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A
NEW METHOD
OF RESOLVING
CUBIC EQUATIONS.

By JAMES IVORY, Esq.

FROM THE TRANSACTIONS OF
THE ROYAL SOCIETY OF EDINBURGH.

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*A NEW METHOD of resolving CUBIC EQUATIONS. By
JAMES IVORY, ESQ. Communicated by JOHN PLAYFAIR,
F.R.S. EDIN. and Professor of Mathematics in the University
of Edinburgh.*

[Read, 6th May 1799.]

1. **I** DIVIDE cubic equations into two varieties or species: the one, comprehending all cubic equations with three real roots; the other, all those with only one real root.

2. LET ϕ denote any angle whatever, and let $\tau = \tan \phi$, the radius being unity: let also $z = \tan \frac{\phi}{3}$: then from the doctrine of angular sections we have

$$\tau = \frac{3z - z^3}{1 - 3z^2},$$

which being reduced to the form of an equation, is

$$z^3 - 3\tau z^2 - 3z + \tau = 0.$$

Now, from what is commonly taught in angular sections, z , in this equation, may denote, not only $\tan \frac{\phi}{3}$, but also $\tan(\frac{\phi}{3} + 120^\circ)$, or $\tan(\frac{\phi}{3} + 240^\circ)$. It is to be remarked, too, that any value whatsoever may be assigned to τ , positive or negative, and without limit or restriction as to magnitude. The equation, then, has three different values of z for every given value of τ ; and it belongs to the species of cubic equations, having three real roots.

A

3. AGAIN

3. AGAIN I assume this expression,

$$\frac{1-\tau}{1+\tau} = \frac{(1-z)^3}{(1+z)^3}.$$

LET there be conceived an equilateral hyperbola, of which the femiaxes are each equal to unity, and let a straight line be drawn to touch the hyperbola at its vertex: Conceive also a straight line to be drawn from the centre, to cut off a sector from the hyperbola itself, or from its opposite, or conjugate hyperbolas, and to intercept a part τ (estimated from the vertex) on the tangent line: And, in like manner, let another straight line be drawn from the centre to cut off another sector, that shall be one-third part of the former sector, and to intercept a part z on the tangent line: Then the relation of τ and z will be as in the expression here assumed, viz.

$$\frac{1-\tau}{1+\tau} = \frac{(1-z)^3}{(1+z)^3}.$$

I SHALL not stop to demonstrate this proposition respecting the hyperbola: it easily follows from the known properties of that curve. I mention it merely with the view of marking the strict analogy that subsists between the two varieties of cubic equations. It is sufficient for our purpose to remark, what is indeed very evident from the nature of the assumed expression, that, whatever value be assigned to τ , z has always one real correspondent value, and only one.

FROM our assumed expression we get

$$\tau = \frac{(1+z)^3 - (1-z)^3}{(1+z)^3 + (1-z)^3} = \frac{3z + z^3}{1 + 3z^2},$$

which being reduced to the form of an equation, is

$$z^3 - 3\tau z^2 + 3z - \tau = 0.$$

THIS equation has only one value of z for every given value of τ ; and it belongs to the species of cubic equations having only one real root.

4. IN

4. IN order to give to the two equations, investigated above, the utmost generality of which they are capable, I write $\frac{\tau}{R}$ for τ , and they finally become,

$$\text{I. } Rz^3 - 3\tau z^2 - 3Rz + \tau = 0,$$

$$\text{II. } Rz^3 - 3\tau z^2 + 3Rz - \tau = 0,$$

in which two equations, R and τ represent any numbers, positive or negative, and altogether unlimited and arbitrary as to magnitude.

I CONSIDER the two preceding equations as the simple cases, or simple forms, of the two species of cubic equations: And the method of resolution I have to propose is, to reduce every cubic equation whatsoever to one or other of these two forms.

THE first of the above forms is an equation belonging to the circle. It expresses the relation between the tangent of an arch, and the tangent of the third part of that arch: and it has, in all cases, three real roots. If we take the angle ϕ , of which the tangent is $\frac{\tau}{R}$, the radius being unity; the three roots of the equation, or the three values of z , are, $\tan \frac{\phi}{3}$, $\tan \left(\frac{\phi}{3} + 120^\circ \right)$, and $\tan \left(\frac{\phi}{3} + 240^\circ \right)$.

