* Truth, Juſtice *and* Gratitude *oblige me to ſay.*

I ſhall only add, That I am again glad of this Opportunity to ſhew the juſt Eſteem I have of your Merit, and the equal Regard I have for your Friendſhip. I am,

SIR,

Your moſt obliged

Humble Servant

John Harris.

THE TRANSLATOR TO THE READER.

After frequent Perusal, and mature Deliberation on this Book; I judge it to be the plainest, shortest, and yet easiest Geometry I have ever seen publish'd: And therefore I thought it very well worth my while to let it appear a sixth Time in our own Language, as it had already done twice before in the Latin Tongue. 'And 'tis so well esteem'd of, by very competent Judges amongst Us, as to be read in our Universities, by Tutors to their Pupils: And also, which is not usual with Books of this Kind, there have been three entire Impressions sold off in a little more than as many Years Time.

As to the Translation; I have by no means obliged my self servilely to follow the French way of Expression; for indeed a literal Version of a Book out of any Language will be scarce intelligible in English. I have therefore all along aimed rather to give you F. Pardies's Sense, than his Words; and have made him speak what I judge he would have done, had he wrote in our Language. I have made no Scruple to add any thing that I saw necessary, *to render him clear and intelligible; and particularly what follows, which was not in some of the former Editions.*

As the second Book of Euclid *about the Power of Lines; The Mensuration of the Surfaces of Solids,* Archimedes *his Proportion of the inscribed Cone and Sphere to the circumscribing Cylinder; the Figure of the 5 regular Bodies; several Additions and Improvements in the Doctrine of Proportion: The Mensuration of the* Frustums *of Pyramids and Cones; some new Properties of a Right-angled Triangle, and of the Circle,* &c. *I have also left out some more of* Pardie's *Propositions, which, on repeated Experience in Teaching, I have found less useful; as also all the Elements of plain Trigonometry, which I had before added to his Ninth Book; because I have publish'd a small Treatise on that Subject by it self; and my chief Aim* now *hath been to lead the Learner into a little more abstracted and concise, tho' a most useful and universal Method of Demonstration; introducing now and then a little* Algebra, *that I thereby engage the Reader in a Love of, and and Value for that most noble and wonderful Science: And to give him a good Foundation to build upon, and a sufficient Rise thereby to carry him into Fluxions, and the new Methods of Investigation and Demonstration, where he will find sufficient Satisfaction. Nor need he be discouraged at the Attempt, for 'tis well known, that I have taught several Persons to understand the elementary Parts of all Mathematicks so well, that they have been able to go on every where, without the Assistance of any Master, in less than a Year's Time.*

PARDIE's

PARDIE's *Advice to those who would Understand Geometry.*

I. THEY ought to enure themselves to consider well the *Figures*, at the same Time as they read the *Propositions*. There will be some Labour and Difficulty at first, but they will break thro' it in two or three Days.

II. They ought not to be discouraged, if they meet with some Things which they do not understand at first; Geometry is not so easie to be attained, as History.

III. If, after they have read and considered attentively any Proposition, they find they don't understand it, let it be passed over, it will probably be intelligible by reading farther, or at least when they have gone over the whole, and have began to read it over a-new. There are indeed many things in Geometry, that will never be well understood at first reading over.

IV. The

IV. The Numbers which are within the Parenthesis, *v. gr.* (3. 14.) shew that the Matter there spoken of, hath been proved elsewhere, *viz.* in this Instance, in the 14th *Article* of the III *Book*: And they ought always to mind the Number of the Article, and to consult the Places referred to, that so they may gain the Demonstration of what they read.

V. When they meet with any Words which they don't understand, they may consult the Table at the End of the Book.

VI. 'Tis good to have a Master at first, to explain to them the Nature and Manner of the Demonstrations; for by that Means they will understand the Thing much easier, and much sooner, than they can do by reading by themselves.

ELE-

ELEMENTS OF *GEOMETRY.*

BOOK I.

Of Lines and Angles.

1. BY the Word *Quantity*, which in the General is the Subject of *Geometry*; we mean that whereby one Thing being compared with another of the ſame Nature, may be ſaid to be Greater or Leſs than, Equal or Unequal to it: As Extenſion, *i. e.* Length, Breadth or Thickneſs, Number, Weight, Time, Motion, and all thoſe things which are capable of being ſo compared as to more or Leſs, are the Object of Geometry.

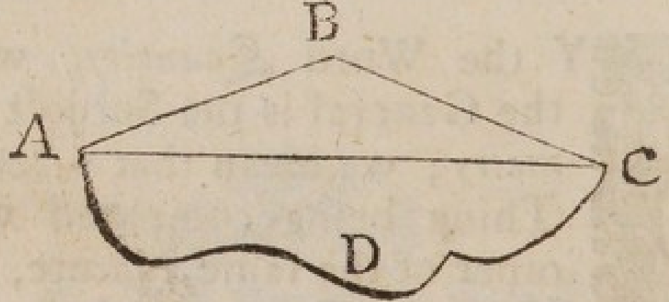
2. We deſign nevertheleſs to conſider now only Extenſion; as being that which ſerves for an *Example* and *Rule* to meaſure all other Quantities by.

3. That Quantity which, being ſuppoſed without any Breadth or Thickneſs, is extended only in Length, is called a *Line*. That which hath both Length and Breadth, (but is ſuppoſed to have no Thickneſs) is called a *Surface*, or *Superficies*: And that which hath Length, Breadth and Thickneſs, is called a *Body*, or *Solid*.

4. A *Point* is that, which is conſider'd as having no manner of Dimenſions; and as being indiviſible in every reſpect. The Ends or Extremities of Lines, as alſo the Middle of them, are Points.

5. There are *Strait Lines*, and there are *Crooked* or *Curved* ones: Alſo there are Even and Plain Surfaces which are called *Planes*; and there are *Crooked* or *Curved* ones: Which like a Vault, (*or the Tilt of a Boat or Waggon*, are *Convex* above, and *Concave* below.

The Generation of Lines may eaſily be conceived to be made by the Motion or Fluxion of a Point, as A.



Which if it move directly from the Term A, to the Term C, or go the neareſt or ſhorteſt Way poſſible, it then forms what Geometers call a *Right*, or *Strait Line*.

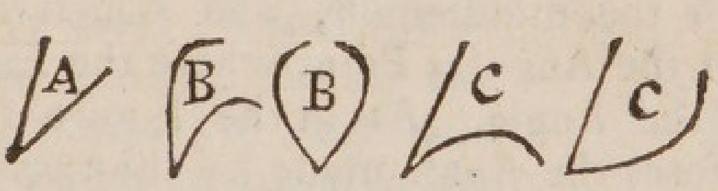
If it go firſt directly to B, and then alſo the neareſt Way to C; it forms two Right Lines, A B and B C, which, taken together, are longer than the Line A C; and conſequently, two Sides of any Triangle muſt be longer than the Third.

If the Point A move not in one or more right or ſtrait Lines towards C, it muſt go crooked, and ſo will form a Curve or crooked Line, as A D C.

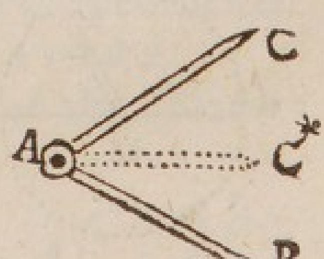
And from hence alſo 'tis plain, That any two Points, moving with equal Velocity, will in the ſame Time generate equal Lines.

6. When two Lines meet in a Point, the *Aperture*, *Diſtance* or *Inclination* between them, is call'd an *Angle*. Which, when the Lines forming it are right or ſtrait ones, is called a *Rectilineal Angle*; as A. But if they are crooked, 'tis called a *Curvilineal One*; as B. And when one is ſtrait and the other crooked, 'tis called a *Mix'd Angle*; as C.

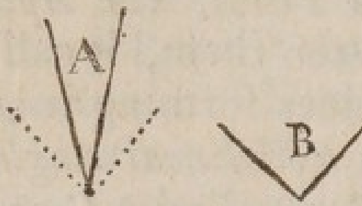
N. B. *The Lines, forming any Angle, are called its Legs.*



7. That Angle is ſaid to be leſs than another, whoſe Legs are more inclined to *(or nearer to)* each other. Let there be two Lines A B and A C meeting in the Point A. If you imagine thoſe Lines to be moveable like the Legs of a Pair of Compaſſes, and yet faſtened together in A, as with a Joint, 'tis eaſie then to conceive, that the further they are opened, or parted from one another, the greater will be the Angle between them: As on the Contrary, the nearer they are brought together, the more they will incline towards each other, and ſo the Angle between them muſt be the leſs.

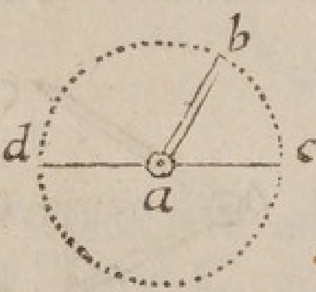


8. It muſt therefore be obſerved, that the Quantity of Angles is by no Means meaſured by the Length of their Legs, but by their Inclination. Thus, *v. gr.* the Angle B is bigger than A; tho' the Legs of the Angle B, are much ſhorter than thoſe of A: But then thoſe of A are much more inclined to each other, than thoſe of B. And to apprehend this the better, imagine the Angle B to be put upon A, as you may conceive by the prick'd Lines about A, which repreſent the Legs of B lying on it. For 'tis plain the Angle A will be eaſily contained within B; and that its Legs are much more inclined to one another, than thoſe of B, and therefore it is leſs than B.



9. An Angle is uſually marked by three Letters, of which the middlemoſt, and which always is placed at the Angular Point where the Lines meet, denotes the Angle. As in the Figure following, *b a c* denotes the Angle made by the two Lines *b a* and *c a* meeting in the Point *a*.

10. If we imagine the Line *a b* faſtened by its End *a*, in the middle of the Line *d c*, but yet ſo as to be moveable to *a*, as on a Center: If then you conceive it to be moved quite round, till it arrive at the Place where it began, the Point *b* will deſcribe a Curve Line, which is uſually called a Circle; but 'tis rather the *Circumference of a Circle*; for properly ſpeaking, the Circle is the *Space* contain'd within the Circumference.



11. Any Part of the Circumference is called an Ark, as *b c*.

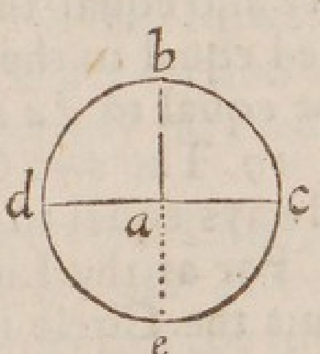
12. The Line *d e* (*paſſing through the Center*) and terminated by the Circumference, is called the *Diameter*,

meter, and divides the Circle into two equal Parts. Alſo every Right Line paſſing thro' the Center *a*, (*and terminated at each End by the Circumference*) divides the Circle into two equal Parts, as will be a Diameter.

13. The Line *a b* or *a c*, or any other drawn from the Center to the Circumference, is called the Radius, or Semi-diameter.

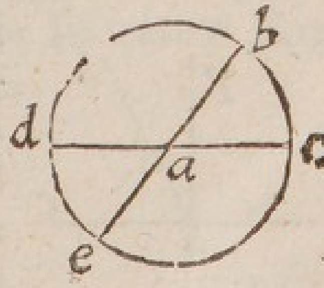
14. All *Radius's* or *Semi-diameters* (*of the ſame or equal Circles*,) are equal. (*As is plain from the Geneſis of a Circle given in* Art. 10.)

15. When the End *b* of the Radius *a b* is equally diſtant from the two Ends of the Diameter *d c*; That is, when the Point *b* is in the very middle of the Semi-circumference *d b c*; then will *b a* make two Angles with *d c* that are called *Right ones:* Which are equal to one another, that is, the Angle *d a b* is equal to *b a c*. And if the Line *b a* be produced below to *e*, it ſhall then (with *dc*) make four Right Angles; and it will be another Diameter; which with the former *dc* will divide the Circle into four equal Parts.

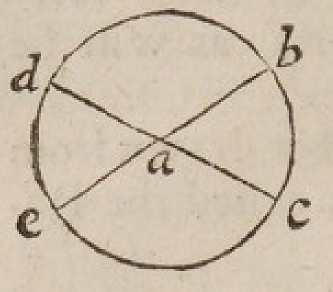


16. Then thoſe Lines are ſaid to be perpendicular one to another, *viz. b a* to *d c*, and *d a* to *b e*.

17. But if *b* be nearer to one End of the Diameter (or Right Line) *d c*, than it is to the other, it is then ſaid to fall on the other obliquely; and it makes with *d c* two Angles that are Unequal: Of which the Leſſer *b a c* is called Acute, and the Greater *d a b* is called Obtuſe.



If the Line *a b* be produced to *e*, it will be a new Diameter, and will make below two other Angles: So that in the whole here will be four Angles; of which thoſe two that touch only in the Angular Point, as *b a c* and *e a d*; as alſo, *d a b* and *e a c*, are called *Vertical*, or *Oppoſite* Angles. But thoſe that have one Leg common to both, as *d a b* and *b a c*; and *b a c* and *e a c* are called *Adjoining or Contiguous Angles.*

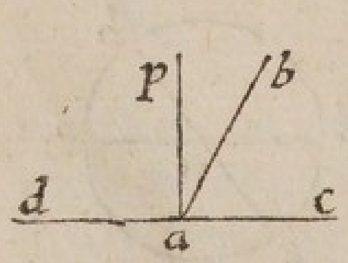


18. Thoſe Angles, which (at equal Diſtances from the Angular Point) are ſubtended by equal Arks, are alſo equal themſelves. As if the Ark *b c* be proved equal to the Ark *d e*, then will the Angle *b a c* be equal to *d a e*.

19. The two *Contiguous Angles*, taken together, are always equal to two Right ones.

For as the Line *d c* is a Diameter, and therefore cuts the Circle into two equal Parts, the two Arks *d b* and *b c*, taken together, will be equal to a Semi-circle. Wherefore the two Angles *d a b* and *b a c*, together, will be equal to two Right ones, becauſe they compleat the whole Semi-circle, as two Right ones do. (*Art.* 15.)

20. So that this Propoſition is of univerſal Truth, *That one Right Line, falling on another, makes the Contiguous Angles* (together) *equal to two Right ones.* For if the Lines are *Perpendicular* to each other, as *p a* is to *d c*. Then 'tis plain the Angles muſt be Right (by the 15.) And if the Line fall obliquely, as *b a* doth, then indeed the Angles are unequal: But as much as the Obtuſe one *d a b* exceeds one Right Angle, by ſo much is the Acute one *b a c* exceeded

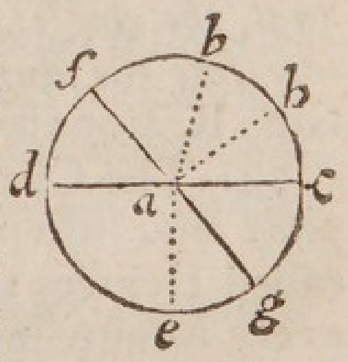


by

by the other Right one. So that the Smallneſs of one is compenſated by the Greatneſs of the other.

21. Hence alſo it follows converſely, for (where-ever the Property is found, there the Thing is, in Geometry,) that if two Angles, which have one Leg common to both, do make Angles equal to two Right ones, their other Legs do make but one Right Line. Let the Angles *d a b* and *b a c* be (*together*) equal to two Right ones. Then I ſay, that the Lines *d a* and *a c* do join ſo together, as to make one Right Line (*vid. Fig. in Art.* 17.) which is clear from what hath been ſaid. For if on the Center *a* you deſcribe a Circle *d b c e*, the two Arks *d b* and *b c* will be equal to a Semi-circle, becauſe the two Angles *d a b* and *b a c* are ſuppoſed to be equal to two Right ones. Wherefore the Lines *d a* and *c a* will make a Diameter, and conſequently be joined into one Right Line.

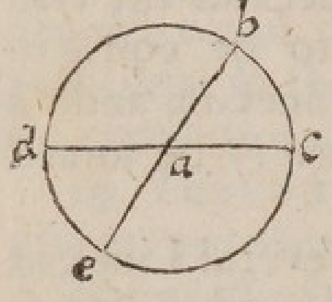
22. If from the Point *a* you draw ſeveral Lines, as *a d*, *a f*, *a b*, *a h*, *a g*, &c. they will make diverſe Angles; and all thoſe Angles taken together, be they more or leſs, will be equal to four Right ones. For 'tis clear, all theſe Angles together do compleat the Circle *d b c e*, whoſe Circumference they divide into as many Arks as there are Angles. Now all theſe together are equal to four Quarters of a Circle; which is as much as to ſay, that all the Angles are equal in the whole to four Right ones; for ſo many Right Angles do alſo compleat the Circle.



AXIOM I.

If to, or from equal Things, you add or subtract Equals, the Sums or Remainders will be equal.

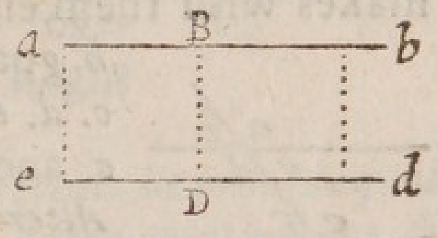
23. The Vertical or opposite Angles are equal. Let there be two Right Lines *d a c* and *b a c* (*crossing or cutting one another in the Point* a,) I say, the Angle *d a e* is equal to *b a c.* For the Ark *b d*, with the Ark *b c*, makes a Semi-circle; and so doth the same Ark *b d* with the Ark *d e.* Therefore the Ark *b c* must be equal to *d e*; because the Ark *d b* continues the same, whether it help to compleat the Semi-circle with *d e*, or *b c* : (*wherefore being taken away from both, it must leave the Ark* d e *equal to* b c. *But if the Arks be equal, the Angles subtended by them must be so too, and therefore the Angle* d a e *is equal to* b a c.) And by the same Reason the Angle *d a b* will be equal to *e a c.*



24. The Circumference of every Circle is (*supposed to be*) divided into 360 equal Parts, which are called Degrees: And every Degree into 60 Minutes, every Minute into 60 Seconds, every Second into 60 Thirds, and so on infinitely. And to determine the Quantity of every Angle, we compute the Degrees that (*the Ark, which is its Measure*) doth contain, *v. gr.* When we speak of an Angle of 90 Deg. we mean a Right Angle; because the Right Augle contains the fourth Part of the whole Circumference, which is 90 Deg. the fourth Part of 360. So an Angle of 60 Deg. is an Angle that contains two Thirds of a Right one.

25. *Degrees are marked either with* Degr. *or usually with a small Cypher over the last Figure, as* 60°.) Minutes with a small Line, as 50′, Seconds with two such, as 20″, Thirds with three such, as 25‴, *&c.* So that 25° 32′ 43″, is to be read, 25 Degrees, 32 Minutes, 43 Seconds.

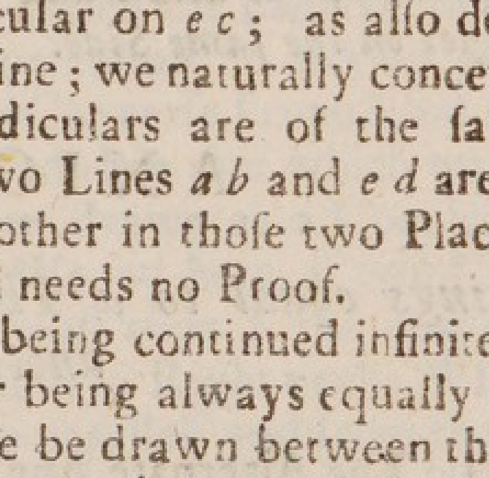
26. Two Lines are said to be parallel, when they run always equi-distant from each other. Thus the two Lines *a b* and *e d* are parallel, if they are equally distant from each other in *a e*, in BD, in *b d*, and in all other Places.



27. This Distance is always measured by a Perpendicular; as if from the Point *a* you imagine the Line *a e* to fall perpendicular on *e c*; as also doth the Line *b d* on the same Line; we naturally conceive that if those two Perpendiculars are of the same Length, or equal; the two Lines *a b* and *e d* are equally distant from each other in those two Places, which is self-evident, and needs no Proof.

28. Two parallel Lines, being continued infinitely, yet can never meet: For being always equally distant, there may any where be drawn between them a Perpendicular equal to *a e* or *b d*, and consequently they can never meet.

29. Two parallel Lines have the same Inclination, one as the other, to any right Line that crosses them both.



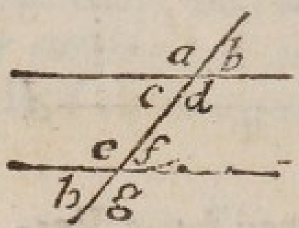
That is, the Angle *a* will always be equal to *b*, and *c* to *d*; for the intersecting Line being supposed inflexible, as is the Case of all Mathematick Lines, it cannot bend to, or from one

Parallel

Parallel more than it doth to or from the other : And neither of these Lines can alter its Position in respect of the Crossing Lines, for then the Parallelism would be destroyed, which contradicts the Supposition.

And this is the first Property of Parallel Lines.

30. Whenever a Right Line cuts two Parallels, it makes with them eight Angles: Of which four *a. b. h. g.* are *external*; and the other four *c. d. e. f.* are *internal.* The Angles *c* and *f*, as also *d* and *e*, are called *Alternate.* The Angles *e* and *a*, as also *f* and *b*, are called the internal, and *opposite on the same Side.* And the Angles *d f*, as also *c* and *e*, are called the *internal Angles on the same Side.*



AXIOM II.

Things equal to a Third, are equal to one another.

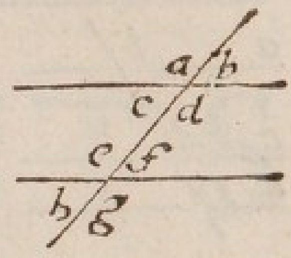
31. The Alternate Angles *c* and *f* must be equal; and also *e* and *d*; for *c* is equal to the Vertical Angle *b*, and *b* is equal to the internal one *f*, by the last *Prop.* Wherefore *c* and *f*, being both equal to *b*, must be equal to one another.

The same may be proved of *c* and *d*, which are both equal to *a*.

32. When

32. When a Line falls on two parallel ones, it makes the internal Angles on the ſame Side equal to two Right ones.

I ſay, the Angle *d* with *f*, is equal to two Right ones: Becauſe *f* is equal to *c* (by 31.) and *c* and *d* together are equal to two Right ones (by 20.) Therefore *f* and *d* together muſt be equal to two Right ones, which was to be prov'd.



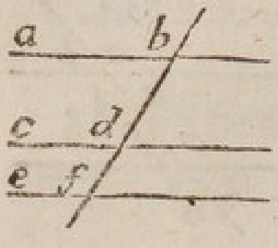
(The ſame Way may c *and* e *together be proved equal to two Right ones; for* c *and* d *taken together are ſo (by* 20. *but* d *is equal to* e (31.) *Therefore* c *and* e *are equal to two Right ones.)*

33. One Propoſition is called the *Converſe* of another; when, after a Concluſion is drawn from ſomething ſuppoſed, in the Converſe Propoſition that Concluſion is ſuppoſed; and then that which was in the other ſuppoſed, is now drawn as a Concluſion from it. For Example: We ſay here, if two Lines are parallel, *(and another croſs them,)* the Angles *d* and *f* together, are equal to two Right ones: Where we ſuppoſe the Lines to be parallel, and from thence conclude thoſe Angles muſt be equal to two Right ones: But the Converſe is thus, If the *internal Angles on the ſame Side*, *d* and *f* together, are equal to two Right ones; Then thoſe Lines are parallel: Where, after we have ſuppoſed the Angles equal to two Right Ones, we conclude the Lines are parallel.

34. Converſe Propoſitions in this Caſe are very true; as that, if a Line cut two other Lines, and makes the alternate Angles equal; Thoſe two Lines are parallel: which I deſire the Reader to remember.

35. If two Lines are parallel to a third Line, they are so to one another.

Let the Line *a b* be parallel to *c d*; and let *e f* also be parallel to the same Line *c d*; I say, *a b* is parallel to *e f*: For if you draw a Line as *b d f* cutting them all Three; the Angle *b* will be equal to *d* (by 31.) and the same *d* will be equal to *f* (by 31.) because *e f* is also parallel to *c d*. Wherefore the Angle *b* must be equal to *f*: Because by Axiom 2. *if two Things are equal to a third, they are so one to another.* But if the Angle be $=f$, then the Line *a b* is parallel to *e f*, (by 34)



ELEMENTS OF *GEOMETRY.*

BOOK II.

Of Triangles.

1. A *Figure* is a Space compaſſed round on all Sides. And if the Lines, which terminate it, are all Right ones, 'tis called a Rectilineal (or Right Lined) Figure: If they are crooked, 'tis called a Curvilineal; and if they are partly Right Lines, and partly Crooked, 'tis called a Mix'd Figure.

2. There are Plane Figures, which are Plane Surfaces, and there are Solid ones, which have three Dimenſions. But we ſpeak here only of Plane Surfaces, or Plane Figures.

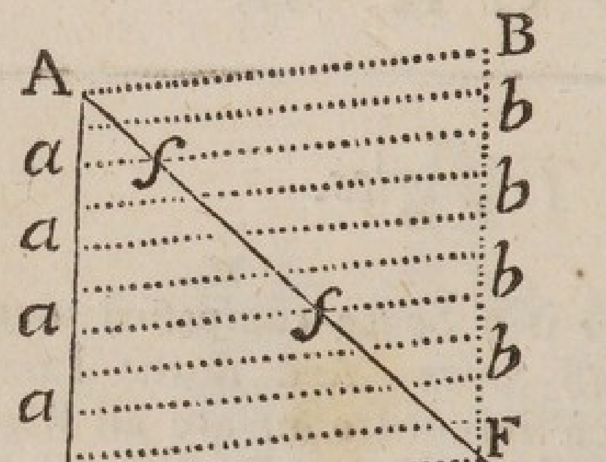
3. All

3. All the Lines which encompaſs any Figure, taken together, make that which is called the Circumference, Perimeter, or the Compaſs of the Figure.

4. Of all Curvilineal or Mix'd Plane Figures, in Common Geometry, we conſider properly only the Circle, or a Part of a Circle terminated on one Side by an Ark, and on the other by one or more Right Lines.

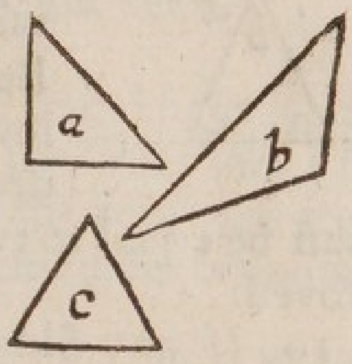
5. Of Rectilineal Figures, the moſt ſimple are Triangles, which are terminated by three Right Lines *(and no more)* making as many Angles.

If a Right Line, (A B) having one of its Ends or Points (as A) in the Vertex or Top of the Angle E A D, be moved downwards, with a Motion always parallel to it ſelf, ſo that the Point A ſhall always keep in, or touch the Line AE, until it come to be all of it within the Legs of the Angle E A D; that is, till it come to be in the Situation E F; that Line ſhall in its Motion continually cut the Line A D, and at length deſcribe the Triangle EAF within the Legs of the Angle; as alſo another equal to it *(AFB)* on the other Side of the Line A D. The Parts of which latter Triangle ſhall continually decreaſe, as thoſe of the former A E F, do continually increaſe. And the Line A B ſhall alſo deſcribe with its whole Length the Quadrilateral Figure A E F B; which will be divided into two equal Parts by the Diagonal Line A F.

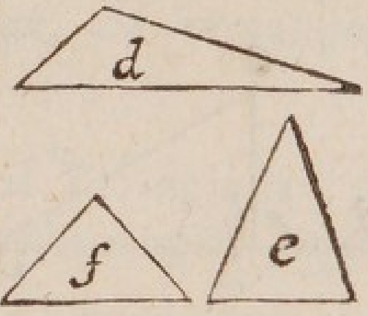


N. B.

N. B. The Line AB may be called the *Describent*, and AE the *Dirigent*, because the latter directs the Motion of the former.



6. A Triangle as *a*, which hath ne Right Angle, is a *Right-an-led* Triangle; if it have one An-le Obtuse, 'tis called an *Obtuse-ngled* one, as *b*; and if all its hree Angles are Acute, 'tis call-d an *Acute-angled* Triangle, s *c*.



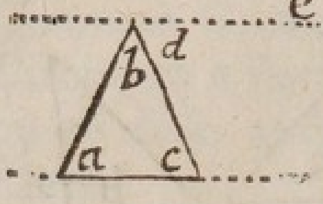
7. If a Triangle have all its hree Sides unequal, 'tis called a *calene*, as *d*. If it hath two Sides qual, 'tis called an *Isosceles*, as *e*; And if all the three Sides are e-qual, 'tis called an *Equilateral* ne, as *f*.

8. When two Sides of a Triangle are consider'd, hey may be called its *Legs*, and the third Side may then be called the *Base*. But any one Side may be called the *Base*, *tho' we usually and most pro-erly call that so, which lies parallel to the Horizon, nd which is next to us.*

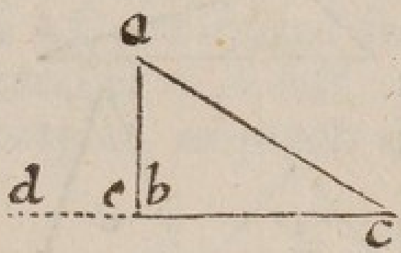
9. In every Triangle, the three Angles, taken to-gether are equal to two Right ones.

Let the Triangle be *a b c*: I say, that the Angle *a* dded to the Angle *c*, added to the Angle *b* (*or the Sum of all three*) are equal to two Right ones. For et *d e* be drawn parallel to the *Base a c*, then will hose two parallel Lines be cut by the Line *b c*; and consequently the alternate Angles *c* and *d* will be equal to each other (by 1. 31.) Moreover the Line *b a* falling on, or cutting the same Parallels *d c* and *a c*, will make the two internal Angles on the

the ſame Side equal to two Right ones; that is, *a* added to *a b e* are equal to two Right Angles (by 1. 32.) But the Angle *a b e* contains the two Angles *b* and *d*. So that the Angle *a* added to *b* added to *d*, will be equal to two Right ones. But *c* being equal to *d*, it will follow, that *a* added to *b* added to *c*, or the Sum of all three together, muſt be equal to two Right ones: Which was to be proved.



10. If any Side of a Triangle be produced, or drawn out, the external Angle will be equal to the two internal oppoſite Angles, (*taken together.*) Let the Triangle be *a b c*, whoſe Baſe *c b* draw out to *d*, by which means a new Angle as *e* will be made, which is called the *External* Angle of that Triangle. Then I ſay, That that external Angle *e*, is equal to both the internal and oppoſite ones *a* and *c*.



For thoſe Angles *a* and *c*, together with *b*, are equal to two Right ones (by the Precedent,) and ſo alſo are *e* and *b*, by (1. 20.) wherefore *e* muſt be equal to *a* added to *c*, becauſe together with *b*, it makes two Right Angles, as they do. Q. E. D.

COROLLARIES.

1. The Sum of the three Angles of all Triangles is the ſame.

2. No Triangle can have above one Right, or Obtuſe Angle.

3. If in any Triangle, one Angle be Right, the other two muſt be Acute.

4. If in any Triangle there be one Angle equal to both the others, that muſt be a Right One.

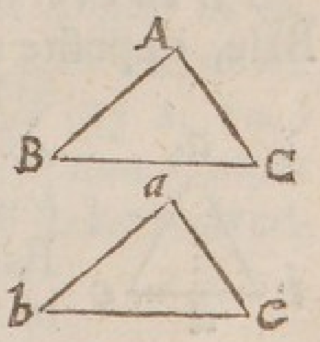
5. If you know the Degrees of one Angle in any Triangle, you know the Sum of the other two; for 'tis what is wanting of 180°, and if the Sum of any two be known, the Quantity of the Remainder is known.

6. Hence if two Triangles have any two Angles reſpectively equal to one another, the remaining Angles muſt alſo be equal.

7. The Angle of an Equilateral Triangle is $\frac{1}{3}$ of two Right Angles, or $\frac{2}{3}$ of one Right Angle, equal to 60°.

8 Hence 'tis very eaſy to Triſect a Right Angle, by making on one of the Legs an Equilateral Triangle.

11. If a Triangle A B C hath two Sides, A B and A C, equal to two other *a b* and *a c* in another Triangle, and if alſo the Angle A be equal to *a*; I ſay, the third Side B C ſhall be equal to *b c*; the Angle B equal to *b*, the Angle C to *c*, and the whole Triangle A B C to *a b c*.



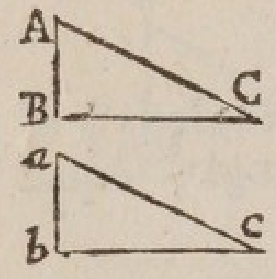
For if we imagine the Triangle *a b c* to be placed upon ABC, ſo that the Side *a b* ſhall lie exactly on its Equal A B: Then muſt the Side *a c* fall on its Equal A C, becauſe the Angle *a* is equal to A, and ſo the Point *c* will fall on C, and *b* upon B, and the whole

Triangle *a b c* on the Triangle A B C; becauſe all things ſo exactly anſwer, that nothing of the upper Triangle can fall beſides the under one.

12. Figures which do thus meet, fit, or anſwer to each other exactly, when they are placed one upon the other, are called Congruous Figures, *Quia mutuo ſibi Congruunt.*

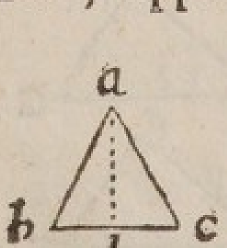
And therefore the third Axiom is, *Quæ ſibi mutuo Congruunt ſunt Æqualia*; *i.e.* Thoſe Figures, which placed one upon another, do anſwer to, and cover one another exactly, are equal.

13. It is alſo true, That if a Triangle hath all its three Sides equal to the three Sides of another Triangle, all the Angles alſo in one, ſhall be equal to thoſe in the other: And all the Space which one Triangle contains, ſhall be equal to that contained in the other: As if A B be equal to *a b*, A C to *a c*, and B C to *b c*: I ſay, that the Angle A ſhall be equal to *a*, B to *b*, and C to *c*; and the whole Triangle A B C, to *a b c*; this needs no other Proof.



14. If the Angle A be equal to *a*, the Angle B to *b*, and the Side A B to *a b*: Then ſhall the Side A C be always equal to *a c*, B C to *b c*; and the whole Triangle A B C to *a b c*: which is eaſy to prove by the precedent Propoſitions.

15. In every *Iſoſceles* Triangle, the Angles at the Baſe, oppoſite to the equal Legs, are equal.



Let the Triangle be *a b c*, whoſe Legs *a b* and *a c* are equal: I ſay, the Angle *b* is alſo equal to *c*. For imagine the Baſe *b c* divided into two equal Parts in the Point *d*, then will the Line *a d* (*which let be drawn*) make of the whole two Triangles, *a b d* and *d a c*, which will have all three Sides in one, equal to thoſe in the other: For

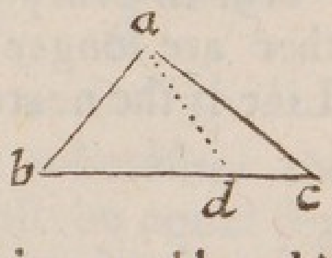
a b

a b is equal to *a c* by the Suppoſition, and *b d* is equal to *d c*, and *a d* is common to both. Wherefore (by 2. 13.) the whole Triangle *b a d* is equal to *d a c*, and the Angle *b* is equal to *c*; which was to be proved.

16. In an *Iſoſceles* Triangle, if a Line drawn from the Angle at the Top do (*biſſect or*) divide the Baſe into two equal Parts, it is both perpendicular to the Baſe, and alſo biſſects the Angle at the Top. For (*vid. Fig. præcedent*) the Angle *a d c* is equal to the Angle *a d b* (by the laſt) and conſequently they muſt be both Right ones; and therefore the Line *a d* is perpendicular to the Baſe *b c* (1. 15.) and the Angle *d a c* will be equal to *d a b* (by the laſt *Prop.*)

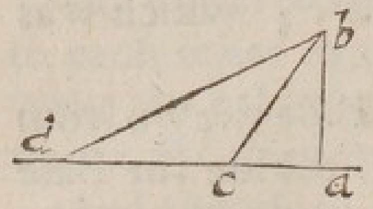
17. In every Triangle the Greater Side is always oppoſite to, or ſubtends the Greater Angle.

In the Triangle *a b c*, let the Side *b c* be longer than *b a*, then I ſay, the Angle *b a c* ſubtended by the Greater Side *b c*, is bigger than the Angle *c*, which is ſubtended by the Leſſer Side. For let *b d* be taken equal to *b a*, then will *a b d* be an *Iſoſceles* Triangle; whoſe Angle *b a d* will be equal to *b d a* (2. 15.) But the Angle *c a b* is bigger than *b a d*; (*The Whole being greater than the Part*) and therefore



muſt be bigger than *b d a* (*which is equal* b a d.) Now the Angle *a d b* is an External Angle in reſpect of the little Triangle *a d c*; and therefore muſt be bigger than the Internal one *c* (by 2. 10.) Wherefore the Angle *b a c* being bigger than *d*, muſt certainly be bigger than *c*; which was to be proved.

18. Of all Lines that can be drawn from a Point given to a Line given, the ſhorteſt is the Perpendicular; and they are all longer, according as they are farther diſtant from it. Let the given Line be *a d*, and the Point given *b*; let *b a* be perpendicular to *d a*; let alſo *b c* and *b d* be drawn. I ſay, that *b a* is the ſhorteſt Line that can poſſibly be drawn from *b*; and (for inſtance) is ſhorter than *b c* (*or any other that can be aſſigned:*) And I ſay alſo, that *b d* is longer than *b c*.



For in the Triangle *b a c*, the Angle *a* is a Right One, and conſequently bigger than either of the other; becauſe they muſt neceſſarily be both Acute (by *Cor.* 3. of *Art.* 10.) Therefore the Side *b c* is longer than *b a* (2. 17.) as ſubtending a greater Angle.

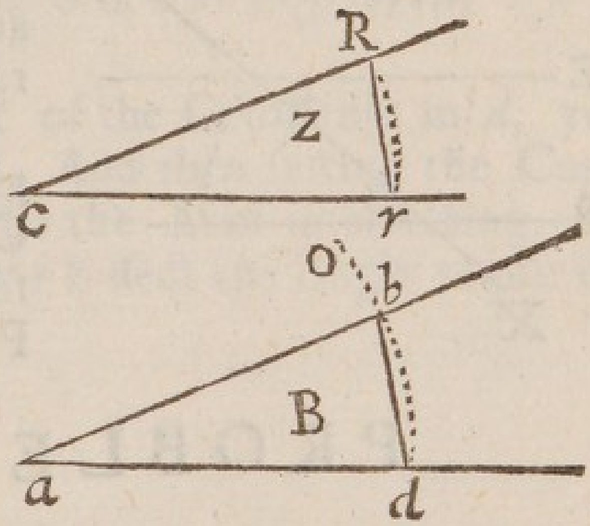
So alſo in the Triangle *d b c*, the Angle *d c b* is Obtuſe, becauſe the Angle *b c a* is Acute: And conſequently the Side *d b* muſt be longer than *c b*, as ſubtending a greater Angle (2. 17.)

