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Sir John Chetwode, Bar^t.
Oakley Hall, Staffordshire.

SAUNDERSON, N

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SELECT PARTS

OF

SAUNDERSON'S ELEMENTS OF ALGEBRA

FOR THE USE OF

STUDENTS AT THE UNIVERSITIES.

THE FOURTH EDITION,

REVISED AND CORRECTED.

Jon: Shetwode 1782

LONDON:
Printed for W. BOWYER and J. NICHOLS,
J. F. and C. RIVINGTON, L. DAVIS,
[illegible], [illegible],
and J. WOODYER.
MDCCLXXVI.

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ADVERTISEMENT.

THE Excellence of Profeſſor SAUNDERSON's *Elements of Algebra* is univerſally acknowledged: But as that WORK contains many CURIOUS and ELEGANT PIECES, which are rather of Advantage and Amuſement to Proficients in the general Science of the Mathematics, than of neceſſary Uſe to Students in Algebra; ſome of the principal Tutors in the Univerſity of *Cambridge* were deſirous of having ſuch Parts ſelected from the Whole, as would give their Pupils a clear and comprehenſive Knowledge of Algebra, without putting them promiſcuouſly to the Expence of purchaſing the original Work, which was publiſhed in Two Volumes, Quarto. The Public is indebted to a Gentleman in

POSTULATA.

BEFORE I enter upon my province, it may not be amiſs to acquaint my young diſciple what preparations he is to make, and what qualifications I expect of him beforehand, that we may neither of us find ourſelves diſappointed afterwards. I expect then that he knows how to add, to ſubtract, to multiply, to divide, to find a fourth proportional, and to extract roots, eſpecially the ſquare root: nay I expect further, that he ſhall not only be able to perform all theſe operations exactly and readily, but alſo that he ſhall be able to apply them upon all common occaſions; in a word, I expect that he be tolerably well ſkilled in common Arithmetick, at leaſt ſo far as relates to whole numbers: for this reaſon it is that I have prefixed a few arithmetical queſtions, wherein he may firſt try his ſtrength and ſkill before he ventures any further; they are for the moſt part very eaſy. I cannot ſay indeed they are the beſt choſen, but they were ſuch as lay in my way when I firſt began this work and was haſtening to matters of greater moment; and I do not ſee but they may, if ſtudied with care and attention, anſwer well enough the end they were intended for. If he finds no difficulty in theſe, he will have little reaſon to doubt of his ſucceſs afterwards; but if he does, he ought then at laſt to become ſenſible of his own defects, and to endeavour to ſupply whatever is wanting, and to correct whatever is amiſs, before he enters himſelf under my conduct; in the mean time he has my leave to hope that I ſhall be leſs upon the reſerve with him when he falls more immediately under my care.

N. B. The *praxis* of the rule of proportion, and of the rule for extracting the ſquare root, not being (properly ſpeaking) of the nature of ſimple *poſtulata*, but rather deducible from the four firſt; I ſhall not fail to demonſtrate theſe rules ſo ſoon as I ſhall find proper opportunities for that purpoſe.

Queſtions for exercife in Multiplication.

Multiplication is taking any one number called the multiplicand as often as is expreſſed by any other number called the multiplicator, and the number produced by this operation is called the product: whence it follows, that the product contains the multiplicand as often as there are units in the multiplicator, and that if a number of a greater denomination is to be reduced to an equivalent number of a leſs, it muſt be done by multiplication. As for example; In a pound ſterling there are 20 ſhillings; therefore in every ſum of money conſiſting of even pounds, there are twenty times as many ſhillings as there are pounds; therefore, if any number of pounds be multiplied by 20, the product will be an equivalent number of ſhillings; and the ſame muſt be obſerved in all other caſes.

QUEST. I.

It is required to reduce 456 pounds 13 ſhillings and 4 pence, into ſhillings, pence, and farthings.

Anſwer. Shillings 9133
Pence 109600
Farthings 438400.

QUEST. 2.

A certain island contains 36 counties, every county 37 parishes, every parish 38 families, and every family 39 persons: I demand the number of parishes, families, and persons in the whole island.

Answer.	Parishes	1332
	Families	50616
	Persons	1974024

QUEST. 3.

In 1730 years, 42 weeks, and 3 days, how many minutes?

N. B. A year consists of 365 days, 6 hours, and an hour of 60 minutes.

Hours in one year	8766
In 1730 years	15165180
In 42 weeks 3 days	7128
In the whole	15172308
Minutes in the whole	910338480.

QUEST. 4.

There is a certain field 102004 feet long, and 102003 feet broad: I demand the number of square feet therein contained?

Answer. 10404714012.

QUEST. 5.

There is a certain floor 24 feet 4 inches broad, and 96 feet 6 inches long: I demand how many square inches are therein contained?

Answer. 338136 square inches.

QUEST. 6.

A certain piece of wood 1 foot 2 inches thick, 3 feet 4 inches broad, and 5 feet 6 inches long, is to be cut into ſmall cubes like dies, each of which is to be a quarter of an inch every way: I demand into how many dies the whole may be reſolved.

Anſwer. The whole may be reſolved into 2365440 dies.

QUEST. 7.

I demand the number of changes that may be rung on 12 bells.

Changes upon	2	bells	2
on	3	bells	6
on	4	bells	24
on	5	bells	120
on	6	bells	720
on	7	bells	5040
on	8	bells	40320
on	9	bells	362880
on	10	bells	3628800
on	11	bells	39916800
on	12	bells	479001600

QUEST. 8.

How many different ways can four common dies come up at one throw?

Anſwer. 1296 ways.

QUEST. 9.

Suppoſe one undertake to throw an ace at one throw with four common dies; what probability is there of his effecting it?

Anſwer. By the laſt queſtion four dies can come up 1296 different ways with and without the ace; and

and by a like computation, they can come up 625 ways without the ace; therefore there are 671 ways wherein one or more of them may turn up an ace; therefore the undertaker has the better of the lay in the proportion of 671 to 625.

QUEST. 10.

There are two inclosures of the same circumference, that is, both inclosed with the same number of pales; but one is a square whose side is 125 feet, and the other an oblong or long square: 124 feet in breadth, and 126 in length: quære *which is the greater close, that is, which,* cæteris paribus, *will bear most grass?*

Answer. The square: for that contains 15625 square feet; whereas the other contains but 15624.

Questions for exercise in Division.

The design of division is, to shew how often one number called the divisor is contained in another called the dividend, and the number that shews this is called the quotient; whence, and from the definition of multiplication already given, I observe, 1*st*, That the divisor multiplied by the quotient, and consequently the quotient multiplied by the divisor, will always be equal to the dividend, provided there be no remainder after the division is over; but if there be, then this remainder added to, or taken into, the product, will give the dividend, which is the best proof of division. 2*dly*, That as the divisor is such a part of the dividend as is expressed by the quotient; so also is the quotient such a part as is expressed by the divisor. Thus 12 divided by 3 quotes 4; therefore 3 is a fourth part, and 4 a third part, of 12. 3*dly*, Hence may a number be found that shall be divisible by any two given numbers whatever without remainders, to wit, by multiplying the two given num-

QUEST. 12.

One lends me 1296 guineas when they were valued at 1l. 1s. and ſixpence a piece: how many muſt I pay him when they are valued at 1l. 1s. apiece?

Anſwer. 1326 guineas 18 ſhillings.

QUEST. 13.

A certain floor 24 feet 4 inches broad, 96 feet 6 inches long, is to be laid at the rate of 12 pence the ſquare foot: I demand what the whole charge will amount to.

Anſwer. The floor contains 338136 ſquare inches, or 2348 ſquare feet and 24 ſquare inches; therefore the whole charge amounts to 117 pounds 8 ſhillings and two pence.

QUEST. 14.

There is a certain cooler 36 inches deep, 42 inches wide, and 72 inches long: I demand its ſolid content in Engliſh gallons.

Note. An Ale gallon is 282 cubic inches.

Anſwer. The veſſel contains 108864 cubic inches, that is, 386 gallons, and 12 cubic inches over.

QUEST. 15.

A cubic foot of water weighs 76 pounds Troy or Roman weight; and air is 860 times lighter than water: I demand the weight of a cubic foot of air.

N. B. A pound Troy contains 12 ounces, one ounce 20 pennyweights, and one pennyweight 24 grains.

Anſwer. A cubic foot of air weighs Troy weight 1 *oz.* 1 *pwt.* 5 *gr.*

QUEST. 16.

The mean time of a lunation, that is, from new moon to new moon, is 29 days 12 hours 44 minutes and 3 ſeconds; and a Julian year conſiſts of 365 days 6 hours: I demand then how many lunations are contained in 19 Julian years.

Hours in a Lunation		708
	Minutes	42524
	Seconds	2551443
Hours in 19 Julian years		166554
	Minutes	9993240
	Seconds	599594400

Lunations 235; and 1 hour 28′ 15″ over.

QUEST. 17.

In what time may all the changes on 12 bells be rung, allowing 3 ſeconds to every round? See Queſtion the 7*th*.

The number of changes on 12 bells 479001600

The time	1437004800	ſeconds,
or	23950080	minutes,
or	399168	hours,

or 45 years 27 weeks 6 days 18 hours.

QUEST. 18.

A General of an army diſtributes 15 pounds 19 ſhillings and 2 pnce halfpenny among 4 captains, 5 lieutenants, and 60 common ſoldiers, in the manner following: Every captain is to have 3 times as much as a lieutenant, and every lieutenant twice as much as a common ſoldier: I demand their ſeveral ſhares.

The

The ſhare of a common ſoldier	3s. 4d. $\frac{3}{4}$
of a lieutenant	6s. 9d. $\frac{1}{2}$
of a captain	1l. 0s. 4d. $\frac{1}{2}$

Queſtions for exercise in the Rule of Three.

And firſt in the Rule of Three Direct.

The rule of proportion, or rule of three, or by ſome the golden rule, is that which teacheth, having three numbers given, to find a fourth proportional, that is, to find a fourth number that ſhall have the ſame proportion to ſome one of the numbers given, as is expreſſed by the other two; and therefore, whenever a queſtion is propoſed wherein ſuch a fourth proportional is required, that queſtion is ſaid to belong to the rule of proportion. Now in queſtions of this nature, eſpecially where the numbers given are not merely abſtract numbers, but are applied to particular quantities, three things are uſually required, to wit, preparation, diſpoſition, and operation.

Firſt as to the preparation, it muſt be obſerved, that of the three numbers given in the queſtion, two will always be of the ſame kind, and muſt be reduced to the ſame denomination, if they be not ſo already; and if the remaining number be of a mixt denomination, that alſo muſt be reduced to ſome ſimple one.

Secondly, in diſpoſing the numbers thus prepared, thoſe two that are of the ſame denomination muſt be made the firſt and third nnmbers in the rule of proportion, and conſequently the remaining number muſt be the ſecond. But here particular care muſt be taken, that of the two numbers that are of the ſame denomination, that be made the third in the rule of proportion, upon which the main ſtreſs of the queſtion lies, or to which the queſtion more immediately relates, or which contains the demand: and the

the place of this number being once known, the other two muſt take their places as above directed. This ordering of the numbers for the operation is commonly called, ſtating of the queſtion.

Laſtly, having thus ſtated the queſtion, multiply the ſecond and third numbers together; divide the product by the firſt, and the quotient thence ariſing will be the fourth number ſought; which fourth number, as well as the remainder, if there be any, muſt always be underſtood to be of the ſame denomination with the ſecond. As for example.

QUEST. 19.

A piece of plate weighing 3 pounds 4 ounces and 5 pennyweights, Troy weight, is valued at 5 ſhillings and 6 pence an ounce; what is the value of the whole?

Here we have three quantities concerned in the queſtion, *viz.* 3 pounds 4 ounces and 5 pennyweights; one ounce; and 5 ſhillings and 6 pence; whereof the two firſt, which are of the ſame kind, muſt be reduced to the ſame denomination, and the laſt to a ſimple one; thus: for one ounce I write 20 pennyweights; for 3 pounds, 4 ounces and 5 pennyweights, 805 pennyweights; and for 5 ſhillings and 6 pence, 66 pence; and ſo the numbers are ſufficiently prepared. In the next place I enquire which of the two numbers 20 and 805, which are of the ſame denomination, is that upon which the main ſtreſs of the queſtion lies, and I find it to be 805; for the main buſineſs of this queſtion is, to enquire into the value of 805 pennyweights of plate; the reſt being no more than *data* in order to diſcover this: So I make 805 my third number, 20 which is a number of the ſame denomination my firſt, and 66 my ſecond, and ſtate the queſtion thus; *If 20 pennyweights of plate be worth 66 pence, what will 805 pennyweights of plate be* *be worth?* Now to anſwer this queſtion, I multiply 805 by 66, and the product is 53130; this I divide by my firſt number 20, and the quotient is 2656, and there remains 10, that is, 10 pence; therefore, to render my quotient more compleat, I bring the remaining 10 pence into 40 farthings, and ſo divide again by 20, and find the quotient to be 2, that is, 2 farthings, without any remainder; ſo the value ſought is 2656 pence 2 farthings; that is, 11 pounds 1 ſhilling and 4 pence halfpenny.

A demonſtration of this Praxis.

Caſe 1ſt. Now to demonſtrate this manner of operation, I ſhall reſume the foregoing queſtion, but at firſt under a different ſuppoſition, as thus; *If one pennyweight of plate coſt* 66 *pence, what will* 805 *pennyweights coſt?* Here nobody doubts but that upon this ſuppoſition, 805 pennyweights will coſt 805 times 66 pence, or 66 times 805, that is, 53130 pence; therefore in all inſtances of this kind, that is, where the firſt number in the rule of proportion is unity, the fourth number muſt be found by multiplying the ſecond and third numbers together.

Caſe 2d. Let us now put the queſtion as it was at firſt ſtated, to wit, *If* 20 *pennyweights of plate be worth* 66 *pence, what will* 805 *pennyweights be worth?* Now upon this ſuppoſition it is eaſy to ſee, that neither 1 pennyweight, nor conſequently 805 pennyweights, will be worth above a 20th part of what they were in the former caſe; and therefore we muſt not now ſay that 805 pennyweights are worth 53130 pence, but a 20th part of that ſum, *viz.* 2656 pence 2 farthings: and as this way of reaſoning will be the ſame in all other inſtances, it follows now, that *In the rule of proportion, let the numbers given be what they will, the fourth number muſt be had by multiplying the ſecond and third numbers together and dividing the product by the firſt.* Q. E. D.

QUEST.

QUEST. 20.

How far will one be able to travel in 7 days 8 hours, at the rate of 13 miles every 4 hours, allowing 12 hours to a travelling day?

Anſwer. 299 miles.

QUEST. 21.

What will 1296 yards of walling amount to, at the rate of 4 ſhillings and 5 pence a rod, a rod being 5 yards and a half?

Anſwer. 52 pounds 8 pence 3 farthings.

QUEST. 22.

In the mint of England *a pound of gold, that is, 11 ounces fine and 1 alloy, is at this time coined into 44 guineas and an half: I demand how much ſterling a pound of pure gold is worth, obſerving that the alloy is valued at nothing.*

Anſwer. 50 pounds 19 ſhillings and 5 pence ½ penny.

QUEST. 23.

What is the annual intereſt of 987 pounds 6 ſhillings and 5 pence, at the rate of 6 per cent.?

Anſwer. 59 pounds 4 ſhillings and 9 pence ½ penny.

QUEST. 24.

The circumference of the Earth according to the French menſuration is 123249600 French feet: I demand the ſame in Engliſh miles.

N. B.

N. B. A thouſand French feet are equivalent to 1068 Engliſh feet; 3 feet make a yard, and 1760 yards make a mile.

Anſwer. 131630573 Engliſh feet,
or 43876857 yards and 2 feet,
or 24930 miles, 57 yards and 2 feet.

QUEST. 25.

Suppoſing all things as in the foregoing queſtion, I demand how long a ſound will be in paſſing from pole to pole. upon a ſuppoſition that a ſound paſſes over 1142 *feet in a ſecond of time.*

Anſwer. 16 hours and 32 ſeconds.

QUEST. 26.

Monſieur Huygens *found that at* Paris, *the length of a pendulum that ſwung ſeconds, was three feet 8 lines and* ½: *I demand its length in Engliſh meaſure.*

Note. A line is $\frac{1}{12}$ part of an inch, and 1000 French half lines are equivalent to 1068 Engliſh half lines, as in the 24th queſtion.

Anſwer. The length in Engliſh meaſure of a pendulum that ſwings ſeconds, is 941 Engliſh half lines; or 39 inches 2 lines and ½.

QUEST. 27.

I demand in how long a time a pipe, that diſcharges 15 *pints in* 2 *minutes* 34 *ſeconds, will fill a ciſtern that is* 36 *inches deep,* 42 *inches wide, and* 72 *inches long.* (ſee queſtion the 14.)

Anſwer. In 31707 ſeconds; or 8 hours 48′ 27″.

For

For as eight pints make a gallon, ſo alſo eight cubic half inches, that is, eight ſmall cubes of half an inch every way make one cubic inch; therefore a pint contains 282 cubic half inches, and fifteen pints 4230; but the whole veſſel contains 108864 cubic inches by queſt. 14; which are equivalent to 870912 cubic half inches; therefore this queſtion ought to be ſtated thus;

If 4230 cubic half inches be diſcharged in 154 ſeconds of time, in what time will 870912 cubic half inches be diſcharged? And the anſwer is,

In 8 hours 48′ 27″ as above.

QUEST. 28.

If a wall 6 feet thick, 9 feet high, and 432 feet long, coſt 720 pounds in building, what will be the price of a wall of the ſame materials, that is 12 feet thick, 18 feet high, and 576 feet long?

In the former wall are contained 23328 cubic feet; in the latter 124416; therefore the anſwer to this queſtion is 3840 pounds.

QUEST. 29.

A certain ſteeple projected upon level ground a ſhadow to the diſtance of 57 yards, when a four-foot ſtaff perpendicularly erected caſt a ſhadow of 5 feet 6 inches: what was the height of the ſteeple?

Anſwer. 41 yards 1 foot 4 inches.

QUEST. 30.

Two Perſons A and B make a joint ſtock; A puts in 372 pounds, and B 496 pounds, for the ſame time; and they gain 114 pounds 2 ſhillings: I demand each man's ſhare of the gain.

Both their ſtocks make 868 pounds; ſay then, if 868 pounds ſtock bring in 114 pounds 2 ſhillings gain, what will 372 pounds, *A's* part of the ſtock, bring in? *Anſwer.* 48 pounds 18 ſhillings for *A's* ſhare of the gain; and this ſubtracted from the whole gain, leaves 65 pounds 4 ſhillings for *B's* ſhare of the gain.

Note. If there be ever ſo many partners, their ſhares of the gain muſt all be found by the rule of proportion, except the laſt, which may be had by ſubtraction; but it would be better to find them all by the rule of proportion, becauſe then, if all the ſhares, when added together, make up the whole gain, it will be an argument that the work is rightly performed.

QUEST. 31.

Two perſons A and B make a joint ſtock; A puts in 496 pounds for 2 months, and B 620 pounds for 3 months; and they gain 456 pounds: What will be each man's ſhare of the gain?

In order to give an anſwer to this queſtion, it muſt be conſidered, that it is the ſame in the caſe of trade, as it is in that of money let out to intereſt, where time is as good as money, that is, whoever lets out 496 pounds for 2 months, is intitled to the ſame ſhare of the whole gain, as if he had let out twice as much, that is 992 pounds, for one month: in like manner, he that lets out 620 pounds for 3 months, has a right to the ſame ſhare of the gain,

as

as if he had let out three times as much, that is, 1860 pounds, for one month: ſubſtitute therefore theſe ſuppoſitions inſtead of thoſe in the queſtion, which may ſafely be done without affecting the concluſion, and then this queſtion will be reduced to the form of the laſt, without any conſideration of the particular quantity of time, thus; *Two merchants* A *and* B *make a joint ſtock;* A *puts in* 992 *pounds, and* B 1860 *pounds for the ſame time; and they gain* 456 *pounds: What will be their reſpective ſhares of the gain?*

Anſwer. A's ſhare will be 158 pounds 12 ſhillings and 2 pence; and *B's*, 297 pounds 7 ſhillings and 10 pence.

QUEST. 32.

If two men in three days will earn 4 *ſhillings, how much will* 5 *men earn in ſix days?*

This and the following queſtion belong to that which they call the double rule of three, wherein 5 numbers are concerned: theſe numbers muſt always be placed as they are in this example, that is, the two laſt numbers muſt always be of the ſame denomination with the two firſt reſpectively, and the number ſought of the ſame denomination with the middle one; then may the queſtion be reduced to the ſingle rule of three two ways, either by expunging the firſt and fourth numbers, or the ſecond and fifth. If you would have the firſt and fourth numbers expunged, you muſt argue thus; two men will earn as much in three days, as one man in two times 3, or 6 days; alſo 5 men will earn as much in 6 days, as one man in thirty days; ſubſtitute therefore this ſuppoſition and this demand, inſtead of thoſe in the queſtion; and it will ſtand thus. If one man in 6 days will earn four ſhillings, how much

much will one man earn in 30 days? Which is as much as to ſay, *If in 6 days a man will earn 4 ſhillings, how much will he earn in 30 days?*

Anſwer. 20 ſhillings.

If you would have the ſecond and fifth numbers expunged, you muſt argue thus: Two men will earn as much in three days, as 3 times two or 6 men in one day; alſo 5 men will earn as much in 6 days, as 30 men in one day; put then the queſtion this way, and it will ſtand thus; If 6 men in one day will earn 4 ſhillings, how much will 30 men earn in one day? That is, *If in any quantity of time 6 men will earn 4 ſhillings, how much will 30 men earn in the ſame time?*

Anſwer. 20 ſhillings, as before.

Whoſoever attends to both theſe methods of extermination, will eaſily fall into a third, which includes both the other, and in practice is much better than either of them; for at the concluſion of both operations, the number ſought was found by multiplying 30 by 4, and dividing the product by 6: Now if he looks back, and traces out theſe numbers, he will find that the number 30 came from the multiplication of the two laſt numbers 5 and 6 together, that 4 was the middle number in the queſtion, and that the diviſor 6 was the product of the two firſt numbers 2 and 3 multiplied together; therefore, *In all queſtions of this nature, if the three laſt numbers be multiplied together, and the product be divided by the product of the two firſt, the quotient will give the number ſought, without any further trouble.*

QUEST. 33.

If for the carriage of three hundred weight 40 miles, I must pay 7 ſhillings and 6 pence, what muſt I pay for the carriage of 5 hundred weight 60 miles?

Anſwer. 225 pence, or 18 ſhillings and 9 pence.

Queſtions in the rule of three Inverſe.

Hitherto we have inſtanced in the rule of three direct; but there is alſo another rule of proportion, called the rule of three inverſe; which, as to the preparation and diſpoſition of its numbers, differs nothing from the rule of three direct, but only in the operation; for whereas there, the fourth number was found, by multiplying the ſecond and third numbers together, and dividing by the firſt; here it is found by multiplying the firſt and ſecond numbers together, and dividing by the third. All that remains then, is to be able to diſtinguiſh, when a queſtion belongs to one rule, and when to the other; in order to which, obſerve the following directions: *If more requires more, or leſs requires leſs, work by the rule of three direct; but if more requires leſs, or leſs requires more, work by the rule of three inverſe.* The meaning whereof is, that if, when the third number is greater than the firſt, the fourth muſt be proportionably greater than the ſecond; or if, when the third number is leſs than the firſt, the fourth muſt be proportionably leſs than the ſecond, the queſtion then belongs to the rule of three direct: But if, when the third number is greater than the firſt, the fourth muſt be leſs than the ſecond; or when the third number is leſs than the firſt, the fourth muſt be greater than the ſecond; in either of theſe caſes, the queſtion belongs to the rule of three inverſe, and muſt be reſolved as above directed.

As

As for example,

QUEST. 34.

If 12 men will eat up a quantity of provifion in 15 days, how long will 20 men be in eating the fame?

This queftion is of fuch a nature, that more requires lefs; for 20 men will confume the fame provifion in lefs time than 12; therefore the queftion belongs to the rule of three inverfe; fo I multiply the firft and fecond numbers together, and divide by the third, and the quotient 9, that is, 9 days, is an anfwer to the queftion.

A demonftration of the rule of three inverfe.

If I was to anfwer this queftion by pure dint of thought, without any rule to direct me, I fhould reafon thus: whatever quantity of provifion lafts 12 men 15 days, the fame will laft 1 man 12 times as long, that is, 12 times 15, or 180 days; but if it will laft 1 man 180 days, it will laft 20 men but the 20th part of that time, that is, 9 days: here then the fourth number was found by multiplying the firft and fecond numbers together; and dividing the product by the third; and the reafon is the fame in all other cafes, where-ever the rule of three inverfe is concerned. *Q. E. D.*

QUEST. 35.

One lends me 372 pounds for 7 years and 8 months, or 92 months: how long muft I lend him 496 pounds for an equivalent?

Anfwer. 5 years, 9 months.

QUEST. 36.

If a ſquare pipe, 4 inches and 5 lines wide, will diſcharge a certain quantity of water in one hour's time; in what time will another ſquare pipe, 1 inch and 2 lines wide, diſcharge the ſame quantity from the ſame current?

The orifice of a ſquare pipe 4 inches 5 lines, or 53 lines wide, contains 2809 ſquare lines; and the orifice of a pipe 1 inch 2 lines, or 14 lines wide, contains 196 ſquare lines. Say then, *If an orifice of 2809 ſquare lines will diſcharge a certain quantity of water in one hour; in what time will an orifice of 196 ſquare lines diſcharge the ſame?*

Anſwer. In 14 hours 19′ 54″.

QUEST. 37.

If 3 men, or 4 women, will do a piece of work in 56 days, how long will one man and one woman be in doing the ſame?

Becauſe of the 3 men, or 4 women, ſome number muſt be found that is diviſible both by 3 and by 4 without remainder; ſuch an one is the number 12, which is the product of 3 and 4 multiplied together; (ſee obſervation the third upon the definition of diviſion :) make then 3 men or 4 women equivalent to 12 boys, and you will have 1 man equivalent to 4 boys, 1 woman to 3 boys, and 1 man and 1 woman to 7 boys, and the queſtion will ſtand thus; *If 12 boys will do a piece of work in 56 days, how long will 7 boys be in doing the ſame?*

Anſwer. 96 days.

QUEST. 38.

If 5 oxen, or 7 colts, will eat up a close in 87 days, in what time will 2 oxen and 3 colts eat up the same?

Answer. In 105 days.

QUEST. 39.

If 2 acres of land will maintain 3 horses 4 days, how long will 5 acres maintain 6 horses?

This question may perhaps, at first sight, be taken to be somewhat of the same nature with the 32d and 33d questions, which belonged to the double rule of three direct; but when it comes to be examined into more narrowly, it will be found to be of a very different nature; for we cannot say here as we did there, that 2 acres will last 3 horses as long as 1 acre will last 6 horses; this would be a very unjust way of thinking, and where-ever it is so, the question ought to be referred to another rule, which they call the double rule of three inverse; the propriety or impropriety of this thought being an infallible criterion whereby to distinguish when a question belongs to one rule, and when to the other. All questions belonging to this rule, as well as those belonging to the other, may be reduced to the single rule of three two ways; either by expunging the first and fourth numbers, or the second and fifth; but then the methods of extermination are different. In questions of this nature, if the first and fourth numbers are to be expunged, the 2 first numbers are to be multiplied by the fourth, and the 2 last by the first; but if the second and fifth numbers are to be expunged, then the two first numbers are to be multiplied by the fifth, and the two last by the second: thus in the question before us, if we would exterminate the firſt and fourth numbers, we muſt multiply the two firſt numbers, that is, 2 and 3, by the fourth, that is, by 5, and ſay, that 2 acres will laſt three horſes juſt as long as 10 acres will laſt 15 horſes; we muſt alſo multiply the 2 laſt numbers, to wit, 5 and 6, by the firſt, that is, by 2, and ſay, that 5 acres will laſt 6 horſes as long as 10 acres will laſt 12 horſes. Uſe now theſe numbers inſtead of thoſe in the queſtion, and it will be changed into this equivalent one; If 10 acres of land will maintain 15 horſes 4 days, how long will 10 acres maintain 12 horſes? Strike out of the queſtion the firſt and fourth numbers, which, being equal, will be of no uſe in the concluſion, and then the queſtion will ſtand thus; *If 15 horſes will eat up a certain piece of ground in 4 days, how long will 12 horſes be in eating up the ſame?*

Anſwer. 5 days; for this queſtion belongs to the rule of three inverſe.

If we would exterminate the ſecond and fifth numbers out of the queſtion, we muſt multiply the two firſt numbers by the fifth, and ſay, that 2 acres will laſt 3 horſes juſt as long as 12 acres will laſt 18 horſes; we muſt alſo multiply the two laſt numbers by the ſecond, and ſay, that 5 acres will laſt 6 horſes as long as 15 acres will laſt 18 horſes: uſe theſe numbers inſtead of thoſe in the queſtion, and it will be changed into this equivalent one: If 12 acres will maintain 18 horſes 4 days, how long will 15 acres maintain 18 horſes? That is, (ſtriking out the ſecond and fifth numbers) *If 12 acres of land will maintain a certain number of horſes 4 days, how long will 15 acres laſt the ſame number?*

Anſwer. 5 days as before; for this queſtion belongs to the rule of three direct.

In

In both these operations, the number sought was at last found by multiplying 15 by 4, and then dividing the product by 12: now whosoever looks back upon the foregoing resolution, and observes how these numbers were formed, he will easily perceive, that the number 4 was the middle term in the question; that the number 15 in both operations was the product of the numbers 3 and 5, which lay next the middle term on each side; and that the divisor 12 was in both cases the product of the extreme numbers 2 and 6: therefore, *In all questions belonging to the double rule of three inverse, where the numbers are supposed to be ordered as in the double rule of three direct, if the three middle numbers be multiplied together, and the product be divided by the product of the two extremes, the quotient of this division will be the number sought.* And thus may all the trouble of expunging be avoided, though I thought it proper to explain that method in the first place, in order to let the learner into the reason of this last theorem, which is founded upon it.

Questions wherein the extraction of the square root is concerned.

QUEST. 40.

There is a certain field, whose breadth is 576 yards, and whose length is 1296 yards: I demand the side of a square field equal to it.

Answer. This field will be equal to a square whose side is 864 yards.

QUEST. 41.

There is a certain inclosure 3 times as long as it is broad, whose area is 46128 square yards: I demand its breadth and length?

The breadth multiplied into the length, that is, the breadth multiplied into 3 times itself, is 46128;

therefore the breadth multiplied into itſelf is 15376; therefore the breadth is 124, and the length 372.

QUEST. 42.

A certain ſociety collect among themſelves a ſum amounting to 15 pounds 5 ſhillings and a farthing, every one contributing as many farthings as there were members in the whole ſociety: I demand the number of members.

Anſwer. 121 members.

THE

INTRODUCTION,

Concerning *Vulgar and Decimal Fractions*.

DEFINITIONS.

Art. 1. A FRACTION, ſimply and abſtractedly conſidered, is that wherein ſome part or parts of an unit are expreſſed: as, if an unit be ſuppoſed to be divided into 4 equal parts, and three of theſe parts are to be expreſſed, it muſt be done by the fraction three fourths, to be written thus $\frac{3}{4}$: here the number 4, which ſhews into how many equal parts the unit is ſuppoſed to be divided, and ſo determines the true value, magnitude, or denomination of thoſe parts, is called the denominator of the fraction; and the number 3, which ſhews how many of theſe parts are conſidered in the fraction, is called the numerator: thus in the fraction $\frac{1}{2}$ or one half, 1 is the numerator, and 2 the denominator: in $\frac{2}{2}$ or two halves, 2 is both numerator and denominator, *&c.*

When a fraction is applied to any particular quantity, that quantity is called the integer to the fraction; thus in $\frac{3}{4}$ of a penny, a penny is the integer;

teger; in three fourths of ſix, the number 6 is the integer; thus in three fourths of five ſixths, the fraction five ſixths is the integer; for though in an abſolute ſenſe it be a fraction, yet here, with reſpect to the fraction three fourths, it is an integer: and thus may one and the ſame quantity, under different ways of conception, be both an integer and a fraction; as a foot is an integer, and a third part of a yard is a fraction, though they both ſignify the ſame thing. When the integer to a fraction is not expreſſed, unity is always to be underſtood. Thus $\frac{3}{4}$ is $\frac{3}{4}$ of an unit; thus when we ſay, $\frac{1}{3}$ and $\frac{1}{4}$ make $\frac{7}{12}$, the meaning is, that if $\frac{1}{3}$ part of an unit, and $\frac{1}{4}$ part of an unit be added together, the ſum will amount to the ſame as if that unit had been divided into 12 equal parts, and 7 of thoſe parts had been taken. Thus again, when we ſay that $\frac{2}{3}$ of $\frac{4}{5}$ are equivalent to $\frac{8}{15}$, we mean, that if an unit be divided into 5 equal parts, and 4 of them be taken, and then this fraction $\frac{4}{5}$ be again divided into 3 equal parts, and two of them be taken, the reſult will be the ſame as if the unit had at firſt been divided into 15 equal parts, and 8 of them had been taken; and whatever is true in the caſe of unity, will be equally true in the caſe of any other integer whatever. Thus if it be true that $\frac{1}{3}$ and $\frac{1}{4}$ of an unit are equal to $\frac{7}{12}$ of an unit, that is, if it be true in general that $\frac{1}{3}$ and $\frac{1}{4}$ added together are equal to $\frac{7}{12}$, it will be as true of any particular integer, ſuppoſe of a pound ſterling, that $\frac{1}{3}$ of a pound, and $\frac{1}{4}$ of a pound when added together, are equal to $\frac{7}{12}$ of a pound; again, if it be true in general that $\frac{2}{3}$ of $\frac{4}{5}$ are equal to $\frac{8}{15}$, it is as true in particular that $\frac{2}{3}$ of $\frac{4}{5}$ of a pound are equivalent to $\frac{8}{15}$ of a pound, &c.

Of proper and improper fractions; and of the reduction of an improper fraction to a whole or mixt number.

2. Fractions are of two ſorts, proper and improper; a proper fraction is that, whoſe numerator is leſs than the denominator, as $\frac{1}{2}$; therefore an improper one is that, whoſe numerator is equal to, or greater than, the denominator, as $\frac{2}{2}$, $\frac{3}{2}$, *&c.*

OBJECTION.

But is there no abſurdity in the ſuppoſition of an improper fraction, as in three halves for inſtance, conſidering that an unit cannot be divided into more than two halves? *Anſwer:* No more than there is in ſuppoſing three halfpence to be the price of any thing, conſidering that a penny cannot be divided into above two halfpence. Theſe fractions therefore are called improper, not from any abſurdity either in the ſuppoſition or in the expreſſion, but becauſe they may be more properly and more intelligibly expreſſed, either by a whole number, or at leaſt by a mixt number conſiſting of a whole number and a fraction; as for example, if the numerator of a fraction be equal to the denominator, as $\frac{4}{4}$, that fraction will always be equivalent to unity, as $\frac{4}{4}$ of an hour, that is, four quarters of an hour, are equivalent to one hour, $\frac{4}{4}$ of a penny, that is, 4 farthings, are equal to one penny, *&c:* and the reaſon is plain; for if an unit be divided into four equal parts, and four of theſe parts be expreſſed in a fraction, the whole unit is expreſſed in that fraction, that is, ſuch a fraction muſt always be looked upon as equal to an unit: therefore if the numerator be double of the denominator, as $\frac{8}{4}$, the fraction muſt be equal to the number 2, becauſe $\frac{8}{4}$ contain $\frac{4}{4}$ or 1 twice; in like manner $\frac{12}{4}$ are equal to, and

and may be more properly expreſſed by, the number 3; $\frac{16}{4}$ by the number 4, &c: and univerſally, as often as the numerator of a fraction contains the denominator, ſo many units is that fraction equivalent to: But to find how often the numerator contains the denominator, is to divide the numerator by the denominator; therefore if the numerator of an improper fraction be divided by the denominator, the quotient, if nothing remains, will be the whole number by which the fraction may be expreſſed; but if any thing remains of this diviſion, then the quotient, together with a fraction whoſe numerator is that remainder, and denominator the diviſor, will be a mixt number, expreſſing the fraction propoſed. Thus $\frac{24}{3}$ are equivalent to the whole number 8, but $\frac{25}{3}$ are equivalent to the mixt number 8 $\frac{1}{3}$, $\frac{26}{3}$ to the mixt number 8 $\frac{2}{3}$, juſt as 24 feet are equal to 8 yards, 25 feet to 8 yards and 1 foot, 26 feet to 8 yards and 2 feet, &c: and this is what we call the reduction of an improper fraction into a whole or mixt number.

The reduction of a whole or mixt number into an improper fraction.

3. As unity may be expreſſed by any fraction of any form or denomination whatever, provided the numerator be equal to the denominator, as $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, &c; ſo the number 2 is reducible to any fraction whoſe numerator is double the denominator, as $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, &c; and ſo is every number reducible to any fraction, whoſe numerator contains the denominator as often as there are units in the number propoſed: therefore whenever a whole number is to be reduced to a fraction whoſe denominator is given, it muſt be multiplied by that given denominator, and the product with that denominator under it, will be the equivalent fraction. Thus, if the number 5 is to be reduced

reduced into halves, that is, into a fraction whose denominator is 2, it must be multiplied by 2, and so you will have 5 equal to $\frac{10}{2}$, just as 5 pence are equivalent to 10 halfpence; if the number 8 is to be reduced into thirds, it must be multiplied by 3, and so you will have 8 equal to $\frac{24}{3}$, just as 8 yards are equal to 24 feet; lastly, if the number 2 is to be reduced into fourths, it will be equal to $\frac{8}{4}$, just as 2 pence are equal to 8 farthings. If the number to be reduced be a mixt number, consisting of a whole number and a fraction, the whole number must always be reduced to the same denomination with the fraction annexed, and the rule will be this: Multiply the whole number by the denominator of the fraction annexed; add the numerator to the product, and the sum with the denominator under it will be the equivalent fraction. Thus the mixt number 5 $\frac{1}{2}$ is equivalent to $\frac{11}{2}$, just as 5 pence halfpenny in money is equivalent to 11 halfpence: This operation carries its own evidence along with it; for the number 5 itself is equal to $\frac{10}{2}$ as above; therefore 5 $\frac{1}{2}$ must be equivalent to $\frac{11}{2}$: again, the number 8 $\frac{2}{3}$ is equal to $\frac{26}{3}$, just as 8 yards and 2 feet over are equivalent to 26 feet; lastly, 2 $\frac{3}{4}$ is reducible to $\frac{11}{4}$, just as two pence and 3 farthings are reducible to 11 farthings.

A LEMMA.

4. *If any integer be assumed, as a pound sterling, and also any fraction, as $\frac{3}{4}$, I say then, that $\frac{3}{4}$ parts of one pound amount to the same as $\frac{1}{4}$ part of 3 pounds.*

To demonstrate this Lemma (which scarce wants a demonstration) I argue thus: If any quantity, greater or less, be always divided into the same number of parts, the greater or less the quantity so divided is, the greater or less will the parts be. Thus $\frac{1}{4}$ of a yard is 3 times as much as $\frac{1}{4}$ of a foot; because a yard is 3 times as much as a foot; and for the

same

ſame reaſon $\frac{1}{4}$ of 3 pounds is 3 times as much as $\frac{1}{4}$ of 1 pound; but $\frac{3}{4}$ of one pound are alſo 3 times as much as $\frac{1}{4}$ of one pound; therefore $\frac{3}{4}$ of 1 pound are equal to $\frac{1}{4}$ of 3 pounds, becauſe both are juſt 3 times as much as $\frac{1}{4}$ of 1 pound. *Q. E. D.*

How to eſtimate any fractional parts of an Integer in parts of a leſſer denomination, and vice versâ.

5. This may be done various ways; but the ſhorteſt and ſafeſt, as I take it, is that which follows: Suppoſe I had a mind to know the value of $\frac{5}{6}$ of a pound; I ſhould argue as in the foregoing lemma, that $\frac{5}{6}$ of one pound are the ſame as $\frac{1}{6}$ of 5 pounds, but the latter is more eaſily taken than the former; therefore I apply myſelf wholly to the latter, to wit, to find the ſixth part of five pounds, thus: 5 pounds, or 100 ſhillings, divided by 6, quote 16 ſhillings, and there remain 4 ſhillings; again, 4 ſhillings, or 48 pence, divided by 6, quote 8 pence, and there remains nothing; therefore the value of 1 ſixth of 5 pounds, or $\frac{5}{6}$ of 1 pound, is 16 ſhillings and 8 pence. Again, ſuppoſe I would know the value of $\frac{6}{7}$ of a pound, I find the value of $\frac{1}{7}$ of 6 pounds thus; 6 pounds, or 120 ſhillings, divided by 7, give 17 ſhillings, and there remains 1 ſhilling; again, 1 ſhilling, or 12 pence, divided by 7, gives 1 penny, and there remain 5 pence; again, 5 pence, or 20 farthings, divided by 7, give 2 farthings, and there remain 6 farthings; laſtly, a ſeventh part of 6 farthings is juſt as much as $\frac{6}{7}$ of 1 farthing, by the lemma: hence I conclude, that $\frac{6}{7}$ of a pound are 17 ſhillings 1 penny 2 farthings and $\frac{6}{7}$ of a farthing: But the value of $\frac{6}{7}$ of a farthing is ſo near to one farthing, that if I would rather admit of a ſmall inaccuracy in my account, than a fraction, I ſhould make the value of $\frac{6}{7}$ of a pound to be 17 ſhillings

1 penny and 3 farthings. Laſtly, ſuppoſe I would know the amount of $\frac{2}{3}$ parts of 17 ſhillings and 6 pence, I ſhould argue thus; $\frac{2}{3}$ parts of 17 ſhillings and 6 pence are equivalent to $\frac{1}{3}$ part of twice as much, that is, to $\frac{1}{3}$ part of 35 ſhillings: but $\frac{1}{3}$ part of 35 ſhillings is 11 ſhillings and 8 pence; therefore $\frac{2}{3}$ parts of 17 ſhillings and 6 pence make 11 ſhillings and 8 pence.

Of the reverſe of this reduction, one ſingle inſtance will ſuffice: Let it then be required to reduce 1 ſhilling 2 pence 3 farthings to fractional parts of a pound: here I conſider, that in 1 pound are 960 farthings; and in 1 ſhilling 2 pence 3 farthings, are 59 farthings; therefore 1 farthing is $\frac{1}{960}$ of a pound; and 1 ſhilling 2 pence 3 farthings are $\frac{59}{960}$ of a pound.

Preparations for further reductions and operations of fractions.

6. All the operations and reductions of fractions are mediately or immediately deducible from the following principle; which is, that *If the numerator of a fraction be encreaſed, whilſt the denominator continues the ſame, the value of the fraction will be encreaſed proportionably: and* vice verſâ. *On the other hand, if the denominator be encreaſed in any proportion, whilſt the numerator continues the ſame, the value of the fraction will be diminiſhed in a contrary proportion; and* vice verſâ. Thus $\frac{2}{3}$ are twice as much as $\frac{1}{3}$, and $\frac{1}{6}$ is but half as much.

From this principle it follows, that if the numerator and demoninator of a fraction be both multiplied, or both divided, by the ſame number, the value of the fraction will not be affected thereby; becauſe, as much as the fraction is encreaſed by multiplying the numerator, juſt ſo much again it will be diminiſhed by multiplying the denominator; and as much as the fraction is diminiſhed by dividing

ing the numerator, juſt ſo much again it will be encreaſed by dividing the denominator. Thus the terms of the fraction $\frac{3}{4}$ being doubled, produce $\frac{6}{8}$, a fraction of the ſame value; and, on the contrary, the terms of the fraction $\frac{6}{8}$ being halved, give $\frac{3}{4}$.

Hence it appears, that every fraction is capable of infinite variety of expreſſion, ſince there is infinite choice of multiplicators, whereby the numerator and denominator of a fraction may be multiplied, and ſo the expreſſion may be changed, without changing the value of the fraction. Thus the fraction $\frac{1}{2}$, if both the numerator and denominator be multiplied by 2, becomes $\frac{2}{4}$; if by 3, $\frac{3}{6}$; if by 4, $\frac{4}{8}$; if by 5 $\frac{5}{10}$; and ſo on *ad infinitum*; all which are nothing elſe but different expreſſions of the ſame fraction: therefore, in the midſt of ſo much variety, we muſt not expect that every fraction we meet with ſhould always be in its leaſt or loweſt terms; but how to reduce them to this ſtate whenever they happen to be otherwiſe, ſhall be the buſineſs of the next article.

The reduction of fractions from higher to lower terms.

7. Whenever a fraction is ſuſpected not to be in its leaſt terms, find out, if poſſible, ſome number that will divide both the numerator and denominator of the fraction without any remainder; for if ſuch a number can be found, and the diviſion be made, the two quotients thence ariſing will exhibit reſpectively the numerator and denominator of a fraction, equal to the fraction firſt propoſed, but expreſſed in more ſimple terms: this is evident from the laſt article. As for example: let the fraction $\frac{10}{15}$ be propoſed to be reduced: here, to find ſome number that will divide both the numbers 10 and 15 without any remainder, I begin with the number 2, as being the firſt whole number that can have any

effect in division; but I find 2 will not divide 15; 3 is the next number to be tried; but neither will that succeed, for it will not divide 10; as for the number 4, I pass that by, because if 2 would not divide 15, much less will 4 do it; the next number I try is 5, and that succeeds; for if 10 and 15 be divided by 5, the quotients will be 2 and 3 respectively, each without remainder; therefore the fraction $\frac{10}{15}$, after being reduced to its least terms, is found to be the same as $\frac{2}{3}$; that is, if an unit be divided into 15 equal parts, and 10 of them be taken, the amount will be the same as if it had been divided into 3 equal parts, and 2 of them had been taken. Secondly, if the fraction proposed to be reduced be $\frac{2520}{7560}$ divide its terms by 2, and you will have the fraction $\frac{1260}{3780}$; divide again by 2, and you will have $\frac{630}{1890}$; divide again by 2, and you will have $\frac{315}{945}$; therefore all further division by 2 is excluded: divide then these last terms by 3, and you will have $\frac{105}{315}$; divide again by 3, and you will have $\frac{35}{105}$; divide by 5, and you will have $\frac{7}{21}$; and lastly, divide by 7, and you will have $\frac{1}{3}$; so that the fraction $\frac{2520}{7560}$, after a common division by 2, 2, 2, 3, 3, 5, 7, is found at last equal to $\frac{1}{3}$. Thirdly, the fraction $\frac{36}{48}$, after a continual division by 2, 2, 3, becomes $\frac{3}{4}$. Fourthly, $\frac{56}{84}$, after a continual division by 2, 2, 7, becomes $\frac{2}{3}$. Fifthly, $\frac{144}{180}$, after a continual division by 2, 2, 3, 3, becomes $\frac{4}{5}$. Sixthly, $\frac{42}{126}$, after a continual division by 2, 3, and 7, becomes $\frac{1}{3}$. Seventhly, $\frac{315}{840}$, after a conti-

continual divifion by 3, 5, 7, becomes $\frac{3}{8}$. Eighthly, $\frac{35}{840}$, after a continual divifion by 5 and 7, becomes $\frac{1}{24}$. Ninthly, $\frac{735}{245}$, after a continual divifion by 5, 7, 7, becomes $\frac{3}{1}$, or 3.

Some perhaps may think themfelves helped in the practice of this rule by the following obfervations:

Firft, that 2 will divide any number that ends with an even number, or with a cypher, as 36, 30, *&c.* and no other.

Secondly, that 5 will divide any number that ends with a 5, or with a cypher, as 75, 70, *&c.* and no other.

Thirdly, that 3 will divide any number, when it will divide the fum of its digits added together: thus 3 will divide 471, becaufe it will divide the number 12, which is the fum of the numbers 4, 7, and 1.

Fourthly, if both the numerator and denominator have cyphers annexed to them, throw away as many as are common to both: thus $\frac{3500}{56000}$ is the fame as $\frac{35}{560}$, or $\frac{7}{112}$, or $\frac{1}{16}$.

After all, there is a certain and infallible rule for finding the greateft common divifor of any two numbers whatever, that have one, whereby a fraction may be reduced to its leaft terms by one fingle operation only. I fhall be forced indeed to poftpone the demonftration of this rule to a more convenient place, not fo much for want of principles to proceed upon, as for want of a proper notation; but the rule itfelf is as follows: Let *a* and *b* be two given numbers, whofe greateft common divifor is required; to wit, *a* the greater, and *b* the lefs: then, dividing *a* by *b* without any regard to the quotient, call the remainder *c*; divide again *b* by *c*, and call the remainder *d*;

then divide c by d, and call the remainder e; then divide d by e, and call the remainder f; and ſo proceed on, till at laſt you come to ſome diviſor, as f, which will divide the preceding number e without a remainder: I ſay then, that this laſt diviſor will be the greateſt common diviſor of the two given numbers a and b. As for example; let a be 1344 and b 582: then, to find the greateſt common diviſor of theſe numbers, I divide a (1344) by b (582) and there remains 180, which I call c; then I divide b (582) by c (180) and there remains 42, which I call d; then I divide c (180) by d (42) and there remains 12, which I call e; then I divide d (42) by e (12) and there remains 6, which I call f; laſtly I divide e (12) by f (6) and there remains nothing: whence I conclude that 6 is the greateſt common diviſor of the two numbers 1344 and 582; and as the quotients by 6 are 224 and 97, it follows, that the fraction $\frac{582}{1344}$, when reduced to its leaſt terms, will be $\frac{97}{224}$. If no common diviſor can be found but unity, it is an argument that the fraction is in its leaſt terms already.

From this and the laſt article it follows, that all fractions that are reducible to the ſame leaſt terms are equal; as $\frac{4}{6}$, $\frac{6}{9}$, $\frac{10}{15}$, &c. which are all reducible to $\frac{2}{3}$; though it does not follow *è converſo*, that all equal fractions are reducible to the ſame leaſt terms; this will be demonſtrated in another place. (See *Elements of Algebra*, Art. 193. page 290, 4to.)

For the better underſtanding of the following article, it muſt be obſerved, that this mark × is a ſign of multiplication, and is uſually read *into:* thus 2 × 3 ſignifies 6, 2 × 3 × 4 ſignifies 24, 2 × 3 × 4 × 5 ſignifies 120, &c.; and in ſome caſes it will be better to put down theſe components or factors, than the character of the number ariſing from their continual multiplication, as in the following article. It ought alſo to be obſerved, that it matters not in what order theſe components are placed; for $2\times 3\times 4\times 5$ ſignifies juſt the ſame as $4\times 5\times 2\times 3$, &c.

The reduction of fractions of different denominations, to others of the ſame denomination.

8. There is another reduction of fractions, no leſs uſeful than the former; and that is, the reduction of fractions of different denominations to others of the ſame denomination, or which have the ſame denominator, without changing their values; which is done as follows: Having firſt put down the fractions to be reduced, in any order, one after another, and beginning with the numerator of the firſt fraction, multiply it, by a continual multiplication, into all the denominators but its own, and put down the product under that fraction; then multiply, in like manner, the numerator of the next fraction into all the denominators but its own, and put down the product under that fraction; and ſo proceed on through all the numerators, always taking care to except the denominator of that fraction whoſe numerator is multiplied. Then, multiplying all the denominators together, put down the product under every one of the products laſt found, and you will have a new ſet of fractions, all of the ſame denomination with one another, and all of the ſame values with their reſpective original ones. As for example; let it be propoſed to reduce the following fractions to the ſame denomination, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$: *1ſt*, The numerator of the firſt fraction is 1, and the denominators of the reſt are, 4, 6, and 8, and $1\times 4\times 6\times 8$ gives 192; therefore I put down 192 under $\frac{1}{2}$. *2dly*, The numerator of the ſecond fraction is 3, and the denominators of the reſt are 6, 8, and 2, and $3\times 6\times 8\times 2$ gives 288; therefore I put down 288 under $\frac{3}{4}$. *3dly*, $5\times 8\times 2\times 4$ gives 320; therefore I put down 320 under $\frac{5}{6}$. *4th*, $7\times 2\times 4\times 6$ gives 336; therefore

fore I put down 336 under $\frac{7}{8}$. Laſtly, $2 \times 4 \times 6 \times 8$, or the product of all the denominators, is 384. This therefore I put down under every one of the numerators laſt found, and ſo have a new ſet of fractions, *viz.* $\frac{192}{384}$, $\frac{288}{384}$, $\frac{320}{384}$, $\frac{336}{384}$, all of the ſame denomination, as appears from the operation itſelf; and all of the ſame value with their reſpective original ones, as will appear preſently; but firſt ſee the work:

$$\frac{1}{2}. \quad \frac{3}{4}. \quad \frac{5}{6}. \quad \frac{7}{8}.$$

$$\frac{192}{384}. \quad \frac{288}{384}. \quad \frac{320}{384}. \quad \frac{336}{384}.$$

A demonſtration of the rule.

All that is to be demonſtrated in this rule is, to prove from the nature of the operation itſelf, that the original fractions ſuffer nothing in their values by this reduction: in order to which, it will be convenient to put down the components of the new numerators inſtead of their proper characters, as in the laſt article; as alſo thoſe of the common denominator, and the work will ſtand thus:

$$\frac{1}{2}. \qquad \frac{3}{4}. \qquad \frac{5}{6}. \qquad \frac{7}{8}.$$

$$\frac{1\times4\times6\times8}{2\times4\times6\times8}. \quad \frac{3\times6\times8\times2}{4\times6\times8\times2}. \quad \frac{5\times8\times2\times4}{6\times8\times2\times4}. \quad \frac{7\times2\times4\times6}{8\times2\times4\times6}.$$

By this method of operation it appears, that the numerator and denominator of the firſt fraction $\frac{1}{2}$, are both multiplied by the ſame number in the reduction, to wit, by $4 \times 6 \times 8$; and therefore that fraction ſuffers nothing in its value, by art. 6. In like manner, the terms of the ſecond fraction $\frac{3}{4}$ are both multiplied by the ſame number $6 \times 8 \times 2$; therefore that fraction can ſuffer nothing in its value; and the ſame may be ſaid of all the reſt. *Q. E. D.*

Other examples to this rule.

$\frac{1}{2}$. $\frac{1}{3}$. $\frac{1}{4}$. $\frac{1}{5}$. $\frac{1}{6}$. $\frac{1}{3}$. $\frac{1}{4}$. $\frac{1}{5}$. $\frac{1}{6}$.

$\frac{360}{720}$. $\frac{240}{720}$. $\frac{180}{720}$. $\frac{144}{720}$. $\frac{120}{720}$. $\frac{120}{360}$. $\frac{90}{360}$. $\frac{72}{360}$. $\frac{60}{360}$.

$\frac{1}{4}$. $\frac{1}{5}$. $\frac{1}{6}$. $\frac{1}{5}$. $\frac{1}{6}$.

$\frac{30}{120}$. $\frac{24}{120}$. $\frac{20}{120}$. $\frac{6}{30}$. $\frac{5}{30}$.

The uſe of this rule will ſoon appear in the addition and ſubtraction of fractions: in the mean time it may not be amiſs to obſerve, that it would be very difficult, if not impoſſible, to compare fractions of different denominations, without firſt reducing them to the ſame. As for inſtance; ſuppoſe it ſhould be aſked, which of theſe two fractions is the greater; $\frac{3}{4}$, or $\frac{5}{7}$; in this view it would be difficult to determine the queſtion; but when I know that $\frac{3}{4}$ are the ſame with $\frac{21}{28}$, and that $\frac{5}{7}$ are the ſame with $\frac{20}{28}$, I know then, that $\frac{3}{4}$ are greater than $\frac{5}{7}$ by a twenty-eighth part of the whole. We now proceed to the four operations of fractions, to wit, their addition, ſubtraction, multiplication, and diviſion: and firſt,

Of the addition of fractions.

9. Whenever two or more fractions are to be added together, let them firſt be reduced to the ſame denomination, if they be not ſo already; and then, adding the new numerators together, put down the ſum with the common denominator under it. In the caſe of mixt numbers, add firſt the fractions together, and then the whole numbers: but if the fractions, when added together, make an improper fraction, reduce it by the 2d art. to a whole or mixt number; and then putting down the fractional part, if there be any, reſerve the whole number for the place of integers.

To

To this rule might be referred (if it had not been taught already in the 5d art.) the reduction of a mixt number into an improper fraction, which is nothing elſe but adding a whole number and a fraction together, and may be done by conſidering the whole number as a fraction whoſe denominator is unity.

Examples of addition of fractions.

1*ſt*, $\frac{3}{10}$ and $\frac{4}{10}$ when added together make $\frac{7}{10}$, for juſt the ſame reaſon as 3 ſhillings and 4 ſhillings when added together make 7 ſhillings.

2*dly*, The fractions $\frac{1}{3}$ and $\frac{1}{4}$ when reduced to the ſame denomination by the laſt art. are $\frac{4}{12}$ and $\frac{3}{12}$, and theſe added together make $\frac{7}{12}$; therefore the fractions $\frac{1}{3}$ and $\frac{1}{4}$ when added together make up the fraction $\frac{7}{12}$.

For a better confirmation of theſe abſtract concluſions, but chiefly to inure the learner to conceive and reaſon diſtinctly about fractions, it may be very convenient to apply theſe examples in ſome particular caſe; as for inſtance, in the caſe of a pound ſterling; and if we do ſo here, we are to try, whether $\frac{1}{3}$ and $\frac{1}{4}$ of a pound, when added together, amount to $\frac{7}{12}$ of a pound, or not: here then we ſhall find by diviſion, that the third part of a pound is 6 ſhillings and 8 pence, and the fourth part 5 ſhillings; and theſe, added together, make 11 ſhillings and 8 pence; therefore $\frac{1}{3}$ and $\frac{1}{4}$ of a pound, when added together, make 11 ſhillings and 8 pence; but by the 5th art. it will be found that $\frac{7}{12}$ of a pound are alſo 11 ſhillings and 8 pence; therefore $\frac{1}{3}$ and $\frac{1}{4}$ of a pound, when added together, make $\frac{7}{12}$ of a pound; and the ſame would have been true in any other inſtance whatever.

3*dly*, $\frac{2}{5}$ and $\frac{3}{8}$, that is, $\frac{16}{40}$, and $\frac{15}{40}$, when added together, make $\frac{31}{40}$, which will alſo be true in the caſe of a pound ſterling; for by the 5th art. $\frac{2}{5}$ of a pound are 8 ſhillings, $\frac{3}{8}$ of a pound are 7 ſhillings and 6 pence, and their ſum is 15 ſhillings and 6 pence; which will alſo be found to be the value of

$\frac{31}{40}$ of a pound; therefore $\frac{2}{5}$ and $\frac{3}{8}$ of a pound, when added together, make $\frac{31}{40}$ of a pound.

4thly, $\frac{2}{3}$ and $\frac{4}{5}$, that is, $\frac{15}{15}$ and $\frac{12}{15}$, when added together, make $\frac{22}{15}$, an improper fraction; which being reduced to a mixt number, by the 2d art. is 1 and $\frac{7}{15}$: let us now try, whether $\frac{2}{3}$ of a pound, and $\frac{4}{5}$ of a pound when added together will make one pound and $\frac{7}{15}$ of a pound over, or not: now $\frac{2}{3}$ of a pound, or 13 ſhillings and 4 pence, added to $\frac{4}{5}$ of a pound, or 16 ſhillings, amount to 1 pound 9 ſhillings and 4 pence: and $\frac{7}{15}$ of a pound are found to be 9 ſhillings and 4 pence; therefore $\frac{2}{3}$ and $\frac{4}{5}$ of a pound, when added together, make one pound and $\frac{7}{15}$ of a pound over.

5thly, $\frac{3}{4}$ and $\frac{5}{6}$, that is, $\frac{18}{24}$ and $\frac{20}{24}$, when added together, make $\frac{38}{24}$, or $\frac{17}{12}$, which will alſo be true in the caſe of a pound ſterling.

6thly, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, that is, $\frac{360}{720}$, $\frac{240}{720}$, $\frac{180}{720}$, $\frac{144}{720}$, $\frac{120}{720}$, when added together, make $\frac{1044}{720}$, that is, $1\frac{9}{20}$; try it in money.

7thly, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$, that is, $\frac{360}{720}$, $\frac{480}{720}$, $\frac{540}{720}$, $\frac{576}{720}$ and $\frac{600}{720}$, when added together, make $\frac{2556}{720}$, that is, $3\frac{11}{20}$.

8thly, The ſum of the mixt numbers $7\frac{1}{3}$ and $8\frac{1}{4}$ is $15\frac{7}{12}$; for the ſum of the fractions is $\frac{7}{12}$ by the ſecond example, and the ſum of the whole numbers is 15.

9thly, $5\frac{2}{3}$ added to $7\frac{4}{5}$ gives $13\frac{7}{15}$; for the ſum of the fractions is $1\frac{7}{15}$; by the fourth example; and the whole number 1, added to the whole numbers 5 and 7, gives 13.

10thly, $8\frac{1}{2}$, $9\frac{2}{3}$, $10\frac{3}{4}$, $11\frac{4}{5}$, $12\frac{5}{6}$, added together, make $53\frac{11}{20}$; for the fractions themſelves make $3\frac{11}{20}$ by the ſeventh example, and the whole number 3 added to the reſt makes 53.

11thly,

11*thly*, The whole number 2 added to the fraction $\frac{3}{4}$ gives $\frac{11}{4}$; for the whole number 2 may be considered as a fraction, whose denominator is unity; now $\frac{2}{1}$ and $\frac{3}{4}$, when reduced to the same denomination, are $\frac{8}{4}$ and $\frac{3}{4}$, which added together make $\frac{11}{4}$.

Thus also may unity be added to any fraction whatever, when subtraction requires it; but better thus: unity may be made a fraction of any denomination whatever, provided the numerator be equal to the denominator, by art. 2d: suppose then I would add unity to $\frac{2}{3}$; I suppose unity equal to $\frac{3}{3}$, and this added to $\frac{2}{3}$ makes $\frac{5}{3}$: again, unity added to $\frac{3}{5}$ makes $\frac{8}{5}$, because $\frac{5}{5}$ and $\frac{3}{5}$ make $\frac{8}{5}$.

Of the subtraction of fractions.

10. Whenever a less fraction is to be subtracted from a greater, they must be prepared as in addition; that is, they must be reduced to the same denomination, if they be not so already; then, subtracting the numerator of the less fraction from that of the greater, put down the remainder with the common denominator under it. In the case of mixt numbers, subtract first the fraction of the lesser number from that of the greater, and then the lesser whole number from the greater; but if, as it often happens, the greater number has the lesser fraction belonging to it, then an unit must be borrowed from the whole number and added to the fraction, as intimated in the close of the last article.

Examples of subtraction in fractions.

1*st*, $\frac{3}{10}$ subtracted from $\frac{4}{10}$ leaves $\frac{1}{10}$, just in the same manner as 3 shillings subtracted from 4 shillings leave 1 shilling.

2*dly*, $\frac{3}{4}$ subtracted from $\frac{5}{6}$, that is $\frac{18}{24}$ subtracted from $\frac{20}{24}$, leaves $\frac{2}{24}$, or $\frac{1}{12}$. So $\frac{3}{4}$ of a pound, or 15 shillings, 15 ſhillings, ſubtracted from $\frac{5}{6}$ of a pound, or 16 ſhillings and 8 pence, leaves $\frac{1}{12}$ of a pound, that is, 1 ſhilling and 8 pence.

3dly, $7\frac{1}{3}$ ſubtracted from $8\frac{1}{2}$, that is, $7\frac{2}{6}$ ſubtracted from $8\frac{3}{6}$, leaves $1\frac{1}{6}$.

4thly, $7\frac{3}{4}$ ſubtracted from $8\frac{1}{4}$, that is, $7\frac{3}{4}$ ſubtracted from $7\frac{5}{4}$, leaves $\frac{2}{4}$, or $\frac{1}{2}$; for here the greater number having the leſs fraction belonging to it, I borrow an unit from the whole number 8, and ſo reduce it to 7; and then this unit, under the name of $\frac{4}{4}$, I add to the fraction $\frac{1}{4}$, and ſo make it $\frac{5}{4}$.

5thly, $7\frac{2}{3}$ ſubtracted from $8\frac{1}{2}$, that is, $7\frac{4}{6}$ ſubtracted from $8\frac{3}{6}$, that is, $7\frac{4}{6}$ ſubtracted from $7\frac{9}{6}$, leaves $\frac{5}{6}$.