THE second of these forms is an equation belonging to the hyperbola. It expresses the relation between the tangents of two hyperbolic sectors, of which the one is triple of the other; and it has, in all cases, only one real root. From the expression assumed (Art. 3.) above, whence this equation was deduced, we get, $\frac{1-z}{1+z} = \sqrt[3]{\frac{1-\tau}{1+\tau}}$: therefore,

$$z = \frac{1 - \sqrt[3]{\frac{1-\tau}{1+\tau}}}{1 + \sqrt[3]{\frac{1-\tau}{1+\tau}}}: \text{ or, writing } \frac{\tau}{R} \text{ for } \tau, z = \frac{1 - \sqrt[3]{\frac{R-\tau}{R+\tau}}}{1 + \sqrt[3]{\frac{R-\tau}{R+\tau}}}: \text{ And}$$

so z is found by extracting the cubic root of a given number.

THE two forms differ from one another only in the signs of their terms. The first and third terms, as well as the second and fourth, have always unlike signs in the first form; but always like signs in the second form. This property respecting the signs of the alternate terms, by which the one equation is essentially distinguished from the other, I shall denominate the "*Characteristic of the form.*"

5. I PROCEED, now, to shew, in what way any proposed cubic equation may be reduced to one or other of the two forms.

LET the proposed equation be,

$$x^3 + Ax^2 + Bx + C = 0,$$

where A, B, C denote any given coefficients, positive or negative. I assume $x = \frac{a+z}{b+z}$: a and b being indeterminate quantities, and z a new unknown quantity. And it is to be observed, that the supposition of $x = \frac{a+z}{b+z}$ is always possible, provided a be not equal to b : for if a be not equal to b , a value may be assigned to z , such, that the fraction $\frac{a+z}{b+z}$ shall be equal to any number whatever, positive or negative. But if $a = b$, the value of $\frac{a+z}{b+z}$ is not altered, whatever number z may denote.

HAVING substituted, and taken away the denominators, we get,

$$(a+z)^3 + A(a+z)^2(b+z) + B(a+z)(b+z)^2 + C(b+z)^3 = 0.$$

THE

THE terms of this expression are now to be evolved and arranged, according to the powers of z : which being done, we shall find,

$$\left. \begin{aligned} & a^3 + 3a^2 \times z + 3a \times z^2 + 1 \times z^3 \\ & + Aa^2b + 2Aab \times z + Ab \times z^2 \\ & + Aa^2 \times z + 2Aa \times z^2 + A \times z^3 \\ & + Bab^2 + Bb^2 \times z \\ & + 2Bab \times z + 2Bb \times z^2 \\ & + Ba \times z^2 + B \times z^3 \\ & + Cb^3 + 3Cb^2 \times z + 3Cb \times z^2 + C \times z^3 \end{aligned} \right\} = 0.$$

IN order to reduce this equation to our forms, we must equate three times the coefficient of z^3 to the coefficient of z , either with the same or different signs: and also the coefficient of z^2 to three times the absolute term, likewise with the same or different signs. For in the forms the coefficient of z^3 and z are R and $\mp 3R$: and the coefficient of z^2 and the absolute term are, -3τ and $\pm \tau$. Now, in the transformed equation above, three times the coefficient of z^3 is $3 + 3A + 3B + 3C$, which I write thus, $(3 + 2A + B) + (A + 2B + 3C)$: And in like manner, for three times the absolute term, I write $(3a^3 + 2Aa^2b + Bab^2) + (Aa^2b + 2Bab^2 + 3Cb^3)$. This being observed, we shall have these two equations for determining a and b :

$$\begin{aligned} \left. \begin{aligned} & 3 + 2A + B \\ & + A + 2B + 3C \end{aligned} \right\} &= \mp \left\{ \begin{aligned} & 3a^3 + 2Aa^2b + Bab^2 \\ & + Aa^2b + 2Bab^2 + 3Cb^3 \end{aligned} \right. \\ \left. \begin{aligned} & 3a + 2Aa + Ba \\ & + Ab + 2Bb + 3Cb \end{aligned} \right\} &= \mp \left\{ \begin{aligned} & 3a^3 + 2Aa^2b + Bab^2 \\ & + Aa^2b + 2Bab^2 + 3Cb^3 \end{aligned} \right. \end{aligned}$$