19. In every Triangle any two Sides taken together are longer than the third; becauſe a Right Line is the neareſt Diſtance between any two Points.

PROBLEM I.

On a Line given a d, *to make an Angle* B, *equal to a given one* Z.

Place the Compasses in *c*, the Vertex of the given Angle, and describe the Ark *R r*; then keeping them at the same distance, set one Foot in *a*, one end of the given Line, and with the other describe the Ark *o b d*; set *R r* from *d* to *b*; and draw *a b*, so shall the Angle *b a d* or B be equal to Z.



For the Legs of each are *Radii* of equal Circles, and the Line *b d* was taken equal to R *r*; wherefore the whole Triangles *c* R *r* and *a b d* must be equal (by 13.) and consequently the Angle *a* equal to *c*.

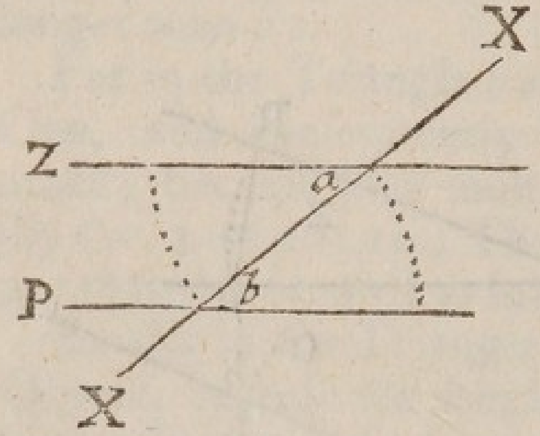
PROBLEM II.

Hence the Practice of making all sorts of Triangles, Equilateral, Isoscelar; or without any given Angles or Sides, will easily appear.

PROBLEM III.

A Right Line, as P, *being given, to draw thro'* a, *a Point given, the Line* Z a, *Parallel to it.*

Through *a* draw any Line, as XX, making any Angle, as *b*, with the given Line; then make the Angle Z *a*X = to *b*, and Z *a* shall be the Parallel sought.

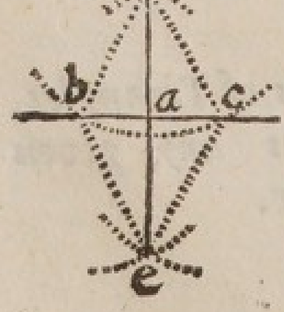


For the Alternate Angles *a* and *b* are equal by Construction; Wherefore Z *a* is Parallel to P *b*, (by 1. 31.) Q.E.D.

PROBLEM III.

To Bissect or Divide a given Line c b *into two equal Parts in the Point* a.

Open the Compasses to more than ½ the Length of *b a c*, and with that distance make at each end of *b a c*, two Pairs of intersecting Arks, as at *e* and *d*: Then drawing the Line *e d*, it will bissect the given Line in *a*.



For the Triangles *b e d* and *d e c* are equal (by 2. 13.) Wherefore the Angle *a b d* = *a d c*. Therefore the Triangles *a b d* and *a d c* will be equal also (by 2. 11.) and consequently *a b* is = to *a c*, Q. E. D.

PROBLEM V.

By much the ſame Method may a Perpendicular, as a d, *be raiſed in the middle of any given Line, or one may be let fall from the Point* e *or* d, *to the given Line* a b c, and the Demonſtration is the ſame in all.

And after the ſame way of Practice may the given Angle b d c *be Biſſected.*

If ſetting one Foot of the Compaſſes in *d*, you take *d b* equal to *d c*. And then ſetting the Compaſſes in *b* and *c*, ſtrike the Arks interſecting each other in *e*; So ſhall *d e* biſſect the Angle requir'd.

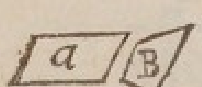
ELEMENTS OF *GEOMETRY.*

BOOK III.

Of Quadrilateral Figures and Polygons.

1. THOSE Figures, whose Sides are four Right Lines, and those making four Angles, are called *Quadrilateral*, or four-sided Figures.

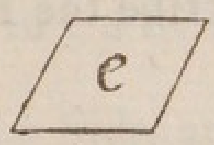
2. When the opposite Sides are parallel, the Quadrilateral Figure is called a *Parallelogram*, as *a*; but if not, 'tis called a *Trapezium*, as B.



3. When the Parallelogram hath all its four Angles Right, 'tis called a *Rectangled Parallelogram*; or for brevity's sake a *Rectangle*, as *c*: And if the Angles are right, and the Sides are all equal, 'tis called a *Square*, as *d*.

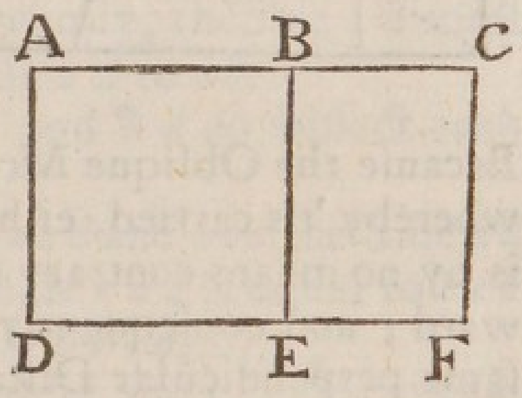
4. If

4. If a Parallelogram hath all its Sides equal, but ts Angles unequal, then 'tis called a *hombus*, as *e*.



5. If a Paralelogram hath neither s Angles nor Sides all equal, 'tis cal-ed a *Romboides*, as *a*.

The Generation of all Paralelogramick Figures vill be eaſily conceived, you ſuppoſe the *Deſcri-ent* A C, to be carried r moved along the Di-gent A D, in a Poſition ways parallel to itſelf its firſt Situation. or then, if the Angle A hich the Deſcribent



akes with the Dirigent, be a right one, and A B e equal to A D: The Figure produced will be a quare. If A C be longer or ſhorter than A D, e Figure will be an Oblong or a Rectangle.

If the Angle at A be oblique, only a Parallelo-ram at large will be deſcribed: Which when the eſcribent is equal to the Dirigent, the Figure will e a *Rhombus*; if unequal to it, a *Rhomboides*.

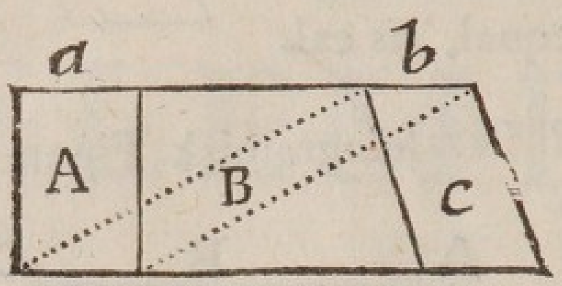
CORALLARIES.

I. Hence 'tis natural to ſuppoſe, that equal Lines oving thro' the ſame or equal Spaces, will de-ribe equal Surfaces.

II. Equal Lines, with uniform or equable Motions *. e.* being neither accelerated nor retarded) in e-ual Times, will deſcribe equal Surfaces: And if ey do thus deſcribe equal Surfaces, it muſt be in qual Times.

III. Hence

III. Hence alſo, if the Line *a* in a given Time deſcribe the Parallelogram A and the equal Line *b* in the ſame Time deſcribe the Oblique Parallelogram B or C, whoſe Perpendicular Altitude is the ſame with that of A: Thoſe Parallelograms will be all three equal one to another.

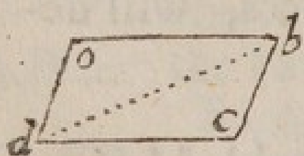


Becauſe the Oblique Motion, which the Line *b* hath, whereby 'tis carried, either to the right or left Hand, is by no means contrary to the direct Motion downward; and conſequently, the Line *b* will move the ſame perpendicular Diſtance in the ſame time, with an equable Motion, whether the latter Motion be impreſſed upon it or not. Wherefore,

IV. All Parallelogramick Figures, with equal Baſes and equal Perpendicular Altitudes, muſt be equal.

6. In every Parallelogram, the oppoſite Angles are equal. Let the Parallelogram be *o c*: I ſay, the Angle *o*, is equal to *c*; for the Angle *o* is equal to the Alternate one *b* (1. 31.) and the External one *b* is equal to the Internal one *c* (1. 31.) wherefore *o* is equal to *c*.

7. A Line, as *d b*, drawn acroſs the Figure from Angle to Angle, is called the Diagonal, and by ſome, the Diameter.



8. Every Parallelogram is divided into two equal Parts by the Diagonal. The Diagonal *b d* divides the Parallelogram *o c*, into the two equal Triangles *o b d* and *b c d*. For, 1. The Angle *o* is equal to *c* (3. 6.) 2. The

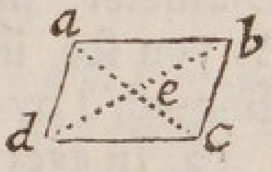
2. The Angle *o b d* is equal to *c d b* (1. 31.) and the Side *b d* is common to both these Triangles; wherefore the Triangle *o b d* is equal to *c d b* (*by* . 14.)

9. In every Parallelogram, the opposite Sides are always equal.

For (*drawing the Diagonal* d b) the whole Triangle *d o b* will be equal to the Triangle *b c d*, by the foregoing *Prop.* And consequently, the Side *c d* must be equal to *o b*, and the Side *o d* to *c b*.

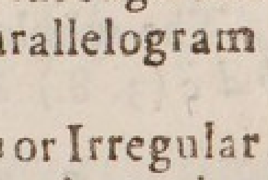
10. Two Diagonals, *a c* and *b d* do, bissect each other in the middle at *e*.

For in the two Triangles *a e d* and *b e c*, the Side *a d* is equal to *b c* (3. 9.) The Angle *e a d* is equal to *e c b* (1. 31.) and moreover the (*Vertical*) Angles *a e d* and *c e b* are equal also (1. 23.) *Wherefore the whole Triangle* a e d *is respectively equal to the Triangle* b e c (2. 14.) And consequently, the Side *d e* is equal to *e b*, and the Side *a e* to the Side *e c*. The two Diagonals therefore bissect each other in the middle. Q. E. D.



11. Every Right Line, as *f g*, passing through the middle of a Diagonal, divides the Parallelogram into two equal Parts.

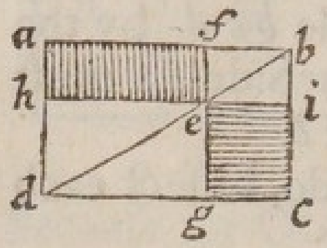
To demonstrate which, the *Trapezium* or Irregular Quadrilateral Figure *f g d a* must be proved equal to the *Trapezium f g c b.* And that is thus done. 1. The Triangle *b e f* is equal to the Triangle *d e g*: For the Side *de* is equal to *e b* by the Supposition; and the Angle *e f b* is equal to *e g d* (1. 31.) and the opposite Angles at *e* are equal; wherefore the Triangle *e f b* is equal to *e d g* (2. 14.) 2. The great Triangle *a b d* is equal to *b d c* (3. 8.) wherefore if from the Triangle *a b d* you take away the little Triangle *f e b*, and instead of it put the Triangle *e d g*



(*which*

(*which is equal to* f e b) you will have the *Trapezium* *f a d g*, which will be equal to the Triangle *a d b*: That is, to juſt one half of the whole Parallelogram (3. 8.) which was to be proved.

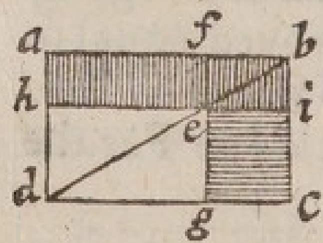
12. If in the Diagonal *d b* you take a Point as *e*, and thro' it draw two Lines *h i* and *f g* parallel to the two Sides of the Parallelogram, it will be divided by them into four leſſer *Parallelograms*, i. e. *f i*, *h g* (which two are called the *Parallelograms about the Diameter*) and *a e*, *e c*; which other two are called the *Complements*. And thoſe two Complements with either of the Parallelograms about the Diameter, make a Figure that is called a *Gnomon.* As you ſee in the Figure, where the *Gnomon* is diſtinguiſh'd by being ſhaded.



13. In every Parallelogram the Complements are equal. We muſt prove that *e a* is equal to *e c*.

DEMONSTRATION.

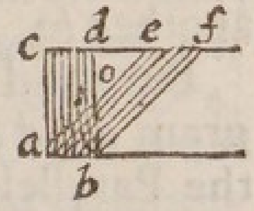
The whole Triangle *a b d* is equal to the whole *b d c* (3. 8.) And the little Triangle *e f b* is (for the ſame Reaſon) equal to *e b i*. And the Triangle *h e d* is alſo (by the ſame) equal to *e d g*. Wherefore if, from the two equal Triangles *a b d* and *b d c*, we take away equal things, *viz*. if from one we take away *e f b* and *d h e*, and from the other *e b i* and *e g d*, there will remain on one Side the Paralellogram *e a*, equal to the Parallelogram *e c*, which remains on the other; which was to be proved.



14. Parallelograms having the ſame Baſe, and being between the ſame Parallels, are equal.

Let

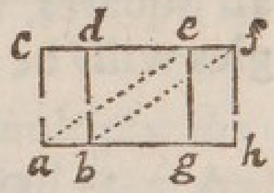
Let there be a Parallelogram *b c*, and another *a f*, ‹oth on the ſame Baſe *a b*; and let the ‿ine *c d*, when produced, be ſuppoſed ‹o paſs by *e f*; ſo that the two Paralle-‹ograms ſhall be between the ſame Pa-‹allels, and terminated by them; that ‹s, between the two Parallels *c f* and *a b*. I ſay then, ‹hat the Paralellogram *c b* is equal to *a f*.



For *c a* is equal to *b d*, and *a e* equal to *b f*, be-‹auſe oppoſite Sides of Parallelograms, and the An-‹les at *c* and *d* equal (by 29. 1.) wherefore the Tri-‹ngle *c a e* is equal to the Triangle *d b f*. Now if ‹rom each of theſe equal Triangles be taken the lit-‹le Triangle *d o e*, and to the Remainders be added ‹he Triangle *a o b*, the Parallelogram *a d* will be e-‹qual to the Parallelogram *a f*. Q. E. D.

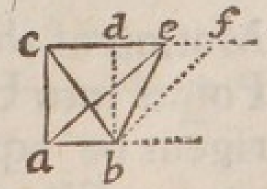
15. Parallelograms on equal Baſes *a b* and *g h*, and ‹etween the ſame Parallels *a h* and *c f*, are equal.

For if we imagine the third Parallelogram *f a* to ‹e drawn; that ſhall be equal to the Parallelogram ‹a d*, becauſe on the ſame Baſe *a b* with ‹t, and between the ſame Parallel ‹Lines *a h* and *c f*. And that Paralle-‹ogram will alſo be equal to *e h*, be-‹cauſe it hath the ſame Baſe *e f* with ‹t (it matters not whether you reckon ‹he Baſe above or below) and it is between the ſame ‹Parallels. Therefore *h e* and *b c*, being both equal ‹to the third Parallelogram *f a*, muſt be equal to ‹each other.



16. *Triangles* on the ſame Baſe *a b*, and being be-‹tween the ſame Parallels *c f* and *a b*, ‹are always equal.

The Triangle *a b c* is equal to *a e b*: ‹Becauſe if you imagin a Line *bd* drawn ‹parallel to *a c*, and another as *bf*, drawn ‹parallel to *a e*; there will be made two Parallelograms



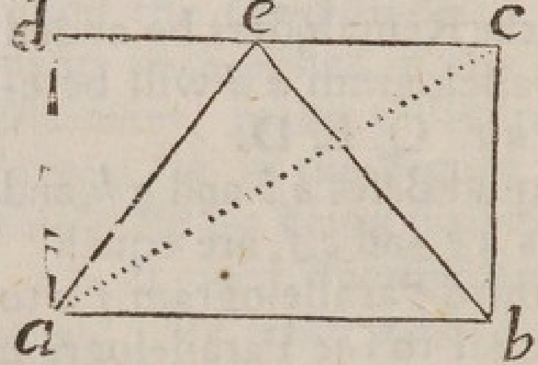
a c d b,

a c d b and *a e f b*; which being on the ſame Baſe, and between the ſame Parallels, will be equal to one another (3. 14.)

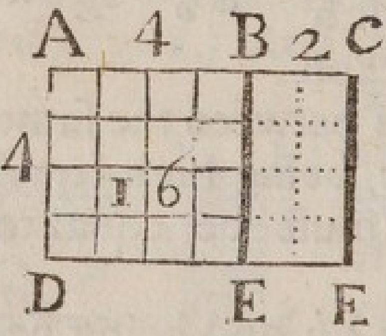
But the Triangle *a b c* is the half of the Parallelogram *a c d b*, and the Triangle *a b e* is the half of the Parallelogram *a e f b* (3. 8.); wherefore, (*ſince the Wholes are equal, the Halves muſt*) and conſequently the Triangle *a c b* is equal to the Triangle *a e b*.

17. Triangles on *equal Baſes*, and between the ſame Parallels, are alſo equal; as is very eaſy to prove from (3. 15.)

18. If a Triangle *a c b* have the ſame Baſe with a Parallelogram, and be alſo between the ſame Parallels, it ſhall be juſt the half of that Parallelogram. For it will ſtill be equal to *a b c*, which is juſt half (3. 8.) of *a b c d*.



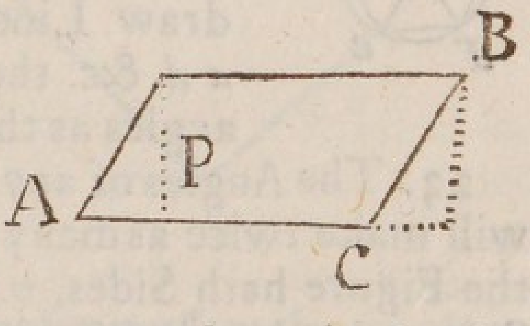
The Menſuration of all Squares, Rectangles, Parallelograms and Triangles will be underſtood from what hath been deliver'd above. If you ſuppoſe,



1. That the Deſcribent A B or A C, before its motion, be divided into any determinate Number of equal Parts; and the Dirigent (now ſuppoſed to ſtand at Right Angles with it) into the ſame or any other Number of ſuch Parts; for then the Motion of the Deſcribent Line, thus mark'd out by Points into Units will deſcribe a Square (if the Dirigent be equal to it) and a Rectangle if it be unequal. Which Square or Rectangle will be divided into as many little Squares as there are Units in the

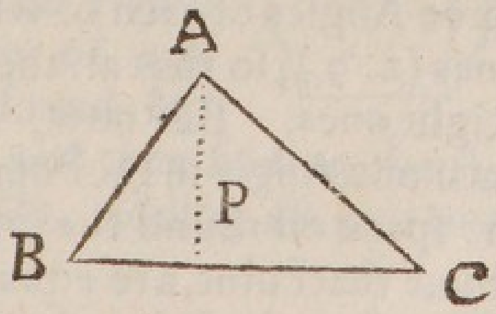
ne Product of the Number of the Divisions, or equal arts in one Line, multiply'd by those in the other; That is A B 4 multiply'd by A D 4, produces 16, ne Square of 4. And A C 6 multiply'd by A D 4, roduces 24; the Rectangle under A C and A D. o that what is a Product in Numbers or in Arithmetick; in Lines, or in Geometry, is called a Rectangle. And therefore you will find the Latin Writers of Geometry, when A C is to be multiply'd by D, not saying *Multiplica*, but *Duc* A C in A D. That is, carry the Line A C along the Dirigent A D, a Normal Position to it, till it come to end, and nen it will form the Rectangle A F = 24; whereore the Area of a Square is found, by multiplying e Side A B into itself.

The Area of any Rectangle, as A F, is found by ultiplying the Side A C y A D.



And since a Rectangle n the same Base and of e same Altitude with a arallelogram is equal to ; to find the Area of ny Parallelogram, as A B, you must multiply the de A C by a Perpendicular, as P, let fall from the her Side to it.

And since every right-lin'd Triangle is the half a Parallelogram or ectangle of the same ase and Altitude: To nd the Area of the Trigle A B C, you must ultiply any Side, as B C, aPerpendicular, as P, t fall to it from an oppoe Angle, and take half the Product: or if either P B happen to be even Numbers, multiply one by



$\frac{1}{2}$ of

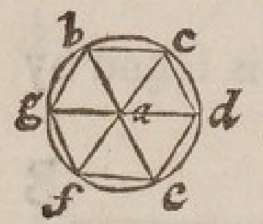
$\frac{1}{2}$ of the other, the Product is the Area of the Triangle.

19. A *Pentagon* is a Figure having five Sides and five Angles.

If all the Sides are equal, and consequently the Angles, 'tis called a *Regular Pentagon.*

20. An *Hexagon* is a Figure of six Sides and Angles, an *Heptagon* of seven, an *Octagon* of eight, *&c.* which are all called *Regular* when they have equal Sides and Angles.

21. A *Polygon* in general signifies any Figure of many Sides and Angles; but no Figure is called by this Name, unless it have more than four or five Sides.



22. Every *Polygon* may be divided into as many Triangles as it hath Sides, if any where within the *Polygon* you take a Point, as *a*, and from thence draw Lines to every Angle *a b*, *a c*, *a d*, &c. they shall make as many Triangles as the Figure hath Sides.

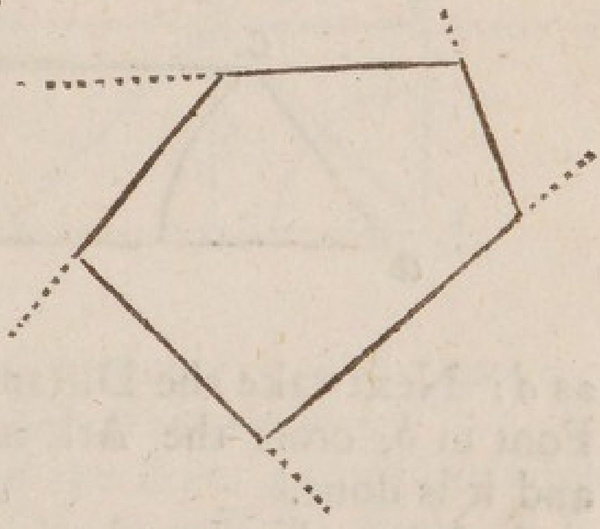
23. The Angles of any *Polygon* taken all together, will make twice as many Right ones, except four, as the Figure hath Sides, *v. gr.* If the *Polygon* have six Sides, the double of that is 12; from whence take four, there remains eight. I say, that all the Angles of that *Polygon*, viz. *b*, *c*, *d*, *e*, *f*, *g*, taken together, are equal to eight Right Angles. For the Lines *a b*, *a c*, *a d*, *&c.* do divide the Figure into six *Triangles*; the three Angles of each of which are equal to two Right ones (2. 9.); so that all their Angles together make 12 Right ones. But now, each of these six *Triangles* hath one Angle in the Point *a*, and by it they compleat the space all round the said Point. And all the Angles about that Point, are equal to four Right ones (1. 22.) Wherefore those four being taken from 12, (*The Sum of the Right Angles of all the six Triangles*) leaves eight, the Sum of the Right Angles of the *Hexagon*, which

which make 8 times 90, or 720 Degrees; and therefore each Angle must be $\frac{1}{6}$ of that, *viz.* 120 Degrees.

So that the Figure hath plainly twice as many Right Angles as it hath Sides, except four; which was to be proved.

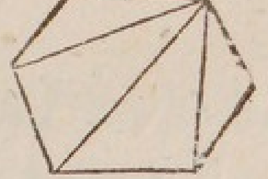
COROLLARY.

All the external Angles of any Right-lined Figure, are equal to just four Right Ones: For drawing out the Sides, as in the Figure, 'tis plain the internal and external Angles together will make twice as many Right Ones as the Figure hath Sides; but the internal Angles are equal to all those, except four (by this *Prop.*)



Wherefore the external Angles must make up these four, and no more.

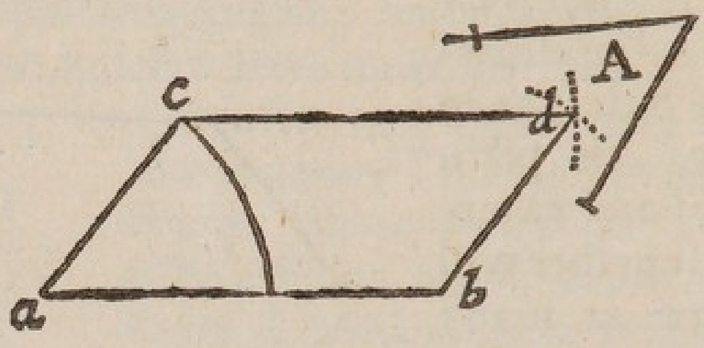
24. A *Polygon* may be divided also into *Triangles*, by drawing Lines from Angle to Angle. But then the Number of the Sides will exceed that of the *Triangles*. And hence the *Area* of any Right-lined Figure may be found, by reducing it into *Triangles*, and then finding the Area of each *Triangle* severally, adding all into one Sum.



PROBLEM I.

On a given Line a b, *to make a Parallelogram, having an Angle equal to a given Angle* A.

Make the Angle *c a b* = A. Then take *a b* in your Compaſſes, and ſetting one Foot in *c*, ſtrike an Ark



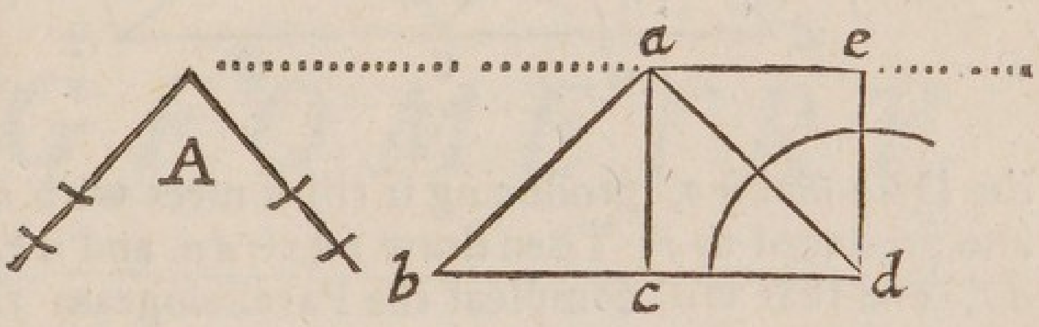
as *d*: Next take the Diſtance *a c*, and placing one Foot in *b*, croſs the Ark in *d*: draw *c d* and *d b*, and it is done.

And thus alſo may the Line *c d* be drawn parallel to *a b*, thro' a Point aſſigned, and any Parallelogram readily be deſcribed.

PROB. II.

A Triangle a b d *being given, to make a Parallelogram equal to it, which ſhall have a given Angle equal to* A.

Biſſect the Baſe of the Triangle in *c*: Make the Angle *c d e* = A, thro' the Vertex *a* draw *a e* parallel to the Baſe *b d*. Make *a e* = *c d*, and draw *a c*. So will *c e* be the Parallelogram required.



For being on but half the Baſe, and of the ſame Height with the Triangle, it will be equal to it, by the 18*th* of this Book, and its Angle *c d e* is equal to A. Q. E. F.

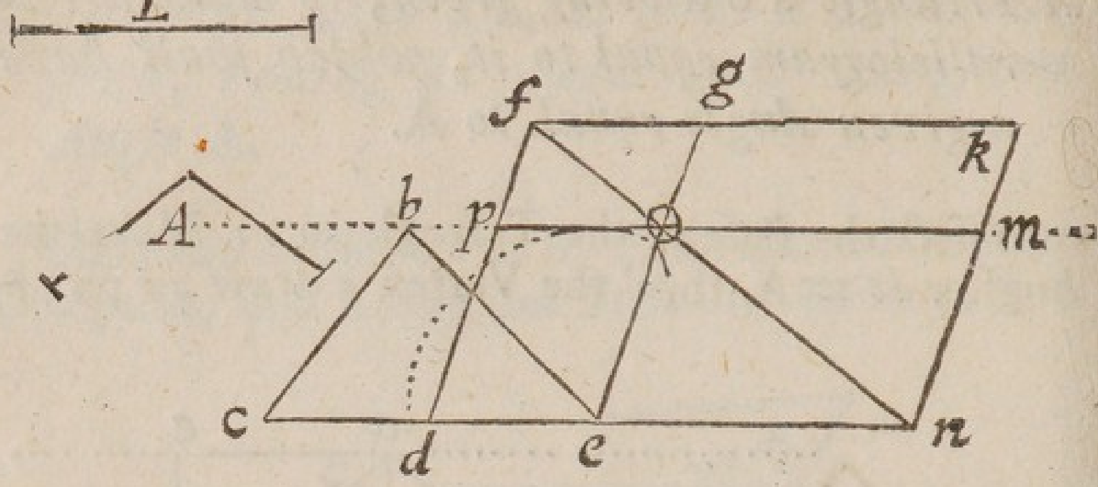
PROB. III.

On a Line given, as L, *to make a Parallelogram equal to a given Triangle* c b e, *and having an Angle equal to an Angle given, as* A.

Make the Parallelogram *d o* equal to the Triangle, and having its Angle *e* = A, by Problem the laſt.

 Then

Then produce *p o*, till *o m* become equal to L, and draw out *d e*, till *e n* be alſo equal to L. Then draw



the Diagonal *n o*, producing it till it meet with *d p*, alſo produced to *f*. Then draw *f k* = *d n*, and *n k* = *d f*, and that will compleat the Parallelogram *f n*; in which the Complement *g m* will be equal to *p e*, (3. 13.) which is equal to the Triangle *c b e*. Q.E.F.

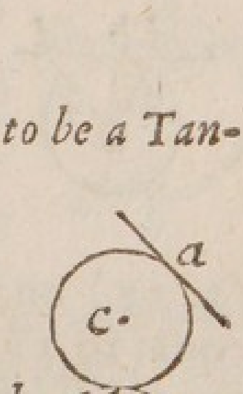
And thus 'tis eaſie to make a Parallelogram equal to any Right-lined Figure given, by reducing that Figure into Triangles, &c.

ELEMENTS
OF
GEOMETRY.

BOOK IV.

Of a Circle.

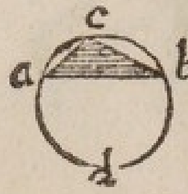
A Line is ſaid to *Touch (or to be a Tangent to)* a Circle, when, though produced both Ways from the Point of Contact, it will only touch it, and not t or enter within it. Thus the Line touches the Circle C, as that Circle doth the Circle D; but *d* enters ithin the Circle, and *cuts* it, *and is called a cant.*



2. If a Right Line enter within a Circle and cut into two Parts, thoſe Parts are called *Segments* : *b* a leſs Segment, and D a greater: That *Part of the*

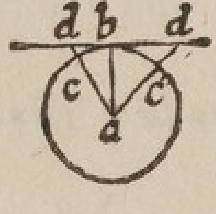
Line cutting the Circle *(and which is within it)* is called a *Chord* as *e f*. And the Parts of the Circle *(or rather Circumference)* cut off, are call'd *Arks*: The Chord with the Ark makes two mix'd Angles, as *e* and *f*. and they are call'd *Angles of a Segment*.

3. If you take a Point, as *c*, in the Ark of any Segment, and from thence draw two Lines *c a* and *c b* (*to the Ends of the Chord*) they ſhall make an Angle *a c b*; which is call'd *an Angle in a Segment*: And that Angle *a c b* is ſaid to *inſiſt* or *ſtand* on *a b d*, the Ark of the other Segment below.

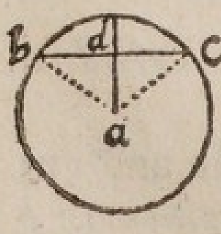


4. A Sector of a Circle is a mix'd Triangle comprehended between two Radii, *a b*, *a c*, and the Ark of the Circle *b c*; 'tis mark'd in the Figure by being ſhaded.

5. If at the End of any Radius, or Semidiameter, *a b*, you draw a Perpendicular, as *d b*, it ſhall touch the Circle but in one Point. And all the Points of the Line *b d* ſhall be without the Circle, *v. g.* I ſay, the Point *d* (or any other aſſignable) is without: For if you draw the Line *a d* from the Center, *and that* ſhall cut the Circle in the Point *c*, that Line *a d* will be longer than *a b*; (2. 17.) and conſequently longer than *a c*, which is equal to *a b* (1. 14.) Wherefore the Point *d* is without the Circle. Q. E. D.



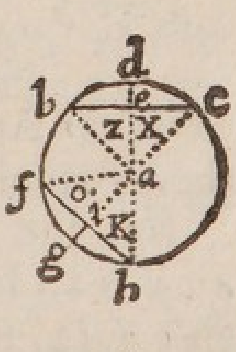
6. A *Chord*, as *b c*, is divided into two equal Parts (or biſſected) by a Perpendicular *d a*, drawn from the Center *a*. For the Triangle *a b c* is an *Iſoſceles*, becauſe *b a* is equal to *c a* (1. 14.) and therefore the Perpendicular *a d* biſſects the Baſe *b c* (2. 16.) The Ark *b c* is alſo by this means biſſected.



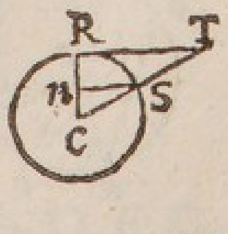
7. Two

7. Two Tangents, *c b* and *c d*, drawn from the same Point without a Circle, are equal one to another. For, draw from the Center to the Points of Contact, *b a* and *a d*; Then will those Lines be Perpendiculars to the Tangents (by 4. 5.) Then if you draw also the Line *b d*, the Angle *a b d* will be equal to *a d b* (2. 15.) Wherefore if from the Right, and (*consequently*) equal Angles *c b a* and *c d a*, you take away the equal One *a b d* and *a d b*, the remaining Angles *c b d* and *c d b* will be equal: Wherefore their opposite Sides must also be equal *by the Converse of* (2. 15.) That is, *c b* is equal to *c d*. Q. E. D.

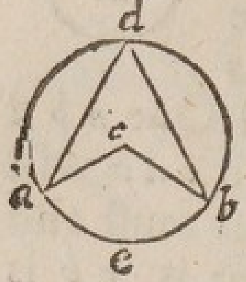
8. Equal Chords, as *b c* and *f h*, do cut off equal Segments *b d c* and *f g h*. And the Perpendiculars *a e* and *a i*, drawn to them from the Center, are also equal, as is easily proved; (saith *Pardie, but he gives us no Demonstration.*) *Yet 'tis plainly thus proved; The Chords and Arks are both bissected by the Perpendiculars* (4. 6.) *And therefore the Sectors* c a d, d a b, f a g *and* g a h, *must be all equal; as also will all the Triangles* x, z, o *and* k, *by* (2. 11.) *Therefore their Doubles will also be equal,* i. e. *The Sector* b a c *will be equal to* f a h: *And the Triangle* b a c *to the Triangle* f a h. *And if these last Triangles are taken from the equal Sectors* h a f *and* b a c, *the Segments* b c d *and* h g f *must remain equal. That the Perpendiculars are equal, is plain from the Equalities of the Triangles* z *and* o, *or* X *and* K.



9. Let there be a Semidiameter R *c*, and a Perpendicular (*to it without the Circle*) R T, another Line cutting the Circle in S, and a Perpendicular (*let fall from thence*) to the Radius R C in *n* (*a Point within the Circle.*) All these Lines have Artifi-

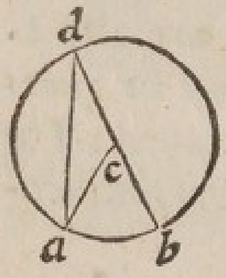


cial Names. The Line TR is called the Tangent of the Ark RS (*which ſuppoſe* 30°) TC is called the Secant of the ſame Ark of 30°, and the Line S *n* is called the (Right) Sine of the ſame Ark. RC is by ſome called the whole Sine, but moſt uſually the Radius. And *n* R is called the Verſed Sine of the ſame Ark.



10. If in the Circumference of a Circle, you take two Points, as *a* and *b*, and from thence draw two Lines to the Centre *c*, and two others to any Point, as *d* in the Circumference; they will make two Angles, of which *a c b* is called an *Angle at the Center*, and *a d b* an *Angle at the Circumference.*

11. The Angle at the Center *a c b* is always double to one at the Circumference *a d b* (*inſiſting with it on the ſame Ark* a b.)

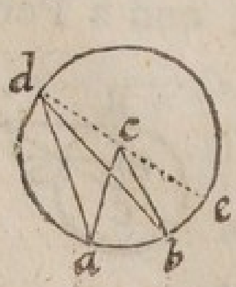


Of which there are three Caſes.

I. If one of the Lines, as *d b*, paſs thro' the Center *c*, then 'tis plain the external Angle *a c b* (2. 10.) will be equal to both the internal and oppoſite Ones *a* and *d* taken together.

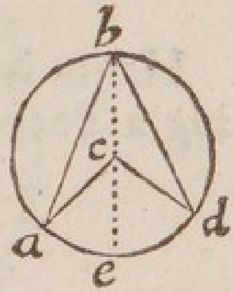
But the two Angles *d* and *a* are equal, becauſe *a c d* is an Iſoſceles Triangle, whoſe Side *a c* is equal to *c d* (2. 15.) Therefore the Angle *c* at the Center being equal to both, is double of either alone: That is, double to *d*. Q. E. D.

II. If neither of the Lines *d b*, *d e* (*which form the Angle at the Circumference*) paſs thro' the Center *c*: (*But fall both on the ſame Side of the Diameter*) Let the Diameter *d c e* be drawn. Then will the whole Angle *a c e* (*at the Center*) be double to the Angle *a d e* (at the Circumference)



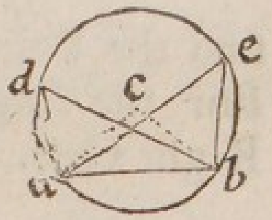
umference) by what was proved in the first Case. lso the Angle *b c e* is double to *b d e*, by the same. Wherefore if from the Angle *a c e*, we take away hat *b c e*, and from the Angle *a d e*, which is the half f *a c e*, we take away *b d e*, which also is the half f *b c e*, the remaining Angle *a d b* must be just the alf of *a c b*. For 'tis as plain as an Axiom, that if ne Quantity be double to another, and you take way from the Bigger, just the Double of what you ake from the other, the Remainder of the Bigger must be double to the Remainder of the Lesser.