6thly, $7\frac{3}{5}$ ſubtracted from 8, that is, $7\frac{3}{5}$ ſubtracted from $7\frac{5}{5}$, leaves $\frac{2}{5}$.

Of the multiplication of fractions.

11. To multiply by a whole number is to take the multiplicand as often as that whole number expreſſes: therefore to multiply by a mixt number is, not only to take the multiplicand as often as the integral part expreſſes, but alſo to take ſuch a part or parts of it over and above, as is expreſſed by the fraction annexed. Thus 10 multiplied by $2\frac{1}{2}$ produces 25: for as $2\frac{1}{2}$ is a middle number between 2 and 3, ſo the product ought to be a middle number between 20 and 30, that is, 25: In like manner 10 multiplied by $1\frac{1}{2}$ produces 15, and being multiplied by $\frac{1}{2}$ produces 5: therefore to multiply by a proper fraction is nothing elſe but to take ſuch a part or parts of the multiplicand, as is expreſſed by that fraction. Certainly to take 10 twice and half of it over, once, and half of it over no times, and half of it over, (which laſt is taking the half of 10), are operations of the ſame kind, and differ only in degree one from another; and therefore, if the two former operations paſs by the name of multiplication, this laſt ought to

do ſo too; and if there be any abſurdity in the caſe, it lies in the name, and not in the thing.

Arithmetic was at firſt employed about whole numbers only, and thus far the name of multiplication was adequate enough, except in the caſe of unity. But it being afterwards conſidered, that no quantity whatever could be called an unit, that was not further diviſible; and conſequently, that there was not only an infinity of fractional numbers below unity, but alſo an infinity of mixt numbers between any two whole numbers whatever; it was judged, rightly enough, that the art of Arithmetic would not be perfect till its operations extended themſelves to this ſort of number alſo; and this being done without changing their names, it was then that the name of multiplication became too ſcanty for the thing ſignified: this therefore ought to be attributed to the unavoidable want of foreſight in the firſt impoſers, and not to any imperfection in the ſcience itſelf. This is no more than the caſe of many other arts and ſciences, that have outgrown their names. Thus Geometry, that originally and properly ſignified no more than the art of ſurveying, is now defined to be a ſcience treating of the nature and properties of all figures, or rather of the different modifications of extenſion and ſpace; ſo that now ſurveying is the leaſt and loweſt part of that ſcience. Thus Hydroſtatics, which originally ſignified no more than the art of weighing bodies in water, or rather the art of finding out the ſpecific gravities of bodies by weighing them in water, is now made the name of a ſcience, which treats of the nature and properties of fluids in general; and the ſeveral properties of air and mercury, ſo far as they are fluids, fall under the conſideration of Hydroſtatics, as properly as thoſe of water.

But perhaps it may be further urged, that to take the half of any quantity, is not to multiply, but to divide it. To which I anſwer; that it is impoſſible

to

to take the half of any quantity without dividing it by 2; and consequently, that to multiply by $\frac{1}{2}$ has the same effect as to divide by 2; but this does not prove that multiplication is the same as division, but only that these two operations, how contrary soever, may be made to do each other's business, which is no mystery to any one who is the least conversant in Arithmetic; and will be further explained in the next article.

A fraction may be multiplied by a whole number two ways; either by multiplying the numerator by that number, or else by dividing the denominator by the same, where such a division is possible: thus if the fraction $\frac{5}{6}$ be to be multiplied by 2, the product will either be $\frac{10}{6}$ by doubling the numerator, or $\frac{5}{3}$ by halving the denominator: this is evident from the 6th art. because a fraction will be equally encreased, whether it be by encreasing the numerator, or by diminishing the denominator.

If a fraction be to be multiplied by a fraction, multiply the numerator and denominator of the multiplicand, by the numerator and denominator of the multiplicator respectively, and the fraction thence arising will be the product sought; thus if it was required to multiply $\frac{4}{5}$ by $\frac{2}{3}$, or (which amounts to the same thing) if it was required to determine how much is $\frac{2}{3}$ of $\frac{4}{5}$, the answer would be $\frac{8}{15}$; and the reason is plain; for $\frac{1}{3}$ of $\frac{4}{5}$ is $\frac{4}{15}$, by the sixth art. because making the denominator three times greater, makes the fraction three times less; but if $\frac{1}{3}$ of $\frac{4}{5}$ be $\frac{4}{15}$, then $\frac{2}{3}$ of $\frac{4}{5}$ ought to be twice as much, that is $\frac{8}{15}$; therefore to determine the amount of $\frac{2}{3}$ of $\frac{4}{5}$, the numerator and denominator of $\frac{4}{5}$ must be multiplied respectively by the numerator and denominator of $\frac{2}{3}$; and the same reason will hold good in all other instances.

If a whole number is to be multiplied by a fraction, either change the multiplicator and multiplicand one for another, and then proceed as above directed; or else

elſe conſider the multiplicand as a fraction whoſe denominator is unity, and ſo proceed according to the rule for multiplying one fraction by another; by which means both rules will be contracted into one. Thus 6, or $\frac{6}{1}$, multiplied into $\frac{2}{3}$, produces $\frac{12}{3}$, or 4.

If the multiplicator, or multiplicand, or both, be mixt numbers, they muſt firſt be reduced to improper fractions by the third art. and then be multiplied according to the general rule.

Examples of multiplication in fractions.

1*ſt*, $\frac{2}{3}$ of $\frac{7}{8}$, multiplying numerators together, and denominators together, is $\frac{14}{24}$, or $\frac{7}{12}$; and ſo we find it in any particular caſe; for $\frac{7}{8}$ of a pound are 17 ſhillings and 6 pence; and $\frac{2}{3}$ of 17 ſhillings and 6 pence, that is (by the 5th art.) $\frac{1}{3}$ of 35 ſhillings, is 11 ſhillings and 8 pence; therefore $\frac{2}{3}$ of $\frac{7}{8}$ of a pound are 11 ſhillings and 8 pence, which will alſo be found to be the value of $\frac{7}{12}$ of a pound.

Here we may obſerve once for all, that whenever two fractions are to be multiplied together, the product will be the ſame, which ſoever it is that multiplies the other, juſt as it is in whole numbers, and for the ſame reaſon; for if $\frac{7}{8}$ be to be multiplied by $\frac{2}{3}$, then the numbers 7 and 8 muſt be reſpectively multiplied by 2 and 3; but if $\frac{2}{3}$ is to be multiplied by $\frac{7}{8}$, then the numbers 2 and 3 muſt be reſpectively multiplied by 7 and 8, which amounts to the ſame thing; whence it follows, that $\frac{2}{3}$ of $\frac{7}{8}$ come to the ſame as $\frac{7}{8}$ of $\frac{2}{3}$: to confirm this, we have ſeen already that $\frac{2}{3}$ of $\frac{7}{8}$ of a pound amount to 11 ſhillings and 8 pence; let us in the next place enquire into the value of $\frac{7}{8}$ of $\frac{2}{3}$ of a pound: now $\frac{2}{3}$ of a pound are 13 ſhillings and 4 pence; and $\frac{7}{8}$ of 13 ſhillings and 4 pence, that is, $\frac{1}{8}$ of 93 ſhillings and 4 pence, is 11 ſhillings and 8 pence; therefore $\frac{2}{3}$ of $\frac{7}{8}$ of a pound are the ſame as $\frac{7}{8}$ of $\frac{2}{3}$ of a pound, ſince both amount to 11 ſhillings and 8 pence.

2dly, $\frac{2}{3}$ of $\frac{5}{6}$ of $\frac{9}{10}$ are $\frac{90}{180}$, or $\frac{1}{2}$; for 2 × 5 × 9 make 90, and 3 × 6 × 10 make 180: thus $\frac{9}{10}$ of a pound are 18 ſhillings; and $\frac{5}{6}$ of 18 ſhillings are 15 ſhillings; and $\frac{2}{3}$ of 15 ſhillings are 10 ſhillings; which are $\frac{1}{2}$ of a pound.

3dly, $\frac{3}{4}$ of $\frac{3}{4}$ of $\frac{3}{4}$ are $\frac{27}{64}$: thus $\frac{3}{4}$ of a pound are 15 ſhillings; and $\frac{3}{4}$ of 15 ſhillings are 11 ſhillings and 3 pence; and $\frac{3}{4}$ of 11 ſhillings and three pence are 8 ſhillings and 5 pence farthing; which will alſo be found to be the value of $\frac{27}{64}$ of a pound.

4thly, The mixt number 6 $\frac{3}{4}$ multiplied by the whole number 7, or the whole number 7 multiplied by the mixt number 6 $\frac{3}{4}$, will produce in either caſe 47 $\frac{1}{4}$: for the mixt number 6 $\frac{3}{4}$ being reduced (by the 3d art.) to an improper fraction, becomes $\frac{27}{4}$; which being multiplied by 7, or $\frac{7}{1}$, makes $\frac{189}{4}$, or, when reduced to a mixt number, 47 $\frac{1}{4}$.

This multiplication may alſo be made another way, thus: $\frac{3}{4}$ multiplied by 7 makes $\frac{21}{4}$, that is, (by the 2d art.) 5$\frac{1}{4}$; put down the fraction $\frac{1}{4}$, and keep the 5 in reſerve; then 6 multiplied by 7 makes 42, which, with the 5 in reſerve, makes 47; therefore the whole product is 47 $\frac{1}{4}$ as before.

5thly, 3 $\frac{3}{4}$ multiplied by 2 $\frac{2}{3}$, that is, $\frac{15}{4}$ multiplied by $\frac{8}{3}$, makes $\frac{120}{12}$, that is, 10: thus 3 $\frac{3}{4}$ of a pound are 3 pounds 15 ſhillings; and twice 3 pounds 15 ſhillings is 7 pounds 10 ſhillings; moreover $\frac{2}{3}$ of 3 pounds 15 ſhillings, or $\frac{1}{3}$ of 7 pounds 10 ſhillings, is 2 pounds 10 ſhillings; and theſe 2 pounds 10 ſhillings, added to the former part of the product, to wit, 7 pounds 10 ſhillings, give 10 pounds for the whole product; therefore 3 $\frac{3}{4}$ of a pound multiplied by 2 $\frac{2}{3}$ make 10 pounds.

6thly,

6thly, 96 ½ multiplied by 24 ⅓, that is, $\frac{193}{2}$, multiplied by $\frac{73}{3}$, gives $\frac{14089}{6}$, that is, (by the 2d art.) 2348 $\frac{1}{6}$.

7thly, 36 ¼ multiplied into itself, that is, $\frac{145}{4}$, multiplied by $\frac{145}{4}$, makes $\frac{21025}{16}$, that is, 1314 $\frac{1}{16}$.

Before I put an end to this article, I do not know whether it will be thought worth my while to take notice of a very abſurd queſtion ſometimes bandied about, wherein it is required to multiply ⅓ of a pound by ½ of a pound: I call this a very abſurd queſtion, becauſe there is no manner of propriety in it; for in the very idea and definition of multiplication, the multiplicator at leaſt is ſuppoſed to be an abſtract number, or fraction; otherwiſe, what can be the meaning of taking the multiplicand as often, or as much of it, as is expreſſed by the multiplicator? If by multiplying ⅓ of a pound by ½ of a pound, be meant no more than multiplying ⅓ of a pound by ½, why is the word pound expreſſed in the multiplicator? and if there be any other meaning in it, why does not the propoſer explain it, ſince it is not expreſſed in the queſtion? Let him tell me what he means by multiplying 1 pound by 1 pound, and I will ſoon undertake to anſwer his queſtion. But if he neither can nor will do this, the queſtion neither deſerves nor is capable of an anſwer. I am not ignorant of another queſtion more frequently uſed than this, and of equal nonſenſe, if cuſtom had not explained it; and that is, to multiply 3 yards by 2 yards, and the like; whereby is meant, I ſuppoſe, to aſſign the number of ſquare yards contained in a rectangled parallelogram, or long ſquare, 3 yards in length, and 2 yards in breadth; but if this be the ſenſe put upon that queſtion by common conſent, that is all the title it has

has to it, there being no ſuch thing either expreſſed, or ſo much as implied, in the terms of the queſtion.

A LEMMA.

12. *Let* n *be any whole number, mixt number, or fraction; I ſay then that the quotient of* n *divided by any fraction is equal to the product of* n *multiplied into the reverſe of that fraction:* as for inſtance,

Let n be divided by $\frac{3}{2}$; I ſay that the quotient of n divided by $\frac{3}{2}$, will be equal to the product of n multiplied by $\frac{2}{3}$: for let q be the quotient of n divided by $\frac{3}{2}$; that is, let q be a number expreſſing how often the fraction $\frac{3}{2}$ is contained in n; then will $\frac{3}{2}$ multiplied by q be equal to n, from the nature of multiplication; but the product of $\frac{3}{2}$ multiplied by q is the ſame with the product of q multiplied by $\frac{3}{2}$; that is, $\frac{3}{2}$ of q, by the laſt article; therefore n is equal to $\frac{3}{2}$ of q; therefore $\frac{1}{3}$ of n is equal to $\frac{1}{2}$ of q; therefore $\frac{2}{3}$ of n are equal to q; but $\frac{2}{3}$ of n is the product of n multiplied by $\frac{2}{3}$; therefore the product of n multiplied by $\frac{2}{3}$ is equal to q; but the quotient of n divided by $\frac{3}{2}$ was q, by the ſuppoſition; therefore the quotient of n divided by $\frac{3}{2}$, is equal to the product of n multiplied by $\frac{2}{3}$. *Q. E. D.*

COROLLARY.

Hence may the rule of diviſion be at any time changed into that of multiplication, only by inverting the terms of the diviſor, and then multiplying inſtead of dividing. The ſame will alſo obtain in whole numbers, if they be conſidered as fractions whoſe denominators are units: thus to divide n by 2, that is, $\frac{2}{1}$, will have the ſame effect as to multiply it by $\frac{1}{2}$, as was hinted in the foregoing article.

Of the divifion of fractions.

13. The divifion of fractions, like all other divifion, is, to find how often one fraction, called the divifor, is contained in another, called the dividend; and that which fhews this, is called the quotient, whether it be a whole number, a mixt number, or a proper fraction: for in fractional divifion the quotient is always intended to be exact, without any remainder, and therefore muft fometimes be a whole number, fometimes a mixt number, and fometimes a proper fraction. Thus, if 18 is to be divided by 6, the quotient will be 3; becaufe 18 contains 6 3 times: but if 21 is to be divided by 6, the quotient will be $3\frac{1}{2}$; becaufe 21 contains 6 three times, and half of it over and above: laftly, if 3 is to be divided by 6, the quotient will be $\frac{1}{2}$; becaufe here the divifor, being greater than the dividend, cannot be fo much as once contained in it, and therefore the quotient in this cafe muft be a proper fraction, that is, $\frac{1}{2}$, fince 3 is juft the half of 6.

A fraction may be divided by a whole number two ways; either by dividing the numerator by that whole number when poffible, or elfe by multiplying the denominator by the fame: thus the half of $\frac{6}{7}$ may be taken, that is, $\frac{6}{7}$ may be divided by 2, either by halving the numerator, and the quotient will be $\frac{3}{7}$, or elfe by doubling the denominator, and then the quotient will be $\frac{6}{14}$, both which amount to the fame thing, by the 6th and 7th articles.

If the divifor be a fraction, the quotient may be had by multiplying the dividend into the inverted divifor, according to the rules of multiplication already laid down: thus if $\frac{4}{5}$ is to be divided by $\frac{2}{3}$, the quotient will be the fame as the product of $\frac{4}{5}$ multiplied by $\frac{3}{2}$, that is, $\frac{12}{10}$, or $1\frac{1}{5}$; the demonftration whereof is contained in the laft article.

And here again, as well as in the eleventh article, we are to obſerve, that if either the diviſor or dividend, or both, be mixt numbers, they muſt be reduced to improper fractions before the general rule can have place; and that, if either or both be whole numbers, they muſt be conſidered as fractions whoſe denominators are units.

From the general rule of diviſion before laid down it follows, that every fraction may be conſidered as the quotient of the numerator divided by the denominator, and that, whether the terms of the fraction under conſideration be whole numbers, or (which ſometimes happens) mixt numbers, or even pure fractions: a demonſtration of this laſt caſe will ſerve for all, ſince mixt numbers may be reduced to fractions, and whole numbers may be conſidered as fractions whoſe denominators are units. Let the fraction propoſed be $\frac{\frac{4}{5}}{\frac{2}{3}}$ I ſay, that this fraction is equal to the quotient ariſing from the diviſion of the numerator $\frac{4}{5}$ by the denominator $\frac{2}{3}$: to demonſtrate which, multiply both $\frac{4}{5}$ the numerator, and $\frac{2}{3}$ the denominator, by $\frac{3}{2}$ the inverted denominator, and the fraction will be changed into this, $\frac{\frac{12}{10}}{1}$, or $\frac{12}{10}$, being of the ſame value with the former, by the 6th art. but the quotient of $\frac{4}{5}$ divided by $\frac{2}{3}$ is alſo $\frac{12}{10}$ as above: therefore the fraction $\frac{\frac{4}{5}}{\frac{2}{3}}$ is equal to the quotient ariſing from the diviſion of the numerator by the denominator: and the ſame way of reaſoning may be uſed in any other inſtance. This conſideration is of very great uſe in Algebra, where quantities are very often ſo generally expreſſed, that there is no other way of repreſenting the quotient, but by a fraction whoſe numerator is the dividend, and denominator the diviſor. Hence alſo we are taught how to reduce a complicated fraction, into a ſimple one, whoſe numerator and denominator are whole numbers, to wit, by dividing the numerator

by

by the denominator: thus we see that $\frac{\frac{4}{5}}{\frac{2}{3}}$ is the same as $\frac{12}{10}$.

Other examples of division in fractions.

1st, $\frac{5}{6}$ divided by $\frac{3}{4}$, or, which is the same thing, $\frac{5}{6}$ multiplied into $\frac{4}{3}$, makes $\frac{20}{18}$, or $1\frac{1}{9}$; which shews that $\frac{3}{4}$ is contained once, and $\frac{1}{9}$ part of it over and above, in $\frac{5}{6}$: for a further confirmation of this, $\frac{5}{6}$ of a pound are 16 shillings and 8 pence; and $\frac{3}{4}$ of a pound are 15 shillings: now 15 shillings are once contained in 16 shillings and 8 pence, and there is 1 shilling and 8 pence over; which 1 shilling and 8 pence is just $\frac{1}{9}$ of 15 shillings. To prevent oversights, the learner is to remember, that it is the terms of the divisor only that are to be inverted, and not those of the dividend: thus to divide $\frac{5}{6}$ by $\frac{3}{4}$ is the same as to multiply $\frac{5}{6}$ into $\frac{4}{3}$, but not the same as to multiply $\frac{6}{5}$ into $\frac{4}{3}$.

2dly, $\frac{9}{10}$ divided by $\frac{1}{3}$, or multiplied into $\frac{3}{1}$, make $\frac{27}{10}$, or $2\frac{7}{10}$, which may be confirmed like the former: for $\frac{9}{10}$ of a pound are 18 shillings; and $\frac{1}{3}$ of a pound is 6 shillings and 8 pence: now 6 shillings and 8 pence are twice contained in 18 shillings, and there are 4 shillings and 8 pence over; which 4 shillings and 8 pence will be found by the 5th art. to be just $\frac{7}{10}$ of 6 shillings and 8 pence.

3dly, The whole number 10 divided by $2\frac{2}{3}$, that is $\frac{10}{1}$ divided by $\frac{8}{3}$, or multiplied into $\frac{3}{8}$, makes $\frac{30}{8}$, or $3\frac{3}{4}$.

4thly, $2\frac{2}{3}$ divided by $\frac{10}{1}$, or $\frac{8}{3}$ divided by $\frac{10}{1}$, or multiplied into $\frac{1}{10}$, makes $\frac{8}{30}$, or $\frac{4}{15}$.

5thly, $16\frac{1}{3}$ divided by $1\frac{1}{6}$, that is, $\frac{49}{3}$ divided by $\frac{7}{6}$, or multiplied into $\frac{6}{7}$, makes $\frac{294}{21}$, or 14.

Further obſervations concerning multiplication and diviſion in fractions.

14. When two fractions are multiplied together, or one is divided by the other, it often happens, that though the original fractions be both in their leaſt terms, yet the product, or quotient from them, ſhall be otherwiſe, and require a further reduction: as for inſtance, the fractions $\frac{5}{6}$ and $\frac{9}{10}$ are both in their leaſt terms; and yet, if they be multiplied together, their product $\frac{45}{60}$ is ſo far from being in its leaſt terms, that it may be reduced to $\frac{3}{4}$: ſo again in diviſion, $\frac{9}{10}$ and $\frac{15}{16}$ are fractions both in their leaſt terms; and yet if the latter be divided by the former, the quotient $\frac{150}{144}$ is reducible to $\frac{25}{24}$. It may not be amiſs, therefore, to enquire into the cauſe of this, and ſee whether the original fractions may be ſo prepared beforehand, as that the product, or quotient, ſhall always come out in its leaſt terms. Firſt then, as to the multiplication of $\frac{5}{6}$ and $\frac{9}{10}$; here it is eaſy to ſee, that the product of $\frac{5}{6}$ and $\frac{9}{10}$ multiplied together, will juſt amount to the ſame as that of $\frac{5}{10}$ into $\frac{9}{6}$, the denominators of the fractions being interchanged; this, I ſay, is certain from the operation itſelf; for the ſame numbers are multiplied together in both caſes; but theſe laſt fractions are far from being in their leaſt terms, the former, $\frac{5}{10}$ being reducible to $\frac{1}{2}$, and the latter $\frac{9}{6}$ to $\frac{3}{2}$; but after theſe new fractions $\frac{5}{10}$ and $\frac{9}{6}$ are reduced to their leaſt terms $\frac{1}{2}$ and $\frac{3}{2}$; their product $\frac{3}{4}$ will be the ſame in value with that of the original fractions, and at the ſame time will be in its leaſt terms. Thus then we ſee that, to have the product in its leaſt terms, care muſt be taken, not only to reduce the original fractions as low as poſſible, but after that, to interchange their denominators, and then again to reduce theſe

new fractions to their leaſt terms, and laſtly, to multiply theſe reduced fractions one into another.

The ſame manner of practice will alſo ſerve for diviſion, after it is reduced to the rule of multiplication: as for example; the quotient of $\frac{15}{16}$ divided by $\frac{9}{10}$, is the ſame with the product of $\frac{15}{16}$ multiplied into $\frac{10}{9}$; and this again is the ſame with the product of $\frac{15}{9}$ multiplied into $\frac{10}{16}$, as above; but becauſe the fractions $\frac{15}{9}$ and $\frac{10}{16}$ are not in their loweſt terms, they muſt be reduced to $\frac{5}{3}$ and $\frac{5}{8}$ before it can be expected that their product $\frac{25}{24}$ ſhould be in its leaſt terms. Thus we have reduced the two compendiums of multiplication and diviſion, not only to one rule inſtead of two, as they are commonly given out, but alſo to ſuch a rule as carries its own evidence along with it.

N. B. What was here done by interchanging the denominators, and keeping the numerators in their places, may as well be done by interchanging the numerators, and keeping the denominators in their places, the reaſon of both being the ſame.

Of the rule of proportion in fractions.

15. The rule of proportion in fractions is ſo much the ſame with the rule of proportion in whole numbers, that nothing more needs to be ſaid of it, except to illuſtrate it by an example or two.

Examples of the rule of proportion in fractions.

1ſt, If $\frac{1}{4}$ give $\frac{1}{5}$, what will $\frac{1}{6}$ give? Here $\frac{1}{5}$ and $\frac{1}{6}$ multiplied together give $\frac{1}{30}$; and this divided by $\frac{1}{4}$, (or multiplied by $\frac{4}{1}$) quotes $\frac{4}{30}$, or $\frac{2}{15}$, which is an anſwer to the queſtion.

2dly, If 2 $\frac{2}{3}$ give 3 $\frac{3}{4}$, what will 4 $\frac{4}{5}$ give? Theſe mixt numbers, being by the 3d art. reduced to improper fractions, will ſtand thus: *If $\frac{8}{3}$ give $\frac{15}{4}$, what will $\frac{24}{5}$ give?* Here $\frac{15}{4}$ and $\frac{24}{5}$ multiplied together

 give

give $\frac{360}{20}$ or 18; and this divided by $\frac{8}{3}$, quotes $6\frac{3}{4}$, which is an anfwer to the queftion.

3dly, If $\frac{1}{2}$ of a yard coft $\frac{1}{3}$ of a pound, what will $\frac{1}{4}$ of an ell coft? Here it muft be obferved, that an ell is $\frac{5}{4}$ of a yard, and confequently that $\frac{1}{4}$ of an ell is $\frac{1}{4}$ of $\frac{5}{4}$ or $\frac{5}{16}$ of a yard; fo that the queftion may be ftated thus: *If $\frac{1}{2}$ of a yard coft $\frac{1}{3}$ of a pound, what will $\frac{5}{16}$ of a yard coft?* Here $\frac{1}{3}$ and $\frac{5}{16}$ multiplied together make $\frac{5}{48}$, and this divided by $\frac{1}{2}$ quotes $\frac{5}{24}$ of a pound, or 4 fhillings and 2 pence; which therefore is an anfwer to the queftion.

The reduction of proportion from fractional to integral terms.

Whenever two fractions are propofed, as $\frac{2}{3}$ and $\frac{4}{5}$, whofe proportion is defired in whole numbers, reduce the fractions firft to the fame denomination by the 8th art. that is, in the prefent cafe, to $\frac{10}{15}$ and $\frac{12}{15}$; then you will have $\frac{2}{3}$ to $\frac{4}{5}$ as $\frac{10}{15}$ is to $\frac{12}{15}$; but $\frac{10}{15}$ is to $\frac{12}{15}$ as 10 to 12, or as 5 to 6; therefore $\frac{2}{3}$ is to $\frac{4}{5}$ as 5 to 6: here we may obferve, that though the finding of the common denominator be neceffary for underftanding the reafon of the rule, yet it is not at all neceffary for the practice of it; for to what purpofe is it to find the common denominator, to throw it away again when we have done? In practice, therefore, multiply the numerator of the fraction which is the firft in the proportion, by the denominator of the fecond, and then the numerator of the fecond fraction by the denominator of the firft, and the two products will exhibit refpectively the proportion of the firft fraction to the fecond in whole numbers, as was evident in the foregoing example.

Of the extraction of roots in fractions.

16. As every fraction is fquared, or multiplied into itfelf, by fquaring both the numerator and denominator

nator (ſee art. 11.), ſo *è converſo* the ſquare root of every fraction will be obtained by extracting the ſquare root both of the numerator and denominator: thus the ſquare of $\frac{3}{4}$ is $\frac{9}{16}$, and the ſquare root of $\frac{9}{16}$ is $\frac{3}{4}$. But here care muſt be taken, whenever the ſquare root of a fraction is to be extracted, that the fraction itſelf be firſt reduced to its ſimpleſt terms, by the 7th art. otherwiſe the fraction may admit of a ſquare root, and yet this root may not be diſcovered: thus, if it was required to extract the ſquare root of the fraction $\frac{18}{32}$, it would be impoſſible to obtain the root either of 18 or 32; and yet when this fraction is reduced to its leaſt terms $\frac{9}{16}$, its ſquare root will be found to be $\frac{3}{4}$.

When the ſquare root of a number cannot be extracted exactly, it is uſual to make an approximation by the help of decimals, or otherwiſe, and ſo to approach as near to the value of the true root as occaſion requires. Now in the caſe of a fraction, if the ſquare root of neither the numerator nor denominator can be exactly obtained, there will be no neceſſity however for two approximations, becauſe ſuch a fraction may be eaſily reduced to another of the ſame value, whoſe denominator is a known ſquare: as for inſtance; ſuppoſe the ſquare root of $46\frac{1}{5}$, or $\frac{231}{5}$ was required: I multiply both the numerator and denominator of this fraction by 5, and ſo reduce it to $\frac{1155}{25}$: Here the denominator 25 is a known ſquare number, whoſe root is 5; and the ſquare root of 1155 is 34 nearly; therefore, the ſquare root of the fraction propoſed is nearly $\frac{34}{5}$, or $6\frac{4}{5}$. But, after all, the beſt way of extracting the ſquare root of a vulgar fraction, is by throwing it into a decimal fraction, as will be ſhewn hereafter.

Note, That whatever has here been ſaid concerning the extraction of the ſquare root in fractions may easily be applied, *mutatis mutandis*, to the extraction of the cube root, &c.

Of decimal fractions.

And first of their notation.

17. A decimal fraction is a fraction whose denominator is 10, or 100, or 1000, or 10000, &c. and this denominator is never expressed, but always understood by the place of the figure it belongs to: for as all figures on the left hand of the place of units rise in their value, according to their distances from it, in a decuple proportion; so all figures on the right hand of the place of units sink in their value in a subdecuple proportion; as for instance; the number 345.6789, where 5 stands in the place of units, is to be read thus; *three hundred forty five, six tenths, seven hundredth parts, eight thousandth parts, nine ten-thousandth parts:* or the decimal parts may be read thus; *six thousand seven hundred eighty nine ten-thousandth parts*; the denominator being ten thousand, because the last figure 9, according to the former way of reckoning, stands in the place of ten-thousandth parts. The reason of this latter way of reading is plain; for $\frac{6}{10}$ are $\frac{6000}{10000}$, and $\frac{7}{100}$ are $\frac{700}{10000}$, and $\frac{8}{1000}$ are $\frac{80}{10000}$, and $\frac{6000}{10000}$, $\frac{700}{10000}$, $\frac{80}{10000}$, and $\frac{9}{10000}$, all added together, make $\frac{6789}{10000}$.

Cyphers are used in the expression of decimals as well as whole numbers, and for the same reason, Thus .067 may be read either *no tenths, six hundredth parts, seven thousandth parts*; or *sixty seven-thousandth parts*. But cyphers on the right hand of a decimal number (if nothing follows them) are as insignificant as cyphers on the left hand of a whole number; and yet cyphers are sometimes placed after decimals, for the

the ſake of regularity, or when we want to increaſe the number of decimal places.

From what has here been ſaid, it will be eaſy to multiply or divide any number by 10, 100, 1000, *&c.* only by removing the ſeparating point towards the right or left hand. Thus the number 345.6789 being multiplied by 10, becomes 3456.789; and being multiplied by 100, becomes 34567.89: and the ſame number 345.6789 being divided by 10, becomes 34.56789; and being divided by 100, becomes 3.456789: thus again, the number 345 being divided by 10000, becomes .0345; for to divide by 10000, is the ſame thing as to remove the ſeparating point 4 degrees towards the left hand, if there be any ſeparating point in the number given; but if there be none, as in the preſent caſe, then to put a ſeparating point four degrees towards the left hand, which in this example cannot be done, but by the help of a cypher in the firſt decimal place.

Of the addition and ſubtraction of decimal fractions.

18. The chief advantage of decimal arithmetic above that of common fractions, conſiſts in this, that in decimals all operations are performed as in whole numbers: this will preſently appear from the ſeveral parts of decimal arithmetic, as they come now to be treated of in order; and firſt of addition and ſubtraction.

Addition and ſubtraction in decimals are performed after the ſame manner as in whole numbers, care being taken, that like parts be placed under one another: as for example, .567 are added to .89 thus;

.89		.89		.890.
.567	ſubtracted thus;	.567;	or thus;	.567.
1.457:		.323;		.323.

Of

Of the multiplication of decimal fractions.

19. Multiplication of decimals is alſo performed as in whole numbers, no regard being had to the decimals as ſuch, till the product is obtained; but then, ſo many decimal places muſt be cut off from the right hand of the product, as are contained both in the multiplicator and multiplicand: as for inſtance; let it be required to multiply 4.56 by 2.3: here, conſidering both factors as whole numbers, I multiply 456 by 23, and find the product to be 10488; but then, conſidering that there was one decimal in the multiplicator, and two in the multiplicand, I cut off three decimal places from the right hand of the product, and the true product ſtands thus; 10.488.

To ſhew the reaſon of this operation, let the two factors be reduced to ſimple fractions according to the common way, and we ſhall have 2.3 equal to $\frac{23}{10}$, and 4.56 equal to $\frac{456}{100}$, and theſe two fractions multiplied together make $\frac{10488}{1000}$; divide by 1000, which is done by cutting off the three laſt figures, according to art. the 17th, and the quotient will be 10.488. Another example may be this: let it be required to multiply 45600 by .23: the product of 45600 multiplied by 23 is 1048800: but as there were two decimals in the given multiplicator, and none in the multiplicand; I cut off two decimal places from the laſt product, and the true product will be found to be 10488.00, or 10488. Laſtly, let it be required to multiply .000456 by .23: here, neglecting the initial cyphers in the multiplicand, I multiply 456 by 23, and the product is 10488: then I conſider, that there were two decimal places in the multiplicator, and ſix in the multiplicand, and conſequently that eight decimal places are to be cut off from the laſt product; but the laſt product conſiſts confifts of only 5 places; therefore I place three cyphers to the left hand, with the feparating point before them, and fo make the true product .00010488.

There are various compendiums of this fort of multiplication to be met with in *Oughtred* and others; but they are fuch as, by a little exercife, any one tolerably well grounded in this part of Arithmetic will eafily difcover of himfelf as they lie in his way.

Of the divifion of decimal fractions.

20. Divifion in decimal fractions is performed, firft by confidering them as whole numbers, and dividing accordingly; and then cutting off from the right hand of the quotient, as many decimal places as the dividend hath more than the divifor. The reafon whereof is manifeft from the 19th article: for fince the divifor and quotient multiplied together are to make the dividend, the divifor and quotient ought to have as many decimal places between them, as there are in the dividend; therefore the quotient alone ought to have as many decimal places as the dividend hath more than the divifor.

Example the *1ft*; Let it be propofed to divide 10.488 by 2.3: here dividing the whole number 10488 by the whole number 23, I find the quotient to be 456: but then confidering that there were 3 decimal places in the dividend, and but one in the divifor, I cut off two places from the right hand of the quotient, and fo make the true quotient 4.56.

Example *2d*; Let it be propofed to divide 5678.9 by .06: here, becaufe there are two decimal places in the divifor, and but one in the dividend, I fupply the deficient place by putting a cypher after the dividend, thus, 5678.90; then dividing the whole number 567890 by the whole number 6 (for fince 6 is now confidered as a whole number, the cypher before it may be neglected), I find the quotient to be 94648, which is not to be funk, becaufe the dividend

vidend was made to have as many decimal places as the divisor; but as this quotient is not exact, if for a greater degree of exactness I would continue it to any number of decimal places, suppose 2, instead of one cypher after the divisor, I would have put three, and then the quotient would have come out 94648.33, and this quotient is much more exact than the former, as lying between 94648.33 and 94648.34: but it ought further to be observed concerning this quotient, that if the division was to be continued *in infinitum*, the figures in the decimal places would be all 3's: this is evident from the work; for the two last dividuals are the same, and therefore they must all be the same.

To reduce a vulgar fraction to a decimal fraction.

21. Since every fraction may be considered as the quotient of the numerator divided by the denominator (see art. 13th,) we have an easy rule for reducing a vulgar fraction to a decimal fraction, which is as follows: put as many cyphers after the numerator as are equal in number to the number of decimal places whereof you intend your reduced fraction to consist, and call these cyphers decimal; and then dividing the numerator by the denominator, the quotient will be a decimal number equal to the fraction first proposed, or perhaps a mixt number, if the fraction proposed was an improper one.

Example 1*st*; Let this fraction $\frac{3}{49}$ be proposed to be reduced to a decimal one consisting of four decimal places; here putting 4 decimal cyphers after the numerator 3, I divide 3.0000 by 49, and the quotient uncorrected is 612: but now considering that there were 4 decimal places in the dividend, and none in the divisor, and consequently that four decimal places are to be cut off from the quotient, whereas it consists but of three; I supply this defect of places by a cypher at the left hand, and so make the quotient .0612.

Example

Example 2*d*; Let this fraction $\frac{7}{16}$ be propoſed to be reduced to a decimal fraction, conſiſting, if poſſible, of ſix places: here dividing 7.000000 by 16, I find the true quotient to be .4375, the two laſt cyphers in the dividend being uſeleſs.

Note. When this diviſion runs *ad infinitum*, it will be impoſſible for the reduction to be exact in a finite number of terms; but an approximation may be made, that ſhall come nearer to the quotient than the leaſt aſſignable difference, by taking more and more terms.

To reduce the decimal parts of any integer to ſuch other parts as that integer is uſually divided into.

22. To explain this rule, and to give an example of it at the ſame time; let .345 of a pound ſterling, that is, three hundred forty five thouſandth parts of a pound, be given to be reduced into ſhillings pence and farthings: here then I obſerve, that as any number of pounds, multiplied by 20, will give as many ſhillings as are equal to the pounds, ſo any decimal parts of a pound, multiplied by 20, will give as many ſhillings, and decimal parts of a ſhilling, as are equivalent to the decimal parts of a pound; and ſo on as to pence and farthings: multiplying therefore .345 by 20, the product is 6 and .900, or 6.9, which ſignifies, that .345 of a pound are equivalent to ſix ſhillings and nine tenths of a ſhilling, which is uſually written thus; 6.9 ſhillings: again, multiplying this laſt decimal .9 by 12 for pence, I find that .9 of a ſhilling are equivalent to 10.8 pence: laſtly, multiplying .8 by 4 for farthings, I find that .8 of a penny are equivalent to 3.2 farthings; as for the .2 of a farthing, I neglect it, there being no lower denomination, or at leaſt not intending to deſcend any lower; and ſo I find .345 of a pound to amount to ſix ſhillings and ten pence three farthings.

To

duce 6 ſhillings 10 pence 3.2 farthings into equivalent decimal parts of a pound: one pound contains 960 farthings, or 9600 tenths of a farthing; and 6 ſhillings 10 pence 3.2 farthings contain 3312 tenths of a farthing; therefore 6 ſhillings 10 pence 3.2 farthings are equivalent to $\frac{3312}{9600}$ of a pound; but $\frac{1}{9600}$ being reduced to a decimal, is, .0001 &c. wherein the firſt ſignificant figure is in the 4th place; therefore I reduce the fraction $\frac{3312}{9600}$, to four decimal places, and they amount to .3450, that is, .345 of a pound; ſo that in this particular caſe three decimal places are ſufficient to expreſs exactly the ſum propoſed.

Of the extraction of the ſquare root in decimal fractions.

24. Having treated of the multiplication and diviſion of decimal fractions, it would be altogether needleſs to ſay any thing concerning the rule of proportion, which is but a particular application of both: therefore I ſhall now paſs on to the extraction of the ſquare root, at leaſt ſo far as it concerns decimal fractions. There are but few ſquare numbers, or ſuch as will admit of an exact ſquare root, in compariſon of the reſt; and therefore, whenever a number is propoſed to have its ſquare root extracted, the artiſt muſt firſt determine with himſelf, to how many decimal places it is proper the root ſhould be continued; and then, by annexing decimal cyphers, if need be, to the right hand of the number propoſed, he muſt make twice as many decimal places there as the root is to conſiſt of; after this, he muſt put a point over the place of units, and then, paſſing by every other figure, he muſt point in like manner all the reſt, both to the right hand, and to the left: by this means

means the number will be prepared, and the ſquare root may be extracted as in whole numbers, provided that ſo many decimal places be cut off from the root when obtained, as were firſt deſigned.

Example 1*ſt*; Let the root of 2345.6 be required to two decimal places. The number, when prepared, ſtands thus, 2345.6000, or as a whole number, thus, 23456000; and its ſquare root, when extracted, will be 4843 nearly; and therefore 48.43 will be the root ſought. To try this root 48.43, multiply it into itſelf, and the four firſt figures of the ſquare will be 2345, which are all true; nor can it be expected any more ſhould be ſo, becauſe there were but four places true in the root, no notice being taken of the reſt; but had the root been extracted true to 5 places, that is, to as many places as the original ſquare conſiſted of, it would then have been 48.431; multiply this number into itſelf, and 5 of the firſt figures of the product, taken with the leaſt error, will be 2345.6, which is the original ſquare itſelf.

Example 2*d*; Let the root of .0023456 be required to 5 decimal places. Here putting a cypher in the place of units to direct the punctuation, thus, 0.0023456000, I extract the ſquare root of 23456000 as of a whole number, and find it to be 4843, as above: but, conſidering that this root is to be ſunk 5 places, I put a cypher to the left hand, and ſo make the true root .04843.

That the ſuppoſed ſquare ought to have twice as many decimal places as the root, is evident, both *à priori*, and *à poſteriori*: *à priori*, becauſe in extracting the ſquare root, two figures are brought down from the ſquare for every ſingle figure gained in the root; and *à poſteriori*, becauſe the root multiplied into itſelf is to produce the ſquare; and therefore, from the nature of multiplication, the ſquare ought to have twice as many decimal places as the root.

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ELEMENTS of ALGEBRA.

BOOK I.

The Definition of Algebra.

Art. 1. I SHALL not here detain the young ſtudent with a long hiſtorical account of the riſe and progreſs of Algebra; nor even ſo much as with either the etymology or ſignification of the word; which would contribute but very little to his information, till he has made a further progreſs in the ſcience itſelf, and whereof he will find enough in Dr. *Wallis* and others. Nor indeed is it a ſubject altogether ſo proper at this time to be inſiſted upon; this art, like many others, having now conſiderably outgrown its name, and being often employed in arithmetical operations very different from what its name imports. All I ſhall advance then, by way of definition, is, that *Algebra*, in the modern ſenſe of the word, *is the art of computing by ſymbols*, that is, generally ſpeaking, by letters of the alphabet; which, for the ſimplicity and diſtinctneſs both of their ſounds and characters, are much more commodious for this purpoſe than any other ſymbols or marks whatever.

In this way of notation, it is uſual to ſubſtitute letters not only for ſuch quantities as are unknown, and conſequently ſuch as cannot well be repreſented otherwiſe, but alſo for known quantities themſelves, in order to keep them diſtinct one from another, and to form general concluſions. As for inſtance; ſuppoſe it was demanded of me, what two numbers are thoſe, whoſe ſum is 48, and whoſe difference is 14: here, if I only put *x*, or ſome other letter, for one of the unknown quantities, and uſe the known ones 48 and 14 as I find them in the problem, I ſhall only come to this particular concluſion, to wit, that the greater number is 31, and the leſſer 17, which numbers will anſwer both the conditions of the problem. But if, inſtead of the known numbers 48 and 14, I ſubſtitute the general quantities *a* and *b* reſpectively, and ſo propoſe the problem thus; *What two numbers are thoſe, whoſe ſum is* a, *and whoſe difference is* b? I ſhall then come to this general concluſion, *viz.* that *Half the ſum of* a *and* b *will be the greater number, and half their difference will be the leſs*: which general theorem will ſuit not only the particular caſe abovementioned, but alſo all other caſes of this problem that can poſſibly be propoſed. How I come by theſe two concluſions, will be ſufficiently ſhewn in the courſe of this work; as alſo many other advantages attending this way of ſubſtituting letters for known quantities, beſides thoſe already mentioned.

What I have here ſaid, was only to illuſtrate in ſome meaſure the definition already given of Algebra, and to ſhew, that letters are there uſed, not ſo much to ſignify particular quantities as ſuch, as to ſignify the relation they have to one another in any problem or computation. From all which it may be obſerved, that letters repreſent quantities in Algebra juſt in the ſame manner as they do perſons in common life, when two or more perſons are diſtinctly to be conſidered, with regard to any compact, law-ſuit, or in any other relation whatever.

N. B.

N. B. A single quantity is sometimes represented by two or more letters, when it is considered as the product of the quantities signified by those letters singly: thus *ab* is the product of the multiplication of *a* and *b*; and *abc* is the product arising from the continual multiplication of *a*, *b*, and *c*. But of this more particularly under the head of multiplication.

Of affirmative and negative quantities in algebra.

2. Algebraic quantities are of two sorts, affirmative and negative: an affirmative quantity is a quantity greater than nothing, and is known by this sign +: a negative quantity is a quantity less than nothing, and is known by this sign —: thus + *a* signifies that the quantity *a* is affirmative, and is to be read thus, *plus a*, or more *a*: — *b* signifies that the quantity *b* is negative, and must be read thus, *minus b*, or less *b*.

The possibility of any quantity's being less than nothing is to some a very great paradox, if not a downright absurdity; and truly so it would be, if we should suppose it possible for a body or substance to be less than nothing. But quantities, whereby the different degrees of qualities are estimated, may be easily conceived to pass from affirmation through nothing into negation. Thus a person in his fortunes may be said to be worth 2000 pounds, or 1000, or nothing, or — 1000, or — 2000; in which two last cases he is said to be 1000 or 2000 pounds worse than nothing: thus a body may be said to have 2 degrees of heat, or one degree, or no degree, or — one degree, or — two degrees: thus a body may be said to have two degrees of motion downwards, or one degree, or no degree, or — one degree, or — two degrees, &c. Certain it is, that all contrary quantities do necessarily admit of an intermediate state, which alike partakes of both extremes, and is best represented by a cypher or 0: and if it is proper to ſay, that the degrees on either ſide this common limit are greater than nothing; I do not ſee why it ſhould not be as proper to ſay of the other ſide, that the degrees are leſs than nothing; at leaſt in compariſon to the former. That which moſt perplexes narrow minds, in this way of thinking, is, that in common life, moſt quantities loſe their names when they ceaſe to be affirmative, and acquire new ones ſo ſoon as they begin to be negative: thus we call negative goods, debts; negative gain, loſs; negative heat, cold; negative deſcent, aſcent, *&c*: and in this ſenſe indeed, it may not be ſo eaſy to conceive, how a quantity can be leſs than nothing, that is, how a quantity under any particular denomination can be ſaid to be leſs than nothing, ſo long as it retains that denomination. But the queſtion is, whether, of two contrary quantities under two different names, one quantity under one name may not be ſaid to be leſs than nothing, when compared with the other quantity, though under a different name; whether any degree of cold may not be ſaid to be further from any degree of heat, than is lukewarmth, or no heat at all. Difficulties that ariſe from the impoſition of ſcanty and limited names, upon quantities which in themſelves are actually unlimited, ought to be charged upon thoſe names, and not upon the things themſelves, as I have formerly obſerved upon another occaſion; ſee introduction, art. 11. In Algebra, where quantities are abſtractedly conſidered, without any regard to degrees of magnitude, the names of quantities are as extenſive as the quantities themſelves; ſo that all quantities that differ only in degree one from another, how contrary ſoever they may be one to another, paſs under the ſame name; and affirmative and negative quantities are only diſtinguiſhed by their ſigns, as was obſerved before, and not by their names; the ſame letter repreſenting both: theſe ſigns therefore in algebra carry the ſame diſtinction along with them as do particles and adjectives ſometimes in common language, as in the words convenient and inconvenient, happy and unhappy, good health and bad health, *&c.*

Theſe affirmative and negative quantities, as they are contrary to one another in their own natures, ſo likewiſe are they in their effects; a conſideration which, if duly attended to, would remove all difficulties concerning the ſigns of quantities ariſing from addition, ſubtraction, multiplication, diviſion, *&c:* for the reſult of working by affirmative quantities in all theſe operations is known; and therefore, like operations in negative quantities, may be known by the rule of contraries.

Before we proceed any further, it may not be amiſs to advertiſe, that if a quantity has no ſign before it, it muſt always be taken to be affirmative; and that if it has no numeral coefficient before it, unity muſt always be underſtood: thus $2a$ ſignifies $+2a$, and a ſignifies $1a$ or $+1a$.

By the numeral coefficient of a quantity, I mean, the number or fraction by which that quantity is multiplied: thus $2a$ ſignifies twice a, or a taken twice, and the coefficient is 2: $\frac{3}{4}a$, or $\frac{3a}{4}$ ſignifies $\frac{3}{4}$ of the quantity a, and the coefficient is $\frac{3}{4}$.

N. B. The ſign of a negative quantity is never omitted; nor the ſign of an affirmative one, except when ſuch an affirmative quantity is conſidered by itſelf, or happens to be the firſt in a ſeries of quantities ſucceeding one another: thus we do not often mention the quantity $+a$, but the quantity a; nor the ſeries $+a-b-c+d$, but the ſeries $a-b-c+d$. We ſhall now conſider the ſeveral operations of algebraic quantities.

Of the addition of algebraic quantities.

3. This article I ſhall divide into ſeveral paragraphs: as,

1*st*, Whenever two or more quantities of the ſame denomination, and which have the ſame ſign before them, are to be added together, put down the ſum of their numeral coefficients with the common ſign before it, and the common denominator after it: thus $+2a$ and $+3a$ added together make $+5a$, for the ſame reaſon as 2 dozen and 3 dozen added together make 5 dozen: thus again, $-3ab$, $-4ab$, and $-5ab$, when added together, make $-12ab$; for the ſame reaſon as ſeveral debts added together make a greater debt.

2*d*, If two quantities of the ſame denomination which have different ſigns before them are to be added together, put down only the difference of their numeral coefficients, with the common denominator after it, and the ſign of the greater quantity before it: for in this caſe, the quantities to be added being contrary one to another, the leſs quantity, on which ſide ſoever it lies, will always deſtroy ſo much of the other as is equal to itſelf. Thus $+5a$ added to $-2a$ makes $+3a$; as if a perſon owes me 5000 pounds upon one account, to whom I owe 2000 upon another, the balance upon the whole will be 3000 pounds on my ſide. If it be objected, that this is ſubtraction, and not addition; I anſwer, that the addition of $-2a$ will at any time have the ſame effect as the ſubtraction of $+2a$: but I deny that the addition of $-2a$ is the ſame, or will have the ſame effect as the ſubtraction of $-2a$. Other examples of this caſe may be theſe; $+7a$ added to $-7a$ gives 0; $+3a$ added to $-12a$ gives $-9a$; $+a$ added to $-5a$ gives $-4a$; $+5a$ added to $-a$ gives $+4a$; $+\frac{1}{3}a$ added to $-\frac{1}{4}a$ gives $+\frac{1}{12}a$, &c.

3*d*, When many quantities of the ſame denomination are to be added together, whereof ſome are affirmative and ſome negative, reduce them firſt to two, by adding all the affirmative quantities together, and all the negative ones, and then to one by the laſt paragraph. Thus $+10a-9a+8a-7a$, when added together,

together, make $2a$; for $+10a$ and $+8a$ make $+18a$, $-9a$ and $-7a$ make $-16a$; and $+18a$ and $-16a$ make $+2a$.

4th, Quantities of different denominations will not incorporate, and therefore cannot otherwiſe be added together, than by placing them in any order one after another, with their proper ſigns before them, except the firſt, whoſe ſign, if affirmative, may be omitted. Thus $+2a$ and $-3b$ and $+4c$ and $-5d$, when added together, make $2a-3b+4c-5d$: thus a and b added together make $a+b$; and hence it is, that whenever two quantities are found with this ſign $+$ betwixt them, it ſignifies the ſum ariſing from the addition of thoſe two quantities together: thus if a ſtands for 7, and b ſtands for 3, $a+b$ will ſtand for 10, and ſo of the reſt: but if $-b$ is to be added to a, the ſum muſt be written down thus, $a-b$; for to add $-b$, is the ſame as to ſubtract $+b$.

5th, Compound quantities, whoſe members are all of different denominations, are likewiſe incapable of being added any other way, than by being placed one after another without altering their ſigns: thus $3a+4b$ added to $5c-6d$ can only make $3a+4b+5c-6d$. But if the members are not all of different denominations, it may then be convenient to place one compound quantity under another, with like parts under like, as far as it can be done, as in the following examples;

$$\begin{array}{ll} a+b & \\ a-b & \\ \hline 2a & * .\dagger \end{array}$$

† For a and a added together make $2a$; and $+b$ and $-b$ added together deſtroy one another, and ſo make 0 or *; which character in Algebra is always uſed to ſignify a vacant place.

$$\begin{array}{l} 2x-3a+4b-5c+6d-7e \quad * \\ 10x+9a-8b-7c-6d \quad * -5f \\ \hline 12x+6a-4b-12c \quad * -7e-5f. \end{array}$$

Note, That in the addition, ſubtraction, and multiplication of compound algebraic quantities, it matters little which way the work is carried on, whether from right or left, or from left to right, becauſe here are no reſerves made for higher places.

Of the ſubtraction of algebraic quantities.

4. Whenever a ſingle algebraic quantity is to be ſubtracted from another quantity, whether ſimple or compound, firſt change the ſign of the quantity to be ſubtracted, that is, if it be affirmative, make it, or at leaſt call it, negative, and *vice verſa*, and then add it ſo changed to the other: for ſince (as was before hinted) the ſubtracting of any one quantity from another, is the ſame in effect as adding the contrary; and ſince changing the ſign of the quantity to be ſubtracted, renders that quantity juſt contrary to what it was before, it is evident, that after ſuch a change it may be added to the other, and that the reſult of this addition will be the ſame with that of the intended ſubtraction. Thus may the rule of ſubtraction, by changing the ſign of the quantity to be ſubtracted, be at any time changed into that of addition, juſt as the rule of diviſion in fractions, by inverting the terms of the diviſor, was changed into that of multiplication. As for example, $+b$ ſubtracted from a leaves $a-b$, becauſe $-b$ added to a makes $a-b$; ſo that $a-b$ may be conſidered either as the ſum of a and $-b$ added together, or as the remainder of $+b$ ſubtracted from a, or as the difference between a and b, or as the exceſs of a above b, all which amount to the ſame thing: as if a ſignifies 7, and b 3, $a-b$ muſt ſtand for 4, and ſo of the reſt.

The rule of ſubtraction here given is univerſal, though there will not be always occaſion to have recourſe to it: for ſuppoſe $3a$ is to be ſubtracted from $7a$, every one's common ſenſe will inform him, that there

there muſt remain $4a$, juſt as threeſcore ſubtracted from ſevenſcore leaves fourſcore.

Other examples of algebraic ſubtraction may be theſe that follow.

1ſt, $7a$ ſubtracted from $5a$ leaves $-2a$, becauſe $-7a$ added to $+5a$ makes $-2a$, by the 2d paragraph of the laſt article.

2d, $9a$ ſubtracted from o leaves $-9a$, becauſe $-9a$ added to o makes $-9a$.

3d, $12a$ ſubtracted from $-3a$ leaves $-15a$, becauſe $-12a$ added to $-3a$ makes $-15a$, by the firſt paragraph of the laſt article.

4th, $-3a$ ſubtracted from $-8a$ leaves $-5a$, becauſe $+3a$ added to $-8a$ makes $-5a$.

5th, $-7a$ ſubtracted from $-3a$ leaves $+4a$, becauſe $+7a$ added to $-3a$ makes $+4a$.

6th, $-6a$ ſubtracted from o leaves $+6a$, becauſe $+6a$ added to o makes $+6a$.

7th, $-5a$ ſubtracted from $+5a$ leaves $+10a$, becauſe $+5a$ added to $+5a$ makes $+10a$.

8th, $-b$ ſubtracted from a leaves $a+b$, becauſe $+b$ added to a makes $a+b$, by the 4th paragraph of the laſt article.

9th, -2 ſubtracted from 7 leaves 9, becauſe $+2$ added to 7 makes 9.

From the firſt of theſe examples it appears, that a greater quantity may be taken out of a leſs, but then the remainder will be negative; juſt as a gameſter that has but 5 guineas about him may loſe 7, but then there will remain a debt of 2 guineas upon him. By the laſt example it appears, that -2 ſubtracted from 7 leaves 9, that is, that if a negative quantity be ſubtracted from an affirmative one, the affirmative quantity will be ſo far from being diminiſhed thereby, that it will be increaſed; a principle which I fear will be found ſomewhat hard of digeſtion, eſpecially by weak conſtitutions: therefore,

therefore, to ſtrengthen my patient as far as lies in my power, I ſhall ſuggeſt to him the following conſiderations:

1ſt, In any ſubtraction, if the remainder and the leſs number added together make the greater, the ſubtraction is juſt: but in our caſe, the remainder 9 added to the leſs number -2 makes the greater number 7; therefore -2 ſubtracted from 7 leaves 9.

2dly, In all ſubtraction whatever, the remainder is the difference betwixt the greater number and the leſs; but the difference between $+7$ and -2 is 9; therefore -2 ſubtracted from $+7$ leaves 9.

3dly, 7 is equal to $\overline{9-2}$ by the ſecond paragraph of the laſt article; therefore -2 ſubtracted from 7 will have the ſame remainder as -2 ſubtracted from $\overline{9-2}$: but -2 ſubtracted from $\overline{9-2}$ leaves 9; therefore -2 ſubtracted from 7 leaves 9. In ſhort, the taking away a defect, in any caſe whatever, will amount to the ſame as adding ſomething real: as if an eſtate be incumbered with a mortgage or a rent-charge upon it, whoever takes off the incumbrance juſt ſo much encreaſes the value of the eſtate.

4thly, The leſs there is taken from 7, the more will be left; if nothing be taken, there will remain 7; therefore if leſs than nothing be taken, there ought to remain more than 7.

5thly, If, after all that has been ſaid, or perhaps all that can be ſaid in this abſtracted way, ſome ſcruples ſtill remain, let us apply the principle we have already advanced, and try whether we ſhall meet with any better ſucceſs that way. Let it then be required to ſubtract the compound quantity $a-2$ from the compound quantity $6a+7$: in order to this, I place a under $6a$, and -2 under 7, and then ſubtract as follows; a from $6a$ and there remains $5a$, -2 from 7 and (if our aſſertion be true) there remains 9; therefore the whole remainder is

$5a+9$. Now I dare appeal to every one's common sense, whether this subtraction be not just: for certain it is, that if a be subtracted from $6a+7$, the remainder will be $5a+7$; and if so, then it is as certain, that if $a-2$ be subtracted, which is less than the former by 2, the remainder will be greater by 2, that is, $5a+9$. But to proceed:

Other examples of the subtraction of compound algebraic quantities may be these.

$a+b$	$*+12$
$a-b$	$3a+7$
$*+2b$‡	$-3a+5$.

‡Thus $7-3$, or 4, subtracted from $7+3$, or 10, leaves twice 3, or 6.

From	$12x$	$+6a$	$-4b$	$-12c$	$*$	$-7e$	$-5f$
Take	$2x$	$-3a$	$+4b$	$-5c$	$+6d$	$-7e$	$-*$
Remains	$10x$	$+9a$	$-8b$	$-7c$	$-6d$	$*$	$-5f$
Proof	$12x$	$+6a$	$-4b$	$-12c$	$*$	$-7e$	$-5f$.

If never a member of the subtrahend be found to be of the same denomination with any member of the number from whence the subtraction is to be made, change the sign of every member of the subtrahend, and then add it to the other. As if $5c-6d$ is to be subtracted from $3a-4b$, first change the sign of $5c-6d$, and make it $-5c+6d$, and then add it to the other, and you will have $3a-4b-5c+6d$ for the remainder.

Of the multiplication of algebraic quantities.

And first, how to find the sign of the product in multiplication, from those of the multiplicator and multiplicand given.

5. Before we can proceed to the multiplication of algebraic quantities, we are to take notice, that if

if the ſigns of the multiplicator and multiplicand be both alike, that is, both affirmative, or both negative, the product will be affirmative, otherwiſe it will be negative: thus + 4 multiplied into + 3, or — 4 into — 3, produces in either caſe + 12: but — 4 multiplied into + 3, or + 4 into — 3 produces in either caſe — 12.

If the reader expects a demonſtration of this rule, he muſt firſt be advertiſed of two things: *firſt*, that numbers are ſaid to be in arithmetical progreſſion when they increaſe or decreaſe with equal differences, as 0, 2, 4, 6; or 6, 4, 2, 0; alſo as 3, 0, — 3; 4, 0, — 4; 12, 0, — 12; or — 12, 0, + 12: whence it follows, that three terms are the feweſt that can form an arithmetical progreſſion; and that of theſe, if the two firſt terms be known, the third will eaſily be had: thus, if the two firſt terms be 4 and 2, the next will be 0: if the two firſt be 12 and 0, the next will be — 12; if the two firſt be — 12 and 0, the next will be + 12, &c.

2*dly*, If a ſet of numbers in arithmetical progreſſion, as 3, 2, and 1, be ſucceſſively multiplied into one common multiplicator, as 4, or if a ſingle number, as 4, be ſucceſſively multiplied into a ſet of numbers in arithmetical progreſſion, as 3, 2, and 1, the products 12, 8, and 4, in either caſe, will be in arithmetical progreſſion.

This being allowed (which is in a manner ſelf-evident), the rule to be demonſtrated reſolves itſelf into four caſes:

1*ſt*, That + 4 multiplied into + 3 produces + 12.

2*dly*, That — 4 multiplied into + 3 produces — 12.

3*dly*, That + 4 multiplied into — 3 produces — 12.

And *laſtly*, that — 4 multiplied into — 3 produces + 12. Theſe caſes are generally expreſſed in ſhort thus: firſt + into + gives +; ſecondly — into + gives

gives —; thirdly + into — gives —; fourthly — into — gives +.

Caſe 1ſt, That + 4 multiplied into + 3 produces + 12, is ſelf-evident, and needs no demonſtration; or, if it wanted one, it might receive it from the firſt paragraph of the third article; for to multiply + 4 by + 3 is the ſame thing as to add 4 + 4 + 4 into one ſum; but 4 + 4 + 4 added into one ſum give + 12, therefore + 4 multiplied into + 3 gives + 12.

Caſe 2d. And from the ſecond paragraph of the 3d art. it might in like manner be demonſtrated, that—4 multiplied into+3 produces—12: but I ſhall here demonſtrate in another way, thus: multiply the terms of this arithmetical progreſſion 4, 0, —4, into + 3, and the products will be in arithmetical progreſſion, as above; but the two firſt products are 12 and 0; therefore the third will be — 12; therefore — 4 multiplied into + 3 produces — 12.

Caſe 3d. To prove that + 4 multiplied into — 3 produces — 12; multiply + 4 into + 3, 0, and — 3 ſucceſſively, and the products will be in arithmetical progreſſion; but the two firſt products are 12 and 0, therefore the third will be — 12; therefore + 4 multiplied into — 3 produces — 12.

Caſe 4th. Laſtly, to demonſtrate, that — 4 multiplied into — 3 produces + 12, multiply — 4 into 3, 0, and — 3 ſucceſſively, and the products will be in arithmetical progreſſion; but the two firſt products are — 12 and 0, by the ſecond caſe; therefore the third product will be + 12; therefore — 4 multiplied into — 3 produces + 12.

Caſ. 2d, $\begin{array}{r} +4,\ \ 0,\ -4 \\ +3,\ +3,\ +3 \\ \hline +12,\ \ 0,\ -12. \end{array}$ *Caſ. 3d*, $\begin{array}{r} +4,\ +4,\ +4 \\ +3,\ \ 0,\ -3 \\ \hline +12,\ \ 0,\ -12. \end{array}$

Caſ. 4th, $\begin{array}{r} -4,\ -4,\ -4 \\ +3,\ \ 0,\ -3 \\ \hline -12\ \ 0,\ +12. \end{array}$

These

These 4 cases may be also more briefly demonstrated thus: + 4 multiplied into + 3 produces + 12; therefore — 4 into + 3, or + 4 into — 3 ought to produce something contrary to + 12, that is, — 12: but if — 4 multiplied into + 3, produces — 12, then — 4 multiplied into — 3 ought to produce something contrary to — 12, that is, + 12; so that this last case, so very formidable to young beginners, appears at last to amount to no more than a common principle in Grammar, to wit, that two negatives make an affirmative; which is undoubtedly true in Grammar, though perhaps it may not always be observed in languages.

Of the multiplication of simple algebraic quantities.

6. These things premised, the multiplication of simple algebraic quantities is performed, first by multiplying the numeral co-efficients together, and then putting down, after the product, all the letters in both factors, the sign (when occasion requires) being prefixed as above directed. Thus $4b$ multiplied into $3a$ produces $12ab$.

Though this kind of language (for it is no more) like all others, be purely arbitrary, yet that a more rational one could not have been invented for this purpose, will appear by the following consideration. If any quantity, as b, is to be multiplied by any number, as 2, 3, or 4, the product cannot be better represented than by $2b$, $3b$, $4b$, &c; therefore if b is to be multiplied by a, the product ought to be called ab: but if b multiplied into a produces ab, then $4b$ multiplied into a ought to produce 4 times as much, that is, $4ab$, lastly, if $4b$ multiplied into a produces $4ab$, then $4b$ multiplied into $3a$ ought to produce 3 times as much, that is, $12ab$.

