AN ELEMENTARY

LECTURE

ON THE

THEORY OF LIFE ASSURANCE,

DELIVERED AT A MEETING OF

THE BIRMINGHAM INSURANCE INSTITUTE,

11TH JANUARY, 1889.

BY

W. J. H. WHITTALL, F.I.A. F.A.S.,

Late Actuary of the Clerical, Medical and General Life Assurance Society.

SECOND EDITION (REVISED)

TO WHICH ARE APPENDED TABLES OF SELECTED COMPOUND INTEREST VALUES; AND ALSO FOUR-FIGURE TABLES OF LOGARITHMS, ANTI-LOGARITHMS, AND RECIPROCALS.

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PREFACE.

THIS Lecture was prepared, by special request, for the Birmingham Insurance Institute shortly after its formation. That Institute is, I think, with the exception of the Manchester Society, the oldest of the now numerous provincial institutes. If the many interesting addresses which have since been given in the provinces had been then in existence, I should never have thought of giving this Lecture the form it took; but being somewhat of a pioneer, I suppose I set out with youthful energy to cover the ground. When the Lecture was printed it met with some approval (it is pleasant now to remember that it brought me a kind letter from Dr. Sprague), but I expect most of my old friends will think that the little thing might now, with the exhaustion of its first edition, have been allowed to die in peace. The publishers, however, are still being asked for copies, and so I agreed, perhaps rather weakly, to issue a revised edition.

No doubt, as regards its scope, the Lecture is open to the criticism that it attempts at once too much and too little. I hope, however, that it may still be found useful to the class for whom it was more particularly intended, viz., the Branch Managers and other Branch Officials who have but limited time and energy for study, but to whom it is so very important, in discussing Life Assurance daily with the general public, to know something of the theory of their business. To such, even if they do not master the whole, it must surely be useful to have a general inkling of the difficult problems which lie behind most pages of the prospectus, and which need much delicacy in daily treatment if the public are not to be misled.

I sincerely thank Mr. Frank L. Collins, F.I.A., the Assistant Actuary of the Clerical, Medical and General Life Assurance Society, for several suggestions, and for his general help in seeing this new edition through the press; and also Mr. O. F. Diver, M.A., F.I.A., and Mr. G. H. Lawton, A.I.A., for kind assistance in looking through the proofs.

W. J. H. W.

18, Airlie Gardens,
November, 1909.

THE THEORY OF LIFE ASSURANCE.

1. Introduction. In choosing a subject on which I could address you this evening, I have been guided by a suggestion made to me that there are many junior members of this Institute who have not at present any extended acquaintance with the theory of Life Assurance, and that an address with an educational tendency would be preferred. I shall therefore endeavour to place before you an outline of some elementary actuarial work, the study of which may, by a slight effort of acquisition, be made really useful in the every-day business of Life Assurance. This should create an appetite for further knowledge by showing how, for a comparatively small outlay of time and work, a certain degree of acquaintance may be reached with a subject which will otherwise remain a sealed book to all who do not aspire to the name of Actuary.

2. Nature of Actuarial Enquiry. What is an Actuary? Some have said a "Scientific Accountant." A former President of the Institute of Actuaries suggested as the best description—a "Scientific Financial Adviser." None of these attempts at definition has been happy, and it will be more profitable to enquire, What is the proper scope of an Actuary's work? For instance, an Accountant deals only with ascertained facts. Not so the Actuary, who deals chiefly with probabilities or contingencies. There is a highly perfected doctrine of compound interest with which the Accountant may possibly have some degree of acquaintance; and there is an abstruse theory of probabilities and life contingencies in which pure mathematicians and modern statisticians are possibly proficient. The real starting point of the Actuary's special pursuit of an Expert in Life Assurance is where he commences to interweave probability with interest. It is, in fact, the study of the problems raised by the practice of Life Assurance that has evolved the Actuary. Some of us think that the evolution of the Actuary might well proceed further. The nation's vital statistics, the national schemes for Old Age Pensions, for Invalidity, and for Unemployment, many problems connected with the Navy and the Army, and the administrative work of the great government departments and local councils together open a vast field of usefulness before the well-trained Actuary. To occupy it effectively he must, however, widen his horizon and enlarge his present function, which is, primarily, that of an Expert in the theory of Life Assurance.

I have said that the Actuary combines interest with probability. I will ask your attention, then, first to some elementary propositions in compound interest and probabilities (including life contingencies), and then to a consideration of some of the more easy problems of the theory of Life Assurance which has been built on the study of these subjects combined.

I.—COMPOUND INTEREST.

3. Amount of a sum of money. I shall endeavour in what follows to assume an acquaintance on your part with no more than the most elementary rules of arithmetic and algebra.

By the "amount" of a sum is meant the amount of money to which the sum will be increased if it be invested at a stated rate of interest for a stated period. In all that follows the interest is assumed, for simplicity, to be re-invested at the end of every year.

If i be the interest of 1 for one year, then

The amount of 1 in 1 year	$=(1+i)$
,, ,, 2 years	$=(1+i)+i(1+i)=(1+i)^2$
,, ,, n ,,	$=(1+i)^n$
,, ,, x in n ,,	$=x(1+i)^n$

Example.—If £1 be invested at 4 per-cent. interest, to what will it amount in 7 years?

Here the amount we want is $(1{\cdot}04)^7$; or in logarithms* $7 \times \log 1{\cdot}04$. Using the handy Table of 4-figure logarithms printed in the Appendix on p. 38, we have

$$\begin{array}{r} \log 1{\cdot}04 = 0{\cdot}0170 \\ 7 \\ \hline 0{\cdot}1190 \end{array}$$

This result, we see by the Table of anti-logarithms at page 40 = log 1·315.

The answer is 1·315, or £1. 6*s.* 4*d.*

On pages 36 and 37 there are printed some Tables of selected Compound Interest functions, and a reference to the 4 per-cent. col. of the first Table shows that the correct amount is, to the fourth place, 1·3159, so that our answer got thus roughly is correct to the nearest penny.

Further Example.—To what sum would £222 amount in 10 years at 5 per-cent?

Of course, in practice, one uses any Tables already calculated that may be available. From the same Table, we see that the amount of 1 in 10 years at 5 per-cent. is 1·629, so the amount we seek is obviously $222 \times 1{\cdot}629$.

$$\begin{array}{r} \log 222 = 2{\cdot}3464 \\ \log 1{\cdot}629 = 0{\cdot}2119 \\ \hline 2{\cdot}5583 = \log 361{\cdot}6 \end{array}$$

Answer: £361. 12*s.* 0*d.*

* It is quite easy to use logarithms without any knowledge of the principles on which they are based. The few simple rules prefixed to the Tables at page 34 will be found sufficient for the present purpose.

4. Present value of a sum of money. From the above it is evident, that at i per 1 interest, 1 is the present value of $(1+i)$ due a year hence.

That is, $\frac{1}{1+i}$ is the present value of 1 due a year hence.

Similarly $\frac{1}{(1+i)^2}$ " " 1 due 2 years hence.

And generally $\frac{1}{(1+i)^n}$ " " 1 " n " "

Example.—What is the present value of £1 due 7 years hence at 4 per-cent.?

Here the result we want is $\frac{1}{(1·04)^7}$, which may also be written $(1·04)^{-7}$; or in logarithms log 1 − 7 log 1·04. Log 1 being 0, we want, in other words, the complement of 7 log 1·04, commonly called the "colog."

In Sec. 3 we saw that 7 log 1·04 = 0·1190. Colog. = $\bar{1}$·8810 = log ·7603. On p. 36 we see the correct value to the 4th place is ·7599, the slight inaccuracy being due to using only 4-figure logarithms. The calculation is still shorter with the aid of the table of reciprocals on page 42. $\frac{1}{(1·04)^7}$ is the reciprocal of $(1·04)^7$. The latter we found to be 1·3159, and this, entered in the Table, gives us directly the result as ·7601.

Answer: About 15*s.* 2*d.*

These examples sufficiently illustrate the principles underlying the calculation of a Table of present values. From the latter the value of any stated sum is obtained by simple multiplication as in the latter example of Sec. 3.

Thus we have two useful results:—

(1) The sum to which 1 will amount in n years at i per 1 interest is $(1+i)^n$.

(2) The sum which will amount to 1 in the same time, or in other words, the present value of 1, is $\frac{1}{(1+i)^n}$, or $(1+i)^{-n}$.

It is usual to put $\frac{1}{1+i} = v$; so that we have the convenient symbols v, v^2, v^3, &c., to denote the present value of 1 due 1, 2, 3, &c., years hence.

5. Discount. Another symbol which we shall have occasion to use later is d, which stands for the *discount* on 1 for one year. Trade discount, as you are aware, is always calculated on the full amount of the payment to become due, *i.e.*, the trade discount at i per 1 on a sum of 1 due a year hence would be $1 \times i = i$. The true discount, however, should be such that the present value of 1 and the discount of 1 will together amount to the 1. That is, we must have $v + d = 1$: whence $d = 1 - v$.

We can also see that the present value of 1 due a year hence together with interest on that present value must amount to 1 at the end of the year, — *i.e.*, $v + iv = 1$, or $iv = 1 - v$, which, as just seen, equals d. Hence the discount on 1 is alternatively equal to iv; and we see that discount may be regarded in a double light, either as

(1) Interest on present value ; or

(2) Present value of interest—that is, interest paid in advance.

These views of discount should be well remembered.

Example.—The discount on 1 due a year hence, at 5 per-cent.,

$$=1-v=1-{\cdot}9524={\cdot}0476.$$

It is seen by inspection that this may be viewed either as 5 per-cent. interest on the present value of ·9524, or as the present value of the ·05 interest due at the end of the year.

6. Perpetuity. Suppose 1 to be invested in a perpetual stock producing i per annum. Then 1 is called the value of a perpetuity of i.

Thus, if the value of a perpetuity of $i = 1$

we have ,, ,, ,, $1 = \dfrac{1}{i}$

And ,, ,, ,, $k = \dfrac{k}{i}$

Example.—If £100 permanent Railway Debenture 4 per-cent. Stock be bought for £133, what rate of interest does it return?

Here £133 is the present value of a perpetuity of £4, *i.e.*,

$$133 = \frac{4}{i}, \text{ or } i = \frac{4}{133} = {\cdot}03. \quad \textit{Answer;}\ 3 \text{ per-cent.}$$

7. Amount of annuity-certain. Now we have seen that in n years 1 will amount to $(1+i)^n$

$\therefore$ the *Increase* of 1 in the n years is $(1+i)^n - 1$

But this increase must obviously be the accumulated amount of the annual interest of i due at the end of every year. Thus the expression gives us the amount of an annuity of i for n years. And the amount of 1 per annum for n years at the same rate is seen to be $\dfrac{(1+i)^n - 1}{i}$. This it is usual to write as $s_{\overline{n|}}$.

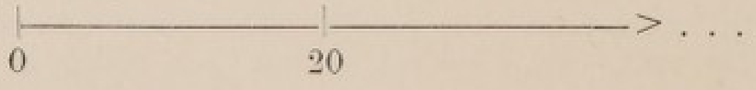
Example.—What is the amount of an annuity of 1 per annum for 7 years at 4 per-cent.?

Here the result we need is $\dfrac{(1{\cdot}04)^7 - 1}{{\cdot}04}$.

In Section 3 we saw that $(1{\cdot}04)^7 = 1{\cdot}3159$: and the answer is $\dfrac{1{\cdot}3159 - 1}{{\cdot}04} = 7{\cdot}8975$.

The Table (p. 37) shows that the amount correct to the fourth place is 7·8983.

8. Present value of annuity-certain. Suppose the following line on which a duration of 20 years has been marked off, to be capable of infinite extension at the right hand end and to represent a perpetuity.



Now if from the value of this whole perpetuity we subtract the present value of a perpetuity deferred 20 years, we shall clearly have the value of the intervening portion—*i.e.* of an annuity-certain for 20 years.

The value of a perpetuity $= \dfrac{1}{i}$

And the value of the same deferred 20 years $= \dfrac{1}{i} \times v^{20}$

Thus the value of an annuity for the first 20 years as required $= \dfrac{1}{i} - \dfrac{v^{20}}{i}$

$$= \frac{1 - v^{20}}{i} = \frac{1 - (1+i)^{-20}}{i}$$

And, generally, the present value of an annuity for n years $= \dfrac{1 - v^n}{i}$

The following is another way of regarding the present value of an annuity-certain. The value of 1 due n years hence, as we have already seen, is $\dfrac{1}{(1+i)^n} = v^n$

∴ The value of an annuity for n years $= v + v^2 + v^3 + \ \ldots\ + v^n$

The sum of this geometrical progression is $\dfrac{1 - v^n}{i}$, an expression similar to that obtained above. It is usually denoted by the convenient symbol $a_{\overline{n}|}$

Example.—What is the present value of an annuity of 1 per annum for 7 years at 4 per-cent.?

The value we seek is $\dfrac{1-(1{\cdot}04)^{-7}}{{\cdot}04}$.

In Section 4 we saw that $(1{\cdot}04)^{-7} = {\cdot}7599$, and the answer is

$$\frac{1 - {\cdot}7599}{{\cdot}04} = \frac{{\cdot}2401}{{\cdot}04} = 6{\cdot}0025.$$

A reference to the Table (p. 37) shows that the present value correct to the fourth place is 6·0021.

9. Annuity that will amount to 1, or Sinking Fund. In Section 7 we saw that the amount of an annuity of 1 for n years is $\dfrac{(1+i)^n - 1}{i} = s_{\overline{n}|}$. From this it is obvious that an annuity of $\dfrac{1}{s_{\overline{n}|}}$ will in n years amount to 1. In other words $\dfrac{1}{s_{\overline{n}|}}$ is the annual Sinking Fund, payable at the end of each year, that will replace 1 at the end of n years.

Example.—£100 is invested in a wasting security that will produce 10 per-cent. for 20 years and then cease. What Sinking Fund, to be accumulated at 4 per-cent., must be set aside annually to replace the £100?

Here $s_{\overline{20}|}$ at 4 per-cent. $= 29{\cdot}7781$ (*see* page 37). This entered in the Table of Reciprocals gives ·03358, and multiplying by 100 we have 3·358 as the result.