III. If the Diameter fall between the Lines forming the Angle at the Circumference: Then will, as before, the Angle *a c e* be double to *a b e* (*by Case* 1. *of this*) and the Angle *e c d* will be double to *e b d* by the same; therefore the whole Angle *a c d* must be double to *a b d*. (*So that in all Cases the Angle at the Center is double to one at the Circumference, if they both stand on the same Ark, or (which is all one) are in the same Segment.*

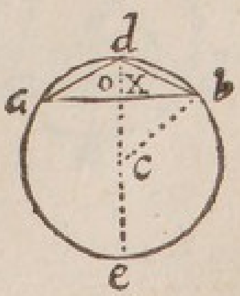


12. All Angles (*in the same Segment or*) insisting on the same Ark *a b*, are equal, let them terminate in any Part of the Circumference whatsoever.

For the Angle *a d b* will be equal to *a e b*, because each is the half of the Angle at the Center *a c b* (4. 11.)

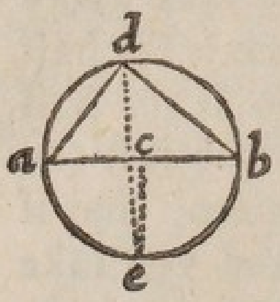


13. An Angle at the Center *b c e*, standing on half of the Ark *a e b*, is equal to the Angle *a d b* at the Circumference, standing on the whole Ark, (*for* c *is equal to twice* x; (by 4. 11.) *and* x *is equal to* o, *that is to half* a b d, (4. 6. and 4. 8.) *Wherefore* c *is equal to* a d b. Q. E. D.



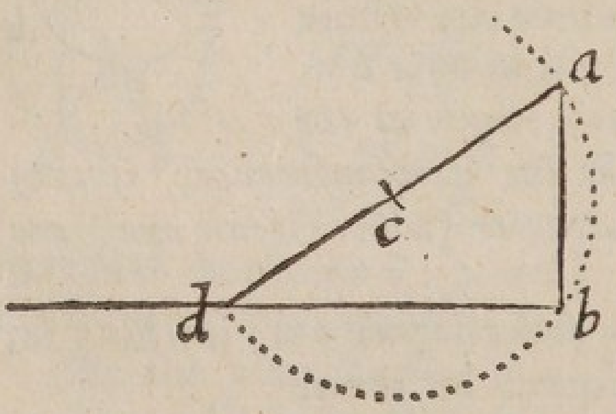
14. The

14. The Angle *a d b* ſtanding on the Semi-circumference *a e b* (*or being in the Semi-circle* a d b) is a Right One. Let *c e* be drawn biſſecting the Semi-circumference *a e b*; then is (by the *Precedent*) the Angle *a c e* at the Center, ſtanding on half a Semi-circle (*or on a Quadrant*) equal to *a d b* at the Circumference, which ſtands on twice that Ark, or on a Semi-circle. But *a c e* is a Right Angle; wherefore *a d b* (it's equal) muſt be ſo too.



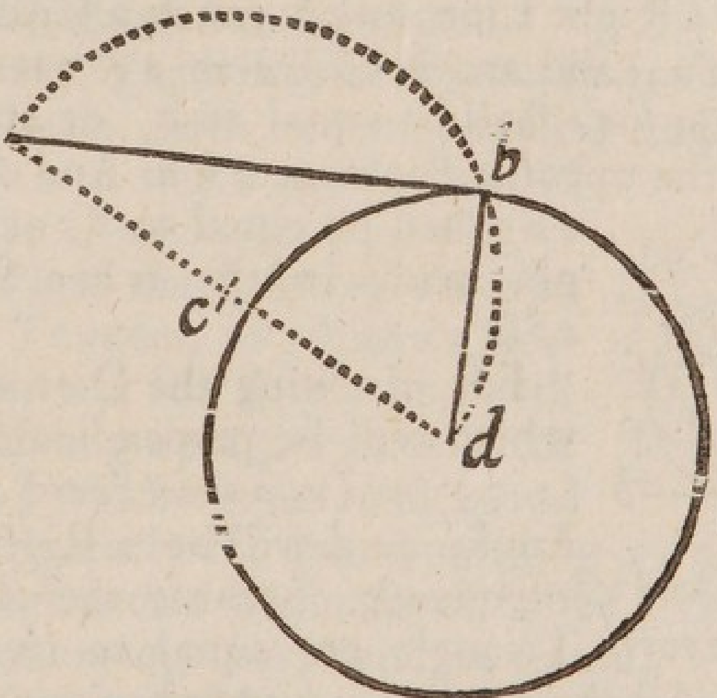
COROLLARY I.

Hence is derived the Common Practice of Erecting a Perpendicular, as *a b*, at the End of a given Line. For opening the Compaſſes to any convenient Diſtance, ſet one Point in *c*, and with the other draw the Ark *d b a*, cutting the given Line in *d*; then a Ruler laid from *d* to *c* ſhall find the Point *a*, which is perpendicularly over *b*: For the Angle *d b a*, being in a Semi-circle, is a Right One.



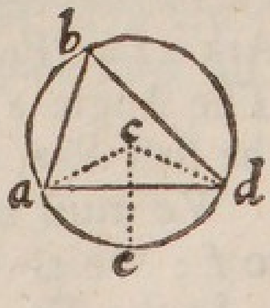
COROL. II.

Hence alſo ariſes this expeditious Practice of rawing from a Point given, as *a*, a Tangent, as *a b*, to given Circle. For joining the Points *a* and *d*, the



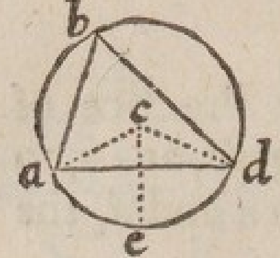
Center of the Circle, biſſect their Diſtance *a d* in the Point *c*: On *c*, as a Center, deſcribe the Semi-circle *b d* : So ſhall *a b* be a true Tangent, becauſe the Angle *a b d* being in a Semi-circle, is a Right One.

15. The Angle *a b d* in a Segment leſs (*than a Semi-circle*) is Obtuſe : Becauſe the Ark *a e d* being more than half the Circumference, its half, the Ark *a e*, muſt be more than 90°; therefore the Angle *a b d*, which is equal to *a c e*, (4. 13.) muſt alſo be more than 90°, that is obtuſe.



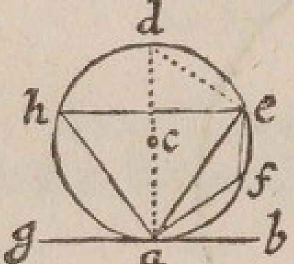
16. The

16. The Angle *a b d* made in a Segment greater than a Semicircle, is *Acute.*



For 'tis equal to the Angle *a c e* (4 13.) whose Measure *a e* being the half of *a e d*, an Ark less than a Semicircle, must be less than 90°. And therefore *a b d* is less than 90°. (*i. e.*) *Acute.*

17. If a Right Line, as *g b*, touch a Circle, as in the Point *a*; and another Line as *a e* cut it there. The Angle *b a e* shall be equal to *h*, or any Angle made in the opposite Segment *a h e*. And the Angle *e a g* shall be equal to *f*, or any Angle made in the other Segment, *e f a*.



For, drawing the Diameter *a d*, which will be perpendicular to *a b*, (4. 9.) (*and also the Line* d e:) The Angle *a e d* will be a Right One ; (4. 14) And consequently, because the three Angles of every Triangle are equal to two Right Ones, (2. 9.) the Angle *e a d*, together with *d*, must make just another Right Angle.

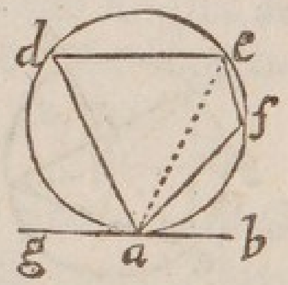
But that Angle *d a e*, together with *e a b*, doth make also a Right One, because the Radius *c a* is perpendicular to the Tangent *a b*; wherefore *take away* e a d *from both*, and then *e a b* will remain equal to *d*; and consequently to *h*, or to any other Angle *in that Segment* a h e, or that stands on the same Ark *e f a*: For all those Angles are equal (by 4. 12.) The Angle *e a b* therefore is equal to *h* ; which is the first Part of the Proposition.

We must next prove the Angle *g a e* to be equal to *f* ; which is the other Part.

In the Triangle *a f e*, all the three Angles *e*, *f* and *a*, are equal to two Right Ones (2. 9) And the Angle *c* is equal to *f a b*, by the first Part of this Proposition

position, for fa may be consider'd as cutting the Circle in the Point a, where ab touches it, and consequently fab will be equal to any Angle that can be made in the opposite Segment $abdef$; and therefore to e. Now the two Angles eaf, and fab (that is e) together with f, are equal to two Right Ones, (2. 9.) and so are af, and fab taken together with gae (1. 20.) Wherefore the Angle f is equal to gae. Which was to be proved.

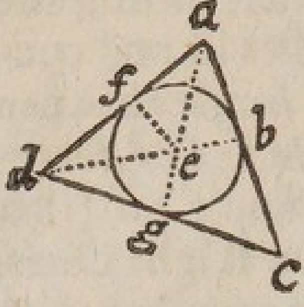
18. Every Quadrilateral Figure, as $defa$, inscribed in a Circle, hath its two opposite Angles taken together (as d added to f) equal to two Right Ones.



For if thro' the Point a, there be drawn a Tangent, as gb, and a Diagonal, as ea; the Angle at f will be equal to gae (4. 17.) and the Angle eab will be equal to d (4. 17.) And consequently the two Angles, gae and eab, being equal to two Right Ones (1. 20.) the Angles d and f taken together, must be so too.

After the same Manner might the other two opposite Angles, daf, and def, be proved equal to two Right Ones, by drawing another Tangent thro' the Point f.

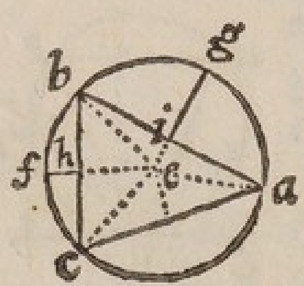
19. The Converse of this Proposition is also manifest; *viz.* That if any Quadrilateral Figure have its opposite Angles equal to two Right Ones, it may then be inscribed in a Circle; that is, a Circle may be made that shall touch or pass thro' all its four angular Points.



20. A Rectilineal Figure is said to be circumscribed about a Circle, when all its Sides touch the Circle without cutting it. Thus the Triangle *d a e* is circumscribed about the Circle *b g f*; because every Side of the Triangle touches the Circle in *b*, *g* and *f*.

21. A Figure is said to be *Inscribed* in a Circle when all its Angles are in the Circumference of that Circle, as the Triangle *a b c*, in the following Figure.

22. Every Triangle, *a b c*, may be inscribed in a Circle; for if two Lines, as *e h* and *e i*, are drawn perpendicularly bisecting the Sides *b a* and *c b*, they will cross or meet each other in the Point *e*, on which, as on a Center, a Circle may be drawn, which shall pass through *b*. And I say also, that that Circle shall pass through *a* and *c*.



For I. The two Triangles *e i b* and *e i a* are equal; because *i b* is equal to *i a* by the Supposition, the Side *e i* is common to both, and the Angles at *i* are Right. Wherefore the Side *e b* is also equal to *e a* (2. 11.)

II. And for the same Reason *the Triangles* e h c *and* e h b *may be proved equal*, and consequently, the Side *e c* also will be equal to *e b* and to *e a*. But if those three Lines are all equal, the Point *e*, where they meet, must be the Center of a Circle of which they are *Radii*: And therefore the Triangle is circumscribed by a Circle. Q. E. D.

And thus may a Circle be made to pass through any three Points, if they be not all in a Right Line.

23. Every

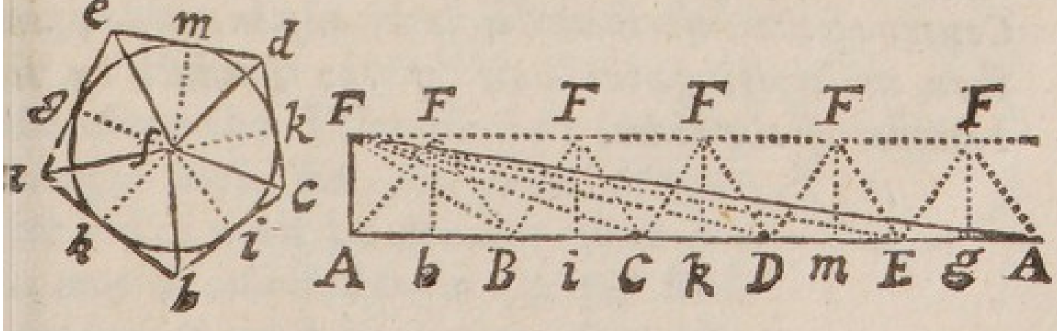
23. Every Triangle may (*have a Circle inscribed* *it, or*) be circumscribed about one. *Vid. Fig. in* *rt.* 20.

For drawing the Lines *a e* and *e d*, bissecting the ngles *a* and *d*, and from the Point *e*, where they ross, letting fall the Perpendiculars (to the Sides the Triangle) *e b*, *e f* and *e g*; I say, that if you raw a Circle on the Center *e* through *b*, that Cir- e shall touch all the Sides of the Triangle in the oints *b*, *f* and *g*.

For I. The two Triangles *a e f* and *a e b* are e- ual, as having the Side *a e* common, the Angles at and *b* Right, and those at *a* equal (by the Suppo- tion:) Wherefore *e b* is equal to *e f*. (2. 14.)

II. By the same Method *e g* may be proved equal so to *e f*, (that is to *e b*) so that these three Lines eing all equal, a Circle will pass through their three xtremities, of which Circle they will be Radii, and eing also all perpendicular to the Sides of the Tri- ngle, the said Sides are Tangents to that Circle (4.) and therefore do circumscribe it (by 4. 18.)

24. Every Polygon circumscribed about a Circle equal to a Rectangled Triangle, one of whose egs shall be the Radius of the Circle, and the o- her the Perimeter (or the Sum of all the Sides) of e Polygon.



Let

Let the Line FA be equal to the Radius *fh*, and to it, at Right Angles, draw the infinite Line ABCD, *&c.* out of which take A*h* equal to *a h*, *h* B equal to *h b*, B *i* equal to *b i*, and *i* C equal to *i c*, &c. So that the whole Line ABCDEA may be equal to the whole Compaſs, or *Perimeter* of the Polygon *a b c d e a*. Alſo draw FF parallel to AA, ſo that all the Perpendiculars, F*h*, F*i*, F*k*, &c. may be equal to the Radius *f h*, or *f i*, &c. 'Tis then plain, that the Triangle A F B will be equal to the Triangle *a f b* in the Polygon, and the Triangle B F C, to *b f c*; and alſo C F D, to *c f d*, &c. So that all theſe Triangles taken together, will be equal to all theſe in the Polygon, or to the whole Polygon.

But the Triangle F A A is equal to all the five Triangles within the Parallels; becauſe drawing the Lines, BF, CF, DF, *&c.* the Triangle FAB will be equal to FAB, FBC to FBC, *&c.* (3. 16.); wherefore the Triangle F A A is equal to the Polygon, which was to be proved.

25. Every regular Polygon is equal to a Rectangled Triangle, one of whoſe Legs is the Perimeter of the Polygon, and the other a Perpendicular drawn from the Center, to one of the Sides of the Polygon. The Proof of which is the ſame as that in the precedent Propoſition; For all the Perpendiculars *f h*, *f i*, *f k*, &c. are equal, *&c. See the laſt Figure.*

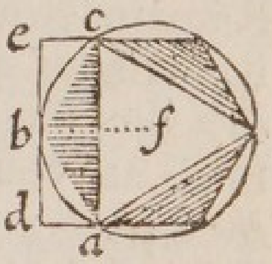
Wherefore the Area of every regular Polygon is found, my multiplying the Perpendicular let fall from the Center of the inſcribed Circle by any one Side; and then multiplying the half of the Product by the Number of the Sides.

26. Every

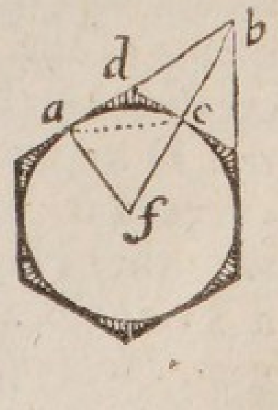
26. Every Polygon circumſcribed about a Circle, is bigger than it; and every Polygon inſcribed, is leſs than the Circle; as is manifeſt, becauſe the thing containing, is always greater than the thing contained.

27. The *Perimeter*, or (as ſome call it, tho' improperly) the Circumference of every Polygon circumſcribed about a Circle, is greater than the Circumference of that Circle; and the Perimeter of every Polygon inſcribed, is leſs.

28. If in any little Segment of a Circle, you inſcribe an Iſoſceles Triangle, as *a b c*; ſo that *a b* be equal to *b c*; I ſay, that Triangle ſhall be greater than half that Segment. For if you draw a Tangent *e b d*, which ſhall be parallel to *c a*; and which ſhall be, as *c a* is, perpendicular to the Radius *b f*; (4. 5.) (4. 6.) And then compleat the Rectangle *a d e c*; that Rectangle will be greater than the whole Segment *a c b*: But the Triangle *a b c*, is the half of that Parallelogram (3. 18.) And therefore muſt be greater than half the Segment *a b c*.



29. Let there be a Tangent *a d b*, a Secant *f c b*, a Chord *a c*, and another Tangent *c d*; I ſay, that the Triangle *d b c* is more than half the mixt Triangle *a c b*, comprehended between the Lines *a b*, *b c*, and the Ark of the Circle *a c*. For in the Triangle *d b c*, the Angle *c*, being a Right one (4. 5.) the Side *d b*, is longer than *d c* (2. 17.) That is, than *d a*; which is equal to *d c* (4. 7.) Therefore the Triangle *b d c* (*having a longer Baſe, but the ſame Height with* a d c) muſt be greater than it; *as may be collected from* (3. 7.) And therefore it muſt be greater than the



half of the whole Triangle *a c b*. But the Triangle *a c b*, is greater than the mixt Triangle, made by the Ark *a c*, and the Right Lines, *a b* and *a c*; and therefore the Triangle *b d c*, (which is more than half of *a c b*) muſt be greater than the half of the mixt Triangle *a b c*. Q. E. D.

30. From theſe two laſt Poſitions, it follows, that by multiplying the Sides of Polygons, you may make them ſo *circumſcribed* about, or *inſcribed* in Circles, that the Difference by which the circumſcribed exceeds, or the inſcribed wants of the Circle, ſhall be as ſmall as you will: Becauſe if from any Quantity whatever, you take more than the half, and from the Remainder more than its half, and again from that Remainder more than its half; you máy by doing this very often, at laſt come to leave a Remainder as ſmall as you pleaſe; as is ſelf-evident. Thus (*See the* 28*th Figure*) after a Triangle is inſcribed in a Circle that ſhall be leſs than it by the three great Segments, you may inſcribe an Hexagon that ſhall exceed the Triangle by thoſe three Segments, but ſhall be leſs than the Circle, by the ſix little Segments that are left white in the Figure.

But thoſe ſix white Segments taken together, do not contain ſo much Space as the half of the three former ſhaded ones, (4. 28.) After this you may alſo inſcribe a *Duodecagon*, which will be leſſer than the Circle by 12 ſmaller Segments; which 12 Segments will ſtill be leſs than the half of the ſix Segments of the Hexagon: And thus may you, by increaſing the Number of Sides of the Polygon, leſſen the Difference by which the circumſcribing Circle exceeds it, as much as you pleaſe. So likewiſe on the other Hand, you might have firſt circumſcribed a Triangle, then an Hexagon, and then a Duodecagon, &c. (*and have made*, that way, *the Difference between the circumſcribing Polygon and the Circle, as ſmall as you would.*)

31. Every Circle is equal to a Rectangled Triangle, one of whoſe Legs is the Radius, and the other a Right Line equal to the Circumference of the Circle. For ſuch a Triangle will be greater than any Polygon inſcribed, and leſs than any Polygon circumſcribed, (by 24, 25, 26, and 27 of this fourth Book.) And therefore muſt be equal to the Circle.

For ſhould it be greater than the Circle, be the Exceſs as little as it will, a Polygon may be circumſcribed, whoſe Difference from the Circle ſhall be yet leſs than the Difference between that Circle and the Rectangled Triangle, and that Polygon will be leſs than the Triangle, which is abſurd. And if it be ſaid that this Rectangled Triangle is leſs than the Circle, an inſcribed Polygon may be made, which ſhall be greater than that Triangle, which is impoſſible.

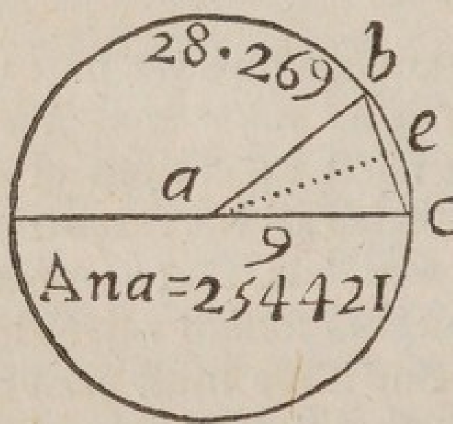
‘ This kind of Demonſtration, which we here uſe, ‘ and which is called *Reductio ad Abſurdum, ſive ad* ‘ *Impoſſibile*, is one of the fineſt Inventions of the ‘ Ancients: And on it is founded all the *Geometry* of ‘ *Indiviſibles*; ſo that I cannot but much wonder ‘ ſome of our Modern Authors ſhould reject it as ‘ indirect and deficient. But if we muſt arrive to ‘ ſuch a Point of Niceneſs, that we can't bear any ‘ Demonſtration, unleſs it be Direct and Poſitive; ‘ 'tis eaſy enough to give this before us ſuch a Turn, ‘ as ſhall render it Regular and Direct.

‘ For this cannot but be admitted as a Principle; ‘ *That if two determinate Quantities* a *and* b *are ſuch,* ‘ *that every other imaginable Quantity, which is great-* ‘ *er or leſs than* a, *is alſo greater or leſs than* b; *theſe* ‘ *two Quantities* a *and* b *muſt be equal.* And this ‘ Principle being granted, which is in a Manner ſelf-‘ evident, it may directly be proved that the Triangle ‘ (before-mention'd) is equal to the Circle: Be-‘ cauſe every imaginable inſcribed Figure, which is 'less than the Circle, is also less than the Triangle: 'And every circumscribed Figure greater than the 'Circle, is also greater than the Triangle.

This is that which is called the Quadrature of (or squaring) the Circle, which consists in finding a Square, Triangle, or any other Rectilineal Figure exactly equal to a Circle. And this would easily be done, could we find a Right Line equal to the Circumference; as is plain from this last Proposition. But such an Equality is not to be found Geometrically.

To find the Area of a Circle.

Since the Circle is equal to a Right-angled Triangle, whose Base is the Radius, and the Perpendicular a Line equal to the Circumference; half the Product of the Radius into the Periphery, will give the Area of the Circle.



In Practice, therefore say, either as 7 : to 22 : : So is the Diameter in Inches equal Parts, &c. to the Circumference, or more nearly and without Division, say, as 1000 is to 3141 : : So is the Radius of any Circle in Inches (suppose 9 Inches) to 28.269, which therefore will be Semi-circumference: And this multiply'd by 9 Radius, gives 254.421, for the Area required.

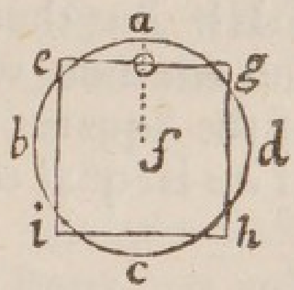
For the Area of a Sector or Segment of any Circle.

Since a Circle may be conceiv'd as an Aggregate of an infinite Number of Isosceles Triangles, whose common Vertex is the Center; any Portion of the Periphery, as *b c*, being considered as a strait Line,

and

nd the Perpendicular *a e* let fall, the Area of the ector must be half the Product of the Ark *b c* into e Radius *a e*; and if from the Sector you take the rea of the Right-lined Triangle *a b c*, there will emain the Area of the Segment *b e c*.

32. If a Right Line could be disposed into the Form f the Circumference of a Circle, it would contain ore Space than any other Figure, or Regular Polygon whatsoever: Suppose the Circumference of e Circle, *a b c d*, to be disposed into the Form of a quare, or into any other Regular Polygon: So that l the Sides *e g*, *g h*, *h i*, and *i e* together may be equal to the Circumference *a b c d*; I say, the Circle is reater than that Square. For the ircle is equal to a Rectangled Triangle, one of whose Legs is the Radius *f a*, and the other the Circumference. And the Polygon is equal



so to such a Triangle, one of whose Legs is the me Circumference *a b c d*, or the Sum of the Sides *e i h*; and the other Leg is the Line *f o* (4. 25.) But the Line *f o* is less than the Radius *f a*, so the second Triangle, which is equal to the Polygon, must less than the first, which is equal to the Circle; d therefore the Square or Polygon must be less an the Circle, which was to be demonstrated.

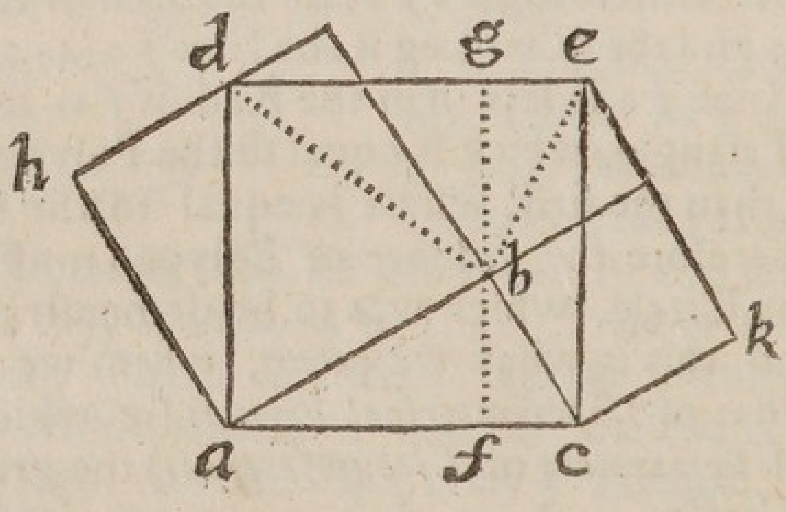
" And this is what we mean, when we usually say, that of *Isoperimetrical Figures* (or which have equal *Perimeters* or *Circumferences*) the greatest is the Circle.

Before we go to Solids, I thought it proper to ve the Learner here, this most noble Theorem of *thagoras*; because, tho' it be indeed demonstrated the sixth Book, yet nearly after *Euclid*'s manner, may also be done here: Thus, In every Right-angled Triangle as *a b c*. The Square of the Hypothenuſe *a c*, is equal to the Sum of the Squares of the Legs *a b* and *b c*; For,

I. The Square of *c a*, is equal to the two Rectangles *d f* and *f e*.

II. The Rectangle *d f* is double of the Triangle *a b d*, being of the ſame Baſe and Altitude; and the Rectangle *f e* is for the ſame Reaſon, double of the Triangle *b e c*. (by 3. 18.)

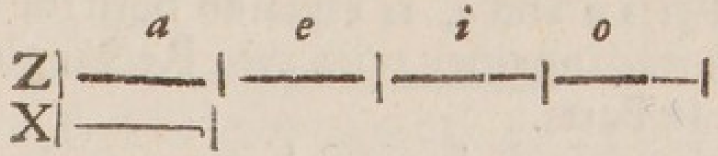
III. But thoſe Triangles, being of the ſame Baſe and Altitude with, will be equal alſo to one half of the Squares *h b* and *b k*: Wherefore the Square of *a c* is equal to the Sum of the Squares of the Legs.



Ihave

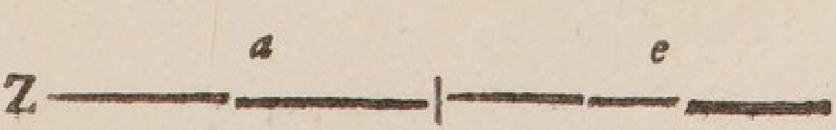
I have here added alſo the Subſtance of the ſecond ook of *Euclid*, about the Power of Lines, *&c.* nd I would adviſe the young Geometrician, before e proceeds any farther, (and if not done already) begin the Study of Algebra; a little of which ill be of excellent Uſe to him in facilitating the emonſtrations in Geometry, and in preparing the ind, and enuring of it to Abſtraction, before he me to the Doctrine of Proportion. And the four ſt Rules of *Addition*, *Subſtraction*, *Multiplication* d *Diviſion* in *Integers* and *Fractions*, will be ſuf-cient to enable him to underſtand the following opoſitions: As alſo the moſt uſeful Ones, which will find added (in this Edition) in all the fol-wing Books of theſe Elements.

I. If there be two Lines Z and X, one of which, z, is divided into any Number of Parts, as into $+e+i+o$. The Rectangle under the two hole Lines $z\,x$, is equal to the Sum of all the ectangles made by x multiplied into the Parts of z.



hat is, $ZX = Xa\,Xe + Xi + Xo$. This is plain, it needs no Proof.

If a Right Line, as Z, be divided into two arts, $a + e$; The Rectangles made by the whole ine, and both its Parts, are equal to the Square f the whole Line.



That is, $za + ze = zz$.
For $za = aa + ae$.
And $ze = ae + ee$.

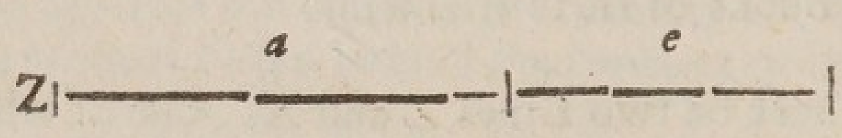
That is,

$$za + ze = aa + 2ae + ee = Q.\overline{a + e}.$$

Q. E. D.

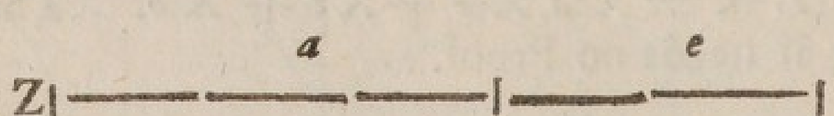
III. Let the Line Z be cut into $a + e$; then shall the Rectangle under the whole Line (Z) and the Part (a) be equal to the Square of that Part a, together with the Rectangle made by the two Parts a and e.

That is, $Za = aa + ae$.



For $Z = a + e$
And $\overline{a + e} \times a = aa + ae$. Q. E. D.

IV. The Square of any Line, as Z, divided into any two Parts a and e, is equal to both the Squares of those Parts, together with two Rectangles made out of those Parts.
That is, $zz = aa + 2ae + ee$.

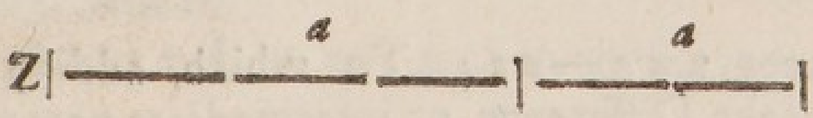


Multiply $a + e$ by itself, and the thing is plain.

$$\begin{array}{r} a+e \\ a+e \\ \hline aa+ae \qquad \\ +ae+ee \\ \hline aa+2ae+ee \\ \hline \end{array}$$

COROLLARY.

Hence 'tis plain that the Square of any Line is qual to four times the Square of its half. For suppose Z to be bissected, then each part will be a, and ultiplying $a + a$ by itself, the thing will plainly ppear.

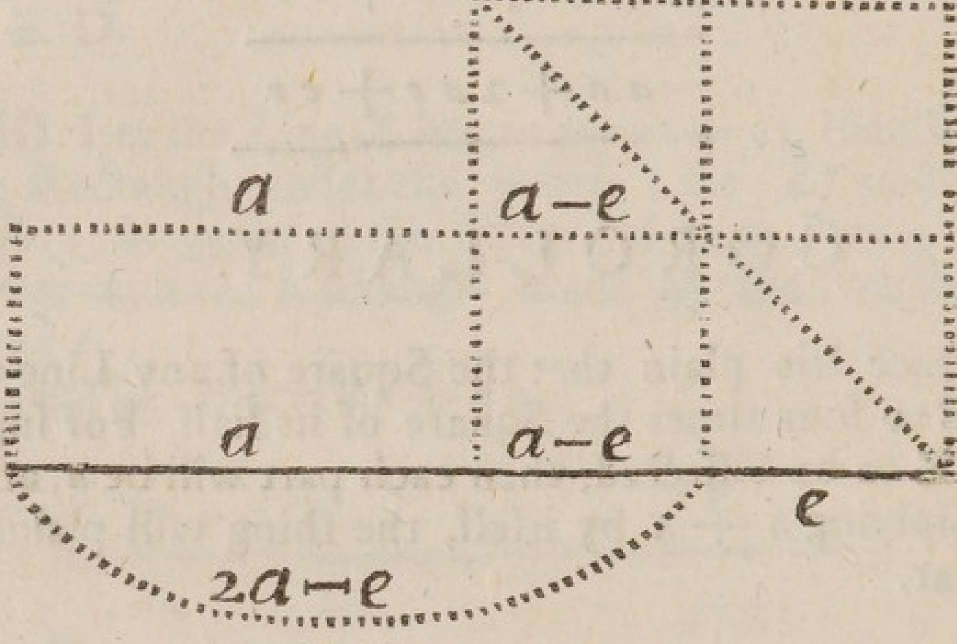


$$\begin{array}{l} a+a \\ a+a \\ \hline aa+aa+aa+aa=4aa. \end{array}$$

V. If a Line be divided into two Parts equally, nd in two other Parts unequally, the Rectangle nder the unequal Parts, together with the Square f (the intermediate Part) the Difference between he equal and unequal Parts, is equal to the Square f half that Line.

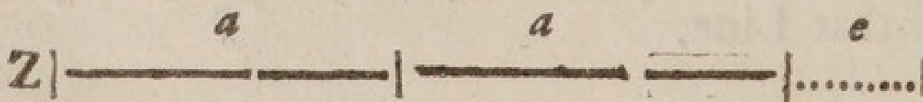
Let

Let the whole Line be $2a$, then each Part will be a. Let the leſſer unequal Part be e, then the intermediate Part will be $a - e$, and the greater unequal Part will be $2a - e$; which multiplied by



e, produces $2ae - ee$; To which adding the Square of the Difference, or intermediate part $a - e$, which is $aa - 2ae + ee$, the Sum will be only aa, the Square of half the Line.

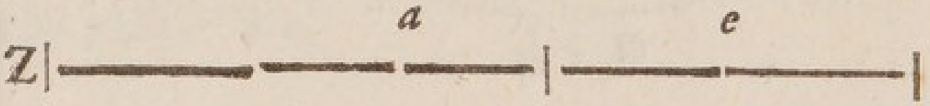
VI. If a Line be biſſected, and then another Right Line be added to it, the Rectangle or Product of the whole augmented Line, multiplied by the Part added, together with the Square of the half Line, is equal to the Square of the half Line and part added, conſider'd as one Line.



Let the firſt Line be $2a$, and the Part added e, then the whole will be $2a + e$; which multiply'd by e, produces $2ae + ee$, and the Square of half the Line

ine aa being added to it, it will be $2ae + ee +$ a, which is equal to the Square of $a + e$, by rop. 4.

VII. If a Quantity or Line be divided any how to two Parts, the Square of the whole added to e Square of one of the Parts, ſhall be equal to two ectangles contained under the whole Line, and that art added to the Square of the other Part.



Let a be one Part, and e the other. The Square the whole and of the leſſer Part e, makes $aa +$ $ae + 2ee$. Then if the whole $a + e$ be multi- ied twice by e, it will produce $2ae + 2ee$; and to this be added the Square of the other Part aa, e Sum will be

$aa + 2ae + 2ee$, equal to the former.

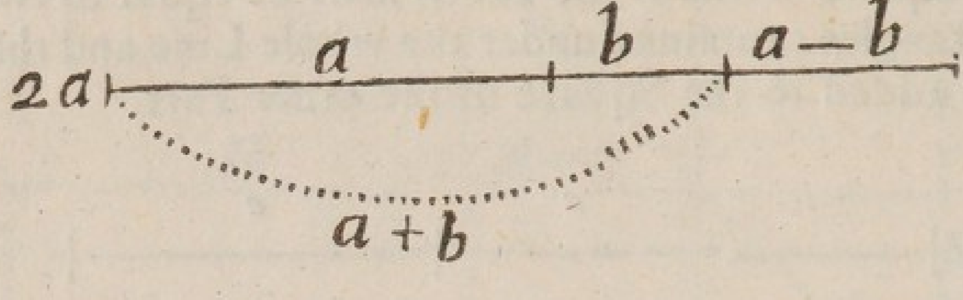
VIII. If a Line be cut any how into two Parts, e Quadruple Rectangle under the whole Line and ne of the Parts added to the Square of the other art, is equal to the Square of the whole and the ther Part added to it, as if it were but one Line.

Let the whole Line be $a + e$, then four times that ultiply'd by e (or the Quadruple Rectangle under at and e) will $4ae + 4ee$; to which adding the quare of the other Part aa, the Sum will be $a + 4ae + 4ee$.

And if you ſquare $a + 2e$, which expreſſes the hole Line, with e added to it, the Product will be e former Sum of $aa + 4ae + 4ee$.

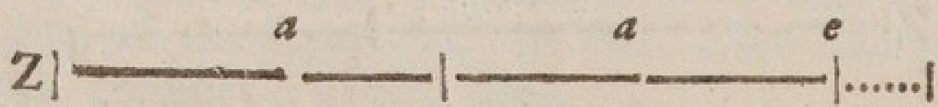
IX. If

IX. If a Line be bissected, and also cut into two other unequal Parts, the Sum of the Squares of the unequal Parts will be double to the Sum of the Squares of the half Line, and of the Difference between the two unequal Parts.



Let the whole Line be $2a$; and the Difference between the equal and unequal Parts b; then the greater unequal Part will be $a + b$, and the lesser $a - b$: The Sum of the Squares of the unequalParts will be $2aa + 2bb$, which is double to the Square of half the Line added to the Square of the Difference. *Q. E. D.*

X. If a Line be bissected, and then another Line added to it; the Square of the whole encreased Line, together with the Square of the Part added, is double the Sum of the Squares of the half Line, and of the half Line and Part added, taken as one Line.