Hence it is, that whenever in Algebra two or more letters are found together, as they stand in a word, without any thing between them, they signify the product arifing from a continual application of the quantities reprefented by them: thus ab fignifies the product of a and b multiplied together; and abc fignifies the product of the quantity ab multiplied into c: thus aa fignifies the product of a multiplied into itfelf, or the fquare of a, and not $2a$; and therefore whoever fhews himfelf unable to diftinguifh betwixt $2a$ and aa, difcovers as great a weaknefs as one that is not able to diftinguifh betwixt 2 dozen and a dozen dozen or 12 times 12.

It is a matter of no great confequence in what order the letters are placed in a product; for ab and ba differ no more from one another than 3 times 4, and 4 times 3: and yet it is convenient that a method be obferved, left like quantities be fometimes taken for unlike; therefore the beft way will be, to give thofe letters the precedency in a product, that have it in the alphabet; except when an unknown quantity is multiplied by fome known one, and then it is ufual to place the known quantity before it.

Note. For the fignification of this mark $\times$, fee introduct. at the clofe of the 7th article. Note alfo, that this mark $=$ is a mark of equality, fhewing that the quantities between which it ftands, are equal to each other, and muft be read as the fenfe requires: thus $2\times6=3\times4=12$ may be read thus; 2×6 equal 3×4 equal to 12: or thus; 2×6 is equal to 3×4, which is equal to 12.

Examples of fimple algebraic multiplication.

1ft, $4ab\times5a=20aab$. 2d, $-5ab\times6bc=-30abbc$.
3d, $6ac\times-7bd=-42abcd$. 4th, $-7a\times-b=+7ab$.
5th, $x\times3x=3xx$. 6th, $-x\times-x=+xx$.
7th, $-5ab\times+3=-15ab$. 8th, $\frac{2}{3}a\times\frac{4}{5}b=\frac{8}{15}ab$.

Diftinctions

Distinctions to be observed betwixt addition and multiplication.

That the young algebraist may not confound the operations of addition and multiplication, as is frequently done; I shall here set down some marks of distinction, which he ought to attend to:

As *first*, a added to a makes $2a$, but a multiplied, into a makes aa.

2dly, a added to o makes a, but a multiplied into o makes o.

3dly, a added to $-a$ makes o, but a multiplied into $-a$ makes $-aa$.

4thly, $-a$ added to $-a$ makes $-2a$, but $-a$ multiplied into $-a$ makes $+aa$.

5thly, a added to 1 makes $a+1$, but a multiplied into 1 makes a.

6thly, $2a$ added to $-3b$ makes $2a-3b$, but $2a$ multiplied into $-3b$ makes $-6ab$.

For a further confirmation of the learner, I have added, by way of exercise in his algebraic language, the following equations; which I desire he would compute after me. Suppose $a=7$, and $b=3$: then we shall have 1st, $a+b=10$. 2dly, $a-b=4$. 3dly, $4a+5b=43$. 4thly, $4a-5b=13$. 5thly, $aa=49$. 6thly, $ab=21$. 7thly, $bb=9$. 8thly, $aaa=343$. 9thly, $aab=147$. 10thly, $abb=63$. 11thly, $bbb=27$. 12thly, $aa+2ab+bb=49+42+9=100$. 13thly, $aa-2ab+bb=49-42+9=16$. 14thly, $aaa+3aab+3abb+bbb=343+441+189+27=1000$. 15thly, $aaa-3aab+3abb-bbb=343-441+189-27=64$.

Of powers and their indexes.

7. Whenever in multiplication a letter is to be repeated oftener than once, it is usual, by way of compendium, to write down the letter with a small figure after

after it, ſhewing how often that letter is to be repeated: thus inſtead of xx we write x^2, inſtead of xxx we write x^3, inſtead of $xxxx$ we write x^4, &c. Theſe products are called powers of x; the figures repreſenting the number of repetitions are called the indexes of thoſe powers; and the quantity x, from whence all theſe powers ariſe, is called the root of theſe powers, or the firſt power of x; x^2 is called the ſecond power of x, x^3 the third power, x^4 the fourth power, &c. *Vieta*, *Oughtred*, and ſome other analyſts, inſtead of ſmall letters uſed capitals; and inſtead of numeral indexes, diſtinguiſhed theſe powers by names: thus *Vieta* in particular called x^2, *X ſquare*; x^3, *X cube*; x^4, *X ſquare-ſquare*; x^5, *X ſquare-cube*; x^6, *X cube-cube*; x^7, *X ſquare-ſquare-cube*, &c: which names *Oughtred* contracted, and wrote them thus; *Xq*, *Xc*, *Xqq*, *Xqc*, *Xcc*, *Xqqc*, &c.: but now theſe names are pretty much out of uſe, except the two firſt, when applied to a line ſquared or cubed.

If we ſuppoſe $x=5$, we ſhall have $2x=10$, $x^2=25$, $3x=15$, $x^3=125$, $4x=20$, $x^4=625$, &c.

The multiplication of theſe powers is eaſy: thus $x^2 \times x^3 = x^5$, becauſe $xx \times xxx = xxxxx$: whence it may be obſerved, that the addition of indexes will always anſwer to the multiplication of powers, provided they be powers of the ſame quantity; for as $2+3=5$, ſo $x^2 \times x^3 = x^5$, &c: but if they be powers of different quantities, their indexes muſt not be added: thus $a^2 \times x^3 = a^2x^3$, and $a^2x^3 \times a^4x^5 = a^6x^8$. And here it muſt be obſerved, that if a number be found between two letters, it muſt always be referred to the former letter; thus a^2x^3 does not ſignify $a \times 2x^3$, but $a^2 \times x^3$.

The multiplication of ſurds.

8. This mark $\surd$ ſignifies the ſquare root of the number to which it is prefixed, and is generally prefixed to numbers whoſe ſquare root cannot be otherwiſe expreſſed, either by whole numbers or fractions:

thus $\sqrt{2}$ ſignifies the ſquare root of 2; $\sqrt{a}$ the ſquare root of a, &c. Theſe roots are commonly called ſurd roots, or irrational roots, becauſe their proportion to unity cannot be expreſſed in numbers.

Whenever two ſurd numbers are to be multiplied together, the ſhorteſt way will be, to multiply the numbers themſelves one into the other without any regard to the radical ſign, and then to prefix the radical ſign to the product. Thus if $\sqrt{a}$ is to be multiplied into $\sqrt{b}$, the product will be $\sqrt{ab}$; which I thus demonſtrate: let $\sqrt{a}=x$, and $\sqrt{b}=y$; then will $x^2=a$, and $y^2=b$, and $x^2y^2=ab$, and $xy=\sqrt{ab}$; but xy, or $x\times y=\sqrt{a}\times\sqrt{b}$ by the ſuppoſition; therefore, $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$. Thus $\sqrt{2}\times\sqrt{3}=\sqrt{6}$.

Theſe multiplications are of conſiderable uſe, not only in matters of ſpeculation, but alſo in practice: for ſuppoſe I had occaſion to multiply the ſquare root of 2 into the ſquare root of 3, if I had not this rule, I muſt firſt extract the root of 2, to what degree of exactneſs I think proper for my purpoſe: then again I muſt extract the root of 3 to the ſame degree of exactneſs, and laſtly I muſt multiply theſe two roots together, before I can obtain the number wanted: but after it is known that $\sqrt{2}\times\sqrt{3}=\sqrt{6}$, the whole operation will then be reduced to the extraction of the root of 6 only: nay it ſometimes happens, that two roots, though both irrational, ſhall have a rational product: thus $\sqrt{2}\times\sqrt{8}=\sqrt{16}=4$, and $\sqrt{ab^2}\times\sqrt{ac^2}=\sqrt{a^2b^2c^2}=abc$.

Of the multiplication of compound algebraic quantities.

9. The multiplication of compound algebraic quantities is performed, firſt by multiplying the multiplicand into every particular member of the multiplicator, and then reducing the whole product into the leaſt compaſs poſſible.

As for example; let it be required to multiply this compound quantity $6x-7a-8b$ into this compound quantity $2x-3a+4b$: here having put down the multiplicand, and the multiplicator under it, and beginning at the left hand (for it is all one which way the operation is carried on), I multiply the whole multiplicand into $2x$, the firſt member of my multiplicator, and the product is $12xx-14ax-16bx$, which I put down: then I multiply the multiplicand into $-3a$, the next member of the multiplicator, and the product is $-18ax+21aa+24ab$; whereof the firſt member $-18ax$, I place under $-14ax$ before found, being of the ſame denomination, for the conveniency of adding; the reſt, to wit, $+21aa+24ab$, I place in the firſt line: this done, I now multiply by $4b$, the laſt member of the multiplicator, and the product is $24bx-28ab-32bb$; whereof I place $24bx$ under $-16bx$, and $-28ab$ under $+24ab$, and the laſt member $-32bb$ I place in the firſt line, as having no quantity of the ſame denomination to join with it: laſtly I reduce the whole product into the leaſt compaſs poſſible; and it ſtands thus: $12xx-32ax+8bx+21aa-4ab-32bb$. See the work:

$$\begin{array}{l} 6x-7a-8b \\ 2x-3a+4b \\ \hline 12xx-14ax-16bx+21aa+24ab-32bb \\ \qquad -18ax+24bx \qquad -28ab \\ \hline \end{array}$$

$$\text{Sum } 12xx-32ax+8bx+21aa-4ab-32bb.$$

Example 2d.

$$\begin{array}{l} 3x+4a-5b \\ 3x-4a+5b \\ \hline 9xx+12ax-15bx-16aa+20ab-25bb \\ \qquad -12ax+15bx \qquad +20ab \\ \hline 9xx \quad * \quad * \quad -16aa+40ab-25bb. \end{array}$$

Example 3d.

$$\begin{array}{l} 6xx-7ax+8aa \\ 2xx-3ax+4aa \\ \hline 12x^4-14ax^3+16a^2x^2-24a^3x+32a^4 \\ \quad -18ax^3+21a^2x^2-28a^3x \\ \quad\quad +24a^2x^2 \\ \hline 12x^4-32ax^3+61a^2x^2-52a^3x+32a^4 \end{array}$$

Example 4th.	Example 5th.	Example 6th.
$a+b$	$a+b$	$a-b$
$a-b$	$a+b$	$a-b$
$aa+ab-bb$	$aa+ab+bb$	$aa-ab+bb.$
$-ab$	$+ab$	$-ab$
aa *$-bb.$	$aa+2ab+bb.$	$aa-2ab+bb.$

N. B. A dash over two or more quantities signifies that all those quantities are to be taken into one conception, or to be considered as making up but one compound quantity: thus $\overline{a+b}\times\overline{c-d}$ does not signify that which arises from multiplying $b\times c$, and then adding $a-d$ to the product, as it might be mistaken without the dash; but it signifies the product of the whole quantity $\overline{a+b}$ multiplied into the whole quantity $\overline{c-d}$.

The proof of compound multiplication.

10. In the 3d example we multiplied $\overline{6xx-7ax+8aa}$ into $\overline{2xx-3ax+4aa}$, and the product amounted to $12x^4-32ax^3+61a^2x^2-52a^3x+32a^4$: let us try this in numbers, and see how it will answer. In order to which, we may suppose a and x equal to any two numbers whatever; but the simplest way of tryal will be to make

make a equal 1, and $x=1$; and then we ſhall have in the multiplicand $6xx=6$, $-7ax=-7$, and $+8aa=+8$, and $6-7+8=7$: therefore the multiplicand is 7: again, in the multiplicator we have $2xx=2$, $-3ax=-3$, $+4aa=+4$, and $2-3+4=3$; therefore the multiplicator is 3: and 7 the multiplicand, multiplied into 3 the multiplicator, gives 21 for the product. Let us now examine the ſeveral parts of the product, as they are here repreſented in letters, and ſee whether they will amount to that number: $12x^4=12$, $-32ax^3=-32$, $+61a^2x^2=+61$, $-52a^3x=-52$, $+32a^4=+32$; and $12-32+61-52+32$ amount to juſt 21. This may ſerve as a proof to the work, though not a neceſſary one: for it is not impoſſible but there may be a conſiſtency this way, and yet the work be falſe; but this will rarely happen, unleſs it be deſigned. But the work may ſtill be confirmed by making $a=1$, and $x=-1$; for then the multiplicand will be $6+7+8=21$; and the multiplicator $2+3+4=9$; and the product $12+32+61+52+32=189$, which is the ſame with the product of 21 the multiplicand, multiplied into 9 the multiplicator.

How general theorems may be obtained by multiplication in Algebra.

11. From theſe algebraic multiplications are derived and demonſtrated many very uſeful theorems in all the parts of Mathematicks; whereof I ſhall juſt give the learner a taſte, and then proceed to another ſubject.

In the fourth example of compound multiplication we found, that $a+b$ multiplied into $a-b$ produced $aa-bb$; whence I infer, that *The ſum and difference of any two numbers multiplied together will give the difference of their ſquares, and* vice verſâ: for a and b will repreſent any two numbers at pleaſure; $a+b$ their ſum, $a-b$ their difference, and $aa-bb$ the difference of their ſquares: thus, if we aſſume any two numbers whatever, ſuppoſe 7 and 3, the difference of their ſquares is 49—9, or 40; and 10 their ſum, multiplied into 4 their difference, makes alſo 40.

But here I am to give notice once for all, that inſtances in numbers ſerve well enough to illuſtrate a general theorem, but they muſt not by any means be looked upon as a proof of it; becauſe a propoſition may be true in ſome particular caſes inſtanced in, and yet fail in others; but whenever a propoſition is found to be true *in ſpeciebus*, that is, in letters or ſymbols, it is a ſufficient demonſtration of it, becauſe theſe are univerſal repreſentations.

In the 5th example it was ſhewn, that $a+b$ multiplied into itſelf produced $aa+2ab+bb$; whence I infer, that *If a number be reſolved into any two parts whatever, the ſquare of the whole will be equal to the ſquare of each part, and the double rectangle, or product of the multiplication of thoſe parts, added together:* thus if the number 10 be reſolved into 7 and 3; 100 the ſquare of 10, the whole, will be equal to 49 the ſquare of 7, and 9 the ſquare of 3, and 42 the double product of 7 and 3 multiplied together: for $49+9+42=100$.

In the 6th example we found, that $a-b$ multiplied into itſelf, produced, $aa-2ab+bb$; whence I infer, that *If from the ſum of the ſquares of any two numbers, be ſubtracted the double product of thoſe numbers, there will remain the ſquare of their difference:* for $aa+bb$ is the ſum of the ſquares of a and b, and $2ab$ is their double product, and $aa-2ab+bb$ was found to be the ſquare of $a-b$, that is, the ſquare of the difference of a and b: thus in the numbers 7 and 3, the ſquare of 7 is 49, the ſquare of 3 is nine, and the ſum of their ſquares is 58; and if from this be ſubtracted the double product 42, the remainder will be 16, the ſquare of 4, that is, the ſquare of the difference of the numbers 7 and 3.

Theſe two laſt theorems are in ſubſtance the fourth and ſeventh propoſitions of the ſecond book of *Euclid*;

Of the division of simple algebraic quantities.

13. The divifion of fimple algebraic quantities, where it is poffible in integral terms, is performed, firft by dividing the numeral coefficient of the dividend by the numeral coefficient of the divifor, and then putting down after the quotient all the letters in the dividend, that are not in the divifor; the fign of the quotient in divifion being determined by thofe of the divifor and dividend, juft in the fame manner as the fign of the product in multiplication is determined by thofe of the multiplicator and multiplicand; that is, if the figns of the divifor and dividend be both alike, whether they be both affirmative, or both negative, the quotient will be affirmative, otherwife it will be negative: thus if the quantity $-12ab$ is divided by $-3a$, the quotient will be $+4b$; which I thus demonftrate: In all divifion whatever, the quotient ought to be fuch a quantity as, being multiplied by the divifor, will make the dividend; therefore, to enquire for the quotient in our cafe, is nothing elfe, but to enquire what number or quantity, multiplied into $-3a$, the divifor, will produce $-12ab$, the dividend. Firft then I afk, what fign multiplied into $-$, the fign of the divifor, will give $-$, the fign of the dividend, and the anfwer is $+$; therefore $+$ is the fign of the quotient: in the next place I enquire, what number multiplied into 3, the coefficient of the divifor, will give 12, the coefficient of the dividend, and the anfwer is 4; therefore 4 is the coefficient of the quotient: laftly I enquire, what letter multiplied into a, the letter of the divifor, will produce ab, the denominator, or literal part of the dividend, and the anfwer is b; therefore b is the letter of the quotient: and thus at laft we have the whole quotient, which is $+4b$. And this way of reafoning will carry the learner through all the other cafes.

Examples of ſimple diviſion in Algebra.

Example 1ſt, $4ab)\ 24abbc\ (6bc.$
2d, $+7)\ -35ab\ (-5ab.$
3d, $-x)\ -3xx\ (+3x.$
4th, $-9ab)\ +72ab\ (-8.$
5th, $-4a^3)\ -60a^8\ (+15a^5.$
6th, $4x^2)\ 60x^9\ (+15x^7.$
7th, $+4a^3x^2)-60a^8x^9\ (-15a^5x^7.$
8th, $b)\ \frac{3}{4}ab\ (\frac{3}{4}a.$
9th, $\frac{2}{3})\ \frac{4}{5}b\ (\frac{6}{5}b.$

Of the notation of algebraic fractions.

Whenever a diviſion according to the foregoing method is found impoſſible, the quotient cannot be otherwiſe expreſſed than by a fraction, whoſe numerator is the dividend, and denominator the diviſor; ſee the introduction, art. 13. As, if it was required to divide a by b, which diviſion is impoſſible according to the foregoing rule, the quotient muſt be expreſſed by this fraction $\frac{a}{b}$, which is uſually read thus, a by b, that is, a divided by b, or the quotient of a divided by b: for in Algebra the word *by* is, generally ſpeaking, appropriated to diviſion, as the word *into* is to multiplication.

If the numerator, or denominator, or both, be compound quantities, the reſpective fractions muſt be written thus $\frac{a+b}{c}$, $\frac{a}{b+c}$, $\frac{a+b}{c-d}$.

If a diviſion be partly poſſible according to the foregoing rules, and partly impoſſible, it muſt be purſued as far as it is poſſible, and the reſt muſt be repreſented by a fraction, as in common diviſion: thus if $ad+bd+c$ was to be divided by d, the quotient would be $a+b+\frac{c}{d}$.

Of

Of proportion in numbers.

15. The rule of proportion in Algebra is fo very little different from the rule of proportion in common arithmetick, that one example of it will be fufficient. Let then the following queftion be put: *If* a *gives* b, *what will* c *give?* Here the fecond and third terms multiplied together produce bc; and the quotient of this, divided by the firft term a, cannot otherwife be expreffed than by the fraction $\frac{bc}{a}$: this is evident from the notation of fractions explained in the 13th article. But as I have hitherto purpofely avoided all confideration of proportion, chufing rather to appeal, upon all occafions, to the common idea every one has or thinks he has of it, than to be more particular, it may not be improper, now we come to reafon more clofely upon things, to enter more diftinctly into the particular nature of proportion, fo far at leaft as it relates to numbers, and fhew wherein it confifts.

According to *Euclid*, four numbers are faid to be proportionable, that is, the firft number is faid to have the fame proportion to the fecond, that the third hath to the fourth; or the firft is faid to be to the fecond, as the third is to the fourth, when the firft number is the fame multiple, part or parts, of the fecond, that the third is of the fourth: but it will be afked perhaps, How can we know, what parts, part, or multiple, any one number is of another? To which I anfwer, by a fraction, whofe numerator is the former number, and denominator the latter: thus the fraction $\frac{2}{3}$ expreffly fhews, that the numerator 2 is two third parts of the denominator 3; for this is certain, that 1 is $\frac{1}{3}$ part of 3, and therefore 2 muft be $\frac{2}{3}$ of it: for the fame reafon the fraction $\frac{12}{8}$ fhews that the number 12 is $\frac{12}{8}$ or $\frac{3}{2}$ of the number 8; and laftly, the fraction $\frac{12}{5}$ fhews that the number

ber 12 is $\frac{12}{4}$ of, or 3 times the number 4; and consequently, that 12 is a multiple of 4, as containing it juſt 3 times without any remainder: therefore, to any one who underſtands fractions, *Euclid's* definition of proportion may be more diſtinctly expreſſed thus: *Four numbers are ſaid to be proportionable, when a fraction whoſe numerator is the firſt number, and denominator the ſecond, is equal to a fraction whoſe numerator is the third number, and denominator the fourth.* Thus 2 is to 3 as 4 is to 6, becauſe $\frac{2}{3}$ is equal to $\frac{4}{6}$; thus 12 is to 8 as 15 is to 10, becauſe $\frac{12}{8}$ equals $\frac{15}{10}$, both being reducible to $\frac{3}{2}$; thus 2 is to 6 as 4 is to 12, becauſe $\frac{2}{6}$ equals $\frac{4}{12}$, for each is equal to $\frac{1}{3}$; laſtly, 6 is to 2 as 12 is to 4, becauſe $\frac{6}{2} = \frac{12}{4} = 3$.

From this idea of proportionality may be demonſtrated a very uſeful theorem in Algebra; which is, that *Whenever four numbers are proportionable, the product of the extreme terms multiplied together will be equal to the product of the two middle terms ſo multiplied:* for let a, b, c, and d, be four proportionable numbers in their order; that is, let a be to b as c is to d; I ſay then that $a\,d$ the product of the extremes will be equal to bc the product of the two middle terms: for ſince a is to b as c is to d, it follows from what has already been laid down, that the fraction $\frac{a}{b}$ is equal to the fraction $\frac{c}{d}$; multiply both the terms of the fraction $\frac{a}{b}$ into d, and both thoſe of the fraction $\frac{c}{d}$ into b (which multiplications may be made without altering the values of the fractions), and then you will have $\frac{a\,d}{b\,d} = \frac{b\,c}{b\,d}$; that is, the quotient of $a\,d$ divided by bd, is equal to the quotient of bc divided by bd; therefore ad muſt be equal to $b\,c$, that is, the

the product of the extremes muſt be equal to the product of the middle terms. *Q. E. D.*

The converſe of this propoſition is alſo true, to wit, that *Whenever we have an equation in numbers, wherein the product of two numbers on one ſide is found equal to the product of two numbers on the other, ſuch an equation may be reſolved into four proportionals, by making the two numbers on either ſide, the extremes; and thoſe on the other ſide, the middle terms:* thus if $ad = bc$; by making a and d the extremes, and b and c the middle terms, we ſhall have a to b as c to d: if this be denied, let a be to b as c to e; then we ſhall have $ae = bc$ by the laſt; but $ad = bc$ by the ſuppoſition; therefore $ae = ad$; therefore e equals d, and a is to b as c is to d. *Q. E. D.*

COROLLARY.

Whence if a, b, and c, be continual proportionals, that is, if a is to b as b is to c, we ſhall have $b^2 = ac$: and *è converſo*, if $b^2 = ac$, then a, b, and c, will be continual proportionals.

The common properties of proportionality in numbers demonſtrated.

16. From what has been delivered in the laſt article, may be demonſtrated all or moſt of the common properties of proportionable numbers with a great deal of eaſe, ſome of the moſt uſeful whereof I ſhall here throw together into one ſingle article, for the reader to peruſe, either at preſent, or hereafter, as he ſhall ſee occaſion.

Firſt then, from what has been ſaid, may the rule of three, which conſiſts in finding a fourth proportional, be moſt diſtinctly demonſtrated: for let a, b and c be three numbers given, in order to find d, a fourth proportional; then ſince a is to b as c is to d, you will have ad the product of the extremes, equal

to

to bc the product of the middle terms; divide both ſides of the equation by a, and you will have $d = \frac{bc}{a}$: which is as much as to ſay, that if three numbers be given, a fourth proportional may be obtained by multiplying the ſecond and third numbers together, and dividing the product by the firſt.

In the rule of three inverſe, let the numbers when diſpoſed according to form be a, b, and c; then whoſoever attentively conſiders the nature of that rule, will eaſily ſee, that the fourth number there ſought for, is not to be a fourth proportional to the three numbers given as they are diſpoſed in the order a, b, c, but as they ſtand in the order c, b, a, or c, a, b, and therefore, in this caſe, the fourth number will be $\frac{ab}{c}$.

Secondly, if two proportions be equal to a third, they muſt be equal to one another, becauſe if two fractions be equal to a third, they muſt be equal to one another: thus if a is to b as c is to d, and c is to d as e is to f, we ſhall have a to b as e to f.

Thirdly, if a is to b as c is to d; then b will be to a as d to c, which is called inverſe proportion: for if a is to b as c is to d, we ſhall have $ad = bc$; make b and c the extremes, and you will have b to a as d to c.

Fourthly, if a is to b as c is to d; we ſhall have, by permutation, a to c as b to d: for ſince a is to b as c is to d, and conſequently $ad = bc$, make a and d the extremes, and c and b the middle terms, and you will have a to c as b to d.

Fifthly, if a is to b as c is to d, and any two multiplicators whatever be aſſumed, as e and f; I ſay then, that ea is to fb as ec to to fd: for ſince a is to b as c is to d, and ſo $ad = bc$; multiply both ſides of the equation by the product ef, and you will have

have $ad \times ef = bc \times ef$; but $ad \times ef = ea \times fd$, and $bc \times ef = fb \times ec$; therefore $ea \times fd = fb \times ec$; make ea and fd extremes, and the proportion will ſtand thus; ea is to fb as ec to fd. In like manner, *mutatis mutandis*, it may be demonſtrated, that if a is to b as c is to d, then $\frac{a}{e}$ will be to $\frac{b}{f}$ as $\frac{c}{e}$ is to $\frac{d}{f}$.

Sixthly, if a is to b as c is to d; then a^2 is to b^2 as c^2 is to d^2: for ſince a is to b as c is to d, and ſo $ad = bc$; ſquare both ſides of the equation, and you will have $a^2 d^2 = b^2 c^2$; make a^2 and d^2 extremes, and you will have a^2 to b^2 as c^2 to d^2. And by taking theſe ſteps backwards, it will alſo appear, that if a^2 is to b^2 as c^2 is to d^2; a is to b as c is to d, and $\sqrt{a}$ is to $\sqrt{b}$ as $\sqrt{c}$ is to $\sqrt{d}$.

Seventhly, if a is to b as c is to d; then by compoſition (as it is called) $\overline{a+b}$ is to b as $\overline{c+d}$ is to d; or $\overline{a+b}$ is to a as $\overline{c+d}$ is to c: for ſince a is to b as c is to d, and conſequently $ad = bc$; add bd to both ſides of the equation, and you will have $ad + bd = bc + bd$; but $ad + bd$ is the product of $\overline{a+b}$ multiplied into d, as is eaſily ſeen; and $bc + bd$ is the product of b multiplied into $\overline{c+d}$; therefore $\overline{a+b} \times d = b \times \overline{c+d}$; make $\overline{a+b}$ and d extremes, and you will have $\overline{a+b}$ to b as $\overline{c+d}$ to d. Again, ſince $bc = ad$, add ac to both ſides, and you will have $ac + bc = ac + ad$, that is, $\overline{a+b} \times c = a \times \overline{c+d}$; make $\overline{a+b}$ and c extremes, and you will have $\overline{a+b}$ to a as $\overline{c+d}$ to c.

Eighthly, if a is to b as c is to d; then by diviſion $\overline{a-b}$ is to b as $\overline{c-d}$ is to d; or $\overline{a-b}$ is to a as as $\overline{c-d}$ to c. This propofition is demonftrated by fubtraction, juft in the fame manner as the laft was by addition.

Ninthly, if to or from two numbers in any given proportion, be added or fubtracted other two numbers in the fame proportion, the fums or remainders will ftill be in the fame proportion with the numbers firft propofed: thus if the numbers c and d be in the fame proportion with the numbers a and b, that is, if as a is to b fo is c to d, and if to or from the former two numbers be added or fubtracted the latter, we fhall have not only $\overline{a+c}$ to $\overline{b+d}$ as a to b, but alfo $\overline{a-c}$ to $\overline{b-d}$ as a to b: for fince, by the fuppofition, a is to b as c is to d; it follows by permutation, that a is to c as b is to d; and by compofition, that $\overline{a+c}$ is to a as $\overline{b+d}$ to b; and again by permutation, that $\overline{a+c}$ is to $\overline{b+d}$ as a is to b: in like manner by permutation and divifion we fhall have $\overline{a-c}$ to $\overline{b-d}$ as a to b.

Tenthly, if there be three numbers a, b, and c, and other three numbers d, e, and f, proportionable to them, and in the fame order, that is, if as a is to b fo d is to e, and as b is to c fo e is to f; I fay then, that, *ex æquo*, the extremes will be in the fame proportion, (*viz.*) that a will be to c as d is to f: for fince by the fuppofition, a is to b as d is to e; by permutation we fhall have a to d as b to e; and for the fame reafon, fince b is to c as e is to f; we fhall have b to e as c to f: fince then a is to d as b to e, and b to e as c to f; it follows from the fecond propofition, that a is to d as c to f; and by permutation, that a is to c as d to f.

Eleventhly, if there be three numbers, a, b, and c, and three other numbers d, e, and f, proportionable to them, but in a contrary order, fo that a is to b as e to f, and b to c as d to e; I fay, that the extremes will ftill be proportionable, to wit, that a will be to c as d to f: for fince a is to b as e to f, we have

$af = be$;

$af=be$; moreover, ſince b is to c as d to e, we have $cd=be$; therefore $af=cd$; make a and f extremes, and you will have a to c as d to f.

N.B. If there be two ſerieſes of numbers, as a, b, c, *&c.*; d, e, f, *&c.*; each ſeries conſiſting of the ſame number of terms; and if all the proportions between contiguous terms in one ſeries be reſpectively equal to all thoſe in the other, that is, each to each, as they ſtand in order; as if a be to b as d to e, and b to c as e to f, *&c.*; then the extreme terms of one ſeries will be proportionable to the extreme terms of the other: for the demonſtration of the tenth propoſition may be extended to as many terms as we pleaſe; and this proportionality of the extremes is ſaid to follow *ex æquo ordinate*, or barely *ex æquo*, that is from a reſpective equality of all the proportions in one ſeries to their correſpondents in the other, in an orderly manner. But if every proportion in one ſeries has an equal proportion to anſwer it in the other, but not in a correſpondent part of the ſeries; as if a be to b as e to f, and b to c as d to e, *&c.*; then though the extremes will ſtill be proportionable, as will be evident by continuing the demonſtration of this eleventh propoſition; yet now the proportionality of the extremes is ſaid to follow *ex æquo perturbate*, that is, from an equality of all the proportions in one ſeries to all thoſe in the other, but in a diſorderly manner.

Twelfthly, if a is to b as c is to d; we ſhall have $\overline{a+b}$ to $\overline{a-b}$ as $\overline{c+d}$ is to $\overline{c-d}$: for ſince a is to b as c is to d, we ſhall have by compoſition, $\overline{a+b}$ to a as $\overline{c+d}$ is to c; we ſhall have alſo by diviſion, $\overline{a-b}$ to a as $\overline{c-d}$ to c; and by inverſion, a to $\overline{a-b}$ as c to $\overline{c-d}$: ſince then we have $\overline{a+b}$ to a as $\overline{c+d}$ to c; and a to $\overline{a-b}$ as c to $\overline{c-d}$, that is, ſince we have three numbers, $\overline{a+b}$, a, and $\overline{a-b}$, and other three numbers proportionable to them in the ſame order, to wit, $\overline{c+d}$, c, and $\overline{c-d}$; it follows *ex æquo* that

that the extremes will be proportionable, that is, that $a+b$ will be to $a-b$ as $c+d$ is to $c-d$.

Thirteenthly, if there be a ſeries of numbers, k, l, m, n, whereof k is to l as a to b, and l to m as c to d, and m to n as e to f; I ſay then that k the firſt term will be to n the laſt, as ace the product of all the other antecedents to bdf the product of all the other conſequents: for k is to l as a to b, by the ſuppoſition; and we ſhall find that a is to b as ace to bce by multiplying extremes and means; therefore k is to l as ace to bce; and for a like reaſon l is to m as bce to bde, and m is to n as bde to bdf; therefore, *ex æquo*, k is to n as ace to bdf.

Of the extraction of the ſquare roots of ſimple algebraic quantities.

17. The extraction of the ſquare root of ſimple algebraic quantities is ſo very eaſy, that it needs not to be inſiſted on. Thus the ſquare root of aa is $+$ or $-a$, the ſquare root of $9aa$ is $+$ or $-3a$, and that of $4aabb$ is $+$ or $-2ab$: this is plain from the definition of the ſquare root; for the ſquare root of any quantity, ſuppoſe of $4aabb$, is that which, being multiplied into itſelf, will produce $4aabb$: now $-2ab$ multiplied into itſelf will produce $4aabb$, as well as $+2ab$, and therefore one quantity is as much its ſquare root as the other.

When the ſquare root of a quantity cannot be extracted, it is uſual to ſignify it by this mark $\surd$: thus $\surd 2aa$ ſignifies the ſquare root of $2aa$; thus $\sqrt{aa-4b}$ ſignifies the ſquare root of the whole quantity $\overline{aa-4b}$; thus $\frac{\sqrt{aa-4b}}{2a}$ ſignifies a fraction whoſe numerator is the ſquare root of the whole quantity $\overline{aa-4b}$, and whoſe denominator is $2a$; thus $\sqrt{\frac{4ab-a^3}{12a}}$ ſignifies the

the ſquare root of the whole fraction, $\frac{4ab-a^3}{12a}$, that is, the ſquare root of both the numerator and denominator.

When the ſquare root of a quantity cannot be extracted, the quantity may ſometimes however be reſolved into two factors, whereof the one is a ſquare, and the other is not; and whenever this is poſſible, the root of the ſquare may be extracted, and the radical ſign may be prefixed to the other factor; thus $12aa$ equals $4aa\times3$; therefore $\sqrt{12aa}\times2a\times\sqrt{3}$.

The ſeveral rules of fractions exemplified in algebraic quantities.

22. Fractions in Algebra are treated juſt in the ſame manner as in common arithmetic, only uſing algebraical inſtead of numeral operations; as will plainly appear from the following examples.

Examples of the reduction of fractions from higher to lower terms, according to introduction art. 7th.

The fraction $\frac{4ab}{6bc}$, dividing both the numerator and denominator by the ſame quantity $2b$, will be reduced to the fraction $\frac{2a}{3c}$, a fraction of the ſame value with the former, but expreſſed in more ſimple terms: whence we may infer, that whenever a common letter or factor is to be found in every member both of the numerator and denominator, it may be cancelled everywhere, without affecting the value of the fraction: thus the fraction $\frac{ac+bc}{cd+ce}$, expunging c, becomes $\frac{a+b}{d+e}$, a fraction of the ſame value. But if there be any one member wherein the factor is not concerned, it muſt not be expunged at all: thus the fraction $\frac{ac+bc}{cd+e}$ cannot be reduced, becauſe the factor c is not to be found in e.

Note, That cancelling here, is not ſubtracting, but dividing: thus to cancel the letter b in the quantity ab, ſo as to reduce it to a, is not to ſubtract b from ab, but to divide ab by b, in which caſe the quotient will be a.

Examples of fractions reduced to the ſame denomination, according to introduction art. 8th.

1*ſt*. The fractions $\frac{a}{2}$, $\frac{b}{3}$, and $\frac{c}{4}$, when reduced to the ſame denomination, will ſtand thus; $\frac{12a}{24}$, $\frac{8b}{24}$, and $\frac{6c.}{24}$ 2*d*. The fractions $\frac{a}{b}$ and $\frac{c}{d}$, ſo reduced, will ſtand thus; $\frac{ad}{bd}$ and $\frac{bc}{bd}$. 3*d*. The fractions $\frac{p}{q}$, $\frac{r}{s}$, $\frac{t}{u}$, and $\frac{x}{y}$, after reduction, will ſtand thus; $\frac{psuy}{qsuy}$, $\frac{qruy}{qsuy}$, $\frac{qsty}{qsuy}$, and $\frac{qsux}{qsuy}$. And here I cannot but obſerve, that now the rule for this reduction demonſtrates itſelf: for in this example it is impoſſible not to ſee, that all theſe fractions, notwithſtanding this reduction, ſtill retain their former values: thus the firſt fraction $\frac{psuy}{qsuy}$, by cancelling common factors, is reduced to $\frac{p}{q}$, its former value; and the ſame may be obſerved of all the reſt: and this example amounts to a demonſtration, becauſe it is comprehended in general terms. But to go on:

on: 4*th*. The fractions $\frac{1}{a}$, $\frac{1}{b}$, and $\frac{1}{c}$, being reduced to the same denomination, become $\frac{bc}{abc}$, $\frac{ac}{abc}$, and $\frac{ab}{abc}$. 5*th*. And lastly, $\frac{1}{a+b}$ and $\frac{1}{a-b}$ when thus reduced, become $\frac{a-b}{aa-bb}$ and $\frac{a+b}{aa-bb}$: for 1 the numerator of the first fraction multiplied into $a-b$, the denominator of the second, makes $a-b$; and 1 the numerator of the second fraction multiplied into $a+b$, the denominator of the first, makes $a+b$; and the product of the two denominators $a+b$ and $a-b$ multiplied together is $aa-bb$, as in the 4th example of the 9th article.

Examples of addition in fractions, according to introduction art. 9th.

1st. These fractions $\frac{a}{2}$, $\frac{b}{2}$ and $\frac{-c}{2}$, when added together, make $\frac{a+b-c}{2}$.

2d. The fraction $\frac{a+b}{2}$ added to the fraction $\frac{a-b}{2}$ makes $\frac{2a}{2}$ or a.

3d. The fractions $\frac{a}{2}$, $\frac{-b}{3}$ and $\frac{+c}{4}$, when added together, make $\frac{12a-8b+6c}{24}$.

4th. The fraction $\frac{a}{b}$ added to the fraction $\frac{c}{d}$ makes $\frac{ad+bc}{bd}$.

5th. a added to $\frac{b}{c}$, that is, $\frac{a}{1}$ added to $\frac{b}{c}$, makes $\frac{ac+b}{c}$.

6th. $\frac{1}{a}$ added to $-\frac{1}{b}$ makes $\frac{b-a}{ab}$.

7th. The fractions $\frac{p}{q}$, $\frac{r}{s}$, $\frac{t}{u}$, and $\frac{x}{y}$, when added together, make $\frac{psuy+qruy+qsty+qsux}{qsuy}$.

8th. $\frac{a}{b}$ added to $\frac{1}{c}$ gives $\frac{ac+b}{bc}$.

9th. $\frac{1}{a+b}$ added to $\frac{1}{a-b}$ gives $\frac{2a}{aa-bb}$. See the 5th example of fractions reduced to the same denomination.

Examples of subtraction in fractions, according to introduction art. 10th.

Note first, If the signs of both the numerator and denominator of any fraction be changed, which is no more than multiplying both terms into -1, the value of the fraction will still remain.

Secondly, The denominator of a fraction is always supposed to be affirmative; and therefore if at any time it happens to be otherwise, it must be made affirmative by changing the signs of both terms.

Thirdly, $+\frac{a}{b}$ and $-\frac{a}{b}$ are the same in effect as $\frac{+a}{b}$ and $\frac{-a}{b}$, as is evident from the nature of division: and sometimes this latter way of notation is more convenient than the former.

Fourthly,

Fourthly, Therefore the ſign of the numerator is the ſign of the whole fraction; and to change the ſign of the former, is the ſame in effect as to change the ſign of the latter.

Fifthly, Whenever one algebraic fraction is to be ſubtracted from another, the ſafeſt way will be to change the ſign of the numerator of the fraction to be ſubtracted, and to place it after the other, and then to reduce them at laſt into one fraction: for if the ſubtraction be deferred till after the reduction is over, one may make a miſtake, and ſubtract the wrong quantity. Thus, 1ſt, $\frac{4b}{5}$ ſubtracted from $\frac{2a}{3}$ gives $\frac{2a}{3}\frac{-4b}{5}=\frac{10a-12b}{15}$.

2d. $\frac{r}{s}$ ſubtracted from $\frac{p}{q}$ gives $\frac{p}{q}\frac{-r}{s}=\frac{ps-qr}{qs}$.

3d. $\frac{b}{c}$ ſubtracted from a gives $\frac{a}{1}\frac{-b}{c}=\frac{ac-b}{c}$.

4th. $\frac{1}{a+b}$ ſubtracted from $\frac{1}{a-b}$ gives $\frac{1}{a-b}\frac{-1}{a+b}$ $=\frac{2b}{aa-bb}$.

Examples of multiplication in fractions.

The multiplication of fractions is performed by multiplying the numerator and denominator of the multiplicand, into the numerator and denominator of the multiplicator reſpectively.

Thus 1ſt. $\frac{p}{q}\times\frac{r}{s}=\frac{pr}{qs}$.

2d. $\frac{3p}{4q}\times\frac{5q}{6r}=\frac{15pq}{24qr}=\frac{5p}{8r}$.

3d. $\frac{a}{b}\times c$ or $\frac{a}{b}\times\frac{c}{1}=\frac{ac}{b}$:

4th. $\frac{a}{b} \times b = \frac{ab}{b} = a.$

5th. $\frac{3a}{4b} \times 20b = \frac{60ab}{4b} = 15a.$

6th. $\frac{4a}{5} \times \frac{7}{8a} = \frac{28a}{40a} = \frac{7}{10}.$

7th. $\frac{3a}{4b} \times \frac{3a}{4b} = \frac{9aa}{16bb}.$

8th. $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}.$

9th. $\overline{a + \frac{b}{c}} \times d$, or $\frac{\overline{ac + b}}{c} \times \frac{d}{1} = \frac{acd + bd}{c}.$

10th. $\overline{d + \frac{e}{f}} \times \frac{g}{b}$, or $\frac{df + e}{f} \times \frac{g}{b} = \frac{dfg + eg}{fb}.$

11th. $\overline{a + \frac{b}{c}} \times \overline{d + \frac{e}{f}}$, or $\frac{ac + b}{c} \times \frac{df + e}{f} =$ $\frac{acdf + ace + bdf + be}{cf}.$

This multiplication might also be performed thus:

$$d + \frac{e}{f}$$
$$a + \frac{b}{c}$$
$$ad + \frac{ae}{f} + \frac{bd}{c} + \frac{be}{cf}.$$

12th. $\overline{a + \frac{b}{c}} \times \overline{a + \frac{b}{c}} = \frac{aacc + 2abc + bb}{cc}:$

Or, $aa + \frac{2ab}{c} + \frac{bb}{cc}$. See the work:

a+

$$a+\frac{b}{c}$$

$$a+\frac{b}{c}$$

$$\overline{aa+\frac{ab}{c}+\frac{bb}{cc}}$$

$$+\frac{ab}{c}$$

$$\overline{aa+\frac{2ab}{c}+\frac{bb}{cc}}$$

Examples of division in fractions.

Divifion in fractions is performed by multiplying the direct terms of the dividend into the inverted terms of the divifor: thus,

1ft. $\frac{r}{s}\Big)\ \frac{p}{q}\ \Big(\frac{ps}{qr}.$ 2d. $\frac{b}{c}\Big)\ \frac{1}{a}\ \Big(\frac{c}{ab}.$

3d. $\frac{1}{c}\Big)\ \frac{a}{b}\ \Big(\frac{ac}{b}.$ 4th. $c\Big)\ \frac{a}{b}\ \Big(\frac{a}{bc}.$

5th. $\frac{1}{c}\Big)\ \frac{ab}{c}\ \Big(\frac{abc}{c}$, or ab.

6th. $d+\frac{e}{f}\Big)\ a+\frac{b}{c}\ \Big(\frac{acf+bf}{cdf+ce}.$

7th. $\sqrt{b})\ \sqrt{a}\ \Big(\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$: for if we make $x=\frac{\sqrt{a}}{\sqrt{b}}$, we fhall have $xx=\frac{a}{b}$, and $x=\sqrt{\frac{a}{b}}$.

I ſhall only give one example more, and that ſhall be of the rule of proportion, as follows: *If* $\frac{a}{b}$ *gives* $\frac{c}{d}$, *what will* $\frac{e}{f}$ *give?* Anſwer, $\frac{bce}{adf}$: for $\frac{c}{d}$ the ſecond number, multiplied into $\frac{e}{f}$ the third, produces $\frac{ce}{df}$; and this divided by the firſt $\frac{a}{b}$ quotes $\frac{bce}{adf}$.

Of equations in Algebra; and particularly of ſimple equations, together with the manner of reſolving them.

23. An equation in Algebra is a propoſition wherein one quantity is declared equal to another, or where one expreſſion of any quantity is declared equal to another expreſſion of the ſame quantity: as when we ſay $\frac{2}{4} = \frac{3}{6}$; where $\frac{2}{4}$ is ſaid to poſſeſs one ſide of the equation, and $\frac{3}{6}$ the other.

An affected quadratic equation is an equation conſiſting of three different ſorts of quantities; one wherein the ſquare of the unknown quantity is concerned, another wherein the unknown quantity is ſimply concerned, and a third wherein it is not concerned at all: as if $xx - 2x = 3$; ſuppoſing x to be an unknown quantity.

If either the term wherein the ſimple power of x is concerned, as $-2x$, or that which is called the abſolute term, to wit, 3, be wanting, the equation is ſtill a quadratic equation, though incompleat. Some indeed there are, who rank this latter ſort of equations under the denomination of ſimple equations; and ſo ſhall we, upon account of their eaſy reſolution; though, properly ſpeaking, a ſimple equation is that wherein ſome ſimple power of the unknown

unknown quantity is concerned, all others being excluded: as if $3x = 6$; $2x + 3 = 4x - 5$, &c.

The uſe of theſe equations is for repreſenting more conveniently and more diſtinctly the conditions of problems, when tranſlated out of common language into that of Algebra. As for example; let it be propoſed to find a number with the following property, to wit, that $\frac{2}{3}$ of it with 4 over may amount to the ſame as $\frac{7}{12}$ of it with 9 over: here, putting x for the unknown quantity, the condition of this problem, when tranſlated out of common language into that of Algebra, will be repreſented by the following equation, to wit, $\frac{2x}{3} + 4 = \frac{7x}{12} + 9$: for $\frac{2}{3}$ of x, that is, $\frac{2}{3}$ of $\frac{x}{1}$ is $\frac{2x}{3}$; therefore $\frac{2x}{3} + 4$ ſignifies $\frac{2}{3}$ of x with 4 over: and ſince this expreſſion, according to the problem, amounts to the ſame with the other, to wit, $\frac{7x}{12} + 9$; hence it is that we pronounce them equal to one another.

Now ſince, in the foregoing equation, as well as in almoſt all others ariſing immediately from the conditions of problems themſelves, the unknown quantity is embarraſſed and entangled with ſuch as are known, the way to diſengage it from ſuch known quantities, ſo that itſelf alone poſſeſſing one ſide of the equation, may be found equal to ſuch as are entirely known on the other, that is, in the preſent caſe, to determine the value of the unknown quantity x, is what is commonly called the reſolution of an equation: for the effecting of which, ſeveral axioms and proceſſes are required; ſome whereof, namely ſuch as moſt frequently occur, I ſhall here put down; the reſt I ſhall take notice of occaſionally, as they offer themſelves.

Of the resolution of simple equations.

AXIOM I.

Whenever a fraction is to be multiplied by a whole number, it will be sufficient to multiply only the numerator by that number, retaining the denominator the same as before. Thus $\frac{4}{5}$ multiplied into 2, gives $\frac{8}{5}$, for the same reason that 4 shillings multiplied into 2 gives 8 shillings: thus in the first example following, $\frac{7x}{12}$ multiplied into 3, gives $\frac{21x}{12}$.

AXIOM 2.

But if the whole number into which the fraction is to be multiplied, be equal to the denominator of the fraction, then throw away the denominator, and the numerator alone will be the product. Thus the fraction $\frac{a}{b}$ multiplied into b, gives $\frac{ab}{b}$ or a: thus in the first example, $\frac{2x}{3}$ multiplied into 3, gives $2x$; and $\frac{21x}{12}$ multiplied into 12, gives $21x$.

AXIOM 3.

If the two sides of an equation be multiplied or divided by the same number, the two products, or quotients, will still be equal to each other. Thus in the first example, where $\frac{2x}{3}+4=\frac{7x}{12}+9$; if both sides of the equation be multiplied into 3, we shall have $2x+12=\frac{21x}{12}+27$; and if again this last equation be multiplied into 12, we shall have $24x+144=21x+324$.

AXIOM

AXIOM 4.

If a quantity be taken from either ſide of an equation, and placed on the other with a contrary ſign, which is commonly called tranſpoſition, the two ſides will be equal to each other. Thus if $7+3=10$, tranſpoſe $+3$, and you will have $7=10-3$: thus if $7-3=4$, tranſpoſe -3, and you will have $7=4+3$: thus if (as in the firſt example) $24x+144=21x+324$, tranſpoſe $21x$, and you will have $24x-21x+144=324$, that is, $3x+144=324$; and if again in this laſt equation you tranſpoſe 144, you will have $3x=324-144=180$.

Tranſpoſition, therefore, as it is here delivered, is nothing but a general name for adding or ſubtracting equal quantities from the two ſides of an equation; in which caſe it is no wonder if the ſums or differences ſtill continue equal to each other. As for inſtance, in this equation $a-b=c$, tranſpoſing $-b$ we have $a=c+b$: and what is this, after all, but adding b to both ſides of the equation? for if b be added to $a-b$, the ſum will be a; and if b be added to c, the ſum will be $c+b$; therefore $a=c+b$: again in the equation $a+b=c$, tranſpoſing $+b$, we have $a=c-b$, which is nothing elſe but ſubtracting b from both ſides of the equation.

The 1ſt Proceſs.

If, when an equation is to be reſolved, fractions be found on one or both ſides, it muſt be freed from them by multiplying the whole equation into the denominators of thoſe fractions ſucceſſively.

The 2d Proceſs.

After the equation is thus reduced to integral terms, if the unknown quantity be found on both ſides the equation, let it be brought by tranſpoſition to one and the ſame ſide, viz. *to that ſide which after reduction will exhibit it affirmative*

The 3d Procefs.

After this, if any loofe known quantities be found on the fame fide with the unknown, let them alfo be brought by tranfpofition to the other fide of the equation.

The 4th Procefs.

If now the unknown quantity has any coefficient before it, divide all by that coefficient, and the equation will be refolved.

The 5th Procefs.

If the whole equation can be divided by the unknown quantity, let fuch a divifion be made, and the equation will be reduced to a more fimple one. Thus in the 16th example you have $615x-7xxx=48x$; divide the whole equation by x, and you will have $615-7xx=48$. In the 13th example you have $\frac{42x}{x-2}=\frac{35x}{x-3}$; divide the whole by x, which is done by dividing only the numerators of the two fractions, and you will have $\frac{42}{x-2}=\frac{35}{x-3}$.

The 6th Procefs.

If at laft the fquare of the unknown quantity, and not the unknown quantity itfelf, appears to be equal to fome known quantity on the other fide of the equation, then the unknown quantity muft be made equal to the fquare root of that which is known. Thus in the 14th example we have $xx=36$; therefore $x=6$, and not 18: in the 15th, we have $xx=64$; therefore $x=8$, the fquare root of 64, and not 32, its half.

Examples of the refolution of fimple equations.

24. This preparation being made, I fhall now give fome examples of the refolution of fimple equations; and

and my first example shall be the equation given in the last article, in order to trace out the number there described.

Example 1.

$$\frac{2x}{3}+4=\frac{7x}{12}+9.$$

In this equation it is plain, that there are two fractions, $\frac{2x}{3}$, and $\frac{7x}{12}$, which must be taken off at two several operations, thus: as 3 is the denominator of the first fraction, multiply the whole equation by 3, and you will have $2x+12=\frac{21x}{12}+27$: again, as the denominator of the remaining fraction is 12, multiply all by 12, and you will have $24x+144=21x+324$; which is an equation free from fractions.

2dly, It must in the next place be considered, that in this last equation $24x+144=21x+324$, the unknown quantity is concerned on both sides, to wit, $24x$ on one side, and $21x$ on the other; transpose therefore $21x$, and you will have $24x-21x+144=324$, that is, $3x+144=324$. If it be asked why I chose to transpose $21x$ rather than $24x$; my answer is, that had $24x$ been transposed, the unknown quantity, or its coefficient at least, after reduction, would have been negative, contrary to the rule in the second process; for, resuming the equation $24x+144=21x+324$, if $24x$ be transposed, we shall have $144=21x-24x+324$, that is, $144=-3x+324$: but even in this case, another transposition will set all right; for if $-3x$ be transposed in this last equation, we shall then have $3x+144+324$ as before: all that can be said then against this last way is, that it creates unnecessary transpositions, which an artist would always endeavour to avoid.

3dly, Having now reduced the equation to a much greater degree of simplicity than before, to wit, $3x+144=324$; because the unknown quantity $3x$ has still

ſtill a looſe quantity, *viz.* 144 joined with it, tranſpoſe that quantity 144 to the other ſide of the equation, and you will have $3x=324-144$, that is, $3x=180$.

N. B. By a looſe quantity I mean ſuch a one as is joined with the unknown by the ſign + or —, and not by way of multiplication, as is the coefficient 3 in the laſt equation.

4thly, By this time the quantity x is very near being diſcovered; for if $3x=180$, it is but dividing all by 3, and we ſhall have $x=60$: 60 therefore is the number deſcribed in the laſt article by this property, to wit, that $\frac{2}{3}$ of it with 4 over, will amount to the ſame as $\frac{7}{12}$ of it with 9 over: and that 60 has this property, will now be eaſily made to appear ſynthetically; for $\frac{2}{3}$ of 60 is 40, and this with 4 over is 44; moreover $\frac{7}{12}$ of 60 is 35, and this with nine over is alſo 44.

N. B. A demonſtration that proves the connection between any number and the property aſcribed to it, is either analytical or ſynthetical: if this connexion is ſhewn by tracing the number from the property, the demonſtration of it is called an analytical demonſtration; but if it is ſhewn by tracing the property from the number, the demonſtration is then ſaid to be ſynthetical.

Example 2.

$$\frac{2x}{3}+12=\frac{4x}{5}+6.$$

Here multiply by 3, and you will have $2x+36=\frac{12x}{5}+18$; multiply again by 5, and you will have $10x+180=12x+90$; tranſpoſe $10x$, and you will have $180=12x-10x+90$, that is, $180=2x+90$, or rather $2x+90=180$, for I generally chooſe to have the unknown quantity on the firſt ſide of the equation: tranſpoſe 90, and you will have $2x=180-90$, that is, $2x=90$; divide by 2, and you will have $x=45$.

The

The Proof.

The original equation was $\frac{2x}{3}+12=\frac{4x}{5}+6$: now if $x=45$, we have $\frac{2x}{3}=30$, and $\frac{2x}{3}+12=42$: again, we have $\frac{4x}{5}=36$, and $\frac{4x}{5}+6=42$: therefore $\frac{2x}{3}+12=\frac{4x}{5}+6$, becauſe the amount of both is 42.

Example 3.

$\frac{3x}{4}+5=\frac{5x}{6}+2$: therefore $3x+20=\frac{20x}{6}+8$; therefore $18x+120=20x+48$; therefore $120=20x-18x+48$, that is, $120=2x+48$; therefore $120-48=2x$, that is, $2x=72$; therefore $x=36$.

The Proof.

The original equation was $\frac{3x}{4}+5=\frac{5x}{6}+2$: now if $x=36$, we ſhall have $\frac{3x}{4}=27$, and $\frac{3x}{4}+5=32$: we ſhall alſo have $\frac{5x}{6}=30$, and $\frac{5x}{6}+2=32$; therefore if $x=36$, we ſhall have $\frac{3x}{4}+5=\frac{5x}{6}+2$.

Example 4.

$\frac{7x}{8}-5=\frac{9x}{10}-8$: therefore $7x-40=\frac{72x}{10}-64$; therefore $70x-400=72x-640$; therefore $-400=72x-70x-640$, that is, $-400=2x-640$, or rather $2x-640=-400$; therefore $2x=640-400$, that is, $2x=240$; and $x=120$.

The

The Proof.

The original equation, $\frac{7x}{8}-5=\frac{9x}{10}-8$; but $x=120$; therefore $\frac{7x}{8}=105$; therefore $\frac{7x}{8}-5=100$: moreover $\frac{9x}{10}=108$; therefore $\frac{9x}{10}-8=100$; therefore $\frac{7x}{8}-5=\frac{9x}{10}-8$.

Example 5.

$\frac{5x}{9}-8=74-\frac{7x}{12}$: therefore $5x-72=666-\frac{63x}{12}$; therefore $60x-864=7992-63x$; therefore $60x+63x-864=7992$, that is, $123x-864=7992$; therefore $123x=7992+864$, that is, $123x=8856$; and $x=72$.

The Proof.

The original equation, $\frac{5x}{9}-8=74-\frac{7x}{12}$; $x=72$; therefore $\frac{5x}{9}=40$; therefore $\frac{5x}{9}-8=32$: again, $\frac{7x}{12}=42$; therefore $74-\frac{7x}{12}=74-42=32$: therefore $\frac{5x}{9}-8=74-\frac{7x}{12}$.

Example 6.

$\frac{x}{6}-4=24-\frac{x}{8}$: therefore $x-24=144-\frac{6x}{8}$ therefore $8x-192=1152-6x$; therefore $8x+6x-192=1152$, that is, $14x-192=1152$; therefore $14x=1152+192$, that is, $14x=1344$; and $x=96$.

The

The Proof.

The original equation, $\frac{x}{6}-4=24-\frac{x}{8}$; $x=96$; $\frac{x}{6}=16$; $\frac{x}{6}-4=12$: again, $\frac{x}{8}=12$; therefore $24-\frac{x}{8}=24-12=12$; therefore $\frac{x}{6}-4=24-\frac{x}{8}$.

Example 7.

$56-\frac{3x}{4}=48-\frac{5x}{8}$: therefore $224-3x=192-\frac{20x}{8}$; therefore $1792-24x=1536-20x$; therefore $1792=1536+24x-20x$, that is, $1792=1536+4x$; therefore $1792-1536=4x$, that is, $4x=256$; and $x=64$.

The Proof.

The original equation, $56-\frac{3x}{4}=48-\frac{5x}{8}$; $x=64$; therefore $\frac{3x}{4}=48$; therefore $56-\frac{3x}{4}=56-48=8$: again, $\frac{5x}{8}=40$; therefore $48-\frac{5x}{8}=48-40=8$; therefore $56-\frac{3x}{4}=48-\frac{5x}{8}$.

Example 8.

$36-\frac{4x}{9}=8$: therefore $324-4x=72$; therefore $324=72+4x$; therefore $324-72=4x$, that is, $4x=252$; and $x=63$.

The Proof.

The original equation, $36 - \frac{4x}{9} = 8$; $x = 63$; therefore $\frac{4x}{9} = 28$; therefore $36 - \frac{4x}{9} = 36 - 28 = 8$.

Example 9.

$\frac{2x}{3} = \frac{176 - 4x}{5}$; therefore $2x = \frac{528 - 12x}{5}$; therefore $10x = 528 - 12x$; therefore $10x + 12x = 528$; that is, $22x = 528$; and $x = 24$.

The Proof.

The original equation, $\frac{2x}{3} = \frac{176 - 4x}{5}$; $x = 24$; therefore $\frac{2x}{3} = 16$: again, $4x = 96$; therefore $176 - 4x = 176 - 96 = 80$; therefore $\frac{176 - 4x}{5} = \frac{80}{5} = 16$; therefore $\frac{2x}{3} = \frac{176 - 4x}{5}$.

Example 10.

$\frac{}{4} + \frac{180 - 5x}{6} = 29$: therefore $3x + \frac{720 - 20x}{6} = 116$; therefore $18x + 720 - 20x = 696$, that is, $720 - 2x = 696$; therefore $720 = 2x + 696$; therefore $720 - 696 = 2x$, that is, $2x = 24$; and $x = 12$.

The

The Proof.

The original equation, $\frac{3x}{4}+\frac{180-5x}{6}=29$; $x=12$; therefore $\frac{3x}{4}=9$; $5x=60$; therefore $180-5x=180-60=120$; therefore $\frac{180-5x}{6}=\frac{120}{6}=20$; therefore $\frac{3x}{4}+\frac{180-5x}{6}=29$.

Example 11.

$$\frac{45}{2x+3}=\frac{57}{4x-5}.$$

Multiply by $2x+3$, and you will have $45=\frac{114x+171}{4x-5}$; multiply by $4x-5$, and you will have $180x-225=114x+171$; therefore $180x-114x-225=171$, that is, $66x-225=171$; therefore $66x=171+225$, that is, $66x=396$; and $x=6$.

The Proof.

The original equation, $\frac{45}{2x+3}=\frac{57}{4x-5}$; $x=6$; therefore $2x=12$; therefore $2x+3=15$; therefore $\frac{45}{2x+3}=\frac{45}{15}=3$: again, $4x=24$; therefore $4x-5=19$; therefore $\frac{57}{4x-5}=\frac{57}{19}=3$; therefore $\frac{45}{2x+3}=\frac{57}{4x-5}$.

Example 12.

$\frac{128}{3x-4} = \frac{216}{5x-6}$: therefore $128 = \frac{648x-864}{5x-6}$; therefore $640x - 768 = 648x - 864$; therefore $-768 = 648x - 640x - 864$, that is, $-768 = 8x - 864$; therefore $+864 - 768 = 8x$, that is, $8x = 96$; and $x = 12$.

The Proof.

The original equation, $\frac{128}{3x-4} = \frac{216}{5x-6}$; $x = 12$; therefore $3x = 36$; therefore $3x - 4 = 32$; therefore $\frac{128}{3x-4} = \frac{128}{32} = 4$: again, $5x = 60$; therefore $5x - 6 = 54$; therefore $\frac{216}{5x-6} = \frac{216}{54} = 4$; therefore $\frac{128}{3x-4} = \frac{216}{5x-6}$.

Example 13.

$\frac{42x}{x-2} = \frac{35x}{x-3}$: divide both numerators by x, and you will have $\frac{42}{x-2} = \frac{35}{x-3}$; therefore $42 = \frac{35x-70}{x-3}$; therefore $42x - 126 = 35x - 70$; therefore $42x - 35x - 126 = -70$, that is, $7x - 126 = -70$; therefore $7x = 126 - 70$, that is, $7x = 56$; and $x = 8$.

The Proof.

The original equation, $\frac{42x}{x-2} = \frac{35x}{x-3}$; $x = 8$; therefore $x - 2 = 6$; $42x = 336$; therefore $\frac{42x}{x-2} = \frac{336}{6} = 56$:

$=56$: again, $x-3=5$; and $35x=280$; therefore $\frac{35x}{x-3}=\frac{280}{5}=56$; therefore $\frac{42x}{x-2}=\frac{35x}{x-3}$.

Example 14.

$\frac{xx-12}{3}=\frac{xx-4}{4}$: therefore $xx-12=\frac{3xx-12}{4}$; therefore $4xx-48=3xx-12$; therefore $4xx-3xx-48=-12$, that is, $xx-48=-12$; therefore $xx=+48-12$, that is, $xx=36$; and $x=6$.

The Proof.

The original equation, $\frac{xx-12}{3}=\frac{xx-4}{4}$; $x=6$; therefore $xx=36$; therefore $xx-12=24$; therefore $\frac{xx-12}{3}=\frac{24}{3}=8$: again, $xx-4=32$; therefore $\frac{xx-4}{4}=\frac{32}{4}=8$; therefore $\frac{xx-12}{3}=\frac{x^2-4}{4}$.

Example 15.

$\frac{5xx}{16}-8=12$: therefore $5xx-128=192$; therefore $5xx=192+128$, that is, $5xx=320$; therefore $xx=64$; and $x=8$.

The Proof.

The original equation, $\frac{5xx}{16}-8=12$; $x=8$; therefore $xx=64$; therefore $5xx=320$; therefore $\frac{5xx}{16}=\frac{320}{16}=20$; therefore $\frac{5xx}{16}-8=20-8=12$.

Example 16.

$615x - 7xxx = 48x$: divide the whole by x, and you will have $615 - 7xx = 48$; therefore $615 = 7xx + 48$; therefore, $615 - 48 = 7xx$, that is, $7xx = 567$; therefore, $xx = 81$; and $x = 9$.

The Proof.

The original equation, $615x - 7xxx = 48x$; $x = 9$; therefore $xx = 81$; therefore $xxx = 729$; $7xxx = 5103$; again, $615x = 5535$; therefore $615x - 7xxx = 5535 - 5103 = 432$: laſtly, $48x = 432$; therefore $615x - 7xxx = 48x$.

THE

THE

ELEMENTS of ALGEBRA.

BOOK II.

Preparations for the ſolution of Algebraic problems.