6. It is manifest, from the manner in which I have written the two equations for determining a and b , that they depend upon

upon the two following more simple equations of the quadratic form :

$$3a^2 + 2Aab + Bb^2 = \mp (3 + 2A + B),$$

$$Aa^2 + 2Bab + 3Cb^2 = \mp (A + 2B + 3C).$$

For these two equations are no other than the two parts of the first of the preceding equations : and if we multiply the first of them by a , and the second by b , we shall have the two parts of the second of the preceding equations.

To determine a and b , I now write $M = 3 + 2A + B$; $N = A + 2B + 3C$; and $ay = b$: thus we have,

$$a^2 \times (3 + 2Ay + By^2) = \mp M,$$

$$a^2 \times (A + 2By + 3Cy^2) = \mp N.$$

MULTIPLY the first equation by N ; the second by M ; subtract the one from the other; and divide by a^2 ; and there will result this equation for y :

$$(3N - AM) + 2(AN - BM)y + (BN - 3CM)y^2 = 0.$$

AND in this equation there is no ambiguity of signs.

IF we suppose $y = 1$, the equation last found is equivalent to the identical equation $MN - MN = 0$. One value of y is therefore unity. But this is precisely the case of $a = b$, which we have noticed above to be inapplicable to the present purpose. Nor is it to be wondered at that this value of y is of no use in the present inquiry: for it is manifest that it does not at all depend on the given quantities A, B, C, M and N , but merely on the peculiar manner in which the equation is constituted. We learn from hence, however, that there is always another value of y , whatever numbers A, B, C, M and N may denote; because every quadratic must have two roots or none at all.

QUADRATIC equations, with one root $= 1$, may be supposed to be thus generated: $(y-1) \times (my-n) = my^2 - (m+n)y + n = 0$,
the

the two roots being 1 and $\frac{n}{m}$: Comparing this formula with our equation, we have $n = 3N - AM$ and $m = BN - 3CM$: And so we get,

$$y = \frac{3N - AM}{BN - 3CM}.$$

Now, $a^2 = \frac{\mp M}{3 + 2Ay + By^2}$ and $b = a \times y$: therefore,

$$a = (BN - 3CM) \times \mp \sqrt{\frac{\mp M}{3(BN - 3CM)^2 + 2A(3N - AM)(BN - 3CM) + B(3N - AM)^2}}$$

$$b = (3N - AM) \times \pm \sqrt{\frac{\mp M}{3(BN - 3CM)^2 + 2A(3N - AM)(BN - 3CM) + B(3N - AM)^2}}$$

THE quantities a and b are therefore found by a single extraction of the square root: and they have each two values, one positive, and the other negative. It is indifferent which of these two values of a and b be taken, provided they are correspondent values, so that $b = a \times y$. It is to be remarked too, that a and b have always real values, on account of the double sign prefixed to M ; for that sign is to be taken that will render the radical quantity positive. And it is to be carefully noted which of the two signs is necessary, that a and b may have real values: because on this depends the characteristic of the reduced equation, and whether it is to be referred to the first or second form, and; consequently, whether it has three roots, or only one. If the sign $-$ is requisite that a and b may have real values, then the reduced equation will have the characteristic of the first form, and will have three roots. But if the sign $+$ is requisite for that end, the reduced equation will have the characteristic of the second form, and will have only one root. All this is manifest from the statement in Art. 5.

7. THE rule, or law, according to which the preceding formulæ for a and b are constituted, is sufficiently simple and perspicuous; and the formulæ are therefore, in that respect, convenient for practice. But in examining the expression in the denominator

denominator of the radical, I find that it is always divisible by M , the quantity in the numerator: and that thus the formulæ may be exhibited in another shape, having this advantage, that it will introduce smaller numbers in the arithmetical operations requisite for computing a and b .