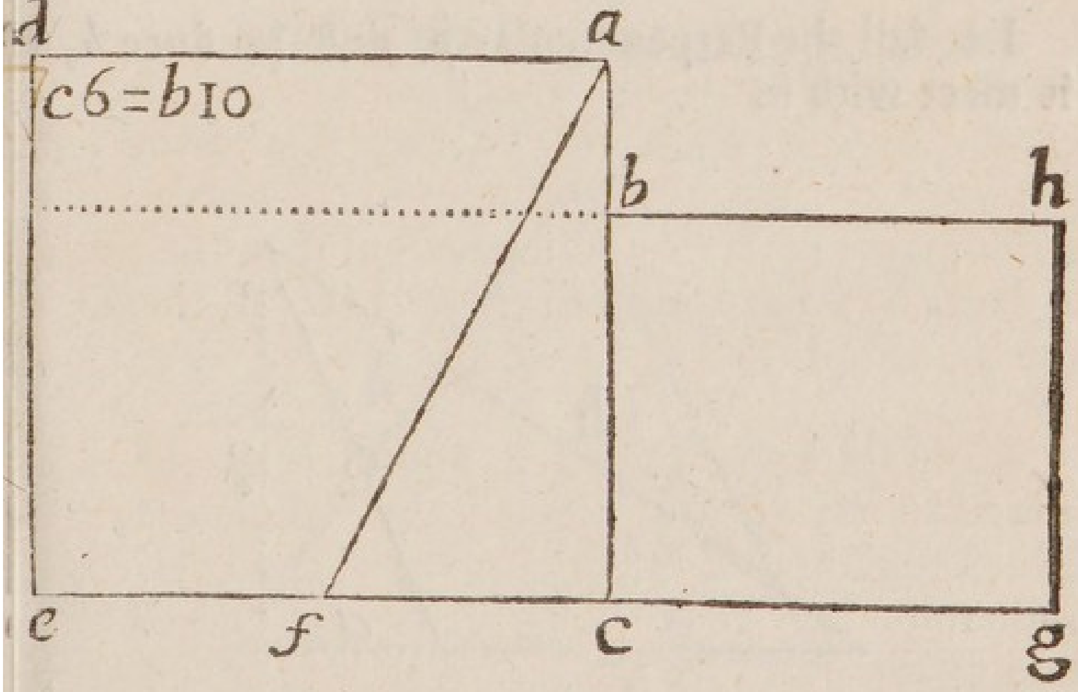
Let the whole Line be $2a$, and the Part added e; then the whole encreased Line will be $2a + e$; and the half Line and Part added will be $a + e$. The Sum of the Squares of $2a + e$, and of e, is $4aa + 4ae + 2ee$; which is plainly double to aa, and $aa + 2ae + ee$, added together. *Q. E. D.*

This

This Problem is also of frequent Use.

PROBLEM.

To divide a Line so, as that the Rectangle under the whole Line a c, *and one Segment* a b, *shall be equal to the Square of the other* b c.



On *a c* make the □ *c d*, whose Base *e c* bissect in and draw *a f*; make *f g* = *a f*, and compleat the □ *b g*, producing *b h* to *k*; Then is *a c* truly divided in *b*; for the Line *e c* being bissected in *f*, and the Part *c g* added to it, the (*by* Prop. 6. *of the Power of Lines*) Rect. *k g* + *f c q* = *f g q* = *f a q* = *a c q* + *f c q*: Wherefore taking *f c q* from both, the Rect. *k g* = *a c q*, and taking the Rect. *k c* from both the Rect. *d b* = □ *b g*; that is Rect. *c a b* = *b g q*. Q. E. F.

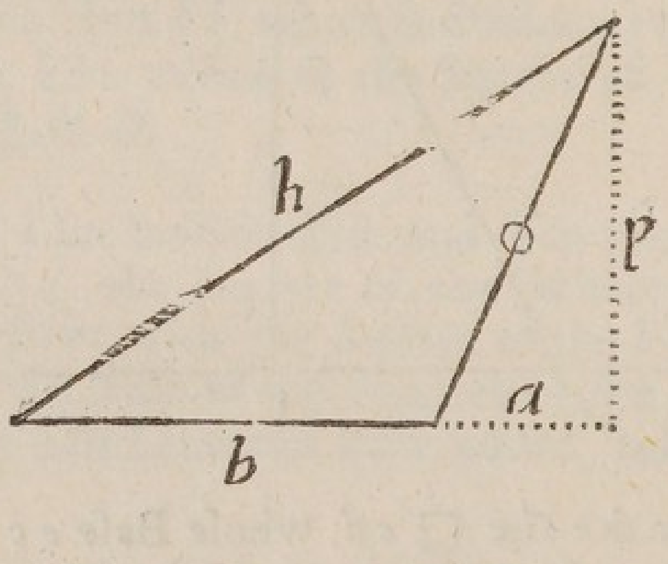
N. B.

N. B. *This is called dividing a Line according to Extream and Mean Proportion; which Proportion cannot be expreſs'd in Numbers.*

PROP. I.

In an Obtuſe-angled Triangle, *the Square of the Side ſubtending the Obtuſe Angle, exceeds the Sum of the Squares of the other two Sides by the double Rectangle,* (2 b a) *under the Baſe, and the part added to it.*

Let fall the Perpendicular p, and produce b, till it meet with it.



DEMONSTRATION.

1. $hh = bb + 2ba + aa + pp$.

2. And $oo = pp + aa$.

3. But $bb + oo = bb + aa + pp$.

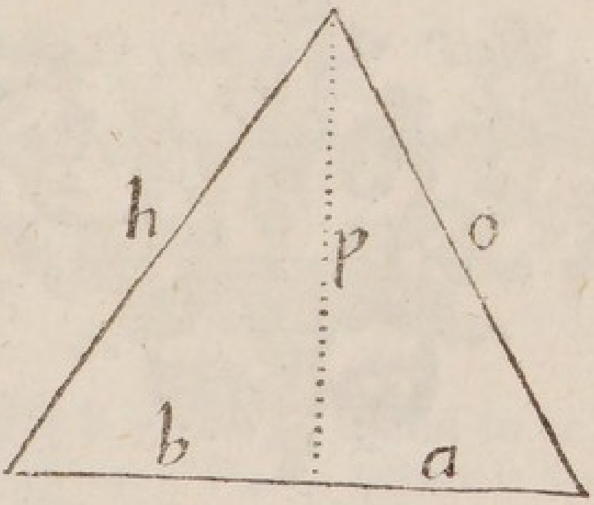
Wherefore hh exceeds the laſt Step by $2ba$. *Q. E. D.*

PROP.

PROP. II.

In an Acute-angled Triangle, *the Square of the Side* (h) *subtending an* Acute Angle, *is less than the Sum of the Squares of the other two Sides, by double the Rectangle under the whole Base,* (b + a) *and the Segment of the Base* (a) *which is next to the* Acute Angle.

Let fall the Perpendicular *p*.



DEMONSTRATION.

1. $hh = bb + pp$.

2. $oo = pp + aa$.

3. $Q.\ \overline{b+a} = bb + 2ba + aa$.

4. $bb + pp + 2aa + 2ab$, is the Sum of the Squares of the Legs.

Wherefore hh is leſs than that by $2aa + 2ab$, which is plainly equal to the double Rectangle under the whole Baſe, and the Part a.

ELE-

ELEMENTS
OF
GEOMETRY.

BOOK V.

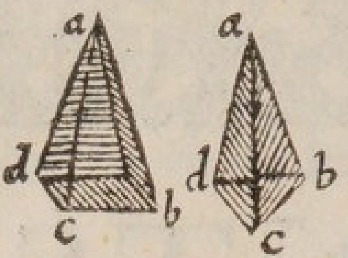
Of Solids.

A Right Line is ſaid to be *Right* upon a Plane, when it ſtands on it at Right Angles, juſt like a Pillar on the Ground, and is inclined no more to any one ſide of the Plane, than to the other.

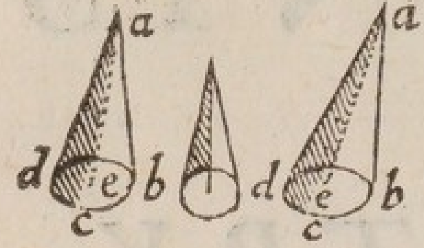
2. Two Planes are parallel to each other, when ll the Perpendiculars that can be drawn between hem, are equal. (That is, *when they every where re equally diſtant.*)

3. One Plane is right or perpendicular to another lane, when, like a well-made Wall, it inclines and ans on one ſide no more than it does on the other.

4. A *ſolid Angle* is made by the meeting of three or more Planes, and thoſe joining in a Point; like the Point of a Diamond well cut.



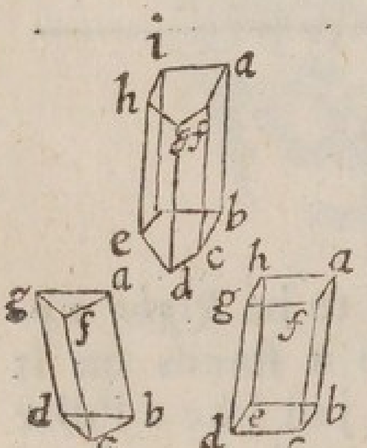
5. If we imagine a Line, as *a b*, fixt above in the Point *a*, to be moved along the Sides of any Polygon *d b c*; that Line by its Motion ſhall deſcribe a Figure that is call'd a *Pyramid.*



6. The *Polygon* is call'd the *Baſe* of the Pyramid.

7. If a Line faſtened, as before, move round a Circle, as *d b c*, it will deſcribe a Cone; and the Circle is its Baſe. And a Line drawn from the Center *e* to *a*, is call'd its *Axis.*

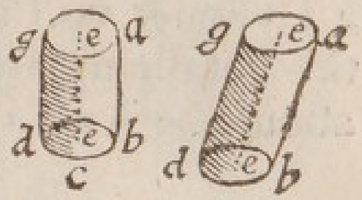
8. If a Line *a b* move uniformerly about two Polygons *g f a* and *d c b*, which are every way equal, having their Sides and Angles mutually parallel and correſponding exactly to one another, as *a f* to *b c*, *f g* to *d c*, &c. then that Line ſhall by its Motion deſcribe a Figure which is call'd a *Priſm*, and the Polygon is its Baſe.



9. If

9. If all the Sides of a Prism be a Parallelogram, then that Prism is call'd a Parallelopiped.

10. If a Line *a b* move uniformly round two equal and parallel Circles, it shall describe or generate a *Cylinder*.



11. The Line joining the Centers *e e*, in the two Bases, is call'd the Axis.

'There is no need of conceiving two Bases, equal, parallel and opposite, for the Genesis of Prisms and Cylinders. For they will be describ'd as well by imagining a Line moving round the Circumference of any plane Figure with a Motion always parallel to it self in its first Position. As if *a b* be supposed to be carried round any of the Bases *d c b*, keeping always the same Angle with the Plane which it first had, it will describe a *Triangular*, *Quinquangular*, or *Circular Prism*, according to the Figure of the Base. And the upper end of the Line will describe a Base (as you may call it) at the Top, equal and parallel to that below.

CORALLARY.

The Solid Content of all Isosceles Prisms and Cylinders (as also of all Parallelopipeds) is had by multiplying their Height into the Area of their Base.

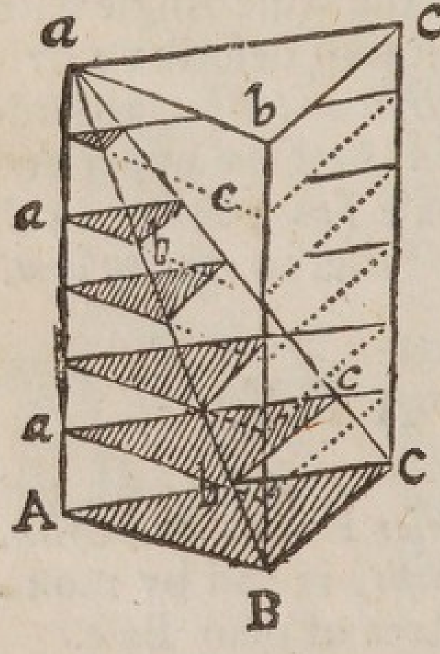
And if they are scalenous Prisms or Cylinders, by multiplying the Base by the perpendicular Altitude.

But after all, this Genesis of Prisms and Pyramids Mr. *Pardie*, respects only their Surfaces. And therefore, the most proper way to conceive the Genesis of all kinds of Prisms, is to imagine a Triangle, Quadrilateral Figure, or Polygon, or the Plane a Circle to be moved in a Position always parallel to itſelf; as ſuppoſe from *h* to *e*, or from *g* to *d* (in the preceding Figures) according to the Directi- on of the Line *h e* or *g d*. Or according to *Euclid*, a Cylinder will be generated by the Revolution of the Parallelogram *g e d e* (*See Fig. in Art* 8.) round about the Axis *a e*.

COROLLARY.

And from hence (as was obſerved before of Lines) 'tis plain that equal Surfaces moved uniformly over equal Places or Intervals, will deſcribe or generate equal Solids.

And as for the Geneſis of Pyramids, ſuppoſe the Triangle *a b c*, to move downwards from the Top of a Plane Angle, determined by the two Planes *a* A B, *a* A C: Let this Motion be always parallel to itſelf, and let the Angular Point of the moving Triangle *a*, be ſuppoſed always to keep in the Line *a* A.

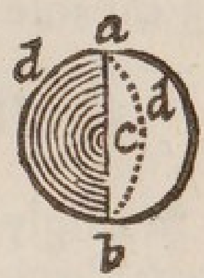


'Tis plain, as this Tri- angle moves farther down- wards, it will ſtill get more and more within the Solid Angle, and at laſt will come to be all of it within it, and to lie in the Poſition A B C, which will be the Baſe of a Triangular Pyramid, whoſe Vertex is at *a*.

The ſame Triangle *a b c*, will alſo, by its Mo- tion, deſcribe another Pyramid, whoſe Baſe ſhall be the Parallelogram *b c* B C, and its Vertex *a*, as before.

13. If

13. If a Semi-circle *a d b* be turned quite round on its Diameter *a b*, it will describe a Sphere or Globe, whose Axis will be *a b* and its Center *c*, the same with the Semi-circle. Every Line passing through the Center *c*, and terminated at each end by the Surface of the Sphere, is called a Diameter, and may be called an Axis.



14. All Lines drawn from the Center *c* to the Surface, are call'd Radii, and are all equal to one another.

To find the Surfaces of Solids.

I. *For all Prisms, Parallelopipeds and Cylinders.*

Find the Perimeter of the Base (which in Practice is done by girting it with a String) and multiply that by the perpendicular Height, the Product is the Surface without the Base, (*i. e.* without the top and bottom Planes) and the Bases may be found by the Rules given in *Plain Mensuration*: The Reason of which is, because a Rectangle of that Form and Dimensions will just cover the outside of the Body.

II. *For Pyramids and Cones.*

The Surface of a Pyramid, is only an Aggregate of Triangles, which therefore must be found severally, and then added up into one Sum.

The Surface of scalenous Cones cannot be found exactly; but for Right Ones multiply the Circumference of the Base by half of the Side of the Cone, the Product is the Area of the Convex Surface. Because the Curve Surface of a Cone is equal to a Triangle, whose Base is the Periphery of the Base, and its Height the Side of the Cone; such a Figure being capable of exactly covering it.

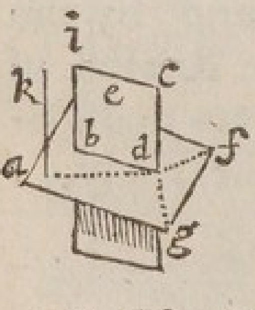
III. *For the Surface of the Sphere.*

Multiply the Diameter by the Periphery of any great Circle, or by such a Circle as hath the Diameter of the Sphere for its Diameter, the Product is the Surface. As appears from what will be prov'd below, after *Art.* 34.

IV. *The Surface of the five Regular Bodies, is easily had, by the Principles of Plain Mensuration.*

15. Two Right Lines if they meet so as to cut or cross each other, are in the same Plane: Wherefore all the Angles and Sides of every Triangle are in the same Plane.

16. If two Planes *e b d* and *a g f* cut or intersect one another, they shall do so in a Right Line, as *b d*; which is call'd their common Section.



17. If a Right Line *d c* be perpendicular to two Lines *d f* and *d g*, which are in the same Plane, that Line is also perpendicular to that Plane.

18. If a Right Line *d c* be perpendicular to three Right Lines *d f*, *d g* and *d a*, they are all three in the same Plane.

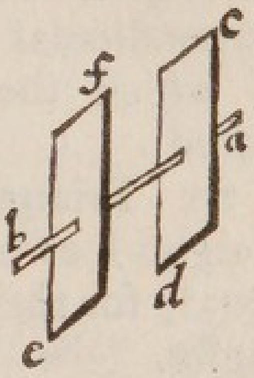
19. If two Lines *d c*, *b i* are perpendicular to the same Plane *f g a*, they will be parallel to one another.

20. If two Lines *d c*, *b i* are parallel, and you draw another Line, from any Point in one to the other, as *b d*, those three will be all in the same Plane.

21. If

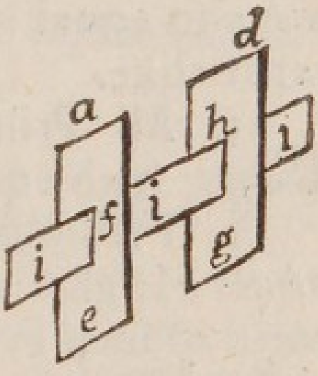
21. If two Lines *d c*, *b i* are parallel to a third *a k*, though that third Line be not in the same Plane with them, yet they shall be parallel to each other.

22. If a right Line *a b* be perpendicular to (*or make any other equal Angles with*) two Planes *f e* and *c d*, those Planes are parallel.



23. If two parallel Planes, *d h g* and *a f e*, are cut by a third *i i i*, the common Sections *f e* and *h g* are parallel.

24. If a Solid Angle be made by three Plane Angles, any two of those are always greater than the third.



All these Propositious are so manifest to one that will but consider them with a little Attention, that 'tis needless to stay to demonstrate them. (And indeed the Solemn and Regular Demonstration of a thing plain in itself, always makes it more obscure.

25. The *Plain Angles*, concurring to make a *Solid* one, taken all together, are always less than four Right ones. For if they should make four Right Angles, they would form a Plane and not an Angle. Wherefore, that they may make a Solid Angle, they must be less than four Right ones.

'Tis a very good way in order to gain a clear Idea of Solids and their Angles, to make the Regular Bodies out of thick Paper or Past-board, and after the Description of every Body, you will see the Figure, which being folded up together, will express the Solid.

26. In all Parallelopipeds, the opposite Planes are equal; as is easy to conceive (from 5. 9.)

27. All Parallelopipeds having equal Bases (and Heights) or being between the same Parallels, are equal, for they are equal Aggregates of equal Parallelograms. (3. 14.)

28. Every Parallelopiped is divided into two equal Triangular Prisms, by a Diagonal Plane, which is perpendicular to its Base: For every Parallelogram of which the Figure is composed, is equally bissected.

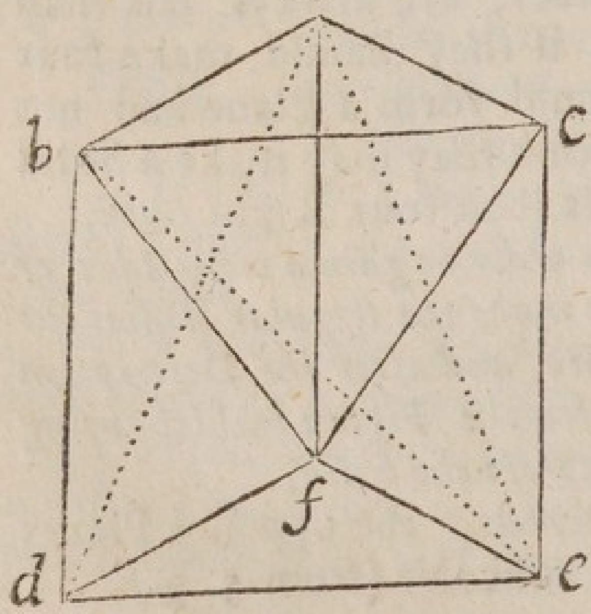
29. Triangular Prisms, having equal Bases (and Heights) or being between the same Parallels, are equal, for they are equal Aggregates of equal Triangles.

30. Pyramids having equal Bases and Heights, are also equal: For they are all supposed to grow taper alike.

31. All Prisms in general, all Cylinders and Cones, with equal Bases and Heights, are equal.

32. Pyramids and Cones on equal Bases, and of equal Heights with Prisms and Cylinders, are one third of such Prisms and Cylinders.

In a Triangular Prism and Pyramid of the same Base and Altitude, it is thus prov'd



The Quadrangular Pyramid *a c e f b* is divided into two equal Triangular Ones, by the Triangular Plane *f b c*, and the Pyramid *f c a b*, is the very same with *b a c f*; and this is equal to the Pyramid *d f e a*: As having an equal Base and the same Altitude with it, and therefore the whole Prism is divided into three equal Pyramids.

And

and ſince all multangular Priſms can be divided into Triangular ones, and that Cylinder is only a Multangular Priſm of infinite Sides, the Propoſition is univerſally true ; *That Pyramids and Cones*, &c.

N. B. A Piece of Cork or Wood, in the Form of a Triangular Priſm, may be cut into three equal Pyramids.

CORALLARY I.

Hence the way of finding the Solidity of a Pyramid or Cone is diſcover'd, viz. *To multiply the Baſe by* $\frac{1}{3}$ *of the Perpendicular Altitude.*

33. Every Sphere is equal to a Cone whoſe Perpendicular Axis is the Radius of the Sphere, and its Baſe a Plane, equal to all the Convex Surface of it.

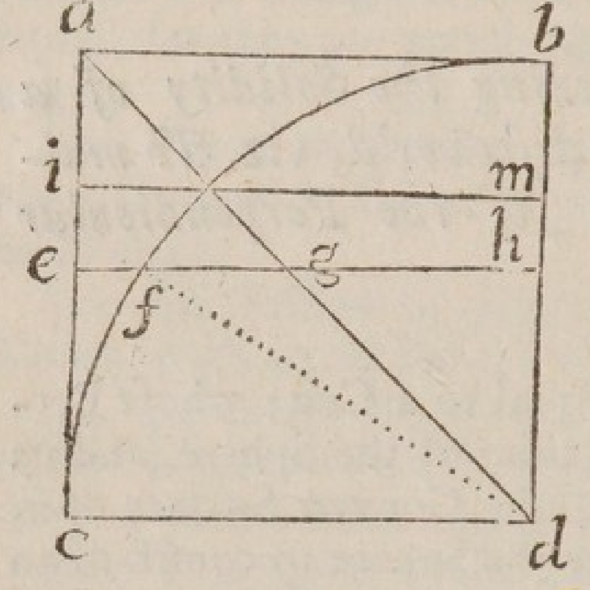
For you may conceive the Sphere to conſiſt of an infinite Number of Cones, whoſe Baſes taken all together compoſe the Surface, and whoſe Vertexes meet all together in the Center of the Sphere : Juſt as a Circle may be imagined to be compoſed of an infinite Number of Iſoſceles Triangles, the Aggregate of whoſe Baſes makes the Circumference, and their common Vertex is at the Center.

COROL.

COROL. II.

Hence the Solidity of the Sphere will be gain'd by multiplying its Surface by $\frac{1}{3}$ of its Radius.

Let the Square ad, the Quadrant cbd, and the Right-angled Triangle abd, be ſuppoſed all three to revolve round the Line bd as an Axis: Then will the Square generate a Cylinder, the Quadrant an Hemiſphere, and the Triangle a Cone, all of the ſame Baſe and Altitude.



I. Then the Square of eh (which is equal to the Square of fd, which is equal to the Square of fh, together with that of hd (or its equal gh) will be equal to the Square of gh ($= hd$) together with the Squares of fh. And ſince Circles are as the Squares of their Diameters (which muſt be now taken for granted, but will be proved in the ſixth Book) the Circle made by the Revolution of eh, muſt be equal to the two Circles made by the Motion of fh and hg. Wherefore,

II. If you take the Circle made by the Revolution of fh from both, there will remain the Circle made by the Motion of gh equal to the Ring deſcrib'd by the Motion of ef: And thus it muſt always be, wherever you draw the Line eh, or im, &c.

 III. There-

III. Therefore the Aggregate of all the Rings made by the Revolution of the *ef*'s, must be equal to that of all the Circles made by the Motion of the *h*'s: (*i. e.*) the Dish-like Solid, formed by the Revolving Rings, will be equal to the Cone formed by the Revolution of the *g h*'s, which are the Elements of the Triangle *a b d*. That is, the Dish-like Solid will be as the Cone is $\frac{1}{3}$ of the circumscribing Cylinder, and consequently the Hemisphere must be $\frac{2}{3}$ of it: Wherefore the Sphere is $\frac{2}{3}$ of the circumscribing Cylinder.

IV. Let then the Radius of the Sphere be $r = d = b\,d$, then the Diameter will be $2\,r$; let the Surface of the Sphere generated by the revolving Semicircle be called S; and that of the Cylinder, formed by the Revolution of $2\,a\,c = 2\,r =$ Diameter, be called *ſ*. Wherefore in what was just now proved, (by *Art.* 33. of this Book) the Expression for the Solidity of the Sphere in this Notation will be $\frac{rS}{3}$ and putting *c* equal to the Circumference of the Base, or for the Periphery of a great Circle of the Sphere, the Curve Surface of the Cylinder (by multiplying the Altitude into the Periphery of the Base) will be $2\,r\,c$; also $\frac{r\,c}{2}$ will be the Area of a great Circle (by *Prop.* 26. of *Book* 4.) and this multiplied by $2\,r$, makes $\frac{2\,r\,r\,c}{2}$, which is the Solidity of the Cylinder (by *Cor. Art.* 11.) Now since *ſ* was put equal to $2\,r\,c =$ to the Curve Surface of the Cylinder $\frac{ſ\,r}{2}$ (by substituting *ſ* for $2\,r\,c$) will be also $=$ to the Solidity of the Cylinder. Now since the Sphere is $= \frac{2}{3}$ of the Cylinder, $\frac{r\,S}{3}$ $= \frac{2}{3}$

$= \frac{2}{3} \frac{r ſ}{2}$. That is $\frac{r S}{3} = \frac{2 r ſ}{6} = \frac{r ſ}{3}$. Wherefore $r S = r ſ$; that is, dividing by r, $S = ſ$, or the Surface of the Sphere, is equal to the Curve Surface of the Cylinder: But the Curve Surface of the Cylinder was $2\ r\ c$.

Wherefore to find the Area of the Surface of either Sphere or Cylinder, you muſt multiply the Diameter ($= 2\ r$) by the Circumference of a great Circle of the Sphere, or by the Periphery of the Baſe. From this Notation alſo $\frac{r c}{2}$, the Area of a great Circle of the Sphere, is plainly $\frac{1}{4}$ of $2\ r\ c$ the Surface of the Sphere. That is, the Surface of the Sphere is Quadruple of the Area of a great Circle of it.

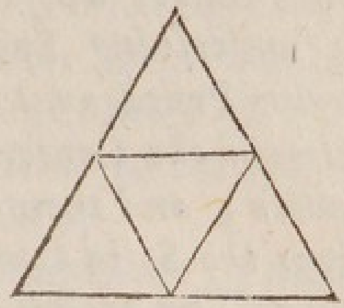
V. Wherefore, to $2\ r\ c$, the Convex Surface of the Cylinder, add $r\ c$ equal to the Area of both its Baſes $\left(\text{each of which is } \frac{r c}{2}\right)$ you will have $3\ r\ c$; which ſhews you that the Surface of the Cylinder (including its Baſes) being $3\ r\ c$, is to the Surface of the Sphere, which is, $2\ r\ c$, as three is to two: Or that the Sphere is $\frac{2}{3}$ of the circumſcribing Cylinder, in Area, as well as Solidity.

34. Of all Solid Figures that can be encompaſs'd or determinated by the ſame Surface, the greateſt is a Spherical One, by *Art.* 13. of this Book, and *Art.* the laſt of Book the 4th.

35. That is call'd a *Regular Body*, whoſe Surface is compoſed of Regular and Equal Figures. And whoſe Solid Angles are all equal, as are——

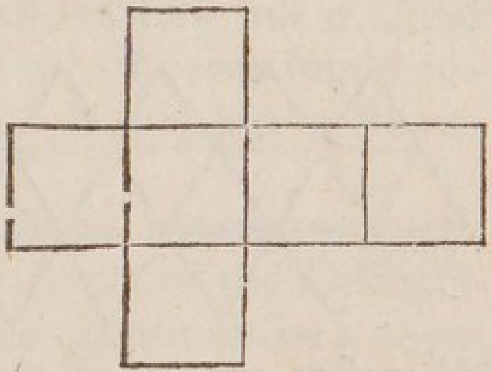
36. The

36. The *Tetrahedron*, which is a Pyramid, com-rehended under four equal and quilateral Triangles; so that s Base is equal to each Side.



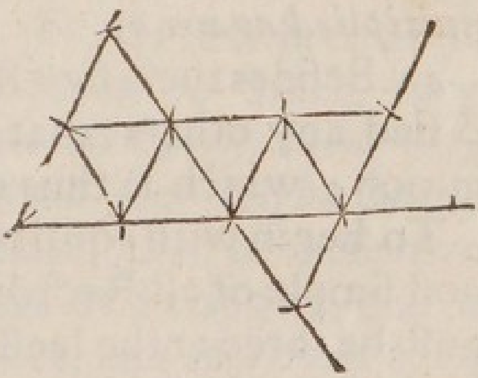
Wherefore its Solidity will be und by multiplying the Base by of the Altitude; which is the eneral way for all Pyramids.

37. The *Hexahedron* or Cube, whose Surface is ompos'd of six equal quares, like Dice which re us'd in play.



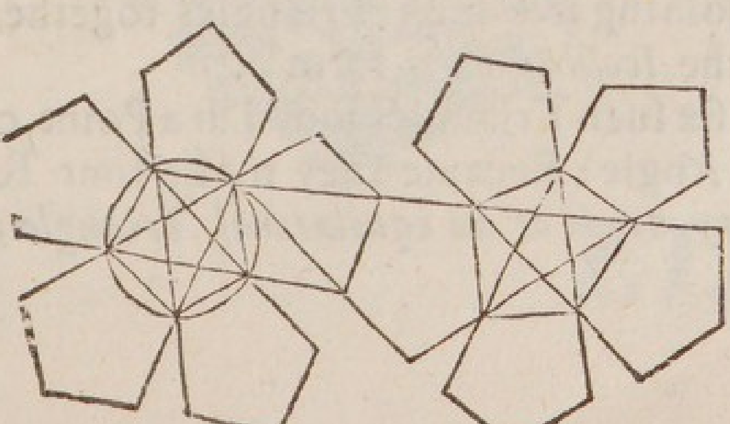
Its Solidity will be found Cor. of Art. 12.

38. The *Octahedron*, which is bounded by eight qual and equilateral Tri-ngles.



This Figure is two Pyra-ids put together at their ases: Wherefore its Soli-ity is had by multiplying he Quadrangular Base of ither (here they are both in'd together in the middle the Figure) by one third of the perpendicular Alti-ude of one of the joined Pyramids, and then doubling he Product.

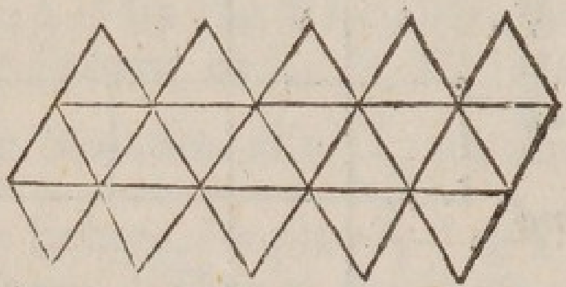
39. The *Dodecahedron*, which is contained under welve equal and equilateral Pentagons.



This

This Figure consists of twelve Pyramids, with Pentagonal Bases, whose common Vertex is the Center of a circumscribing Sphere: Wherefore any one of these twelve Pentagonal Bases multiply'd by $\frac{1}{3}$ of the Distance between the Center of that Base, and the Center of the Sphere; and then that Product multiplied by twelve, gives the Solid Content of this Regular Body.

40. The *Icosihedron*, consisting of twenty equal and equilateral Triangles.



This Figure is composed of 20 Triangular Pyramids, all equal to one another, and whose Vertex is the Center of a circumscribing Sphere: Wherefore any one of the 20 Triangular Faces, multiplied by $\frac{1}{3}$ of the Distance between the Center of the Face and the Center of the Sphere, and that Product multiplied again by 20, gives its Solid Content.

41. Besides these five *Regular Bodies*, 'tis not possible to find any others that shall correspond to the Definition; which is thus demonstrated.

To begin with equilateral Triangles, which are the most simple of all Rectilineal Figures. Of these there must be three at the least to make a Solid Angle, and three of them join'd together will just make the *Tetrahedron*. For those three Triangles meeting in a Point do form a Triangular Base similar and equal to the Sides; as appears by the bare Composition of the Figure. Four Triangles join'd together in a Point make the Angle of the *Octahedron*.

By joining five such Triangles together, the Angle of the *Icosihedron* is form'd.

But six such Triangles join'd in a Point can't make a Solid Angle: Because they make four Right Ones (*for every Angle of an equilateral Triangle is $\frac{2}{3}$ of two*

or

- $\frac{2}{3}$ *of one Right Angle, either of which Fractions mul-plied by ſix, gives four right Angles.*) Whereas every olid Angle is made up of ſuch plane Angles as all ogether muſt be leſs than four Right ones (5. 25.) o that with Triangles 'tis impoſſible to form any ore Regular Bodies than theſe three.

Next, if you take *Squares* and join three of them ogether, they will make the Angle of the Cube: nd there can no other Regular Body but a Cube e made with Squares, for four Squares join'd to-ether, will not make a Solid Angle, but a Plane. 5. 25.)

If you join the Angles of three Pentagons toge-her, you will conſtitute the Angle of the Dodeca-edron: But four ſuch Angles cannot make a Solid One.

And laſtly, *Three Hexagons* joined together do ake juſt four Right Angles, and therefore they annot make a Solid Angle: And as for three *Hep-agons*, or other Figures of yet more Sides, they an much leſs do it; (*becauſe their Angles being very btuſe, three of them will exceed four Right Ones.*) o that upon the whole 'tis plain, that of theſe five *egular Bodies*, three are made of Triangles, one of quares, and one of Pentagons, *and there can be no her.*

ELE-

ELEMENTS
OF
GEOMETRY.

BOOK VI.

Of Proportion.

1. WHEN we ſpeak of *Magnitude*, and ſay, that any Quantity is great, we always make a Compariſon between that Quantity and ſome other of the ſame Nature, in reſpect to which we ſay that it is Great.

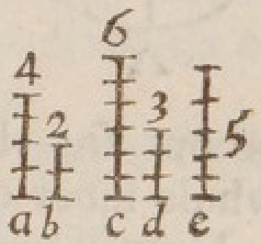
Thus we ſay of an *Hill*, that 'tis Little; or of a *Diamond*, that 'tis Large; becauſe we compare that Hill with others that are Higher, and in reſpect of them 'tis Little; and we compare that Diamond with others that are Little, and in reſpect of them, we ſay 'tis a Large one.

2. When we conſider one Quantity in reſpect of another, to ſee what Magnitude it hath in compari-ſon

ſon of that other: The Magnitude ſo found, is call'd its *Ratio* or *Reaſon*; tho' it would be more intelligible if it were call'd Compariſon.

3. That Quantity which is compar'd with another is call'd the Antecedent; and *that other* with which it is compar'd, is call'd the Conſequent.

4. When we conſider four Quantities, and compare them (by Pairs) two with two; as *a* 4 with *b* 2, and *c* 6 with *d* 3. If we find that *a* hath as much Magnitude (or is as big) in compariſon of *b*, as *c* hath in compariſon of *d*; (i. e. *When we find that* a *is contained in, or doth contain* b : : *as often as* c *is contained in, or doth contain* d): Then we ſay, that their Ratio's are equal; that is, the Ratio that *a* hath to *b*, is equal to the Ratio of *c* to *d*: For as *a* is twice as big as *b*; ſo *c* is twice as big as *d*.



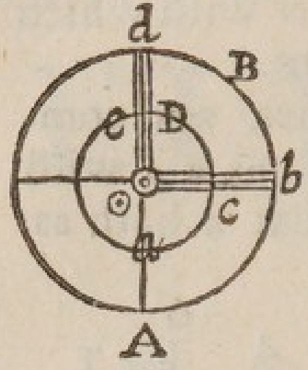
5. But if *a* 4 hath more Magnitude in reſpect of *b*, than *c* 6 hath in reſpect of *e* 5. That is, if as *a* 4 is twice as big as *b* 2, *c* 6 be found not to be twice as big as *e* 5: Then the Ratio's are unequal: And we ſay, *a* hath a greater Ratio to *b*, than *c* hath to *e*. So that to have a greater Ratio, is nothing but to have more Magnitude, or to be bigger, in reſpect of a ſecond Term, than a third is in reſpect of a fourth.

6. The Equality of Ratio's is called Proportion; and when we find that of four Quantities or Numbers, the firſt hath as much Magnitude *(or is as big)* in reſpect of the ſecond; as the third is in reſpect of the fourth; then we ſay, that thoſe four Quantities are *Proportionals*.

The better to make the Myſteries of Proportion comprehended, which paſs for the moſt difficult things in Geometry, *as unqueſtionably they are moſt important*, *I will explain them by an Example; which (in my Opinion)*

nion) will render all those things very intelligible, which otherwise appear very perplex'd.

7. Let us imagine the Circle *b* A *d* to be describ'd by the Motion of the Line *o b*, round the Center *o*: And at the same time, let the Circle *c a e* be described by the Motion of a Point *c*, in the Line *o b*: Let us suppose also that the Line *o b* be moved once round again, and at last to stand in the Position *o d*. Let the Ark *d* B *b* be called B, and the Ark *e* D *c*, be called D. Let A be put for the whole outer Circle, and *a* for the whole inner one.



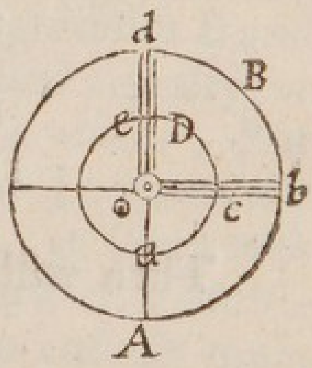
Now if we compare the whole Circle A with its Ark B, and the whole other Circle *a* with its Ark D. We shall find plainly, that the Circle A is just as big in respect of the Ark B, as the inner one *a* is in respect of the Ark D; and therefore if B be a fourth, or any other Part of the Circle A, D also will be a fourth, or the same proportional Part of its Circle *a*. Which we usually express by saying, as A is to B, so is *a* to D. And write it thus, A : B : : *a* : D, or 24 : 6 : : 8 : 2.

8. If you should change the Order of the Terms, and compare B with A, and D with *a*; you will find plainly that B : A : : D : *a*. So that supposing A : B : : *a* : D, we cannot but presently conclude by inverse Proportion, that B : A : : D : *a*.