Art. 25. IN ſolving the following problems, I ſhall make uſe of a ſort of mixt Algebra, uſing letters only in repreſenting unknown quantities, and numbers for ſuch as are known. This method, as I take it, will be the beſt to begin with: but afterwards, when my young ſcholar has been ſufficiently exerciſed in this way, I ſhall then introduce him into pure Algebra, which he will find much more extenſive than the former, not only as it enables him analytically to find out general ſolutions, taking in all the particular caſes that can be propoſed in the problem to which the ſolution belongs, but alſo as it enables him afterwards to demonſtrate the ſame ſolutions or theorems ſynthetically.

And becauſe I am not yet to ſuppoſe him ſkilled in any of the mathematical ſciences, I ſhall draw my problems, generally ſpeaking, from numbers, either conſidered abſtractedly, or elſe as they relate to common life.

If a problem be juſtly propoſed, it ought to have as many independent conditions comprehended in it, expreſsly or implicitly, as there are unknown quantities to be diſcovered by them; and it muſt be the chief buſineſs of an Algebraiſt, to ſearch out, ſift and diſtinguiſh theſe conditions one from another, before ever he enters upon the ſolution of his problem.

I ſaid, that ſo many conditions ought to be comprehended in the problem expreſsly or implicitly, becauſe it may happen, that a condition may not be expreſſed in a problem, and yet be implied in the nature of the thing; thus in the 44th problem, where ſeveral rods are to be ſet upright in a ſtreight line, at certain intervals one from another, it is implied, though not expreſſed, that the number of intervals muſt be leſs than the number of rods by unity.

Sometimes a condition may be introduced into a problem, that includes two or more conditions; as when we ſay, four numbers are in continual proportion, we mean, not only that the firſt number is to the ſecond as the ſecond is to the third, but alſo, that the ſecond number is to the third as the third is to the fourth.

Whenever a problem is propoſed to be ſolved algebraically, the Algebraiſt muſt ſubſtitute ſome letter of the alphabet for the unknown quantity: and if there be more unknown quantities than one, the reſt muſt receive their names from ſo many conditions of the problem: and if the problem be juſtly ſtated and examined, there will ſtill remain a condition at laſt, which, tranſlated into algebraic language, will afford him an equation, the reſolution whereof will give the unknown quantity for which the ſubſtitution was made; and when this unknown quantity is once diſcovered, the reſt will be eaſily diſcovered by their names. Suppoſe there are four unknown quantities in a problem; then there ought to be four conditions: now the firſt unknown quantity receives its name arbitrarily without any condition; therefore the other three

three muſt take up three of the conditions of the problem for their names; and the fourth condition will ſtill be left to furniſh out an equation.

The learner muſt here be very careful to make no poſitions but what are ſufficiently juſtifiable, either from the expreſs conditions of the problem, or from the nature of the thing; all the liberty he is allowed in caſes of this nature is, that he is not obliged to draw out the conditions in the ſame order as they are given him in the problem, but may make uſe of them in ſuch an order as he thinks will be moſt convenient for his purpoſe; provided that he does not make uſe of the ſame condition twice, except in company with others that have not been conſidered.

My method in the forty-four following problems will be, to put down the anſwer immediately after the problem, and then the ſolution: for, in my opinion, this way of putting down the anſwer firſt, will not only ſerve to illuſtrate the following ſolution, but may alſo ſerve to fix the problem more firmly in the minds of young beginners, who are but too apt to neglect it, and to ſubſtitute chimerical notions of their own, that are not to be juſtified, either from the conditions of the problem, or common ſenſe.

After the learner has run over ſome of theſe problems, and has got a tolerable inſight into the method of their reſolution, it will be very proper for him to begin again, and to attempt the ſolution of every problem himſelf, and not to have recourſe to the ſolutions here given, but in caſes of abſolute neceſſity: but after the work is over, he may then compare his own ſolution with that which is here given, and may alter or reform it as he thinks fit.

The ſolution of ſome problems producing ſimple equations.

PROBLEM I.

26. *What two numbers are thoſe, whoſe difference is* 14, *and whoſe ſum, when added together, is* 48?

Anſ.

Anſ. The numbers are 31 and 17: for $31-17=14$; and $31+17=48$.

Solution.

In this problem there are two unknown quantities, to wit, the two numbers ſought; and there are two conditions; firſt, that the leſs number when ſubtracted from the greater muſt leave 14; and ſecondly, that the two numbers when added together muſt make 48: therefore I put x for the leſs number; and to find a name for the greater, I have recourſe to the firſt condition of the problem, which informs me, that the difference betwixt the two numbers ſought is 14; therefore, if I call the leſs number x, I ought to call the greater $x+14$: thus then I have got names for both my unknown quantities, and have ſtill a condition in reſerve for an equation, which is the ſecond: now according to this ſecond condition, the two numbers ſought, when added together, muſt make 48; therefore x and $x+14$ when added together muſt make 48; but x and $x+14$ when added together make $2x+14$; whence I have this equation, $2x+14=48$; therefore $2x=48-14=34$; therefore x, or the leſs number$=17$, and $x+14$, or the greater number$=31$, as above.

In our ſolution of this problem, the notation was drawn from the firſt condition, and the equation from the ſecond; but the notation might have been drawn from the ſecond condition, and the equation from the firſt, thus: put x for the leſs number ſought; then becauſe the ſum of both the numbers is 48, if you ſubtract the leſs number x from 48, the remainder $48-x$ will be the greater number, ſo that the two numbers ſought will be x, and $48-x$: ſubtract the former number from the latter, and the remainder or difference will be $48-2x$; but, according to the firſt condition of the problem, this difference ought to be 14; therefore $48-2x=14$: reſolve this equation, and you will have $x=17$, and $48-x=31$, as above.

Problem

PROBLEM 2.

27. *Three perſons,* A, B *and* C, *make a joint contribution, which in the whole amounts to 76 pounds: of this,* A *contributes a certain ſum unknown;* B *contributes as much as* A, *and* 10 *pounds more; and* C *as much as both* A *and* B *together: I demand their ſeveral contributions.*

Anſ. *A* contributes 14 pounds, *B* 24, and *C* 38: for $14+10=24$, and $14+24=38$, and $14+24+38=76$.

SOLUTION.

In this problem there are three unknown quantities, and there are three conditions for finding them out; firſt, that the whole contribution amounts to 76 pounds; ſecondly, that *B* contributes as much as *A*, and 10 pounds more; and laſtly, that *C* contributes as much as both *A* and *B* together.

Theſe things being ſuppoſed, I firſt put x for *A's* contribution; then ſince, according to the ſecond condition, *B* contributes as much as *A*, and 10 pounds more, I put $x+10$ for *B's* contribution; laſtly, ſince *C* contributes as much as both *A* and *B* together, I add x and $x+10$ together, and ſo put down the ſum $2x+10$ for *C's* contribution: thus have I got names for all my unknown quantities, and there remains ſtill one condition unconſidered for my equation, which is, that all the contributions added together make 76 pounds; therefore I add x, and $x+10$, and $2x+10$ together, and ſuppoſe the ſum $4x+20=76$; therefore $4x=76-20=56$; therefore x, or *A's* contribution, equals 14; $x+10$, or *B's* contribution, equals 24; and $2x+10$, or *C's* contribution, equals 38, as above.

PROBLEM 3.

28. *Suppoſe all things as before, except that now, the whole contribution amounts to* 276 *pounds; that of this,*

this, A *contributes a certain ſum unknown; that* B *contributes twice as much as* A, *and* 12 *pounds more; and* C *three times as much as* B, *and* 12 *pounds more: I demand their ſeveral contributions,*

Anſ. *A* contributes 24 pounds, *B* 60, and *C* 192: for $24\times2+12=60$; and $60\times3+12=192$; and $24+60+192=276$.

Solution.

Put x for *A's* contribution; then, becauſe *B* contributes twice as much as *A*, and 12 pounds more, *B's* contribution will be $2x+12$; therefore, if *C* had contributed juſt three times as much as *B*, his contribution would have amounted to $6x+36$; but, according to the problem, *C* contributes this, and 12 pounds more; therefore *C's* contribution is $6x+48$; add theſe contributions together, to wit, x, $2x+12$, and $6x+48$, and you will have $9x+60=276$: therefore $9x=276-60=216$; and x, or *A's* contribution, equals 24; whence $2x+12$, or *B's* contribution, equals 60; and $6x+48$, or *C's* contribution, equals 192, as above.

Advertisement.

I know not whether it may not be thought impertinent here to put the learner in mind, that after x was found equal to 24, the other two unknown quantities, $2x+12$, and $6x+48$ were found, by ſubſtituting 24 inſtead of x.

Problem 4.

29. *One begins the world with a certain ſum of money, which he improved ſo well by way of traffick, that, at the year's end, he found he had doubled his firſt ſtock, except an hundred pounds laid out in common expences; and ſo he went on every year doubling the laſt year's ſtock, except a hundred a year expended as before; and at the end of three years, found himſelf juſt three times as rich as at firſt: What was his firſt ſtock?*

Anſ. 140 pounds: for the double of this is 280, and 280—100=180 pounds at the end of the firſt year; the double of this laſt is 360, and 360—100= 260 pounds at the end of the ſecond year; again, the double of this is 520, and 520—100=420 pounds at the end of the third year; and 420 pounds is juſt three times as much as 140 pounds, his firſt ſtock.

SOLUTION.

Put x for his firſt ſtock, that is, let x be the number of pounds he began with; then the double of this is $2x$, and therefore he will have $2x-100$ at the end of the firſt year; the double of this is $4x-200$; therefore he will have $4x-200-100$, that is, $4x-300$ at the end of the ſecond year; the double of this is $8x-600$; therefore he will have $8x-600-100$, that is, $8x-700$ at the end of the third year; but according to the problem, he ought to have three times his firſt ſtock, that is, $3x$, at the end of the third year; therefore $8x-700=3x$; therefore $8x-3x-700=0$, that is, $5x-700=0$; therefore $5x=700$; and x, or his firſt ſtock, equals 140, as above.

To this problem I ſhall add another of a like kind for the learner to ſolve himſelf.

One goes with a certain quantity of money about him to a tavern, where he borrows as much as he had then about him, and out of the whole, ſpends a ſhilling; with the remainder he goes to a ſecond tavern, where he borrows as much as he had then left, and there alſo ſpends a ſhilling; and ſo he goes on to a third, and a fourth tavern, borrowing and ſpending as before; after which he had nothing left: I demand how much money he had at firſt about him.

Anſ. $\frac{15}{16}$ of one ſhilling, that is, 11 pence farthing.

PROBLEM 5.

30. *One has ſix ſons, each whereof is four years older than his next younger brother; and the eldeſt is three times as old as the youngeſt: What are their ſeveral ages?*

Anſ.

Anſ. 10, 14, 18, 22, 26, 30: for 30, the age of the eldeſt, will then be juſt three times 10, that is, three times the age of the youngeſt.

SOLUTION.

For their ſeveral ages put x, $x+4$, $x+8$, $x+12$, $x+16$, $x+20$; then according to the problem $x+20$ the age of the eldeſt, ought to be equal to $3x$, that is, three times the age of the youngeſt; ſince then $3x=x+20$, we ſhall have $3x-x=20$, that is, $2x=20$, and $x=10$, as above.

PROBLEM 6.

31. *There is a certain fiſh whoſe head is 9 inches; the tail is as long as the head and half the back; and the back is as long as both the head and the tail together: I demand the length of the back, and of the tail.*

Anſ. The length of the back is 36 inches, and that of the tail 27: for $27=9+\frac{36}{2}$; and $36=9+27$.

SOLUTION.

For the length of the back put x; then will x be equal to the length of both head and tail together, by the ſuppoſition; therefore if from x, the length of the head and tail together, you ſubtract 9, the length of the head, there will remain $x-9$ for the length of the tail; but according to the problem, the tail is as long as the head and half the back; therefore $x-9=\frac{x}{2}+9$; therefore $2x-18=x+18$; therefore $2x-x-18=18$, that is, $x-18=18$; and x, the length of the back, equals $18+18=36$; therefore $x-9$, the length of the tail, equals 27, as above.

PROBLEM 7.

32. *One has a leaſe for 99 years; and being aſked how much of it was already expired, anſwered, that two thirds of the time paſt was equal to four fifths of the time to come; I demand the times paſt, and to come.*

Anſ.

Anſ. The time paſt was 54 years; and the whole term of years was 99; therefore the time to the expiration of the leaſe was 45 years: now $\frac{2}{3}$ of 54 is 36; and $\frac{4}{5}$ of 45 is 36.

SOLUTION.

Put x for the time paſt; then, ſince the whole term of years was 99, if x the time paſt be ſubtracted from 99 the whole time, there will remain $99-x$ for the time to come; but $\frac{2}{3}$ of the time paſt is $\frac{2x}{3}$; and $\frac{4}{5}$ of the time to come is $\frac{4}{5}$ of $\frac{99-x}{1}=\frac{396-4x}{5}$; therefore $\frac{2x}{3}=\frac{396-4x}{5}$; therefore $2x=\frac{1188-12x}{5}$; therefore $10x=1188-12x$; therefore $10x+12x=1188$, that is, $22x=1188$; and x the time paſt $=54$ years; therefore $99-x$ the time to come equals 45 years.

To this problem I ſhall add two others of the ſame nature, without any ſolution.

Firſt, *To divide the number* 84 *into two ſuch parts, that three times one part may be equal to four times the other.*

Anſ. The parts are 48 and 36: for in the firſt place, $48+36=84$; and in the next place, three times $48=144=$ four times 36.

Second, *To divide the number* 60 *into two ſuch parts, that a ſeventh part of one may be equal to an eighth part of the other.*

Anſ. The parts are 28 and 32; for in the firſt place, $28+32=60$; and in the next place, $\frac{1}{7}$ of 28 equals $4=\frac{1}{8}$ of 32.

PROBLEM 8.

33. *It is required to divide the number* 50 *into two ſuch parts, that* $\frac{3}{4}$ *of one part being added to* $\frac{5}{6}$ *of the other, may make* 40.

Anſ.

Anſ. The parts are 20 and 30: for in the firſt place, $20+30=50$; and in the next place, $\frac{3}{4}$ of 20, which is 15, added to $\frac{5}{6}$ of 30, which is 25, makes 40.

SOLUTION.

Put x for one part, and conſequently $50-x$ for the other part; then we ſhall have $\frac{3}{4}$ of $x=\frac{3x}{4}$, and $\frac{5}{6}$ of $50-x=\frac{250-5x}{6}$; but, according to the problem, theſe two added together ought to make 40; whence we have this equation, $\frac{3x}{4}+\frac{250-5x}{6}=40$: multiply by 4, and you will have $3x+\frac{1000-20x}{6}=160$; multiply again by 6, and you will have $18x+1000-20x=960$, that is, $1000-2x=960$; therefore $1000=2x+960$; and $1000-960=2x$, that is, $2x=40$; and x, which is one of the parts ſought, will be 20; whence $50-x$ or the other part will be 30, as above.

Other two problems of the ſame nature.

Firſt: *It is required to divide the number* 20 *into two ſuch parts, that three times one part being added to five times the other may make* 84.

Anſ. The parts are 8 and 12: for $8+12=20$; and $8\times3+12\times5$, that is, $24+60=84$.

Second: *It is required to divide the number* 100 *into two ſuch parts, that if a third part of one be ſubtracted from a fourth part of the other, the remainder may be* 11.

Anſ. The parts are 24 and 76: for firſt, 24 added to 76 makes 100; and ſecondly; $\frac{1}{3}$ part of 24, which is 8, ſubtracted from $\frac{1}{4}$ of 76, which is 19, leaves 11.

PROBLEM

PROBLEM 9.

34. *Two perſons* A *and* B *engage at play*; A *has* 72 *guineas and* B 52 *before they begin*; *and after a certain number of games won and loſt between them*, A *riſes with three times as many guineas as* B: *I demand how many guineas* A *won of* B.

Anſ. 21: for $72+21=93$; and $52-21=31$; and $93=31\times3$.

SOLUTION.

Put x for the number of guineas *A* won of *B*, and conſequently that *B* loſt; then will *A's* laſt ſum be $72+x$, and *B's* laſt ſum $52-x$: now, according to the problem, *A's* laſt ſum is three times as much as *B's* laſt ſum; that is, three times $\overline{52-x}$, or $156-3x$; therefore $72+x=156-3x$; therefore $72+x+3x=156$, that is, $72+4x=156$; therefore $4x=156-72=84$; therefore x, the money *A* won of *B*, equals 21 guineas, as above.

PROBLEM 10.

35. *One meeting a company of beggars, gives to each four pence, and has ſixteen pence over*; *but if he would have given them ſix pence apiece, he would have wanted twelve pence for that purpoſe*: *I demand the number of perſons.*

Anſ. 14: for $14\times4+16=72=14\times6-12$.

SOLUTION.

Put x for the number of perſons; then if he gives them four pence apiece, the number of pence given will be four times as many as the number of perſons, that is, $4x$; therefore $4x+16$ will expreſs all the money he had about him; and ſo alſo will $6x-12$ by a like way of reaſoning; therefore $4x+16=6x-12$; therefore $16=6x-4x-12=2x-12$; therefore $2x=16+12=28$; and x, the number of perſons equal 14, as above.

I PROBLEM

Problem 11.

36. *What two numbers are those, whose difference is 4, and the difference of those squares is 112?*

Anſ. 12 and 16: for $16-12=4$, and $16\times16-12\times12$, that is, $256-144=112$.

Solution.

The leſs number, x.
The greater, $x+4$.

$$\begin{array}{r} x+4 \\ x+4 \\ \hline xx+4x+16 \\ +4x \\ \hline \end{array}$$

The ſquare of the greater, $xx+8x+16$
The ſquare of the leſs, xx
The difference of their ſquares, $*\quad 8x+16$;

whence $8x+16=112$; therefore $8x=112-16=96$; therefore x the leſs number equals 12, and $x+4$ the greater equals 16, as above.

Problem. 12.

37. *What two numbers are those, whereof the greater is three times the leſs, and the ſum of whoſe ſquares is five times the ſum of the numbers?*

Anſ. The numbers are 6 and 2, whoſe ſum is 8: now $6=3$ times 2; and $6\times6+2\times2=40=5$ times 8.

Solution.

The leſs number	x.
The greater,	$3x$.
Their ſum,	$4x$.
The ſquare of the leſs,	xx.
The ſquare of the greater,	$9xx$.
The ſum of their ſquares,	$10xx$.

But, according to the problem, the ſum of their ſquares is 5 times the ſum of their numbers, that is, 5 times $4x$ or $20x$; therefore $10xx=20x$; and $10x=20$;

=20; and x the leſs number = 2; whence $3x$ the greater equals 6, as above.

PROBLEM 13.

38. *What two numbers are thoſe, whereof the leſs is to the greater as 2 to 3, and the product of whoſe multiplication is 6 times the ſum of the numbers?*

Anſ. The numbers are 10 and 15, whoſe ſum is 25: for 10 is to 15 as 2 to 3. This will be plain by putting the queſtion thus; if 2 gives 3, what will 10 give? for the anſwer will be 15: theſe numbers will alſo anſwer the ſecond condition of the problem; for $10 \times 15 = 150 = 25 \times 6$.

SOLUTION.

Put x for the leſs number; then to find the greater number ſay, if 2 gives 3, what will x give? and the anſwer is $\frac{3x}{2}$; therefore, if x ſtands for the leſs number, the greater number will be $\frac{3x}{2}$, their ſum will be $\frac{x}{1} + \frac{3x}{2}$, or $\frac{2x+3x}{2}$, or $\frac{5x}{2}$; and the product of their multiplication $x \times \frac{3x}{2}$, or $\frac{3xx}{2}$; but, according to the problem, the product of their multiplication ought to be ſix times the ſum of their numbers, that is, ſix times $\frac{5x}{2}$, or $\frac{30x}{2}$; therefore $\frac{3xx}{2} = \frac{30x}{2}$; and $3x^2 = 30x$; and $3x = 30$; and x the leſs number equals 10; therefore $\frac{3x}{2}$ the greater number equals 15, as above.

SOLUTION.

For the greater number I put x; then, had their ſum been 108, I ſhould for the other number have put $108-x$; but it is not the ſum of their addition, but the product of their multiplication, that is equal to 108; therefore, if one number be called x, the other will be $\frac{108}{x}$, which I thus demonſtrate: let y be the other number; then will $x\times y$ or $xy=108$ by the ſuppoſition; divide both ſides of the equation by x, and you will have $y=\frac{108}{x}$; as was to be demonſtrated. This being admitted, the difference between the greater number x, and the leſs $\frac{108}{x}$, is $x-\frac{108}{x}$; and their ſum is $x+\frac{108}{x}$: but, by the condition of the problem, this ſum ought to be equal to twice the difference, that is, to twice $\overline{x-\frac{108}{x}}$ or $2x-\frac{216}{x}$; therefore $2x-\frac{216}{x}=x+\frac{108}{x}$; therefore $2xx-216$ $=xx+108$; therefore $2xx-xx-216=108$, that is, $xx-216=108$; therefore $xx=108+216=324$; therefore x the greater number equals 18, and $\frac{108}{x}$ the leſs equals 6, as above.

PROBLEM 16.

41. *It is required to divide the number* 48 *into two ſuch parts, that one part may be three times as much above* 20, *as the other wants of* 20.

Anſ. The two parts are 32 and 16: for $32+16=48$; moreover 32 is 12 above 20, and 16 wants 4 of 20, and 12 is three times 4.

SOLUTION.

Put x for the leſs number ſought; then will $48-x$ be the greater, and the exceſs of this greater above 20 will be $28-x$, as is evident by ſubtracting 20 from $48-x$: again, the exceſs of 20 above the leſs number (which is, what the leſs number wants of 20) is $20-x$; and according to the problem, the former exceſs is three times the latter, that is, three times $\overline{20-x}$; or $60-3x$; whence we have this equation, $28-x=60-3x$; therefore $28-x+3x=60$, that is, $28+2x=60$; therefore $2x=60-28=32$; therefore x the leſs part$=16$, and $48-x$ the greater $=32$, as above.

Another ſolution of the foregoing problem.

Put x for what the leſs number wants of 20; then will the leſs number be $20-x$, the greater $20+3x$, and their ſum $40+2x$; but, by the problem, their ſum is 48; therefore $40+2x=48$; therefore $2x=48-40=8$; therefore $x=4$; whence $20-x$ the leſs number $=16$, and $20+3x$ the greater$=32$.

PROBLEM 17.

42. *One has three debtors,* A, B, *and* C, *whoſe particular debts he has forgot; but thus much he could remember from his account, that* A's *and* B's *debts together amounted to* 60 *pounds;* A's *and* C's *to* 80 *pounds; and* B's *and* C's *to* 92 *pounds: I demand the particulars.*

Anſ. A's debt was 24 pounds, *B's* 36, and *C's* 56: for $24+36=60$, $24+56=80$, and $36+56=92$.

SOLUTION.

Put x for *A's* debt; then, becauſe *A's* and *B's* together made 60 pounds, *B's* debt will be $60-x$; again, becauſe *A's* and *C's* together made 80 pounds, *C's* debt muſt be $80-x$: now ſince, according to the

 problem,

problem, *B's* and *C's* debts when added together make 92 pounds, I add $60-x$, and $80-x$ together, and suppose the sum $140-2x=92$; whence $2x+92=140$; and $2x=140-92=48$; and x, that is, *A's* debt, $=24$ pounds: whence $60-x$, or *B's* debt, $=36$ pounds; and $80-x$, or *C's*, is 56 pounds, as above.

PROBLEM 18.

43. *One being asked how many teeth he had remaining in his head, answered, Three times as many as he had lost; and being asked how many he had lost, answered, As many as, being multiplied into* $\frac{1}{6}$ *part of the number left, would give all he ever had at first: I demand how many he had lost, and how many he had left?*

Ans. He had lost 8, and had 24 left: for then 24 the number left, will be equal to 3 times 8, the number lost; and moreover 8 the number lost, multiplied into 4, that is, into $\frac{1}{6}$ part of 24 the number left, will give $32=24+8$, all he ever had at first.

SOLUTION.

Teeth lost,	x.
left,	$3x$.
In all,	$4x$.

$\frac{1}{6}$ part of the number left $\frac{3x}{6}$, or $\frac{x}{2}$; this, multiplied into the number lost, makes $\frac{x}{2}\times x$ or $\frac{xx}{2}$; but, according to the problem, this product is equal to all he ever had at first; whence $\frac{xx}{2}=4x$; and $xx=8x$; and x, the number lost, $=8$; whence $3x$, the number left, $=24$, as above.

PROBLEM 19.

44. *One rents 25 acres of land at 7 pounds 12 shillings* per annum; *which land consists of two sorts, the bet-*

ter ſort he rents at 8 *ſhillings* per *acre, and the worſe at* 5: *I demand the number of acres of each ſort.*

Anſ. He had 9 acres of the better ſort. and 16 of the worſe: for 9 times 8 ſhillings $=72$ ſhillings; and 16 times 5 ſhillings $=80$ ſhillings; and $72+80=152$ ſhillings $=7$ pounds 12 ſhillings.

Solution.

Put x for the number of acres of the better ſort; then will $25-x$ be the number of acres of the worſe, ſort, becauſe both together make 25 acres: moreover, ſince he paid 8 ſhillings an acre for the better ſort, he muſt pay 8 times as many ſhillings as he had acres, that is, $8x$: and ſince he paid 5 ſhillings an acre for the worſe ſort, he muſt pay 5 times as many ſhillings as he had acres of this ſort, that is, $\overline{25-x}\times 5$, or $125-5x$: put both theſe rents together, and they will amount to $8x+125-5x$, or $3x+125$ ſhillings; but they amount to 152 ſhillings by the ſuppoſition; therefore $3x+125=152$; therefore $3x=152-125=27$; therefore x, the number of acres of the better ſort, $=9$, and $25-x$, the number of the worſe ſort, $=16$, as above.

Problem 20.

45 *One hires a labourer into his garden for* 36 *days upon the following conditions; to wit, that for every day he laboured, he was to receive two ſhillings and ſixpence; and for every day he was abſent, he was to forfeit one ſhilling and ſixpence: now at the end of the* 36 *days, after due deductions made for his forfeitures, he received clear* 2 *pounds* 18 *ſhillings: I demand how many days he laboured, and how many he was abſent.*

Anſ. He laboured 28 days, and loitered 8: for 28 half-crowns amount to 3 pounds 10 ſhillings due to him for wages; and 8 eighteenpences amount to 12 ſhillings due from him in forfeitures; and this latter ſum ſubtracted from the former, leaves 2 pounds 18 ſhillings to be received clear.

Solution.

SOLUTION.

Put x for the number of days he laboured; then will $36-x$ reprefent the number of days he was abfent: again, fince he was to receive 30 pence for every day he laboured, the pence due to him in wages will be $30\times x$, or $30x$; and fince he was to forfeit 18 pence for every day he was abfent, the pence due from him in forfeitures will be $18\times\overline{36-x}$, or $648-18x$: fubtract now $648-18x$, the pence due from him in forfeitures, from $30x$, the pence due to him for wages; or, which is all one, add $18x-648$ to $30x$, and there arifes $48x-648$, the pence to be received clear: but he received clear 2 pounds 18 fhillings, or 696 pence, by the fuppofition; therefore $48x-648=696$; therefore $48x=648+696=1344$; therefore x, the number of days he laboured, $=28$; and $36-x$, the number of days he loitered, $=8$, as above.

PROBLEM 22.

47. *One lets out a certain fum of money at 6 per cent. fimple intereft; which intereft in 10 years time wanted but 12 pounds of the principal: What was the principal?*

Anf. The principal was 30 pounds, and the intereft 18 pounds $=30-12$: for as 100 pounds principal is to its annual intereft 6 pounds, fo is 30 pounds principal to its annual intereft 1.8 pounds; and therefore its 10 years intereft will be 18 pounds.

SOLUTION.

Put x for the number of pounds in the principal; then, to find its intereft for one year, fay, if 100 pound principal give 6 pounds intereft, what will x principal give? and the anfwer will be $\frac{6x}{100}$; this will be the intereft of x for one year, and therefore its intereft interest for ten years will be $\frac{60x}{100}$, or $\frac{6x}{10}$, or $\frac{3x}{5}$: but, according to the problem, this interest is to be $x-12$; for it is to want just 12 pounds of the principal, by the supposition; therefore $x-12=\frac{3x}{5}$; therefore $5x-60=3x$; therefore $5x-3x-60=0$, that is, $2x-60=0$; therefore $2x=60$, and x the principal $=30$ and $\frac{3x}{5}$ the 10 years interest $=18$ pounds, as above.

Problem 23.

48. *One lets out 98 pounds in two different parcels; one at 5, the other at* 6 per cent. *simple interest; and the interest of the whole in 15 years amounted to 81 pounds: What were the two parcels?*

Ans. The parcel at 5 *per cent.* was 48 pounds, and the other at 6 *per cent.* was 50 pounds: for in the first place, $48+50=98$; and moreover, the annual interest of 48 pounds at 5 *per cent.* amounts to 2 pounds 8 shillings; and the annual interest of 50 pounds at 6 *per cent.* is 3 pounds; therefore the whole interest amounts to 5 pounds 8 shillings in one year; and consequently to 81 pounds in 15 years.

Solution.

Put x for the number of pounds in the parcel at 5 *per cent.* and consequently $98-x$ for the number of pounds in the other parcel at 6 *per cent.*; then, to find the annual interest of x, say, if 100 pounds principal give 5 pounds interest, what will x give? and the answer will be $\frac{5x}{100}$: again, for the other parcel, say, if 100 pounds principal give 6 pounds interest, what will $98-x$ give? and the answer will be $\frac{588-6x}{100}$: add these two interests together, to wit,

$\frac{5x}{100}$

$\frac{5x}{100}$ and $\frac{588-6x}{100}$, and the ſum will be $\frac{5x+588-6x}{100}$ that is, $\frac{588-x}{100}$ this is the intereſt of the two parcels for one year; and therefore, in 15 years time, the intereſt muſt amount to $\frac{8820-15x}{100}$; but it amounts to 81 pounds, by the ſuppoſition; therefore $\frac{8820-15x}{100}=81$; therefore $8820-15x=8100$; therefore $8820=15x+8100$; therefore $15x=8820-8100=720$; therefore x, the parcel at 5 *per cent.* $=48$ pounds; and $98-x$, the parcel at 6 *per cent.* $=50$ pounds, as above.

PROBLEM 24.

49. *A gentleman hires a ſervant for a year*, or 12 *months, and was to allow him for his wages ſix pounds in money, together with a livery cloak of a certain value agreed upon: but after ſeven months, upon ſome miſdemeanor of the ſervant, he turns him off, with the aforeſaid cloak and* 50 *ſhillings in money; which was all that was due to him for that time: I demand the value of the cloak.*

Anſ. The value of the cloak was 48 ſhillings: for then his whole wages for 12 months would be 168 ſhillings; and by the rule of proportion, his wages for 7 months would be 98 ſhillings; whence ſubtracting 48 ſhillings, the value of the cloak, there would remain 50 ſhillings due to him in money.

SOLUTION.

Put x for the value of the cloak in ſhillings; then will his whole wages for 12 months be $x+120$; and his wages for 7 months, may be found by the golden rule, ſaying, as 12 is to 7, ſo is $x+120$ to $\frac{7x+840}{12}$; but,

but, according to the problem, his wages for 7 months was the cloak and 50 ſhillings in money, that is, $x+50$; therefore $x+50=\frac{7x+840}{12}$; therefore $12x+600=7x+840$; therefore $12x-7x+600=840$, that is, $5x+600=840$; therefore $5x=840-600=240$; therefore x, the value of the cloak in ſhillings, is 48, as above.

Problem 25.

50. *One diſtributes 20 ſhillings among 20 people, giving 6 pence apiece to ſome, and 16 pence apiece to the reſt: I demand the number of perſons of each denomination.*

Anſ. There were 8 perſons who received 6 pence apiece; and 12 who received 16 pence apiece: for in the firſt place, $8+12=20$ perſons; and ſince 8 ſixpences are equivalent to 4 ſhillings, and 12 ſixteenpences to 16 ſhillings, we ſhall have in the next place, $4+16=20$ ſhillings.

Solution.

Put x for the number of perſons who received 6 pence apiece; then, ſince there were 20 perſons in all, $20-x$ will be the number of thoſe who received ſixteenpence apiece: the number of pence received by the former company will be $6x$; and the number of pence received by the latter will be $\overline{20-x}\times16$, that is, $320-16x$; and therefore the whole number of pence received will be $6x+320-16x$, or $320-10x$; but, according to the problem, there was received in the whole, 20 ſhillings, or 240 pence; therefore, $320-10x=240$; therefore $10x+240=320$; therefore $10x=320-240=80$; therefore x, the number of perſons who received ſixpence apiece, is 8, and conſequently $20-x$, the number of the reſt, is 12, as above.

Problem

PROBLEM 26.

51. *It is required to divide 24 ſhillings into 24 pieces, conſiſting only of ninepences and thirteenpencehalf-pennies.*

Anſ. There muſt be 8 ninepences, and 16 thirteen-pencehalfpennies; for in the firſt place, $8+16=24$ pieces; and ſince 8 ninepences are equivalent to 6 ſhillings, and 16 thirteenpencehalfpennies to 18 ſhil-lings, we have in the next place $6+18=24$ ſhillings.

SOLUTION.

Put x for the number of ninepences, and conſe-quently $24-x$ for the number of thirteenpencehalf-pennies: now the number of halfpence equivalent to the former is $18x$, becauſe there are 18 halfpence in every ninepence; and the number of halfpence equiva-lent to the latter is $\overline{24-x}\times 27$, or $648-27x$, be-cauſe there are 27 halfpence in every thirteenpence-halfpenny piece: therefore the number of halfpence equivalent to the whole will be $18x+648-27x$, that is, $648-9x$; but, according to the problem, the whole amounts to 24 ſhillings, or 576 halfpence; there-fore $648-9x=576$; therefore $9x+576=648$; there-fore $9x=648-576=72$; therefore x, the number of ninepences, is 8; and $24-x$, the number of thirteen-pencehalfpennies, is 16, as above.

PROBLEM 27.

52. *Two perſons,* A *and* B, *travelling together,* A *with* 100, *and* B *with* 48 *pounds about him, met a com-pany of robbers, who took twice as much from* A *as from* B, *and left* A *thrice as much as they left* B: *I demand how much they took from each.*

Anſ. They took 44 pounds from *B*, and twice as much, that is, 88 pounds from *A*, ſo they left *B* 4 pounds, and *A* 12 pounds, which is 3 times 4.

SOLUTION.

Solution.

Taken from B, x.
from A, $2x$.
Left B, $48-x$.
Left A, $100-2x$.

But, according to the problem, they left A three times as much as they left B, that is, three times $48-x$, or $144-3x$; therefore $100-2x=144-3x$; therefore $100-2x+3x=144$, that is, $100+x=144$; therefore x, the ſum taken from B, $=144-100=44$; and $2x$, or 88, is the ſum taken from A, as above.

Problem 30.

55. *There are two places 154 mils diſtant from each other; from whence two perſons ſet out at the ſame time with a deſign to meet, one travelling at the rate of 3 miles in two hours, and the other at the rate of 5 miles in 4 hours: I demand how long and how far each travelled before they met.*

Anſ. As our travellers were ſuppoſed both to ſet out at the ſame time, and they muſt both meet at the ſame time, it follows, that each muſt perform his journey in the ſame time; I ſay then, that each performed his journey in 56 hours: for if in 2 hours the firſt travelled 3 miles, in 56 hours he muſt travel 84 miles, by the rule of proportion: in like manner, if in 4 hours the ſecond travels 5 miles, in 56 hours he muſt travel 70 miles; and $84+70=154$ miles, the whole diſtance.

Solution.

Put x for the number of hours each travelled; then, to find how many miles the firſt travelled, ſay, if in 2 hours he travelled 3 miles, how far did he travel in x hours? and the anſwer is $\frac{3x}{2}$; then for the other ſay, if in 4 hours he travelled 5 miles, how far did he travel in x hours? and the anſwer is $\frac{5x}{4}$, therefore both their journies put together make $\frac{3x}{2}$ $+\frac{5x}{4}$; but they both travelled the whole diſtance, 154 miles; therefore $\frac{3x}{2}+\frac{5x}{4}=154$; therefore $3x$ $+\frac{10x}{4}=308$: therefore $12x+10x$, that is, $22x$ $=1232$; therefore x, the number of hours each travelled, $=56$; therefore $\frac{3x}{2}$, the number of miles the firſt travelled, $=84$; and $\frac{5x}{4}$, the number of miles the ſecond travelled, $=70$, as above.

PROBLEM 31.

56. *One ſets out from a certain place, aud travels at the rate of 7 miles in 5 hours; and 8 hours after, another ſets out from the ſame place, and travels the ſame road at the rate of 5 miles in 3 hours: I demand how long and how far the firſt muſt travel before he is overtaken by the ſecond.*

Anſ. The firſt muſt travel 50 hours, and conſequently 70 miles; the ſecond muſt travel 50—8, or 42 hours, and conſequently alſo 70 miles: ſince then they both ſet out from the ſame place, and the ſecond traveller has now travelled as far as the firſt, he muſt have overtaken the firſt.

SOLUTION.

Put x for the number of hours the firſt travelled, and conſequently $x-8$ for the number of hours wherein the ſecond travelled: then, to find the miles travelled

travelled by the firſt, ſay, if in 5 hours he travels 7 miles, how far will he travel in x hours? and the anſwer is $\frac{7x}{5}$; then for the other ſay, if in 3 hours he travelled 5 miles, how far will he travel in $x-8$ hours, and the anſwer is $\frac{5x-40}{3}$; but as theſe two travellers both ſet out from the ſame place, and muſt come together at the ſame place, it follows, that they muſt both travel the ſame length of ſpace; therefore $\frac{5x-40}{3}=\frac{7x}{5}$; therefore $5x-40=\frac{21x}{5}$; therefore $25x-200=21x$; therefore $25x-21x-200=0$, that is, $4x-200=0$; therefore $4x=200$; and x, the hours travelled by the firſt, $=50$; whence $x-8$, the hours travelled by the ſecond, $=42$; $\frac{7x}{5}$, the miles travelled by the firſt, $=70$; and $\frac{5x-40}{3}$, the miles travelled by the ſecond, $=70$, as above.

PROBLEM 36.

61. *A ſhepherd driving a flock of ſheep in time of war, meets a company of ſoldiers who plunder him of half his flock, and half a ſheep over; the ſame treatment he meets with from a ſecond, a third, and a fourth company, every ſucceeding company plundering him of half the flock the laſt had left, and half a ſheep over; inſomuch that at laſt he had but 7 ſheep left: I demand how many he had at firſt.*

Anſ. His flock at firſt conſiſted of 127 ſheep; and if the firſt company had only robbed him of half his flock, they would have left him $63\frac{1}{2}$ ſheep; but as they plundered him of half his flock, and half a ſheep over, they left him only 63 ſheep; in like manner the ſecond company left him 31, the third 15, and the fourth 7.

N. B.

N. B. Before I enter upon the ſolution of this problem, I muſt put the learner in mind of what he has been told before, (introduction, art. 13.) to wit, that a fraction may be halved two ways, either by halving the numerator, or doubling the denominator.

SOLUTION.

Put x for the number of his firſt flock; then, had the firſt company only taken half his flock, they would have left him the other half, *viz.* $\frac{x}{2}$; but they took half his flock and half a ſheep over; therefore they left him juſt ſo much leſs, to wit, $\frac{x}{2}-\frac{1}{2}$, or $\frac{x-1}{2}$: again, had the ſecond company only taken half what remained, they would have left him half, to wit, $\frac{x-1}{4}$; but by taking half a ſheep more, they left him $\frac{x-1}{4}-\frac{1}{2}$, that is, $\frac{2x-2-4}{8}$, or $\frac{2x-6}{8}$, or $\frac{x-3}{4}$; in like manner the third company left $\frac{x-3}{8}-\frac{1}{2}$, or $\frac{2x-6-8}{16}$, or $\frac{2x-14}{16}$, or $\frac{x-7}{8}$; and the laſt company left him $\frac{x-7}{16}-\frac{1}{2}$, or $\frac{x-15}{16}$; but they left him 7 ſheep, by the ſuppoſition; therefore $\frac{x-15}{16}=7$; and $x-15=112$; and x his firſt number $=127$, as above.

PROBLEM 37.

62. *One buys a certain number of eggs, half whereof he buys in at 2 a penny, and the other half at 3 a*

penny; these he afterwards sold out again at the rate of 5 for twopence, and, contrary to his expectation, lost a penny by the bargain: what was the number of his eggs?

Ans. The number of his eggs was 60; half whereof at two a penny cost him 15 pence; and the other half at three a penny, ten pence; and the whole 25 pence: but 60 eggs sold out at 5 for two pence, would only bring him in 24 pence, as appears by the rule of proportion; therefore he lost a penny by the bargain.

SOLUTION.

Put x for the number of eggs; then say, if 2 eggs cost one penny, what will $\frac{x}{2}$ one half of his eggs cost? and the answer will be $\frac{x}{4}$; and for the same reason the other half at three a penny will cost him $\frac{x}{6}$; so that for the whole he must pay $\frac{x}{4}+\frac{x}{6}$, or $\frac{5x}{12}$: again say, if five eggs were sold for two pence, what were x eggs sold for? and the answer will be $\frac{2x}{5}$; therefore $\frac{2x}{5}$ will be the number of pence he received for his eggs; subtract this from $\frac{5x}{12}$, the pence he paid for them, and the remainder $\frac{5x}{12}-\frac{2x}{5}$, or $\frac{x}{60}$ will be his loss; but by the supposition, he lost one penny; therefore $\frac{x}{60}=1$; and x the number of eggs will be 60, as above.

PROBLEM

PROBLEM 39.

64. *It is required to divide the number* 90 *into two ſuch parts, that one part may be to the other as* 2 *to* 3.

Anſ. The numbers are 36 and 54: for in the firſt place, $36+54=90$; and in the next place, if both 36 and 54 be divided by 18, the quotients will be 2 and 3; whence I infer, that 36 is to 54 as 2 to 3; for a common diviſion by the ſame number cannot alter the proportion of the numbers divided; and therefore if, after this common diviſion, the quotients be to one another as 2 to 3, the dividends muſt be alſo in the ſame proportion.

SOLUTION.

Put x for the leſs part, and $90-x$ for the other; then will x be to $90-x$ as 2 to 3, by the ſuppoſition; but by art. 15, whenever there are four proportionals, the product of the extremes will be equal to the product of the middle terms; here the extremes are x and 3, whoſe product is $3x$; and the middle terms are $90-x$ and 2, whoſe product is $180-2x$; therefore $3x=180-2x$; therefore $5x=180$; and x, the leſs part, $=36$; and $90-x$, the greater, $=54$, as above.

PROBLEM 41.

66. *What number is that, which, being ſeverally added to* 36 *and* 52, *will make the former ſum to the latter as* 3 *to* 4?

Anſ. The number is 12: for $36+12$ is to $52+12$, as 48 is to 64, as $\frac{48}{16}$ is to $\frac{64}{16}$, as 3 to 4.

SOLUTION.

Put x for the number ſought, and you will have this proportion; $36+x$ is to $52+x$ as 3 to 4. Whence by multiplying extremes and means you will have $144+4x=156+3x$; therefore $144+x=156$; therefore x the number ſought $=12$, as above.

Problem 42.

67. *A bookbinder ſells me two paper books, one containing 48 ſheets for 3 ſhillings and 4 pence, and another containing 75 ſheets for 4 ſhillings and 10 pence, both bound at the ſame price, and both of the ſame ſort of paper: I demand what he allows himſelf for binding.*

Anſ. He reckoned 8 pence for binding; ſo that the price of the paper of the firſt book was 32 pence, and the price of the paper of the latter 50 pence: now if this anſwer be juſt, the two prices ought to bear the ſame proportion to one another as the two quantities of paper; and ſo we ſhall find them: for 32 pence are to 50 pence as $\frac{32}{2}$ are to $\frac{50}{2}$, that is, as 16 to 25; and 48 ſheets are to 75 ſheets as $\frac{48}{3}$ are to $\frac{75}{3}$, that is alſo, as 16 to 25.

Solution.

Put x for the number of pence reckoned for binding; then we ſhall have $40-x$ for the price of the paper in the firſt book, and $58-x$ for the price of the paper in the ſecond book; and $40-x$ will be to $58-x$ as 48 to 75; multiply extremes and means, and you will have this equation, $2784-48x=3000-75x$; therefore $2784+27x=3000$; therefore $27x=216$; and x the number of pence reckoned for binding $=8$, as above.

Problem 43.

68. *What number is that, which, being ſeverally added to 15, 27, and 45, will give three numbers in continual proportion.*

N. B. Three numbers are ſaid to be in continual proportion, when the firſt is to the ſecond as the ſecond is to the third.

Anſ. The number ſought is 9: for $15+9=24$; and $27+9=36$; and $45+9=54$; and 24 is to 36 as 36 is to 54: for 24 is to 36 as $\frac{24}{12}$ is to $\frac{36}{12}$, that is, as

2 to

2 to 3; and 36 is to 54 as $\frac{36}{18}$ is to $\frac{54}{18}$, that is alſo, as 2 to 3.

SOLUTION.

Put x for the number ſought; then we ſhall have this proportion, $x+15$ is to $x+27$, as $x+27$ is to $x+45$; where the two middle terms are $x+27$ and $x+27$: multiply extremes and means, and you will have this equation, $xx+60x+675=xx+54x+729$; therefore $60x+675=54x+729$; therefore $6x+675=729$; therefore $6x=54$; and x the number ſought $=9$, as above.

Of the method of reſolving problems wherein more unknown quantities than one are concerned, and repreſented by different letters.

70. Hitherto we have uſed but one ſingle letter in every problem for ſome one unknown quantity in it; and if there were more, the reſt received their names from the conditions of the problem; but in caſes of a more complicated nature, where many unknown quantities are linked and entangled in one another, this method will be found very difficult; and therefore, in ſuch caſes, the Algebraiſt is allowed to uſe as many different letters as he has unknown quantities, provided he finds out as many independent equations for diſcovering their values; ſee art. 92: for though in every equation wherein more unknown quantities than one are concerned, they hinder one another from being found out, yet if as many fundamental equations at firſt be given as there are unknown quantities, it will not be difficult, in many caſes, from theſe to derive others that are more ſimple, till at laſt you come to an equation wherein but one only unknown quantity is concerned, in which caſe all the reſt are ſaid to be exterminated.

Whenever two or more equations are propoſed, involving as many unknown quantities, theſe equations muſt firſt be prepared by freeing them from fractions where-ever there are any, and by ordering every particular equation ſo, that all the unknown quantities may poſſeſs one ſide of the equation, and ſuch as are known the other; or elſe, that all the quantities may poſſeſs one ſide of the equation, and a cypher the other; it will be alſo convenient, that in every particular equation, the unknown quantities be placed in the ſame order.

In laying down rules for exterminating unknown quantities, I ſhall begin with the ſimpleſt caſe firſt, which is that of two equations, and two unknown quantities; and when I have given as many examples as ſhall be thought proper in this caſe, I ſhall then proceed to others where more unknown quantities are to be exterminated.

But here I muſt not forget to advertiſe the reader, that, as I am now treating of ſimple equations, and problems producing ſuch equations, I ſhall not meddle with any caſes of extermination which lead to equations of higher forms: when I come to treat of quadratic equations, I may then perhaps add ſomething further upon this ſubject; but to undertake to explain all the various methods of exterminating unknown quantities would be an endleſs taſk, and a moſt intolerably laborious and tedious one both to the writer and the reader, whom I cannot yet ſuppoſe to be ſo far gone in Analytics, as to be willing to purchaſe this ſort of knowledge at any rate.

Let then x and y be two unknown quantities to be found out by the help of the two following equations, $4x-5y=2$, and $6x-7y=4$: or the queſtion may be ſtated thus: if $4x-5y=2$, and $6x-7y=4$, what are x and y? Now as theſe equations want no preparation, put them down one under another; then upon a bye piece of paper multiply the firſt equation ($4x-5y=2$) by 6 the coefficient of x in the ſecond equation, and the product will give this equation, $24x-30y=12$; again, multiply the ſecond equation ($6x-7y=4$) by

4,

4, the coefficient of x in the firſt equation, and the product gives $24x-28y=16$; ſubtract now either of theſe two laſt equations from the other, and x will be exterminated: I chooſe in the preſent caſe to ſubtract the former equation from the latter, that the coefficient of y after ſubtraction may be affirmative, thus;

$$\begin{array}{r} 24x-28y=16 \\ 24x-30y=12 \\ \hline *+2y=4. \end{array}$$

From this ſubtraction you have the following equation, $2y=4$, which put down under the two firſt equations to make a third; then reſolve this third equation $2y=4$, and you will have $y=2$, which put down under the reſt for a fourth equation.

Having thus found the value of $y=2$, put this value inſtead of y in the more ſimple of the two firſt equations, ſuppoſe in the equation $4x-5y=2$, and you will have $4x-10=2$; whence $4x=12$, and $x=3$, which put down for a fifth equation, and the work is done; for x is now found equal to 3, and y equal to 2, and theſe numbers three and two being ſubſtituted for x and y reſpectively, will anſwer both the conditions of the queſtion, that is, you will have $4x-5y=12-10=2$, and $6x-7y=18-14=4$.

1ſt Equ.	$4x-5y=2.$
2d,	$6x-7y=4.$
3d,	$*+2y=4.$
4th,	$*\quad y=2.$
5th,	$x\quad *=3.$

The coefficients of x, the quantity to be exterminated in the two firſt equations, were 4 and 6: now, as theſe numbers admit of a common diviſor without any remainder, namely 2, divide them both by 2, and the quotients will be 2 and 3; uſe now theſe numbers 2 and 3 inſtead of 4 and 6, and the operation, as well as the equation reſulting from it, will become more ſimple: for the firſt equation multiplied by 3 inſtead of 6, gives $12x-15y=6$; and the ſecond equation multiplied by 2 inſtead of 4, gives $12x-14y=8$; and the difference of theſe two equations is $y=2$.

Another way of exterminating the unknown quantity x is as follows: find out the value of x in reſpect of y, in the more ſimple of the two firſt equations; then, ſubſtituting this value inſtead of x in the other equation, you will have an equation, wherein y alone is concerned: thus in the foregoing example, the firſt equation was $4x-5y=2$, therefore $4x=5y+2$, and $x=\frac{5y+2}{4}$; ſubſtitute now this value $\left(\frac{5y+2}{4}\right)$ inſtead of x in the ſecond equation, $6x-7y=4$, by making $6x=\frac{30y+12}{4}$, and you will have this equation, $\frac{30y+12}{4}-7y=4$; therefore $30y+12-28y=16$; therefore $2y+12=16$; whence $2y=4$, and $y=2$; and x, or $\frac{5y+2}{4}=3$, as before.

N.B. 1*ſt*, What has here been ſaid concerning the extermination of the quantity x, may as well be applied to the other quantity y, except that its coefficients 5 and 7 will not admit of a common diviſor, as did the numbers 4 and 6.

2*dly*, Of the two different ways of extermination here laid down, ſometimes one will be found more expeditious, and ſometimes the other, as will appear by the following problems.

3*dly*, In the caſe of two unknown quantities, if the value of either of them can be had in integral terms in both equations, equate the two values one to the other, and you will have the other unknown quantity, by means whereof the firſt will alſo be known; and this makes a third way of extermination, whereof there

there are ſo many examples in the following problems, that nothing more needs here to be ſaid of it.

Whenever two quantities, as x and y, are multiplied together to produce a third, xy, the two multiplicants x and y are called factors, or efficients, in which caſe, each is ſaid to be the other's coefficient: thus, in the quantity xy, x is ſaid to be the coefficient of y, and y the coefficient of x; therefore, if in any quantity wherein x is concerned as an efficient, its coefficient be deſired; divide that quantity by x, and the quotient will be the coefficient: thus if the quantity $12x-yx$ be divided by x, the quotient is $12-y$; therefore in the quantity $12x-yx$, the coefficient of x is $12-y$.

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The reader muſt now no longer expect to have all ſimple equations reſolved to his hand, as hitherto has been done. If, after ſixteen examples of ſimple equations reſolved, and the ſolution of forty-four Algebraic problems, he be ſtill at a loſs how to reduce a ſimple equation, it muſt proceed from a weakneſs that either admits of no cure or deſerves none.

PROBLEM 45.

71. *What two numbers are thoſe, the product of whoſe multiplication is* 144, *and the quotient of the greater divided by the leſs is* 16?

SOLUTION.

Put x for the greater number, and y for the leſs; and the queſtion when abſtracted from words will ſtand thus: if $xy=144$, and $\frac{x}{y}=16$, what are x and y?

The firſt of theſe equations wants no preparation, and therefore may be put down thus;

Equ. 1ſt, $xy \quad * = 144$.

The

The ſecond equation, when prepared according to the laſt art. will ſtand thus;

Equ. 2d, $x-16y=0$.

Multiply the firſt equation by 1, the ſuppoſed coefficient of x in the ſecond, and the quotient not being altered by ſuch a multiplication, will be $xy * = 144$; multiply alſo the ſecond equation by y, which according to the foregoing art. is the coefficient of x in the firſt, and you will have $xy-16yy=0$; ſubtract this latter product from the former, and you will have, Equ. 3d, $* \ 16yy=144$; whence

Equ. 4th, $* \quad y=3$.

Subſtitute now 3 inſtead of y, or $3x$ inſtead of xy in the firſt equation, and you will have $3x=144$, and conſequently,

Equ. 5th, $x \quad * = 48$.

So that the numbers at laſt are found to be 48 and 3; and they will anſwer the conditions of the queſtion: for $48 \times 3 = 144$, and $\frac{48}{3} = 16$.

$$\begin{array}{llcr} \text{Equ. 1ſt,} & xy & * & =144. \\ \text{2d,} & x- & 16y & =0. \\ \text{3d,} & * & 16yy & =144. \\ \text{4th,} & * & y & =3. \\ \text{5th,} & x & * & =48. \end{array}$$

Another ſolution of the foregoing problem, from the laſt article.

Having found from the ſecond equation that $x = 16y$; put $16y$ for x, or $16yy$ for xy in the firſt equation, and you will have $16yy = 144$; whence y and x may be found as before.

Problem 46.

72. *It is required to find two numbers with the following properties, to wit, that the firſt with half the ſecond may make* 20; *and moreover, that the ſecond with a third part of the firſt may make* 20.

Solution.

SOLUTION.

Put x for the firſt number, and y for the ſecond, and the fundamental equations will be $x+\frac{y}{2}=20$, and $y+\frac{x}{3}=20$; which being prepared according to art. 70, will ſtand thus;

Equ. 1ſt, $2x+y=40$.
Equ. 2d, $x+3y=60$.

Subtract the firſt equation from twice the ſecond, and you will have

Equ. 3d, $*\ 5y=80$; whence
Equ. 4th, $*\ y=16$.

Put 16 inſtead of y in the firſt equation, and you will have $2x+16=40$, whence

Equ. 5th, $x\ *=12$.

Therefore the numbers ſought are 12 and 16, and not 16 and 12, though 16 was found firſt; becauſe $x=12$ was put for the firſt number. That theſe numbers will anſwer the conditions of the queſtion is plain: for $12+\frac{16}{2}$ or $12+8=20$; and $16+\frac{12}{3}$, or $16+4=20$.

Another ſolution from art. 70.

Having found from the ſecond equation that $x=60-3y$, put $60-3y$ for x, or $120-6y$ for $2x$ in the firſt equation, and you will have $120-6y+y=40$; whence $y=16$, as before.

PROBLEM 47.

73. *One exchanges 6 French crowns and two French dollars for 45 ſhillings; and at another time 9 crowns and 5 dollars of the ſame coin for 76 ſhillings: I demand the diſtinct values of a crown and of a dollar.*

SOLUTION.

Put x and y for the number of ſhillings a crown and a dollar are reſpectively worth, and the equations will ſtand thus;

Equ.

Equ. 1ſt, $6x+2y=45$.
Equ. 2d, $9x+5y=76$.

Subtract 3 times the firſt equation from twice the ſecond, and you will have

Equ. 3d, * $4y=17$; whence
Equ. 4th, * $y=4\frac{1}{4}$ ſhillings;

that is, 4 ſhillings and 3 pence; put now $4\frac{1}{4}$ for y or $8\frac{1}{2}$ for $2y$ in the firſt equation, and you will have $6x+8\frac{1}{2}=45$, and $6x=36\frac{1}{2}$, and

Equ. 5th, x * $=6\frac{1}{12}$;

that is, $6\frac{1}{12}$ ſhillings, or 6 ſhillings and a penny; therefore the value of a crown was 6 ſhillings and a penny, and that of a dollar 4 ſhillings and 3 pence; and theſe values will anſwer the conditions of the queſtion; for, at this rate, 6 crowns will amount to 36 ſhillings and 6 pence, 2 dollars to 8 ſhillings and 6 pence, and the whole to 45 ſhillings; moreover, 9 crowns will amount to 54 ſhillings and 9 pence, 5 dollars to 21 ſhillings and 3 pence, and the whole ſum to 76 ſhillings.

Problem 48.

74. *It is required to find two ſuch numbers, that half the firſt together with a third part of the ſecond may make* 32; *and moreover, that a fourth part of the firſt together with a fifth part of the ſecond may make* 18.

Solution.

Put x and y for the two numbers, and the fundamental equations will be $\frac{x}{2}+\frac{y}{3}=32$, and $\frac{x}{4}+\frac{y}{5}=18$; which equations, when duly prepared, will ſtand thus;

Equ. 1ſt, $3x+2y=192$.
Equ. 2d, $5x+4y=360$.

Subtract 5 times the firſt equation from 3 times the ſecond, and you will have

Equ. 3d, * $2y=120$; whence
Equ. 4th, * $y=60$;

whence,

whence, and from the first equation, you will have $3x+2y$, or $3x+120=192$, which gives

Equ. 5th, $x \quad * = 24$.

So the numbers are 24 and 60; and they will answer the conditions of the question: $\frac{24}{2}+\frac{60}{3}$, that is, $12+20=32$; and moreover, $\frac{24}{4}+\frac{60}{5}$, that is, $6+12=18$.

PROBLEM 49.

75. *Two persons* A *and* B *were talking of their ages: says* A *to* B, *7 years ago I was just three times as old as you were, and 7 years hence I shall be just twice as old as you will be: I demand their present ages.*

SOLUTION.

Let a and b represent the present ages of A and B respectively; then their ages 7 years ago were $a-7$ and $b-7$, and their ages 7 years hence will be $a+7$ and $b+7$; whence, and from the conditions of the problem, may be derived the two following fundamental equations:

$$a-7=\overline{b-7}\times 3=3b-21, \text{ and}$$
$$a+7=\overline{b+7}\times 2=2b+14.$$

From the former of these two equations, to wit, $a-7=3b-21$, we have $a=3b-14$; from the second equation, to wit, $a+7=2b+14$, we have $a=2b+7$; therefore $3b-14=2b+7$, since both are equal to a; whence $b=21$, and $2b+7$, or $a=49$.

A therefore was 49 years old, and B 21 years old; which is true: for then, 7 years before, A's age would be 42, and B's 14; and 42 is three times 14: on the other hand, 7 years after, A's age would be 56, and B's 28; and 56 is twice 28.

PROBLEM 50.

76. *A jockey has two horses,* A *and* B, *whose values are sought: he has also two saddles, one valued at 12 pounds, the other at 2: now if he sets the better*

saddle

ſaddle upon A, *and the worſe ſaddle upon* B, A *will then be worth twice as much as* B; *but on the other hand, if he ſets the better ſaddle upon* B, *and the worſe ſaddle upon* A, B *will then be worth three times as much as* A: *I demand the values of the horſes.*

Solution.

Let a and b repreſent the prices of the two horſes *A* and *B* reſpectively in pounds; then if the better ſaddle be ſet upon *A*, and the worſe upon *B*, *A* will be worth $a+12$, and *B* will be worth $b+2$, and the firſt fundamental equation will be $a+12=\overline{b+2}\times2=2b+4$; on the other hand, if the better ſaddle be ſet upon *B*, and the worſe upon *A*, then *B* will be worth $b+12$, and *A* will be worth $a+2$, and the ſecond fundamental equation will be $b+12=\overline{a+2}\times3=3a+6$: in the firſt fundamental equation, where $a+12=2b+4$, we have $a=2b-8$; ſubſtitute therefore $2b-8$ inſtead of a, or rather $6b-24$ inſtead of $3a$, in the ſecond fundamental equation (which is $3a+6=b+12$), and you will have $6b-24+6$, that is, $6b-18=b+12$; whence $b=6$, and $2b-8$, or $a=4$: *A* then was valued at 4 pounds, and *B* at 6, and they will anſwer the conditions of the queſtion, as any one may eaſily try.

Problem 51.

77. *There is a certain fraction, which if an unit be added to the numerator, will be equal to* $\frac{1}{3}$; *but if on the contrary an unit be added to the denominator, the fraction will then be equivalent to* $\frac{1}{4}$: *I demand the numerator and denominator of the fraction.*

Solution.

Call the fraction $\frac{x}{y}$, and you will have theſe two fundamental equations, $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$: the

former

former of these equations, when reduced, gives $y=3x+3$, and the latter gives $y=4x-1$; therefore $4x-1=3x+3$, because both are equal to y; whence x the numerator of the fraction is 4; and $3x+3$, or y, the denominator is 15; and the fraction itself is, $\frac{4}{15}$; which if an unit be added to the numerator, will be $\frac{5}{15}$, or $\frac{1}{3}$; but if an unit be added to the denominator, it will be $\frac{4}{16}$, or $\frac{1}{4}$.

PROBLEM 52.

78. *There is a certain fishing rod consisting of two parts, whereof the upper part is to the lower as 5 to 7; and moreover 9 times the upper part, together with 13 times the lower, is equal to 11 times the whole rod and 36 inches over: I demand the length of the two parts.*

SOLUTION.

Put x for the length of the upper part in inches, and y for the lower; then will $x+y$ be the length of the whole rod, and since x is to y as 5 to 7 *ex hypothesi*, by multiplying extremes and means according to art. 15, you will have $7x=5y$ for a fundamental equation: again, as 9 times the upper part, together with 13 times the lower, is equal to 11 times the whole rod, and 36 inches over, you have $9x+13y=11x+11y+36$ for a second fundamental equation: the latter of these two equations gives $x=y-18$, and consequently $7x=7y-126$; substitute this value instead of $7x$, in the first fundamental equation, where $7x=5y$, and you will have $7y-126=5y$; whence $y=63$; and $y-18$, or $x=45$.

The upper part therefore was 45 inches, and the lower 63, as will appear upon trial.

PROBLEM 53.

79. *One lays out 2 shillings and sixpence in apples and pears, buying his apples at four, and his pears at five a penny; and afterwards accommodates his neighbour with half his apples and one third part of his pears*

 for

for thirteenpence, which was the price he bought them at: I demand how many he bought of each ſort.

SOLUTION.

Put x for the number of apples, and y for the number of pears; then if 4 apples coſt one penny, x will coſt $\frac{x}{4}$ pence; and for the ſame reaſon y will coſt $\frac{y}{5}$ pence, and you will have $\frac{x}{4}+\frac{y}{5}=30$ for a firſt fundamental equation: again, the price of $\frac{x}{2}$, half of his apples, will be $\frac{x}{8}$ and the price of $\frac{y}{3}$, a third part of his pears, will be $\frac{y}{15}$; and you will have $\frac{x}{8}+\frac{y}{15}=13$ for a ſecond fundamental equation. Hence,

Equ. 1ſt, $5x+4y=600$.
Equ. 2d, $15x+8y=1560$.

Subtract the ſecond equation from three times the firſt, according to art. 70, and you will have

Equ. 3d, * $4y=240$; whence
Equ. 4th, * $y=60$.

Subſtitute now 60 for y, that is, 240 for $4y$ in the firſt equation $5x+4y=600$, and you will have $5x+240=600$; whence

Equ. 5th, x * $=72$.

Therefore the number of apples was 72, and the number of pears 60, as will appear upon trial.

PROBLEM 57.

83. *A certain company at a tavern found, when they came to pay their reckoning, that if they had been three more in company to the ſame reckoning, they might have paid one ſhilling apiece leſs than they did; and that, had they been two fewer in company, they muſt have paid one ſhilling apiece more than they did; I demand the number of perſons, and their* quota.

SOLUTION.

SOLUTION.