I WRITE $Q \times M = 3(BN - 3CM)^2 + 2A(BN - 3CM)(3N - AM) + B(3N - AM)^2$, and evolving by actual multiplication

$$\begin{aligned} Q \times M = & 3B^2 \cdot N^2 - 18BC \cdot MN + 27C^2 \cdot M^2 \\ & + 6AB \cdot N^2 - 2A^2B \cdot MN \\ & - 18AC \cdot MN + 6A^2C \cdot M^2 \\ & + 9B \cdot N^2 - 6AB \cdot MN + BA^2 \cdot M^2 \end{aligned}$$

Now, the coefficient of N^2 is, $3B^2 + 6AB + 9B = 3B(B + 2A + 3) = 3B \times M$: dividing therefore by M , we get,

$$\begin{aligned} Q = & 3B \times N^2 - (18BC + 2A^2B + 18AC + 6AB) \times N \\ & + (27C^2 + 6A^2C + BA^2) \times M. \end{aligned}$$

And if for N and M we substitute their values $(A + 2B + 3C)$ and $(3 + 2A + B)$, we shall find,

$$Q = 12B^3 + 81C^3 - 54ABC + 12A^3C - 3A^2B^2$$

all the other terms destroying one another, except these five.

ALL the terms in this value of Q , being divisible by 3, I change Q , and put

$$Q = 4B^3 + 27C^3 - 18ABC + 4A^3C - A^2B^2$$

$$\text{or, } Q = (4B^3 + 27C^3) - 2AC(9B - A^2) + A^2(2AC - B^2)$$

And since $3QM$ is now equal to the denominator of the radical in the preceding formulæ for a and b , we have the following new formulæ,

$$a = \pm \frac{BN - 3CM}{\sqrt{\mp 3Q}}$$

$$b = \pm \frac{3N - AM}{\sqrt{\mp 3Q}}$$

WHAT

WHAT was remarked above, with regard to the double sign prefixed to M, is now to be applied to the double sign prefixed to Q.

8. THE preceding investigation supplies us with the following rule, or criterion, by which to determine, whether any proposed cubic equation has three real roots or not:

THE proposed equation will have three real roots, when the amount of the expression

$$(4B^3 + 27C^2) - 2AC(9B - A^2) + A^2(2AC - B^2)$$

is negative: But if this expression is positive, the equation will have only one real root: And, (as will afterwards be shewn), when the amount of the expression is $= 0$, the equation will have two equal roots.

9. HAVING now found a and b , if we substitute $\frac{a+z}{b+z}$ for x in the proposed equation, we shall have an equation for z that will come under one of our two forms, and from which z (and consequently the root or roots of the proposed equation) may therefore be found. But such substitution is not necessary. For if we go back to Art. 5. and compare the transformed equation with the forms, we shall find $R = 1 + A + B + C$, and $3R = M + N$: also $-3r = (3a + 2Aa + Ba) + (Ab + 2Bb + 3Cb) = Ma + Nb$: Therefore

$$\frac{r}{R} = -\frac{Ma + Nb}{M + N}.$$

Whence the value or values of z are found by what is observed in Art. 4.

10. I SHALL now give the result of the whole analysis in the form of a general rule for the resolution of cubic equations, and add a few examples by way of illustration.

LET the proposed equation be

$$x^3 + Ax^2 + Bx + C = 0.$$

B

1. COM-

1. COMPUTE, $M = 3 + 2A + B$

$$N = A + 2B + 3C$$

$$m = BN - 3CM$$

$$n = 3N - AM,$$

$$\text{And } Q = (4B^3 + 27C^3) - 2AC(9B - A^2) + A^2(2AC - B^2)$$

AND let it be carefully noted under which of the two following cases the equation comes :

CASE I. When Q is negative.

CASE II. When Q is positive.

To these two cases a third may be added, viz. when $Q = 0$: but of this case I shall treat in one of the following examples.