Answer: £3. 7*s*. 2*d*., leaving a balance of £6. 12*s*. 10*d*. annually to the investor by way of interest.

NOTE.—In this case the correct answer is 3·35818. To be sure of exactitude a Table of Reciprocals to more figures than ours, or a Table of 7-figure logarithms, should be used.

Of late years there has been a great increase in the number of Sinking Fund or Redemption Assurances granted by Life Offices. In these cases, following the analogy of premiums on other assurances, the payments are made at the beginning of each year, instead of at the end as in the foregoing example. Now $\frac{1}{s_{\overline{n}|}}$ payable at the end of a year is the same thing as $v \times \frac{1}{s_{\overline{n}|}}$ payable at the beginning. Hence the annual premium for a Sinking Fund Policy of 1 payable in n years is $\frac{v}{s_{\overline{n}|}} = \frac{1}{(1+i)s_{\overline{n}|}}$. The last expression is easily seen to be $= \frac{1}{s_{\overline{n+1}|} - 1}$, from which any net premium can be directly obtained.

Example.—What is the net premium for securing £100 at the end of 20 years at 3 per-cent?

Here $s_{\overline{21}|} = 28·6765$. Reciprocal of 27·6765 = ·03612.

Answer: £3. 12*s*. 3*d*., which must be considered approximate, as before mentioned.

10. Annuity that 1 will purchase. As $a_{\overline{n}|}$ (*see* Section 8) is the present value of 1 for n years, it is obvious that 1 is the present value of $\frac{1}{a_{\overline{n}|}}$ payable for n years, and would be the net purchase money of an annuity of $\frac{1}{a_{\overline{n}|}}$ if an office were asked to grant one. This does not happen often, but the formula is frequently required for quoting the terms of loans, to public bodies and others, which are to be repaid by equal annual instalments of principal and interest combined—that is, by way of annuity.

Example.—A corporation borrows £1,000 at 4 per-cent. repayable by annuity in 30 years. What will the annual payment be?

Here $a_{\overline{30}|}$ at 4 per-cent. = 17·292. Reciprocal by our Table = ·05785. Approximate answer £57. 17*s*. 0*d*.

It is interesting to reconcile this formula with that in the last Section. A capital of 1 produces an annuity-certain for n years of $\frac{1}{a_{\overline{n}|}}$. Of the latter sum i is needed each year for interest, leaving a balance of $\frac{1}{a_{\overline{n}|}} - i$ to be accumulated by way of sinking fund for replacement of the capital of 1. But

$$\frac{1}{a_{\overline{n}|}} - i = \frac{i}{1 - \frac{1}{(1+i)^n}} - i = \frac{i}{(1+i)^n - 1} = \frac{1}{s_{\overline{n}|}},$$

which is the expression for the sinking fund to replace 1 we have already obtained in Sec. 9.

Specimen Portion of a Life Table, based upon the Institute of Actuaries HM (Healthy Males) Table.

Age x	No. Living l_x	No. Dying d_x	Probability of Living 1 Year $p_x = \frac{l_{x+1}}{l_x}$	Probability of Dying in 1 Year $q_x = \frac{d_x}{l_x}$	Σl_{x+1}	$e_x = \frac{\Sigma l_{x+1}}{l_x}$	$L_x = \frac{1}{2}d_x + l_{x+1} = \frac{1}{2}(l_x + l_{x+1})$	$T_x = \Sigma L_x$	$\mathring{e}_x = \frac{\Sigma L_x}{l_x} = \frac{1}{2} + e_x$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
70	381	24	·937	·063	3,049	8·00	369	3,239·5	8·50
1	357	24	·933	·067	2,692	7·54	345	2,870·5	8·04
2	333	25	·925	·075	2,359	7·08	320·5	2,525·5	7·58
3	308	25	·919	·081	2.051	6·66	295·5	2,205	7·16
4	283	26	·903	·092	1,768	6·25	270	1,909·5	6·75
75	257	25	·903	·097	1,511	5·88	244·5	1,639·5	6·38
6	232	25	·892	·108	1,279	5·51	219·5	1,395	6 01
7	207	24	·884	·116	1,072	5·18	195	1,175·5	5·68
8	183	23	·874	·126	889	4·86	171·5	980·5	5·36
9	160	21	·869	·131	729	4·56	149·5	809	5·06
80	139	20	·856	·144	590	4·24	129	659·5	4·74
1	119	19	·840	·160	471	3·96	109·5	530·5	4·46
2	100	17	·830	·170	371	3·71	91·5	421	4·21
3	83	15	·819	·181	288	3·47	75·5	329·5	3·97
4	68	14	·794	·206	220	3·24	61	254	3·74
85	54	11	·796	·204	166	3·07	48·5	193	3·57
6	43	9	·791	·209	123	2·86	38·5	144·5	3·36
7	34	8	·764	·235	89	2·62	30	106	3·12
8	26	6	·769	·231	63	2·42	23	76	2·92
9	20	5	·750	·250	43	2·15	17·5	53	2·65
90	15	4	·733	·267	28	1·87	13	35·5	2·37
1	11	3	·727	·273	17	1·55	9·5	22·5	2·05
2	8	3	·625	·375	9	1·13	6·5	13	1·63
3	5	2	·600	·400	4	·80	4	6·5	1·30
4	3	2	·334	·666	1	·33	2	2·5	·83
95	1	1	·000	1·000	0	...	·5	·5	·50

NOTE.—The symbol Σ is used to denote the summation of the values of a function, from the age indicated in the suffix up to the end of the Table.

II.—PROBABILITIES AND LIFE CONTINGENCIES.

11. The Construction of a Life Table. We will first consider the construction of an ordinary Life Table, taking as a guide the specimen portion of the H^M Table which is now exhibited. The H^M Table is now largely superseded by the more modern O^M and other Tables, but it is retained here as its use is restricted to purposes of illustration.

It will be noticed that, for the sake of brevity and simplicity, this Table of ours begins at age 70; but the principles involved are exactly the same as if the Table commenced at birth or any subsequent age. Suppose, then, that 381 persons aged 70 were observed till death, and it were found that there died of them (d_x)

In the 1st year 24
,, 2nd ,, ... 24 and so on

as set forth in Col. 3 of the Table. We should here have all the materials for the construction of our limited mortality Table; and the column "No. living," or l_x, would in this case be formed by deducting from the No. living at any previous age the number who had died in the intervening year.

Of course, in the world of facts, the material for a mortality Table from birth to old age does not exist in the simple form here supposed. The observations for a Table are usually compressed within a limited time. They may be recorded by many different methods, and there is a great variety of classes from which they may be drawn. Finally, when the *data* are complete, the construction and graduation of the Tables are matters so complicated as to call for the highest skill of the greatest actuaries. It will be seen that we are here touching the fringe of a large subject, and that the use of Mortality Tables needs, where important interests are involved, considerable judgment and experience.

12. Curtate Expectation of Life. Passing over columns 4 and 5 for the moment, we come to column 6, headed $\Sigma\, l_{x+1}$, which has been formed by adding column 2, the "No. living" at each age, continuously from the bottom, but in such a manner that opposite age 70, for example, stands the sum of the Nos. living, not from age 70, but from age 71 to the end of the Table. This column shows, therefore, the total number of complete and unbroken years of life which will be enjoyed by the number living at any age, and does not include the portions of a year which remain to be enjoyed by those who die before attaining the next age. If now we divide this total number of unbroken years of life at any age by the number living at that age, we arrive at the *average* number of integral years of life which the persons at that age will, taken one with another, enjoy. This result is called the *Curtate Expectation of Life*, and the values of this function are inserted in column 7 under the symbol e_x. Thus at age 72 there are 333 persons who may be expected to enjoy 2359 unbroken years of life. This gives a curtate expectation of 7·08 years; so that persons of that age might, one with another expect, according to our Table, to keep seven more birthdays.

13. Complete Expectation of Life. The method of ascertaining the curtate expectation of life takes no account, as before mentioned, of the portions of years of life which are enjoyed in the years of death, and if we assume the deaths to be evenly distributed throughout the year, the persons dying in any particular year will enjoy on the average six months of life more than is shown above. Thus at age 70, instead of taking into account, in respect of the first year, merely the 357 years completed by the survivors, we should add to them 12 years, being six months for each of the number dying in that year. The total is 369, as appears by column 8. The general expression, then, for the total years of life between one age and the next is $\frac{1}{2}d_x + l_{x+1}$, which is the same thing as $\frac{1}{2}(l_x - l_{x+1}) + l_{x+1} = \frac{1}{2}(l_x + l_{x+1})$; and this is usually denoted by L_x. If, now, column 8 be summed from the bottom as shown in column 9, headed T_x, we have at each age a full estimate of the number of years of life which will be passed through by the body of persons existing at that age. The numbers in this last column being divided at each age by those numbers living we arrive at the results shown in column 10, headed $\mathring{e}_x$. This means the *complete expectation of life.* For instance, at age 72 there are 333 persons who may be expected to enjoy amongst them a total number of years of life of 2525·5, giving a complete expectation of 7·58 years. You will notice that each value in the last column is, as it should be, ·5 of a year, *i.e.*, six months in excess of the corresponding curtate expectation of life. When we speak simply of the "expectation of life", we mean the *complete* expectation.

It is important that we should always bear in mind, when speaking of this function, the nature of the calculation by which its value is found and which you will see to be typified in the following formula :—

$$\mathring{e}_x = \frac{T_x}{l_x} = \frac{\Sigma L_x}{l_x} = \frac{\frac{1}{2}(l_x + l_{x+1}) + \frac{1}{2}(l_{x+1} + l_{x+2}) + \&c.}{l_x}$$

$$= \frac{1}{2} + \frac{l_{x+1} + l_{x+2} + l_{x+3} + \&c.}{l_x}$$

14. Most probable duration of life. Many suppose that the "expectation" exhibits the most probable duration of a lifetime. To find the latter we must first seek the most probable age at death ; and this is, pretty obviously, the age at which the largest number of deaths occur. Looking at our Table we see that the largest number (26) die at age 74, so that four years is the "most probable duration" of life at 70, in the sense that death is more likely to occur at 74 than any other particular age, whereas the "expectation" by column 10 is $8\frac{1}{2}$ years.

15. "Vie Probable." Then, again, the expectation of life has been confused with what the French call "*Vie Probable*", or the age which a person has an even chance of attaining. Now to arrive at the latter age we need to know the age at which the numbers living are reduced by one-half. For instance, referring again to the Table we see that the 381 persons aged 70 will be reduced to 190 by the age of about $77\frac{3}{5}$ years; and this gives $7\frac{3}{5}$ years as the after-lifetime which the man of 70 has an even chance of enjoying, as against the $8\frac{1}{2}$ years "expectation." The latter is simply the average number of years which persons of a given attained age will, one with another, live through, if they die according to the given Mortality Table.

16. The Life Table represents a stationary population. We will now further consider some of the secondary but highly practical uses which can be made of a Life Table from the statistical point of view, that is, before questions of interest and monetary values are introduced. In this connection it is first important to mention that where a question of population is concerned the Life Table represents a stationary population; that is to say, when the l_x persons now aged x become l_{x+1} persons aged $x+1$ their places are assumed to be taken by another l_x persons who concurrently attain age x. In applying a Life Table, therefore, to the consideration of questions of population and communities, we must remember that the Table does not strictly represent the actual position. A population is never stationary, but is constantly being affected by increase of births over deaths, or the reverse, or by immigration or emigration; subject, however, to these reservations, Col. T_x, which has hitherto only been used for ascertaining the expectation of life, may be regarded as showing at any age x, the total number of the population of age x and upwards. To test this, imagine a stationary community which is subject to the mortality of the Table of which our specimen portion forms part, and in which the births do not necessarily occur annually on a fixed day, as postulated in the Life Table, but are distributed, like the deaths, evenly throughout the year.* Let us now see what is found if a census of this community be taken on any given day. Commence with the babies. They will be found of all ages from 0 to 1, but they will not add up to l_0, the number with which the Mortality Table commences; for some have died. On the average they are half way through their first year of life, so that $\frac{1}{2}d_0$ are already dead, and the number enumerated at age 0 will be $l_0-\frac{1}{2}d_0=l_0-\frac{1}{2}(l_0-l_1)=\frac{1}{2}(l_0+l_1)$, which agrees with the formula at the head of the column L_x. Similarly the enumerators would not find 381 persons, aged 70, for instance. On the average, those calling themselves 70 would be $70\frac{1}{2}$ years old, and of those dying between 70 and 71, one-half would, one with another, be already dead, so that the enumerators would be able to record only the remainder, being those who had survived the first six months of the year of age from 70 to 71. That is, they would record the 369 persons appearing in Col. 8 (L_x) of our Table. This column being added up from the bottom to form Col. T_x, the latter may properly be said to

* Having spoken of a "year", one ought to define it strictly. In mortality investigations there are known to actuaries many kinds of years, such as "calendar years", "financial years", "policy years", "life years", and so forth. For the present purpose we must picture a "census year", of which the census day would be the central point; such as would be an ordinary calendar year if the census day were July 1st.

represent at any age x, subject to the operation of the law of averages and to the reservations previously made, the total number of the population of age x and upwards, such as would be recorded at a census.

17. Examples of Use of Population Column. That portion of the imaginary community represented by our sectional Life Table is, as we have seen, constantly being recruited by 381 entrants at the age of 70. What will be the total standing population over that age? Col. T_x shows us that the number is 3,239, of whom 809 will be aged 79 and upwards.

Mr. George King, in the Institute Text-Book, applies column T_x to answer a number of interesting questions, such as the number of soldiers of a certain age which a population recruited by a certain number of births will support; the number of new clerks and superannuations involved in carrying out a pension scheme; and the strain on an annuity society, &c. Reverting, for the sake of simplicity, to our small Table, suppose that under a national pension scheme 50,000 persons became annually entitled, at the age of 70, to a pension of 5*s.* per week, to be raised to 7*s.* per week on attainment of age 75. In course of years what would be the total number of beneficiaries under each class?