Put x for the number of perſons, and y for the number of ſhillings every one actually paid; now if 4 perſons are to pay 5 ſhillings apiece, the whole reckoning muſt be 4×5 or 20 ſhillings; therefore if x perſons are to pay y ſhillings apiece, the whole reckoning muſt be $y \times x$ or xy ſhillings: this being laid down, ſuppoſe them now to be three more in company; then will the number of perſons be $x+3$; and to find what every particular perſon ought to pay in this caſe, the whole reckoning xy muſt be divided by $x+3$, the number of perſons, and the quotient $\frac{xy}{x+3}$ will be every one's particular reckoning; but according to the problem, every one's particular reckoning in this caſe would have been one ſhilling leſs than it actually was, that is, $y-1$; therefore $\frac{xy}{x+3} = y-1$; in like manner the ſecond condition of the problem furniſhes this equation, $\frac{xy}{x-2} = y+1$: the firſt of theſe equations, to wit, $\frac{xy}{x+3} = y-1$, being reduced, gives $x = 3y-3$; and the ſecond equation, to wit, $\frac{xy}{x-2} = y+1$ being reduced gives $x = 2y+2$; therefore $3y-3 = 2y+2$, and $y = 5$; whence $2y+2$, or $x = 12$.

So there were 12 perſons in company, their reckoning 5 ſhillings apiece, and their whole reckoning 3 pounds, or 60 ſhillings; which anſwers the conditions of the queſtion: for $\frac{60}{15} = 4$, and $\frac{60}{10} = 6$.

PROBLEM 61.

88. *What two numbers are thoſe, whoſe ſum is twice, and the product of whoſe multiplication is twelve times their difference?*

 SOLUTION.

Solution.

Put x for the greater number, and y for the lefs; then will their difference be $x-y$, their fum $x+y$, and the product of their multiplication xy or yx; and the equations will be $x+y=2x-2y$, and $yx=12x-12y$; whence

Equ. 1ft, $x-3y=0$.

Equ. 2d, $12x-yx-12y=0$.

Multiply the firft equation by $12-y$, which, by art. 70, is the coefficient of x in the fecond, and the product will be $12x-yx-36y+3yy=0$; fubtract this equation from the fecond, and you will have

Equ. 3d, $24y-3yy=0$; whence

Equ. 4th, $y=8$; and

Equ. 5th, $x=24$.

And the numbers 24 and 8 will anfwer the conditions.

Otherwife thus: by the firft equation $x=3y$, and $4x=12y$; fubftitute $4x$ for $12y$ in the fecond equation, and you will have $12x-yx-4x=0$; divide by x, and you will have $12-y-4=0$, and $y=8$, and x or $3y=24$, as before.

Problem 62.

89. *What two numbers are thofe, whofe difference, fum and product are to each other as are the numbers two, three and five refpectively; that is, whofe difference is to their fum as two to three, and whofe fum is to their product as three to five?*

Solution.

Put x for the greater number, and y for the lefs; then will their difference be $x-y$, their fum $x+y$, and their product yx; and we fhall have thefe two proportions productive of two equations, 1ft, $x-y$ is to $x+y$ as 2 to 3, whence $3x-3y=2x+2y$; 2d, $x+y$ is to yx as 3 to 5, whence $3yx=5x+5y$: the refolution follows;

Equ.

Equ. 1st, $x-5y=0$.

Equ. 2d, $3yx-5x-5y=0$.

Multiply the first equation by $3y-5$, the coefficient of x in the second, and the product will be $3yx-5x-15yy+25y=0$; subtract this from the second equation, and you will have,

Equ. 3d, $15yy-30y=0$; whence

Equ. 4th, $y=2$, and

Equ. 5th, $x=10$.

And the numbers 10 and 2 will answer the conditions of the problem.

Otherwise thus: by the first equation $x=5y$; substitute therefore x instead of $5y$ in the second, and you will have $3yx-5x-x=0$; divide by x, and you will have $3y-5-1=0$, and $y=2$, as before.

PROBLEM 63.

90. *It is required to find two numbers such, that if their difference be multiplied into their sum, the product will be five; but if the difference of their squares be multiplied into the sum of their squares, the product will be sixty-five.*

SOLUTION.

Put x for the greater number, and y for the less; then will their difference be $x-y$, their sum $x+y$, and the product of their sum and difference multiplied together will be x^2-y^2, by art. 11; then will $x^2-y^2=5$ by the supposition, and $x^2=5+yy$; square both sides, and you will have $x^4=25+10y^2+y^4$: again, the difference of the squares of the two numbers sought is x^2-y^2, and the sum of their squares x^2+y^2, and the product of these two x^4-y^4; therefore $x^4-y^4=65$ by the supposition, and $x^4=65+y^4$; but x^4 was before found equal to $25+10y^2+y^4$; therefore $25+10y^2+y^4=65+y^4$; whence $y^2=4$, and $y=2$; substitute now 4 for y^2 in the first fundamental equation, which was $x^2-y^2=5$, and you will have $x^2-4=5$, and $x=3$; therefore the numbers sought are 3 and 2, which will answer the conditions.

Problem 65.

93. *Three persons,* A, B *and* C *were talking of their money; says* A *to* B *and* C, *Give me half of your money, and I shall have* d; *says* B *to* A *and* C, *Give me a third part of your money, and I shall have* d; *says* C *to* A *and* B, *Give me a fourth part of your money, and I shall have* d. *How much money had each?*

N. B. The letter d is here supposed to supply the place of some known quantity, which is left undetermined till the calculation is over.

Solution.

Let a, b and c represent the money of A, B and C respectively, and we shall have these three fundamental equations;

$$a+\frac{b+c}{2}=d;$$

$$b+\frac{a+c}{3}=d; \text{ and}$$

$$c+\frac{a+b}{4}=d.$$

These equations, after due preparations according to art. 70, will stand thus;

Equ. 1st, $2a+b+c=2d.$
Equ. 2d, $a+3b+c=3d.$
Equ. 3d, $a+b+4c=4d.$

Subtract the first equation from twice the second, and you will have

Equ. 4th, * $5b+c=4d.$

Subtract the third equation from the second, and you will have

Equ. 5th, * $2b-3c=-d.$

Subtract five times the fifth equation from twice the fourth, and you will have

Equ. 6th, * * $17c=13d.$

Equ. 7th, * * $c=\frac{13d}{17}.$

Put

Put this value for c in the fourth equation, and you will have $5b+c$, that is, $5b+\frac{13d}{17}=4d$; therefore $85b+13d=68d$, therefore $85b=55d$, and $b=\frac{55d}{85}=\frac{11d}{17}$; therefore

Equ. 8th, $* \quad b \quad * = \frac{11d}{17}$.

Put now the two values of b and c already found, instead of b and c in the first equation, and you will have $2a+b+c$, that is, $2a+\frac{11d+13d}{17}$, or $2a+\frac{24d}{17}=2d$; whence $34a+24d=34d$; and $34a=10d$, and $a=\frac{10d}{34}=\frac{5d}{17}$; therefore

Equ. 9th, $a \quad * \quad * = \frac{5d}{17}$.

So that the numbers are at last found to be $a=\frac{5d}{17}$, $b=\frac{11d}{17}$, and $c=\frac{13d}{17}$; whence it follows, that if any number be put for d, that will admit of the number 17 for a divisor, the quantities a, b and c will come out in whole numbers: as if d be made equal to 17, the quantities a, b and c will be, 5, 11 and 13 respectively; and the numbers will answer the conditions of the problem; for $5+\frac{11+13}{2}$, or $5+12=17$; $11+\frac{5+13}{3}$, or $11+6=17$; $13+\frac{5+11}{4}$, or $13+4=17$.

Advertisement. I hope the reader does not need to be told, that the numbers a, b and c must always be understood to be of the same denomination with the number d; as, if the number d signifies so many guineas,

 the

the numbers a, b and c muſt alſo ſignify guineas; if ſhillings, ſhillings; if pence, pence;

Equ. 1ſt, $2a+b+c=2d$.	Equ. 6th, $* * 17c=13d$.
2d, $a+3b+c=3d$.	7th, $* * c=\frac{13d}{17}$.
3d, $a+b+4c=4d$.	8th, $* b *=\frac{11d}{17}$.
4th, $* 5b+c=4d$.	9th, $a * *=\frac{5d}{17}$.
5th, $*2b-3c=-d$.	

A Scholium.

94. Of the foregoing equations, the firſt, ſecond and third, wherein the quantity a is concerned, may be called equations of the firſt rank; the fourth and fifth, wherein the quantity b is concerned, and out of which the quantity a is excluded, may be called equations of the ſecond rank; the ſixth, wherein c is concerned, and out of which both a and b are excluded, may be called an equation of the third rank; and ſo on, were there ever ſo many unknown quantities.

Whenever the equations of any particular rank are given or found, in order to derive from thence equations of an inferior rank, the Analyſt is at liberty to combine theſe firſt equations by pairs as he pleaſes, provided he does but obſerve theſe two things; firſt, that every equation of the given rank be ſome time or other coupled with ſome other equation of the ſame ſet, ſo as that no equation be left out of the account; ſecondly, that in every particular combination, one of the equations be ſuch as was never made uſe of in any combination before, and the other ſuch as hath been concerned in ſome combination before, excepting the firſt pair. It is not to be denied but that the artiſt may, if he pleaſes, vary ſometimes from this laſt precept; but if he always obſerves it, it will be altogether as well.

THE

ELEMENTS OF ALGEBRA.

BOOK III.

Of the composition and resolution of a square raised from a binomial root.

101. HITHERTO we have been chiefly concerned in simple equations: it is now high time to apply ourselves to the resolution of quadratics; in order to which, something must be said concerning the nature of a binomial, upon which that resolution entirely depends.

Now a binomial (at least as it is here used) is a quantity consisting of two parts or members, connected together by the sign + or —, as $x+a$, $x-a$, $x+\frac{b}{2}$, $x-\frac{b}{2}$; and a square raised from a binomial root is nothing else but the square of such a quantity: thus the square of $x+\frac{b}{2}$ is $xx+bx+\frac{bb}{4}$, and that of $x-\frac{b}{2}$ is $xx-bx+\frac{bb}{4}$.

$$x+\frac{b}{2} \qquad\qquad x-\frac{b}{2}$$

$$x+\frac{b}{2} \qquad\qquad x-\frac{b}{2}$$

$$x^2+\frac{bx}{2}+\frac{bb}{4} \qquad\qquad x^2-\frac{bx}{2}+\frac{bb}{4}$$

$$+\frac{bx}{2} \qquad\qquad -\frac{bx}{2}$$

$x^2+\frac{2bx}{2}+\frac{bb}{4}$; that is, $x^2-\frac{2bx}{2}+\frac{bb}{4}$; that is,

$$x^2+bx+\frac{bb}{4} \qquad\qquad x^2-bx+\frac{bb}{4}.$$

The difference betwixt these two squares arises from the different sign of b; and that only affects the second member; for the third member $\frac{bb}{4}$ will be the same, whether the quantity b be affirmative or negative; therefore, if those cases be thrown into one, it will stand thus: *The square of* $x\pm\frac{b}{2}$ *is* $xx\pm bx+\frac{bb}{4}$; *to wit,* $+bx$ *when the root is* $x+\frac{b}{2}$, *and* $-bx$ *when the root is* $x-\frac{b}{2}$. Now of the three members that compose this square, the first xx is the square of x, the second $\pm bx$ is the root of that square multiplied into the coefficient $\pm b$; for the root of xx is x, and $x\times\pm b=\pm bx$; the third and last member $\frac{bb}{4}$, is the square of $\pm\frac{b}{2}$, that is, the square of half the coefficient of the second member; whence may be deduced the two following observations.

Obseevation

OBSERVATION I.

Whenever we meet with a quantity consisting of two members, as $xx \pm bx$, *whereof one, as* xx, *is a square, and the other* $\pm bx$ *is the root of that square multiplied into some given coefficient* $\pm b$; *whenever I say we meet with such a quantity, it may be considered as an imperfect square raised from a binomial root, and may easily be compleated by adding* $\frac{bb}{4}$, *that is, by adding the square of half the coefficient of* x *in the second term:* thus $xx + 6x$ when compleated becomes $xx + 6x + 9$; $xx - 8x$ when compleated becomes $xx - 8x + 16$; $xx + 3x$ when compleated becomes $xx + 3x + \frac{9}{4}$; for here the coefficient being 3, its half will be $\frac{3}{2}$, and the square of this will be $\frac{9}{4}$: again, $xx + \frac{2x}{3}$ when compleated becomes $xx + \frac{2x}{3} + \frac{1}{9}$; for here the second term is $\frac{2x}{3}$, and therefore the coefficient of x is $\frac{2}{3}$ by art. 70; but the half of $\frac{2}{3}$ is $\frac{1}{3}$, and the square of this is $\frac{1}{9}$: again, $xx - \frac{5x}{6}$ when compleated becomes $xx - \frac{5x}{6} + \frac{25}{144}$; for here the coefficient is $-\frac{5}{6}$, whose half is $-\frac{5}{12}$, and the square of this is $+\frac{25}{144}$: lastly, $xx - \frac{bx}{a}$ when compleated becomes $xx - \frac{bx}{a} + \frac{bb}{4aa}$; for here the coefficient is $-\frac{b}{a}$, its half $-\frac{b}{2a}$, and the square of this is $\frac{bb}{4aa}$.

OBSERVATION 2.

In the second place it may be observed, that the root of such a square when compleated, that is, the root of $xx \pm bx + \frac{bb}{4}$ *will always be* $x \pm \frac{b}{2}$ *that is, it will always be the square root of the first member, together with half the coefficient, of the second:* thus the square root of $xx + 6x + 9$ will be $x + 3$; that of $xx - 8x + 16$ will be $x - 4$; that of $xx + 3x + \frac{9}{4}$ will be $x + \frac{3}{2}$; that of $xx + \frac{2x}{3} + \frac{1}{9}$ will be $x + \frac{1}{3}$; that of $xx - \frac{5x}{6} + \frac{25}{144}$ will be $x - \frac{5}{12}$; and lastly, that of $xx - \frac{bx}{a} + \frac{bb}{4aa}$ will be $x - \frac{b}{2a}$.

The common form to which all quadratic equations ought to be reduced in order to be resolved.

102. Since an affected quadratic equation, as we have elsewhere defined it (art. 23,) is an equation consisting of three different sorts of quantities; one sort wherein the square of the unknown quantity is concerned, another sort wherein the unknown quantity is simply concerned, and a third sort wherein it is not concerned at all; it follows, that all quadratic equations whatever may be reduced to this form, *viz.* $Axx = Bx + C$; wherein A, B and C denote known integral quantities whether affirmative or negative, and x the quantity unknown, the sign $+$ on the latter side of the equation $Bx + C$, signifying no more than that the two quantities Bx and C are to be added together according to the common rules of addition, whether they be both affirmative or both negative, or one affirmative and the other negative: this will easily be allowed, if it be considered, that quadra-

tic equations, like all others, may be freed from fractions after the ſame manner as ſimple equations; and when that is done, there needs no more at moſt, than a bare tranſpoſition of the terms to reduce them to the form above deſcribed: we ſhall however give ſome examples of the reduction of quadratic equations to this form, amongſt thoſe that follow.

A general theorem for reſolving all quadratic equations.

103. This preparation being made, let now ſome general quadratic equation be propoſed to be reſolved, with which all particular equations may afterwards be compared, and by means whereof thoſe equations may be more readily reſolved; as for example, let the general equation in the laſt article be propoſed, to wit, $Axx = Bx + C$; and let it be propoſed to find the value or values of x in this equation; here, tranſpoſing Bx, I have $Axx - Bx = C$; and then dividing by A in order to free xx the higheſt power of x from its coefficient, I have $xx - \frac{Bx}{A} = \frac{C}{A}$; this done, I conſider the firſt ſide $xx - \frac{Bx}{A}$ as an imperfect ſquare raiſed from a binomial root; and accordingly I compleat that ſquare by art. 101, to wit, by adding $\frac{BB}{4AA}$, that is, by adding the ſquare of half the coefficient of the ſecond term; but if $\frac{BB}{4AA}$ muſt be added to the firſt ſide of the equation to compleat the ſquare, it muſt alſo be added to the other ſide to preſerve the equality; otherwiſe, by an unequal addition, the equation would be deſtroyed: this equal addition then being made, the equation will ſtand thus,

thus, $xx - \frac{Bx}{A} + \frac{BB}{4AA} = \frac{BB}{4AA} + \frac{C}{A}$; but the two fractions $\frac{BB}{4AA}$ and $\frac{C}{A}$, when thrown into one, give $\frac{ABB+4AAC}{4AAA}$, which, dividing by A, gives $\frac{BB+4AC}{4AA}$; therefore $xx - \frac{Bx}{A} + \frac{BB}{4AA} = \frac{BB+4AC}{4AA}$; therefore the ſquare root of one ſide will be equal to the ſquare root of the other; but the ſquare root of the fraction $\frac{BB+4AC}{4AA}$, at leaſt as it here ſtands in letters, cannot be extracted, becauſe, though the denominator $4AA$ be a ſquare, yet there is no literal quantity whatever which being multiplied into itſelf will produce $BB+4AC$; therefore, to put this numerator into the form of a ſquare, let us ſuppoſe $BB+4AC=ss$; and then the equation will ſtand thus, $xx - \frac{Bx}{A} + \frac{BB}{4AA} = \frac{ss}{4AA}$; but the ſquare root of $xx - \frac{Bx}{A} + \frac{BB}{4AA}$ is $x - \frac{B}{2A}$, by art. 101; and the ſquare root of $\frac{ss}{4AA}$ is $\pm\frac{s}{2A}$ for a reaſon formerly given, to wit, becauſe $\frac{-s}{2A}$ when multiplied into itſelf will produce $\frac{+ss}{4AA}$ as well as $\frac{+s}{2A}$; and therefore, by the very definition of the ſquare root, the former quantity has as good a right to be ſtiled the ſquare root of $\frac{ss}{4AA}$ as the latter; therefore this equation will now be reduced to a ſimple one, and will

will ſtand thus, $x - \frac{B}{2A} = \pm \frac{s}{2A}$; therefore $x = \frac{B \pm s}{2A}$, that is, $x = \frac{B + s}{2A}$, and $x = \frac{B - s}{2A}$. *Q. E. I.*

Thus we ſee that every quadratic equation neceſſarily admits of two numbers or roots (as they are called) which will equally anſwer the condition of the equation, that is, either of which being put equal to x, will make the two ſides of the equation equal one to the other; and theſe two roots, in all arts and ſciences where quadratic equations are concerned, are of equal eſtimation, whether they be affirmative or negative, or one be affirmative and the other negative: as for example, in Geometry, if a line drawn from any point towards the right hand be conſidered as affirmative, a line drawn from the ſame point to the left hand ought to be conſidered as negative; for let *AB* be any line drawn from the fixt point *A* to the point *B* on the right hand, and then imagine the point *B* to move towards *A*; here then it is plain that the nearer *B* approaches towards *A*, the leſs will be the affirmative line *AB*; when the point *B* coincides with *A*, the line *AB* muſt be looked upon as nothing, and therefore, when the point *B* by a continuation of its motion has paſſed through *A*, ſo as to lie on the left hand of *A*, the line *AB* ought now to be looked upon as negative, having paſſed from ſomething through nothing into negation; and yet a line of this negative kind is as true a line as any of the affirmative kind; and therefore the negative roots of quadratic equations, which exhibit negative lines, ought to be of equal eſtimation with the affirmative roots that exhibit affirmative lines; and the ſame will be the caſe (I ſay) of all other arts and ſciences where quadratic equations are concerned: but in common life, where negative quantities have no place, the affirmative roots of quadratic equations are only allowed of in the reſolution

lution of problems, the negative ones being for the most part excluded.

N. B. 1*st*, The root of any quantity whether in numbers or letters, that cannot be expressed, is called a surd: thus $\sqrt{3}$ is a surd, and so also is $\sqrt{BB+4AC}$; and it was for this reason, that I made $\sqrt{BB+4AC} = s$, or, which is all one, $BB+4AC=ss$.

2*dly*, The quantity C and consequently $4AC$ will sometimes be negative; in which case the quantity ss, or $BB+4AC$ must be looked upon as the sum of the affirmative quantity BB and the negative one $4AC$ when added together according to the common rules of addition.

3*dly*, In many of the following examples, the learner must be very careful to form a right estimation of negative quantities: thus for instance, if x, that is, $+x=-3$, he must make $4x$, or $+4\times-3=-12$; but he must make $-4x$, or $-4\times-3=+12$; so likewise $-x$, or $-1x$, or -1×-3 must be made equal to $+3$, &c.

A synthetical demonstration of the foregoing theorem.

104. In the last article it was demonstrated analytically, that if Axx be equal to $Bx+C$, then x must necessarily be equal both to $\frac{B+s}{2A}$, and to $\frac{B-s}{2A}$, supposing ss to be equal to $BB+4AC$. Now it may not be improbable but that the learner, especially if he has any taste or genius. may have a curiosity to see the same demonstrated again synthetically, that is, to see it demonstrated, that if x be made equal to $\frac{B+s}{2A}$, or $\frac{B-s}{2A}$, then Axx must necessarily be equal to $Bx+C$: it is therefore to gratify the learner in this particular, that I have added the following demonstration.

CASE 1st.

Let $x = \frac{B+s}{2A}$; then you will have $xx = \frac{BB+2Bs+ss}{4AA}$; multiply both sides by A, and you will have Axx (or one side of the general equation) equal to $\frac{BB+2Bs+ss}{4A}$; for a fraction may be multiplied by dividing the denominator, as well as by multiplying the numerator: again, since $x = \frac{B+s}{2A}$, you will have $Bx = \frac{BB+Bs}{2A}$; double both the numerator and denominator of this last fraction, which will not affect the value of the fraction, and you will have $Bx = \frac{2BB+2Bs}{4A}$; therefore $Bx+C = \frac{2BB+2Bs}{4A} + \frac{C}{1} = \frac{2BB+2Bs+4AC}{4A} = \frac{BB+2Bs+BB+4AC}{4A} = \frac{BB+2Bs+ss}{4A}$, because $BB+4AC=ss$ by the supposition; therefore $Axx=Bx+C$, since each side is equal to the same quantity $\frac{BB+2Bs+ss}{4A}$.

CASE 2d.

Let now $x = \frac{B-s}{2A}$, and you will have $xx = \frac{BB-2Bs+ss}{4AA}$, and Axx (or the first side of the general equation) $= \frac{BB-2Bs+ss}{4A}$: again, $Bx = \frac{BB-Bs}{2A} = \frac{2BB-2Bs}{4A}$;

$-24=1$; therefore $s=1$, $\frac{B+s}{2A}=\frac{5+1}{12}=\frac{1}{2}$, $\frac{B-s}{2A}$ $=\frac{5-1}{12}=\frac{1}{3}$; therefore the two roots of this equation $6xx=5x-1$ are $\frac{1}{2}$ and $\frac{1}{3}$. The resolution of this equation in numbers, without the general theorem, is as follows: Equation, $6xx=5x-1$; therefore $6xx-5x$ $=-1$, and $xx-\frac{5x}{6}=-\frac{1}{6}$; where $xx-\frac{5x}{6}$ may be considered as the two first members of a square raised from a binomial root; the coefficient of the second term is $\frac{-5}{6}$, its half $\frac{-5}{12}$, and the square of this $\frac{+25}{12\times12}$, which expression I choose to make use of rather than $\frac{+25}{144}$ for a reason that will presently be seen; add now $\frac{25}{12\times12}$ to both sides, that is, to one side to compleat the square, and to the other to preserve the equality, and you will have $xx-\frac{5x}{6}+\frac{25}{12\times12}=\frac{-1}{6}+\frac{25}{12\times12}$; here now it is certain that the fractions $\frac{-1}{6}$ and $\frac{+25}{12\times12}$ must be reduced to the same denomination in order to be added together into one sum; but if this be done the common way, it will be impossible to obtain the square root of that sum without a further reduction; therefore, to avoid this, I enquire what number the denominator 6 must be multiplied by to make it 12×12 the same with the other denominator, and the answer in this case, as well as in all others of this kind, will be very easy; for $2\times6=12$, and therefore $12\times2\times6$, or $24\times6=12\times12$; therefore I multiply both the numerator and denominator of the fraction $\frac{-1}{6}$ into 24, and so have $\frac{-24}{12\times12}$; and this added to the other fraction $\frac{+25}{12\times12}$ gives $\frac{+1}{12\times12}$; and now the equation will be $xx-\frac{5x}{6}+\frac{25}{12\times12}=\frac{1}{12\times12}$; extract the root of both sides, and you will have $x-\frac{5}{12}=\pm\frac{1}{12}$, whence $x=\frac{5\pm1}{12}$; but $\frac{5+1}{12}=\frac{1}{2}$, and $\frac{5-1}{12}=\frac{1}{3}$; therefore $x=\frac{1}{2}$, or $\frac{1}{3}$.

This may also be proved synthetically thus: let $x=\frac{1}{2}$, then you will have $xx=\frac{1}{4}$, and $6xx=\frac{6}{4}$, or $1\frac{1}{2}$: again, $5x=\frac{5}{2}=2\frac{1}{2}$; therefore $5x-1=1\frac{1}{2}$; therefore $6xx=5x-1$, since each equals $1\frac{1}{2}$.

Let us now suppose $x=\frac{1}{3}$, and you will have $xx=\frac{1}{9}$, and $6xx=\frac{6}{9}$, or $\frac{2}{3}$: on the other hand you will have $5x=\frac{5}{3}$ or $1\frac{2}{3}$; therefore $5x-1=\frac{2}{3}$; therefore $6xx=5x-1$: these two fractions therefore will answer the condition of the equation; and there are no other numbers beside these, whether whole numbers or fractions, that will do it.

EXAMPLE 2.

Let the equation to be resolved be $24x-2xx=xx+45$. Here transposing $-2xx$ we have $3xx+45=24x$, whence $3xx=24x-45$; and thus we have reduced the equation proposed to the form of the general one in art. 102; wherefore applying that general equation to this particular one, the resolution, by art. 103, will be as follows: $A=3$, $B=24$, $C=-45$, $BB=576$, $4AC=-540$, $ss=576-540=36$, $s=6$, $\frac{B+s}{2A}=5$, $\frac{B-s}{2A}=3$; therefore $x=5$, or 3; and this will further easily appear by substituting 5 or 3 for x in in the original equation thus; $x=5$; therefore $24x=120$; $xx=25$; therefore $24x-2xx=120-50=70$, which is one ſide of the equation: on the other ſide we have $xx+45=25+45=70$; therefore $24x-2xx=xx+45$. Again, let $x=3$, then we ſhall have $24x=72$, and $xx=9$, and $24x-2xx=54$: on the other hand, $xx+45=54$; therefore $24x-2xx=xx+45$.

N. B. This laſt equation when reduced to the form of the general one in art. 102, ſtood thus; $3xx=24x-45$: but this equation might have been reduced to a more ſimple one of the ſame form by dividing the whole by 3, and then the equation would have ſtood thus, $xx=8x-15$: in which caſe we ſhould have had $A=1$, $B=8$, $C=-15$, $BB=64$, $4AC=-60$, $ss=4$, $s=2$, $\frac{B+s}{2A}=5$, $\frac{B-s}{2A}=3$, as before: the ſolution of the foregoing equation in the common way is this, $xx-8x=-15$; therefore compleating the ſquare, $xx-8x+16=1$; therefore extracting the ſquare root, $x-4=\pm 1$; therefore $x=+4\pm 1=5$, or 3.

EXAMPLE 3.

Let the equation to be reſolved be $72x-2xx+144=3xx-8x+444$. Hence by tranſpoſitions we have $72x+144=5xx-8x+444$, and $80x+144=5xx+444$, and $5xx=80x-300$, and $xx=16x-60$; which equation being reſolved like that in the laſt example, gives $x=10$, or 6; which may alſo be eaſily ſeen by ſubſtituting 10 or 6 for x in the original equation.

EXAMPLE 4.

Let the equation to be reſolved be $28x-xx=115$. Here we have $xx+115=28x$, and $xx=28x-115$; which equaton being reſolved like that in the ſecond example, gives $x=23$, or 5; the proof whereof is eaſy.

Example 5.

Let the equation to be resolved be $\frac{120}{x}-5=\frac{120}{x+4}$; therefore $120-5x=\frac{120x}{x+4}$; therefore $100x-5xx+480=120x$; therefore $5xx+120x=100x+480$; therefore $5xx=-20x+480$; therefore (dividing by 5) $xx=-4x+96$; therefore in this case, $A=1$, $B=-4$, $C=96$, $BB=16$, $4AC=384$, $ss=16+384=400$, $s=20$, $\frac{B+s}{2A}=\frac{-4+20}{2}=8$, $\frac{B-s}{2A}=\frac{-4-20}{2}=-12$; therefore in this equation, $x=8$, or -12: the proof is thus; let $x=+8$; then $\frac{120}{x}=15$, and $\frac{120}{x}-5=10$: again, $x+4=12$, and $\frac{120}{x+4}=10$; therefore $\frac{120}{x}-5=\frac{120}{x+4}$. Again, let $x=-12$, then $\frac{120}{x}=-10$; therefore, $\frac{120}{x}-5=-10-5=-15$: on the other hand, $x+4=-12+4=-8$; therefore $\frac{120}{x+4}=\frac{120}{-8}=-15$; therefore $\frac{120}{x}-5=\frac{120}{x+4}$. The resolution in the common way is this; $xx=-4x+96$; therefore $xx+4x=96$; therefore $xx+4x+4=100$; therefore $x+2=\pm 10$; therefore $x=-2\pm 10=+8$, or -12.

Example 6.

Let the equation to be resolved be $2xx+3x=65$; therefore $2xx=-3x+65$; therefore in this case, $A=2$, $B=-3$, $C=65$, $BB=9$, $4AC=520$, $ss=529$, $s=23$, $\frac{B+s}{2A}=\frac{-3+23}{4}=5$, $\frac{B-s}{2A}=\frac{-3-23}{4}=-$

$-6\frac{1}{2}$: therefore in this equation, $x=+5$, or $-6\frac{1}{2}$: that $x=+5$ will eafily be feen; and that $x=-6\frac{1}{2}$: or that $-6\frac{1}{2}$ being fubftituted for x will make $2xx+3x=65$, I thus demonftrate: $x=-6\frac{1}{2}=\frac{-13}{2}$; therefore $xx=\frac{+169}{4}$; therefore $2xx=\frac{169}{2}$; and $+3x=+3\times\frac{-13}{2}=\frac{-39}{2}$: therefore $2xx+3x=\frac{169-39}{2}=\frac{130}{2}=65$. The refolution in numbers; $2xx+3x=65$: therefore $xx+\frac{3x}{2}+\frac{9}{4\times4}=\frac{65}{2}+\frac{9}{4\times4}=\frac{520+9}{4\times4}=\frac{529}{4\times4}$; therefore $x+\frac{3}{4}=\pm\frac{23}{4}$; therefore $x=\frac{-3\pm23}{4}=+5$, or $-6\frac{1}{2}$.

EXAMPLE 7.

Let the equation to be refolved be $9xx-x=140$; therefore $9xx=1x+140$. Here $A=9$, $B=1$, $C=140$, $BB=1$, $4AC=5040$, $ss=5041$, $s=71$, $\frac{B+s}{2A}=4$; $\frac{B-s}{2A}=-3\frac{8}{9}$; therefore $x=+4$, or $-3\frac{8}{9}$: the latter cafe I thus demonftrate; $x=-3\frac{8}{9}=\frac{-35}{9}$; therefore $xx=\frac{+1225}{81}$; therefore $9xx=\frac{1225}{9}$: again, $-1x$, that is, $-1\times\frac{-35}{9}=\frac{+35}{9}$, therefore $9xx-x=\frac{1225+35}{9}=\frac{1260}{9}=140$. In numbers thus;

$9xx-1x=140$; therefore $xx-\frac{1x}{9}=\frac{140}{9}$; therefore $xx-\frac{1x}{9}+\frac{1}{18\times18}=\frac{140}{9}+\frac{1}{18\times18}=\frac{5040+1}{18\times18}=\frac{5041}{18\times18}$; extract the root of both sides, that is, of $xx-\frac{1x}{9}+\frac{1}{18\times18}$ on one side, and of $\frac{5041}{18\times18}$ on the other, and you will have $x-\frac{1}{18}=\pm\frac{71}{18}$; whence $x=+4$, or $-3\frac{8}{9}$.

EXAMPLE 8.

Let the equation to be resolved be $\frac{45}{2x+3}+\frac{116}{4x+5}=7$; therefore $45+\frac{232x+348}{4x+5}=14x+21$; therefore $180x+225+232x+348=56xx+154x+105$; that is, $412x+573=56xx+154x+105$; therefore $258x+573=56xx+105$; therefore $56xx=258x+468$; therefore (dividing by 2) you have $28xx=129x+234$; which equation being compared with the general one exhibited in art. 103. gives $A=28$, $B=129$, $C=234$, $BB=16641$, $4AC=26208$, $ss=42849$, $s=207$, $\frac{B+s}{2A}=6$, $\frac{B-s}{2A}=-1\frac{11}{28}$; therefore in this equation $x=+6$, or $-1\frac{11}{28}$; both which I thus demonstrate: first $x=6$; therefore $2x+3=15$; therefore $\frac{45}{2x+3}=3$; moreover, $4x+5=29$; therefore $\frac{116}{4x+5}=4$; therefore $\frac{45}{2x+3}+\frac{116}{4x+5}=3+4$
$=7$;

$=7$: secondly, $x=-1\frac{11}{28}=-\frac{39}{28}$; therefore $2x$ $=\frac{-39}{14}$; therefore $2x+3=\frac{-39}{14}+\frac{3}{1}=\frac{3}{14}$; therefore $\frac{45}{2x+3}$ is the quotient of $\frac{45}{1}$ divided by $\frac{3}{14}$; but this quotient, according to the rules of fractional division, is $\frac{630}{3}=210$; therefore $\frac{45}{2x+3}$ $=210$: again, $4x=\frac{-39}{7}$; therefore $4x+5=$ $\frac{-39}{7}+\frac{5}{1}=\frac{-4}{7}$; therefore $\frac{116}{4x+5}$ is the quotient of $\frac{116}{1}$ divided by $\frac{-4}{7}$; but this quotient is $\frac{-812}{4}$, or -203; therefore $\frac{116}{4x+5}=-203$; therefore $\frac{45}{2x+3}$ $+\frac{116}{4x+5}=210-203=7$.

The resolution of this equation in the common way is as follows; $56xx-258x=468$; therefore $xx-\frac{258x}{56}=\frac{468}{56}$: here the coefficient of the second term is $-\frac{258}{56}$, its half $-\frac{129}{56}$, and the square of this $\frac{16641}{56\times56}$; add this square to both sides, and you will have $xx-\frac{258x}{56}+\frac{16641}{56\times56}=\frac{468}{56}+\frac{16641}{56\times56}$ $=\frac{26208+16641}{56\times56}=\frac{42849}{56\times56}$; extract the square root of both sides, that is, of $xx-\frac{258x}{56}+\frac{16641}{56\times56}$ on on one side, and of $\frac{42849}{56\times56}$ on the other, and you will have $x-\frac{129}{56}=\pm\frac{207}{56}$; whence $x=+6$, or $+1\frac{11}{28}$.

Example 9.

Let the equation be $15x-xx=56$; then this equation being resolved by the general theorem gives $x=8$, or 7; and in the common way it is thus resolved; $15x-xx=56$; change all the signs to make xx affirmative, and you will have $xx-15x=-56$; whence $xx-15x+\frac{225}{4}=-56+\frac{225}{4}=\frac{1}{4}$; therefore $x-\frac{15}{2}=\pm\frac{1}{2}$, and $x=8$, or 7; but what I chiefly intend by this example is, to shew, that in resolving a quadratic equation by the general theorem there is no necessity of making any transposition to exhibit xx affirmative when it would otherwise have been negative; as for instance, in the equation here proposed we had $15x-xx=56$; transpose $15x$, and you will have $-xx$, that is, $-1xx=-15x+56$; let this equation be referred to the general one in art. 102, and resolved by the general theorem in art. 103, and you will have $A=-1$, $B=-15$, $C=56$, $BB=225$, $4AC=-224$, $ss=1$, $s=1$ $\frac{B+s}{2A}=\frac{-15+1}{-2}=\frac{-14}{-2}=+7$, $\frac{B-s}{2A}=\frac{-15-1}{-2}=+8$.

How the learner is to proceed when the roots of a quadratic equation are inexpressible.

106. As there are but few square numbers in comparison of the rest, and as all quadratic equations are

are resolved by extracting the square root, it follows, that there are but few quadratic equations capable of an exact numeral solution in comparison of those that are not: but as the square root may be extracted to any degree of exactness we please, the resolution of a quadratic equation, which depends upon it, may also be performed to any degree of accuracy whatever; as will appear by the following example.

EXAMPLE 10.

Let the equation be $xx-4x+1=0$, or $xx=4x-1$. Here $A=1$, $B=4$, $C=-1$, $BB=16$, $4AC=-4$, $ss=12$, $s=\sqrt{12}$, $\frac{B+s}{2A}=\frac{4+\sqrt{12}}{2}$, and $\frac{B-s}{2A}=\frac{4-\sqrt{12}}{2}$; therefore $x=\frac{4+\sqrt{12}}{2}$, or $\frac{4-\sqrt{12}}{2}$: but let us enquire in the next place, whethere these two fractions are not capable of being reduced to more simple terms; first then it is plain that $\frac{4}{2}=2$, and I say further that $\frac{\sqrt{12}}{2}=\sqrt{3}$; for $12=3\times4$; therefore $\sqrt{12}=\sqrt{3}\times\sqrt{4}=\sqrt{3}\times2$; therefore $\frac{\sqrt{12}}{2}=\sqrt{3}$; whence it follows, that $x=2+\sqrt{3}$, or $2-\sqrt{3}$; but $\sqrt{3}$ extracted to three decimal places gives 1.732: therefore $2+\sqrt{3}=3.732$, and $2-\sqrt{3}=.268$; therefore $x=$ (nearly) 3.732, or .268, as will be further evident from the proof following: first $x=3.732$; therefore $xx=13.927824$; and $4x=14.928$; therefore $4x-xx=1.000176$; therefore $xx-4x=-1.000176$; therefore $xx-4x+1=-.000176=0$ very nearly; secondly, let $x=.268$ and you will have $xx=.071824$ and $4x=1.072$, and $4x-xx=1.000176$; therefore $xx-4x=-1.000176$; therefore $xx-4x+1=-.000176=0$ very nearly;

therefore

therefore in both cases, the condition of the equation is answered to as many figures or cyphers, as is equal to the number of decimal places to which the square root of 3 was extracted.

It may seem to some perhaps a paradox to assert, that though the two surd values of the unknown quantity found in this and the like cases, are not to be expressed in numbers, yet they may be demonstrated to be just: Thus I shall demonstrate, that if either of the two values of x found in the last case, to wit, $2+\sqrt{3}$, or $2-\sqrt{3}$, be substituted for x, we shall have this equation $xx-4x+1=0$, which was the equation there proposed: in order to this, make $\sqrt{3}=s$; and first, let $x=2+\sqrt{3}$, or $2+s$; and we shall have $xx=4+4s+ss$, and $-4x=-8-4s$; and $xx-4x=4+4s+ss-8-4s=ss-4$; but if $s=\sqrt{3}$, $ss=3$, and $ss-4=-1$; therefore, $xx-4x=-1$, and $xx-4x+1=0$: secondly, let $x=2-\sqrt{3}$, or $2-s$, and we shall have $xx=4-4s+ss$, and $-4x=-8+4s$, and $xx-4x=ss-4=-1$, as before; whence $xx-4x+1=0$.

Of impossible roots in a quadratic equation, and whence they arise.

107. The roots of quadratic equations are not only very often inexpressible, but sometimes even impossible, as will appear by the following example.

Example II.

Let the equation be $xx-4x+6=0$, or $xx=4x-6$. Here $A=1$, $B=4$, $C=-6$, $BB=16$, $4AC=-24$, $ss=-8$, $s=\sqrt{-8}$, $\frac{B+s}{2A}=\frac{4+\sqrt{-8}}{2}$, $\frac{B-s}{2A}=\frac{4-\sqrt{-8}}{2}$; but $\frac{4}{2}=2$, and $-8=-2\times+4$; there-

fore

fore $\sqrt{-8} = \sqrt{-2} \times \sqrt{+4} = \sqrt{-2} \times 2$; therefore $\frac{\sqrt{-8}}{2} = \sqrt{-2}$; therefore in this equation, $x = 2 + \sqrt{-2}$, or $2 - \sqrt{-2}$; but as no quantity whatever, either affirmative or negative, being multiplied into itſelf, will produce a negative, it follows, that $\sqrt{-2}$ is not only an inexpreſſible quantity, but alſo an impoſſible one; and conſequently, that the two values of x in this equation $2 + \sqrt{-2}$ and $2 - \sqrt{-2}$ will both be impoſſible.

N. B. Though the roots of this laſt equation be impoſſible in their own natures, yet they may be abſtractedly demonſtrated to be juſt, as in the laſt article, by making $s = \sqrt{-2}$, and conſequently $ss = -2$.

From what has been ſaid concerning impoſſible roots, it appears that one root of a quadratic equation can never be impoſſible alone, but that they muſt either be both poſſible or both impoſſible: for it appears from the reſolution of the laſt equation, that the impoſſibility of the roots flows from the impoſſibility of the quantity s, or of the ſquare root of ss when it is negative; now when s is poſſible, both the roots of the equation $\frac{B+s}{2A}$ and $\frac{B-s}{2A}$ will be poſſible; on the other hand, when s is impoſſible, both the roots muſt neceſſarily be impoſſible.

Since the poſſibility or impoſſibility of the two roots of a quadratic equation depends upon the quantity ss being affirmative or negative, it follows, that when ss and conſequently s equals nothing, the roots will be in the limit between poſſible and impoſſible: now if $s = 0$, we ſhall have $\frac{B+s}{2A} = \frac{B}{2A}$, and $\frac{B-s}{2A} = \frac{B}{2A}$; therefore the two unequal roots of a quadratic equation grow nearer and nearer to a ſtate of equality as they grow nearer and nearer to a ſtate of impoſſibility,

 but

but do not come to be equal till they come to the limit between possibility and impossibility.

How to find the sum and product of two roots of a quadratic equation without resolving it: also how to generate a quadratic equation that shall have any two given numbers whatever for its roots.

108. *In a quadratic equation of this general form, to wit,* $Axx=Bx+C$, *the sum of the roots will always be* $\frac{B}{A}$, *and the product of their multiplication* $\frac{-C}{A}$: for the roots of such an equation were $\frac{B+s}{2A}$ and $\frac{B-s}{2A}$ the sum whereof is $\frac{2B}{2A}$, or $\frac{B}{A}$; and if these two roots be multiplied together, their product will amount to $\frac{BB-ss}{4AA}$; but $ss=BB+4AC$ as was formerly supposed, art. 103; therefore $ss-BB=4AC$, and $BB-ss=-4AC$; therefore $\frac{BB-ss}{4AA}$, or the product of the two roots, equals $\frac{-4AC}{4AA}=\frac{-C}{A}$.

Therefore if $A=1$, that is, if the equation be $xx=Bx+C$, the sum of the roots will be B, and their product $-C$; that is, as the equation now stands, the sum of the roots will be the coefficient of the unknown quantity on the second side of the equation, and their product, what we call the absolute term, with its sign changed.

Hence we have an easy way to form a quadratic equation whose roots shall be any two given numbers whatever: as for instance, suppose I would have a quadratic equation whose roots shall be the two numbers 3 and

4;

4; here it is plain that the ſum of the two numbers 3 and 4 is 7, and that the product of their multiplication is 12; therefore I form an equation whereof one ſide is xx, and the other ſide is $7x-12$, to wit, $xx=7x-12$; and the roots of this equation will be the given numbers 3 and 4, as will appear from the reſolution: if I intend the two roots to be 3 and -4, their ſum will be -1, and the product of their multiplication -12, and the equation $xx=-x+12$: if the roots are to be -3 and $+4$, their ſum will be $+1$, the product of their multiplication -12, and the equation $xx=x+12$: laſtly, if the roots are to be -3 and -4, their ſum will be -7, the product of their multiplication $+12$, and the equation $xx=-7x-12$. I ſhall demonſtrate one general caſe according to the reſolution given in art. 103, which will be ſufficient to ſhew the way to all the reſt: let then the roots propoſed be p and q, whoſe ſum is $p+q$, and the product of whoſe multiplication is pq; and the equation will be $xx=\overline{p+q}\times x-pq$; now if this equation be referred to the general one, we ſhall have $A=1$, $B=\overline{p+q}$, $C=-pq$, $BB=pp+2pq+qq$, $4AC=-4pq$, $ss=pp-2pq+qq$, $s=p-q$, $\frac{B+s}{2A}=\frac{p+q+p-q}{2}=\frac{2p}{2}=p$, $\frac{B-s}{2A}=\frac{p+q-p+q}{2}=\frac{2q}{2}=q$; therefore the two roots of this equation are p and q. *Q. E. D.*

I think I ought not to omit here, that if any one has a mind to form a quadratic equation with any two given impoſſible roots whatever (if I may be allowed the expreſſion), it may be done by the foregoing rule, provided that theſe impoſſible roots be in ſuch a form as is proper for a quadratic equation: as for example, ſuppoſe I would form a quadratic equation with theſe two impoſſible roots, to wit, $2+\sqrt{-3}$ and $2-\sqrt{-3}$, I put ss for -3; for though no poſſible quantity multiplied into itſelf can produce a negative, yet an impossible one may, that being the very thing wherein the impossibility consists; making then $ss=-3$, I have $s=\sqrt{-3}$, and so the two roots of the equation will now be $\overline{2+s}$, and $\overline{2-s}$; the sum of these two roots is 4, and the product of their multiplication $4-ss$; but if $ss=-3$, $-ss=+3$, and $4-ss=4+3=7$; therefore the equation with these roots will be $xx=4x-7$: and this will be further evident by the resolution; for if $xx=4x-7$, that is, if $xx-4x=-7$, we shall have $xx-4x+4=-3$, and $x-2=\pm\sqrt{-3}$, and $x=2+\sqrt{-3}$, or $2-\sqrt{-3}$.

How to determine the signs of the possible roots of a quadratic equation without resolving it.

109. If all the terms of a quadratic equation be thrown on one side of the equation, so as to be made equal to nothing; and if the term wherein xx, the square of the unknown quantity is concerned, be made the first, that wherein x, the simple power is concerned, be made the second, and the absolute term, as it is called, be made the third; the number of affirmative and negative roots in such an equation may be found by the following rule, to wit, *As often as the signs are changed in passing through all the terms from the first to the last, of so many affirmative roots will the equation consist; but as often as the signs are the same, so many negative roots will be found in the equation.* This is true in all equations whatever, though at present we shall only demonstrate it in the case of a quadratic equation: but first we shall give the following explication of the rule.

Case 1.

Let the equation be $axx-bx+c=0$. Here there are two changes in passing through the terms from the first to the last, to wit, from $+axx$ to $-bx$, and from

from $-bx$ to $+c$; therefore the roots of this equation are both affirmative.

CASE 2.

Let the equation be $axx-bx-c=0$. Here from $+axx$ to $-bx$ is one change, and from $-bx$ to $-c$ is none; therefore this equation confifts of an affirmative and a negative root.

CASE 3.

Let the equation be $axx+bx-c=0$. Here in paffing from $+axx$ to $+bx$, there is no change of fign, but in paffing from $+bx$ to $-c$ there is a change; therefore this equation alfo confifts of an affirmative and a negative root.

CASE 4.

Laftly, let the equation be $axx+bx+c=0$. Here there are no changes, and confequently the roots of this equation are both negative. All thefe cafes I fhall demonftrate in the following manner.

CASE 1.

Let the equation be $axx-bx+c=0$, or $axx=bx-c$. Here the product of the two roots is $\frac{c}{a}$ by the laft article, that is, the product of the two roots is an affirmative quantity, and therefore thofe roots muft either be both affirmative or both negative; but they cannot be both negative, becaufe their fum is $\frac{+b}{a}$, by the fame article; therefore they muft both be affirmative.

CASE 2.

Let the equation be $axx-bx-c=0$, or $axx=bx+c$. Here the product of the two roots is $-\frac{c}{a}$, and confequently ſequently thoſe roots muſt be of different kinds, one affirmative and the other negative; and becauſe their ſum, $+\frac{b}{a}$, is an affirmative quantity, it is an argument that the greater root is affirmative.

Case 3.

Let the equation be $axx+bx-c=0$, or $axx=-bx+c$. Here again the product of the two roots is $\frac{+c}{a}$, which argues one root to be affirmative and the other negative; and becauſe their ſum $\frac{-b}{a}$ is a negative quantity, it is an indication that of theſe two roots, the greater is the negative one.

Case 4.

Laſtly, let the equation be $axx+bx+c=0$, or $axx=-bx-c$. Here the product of the two roots is $+\frac{c}{a}$ an affirmative quantity; therefore the roots are either both affirmative or both negative; but they cannot be both affirmative, becauſe their ſum $\frac{-b}{a}$ is negative; therefore they muſt both be negative.

Impoſſible roots excluded out of the foregoing rule.

The rule here given for determining the number of affirmative and negative roots relates only to poſſible roots; for impoſſible ones cannot be ſaid to belong to any claſs, either of affirmatives or negatives; nay, ſo capricious are they in this reſpect, that in one and the ſame equation, the very ſame impoſſible roots ſhall ſometimes appear under one form, and ſometimes under the other: as for example, this equation $xx+3=0$ may be filled up two ways without affecting,

ing either the equation or its roots; to wit, either thus, $xx - 0x + 3 = 0$, the roots of which equation according to the foregoing rule are both affirmative; or thus, $xx + 0x + 3 = 0$, the roots of which equation, though it be the ſame with the other, and differs only in form, are both negative: the reaſon of this abſurdity is, that the two roots of the equation $xx + 3 = 0$ are impoſſible, and occaſioned this confuſion by putting on one ſhape in one equation, and another ſhape in the other: this will further appear from the reſolution; for if $xx + 3 = 0$, we have $xx = -3$, and $x = +\sqrt{-3}$, or $-\sqrt{-3}$, which are both impoſſible quantities. Again, the equation $x^3 - 3 = 0$ may be filled up various ways; as thus, $x^3 - 0x^2 + 0x - 3 = 0$, in which equation, according to the foregoing rule, there are three affirmative roots; or thus, $x^3 - 0x^2 - 0x - 3 = 0$, in which equation, there is but one affirmative root and two negative ones: hence an experienced Analyſt would immediately conclude (as is really the caſe) that two of the roots of the equation $x^3 - 3 = 0$ were impoſſible, and that they ſtood for affirmative quantities in the former way of putting the equation, and for negative ones in the latter. This will further appear, when we come to treat of cubic equations.

Of biquadratics, and other equations in the form of quadratics.

110. Thus much for the reſolution, nature, and properties of a quadratic equation: I ſhall only add an example or two more of other equations that ſometimes put on the form of quadratics, and have done.

Example 12.

Let the equation to be reſolved be, $\frac{1600}{xx} + xx = 116$; therefore $1600 + x^4 = 116xx$; therefore $x^4 = 116xx - 1600$. This equation is, properly ſpeaking,

 a bia biquadratic, that is, an equation wherein the fourth power of the unknown quantity is concerned: now as every poſſible quadratic equation has two roots, which will equally anſwer the condition thereof, ſo a cubic equation, that is, an equation that riſes to the third power of the unknown quantity, may have three ſuch roots, a biquadratic four, &c.: but the equation $x^4 = 116xx - 1600$, though it be a biquadratic, and admits of four roots, yet it is in the form of a quadratic, if we conſider xx as the unknown quantity; in which caſe x^4 muſt be looked upon as the ſquare of the unknown quantity, and the equation muſt be referred to the general one in art. 103, thus; $A = 1$, $B = 116$, $C = -1600$, $BB = 13456$, $4AC = -6400$, $ss = 7056$, $s = 84$, $\frac{B+s}{2A} = 100$ $\frac{B-s}{2A} = 16$; therefore in this equation, $xx = 100$, or 16: now if $xx = 100$, we ſhall have $x = +$ or -10; if $x^2 = 16$, we ſhall have $x = +$ or -4; therefore the four roots of this biquadratic equation are, $+10$, -10, $+4$ and -4: but though in this equation x has four ſignifications, xx has but two, *viz.* 100 and 16, either of which being ſubſtituted inſtead of xx in the original equation, will anſwer that equality, as may eaſily be tried.

N. B. Whenever of the four roots of a biquadratic equation any two are equal and contrary to the other two, the equation will be in form of a quadratic, and may be reſolved accordingly.

EXAMPLE 13.

Let the equation be $\frac{576}{xx} - xx = 55$: here we have $576 - x^4 = 55xx$, and $x^4 + 55xx = 576$, and $x^4 = -55x^2 + 576$; therefore, according to the general equation in art. 103, $A = 1$, $B = -55$, $C = 576$, $BB = 3025$, $4AC = 2304$, $ss = 5329$, $s = 73$, $\frac{B+s}{2A} = 9$, $\frac{B-s}{2A} = -64$;

64; therefore in this equation, $xx = +9$ or -64: if $xx = +9$, $x = +$ or -3; if $xx = -64$, x will be equal to $+\sqrt{-64}$, or $-\sqrt{-64}$, both which values are impoſſible; ſo that in this equation x has but two values, $+$ or -3, the other two being impoſſible; and xx has two values, to wit, $+9$ and -64, which are both poſſible, and which, being ſubſtituted inſtead of xx into the original equation, will anſwer that equality. From this example it is eaſy to ſee, that a biquadratic equation may have four roots, and never can have more; yet it may ſometimes have fewer, upon the account of ſome of its roots becoming impoſſible; nay inſtances might eaſily be given wherein all the roots of a biquadratic equation are impoſſible.

If any one diſapproves of the reſolutions here given, he may perhaps reliſh the following better: let the equation be $Ax^4 = Bx^2 + C$; here putting z for xx, and conſequently zz for x^4, the equation will be changed into this common quadratic, $Azz = Bz + C$; which being reſolved, z or xx, and conſequently x itſelf will be known: ſuppoſe the equation to be $Ax^6 = Bx^3 + C$; here putting z for x^3, the equation will be changed into a quadratic, as before, to wit, $Azz = Bz + C$, the reſolution whereof will give z for x^3, and conſequently x by an extraction of the cube root: laſtly, let the equation be $Ax = B \times \sqrt{x} + C$; here putting zz for x, and z for $\sqrt{z}$, the equation will be $Azz = Bz + C$, as before; whence z, and conſequently zz or x will be known.

The ſolution of ſome problems producing quadratic equations.

PROBLEM 69.

111. *It is required to divide the number* 60 *into two ſuch parts, that the product of their multiplication may amount to* 864.

 SOLUTION.

SOLUTION.

Put x for one of the parts; then will the other part be $60-x$, and the product of their multiplication will be $60x-xx$; whence the equation will be $60x-xx=864$: therefore $xx+864=60x$, and $xx=60x-864$: this equation, compared with the general one in art. 103, gives $A=1$, $B=60$, $C=-864$, $BB=3600$, $4AC=-3456$, $ss=144$, $s=12$, $\frac{B+s}{2A}=36$, $\frac{B-s}{2A}=24$; therefore the parts sought are 24 and 36; which upon trial will anſwer the conditions of the problem.

Obſervations upon the foregoing problem.

OBSERVATION 1ſt.

In this problem we may clearly ſee the neceſſity of the unknown quantity's having ſometimes two diſtinct values in one and the ſame equation: for here, if I put x for the greater part of 60, the leſs will be $60-x$, and the equation will be $60x-xx=864$: ſuppoſe now I put x for the leſs part; then the greater will be $60-x$, and the equation will ſtill be $60x-xx=864$; therefore, whether x be put for the greater or the leſs part, we ſtill fall into the ſame equation $60x-xx=864$; whence I infer, that this equation muſt either give us both the parts ſought, or neither; ſince no reaſon can be ſhewn why it ſhould give us one part rather than the other.

OBSERVATION 2d.

Hence alſo we ſee the neceſſity ſometimes of impoſſible roots, to wit, when the caſes of problems to be ſolved by them become impoſſible: as for inſtance, if any number, as 60, be divided into two parts, the nearer the two parts approach towards an equality, the greater will be the product of their multiplication; and therefore, if the parts be equal, the product will be the greatest possible: thus if the parts be 24 and 36, the product will be 864; if they be 25 and 35, the product will be 875; if 30 and 30, the product will be 900, which will be the greatest possible: let us now for once put an impossible case, and let it be required to divide the number 60 into two such parts that the product of their multiplication may amount to 901; here the equation will be $60x-xx=901$; which being resolved according to art. 103, gives $x=\frac{60+\sqrt{-4}}{2}$, or $\frac{60-\sqrt{-4}}{2}$; but these values of x may be reduced to more simple terms thus; $-4=-1\times+4$; therefore $\sqrt{-4}=\sqrt{-1}\times\sqrt{+4}=\sqrt{-1}\times2$; therefore $\frac{\sqrt{-4}}{2}=\sqrt{-1}$; but $\frac{60}{2}=30$; therefore the two parts sought are $30+\sqrt{-1}$, and $30-\sqrt{-1}$, both which are impossible upon the account of the impossibility of $\sqrt{-1}$; and yet these two parts abstractedly considered will answer the conditions of the problem; for if $\sqrt{-1}$ be made equal to s, the two parts will be $30+s$ and $30-s$ whose sum is 60, and the product of whose multiplication is $900-ss$; but if $s=\sqrt{-1}$, we shall have $ss=-1$, and $-ss=+1$, and $900-ss=901$; therefore the product of the two parts, $30+\sqrt{-1}$, and $30-\sqrt{-1}$, amount to 901, as was required.

OBSERVATION 3d.

Lastly, we here also see the necessity of both the roots of a quadratic equation becoming impossible at once. Two impossible quantities added together, may sometimes make a possible one, because one quantity may be as much impossible one way as the other is the contrary way: thus the two impoſſible quantities $30+\sqrt{-1}$ and $30-\sqrt{-1}$ being added together make 60, the impoſſible ſurds $+\sqrt{-1}$ and $-\sqrt{-1}$ deſtroying one another; but a poſſible and an impoſſible quantity when added together can never make a poſſible one; and therefore the two parts of 60 in this problem muſt either be both poſſible, or both impoſſible.

Problem 70.

112. *There are three numbers in continual proportion, whereof the middle term is ſixty, and the ſum of the extremes one hundred twenty-five: What are the extremes?*

Solution.

For the extremes put x and $125-x$, and you will have this proportion; x is to 60 as 60 is to $125-x$, whence, by multiplying extremes and means, you have this equation, $125x-xx=3600$, or $xx+3600=125x$, or $x^2=125x-3600$: here then $A=1$, $B=125$, $C=-3600$, $BB=15625$, $4AC=-14400$, $ss=1225$, $s=35$, $\frac{B+s}{2A}=80$, $\frac{B-s}{2A}=45$, therefore in this equation, $x=45$, or 80; but x repreſents either extreme, becauſe, which extreme ſoever x is put for, the other will be $125-x$, and the ſame equation will ariſe, to wit, $125x-xx=3600$; therefore the two extremes are 45 and 80; and they will anſwer the conditions of the problem; for 45 is to 60 as $\frac{45}{15}$ is to $\frac{60}{15}$, that is, as 3 to 4; and 60 is to 80 as $\frac{60}{20}$ is to $\frac{80}{20}$, which is alſo as 3 to 4.

Problem 71.

113. *It is required, having given the ſum or the difference of two numbers, together with the ſum of their ſquares, to find the numbers.*

Solution.

SOLUTION.

Case 1st. Let the ſum of the numbers ſought be 28, and the ſum of their ſquares 400; then putting x and $28-x$ for the two numbers ſought, the ſquare of the former will be xx, the ſquare of the latter $784-56x+xx$, and the ſum of their ſquares $2xx-56x+784=400$; and the ſame equation will ariſe, whether x be made to ſtand for one number or the other; therefore the two values of x in this equation will be the two numbers ſought; but if $2xx-56x+784=400$, we ſhall have $2xx-56x=-384$; divide the whole by 2 for a more ſimple equation, and you will have $xx-28x=-192$; and $xx=28x-192$; which equation being reſolved according to art. 103, gives $x=12$, or 16; therefore 12 and 16 are the two numbers ſought.

Case 2d. Let now the difference of two numbers be given, ſuppoſe 4, and let the ſum of their ſquares be 400, as before; then, putting x for the leſs number, and $x+4$ for the greater, the ſum of their ſquares will be $2xx+8x+16=400$; whence $2xx+8x=384$, $xx+4x=192$, $xx+4x+4=196$, $x+2=\pm 14$, $x=+12$ or -16; now it cannot be ſuppoſed that $+12$ and -16 are the two numbers required in the problem, for their difference is 30, not 4; neither ought it to be expected; for when x was put for the leſs number, and $x+4$ for the greater, the equation was $2xx+8x+16=400$; but if x be put for the greater number, and conſequently $x-4$ for the leſs, the equation will be $2xx-8x+16=400$, different from the former; ſince then a different equation ariſes according as x is put for the greater or leſs number, it cannot be expected that one and the ſame equation ſhould give both: the true ſtate of the caſe is this; there are two pairs of numbers which will equally ſolve this queſtion, and the equation $2xx+8x+16=400$ gives the leſſer number of each pair; for if we make $x=12$, and $x+4=16$, the numbers 12 and 16 will solve the problem; on the other hand, if we make $x=-16$, we shall have $x+4=-12$, and the numbers -16 and -12 will equally solve the problem; for their difference is $+4$, and the sum of their squares $+400$: here then we may observe, that affirmative and negative solutions of problems are of equal estimation in the nature of things, though perhaps not amongst men, the narrowness of our minds contracting our views; but truth does justice alike to all: certainly negative numbers differ no more from affirmative ones, than affirmative ones do from one another, which is in degree, not in kind; and therefore, in the nature of things, negative quantities ought no more to be excluded out of the scale of number than affirmative ones, though in common life they are set aside.

PROBLEM 72.

114. *What two numbers are those, whose sum is seventeen, and the sum of their cubes one thousand three hundred forty-three?*

SOLUTION.

For the two numbers sought put x and $17-x$, and the cube of the former will be xxx, and the cube of the latter $4913-867x+51xx-xxx$, as appears from the following computation:

$$\begin{array}{l} 17-x \\ 17-x \\ \hline 289-17x+xx \\ \quad -17x \\ \hline 289-34x+xx \\ 17-x \\ \hline 4913-578x+17xx-x^3 \\ \quad -289x+34xx \\ \hline 4913-867x+51xx-x^3. \end{array}$$

Therefore

Therefore the ſum of theſe two cubes will be $51xx - 867x + 4913 = 1343$, and the equations will be the ſame, whichſoever of the two numbers ſought x is made to ſtand for; but if $51xx - 867x + 4913 = 1343$, we ſhall have $51xx - 867x = -3570$; divide the whole by 51, which, though not neceſſary, is however convenient, to render the equation more ſimple, ſince it may be done without fractions, and you will have, $xx - 17x = -70$; which, being reduced as in art. 103, gives $x = 7$, or 10; therefore 7 and 10 are the two numbers ſought.

PROBLEM 73.

115. *Let there be a ſquare whoſe ſide is a hundred and ten inches; it is required to aſſign the length and breadth of a rectangled parallelogram or long ſquare, whoſe perimeter ſhall be greater than that of the ſquare by four inches, but whoſe area ſhall be leſs than the area of the ſquare by four ſquare inches.*

N. B. By the perimeter of a plain figure is meant the length of a line that will encompaſs it round; ſo that the perimeter of a ſquare is equal to four times its ſide; and the perimeter of a rectangled parallelogram is equal to twice its length and twice its breadth added together.

SOLUTION.

Since the ſide of the given ſquare is 110 inches, its area will be 12100 ſquare inches; therefore the area of the parellelogram ſought will be 12096 ſquare inches: again, the perimeter of the given ſquare is 440 inches; therefore the perimeter of the parallelogram ſought muſt be 444 inches; therefore half its perimeter, or its length and breadth added together, muſt be 222 inches; therefore, if either the length or breadth be called x, the other will be $222 - x$, and the area will be $222x - xx = 12096$; which equation reſolved according to art. 103, will give $x = 96$, or 126; therefore the breadth of the parallelogram ſought

sought must be 96 inches, and the length 126 inches: and these numbers will answer the conditions of the question; for twice the length will be 252, twice the breadth 192, and the whole perimeter 444; moreover 126×96, or the area, will be 12096, as the problem requires.

Scholium.