2. COMPUTE also, $a = \frac{\pm m}{\sqrt{\mp 3Q}}$

$$b = \frac{\pm n}{\sqrt{\mp 3Q}} = \frac{an}{m}$$

$$\text{and } \tau = -\frac{Ma + Nb}{M + N}.$$

3. THEN, if the equation comes under case I : Find the angle ϕ , of which the tangent is τ , the radius being unity : take $z = \tan \frac{\phi}{3}$, $z = \tan \left(\frac{\phi}{3} + 120^\circ \right)$, and $z = \tan \left(\frac{\phi}{3} + 240^\circ \right)$: And the three roots of the equation will be found, by substituting these values of z in the formula $x = \frac{a \pm z}{b \mp z}$.

4. BUT if the equation comes under Case II. we must compute

$$z = \frac{1 - \sqrt[3]{\frac{1 - \tau}{1 + \tau}}}{1 + \sqrt[3]{\frac{1 - \tau}{1 + \tau}}}$$

And the only root of the equation will be found, by substituting this value of z in the formula $x = \frac{a \pm z}{b \mp z}$.

THE

THE following examples are chiefly taken from Dr HUTTON'S "*Treatise Mathematical and Philosophical.*" Tract 5.

II. EXAMPLE 1. Let the proposed equation be

$$x^3 - 6x^2 + 11x - 6 = 0.$$

Here, $A = -6$, $B = +11$, $C = -6$: therefore

$$M = +2$$

$$N = -2$$

$$m = +14$$

$$n = +6$$

$$Q = -4$$

So that we have here Case I.

$$a = \frac{14}{2\sqrt{3}} = \frac{7}{\sqrt{3}}$$

$$b = \frac{6}{14} \times a = \sqrt{3}$$

$$\text{and } r = -\frac{2 \cdot \frac{7}{\sqrt{3}} - 2 \cdot \sqrt{3}}{2 - 2} = \infty$$

Now, $\tan 90^\circ = \infty$, and also $\tan 270^\circ = \infty$: And we may take either of these angles for ϕ : Take $\phi = 90^\circ$, then,

$$z = \tan \frac{\phi}{3} = \tan 30^\circ = +\frac{1}{\sqrt{3}}$$

$$z = \tan \left(\frac{\phi}{3} + 120^\circ \right) = \tan 150^\circ = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$$

$$z = \tan \left(\frac{\phi}{3} + 240^\circ \right) = \tan 270^\circ = \infty$$

$$\text{THEREFORE, } x = \frac{\frac{7}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{\sqrt{3} + \frac{1}{\sqrt{3}}} = \frac{8}{4} = 2$$

$$x = \frac{\frac{7}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{\sqrt{3} - \frac{1}{\sqrt{3}}} = \frac{6}{2} = 3$$

$$x = \frac{\frac{7}{\sqrt{3}} + \infty}{\sqrt{3} + \infty} = 1.$$

And so the three roots are 1, 2, and 3.

SINCE $M + N = 3(1 + A + B + C)$: it is obvious, that when $M + N = 0$, we shall also have $1 + A + B + C = 0$; and so one root of the equation is unity. It is manifest too, that, in this case, the transformed equation for z becomes simply, $3z^2 - 1 = 0$: whence $z = \pm \frac{1}{\sqrt{3}}$; and consequently the other

two roots of the equation are, $x = \frac{a + \frac{1}{\sqrt{3}}}{b + \frac{1}{\sqrt{3}}}$, and $x = \frac{a - \frac{1}{\sqrt{3}}}{b - \frac{1}{\sqrt{3}}}$.

Or, writing for a and b , their values, $\frac{m}{\sqrt{-3}Q}$ and $\frac{n}{\sqrt{-3}Q}$, the two roots are, $x = \frac{m + \sqrt{-3}Q}{n + \sqrt{-3}Q}$ and $x = \frac{m - \sqrt{-3}Q}{n - \sqrt{-3}Q}$.