Now we have seen that 381 recruited annually will support a community (*see* Col. T_x) of 3,239 on the assumption that the population is subject to the rate of mortality on which the Table is based. Hence, by rule of three,

$$\text{The total number of beneficiaries} = \frac{T_{70}}{l_{70}} \times 50{,}000 = \frac{3{,}239}{381} \times 50{,}000$$
$$= 425{,}066$$

Similarly—

$$\text{The number of Pensions @ 5s.} = \frac{T_{70} - T_{75}}{l_{70}} \times 50{,}000$$
$$= \frac{3{,}239 - 1{,}639}{381} \times 50{,}000$$
$$= 209{,}974$$

$$\text{And the number of Pensions @ 7s.} = \frac{T_{75}}{l_{70}} \times 50{,}000 = \frac{1{,}639}{381} \times 50{,}000$$
$$= 215{,}092$$

It will of course be remembered that the use of the H^M Table for the above or any other example is no indication of its suitability in actual practice.

18. Some elementary propositions in probability. We have already touched on probability. Before finishing what I have to say on the use of the Life Table, it is necessary to devote a few words to the most elementary propositions of this subject.

If an event may happen in α ways and fail in β ways, so that the total number of possible ways is $\alpha + \beta$, and each way is equally likely, then the probability of happening is $\frac{\alpha}{\alpha + \beta}$; and the probability of failure $\frac{\beta}{\alpha + \beta}$.

The simplest possible illustration of this may be found in the Life Table itself. Taking l_x persons aged x, the event of survival until age $x+1$ will happen in l_{x+1} cases: it will fail in d_x, or $l_x - l_{x+1}$, cases. Hence the probability of a life aged x living to age $x+1$ is equal to $\frac{l_{x+1}}{l_x}$; and this is denoted by p_x. Similarly, the probability of a life aged x dying before attaining age $x+1$ is equal to $\frac{d_x}{l_x}$; and this is denoted by q_x. Thus at age 70, there are 381 persons living, of whom 357 will be alive at the end of the year, and 24 dead. The probability, therefore, that a person aged 70 will be alive at the end of the year is $\frac{357}{381} = \cdot 937$; and that he will be dead $\frac{24}{381} = \cdot 063$. We can in this way complete our Life Table in respect of columns 4 and 5, calling the former series p_x and the latter q_x.

Now the probability of the happening and that of the failure must always together make certainty, therefore $p_x + q_x$ must always equal unity; and q_x, the probability of death, may thus be represented by $1 - p_x$.

If there be (say) three independent events, the probabilities of which happening are respectively p_1, p_2, p_3, then—

The probability of all happening... $= p_1 \times p_2 \times p_3$

The probability of all failing ... $= (1-p_1)(1-p_2)(1-p_3)$

The probability that *all* will not happen (*i.e.*, that one at least will fail) ... ... $= 1 - p_1 \times p_2 \times p_3$

The probability that *all* will not fail (*i.e.*, that at least one will happen) ... ... $= 1 - (1-p_1)(1-p_2)(1-p_3)$

The probability that the first will happen and the others fail... $= p_1(1-p_2)(1-p_3)$.

These simple formulas will enable us to answer several interesting questions with the aid of a Life Table.

Examples.—A husband is aged 73 and his wife 71, they having been married 45 years. What is the probability that they will live to enjoy their golden wedding?

The probability of a person aged 73 living 5 years $= \frac{l_{78}}{l_{73}} = \frac{183}{308}$

" " " " 71 " 5 " $= \frac{l_{76}}{l_{71}} = \frac{232}{357}$

Required probability $= \frac{183}{308} \times \frac{232}{357} = \frac{42,456}{109,956} = \frac{2}{5}$ nearly.

Therefore, of 5 chances in all, 2 are in favour of the happening of the event and 3 are against it. Hence, if you desired to bet, the odds would be 3 to 2 against it.

Again, what is the probability that either both *or one* of the said persons will survive the 5 years? In other words, we want the probability that both will *not* die.

Here the probability that 73 will die $= \frac{l_{73}-l_{78}}{l_{73}}$

and ,, ,, 71 ,, ,, $= \frac{l_{71}-l_{76}}{l_{71}}$

∴ ,, ,, both ,, ,, $= \frac{l_{73}-l_{78}}{l_{73}} \times \frac{l_{71}-l_{76}}{l_{71}}$

and ,, ,, both will *not* die $= 1 - \frac{l_{73}-l_{78}}{l_{73}} \times \frac{l_{71}-l_{76}}{l_{71}}$

$$= 1 - \frac{308-183}{308} \times \frac{357-232}{357} = 1 - \frac{125}{308} \times \frac{125}{357}$$

$$= 1 - \frac{1}{7} \text{ (nearly)} = \frac{6}{7}.$$

That is, 6 chances are in favour of the event and one against it, the odds being therefore 6 to 1 on the event.

A, B and C are aged respectively 70, 72 and 74. What is the probability that in 4 years time A will be still living and B and C dead?

Here the probability of A living 4 years $= \frac{l_{74}}{l_{70}}$

,, ,, B dying in the 4 years $= 1 - \frac{l_{76}}{l_{72}}$

and ,, ,, C ,, ,, 4 ,, $= 1 - \frac{l_{78}}{l_{74}}$

Required probability $= \frac{l_{74}}{l_{70}} \times \left(1 - \frac{l_{76}}{l_{72}}\right) \times \left(1 - \frac{l_{78}}{l_{74}}\right)$

$$= \frac{283}{381} \times \frac{333-232}{333} \times \frac{283-183}{283}$$

$$= \frac{283}{381} \times \frac{101}{333} \times \frac{100}{283} = \frac{10{,}100}{126{,}873} = \frac{2}{25} \text{ nearly,}$$

showing odds of 23 to 2 against the supposed event.

III.—THE COMBINATION OF INTEREST WITH PROBABILITY.

19. Endowment. Suppose now that 381 persons aged 70 agree to subscribe to provide an endowment of £1 for each one who survives to 80, how much must each pay, taking interest at 3 per-cent.? In 10 years £139 will be required. The present value of that £139 due 10 years hence, as we have seen, is $139 \times v^{10}$. There are 381 subscribers to make up this sum, and the share each has to pay is therefore $\frac{139 \times v^{10}}{381}$. The table of values of v gives us $v^{10} = \cdot 7441$ and the answer will be found to be ·2715, or about 5*s.* 5*d.* This, then, is the net premium at 3 per-cent. for an endowment of 1 on a life aged 70, payable at age 80 if then alive.

Putting the case generally we say the value at age x of an endowment payable n years hence, is $\frac{v^n \, l_{x+n}}{l_x}$. Looking at the matter in a different way, we see that the probability of any one person aged x receiving the endowment at age $x+n$ is $\frac{l_{x+n}}{l_x}$, whilst the value of the endowment, if it were a certainty, would be v^n. Multiplying this value into the chance of receiving it, we have for the value of the endowment $\frac{v^n l_{x+n}}{l_x}$ as before. Here then, is our first combination of Compound Interest and Probability, a step which may be considered to constitute the real threshold of actuarial study.

20. Life annuity. If a person aged 70 desired to secure, instead of an isolated endowment at age 80, a succession of endowments of 1 for every year that he survived, what would the value be? Proceeding as in the last section, and multiplying the present value of each payment by the chance of receiving it, we have

$v\frac{l_{71}}{l_{70}} + v^2\frac{l_{72}}{l_{70}} + v^3\frac{l_{73}}{l_{70}} + v^4\frac{l_{74}}{l_{70}} + v^5\frac{l_{75}}{l_{70}}$ + &c., up to the end of the Table.

$$\text{This} = \frac{vl_{71} + v^2 l_{72} + \ldots \ \&c.}{l_{70}}$$

and gives us the value of an annuity of 1 on the life of a person aged 70, denoted by a_{70}. Thus, for the general formula for the value of a life annuity, we have

$$a_x = \frac{1}{l_x}(vl_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \&c.)$$

Example.—What annuity, payable annually and not apportionable to date of death, could be given to a male aged 90 in exchange for £100, taking interest at 3 per-cent. and disregarding expenses? In this example I have taken age 90 for the sake of brevity in making the calculation. The principles are the same in all cases.

We first need to find a_{90}, the formula for which is

$$\frac{1}{l_{90}}(vl_{91} + v^2 l_{92} + v^3 l_{93} + v^4 l_{94} + v^5 l_{95})$$

$$= \frac{1}{15}(\cdot 9709 \times 11 + \cdot 9426 \times 8 + \cdot 9151 \times 5 + \cdot 8885 \times 3 + \cdot 8626 \times 1)$$

$$= \frac{26 \cdot 3243}{15} = 1 \cdot 7550.$$

The reciprocal of this, viz., ·5698, gives us the annuity at age 90 which 1 will purchase, so for £100 the annuity given would be £57.

Simple as is the mode of calculating the value of a life annuity, there are few things which it is more important to understand thoroughly, for the annuity-value enters, directly or indirectly, into most actuarial formulas. Of course, in practice, one always takes the values of a_x from a Table; and the values for a Table are calculated much more briefly than in the example here given from first principles.

21. Annuity-certain and life annuity compared. Many people suppose that the value of a life annuity is the same as that of an annuity-certain for the term of the expectation of life. This is not so. It would be, perhaps, outside the scope of this lecture to give the strict algebraical proof, but the *rationale* of the distinction is interesting. Suppose the expectation to be n years. The value of an annuity-certain for n years is, as we have seen, $v+v^2+v^3+\ \ldots\ v^n$. The life annuity-value is

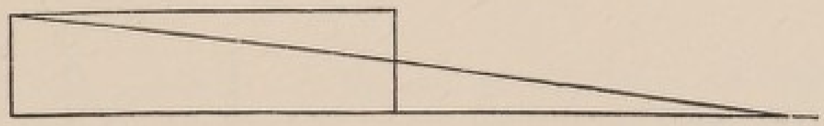
$$v\frac{l_{x+1}}{l_x}+v^2\frac{l_{x+2}}{l_x}+v^3\frac{l_{x+3}}{l_x}+\text{\&c.}$$

and we may conceive this life annuity as an annuity-certain, lasting to the end of the Life Table, with gradually diminishing payments consisting of

$$\frac{l_{x+1}}{l_x}, \frac{l_{x+2}}{l_x}, \frac{l_{x+3}}{l_x}, \ldots \text{\&c.}$$

But these fractions together represent the expectation (curtate) itself. Hence we see that the total amount of money paid on the average in respect of the life annuity is the same as would be paid under the annuity-certain for the term of the curtate expectation. But in the case of the life annuity the payments are spread over a longer period; and the aggregate discounted value of these payments must therefore be less than the value of the certain payments, the last of which is only n years hence.

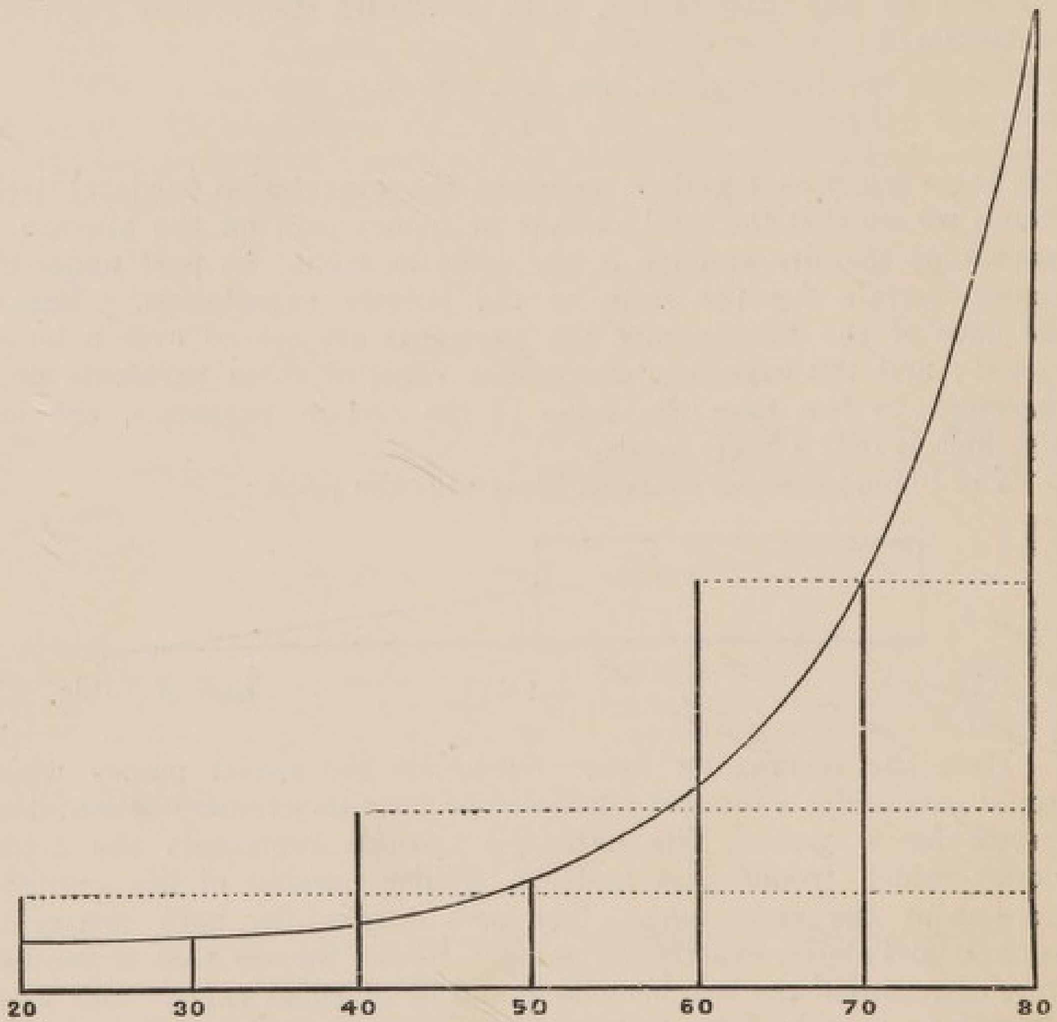
The following small diagram illustrates the point:



Age x $x+n$ End of Table.