This problem shews how grossly they are mistaken who think to estimate the areas or magnitudes of plain figures by their perimeters, as if such figures were greater or less in proportion as their perimeters were so; whereas here we see, that the perimeter of one figure may be greater than that of another by four inches, and at the same time its area may be less than the area of that other by four square inches. This error, it is true, does not obtain but in low and vulgar minds, nor there neither any longer than whilst it continues to be a matter of mere speculation, and truth and falshood are equally indifferent to them: for whenever men come to apply their notions, and find it their interest not to be mistaken, then it is, and frequently not till then, that they begin to look about them, correct their errors, and entertain more just and accurate notions of things. The greatest part of mankind have a natural aversion to abstract thinking, and, where their interest is not concerned, will rather submit their opinions to humour, caprice, and custom, or be content to be without any opinions at all, than they will examine strictly into the nature of things.

Problem 74.

116. *One buys a certain number of oxen for eighty guineas; where it must be observed, that if he had bought four more for the same money, they would have come to him a guinea apiece cheaper: What was the number of oxen?*

 Solution.

SOLUTION.

For the number of oxen put x; then to find the price of a single ox, say, if x oxen cost 80 guineas, what will one ox cost? and the answer is $\frac{80}{x}$; and for the same reason, if he had bought 4 more, that is, $x+4$ for the same money, the price of an ox would have been $\frac{80}{x+4}$; but, according to the problem, the latter price is less than the former by one guinea; whence we have this equation $\frac{80}{x}-1=\frac{80}{x+4}$, therefore $80-x=\frac{80x}{x+4}$; therefore $\overline{80-x}\times\overline{4+x}$ or $320+76x-xx=80x$; therefore $xx+80x=76x+320$; therefore $xx=-4x+320$. Here then $A=1$, $B=-4$, $C=320$, $BB=16$, $4AC=1280$, $ss=1296$, $s=36$, $\frac{B+s}{2A}=16$, $\frac{B-s}{2A}=-20$; therefore $x=+16$, or -20; therefore the number of oxen was 16, the negative root -20 having no place in this problem; and this number 16 answers the condition of the problem; for if 16 oxen cost 80 guineas, one will cost 5 guineas: but if 20 oxen cost 80 guineas, one will cost 4 guineas.

N. B. The equation $\frac{80}{x}-1=\frac{80}{x+4}$, gave $x=+16$ or -20, not because the number -20 would solve the problem, but because it would solve the equation; for if we make $x=-20$, we shall have $\frac{80}{x}=-4$, and $\frac{80}{x}-1=-5$; on the other side, we shall have $x+4=-16$, and $\frac{80}{x+4}=-5$; therefore

if

if x be made equal to -20, we shall have $\frac{80}{x} - 1 = \frac{80}{x+4}$, because both sides are equal to -5; and so in all other cases we shall always find, that the several roots of an equation will be such as will equally solve that equation, though perhaps they may not be equally proper to solve the problem from whence the equation was deduced: but of this more in another place.

PROBLEM 75.

117. *A certain company at a tavern had a reckoning of seven pounds four shillings to pay; upon which two of the company sneaking off, obliged the rest to pay one shilling apiece more than they should have done: What was the number of persons?*

SOLUTION.

For the number of persons put x; then to find the number of shillings every man should have paid, say, if x persons were to have paid 144 shillings, what must one man have paid? and the answer is $\frac{144}{x}$; therefore $\frac{144}{x}$ is the number of shillings every man should have paid; and for the same reason $\frac{144}{x-2}$ is the number of shillings every man did pay; but, according to the problem, this latter reckoning is greater than the former by one shilling; whence the equation will be $\frac{144}{x} + 1 = \frac{144}{x-2}$; therefore $144 + x = \frac{144x}{x-2}$; therefore $\overline{x-2} \times \overline{144+x}$, or $xx + 142x - 288 = 144x$; therefore $xx - 288 = 2x$; therefore $xx = 2x + 288$. Here then $A = 1$, $B = 2$, $C = 288$, $BB = 4$, $4AC = 1152$, $ss = 1156$, $s = 34$,

$s = 34, \frac{B+s}{2A} = 18, \frac{B-s}{2A} = -16$; therefore $x = +18$, or -16; but negative roots have no place in this ſort of problems; therefore the number of perſons was 18, which anſwers the condition; for $\frac{144}{18} = 8$, and $\frac{144}{16} = 9$.

PROBLEM 76.

118. *What number is that, which being added to its ſquare root will make two hundred and ten?*

SOLUTION.

For the number ſought put xx; then will its ſquare root be x, and the equation will be $xx + x = 210$, or $xx = -x + 210$; where $A = 1$, $B = -1$, $C = 210$, $BB = 1$, $4AC = 840$, $ss = 841$, $s = 29$, $\frac{B+s}{2A} = 14$, $\frac{B-s}{2A} = -15$; therefore $x = +14$, or -15; therefore xx or the number ſought equals 196 or 225, ſuppoſing the ſquare root of 225 to be -15; and either of theſe two numbers will anſwer the condition; for $196 + 14 = 210$, and $225 - 15 = 210$.

PROBLEM 77.

119. *What two numbers are thoſe, the product of whoſe multiplication is one hundred ninety two, and the ſum of whoſe ſquares is ſix hundred and forty?*

SOLUTION.

For the two numbers ſought put x and $\frac{192}{x}$; then will the ſquare of the former be xx, and that of the latter $\frac{36864}{xx}$, and the ſum of their ſquares will be $xx + \frac{36864}{xx}$ $= 640$;

$=640$; which equation will be the ſame, whichſoever of the two numbers ſought x is made to ſtand for; but if $xx+\frac{36864}{xx}=640$, we ſhall have $x^4+36864=640xx$; and $x^4=640x^2-36864$: here then $A=1$, $B=640$, $C=-36864$, $BB=409600$, $4AC=-147456$, $ss=262144$, $s=512$, $\frac{B+s}{2A}=576$, $\frac{B-s}{2A}=64$; therefore $xx=576$, or 64; therefore $x=+$ or -24, or $+$ or -8; therefore the two numbers ſought are 8 and 24.

Problem 78.

120. *One lays out a certain ſum of money in goods, which he ſold again for twenty-four pounds, and gained as much* per cent. *as the goods coſt him: I demand what they coſt him.*

N. B. One's gain *per cent.* is ſo much as he gains, every hundred pounds he lays out; or if he does not lay out ſo much as a hundred pounds, his gain *per cent.* however, is ſo much as he would have gained if he had laid out a hundred pounds with the ſame advantage: thus if he lays out 20 pounds and gains 2 pounds, he is ſaid to make 10 *per cent.* of his money, becauſe 20 pounds is to 2 pounds as 100 pounds is to 10 pounds.

Solution.

Put x for the money laid out, and the gain will be $24-x$; ſay then, by the golden rule, if in laying out x he gained $24-x$, what would he have gained if he had laid out 100 pounds to the ſame advantage? and the anſwer will be $\frac{2400-100x}{x}$; therefore $\frac{2400-100x}{x}$ will be his gain *per cent.*; but, according to the problem, this gain is equal to x, the money laid out; therefore $x=\frac{2400-100x}{x}$, and $xx=2400-100x$: here

here then $A=1$, $B=-100$, $C=2400$, $BB=10000$, $4AC=9600$, $ss=19600$, $s=140$, $\frac{B+s}{2A}=20$, $\frac{B-s}{2A}=-120$; therefore the money laid out was 20 pounds; therefore his gain *per* 20 was 4 pounds; therefore his gain *per cent.* was 20 pounds, equal to the money laid out.

PROBLEM 79.

121. *One lays out thirty-three pounds fifteen ſhillings in cloth, which he ſold again for forty eight ſhillings* per *piece, and gained as much in the whole as a ſingle piece coſt: I demand how he bought in his cloth* per *piece.*

SOLUTION.

Put x for the number of ſhillings every ſingle piece was bought for, and the gain *per* piece will be $48-x$; ſay then, by the rule of proportion, if in laying out x he gained $48-x$, what did he gain in laying out 33 pounds 15 ſhillings, or 675 ſhillings? and the anſwer will be $\frac{32400-675x}{x}$; therefore $\frac{32400-675x}{x}$ will be his whole gain; but, according to the problem, the whole gain was equal to x, the money given for a ſingle piece; therefore $x=\frac{32400-675x}{x}$; therefore $xx=32400-675x$; therefore $A=1$, $B=-675$, $C=32400$, $BB=455625$, $4AC=129600$, $ss=585225$, $s=765$, $\frac{B+s}{2A}=45$, $\frac{B-s}{2A}=-720$; therefore $x=+45$, or -720; therefore the money every ſingle piece was bought for was 45 ſhillings, and the gain *per* piece was 3 ſhillings; but if 45 ſhillings gains 3 ſhillings, 33 pounds 15 ſhillings, or 675 ſhillings, will gain 45 ſhillings; therefore the whole gain was 45 ſhillings, equal to the money given for a ſingle piece.

N. B.

N. B. It is not impoſſible but that ſometimes two different problems may produce one and the ſame equation; and then the equation muſt provide equally for both: therefore, in ſuch a caſe, though the equation has two roots, and both affirmative, yet it muſt not be expected that both roots ſhould equally ſerve for the ſolution of one problem, and that there ſhould be no ſolution left for the other; we ought rather to conclude, whenever an equation gives two roots, and both affirmative, whereof one only will ſolve the problem that produced the equation, we ought, I ſay, rather to conclude, that the other root is for the ſolution of ſome other problem producing the ſame equation; a curious inſtance whereof we have in the two following problems.

Problem 80.

122. *Two travellers,* A *and* B, *ſet out from two places* C *and* D *at the ſame time,* A *from* C *bound for* D, *and* B *from* D *bound for* C; *when they met and had computed their travels, it was found, that* A *had travelled thirty miles more than* B, *and that, at their rate of travelling,* A *expected to reach* D *in four days, and* B *to reach* C *in nine days: I demand the diſtance between the two places* C *and* D.

Solution.

Put x for the number of miles between C and D; then it is plain that A and B both together had travelled x miles when they met; therefore as much as the miles travelled by A exceeded $\frac{x}{2}$, juſt ſo much did the miles travelled by B come ſhort of $\frac{x}{2}$; but, by the ſuppoſition, A's miles exceeded thoſe of B by 30; therefore A muſt have travelled $\frac{x}{2}+15$ or $\frac{x+30}{2}$ miles;

and

and *B* muſt have travelled $\frac{x}{2}-15$ or $\frac{x-30}{2}$ miles; therefore the remaining part of *A*'s journey is $\frac{x-30}{2}$ miles, which he expects to perform in four days, and the remaining part of *B*'s journey is $\frac{x+30}{2}$ miles, which he expects to perform in 9 days: theſe things being allowed, let us now enquire into the number of days each hath travelled already; and firſt for *A* ſay, if *A* expects to travel $\frac{x-30}{2}$ miles in 4 days, in how many days did he travel $\frac{x+30}{2}$ miles? and the anſwer is $\frac{4\times\frac{x+30}{2}}{\frac{x-30}{2}}=\frac{4\times\overline{x+30}}{x-30}$; then for *B* ſay, if *B* expects to travel $\frac{x+30}{2}$ miles in 9 days, in how many days did he travel $\frac{x-30}{2}$ miles? and the anſwer is $\frac{9\times\overline{x-30}}{x+30}$; therefore *A* hath travelled $\frac{4\times\overline{x+30}}{x-30}$ days, and *B* $\frac{9\times\overline{x-30}}{x+30}$ days from the time of their firſt ſetting out: but as they both ſet out at the ſame time, and are now met, they muſt both have travelled the ſame number of days; therefore $\frac{4\times\overline{x+30}}{x-30}=\frac{9\times\overline{x-30}}{x+30}$: multiply both ſides of the equation into $\overline{x-30}$, and you will have $4\times\overline{x+30}=\frac{9\times\overline{x-30}\times\overline{x-30}}{x+30}$; again multiply by $\overline{x+30}$, and you will have $4\times\overline{x+30}\times\overline{x+30}$

$\overline{x+30}=9\times\overline{x-30}\times\overline{x-30}$; extract the square root of both sides, and you will have $\pm 2\times\overline{x+30}$ $=\pm 3\times\overline{x-30}$: this general equation resolves itself into four particular ones, *viz.*

1st, $+2\times\overline{x+30}=+3\times\overline{x-30}$.
2d, $+2\times\overline{x+30}=-3\times\overline{x-30}$.
3d, $-2\times\overline{x+30}=+3\times\overline{x-30}$.
4th, $-2\times\overline{x+30}=-3\times\overline{x-30}$.

But as the two last of these equations give but the same values as the two former, I shall only make use of the two former, thus;

1st, Suppose $+2\times\overline{x+30}=+3\times\overline{x-30}$, then we shall have $2x+60=3x-90$, and $x=150$.

2dly, Suppose $+2\times\overline{x+30}=-3\times\overline{x-30}$, then we shall have $2x+60=-3x+90$, and $x=6$; therefore the distance between the two places *C* and *D* must either be 150 miles, or 6 miles; but 6 miles it cannot be, because when *A* came up to *B*, he had travelled 30 miles more than *B*, and had not yet reached *D*; therefore the distance between the two places *C* and *D* must be 150 miles; which will satisfy the problem; for then *A* must have travelled $75+15$, or 90 miles, and *B* $75-15$, or 60 miles, from the time of their setting out; therefore *A* has 60 miles, and *B* 90 to travel; but if *A* could travel 60 miles in 4 days, he must, at the same rate, have travelled 90 miles in 6 days; and if *B* could travel 90 miles in 9 days, he must have travelled 60 miles also in 6 days; therefore they both travelled the same number of days from the time of their first setting out to the time of their meeting, as the problem requires.

Problem 81.

123. *Two travellers* A *and* B *set out from two places* C *and* D *at the same time;* A *from* C *with a design to pass*

pass through D, *and* B *from* D *with a design to travel the same way: after* A *had overtaken* B, *and they had computed their travels, it was found, that they had both together travelled thirty miles, that* A *had passed through* D *four days before, and that B, at his rate of travelling, was a nine days journey distant from* C: *I demand the distance between the two places* C *and* D.

SOLUTION.

Put x for the number of miles from C to D; then it is plain, that A must have travelled more miles than B by x; but they both together travelled 30 miles, by the supposition; therefore as much as A's miles exceeded 15, just so much B's miles came short of 15: but the whole difference was x, as above; therefore A must have travelled $15+\frac{x}{2}$ or $\frac{30+x}{2}$ miles, and B must have travelled $15-\frac{x}{2}$ or $\frac{30-x}{2}$ miles; therefore A's distance from D, after he had overtaken B, was $\frac{30-x}{2}$ miles, which he had travelled in 4 days, and B's distance from C was $\frac{30+x}{2}$ miles, which by the problem he could travel in 9 days; therefore, to find how many days each had travelled already, say, if A hath travelled $\frac{30-x}{2}$ miles from D in 4 days, in how many days did he travel $\frac{30+x}{2}$ miles since his departure from C? and the answer is $\dfrac{4\times\frac{20+x}{2}}{\frac{30-x}{2}}=\dfrac{4\times\overline{30+x}}{30-x}$; again

 say,

ſay, if *B* could travel $\frac{30+x}{2}$ miles, the whole diſtance from *C*, in 9 days, in how many days did he travel $\frac{30-x}{2}$ miles ſince his ſetting out from *D*? and the anſwer is $\frac{9\times\overline{30-x}}{30+x}$; but as they both ſet out at the ſame time, and *A* has now overtaken *B*, they muſt both have travelled the ſame number of days; therefore we have this equation, $\frac{4\times\overline{30+x}}{30-x}=\frac{9\times\overline{30-x}}{30+x}$: multiply both ſides into $\overline{30-x}$, and you will have $4\times\overline{30+x}=\frac{9\times\overline{30-x}\times\overline{30-x}}{30+x}$; again multiply by $\overline{30+x}$, and you will have $4\times\overline{30+x}\times\overline{30+x}=9\times\overline{30-x}\times\overline{30-x}$; but the product of $\overline{30-x}\times\overline{30-x}$ differs nothing from the product of $\overline{x-30}\times\overline{x-30}$, as will appear upon tryal, and will be further evident from hence, that $\overline{30-x}$ and $\overline{x-30}$ differ no more from one another than an affirmative quantity does from an equal negative one, and therefore each multiplied into itſelf muſt give the ſame product, therefore the equation as it now ſtands is, $4\times\overline{x+30}\times\overline{x+30}=9\times\overline{x-30}\times\overline{x-30}$; but this equation is the ſame with the equation deduced from the laſt problem, which juſtifies what I obſerved before, art. 121, that different problems may produce the ſame equation; therefore the two roots of this equation will be 6 and 150, as in the laſt article; therefore the diſtance between the two places *C* and *D* muſt either be 6 miles, or 150 miles; but 150 miles it cannot be, becauſe, after *A* had paſſed from *C* beyond *D*, and at laſt had overtaken *B*, they had both travelled but 30

miles;

miles; therefore the diſtance from *C* to *D* muſt be 6 miles; and this number will anſwer the conditions of the problem; for then *A*, when he had overtaken *B*, had travelled 15+3 or 18 miles, and *B* 15—3 or 12 miles; therefore *A* had got 12 miles beyond *D* in 4 days time, and *B* was 18 miles diſtant from *C*, which he could travel in 9 days; but at the rate of 12 miles in 4 days, *A* muſt have performed his 18 miles journey in 6 days; and at the rate of 18 miles in 9 days, *B* muſt have performed his 12 miles journey alſo in 6 days; therefore, from the time of their firſt ſetting out to the time of *A's* overtaking *B*, they had both travelled the ſame number of days, as the problem requires; therefore the ſuppoſition whereupon this calculation was founded, to wit, that the diſtance of *C* from *D* was 6 miles, is juſt.

N. B. The ſolutions here given of the two laſt problems are, in my opinion, the moſt natural, though ſomewhat different from the reſt.

A LEMMA.

124. *The ſum of a ſeries of quantities in arithmetical progreſſion may be had by adding the greateſt and leaſt terms together, and then multiplying either half that ſum by the whole number of terms, or the whole ſum by half the number of terms, or laſtly, by multiplying the whole ſum into the whole number of terms, and then taking half the product:* thus in the ſeries 2, 4, 6, 8, 10, 12, where the leaſt term is 2, the greateſt 12, their ſum 14, and the number of terms 6; the ſum of all the terms taken together will be 7×6, or 14×3, or $\frac{14\times6}{2}=42$. This will beſt appear by writing down the ſeries 2, 4, 6, 8, 10, 12, and then by writing down over it the ſame ſeries inverted, 12, 10, 8, 6, 4, 2: for, if this be done, 2, the firſt term of the lower ſeries, added to 12, the firſt term of the up-

per ſeries (which is the ſame as the greateſt and leaſt terms of the ſame ſeries added together) will make 14; in like manner, every term of the lower ſeries added to the next above it will make 14; therefore both the ſerieſes together will be equal to 14 as often taken as there are terms in either ſeries, that is, 6 times 14, or 84; therefore either ſeries taken alone will be equal to 42.

12	10	8	6	4	2
2.	4.	6.	8.	10.	12.
14.	14.	14.	14.	14.	14.

The deſign of this lemma is, to add the terms of a ſeries together, where only the greateſt and leaſt terms and the number of terms are known, or ſuppoſed to be known; the intermediate terms being either not aſſigned, or too many to be ſummed up by a continual addition.

Problem 82.

125. *A traveller, as* A, *ſets out from a certain place, and travels one mile the firſt day, two miles the ſecond day, three the third, four the fourth, &c; and five days after, another, as* B, *ſets out from the ſame place, and travels the ſame road at the rate of twelve miles every day: I demand how long and how far* A *muſt travel before he is overtaken by* B.

Solution.

Put x for the number of days A travelled before he was overtaken by B; then, to find an expreſſion for the number of miles travelled by him in that time, I obſerve that in three days A travelled over $1+2+3$ miles, that is, he travels over a ſeries of miles in arithmetical progreſſion, whereof the number of terms is 3, the greateſt term 3, and the leaſt term 1; in four

four days he travels over a ſeries whereof the number of terms is 4, the greateſt term 4, and the leaſt 1; therefore, univerſally, in any number x of days, he muſt travel over a ſeries of miles in arithmetical progreſſion, whereof the number of terms is x, the greateſt term x, and the leaſt term 1; but the ſum of the extremes of this ſeries is $x+1$, which, multiplied by x the number of terms, gives $xx+x$, the half whereof is $\frac{xx+x}{2}$; therefore, by the lemma foregoing, $\frac{xx+x}{2}$ will be the ſum of this ſeries, and conſequently the miles travelled by A before he was overtaken: again, if A travel x days, B muſt have travelled $x-5$ days, which at the rate of 12 miles a day, gives $12x-60$ for the miles travelled by B when he overtook A; but as they both ſet out from the ſame place, and are now got together, they muſt have travelled the ſame number of miles; whence we have this equation, $\frac{xx+x}{2}=12x-60$; therefore $xx+x=24x-120$; therefore $xx=23x-120$; compare this equation with the general one in art. 103, and you will have $A=1$, $B=23$, $C=-120$, $BB=529$, $4AC=-480$, $ss=49$, $s=7$, $\frac{B+s}{2A}=15$, $\frac{B-s}{2A}=8$; therefore $x=8$, or 15: now, for the better application of theſe roots to the ſolution of this problem, it muſt be obſerved, that the problem is more limited than the equation deduced from it; juſt as if, in tranſlating out of one language into another, the terms of the latter, inſtead of being adequate to thoſe of the former, ſhould be found to be of a more extenſive ſignification: in the problem it is only ſuppoſed that B overtakes A, whereas in the equation it is ſuppoſed that A and B are got both together by having travelled the ſame number of miles from their firſt ſetting out, without ſpecifying whether this ariſes from *B's* overtaking *A*, or from *A's* overtaking *B*; both which in this caſe muſt neceſſarily happen in the courſe of their travels, provided they be but continued long enough for that purpoſe: for ſince at firſt *B* is the ſwifter traveller, whenever they come together, it muſt ariſe from *B's* overtaking *A*, which happens after *A* has travelled 8 days; then, if we ſuppoſe them ſtill to continue their travels, *B* paſſes by *A*, and continues before him for ſome time; but after 12 days, *A* becomes the ſwifter traveller, and muſt neceſſarily come up to *B* again after he has travelled 15 days: therefore though the two roots, 8 and 15, will both anſwer the condition of the equation, yet but one of them, to wit, 8, will anſwer the condition of the problem; and that both of them will anſwer the condition of the equation, will be evident as follows.

In 8 days *A* travels over a ſeries of miles whereof the number of terms is 8, the greateſt 8, and the leaſt 1; the ſum of which ſeries is 36 miles; but when *A* has travelled 8 days, *B* muſt have travelled 3 days, during which time, at the rate of 12 miles a day, he alſo muſt have travelled 36 miles; therefore after *A* hath travelled 8 days, *A* and *B* muſt neceſſarily find themſelves together: again, in 15 days, *A* muſt have travelled over a ſeries of miles, whereof the number of terms is 15, the greateſt 15, the leaſt 1, and the ſum 120 miles; but when *A* had travelled 15 days, *B* muſt have travelled 10 days, which at 12 miles a day gives alſo 120 miles; therefore now again *A* and *B* muſt find themſelves together; and conſequently 8 and 15 equally anſwer the ſuppoſition contained in the equation.

N. B. If we ſuppoſe *B* after 5 days to have begun to follow *A*, and to have travelled only 10 miles a day, he could never have overtaken *A*, nor *A* him, ſo that in this caſe both the roots would have become impoſſible, as will be found by the reſolution of an equation founded upon this ſuppoſition.

PROBLEM

PROBLEM 83.

126. *It is required to divide the number ten into two ſuch parts, that the product of their multiplication being added to the ſum of their ſquares, may make ſeventy-ſix.*

SOLUTION.

The two parts ſought, x and $10-x$.
The product of their multiplication, $10x-xx$.
The ſum of their ſquares, $2xx-20x+100$.
The product of their multiplication added to the ſum of their ſquares, $\Big\}$ $x^2-10x+100=76$.

Whence $x=4$, or 6; but this equation will be the ſame, which part ſoever x is put for; therefore the two parts ſought are 4 and 6.

PROBLEM 84.

127. *It is required to find two numbers with the following properties, to wit, that twice the firſt with three times the ſecond may make ſixty, and moreover, that twice the ſquare of the firſt with three times the ſquare of the ſecond may make eight hundred and forty.*

SOLUTION.

For the two numbers ſought put x and y, and we ſhall have

Equ. 1ſt, $2x+3y=60$, and
Equ. 2d, $2x^2+3y^2=840$.

From the firſt equation, $2x+3y=60$, we have

Equ. 3d, $x=\frac{60-3y}{2}$; and by ſquaring both ſides we have

Equ. 4th, $xx=\frac{3600-360y+9yy}{4}$.

From the ſecond equation, $2xx+3yy=840$, we have

Equ.

$$\text{Equ. 5th, } xx = \frac{840 - 3yy}{2}$$

Compare the two values of xx in the fourth and fifth equations, which muſt neceſſarily be equal one to the other, and you will have $\frac{3600 - 360y + 9yy}{4} = \frac{840 - 3yy}{2}$; multiply both ſides into 2, by halving the denominators, and you will have $\frac{3600 - 360y + 9yy}{2} = 840 - 3yy$; therefore $3600 - 360y + 9yy = 1680 - 6yy$; therefore $3600 - 360y + 15yy = 1680$; therefore $15yy - 360y = -1920$; therefore $15yy = 360y - 1920$; divide by 15 for a more ſimple equation, and you will have $yy = 24y - 128$; whence $y = 8$, or 16: ſuppoſe $y = 8$, then ſince by the third equation $x = \frac{60 - 3y}{2}$, we ſhall have $x = 18$; ſuppoſe $y = 16$, then we ſhall have x or $\frac{60 - 3y}{2} = 6$; therefore there are two pair of numbers that will equally anſwer the conditions of this problem, to wit, 18 and 8, and alſo 6 and 16: for a proof, let us firſt ſuppoſe the numbers to be 18 and 8; and we ſhall have twice the firſt number with three times the ſecond $= 36 + 24 = 60$; and twice the ſquare of the firſt together with three times the ſquare of the ſecond equal to $648 + 192 = 840$: ſecondly, let us ſuppoſe the numbers to be 6 and 16; and we ſhall have twice the firſt with three times the ſecond equal to $12 + 48 = 60$; and twice the ſquare of the firſt with three times the ſquare of the ſecond equal to $72 + 768 = 840$.

PROBLEM 85.

128. *To find four numbers in continual proportion, and ſuch, that the ſum of the two middle terms may be eighteen, and that of the extremes twenty-ſeven.*

Note,

Note, Four numbers are ſaid to be in continual proportion, when the firſt is to the ſecond as the ſecond is to the third, and the ſecond is to the third as the third is to the fourth.

SOLUTION.

For the two middle terms put x and y, without intending which is to be the greater; then the extreme next to x may be found by ſaying, as y is to x ſo is x to $\frac{xx}{y}$, and the extreme next to y may be found by ſaying, as x is to y, ſo is y to $\frac{yy}{x}$; therefore the extremes are $\frac{xx}{y}$ and $\frac{yy}{x}$, and their ſum $\frac{x^3+y^3}{xy}$; therefore the fundamental equations are 1ſt, $x+y=18$, or $x=18-y$; and 2dly, $\frac{x^3+y^3}{xy}=27$, or $x^3+y^3=27xy$; inſtead of x in this equation put $18-y$, its value in the laſt, and you will have $x^3=5832-972y+54y^2-y^3$; therefore $x^3+y^3=5832-972y+54yy$; you will alſo have $27xy$ or $27y\times\overline{18-y}=486y-27yy$; therefore $5832-972y+54yy=486y-27yy$; tranſpoſe $486y-27yy$, and you will have $81yy-1458y+5832=0$; divide all by 81, which may be done without a fraction, and you will have $yy-18y+72=0$; which equation being reſolved, either by the general theorem or any other way, gives $y=6$, or 12; and ſince the equation will be the ſame, whichſoever of the two middle terms y ſtands for, it follows, that the two middle terms are 6 and 12; whence the extreme next to 6 is 3, and that next to 12 is 24; and the numbers are either 3, 6, 12, and 24, or 24, 12, 6, and 3, for either way they will anſwer the conditions of the problem.

PROBLEM

Problem 86.

129. *There are three numbers in continual proportion, whose ſum is nineteen, and the ſum of their ſquares one hundred thirty-three: What are the numbers?*

Solution.

For the three numbers ſought put x, y and z; then ſince, by the firſt condition, x is to y as y is to z, by multiplying extremes and means we have $yy=xz$; again, by the ſecond condition of the problem, we have $x+y+z=19$, and $19-y=x+z$, and (ſquaring both ſides) $361-38y+yy=xx+2xz+zz$; ſubtract yy from one ſide of the equation, and its equal xz from the other, and you will have $361-38y=x^2+xz+z^2=x^2+y^2+z^2=133$ by the third condition of the problem: having thus expunged both x and z at once, reſolve the equation $361-38y=133$, and you will have y the middle term equal to 6, and $19-y$, or the ſum of the extremes, $=13$; therefore the problem propoſed is now reduced to this, *viz.* *Of three numbers in continual proportion, whereof ſix the middle term, and thirteen the ſum of the extremes, are given, to find the extremes:* this problem is of the ſame nature with that in art. 112, and, being reſolved, gives 4 and 9 for the extremes; therefore the three numbers ſought are 4, 6, and 9, or 9, 6, and 4.

Problem 87.

130. *To find two numbers ſuch, that their difference multiplied into the difference of their ſquares ſhall make thirty-two, but their ſum multiplied into the ſum of their ſquares ſhall make two hundred ſeventy-two.*

Solution.

For the two numbers ſought put x and y; and the firſt fundamental equation will be $\overline{x-y}\times\overline{x^2-y^2}$, or $\overline{x-y}\times\overline{x-y}\times\overline{x+y}$, or $\overline{x^2-2xy+y^2}\times\overline{x+y}=32$; therefore Equ.

$$\text{Equ. 1st, } x^2-2xy+y^2=\frac{32}{x+y}.$$

The second fundamental equation is, $\overline{x+y}\times\overline{x^2+y^2}=272$; therefore

$$\text{Equ. 2d, } x^2+y^2=\frac{272}{x+y}.$$

From twice the second equation subtract the first, that is, from $2x^2 * + 2y^2=\frac{544}{x+y}$

subtract $x^2-2xy+y^2=\frac{32}{x+y}$

and you will have $x^2+2xy+y^2=\frac{512}{x+y}$,

that is, $\overline{x+y}^2=\frac{512}{x+y}$; therefore $\overline{x+y}^3=512$, and $x+y=\sqrt[3]{512}$, or the cube root of $512=8$: thus we have got the sum of the two numbers sought, to wit, 8; whence their difference may be found by the first equation, thus; $x^2-2xy+yy=\frac{32}{x+y}$, that is, $\overline{x-y}^2=\frac{32}{8}=4$; therefore $x-y$, or the difference of the two numbers sought, equals 2; therefore the problem proposed is now reduced to this; *Having given eight the sum, and two the difference of the two numbers* x *and* y, *to find those numbers*; and by art. 26 we shall have $x=5$, and $y=3$; which numbers will answer the conditions of the question.

N. B. After we had found $x+y$, the sum of the numbers equal to 8, we might have found the sum of their squares by the second equation, which gave $x^2+y^2=\frac{272}{x+y}=\frac{272}{8}=34$; and then the problem would have been reduced to this; *What two numbers are those, whose sum is eight, and the sum of their squares thirty-four?*

four? which would have produced a quadratic equation, as in art. 113, whose two roots would have been 5 and 3, as before.

Problem 88.

131. *To find two numbers such, that their difference added to the difference of their squares may make fourteen, and their sum added to the sum of their squares may make twenty-six.*

Solution.

For the two numbers sought put x and y, and you will have the two following equations;

Equ. 1st, $x-y+x^2-y^2=14$.

Equ. 2d, $x+y+x^2+y^2=26$.

Add these two equations together, and you will have $2xx+2x=40$, $xx+x=20$, and $x=+4$, or -5; again, subtract the first equation from the second, and you will have $2yy+2y=12$, $yy+y=6$, and $y=+2$, or -3; and as these two values of y were obtained without any manner of dependence upon those of x, it is plain that either of the values of x may be joined with either of the values of y; and so we have no fewer than four pairs of numbers which will equally satisfy the conditions of the equations, to wit, $+4$ and $+2$, $+4$ and -3, -5 and $+2$, -5 and -3; but it is the first pair only, which, consisting of affirmative numbers, is proper for the solution of the problem, thus; the difference of 4 and 2 is 2, the difference of their squares 12, and $2+12=14$; again, the sum of 4 and 2 is 6, the sum of their squares 20, and $6+20=26$: let us see however how the other pairs will satisfy the conditions of the equations; make then x equal to 4, y, that is, $+y=-3$, and you will have $-y=+3$; whence $x-y=4+3=7$, $x^2-y^2=16-9=7$, and $7+7=14$; again, $x+y=4-3=1$, and $x^2+y^2=16+9=25$, and $1+25=26$: in the next place, make $x=-5$, and $y=+2$, then we shall

6 have

have $x-y=-5-2=-7$, $x^2-y^2=25-4=21$, and $-7+21=14$; again, $x+y=-5+2=-3$, and $x^2+y^2=25+4=29$, and $-3+29=26$: laftly, make $x=-5$, and $y=-3$, and you will have $x-y=-5+3=-2$, and $x^2-y^2=25-9=16$, and $-2+16=14$; again, $x+y=-5-3=-8$, and $x^2+y^2=25+9=34$, and $-8+34=26$.

PROBLEM 89.

132. *What two numbers are thofe, whofe fum, when added together, is equal to their product when multiplied together; and this fum or product, when added to the fum of their fquares, makes twelve?*

SOLUTION.

For the two numbers fought put x and y, and the fundamental equations will be 1ft, $x+y=xy$; and fecondly, $x+y+x^2+y^2=12$: in the firft of thefe fundamental equations, where $x+y=yx$, we have $yx-x=y$; but $yx-x$ is the product of $\overline{y-1}\times x$, or of $x\times\overline{y-1}$; therefore $x\times\overline{y-1}=y$, and $x=\frac{y}{y-1}$; but if inftead of x, this value be fubftituted into the fecond fundamental equation, the equation will rife to a biquadratic, for the refolution whereof no rules have hitherto been given; therefore, to extricate ourfelves out of this difficulty, it will be proper to have recourfe to fome other artifice, by trying other pofitions, as thus; for the fum of the two numbers fought put z; then will z be alfo the product of their multiplication, by the fuppofition; and fince this product z added to the fum of their fquares gives 12, the fum of their fquares will be $12-z$; but every one knows, that if to the fum of the fquares of any two numbers be added their double product, there will arife the fquare of their fum; therefore $12-z+2z$, or $12+z=z^2$; which equation being refolved, gives $z=+4$, &c.;

&c.; and therefore the queſtion is now reduced to this; *What two numbers are thoſe, whoſe ſum is four, and the product of whoſe multiplication is four?* for the numbers ſought, put x and $4-x$, and you will have $4x-xx=4$; and changing the ſigns, $xx-4x=-4$; and compleating the ſquare, $xx-4x+4=0$; and extracting the ſquare root, $x-2=\pm 0$; whence $x=2$, or 2, for the roots of this equation are equal; therefore 2 and 2 are the numbers deſired in the queſtion; and they will anſwer the conditions; for in the firſt place, $2+2=4=2\times 2$; and in the next place, 4 the ſum of 2 and 2, being added to 8, the ſum of their ſquares, gives 12.

Corollary.

From our firſt attempt to ſolve this problem we may learn thus much however, that if any number whatever be made equal to y, then theſe two numbers y and $\frac{y}{y-1}$ will always have this property, that their ſum when added together will be equal to their product when multiplied together; thus if $3=y$, and conſequently $\frac{3}{2}=\frac{y}{y-1}$, we ſhall have $3+\frac{3}{2}=4\frac{1}{2}$, and $3\times\frac{3}{2}$ or $\frac{9}{2}=4\frac{1}{2}$; whence it follows, that this problem cannot be ſolved in whole numbers in any other caſe than that we have here put.

Problem 90.

133. *What two numbers are thoſe, whoſe ſum added to the product of their multiplication makes thirty-four, and the ſame ſum ſubtracted from the ſum of their ſquares leaves forty-two.*

Solution.

Here, to avoid all difficulties that would otherwiſe ariſe, put z for the ſum of the two numbers ſought; then, ſince this ſum added to the product of their multiplication makes 34, the product of their multiplication will be $34-z$; but this sum z subtracted from the sum of their squares, leaves 42; therefore the sum of their squares is $42+z$; to this add their double product $68-2z$, and you will have $110-z=z^2$; whence $z=+10$, &c. and $34-z=24$; therefore now the question is, *What two numbers are those, whose sum is ten, and the product of their multiplication twenty-four?* and by art. 111, the two numbers sought are 4 and 6.

Whoever would see more questions of this nature, may consult *Bachet's* comment upon the 33d question of the first book of *Diophantus's* Arithmetics.

N. B. Having now done with quadratic equations, at least for a time, it may perhaps be expected that, according to order of method I should proceed on to equations of higher forms: but I shall take the liberty for once to dispense with that method; not but that I intend (God willing) to treat fully and distinctly of these equations hereafter; but in the mean time I think it more adviseable to employ the reader's thoughts in some other things, which I take be of much greater importance, and more proper for his information.

THE

ELEMENTS OF ALGEBRA.

BOOK IV.

Of general problems, and general theorems deduced from them; together with the manner of applying and demonſtrating theſe theorems ſynthetically.

The deſign of this fourth book more fully explained.

Art. 134. HITHERTO my young Analyſt has been indulged for the moſt part in a ſort of mixt Algebra, where letters were put only for unknown quantities: but if he would reaſon abſtractedly upon his problems, and draw general concluſions from them, he muſt put letters not only for his unknown quantities, but alſo for ſuch as are known; and ſo propoſe and ſolve his problems indefinitely. By this means, in the firſt place, he will obtain indefinite anſwers, which in many caſes are much preferable to more particular ones, as they ſuit and ſolve all particular caſes to which they are applicable; and in the next place he will be able to prove his work ſynthetically; which will not only confirm his former *analyſis*, but

but will alſo further inure and reconcile him to the operations of ſymbolical or ſpecious Arithmetic; and ſo render him entire maſter of this ſort of computation. A ſufficient ſpecimen of this ſort of reaſoning, both in the analytical and ſynthetical way, has already been given in our general theorem for the reſolution of a quadratic equation, ſo that no more needs be ſaid by way of preparation; it remains therefore now, that we look back upon ſome of the problems already ſolved, and ſhew how to ſolve them over again in general terms, as follows:

PROBLEM 1. (See art. 26.)

135. *What two numbers are thoſe, whoſe ſum is* a, *and difference* b?

SOLUTION.

Put x for the leſs number; then will the greater be $x+b$, and their ſum $2x+b=a$; whence $2x=a-b$, and x (the leſs number) will be $\frac{a-b}{2}$; whence $x+b$, (the greater number) will be $\frac{a-b}{2}+\frac{b}{1}=\frac{a-b+2b}{2}$ $=\frac{a+b}{2}$; ſo the greater number is found to be $\frac{a+b}{2}$, and the leſs $\frac{a-b}{2}$; where a and b are left undetermined till ſome particular caſe of this problem is propoſed to be compared with the general one; and then the quantities a and b will not only be determined in that caſe, but the problem may be ſolved by the general theorem without any further *analyſis*. As for example, let it be propoſed, as in art. 26, to find two numbers whoſe ſum is 48, and difference 14: here it is plain that a in the general problem anſwers to 48 in the particular caſe, and b to 14; whence $\frac{a+b}{2}$ (or the greater number) $=\frac{48+14}{2}=\frac{62}{2}=31$,

 and

and $\frac{a-b}{2}$ (or the leſs number) $= \frac{48-14}{2} = \frac{34}{1} =$ 17; ſo that the numbers ſought are 31 and 17; which will anſwer the conditions of the queſtion. Again, ſuppoſe we were to find two numbers whoſe ſum is 35, and whoſe difference is 9: in this caſe it is plain that a and b have other ſignifications; for here $a=35$, and $b=9$, and therefore $\frac{a+b}{2}$ (or the greater number) will be 22, and $\frac{a-b}{2}$ (or the leſs number) will be 13.

Theſe theorems are capable of being tranſlated out of Algebraic language into any other; though to no great purpoſe that I know of, to ſuch as underſtand any thing of ſymbolical Arithmetic; for, in my opinion, they appear much more diſtinct as they are, and leſs liable to ambiguity. The foregoing problem, together with the anſwer belonging to it, being tranſlated into common Engliſh, will ſtand thus:

Problem.

It is required, having given the ſum and difference of any two numbers, to find the numbers themſelves.

Anſ. 1ſt. *Add the difference to the ſum, and half the aggregate will be the greater number.* 2dly, *Subtract the difference from the ſum, and half the remainder will be the leſs number.*

That this is a true tranſlation, is plain: for what is $\frac{a+b}{2}$ but half the aggregate of the ſum and difference added together? and what is $\frac{a-b}{2}$ but half the remainder, after the difference is ſubtracted from the ſum?

We come now, in the laſt place, to examine this theorem as it ſtands in general terms, and to try whether it will anſwer the conditions of the problem in the letters themſelves. It was propoſed to find two numbers, whoſe ſum is a, and whoſe difference is b; and the anſwer was, that the greater number was $\frac{a+b}{2}$, and the leſs $\frac{a-b}{2}$: now that this is a true anſwer, will be evident from a bare addition and ſubtraction of the numbers themſelves, without any other principles; for if $\frac{a+b}{2}$ be added to $\frac{a-b}{2}$, their ſum will be $\frac{2a}{2}$ or a, which anſwers the firſt condition of the problem; and if $\frac{a-b}{2}$ be ſubtracted from $\frac{a+b}{2}$, the remainder will be $\frac{2b}{2}$ or b, which anſwers the ſecond condition.

This is that which is called a ſynthetical demonſtration, and doubtleſs ſhews the truth of the theorem to which it belongs, as well as the *analyſis* whereby that theorem was inveſtigated; but not ſo much to the ſatisfaction of the mind: for a ſynthetical demonſtration only ſhews that a propoſition is true; whereas an analytical one ſhews not only that a propoſition is true, but why it is ſo; places you in the condition of the inventor himſelf, and unveils the whole myſtery. Synthetical demonſtrations uſually require fewer principles than analytical ones, as will evidently appear, by comparing both, in this very example; and this I take to be the reaſon why the ancients, generally ſpeaking, choſe to demonſtrate their propoſitions this way; not with a deſign to conceal their *analyſis*, as ſome have, unjuſtly enough, imagined; but becauſe this ſort of demonſtration required fewer principles to proceed upon, and thoſe too, ſuch as were commonly known.

Problem 2.

136. *What three numbers are those, whereof the sum of the first and second is* a, *that of the first and third* b, *and that of the second and third* c?

Solution.

Put x for the first number sought; then will the second number be $a-x$, because the first and second numbers together make a; for a like reason the third number will be $b-x$, because the first and third together make b: add now the second and third numbers together, and you will have $a+b-2x=c$; therefore $2x+c=a+b$; therefore $2x=a+b-c$; and x (or the first number) $=\frac{a+b-c}{2}$; subtract now the first number $\frac{a+b-c}{2}$ from a, or, which is all one, add $\frac{-a-b+c}{2}$ to a, and you will have the second number equal to $\frac{-a-b+c}{2}+\frac{a}{1}=\frac{-a-b+c+2a}{2}=\frac{a-b+c}{2}$; again, subtract the first number $\frac{a+b-c}{2}$ from b, and you will have the third number equal to $\frac{-a-b+c}{2}+\frac{b}{1}=\frac{-a+b+c}{2}$; and thus we have all the three numbers sought, to wit,

The first, $\frac{a+b-c}{2}$,

The second, $\frac{a-b+c}{2}$,

The third, $\frac{-a+b+c}{2}$.

To apply this general solution to some particular case, I shall make use of that in art. 42, where it was required

required to find three ſuch numbers, that the ſum of the firſt and ſecond may make 60, that of the firſt and third 80, and that of the ſecond and third 92: in this caſe it is plain that $a=60$, $b=80$, and $c=92$; therefore $\frac{a+b-c}{2}$ or the firſt number will be 24; $\frac{a-b+c}{2}$ or the ſecond number will be 36; and $\frac{-a+b+c}{2}$ or the third number will be 56; which numbers upon tryal will be found to be ſuch as the problem requires. But that the theorems here given are not only true in this particular caſe, but are univerſally ſo, will beſt appear from the ſynthetical demonſtration following.

1ſt, The firſt number $\frac{a+b-c}{2}$, and the ſecond number $\frac{a-b+c}{2}$ being added together make $\frac{2a}{2}$ or a, according to the firſt condition, the other quantities deſtroying one another.

2dly, The firſt number $\frac{a+b-c}{2}$, and the third number $\frac{-a+b+c}{2}$ being added together make $\frac{2b}{2}$ or b, according to the ſecond condition.

Laſtly, The ſecond number $\frac{a-b+c}{2}$ and the third number $\frac{-a+b+c}{2}$ being added together make $\frac{2c}{2}$ or c, according to the third condition.

This problem may alſo be ſolved ſomewhat more elegantly thus: put s for the unknown ſum of all the three numbers ſought: then if c, the ſum of the ſecond and third numbers, be ſubtracted from s, the ſum of all three, there will remain the firſt number

 equal

equal to $s-c$; in like manner b, the ſum of the firſt and third numbers, ſubtracted from s, the ſum of all three, leaves the ſecond number equal to $s-b$; and a, the ſum of the firſt and ſecond numbers, ſubtracted from s, the ſum of all three, leaves the third number equal to $s-a$; add now all theſe three numbers together, to wit, $s-c$, $s-b$ and $s-a$, and the ſum will be $3s-a-b-c$; but the ſum is s, by the ſuppoſition; therefore, $3s-a-b-c=s$; and $s=\frac{a+b+c}{2}$, whence we have the following theorem:

Make $\frac{a+b+c}{2}=s$; *then if the numbers* a, b *and* c *be taken backwards, and ſubtracted ſeverally from* s, *the three remainders* $\overline{s-c}$, $\overline{s-b}$, *and* $\overline{s-a}$ *will be the three numbers ſought, in order as they are ſuppoſed in the problem.* Thus if $a=60$, $b=80$, and $c=92$, as before, we ſhall have $\frac{a+b+c}{2}$ or $s=116$; whence the firſt number will be $116-92$ or 24, the ſecond $116-80$ or 36, and the third $116-60$ or 56.

Scholium.

What three numbers are thoſe, whereof the product of the firſt and ſecond is a, *that of the firſt and third* b, *and that of the ſecond and third* c?

Solution.

Put p for the product of all the three numbers; then ſince c is the product of the two laſt, we ſhall have the firſt number equal to $\frac{p}{c}$; for a like reaſon the ſecond equals $\frac{p}{b}$, and the third equals $\frac{p}{a}$, and the product of all three equals $\frac{p^3}{abc}=p$; therefore $p^2=abc$, and $p=\sqrt{abc}$.

Demon-

Demonstration.

$\frac{p}{c}\times\frac{p}{b}$, or the product of the first and second numbers, is, $\frac{p^2}{bc}=\frac{abc}{bc}=a$: and so of the rest.

Problem 3.

137. *It is required to find two numbers whose difference is* b, *and the difference of whose squares is* a.

Solution.

Put x for the less number, and consequently $x+b$ for the greater; then will the square of the less number be xx, that of the greater $xx+2bx+bb$, and the difference of their squares $2bx+bb=a$; therefore $2bx=a-bb$, and x (the less number) $=\frac{a-bb}{2b}$; whence $x+b$ (the greater) $=\frac{a-bb}{2b}+\frac{b}{1}=\frac{a-bb+2bb}{2b}=\frac{a+bb}{2b}$.

To apply this general solution, let it be required to find two numbers whose difference is 4, and the difference of whose squares is 112: here $a=112$, $b=4$, $bb=16$, $\frac{a-bb}{2b}=12$, $\frac{a+bb}{2b}=16$; therefore the numbers are 12 and 16. The general demonstration is as follows: if the less number $\frac{a-bb}{2b}$ be subtracted from the greater $\frac{a+bb}{2b}$, their difference will be $\frac{2bb}{2b}$ or b, according to the first condition of the problem; again, the square of the less number $\frac{a-bb}{2b}$ is

$aa-$

$\frac{aa - 2abb + b^4}{4bb}$, and the ſquare of the greater $\frac{a+bb}{2b}$ is $\frac{aa+2ab^2+b^4}{4bb}$; ſubtract the ſquare of the leſs from that of the greater, and you will have the difference of their ſquares $= \frac{4abb}{4bb} = a$, as the ſecond condition requires.

Problem 4.

138. *Let* r *and* s *be two given multiplicators, whereof* r *is the greater; it is required to divide a given number as* a *into two ſuch parts, that the greater part when multiplied into the leſs multiplicator may be equal to the leſs part when multiplied by the greater multiplicator.*

Solution.

Put x for the greater part, and $a-x$ for the leſs; then will the greater part multiplied into the leſs multiplicator be sx, and the leſs part multiplied into the greater multiplicator will be $ar-rx$: but according to the problem, theſe products are to be equal; therefore $sx=ar-rx$, and $rx+sx=ar$, but $rx+sx$ is $x \times \overline{r+s}$; therefore $x + \overline{r+s} = ar$; and x (the greater of the two parts ſought) $= \frac{ar}{r+s}$; whence $a-x$, (the leſs part) equal $\frac{a}{1} - \frac{ar}{r+s} = \frac{ar+as-ar}{r+s}$ $= \frac{as}{r+s}$; ſo the greater part ſought is $\frac{ar}{r+s}$, and the leſs $\frac{as}{r+s}$.

The Application.

To apply this canon, let it be required to divide 84 into two ſuch parts, that five times one part may be be equal to feven times the other: here $a=84$, r the greater multiplicator $=7$, $s=5$, $\frac{ar}{r+s}=\frac{7\times84}{12}=49$, $\frac{as}{r+s}=\frac{5\times84}{12}=35$; therefore the greater part is 49, and the lefs 35; and they will anfwer the conditions; for firft, $49+35=84$; and fecondly, $49\times5=245=35\times7$. Again, let it be required to divide 99 into two fuch parts, that $\frac{2}{3}$ of one part may be equal to $\frac{4}{5}$ of the other: here $a=99$, $r=\frac{4}{5}$, $s=\frac{2}{3}$, $r+s=\frac{22}{15}$, $\frac{r}{r+s}=\frac{\frac{4}{5}}{\frac{22}{15}}=\frac{6}{11}$, $\frac{s}{r+s}=\frac{\frac{2}{3}}{\frac{22}{15}}=\frac{5}{11}$, $\frac{ar}{r+s}=99\times\frac{6}{11}=54$, $\frac{as}{r+s}=99\times\frac{5}{11}=45$; fo the two parts are 54 and 45; which is true; for firft, $54+45=99$; and fecondly, $\frac{2}{3}$ of $54=36=\frac{4}{5}$ of 45.

As to the demonftration of this general folution, it muft be obferved that in this problem there are two conditions; firft, that the two parts, when added together, muft make a; and fecondly, that the greater part multiplied into the lefs multiplicator muft be equal to the lefs part multiplied into the greater multiplicator: as to the firft of the conditions, it is certain that the parts $\frac{ar}{r+s}$ and $\frac{as}{r+s}$ when added together will make $\frac{ar+as}{r+s}$; but $ar+as=a\times\overline{r+s}$, therefore $\frac{ar+as}{r+s}=a\times\frac{r+s}{r+s}=a\times1=a$: as to the fecond condition, if the greater part $\frac{ar}{r+s}$ be multiplied into s, the lefs multiplicator, the product will be $\frac{ars}{r+s}$; and again, if the lefs part $\frac{as}{r+s}$ be multiplied into r, the greater multiplicator, the product will

will alſo be $\frac{ars}{r+s}$; therefore the two products are equal, as the problem requires; and ſo the conditions are both ſatisfied. *Q. E. D.*

N. B. If any one has a mind to throw the foregoing theorem into words, it may eaſily be done, and in ſuch a manner as almoſt to carry its own evidence along with it; for by the rule of proportion, $r+s$ is to r as a to $\frac{ar}{r+s}$; and $r+s$ is to s as a to $\frac{as}{r+s}$; therefore, *As the ſum of the two multiplicators is to the greater or leſs multiplicator, ſo is the ſum of the two parts ſought to the greater or leſs part:* and this, I ſay, is pretty evident; for had $\overline{r+s}$ been the number to be divided, the parts would certainly have been r and s; therefore if a greater or leſs number than $\overline{r+s}$ is to be divided, the parts ought to be greater or leſs than r and s in the ſame proportion.

Problem 5.

139. *Let* r *and* s *be two given multiplicators, whereof* r *is the greater; it is required to divide a given number as* a *into two ſuch parts, that* r *times one part being added to* s *times the other may make ſome other given number, as* b.

Solution.

Put x for the part that is to be multiplied by r, and conſequently $a-x$ for the other part that is to be multiplied by s, and the products will be rx and $as-sx$, and their ſum will be $rx+as-sx=b$; therefore $rx-sx=b-as$, that is, $x\times\overline{r-s}=b-as$; therefore x (the part to be multiplied by r) $=\frac{b-as}{r-s}$; therefore $a-x$ (the part to be multiplied by s) $=$ $\frac{a}{1}-\frac{b+as}{r-s}=\frac{ar-as-b+as}{r-s}=\frac{ar-b}{r-s}$.

The

The APPLICATION.

Let it be required to divide 20 into two ſuch parts, that three times one part being added to five times the other may make 84: here $a=20$, $b=84$, $r=5$, $s=3$, $as=60$, $b-as=24$, $\frac{b-as}{r-s}$ (or the part to be multiplied by 5) $=\frac{24}{2}=12$, $ar=100$, $ar-b=16$; $\frac{ar-b}{r-s}$ (or the part to be multiplied by 3) $=\frac{16}{2}=8$; therefore the parts ſought are 8 and 12; for firſt, $8+12=20$; and ſecondly, three times 8 + five times $12=84$.

Again, let it be required to divide 100 into two ſuch parts, that $\frac{3}{4}$ of one part being ſubtracted from $\frac{5}{6}$ of the other, may leave 39: here it muſt be obſerved, that to ſubtract $\frac{3}{4}$ of any one quantity from another, is the ſame as to add $\frac{-3}{4}$ of it; therefore this problem when reduced to the form of the general one, will ſtand thus: *To divide a hundred into two ſuch parts, that* $\frac{-3}{4}$ *of one part being added to* $+\frac{5}{6}$ *of the other may make thirty-nine.* Here $a=100$, $b=39$, $r=\frac{5}{6}$, $s=\frac{-3}{4}$, $r-s=\frac{5}{6}+\frac{3}{4}=\frac{19}{12}$, $as=\frac{-300}{4}=-75$, $b-as=39+75=114$, $\frac{b-as}{r-s}=\frac{114}{\frac{19}{12}}=72$, $ar=\frac{500}{6}=\frac{250}{3}$, $ar-b=\frac{250}{3}-\frac{39}{1}=\frac{133}{3}$, $\frac{ar-b}{r-s}=\frac{\frac{133}{3}}{\frac{19}{12}}=28$; ſo the two parts are 28 and 72: for $28+72=100$; and moreover $\frac{3}{4}$ of 28, that

that is, 21, ſubtracted from $\frac{5}{6}$ of 72, that is, from 60, leaves 39.

The general Demonstration.

The two parts $\frac{ar-b}{r-s}$ and $\frac{b-as}{r-s}$ when added together, make $\frac{ar-b+b-as}{r-s}=\frac{ar-as}{r-s}=a\times\frac{r-s}{r-s}=a$: again, the part $\frac{b-as}{r-s}$ being multiplied into r, its proper multiplicator, gives $\frac{br-ars}{r-s}$, and the other part $\frac{ar-b}{r-s}$, multiplied into the other multiplicator s, gives $\frac{ars-bs}{r-s}$; add theſe two products together, and they will make $\frac{br-ars+ars-bs}{r-s}=\frac{br-bs}{r-s}=b$. *Q. E. D.*

If any one hereafter ſhall think me too conciſe in the ſolutions of theſe general problems, he muſt have recourſe to the particular ones in the articles I ſhall refer him to, which he will find explained more at large: and as to the application of theſe general ſolutions to thoſe particular caſes, it is to be preſumed that by this time the learner will be able in ſome meaſure to perform that part himſelf; and therefore I ſhall for the future leave it to him, except where I ſhall think my aſſiſtance may be of any uſe.

Problem 6. (See art. 35.)

140. *One meeting a company of beggars, gives to each* p *pence, and has* a *pence over; but if he would have given them* q *pence apiece, he would have found he had wanted* b *pence for that purpoſe: What was the number of perſons?*

SOLUTION.

The number of perſons, x.

Pence given, px.

Pence in all, $px+a$.

The pence that would have been given upon the other ſuppoſition, qx.

Another expreſſion for the number of pence in all, $qx-b$.

Equ. $qx-b=px+a$; therefore $qx-px-b=a$; therefore $qx-px=a+b$; therefore x (the number of perſons) $=\frac{a+b}{q-p}$.

DEMONSTRATION.

If the number of perſons be $\frac{a+b}{q-p}$, then the pence given will be $\frac{ap+bp}{q-p}$, and the pence in all will be $\frac{ap+bp}{q-p}+\frac{a}{1}=\frac{ap+bp+aq-ap}{q-p}=\frac{aq+bp}{q-p}$: again the number of pence that would have been given upon the ſecond ſuppoſition is $\frac{aq+bq}{q-p}$; and therefore the other expreſſion for the number of pence in all will be $\frac{aq+bq}{q-p}-\frac{b}{1}=\frac{aq+bp}{q-p}$; and the perfect agreement between this account and the former is an infallible argument that the number of perſons was rightly aſſigned.

PROBLEM 7. (See art. 64.)

141. *It is required to divide a given number as* a *into two ſuch parts, that one part may be to the other as* r *to* s.

SOLUTION.

Solution.

The two parts ſought, x and $a-x$.

Proportion, x is to $a-x$ as r to s.

Equation, $sx = ar - rx$; therefore $rx + sx = ar$; therefore x (or the firſt number) $= \frac{ar}{r+s}$; therefore $a-x$ (or the ſecond number) $= \frac{a}{1} - \frac{ar}{r+s} = \frac{as}{r+s}$; therefore the two numbers are $\frac{ar}{r+s}$ and $\frac{as}{r+s}$.

Demonstration.

1ſt, The two numbers $\frac{ar}{r+s}$ and $\frac{as}{r+s}$ when added together make $\frac{ar+as}{r+s} = a$.

2dly, The firſt number $\frac{ar}{r+s}$ is to the ſecond number $\frac{as}{r+s}$ as ar is to as; becauſe throwing away the common denominator is no more in reality than multiplying both fractions by it; and every one knows, that the multiplication of two quantities by the ſame number, makes no alteration in the proportion they bore one to the other: again, ar is to as (dividing both by a) as r to s; for it is well known that a common diviſion affects proportion no more than a common multiplication: ſince then the firſt number is to the ſecond as ar to as, and ar is to as as r to s, it follows, that the firſt number is to the ſecond as r to s. *Q. E. D.*

Problem 8. (See art. 66.)

142. *What number is that, which being ſeverally added to two given numbers,* a *a greater number, and* b *a leſs, will make the former ſum to the latter as* r *to* s? *therefore* r *muſt be greater than* s?

Solution.

SOLUTION.

The number ſought, x.

Proportion, $a+x$ is to $b+x$ as r to s.

Equation, $br+rx=as+sx$; therefore $br+rx-sx=as$; therefore $rx-sx=as-br$; therefore $x=\frac{as-br}{r-s}$.

DEMONSTRATION.

The number $\frac{as-br}{r-s}$ being added to a, gives $\frac{ar-br}{r-s}$ and the ſame number being added to b, gives $\frac{as-bs}{r-s}$; now $\frac{ar-br}{r-s}$ is to $\frac{as-bs}{r-s}$ as $ar-br$ is to $as-bs$, that is, as $r\times\overline{a-b}$ is to $s\times\overline{a-b}$, that is, as r to s. *Q. E. D.*

SCHOLIUM.

This problem was to find a number, which, being ſeverally added to a and b, will make the former ſum to the latter as r to s; let us now change the numbers a and b one for another, as alſo the numbers r and s one for another, and then the problem will ſtand thus: *To find a number, which, being ſeverally added to* b *and* a, *will make the former ſum to the latter as* s *to* r: but the condition of this problem is exactly the ſame with that of the former, and therefore the anſwer ought ſtill to be the ſame; that is, as changing a and b one for another, and r and s one for another, had no effect upon the problem, but left it entirely the ſame as at firſt; ſo if the expreſſion of the number ſought be juſt, the changing of a and b one for another, and of r and s one for another, ought to make no alteration in that expreſſion, and the number ſought ought ſtill to be the ſame; for truth will always be conſiſtent with herſelf. Let us try this however, and ſee what will be the effect of ſuch a change: now the number ſought was $\frac{as-br}{r-s}$; but upon this change, *as* becomes *br*, and *br* becomes *as*, and $r-s$ becomes $s-r$, and the whole expreſſion will be turned into this, $\frac{br-as}{s-r}$; but $\frac{br-as}{s-r}$ is the ſame as $\frac{as-br}{r-s}$; for changing the ſign of both the numerator and denominator of any fraction, no more affects the value of that fraction, than in diviſion the changing of the ſign both of the diviſor and dividend affects the value of the quotient: thus then we find, that the changing of *a* and *b* one for another, and of *r* and *s* one for another, no more affects the theorem for determining the number ſought, than it did the problem from whence it was derived.

Problem 9.

143. *It is required to divide a given number as* a *into two ſuch parts, that the exceſs of one part above another given number as* b, *may be to what the other wants of* b, *as* r *to* s; *ſuppoſing* r *greater than* s.

Solution.

Put *x* for the greater part, and $\overline{a-x}$ for the leſs; then the exceſs of *x* above *b* will be $x-b$; and the exceſs of *b* above $\overline{a-x}$ will be $x-a+b$, as appears by ſubtracting $\overline{a-x}$ from *b*; but by the problem, the former exceſs is to the latter as *r* to *s*; therefore $x-b$ is to $x-a+b$ as *r* to *s*; multiply extremes and means, and you will have $sx-bs=rx-ar+br$; therefore $rx-sx=ar-br-bs$, and *x* (the greater part) $=$ $\frac{ar-br-bs}{r-s}$; therefore $a-x$ (the leſs part) $=\frac{a}{1}$ $\frac{-ar+br+bs}{r-s}=\frac{br+bs-as}{r-s}$; ſo the greater part is

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$\frac{ar-br-bs}{r-s}$, and the leſs part $\frac{br+bs-as}{r-s}$.

EXAMPLE.

Let it be required (as in art. 41,) to divide the number 48 into two ſuch parts, that one part may be three times as much above 20 as the other wants of 20: here $a=48$, $b=20$, $r=3$, $s=1$; for to ſay that the exceſs muſt be three times the defect, is no other than to ſay, that the exceſs muſt be to the defect as 3 to 1; the reſt is eaſy.

The GENERAL DEMONSTRATION.

1ſt, The greater part $\frac{ar-br-bs}{r-s}$, and the leſs part $\frac{br+bs-as}{r-s}$ being added together make $\frac{ar-as}{r-s}$ $=a$: again, the exceſs of the greater part above b, is $\frac{ar-br-bs}{r-s}-\frac{b}{1}=\frac{ar-br-bs-br+bs}{r-s}=\frac{ar-2br}{r-s}$, and the exceſs of b above the leſs part, which is what the leſs part wants of b, is $\frac{b}{1}-\frac{br-bs+as}{r-s}=$ $\frac{br-bs-br-bs+as}{r-s}=\frac{as-2bs}{r-s}$; therefore the exceſs of one part above b is to what the other wants of b, as $\frac{ar-2br}{r-s}$ is to $\frac{as-2bs}{r-s}$, that is, as $ar-2br$ is to $as-2bs$, that is, as $r\times\overline{a-2b}$ is to $s\times\overline{a-2b}$, or as r to s, *Q. E. D.*

PROBLEM 10. (See art. 55.)

144. *There are two places, whoſe diſtance from each other is* a, *and from whence two perſons ſet out at the ſame time with a deſign to meet, one travelling at the rate of* p *miles in* q *hours, and the other at the rate of* r *miles in* s *hours: I demand how long and how far each travelled before they met.*

 SOLUTION.

SOLUTION.

The number of hours travelled by each, x.

Miles travelled by the first, $\frac{px}{q}$.

By the second, $\frac{rx}{s}$.

By them both, $\frac{px}{q}+\frac{rx}{s}$.