12. EXAMPLE 2. Let there be proposed

$$x^3 - 6x^2 + 9x - 2 = 0.$$

Here $A = -6$; $B = +9$; $C = -2$: And hence,

$$M = 0$$

$$N = +6$$

$$m = +54$$

$$n = +18$$

$$Q = -108.$$

And we have here again Case I.

$$a = +3$$

$$b = +1$$

$$r = -1$$

Therefore $\phi = \text{Arc. tan } -1 = 145^\circ$, therefore,

$$z = \tan \frac{\phi}{3} = \tan 45^\circ = 1,$$

$$z = t. \left(\frac{\phi}{3} + 120^\circ \right) = t. 165^\circ = -t. 15^\circ = -(2 - \sqrt{3}),$$

$$z = t. \left(\frac{\phi}{3} + 240^\circ \right) = t. 285^\circ = +t. 75^\circ = -(2 + \sqrt{3});$$

$$\text{THEREFORE, } x = \frac{3 + 1}{1 + 1} = \frac{4}{2} = 2;$$

$$x = \frac{3-2+\sqrt{3}}{1-2+\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{2} = 2 + \sqrt{3};$$

$$x = \frac{3-2-\sqrt{3}}{1-2-\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{2} = 2 - \sqrt{3};$$

And the three roots are 2, $2 + \sqrt{3}$, and $2 - \sqrt{3}$.

13. I TAKE the following example to illustrate the case of two equal roots.

EXAMPLE 3. Let the equation be

$$x^3 - 7x^2 - 5x + 75 = 0.$$

Here $A = -7$: $B = -5$: $C = +75$. Therefore,

$$M = -16$$

$$N = +208$$

$$m = +2560$$

$$n = +512$$

$$Q = 0.$$

SINCE, then, $Q = 0$, it is manifest that a , b , and τ will be all infinitely great. But though a and b be infinite, it is to be remarked that $\frac{a}{b} = \frac{m}{n}$.

SINCE $\tau = \infty$, we have $\phi = 90^\circ$, or $\phi = 270^\circ$: Take $\phi = 90^\circ$: Then,

$$z = \tan \frac{\phi}{3} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$z = \tan \left(\frac{\phi}{3} + 120^\circ \right) = \tan 150^\circ = -\tan 30^\circ = -\frac{1}{\sqrt{3}}$$

$$z = \tan \left(\frac{\phi}{3} + 240^\circ \right) = \tan 270^\circ = \infty$$

We have then, in the first place, for two roots,

$$x = \frac{a + \frac{1}{\sqrt{3}}}{b + \frac{1}{\sqrt{3}}}$$

$$x = \frac{a - \frac{1}{\sqrt{3}}}{b - \frac{1}{\sqrt{3}}}$$

And

And since a and b are infinitely great, it is manifest that these two roots are equal to one another, and each $= \frac{a}{b} = \frac{m}{n}$.

WE can infer nothing with regard to the value of the third root, derived from the infinite tangent, unless we can ascertain the proportion which that infinite tangent bears to the infinitely great quantities a and b . The general relation of $\tau = \tan \phi$, and $z = \tan \frac{\phi}{3}$ is thus expressed $\tau = \frac{3z - z^3}{1 - 3z^2}$: and τ becomes infinitely great; 1st, When $1 - 3z^2 = 0$; 2dly, When z is infinitely great.

Now the values of z , derived from the equation $1 - 3z^2 = 0$, are $z = +\frac{1}{\sqrt{3}}$, and $z = -\frac{1}{\sqrt{3}}$: And these are precisely the values of z used above in determining the two equal roots.

AGAIN, we have $\frac{\tau}{z} = \frac{3 - z^2}{1 - 3z^2}$: And it is manifest that the greater z is, the nearer $\frac{\tau}{z}$ approaches to $\frac{1}{3}$: so that, ultimately, when τ and z are greater than any finite magnitudes, we have $\frac{\tau}{z} = \frac{1}{3}$ and $z = 3\tau$. But $\tau = \frac{Ma - Nb}{M + N}$; therefore,

$$z = \frac{-3Ma - 3Nb}{M + N} \text{ and } x = \frac{a + z}{b + z} = \frac{a - \frac{3Ma + 3Nb}{M + N}}{b - \frac{3Ma + 3Nb}{M + N}}. \text{ If now we}$$

write $\frac{an}{m}$ for b , in this expression, the infinite quantity a may be thrown out by division, and we shall have the value of the root in finite quantities only. The expression being properly reduced, we shall have, $x = \frac{2Mm + N(3n - m)}{M(3m - n) + 2Nn}$.