Here the rectangular figure represents the actual money which would pass under a number (sufficient to form an average) of annuities-certain for n years. The extended triangle represents the actual money which would pass under a similar number of life annuities granted at the same time. The area covered by both figures is, as explained above, exactly the same. Hence we see that if interest were disregarded the two benefits would be of equal value. But when interest is brought into account, then we see that the life annuity must necessarily be the smaller in present value.

22. Natural risk premium for assurance. We will now consider the value of a benefit payable in the event of the death of a person, instead of in the event of his survival; and we will commence with the value of a benefit payable a year hence, provided a person now alive be then dead. The probability of a person aged x dying in the first year is $\frac{d_x}{l_x}$. If we multiply the chance of the money being paid by (as before) the ratio of its present value if it *be* paid, we have $v\times\frac{d_x}{l_x}$. This, then, is the net premium x must pay for an assurance of 1 for 1 year payable at the end of the year of death—that being the basis upon which most assurance premiums are calculated. If he survived the first year and, having attained $x+1$, desired to assure for the second year, the net premium would be similarly, $v \times \frac{d_{x+1}}{l_{x+1}}$. Now if we were to proceed thus to the end of the Table, we should arrive at a series of premiums for term assurances of one year each which would gradually increase towards the end of life, and would, if set out graphically, assume a form which may be roughly illustrated thus:



In this diagram the gradually increasing vertical lines drawn from the base to the curve represent the risk premiums above described. The height of the lowest dotted line represents the level or continuous premium for age 20 at entry; whilst that of the middle and upper dotted lines represents the same for ages 40 and 60 at entry respectively.

The scale of increasing premiums indicated in this diagram is representative of the scheme of mutual assessment assurance which has been largely practised in America, and it was inevitably found that the premium became intolerably heavy in old age. A Society having no hold on its members, there was a tendency in healthy ones to leave it when the rates of premium became high. This in turn tended to make the payments of the remaining old and less healthy members more intolerable still, because the survivors were "assessed" for whatever contributions might be needed to meet the expenses and growing claims. The result was that, one after another, the Societies broke up and disappeared. When this lecture was first delivered assessment assurance was still largely practised, and was even getting some hold in this country; but the actuarial warnings issued against it have been so abundantly justified by the result that a system certain to bring discredit on legitimate Life Assurance has itself become thoroughly discredited and is now scarcely heard of.

23. "Level" Premium desirable. A system of assurance based on the premium for the mere current risk of death has never commended itself to responsible persons in this country, and in the earliest times the pioneer Actuaries were called upon to quote a premium which would ASSURE, *i.e.*, a so-called "level" premium like that represented by the level dotted line in the above diagram, which could never be increased after the assurers had made their contract. The mode of ascertaining this may now be usefully considered.

24. Single Premium for an Assurance. The probability of a person aged x dying in the first year has been seen to be $\frac{d_x}{l_x}$. The probability that the same person aged x will die in the second year is $\frac{d_{x+1}}{l_x}$, and in the third year $\frac{d_{x+2}}{l_x}$, and so on.

If for every year from x to the end of the Table we multiply the chance of the money being paid by (as before) the ratio of its present value if it *be* paid, we have in the total the value of an assurance of 1 (A_x) for the whole life of x.

That is, we have

$$A_x = v\frac{d_x}{l_x} + v^2\frac{d_{x+1}}{l_x} + v^3\frac{d_{x+2}}{l_x} + \&c.$$

$$= \frac{v(l_x - l_{x+1}) + v^2(l_{x+1} - l_{x+2}) + v^3(l_{x+2} - l_{x+3}) + \&c.}{l_x}$$

$$= v\left\{1 + \frac{1}{l_x}(vl_{x+1} + v^2 l_{x+2} + \ldots)\right\} - \frac{1}{l_x}(vl_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \ldots)$$

$$= v(1 + a_x) - a_x \text{ (compare Section 20)}$$

$$= (1 - d)(1 + a_x) - a_x, \text{ since } v = 1 - d$$

$$= 1 - d(1 + a_x)$$

This last formula admits of simple self-evident reasoning. If 1 be payable at once its value is 1. But if its payment be deferred until the death of a person now aged x, its value is 1 less such a sum as will provide interest until the life fall in, namely, $d(1 + a_x)$. Note that the deduction is not an ordinary annuity (a_x) of interest payable at the end of each year (i); but an annuity *due* $(1 + a_x)$ of d, which is interest paid in advance. Interest in advance must be provided for, because the final payment of a_x is made at the end of the year preceding that in which death occurs; whereas A_x is the value of 1 payable at the end of the year in which death happens. The "value" of an assurance thus obtained is in other words the Net Single Premium for the same.

Example.—What is the net single premium for an Assurance of £100 on a male aged 90, by the H[M] Table, at 3 per-cent.?

In Section 20 we found that, according to these *data*, $a_{90}=1{\cdot}755$; and the Table on p. 36 shows us that $d=1-v={\cdot}0291$. Thus we have $1-d(1+a_{90})=1-{\cdot}0291 \times 2{\cdot}755$, which by logarithms is quickly found to be $1-{\cdot}0802={\cdot}9198$.

Answer: £92.

25. Annual Premium for an Assurance. Having found A_x the single premium for a whole life assurance, we can easily find P_x the annual premium. Now P_x must be such that one payment immediately, together with the value of an annuity of P_x for the remainder of x's life life must be equal to A_x. Hence, we have—

$$P_x + P_x.a_x = A_x \text{ and } P_x = \frac{A_x}{1+a_x}$$

$$= \frac{1-d(1+a_x)}{1+a_x} = \frac{1}{1+a_x} - d$$

Example.—What is the net annual premium corresponding to the single premium in the last example?

Here we can either divide $A_{90}(={\cdot}9198)$ by $1+a_{90}(=2{\cdot}755)$ or we can calculate the premium from the direct formula $\frac{1}{1+a_{90}} - d = \frac{1}{2{\cdot}755} - {\cdot}0291$. With a Table of Reciprocals (*see* p. 42) the latter is simpler, and gives ${\cdot}3629-{\cdot}0291={\cdot}3338$.

Answer: £33. 8*s*.

It is clear from what we have seen that the premium for an assurance in no way depends on the expectation of life, as many people suppose, for the formula involves an estimate being made of the probability of the life dropping in each year up to the end of the Table, and the due valuing at interest of each such chance. We have further seen, however, that it is not necessary in most cases to calculate these probabilities in detail, but that if we have already ascertained the value of an annuity on the life of x, the value of an assurance, and the annual premium for the same, are easily ascertained from the annuity-value.

IV.—SOME PROBLEMS IN LIFE ASSURANCE.

26. Practical considerations must be kept in view. The calculation of a set of net premiums for assurance is a simple and mechanical process when once the table of mortality and rate of interest have been determined upon. It is when these premiums require to be loaded and adapted to office use that the Actuary's real trouble and responsibility commence. Amongst the points he must consider are the future of the money market; the probable classes who will enter the Society; the constitution of the Society; the mode in which it will divide its bonuses; the expense at which it will conduct its business, and the probable extent of its new business; and he must not lose sight of the rates of the offices who will compete with his Society. From the bare enumeration of some of these considerations, you will see how impossible it is to satisfactorily forecast the future and the need, at least, for safe margins.

27. The value of a Policy. I think I need scarcely stop to show that a necessary condition of the receipt of a level premium—*i.e.*, a premium which is at the outset in excess of the measure of the current risk of death—is that a life assurance policy gradually acquires a value. This is represented by the reserve which every office holds against its liabilities and which is estimated to be sufficient, together with the future premiums to be received, to meet those liabilities as and when they arise. Having considered the Annuity-Values, Single Premiums and Annual Premiums, we can now ascertain how this "value" of a policy may be estimated. A person assures for 1 at the age of x under a net annual premium of P_x. What will be the value of the policy at age $x+n$? Let ${}_nV_x$ be this value. Now ${}_nV_x$ must be such that it will be sufficient, together with the value of the premiums remaining to be paid in future, to provide the sum of 1 at the death of the assured, who is now aged $x+n$. The value of the future premiums being $P_x(1+a_{x+n})$ and the value of 1 at death of $x+n$ being A_{x+n}, we have the equation

$${}_nV_x + P_x(1+a_{x+n}) = A_{x+n},$$

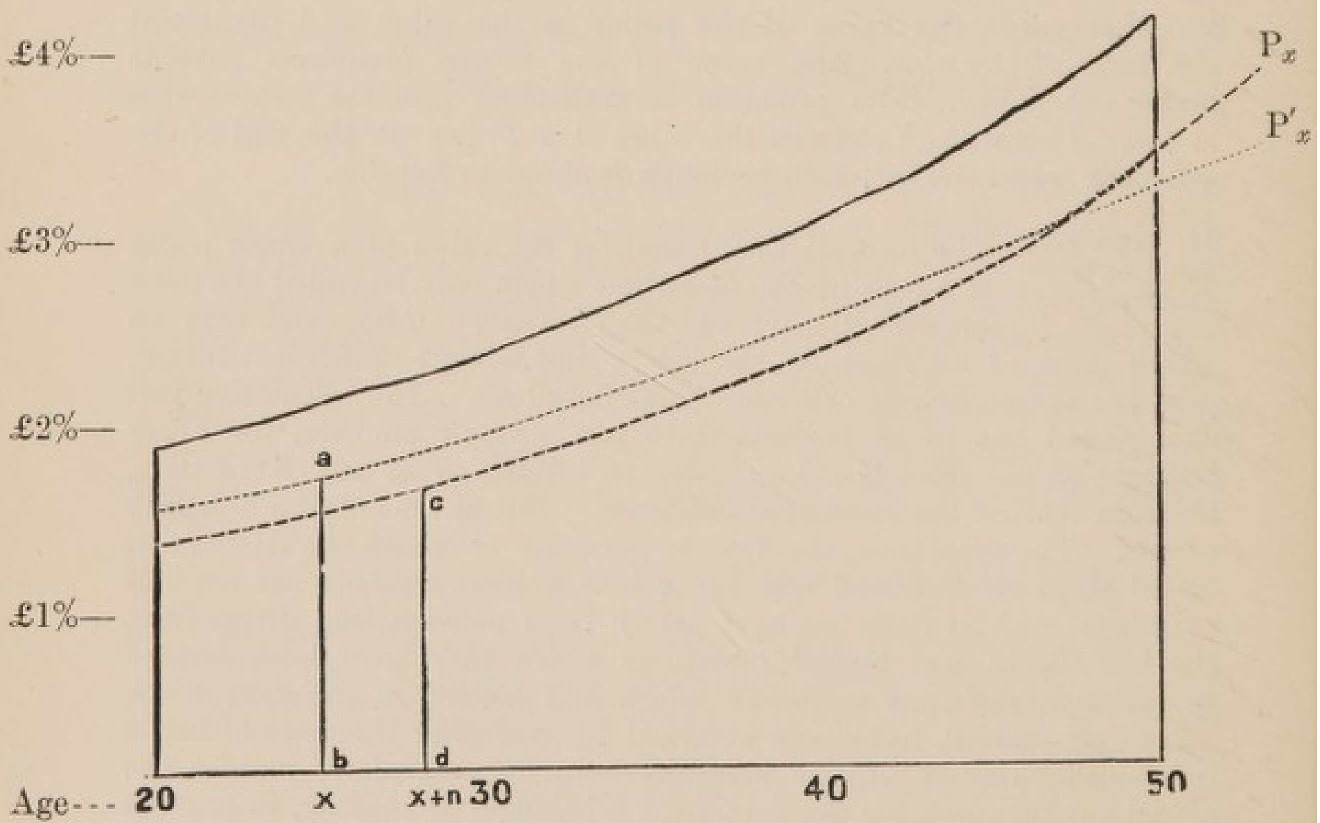
or

$${}_nV_x = A_{x+n} - P_x(1+a_{x+n}).$$

In other words, the value of the policy is the value of 1 payable at the death of $(x+n)$ *less* the value of the future premiums payable under the policy. The premium is multiplied into the annuity-due $(1+a_{x+n})$ because ${}_nV_x$ means the value of a Policy at the end of the nth year, when the $(n+1)$th premium is about to fall due.

28. The valuation of a Society. The formula thus found for the value of a single policy is a type of the operation which may be called the most important part of the Actuary's duty, and that on which most of all depends the safety and honour of his institution, namely: the periodical valuation of its liabilities. The calculation just exemplified has to be performed, in one way or another, for every contract on the office books, in order to obtain, as at any fixed date, the sum total of the calculated liabilities. But at the outset a difficulty occurs. The premiums, the future payment of which the Actuary is called upon to discount and bring into present account, are not net premiums such as those we have lately been constructing direct from the Life Table, but loaded premiums which may have been formed from several different mortality tables and loaded in different ways. They may, indeed, have been obtained by averaging the rates of half a dozen other offices. Every premium contains, therefore, besides the actual net premium for the assurance, a balance, or loading, to provide for expenses of management, for commission, for bonuses and for general safety; and it will not do to discount and bring into present account the portions of those premiums which, when the time for their receipt arrives, will be required for such extraneous purposes as those just referred to, and will not be, in any sense, available towards providing the actual assurance itself. What premium then, must the Actuary value? I will refer to-night to two only of the many methods of valuation which are at times adopted.

29. Office premium valuation. First, as seems most natural, we will consider that in which the Actuary has regard to the actual office premiums which the assured have contracted to pay. From these is usually thrown off a fixed percentage to represent the loading of which we have been speaking, so that we get down to that portion of the premium which is regarded as sufficient to provide the sum assured, and which for the moment we will call the net premium. Having, then, decided upon an appropriate table of mortality and rate of interest by which to make the valuation, the Actuary proceeds to value on the one hand the sums assured which have to be paid, and on the other the net or valuation premiums which have to be received. Subtracting the latter from the former, we ascertain the calculated liability and thus know whether the office exhibits a surplus or a deficiency. This method of valuation has become in a sense discredited by a tendency which it possesses to produce so called "negative values." To enable you to see what these are and how they arise, let us look a little more closely into the *modus operandi*. The following curve is intended to represent an average scale of office premiums at successive ages.