9. If you change them so as to compare Antecedent with Antecedent, and Consequent with Consequent, you will find Alternately, that A : *a* : : B : D. And this is very plain; for if the whole Circle A be double, triple of, or in any other Proportion, to the Circle *a*, the Ark B must be also double, triple of, or in the same Proportion to the Ark D; for Aliquot Parts will be as their Wholes. This I say is plain,

plain, because the two Circles A and a are describ'd by the Motion of the Line $o\ c\ b$; so that while b describes the Circle A, c describes the inner Circle a; and while b describes the Ark B; c also describes the Ark D. And this by one common circular Motion; only the Point c moving much slower than the Point b, describes a Circle much less, in proportion to the slowness of its Motion: Thus also when the Point b shall have described the Ark B, the Point c in like manner will have describ'd the Ark D, which will be much less than B; in Proportion to the slowness of its Motion; in Numbers 24 : 8 :: 6 : 2.

10. If we compare the Differences between the Antecedents and Consequents, with their Consequents; as for Instance, A less B with B, and a less D with D, we shall find they also are proportional: And that A less B : B :: a less D : D : 18 : 6 :: 6 : 2.



For 'tis manifest that the Ark A d (which is A less B) is to B as the Ark $c\ a\ e$ (which is a less D) is to D. And this is call'd *Proportion by Division.*

11. If we add the Antecedents and Consequents together; we shall find that A more B, is to B :: as a more D is to D. Which is call'd *Composition.* In Numbers 30 : 6 :: 10 : 2.

12. And if we would say, that A A less B :: a : a less D. This kind of Proportion s call'd *Conversion.* You may also infer by way of *mixing* the Terms, as some call it, That A + B : A — B :: a + D : a — D, or that A + B a + D :: A — B : a — D, &c. That 30 : 18 :: 10 : 6 and 30 : 10 :: 18 : 6, &c.

And it will be very convenient for the Learner to inure himself to all the Changes and Varieties of Proportion, and to have them ready in his Mind; because a great many Propositions in Geometry, as they have been delivered by the Ancients, and purſued by the Moderns that have trod in their Steps, are demonſtrated by Compoſition, Diviſion, Alternation, and intermixing of Proportion.

13. If never ſo many Quantities are thus proportional: It will be as any one Antecedent to its Conſequent :: So is the Sum of all the Antecedents to the Sum of all the Conſequents. *v. gr.*

If 4 : 12 :: 2 : 6, :: 3 : 9 :: 5 : 15: then ſhall 14 : 42 :: 4 : 12.

14. If *a* : *b* :: *c* : *d*.
4 : 12 :: 3 : 9, and alſo,

b : *f* :: *d* : *g*.
12 : 36 :: 9 : 27.

Then will it be by Proportion of Equality.

a : *f* :: *c* : *g*.
4 : 36 :: 3 : 27.

The Reaſon of which is plain, if you conſider, That ſince *b* : *f* :: *d* : *g* : *f* and *g* muſt needs be either *ſimilar aliquot Parts*, or *Equimultiples* of *b* and *d*. And therefore ſince *a* and *c* are to *b* and *d*, in the ſame Ratio as *b* and *d* are to *f* and *g*, *a* muſt alſo be in the ſame Ratio to *f*, the *Part* or *Multiple* of *b* :: as *c* is to *g*, the *Part* or *Multiple* of *d*.

If *a* : *b* :: *c* : *d*.
24 : 4 :: 9 : 3, and then,

b : *f* :: *h* : *c*.
4 : 2 :: 18 : 9.

Then

Then will $a : f :: h\ d.$

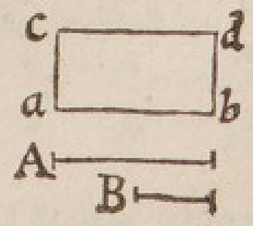
$12 : 2 :: 18 : 3.$

Which is called *Proportio ex æquo perturbata*; and this muſt be true: Becauſe 12 containing 4 as oft as 9 contains 3, and 4 containing 2, as oft as 18 contains 9; 12 muſt contain 2 as often as 18 contains 3. Wherefore this is only the orderly Proportion of Equality diſturbed, and therefore is by ſome called *Inordinate Proportion.*

15. If B be taken as often as D, *ex. gr.* 3 B and 3 D, we may conclude that B : D :: 3 B : 3 D, or as 10 B to 10 D, alſo $12\frac{1}{2}$ B, to $12\frac{1}{2}$ D. And ſo on in whatever Proportion the two Magnitudes B and D are multiplied, ſo they are multiplied equally, *or that you take one as often as you take the other.* For then there will be the ſame Proportion between the Magnitudes thus equally multiplied, as there was between the ſimple Magnitudes, before ſuch Multiplication. And theſe Magnitudes, thus equally multiplied, are call'd *Equimultiples* of the ſimple Magnitudes B and D; and we ſay that *Equimultiples are in the ſame Proportion as ſuch ſimple Magnitudes, out of which they are compounded.*

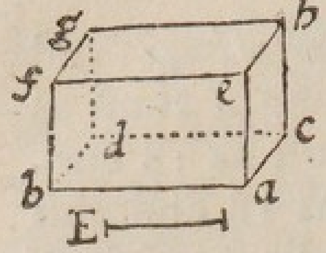
16. If B be divided in the ſame Manner as D is; and *ex. gr.* you take a fourth Part of B, and the like of D, or the tenth, or any other Part of B, and the ſame of D. Then will theſe Parts be proportional to their Wholes, B : D : : $\frac{1}{4}$ B (or $\frac{1}{10}$ B) is to $\frac{1}{4}$, or $\frac{1}{10}$ D, All which is ſelf-evident.

17. To multiply one Line by another is to make a Rectangled Parallelogram, whoſe two contiguous Sides ſhall be the two Lines given. Thus, if you multiply the Line A by B, 'tis the ſame thing as to make the Rectangle *a b c d*; whoſe Side *a b* is equal to A, and *a c* to *b*.



 18. To

18. To multiply a Rectangle, or any other Surface by a Right Line, is to make a Rect-angled Parallelopiped (*or Prism*) (5. 9.) whose Base shall be the Surface given, and its perpendicular Height the Line given.

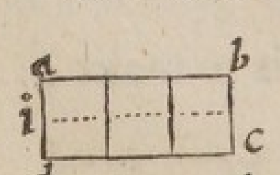


Thus to multiply the Surface *a b d c* by the Line E, is the same thing as to make a Solid *a b f g h e*, whose Base is the Surface given *a d*, and its Height *a e* or *b f*, equal to E, the Line given.

19. All Magnitudes may be expreſs'd by Lines: As if one Magnitude be double or triple of another; or in any other *Ratio*, two Lines may easily be taken, of which one shall be double or triple of the other, or in any other like Proportion with those Magnitudes: So for Instance, to express two times, as one Hour and two Hours; or two Velocities, of which one shall be double to the other; you need only take two Lines, as *a* double of *b*; and then you may say that *a* represents two Hours or Velocities, and *b* answers to one of each; and then you may proceed to compute with those two Lines, as with the Hours and Velocities themselves, &c.

20. To know the Proportion of Rectangles, the Ratio of the Length of one, to the Length of the other, and moreover the *Ratio* of the Breadth of one, to the Breadth of the other must be known.

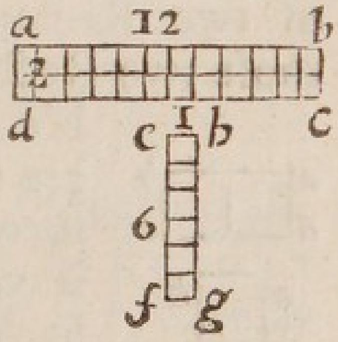
For Example; To know what Proportion the Rectangle *a c* hath to *e g*: 'Tis not enough only to know that the Length *a b* is triple of *e h*; but it must be known also, that *a d* is double of *e f*. For if *a i* be taken equal to *e f*, the Rectangle *b i* will be triple of *e g*, because *a b* is triple of *e h*, and *a i* equal to *e f*. And moreover, because *i d* is also equal to *a i*, or *e f* (for



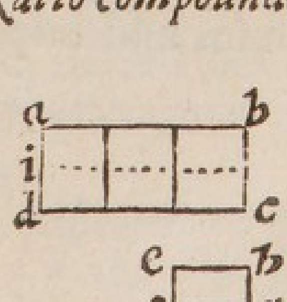
a d

a d is ſuppoſed to be double of *a i*, and of *e f*) the Rectangle *i c* ſhall alſo be triple of *e g*; ſo that the whole Rectangle *a c* is twice triple of the Rectangle *e g*; that is, ſextuple of it, or containing it ſix times. And what we ſay now only of the double or triple *Ratio* of their Breadths and Lengths, is alſo to be underſtood of any other *Ratio*, be it what it will: For if *a b* be quadruple of *e b*, and *a d* triple of *e f*, the Rectangle *a c*, will be three times quadruple of the Rectangle *e g*; that is duodecuple of it, or doth contain it twelve times.

But if *a b* be duodecuple of *e b*, and at the ſame time *e f* be triple of *a d*, then there is a certain Compenſation made: For if Reſpect were had to their Breadths *a b* and *e b* only, the Rectangle *a c* would exceed the other, nay indeed contain it 12 times: Neverthelefs this Exceſs is loſt (*in ſome Meaſure*) in reſpect of their Altitudes or Heights *a d* and *e f*, which if only conſider'd, the Rectangle *e g* would be triple of *a c*. But then when we come to compare theſe ſeveral Exceſſes and Deficiencies together; we ſhall find that the Rectangle *a c* being one way 12 times greater, and the other way three times leſs than *e g*, will be at laſt but only four times as great.



21. And this is what we mean, when we ſay, that all Rectangles are to each other in a *Ratio compounded* of that of their Sides; for if *a b* be triple of *e b*, and *a d* double of *e f*, the Rectangle *a c*, ſhall be to the Rectangle *e g* in a *Ratio* compounded of the triple and the double, that is, it ſhall be twice double, or twice triple, or in one Word ſextuple. So alſo if *a b* were quadruple of *e b*, and *a d*

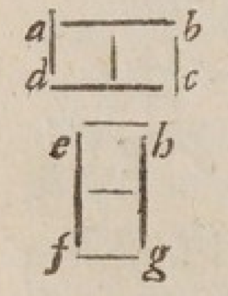


triple of *ef*; the Rectangle *ac* would then be to *eg* in a *Ratio* compounded of the quadruple and the triple; so that it would have been three times quadruple, or four times triple, or in one Word duodecuple of *eg*.

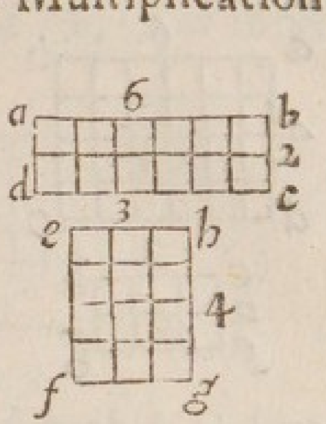
Moreover, if *ab* were duodecuple of *eh*, and *ad* subtriple of *eh*, (that is, if *ef* be triple of *ad*) the *Ratio* of the Rectangle *ae* to *eg* would be compounded of the duodecuple and subtriple *Ratio*; so that *ac* would have been 12 times subtriple of, or in one Word quadruple of *eg*.

If you take the third Part of a Crown 12 times, it will make, or be equal to four whole Crowns: So that four Crowns are 12 times subtriple of one Crown; that is, do make 12 Thirds of a Crown.

22. From whence it will appear, that if the Sides of two Rectangles are reciprocally proportional, those two Rectangles are equal: For if *ab* be double to *eh*, and reciprocally *hg* be double to *cb*: Or if *ab* be triple of *eh*, and then *hg* be triple of *bc*; or in a Word, if whatever *Ratio* *ab* hath to *eh*, *hg*, hath back again the same *Ratio* to *bc*, 'tis plain, that as much as the first Rectangle *ac* exceeds the other in Length, just so much is it exceeded by the other in Breadth; so that the Length of one compensates for the Breadth of the other, and consequently they must be equal. And from hence is deduced this most useful and important Proposition; That,



23. If four Quantities *(or Numbers)* be proportional, the Product arising from the Multiplication of the two middle Terms, is always equal to that which is made by the Multiplication of the two Extreams. As if *a b* : *e h* : : *h g* : *c*.



I say, from the Multiplication of the Extreams *a b* by *b c* there is produced the Rectangle *a c*: and by multiplying the middle Terms *e h* and *h g*, there is produced the Rectangle *e g*; and those two Rectangles *a c* and *e g* are equal. (6. 22.) *Because, as much longer as* a b *is than* e h, *just so much longer is* h g *than* b c). On which is founded the Reason of the Golden Rule.

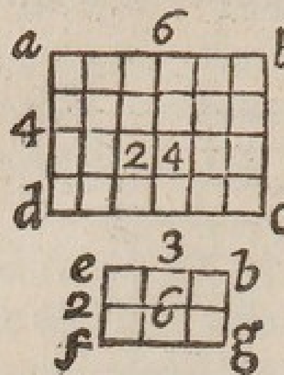
COROLLARY.

Hence, if in two Ranks of *Discrete* Proportionals, the four middle Terms are the same: As if *a* : *b* : : *c* : *d*, and then also *e* : *b* : : *c* : *f*. I say it will be as *a* : *e* : : So will reciprocally *f*, be to *d*: For since the middle Terms are the same in both, the Rectangle *a d* will be equal to *e f*, and consequently their Sides must be reciprocally proportional; that is *a* : *e* : : *f* : *d*.

What is thus done by Lines and Rectangles, may be done by any Quantity whatsoever; because all Quantities can be express'd by Lines, and all Multiplications of Magnitudes by Multiplications of Lines, *i. e.* by Rectangles. (6. 24.)

24. When Rectangles have their Sides directly proportional, so that *a b* : *e h* : : *a d* : *e f*, then is the Rectangle *a c* to the Rectangle *e g*, in a *Duplicate Ratio*, to that of their Sides: For the *Ratio* of *a c* to *e g*,

e g, is compounded of the *Ratio* of *a b* to *e h*, and of the *Ratio* of *a d* to *e f* (6. 26.). But the *Ratio* of *a b* to *e h* is in this Case (by the Supposition) the same as the *Ratio* of *a d* to *e f*; so that to gain the *Ratio* which the Rectangle *a c* hath to *e g*, we need only take *twice the Ratio* of *a b* to *e h*. For Example, if as here *a b* be double to *e h*, and *a d* double to *e f*, the Rectangle *a c* shall be twice double, that is, quadruple of the Rectangle *e g*. And if *a b* had been triple of *e h*, and consequently *a d* triple of *e f*: Then the Rectangle *a c* would have been three times triple, that is nine times as big as *e g*; Or if *a b* had been quadruple of *e h*, *a c* would have been 16 times as great as *e g*.



25. If a third Line be taken as *n o*; and it be so proportional that *a b* : *e h* :: *e h* : *n o*. Then shall the two Rectangles *a c* and *e g* be to one another, as the two Lines *a b* and *n o*: (*vid. Fig. Preced.*)

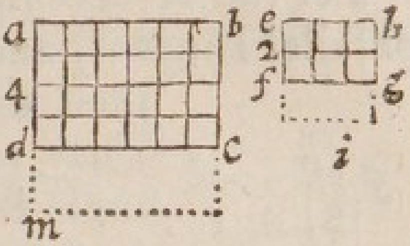
n|.....|*o*

For *a b* is to *n o* in a duplicate *Ratio* of *a b* to *e h*. And if *a b* had been (as it is double,) triple or quadruple of *e h*: Then would *a b* have been in a *Ratio* three times triple; or four times quadruple of (*that is nine or 16 times as great as*) the *third Proportional* n o.

26. Those Rectangles which have their Sides thus proportional: That *a b* : *e h* :: *a d* : *e f*, are called *Similar*, whose *Homologous Sides* are those which answer each to other in the Proportion, as *a b* and *e h*, or *a d* and *e f*: For as *a b* is the greatest Side of the Rectangle *a c*, so *e h* is also the greatest Side of the Rectangle *e g*.

27. All

27. All Squares are similar Rectangles. For 'tis plain that if *a b* be double or triple of *e h*, *a m* must also be double or triple of *h i*: Because *a m* is equal to *a b*, and *h i* to *e h*.



28. All similar Rectangles are to each other as the Squares of their Homologous Sides. I say the Rectangle *a c* is to the Rectangle *e g* : : as the Square *b m* to the Square *e i*. For as well Squares as Rectangles are to one another in a duplicate *Ratio* of *a b* to *e h* (6. 29. 30.)

29. To know the *Ratio* between two solid Rectangles or *Parallelopipeds*, there ought to be known the several *Ratio*'s that their Bases and Heights have to each other; because the *Ratio* of one Solid to another is compounded of the *Ratio*'s of their Lengths and Breadths and Thicknesses or Heights; as is easie to conceive, if that be well understood which hath been said about the Proportions of Rectangles. For if one Parallelopiped hath its Base double to the Base of another, and its Height, triple of the Height of the other: The former will be twice triple, or three times double, or in one Word sextuple of the latter.

30. If the *Bases* of two Parallelopipeds be *Reciprocally* as their *Heights*, those Parallelopipeds are equal: Which is proved by the 27th of this Book; for as much as one exceeds the other in Breadth and Length, so much doth the other exceed it in Height.

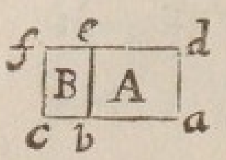
31. When Parallelopipeds have all their Sides proportional, they are called *Similar*; and they are in a *Triplicate Ratio* of their Sides, as it hath been proved of Rectangles, that they are in a *Duplicate Ratio* of their Sides.

32. Similar Parallelopipeds are to one another as the Cubes of their Homologous Sides; for both Cubes and Parallelopipeds are in a *Triplicate Ratio* of their Homologous Sides.

34. All

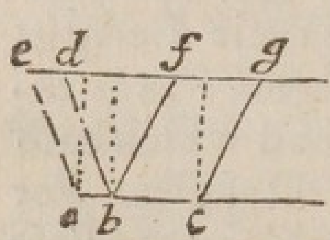
34. All Rectangles, having the ſame or equal Heights, are to one another as their Baſes, and having the ſame Baſes their Heights are equal.

Let the Rectangles A and B be between the ſame parallel Lines *d f* and *c a*; ſo that *a d* be equal to *c f*: then do I ſay, that A : B :: *a b* : *b c*.



That the Rectangle A is to the Rectangle B, as the Baſe *a b* to the Baſe *b c*: And that if, for Inſtance, *a b* be double to *b c*, then ſhall A be double to B. For A is nothing but the Line *b a* multiply'd by *d a*. (6. 17.); and B is nothing but the Line *c b* multiply'd by the ſame Line *a d*, or (which is all one) *b e* or *f c*. Wherefore (6. 15.) A : B :: *a b* : *b c*.

35. All *Parallelograms* which are between the ſame Parallels (*or which have the ſame Height*) are as their Baſes. I ſay the Parallelogram *e b* is to the Parallelogram *b g* :: as the Baſe *a b* is to the Baſe *b c*. For having made the two prick'd Rectangles on the ſame Baſes, thoſe will be equal to the Parallelograms, (by 3. 14.) But thoſe Rectangles are as their Baſes (by the Precedent). Wherefore the Parallelograms muſt alſo be as their Baſes; That is *e b* : *b g* :: *a b* : *b c*.



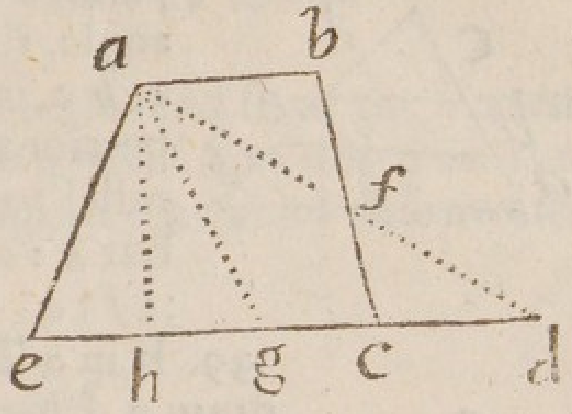
36. All Triangles (*which have the ſame Heights*) or are between the ſame Parallels, as are their Baſes; For they are Halves of Parallelograms (3. 8.)

37. When Triangles (as thoſe in the following Figure) have their Baſes on one and the ſame Line, and their *Vertices* or Tops meeting in the ſame Point: They are taken to be between the ſame Parallels, as *a d e* and *c d e*, and *a d e* and *b d e* (*becauſe they have the ſame perpendicular Height.*)

PRO-

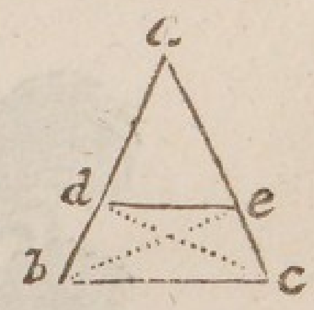
PROBLEM I.

Hence may a Trapezium as *a b c e*, whose two sides *a b* and *e c* are parallel, be divided into any given *Ratio*.



For take *c d* = *a b* and draw *a d*, then will the Triangles *a b f* and *f c d* be equal (by 14. 2.) and consequently the Triangle *e a d* = Trapez. *e a b c*. Wherefore if you divide *e d* the Base of the Triangle *e a d* into any Number of Parts, or according to any *Ratio*, Lines, drawn from the Vertex to such Divisions of the Base, will divide the Triangle *e a d*, and consequently the Trapezium, in the same *Ratio*.

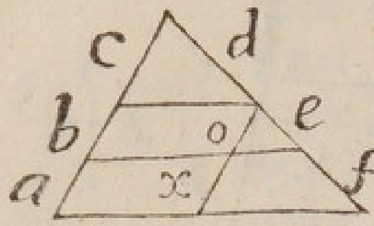
38. If in any Triangle a Line be drawn parallel to the Base, that Line shall cut the Legs proportionally. Let the Triangle be *a b c*, and let the Line *d e* be parallel to *b c*.



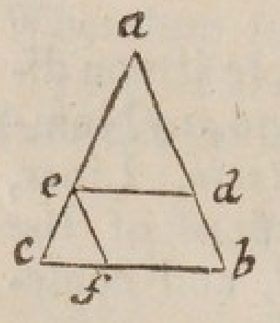
I say that *a d* : *a e* :: *a b* : *a c* :: *d b* : *e c*, &c. Draw the Lines *d c* and *e b*, then shall the Triangle *e c d* be to *e a d*, as the Base *e c* is to *a e*. (6. 40. 46.) So also the Triangle *d e b* is to *e a d* :: as the Base *d b* is to *d a*. But the Triangle *e c d* is equal to *d e b* (3. 15.); wherefore the Triangle *b d e* (or *c e d*) is to the Triangle *e a d* :: as *b d* is to *d a* : : or as *c e* to *e a*. Therefore also must *b d* : *d a* : : *c e* : *e a*, because both the *Ratio* of *b d* : *d a*, and also that of *e c* : *e a*, are the very same with that of the Triangle *b e d* or *c e d*, to the Triangle *a d e*.

COROL-

COROLLARY.



If many Lines are drawn parallel to the Base of any Triangle, the Segments of the Sides a, $b\ c$, and d, e, f, will be proportional, for drawing $o\ x$ parallel to $a\ b\ c : b = o$ and $a = x$, but $x : o :: f : e$: Wherefore $a : b :: f : e$. Q. E. D.



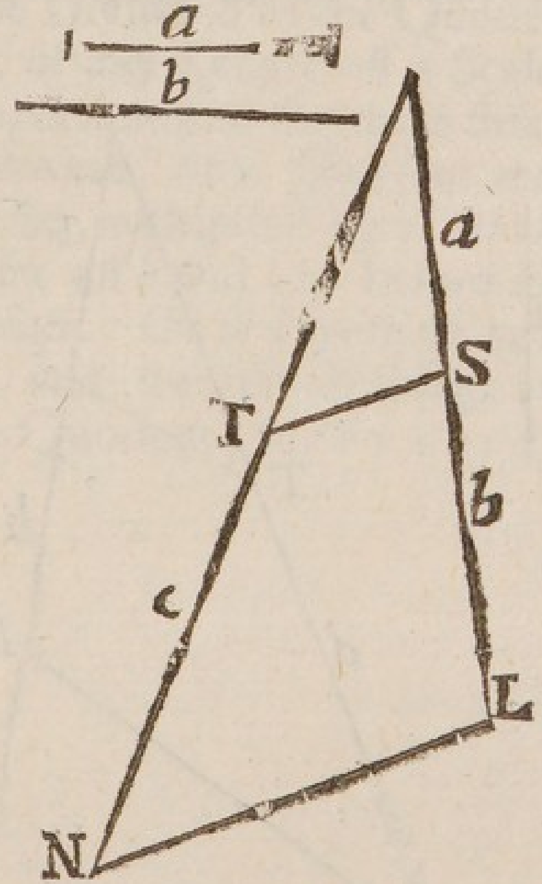
39. If in a Triangle, as $a\ c\ b$, you draw a Line $d\ e$ parallel to the Base $c\ b$, I say, that $e\ d : c\ b :: a\ e : a\ c$: or as $a\ d : a\ b$. For drawing $e\ f$ Parallel to $a\ b$; $f\ b$ will be equal to $e\ d$. (3. 9.) But by the Precedent $f\ b : c\ b :: a\ e : a\ c$. Wherefore $e\ d$: ($f\ b$) $c\ b :: a\ e : a\ c$, or as $a\ d$ to $a\ b$.

PRO-

PROBLEM I.

wo Lines a *and* b *being given, to find* c, *a third Proportional to them.*

Make any Rectilineal Angle, and from the Vertex
 Top of it, set the two given Lines down on the
egs, as you see in the Figure. Set also *b* downward



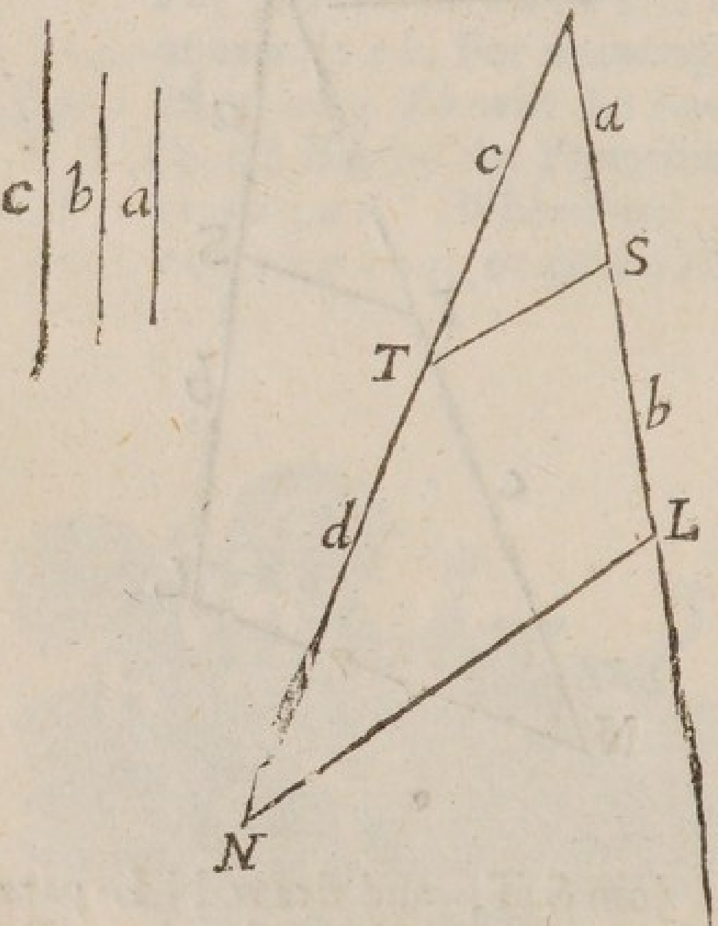
om S to L, join S T, and draw N L parallel to
; so shall T N be *c*, the Line sought; For *a* : *b*
: *b* : *c*, by this Proposition.

PRO-

PROB. II.

If three Lines, as a, b, *and* c, *had been given, to find a fourth Proportional (as* d) *to them, or to work the Rule of Three in Lines, you must proceed thus.*

Set the two first Lines *a* and *b* from the Vertex down on the same Leg; and then set *c* the third Line, from the Vertex on the other Leg: Draw the



Line TS, and thro' the Point L draw LN parallel to it; So shall TN be equal to *d*, the fourth Proportional sought; for *a* : *b* :: *c* : *d*, by the precedent Propositions.

PROBLEM III.

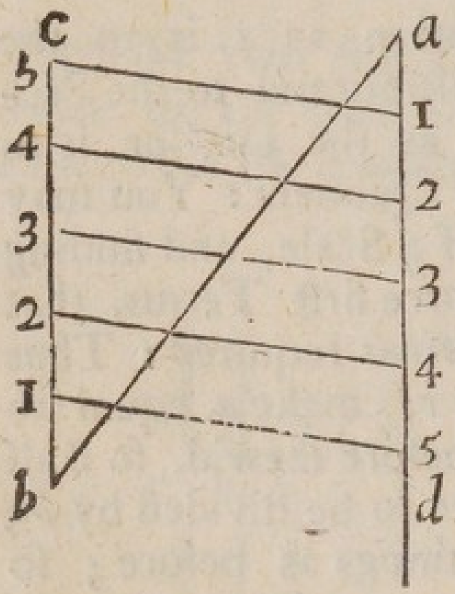
And this way 'tis very easy to find a Line that shall express the Product *of any two Numbers or Quantities: Or the* Quotient *of one divided by the other.*

For since in all Multiplication, as 1. is to the Multiplicator : : So is the Multiplicand to the Pro-luct: And since in Division, as the Divisor is to : : So is the Dividend to the Quotient: You may ake your 1. of any Length off a Scale, and finding . fourth Proportional to the three first Terms, that hall be a *Product*, or a *Quotient* required: Thus f b were to be multiplied by c, make a equal to Jnity, and set off b and c as before shew'd, so shall ! be the *Product*. Or if d were to be divided by b; ake $a = 1$, and set off all things as before; so hall c be the Quotient; for $b : a :: d : c$.

PROBLEM IV.

To divide a given Line a b *into any Number of equal Parts: As ſuppoſe into Six.*

Make at *a* and *b* any two equal Angles, and on the Legs *d a* and *c b* run with a Pair of Compaſſes five equal Diviſions (for they muſt be always one leſs in Number than the required Diviſion or Parts of the Line given) drawing alſo Lines acroſs from one Point to the other, as you ſee in the Figure; ſo ſhall thoſe Lines divide the given Line *a b* into the ſix Parts required: For the croſſing Lines being parallel one to another, muſt divide *a b* in the ſame Proportion as *a d* and *b c* are divided.

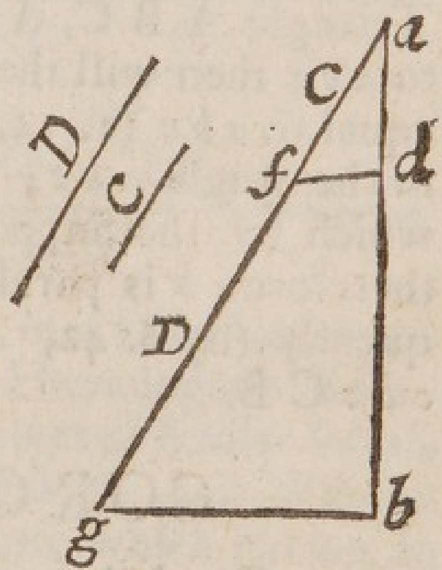


PRO-

PROBLEM V.

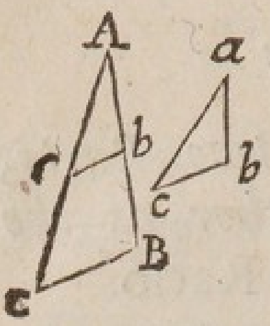
To divide a given Line a b *into two Parts, ſo that they ſhall be to each other as the Line* C *to* D; *or in any given* Ratio.

Make any Angle with the given Line *a b*, and ſet the Line *c* from its Vertex *a* to *f*. And ſet the Line D from *f* to *g*; draw the Line *g b*, and thro' *f*, a Parallel to it, as *f d*: So ſhall the Point *d* divide *a b* in the Ratio required: For C : D :: *a d* : *d b*.



And much the ſame way may you cut off from any given Line *a b* any Part or Parts required; as ſuppoſe $\frac{2}{5}$.

Make any Angle as *g a b* as before, and ſet on the Leg *a g*, *a g* equal to five Parts taken off from any Scale: Then ſet two ſuch Parts from *a* to *f*, join *g b*, and draw *f d* parallel to it; ſo ſhall *a d* be equal to $\frac{2}{5}$ of *a b*.

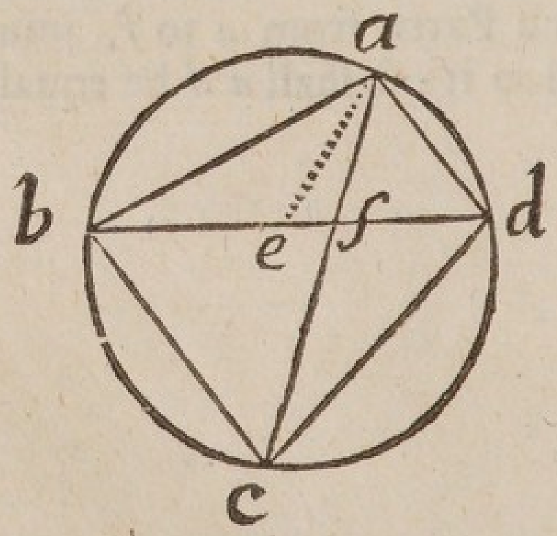


38. Those Triangles are called *Like* or *Similar*, which have all their three Angles respectively equal to one another, *or which are Equiangular:* v. gr. If the Angle A be equal to *a*, the Angle B to *b*, and C to *c*, then the whole Triangle A B C is *Like* or *Similar* to the Triangle *a b c*.

39. All similar Triangles have their Sides about the equal Angles proportional. I say, A B : *a b* :: A C : *a c* :: B C : *b c*, &c. For take in the greater Triangle A B C, A *b* equal to *a b*, and A *c* equal to *a c*; then will the Triangle A *b c* be every way equal to *a b c* (2. 11.) and the Angle A *b c* is equal to the Angle *a b c*; wherefore it will be also to B, which by the Supposition was equal to *a b c*, and therefore *c b* is parallel to C B (1. 31.) and consequently (by 6. 42, 43.) A *b* : A B :: A *c* : A C :: *c b* : C B.

COROLLARY I.

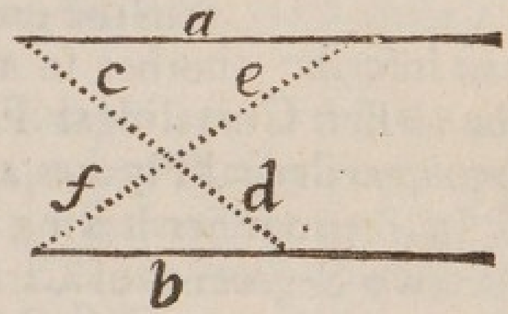
If a Quadrilateral Figure, as *a b c d*, be inscribed in a Circle, the Rectangle under the Diagonals *b d* and *a c*, is equal to both the Rectangles under the opposite Sides: That is, $ac \times bd = ba \times cd + ad \times bc$, make the Angle $bae = cad$; and then adding the Angle *e a f* to both, the Angle $baf = ead$: And the △ *a a d* similar to △ *b a f*: Then will *a c* : *c d* :: *b a* : *b e*. (by this Prop.) Wherefore $ac \times be = cd \times ba$. Again also



so $ad : de :: ac : cb$. Wherefore $ad \times cb = de \times ac$. But $ac \times be + ac \times ed = ac \times bd$. Wherefore $ac \times bd = ba \times dc + da \times bc$. Q. E. D.

COROL. II.

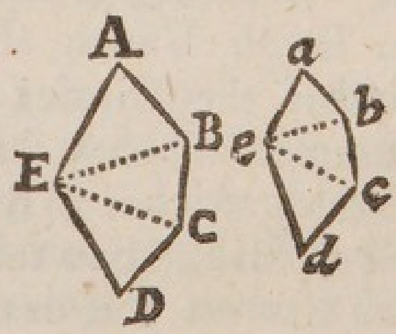
The Segments of Lines intersecting each other between two Parallels, are proportional:



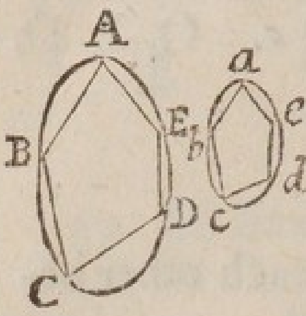
That is, $c : d :: e : f$, for by similar Triangles $c : e :: d : f$; wherefore alternately $c : d :: e : f$. Wherefore $cf = de$.

40. All similar Triangles are in a Duplicate *Ratio* of, or as the Squares of their Homologous Sides; for similar Triangles are the Halves of similar Parallelograms, wherefore they must be as their Wholes.

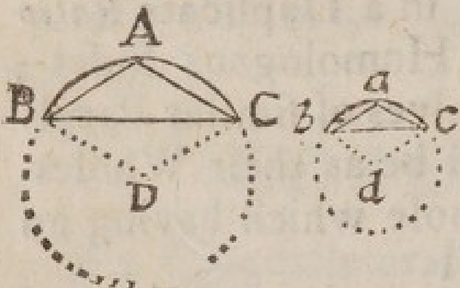
41. Similar Polygons are those which having an equal Number of Sides, have all the several Angles in one, equal to those in the other, and also the Sides about those equal Angles proportional. As if the Angle A be equal to *a*, B to *b*; and moreover AB : *ab* :: BC : *bc* :: CD : *cd*; then those two Polygons are similar.



42. And among *curvilineal* or *mixt Figures*, those are *similar* in which you may inscribe, or about which you may circumscribe similar Polygons; so that any Polygon being inscribed or circumscribed about one Figure, you may inscribe or circumscribe a similar one about the other. For instance, if having inscribed any Polygon, as A B C D E, in the greater *curvilineal* Figure you can inscribe another in all respects similar to it in the lesser Curvilineal Figure *a b c d e*, then those two curvileneal Figures are similar.



In like manner having taken two mixt Figures, as the two Segments of Circles B A C and *b a c*; and having inscribed in one any Triangle at pleasure, as B A C; if then you can inscribe in the other Segment another Triangle *b a c*, that shall be similar to the former; then shall those two *Segments* be similar Figures. And if the Circles of which they are Segments be compleated, they shall be similar Parts of those two Circles; so that if BAC be a third Part of its Circle, *b a c* shall also be a third Part of its Circle: And if to the Centers you draw the Lines B D and C D, and also *b d* and *c d*; the Angles D and *d* shall be equal. (See 4. 11. and the following Propositions.)