WHEN, therefore, $Q = 0$, two roots are equal to one another, and each $= \frac{m}{n}$: And we have this formula by which to compute the third root, $x = \frac{2Mm + N(3n - m)}{2Nn + M(3m - n)}$.

APPLYING

APPLYING this rule to our example, we have:

for the two equal roots, $x = \frac{m}{n} = \frac{2560}{512} = 5$

for the third root, $x = \frac{-81920 - 212092}{-114688 + 212092} = \frac{-294012}{+98304} = -3$

14. EXAMPLE 4. Let the equation be

$$x^3 - 7x^2 + 18x - 18 = 0.$$

Here $A = -7$; $B = +18$; $C = -18$. Therefore,

$$M = +7$$

$$N = -25$$

$$m = -72$$

$$n = -26$$

$$Q = +72.$$

So that we have here Case II.

$$a = \frac{-72}{\sqrt{216}} = \frac{-36}{3\sqrt{6}}$$

$$b = \frac{-26}{\sqrt{216}} = \frac{-13}{3\sqrt{6}}$$

$$\tau = +\frac{73}{54\sqrt{6}}$$

$$\text{Therefore, } \sqrt[3]{\frac{1-\tau}{1+\tau}} = \sqrt[3]{\frac{54\sqrt{6}-73}{54\sqrt{6}+73}} = \frac{2\sqrt{6}-1}{2\sqrt{6}+1}$$

$$\text{HENCE } z = \frac{1 - \sqrt[3]{\frac{1-\tau}{1+\tau}}}{1 + \sqrt[3]{\frac{1-\tau}{1+\tau}}} = \frac{1}{2\sqrt{6}}$$

$$\text{and } x = \frac{-\frac{36}{3\sqrt{6}} + \frac{1}{2\sqrt{6}}}{-\frac{13}{2\sqrt{6}} + \frac{1}{2\sqrt{6}}} = \frac{-72+3}{-26+3} = \frac{-69}{-23} = +3.$$

And 3 is the only root of the equation.

15. WHEN τ is a furd as $\frac{p}{q\sqrt{r}}$, the value of z (in Case II.) always involves radicals of this form, $\sqrt[3]{\frac{q\sqrt{r}-p}{q\sqrt{r}+p}}$; out of which the root may sometimes be extracted; and so the value of z will be expressed by a furd of the same kind as τ .

THE

THE method I have followed to find when this can be done, being very simple and easy in practice, I shall here briefly describe it.

WE have $\tau = \frac{3z + z^3}{1 + 3z}$;

write $\frac{p}{q\sqrt{r}}$ for τ , $\frac{\mu}{v\sqrt{r}}$ for z : then

$$\frac{p}{q\sqrt{r}} = \frac{3r\mu v + \mu^3}{(rv^3 + 3\mu^2v)\sqrt{r}},$$

whence these two equations are formed.

$$p = 3r\mu v + \mu^3$$

$$q = rv^3 + 3\mu^2v,$$

from which it is manifest, that μ is a divisor of p , and v a divisor of q . I seek then amongst the divisors of p for a number μ , and amongst the divisors of q for a number v , that will satisfy the two equations above: or rather, that will satisfy these two following,

$$v^2 = \frac{p - \mu^3}{3r\mu},$$

$$\mu^2 = \frac{q - r^3}{3v}.$$

If two such numbers are to be found amongst the divisors of p and q , then will $z = \frac{\mu}{v\sqrt{r}}$: but if not, we are to conclude that the value of z cannot be expressed this way.

THUS, in the last example $\tau = \frac{73}{54\sqrt{6}}$, it is manifest that 73 admits no divisor but 1: therefore $\mu = 1$, and on trial I find $v = 2$, which two numbers satisfy the two equations, and therefore $z = \frac{1}{2\sqrt{6}}$.