Equation, $\frac{px}{q}+\frac{rx}{s}=a$; therefore $px+\frac{qrx}{s}=aq$; therefore $psx+qrx=aqs$; therefore x (or the number of hours travelled by each) $=\frac{aqs}{ps+qr}$: now to find how many miles the first travelled, say, if in q hours he travelled p miles, how many will he travel in a number of hours equal to $\frac{aqs}{ps+qr}$? for a fourth number, I multiply the third number $\frac{aqs}{ps+qr}$ by the second p, and the product is $\frac{apqs}{ps+qr}$; this again I divide by the first number q, and the quotient is $\frac{aps}{ps+qr}$; for dividing the numerator divides the whole fraction: by the same way of reasoning, the number of miles travelled by the other will be found to be $\frac{aqr}{ps+qr}$: therefore the whole number of miles travelled by them both is $\frac{aps+aqr}{ps+qr}=a$, which demonstrates the solution.

EXAMPLE.

Let the distance of the two places be 154 miles; let the first travel at the rate of 3 miles in 2 hours, and

and the second after the rate of 5 miles in 4 hours; then we shall have $a=154$, $p=3$, $q=2$, $r=5$, $s=4$, $ps=12$, $qr=10$, $ps+qr=22$, $\frac{aqs}{ps+qr}=\frac{154\times2\times4}{22}$ $=56$, $\frac{aps}{ps+qr}=\frac{154\times3\times4}{22}=84$, $\frac{aqr}{ps+qr}=\frac{154\times2\times5}{22}$ $=70$: therefore each travelled 56 hours; the first travelled 84 miles, and the other 70.

Scholium.

If in the foregoing problem we change p into r and q into s, and *vice versa*, the consequence will be, that the first traveller will now travel at the same rate as the second did before, and the second at the same rate as the first did before; but the motion whereby these two travellers approach towards each other will still be the same, and therefore the time this motion is performed in, that is, the time that each travelled, must still be the same: let us then make the changes abovementioned, first in the expression of the time, and see whether that expression will still continue the same; then let us make the same changes in the two expressions of the miles, and see whether by this means these expressions will not be converted each into the other: first then, the expression of the time, which is $\frac{aqs}{ps+qr}$, by changing p into r, and q into s, and *vice versa*, becomes $\frac{asq}{rq+sp}$, which is the same as $\frac{aqs}{ps+qr}$, therefore the expression of the time suffers no alteration by these changes: secondly, the number of miles travelled by the first was $\frac{aps}{ps+qr}$, which, after the changes abovementioned, becomes $\frac{arq}{rq+sp}$, which is the same as $\frac{aqr}{ps+qr}$, the miles travelled by the second;

and therefore, *è converso*, the expression $\frac{aqr}{ps+qr}$ will be changed into the expression $\frac{aps}{ps+qr}$; and thus will the case of the first traveller be changed into that of the second, and *vice versa*.

PROBLEM 15. (See art. 38.)

149. *What two numbers are those, whereof the greater is to the less as* p *to* q, *and the product of their multiplication is to their sum as* r *to* s?

SOLUTION.

Put x for the less number, and the greater will be found by saying, as q is to p, so is x the less number to $\frac{px}{q}$ the greater; whence their sum will be $\frac{px}{q}+\frac{x}{1}$ $=\frac{px+qx}{q}$: on the other hand, if the greater number $\frac{px}{q}$ be multiplied into x, the product will be $\frac{pxx}{q}$; therefore the product of these two numbers will be to their sum as $\frac{pxx}{q}$ is to $\frac{px+qx}{q}$, that is, as px to $p+q$; but according to the problem, the product is to the sum as r to s; therefore px is to $p+q$ as r to s; whence we have this equation, $psx=pr+qr$; and x (the less number sought) $=\frac{pr+qr}{ps}$; therefore $px=\frac{pr+qr}{s}$; for dividing the denominator multiplies the whole fraction; therefore $\frac{px}{q}$ (or the greater number) $=\frac{pr+qr}{qs}$.

DEMON-

DEMONSTRATION.

1st, The greater number is to the less as $\frac{pr+qr}{qs}$ is to $\frac{pr+qr}{ps}$; divide $pr+qr$ by itself, and the quotient will be 1; so that we may now say, that the greater number is to the less as $\frac{1}{qs}$ is to $\frac{1}{ps}$, that is, as $\frac{1}{q}$ is to $\frac{1}{p}$, that is, as $\frac{p}{q}$ is to 1, that is, as p is to q.

2dly, The greater number $\frac{pr+qr}{qs}$ and the less $\frac{pr+qr}{ps}$ being added together make $\frac{pprs+pqrs+pqrs+qqrs}{pqss}$ $=\frac{pprs+2pqrs+qqrs}{pqss}$; but $pp+2pq+qq=\overline{p+q}^2$; therefore the sum of the two numbers sought is $\frac{rs\times\overline{p+q}^2}{pqss}$.

3dly, The greater number $\frac{pr+qr}{qs}$ multiplied into the less $\frac{pr+qr}{ps}$ produces $\frac{rr\times\overline{p+q}^2}{pqss}$.

4thly, Therefore the product of the two numbers sought is to their sum as $\frac{rr\times\overline{p+q}^2}{pqss}$ is to $\frac{rs\times\overline{p+q}^2}{pqss}$, that is, as rr is to rs, or as r to s. *Q. E. D.*

PROBLEM 17.

151. *What two numbers are those, the product of whose multiplication is* p, *and the quotient of the greater divided by the less is* q?

Solution.

Put x for the greater number, and consequently $\frac{p}{x}$ for the less; then will the quotient of the greater divided by the less be $\frac{xx}{p}$; but, according to the problem, this quotient ought to be q; therefore $\frac{xx}{p}=q$; and $xx=pq$, and x (the greater number sought) $=\sqrt{pq}$: again, since $xx=pq$, we have $\frac{pp}{xx}=\frac{pp}{pq}=\frac{p}{q}$; and $\frac{p}{x}$ (or the less number sought) $=\sqrt{\frac{p}{q}}$; so that the greater of the two numbers sought is $\sqrt{pq}$, and the less $\sqrt{\frac{p}{q}}$.

Example.

Let the product of the two numbers sought be 144, and the quotient of the greater divided by the less 16; then we shall have $p=144$, $q=16$, $pq=144\times16$, $\sqrt{pq}=12\times4=48$; $\frac{p}{q}=\frac{144}{16}$, $\sqrt{\frac{p}{q}}=\frac{12}{4}=3$; therefore the numbers are 48 and 3.

Demonstration.

1st, pq multiplied into $\frac{p}{q}$ gives $\frac{ppq}{q}=pp$; therefore $\sqrt{pq}$ multiplied into $\sqrt{\frac{p}{q}}$ gives p.

2dly, pq being divided by $\frac{p}{q}$ gives $\frac{pqq}{p}=qq$; therefore $\sqrt{pq}$ being divided by $\sqrt{\frac{p}{q}}$ gives q. *Q. E. D.*

Problem

PROBLEM 21. (See art. 130.)

155. *What two numbers are those, whose difference being multiplied into the difference of their squares will make* a, *and whose sum being multiplied into the sum of their squares will make* b?

SOLUTION.

For the two numbers sought put x and y; then according to the first supposition, $\overline{x-y}\times\overline{x^2-y^2}$, or $\overline{x-y}\times\overline{x-y}\times\overline{x+y}$, or $\overline{xx-2xy+y^2}\times\overline{x+y}=a$; therefore

$$\text{Equ. 1st, } x^2-2xy+y^2=\frac{a}{x+y}.$$

Again, according to the second supposition, $\overline{x+y}\times\overline{x^2+y^2}=b$; therefore

$$\text{Equ. 2d, } x^2+y^2=\frac{b}{x+y}.$$

From twice the second equation subtract the first,

that is, from $\quad 2x^2 * +2y^2=\dfrac{2b}{x+y}$

subtract $\quad x^2-2xy+y^2=\dfrac{a}{x+y}$,

and there will remain $x^2+2xy+y^2=\dfrac{2b-a}{x+y}$,

that is, $\overline{x+y}^2=\dfrac{2b-a}{x+y}$; therefore $\overline{x+y}^3=2b-a$; make $2b-a=r^3$, that is, put r for the cube root of $2b-a$, and you will have

$$\text{Equ. 3d, } x+y=r.$$

Again, in the first equation we had $x^2-2xy+y^2=\dfrac{a}{x+y}=\dfrac{a}{r}$, that is, $\overline{x-y}^2=\dfrac{a}{r}$ make $\dfrac{a}{r}=ss$, that is, put s for the ſquare root of $\frac{a}{r}$, and you will have

Equ. 4th, $x-y=s$.

Add the third and fourth equations together, and you will have $2x=r+s$, and $x=\frac{r+s}{2}$; ſubtract the fourth equation from the third, and you will have $2y=r-s$, and $y=\frac{r-s}{2}$; whence we have the following canon:

Make $2b-a=r^3$, *and* $\frac{a}{r}=s^2$, *and the numbers ſought will be* $\frac{r+s}{2}$, *and* $\frac{r-s}{2}$.

Demonstration.

The difference of the numbers $\frac{r+s}{2}$ and $\frac{r-s}{2}$ is s, and the difference of their ſquares is rs, as is eaſily tried; therefore the difference of the numbers multiplied into the difference of their ſquares is $rss=\frac{ra}{r}=a$: again, the ſum of the numbers $\frac{r+s}{2}$ and $\frac{r-s}{2}$ is r, and the ſum of their ſquares is $\frac{r^2+s^2}{2}$; therefore the ſum of the numbers multiplied into the ſum of their ſquares is $\frac{r^3+rss}{2}$; but $r^3=2b-a$ by the canon, and $rss=a$ by the ſame; therefore the ſum of the numbers multiplied into the ſum of their ſquares is $\frac{2b-a+a}{2}=b$. *Q. E. D.*

Problem

PROBLEM 22.

156. *Out of a common pack of fifty two cards, let part be distributed into several distinct parcels or heaps in the manner following: upon the lowest card of every heap let as many others be laid as are sufficient to make up its number twelve; as, if four be the number of the lowest card, let eight others be laid upon it; if five, let seven; if* a, *let twelve* — a, &c.: *It is required, having given the number of heaps, which we shall call* n, *as also the number of cards still remaining in the dealer's hand, which we shall call* r. *to find the sum of the numbers of all the bottom cards put together.*

SOLUTION.

Let a, b, c, &c. express the number of the bottom card in the several heaps: then will $12-a$ express the number of all the cards lying upon the bottom card of the first heap, that is, the number of all the cards of the first heap except the lowest, will be $12-a$; therefore $13-a$ will be the number of all the cards in the first heap; for the same reason, $13-b$ will be the number of all the cards in the second heap; and $13-c$ the number of all those in the third, and so on; therefore the number of all the cards in all the heaps will be $13\times n-a-b-c$ &c.: make $a+b+c$ &c. (or the sum of the number of all the bottom cards) $=x$, and then we shall have the numbers of all the cards drawn out into heaps $=13n-x$; but these, together with r, the number of cards undrawn out, make up the whole pack 52; therefore we have this equation, $13n-x+r=52$; therefore $x+52=13n+r$; therefore $x=13n-\underline{52}+r$; but $52=13\times 4$; therefore $13n-52=13\times\overline{n-4}$; therefore $x=13\times\overline{n-4}+r$; in words thus; *From the number of heaps subtract four; multiply the rest by thirteen; and this product, added to the number of cards still remaining in the dealer's hand,* will

will give the sum of the numbers of all the bottom cards put together: as for example, let there be three heaps, and thirty cards remaining; now 4 subtracted from 3 leaves —1; this multiplied by 13 gives —13, and this product added to 30, the number of cards remaining, gives 17 for the sum of the numbers of all the bottom cards.

A more universal theorem is as follows;

Let n *be the number of heaps as before,* p *the number of cards in a pack; let as many cards be laid upon the lowest of every heap as are sufficient to make up its number* q; *and lastly, let* r *be the number of remaining cards as before; and the sum of the numbers of all the bottom cards will be found to be* $\overline{q+1}\times n+\overline{r-p}$.

PROBLEM 24. (See art. 111.)

158. *What two numbers are those, whose sum is* a, *and the product of whose multiplication is* b?

SOLUTION.

The two numbers sought, x and $a-x$.

The product of their multiplication, $ax-xx=b$; whence, changing the signs, $xx-ax=-b$, and completing the square, $xx-ax+\frac{aa}{4}=\frac{aa}{4}-b=\frac{aa-4b}{4}$ $=\frac{ss}{4}$; extract the square root of both sides, that is, of $x^2-ax+\frac{aa}{4}$ on one side, and of $\frac{ss}{4}$ on the other, and you will have $x-\frac{a}{2}=\pm\frac{s}{2}$, and $x=\frac{a\pm s}{2}$; whence the following canon:

Make $aa-4b=ss$, *and the greater number will be* $\frac{a+s}{2}$, *and the less number* $\frac{a-s}{2}$.

The

The SYNTHETICAL DEMONSTRATION.

1ft, $\frac{a+s}{2}$ added to $\frac{a-s}{2}$ gives $\frac{2a}{2}$ or a.

2dly, $\frac{a+s}{2}$ multiplied into $\frac{a-s}{2}$ gives $\frac{aa-ss}{4}=$ (by fubftituting $-aa+4b$ inftead of $-ss$) $\frac{aa-aa+4b}{4}$ $=\frac{4b}{4}=b$. *Q. E. D.*

An example to the foregoing canon.

What two numbers are thofe, whofe fum is twenty-five, and the product of whofe multiplication is 144? Here $a=25$, $b=144$, $aa-4b$ or $ss=49$, $s=7$, $\frac{a+s}{2}$ $=16$, $\frac{a-s}{2}=9$; fo the numbers are 9 and 16.

PROBLEM 25. (See art. 113.)

159. *What two numbers are thofe, whofe fum is* a, *and the fum of their fquares* b?

SOLUTION.

The two numbers fought, x and $a-x$.
The fquare of the former, xx.
The fquare of the latter, $aa-2ax+xx$.
The fum of their fquares $aa-2ax+2xx=b$; therefore $2xx-2ax=b-aa$, and $xx-ax=\frac{b-aa}{2}$, and $xx-ax+\frac{aa}{4}=\frac{aa}{4}+\frac{b-aa}{2}=\frac{2b-aa}{4}=\frac{ss}{4}$; extract the fquare roots, that is, the root of $xx-ax+\frac{aa}{4}$ on

on one side, and of $\frac{ss}{4}$ on the other, and you will have $x-\frac{a}{2}=\pm\frac{s}{2}$, and $x=\frac{a\pm s}{2}$; whence the following canon:

Make 2b—aa=ss, *and you will have* $\frac{a+s}{2}$ *for the greater number, and* $\frac{a-s}{2}$ *for the less.*

DEMONSTRATION.

1st, $\frac{a+s}{2}$ added to $\frac{a-s}{2}$ gives a.

2dly, The square of $\frac{a+s}{2}$ is $\frac{aa+2as+ss}{4}$; the square of $\frac{a-s}{2}$ is $\frac{aa-2as+ss}{4}$; and therefore the sum of their squares is $\frac{2aa+2ss}{4}=\frac{aa+ss}{2}=$ (by the canon) $\frac{aa+2b-aa}{2}=b$. *Q. E. D.*

An example to the foregoing canon.

What two numbers are those, whose sum is 28, and the sum of their squares 400? Here $a=28$, $b=400$, $2b-aa$ or $ss=16$, $s=4$. $\frac{a+s}{2}=16$, $\frac{a-s}{2}=12$; therefore the numbers are 12 and 16.

PROBLEM 26. (See art. 114.)

160. *What two numbers are those, whose sum is* a, *and the sum of their cubes* b?

SOLUTION.

The two numbers sought, x and $a-x$.
The cube of the former, x^3.

The

The cube of the latter, $a^3 - 3a^2x + 3ax^2 - x^3$.

The sum of their cubes, $a^3 - 3a^2x + 3ax^2 = b$; therefore $3ax^2 - 3a^2x = b - a^3$; divide by $3a$, and you will have $xx - ax = \frac{b - a^3}{3a}$, and $xx - ax + \frac{aa}{4} = \frac{aa}{4} + \frac{b - a^3}{3a} = \frac{4b - a^3}{12a} = \frac{1}{4} \times \frac{4b - a^3}{3a} = \frac{ss}{4}$; extract the square root of both sides, that is, of $xx - ax + \frac{aa}{4}$ on one side, and of $\frac{ss}{4}$ on the other, and you will have $x - \frac{a}{2} = \pm \frac{s}{2}$ and $x = \frac{a \pm s}{2}$; whence the following canon:

Make $\frac{4b - a^3}{3a} = ss$, *and you will have* $\frac{a + s}{2}$ *for the greater number, and* $\frac{a - s}{2}$ *for the less.*

DEMONSTRATION.

1st, $\frac{a + s}{2}$ added to $\frac{a - s}{2}$ gives a.

2dly, The cube of $\frac{a + s}{2}$ is $\frac{a^3 + 3a^2s + 3as^2 + s^3}{8}$, and the cube of $\frac{a - s}{2}$ is $\frac{a^3 - 3a^2s + 3as^2 - s^3}{8}$; therefore, the sum of their cubes is $\frac{2a^3 + 6as^2}{8} = \frac{a^3 + 3ass}{4} = \frac{a^3 + 4b - a^3}{4}$ by the canon, $= b$. *Q. E. D.*

An example to the foregoing canon.

What two numbers are those, whose sum is 7, and the sum of their cubes 133? Here $a = 7$, $b = 133$,

$4b - a^3$

$\frac{4b-a}{3a}$ or $ss=9$, $s=3$, $\frac{a+s}{2}=5$, $\frac{a-s}{2}=2$; so the numbers are 5 and 2.

PROBLEM 27.

161. *It is required to find two numbers whose difference is* d, *and which, dividing a given number as* a, *will have two quotients whose difference is* b.

SOLUTION.

The two numbers sought, x and $x+d$.

The two quotients $\frac{a}{x}$ and $\frac{a}{x+d}$.

Their difference, $\frac{a}{x}-\frac{a}{x+d}=\frac{ad}{xx+dx}=b$; therefore $bxx+bdx=ad$, and $xx+dx=\frac{ad}{b}$; therefore $xx+dx+\frac{dd}{4}=\frac{ad}{b}+\frac{dd}{4}=\frac{1}{4}\times\overline{\frac{4ad}{b}+dd}=\frac{ss}{4}$; extract the square root of $xx+dx+\frac{dd}{4}$ on one side, and of $\frac{ss}{4}$ on the other, and you will have $x+\frac{d}{2}=\pm\frac{s}{2}$, whence $x=\frac{s-d}{2}$ or $\frac{-s-d}{2}$; set aside the negative root, and you will have x (the less divisor) $=\frac{s-d}{2}$, and $x+d$ (the greater) $=\frac{s-d}{2}+\frac{d}{1}=\frac{s+d}{2}$; and we shall have the following canon.

Make $\frac{4ad}{b}+dd=ss$, *and you will have* $\frac{s+d}{2}$ *for the greater divisor, and* $\frac{s-d}{2}$ *for the less.*

N. B.

N. B. That $\frac{s-d}{2}$ is an affirmative quantity, is evident from hence, that $ss = \frac{4ad}{b} + dd$; therefore ss is greater than dd, and s greater than d; therefore $\frac{s-d}{2}$ is affirmative.

The demonſtration of the canon.

1ſt, If the leſs diviſor $\frac{s-d}{2}$ be ſubtracted from the greater $\frac{s+d}{2}$, the remainder will be d; therefore the difference of the diviſors is d.

2dly, If the dividend a be ſeverally divided by the two diviſors $\frac{s-d}{2}$ and $\frac{s+d}{2}$, the two quotients will be $\frac{2a}{s-d}$ and $\frac{2a}{s+d}$ reſpectively, whereof the former will be the greater, as having a leſs denominator; therefore the difference of the quotients is $\frac{2a}{s-d} - \frac{2a}{s+d}$

$$= \frac{2as+2ad-2as+2ad}{ss-dd} = \frac{4ad}{ss-dd} = \frac{4ad}{\frac{4ad}{b}} \text{ by the canon,}$$

$= b$. *Q. E. D.*

An example to the foregoing canon.

Let it be required to find two diviſors whoſe difference is 1, and which, dividing a given number as 144, will have two quotients whoſe difference is 2. Here $a=144$, $b=2$, $d=1$, $\frac{4ad}{b} + dd$ or $ss = 289$,

 $s=17$,

$s=17, \frac{s+d}{2}=9, \frac{s-d}{2}=8$; therefore the divisors are 8 and 9, and the quotients 18 and 16.

Scholium.

If in this last problem we had put x for the greater quantity, and $x-d$ for the less, the equation would have been $\frac{a}{x-d}-\frac{a}{x}=b$, or $\frac{ad}{xx-dx}=b$, which is different from the former; and therefore it could not be expected that, in that equation, the two roots should be the numbers sought, but rather the two different values of x, the lesser of them.

Problem 28. (See art. 118.)

162. *What number is that, which, being added to its square root, will make* a?

Solution.

Put xx for the number sought, and you will have this equation, $xx+1x=a$; therefore $xx+1x+\frac{1}{4}=a+\frac{1}{4}=\frac{4a+1}{4}=\frac{ss}{4}$ therefore $x+\frac{1}{2}=\pm\frac{s}{2}$; therefore $x=\frac{s-1}{2}$ or $\frac{-s-1}{2}$: If x be made $=\frac{s-1}{2}$, you will have $xx=\frac{ss-2s+1}{4}$; if x be made equal to $\frac{-s-1}{2}$, you will have $xx=\frac{ss+2s+1}{4}$; whence the following canon:

Make $4a+1=ss$, *and the number sought will be* $\frac{ss-2s+1}{4}$ or $\frac{ss+2s+1}{4}$, *according as the square root to be added is taken affirmatively or negatively.*

Demon-

DEMONSTRATION.

Case 1st, If to the number $\frac{ss-2s+1}{4}$ be added its affirmative square root $\frac{s-1}{2}$, or $\frac{2s-2}{4}$, the sum will be $\frac{ss-1}{4}=a$, by the canon.

Case 2d, If to the number $\frac{ss+2s+1}{4}$ be added its negative square root $\frac{-s-1}{2}$ or $\frac{-2s-2}{4}$, the sum will again be $\frac{ss-1}{4}=a$, as before. *Q. E. D.*

PROBLEM 31.

165. *What two numbers are those, whose sum added to the sum of their squares is* a, *and whose difference added to the difference of their squares is* b?

SOLUTION.

Put x and y for the two numbers sought, and the fundamental equations will be 1st, $x+y+x^2+y^2=a$; 2dly, $x-y+x^2-y^2=b$; which equations when reduced to order will stand thus;

Equ. 1st, $xx+x+yy+y=a$.

Equ. 2d, $xx+x-yy-y=b$.

Add these two last equations together, and you will have $2xx+2x=a+b$; whence $xx+1x=\frac{a+b}{2}$, and $xx+1x+\frac{1}{4}=\frac{a+b}{2}+\frac{1}{4}=\frac{2a+2b+1}{4}=\frac{rr}{4}$; extract the root of $xx+1x+\frac{1}{4}$ on one side, and of $\frac{rr}{4}$ on the other, and you will have $x+\frac{1}{2}=\frac{r}{2}$ and $x=\frac{r-1}{2}$; again, subtract the second equation from the first, and you will have $2y^2+2y=a-b$; and $y^2+y=\frac{a-b}{2}$, and $y^2+1y+\frac{1}{4}=\frac{2a-2b+1}{4}=\frac{ss}{4}$; whence $y+\frac{1}{2}=\frac{s}{2}$, and $y=\frac{s-1}{2}$; whence the following canon:

Make $2a+2b+1=rr$, *and* $2a-2b+1=ss$, *and you will have the greater number equal to* $\frac{r-1}{2}$, *and the less number* $=\frac{s-1}{2}$.

DEMONSTRATION.

The sum of $\frac{r-1}{2}$ and $\frac{s-1}{2}$ is $\frac{r+s-2}{2}$, or $\frac{2r+2s-4}{4}$.

The square of $\frac{r-1}{2}$ is $\frac{r^2-2r+1}{4}$.

The square of $\frac{s-1}{2}$ is $\frac{s^2-2s+1}{4}$.

therefore the sum of their squares is $\frac{r^2+s^2-2r-2s+2}{4}$; add to this the sum of the numbers above found, to wit, $\frac{2r+2s-4}{4}$, and you will have the sum of the numbers added to the sum of their squares equal to $\frac{r^2+ss-2}{4}$; but $r^2+s^2=4a+2$ by the canon; therefore $rr+ss-2=4a$, and $\frac{r^2+s^2-2}{4}$, or the sum of the numbers added to the sum of their squares, equals a: again, the difference of $\frac{r-1}{2}$ and $\frac{s-1}{2}$ is

$r-$

$\frac{r-s}{2}$ or $\frac{2r-2s}{4}$; and the difference of their ſquares is $\frac{r^2-s^2+2s-2r}{4}$; therefore the difference of the numbers added to the difference of their ſquares is $\frac{r^2-s^2}{4}=\frac{4b}{4}$ by the canon, $=b$. *Q. E. D.*

An example to the foregoing canon.

Let the ſum of the numbers added to the ſum of their ſquares be 26, and their difference added to the difference of their ſquares 14; and we ſhall have $a=26$, $b=14$, $2a+2b+1$ or $rr=81$, $r=9$, $\frac{r-1}{2}=4$, $2a-2b+1$ or $ss=25$, $s=5$, $\frac{s-1}{2}=2$; and ſo the numbers ſought will be 4 and 2.

PROBLEM 32.

166. *What two numbers are thoſe, the ſum of whoſe ſquares is* a, *and the product of their multiplication* b?

SOLUTION.

For the two numbers ſought put x and $\frac{b}{x}$, and the ſum of their ſquares will be $x^2+\frac{b^2}{x^2}=a$; therefore $x^4+b^2=ax^2$; therefore $x^4-ax^2=-bb$, and $x^4-axx+\frac{aa}{4}=\frac{aa}{4}-bb=\frac{aa-4bb}{4}=\frac{ss}{4}$; extract the ſquare root of $x^4-ax^2+\frac{aa}{4}$ on one ſide, and of $\frac{ss}{4}$ on the other, and you will have $x^2-\frac{a}{2}=\pm\frac{s}{2}$, and $x^2=\frac{a\pm s}{2}$; and ſince this equation will be the ſame,

which

which ſoever of the unknown quantities *x* is made to ſtand for, you will have the following canon:

Make aa—4bb=ss, *and you will have the ſquare of the greater number equal to* $\frac{a+s}{2}$, *and the ſquare of the leſs equal to* $\frac{a-s}{2}$.

DEMONSTRATION.

If the ſquare of the greater number, which is $\frac{a+s}{2}$, be added to the ſquare of the leſs number, which is $\frac{a-s}{2}$, the ſum of their ſquares will be $\frac{2a}{2}$ or a: again, if the ſquare of the greater number, which is $\frac{a+s}{2}$, be multiplied into the ſquare of the leſs number, which is $\frac{a-s}{2}$, the product of theſe two ſquares will be $\frac{aa-ss}{4}=\frac{aa-aa+4bb}{4}$ by the canon, $=\frac{4bb}{4}=bb$; but if the ſquare of the greater number multiplied into the ſquare of the leſs gives bb, then the greater number multiplied into the leſs will give b. *Q. E. D.*

An example to the foregoing canon.

Let the ſum of the ſquares of the two numbers ſought be 400, and the product of their multiplication 192; then you will have $a=400$, $b=192$, a^2-4b^2 or $s^2=12544$, $s=112$, $\frac{a+s}{2}$ or the ſquare of the greater number $=256$, $\frac{a-s}{2}$ or the ſquare of the leſs number $=144$; therefore the greater number is 16, and the leſs 12.

THE

THE ELEMENTS OF ALGEBRA.

BOOK V.

In what cases a problem may admit of many answers.

Art. 168. IT has already been observed, that if in any problem the number of independent conditions be equal to the number of unknown quantities, such a problem will admit but of one solution; or if it admits of more, they will however be so determined as to leave no room for arbitrary positions: but if the conditions be fewer in number than are the unknown quantities, those that are wanting may then be supplied by the Analyst himself at pleasure; and as there is infinite choice, it is no wonder if in such a case a problem admits of an infinite number of answers, especially where fractions are taken into that number; but if the problem relates to whole numbers only, then the number of answers will sometimes be finite and sometimes infinite, as the nature of the problem will bear. This will be sufficiently illustrated by the two following examples:

EXAMPLE I.

Let it be required to find two numbers whose sum is equal to ten times their difference.

Here putting x and y for the two numbers sought, it is plain that in this case we have but one condition, and consequently but one equation, to wit, $x+y=10x-10y$, which equation being reduced, gives $x=\frac{11y}{9}$; and this is all the problem requires. Here then it is plain that the Analyst is entirely at liberty to substitute whatever whole number, mixt number, or proper fraction, he pleases for y, provided he does but make $x=\frac{11y}{9}$; and the two quantities x and y will solve the problem. As for instance, let $\frac{1}{2}$ be put for y; then will x or $\frac{11y}{9}$ be $\frac{11}{18}$, and those two fractions $\frac{11}{18}$ and $\frac{1}{2}$ or $\frac{9}{18}$ will solve the problem; for their difference is $\frac{1}{9}$, and their sum $\frac{10}{9}$. But if it be intended that x and y shall both be whole numbers, then such a whole number must be substituted for y as will admit of 9 for a divisor without a remainder: but of such whole numbers there is infinite choice, as 9, 18, 27, 36, &c.; therefore this question is capable of an infinite number of answers, both in whole numbers and fractions.

EXAMPLE 2.

Let it now be required to find two numbers x *and* y, *the product of whose multiplication is equal to ten times their difference.*

Here the equation will be $yx=10x-10y$, which being reduced, gives $x=\frac{10y}{10-y}$. Here it is plain that y must be less than 10; for if y was equal to 10, the

the fraction $\frac{10y}{10-y}$ would be infinite, as will be shewn in another place; and if y be greater than 10, then $10-y$, and consequently $\frac{10y}{10-y}$ will be a negative quantity, whereas the problem may be supposed to relate to affirmative quantities only: however, as there is infinite choice of fractions between 0 and 10, and as any of these may be substituted for y, the problem will still be capable of an infinite number of solutions, if fractions may be admitted; but if it be required that x and y be both whole numbers, then there cannot be above nine such numbers that can be put for y; nor perhaps all these neither, as remains in the next place to be shewn. Now to find what whole number being put for y will bring out x a whole number also, I reduce the quantity $\frac{10y}{10-y}$ to a more simple one, by dividing $10y$ by $10-y$, or rather by $-y+10$, beginning with $-y$ thus: $10y$ divided by $-y$ quotes -10, which I put down in the quotient; then multiplying the divisor $-y+10$ by -10 the quotient, I find the product to be $+10y-100$, which being subtracted from $10y$ the dividend, leaves 100 for a remainder; but not intending to carry on the division any farther, I represent the rest of the quotient by the fraction $\frac{100}{10-y}$; so $x=\frac{100}{10-y}-10$; therefore, that x may be a whole number, it is necessary that $\frac{100}{10-y}$ be a whole number; but this will be impossible, unless $10-y$ be some one of the divisors of 100, I mean such a number as will divide 100 without remainder: I enquire therefore in the next place, how many such divisors 100 will admit of that are under 10; for so long as y is any thing, $10-y$ must be less than 10; and I find four such divisors,

to

to wit, 1, 2, 4 and 5; therefore if $10-y$ be put equal to any of theſe, x or $\frac{100}{10-y}-10$ muſt come out a whole number; and it muſt alſo come out affirmative; for ſo long as $10-y$ is greater than nothing and leſs than 10, $\frac{100}{10-y}$ will always be greater than $\frac{100}{10}$, that is, than 10, and conſequently $\frac{100}{10-y}-10$ or x will be affirmative. Let us then ſuppoſe firſt, $10-y=1$, and we ſhall have $y=9$, and $\frac{10y}{10-y}$ or $x=90$. 2dly, if $10-y=2$, we ſhall have $y=8$, and $x=40$. 3dly, if $10-y=4$, we ſhall have $y=6$, and $x=15$. Laſtly, if $10-y=5$, we ſhall have $y=5$, and $x=10$: therefore this queſtion admits of 4 ſolutions in whole numbers, to wit, 90 and 9, 40 and 8, 15 and 6, and 10 and 5; all which equally anſwer the condition of the problem, as will appear upon trial.

THE

THE

ELEMENTS OF ALGEBRA.

BOOK VII.

OF PROPORTION.

Of the necessity of resuming the doctrine of proportion, and removing some difficulties which seem to attend it as delivered in the Elements.

Art. 264. IN the 15th and 16th articles of this treatise I have laid down as clearly, and yet as succinctly, as I was able, the doctrine of proportion so far as it relates to numbers and commensurable quantities, whereof any one may be considered as some multiple part or parts of another of the same kind; and it served well enough all the purposes it was designed for. But being in the next book to apply Algebra to Geometry, and so to consider proportion as it relates to magnitudes in general whether commensurable or incommensurable, I should come short of the ἀκρίβεια *geometrica*, was I not to resume this subject, and to consider it now in its full extent as it is laid down in the fifth book of the elements of Geometry.

metry. I might indeed have excufed myfelf from this part of my tafk, and fhould have been very glad to have done it, by referring the reader at once to the elements themfelves without any further affiftance; but I could not withftand fome reafons drawn from experience, which to me feemed to plead very powerfully to the contrary.

I frequently obferve, that moft of thofe who fet themfelves to read *Euclid*, when they come at the fifth book, which treats of proportion, either entirely pafs it by as containing fomething too fubtil to be comprehended by young beginners, or elfe touch fo very flightly upon it as to be little the better for it; and thus the doctrine of proportion (which is certainly the moft extenfive, and confequently the moft ufeful, part of the Mathematics) is either taken for granted, or at beft but partially underftood by them. The fchemes there made ufe of are fcarce bold enough, I had almoft faid, fcarce complicated enough, to affect the imagination fo ftrongly as is neceffary to fix the attention.

The firft, fecond, third, fifth and fixth propofitions are felf-evident, as well as fome others, and upon that very account create an impatient reader much greater uneafinefs than if they were farther removed from common fenfe; becaufe the truths from whence thefe propofitions are deduced are not fo diftinct from the propofitions themfelves as in many other cafes. But it ought to be confidered, that the perfection of all arts and fciences in general, and of Geometry in particular, is, to fubfift upon as few firft principles or axioms as is poffible; and therefore, whenever a propofition, how evident foever it may appear in itfelf, can be deduced from any that is gone before, it ought by all means to be fo deduced, and not to be made a firft principle, and fo unneceffarily to increafe their number.

The defign of a geometrical demonftration is not fo much to illuftrate the propofition to which it is annexed, or to render it more evident than it would have been without it (though this ought certainly to be done where-ever the nature of things will permit) as it is to ſhew the neceſſary connection the propoſition to be demonſtrated has with ſome previous truth already admitted or proved, ſo as to ſtand and fall together, whether ſuch previous truths be more or leſs evident than the propoſition to be demonſtrated: I ſay more or leſs evident; for it is not uncommon in the courſe of *Euclid's* geometry to meet with propoſitions demonſtrated from others that are leſs evident than themſelves. For an inſtance of this we need go no farther than the twentieth propoſition of the firſt book, where it is demonſtrated that *in every triangle any two ſides taken together are greater than the third:* now it is certain that this propoſition is more evident than that the external angle is greater than either of the internal and oppoſite ones; and yet the former, by the help of the 19th propoſition, is demonſtrated from the latter.

But there is another reaſon to be given for demonſtrating ſelf-evident propoſitions in many caſes, and particularly in this fifth book of the elements. A propoſition may ſometimes be taken to be ſelf-evident according to our narrow and ſcanty notions of things, which, when better underſtood, will be found to be otherwiſe. Theſe propoſitions, to wit, that *equal quantities will have the ſame proportion to a third*, that *of two unequal quantities the greater will have a greater proportion to a third than the leſs*, and ſome others of the ſame ſtamp in the fifth book, are ſuch as will paſs with moſt for ſelf-evident propoſitions; and ſo they are without all doubt according to the common conception of proportionality; but when they come to be examined according to the juſter and more extenſive idea *Euclid* has given of it, I fear they will both, and the latter more eſpecially, be found to want demonſtration.

In

In a perfect and regular ſyſtem of elementary Geometry, ſuch a one as that of *Euclid* may be ſuppoſed to be, or at leaſt to have been, certain properties of lines, angles, and figures, are to be laid down, and thoſe of the ſimpleſt kind, for definitions; from whence, and from one another, all the reſt are to be derived with the utmoſt rigour, without the leaſt appeal even to common ſenſe. Common ſenſe is by no means to be made the ſtandard of any geometrical truths whatever, except firſt principles: its province muſt be only to judge whether a propoſition be duly demonſtrated according to the rules already preſcribed, that is, whether the neceſſary connection it has with any previous truth be clearly and diſtinctly made out; when that is done, nothing remains but to paſs ſentence. Whilſt the ſcience continues thus circumſcribed, no miſtakes, no diſputes, can ariſe concerning its boundaries; but whenever theſe come to be tranſgreſſed, ſuch a looſe will be given to Geometry that it would be impoſſible to agree upon any others whereby to reſtrain it.

Thus much I thought proper to lay down concerning the nature of a geometrical demonſtration, that young ſtudents may not ſometimes think themſelves diſappointed, or not proceed with that coolneſs and judgment abſolutely neceſſary to conduct them through the elements of Geometry.

But as to the matter in hand, there is another difficulty ſtill behind, which I believe is often a greater diſcouragement to young beginners in their entrance into the doctrine of proportion, than any which have hitherto been alledged, and that is the difficulty of underſtanding and applying *Euclid's* definition of proportionable quantities. But, to take away all excuſe from this quarter, I have here annexed a ſmall diſſertation, conducing (as I take it) to clear up that definition. It is an extract out of ſome looſe papers I have by me; and therefore the reader muſt not be ſurprized if he finds ſome things repeated here which have

have already been mentioned in another part of this book.

A vindication of the fifth definition of the fifth book of EUCLID's *elements.*

262. *N. B.* For a more diſtinct underſtanding of what follows, it muſt be obſerved, that *By a part, in the ſenſe of the fifth book of* Euclid, *is meant an aliquot part, and not a part as part related to ſome whole.* Thus 3 is a part of 12 in *Euclid's* ſenſe, as being juſt four times contained in it; and though 9 be a part of 12 in the ſame ſenſe as the part is diſtinguiſhed from the whole, yet 9 in *Euclid's* ſenſe is not a part, but parts of 12, as being three fourth parts of it.

1ſt. *If two quantities* A *and* B *be commenſurable, then* A *muſt neceſſarily be either ſome multiple, or ſome part, or ſome parts, of* B. For if *A* and *B* be commenſurable, then either *B* muſt meaſure *A*, or *A* muſt meaſure *B*, or they muſt both be meaſured by ſome third quantity: if *B* meaſures *A* any number of times, ſuppoſe 3 times, then *A* will be equal to 3 times *B*, and conſequently will be a multiple of *B*: if *A* meaſures *B* any number of times, ſuppoſe 3 times, then *A* will be a third part of *B*, and conſequently will be a part of *B*: if *A* and *B* do not meaſure one the other, let *C* meaſure them both, and let *C* be contained exactly in *A* 3 times and in *B* 4 times: then will a third part of *A* be equal to a fourth part of *B*, as being both equal to *C*; multiply both ſides of the equation by 3, and you will have $\frac{3}{3}$ of *A*, or *A* equal to $\frac{3}{4}$ of *B*; therefore in this caſe *A* is ſaid to be parts of *B*.

2dly. *If two quantities* A *and* B *are incommenſurable, then* A *can neither be any multiple of* B, *nor any part or parts of it.* For if *A* was any multiple of *B*, then *B* would meaſure both itſelf and *A*, which contradicts the ſuppoſition of their incommenſurability: in like manner, if *A* was any part of *B*, then *A* would meaſure both itſelf and *B*: in the laſt place I ſay that neither

ther can *A* be any parts of *B*; for if *A* was any parts of *B*, ſuppoſe, $\frac{3}{4}$ of *B*, then $\frac{1}{4}$ of *B* would meaſure both *A* and *B*, which ſtill contradicts the ſuppoſition: *A* indeed may be greater or leſs than ſome part or parts of *B*, but can never be equal to any; ſo ſubtil is the compoſition of continued quantity. As for inſtance; it is demonſtrated in art. 201, that the ſide and diagonal line of a ſquare are incommenſurable to each other: let then *A* be the diagonal of a ſquare whoſe ſide is *B*, and the ſquare of *A* will be to the ſquare of *B* as 2 to 1, as is evident from the 47th of the firſt book of *Euclid*; therefore *A* will be to *B* as the ſquare root of 2 is to 1; but the ſquare root of 2 is 1.414 &c. that is, $\frac{14}{10}$, or more nearly $\frac{141}{100}$, or more nearly ſtill $\frac{1414}{1000}$: whence it follows, that if the ſide of a ſquare be divided into 10 equal parts, the diagonal will contain more than 14 of theſe parts, but not ſo much as 15 of them; if the ſide be divided into 100 equal parts, the diagonal will contain above 141 of ſuch parts, but not 142; if the ſide be divided into 1000 equal parts, the diagonal will contain above 1414 of ſuch parts, but not 1415; and ſo on *ad infinitum*: therefore the diagonal of a ſquare can never be exactly expreſſed by parts of the ſide, any more than the ſide can by parts of the diagonal. The ſide may indeed be ſet off upon the diagonal, and ſo be conſidered as part of it, ſo far as part of the whole; but the ſide can never be exactly expreſſed by any number of aliquot parts of the diagonal, be theſe parts ever ſo ſmall. Limits may be found and expreſſed by parts of the diagonal as near as poſſible to each other, between which the ſide ſhall always conſiſt, and by which it may be expreſſed to any degree of exactneſs except perfect exactneſs*. And thus alſo may approximations be made in the expreſſions of

* See the Quarto Edition, p. 306.

many

many other incommensurable quantities one by another.

3dly, From the last section it appears, that *If two quantities* A *and* B *be incommensurable, no multiple of one can ever be equal to any multiple of the other.* For if, for instance, $4A$ could be equal to $3B$, then (dividing by 4) A will be found to be just $\frac{3}{4}$ of B, contrary to what has been above demonstrated.

4thly, *If four quantities* A, B, C *and* D *be such, that* A *is the same part or parts of* B *that* C *is of* D, *then are those four quantities* A, B, C *and* D *said to be proportionable, or* A *is said to have the same proportion to* B *that* C *hath to* D. Thus if A be a fourth part of B, and C a fourth part of D, then A will be the same part of B that C is of D, and they will be proportionable. Thus again, if $A=\frac{3}{4}B$, and $C=\frac{3}{4}D$, or if $A=\frac{8}{4}B$ or $2B$, and $C=\frac{8}{4}D$ or $2D$, or if $A=\frac{11}{4}B$, and $C=\frac{11}{4}D$, in all these instances (comprehending multiples under the notion of parts) A may be said to be the same parts of B that C is of D; and therefore, according to this definition, A hath the same proportion to B that C hath to D, which is true, and the mark of proportionality here given is infallible, but not adequate to our idea of it; for though this mark be never found without proportionality, yet proportionality is often found without this mark. Proportionality is often found among incommensurables; but it can never be tried or proved by the marks here given. I believe nobody ever doubted that the side of one square hath the same proportion to its diagonal that the side of any other square hath to its diagonal; and therefore A may have the same proportion to B that C hath to D, though A be incommensurable to B, and C to D: yet who can say in this case, that A is the same part or parts of B that C

is of *D*, when it has already been ſhewn, that *A* is no part or parts of *B*, nor *C* of *D*? This way therefore of defining proportionable quantities by a ſimilitude of aliquot parts, cannot (in ſtrictneſs of Geometry) be laid down as a proper foundation, ſo as from thence to derive all the other properties of proportionality: for ſince theſe properties are to be applied to incommenſurable as well as commenſurable quantities, it is fit they ſhould be deduced from a fundamental property that relates equally to both.

5thly, In order then to eſtabliſh a more general character of proportionality, I ſhall aſſume the following principle, which equally relates to commenſurable and incommenſurable quantities; and which, I believe, there is no one who has a juſt idea of proportionality, which way ſoever he may chooſe to expreſs it, or whether he can expreſs it or not, but will eaſily allow me, which is, that *If four quantities* A, B, C *and* D *be proportionable, that is, if* A *has the ſame proportion to* B *that* C *hath to* D, *it will then be impoſſible for* A *to be greater than any part or parts of* B, *but* C *muſt alſo be greater than a like part or parts of* D; *or for* A *to be equal to any part or parts of* B, *but that* C *muſt alſo be equal to a like part or parts of* D; *or for* A *to be leſs than any part or parts of* B, *but that* C *muſt alſo be leſs than a like part or parts of* D. Thus if *A* hath the ſame proportion to *B* that *C* hath to *D*, it will then be impoſſible for *A* to be greater than, equal to, or leſs than $\frac{14}{10}$ of *B*, but *C* muſt alſo be greater than, equal to, or leſs than $\frac{14}{10}$ of *D*. This principle, I ſay, is ſo very clear that nothing more needs to be ſaid of it, either by way of explication or demonſtration: and if by the help hereof I can demonſtrate the converſe, we ſhall then have a general mark of proportionality as extenſive as proportionality itſelf. Now the converſe of the foregoing proportion is this; *If there be* *be four quantities* A, B, C *and* D, *and if the nature of these quantities be such, that* A *cannot possibly be greater than, equal to, or less than, any part or parts of* B, *but at the same time* C *must necessarily be greater than, equal to, or less than, a like part or parts of* D, *let the number or denomination of these parts be what they will; I say then, that* A *must necessarily have the same proportion to* B *that* C *hath to* D. If this be denied, let some other quantity E have the same proportion to D that A hath to B, that is, let A, B, E and D be proportionable quantities; then imagining the quantity D to be divided into any number of equal parts, suppose 10, let E be greater than 14 of these parts and less than 15, that is, let E be greater than $\frac{14}{10}$ and less than $\frac{15}{10}$ of D; then must A necessarily be greater than $\frac{14}{10}$ and less than $\frac{15}{10}$ of B: this is evident from the concession already made, since A is supposed to have the same proportion to B that E hath to D. But if A be greater than $\frac{14}{10}$ and less than $\frac{15}{10}$ of B, then C must be greater than $\frac{14}{10}$ and less than $\frac{15}{10}$ of D by the *hypothesis*; the relation between A, B, C and D being supposed to be such, that A cannot be greater or less than any part or parts of B, but C accordingly must be greater or less than a like part or parts of D. Therefore we are now advanced thus far, that if E lies between $\frac{14}{10}$ and $\frac{15}{10}$ of D, C must also necessarily lie betwixt the same limits; now the difference betwixt $\frac{14}{10}$ and $\frac{15}{10}$ of D is $\frac{1}{10}$ of D; therefore the difference betwixt C and E, which lie both between these two limits,

limits, muſt be leſs than $\frac{1}{10}$ of *D*. This is upon a ſuppoſition that the quantity *D* was at firſt divided into 10 equal parts; but if inſtead of 10 we had ſuppoſed it to have been divided into 100, or 1000, or 10000 equal parts (which ſuppoſitions could not have affected the quantities *C* and *E*,) the concluſion would then have been, that the difference betwixt *C* and *E* would have been leſs than the hundredth, or thouſandth, or ten thouſandth part of *D*; and ſo on *ad infinitum:* therefore the difference between *C* and *E* (if there be any difference) muſt be leſs than any part of *D* whatever; therefore the difference between *C* and *E* is only imaginary, and not real; therefore in reality *C* is equal to *E*. Since then *C* is equal to *E*, and that *A* is to *B* as *E* is to *D*, the conſequence muſt be that *A* is to *B* as *C* is to *D*. *Q. E. D.*

Here then we have a proper characteriſtic of proportionality which always accompanies it, and, on the other hand, is never to be found without it, to wit, that four quantities may be ſaid to be proportionable, the firſt to the ſecond as the third is to the fourth, when the firſt cannot be greater than, equal to, or leſs than, any part or parts of the ſecond, but the third muſt accordingly be greater than, equal to, or leſs than, a like part or parts of the fourth: or thus; *Four quantities may be ſaid to be proportionable as above, when the firſt cannot be contained between two limits expreſſed by any parts of the ſecond, how near ſoever theſe limits may approach to each other, but the third muſt neceſſarily be contained between the limits expreſſed by like parts of the fourth.*

6thly, Had *Euclid* ſtopped here, without refining any further upon the criterion of proportionality delivered in the laſt ſection (for I dare venture to affirm, he was no ſtranger to it,) I doubt not but it would have given much greater ſatisfaction to the generality of his diſciples, eſpecially thoſe of a leſs delicate taſte, than

than that which he advances in the fifth book of his elements, as being more closely connected with the common idea of proportionality : but it was easy to see, that in demonstrating several other affections of proportionable quantities upon this scheme, there would then be frequent occasion for taking such and such parts of magnitudes, as there is now for taking such and such multiples of them, the *praxis* of which partition had no where as yet been taught by *Euclid* ; nay, he rather seems to have determined, as far as possible, to avoid it, and that upon no ill grounds neither; for the use of whole numbers is in all cases justly esteemed more natural and more elegant than that of fractions, and the multiplication of quantities has always been looked upon as more simple in the conception than the resolution of them into their aliquot parts. It is for this reason that *Euclid* never shews how to multiply a line or any other quantity whatever, assuming the *praxis* thereof as a sort of *postulatum* ; whereas in the ninth proposition of the sixth book of his elements he shews how to cut off any aliquot part of any given line whatever. Upon these and such like considerations it was that *Euclid* resolved to advance his characteristic property of proportionality one step higher, by substituting multiples instead of aliquot parts in such a manner as we shall now describe ; and we shall at the same time demonstrate the justness of his definition from what has been already laid down in the last section. The proposition to be demonstrated shall be this : *If there be four quantities* A, B, C *and* D, *whereof* EA *and* EC *are any equimultiples of the first and third, and* FB *and* FD *are any other equimultiples of the second and fourth* ; *and if now these quantities are of such a nature, that* EA *cannot be greater than, equal to, or less than,* FB, *but at the same time* EC *must necessarily be greater than, equal to, or less than,* FD, *when compared respectively, be the multiplicators* E *and* F *what they will : I say then that* A *must necessarily have the same proportion to* B *that* C

 hath

hath to D. Now that four quantities may be under ſuch circumſtances as are here deſcribed, can be queſtioned by no one who has with any attention conſidered the nature of proportionable quantities: for ſuppoſe *A* to be the diameter and *B* the circumference of any circle, and *C* to be the diameter and *D* the circumference of any other circle; who doubts but that twenty-two times the diameter of one circle will be greater than, equal to, or leſs than, ſeven times the circumference, according as twenty-two times the diameter of the other circle is greater than, equal to, or leſs than, ſeven times the circumference of that circle? I now proceed to the demonſtration of the propoſition.

If it be denied that *A* is to *B* as *C* is to *D*, let *A* be to *B* as *G* is to *D*; and then, ſuppoſing *D* to be divided into 10 equal parts, let G be greater than 14 of theſe parts, and leſs than 15: then ſince by the ſuppoſition *A* is to *B* as *G* is to *D*, we ſhall have *A* greater than $\frac{14}{10}$ and leſs than $\frac{15}{10}$ of *B*; therefore 10*A* will be greater than 14*B* and leſs than 15*B*; but by the *hypotheſis*, no multiple of *A* can be greater or leſs than any multiple of *B*, but the ſame multiple of *C* muſt be greater or leſs than the ſame multiple of *D*; therefore 10*C* is greater than 14 *D* and leſs than 15*D*; therefore *C* is greater than $\frac{14}{10}$ and leſs than $\frac{15}{10}$ of *D*; therefore if *G* be a quantity between $\frac{14}{10}$ and $\frac{15}{10}$ of *D*, *C* muſt alſo be a quantity between the ſame limits; therefore the difference betwixt *C* and *G* muſt be leſs than $\frac{1}{10}$ of *D*. This is upon a ſuppoſition that *D* was divided into 10 equal parts; but *C* and *G* will be the ſame, into what number of parts ſoever we ſuppoſe *D* to be divided; therefore if we ſuppoſe *D* to be divided into

100,

100, 1000, or 10000 equal parts, *&c.* the difference betwixt *C* and *G* might have been ſhewn to be leſs than the hundredth, or the thouſandth, or the ten thouſandth part of *D*; and ſo on *ad infinitum*; therefore *C* and *G* are equal, as was ſhewn in the 5th ſection. Since then *A* cannot be greater than, equal to, or leſs than, any part or parts of *B*, but *G* muſt be greater than, equal to, or leſs than, a like part or parts of *D*, becauſe *A* is to *B* as *G* is to *D*; and ſince *G* cannot be greater than, equal to, or leſs than, any part or parts of *D*, but *C* muſt be greater than, equal to, or leſs than, the ſame part or parts of *D*, becauſe *G* and *C* are equal; it follows *ex æquo*, that *A* cannot be greater than, equal to, or leſs than, any part or parts of *B*, but that *C* muſt accordingly be greater than, equal to, or leſs than, a like part or parts of *D*; and conſequently that *A* is to *B* as *C* is to *D*, according to the mark of proportionality given in the laſt ſection. *Q. E. D.*

Four quantities then may be ſaid to be proportionable, the firſt to the ſecond as the third to the fourth, when no equimultiples whatever can be taken of the firſt and third, but what muſt either be both greater than, or both equal to, or both leſs than, any other equimultiples that can poſſibly be taken of the ſecond and fourth, when compared reſpectively.

7thly, As number is a diſcrete, and not a continued quantity, there is ſuch a thing as a *minimum* in the parts of number, whereas in thoſe of extenſion there is none; whence it follows, that the parts of number muſt neceſſarily be more diſtinct, and for that reaſon more aſſignable, than are the parts of extenſion. Again, as all numbers are commenſurable by unity, every number may be conceived either as ſome multiple, or ſome part, or ſome parts, of every other. Hence it is that *Euclid*, defining proportionable numbers, makes uſe of the definition given in the 4th ſection; ſo unwilling was he to recede from the common notion of proportionable quantities, whenever the ſubject he treated of would bear it.

Of the ſeventh definition of the fifth book of Euclid.

263. If it be allowed to be a ſufficient mark of the proportionality of four quantities, when they are ſo related to one another in their own natures, that no equimultiples can be taken of the firſt and third, but what muſt either be both greater than, or both equal to, or both leſs than, any other equimultiples that can poſſibly be taken of the ſecond and fourth; then wherever it happens, or may happen otherwiſe, there can be no proportionality. As for inſtance, *If in comparing equimultiples of the firſt and third with other equimultiples of the ſecond and fourth, there be any caſes wherein the firſt multiple ſhall be greater than the ſecond, and yet the third not greater than the fourth; or wherein the firſt multiple ſhall be leſs than the ſecond, and the third not leſs than the fourth; then the firſt quantity will not have the ſame proportion to the ſecond that the third hath to the fourth, but either a greater as in the former caſe, or a leſs as in the latter.* Nay, and I may add further, that *if of four quantities, the firſt hath a greater proportion to the ſecond than the third hath to the fourth, there muſt be caſes exiſting, whether thoſe caſes can be aſſigned or not, wherein of equimultiples of the firſt and third, and of other equimultiples of the ſecond and fourth, the firſt multiple ſhall exceed the ſecond, and yet the third ſhall not exceed the fourth:* for if no ſuch caſes were poſſible, then the firſt quantity muſt either have the ſame proportion to the ſecond that the third hath to the fourth, or a leſs: both which are contrary to the ſuppoſition. Thus we have found the fifth and ſeventh definitions of the fifth book of the elements both of a piece.

A queſtion

A question arising out of the foregoing article.

264. This is all that was neceſſary to be obſerved concerning the foregoing definitions; but if, having given four quantities *A*, *B*, *C* and *D*, whereof *A* hath a greater proportion to *B* than *C* hath to *D*, any one, for his own private ſatisfaction, would know how to find ſuch equimultiples of *A* and *C*, and ſuch other equimultiples of *B* and *D*, that *A's* multiple ſhall exceed that of *B*, and at the ſame time *C's* multiple ſhall not exceed that of *D*, it muſt be done thus: If the quantities *A*, *B*, *C* and *D* be commenſurable, let their ratios be expreſſed by numbers: as for inſtance, let *A* be to *B* as 7 to 5, and let *C* be to *D* as 4 to 3; then will 4 and 3, the numeral expreſſions of the leſſer ratio, be the multiplicators required, if of the terms *A* and *B*, the greater term *A* be multiplied into the leſſer multiplicator 3, and the leſſer term *B* into the greater multiplicator 4; for then $3A$ (21) will be greater than $4B$ (20), and yet $3C$ (12) will not be greater than $4D$ (12), for the two laſt multiples are equal. But if ſuch multiples be required, that the firſt multiple ſhall be greater than the ſecond, and at the ſame time the third multiple ſhall be leſs than the fourth, then ſome intermediate fraction muſt be taken between $\frac{7}{5}$ and $\frac{4}{3}$, and the terms of ſuch a fraction will be the multiplicators required. As for inſtance, throwing the extreme fractions into decimals, we have $\frac{7}{5}=1.4$, and $\frac{4}{3}=1.34-$; therefore if any decimal fraction be taken between 1.4 and 1.34, ſuch a fraction being reduced to integral terms will give the multiplicators required. Let us aſſume 1.375, that is $\frac{1375}{1000}$ or $\frac{11}{8}$; then will $8A$ (56) be greater than $11B$ (55), and at the ſame time $8C$ (32) will be leſs than $11D$ (33).

If the quantity *A* be incommenſurable to *B*, or *C* to *D*, or both to both, find however, by ſcholium the

the ſecond in art. 179*, ſuch numbers as will expreſs theſe ratios as accurately as occaſion requires. As let the ratio of the number E to the number F be nearly the ſame with that of A to B, and let the ratio of the number G to the number H be nearly the ſame with that of C to D; then if either of theſe ratios, to wit, the ratio of E to F, or the ratio of G to H, lie between the ratios of A to B and of C to D, the terms of the intermediate ratio will make proper multiplicators; but if neither of theſe caſes happen, ſome intermediate fraction muſt be taken between the two fractions $\frac{E}{F}$ and $\frac{G}{H}$.

Having thus prepared my young ſtudent for *Euclid's* doctrine of proportion, partly by ſetting him right in his notions of things, and partly by removing out of his way all that rubbiſh which ſeemed to block up his entrance to it; I hope I ſhall now be able to conduct him through the whole with a great deal of eaſe, and that he will meet with fewer difficulties in reading the following propoſitions than an equal number in any other part of the elements: and yet all I have done herein has been only to mitigate, as far as I thought proper, the rigour and ſeverity of the author's manner of writing, and to render his demonſtrations more eaſy to the imagination, which the compiler in his whole ſyſtem ſeems to have had no great tenderneſs for: but, whatever I have done elſe, I have taken care to preſerve the force of the demonſtrations, and I hope, in a great meaſure, their elegancy too. I have uſed no algebraic computations in demonſtrating theſe propoſitions, except what may be juſtified by the antecedent ones; as well knowing that theſe principles were never intended to depend upon arithmetical operations, but rather arithmetical operations upon them. I have however, for the reader's eaſe, made uſe of the ſimpleſt algebraic notation. Thus A, B, C, D ſignify magnitudes of any kind whatever; E, F, G, H

* See the Quarto Edition, p. 285.

always

always ſignify whole numbers, unleſs where notice is given to the contrary; $A+B$ ſignifies the ſum of any two homogeneous magnitudes A and B; $A-B$ their difference, or the exceſs of A above B; EA and FB ſignify any two multiples of A and B, the multiplicators being E and F; &c. I have ſometimes alſo uſed very eaſy conſequences of this notation; as that if $A-B$ be added to B, the ſum will be A, which indeed is a general axiom, and ſaying no more than that if to any magnitude be added the exceſs of a greater above it, the ſum will be the greater magnitude.

The Fifth Book of *EUCLID's* ELEMENTS.

DEFINITIONS.

265. 1. *A leſſer magnitude is ſaid to be a part of a greater, when the leſſer meaſures the greater.*

2. *A greater magnitude is ſaid to be a multiple of a leſs, when the greater is meaſured by the leſs.*

Note. Our language is not nice enough to expreſs theſe two definitions as they are in the Greek and Latin.

We may further obſerve, that by theſe two definitions every ſimple quantity is excluded from being conſidered either as a part or multiple of itſelf; for to be a part, in this ſenſe, is to be leſs than that whereof it is a part, and to be a multiple is to be greater than that whereof it is a multiple.

3. *Ratio is that mutual relation two homogeneous quantities are in, when compared together in reſpect to their quantity.* Thus the exceſs of 2 above 1 is equal to the exceſs of 4 above 3, and yet the ratio of 2 to 1 is greater than the ratio of 4 to 3; that is, 2 has more magnitude when compared with 1 than 4 hath when compared with 3; ſince 2 is double of 1, and 4 is not double of 3. But on the other hand, 3 hath a greater ratio to 4 than 1 hath to 2, becauſe 3 hath more magnitude in compariſon of 4 than 1 hath in compa-

comparifon of 2; for 3 is more than the half of 4, whereas 1 is but juft the half of 2.

4. *All quantities are faid to be in fome ratio or other, when they are capable of being fo multiplied as to exceed one another.*

Note. By this definition, 1ft, All heterogeneous quantities are excluded from having any ratio one to another, becaufe heterogeneous quantities are fuch, that their multiples are no more capable of comparifon as to excefs and defect, than the quantities themfelves: a yard can never be multiplied till it exceeds an hour, &c. 2dly, All infinitely fmall quantities are hereby excluded from having any ratio to finite ones, becaufe the former can never be fo multiplied as to exceed the latter.

5. *Magnitudes are faid to be in the fame ratio, the firft to the fecond as the third to the fourth, when no equimultiples can be taken of the firft and third, but what muft either be both greater than, or both equal to, or both lefs than, any other equimultiples that can poffibly be taken of the fecond and fourth.*

Note. This and the feventh definition have been explained already.

6. *Magnitudes in the fame ratio may be called proportionals.*

7. *If there be four quantities, whereof equimultiples are taken of the firft and third, and other equimultiples of the fecond and fourth; and if any cafe can be affigned, wherein the multiple of the firft fhall be greater than the multiple of the fecond, and at the fame time the multiple of the third fhall not be greater than the multiple of the fourth; then of thefe four quantities, the firft is faid to have a greater ratio to the fecond than the third hath to the fourth.*

8. *Proportion confifts in a fimilitude of ratios.*

9. *Proportion cannot be expreffed in fewer than three terms*: as when we fay that *A* is to *B* as *B* is to *C*.

10. *Whenever three quantities are continual proportionals, the firft is faid to be to the third in a duplicate*

ratio

ratio of the firſt to the ſecond: and on the other hand, the firſt is ſaid to be to the ſecond in a ſubduplicate ratio of the firſt to the third.

11. *If four quantities be continual proportionals, the firſt is ſaid to be to the fourth in a triplicate ratio of the firſt to the ſecond; and ſo on.*

12. *The antecedents of all proportions are called homologous terms; and ſo alſo are the conſequents: but antecedents and conſequents conſidered together, are never called homologous terms, but heterologous.*

Note. Theſe three laſt definitions, though placed here, have nothing to do in the following fifth book, but in the ſixth.

13. *Alternate proportion is, when four quantities being proportionable, the firſt to the ſecond as the third to the fourth, it is concluded, that the firſt is to the third as the ſecond to the fourth*; the juſtneſs of which concluſion, as well as of all the reſt that follow, will be ſufficiently made out in the following propoſitions:

14. *Inverſe proportion is, when four quantities being proportionable, the firſt to the ſecond as the third to the fourth, it is concluded, that the ſecond is to the firſt as the fourth to the third.*

15. *Compoſition of proportion is, when four quantities being proportionable, the firſt to the ſecond as the third to the fourth, it is concluded, that the ſum of the firſt and ſecond is to the ſecond as the ſum of the third and fourth is to the fourth.*

16. *Diviſion of proportion is, when four quantities being proportionable, the firſt to the ſecond as the third to the fourth, it is concluded, that the exceſs of the firſt above the ſecond is to the ſecond as the exceſs of the third above the fourth is to the fourth.*

17. *Converſion of proportion is, when four quantities being proportionable, the firſt to the ſecond as the third to the fourth, it is concluded, that the firſt is to the exceſs of the firſt above the ſecond as the third is to the exceſs of the third above the fourth.*

18. *If*

18. *If ever so many quantities in one series be compared with as many in another; and if from all the ratios in one being equal to all those in the other, either in the same or a different order, it be concluded, that the extremes in one series are in the same proportion with the extremes in the other, this proportionality of the extremes is said to follow* ex æquo, *or* ex æqualitate rationum.