Suppose that 20 per-cent. be the loading thrown off these office premiums; then the dotted curve will represent the remaining net premiums which are to be valued. Suppose further that the valuation be made by data which are not stringent, say the Carlisle Table and 4 per-cent. interest; and that the curve of dashes represents the "pure"* premiums for successive ages, which we should arrive at if we were to construct a set of premiums according to the Carlisle Table with 4 per-cent. interest by the method previously explained in Section 25. We proceed to value at age $x+n$ an insurance effected at age x; and in order to do so, we must, as explained in Section 27, deduct from the value of the sum assured the value of the future net premiums. That is, calling this net premium P'_x, we have to find the value of the expression $A_{x+n} - P'_x(1+a_{x+n})$; of which A_{x+n} represents the liability, and $P'_x(1+a_{x+n})$, the value of the future premiums, represents the asset side of the account. Now in the above circumstances we shall almost certainly have here a case in which the net premium to be valued ($ab = P'_x$) is greater than the pure premium ($cd = P_{x+n}$ say) at the valuation age according to the data used in the valuation. Then inasmuch as P'_x is $> P_{x+n}$ we shall have $P'_x(1+a_{x+n}) > P_{x+n}(1+a_{x+n})$. But $P_{x+n}(1+a_{x+n}) = A_{x+n}$, as we saw in Section 25; therefore the asset side of the valuation comes out greater than the liability and this is what is technically termed a negative value.

Looked at in another way, we see that if we deduct from the present value of the sum assured at age $x+n$ the value of a future annual payment of the corresponding pure premium for the *same* age (*cd* in the diagram), the difference or liability will be nothing. If, however, we deduct the value of a future annual payment of the corresponding pure premium for a *younger* age than $x+n$, which premium would be *smaller* than *cd*, there will be a positive difference or liability. But if we deduct the value of a future payment which has been arbitrarily obtained and which happens to be *larger* than *cd* (such as *ab*), the difference will be a negative one constituting a negative liability, or asset, instead of the positive liability to which every contract should give occasion. It is this tendency to the production of negative values which has partly discredited this mode of valuation.

It is, however, easy to see that, either by making the arbitrary deduction larger, or by making the bases of valuation more stringent and thus increasing the theoretical pure premiums according to those bases, or by a combination of both processes, the dotted curve might be made to fall below the dashed curve at all ages, and the effect of this would be, not only to prevent negative values, but to make even a stronger reserve than would ensue from a pure premium valuation according to the bases adopted.

In weighing the effect of this method of valuation in any particular case it is of the utmost importance to know (1) that the table of mortality and interest assumed in the valuation are sufficiently stringent, (2) that the percentage of the premiums reserved is sufficiently large and, generally (3) that no negative values enter

* The expression "net" premiums would in the ordinary way be more usually applied here; but that term has already been taken in this section, as the most appropriate phrase, to signify the reduced premiums brought into valuation.

into it in the mode explained above. Subject to these most important qualifications, the method is one for which much may be said. Its great advantage is that one deals with the office premiums, which are accomplished facts. You have contracted to receive those premiums, and therefore they form an all-important factor in the future. For this reason the method under discussion is the only proper one to be used in cases of insolvency or transfer, but for the purposes of a prosperous going concern Actuaries frequently prefer to shut their eyes to the premiums actually payable, and value only the theoretical pure premiums according to the bases assumed, *i.e.*, the premium which I represented above by the dashed curve.

30. Pure premium valuation. This method, called the net or pure premium method, which for reasons already stated it is better to designate here by the latter term, is the only other mode to which I shall refer to-night. Now if the same table of mortality and rate of interest be used in the valuation as were used for constructing the premiums, the latter having been loaded 25 per-cent. for office purposes, the net premiums to be valued would be 80 per-cent. of the office premiums and would be represented by our dotted curve in the last diagram. A little consideration will show that in this event $P_{x+n}(1+a_{x+n})$ must always be greater than $P_x(1+a_{x+n})$; and that negative values are therefore impossible. The valuation is, in fact, a pure premium one; and with this recommendation, that the 20 per-cent. thrown off for safety, and for future expenses and profits, is reserved untouched in the valuation, and will always be available for its purpose.

It is seldom, however, that this is the case in practice. An office does not frequently alter its scale of premiums, but the developments of actuarial science may not infrequently necessitate a change in the valuation basis, and the Actuary may thus have a pure premium to value which bears no fixed relation to the office premium. This is illustrated in the last diagram where the dashed curve may be taken to represent the net premium (P_x) to be brought into the valuation; and this brings us to the distinguishing feature of the pure premium method, which is that the net or valuation premium must always correspond with the other data of the valuation. In other words, the valuation must be made by the formula $A_{x+n} - P_x(1+a_{x+n})$ where P_x is taken at the same rates of mortality and interest as A_{x+n} and a_{x+n}. As the valuation premium is thus always larger at age $x+n$ than at age x, every policy must rank as a liability, without any special care being taken to exclude negative policy values.

But there is an anomaly of another sort, viz., that as the pure premium, which is also the valuation premium, bears no fixed ratio to the office premium, the margin remaining for future expenses and profits is more or less a matter of chance. For instance, if the pure premium were much below the office premium at young ages at entry, but greater at high ages at entry, the result would be this, that if the office assured many young lives and no old ones, its margin would be large: while if it assured no young lives but many old ones, its margin would be very small. It is, therefore, very important to note the margin of reserve for future contingencies, by way of comparison with the valuation data. I say by way of comparison with the data because a noteworthy thing is, that as we make the data more stringent, *i.e.*, increase the effect of mortality and decrease the rate of interest, and consequently increase the pure premiums, we raise our dashed line (P_x) still nearer to the office premium and correspondingly decrease the margin of premiums to be reserved. A small margin thus accounted for would be satisfactory; but a small margin found in combination with weak valuation *data* would point to the office premiums being insufficient, at any rate at the ages at which the assurances were generally effected. Thus, while an increase in the proportion of premium income reserved would be an unfailing sign of increased strength under the mode of valuation first described, it would, taken by itself, indicate a decrease of stringency under the pure premium method.

31. Sources of Surplus. A little study of the different modes of valuation on the lines we have just been following enables one to see pretty clearly from what sources the surplus of a Life Office comes. One chief item of profit is that arising from the current business, *i.e.*, the gain from a favourable mortality and the unused portion of the free margin of the premiums which has been reserved for expenses and profits, always assuming that the valuation data are not so weak as to lead to the margin being encroached upon by the claims. Care in the selection of risks and a moderate ratio of expenditure in comparison with the ratio of the premiums reserved are, therefore, vital factors in the production of profit. Where an office is suffering from a heavy mortality or is habitually spending in commission and expenses as large a ratio of its premiums as was reserved in the valuation, then it is clear that little current profit is being made. The other chief source of profit is from the interest on the Assurance Fund which has been received in excess of that anticipated in the valuation. In an old office valuing at a low rate of interest and receiving a good rate on its investments, this item of profit would be very large—perhaps contributing one-third or even one-half of the total surplus.

	Office A.	Office B.
Valuation Rate ...	$2\frac{1}{2}$ %	$3\frac{1}{2}$ %
Interest earned...	$3\frac{3}{4}$ %	$4\frac{1}{4}$ %
Proportion of Premiums reserved...	$17\frac{1}{2}$ %	$22\frac{1}{2}$ %

To illustrate the usefulness of investigating the sources of surplus and to show that the amount of a surplus actually declared is not of itself all important, I give imaginary but not impossible particulars of two offices valuing by the net premium method and charging about equal rates of office premiums. Now I see nothing here to prevent office B showing good bonus results, perhaps almost as good as office A, for the time being. It makes up for its smaller excess of interest by drawing a larger margin from every premium received. Yet office A is incomparably stronger than Office B and therefore better equipped for maintaining or increasing its bonuses in time to come.

32. Bonuses. When the surplus has been ascertained, upon what principles should it be distributed? A generation ago there was a bewildering number of systems in use, and the subject, which is very technical theoretically, was for years much discussed by Actuaries. Speaking generally, the systems could be classified in three categories, viz., those in which the surplus is converted into Reversionary Bonuses and then allotted in proportion to the Sums Assured; those in which cash apportionments were made in proportion to the premiums or loading paid; and those in which an attempt was made to analyse the sources of the surplus and apportion it correctly amongst the contributors. The "contribution method", as the last is named, is equitable, using the word in the restricted sense of returning to a person his own contribution towards the profits. It has, however, never been very popular in this country, and apparently is not becoming more so. The first-named system, especially as modified into the well-known "Compound Reversionary" method, under which existing bonuses count for apportionment as well as Sum Assured, is tending to prevail. Its advantages are that the bonuses increase with age, which is popular, that it is simple to understand and to apply, and finally that it is fairly equitable where the premiums and valuation data are suitably adjusted to it. The other and miscellaneous methods were often inequitable, but they have tended to disappear through amalgamations. Of those now remaining most have been modified by adjusting either the system, or the premiums, or both; and I doubt whether any important office is now open to serious criticism for inequitable distribution of surplus.

I cannot venture into further details here, but there are some general considerations which should be borne in mind by all concerned. One is that general excellence is much more important than supposed equity in the Bonus System. Inequity as between young entrants and old ones is material, but that is more a question of premiums and tends to adjust itself; for otherwise the office would suffer through "selection" on the part of the public. But as between recent entrants and old policyholders, it is best not to dogmatize about equity. Office A may favor the old people, and say with justice it is trying to redress the misfortune of old age. Office B may favor the young ones, and say with equal justice that it is but emphasizing the greater benefit that accrues anyway on early deaths. Taking the broadest view, most systems are equitable in which the various entrants have, at starting, fairly equal chances of securing the benefits.

33. Surrender-Values. To see that a Life Office must gradually accumulate a reserve against its accruing liabilities is also to see that on being relieved from those liabilities it can well afford to return at least a portion of the past over-payments. Hence every office pays surrender values on purchasing its own policies; but inasmuch as it is justifiable to fine a person who for his own purposes wishes to terminate a contract into which he has deliberately entered, it is the practice not to pay the full value of a policy on withdrawal. Amongst other reasons for this practice may be mentioned the fact that good lives have the option of withdrawing, perhaps leaving bad ones behind, and the consideration that vacancies caused by withdrawal will have to be filled up, possibly at considerable expense to the office.

But first there is the important question: How is the full value of the policy itself to be arrived at for purpose of surrender? Where the actual office premium payable enters into the calculation of the office reserve, as in the first of the two methods of valuation we have discussed, there appears to be no alternative, and the basis of the surrender value will be the office reserve held against the policy in question. Where, however, as in most cases, the pure premium method of valuation is in use, it is a fair question for argument whether the office reserve is the proper basis. It may be said that in knowing the age of the assured, the amount assured and the premium payable, the office has all the data for determining at what price it should buy the policy, and that the calculated reserve which it chooses for its own purposes to set up against the risk has nothing to do with the case. On the other hand it may be urged that an office cannot return what it does not possess, and that there is no reason beyond the general principle of deduction already mentioned, why it should give less. The system of adopting the reserve as the basis of surrender-values further has the sanction of a great many, if not the majority of Actuaries; and if it be objected that by this method a withdrawer is admitted to share in a fund which may have been artificially enlarged for purposes of safety, yet there is the obvious reply that the withdrawer's own contributions have in past years been used towards the creation of that fund, and that he is on withdrawal equitably entitled to his share of it.

In practice, most offices make a deduction of a fixed percentage from the full value of the policy. This plan has the disadvantage that it fines most heavily those who should be dealt with most leniently, viz., the older policyholders. A graduated deduction such as would be brought about by valuing the policy as an ordinary reversion at a higher rate of interest, or by increasing the value of the future premiums instead of diminishing the net value of the policy, might be more strictly defensible. But a still better graduated deduction is obtained by slightly varying the Reserve formula and allowing on surrender, say,

$$A_{x+n} - P_{x+1}\,(1 + a_{x+n})$$

calculated by the Valuation Table of Mortality, but at a higher rate of interest. The office impounds, as it were, the whole of the first premium, and the value begins to accrue as from age $x+1$. The value, further reduced by the higher rate of interest, thus somewhat punishes recent entrants who quickly surrender; but it ultimately approximates to, though it can never exceed, the Reserve value itself, and may thus be regarded as an actuarial modification of that function.