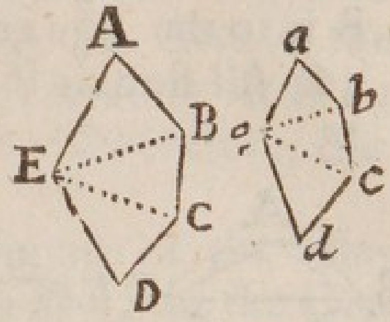
COROLLARY.

The Peripheries of Circles are as their Diameters: For as BA : *ba* :: B D : *bd* :: 2 B D : 2 *b d* :: so it will be of every side of the inscribed or circumscribed Polygon; wherefore the Sum of them all, that is the Peripheries, must be in the same *Ratio*.

43. All

43. All Circles are ſimilar Figures.

44. All ſimilar Polygons may be divided into an equal Number of ſimilar Triangles. Let the ſimilar Polygons be A B C D E and *a b c d e*; and let the firſt be divided into Triangles by the Lines E B and E C (3. 14.) I ſay that if the other be alſo divided into Triangles by the Lines *e b* and *e c*, all the Triangles in one ſhall be (*reſpectively*) ſimilar to thoſe in the other.



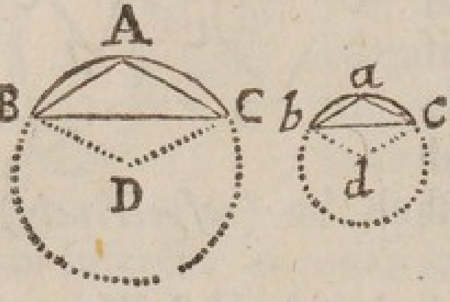
For inſtance, I ſay the Triangle *a b e* is ſimilar to A B E: for the Angle *a* is equal to A, (by the Suppoſition) and alſo A B : *a b* : : A E : *a e* (by the ſame;) wherefore the Triangle A B E is ſimilar to *a b e*. (6. 46.) Again, the Angle E B C may be proved equal to *e b c*; becauſe the Angle A B C is (by the Suppoſition) equal to *a b c*, and it was proved (*in the laſt Step, where the Triangle* A B E *was proved ſimilar to* a b e) that the Angle *a b e* is equal to A B E; wherefore from equal things taking away equal, the Angle E B C remains equal to the Angle *e b c*. In like manner the Angle *e c b* is prov'd equal to E C B, and conſequently (6. 45.) the whole Triangle *e b c* will be ſimilar to E B C; and ſo of the reſt.

Hence the Practice of making on a Line given a Polygon ſimilar to one aſſigned is derived. For dividing the given Polygon into Triangles, make a Figure, conſiſting of a like Number of ſimilar Triangles, on the given Line.

45. All ſimilar Polygons are to one another in a Duplicate *Ratio* of, or as the Squares of their Homologous Sides. I ſay, as the Square of A B : Is to the Square of *a b* : : So is the whole Polygon A B C D E : To the Polygon *a b c d e*. For ſince all the Triangles in one Polygon are ſimilar to thoſe

in the other (6. 51.) All in one Polygon will be to all thoſe in the other in a duplicate *Ratio* of any of their Homologous Sides; that is, as the Square of A B is to the Square of *a b*.

46. All ſimilar Figures, even Curvilineal ones, are to one another as the Squares of any Side of any ſimilar Figures, which can be inſcribed or circumſcribed about them, *v. gr.* Let there be two Circles, in which are inſcribed two ſimilar Triangles *a b c* and A B C: I ſay, the whole Circle A B C, is to the Circle *a b c* : : So is the Square of B C, to the Square of *b c*, or, which is the ſame thing, as the Square of the Radius B D to the Square of the Radius *b d*. For in or about the Circle *a b c* may be inſcribed or circumſcribed any Polygon you pleaſe (or at leaſt ſuch an one may be imagined) (4. 30.) But every Polygon inſcribed in *a b c* will have a leſs *Ratio* to the Circle A B C, than the Square of *b c* hath to the Square of B C: and every one circumſcribed about *a b c* will have a greater *Ratio* to the Circle A B C, as is eaſy to prove by the Precedent, and from what hath been ſaid of Circles in the fourth Book. Wherefore all ſimilar Figures, &c.



COROLLARY I.

I. Circles are to each other as the Squares of their *Radii* or Diameters: for ſuppoſe a Circle whoſe Radius is *r*, and then another Circle greater, or leſs than that; and call its Radius R, then will its Diameter be 2 R: then whatever the *Ratio* of the Diameter (2 R) be to the Periphery, let it be expreſſed by by the Letter *e*, then will 2 R*e* (or *e* times the Diameter) be the Periphery; and half of this, *viz.* R*e*, multiplied by R will be the Area, *viz.* R R *e*. And by the same Method of reasoning, the Area of the other Circle will be *r r e*. But certainly R R *e* : *r e* :: R R : *r r* :: 4 R R : 4 *r r*. Wherefore, &c.

II. Hence 'tis plain, that the Square of the Diameter of any Circle is to the Area of it, as the Diameter is to $\frac{1}{4}$ Part of the Periphery.

For 4 R R : R R *e* :: 2 R : $\frac{2\,\mathrm{R}\,e}{4}$ ($= \frac{1}{2}$ R *e*.)

As is plain by multiplying the Extreams and mean Terms by one another.

III. Hence also 'tis plain (again) that the Peripheries of Circles are as their Diameters.

That is, 2 R : 2 *r* :: 2 R *e* : 2 *r e*.

IV. And since the Area of every Circle is *r r e*, that is, the Product of the Square of the Radius multiplied into the Name of the Ratio, between its Diameter and Periphery.) A very ready way (for common use) to find the Area of a Circle whose Radius is given, will be to multiply the Square of the Radius into this or such like Decimal 3. 1. Thus suppose the Radius 9 Inches : 81 × 3. 1 = 251. 1. which is very nearly the Area in square Inches, tho' something less.

47. All this may be apply'd to Solids. And therefore *similar Solids* are *such*, as have their Angles all equal, and the Sides about those Angles Proportional; or *(if they are of a spherical or of any spheroidical Figure) such*, as can have similar Solids inscrib'd or circumscrib'd in or about them, &c.

48. Simi-

48. Similar Solids are to one another in a (*Triplicate Ratio* of, or) as the *Cubes* (*of their Homologous Sides*, &c.) See 6. 36, 37, *&c.*

(*And therefore all Spheres must be to one another as the Cubes of their Diameters*, &c.) Which may be easily thus proved; the Solidity of the Sphere may be expressed after this manner; by what is said, in the Corallaries in *p.* 75, 76.

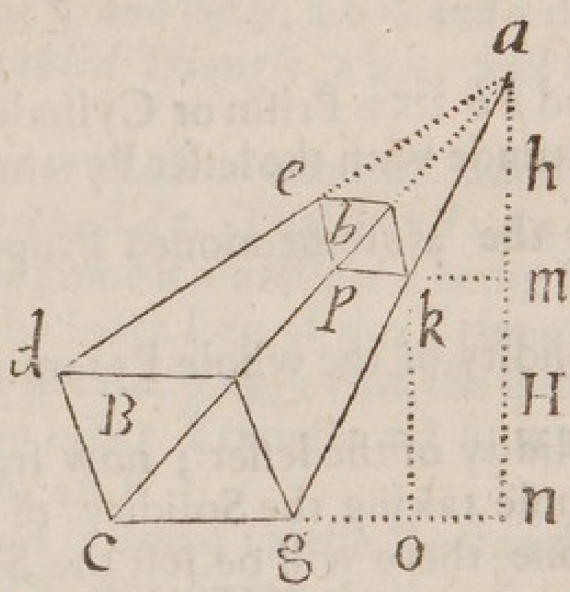
The Area of a great Circle of the Sphere, whose Radius is R or r, being RRe or rre (by *Cor.* 1. *Art.* 53.) 4 times that will be the Surface of each Sphere; that is, $4RRe$ the Surface of the greater and $4rre$ the Surface of the lesser; and multiplying the Surface by $\frac{1}{3}$ of the Radius, the Solidities will be $\frac{4RRRe}{3}$ and $\frac{4rrre}{3}$: Which two Quantities being multiplied and divided by the same, will be in the same *Ratio*, when without such Multiplication and Division: That is $\frac{4RRRe}{3} : \frac{4rrre}{3} :: RRR : rrr$. That is, Spheres are as the Cubes of their Radii, and consequently, as the Cubes of their Diameters. Q. E. D.

PRO-

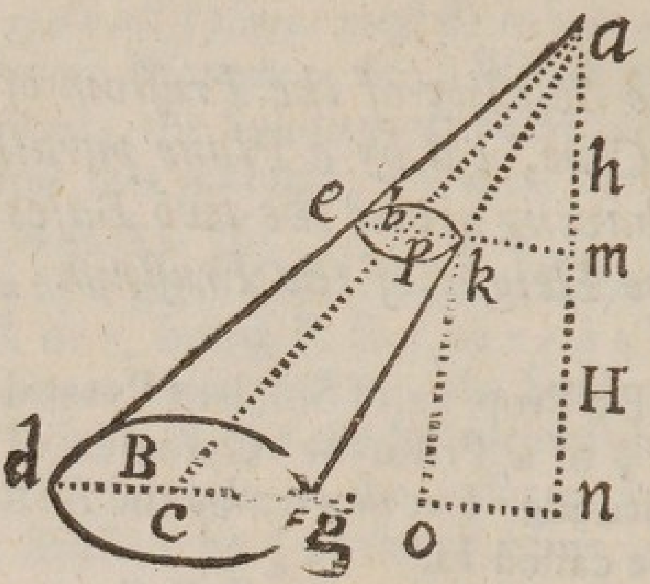
PROBLEM.

To find the Solidity of the Frustum of a Pyramid or Cone, cut by a Plane parallel to the Base, having given the two Bases together with the Height of the Frustum.

Solution. By Prop. 32. of Solids, a Pyramid or Cone is equal to $\frac{1}{3}$ of a Prism or Cylinder of the same Base and Altitude. Let $m\,n = k\,o$ the Altitude of the Frustum, be called H; and $m\,a$ the Height of the Top-piece wanting h; the Greater Base of the Frustum B, and the Lesser b; the Triangles $a\,p\,k$ and $a\,c\,g$ are similar, ($c\,g$ and $p\,k$ being parallel *ex Hyp.*) wherefore $c\,g : g\,a :: p\,k : k\,a$, and alternately $c\,g : p\,k :: g\,a : k\,a$. But $g\,a : k\,a :: n\,a : m\,a$ (from the Similarity of the Triangles, $g\,a\,n$ and $k\,a\,m$;) wherefore *ex equo* $c\,g : p\,k :: n\,a.\ m\,a$; and by Division $c\,g - p\,k : p\,k :: n\,a - m\,a : (= m\,n)\ m\,a$; which put into Symbols (putting $c\,g$ the Side of the Base $=$ S, and $p\,k = s$) will stand thus $S - s : s :: H : \dfrac{SH}{S-s} = h$. Wherefore having found the Height of the little Pyramid or Cone which is wanting, I say, having found it in known Terms, it will be easy to find the Solidity of the Frustum; for multiplying the Base B into the whole Height $H + h$, the



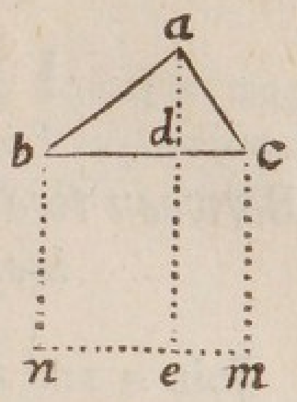
the Product $BH + Bb =$ a Prism of the same Base and Altitude with the whole Pyramid or Cone;



and $bh =$ a Prism or Cylinder of the same Base and Altitude with the lesser Pyramid or Cone. Wherefore, by the aforementioned Proposition, $\frac{BH + Bb}{3} =$ Solidity of the whole Pyramid or Cone, and $\frac{bh}{3} =$ Solidity of the lesser; now from the Solidity of the whole taking the Solidity of the lesser Pyramid or Cone, there will be left the Solidity of the Frustrum required, *viz.* $\frac{BH + Bb - bh}{3} =$ Solidity of the Frustrum.

The Theorem in Words is this: *Multiply the greater Base by the whole Height, and from the Product substract the upper Base multiply'd by the Height of the Top-piece wanting, and* $\frac{1}{3}$ *of the Remainder will give the Frustrum.*

49. If in a Rectangle Triangle *a b c*, a Line as *a d* be drawn from the Vertex or Top of the Right Angle, perpendicular to the Base, Hypothenuse, or longest Side *b c*, it shall divide the Triangle *a b c* into two other Rectangled ones, *a b d* and *d a c*, which will be similar to each other, and to the whole *b a c*. For, 1. All the three Triangles have one Right Angle. 2. The Triangles *a b c* and *a b d* have the Angle *b* common to both: Wherefore they are similar (6. 45.) 3. The Triangles *a b c* and *a d c* have also the Angle *c* common to both: therefore they two are similar; *and lastly*, a b d *and* a d c *being both similar to one third Triangle* a b c, *will be so to each other.*)



50. The Perpendicular *a d* is a mean or middle Proportional between *b d* and *d c*. That is, *c d* : *d a* : : *d a* : *d b*. For the Triangles *c d a* and *b d a* being similar (by the last) *c d* (the lesser Leg of the Triangle *c d a*) shall be to *a d* (the greater Leg) : : As the same *a d* (the lesser Leg of the other Triangle *a d b*) is to *b d* the greater Leg. (6. 46.)

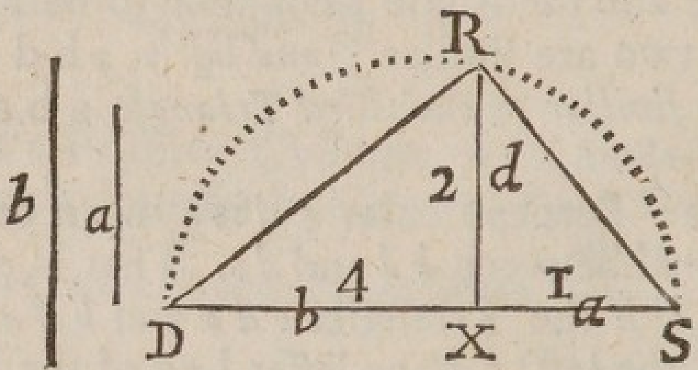
51. The Square of *a d* is equal to the Rectangle made between *c d* and *d b*. For, since *c d* : *d a* : : *d a* : *d b*, (by the last) the Rectangle of the Extreams *c d* and *d b* is equal to the Rectangle of the mean Terms *d a* and *d a* (6. 28) But the two sides of that Rectangle being equal, because 'tis only *d a* taken twice; that Rectangle must be the Square of *d a*; and so it may be laid down as an universal Theorem, that &c. *the Square of the Perpendicular drawn from the Vertex of any Rectangle Triangle to the Hypothenuse, is equal to the Rectangle under the Segments of that Hypothenuse.*

52. The Square of a mean Proportional is always equal to the Rectangle of the Extreams.

PROBLEM I.

Between two given Lines a *and* b, *to find a mean Proportional, as* d.

Join *a* and *b* both in one Line, which make the Diameter of a Circle; and then at the Point *x*, where the given Lines join, erect a Perpendicular as *d*; that



shall be the mean Proportional required. For the Angle DRS being a Right one (as being in a Semicircle) $b : d :: d : a$, by Prop. 57.

PROB. II.

And thus may you find a Line equal to the Square Root of any Number or Quantity, by finding a mean Proportional between it and 1. For if $b = 4$, and $a = 1$; then will $d = 2$, equal to the Square Root of *b*.

PROB. III.

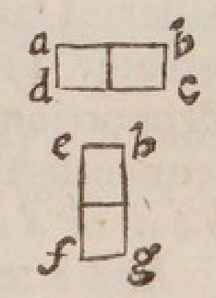
Thus alſo may a Square be found equal to any Rectangle given, by finding a mean Proportional between its Sides, which ſhall be the Side of the Square required.

PROB. IV.

To find a Square equal to any Triangle.

Find a mean Proportional between a Perpendicular let fall from any Angle to an oppoſite Side, and the half of that Side ; and that ſhall be the Side of the Square required.

53. *A Rectangle being given to make another Rectangle equal to it, which ſhall have a Length given.* Let the Rectangle given be *a c*, and let it be required to make another equal to it, the Length of one of whoſe ſides ſhall be the Line *e f:* Here are now three Lines given, *viz.* *a b* and *b c* (which are the ſides of the Rectangle given) and *e f*, which muſt be one ſide of the Rectangle required. Therefore a fourth Line muſt be found which ſhall be the other ſide of the Rectangle ſought : which is done by finding a fourth Proportional to the three given Lines (6. 43.) which let be *e h*. So that *e f*: *a b* :: *b c*: *e h*; and then I ſay, the Rectangle *f h* is equal to *d b*, and is the Rectangle required. (5. 27.)



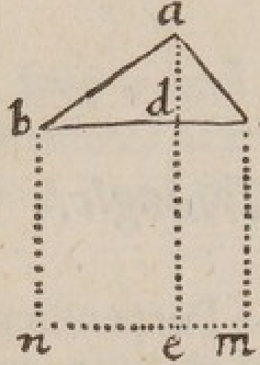
N. B. *This is called* Application *of the Rectangle, equal to a Right Line* a b, vid. Eucl. p. 6. e. 6.

I

54. To

54. To expreſs a Rectangle, you need uſe but three Letters, *v. gr.* When we ſay the Rectangle *b d c*: We mean a Rectangle, one of whoſe Sides is *b d*, and the other *d c*. But if we ſay the Rectangle *b c d*, we then mean a Rectangle, one of whoſe Sides is *b c*, and the other *c d*.

55. In every Rectangle Triangle the Square of the Hypothenuſe is equal to the (*Sum of the*) Squares of the two other Sides (*or to the Sum of the Squares of the Legs.*)



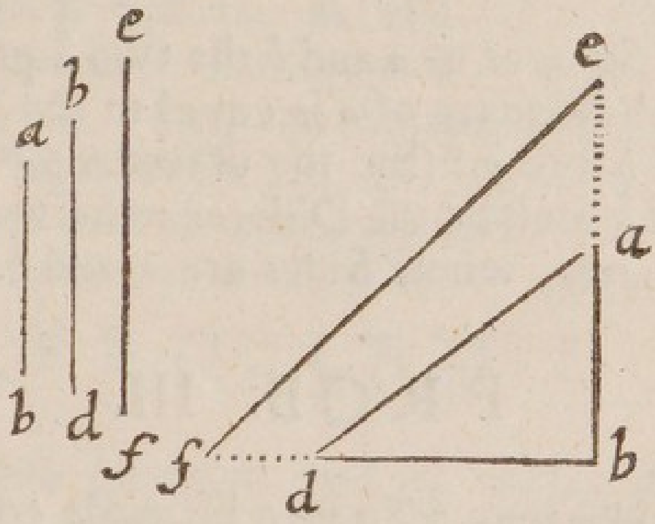
Let the Square *b m*, be divided by the Perpendicular *a d e* into the two Rectangles *d m* and *d n*. I ſay that the Rectangle *dm* is equal to the Square of *a c*, and the Rectangle *d n*, to the Square of *a b*: and that by conſequence the whole Square *b m* is equal to the Sum of the Squares of *a b* and *a c*. For, 1. The two Triangles *a d c* and *a b c* being ſimilar (6. 56.) *dc* : *c a* (in the leſſer Triangle *dc a*) :: as the ſame *a c* : *c b* in the greater Triangle *a c b*. Wherefore *a c* is a mean Proportional between *d c* and *c b* (or *c m*) and conſequently the Square of *a c* is equal to the Rectangle *b c d*, or *d c m*, that is *d m*.

And after the very ſame manner may *a b* be prov'd to be a mean Proportional between *b d* and *b c* (that is *b n*, &c.) (*for the Triangle* a b d *being ſimilar to* a b c; d b *the leſſer Side in one will be to* b a *the greater Side, as that* b a (*now the leſſer Side in the other Triangle* a b c) *is to* b c *the greater Side: That is,* d b : b a :: b a : b c, (*or* b n) *and conſequently the Square of* a b *is equal to the Rectangle* d b n, *or* d n. *And ſo both the Squares together, of* b a *and* a c, *or their Sum is equal to the Square of the Hypothenuſe.* Q. E. D.

PRO-

PROBLEM I.

Hence any two, or more Squares may eaſily be added together into one Sum. Let *a b*, *b d*, and *e f*, be the Sides of three given Squares, place *a b* and *b d*

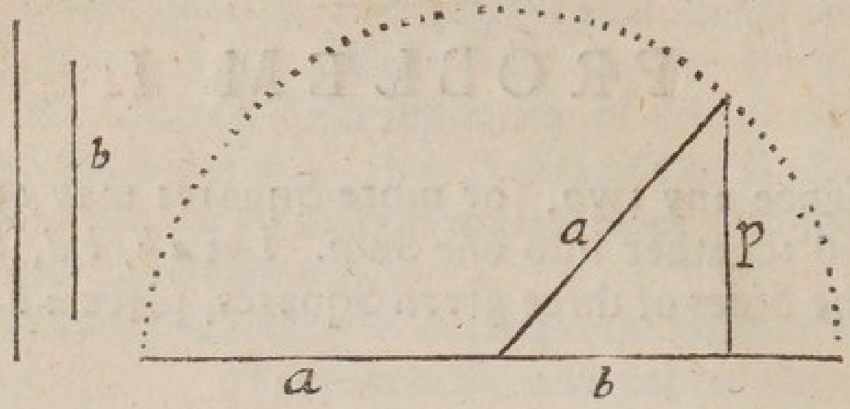


at Right Angles, and draw the Hypothenuſe *a d*, whoſe Square will be equal to the Sum of the Square, of *d b* and *a b*. Then ſet *d a* from *b* to *e*, and the given Line *e f*, from *b* to *f*; So ſhall the Hypothenuſe *f e*, be the Side of a Square equal to the Sum of the three given Squares.

PROB. II.

Or if two Squares be given, you may ſubtract one from the other, and find a Square equal to the Difference between them.

Let *a* and *b* be the Sides of the given Squares; make (the longeſt Line) the Radius of a Circle, and ſet *b* from the Center on the ſame Right Line with *a*; at the End of *b*, erect the Perpendicular *p*, which will be the Side of a Square equal to the Difference be-



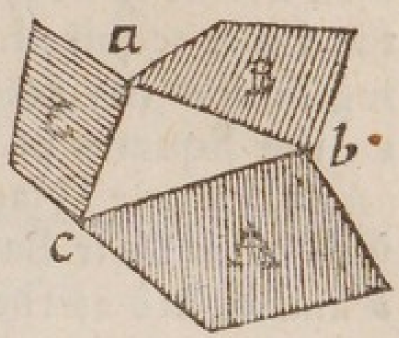
tween the Squares of *a* and *b* the two Squares given: for ſince the Square of *a* is equal to the Sum of the Square of *b* and *p*: (by the precedent *Prop.*) The Square of *p* muſt be the Difference between the two given Squares, whoſe Sides are *a* and *b*.

PROB. III.

Hence alſo may a Square be made equal to any given Polygon, or irregular Right-lined Figure: By reducing the Figure into Triangles; finding Squares equal to thoſe Triangles; and then one Square at laſt equal to the Sum of all thoſe Squares.

Or by making Rectangles equal to thoſe Triangles, which ſhall have all the ſame Height; then joining thoſe Rectangles together, ſo as to make one great one equal to them all; and laſtly, make a Square equal to that Rectangle.

59. If upon the three Sides of a Rectangled Triangle are made three similar Figures, and those similarly posited, the greatest shall be equal to the other two.

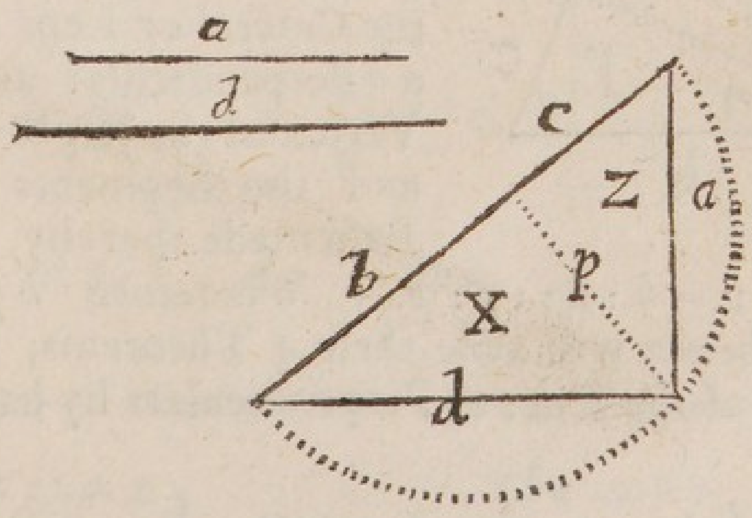


For the three Figures being similar are as the Squares of their Homologous Sides (6. 53). And therefore the Figure A shall be to B and C, as the Square of *b c* is to the Squares of *a b* and *a c*. But the Square of *b c* is equal to those two Squares (*by the last*) therefore (*the Figure* A *is equal to both* B *and* C *together*.

PROBLEM I.

To find two Lines b *and* c, *which shall have the same Ratio to one another, as two given Squares, Similar Triangles, Similar Polygons, or Circles.*

Let *a* and *d* be the Sides of the two given Squares, Triangles, Polygons; or the Diameters or Radius's of the Circles given: Set them at Right Angles to

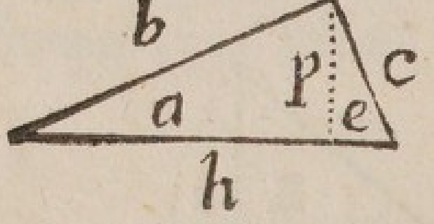


one another, as you see, and draw the Hypothenuse $b+c$, to which let fall the Perpendicular p, which shall divide the Hypothenuse into two Parts b and c, the Lines required. For the Triangles Z and X being similar (by 56. of the 6.) will be to one another as the Squares of their homologous Sides a and d, (6.47.) These Triangles also having the same Height, will be as their Bases (6. 42) wherefore their Bases b and c, are as the Squares of d and a. Q.E.D.

PROBLEM II.

This Problem may be inverted thus; To make two Squares, Triangles, &c. having the Ratio of two given Lines, b *and* c, *or in any given Ratio.*

Join the Lines into one continued Line, and then make that the Diameter of a Circle, from the Point where b and c join; erect a Perpendicular to the Curve as p, then draw d and a, and they shall be the Sides of the Squares, Triangles, similar Polygons, or the Diameters of the Circles required.



In a Right-angled Triangle let the *Hypothenuse* be h, the *Catheti* or Legs b and c, a Perpendicular from the Vertex of the Right Angle p, and the Segments of the Base made thereby, a and e.

Then 1. $h:b::c:p$. Wherefore $hp=bc$, from whence will arise these 4 Theorems, for finding any of the Sides or Perpendiculars by having the rest.

$$\left.\begin{array}{l} 1.\ h = \frac{bc}{p} \\ 2.\ b = \frac{hp}{c} \end{array}\right\} \qquad \begin{array}{l} 3.\ c = \frac{hp}{b} \\ 4.\ p = \frac{bc}{h} \end{array}$$

$$\left.\begin{array}{r} 2.\ a.\ b :: b.\ h \\ e.\ c. :: c.\ h \end{array}\right\} \text{ Wherefore } \left\{\begin{array}{l} bb = ah \\ cc = eh \end{array}\right\}$$

Whence will arise these two Theorems for finding the Segments from the 3 Sides.

$$1.\ \frac{bb}{h} = a. \qquad 2.\ \frac{cc}{h} = e.$$

$$\left.\begin{array}{r} 3.\ a.\ p :: b.\ c \\ e.\ p :: c.\ b \end{array}\right\} \text{ Wherefore } \left\{\begin{array}{l} ac = pb \\ eb = pc \end{array}\right\}$$

From whence these Theorems will arise for finding the Segments, the Sides or the Perpendiculars.

$$1.\ \frac{pb}{c} = a. \qquad 2.\ \frac{pc}{b} = e.$$

$$3.\ \frac{ac}{p} = b. \qquad 4.\ \frac{eb}{p} = c.$$

$$5.\ \frac{ac}{h} = p. \qquad 6.\ \frac{eb}{c} = p.$$

$$\left.\begin{array}{r} 4.\ b.\ p :: c.\ e \\ c.\ p :: b.\ a \end{array}\right\} \text{ Wherefore } \left\{\begin{array}{l} be = pc \\ ca = pb \end{array}\right\}$$

And consequently,

$$1.\ e = \frac{pc}{b}. \qquad 2.\ a = \frac{ph}{c}.$$

5. And since $e = \frac{pc}{b}$ and (by 3) $\frac{cc}{h} = e$.

Therefore $\frac{pc}{b} = \frac{cc}{h}$. Wherefore $pch = bcc$:

and dividing both by c, $ph = bc$.

That is, the Rectangle under the Legs, is equal to that of the Perpendicular into the Hypothenuse, &c.

For, by proceeding after this Method, the Reader may easily discover many such Propositions as these: Which I leave to exercise his Skill and Diligence this way.

I. That the Rectangle under either Leg of a right angled △, and the opposite Segment of the Base, is equal to that under the Perpendicular into the other Leg.

II. The Square of the Hypothenuse is to that of either Leg :: as the Rectangle under the Hypothenuse, and the Segment of it, opposite to that Leg, is to the Square of the Perpendicular.

III. The Solid under the Perpendicular into the Rectangle of the Legs, is equal to that under the Hypothenuse into the Rectangle of its Segments.

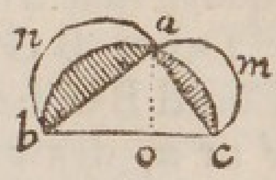
IV. The Square of the Perpendicular is to the Square of any Leg, as the Segment opposite to that Leg, is to the whole Hypothenuse.

V. Th

V. The Square of one Leg into the opposite Segment of the Hypothenuse equal to the Square of the other into its opposite Segment. Wherefore,

VI. The Squares of the Sides are as those Segments.

56. If on the Hypothenuse *b c* of a Rectangled Triangle, there be made a Semicircle *b a c*, and on the other two Sides *a b* and *a c*, two more Semicircles *b n a* and *a m c*, that great Semicircle will be equal to the other two (by the last *Proposition.*) And if from the greater Semicircle, and the two lesser ones, you take away what is common to both; which are the two shaded Segments *a b* and *a c*; what remains of each must be equal, *i. e.* the Triangle *a b c* is equal to both the Lunes *b n a* and *a m c.*



And this is the Quadrature of the *Lunes* of *Hippocrates* of *Scio.*

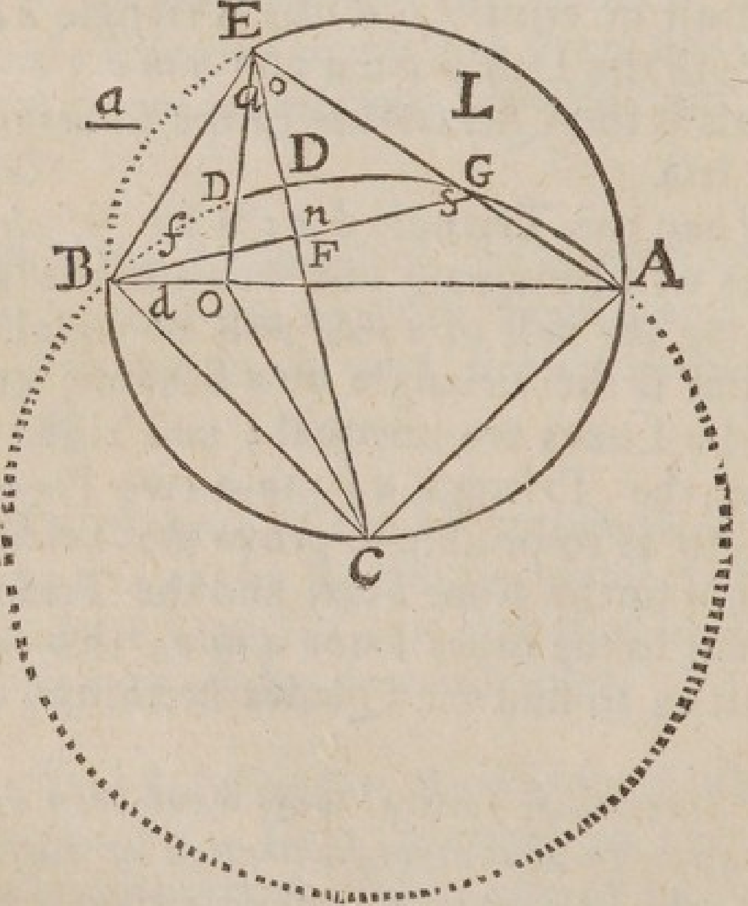
57. When the Triangle *b a c* is an *Isosceles*, then the Lunes will be equal, and then also the Triangle *a b o*, being the half of *a b c*, will be equal to each Lune. But if the Triangle be a Scalene, as in this Figure, the Lunes are unequal; and 'tis as difficult to divide the Triangle *a b c* into two Parts by the Line *a o*, so as to be able to prove the Triangle *a b o* to be equal to the Lune *b n a*, and the Triangle *o a c* to be equal to the other Lune *a m c*; this is, I say, as difficult as to find the Quadrature of the Circle.

N. B. *Since this, several ways have been discover'd of squaring any assigned Portion of these Lunes.* (*See the Philosophical Transactions*, N. 259. pag. 4, 11.) *Of which this is one.*

Let there be a greater Circle B G A C, on whose quadrantal Arch B A, let the Lune be BEAGB, or L, be drawn by describing the semicircular Arch BEA, which is one half of the lesser Circle B C A E. Let then a Line, as C E, be drawn from the Center of the greater Circle, cutting off any Portion or Segment of the Lune, as B E D: 'Tis required to square that Segment.

Draw B G at right Angles to E C; So shall the Chord B G be perpendicularly bissected in the Point F or *n*, draw also B E and E G A. I say, that the Right-Lined Triangle B E F, is equal to the Part of the Lune B E D.

For FG being equal to FB, EF common to both, and the Angles at F equal, because both Right, the



Triangle EFG will be equal to BEF: Wherefore the Angle *o* being equal to *a*, they must be both Semi-right; And consequently, *f* and S must be also

Se-

Semi-right: Therefore the three Triangles E B G, E B F and E F G, muſt be each one the half of a Square. And conſequently, G B : E B :: $\sqrt{}$: 2 : $\sqrt{}$: 1. for the Square of G B is double the Square of E B; and ſince ſimilar Segments are as the Squares of their Chords, the Segments B G muſt be double of B E: Wherefore the half of one will be equal to all the other; that is, B D F equal to the Segment B E. And therefore the Rectilineal Triangle B E F, exceeding the Portion of the Lune by the half Segment B E F, and falling ſhort of the Lune by the Segment B E, which is equal to that former half Segment B D F, the Triangle is exactly equal to the Portion of the Lune. Q. E. D.

And the Ground of all is this, that the Angle BCE being at the Center of one Circle, and at the Circumference of the other, muſt divide the Quadrantal Arch B G A, in the ſame Proportion as it doth the Semi-circular one B E A: On which depends the Equality of the Segments B E, and B D F.

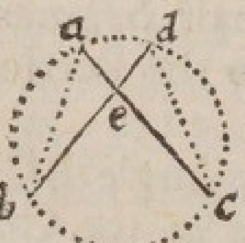
And ſince the Triangle B C A is equal to the Lune L, (as is apparent by taking the common Segment B G A B, from the Semi-circle B E A B, and from the Quadrant B G A C.) It will be eaſie to take from thence a Part, as the Triangle B O C, equal to the aſſigned Portion of the Lune. For having let fall a Perpendicular from E, to find the Point O, draw OC; and then will the Triangle B O C, be equal to the Triangle B E F, before proved equal to the Segment of the Lune. For the Triangles B C A and B E F are ſimilar, as being each the half of a Square: And therefore the former to the latter will be as the Square of B A, to the Square of B E (6. 47.) their homologous Sides. That is, as B A is to B O (6. 25.) for B E is a mean Proportional between B A and B O. Farther, the Triangle B A C, having the ſame Height with B O C, will be to it as the Baſe A B to BO.

Where-

Wherefore the two Triangles BEF and BCO, being proved to have the same Ratio to one and the same thing, must be equal. Q. E. D.

And therefore to divide the Lune according to any given Ratio, you need only divide the Diameter AB, according to that Ratio in the Point O, and from thence erect a Perpendicular to find the Point E: then draw EC, which shall cut off the assigned Portion of the Lune.

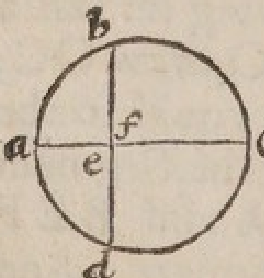
58. Two Chords cutting or crossing each other in a Circle, have the Segments *reciprocally Proportional.*



I say, that *a c* : *b e* :: *d e* : *e c*, and consequently the Rectangle *a e c* is equal to the Rectangle *d e b*.

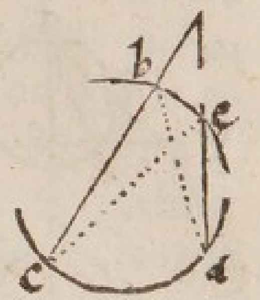
For draw the prick'd Lines *a b* and *d c*, and the two Triangles *a b e* and *d c e* will be Similar: Because 1. The Vertical or opposite Angles at *e* are equal (1. 23.) 2. The Angle *b* is equal to *c*, because standing both on the same Ark *a d*, and being in the same Segment (4. 12.) wherefore the two Triangles are Similar, and consequently *a e*. *e b* :: *d e*. *e c*. (6. 46.) Q. E. D.

59. If *a c* be the Diameter of a Circle, and *b d* a Perpendicular to it, *d e* or *b e* will be a mean Proportional between the Segments of the Diameter *a e* and *e c*. Because *d e* is equal to *e b* (by 4. 6.) and therefore since (*by the last*) the Rectangle *b e d* (that is *b e* Square) is equal to *a e c*, as the Rectangles of the Parts of all crossing Chords are; the Line *b e* or *e d*, must be a mean Proportional between *a e* and *e c*. Q. E. D.

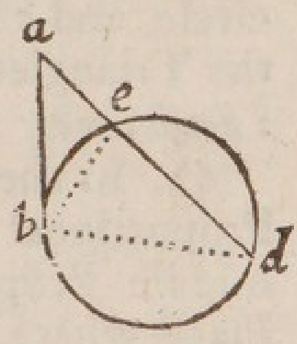


60. Two

60. Two Lines *a c* and *a d*, drawn from a Point *a*, without a Circle, to the internal and opposite Part of its Circumference; are to each other *Reciprocally* as their external Segments. I say, *a c*, *a d* : : *a e* : *a b*. and consequently the Rectangle *c a b* is equal to *d a e*. For supposing the Lines *c e* and *b d* to be drawn, the Triangles *a e c* and *a d b* will be similar, because the Angle *a* is common to both, and the Angle *c* is equal to *d*, because standing on the same Ark *b e* (4. 12.) wherefore *d a* : *a b* : : *c a* : *a e*; and alternately, *d a* : *c a* : : *a b* : *a e*. and by Inversion *c a* : *d a* : : *a e* : *a b* (6. 45.) And therefore the Rectangle *c a b* is equal to *d a c*. Q. E. D.