16. THE same method applies also to Case I. For, in this case,

$$\tau = \frac{3z - z^3}{1 - 3z^2}$$

and

and substituting $\frac{p}{q\sqrt{r}}$ for τ , and $\frac{\mu}{v\sqrt{r}}$ for z , we derive these two equations,

$$p = 3r\mu v^2 - \mu^3$$

$$q = rv^3 - 3\mu^2 v,$$

whence it is manifest that μ is a divisor of p , and v a divisor of q , as before. It will be easier for trial to write the equations thus :

$$v^2 = \frac{p + \mu^3}{3r\mu}$$

$$\mu^2 = \frac{rv^3 - q}{3v}$$

AND let it be observed, that we may here give to μ and v any signs consistent with the condition, that $\frac{p + \mu^3}{3r\mu}$ and $\frac{rv^3 - q}{3v}$ (the values of v^2 and μ^2) are positive numbers.

IT is to be remarked too, that, in this case, z has three values. If, however, we can find one value this way, the two others are readily obtained. For if v be one value of z , the two other values are $\frac{v + \sqrt{3}}{1 + v\sqrt{3}}$ and $\frac{v - \sqrt{3}}{1 + v\sqrt{3}}$: because these values are the tangents of two arches that differ from the arch of which v is the tangent by 120° and 240° .

THOUGH this is a matter more curious than useful, I shall add one more example for the sake of illustration.

LET the equation be

$$x^3 - 39x^2 + 479x - 1881 = 0.$$

Here $A = -39$; $B = +479$; $C = -1881$: And

$$M = +404$$

$$N = -4724$$

$$m = +16976$$

$$n = +1584$$

$$Q = -25600$$

And the equation belongs to Case I.

C

$$a = \frac{16976}{160\sqrt{3}} = \frac{1061}{10\sqrt{3}}$$

$$b = \frac{1584}{160\sqrt{3}} = \frac{99}{10\sqrt{3}}$$

$$c = \frac{-39032}{-43200\sqrt{3}} = -\frac{4879}{5400\sqrt{3}}$$

To find whether z can be expressed by a surd as $\frac{\mu}{v\sqrt{r}}$: I write $p = -4879$, $q = 5400$, and $r = 3$; and I have these two equations,

$$v^2 = \frac{-4879 + \mu^2}{9\mu}$$

$$\mu^2 = \frac{3v^3 - 5400}{3v}$$

I find, that 7, 17, and 41, are divisors of 4879. It is evident that + 7 would give v^2 negative: I therefore try - 7, and it does not succeed: Neither does + 17; but trying - 17, I find $v^2 = 64$, and consequently $v = \pm 8$: And - 8 is found to succeed in the other equation: therefore $\mu = -17$, $v = -8$, and $z = \frac{17}{8\sqrt{3}}$.

We have then, $z = \frac{17}{8\sqrt{3}} = v$

$$z = \frac{v - \sqrt{3}}{1 + v\sqrt{3}} = \frac{-7}{25\sqrt{3}}$$

$$z = \frac{v + \sqrt{3}}{1 - v\sqrt{3}} = \frac{41}{-9\sqrt{3}}$$

$$\text{THEREFORE, } x = \frac{\frac{1061}{10\sqrt{3}} + \frac{17}{8\sqrt{3}}}{\frac{99}{10\sqrt{3}} + \frac{17}{3\sqrt{3}}} = \frac{8488 + 170}{792 + 170} = \frac{8658}{962} = 9$$

$$x = \frac{\frac{1061}{10\sqrt{3}} - \frac{7}{25\sqrt{3}}}{\frac{99}{10\sqrt{3}} - \frac{7}{25\sqrt{3}}} = \frac{5305 - 14}{495 - 14} = \frac{5291}{481} = 11$$

$$x = \frac{\frac{1061}{10\sqrt{3}} - \frac{41}{9\sqrt{3}}}{\frac{99}{10\sqrt{3}} - \frac{41}{9\sqrt{3}}} = \frac{9549 - 410}{891 - 410} = \frac{9139}{481} = 19.$$

And so the three roots are, 9, 11, and 19.

FINIS.