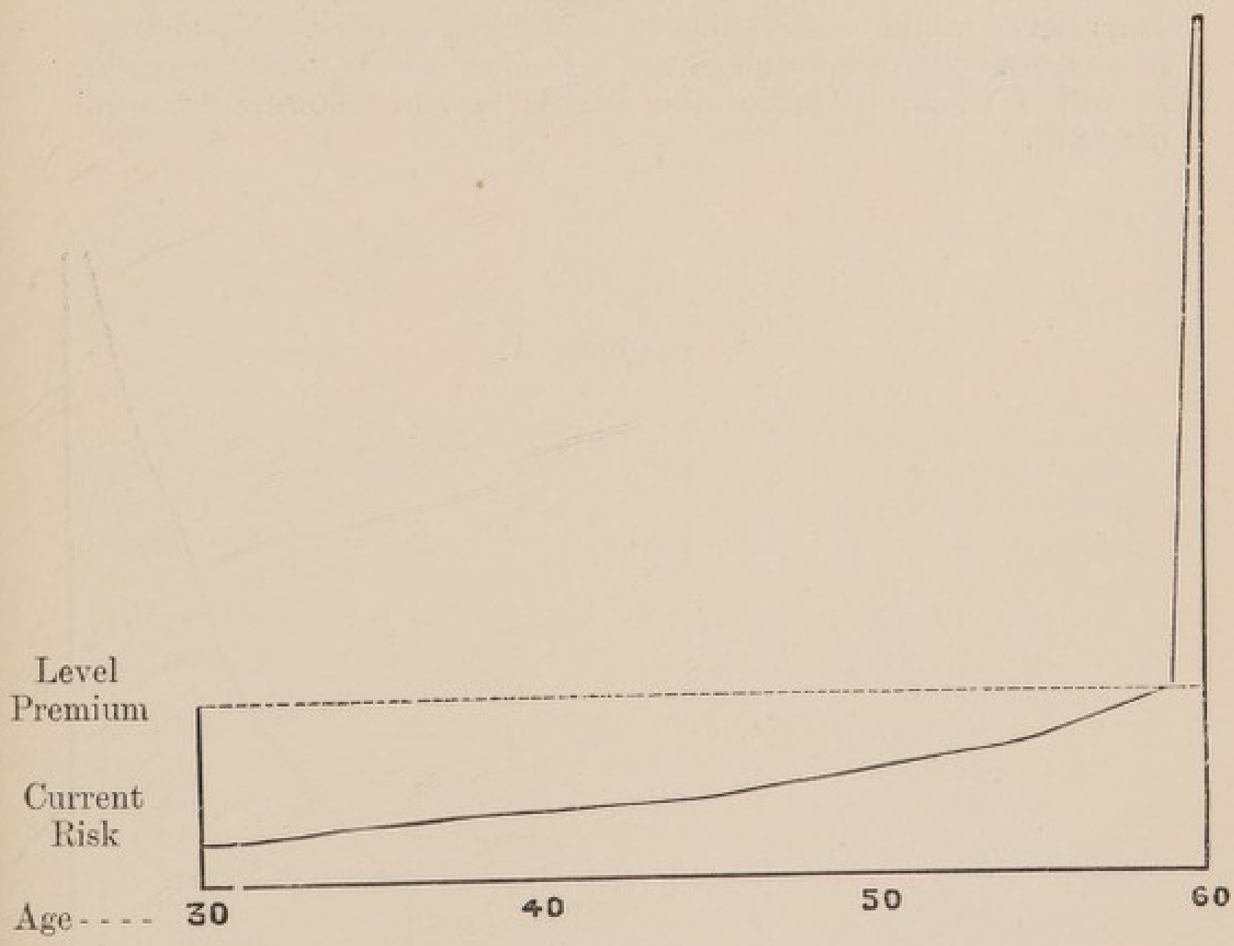
As no practical Actuary would dream of exceeding the Reserve Value, and as the most forceful objections to its adoption as a basis for surrender-values are removed by this latter method, it has naturally received high sanction. Of course it is generous, and if the rate of interest be not taken too high it will enable the office to give more for its policies than they would fetch in the open market; but this is no more than it ought to be able to do. It likewise lends itself to meet certain other practical considerations which modern developments compel us to bear in mind. These are "quick change" days, and a policy is often converted into a Paid-up Policy, or an Endowment Assurance, or an Old Age Pension, only to be re-converted a little later, perhaps, into a Cash Surrender-Value. If these operations are to be conducted with satisfaction to all concerned, it is obvious that the various formulas must be correlated and made consistent; and my own experience has led me to the conclusion that it is necessary for the Actuary to construct some arbitrary system, based on defensible general principles, that will reconcile the interests of his office with the equitable claims of the assured.

34. Endowment Assurances. Before terminating my remarks, which have, I fear, strayed considerably beyond their proper limits, I should like to say a few words about Endowment Assurances. Now a whole life policy is nothing else than an Endowment Assurance payable at the limiting age of the Table, or at previous death; and it is therefore natural that the expressions for the single and annual premiums for an Endowment Assurance should be analogous to those for a whole life policy. Thus if $A_{x.\overline{n|}}$ and $P_{x.\overline{n|}}$ be respectively used to denote the single and annual premiums for an Endowment Assurance on a life aged x payable at age $x+n$ and $|_{n-1}a_x$ be the value of a temporary annuity on the life of x, to last for $n-1$ years, then—

$$A_{x.\overline{n|}} = 1 - d\,(1 + |_{n-1}a_x)\text{; and } P_{x.\overline{n|}} = \frac{1}{1 + |_{n-1}a_x} - d$$

These expressions are as self-evident as the corresponding expressions for a Whole Life Assurance. Thus taking the single premium, we may say that if 1 be payable now it is worth 1; but if its payment be deferred until x attain $x+n$ or previously die, then its value is 1 less such a sum as will provide an annuity of the annual interest of 1 until the sum of 1 be payable: *i.e.*, from 1 we must deduct, for reasons already explained in Section 24, the value of a temporary annuity *in advance* of interest payable *in advance* for n years, whence we get the above expression $1 - d\,(1 + |_{n-1}a_x)$. The annual premium is of course obtained by dividing the single premium by $1 + |_{n-1}a_x$.

An Endowment Assurance, it will thus be seen, is in effect a whole life policy with an arbitrary age at maturity substituted for the limiting age of the table. For example, in the case of an Endowment Assurance payable at age 60, no part of the Life Table above age 60 enters into the calculation in any way, and if the business of a particular office were to consist wholly of endowments payable at age 60, a Life Table entirely cut off after that age would answer all its purposes. Let us consider some of the consequences of working by this limited Table. The whole of the heavy mortality at age 60 and upwards to the end of the Table occurs or is theoretically supposed to occur at age 59—the current rate of mortality for every age below 59, remaining, of course, the same as in the complete Table. The value of the benefit to be received under it is much larger than under the whole Life Table, whilst the term during which the annual premium for such increased benefit is payable is considerably shorter. Hence the annual premium is very much larger than when based on the whole Life Table. Now we saw that the ordinary whole life level premium consisted of (1) the premium for the current risk, and (2) an excess to be accumulated to meet the heavy mortality rate of old age. The Endowment Assurance premium we are now considering contains (1) the same contribution to current claims as the whole life premium, and (2) a very much larger excess payment to be accumulated to provide for the exodus (or theoretical deaths) of all the surviving policyholders at age 60. The following diagram shows, as in the case of the whole life premiums, the risk premiums and the level premiums. It is not drawn to scale, and is only intended to roughly impress on the mind how large a portion of the level premium for an Endowment Assurance is not paid for assurance at all, but is only received to be accumulated for the payment of the theoretical claim at the end of the chapter.



Our business being primarily Insurance and not Investment, it is obvious that persons who are our customers during the early portion of life while the current risk is small and then all die off theoretically at the early age of 60, are in a different category from those who will consent to stay on.

35. Some practical considerations respecting Endowment Assurances. If this question were merely one of technical interest, it would be of secondary importance ; but unfortunately it has serious practical bearings. In the first place it is the persistent old lives who give the offices a chance of reaping those handsome profits from longevity which are the reward of care in the selection of risks. Then every policyholder who leaves an office has to be replaced with constantly increasing difficulty and expense. There are further drawbacks in the facts that the Endowment Assurance premiums unduly swell the premium income of an office and have to bear a proportion of the expenses which from their nature they are ill-fitted to bear. Danger lurks, too, in the not ancommon practice of giving to Endowment Assurances the same ratio of profit as appertains to whole life policies. Unless the premiums have been specially loaded to provide such bonus, the effect will be to decrease the general ratio of profit for the benefit of a certain class. Then again, there is a tendency to accept lives at ordinary rates under Endowment Assurances which would not be admitted under the whole life scale ; or, if a surcharge be made, to rate the life for calculating the Endowment Assurance premium at the same age as would be applicable for a whole life policy. Now, except in comparatively rare instances the reasoning here is totally incorrect, as will, I think, be made clear to you by the following two small diagrams.

Diagram A.

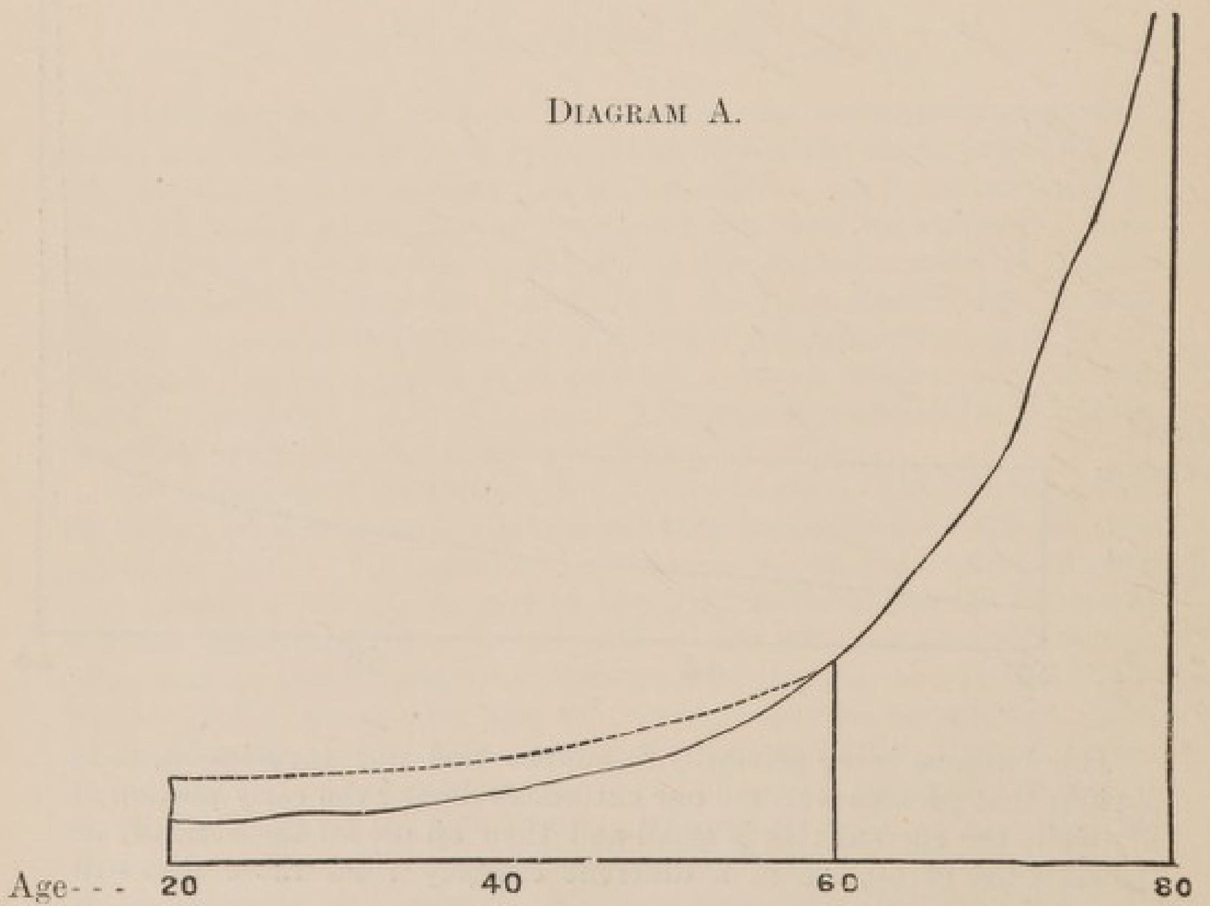
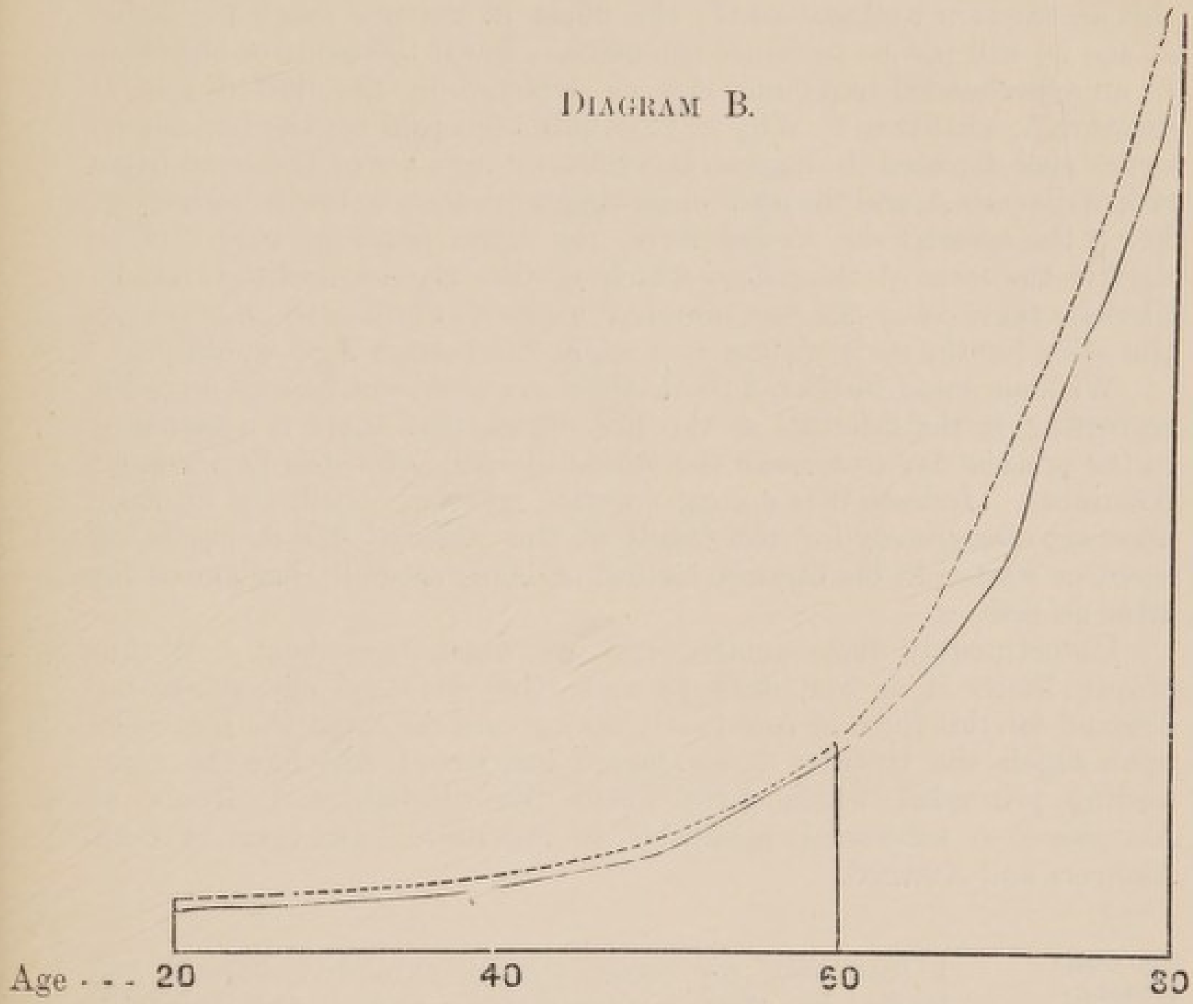


Diagram B.