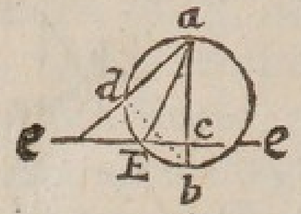


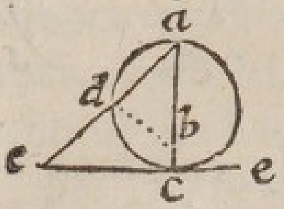
If one Line as *a b*, touch a Circle, as in the Point *b*, and another Line *a d*, drawn from the same Point *a*, do cut it; then is *a b* (*the Tangent*) a mean Proportional between *a d* and *a e* (i. e. *between the whole Secant, and the Part of it without the Circle.*)

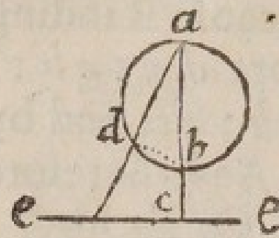


For drawing the Lines *b e* and *b d*, the Triangles *a e b* and *b a d* will be similar, because the Angle *a*, is common to both, and the Angle *a b e* (made by the Tangent, and Secant *e b*) is equal to *d* (*an Angle in the opposite Segment*) (4. 17.) therefore they are similar, and consequently *e a* (in the little Triangle) will be to *a b* : : as that same *a b* is to *a d*, in the greater Triangle : i. e. *e a* : *a b* : : *a b* : *a d*. Q. E. D.

61. Let there be a Diameter *a b* cut in *c* by an Infinite Perpendicular *e e*, whether within the Circle, as in *Fig.* 1. at the Circumference, as in *Fig.* 2. or without the Circle, as in *Fig.* 3. Let there be drawn also from the Point *a*, any Right Line, as *a e*, cutting the Perpendicular in *e*, and the Circle in *d*. I say, it shall always be as *a d* : *a c* : : *a b* : *a c*.





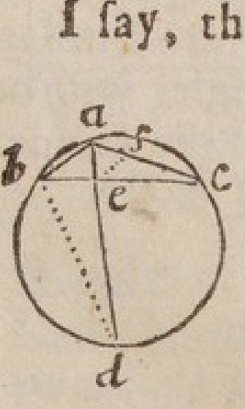


For drawing the Line *b d*, there will be made two Triangles that are similar, as *e a c* and *d a b*; which will be so, because they have one Angle as *a*, common to both, and the Angle *d* equal to *c*, because both are Right ones (for *d* is Right by 4. 14.) as being an Angle in a Semicircle, and *c* is Right by the Supposition. Wherefore the Triangles are similar, and consequently *a d* : *a c* : : *a b* : *ae*. Q. E D.

62. In the second Figure, *a d* is always a mean Proportional between *a e* and *a d*; in the first, the middle Proportional is *a* E, drawn from *a*, to the Place where the Line *e c* cuts the Circle.

63. If of a Triangle inscribed in a Circle, the Angle *b a c* be bissected by the Line *a e d*.

I say, then *b a* : *ae* : : *a d* : *a c*. For drawing the Line *e b*, there will be made two Triangles *a b d* and *a e c*, which are similar; because the Angle *d* is equal to *c* (4. 12.) as (*being in the same Segment*) or insisting on the same Ark, and *b a d* is equal to *e a c*, by the *Supposition*. Wherefore the Triangles are similar, and consequently *b a* : *a d* : : *a e* : *a c*. (and *therefore alternately* b a : a e : : a d : a c.) Q. E. D.

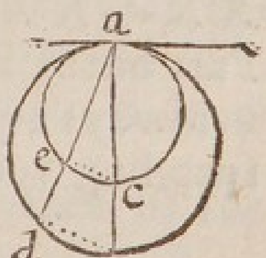


64. When

64. When the Angle at the Vertex is thus bisse-ꞏcted, the Segments of the Base bc are alſo propor-tional to the Legs of the Triangle (*i. e.*) $be : ec :: ba : ae$. For ſuppoſing ef drawn parallel to ba. Then will $ba : ac :: ef : fc$ (6. 40.) But ef is e-qual to af; becauſe the Angle aef is equal to eab, (*as being alternate Angles* 1. 31.) and conſequently to eaf (*by the Suppoſition*) wherefore [the Triangle aef is an *Iſoſceles* (2. 15.) And therefore inſtead of putting of it as before $ba : ac :: ef : fc$, we may ſay $ba : ac :: af : fc$. But as $af : fc ::$ ſo is $be : ec$ (6. 42.) wherefore $ba : ac :: be : ec$. Or, which is all one, $bc : ec :: ba : ac$. Q. E. D.

N. B. *This Propoſition is Univerſal; and if any Angle of a Triangle be biſſected, the Legs about that Angle are proportional to the Segments of the oppoſite Side made by the Line biſſecting the Angle.)*

65. If two Circles touch one another (*in a Point within*) as a, and if to that Point you draw a Tangent and a Perpen-dicular acb (which will paſs thro' both their Centers) (4. 5.) and if alſo you draw any Secant from the ſame Point, as aed. I ſay, it will always be $ae : ad :: ac : ab$. For having drawn the Lines ec and db, the Triangles aec and abd will be ſi-milar, as having the Angle at a, common; and e and d both right ones; (by 4. 14.) and conſequent-ly $ae : ad :: ac : ab$. Q. E. D.

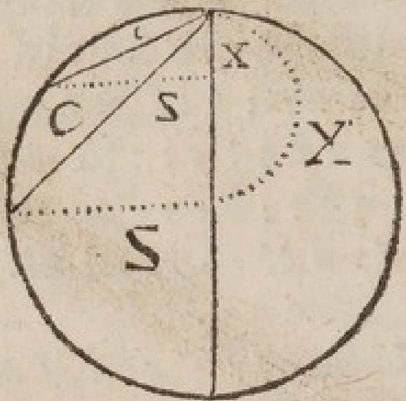


66. The Ark ec is to the Ark db, as the whole Circle aec, to the whole Circle adb. (6. 49. and 5. 11.)

PROP.

PROP.

67. If two (or more) Chords, as c, C, iſſue from the ſame End of any Diameter of a Circle; *Their Squares ſhall be directly, as the Verſed Sines* x X.



And ſhall alſo be equal to the Rectangles under the Diameter and ſuch Verſed Sines.

Let fall the right Sines s and S. Then will $cc = ss + xx$, and $CC = SS + XX$, and if the Diameter be called D, its Parts will be x and $D - x$, X and $D - X$. But $ss = x, D - xx$, and $SS = DX - XX$ (by 66 of this Book) wherefore ſubſtitute thoſe two laſt Quantities inſtead of the Equals ss and SS; and you will have $cc = Dx - xx + xx$ (that is) Dx and $CC = DX - XX + XX$ (that is) DX which proves the latter Part of the Propoſition, that the Square of the Chord is always equal to the Rectangle under the correſponding verſed Sine, and the whole Diameter.

And 'tis plain that,

$$Dx . DX :: x . X. \quad \text{Q. E. D.}$$

PROP.

68. A Circle whoſe Area is equal to the Convex Surface of a given Cone, will have its Radius a mean Proportional between the Side of the Cone and Radius of its Baſe.

Let

Let the Side of the Cone be $= a$, the Radius of the Base $= r$; then the Diameter will be $2\,r$, and the Periphery $= 2\,r\,e = c$. But half the Periphery into the Side of the Cone is $=$ to the Convex Surface of the Cone (by ....) that is, $a\,r\,e$ expresses the the *Area* of the Cone. Now since $\sqrt{:a\,r}$ is a mean Proportional between a and r (for $a . \sqrt{:a\,r} :: \sqrt{:a\,r}$ r) I imagine $\sqrt{:a\,r}$ to be the Radius of the Circle whose Area $=$ Area of the Cone. Then will its Diameters be $2\sqrt{:a\,r}$, and its Periphery $2\sqrt{:a\,r\,e}$: and by Multiplication of $2\sqrt{:a\,r\,e}$, the Periphery into $\frac{1}{2}\sqrt{:r\,a}$ the half Radius: or $\sqrt{:a\,r\,e}$ into $\sqrt{:r\,a}$ the Radius of the Circle will be $a\,r\,e = b$, the Surface of the Cone. Q. E. D.

69. The Convex Surface of a right Cone is to the Area of its Base :: as the Side of the Cone is to the Radius of the Base.

For since the Convex Surface of the Cone, (by what is said after 14. Book 4.) is equal to a Triangle whose Base is equal to the Periphery of the Circular Base of the Cone, and its Height the Side of that Cone, call the Periphery c, and the Side of the Cone a, then will $\frac{a\,c}{2}$ express the Area of the Convex Surface, and the Area of the Base will be $\frac{r\,c}{2}$. (by Art. 26. Book 4.) But there is no Doubt but $\frac{a\,c}{2} : \frac{r\,c}{2} :: a . r$. Wherefore, &c.

70. A Circle whose Radius is equal to the Diameter of the Sphere, will have its Area equal to the Sphere's Surface.

Let

Let the Radius of ſuch a Circle be $2r$, then its Diameter is $4r$, and its Periphery will be $4re$, and by multiplying that by r = half of Radius, the Area is $4rre$. Let then the Radius of the Sphere be r, then will its Diameter be $2r$, and the Periphery of a great Circle $2re$, which being multiplied by the Radius r, makes $2rre$; the half of which is rre, the Area of a great Circle; but the Area of 4 ſuch Circles is equal to the Sphere's Surface (by *Cor.* V. *p.* 76.) that is, $4rre$ = to the Sphere's Surface; which was above proved equal to the Area of the Circle, whoſe Radius was equal to the Sphere's Diameter. Wherefore, *&c.*

ELEMENTS OF *GEOMETRY.*

BOOK VII.

Of Incommensurables.

A Lesser Quantity is said to *measure* a greater, when being taken a certain number of Times, it is exactly equal to the greater. *V. gr.* Suppose a Fathom to contain six Feet; then may one Foot be said to *measure* that Fathom, because being taken or repeated six times, it will be exactly equal to the Fathom.

2. The Quantity which is thus a *Measure* to a greater Quantity, is called a *Part* of that greater; and the greater Quantity is call'd the *Multiple* of the lesser. So a Foot is the *Part* of a Fathom, and a Fathom is the *Multiple* of a Foot.

3. If you take the Quantity (*of a common* French *Pace*) which is two Foot and half, and try with that

to meaſure a Fathom, you cannot do it: Becauſe if you add that Pace only twice, it will make but five Foot, which are leſs than the Fathom; and if you take it three times, it makes ſeven Foot and half, which are more than the Fathom; ſo that this Quantity of two Foot and half cannot meaſure the Fathom, and therefore properly ſpeaking is not a *Part* of it. But nevertheleſs they may be ſaid to be *Parts* of the Fathom, becauſe this Quantity contains five half Feet; for an half Foot is a *Part* of a Fathom, becauſe being taken 12 times, it will juſt meaſure it; ſo therefore this Place contains *Parts* of the Fathom, becauſe it contains five half Feet, which are $\frac{5}{12}$, that is five twelfths of a Fathom.

4. When two Quantities are ſuch, that a third can be found which ſhall be an (*Aliquot or Even*) Part of both, that is, which ſhall *meaſure* them both exactly: Then thoſe Quantities are ſaid to be commenſurable: As for Inſtance, a Pace and a Fathom are two commenſurable Quantities, becauſe we can find a third Quantity, *viz.* half a Foot, which will meaſure them both; For if the half Foot be taken five times, it makes the Pace, and taken 12 times, it makes the Fathom.

5. But when it is not poſſible to find any third Quantity which can meaſure two others, then thoſe two Quantities are called Incommenſurables.

6. Commenſurable Quantities are as *Number to Number*, that is, thoſe Quantities can be expreſſed by Numbers, ſo that as one Quantity is to the other, ſo ſhall one Number be to the other. Thus a Line of ſix Foot or a Fathom, and a Line of two Foot and a half, as a Pace, are to one another as Number to Number. For half a Foot meaſuring them both, the latter by being taken 5 times, and the former by being taken 12 times; it's plain that one Line contains 5 half Feet, and the other 12, and therefore they are as 5 to 12, or as Number to Number. 7. If

7. If two Quantities are not as Number to Number, that is, if it be impoſſible to expreſs their Magnitudes by two Numbers, they are Incommenſurable: As is plain from the laſt.

8. We ought then to ſee whether there are in Reality any ſuch Quantities whoſe Magnitude cannot be expreſs'd by Numbers, and if there be any ſuch, we muſt ſay that there are *Incommenſurable Quantities*.

9. A *plane Number* is that which may be produced by the Multiplication of two Numbers *(one into another) v. g.* 6 is a plane Number, becauſe it may be produced by the Multiplication of 3 by 2: For twice 3 makes 6: So alſo 15 is a plane Number, ariſing from 5 being multiplied by 3; and 9 is a plane Number, produced by the Multiplication of 3 by 3.

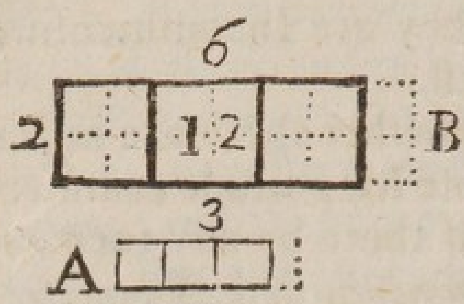
10. Thoſe Numbers which, being multiplied one by another, do produce a plane Number, are called the *Sides* of that Plane, as 2 and 3 are the Sides of the Plane 6; and 3 and 5 are the Sides of 15.

11. If we imagine Units to be little Squares, thoſe Squares may be formed into a Rectangle, if their Number be a Plane. *V. gr.* 12 Squares may be placed in the form of a Rectangle, one of whoſe Sides may be 6 and the other 2, and 48 will make a Rectangle whoſe two Sides may be 12 and 4. See the following Figures B and C.

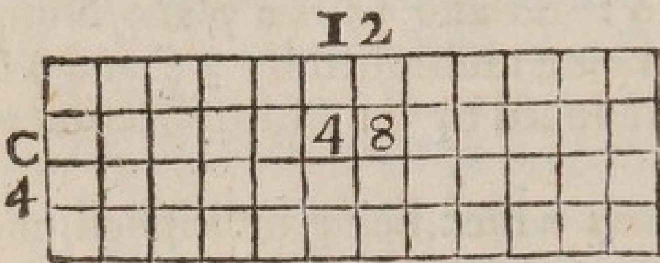
12. *A ſquare Number* is a Plane, whoſe Sides are equal; as 4 ariſing from the Multiplication of 2 by 2; as 9, the Product of 3 by 3: And 16 made by 4 multiplied by 4, *&c.*

13. A ſquare Number may be ranged into the form of a Square, and that Number which can be ranged into the form of a Square, is a ſquare Number, and that which cannot be ranged into the form of a Square, is not a ſquare Number.

14. *Similar plane Numbers* are those which may be ranged into the Form of similar Rectangles; that is, into Rectangles, whose Sides are proportional; such are 12 and 48; For the Sides of 12 are 6 and 2 (*See Fig.* B) and the Sides of 48 are 12 and 4 (*See Fig.* C.) But 6 : 2 : : 12 : 4, and therefore those Numbers are similar.



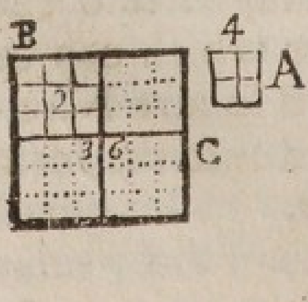
15. All square Numbers are similar Planes (6. 32.)



16. Every Number may be placed in the Form of a Right Line, and in that Disposition may be taken for a Plane. Thus 3 (*in Fig.* A.) may be conceived as a Plane similar to 12 or B; For the Sides of the Plane 3, are 1 and 3, (because once 3 is 9) and the Sides of 12 are 2 and 6. But as 1 : 3 : : 2 : 6.

17. There are Numbers which are not *similar Planes*: As if you examine from 1 to 10, you will find indeed that 1, 4, 9, being Squares are similar, and so are 2 and 8, which have one Side double to the other. But the rest as 3, 5, 6, 7, are by no means similar Planes.

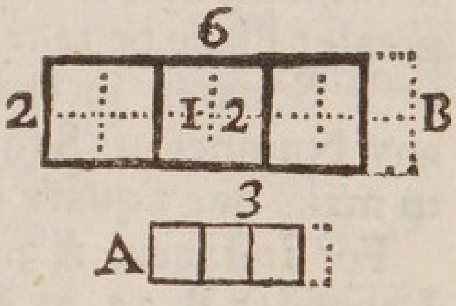
17. If one square Number be multiplied by another, the Product will be a third square Number.



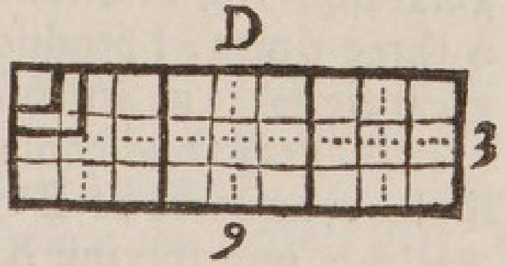
Thus A 4, and B 9, being both Squares do, when multiplied into one another, produce the Number 36 or C: And I say that third Number is a Square. For the Meaning of multiplying B by A, is take B as often as there are Units in A. But

But I may consider the whole Number B 9. as one only Square, and I can take that as often as there are Units or little Squares in A. And as the Units in A are ranged into a Square, so I can range the Square B as often into a square Form, just as if it were an Unit. So that there will be four such Squares of B, which, being placed as you see in the Figure, will make the Square C or 36.

19. If two Numbers are similar Planes, the greater may be divided into as many Squares as there are Units in the lesser. A, 3. and B, 12. are similar Planes; so that the Side 3. is to 6 : : as the Side 1. is to 2. Wherefore I can divide the Plane B, 12 into 3 Squares placed just in such a manner as those 3 little Squares in the Plane A. And every one of the great Squares of B shall answer to 4 of those in A.



So also if the Planes had been 8 and 72; I can divide 72 into eight Squares of which every one shall contain 9 of those in the lesser Plane 8. The same would come to pass also, if either one, or both the Numbers had been Fractions. As if A contain 3 and $\frac{1}{2}$, and B 14. I can divide 14 into three Squares and half, disposed just like those in A. as may be seen by the Partitions in the Figure, and by the half Square added in prick'd Lines.

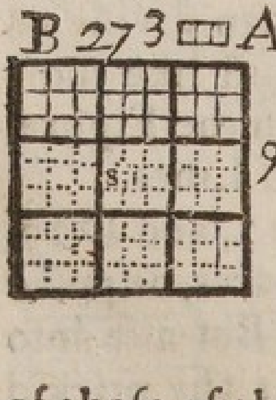


In like Manner if the Planes were B 12, and D 27, I can divide 27, not only into three Squares, disposed after the same Manner as those in A: But also into 12 Squares, so ranged as those in B, as the prick'd Lines in the Figure D do shew. The Way to do which is to divide the Sides of the greater Plane into as many Parts as the homologous Sides of the lesser Plane are divided into; the Figure ſhews the thing, and makes it eaſie.

20. Thoſe plain Numbers which can be ſo divided as that there are as many Squares in the greater Plane, as there are Units in the leſſer, are ſimilar; this is the Converſe of the former.

21. Two ſimilar plane Numbers, multiplied one into another, do produce a Square. For having divided the greater Plane into as many Squares as there are Units in the leſſer (7. 19.) One Plane will be multiplied by the other, if the greater Squares of the greater Plane be taken as often as there are Units or little Squares in the leſſer Plane: But to multiply any Number of Squares, by the ſame Number, is to make one Square out of all thoſe Squares.

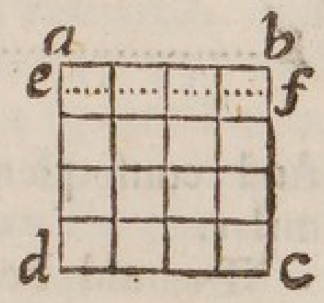
For Inſtance, A 3. and B 27. being ſimilar Planes, I conſider B. 27. as a Plane compos'd of three great Squares, as A 3. is a Plane compos'd of three Units, or three little Squares. So that if I take all theſe three great Squares, as often as there are Units in A, that is three times; I produce then three times three ſuch great Squares as are in B, that is, 9 ſuch Squares; of which every one contains 9 of thoſe in A, and all theſe 9 Squares of B contain 81 of thoſe of A; ſo that A 3. multiplying B 27. produces 81. which is a Number of the leſſer Squares rang'd into a ſquare Figure; and by conſequence a ſquare Number (7. 13.) In like Manner if the Planes were B. 12. and D. 27. I divide 27 into 12 Squares, which I multiply by 12. and there are produced 144 greater Squares rang'd in the Form of a Square, which do contain in all 324 of thoſe of the leſſer Plane. (N. B. *To divide 27 into 12 Squares, each Square muſt be 2.25. (or two and a Quarter) as you may ſee it is in the Figure* D. N° 19.)



22. If two Plane Numbers are similar, after what Form soever you range one, the other also may be so disposed. Let 3 and 12 be similar Planes. If 12 be so rang'd in a Right Line that will make a Rectangle, one of whose Sides shall be 12, and the other 1. I say that 3 may be so disposed as to make a similar Rectangle, one of whose Sides will be 6, and the other the half of one, *&c.*

23. If one Number divide another that is a square one, a third shall be produced which will be a Plane similar to the Divisor.

Let there be a Square *a c* 16, and let it be divided by any Number, as suppose by 8, which is done if you take the eighth Part of the Side *a d*, viz. *a e*, and thro' *e* draw the Parallel *e f*: For by that means you will have the Plane *a f*, which will be the eighth Part of the Square *a c*. But to divide a Number or a Plane by 8, is to take the eighth Part of that Number or Plane.



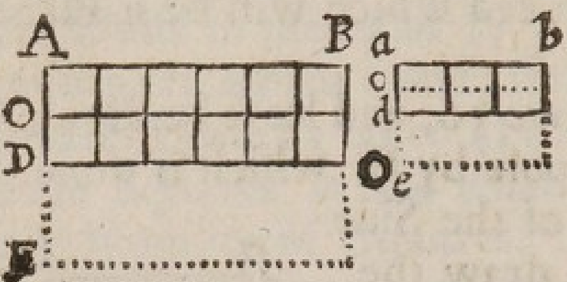
I say the Plane *a f* is similar to 8; for 8 being ranged into a Right Line, so as to make a Rectangle, one of whose Sides shall be 8, and the other 1, shall be similar to it, because *a e* was taken the eighth Part of *a d* or *a b*: Wherefore as 8 : 1 :: (which are the Sides of the Plane 8 the Divisor) so shall *a b* : *a e* (which are the Sides of the Plane of the Quotient arising when the Square *a c* was divided by 8.) Therefore if one Number divide another that is a Square, *&c.* Q. E. D.

24. If two Planes multiplying one another do produce a Square, whose Planes are similar.

25. Two Plane Numbers which are not similar, if they are multiplied into one another, cannot produce a Square. These two Propositions are Consectaries from the foregoing ones.

26. If two Numbers are ſimilar Planes, their *Equi-multiples* and any of their (*reſpectively*) equal Parts, are alſo ſimilar Planes. Let the Planes be *a b c d.* 3. and ABCD. 12. ſo that *a b* : AB :: *b c* : BC. I ſay, if you take the double of the one, and the double of the other (or any other Equi-multiple, be it what you pleaſe) thoſe Doubles ſhall be ſimilar.

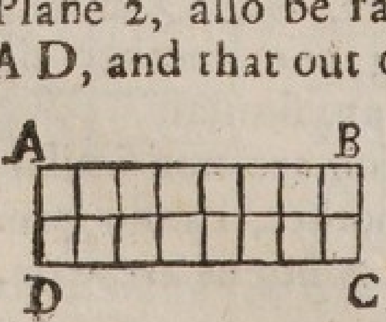
For having taken *a e* double to *a d*, and AE double to AD : in order to make the Plane *b e* double to *b d*, and BE double to BD : 'Tis clear that *a d* : AD :: *a e* : AE. But *a d* : AD :: *a b* : AB. Wherefore alſo *a e* : AE :: *a b* : AB. And conſequently the Planes *b e* and BE are ſimilar.



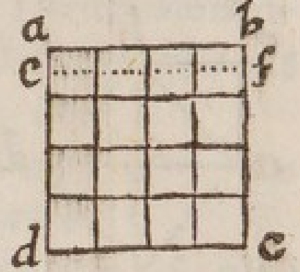
'Twould be the ſame thing had you taken their Halves *b o* and BO, or any other equal Parts of each.

27. If two Numbers are not ſimilar Planes, their Equi-multiples, and all their (*reſpectively*) equal Parts will alſo be not ſimilar, which follows from the laſt.

28. Between any two ſimilar plane Numbers whatſoever, there is to be found a mean Proportional. Let the two Numbers be 2 and 8, I ſay it is poſſible th find a Number which ſhall be a mean Proportional between them. For if we imagine the Plane 8 to be ranged in a Right Line AB, and the Plane 2, alſo be ranged in another Right Line, as AD, and that out of thoſe two Right Lines there be formed the Plane AC, 16, That Plane AC, 16. will be produced by the Multiplication of the two Numbers 2 and 8 (6. 17, and the following Propoſitions) and conſequently the Number of the little



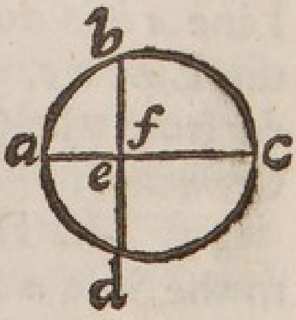
tle Squares of the whole Plane A C : 16, shall be a square Number (7. 21.) and they may be ranged into the Form of a Square (7. 13.) Let them then be dispoſed into the Square *a c*. So shall the Square *a c* be equal to the Plane A C, for 'tis only the ſame Number



diſpos'd or rang'd after another Manner. Wherefore (6. 59.) the Side *a b* 4 ſhall be a mean Proportional between A D 2, and A B 3.

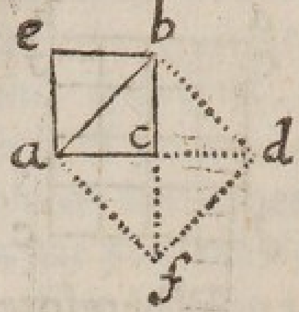
29. Between two Numbers non-ſimilar a mean Proportional can't be found. Let the Numbers be 4, and 6. Range each of them into a Right Line, and multiply them, they will produce the Plane 24. But this Plane 24 is not a ſquare Number (7. 25.) and conſequently cannot be ranged into a ſquare Form. Wherefore 'tis impoſſible to have any *Mean* between 4, and 6. For ſuch a pretended mean Proportional muſt, multiplied by it ſelf, produce a Square, which (as hath been prov'd elſewhere) will be equal to the Plane made between 4, and 6. (6. 59.) which is impoſſible, becauſe this Plane 24, made out of 4 and 6, is not a ſquare Number.

30. Let there be two Lines *a e* and *e c*, ſo to one another, as one Number to another non-ſimilar. *V. gr.* as 1. to 2. Let alſo *e b* be a mean Proportional, ſo that *a c* : *e b* : : *e b* : *e c*. I ſay, that *e b* is *incommenſurable* with the two Extreams *a e* and *e c*. For *a e* and *e c*, being as 1 to 2, (*i. e.*) as Numbers



non-ſimilar (by the Suppoſition) as alſo are all their Equimultiples (7. 27.) 'tis impoſſible to find a mean Proportional between *a e* and *e c* (by the Precedent) and conſequently *e b* cannot be to *a e*, or to *e c*, as Number to Number. Wherefore it is incommenſurable with them.

31. The Diameter of a Square *a b* is Incommensurable to the Side *a c*. For taking *a d* double to *a c*, and making the Triangle *a b d*, it shall be similar to the Triangle *a b c*; because *c d* being equal to *c b*, the Angle *d* is equal to *c b d* (2. 15.) and the Angle *d* must be a Semi-right one as well as *c a b*; wherefore *a b d* is a right Angle; and consequently *a c* : *a b* : : *a b* : *a d*. That is, *a b* is a mean Proportional between *a c* 1, and *a d* 2, and therefore Incommensurable (*by the Precedent.*)



COROLLARY.

Hence 'tis impossible to express one Square that shall be Double of another in rational Numbers.

32. The *Power* of a Line is the Square which is made upon it. Thus the Power of the Line *a c* (*Fig. preced.*) is the Square *a e b c*; and the Power of the Line *a b* is the Square *a b d f*. And we say that Line *a b* is double in Power (in Latin *bis potest*) to the Line *a c*, which is a manner of speaking borrowed from the *Greeks*, and generally receiv'd amongst *Geometers*.

33. The Diameter *a b* is *Commensurable in Power* to the Side *a c*: That is, its Square *a b d f* is Commensurable to the Square *a e b c*, for 'tis indeed double to it.

34. But

34. But if you take *a o*, a mean Proportional between *a b* and *a c*, that mean *a o* ſhall be Incommenſurable to them even in Power; *i. e.* the Square of *a o* is Incommenſurable to the Square of *a b*, or to the Square of *a e*, for the Square of *a c* to the Square of *a b*, is in a Duplicate *Ratio* of *a c* to *a o* (6. 22); that is, as *a c* to *a b* (6. 30.) But *a c* is Incommenſurable to *a b* (7. 31.) wherefore the Square of *a o* is Incommenſurable to the Square of *a o*.

a —— *o*

35. There is a *Second Power* of a Line which is called the Cube, which is made by multiplying the Square by that firſt Line, or Root.

36. If two mean Proportionals *a n* and *a m* be taken betwen *a c* and *a b*; ſo that *a c* *a n* :: *a m*, *a b*; the Line *a n* will be Incommenſurable in this ſecond Power to *a c* (*i. e.*) The Cube of *a c* will be Incommenſurable to the Cube of *a n*, becauſe the Cube of *a c* to the Cube of *a n* is in a *Triplicate Ratio* of the Side *a c*, to the Side *a n*; *i. e.* as *a c* to *a b*. But *a c* and *a b* are Incommenſurable, wherefore, &c. However *a c* and *a b* are Commenſurable in the ſecond Power, for the Cube of *a b* is double to the Cube of *a c*.

a —— *n*

a —— *m*

37. 'Tis eaſy to apply to *Solid Numbers* what hath here been ſaid of Plane ones: And thoſe are called *Solid Numbers*, which ariſe from the Multiplication of a Plane *Number* by any other whatſoever. *V. gr.* 18. is a ſolid Number made of 6 (which is a Plane) multiplied by 3; or of 9 multiplied by 2.

38. *Similar Solid* Numbers are thoſe, whoſe little Cubes may be ſo ranged, as to make ſimilar and rectangular Parallelopipeds.

39. *Cubick Numbers* are ſuch as can be ranged into the Form of Cubes as 8. or 27, whoſe *Sides* are 2 and 3, and their *Baſes* 4 and 9.

40. Every

40. Every cubick Number, multiplying another cubick Number, produces a third cubick Number.

41. Between two similar solid Numbers there may be found two mean Proportionals.

That which hath been demonstrated, in respect to Plane Numbers, may be applied to Solids.

42. These Demonstrations by which 'tis proved that there are incommensurable Lines and Magnitudes shew also that a *Continuum* is not compos'd of finite Points: For if the Diameter as well as the Side of a Square were compos'd of finite Points, a Point would measure both the Side and the Diameter, for that Point would be found a certain Number of Times in the Side, and another determinate Number of Times in the Diameter, which the preceding Propositions prove impossible.

43. Because in a Rectangle Triangle the Square of the Hypothenuse is equal to the Sum of the Squares of the Legs; (6. 61.) we have always used this Triangle for the Discovery of Incommensurables. For if all the three Sides are commensurable, they may all three be express'd by three Numbers, and then the Square of the greatest Number will be equal to the Sum of the Squares of the other two. As if the greatest Side or Hypothenuse be 5 Feet, the least Side 3, and the middle one 4: The Square of 5 will be 25, the Square of 3, 9, and the Square of 4 will be 16: And 9 and 16 added together do make the great Square 25. But if the least Side of such a Triangle be 2, and the middle one 3, then the greatest Side cannot be express'd in Numbers, because the Square of the least Side 4, added to the Square of the middle Side 9, makes 13, which express the Square of the greatest Side. But as that Number 13 is not a square Number, so its Side or Root cannot be express'd by any Number.

44. At all times Men have been sollicitous to find out some Method of discovering proper Numbers to

ex-

expreſs the three Sides of a Rectangle Triangle, ſo as to be aſſured that all the three Sides are Commenſurable. Therefore I here ſhew you ſuch a Method, by which you may find out all the poſſible Numbers that are proper for this Purpoſe.

45. If you take any two Numbers (even Unity it ſelf) differing but by an Unit, and add the Squares of them together, the Sum will be a Number which ſhall be the Root of a Square equal to two Squares; And that Number will expreſs the greateſt Side of a Rectangle Triangle, whoſe middle Side ſhall be that Number leſſen'd by Unity, and the leaſt Side ſhall be the Sum of the two firſt Numbers. *V. gr.* Having taken 1 and 2, and ſquared each of them, you have 1 and 4; Add thoſe two Squares together, and the Sum is 5. I ſay 5 will expreſs the greateſt Side, and then 4 will be the middle one, and 3 the leaſt; and 25 the Square of the Hypothenuſe, will be equal to the Sum of the other two Squares. In like manner if you take 2 and 3, and add the Squares 4 and 9 together, the Sum is 13. Then I ſay, will 13, 12 and 5 be three Sides of a Rectangle Triangle; ſo that 169, the Square of 13, ſhall be equal to 144, and 25, the Squares of 12 and 5. Moreover if you take 3 and 4; The Sum of their Squares 9 and 16, makes 25, wherefore I ſay 25 may be the greateſt Side of a Rectangle Triangle, whereof 24 will be the middle Side, and 7 the leaſt Side.

It muſt be obſerv'd alſo, that the Equimultiples of any 3 Numbers thus found will do the ſame thing: Thus, having found 5, 4 and 3, their doubles 10, 8 and 6, will repreſent the three Sides of a Rectangle Triangle, ſo that 100, the Square of 10, ſhall be equal to the Sum of 64, and 36 the two Squares of 8 and 6. And their Triples alſo 15, 12 and 9, will do the ſame thing: For any one may ſee that all theſe Numbers, ſtill having the ſame Proportion, do as it

were

were constitute but one only Triangle, *viz.* that which is expreſs'd by 5, 4, and 3. And therefore all thoſe Numbers may be taken for the ſame.

N. B. *The three Sides of a Rectangled Triangle will then only be commenſurable, when they are in this Proportion,* viz. *as* $aa + ee$, $aa - ee$, *and* $2ae$. *That is, the Sum of two Square Numbers, the Difference of their Squares, and the double Rectangle of their Roots.*

E L E-

ELEMENTS OF *GEOMETRY.*

BOOK VIII.

Of Progressions *and* Logarithms.

1. P*Rogression* is a *Series* or Rank of Quantities which keep between one another any kind of similar Relation or Proportion; and every one of the Quantities is called a *Term*.

2. When the Terms which so follow one another do equally increase or decrease, the Progression is called *Arithmetical*; as are all Numbers proceeding according to the natural Order of the Figures, as 1, 2, 3, 4, 5, 6, *&c.* As also all odd Numbers, as 1, 3, 5, 7, 9, 11, *&c.* or as 4, 8, 12, 16. or as 20, 15, 10, and the like.

3. Arithmetical Progression may be increased infinitely, but not diminished.

4. If

4. If in an Arithmetical Progression there be four Terms, the Difference between the two first of which is equal to the Difference between the other two, those four Terms are said to be *Arithmetically Proportional*: As in the Progression of the natural Numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9. *&c.* If you take four as 2, 3 : : : 9, 10, (*This Mark* : : : *I shall for the future use to signifie Arithmetical Proportion*) there will be the same Arithmetical Proportion between 2 and 3, as there is between 9 and 10; that is, 10 exceeds 9, as much as 3 doth 2: So also 3 : 5 : : : 8 : 10, are in Arithmetical Proportion; and so are 1 : 5 : : : 5 : 9, where 5 being taken twice, is an Arithmetical mean Proportional between 1 and 9.

5. In Arithmetical Proportion the Aggregate or Sum of two Extreams is equal to the Aggregate of the two Means, as in 2 : 3 : : : 9 : 10. the Sum of 2 and 10 is equal to the Sum of 3 and 9, that is 12; so also in 3 : 5 : : : 8 : 10. The Aggregate of 3 and 10 is 13, which is equal to the Aggregate of 5 and 8. And the Reason of this is self-evident. For tho' 10 exceeds 8, yet that which is added to 8, (*viz.* 5.) doth just as much exceed 3, which is added to 10, and so there necessarily arises an Equality between them.

6. The Sum of the first and last Terms in any Arithmetical Proportion, is equal to the Sum of the second and the last save one; or to the Sum of the *third from the first Term, added to the third, accounted backward from the last*, &c. as in the first Example, 1 and 9 make 10, and so do 2 and 8, 3 and 7, or 6 and 4 always make 10. And in the middle remains 5, which being taken twice (as if it were equivalent to the Terms, because 'tis equally distant from the first and last Term) makes also 10.

7. If you add the first Term to the last, and multiply that Sum by half the Number of the Terms, the Product shall be equal to the Aggregate or Sum

of

of all the Terms. As in the former Example, 1 added to 9, makes 10, and 10 multiplied by $4\frac{1}{2}$ (or 4, 5) for there are 9 Terms, produces 45, which is the Sum of all the Terms from 1 to 9. As is manifest from the Precedent.

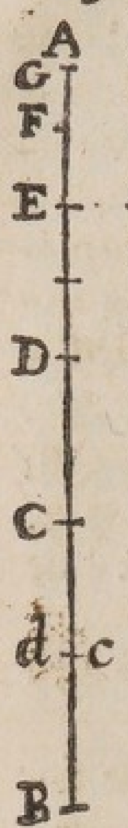
8. When the Terms of the Progression are continual Proportionals; that is, when the first is to the second, as that is to the third Term, as the third is to the fourth, and as the fourth is to the fifth, *&c.* then the Progression is call'd *Geometrical*, as 1, 2, 4, 8, 16, 32 :: ; Or as 1, 3, 9, 27, 81 :: or again, as 3, 12, 48, 192, 768, or *descending*, as 8, 4, 2, 1 ::; or lastly as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{84}$, *&c.*

9 Geometrical Progression may be encreas'd and diminish'd infinitely.

10. When the Progression begins with 1, the second Term is call'd the *Root*, *Side* or first *Power*; the *third* is call'd the *Square* or second *Power*; the *fourth*, the *Cube* or third *Power*; the *fifth*, the *Biquadrate* or *fourth* Power; the *sixth*, the *Sur-solid* or *fifth Power*; the *seventh* the *Quadrato-Cube* or *sixth Power*, &c.