In each of these diagrams the normal rate of mortality is represented by the curved line. In the first the dotted line represents an increased mortality (such as that arising from a consumptive tendency), which is heavy at young ages and wears itself out by about 60—the lives after that age being as good as the normal ones. In the second case the dotted line represents an increased mortality (such as would apply to gout cases or certain simple forms of heart disease), where the mortality in early years is nearly as low as the normal rate, but later in life, say after 60, causes a great many premature claims.

Now when we accept a bad life at the ordinary rate under an Endowment Assurance, we assume that the mortality will follow the normal line depicted in the diagrams. If the extra mortality be of the type depicted in Diagram B no great harm is done, but if it be of the type in Diagram A, then obviously the office will run a much greater risk than it is being paid for. When, on the other hand, we rate a bad life up, say, from age 30 to age 40 for a whole life policy, and then calculate the equivalent premium for an Endowment Assurance as though his future will be the same as that of a man of 40, the assumption underlying our action is that the extra risk is mainly stored up in his case until late in life, somewhat as depicted by the dotted line in diagram B. If the ground of objection accord with this assumption well and good: the effect of cutting short the Table at age 60 will not be to injure the office. But if the cause of objection be an apprehended imminent risk as depicted by the dotted line in diagram A, what then? Why we have only been paid for the little or no extra risk depicted in diagram B, whilst we have borne the large extra risk of diagram A, and the office must therefore stand to lose considerably. For if the assured die in the term, the claim must be paid. If he survive the term of the policy, which is also the term of the risk, he likewise takes away the sum assured without waiting to help recoup the office for the early claims, as a whole life policyholder would do.

Without going further, I think there are sufficient reasons here for regretting, in the interests of the life offices, that there is a tendency in the present day to forsake the whole life policy for the Endowment Assurance. Indeed, it is a great question whether it will not in many cases act disastrously for the family of the Assured the money being spent or wasted in his lifetime instead of being carefully harboured for after necessities.

Unfortunately facts remain and we must face them. Within proper limits it is our duty to meet the increased and increasing demand for this form of Assurance, taking care only that the principles upon which the business is conducted are sound, and like the great leading principles which have made the old-fashioned forms of Assurance so successful, conducive to the mutual interests of both Assurers and Assured.

36. Conclusion. I fear that my remarks have extended beyond the limits we first contemplated; and I am painfully conscious of being open to your criticism on the ground of having been at once diffuse and wanting in thoroughness where it might have been more profitable to give fuller explanations within narrower limits. Some, in fact, may be inclined to remind me of Pope's admonition—

> "A *little learning* is a dangerous thing;
> Drink deep or taste not the Pierian spring:
> There shallow draughts intoxicate the brain
> And drinking largely sobers us again."

Still, if I have endeavoured to show that there are many exact problems which a very moderate exercise of our abilities will enable us to solve, I hope I have also succeeded in suggesting that there are many general questions, especially those relating to Valuations, Bonuses, Surrender-Values, and Endowment Assurances, which, though often simple on the surface and interesting to discuss in an elementary way, are yet extremely complicated in their ultimate bearings. In regard to such we must try to "drink largely"; and in any case it behoves us all, as ambassadors to the public of honourable institutions, to exercise a wise care, so that no hasty or incorrect generalisation, no unfair argument, no ungenerous criticism, no ill-founded suggestion may prejudice the general cause in the progress of which we all have so great an interest.

APPENDIX.

COMPOUND INTEREST TABLES.

NOTE ON THE USE OF LOGARITHMS.

The integer part of a logarithm is called the *Index*, or *Characteristic;* the decimal part is called the *Mantissa*.

In seeking the logarithm of any four figures, find by inspection in the following table the *mantissa* for the first three figures. To this add the number found in the "Proportional Parts" columns in respect of the fourth figure.

The *Index* of the logarithm of any number greater than 1 and less than 10 is 0. For instance, log 6·422 = 0·8076. The indices for other numbers derived from the same figures depend on whether 6·422 is taken into positive or negative multiples of 10, as follows :—

Positive Multiples.		*Negative Multiples.*		
For log 64·22,	Index is 1.	For log ·6422,	Index is −1,	written $\bar{1}$
,, 642·2	,, 2.	,, ·06422	,, −2,	,, $\bar{2}$
,, 6422·	,, 3.	,, ·006422	,, −3,	,, $\bar{3}$

It is important to note that the *mantissa* is always positive. It is only the *index* that may be positive or negative. For example, divide the logarithm $\bar{1}$·2164 by 2. The quotient is not − ·6082. The logarithm should be mentally altered to − 2 + 1·2164 ; and this, when divided by 2, gives the correct quotient, viz., − 1 + ·6082 or $\bar{1}$·6082.

To find the number corresponding to a given logarithm, the Table of Logarithms may be used in the reverse manner ; but a table of anti-logarithms, if available, is easier. The latter table is "entered" (as it is called) with the first three figures of the *mantissa* of the given logarithm, and to the result is added, as before, the amount for the fourth figure found under "Proportional Parts." The *index* of the given logarithm is then used to determine the place of the decimal point in the result, according to the rule already stated for ascertaining the *index* from a given number.

From elementary algebra we know that when numbers having the same root are to be multiplied or divided, the indices or exponents are added or subtracted respectively. Logarithms having the same base are strictly analogous ; so that when numbers have to be multiplied or divided their logarithms are simply added or subtracted. Involution and Evolution may be regarded as extensions of multiplication and division. For example, 9^4 means $9 \times 9 \times 9 \times 9$, or the logarithm of 9 added four times. These remarks will help to explain the following rules :—

1.—*To multiply together two or more numbers.*

Add together the logarithms of the several numbers, and the sum will be the logarithm of their product.

2.—*To divide one number by another.*

Subtract the logarithm of the divisor from that of the dividend, and the remainder is the logarithm of the quotient.

3.—*To find any power of a given number.*

Multiply the logarithm of the given number by the exponent of the power, and the product is the logarithm of the power required.

4.—*To find any root of a given number.*

Divide the logarithm of the given number by the index of the root, and the quotient is the logarithm of the root.

NOTE.—It may here be explained briefly that common logarithms, having 10 for a base, are obtained as follows. We know that $10^2 = 100$. Here 2 is called the logarithm of 100 to the base 10; and as $10^0 = 1$, and $10^1 = 10$, we have also $0 = \log 1$, and $1 = \log 10$. Let us interpolate a few other logarithms between these last two. First find the square root of 10, viz., 3·162. This gives us $\sqrt{10} = 10^{\frac{1}{2}} = 10^{\cdot 5} = 3{\cdot}162$; whence $\cdot 5 = \log 3{\cdot}162$. Then the square root of 3·162, which will be the fourth root of 10, $= 1{\cdot}778$. This gives us $10^{\frac{1}{4}} = 10^{\cdot 25} = 1{\cdot}778$; whence $\cdot 25 = \log 1{\cdot}778$. Similarly, the cube root of 10 is 2·154, and the ninth root of 10 (being the cube root of 2·154) is 1·291. Thus we have $10^{\frac{1}{3}} = 10^{\cdot 3333} = 2{\cdot}154$ and $10^{\frac{1}{9}} = 10^{\cdot 1111} = 1{\cdot}291$; whence $\cdot 3333 = \log 2{\cdot}154$ and $\cdot 1111 = \log 1{\cdot}291$. Collecting these results, we have

log 1·000 = 0·0000	log 2·154 = 0·3333
log 1·291 = 0·1111	log 3·162 = 0·5000
log 1·778 = 0·2500	log 10·000 = 1·0000.

This rudimentary table sufficiently illustrates the principles underlying the construction of a Table of Logarithms, though the computation of a complete table, to several places of decimals, needs great skill and entails enormous labour. It will be seen that only the logarithms for numbers between 1 and 10 need be computed, inasmuch as all other numbers are obtained therefrom by being multiplied into positive or negative integral powers of 10, such as, for example, $10^3 = 1000$, or $10^{-3} = \cdot 001$, giving $\log 1000 = 3{\cdot}0000$ and $\log \cdot 001 = \bar{3}{\cdot}0000$. In other words, there is in the case of all such other numbers (under the multiplication and division rules above stated) an unexpressed additional or supplementary logarithm corresponding to the required power of 10; and this must always be a positive or negative integer. It will form, in fact, the *index* of the completed logarithm, and in no way affects the *mantissa*.

COMPOUND INTEREST TABLE.

	Amount of 1: viz., $(1+i)^n$			Present Value of 1: viz., v^n			
n	3%	4%	5%	3%	4%	5%	n
1	1·0300	1·0400	1·0500	·9709	·9615	·9524	1
2	1·0609	1·0816	1·1025	·9426	·9246	·9070	2
3	1·0927	1·1249	1·1576	·9151	·8890	·8638	3
4	1·1255	1·1699	1·2155	·8885	·8548	·8227	4
5	1·1593	1·2167	1·2763	·8626	·8219	·7835	5
6	1·1941	1·2653	1·3401	·8375	·7903	·7462	6
7	1·2299	1·3159	1·4071	·8131	·7599	·7107	7
8	1·2668	1·3686	1·4775	·7894	·7307	·6768	8
9	1·3048	1·4233	1·5513	·7664	·7026	·6446	9
10	1·3439	1·4802	1·6289	·7441	·6756	·6139	10
11	1·3842	1·5395	1·7103	·7224	·6496	·5847	11
12	1·4258	1·6010	1·7959	·7014	·6246	·5568	12
13	1·4685	1·6651	1·8856	·6810	·6006	·5303	13
14	1·5126	1·7317	1·9799	·6611	·5775	·5051	14
15	1·5580	1·8009	2·0789	·6419	·5553	·4810	15
16	1·6047	1·8730	2·1829	·6232	·5339	·4581	16
17	1·6528	1·9479	2·2920	·6050	·5134	·4363	17
18	1·7024	2·0258	2·4066	·5874	·4936	·4155	18
19	1·7535	2·1068	2·5270	·5703	·4746	·3957	19
20	1·8061	2·1911	2·6533	·5537	·4564	·3769	20
21	1·8603	2·2788	2·7860	·5375	·4388	·3589	21
22	1·9161	2·3699	2·9253	·5219	·4220	·3418	22
23	1·9736	2·4647	3·0715	·5067	·4057	·3256	23
24	2·0328	2·5633	3·2251	·4919	·3901	·3101	24
25	2·0938	2·6658	3·3864	·4776	·3751	·2953	25
26	2·1566	2·7725	3·5557	·4637	·3607	·2812	26
27	2·2213	2·8834	3·7335	·4502	·3468	·2678	27
28	2·2879	2·9987	3·9201	·4371	·3335	·2551	28
29	2·3566	3·1187	4·1161	·4243	·3207	·2429	29
30	2·4273	3·2434	4·3219	·4120	·3083	·2314	30
31	2·5001	3·3731	4·5380	·4000	·2965	·2204	31
32	2·5751	3·5081	4·7649	·3883	·2851	·2099	32
33	2·6523	3·6484	5·0032	·3770	·2741	·1999	33
34	2·7319	3·7943	5·2533	·3660	·2636	·1904	34
35	2·8139	3·9461	5·5160	·3554	·2534	·1813	35
36	2·8983	4·1039	5·7918	·3450	·2437	·1727	36
37	2·9852	4·2681	6·0814	·3350	·2343	·1644	37
38	3·0748	4·4388	6·3855	·3252	·2253	·1566	38
39	3·1670	4·6164	6·7048	·3158	·2166	·1491	39
40	3·2620	4·8010	7·0400	·3066	·2083	·1420	40
41	3·3599	4·9931	7·3920	·2976	·2003	·1353	41
42	3·4607	5·1928	7·7616	·2890	·1926	·1288	42
43	3·5645	5·4005	8·1497	·2805	·1852	·1227	43
44	3·6715	5·6165	8·5572	·2724	·1780	·1169	44
45	3·7816	5·8412	8·9850	·2644	·1712	·1113	45
46	3·8950	6·0748	9·4343	·2567	·1646	·1060	46
47	4·0119	6·3178	9·9060	·2493	·1583	·1009	47
48	4·1323	6·5705	10·4013	·2420	·1522	·0961	48
49	4·2562	6·8333	10·9213	·2350	·1463	·0916	49
50	4·3839	7·1067	11·4674	·2281	·1407	·0872	50

COMPOUND INTEREST TABLE.

COMPOUND INTEREST TABLE.

	Amount of 1 per Annum: viz., $s_{\overline{n}\vert}$			Present Value of 1 per Annum: viz., $a_{\overline{n}\vert}$			
n	3%	4%	5%	3%	4%	5%	n
1	1·0000	1·0000	1·0000	0·9709	0·9615	0·9524	1
2	2·0300	2·0400	2·0500	1·9135	1·8861	1·8594	2
3	3·0909	3·1216	3·1525	2·8286	2·7751	2·7232	3
4	4·1836	4·2465	4·3101	3·7171	3·6299	3·5460	4
5	5·3091	5·4163	5·5256	4·5797	4·4518	4·3295	5
6	6·4684	6·6330	6·8019	5·4172	5·2421	5·0757	6
7	7·6625	7·8983	8·1420	6·2303	6·0021	5·7864	7
8	8·8923	9·2142	9·5491	7·0197	6·7327	6·4632	8
9	10·1591	10·5828	11·0266	7·7861	7·4353	7.1078	9
10	11·4639	12·0061	12·5779	8·5302	8·1109	7·7217	10
11	12·8078	13·4864	14·2068	9·2526	8·7605	8·3064	11
12	14·1920	15·0258	15·9171	9·9540	9·3851	8·8633	12
13	15·6178	16·6268	17·7130	10·6350	9·9856	9·3936	13
14	17·0863	18·2919	19·5986	11·2961	10·5631	9·8986	14
15	18·5989	20·0236	21·5786	11·9379	11·1184	10·3797	15
16	20·1569	21·8245	23·6575	12·5611	11·6523	10·8378	16
17	21·7616	23·6975	25·8404	13·1661	12·1657	11·2741	17
18	23·4144	25·6454	28·1324	13·7535	12·6593	11·6896	18
19	25·1169	27·6712	30·5390	14·3238	13·1339	12·0853	19
20	26·8704	29·7781	33·0660	14·8775	13·5903	12·4622	20
21	28·6765	31·9692	35·7193	15·4150	14·0292	12·8212	21
22	30·5368	34·2480	38·5052	15·9369	14·4511	13·1630	22
23	32·4529	36·6179	41·4305	16·4436	14·8568	13·4886	23
24	34·4265	39·0826	44·5020	16·9355	15·2470	13·7986	24
25	36·4593	41·6459	47·7271	17·4131	15·6221	14·0939	25
26	38·5530	44·3117	51·1135	17·8768	15·9828	14·3752	26
27	40·7096	47·0842	54·6691	18·3270	16·3296	14·6430	27
28	42·9309	49·9676	58·4026	18·7641	16·6631	14·8981	28
29	45·2189	52·9663	62·3227	19·1885	16·9837	15·1411	29
30	47·5754	56·0849	66·4388	19·6004	17·2920	15·3725	30
31	50·0027	59·3283	70·7608	20·0004	17·5885	15·5928	31
32	52·5028	62·7015	75·2988	20·3888	17·8736	15·8027	32
33	55·0778	66·2095	80·0638	20·7658	18·1476	16·0025	33
34	57·7302	69·8579	85·0670	21·1318	18·4112	16·1929	34
35	60·4621	73·6522	90·3203	21·4872	18·6646	16·3742	35
36	63·2759	77·5983	95·8363	21·8323	18·9083	16·5469	36
37	66·1742	81·7022	101·6281	22·1672	19·1426	16·7113	37
38	69·1594	85·9703	107·7095	22·4925	19·3679	16·8679	38
39	72·2342	90·4092	114·0950	22·8082	19·5845	17·0170	39
40	75·4013	95·0255	120·7998	23·1148	19·7928	17·1591	40
41	78·6633	99·8265	127·8398	23·4124	19·9931	17·2944	41
42	82·0232	104·8196	135·2318	23·7014	20·1856	17·4232	42
43	85·4839	110·0124	142·9933	23·9819	20·3708	17·5459	43
44	89·0484	115·4129	151·1430	24·2543	20·5488	17·6628	44
45	92·7199	121·0294	159·7002	24·5187	20·7200	17·7741	45
46	96·5015	126·8706	168·6852	24·7754	20·8847	17·8801	46
47	100·3965	132·9454	178·1194	25·0247	21·0429	17·9810	47
48	104·4084	139·2632	188·0254	25·2667	21·1951	18·0772	48
49	108·5406	145·8337	198·4267	25·5017	21·3415	18·1687	49
50	112·7969	152·6671	209·3480	25·7298	21·4822	18·2559	50

COMPOUND INTEREST TABLE.

TABLE OF LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	Proportional Parts.								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

TABLE OF LOGARITHMS.

TABLE OF LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	Proportional Parts. 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

TABLE OF LOGARITHMS.

TABLE OF ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	Proportional Parts. 1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

TABLE OF ANTILOGARITHMS.

TABLE OF ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	Proportional Parts. 1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TABLE OF ANTILOGARITHMS.

RECIPROCALS OF NUMBERS

FROM 1000 TO 9999.

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11	9091	9009	8929	8850	8772	8696	8621	8547	8475	8403	8	15	23	30	38	45	53	61	68
12	8333	8264	8197	8130	8065	8000	7937	7874	7813	7752	6	13	19	26	32	38	45	51	58
13	7692	7634	7576	7519	7463	7407	7353	7299	7246	7194	5	11	16	22	27	33	38	44	49
14	7143	7092	7042	6993	6944	6897	6849	6803	6757	6711	5	10	14	19	24	29	33	38	43
15	6667	6623	6579	6536	6494	6452	6410	6369	6329	6289	4	8	13	17	21	25	29	33	38
16	6250	6211	6173	6135	6098	6061	6024	5988	5952	5917	4	7	11	15	18	22	26	29	33
17	5882	5848	5814	5780	5747	5714	5682	5650	5618	5587	3	6	10	13	16	20	23	26	29
18	5556	5525	5495	5464	5435	5405	5376	5348	5319	5291	3	6	9	12	15	17	20	23	26
19	5263	5236	5208	5181	5155	5128	5102	5076	5051	5025	3	5	8	11	13	16	18	21	24
20	5000	4975	4950	4926	4902	4878	4854	4831	4808	4785	2	5	7	10	12	14	17	19	21
21	4762	4739	4717	4695	4673	4651	4630	4608	4587	4566	2	4	7	9	11	13	15	17	20
22	4545	4525	4505	4484	4464	4444	4425	4405	4386	4367	2	4	6	8	10	12	14	16	18
23	4348	4329	4310	4292	4274	4255	4237	4219	4202	4184	2	4	5	7	9	11	13	14	16
24	4167	4149	4132	4115	4098	4082	4065	4049	4032	4016	2	3	5	7	8	10	12	13	15
25	4000	3984	3968	3953	3937	3922	3906	3891	3876	3861	2	3	5	6	8	9	11	12	14
26	3846	3831	3817	3802	3788	3774	3759	3745	3731	3717	1	3	4	6	7	8	10	11	13
27	3704	3690	3676	3663	3650	3636	3623	3610	3597	3584	1	3	4	5	7	8	9	11	12
28	3571	3559	3546	3534	3521	3509	3497	3484	3472	3460	1	2	4	5	6	7	9	10	11
29	3448	3436	3425	3413	3401	3390	3378	3367	3356	3344	1	2	3	5	6	7	8	9	10
30	3333	3322	3311	3300	3289	3279	3268	3257	3247	3236	1	2	3	4	5	6	7	9	10
31	3226	3215	3205	3195	3185	3175	3165	3155	3145	3135	1	2	3	4	5	6	7	8	9
32	3125	3115	3106	3096	3086	3077	3067	3058	3049	3040	1	2	3	4	5	6	7	8	9
33	3030	3021	3012	3003	2994	2985	2976	2967	2959	2950	1	2	3	4	4	5	6	7	8
34	2941	2933	2924	2915	2907	2899	2890	2882	2874	2865	1	2	3	3	4	5	6	7	8
35	2857	2849	2841	2833	2825	2817	2809	2801	2793	2786	1	2	2	3	4	5	6	6	7
36	2778	2770	2762	2755	2747	2740	2732	2725	2717	2710	1	2	2	3	4	5	5	6	7
37	2703	2695	2688	2681	2674	2667	2660	2653	2646	2639	1	1	2	3	4	4	5	6	6
38	2632	2625	2618	2611	2604	2597	2591	2584	2577	2571	1	1	2	3	3	4	5	5	6
39	2564	2558	2551	2545	2538	2532	2525	2519	2513	2506	1	1	2	3	3	4	4	5	6
40	2500	2494	2488	2481	2475	2469	2463	2457	2451	2445	1	1	2	2	3	4	4	5	5
41	2439	2433	2427	2421	2415	2410	2404	2398	2392	2387	1	1	2	2	3	3	4	5	5
42	2381	2375	2370	2364	2358	2353	2347	2342	2336	2331	1	1	2	2	3	3	4	4	5
43	2326	2320	2315	2309	2304	2299	2294	2288	2283	2278	1	1	2	2	3	3	4	4	5
44	2273	2268	2262	2257	2252	2247	2242	2237	2232	2227	1	1	2	2	3	3	4	4	5
45	2222	2217	2212	2208	2203	2198	2193	2188	2183	2179	0	1	1	2	2	3	3	4	4
46	2174	2169	2165	2160	2155	2151	2146	2141	2137	2132	0	1	1	2	2	3	3	4	4
47	2128	2123	2119	2114	2110	2105	2101	2096	2092	2088	0	1	1	2	2	3	3	4	4
48	2083	2079	2075	2070	2066	2062	2058	2053	2049	2045	0	1	1	2	2	3	3	3	4
49	2041	2037	2033	2028	2024	2020	2016	2012	2008	2004	0	1	1	2	2	2	3	3	4
50	2000	1996	1992	1988	1984	1980	1976	1972	1969	1965	0	1	1	2	2	2	3	3	4
51	1961	1957	1953	1949	1946	1942	1938	1934	1931	1927	0	1	1	2	2	2	3	3	3
52	1923	1919	1916	1912	1908	1905	1901	1898	1894	1890	0	1	1	1	2	2	3	3	3
53	1887	1883	1880	1876	1873	1869	1866	1862	1859	1855	0	1	1	1	2	2	2	3	3
54	1852	1848	1845	1842	1838	1835	1832	1828	1825	1821	0	1	1	1	2	2	2	3	3

NOTE.—Numbers in difference columns to be subtracted, not added.

RECIPROCALS OF NUMBERS

FROM 1000 TO 9999.

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55	1818	1815	1812	1808	1805	1802	1799	1795	1792	1789	0	1	1	1	2	2	2	3	3
56	1786	1783	1779	1776	1773	1770	1767	1764	1761	1757	0	1	1	1	2	2	2	3	3
57	1754	1751	1748	1745	1742	1739	1736	1733	1730	1727	0	1	1	1	2	2	2	2	3
58	1724	1721	1718	1715	1712	1709	1706	1704	1701	1698	0	1	1	1	1	2	2	2	3
59	1695	1692	1689	1686	1684	1681	1678	1675	1672	1669	0	1	1	1	1	2	2	2	3
60	1667	1664	1661	1658	1656	1653	1650	1647	1645	1642	0	1	1	1	1	2	2	2	3
61	1639	1637	1634	1631	1629	1626	1623	1621	1618	1616	0	1	1	1	1	2	2	2	2
62	1613	1610	1608	1605	1603	1600	1597	1595	1592	1590	0	1	1	1	1	2	2	2	2
63	1587	1585	1582	1580	1577	1575	1572	1570	1567	1565	0	0	1	1	1	1	2	2	2
64	1563	1560	1558	1555	1553	1550	1548	1546	1543	1541	0	0	1	1	1	1	2	2	2
65	1538	1536	1534	1531	1529	1527	1524	1522	1520	1517	0	0	1	1	1	1	2	2	2
66	1515	1513	1511	1508	1506	1504	1502	1499	1497	1495	0	0	1	1	1	1	2	2	2
67	1493	1490	1488	1486	1484	1481	1479	1477	1475	1473	0	0	1	1	1	1	2	2	2
68	1471	1468	1466	1464	1462	1460	1458	1456	1453	1451	0	0	1	1	1	1	2	2	2
69	1449	1447	1445	1443	1441	1439	1437	1435	1433	1431	0	0	1	1	1	1	2	2	2
70	1429	1427	1425	1422	1420	1418	1416	1414	1412	1410	0	0	1	1	1	1	1	2	2
71	1408	1406	1404	1403	1401	1399	1397	1395	1393	1391	0	0	1	1	1	1	1	2	2
72	1389	1387	1385	1383	1381	1379	1377	1376	1374	1372	0	0	1	1	1	1	1	2	2
73	1370	1368	1366	1364	1362	1361	1359	1357	1355	1353	0	0	1	1	1	1	1	2	2
74	1351	1350	1348	1346	1344	1342	1340	1339	1337	1335	0	0	1	1	1	1	1	1	2
75	1333	1332	1330	1328	1326	1325	1323	1321	1319	1318	0	0	1	1	1	1	1	1	2
76	1316	1314	1312	1311	1309	1307	1305	1304	1302	1300	0	0	1	1	1	1	1	1	2
77	1299	1297	1295	1294	1292	1290	1289	1287	1285	1284	0	0	0	1	1	1	1	1	1
78	1282	1280	1279	1277	1276	1274	1272	1271	1269	1267	0	0	0	1	1	1	1	1	1
79	1266	1264	1263	1261	1259	1258	1256	1255	1253	1252	0	0	0	1	1	1	1	1	1
80	1250	1248	1247	1245	1244	1242	1241	1239	1238	1236	0	0	0	1	1	1	1	1	1
81	1235	1233	1232	1230	1229	1227	1225	1224	1222	1221	0	0	0	1	1	1	1	1	1
82	1220	1218	1217	1215	1214	1212	1211	1209	1208	1206	0	0	0	1	1	1	1	1	1
83	1205	1203	1202	1200	1199	1198	1196	1195	1193	1192	0	0	0	1	1	1	1	1	1
84	1190	1189	1188	1186	1185	1183	1182	1181	1179	1178	0	0	0	1	1	1	1	1	1
85	1176	1175	1174	1172	1171	1170	1168	1167	1166	1164	0	0	0	1	1	1	1	1	1
86	1163	1161	1160	1159	1157	1156	1155	1153	1152	1151	0	0	0	1	1	1	1	1	1
87	1149	1148	1147	1145	1144	1143	1142	1140	1139	1138	0	0	0	1	1	1	1	1	1
88	1136	1135	1134	1133	1131	1130	1129	1127	1126	1125	0	0	0	1	1	1	1	1	1
89	1124	1122	1121	1120	1119	1117	1116	1115	1114	1112	0	0	0	1	1	1	1	1	1
90	1111	1110	1109	1107	1106	1105	1104	1103	1101	1100	0	0	0	1	1	1	1	1	1
91	1099	1098	1096	1095	1094	1093	1092	1091	1089	1088	0	0	0	0	1	1	1	1	1
92	1087	1086	1085	1083	1082	1081	1080	1079	1078	1076	0	0	0	0	1	1	1	1	1
93	1075	1074	1073	1072	1071	1070	1068	1067	1066	1065	0	0	0	0	1	1	1	1	1
94	1064	1063	1062	1060	1059	1058	1057	1056	1055	1054	0	0	0	0	1	1	1	1	1
95	1053	1052	1050	1049	1048	1047	1046	1045	1044	1043	0	0	0	0	1	1	1	1	1
96	1042	1041	1040	1038	1037	1036	1035	1034	1033	1032	0	0	0	0	1	1	1	1	1
97	1031	1030	1029	1028	1027	1026	1025	1024	1022	1021	0	0	0	0	1	1	1	1	1
98	1020	1019	1018	1017	1016	1015	1014	1013	1012	1011	0	0	0	0	1	1	1	1	1
99	1010	1009	1008	1007	1006	1005	1004	1003	1002	1001	0	0	0	0	0	1	1	1	1

NOTE.—Numbers in difference columns to be subtracted, not added.