11. If (*in such a Progression*) you take four Terms, the former two of which are as much distant from each other, as the two latter are: Those are simply Proportional, and the Rectangle of their Extreams is equal to that of their two middle Terms.

12. Let the Quantity AB be so divided in C, D, E and F, *&c.* that AB : AC :: AC : AD :: AD : AE, *&c.* Then I say, BC : CD : DE : EF, *&c.* are in continual Geometrical Proportion; and also that AB : AC : : BC : CD :: CD : DE, *&c.* for because AB : AC :: AC : AD, it will follow by Division of Proportion, that AB : less AC : (that is CB :) AC :: as AC less AD : (that is DC) AD, and consequently alternately CB : CD :: AC : AD, or as AB : AC, and so of all others it may be proved :: DC: ED :: EF :: GF, *&c.*



13. Let there be a Progreſſion of Quantities in a Right-line BC, CD, DE, EF, *&c.* let C*d* be equal to the *ſecond* Term CD, that ſo we may have B*d* the Difference between the *firſt* and *ſecond* Terms: And let it be made as B*d* : BC : : BC, to a fourth Line, *viz.* BA. I ſay, that if the Number of the Terms BC : CD : DE, *&c.* be finite, tho' never ſo great, all thoſe Terms taken together, although there be an hundred thouſand Millions of them, ſhall be leſs than BA. But if we ſuppoſe the Progreſſion infinite, or that the Terms are infinitely many, then ſhall all of them taken together be exactly equal to BA. For ſince by the Suppoſition B*d*, (that is BC leſs C*d* or CD) is to BC : : BC, (that is AB leſs AC) AB, it may eaſily be found that as BC : CD : : AB : AC : : AC : AD, *&c.* and conſequently all the Terms CD, DE, EF, *&c.* will always be found within, or be hither the Point A. To which it approaches the nearer, the more the Number of the Terms is increas'd. So that we ſee plainly, that all theſe Terms (which in Books are uſually call'd *Parts Proportional*) tho' they be actually infinite, cannot make an infinite Length, becauſe they will be all included within the Line BA.

14. This Demonſtration will appear much more eaſie and ſenſible by the Example of a particular Progreſſion, where the Terms are in a double *Ratio* *v. gr.* Let CB be double to DC, and DC double to DE, *&c.* For if the Number of the Terms be here finite, tho' it be an hundred thouſand Millions, and you take the laſt and leaſt Term, for Example FE, and add to it another Quantity, as ſuppoſe AF, equal to it: It is then plain, that EA muſt be equal to the Term ED, which is the laſt ſave one; For ED is double EF by the Suppoſition. (the *Ratio*

being

being every where double) and E A is alſo double to E F by the Conſtruction, it having been made ſo, by taking F A equal E F. In like manner A E with D E, that is A D, ſhall be equal to the following Term C D, and at laſt A C will be equal to B C. So that from hence it appears, that the firſt or greateſt Term is always equal to all the others taken together, provided there be added to them but a Quantity equal to the laſt and leaſt Term; but if nothing be added, the firſt Term is always greater than the Sum of them all.

If theſe Terms are ſuppos'd to be actually infinite, then the greateſt BC will be exactly equal to all thoſe infinite others taken together CD, DE, EF, *&c.* For any one may eaſily diſcern, that the more there are of ſuch Terms, the more you approach towards A. by cutting off ſtill the half of the Remainder: But when any Quantity is thus leſſen'd by half, and the Remainder again by half, and then the half of that third Remainder taken, and ſo on: 'Tis plain, that by ſuppoſing the Diminution to be made an infinite number of Times, nothing at laſt will remain.

This alſo might be demonſtrated by a *Reduction ad Impoſſibile*, by ſhewing that all thoſe infinite Terms, taken together, are neither greater nor leſs than A B.

15. Hence may the Difficulties raiſed by the Schoolmen againſt the *(Infinite)* Diviſibility of a *Continuum* be ſolved, tho' to Perſons ignorant of *Geometry* they appear unſolvible: But indeed at the Bottom they are nothing but meer Paralogiſms.

16. If two Progreſſions are ſuppoſed, one Geometrical beginning with 1, and the other Arithmetical beginning with 0, ſo that the Terms in one ſhall be placed over, and anſwer reſpectively to thoſe in the

 other;

other; The Arithmetical ones are called *Logarithms*, *Exponents*, (or *Indexes*) as in the following Ranks.

0.	1.	2.	3.	4.	5.	6.	7.	8.
1.	2.	4.	8.	16.	32.	64.	128.	256.

17. That which is produced in a Geometrical Progression by Multiplication and Division, is effected in the *Logarithms* by Addition and Subtraction: As, if having three Numbers given; 2 : 8 :: 64; You would find a fourth Proportional to them in Geometrical Progression: You must multiply 64, by 8, (which are the two middle Terms). For the Product 512 shall be equal to the Product made by 2, and the fourth Number sought, they being the two Extreams of four Proportionals. And to find this fourth Number, you need only divide 512 by 2, and the Quotient will be 256. So that 2 : 8 : : 64 : 256, and 64 and 256 will be just as far distant from one another in the Order of the Progression, as 2 and 8 are.

But if instead of the Geometrical Numbers 2 : 8 :: 64, you had used their Logarithms 1 : 3 : : 6, which answer to them in the Progression, and were minded to find a fourth Logarithm, then you must have added 3 and 6, which makes 9, and from thence have subtracted 1, there would remain 8. The Logarithm answering to the Geometrick Number 256.

18. So also, if there be two Geometrick Numbers 4 and 8, to which the Logarithms 2 and 3 do answer; by multiplying 4 by 8, you produce 32; the Number under the Logarithm 5, which is the Sum of the Logarithms of 2 and 3.

19. In like manner by multiplying 16 by it self, there will be produced 256, which stands under the Logarithm of 8, the Sum of 4 added to it self.

20. So if the Geometrical Number were required that ſhall anſwer to, or ſtand under, the Logarithm 16, you muſt take 256, which ſtands under 8, and multiply it by it ſelf, and it will produce 65536, the Number required.

21. If moreover the Geometrical Number anſwering to the Logarithm 23 were requir'd, you may take any two Logarithms, whoſe Sum is 23, as ſuppoſe 7 and 16, and multiplying the Geometrical Number under them, *viz.* 128 and 65536 one by another, the Product will be 8388608. The Number which ought to ſtand under the Logarithm 23, or in the 23d Place of a Series of Geometrical Proportionals, beginning from 1.

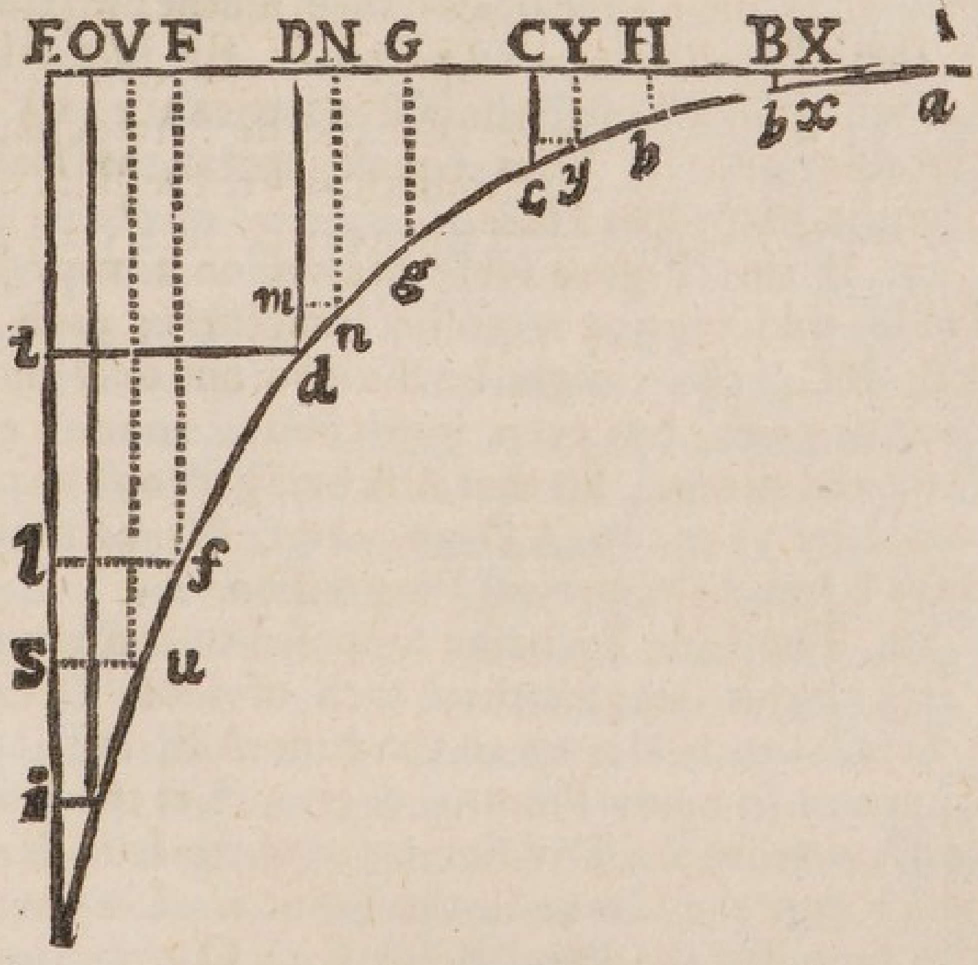
22. From hence appears the Way of anſwering that ordinary Queſtion, how much a Horſe would coſt, if bought on this Condition: That for the firſt Nail in his Shoe a Farthing were to be paid, for the ſecond Nail two Farthings, for the third Nail four Farthings, for the fourth Nail eight Farthings, and ſo on, ſtill doubling for 24 times: For the 23d Place in ſuch a Progreſſion would be the laſt Number 8388608 Farthings, which, being reduced, is 8738 *l.* 2 *s.* 8 *d.* and being doubled according to (8. 14.) gives the whole Price of the Horſe 17476 *l.* 5 *s.* 4 *d.*

23. Where two compleat Progreſſions are fitted ſo as to anſwer one to another, the Geometrical to the Arithmetical; as ſuppoſe in Tables for that purpoſe calculated in Books, there abundance of Pains and Labour is ſpared, in finding the Geometrical Numbers: For Inſtance, let thoſe three Numbers 32, 64, 128 be given, and that a fourth Proportional were required: Inſtead of multiplying 64 by 128, and dividing the Product by 32 (which Way is very tedious in great Numbers); you need only take the Logarithms of the three given Numbers, *viz.* 32, 64,

128. and adding the 2d and 3d together, from their Sum ſubtract the firſt, the Difference will be the Logarithm of the correſponding Geometrick Number 256.

24. But becauſe in ſuch a Geometrical Progreſſion all Numbers will not be found, this Medium hath been diſcovered; they have calculated two Progreſſions, one of which contains all Numbers 1, 2, 3, 4, 5, 6, 7, 8, &c. which ſeems to be an Arithmetical Progreſſion, but yet hath in reality the Properties of a Geometrical one. And the other which contains Numbers in Appearance the moſt irregular, is nevertheleſs a true Arithmetical Progreſſion. See here a Line, which will diſcover perfectly all theſe Myſteries.

25. Let the Right-line A E be divided into the equal Parts A B, B C, C D, D E, *&c.* from the Points A, B, C, D, E, *&c.* let the Lines A *a*, B *b*, C *c*, D *d*, and E *e*, be drawn all (*perpendicular to* AE, *and conſequently*) parallel to one another: And let them be all in a Geometrical Progreſſion; As let A *a* be 1, B *b* 10, C *c* 100, D *d* 1000,



E *e* 10000, *&c.* Then ſhall we have two Progreſſions of Lines, the one Arithmetical, and the other Geometrical: For the Lines A B, A C, A D, A E, are in Arithmetical Progreſſion, or as 1, 2, 3, 4, 5, *&c.* and ſo do repreſent the Logarithms; to which the Geometrical Lines A *a*, B *b*, C *c*, *&c.* do correſpond.

26. Let each of the equal Parts ED, DC, CB, &c. be divided equally again in F, G, H, and let the Parallels F*f*, G*g*, &c. be drawn, and be mean Proportionals between the Collateral ones; that is, E*e*: F*f*: : F*f*: D*d*: : D*d*: G*g*. Let there also be more mean Proportionals, drawn from the middle of each Sub-division EF, FD, DG, and so on, till these Parallel Lines growing very numerous, have at last but a very small Distance from each other; then imagine a Curve Line drawn thro' all the Extremities of these Parallels as *e o u f d g h a*: By this Means you will gain a Line, whose Properties are very considerable, and its Uses equally great, as shall be shewn in its proper Place.

27. If this Figure were drawn on a very large Table, and with a requisite Exactness; each Part AB, BC, &c. might be divided not only into an 100, or 1000, but even into 10000, 100000 equal Parts and more. So that AB being 1000000, AC would be 2000000, AD 3000000, &c. as must always be an Arithmetical Progression.

28. The Line E*e* being supposed to contain 1000 Parts, let us imagine thro' each of those Divisions a Parallel to be drawn to the Line AE, cutting the Curve in so many Points, *v. gr.* Let the Line *i o* be drawn thro the Division 9900 of the Line E*e* and which cuts the Curve in the Point *o*. Let there be also supposed the Parallel (to E*e*) O*o*, cutting the Line AE in the Division 399563. Then any one may know that 399563 is the Logarithm of the Number 90000, In like manner if S*u* passed thro' the Division 9000 of the Line E*e*, and the Line *u v* were drawn cutting AE in 395424, then would that Line *u v* be the Logarithm of 9000, &c.

29. So that by this means a Table of Logarithms from 1 to 10,000 may easily be made; and farther, by producing the Line AE.

30. Note

30. Note, to obtain all the Logarithms from 1 to 10000; 'twill be enough to seek the Logarithms from 1000 to 10000: That is (having drawn the Parallel *dt*) to take the Logarithms of all the Divisions from *t* to *e*, which Logarithms are all contained between E and D. For by this you will have the Logarithms of all the Parts that are between *t* and E; and whose Logarithms lie between D and A: For Example, since O *o* is 9900 Parts, and its Logarithm 399563, the same Number may be taken for the Logarithm of 990, which is N *n*; as also of the Number Y*y* 99, changing only the first Figure 3. Because, according to the Composition of this Line, O N or N Y ought to be equal to E D or D C, as one may easily prove. So that O N or N Y will contain 100,000; and because A O is 399563, subtracting O N 100,000, there will remain 299563, for A N, from whence also taking 100,000, there will rest 199563 for A Y. And after the same manner, having A Y 3995424 for the Logarithm of V *u* which is 9000; you may have also 095424 for the Logarithm of X *x* which is 9. Or 195424 for the Logarithm of 90, or 29524 for the Logarithm of 900.

31. All this may be reduced to Practice for Calculation, without actually drawing these Figures, but only imagining them to be drawn. For by the Rules of Common Arithmetick we may find out F*f*, the mean Proportional between D*d* and E*e*, and after that, another Mean between D*d* and F*f*, or betwen F*f* and E*e*, &c. But what we have here explained is sufficient to gain as much Knowledge as is necessary for us to have of the Nature and Composition of Logarithms: There being no Need for us to undergo the Labour of calculating Tables of Logarithms; since 'tis already so well and so often done to our Hands. God, for the Publick Good, having raised some Persons, whom he has pleased to endow with sufficient Patience to surmount so tedious and

laborious a Work, as one would think to be inſuperable. For we know that above 20 Men were engaged in ſuch a Calculation, for above 20 Years together, with indefatigable Induſtry and Aſſiduity.

(Pardie *ſpeaks here a little Covertly, ſeeming willing to inſinuate that this moſt uſeful and admirable Work was done firſt in his own Country, whereas the Logarithms were the Invention of my Lord* Neper *a* Scotch *Baron, and the firſt* Tables *were calculated by him with the Aſſiſtance of our Countryman, Mr.* Henry Briggs.)

Of late ſeveral Improvements have been made in this Matter : As by *Nicholas Mercator*, of which ſee Dr. *Wallis*'s Thoughts, in *Philoſoph. Tranſact.* 38. John Gregory *hath given us a Way to find Logarithms to* 25 *Places by help of the Hyperbola. But Doctor* Halley, *in* Philoſ. Tranſact. N° 216. *ſhews a Way from the bare Conſideration of Numbers, and withall by the Help of Mr.* Newton's *Way to find the* Unciæ *of the Numbers of a Binomial Power,* &c. *By which you may find compendiouſly the Logarithms of all Numbers to above* 30 *Places. And he gives there ſeveral Series for this Purpoſe, ſome univerſal, and ſome appropriated to a peculiar ſort of Logarithms.*

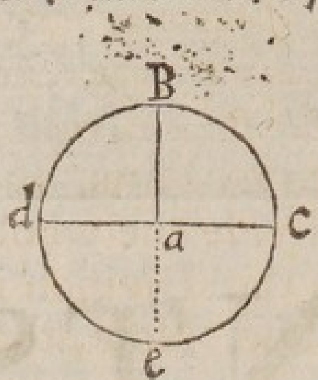
ELEMENTS
OF
GEOMETRY.

BOOK IX.

Problems *or* Practical Geometry.

THAT Propoſition is called a *Problem* in (*Geometry*) which teaches us how to do any Thing, and demonſtrates alſo the Practice of it: Whereas *Theorems* are ſpeculative Propoſitions, in which are conſidered the Affections and Properties of Things *already done*.

2. *To divide a Circle into four and into six; and all Arks into two equal Parts.* To divide it into four, draw two Lines as *d a c* and *B a e* at Right Angles to each other. To divide it into eight Parts; bissect the four Arks B *c*, *c e*, &c. which is done by striking (without the Ark B *c*) two other Arks, with the same opening from the Points B, and C, for if a Line be drawn from the Point where those Arks cross each other, to the Center *a*, it shall bissect the Ark BC. The like is to be done for the other Arks.



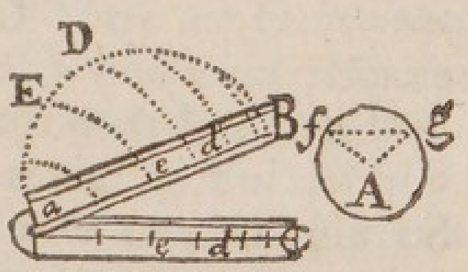
To divide a Circle into six equal Parts; you need only take the Length of the Radius; and applying it six times about the Circle, it will exactly divide the Circumference into six equal Parts, and thus by a new Bissection, may a Circle be divided into 12, 24, 48, or into any Number of equal Parts, &c.

3. *To divide a Circle into five, into fifteen, and into obter equal Parts.* This may be done thus; (as I demonstrate in *Algebra*) Make a Rectangled Triangle, one of whose Legs shall be the Radius of the Circle, and the other half the Radius. From the Hypothenuse of this Triangle, take half the Radius, the Remainder shall be the Chord of 36 *deg.* and the Side of a Decagon. Double that Ark, you have the Ark of 72 *deg.* (*whose Chord is the Side of a Pentagon*) and it is the fifth Part of the Circumference; and the same Chord shall be also the Hypothenuse of a Rectangled Triangle, one of whose Sides is the Radius, and the other the Side of a Decagon. And as by the last was found the Chord of 60 *deg.* so by subtracting the Chord of 36 *deg.* from 60 *deg.* you may have the Chord of 24 *deg.* which is the 15th Part of the Circumference. But for Practece, the shortest and surest Way is, by repeated Trials with the Compasses to find a

Distance

Distance that will go precisely five times about the Circle: Then divide, after the same manner (by Trials) that Distance into three equal Parts exactly. So shall you gain a Chord that will divide the Circumference into 15 equal Parts, and then dividing each of those 15 Chords into four equal Parts, and each of those into six; you will divide the whole Circumference into 360 *deg.* And this Division is most commodious for Practice and Use. Note, that the Way to divide a Circle into 3, 5, 7, or into any other odd Number of Parts, is not yet found Geometrically; Geometrically I say, that is, by making Use only of a strait Line and Circle.

" This Division of a Circle into 360 *deg.* is very " useful, when a Person understands how to use the " *Compasses of Proportion* (or " *Sector.*) 'Tis so called, be- " cause 'tis a kind of Com- " passes with broad Legs: " As *a* B, *a* C, on which are " described divers Lines " and Divisions, but those,



" which are most in Use, are of two Sorts. On one " Side of this Sector, and on each Left, is a Line " *a e* B and *a e* C, which serves to divide a Circle " into 360 *deg.* at one, and also to take at any " time as many Degrees as you please: And this " Line on the Sector is thus divided.

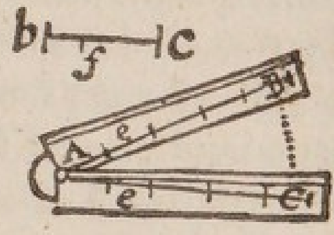
4. *To divide and graduate the Sector, that it may serve for the Division of a Circle.* Imagine a Semicircle *a* E D B accurately divided into 180 *deg.* if then from the Point *a*, as from a Center, you transfer the Divisions of the Semicircle into a Line *a* B. *v. gr.* If from E, 60 *deg.* you draw the Ark E *e*, and if from D 90 *deg.* in the Semicircle, you draw the Ark D *d*, *&c.* Then ought 60 *deg.* on that Leg of the Sector, to be placed at the Point *e*, and 90 *deg.*

deg. at the Point *d*, *&c.* And if you transfer the ſame Degrees after the ſame Manner into the other Leg *a* C, you will graduate the Lines *a* B and *a* C, (on the Sector) as they ought to be for this Purpoſe, *and will they be two ſimilar Lines of Chords.*

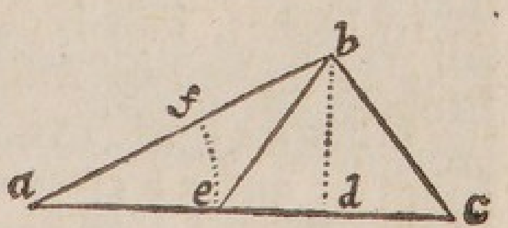
5. *To explain the Uſe of the Sector as far as it ſerves for the Diviſion of a Circle.* Let there be a Circle given A*f*; take with your Compaſſes the Radius A*f*, and (*keeping that Diſtance*) ſet one Foot of them in *e* or 60 *deg.* on one Leg of the Sector; move the other Leg of the Sector to and fro ſo long, till the other Point of the Compaſſes falls exactly on *e* or 60 *deg.* in that Leg of the Sector: So that the Diſtance *e e* be exactly equal to the Radius A*f*: Then if you would have readily 90 *degrees* of that Circle; (*letting the Sector lie ſtill, and always keeping the ſame Angle*) Open your Compaſſes till the Points fall exactly on *d* and *d*, or 90 *deg.* on each Side of the Sector: And then that Diſtance transferred into the Circle, in *f*, *g*, gives you the Ark of 90 *deg.* *f*, *g*. So alſo if you would have had 35 *deg.* you need only apply your Compaſſes to 35 *deg.* and 35 *deg.* on each Leg of the Sector in the Lines (*of Chords*) *a* B and *a* C: and that Diſtance transferred into the Circle, ſhall cut off the Ark of 35 *deg.* and thus may you proceed to find any Degrees you pleaſe. All which is grounded on the 42, 43, 49 and 50 Propoſitions of the VI. Book. For ſince all Circles are ſimilar Figures, (6. 50.) the Chord *f g* will be to the Radius *f* A : : as the Chord of *d d* to the Radius *e e*; that is, as *a d* is to *d e*. Now 'tis plain, from what hath been proved elſewhere, that the Triangles *a d d* and *a e e* are ſimilar; and therefore *d d* : *e e* : : *a d* : *a e*. But *d d* is by the Conſtruction equal to *f g*, and *e e* to A *f*, wherefore *f g* : A *f* : : *d a* : *a e*. Q.E.D.

6. To

6. *To divide the Line of* equal Parts *or* Lines *on the Sector, for the Division of any Right-lines given.* There being two Right-lines drawn from the Center of the Sector on the Legs as *a* B and *a* C: Let each be divided into 100 or 200 equal Parts: And then they will ſerve to divide any Line given, into any Number of equal Parts: As for Inſtance, let the Line given be *c b*, and that you were required to take $\frac{25}{97}$ Parts of it. Now to divide the whole Line *c b* into 97 equal Parts, and then to take 25 of them according to the common Way of dividing Lines, would be very tedious: But by the Sector 'tis done eaſily and ſpeedily thus; Take the Length of the whole Line *c b* in your Compaſſes, and fit or apply it over in your Sector between 97 and 97 in each Leg, from B, ſuppoſe to C. Then letting the Sector lie open'd at that Angle, take in your Compaſſes, the Diſtance between 25 and 25 in each Leg, or between *e* and *e*, which transfer into the given Line from *b* to *f*; ſo ſhall *b f* be juſt $\frac{25}{97}$ of the whole Line *c b*: As is plain from the Triangles A B C and A *e e* being ſimilar.

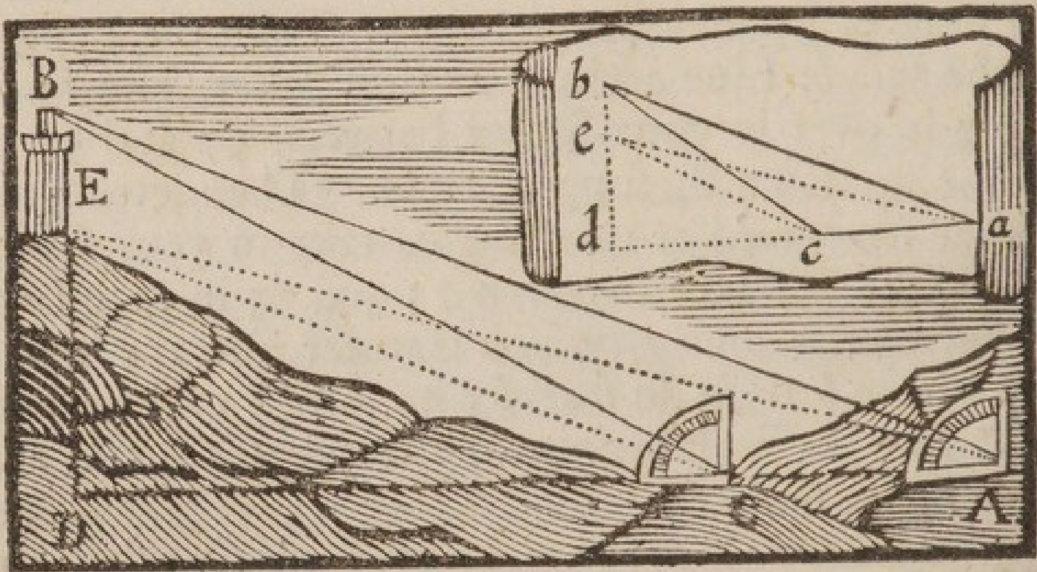


7. *On a Line given to make an Angle that ſhall contain any Number of Degrees aſſign'd.* Let the Line given be *a c*, on which 'tis required to make an Angle of 30 *deg.* From the Point *a*, as from a Center, ſtrike the Ark *f e*, from which take by the Sector, or otherwiſe, 30 *deg.* from *e* to *f*; then through *f* draw the Line *a f*, which with the Line *a c* will make an Angle of 30 *deg.*



8. *Having*

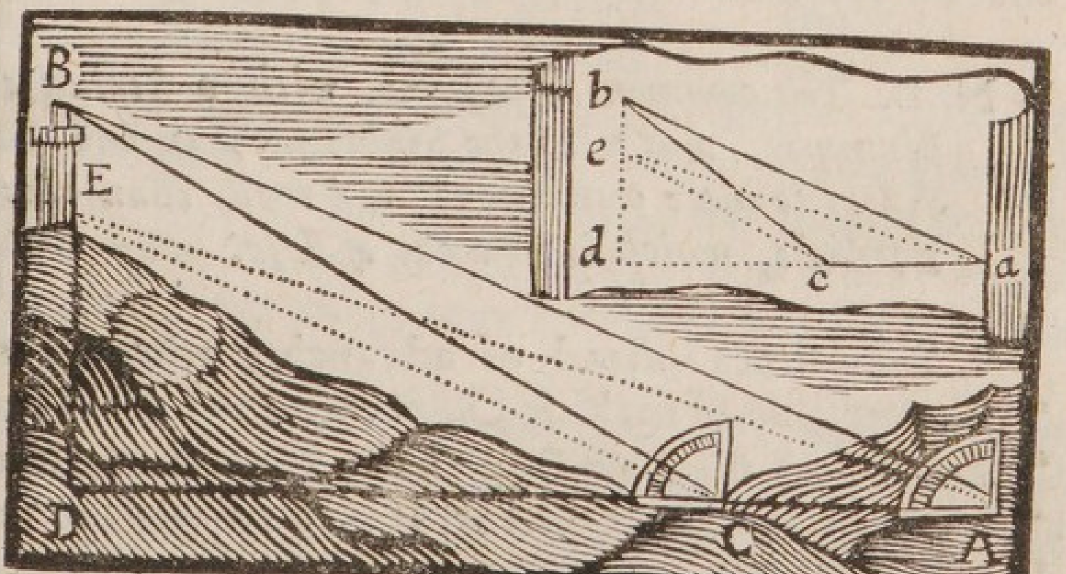
8. *Having the Angles of any Triangle and one Side given to find the other two Sides.* Suppose you are told there is a certain Triangle some where, whose Base A C is 10 Fathom; and that the two Angles at the Base are A C B 150 *deg.* and C A B 20 *deg.* (and consequently the remaining Angle at the Vertex or Top must be 10 *deg.* for the Sum of 150, 20 and 10, is just 180 *deg.* which is two Right Angles). You are required to tell how many Fathom there are in the other Sides A B and A C. Make on Paper, or rather on fine Pasteboard, a Triangle *a b c* similar to the propos'd one, after this manner. Take a Base at pleasure *a c*, and from any Scale of equal Parts let it



be 10 Inches, half Inches, &c. in Length. On this Line *a e* make two Angles, one *c a b* of 20 *deg.* and the other *a c b* of 150 (9. 7) Then will the two Lines *a c* and *c b* cross one another, when produced in the Point *b*. Then measure *(on the same Scale you took the Base* a c *from)* how many *Inches*, &c. the Lines *a b* and *c b* are in Length; And you may be assured that there are just so many Fathom in the Lines A B and C B sought, as you find Inches, &c. or any equal Parts, in the Lines *a b* and *c b*. For since the Triangles are equiangular, they are similar, and therefore *a c* : *a b* : : A C : A B, &c.

9. To

9. *To measure Distances, Heights, Depths, and in general, the Dimensions and Magnitudes of all remote and inaccessible Places.* If on the Top of any Hill appearing at a Distance, there were a Tower, as BE, and its Distance from us and its Height, were required: You must first with some Instrument (as with a *Quadrant*, that is the fourth Part of a Circle divided into 90 *deg.* and furnished with a Ruler, or Label with Sights, and moveable on the Center) you must, I say, with some such Instrument, take two Angles at two several Stations in this manner: If you are in the Station A, place your Instrument so, that one Side of it may answer exactly to the Horizontal Line A D; and keep it without raising or depressing it in this Position. Then place your Eye at A, (that is at the Center of the Instrument) and turn the Label till it point to the Top of the Tower B, and that



looking through the Sights you can see the Top of the Tower exactly; then will the Label cut in the Limb of the Quadrant the Degrees of the Angle B A D, for the Limb is supposed to be graduated for this Purpose: Then change your Station, moving in a Right-line forwards 10 Fathom (or it might have been any other Distance, and backward as well as forward) to C, and there take after the same manner the Angle BCD: By which means you will have alſo the Angle BCA, becauſe thoſe two together make 180 *deg.* or two Right ones. So that in the Triangle ABC you have now found the Baſe AC, which is 10 Fathom, and alſo the two Angles at the Baſe; and conſequently the Sides CB, and AB, may be known: (9. 18.) And then you may have the Height DB, or the Diſtance AD, if you make a little Triangle ſimilar to that, and there from the Point *b*, let fall a Perpendicular *bd*, to the Baſe Line AC continued to *d*. For BD, or AD will be juſt as many Fathoms as *bd*, and *ad* will be equal Parts meaſur'd on the Scale, (as in the laſt.) And if after you have thus gain'd the Height BD, you find, by the ſame Method, the Height ED alſo, you may (*by Subtracting this Altitude from the former*) find the Height of the Tower EB.

N. B. *The common Quadrant, with a String and Plummet, and with the Sights fix'd on one of its Sides, is more convenient and ready than this of* Pardie's, *which is now out of Uſe.*

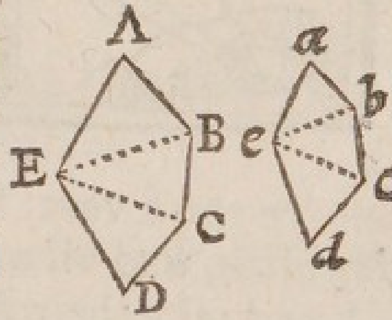
" Sometimes inſtead of advancing towards the " Tower, and of making Obſervations of the Height " below, or of thoſe Angles the viſual Rays make " with the Horizontal Line, it is convenient to take " two Stations ſide-ways of each other; But it " comes all to the ſame, and the Practices in rea" lity are not at all different.

" And by this Means, as any one may ſee, may " all imaginable Heights and Diſtances, and other " Dimenſions be taken; provided we can but come " to obſerve their Extremities, from two different " Places. I ſhall not ſtay now to deſcribe the par" ticular Ways of doing this, nor to enumerate the " great Advantages that would accrue from the Uſe

" of

"of Telescopical Sights fix'd on the Label, or on the
"Side of the Instrument used in taking Angles;
"which indeed is an Invention of inestimable Bene-
"fit to Surveyors.

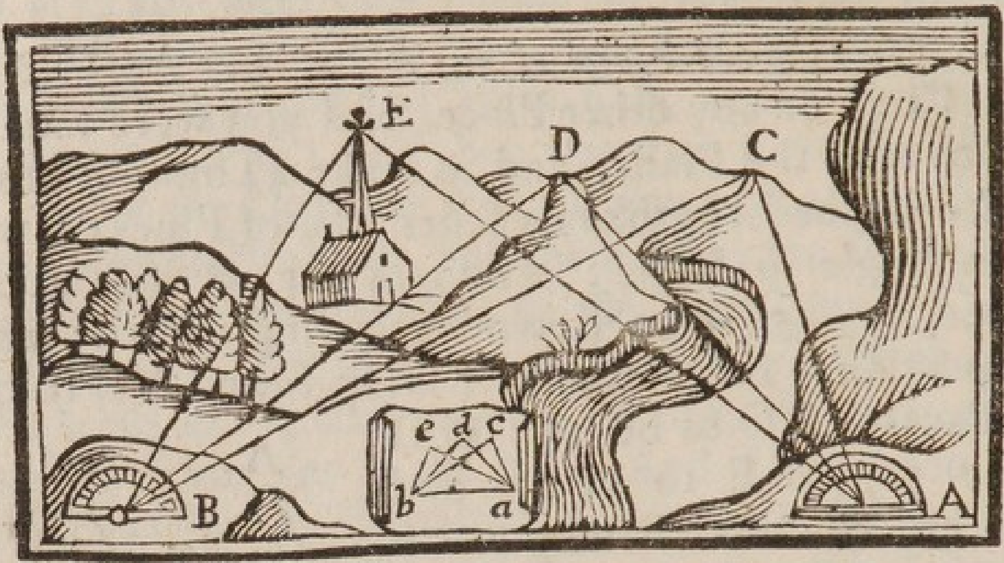
10. *To take the Plane of any Place.* Let ABCDE be a City, or any other Place, and you were required to take the Plane, and to make a Draught of it. Take all the Length of its Sides, and of Lines drawn from Angle to Angle: And transfer all these upon Paper, laying them down according to their true Proportion. For Instance, having found A B to be 30 Paces, BC to be 59, CD to be 50, B E to be 67, and A E 49, &c. and having ready drawn on Paper, a plain Scale divided into 100 equal Parts. Make the Line *a b* 30 of such Parts; *b e*, 67; and *a e*, 49, then those Lines drawn and join'd together will make the Triangle *a b e* every way similar to the Triangle ABE. And if you go on thus, and make the Triangle *b e c*, similar to BEC, &c. you will form the Figure *a b c d e* every way similar to the Plane of the Place ABCDE.



11. But if you cannot get into the Place to survey it, and to measure the Distance between the Angles EB and EC, you must take the several Angles of the Plane, and transfer them into your Draught; so that if the Angle BAE be 66 *deg.* the Angle *b a e* must also be 66 *deg.* and so of all the rest.

12. *To make a Draught of any City or Country.* Ascend up into any two elevated Places, from whence you can plainly see the City or Country, whose Delineation you would make. And having with you a Quadrant, whole Circle, or Semicircle well divided into Degrees, together with its Label (with Sights) and its Center: Place your Instrument at A, and

ſo that one of its Sides may lie in a Line between A and B, which done, and the Inſtrument fix'd there,



obſerve the ſeveral Steeples, eminent Houſes, Towers, Hills, and all other remarkable Places, as EDC, &c. and take their Angles with the Label and Sights, and write them all down to help your Memory. Thus, let the Angle C A B be 50 *deg.* 30 *min.* the Angle D A B 45 *deg.* 8 *min.* &c. Proceed after the ſame manner at the Station B; noting down the Angle ABC to be 40 *deg.* 10 *min.* the Angle A B D 47 *deg.* 28 *min.* &c. After which, draw on Paper any Line at Pleaſure, as *a b*, and make, at each End of it, Angles equal to thoſe which you found, *c a b* equal to C A B, *d a b* equal to D A B, and *a b c* equal to A B C, &c. And by this Means you will have the Points *c*, *d* and *e*, &c. which will be in the ſame Poſition to one another as the Steeples, or other eminent Places C D E, &c. are. And thus having drawn the moſt conſpicuous and principal Places, the reſt may eaſily be taken by the Eye: But to make this Operation very exact, 'tis convenient to take the Angles alſo at a third or fourth Station, and then, if they all agree, any one will know that the Work was well done.

F I N I S.

A
TABLE
Of the Words and Terms of Arts explained in this GEOMETRY.

N. B. The first Figure shews the Book, the second the Article.

Concave

Mean

The Characters, Marks, Signs or Symbols here used, are only these.

= EQual to.
+ More, or Adding.
— Less, or Subtracting.
∷ The Mark of four Quantities being discretely proportional Geometrically.
∺ The Mark for Continual Proportion, or Geometrick Progression.
∷: The Mark for Arithmetical Proportion.
× The Mark for Multiplication.
□ Square.
▯ Rectangle.
△ Triangle.
∠ Angle.
∥ Parallel.

=p=