19. *If all the ratios in one series be equal to all those in the other, and in the same order, this is called ordinate proportion; and the extremes in this case are said to be proportionable* ex æquo ordinate, *or barely* ex æquo.

20. *If all the ratios in one series be equal to all those in the other, but not in the same order, this is called inordinate proportion; and the extremes are said to be proportionable* ex æquo perturbate.

Thus if *A*, *B* and *C* in one series be compared with *D*, *E* and *F* in another; and if *A* is to *B* as *D* to *E*, and *B* to *C* as *E* to *F*, this is called ordinate proportion, and *A* is said to be to *C* as *D* to *F ex æquo ordinate*, or barely *ex æquo*: but if *A* is to *B* as *E* to *F*, and *B* to *C* as *D* to *E*, this is called inordinate proportion, and *A* is said to be to *C* as *D* to *F ex æquo perturbate.*

PROPORTION I.

266. *If there be ever so many homogeneous quantities,* A, B, C, *whereof* EA, EB, EC *are equimultiples respectively; I say then, that the sum* $\overline{EA+EB+EC}$ *will be the same multiple of the sum* $\overline{A+B+C}$ *that* EA *is of* A, *or* EB *of* B, &c.

For the multiples *EA*, *EB* and *EC* may be considered as so many distinct heaps or parcels, whereof *EA* consists wholly of *A's*, *EB* of *B's*, and *EC* of *C's*; and since the number of *A's* in *EA* is the same with the number of *B's* in *EB*, or of *C's* in *EC*, it follows, that as often as *A* can be singly taken out of *EA*, or

B out of *EB*, or *C* out of *EC*, juſt ſo often may the whole ſum $\overline{A+B+C}$ be taken out of the whole ſum $\overline{EA+EB+EC}$; therefore the ſum $\overline{EA+EB+EC}$ is the ſame multiple of the ſum $\overline{A+B+C}$ that *EA* is of *A*, or *EB* of *B*, &c. *Q. E. D.*

PROPOSITION 2.

267. *If* EA *and* EB *be equimultiples of any two quantities whatever* A *and* B, *and if* FA *and* FB *be alſo equimultiples of the ſame; I ſay then that the ſum* $\overline{EA+FA}$ *will be the ſame multiple of* A *that the ſum* $\overline{EB+FB}$ *is of* B.

For ſince the number of *A's* in *EA* is the ſame with the number of *B's* in *EB*; and ſince alſo the number of *A's* in *FA* is the ſame with the number of *B's* in *FB*, add equals to equals, and the number of *A's* in $\overline{EA+FA}$ will be the ſame with the number of *B's* in $\overline{EB+FB}$, that is, the ſum $\overline{EA+FA}$ will be the ſame multiple of *A* that the ſum $\overline{EB+FB}$ is of *B*. *Q. E. D.*

PROPOSITION 3.

268. *If* EA *and* EB *be equimultiples of any two quantities whatever* A *and* B, *and if* 3EA *and* 3EB *be any equimultiples of* EA *and* EB; *I ſay then, that* 3EA *and* 3EB *will alſo be equimultiples of* A *and* B.

This is evident from the laſt propoſition: for ſince *EA* and *EB* are equimultiples of *A* and *B*; and ſince *EA* and *EB* are again equimultiples of the ſame, it follows from that propoſition, that the ſum 2*EA* is the ſame multiple of *A* that the ſum 2*EB* is of *B*: again, ſince 2*EA* and 2*EB* are equimultiples of *A* and *B*, and ſince *EA* and *EB* are other equimultiples of the ſame, the ſum 3*EA* is the ſame multiple of *A* that the ſum 3*EB* is of *B*; and ſo on *ad infinitum*. *Q. E. D.*

PROPOSITION 4.

169. *If four quantities* A, B, C *and* D *be proportionable,* A *to* B *as* C *to* D, *and if* EA *and* EC *be any equimultiples of the first and third, and* FB *and* FD *any other equimultiples of the second and fourth; I say then that these multiples will also be proportionable, provided they be taken in the same order as the proportionable quantities whereof they are multiples; that is, that* EA *will be to* FB *as* EC *is to* FD.

For let 3 *EA* and 3 *EC* be any equimultiples of *EA* and *EC*, and let 2*FB* and 2*FD* be any other equimultiples of *FB* and *FD*: then since 3*EA* and 3*EC* are equimultiples of *EA* and *EC*, and since *EA* and *EC* are equimultiples of *A* and *C*, it follows from the last proposition that 3*EA* and 3*EC* are equimultiples of *A* and *C*; and for the same reason 2*FB* and 2*FD* are also equimultiples of *B* and *D*. Since then, *ex hypothesi*, *A* is to *B* as *C* is to *D*; and since 3*EA* and 3*EC* are equimultiples of *A* and *C*, and 2*FB* and 2*FD* are also other equimultiples of *B* and *D*, it follows from the fifth definition, that 3*EA* cannot be greater than, equal to, or less than, 2*FB*, but 3*EC* must also be greater than, equal to, or less than, 2*FD*. Again, since we have four quantities *EA*, *FB*, *EC*, *FD*, whereof 3*EA* and 3*EC* represent any equimultiples of the first and third, and 2*FB* and 2*FD* any other equimultiples of the second and fourth; and since 3*EA* cannot be greater than, equal to, or less than 2*FB*, but 3*EC* must in like manner be greater than, equal to, or less than 2*FD*, it follows from the fifth definition, than these four quantities *EA*, *FB*, *EC*, *FD* are proportionable; that *EA* is to *FB* as *EC* to *FD*. *Q. E. D.*

SCHOLIUM.

To this place is usually referred the inversion of proportion (though why to this, rather than to any other, I know not;) that is, that *if four quantities*

be

be proportionable, they will alſo be inverſely proportionable: as if A *be to* B *as* C *is to* D, *then* B *will be to* A *as* D *to* C. For let *EA* and *EC* be any equimultiples of *A* and *C*, and let *FB* and *FD* be any other equimultiples of *B* and *D*; and firſt let us ſuppoſe *FB* to be greater than *EA*; then will *EA* be leſs than *FB*: and becauſe *A* is to *B* as *C* is to *D*, *EC* will alſo be leſs than *FD* by the fifth definition; and therefore *FD* will be greater than *EC*: thus then we ſee that if *FB* be greater than *EA*, *FD* will alſo be greater than *EC*. And after the ſame manner it may be demonſtrated, that if *FB* be equal to, or leſs than *EA*, *FD* in like manner will be equal to, or leſs than *EC*. Since then we have four quantities *B*, *A*, *D*, *C*, whereof *FB* and *FD* are equimultiples of the firſt and third, and *EA* and *EC* are other equimultiples of the ſecond and fourth; and ſince *FB* cannot be greater than, equal to, or leſs than *EA*, but *FD* muſt accordingly be greater than, equal to, or leſs than *EC*, it follows from the fifth definition, that theſe four quantities *B*, *A*, *D*, *C*, muſt be proportionable; that *B* muſt be to *A* as *D* to *C*. *Q. E. D.*

PROPOSITION 5.

270. *If* A *and* B *be any two homogeneous quantities, whereof* A *is the greater, and whereof* EA *and* EB *are equimultiples reſpectively; I ſay then that the difference* $\overline{EA - EB}$ *will be the ſame multiple of the difference* $\overline{A-B}$ *that* EA *is of* A, *or* EB *of* B.

If this be denied, let *G* be the ſame multiple of $\overline{A-B}$ that *EA* is of *A*, or *EB* of *B*; then we ſhall have two quantities $\overline{A-B}$ and *B*, whoſe ſum is *A*, and whereof *G* and *EB* are equimultiples reſpectively; therefore, by the firſt propoſition, the ſum $\overline{G+EB}$ will be the ſame multiple of the ſum *A* that *EB* is of *B*: but *EA* is alſo the ſame multiple of *A* that *EB* is of *B*; therefore $\overline{G+EB}$ is the ſame multiple of A that EA is of A; therefore $\overline{G+EB}$ muſt be equal to EA; take EB from both ſides, and G will be equal to $\overline{EA-EB}$: but G was the ſame multiple of $\overline{A-B}$ that EA was of A, or EB of B; therefore $\overline{EA-EB}$ will be the ſame multiple of $\overline{A-B}$ that EA is of A, or EB of B. *Q. E. D.*

PROPOSITION 6.

271. *If from* EA *and* EB, *equimultiples of any two quantities* A *and* B, *be ſubtracted* FA *and* FB *any other equimultiples of the ſame; the remainders* $\overline{\text{EA}-\text{FA}}$ *and* $\overline{\text{EB}-\text{FB}}$ *will either be equal to the quantities* A *and* B *reſpectively, or they will be equimultiples of them.*

CASE I.

In the firſt place, let the remainder $\overline{EA-FA}$ be equal to A; I ſay then that the other remainder $\overline{EB-FB}$ will alſo be equal to B. For ſince FA is the ſame multiple of A that FB is of B, it follows from the nature of multiples, that $\overline{FA+A}$ will be the ſame multiple of A that $\overline{FB+B}$ is of B: but A is equal to $\overline{EA-FA}$; and adding FA to both ſides we have $\overline{FA+A}=EA$; therefore inſtead of ſaying as before, that $\overline{FA+A}$ is the ſame multiple of A that $\overline{FB+B}$ is of B, we may now ſay that EA is the ſame multiple of A that $\overline{FB+B}$ is of B: but EA is the ſame multiple of A that EB is of B; therefore EB is the ſame multiple of B that $\overline{FB+B}$ is of B; therefore EB is equal to $\overline{FB+B}$; ſubtract FB from both ſides, and you will have $\overline{EB-FB}=B$. *Q. E. D.*

CASE 2.

Let us now ſuppoſe the remainder $\overline{EA-FA}$ to be ſome multiple of A; for if A meaſures both EA and FA, it muſt meaſure $\overline{EA-FA}$; and ſo $\overline{EA-FA}$ muſt be ſome multiple of A; and for the ſame reaſon, the other remainder $\overline{EB-FB}$ muſt be ſome multiple of B: I ſay then in the next place, that $\overline{EB-FB}$ muſt be the ſame multiple of B that $\overline{EA-FA}$ is of A. If this be denied, let G be the ſame multiple of B that $\overline{EA-FA}$ is of A; then ſince $\overline{EA-FA}$ and G are equimultiples of A and B, and ſince FA and FB are alſo other equimultiples of the ſame, it follows from the ſecond propoſition, that the ſum $\overline{EA-FA+FA}$ will be the ſame multiple of A that $\overline{G+FB}$ is of B: but $\overline{EA-FA+FA}=EA$; therefore EA is the ſame multiple of A that $\overline{G+FB}$ is of B: but EA is the ſame multiple of A that EB is of B; therefore EB is the ſame multiple of B that $\overline{G+FB}$ is of B; therefore EB is equal to $\overline{G+FB}$; therefore $\overline{EB-FB}$ is equal to G: but G was the ſame multiple of B that $\overline{EA-FA}$ is of A by the ſuppoſition; therefore $\overline{EB-FB}$ is the ſame multiple of B that $\overline{EA-FA}$ is of A. *Q. E. D.*

SCHOLIUM.

As in the ſecond definition it was provided that no ſimple quantity be conſidered as a multiple of itſelf, ſo in this propoſition care is taken that no two ſimple quantities be conſidered as equimultiples of themſelves; which indeed is but a conſequence of that definition, and is the reaſon why this propoſition reſolves itſelf into two caſes.

For a better underſtanding and remembering the ſtructure of the ſix foregoing propoſitions, it may be obſerved, that the two laſt propoſitions are nothing elſe but the two firſt with their ſigns changed. In the firſt propoſition it was demonſtrated, that the ſum $\overline{EA}+\overline{EB}$ is the ſame multiple of the ſum $\overline{A+B}$ that EA is of A, or EB of B: in the fifth propoſition it is demonſtrated, that the difference $\overline{EA}-\overline{EB}$ is the ſame multiple of the difference $\overline{A-B}$ that EA is of A or EB of B. Again, in the ſecond propoſition it was demonſtrated, that the ſum $\overline{EA+FA}$ is the ſame multiple of A that the ſum $\overline{EB+FB}$ is of B; and in the ſixth it is demonſtrated that the remainder $\overline{EA}-\overline{FA}$ is the ſame multiple of A that the remainder $\overline{EB}-\overline{FB}$ is of B.

PROPOSITION 7.

272. *If two equal quantities* A *and* B *be compared with a third as* C, *I ſay then, that both* A *and* B *will have the ſame proportion to* C; *and* vice verſa, *that* C *will have the ſame proportion both to* A *and to* B.

For taking any equimultiples of A and B, ſuppoſe $3A$ and $3B$, and any other multiple of C, ſuppoſe $5C$, it is plain that $3A$ muſt be equal to $3B$, becauſe A is equal to B: but if $3A$ be equal to $3B$, then it will be impoſſible for $3A$ to be greater than, equal to, or leſs than $5C$, but $3B$ muſt accordingly be greater than, equal to, or leſs than the ſame $5C$; therefore we have four quantities A, C, B and C, whereof $3A$ and $3B$ repreſent any equimultiples of the firſt and third, and $5C$ and $5C$ any other equimultiples of the ſecond and fourth; and ſince the firſt multiple $3A$ cannot be greater than, equal to, or leſs than the ſecond $5C$, but the third multiple $3B$ muſt accordingly be greater than, equal to, or leſs than the fourth $5C$, it follows from the fifth definition, that theſe four quantities

quantities A, C, B and C are proportionable, A to C as B to C. *Q. E. D.*

Again, ſince $3A$ is equal to $3B$, it will be impoſſible for $5C$ to be greater than, equal to, or leſs than $3A$, but the ſame $5C$ muſt alſo be greater than, equal to, or leſs than $3B$; therefore we have four quantities C, A, C and B, whereof $5C$ and $5C$ repreſent any equimultiples of the firſt and third, and $3A$ and $3B$ any other equimultiples of the ſecond and fourth; and ſince the firſt multiple $5C$ cannot be greater than, equal to, or leſs than the ſecond $3A$, but the third multiple $5C$ muſt alſo be greater than, equal to, or leſs than the fourth $3B$, it follows from the fifth definition, that theſe four quantities C, A, C and B muſt be proportionable, C to A as C to B. *Q. E. D.*

PROPOSITION 8.

273. *If two unequal quantities* A *and* B, *whereof* A *is the greater, be compared with a third as* C, *I ſay then that* A *will have a greater proportion to* C *than* B *hath to* C; *but that, on the other hand,* C *will have a greater proportion to* B *that it hath to* A.

For ſince by the ſuppoſition, A is greater than B, $\overline{A-B}$ will be the exceſs of A above B; and by the fifth propoſition, if EB be any multiple of B, $\overline{EA-EB}$ will be the ſame multiple of $\overline{A-B}$: multiply then theſe two quantities B and $\overline{A-B}$ alike, till of the equimultiples thence ariſing, the leſs ſhall be greater than C; then will the other be much greater; let theſe equimultiples be $3B$ and $\overline{3A-3B}$, each being greater than C: laſtly multiply C till you come to a multiple of it that ſhall be the next greater than $3B$, which multiple let be $5C$; then it is plain that $3B$ cannot be leſs than $4C$; for if it was, then $4C$, and not $5C$ would be the next multiple of C greater than $3B$, contrary to the ſuppoſition. Since then $3B$ cannot be leſs than $4C$; it follows, that if to $3B$ be added a greater quantity, and to $4C$ a lefs, the former fum will be greater than the latter : but $3A—3B$ is greater than C by the conftruction ; add then $3A—3B$ to $3B$, and C to $4C$, and you will have $3A$ greater than $5C$: but $3B$ is lefs than $5C$ by the conftruction ; therefore we have four quantities A, C, B and C, whereof $3A$ and $3B$ are equimultiples of the firft and third, and $5C$ and $5C$ are other equimultiples of the fecond and fourth ; and fince the firft multiple $3A$ is greater than the fecond $5C$, and at the fame time the third multiple $3B$ is not greater than the fourth $5C$, but lefs, it follows from the feventh definition, that of the four quantities A, C, B and C, A hath a greater proportion to C than B hath to C. *Q. E. D.*

Again, fince we have four quantities C, B, C and A, whereof $5C$ and $5C$ are equimultiples of the firft and third, and $3B$ and $3A$ are other equimultiples of the fecond and fourth ; and fince the firft multiple $5C$ is greater than the fecond $3B$, and at the fame time the third multiple $5C$ is not greater than the fourth $3A$, but lefs, it follows from the feventh definition, that of the four quantities C, B, C and A, C hath a greater proportion to B than C hath to A. *Q. E. D.*

Proposition 9.

274. *If two quantities* A *and* B *have both the fame proportion to a third as* C, *or if* C *hath the fame proportion to both* A *and* B; *in either of thefe cafes* A *and* B *muft be equal to each other.*

For fhould either of them be greater than the other, fhould A be greater than B, then by the laft propofition, A muft have a greater proportion to C than B hath to C, contrary to the firft fuppofition ; and C muft have a greater proportion to B than it hath to A, contrary to the fecond fuppofition ; therefore A and B muft be equal to each other. *Q. E. D.*

PROPOSITION 10.

275. *If of three quantities* A, B *and* C, A *hath a greater proportion to* C *than* B *hath to* C, *or if* C *hath a greater proportion to* B *than it hath to* A; *in either of these cases* A *must be greater than* B.

For was *A* equal to, or less than *B*, then either *A* must have the same proportion to *C* that *B* hath to *C*, as in the seventh proposition, or a less as in the eighth, both which contradict the first supposition: and again, was *A* equal to, or less than *B*, then either *C* must have the same proportion to *A* that it hath to *B*, as in the seventh proposition, or a greater as in the eighth, both which contradict the second supposition; therefore *A* must be greater than *B*. *Q. E. D.*

PROPOSITION 11.

276. *If two ratios be the same with a third, they must be the same with one another: as if the ratio of* A *to* a *and the ratio of* C *to* c *be both the same with the ratio of* B *to* b, *then the ratio of* A *to* a *will be the same with the ratio of* C *to* c: *or thus; If* A *be to* a *as* B *to* b, *and* B *to* b *as* C *to* c; *I say then that* A *will be to* a *as* C *to* c.

For taking any equimultiples of the antecedents, suppose $3A$, $3B$, $3C$; and any other equimultiples of the consequents, suppose $2a$, $2b$, $2c$, let $3A$ be greater than $2a$; then since by the supposition *A* is to *a* as *B* to *b*, and $3A$ is greater than $2a$, $3B$ must be greater than $2b$ by the fifth definition: again, since *B* is to *b* as *C* to *c*, and $3B$ is greater than $2b$, $3C$ must be greater than $2c$: thus then we see that if $3A$ be greater than $2a$, $3C$ must necessarily be greater than $2c$: and in like manner it may be demonstrated that if $3A$ be equal to, or less than $2a$, $3C$ will accordingly be equal to, or less than $2c$. Since then we have four quantities *A*, *a*, *C* and *c*, whereof $3A$ and $3C$ represent any equimultiples of the first and third, and $2a$ and $2c$ any other equimultiples of the ſecond and fourth; and ſince $3A$ cannot be greater than, equal to, or leſs than $2a$, but $3C$ muſt accordingly be greater than, equal to, or leſs than $2c$, it follows from the fifth definition that theſe four quantities A, a, C and c muſt be proportionable, A to a as C to c. *Q. E. D.*

PROPOSITION 12.

277. *If ever ſo many quantities* A, B, C *in one ſeries be proportionable to as many* a, b, c *in another, that is,* A *to* a *as* B *to* b *as* C *to* c; *I ſay then, that as any one antecedent is to its conſequent, ſo will the ſum of all the antecedents be to the ſum of all the conſequents; that is, as* A *is to* a *ſo will* $\overline{A+B+C}$ *be to* $\overline{a+b+c}$: *or if we ſuppoſe* $A+B+C=S$, *and* $a+b+c=s$, *I ſay then that as* A *is to* a *ſo will* S *be to* s.

For taking any equimultiples of the antecedents, ſuppoſe $3A$, $3B$, $3C$, and any other equimultiples of the conſequents, ſuppoſe $2a$, $2b$, $2c$, let $3A$ be greater than $2a$; then ſince A is to a as B to b, and $3A$ is greater than $2a$, $3B$ muſt be greater than $2b$ by the fifth definition: again, ſince B is to b as C to c, and $3B$ is greater than $2b$, $3C$ muſt be greater than $2c$; therefore if $3A$ be greater than $2a$, not only $3B$ will be greater than $2b$, but alſo $3C$ will be greater than $2c$, and conſequently the whole ſum $\overline{3A+3B+3C}$ will be greater than the whole ſum $\overline{2a+2b+2c}$: but by the firſt propoſition, the ſum $\overline{3A+3B+3C}$ is the ſame multiple of the ſum $\overline{A+B+C}$ or S that $3A$ is of A; therefore $\overline{3A+3B+3C}=3S$; and for the ſame reaſon $\overline{2a+2b+2c}=2s$; therefore we may now ſay that if $3A$ be greater than $2a$, $3S$ will be greater than $2s$: and after the ſame manner might it be demonſtrated, that if $3A$ be equal to, or leſs than

2a, $3S$ will be equal to, or leſs than $2s$. Since then we have four quantities A, a, S and s, whereof $3A$ and $3S$ repreſent any equimultiples of the firſt and third, and $2a$ and $2s$ any others of the ſecond and fourth; and ſince $3A$ cannot be greater than, equal to, or leſs than $2a$ but $3S$ muſt in like manner be greater than, equal to, or leſs than $2s$, it follows from the fifth definition that theſe four quantities A, a, S and s muſt be proportionable, A to a as S to s. *Q. E. D.*

PROPOSITION 13.

278. *If* A *hath the ſame proportion to* a *that* B *hath to* b, *but* B *hath a greater proportion to* b *than* C *hath to* c; *I ſay then that* A *hath a greater proportion to* a *than* C *to* c.

For ſince by the ſuppoſition B is to b in a greater proportion than C to c, it follows from the ſeventh definition that there are equimultiples of B and C, and others again of b and c, of ſuch a nature that B's multiple ſhall exceed that of b, and at the ſame time C's multiple ſhall not exceed that of c: let then $3B$ exceed $2b$, and let $3C$ not exceed $2c$; then ſince A is to a as B to b, and $3B$ exceeds $2b$, $3A$ muſt neceſſarily exceed $2a$ by the fifth definition; therefore we have four quantities A, a, C and c, whereof $3A$ and $3C$ are equimultiples of the firſt and third, and $2a$ and $2c$ are other equimultiples of the ſecond and fourth; and ſince $3A$ exceeds $2a$ when $3C$ does not exceed $2c$, it follows from the ſeventh definition that of theſe four quantities A, a, C and c, A hath a greater proportion to a than C hath to c. *Q. E. D.*

PROPOSITION 14.

279. *If four homogeneous quantities be proportionable, the firſt to the ſecond as the third to the fourth; I ſay then that the ſecond will be greater than, equal to, or leſs than the fourth, according as the firſt is greater*

than,

than, equal to, or less than the third: as if A *be to* B *as* C *is to* D; *I say then that* B *will be greater than, equal to, or less than* D, *according as* A *is greater than, equal to, or less than* C.

CASE 1.

Let *A* be greater than *C*: I say then that *B* will be greater than *D*. For since *A* is greater than *C*, *A* will have a greater proportion to *B* than *C* hath to *B* by the eighth proposition: again, since *C* is to *D* as *A* to *B*, and *A* hath a greater proportion to *B* than *C* hath to *B*, it follows from the last proposition that *C* is to *D* in a greater proportion than *C* to *B*; therefore by the tenth proposition *B* is greater than *D*. *Q. E. D.*

CASE 2.

Let now *A* be less than *C*: I say then that *B* will be less than *D*. For if *A* be less than *C*; then *C* will be greater than *A*: since then *C* is to *D* as *A* is to *B* *ex hypothesi*, and *C* is greater than *A*, it follows from the last case that *D* will be greater than *B*; and therefore *B* will be less than *D*. *Q. E. D.*

CASE 3.

Lastly, let *A* be equal to *C*: I say then that *B* will be equal to *D*. For since *A* is equal to *C*, *A* will be to *B* as *C* is to *B* by the seventh proposition; but *C* is to *D* as *A* to *B* by the supposition; therefore *C* is to *D* as *C* is to *B* by the eleventh proposition; therefore *B* and *D* are equal by the ninth. *Q. E. D.*

PROPOSITION 15.

280. *Parts are in the same proportion with their respective equimultiples. Let* A *and* a *be any two homogeneous quantities, whereof* 3A *and* 3a *represent any equimultiples respectively; I say then, that* A *will be to* a *as* 3A *to* 3a.

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For take B and C both equal to A, and alſo b and c both equal to a; then by the ſeventh propoſition we ſhall have A to a as B to b as C to c; therefore by the twelfth propoſition we ſhall have A to a as $\overline{A+B+C}$ to $\overline{a+b+c}$: but in this caſe $\overline{A+B+C}=3A$, and $\overline{a+b+c}=3a$; therefore A is to a as $3A$ is to $3a$, *Q. E. D.*

PROPOSITION 16.

281. *If four homogeneous quantities be proportionable, the firſt to the ſecond as the third to the fourth; I ſay then that they will alſo be alternately proportionable, that is, the firſt to the third as the ſecond to the fourth: as if* A *be to* B *as* C *to* D; *I ſay then that* A *will be to* C *as* B *to* D.

For, taking any equimultiples of A and B, ſuppoſe $3A$ and $3B$, and any others of C and D, ſuppoſe $2C$ and $2D$; ſince $3A$ is to $3B$ as A to B by the laſt, and A is to B as C to D by the ſuppoſition, and C is to D as $2C$ to $2D$ by the laſt; it follows from the 11th propoſition that $3A$ is to $3B$ as $2C$ to $2D$; therefore by the 14th propoſition, $3A$ cannot be greater than, equal to, or leſs than $2C$, but at the ſame time $3B$ muſt be greater than, equal to, or leſs than $2D$. Since then we have four quantities A, C, B and D, whereof $3A$ and $3B$ repreſent any equimultiples of the firſt and third, and $2C$ and $2D$ any other equimultiples of the ſecond and fourth; and ſince $3A$ cannot be greater than, equal to, or leſs than $2C$, but $3B$ muſt accordingly be greater than, equal to, or leſs than $2D$, it follows from the fifth definition that theſe four quantities A, C, B and D, muſt be proportionable, A to C as B to D. *Q. E. D.*

Note, Alternate proportion can have no place, except where all the quantities A, B, C and D, are of the ſame kind: for if A and B were of one kind, and C and D of another, how would it be poſſible for the quantities

quantities A and C, or B and D, to have any proportion one to another, much leſs the ſame?

PROPOSITION 17.

282. *If four quantities* A, B, C *and* D, *whereof* A *is greater than* B, *and* C *greater than* D, *be proportionable,* A *to* B *as* C *to* D; *I ſay then that* $\overline{A-B}$ *will be to* B *as* $\overline{C-D}$ *is to* D, *which is called proportion by diviſion.*

For let $3A$, $3B$, $3C$ and $3D$, be any equimultiples of the quantities A, B, C and D; then will $\overline{3A-3B}$ and $\overline{3C-3D}$ be like multiples of $\overline{A-B}$ and $\overline{C-D}$. Again, let $2B$ and $2D$ be any other equimultiples of B and D, and let $3A-3B$ be greater than $2B$; then if $3B$ be added to both ſides, we ſhall have $3A$ greater than $5B$; and becauſe A is to B as C is to D, we ſhall have, by the fifth definition, $3C$ greater than $5D$; take $3D$ from both ſides, and you will have $\overline{3C-3D}$ greater than $2D$; therefore if $\overline{3A-3B}$ be greater than $2B$, $\overline{3C-3D}$ muſt be greater than $2D$: and by a like proceſs it may be demonſtrated, that if $\overline{3A-3B}$ be equal to, or leſs than $2B$, $\overline{3C-3D}$ will be equal to, or leſs than $2D$. Since then we have four quantities, $\overline{A-B}$, B, $\overline{C-D}$ and D, whereof $\overline{3A-3B}$ and $\overline{3C-3D}$ repreſent any equimultiples of the firſt and third, and $2B$ and $2D$ any other equimultiples of the ſecond and fourth; and ſince $\overline{3A-3B}$ cannot be greater than, equal to, or leſs than $2B$, but at the ſame time $\overline{3C-3D}$ muſt accordingly be greater than, equal to, or leſs than $2D$, it follows from the fifth definition that theſe four quantities $\overline{A-B}$, B, $\overline{C-D}$ and D muſt be proportionable, $\overline{A-B}$ to B as $\overline{C-D}$ to D. *Q. E. D.*

PROPOSITION 18.

283. *If four quantities* A, B, C *and* D *be proportionable,* A *to* B *as* C *to* D; *I ſay then that* $\overline{A+B}$ *will be to* B *as* $\overline{C+D}$ *to* D, *which is called proportion by compoſition.*

If this be denied, that $A+B$ is to B as $C+D$ is to D, it muſt then be allowed that $A+B$ is to B as $C+D$ is to ſome quantity either greater or leſs than D; ſuppoſe to a greater, and call it E; then ſince E is by the ſuppoſition greater than D, if $C-E$ be added to both ſides, we ſhall have C greater than $\overline{C+D-E}$. This being obſerved, let us begin again, and ſuppoſe $A+B$ to B as $C+D$ to E; then we ſhall have *dividendo* (that is, by the laſt propoſition) $\overline{A+B-B}$ to B as $\overline{C+D-E}$ to E; but $\overline{A+B-B}$ is equal to A; therefore A is to B as $\overline{C+D-E}$ is to E; but A is to B as C is to D by the ſuppoſition; therefore C is to D as $\overline{C+D-E}$ is to E; but of theſe four proportionals C, D, $\overline{C+D-E}$ and E, it has been proved that the firſt is greater than the third, that C is greater than $\overline{C+D-E}$; therefore, by the fourteenth, the ſecond muſt be greater than the fourth, that is, D muſt be greater than E; therefore E muſt be leſs than D; therefore if $A+B$ be to B as $C+D$ is to any quantity greater than D, that quantity muſt alſo be leſs than D, which is impoſſible; therefore it is impoſſible for $A+B$ to be to B as $C+D$ is to any quantity greater than D: and by a like proceſs it may be demonſtrated, that it is as impoſſible for $A+B$ to be to B as $C+D$ is to any quantity leſs than D; therefore $A+B$ muſt be to B as $C+D$ is to D. *Q. E. D.*

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PROPOSITION 19.

284. *If from two quantities* A *and* B *in any proportion be subtracted other two* C *and* D *in the same proportion; I say then that the remainders* A—C *and* B—D *will still be in the same proportion, that is, that* A—C *will be to* B—D *as* A *to* B *or as* C *to* D.

For since by the supposition *A* is to *B* as *C* is to *D*, we shall have *permutando* (that is, by the sixteenth proposition,) *A* to *C* as *B* to *D*; and *dividendo*, *A—C* to *C* as *B—D* to *D*; and again *permutando*, *A—C* to *B—D* as *C* is to *D*; but *A* is to *B* as *C* is to *D*; therefore *A—C* is to *B—D* as *A* to *B*. *Q. E. D.*

SCHOLIUM.

Here Doctor *Gregory* in his manuscript copy finds a corollary demonstrating that illation called conversion of proportion; but because it is difficult to make sense of that demonstration, I chuse rather to insert his own demonstration of the same proposition, which is as follows:

If four quantities A, B, C *and* D, *be proportionable*, A *to* B *as* C *to* D; *I say then that* A *is to* A—B *as* C *is to* C—D, *which is called conversion of proportion.* For since by the supposition *A* is to *B* as *C* is to *D*, we shall have *dividendo*, *A—B* to *B* as *C—D* to *D*; and *invertendo*, *B* to *A—B* as *D* to *C—D*; and *componendo*, *B+A—B* to *A—B* as *D+C—D* to *C—D*, that is, *A* to *A—B* as *C* to *C—D*. *Q. E. D.*

As to the foregoing nineteenth proposition I shall further observe, that as in that proposition, by division of proportion it was demonstrated, that if from two quantities *A* and *B* in any proportion be subtracted two others *C* and *D* in the same proportion, the remainders *A—C* and *B—D* will still be in the same proportion with *A* and *B*; so by composition of proportion it may be demonstrated, that if to two quantities *A* and *B* in any proportion be added two others *C* and *D* in the ſame proportion, the aggregates *A*+*C* and *B*+*D* will ſtill be in the ſame proportion with *A* and *B*; but this has already been demonſtrated, being a particular caſe of the twelfth propoſition.

PROPOSITION 20.

285. *If there be three quantities* A, B *and* C *in one ſeries, and three others* D, E *and* F *in another, and if the proportions in one ſeries be the ſame with the proportions in the other when taken in the ſame order, as if* A *be to* B *as* D *is to* E, *and* B *to* C *as* E *to* F; *I ſay then that* A *cannot be greater than, equal to, or leſs than* C *in one ſeries, but accordingly* D *muſt be greater than, equal to, or leſs than* F *in the other.*

For let *A* be greater than *C*; then it is plain from the eighth propoſition that *A* muſt have a greater proportion to *B* than *C* hath to *B*; but *A* is to *B* as *D* to *E* by the ſuppoſition, and *C* is to *B* as *F* to *E*, becauſe by the ſuppoſition *B* is to *C* as *E* to *F*; therefore *D* hath a greater proportion to *E* than *F* hath to *E*; therefore *D* is greater than *F* by the tenth propoſition; therefore if *A* be greater than *C*, *D* muſt be greater than *F*: and after the ſame manner it may be demonſtrated, that if *A* be equal to, or leſs than *C*, *D* muſt accordingly be equal to, or leſs than *F*; therefore *A* cannot be greater than, equal to, or leſs than *C*, but accordingly *D* muſt be greater than, equal to, or leſs than *F*. *Q. E. D.*

PROPOSITION 21.

286. *If there be three quantities* A, B *and* C *in one ſeries, and three others* D, E *and* F *in another, and if the proportions in one ſeries be the ſame with the proportions in the other, but in a different order, as if* A *be to* B *as* E *is to* F, *and* B *to* C *as* D *is to* E; *I ſay ſtill that* A *cannot be greater than, equal to, or leſs than* C, *but accordingly* D *muſt be greater than, equal to, or leſs than* F.

For let *A* be greater than *C*; then by the eighth proposition *A* must have a greater proportion to *B* than *C* hath to *B*: but *A* is to *B* as *E* is to *F* by the supposition, and *C* is to *B* as *E* to *D*, because by the supposition *B* is to *C* as *D* to *E*; therefore *E* hath a greater proportion to *F* than it hath to *D*; therefore *D* must be greater than *F* by the tenth proposition; therefore if *A* be greater than *C*, *D* must be greater than *F*: and by a like way of reasoning, if *A* be equal to, or less than *C*, *D* will accordingly be equal to, or less than *F*; therefore *A* cannot be greater than, equal to, or less than *C*, but accordingly *D* must be greater than, equal to, or less than *F*. *Q. E. D.*

PROPOSITION 22.

287. *If there be three quantities* A, B *and* C *in one series, and three others* D, E *and* F *in another, and if the proportions in one series be the same with the proportions in the other when taken in the same order; I say then that the extremes in one series will be in the same proportion with the extremes in the other: as if* A *be to* B *as* D *is to* E, *and* B *to* C *as* E *to* F; *I say then that* A *will be to* C *as* D *to* F.

Note, For avoiding a multiplicity of words, this consequence is said to follow *ex æquo ordinate*, or *ex æquo*: see the eighteenth and nineteenth definitions.

Take any equimultiples of *A* and *D*, suppose 4*A* and 4*D*, and any others of *B* and *E*, suppose 3*B* and 3*E*, and lastly any others of *C* and *F*, as 2*C* and 2*F*; then since by the supposition *A* is to *B* as *D* is to *E*, it follows from the fourth proposition that 4*A* will be to 3*B* as 4*D* to 3*E*: again, since by the supposition *B* is to *C* as *E* to *F*, it follows from the same fourth proposition that 3*B* will be to 2*C* as 3*E* to 2*F*: so that we have three quantities, to wit 4*A*, 3*B*, 2*C* in one series, and three others, to wit 4*D*, 3*E* and 2*F* in another; and it has been shewn that the proportions in one series are the same with the proportions

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in the other when taken in the fame order, that is, $4A$ is to $3B$ as $4D$ to $3E$, and $3B$ to $2C$ as $3E$ to $2F$; therefore, by the twentieth propofition, $4A$ cannot be greater than, equal to, or lefs than $2C$, but $4D$ muft accordingly be greater than, equal to, or lefs than $2F$. Since then we have four quantities A, C, D and F, whereof $4A$ and $4D$ reprefent any equimultiples of the firft and third, and $2C$ and $2F$ any other equimultiples of the fecond and fourth; and fince $4A$ cannot be greater than, equal to, or lefs than $2C$ but accordingly $4D$ muft be greater than, equal to, or lefs than $2F$, it follows from the fifth definition, that thefe four quantities A, C, D and F are proportionable, A to C as D to F. *Q. E. D.*

COROLLARY.

In like manner, if there be ever fo many quantities A, B, C, G, &c. *in one feries, and as many others* D, E, F, H, &c. *in another, and if* A *be to* B *as* D *is to* E, *and* B *to* C *as* E *to* F, *and* C *to* G *as* F *to* H, &c. *the confequence with refpect to the extremes will ftill be the fame, that is,* A *will be to* G *as* D *to* H: for it has been proved already that A is to C as D to F; and by the fuppofition C is to G as F to H; therefore, *ex æquo*, A will be to G as D to H.

PROPOSITION 23.

288. *If there be three quantities* A, B *and* C *in one feries, and three others* D, E *and* F *in another, and if the proportions in one feries be the fame with the proportions in the other, but in a different order; I fay that the extremes in one feries will ftill be in the fame proportion with the extremes in the other: as if* A *be to* B *as* E *is to* F, *and* B *to* C *as* D *to* E; *I fay ftill that* A *will be to* C *as* D *to* F.

Note. This confequence is faid to be *ex æquo perturbate.*

Take any equimultiples of A, B and D, fuppofe $3A$, $3B$ and $3D$, and any others of C, E and F, fuppofe $2C$, $2E$ and $2F$, and the reafoning is as follows: $3A$ is to $3B$ as A to B by the fifteenth, and A is to B as E to F by the fuppofition, and E is to F as $2E$ to $2F$ by the fifteenth; therefore $3A$ is to $3B$ as $2E$ is to $2F$ by the eleventh: again, B is to C as D to E by the fuppofition; therefore $3B$ will be to $2C$ as $3D$ to $2E$ by the fourth: fince then we have three quantities, to wit, $3A$, $3B$ and $2C$ in one feries, and three others, to wit, $3D$, $2E$ and $2F$ in another, and fince the proportions are the fame in both feriefes, but in a different order, that is, fince $3A$ is to $3B$ as $2E$ to $2F$, and $3B$ is to $2C$ as $3D$ to $2E$, it follows from the twenty-firft propofition, that $3A$ cannot be greater than, equal to, or lefs than $2C$, but $3D$ muft accordingly be greater than, equal to, or lefs than $2F$: again, fince we have four quantities A, C, D and F, whereof $3A$ and $3D$ reprefent any equimultiples of the firft and third, and $2C$ and $2F$ any others of the fecond and fourth, and fince $3A$ cannot be greater than, equal to, or lefs than $2C$, but $3D$ muft accordingly be greater than, equal to, or lefs than $2F$, it follows from the fifth definition that thefe four quantities A, C, D and F are proportionable, A to C as D to F. *Q. E. D.*

PROPOSITION 24.

289. *If there be fix quantities* A, B, C, D, E, F, *whereof* A *is to* B *as* C *is to* D, *and* E *is to* B *as* F *to* D; *I fay then that* A+E *will be to* B *as* C+F *to* D.

For fince by the fuppofition E is to B as F to D, we fhall have, *invertendo*, B to E as D to F. Since then A is to B as C is to D by the fuppofition, and B is to E as D to F, it follows *ex æquo*, that A is to E as C to F; whence, *componendo*, $A+E$ will be to E as $C+F$ is to F: again, fince $A+E$ is to E as $C+F$ is to F, and E is to B as F to D by the fuppofition, it follows again *ex æquo*, that $A+E$ is to B as $C+F$ to D. *Q. E. D.*

LEMMA.

290. *If four quantities* A, B, C *and* D *be proportionable,* A *to* B *as* C *to* D; *I ſay then that* A *cannot poſſibly be greater than, equal to, or leſs than* B, *but that* C *will accordingly be greater than, equal to, or leſs than* D.

That this lemma is ſelf-evident according to the common notion of proportionality, or even upon the plan of the fifth definition, were ſimple quantities allowed to be conſidered as equimultiples of themſelves, is what I ſuppoſe will ſcarcely be denied: but this the name of multiple and equimultiple will by no means admit of, and therefore care has been taken to provide againſt it, as may be ſeen in my obſervations on the ſecond definition, and at the end of the ſixth propoſition: therefore, as the doctrine of proportion here ſtands, this lemma ought certainly to be demonſtrated; and the author's taking it for granted in the demonſtration of the next propoſition following, where he might with ſo much eaſe have avoided it, is not ſo much an argument of its ſelf-evidency, as that he had demonſtrated it ſomewhere before in this fifth book, but that it is now loſt. *Commandine*, from the fourteenth of this book, has demonſtrated one particular caſe of this propoſition, that is, where the quantities *A*, *B*, *C* and *D* are all of a kind: but this propoſition is no leſs true when the quantities *A* and *B* are of one kind, and *C* and *D* of another. This *Clavius* very well obſerves, and endeavours to demonſtrate this propoſition in this more extended ſenſe (ſee his ſcholium to the fourteenth propoſition of the fifth book;) but whether this demonſtration of his amounts to any more than proving *idem per idem*, let them that read it judge. The demonſtration I ſhall here give of it is as follows:

I am to demonſtrate that if *A* be to *B* as *C* is to *D*; then *A* cannot poſſibly be greater than, equal to, or lefs than *B*, but accordingly *C* muft be greater than, equal to, or lefs than *D*.

CASE 1.

Let *A* be greater than *B*; I fay then that *C* muft be greater than *D*. For fince *A* is greater than *B*, multiply the excefs *A—B* to a multiple greater than *B*, and let this multiple be $\overline{3A-3B}$; then fince $\overline{3A-3B}$ is greater than *B*, if 3*B* be added to both fides, we fhall have 3*A* greater than 4*B*: again, fince *A* is to *B* as *C* is to *D*, and 3*A* is greater than 4*B*, we fhall have, by the fifth definition, 3*C* greater than 4*D*; therefore 3*C* muft be much greater than 3*D*, and *C* muft be greater than *D*. *Q. E. D.*

CASE 2.

Let now *A* be lefs than *B*; I fay then that *C* muft be lefs than *D*. For fince *A* is to *B* as *C* is to *D*, we fhall have, *invertendo*, *B* to *A* as *D* to *C*; but *B* is greater than *A*, becaufe by the fuppofition *A* is lefs than *B*; therefore *D* muft be greater than *C* by the laft cafe; therefore *C* muft be lefs than *D*. *Q. E. D.*

CASE 3.

Laftly, let *A* be equal to *B*; I fay than that *C* muft be equal to *D*. For fince *C* is to *D* as *A* is to *B*, fhould *C* be greater or lefs than *D*, *A* would accordingly be greater or lefs than *B* by the two laft cafes; but *A* is neither greater nor lefs than *B* by the fuppofition; therefore *C* is neither greater nor lefs than *D*; therefore *C* is equal to *D*. *Q. E. D.*

PROPOSITION 25.

291. *If four quantities* A. B, C *and* D *be proportionable,* A *to* B *as* C *to* D; *I fay then that the fum of the greateft and leaft terms put together will be greater than the fum of the other two.*

Let

Let A be the greateſt of all; then, ſince A is to B as C is to D, and B is leſs than A, D will be leſs than C by the lemma: again, ſince A is to B as C is to D, and C is leſs than A, D will be leſs than B by the fourteenth; therefore if A be the greateſt of all, D, which is leſs than either A, B or C, will be the leaſt of all, and ſo the ſum of the greateſt and leaſt terms added together will be $A+D$; therefore the ſum of the other two will be $B+C$. We are now then to prove that the ſum $A+D$ is greater than the ſum $B+C$, which is thus done: It has been demonſtrated in the nineteenth propoſition, that if from two quantities A and B in any proportion whatever, be ſubtracted other two C and D in the ſame proportion, the remainder $A-C$ will be to the remainder $B-D$ as A to B; but A is greater than B by the ſuppoſition; therefore $A-C$ muſt be greater than $B-D$ by the lemma; add $C+D$ to both ſides, and you will have $A+D$ greater than $B+C$. *Q. E. D.*

COROLLARY.

If three quantities A, B *and* C *be in continual proportion,* A *to* B *as* B *to* C; *I ſay then that the ſum of the extremes will be greater than twice the middle term, that* A+C *will be greater than* B+B *or* 2B.

Of the COMPOSITION and RESOLUTION of RATIOS.

N. B. As numbers are quantities whereof we have more diſtinct ideas than of any other quantities whatever, and as all ratios muſt be reduced to thoſe of numbers before we can make any conſiderable uſe of their compoſition and reſolution in computing the quantities of time, ſpace, velocity, motion, force, &c. I ſhall confine myſelf chiefly to this ſort of ratios in what I have to deliver in the following articles.

DEFINITION I.

292. *In comparing ratios, that ratio is said to be greater than, equal to, or less than another, whose antecedent hath a greater, or an equal, or a less proportion to its consequent than the other's antecedent hath to its consequent.* Thus the ratio of 6 to 3 is said to be greater, and the ratio of 4 to 3 less than the ratio of 5 to 3; thus again the ratio of 6 to 3 is said to be greater, and the ratio of 6 to 5 less than the ratio of 6 to 4, &c. *Therefore whenever two ratios are to be compared whose antecedents and consequents are both different, it will be proper to reduce them to the same antecedent or to the same consequent before the comparison is made.* As for instance; suppose any one would know which of these two ratios is the greater, to wit, the ratio of 7 to 5, or the ratio of 4 to 3: to know this, it will be proper to set off one of the ratios: suppose that of 4 to 3, from 7 the antecedent of the other (by which phrase I mean no more than finding a number to which 7 hath the same proportion that 4 hath to 3;) and this may be done by saying, as 4 is to 3, so is 7 to $\frac{21}{4}$, or $5\frac{1}{4}$: thus then it appears that the proportion of 4 to 3 is the same with the proportion of 7 to $5\frac{1}{4}$; so that now the question turns upon this, which of these two ratios is the greater, that of 7 to 5, or that of 7 to $5\frac{1}{4}$? and the answer is ready, to wit, that the ratio of 7 to 5 is the greater ratio, by the eighth proposition of the fifth book of the elements; therefore the ratio of 7 to 5 is greater than the ratio of 4 to 3. Again, suppose I would compare the ratio of 3 to 4 with the ratio of 5 to 7; then I would set off the ratio of 3 to 4 from 5, by saying, as 3 is to 4, so is 5 to $\frac{20}{3}$, or $7 - \frac{1}{3}$; whereby it appears that the ratio of 3 to 4 is the same with the ratio of 5 to $7-\frac{1}{3}$; but the proportion of 5 to $7-\frac{1}{3}$ is greater than the propor-

proportion of 5 to 7, as is evident from the eighth proposition of the fifth book of the elements, and also from the very nature of ratios, the number 5 having more magnitude when compared with $7-\frac{1}{3}$ than it hath when compared with 7; therefore the ratio of 3 to 4 is greater than the ratio of 5 to 7.

There is also another way of comparing ratios, by turning their terms into fractions, making the antecedents numerators, and the consequents denominators. Thus the ratio of A to B is greater than, equal to, or less than the ratio of C to D, according as the fraction $\frac{A}{B}$ is greater than, equal to, or less than, the fraction $\frac{C}{D}$: for the ratio of $\frac{A}{B}$ to 1 is greater than, equal to, or less than, the ratio of $\frac{C}{D}$ to 1, according as the fraction $\frac{A}{B}$ is greater than, equal to, or less than, the fraction $\frac{C}{D}$; this is evident from what has been laid down already: but the ratio of $\frac{A}{B}$ to 1 is the same with the ratio of A to B, and the ratio of $\frac{C}{D}$ to 1 is the same with the ratio of C to D; therefore the ratio of A to B is greater than, equal to, or less than, the ratio of C to D according as the fraction $\frac{A}{B}$ is greater than, equal to, or less than, the fraction $\frac{C}{D}$. But this way of representing ratios by fractions, though it may serve well enough for comparing them as to greater and less, yet it ought not by any means to be admitted in general, because these representatives are not in the same proportion with the ratios represented by them: thus the fraction $\frac{6}{2}$ is double of the fraction $\frac{3}{2}$, but yet it must by no means be concluded from thence that the ratio of 6 to 2 is double of the ratio of 3 to 2; for it will be found hereafter that the ratio of 9 to 4 is double of the ratio of 3 to 2. For my own part, I never was a favourer of representing ratios by fractions, or even fraction-wise, as is done by *Barrow* and others; not only for the reasons above given, but also because that this way of representing ratios is very apt to mislead beginners into wrong conceptions of their composition and resolution.

Definition 2.

293. *In a series of quantities of any kind whatsoever increasing or decreasing from the first to the last, the ratio of the extremes is said to be compounded of all the intermediate ratios.* Thus if *A*, *B*, *C*, *D* represent any number of quantities put down (or imagined to be put down) in a series, the ratio of *A* to *D* is said to be compounded of, or to be resolvable into these ratios, to wit, the ratio of *A* to *B*, the ratio of *B* to *C*, and the ratio of *C* to *D*: or thus; *If* A *and* D *be any two quantities, and if* B, C, &c. *represent any number of other intermediate quantities interposed at pleasure between* A *and* D, *the ratio of* A *to* D *is said by this interposition to be resolved into the ratios of* A *to* B, *of* B *to* C, *and of* C *to* D.

A, *B*, *C*, *D*,
48, 40, 30, 15,

This is no proposition to be proved, but a definition laid down of what Mathematicians commonly mean by the composition and resolution of ratios, which is certainly no more than what they mean by composition and resolution in the case of any other *continuum* whatever. As for instance; suppose the letters *A*, *B*, *C*, *D*, instead of representing quantities, to represent so many distinct points placed in a right line one after another, whether at equal or unequal distances it matters not: who then would scruple to say that the whole interval *AD* consisted of the intervals *AB*, *BC*, *CD*, as of its parts? Or, if the points *A* and

A and *D* be the extremities of a line, and any number of points *B*, *C*, *&c.* be marked at pleaſure upon it; who will not ſay that the line *AD* is by theſe points reſolved or diſtinguiſhed into the parts *AB*, *BC*, *CD*, *&c.*? This is the caſe in the compoſition and reſolution of lines; and I ſee no difference when applied to the compoſition and reſolution of ratios, except that here the whole and all its parts are lines, and there the whole and all its parts are ratios.

If *A*, *B*, *C*, *D*, *&c.* ſignify quantities, the ratio of *A* to *B* begins at *A* and terminates in *B*; the ratio of *B* to *C* begins at *B* where the former left off, and terminates in *C*; and the ratio of *C* to *D* begins at *C* and terminates in *D*: why then ſhould not theſe continued ratios be conceived as parts conſtituting the whole ratio of *A* to *D*? That ratios are capable of being compared as to greater and leſs, and that one ratio may be greater than, equal to, or leſs than another, we have ſeen already; and if ſo, why ſhould not ratios be allowed to have quantity as well as all other things that are capable of being ſo compared? but if ratios have quantity, they muſt have parts, and theſe parts muſt be of the ſame nature with the whole, becauſe ratios are not capable of being compared with any thing but ratios: therefore I do not ſee but that the idea I have here given of the compoſition and reſolution of ratios is as juſt and as intelligible as it is when applied to any other compoſition or reſolution whatſoever.

To proceed then: let *A*, *B*, *C*, *D* be points in a right line as before; let the line *AB* be equal to any line *Rr*, let *BC* be equal to ſome other line *Ss*, and *CD* to the line *Tt*; then it will not only be proper to ſay that the line *AD* is equal to the three lines *AB*, *BC*, *CD*, but alſo that the ſame line *AD* is equal to the three lines *Rr*, *Ss* and *Tt* put together: and the ſame conſideration is ſtill applicable to ratios; for ſuppoſing *A*, *B*, *C*, *D*, again to ſignify quantities, as alſo *R*, *S*, *T*, *r*, *s*, *t*; let *A* be to *B* as *R* to *r*, let *B* be

to

to *C* as *S* to *s*, and let *C* be to *D* as *T* to *t*; then it is usual amongst Mathematicians not only to consider the ratio of *A* to *D* as compounded of the lesser ratios of *A* to *B*, of *B* to *C*, and of *C* to *D*, but also as compounded of the ratios of *R* to *r*, of *S* to *s*, and of *T* to *t*. All this will be very intelligible, if we attend to the series already described; for there the ratio of 48 to 15 was compounded of the ratio of 48 to 40, of 40 to 30, and of 30 to 15; but 48 is to 40 as 6 to 5, and 40 is to 30 as 4 to 3, and 30 is to 15 as 2 to 1; therefore it is as proper to consider the ratio of 48 to 15 as compounded of the ratios of 6 to 5, of 4 to 3, and of 2 to 1, as it is to consider it as compounded of the ratios of 48 to 40, of 40 to 30, and of 30 to 15.

DEFINITION 3.

294. *As when a line is divided into any number of equal parts, the whole line is said to be such a multiple of any one of these parts as is expressed by the number of parts into which the whole is supposed to be divided; so in a series of continual proportionals, where the intermediate ratios are all equal to one another, and consequently to some common ratio that indifferently represents them all, the ratio of the extremes is said to be such a multiple of this common ratio as is expressed by the number of ratios from one extreme to the other.* Thus 9, 6 and 4 are continual proportionals whose common ratio is that of 3 to 2; for 9 is to 6 as 3 to 2, and 6 is to 4 as 3 to 2; therefore, in this case, the ratio of 9 to 4 is said to be the double of the ratio of 3 to 2; and on the other hand, the ratio of 3 to 2 is said to be the half of the ratio of 9 to 4; but the common expression is, that 9 is to 4 in a duplicate ratio of 3 to 2, and 3 is to 2 in a subduplicate ratio of 9 to 4. Again, 27, 18, 12 and 8 are in continual proportion, whose common ratio is that of 3 to 2; therefore 27 is to 8 in a triplicate ratio of 3 to 2, and 3 is to 2 in a subtriplicate ratio of 27 to 8. Lastly, 81, 54, 36, 24

and

and 16 are continual proportionals, whose common ratio is that of 3 to 2; therefore 81 is to 16 in a quadruplicate ratio of 3 to 2, and 3 is to 2 in a subquadruplicate ratio of 81 to 16. By these instances we see that one ratio may not only be greater or less than another, but a multiple, or an aliquot part of another; nay there is no proportion can be assigned which some one ratio may not have to another: thus the ratio of 81 to 16 is found to be to the ratio of 27 to 8, as 4 to 3, because the former ratio contains the ratio of 3 to 2 four times, and the latter three times; thus again, the ratio of 27 to 8 is to the ratio of 9 to 4, as 3 to 2, because the former contains the ratio of 3 to 2 three times, and the latter twice; whereby it appears that proportion is competible even to ratios themselves, as well as to all other continued quantities whatever. But though all ratios are in some certain proportion one to another, yet this proportion cannot always be expressed; I mean, when the quantities of ratios are incommensurable to one another; for ratios may be incommensurable as well as any other continued quantities of what kind soever: thus the ratio of 4 to 3 is incommensurable to the ratio of 3 to 2; which is the case of most ratios, though not of all. If all ratios were commensurable to one another, their logarithms would be so too; and so the logarithms of all the natural numbers might be accurately assigned; whereas from other principles we know to the contrary, as will be seen when we come to treat particularly of logarithms.

N. B. The best way of representing the quantities of ratios, that I know of, is by *Gunter's* line, where as many of the natural numbers as can be placed upon it are disposed, not at equal distances one from another, but at distances proportionable to the ratios they are in one to another. Thus the distance between 1 and 2 is equal to the distance between 2 and 4, because the ratio of 1 to 2 is equal to the ratio of

2 to

2 to 4: thus again, the diſtance between 4 and 9 is double the diſtance between 2 and 3, becauſe the ratio of 4 to 9 is double the ratio of 2 to 3; and ſo of the reſt.

Of the addition of ratios.

295. Since all ratios are quantities, as has been ſhewn in the three laſt articles, it follows, that they alſo as well as all other quantities muſt be capable of addition, ſubtraction, multiplication, and diviſion: to treat then of theſe operations in their order, I ſhall begin firſt with addition.

If the ratios to be added to be continued ratios, that is, if they lie in a ſeries wherein the antecedent of every ſubſequent ratio is the ſame with the conſequent of the ratio that went immediately before, their addition is beſt performed by throwing out all the intermediate terms: thus the ratios of A to B, of B to C, and of C to D, when added together, make up the ratio of A to D, as was ſhewn in the 293d article.

Therefore, if the ratios to be added be diſcontinued, it will be proper to continue them from ſome given antecedent, ſuppoſe from unity, before they can be added, thus: let the ratio of A to B, the ratio of C to D, and the ratio of E to F, be propoſed to be added into one ſum: now the ratio of A to B ſet off from 1 reaches to $\frac{B}{A}$ becauſe A is to B as 1 to $\frac{B}{A}$; the next ratio of C to D ſet off from $\frac{B}{A}$ reaches to $\frac{BD}{AC}$; and the laſt ratio of E to F ſet off from $\frac{BD}{AC}$ reaches to $\frac{BDF}{ACE}$; therefore the ratios of A to B, of C to D, and of E to F, when added together, make the ratio of 1 to $\frac{BDF}{ACE}$, which is the same with the ratio of *ACE* to *BDF*; whence we have the following canon:

Multiply first the antecedents of all the ratios proposed together, and then their consequents, and the ratio of the products thence arising will be the sum of the ratios proposed.

That the ratio of *A* to *B*, of *C* to *D*, and of *E* to *F*, all together constitute the ratio of *ACE* to *BDF*, may be further confirmed by setting them off from *ACE* and from one another thus: the ratio of *A* to *B* set off from *ACE* reaches to *BCE*; in the next place the ratio of *C* to *D* set off from *BCE* reaches to *BDE*; and lastly the ratio of *E* to *F* set off from *BDE* reaches to *BDF*; therefore all these ratios together constitute the ratio of *ACE* to *BDF*. An example in numbers take as follows: let it be required to add these four ratios together, *viz.* the ratio of 2 to 3, the ratio of 4 to 5, the ratio of 6 to 7, and the ratio of 8 to 9. Here the product of the antecedents is $2 \times 4 \times 6 \times 8 = 384$, and the product of the consequents is $3 \times 5 \times 7 \times 9 = 945$; therefore the sum of all the ratios proposed is the ratio of 384 to 945; and the proof is easy: for the ratio of 2 to 3 reaches from 384 to 576; the ratio of 4 to 5 reaches from 576 to 720; the ratio of 6 to 7 reaches from 720 to 840; and the ratio of 8 to 9 reaches from 840 to 945; therefore the ratios of 2 to 3, of 4 to 5, of 6 to 7, and of 8 to 9, reach from 384 to 945.

From what has here been said concerning the addition of ratios, may easily be understood an expression so frequent among Mechanical and Philosophical writers; as when they say that *A* is to *B* in a ratio compounded of the ratio of *C* to *D*, and of the ratio of *E* to *F*; whereby they mean no more than that the ratio of *A* to *B* is equal to the sum of the ratios of *C* to *D*, and of *E* to *F*; or that *A* is to *B* as *CE* to *DF*.

According to the Mathematicians, every ratio is either a *ratio majoris inæqualitis*, or a *ratio æquali-*

tatis,

tatis, or a *ratio minoris inæqualitatis*, which takes in all ſort of ratios: for by a *ratio majoris inæqualitatis* they mean the ratio that any greater quantity hath to a leſs; by a *ratio minoris inæqualitatis* they mean the contrary, that is, the ratio of a leſſer quantity to a greater; and therefore by a *ratio æqualitatis* they mean the ratio (if it may be called ſo) that every quantity hath to its equal. If we diſtinguiſh ratios according to the effects they have in compoſition, then every *ratio majoris inæqualitatis* ought to be looked upon as affirmative, becauſe ſuch ratios always increaſe thoſe to which they are added; on the other hand, the *rationes minoris inæqualitatis* ought to be conſidered as negative, becauſe theſe always diminiſh the ratios to which they are added; therefore the *ratio æqualitatis* ought to be looked upon as having no magnitude at all, becauſe ſuch ratios have no effect in compoſition. Thus if to the ratio of 5 to 3 be added the ratio of 3 to 2, the ſum will be the ratio of 5 to 2, as above; but the ratio of 5 to 2 is greater than the ratio of 5 to 3; therefore the ratio of 3 to 2 ought to be looked upon as affirmative, becauſe it increaſes the ratio to which it is added: on the other hand, if to the ratio of 5 to 3 be added the ratio of 3 to 4, the ſum will be the ratio of 5 to 4, which is leſs than the ratio of 5 to 3, and therefore the ratio of 3 to 4 is negative: laſtly, if to the ratio of 5 to 3 be added the ratio of 3 to 3, the ſum will ſtill be the ratio of 5 to 3; therefore the ratio of 3 to 3 is nothing.

Whenever a ratio is to be reſolved into two others by any arbitrary interpoſition of an intermediate term, it may be thought however that this intermediate term ſhould be ſome intermediate magnitude between the terms of the ratio to be reſolved; and ſo we ſuppoſed it in the 293d article: but that reſtriction was only ſuppoſed to prevent unſeaſonable objections that might otherwiſe ariſe about it; for there is no neceſſity that the intermediate term ſhould be of an intermediate magni-

magnitude betwixt the extremes, if we allow of negative ratios; for the ratio of 5 to 4 (for inſtance) may be reſolved into the two ratios of 5 to 3 and of 3 to 4, though the intermediate term 3 be out of the limits of 5 and 4. This I ſay is plain; for though the ratio of 5 to 3, which is one of the parts, be greater than the ratio of 5 to 4, yet the ratio of 3 to 4, which is the other part, is negative, and qualifies the other in the compoſition, ſo as to reduce it to the ratio of 5 to 4: ſo 9 may be looked upon as a part of 7, provided the other part be —2.

COROLLARY.

If there be a ſeries of quantities A, B, C, D, *whereof* A *is to* B *as* R *to* r, *and* B *is to* C *as* S *to* s, *and* C *is to* D *as* T *to* t; *I ſay then that* A *will be to* D *as* RST, *the product of all the antecedents, to* r s t *the product of all the conſequents.* For by the art. 293, the ratio of *A* to *D* is compounded of the ratios of *R* to *r*, of *S* to *s*, and of *T* to *t*; and theſe ratios, when thrown into one ſum, conſtitute the ratio of *RST* to *rst*; therefore *A* is to *D* as *RST* to *rst*.

Of the ſubtraction of ratios.

296. *The ſubtraction of ratios one from another, when both have the ſame antecedent, or both the ſame conſequent, is obvious enough: thus the ratio of* A *to* B *ſubtracted from the ratio of* A *to* C *leaves the ratio of* B *to* C; *and the ratio of* B *to* C *ſubtracted from the ratio of* A *to* C *leaves the ratio of* A *to* B: this I ſay is obvious, becauſe (according to art. 293) the ratio of *A* to *C* contains the ratios of *A* to *B* and of *B* to *C*; and therefore, if either part be taken away, there muſt remain the other.

But if the two ratios, whereof one is to be ſubtracted from the other, have neither the ſame antecedent nor the ſame conſequent, it will be proper then to reduce them to the ſame antecedent, by ſetting off the ratio to be subtracted from the antecedent of the other, thus: let it be required to subtract the ratio of C to D from the ratio of A to B: now the ratio of C to D set off from A reaches to $\frac{AD}{C}$; therefore to subtract the ratio of C to D from the ratio of A to B is the same as to subtract the ratio of A to $\frac{AD}{C}$ from the ratio of A to B; but the ratio of A to $\frac{AD}{C}$ subtracted from the ratio of A to B, a ratio of the same antecedent, leaves the ratio of $\frac{AD}{C}$ to B, or of AD to BC; therefore the ratio of C to D subtracted from the ratio of A to B leaves the ratio of AD to BC. The rule then is as follows:

Whenever one ratio is to be subtracted from another, change the sign of the ratio to be subtracted by inverting its terms, and then the sum of this new ratio added to the other will be the same with the remainder of the intended subtraction. Thus to subtract the ratio of C to D from the ratio of A to B is the same as to add the ratio of D to C to the ratio of A to B; but the ratio of D to C added to the ratio of A to B gives the ratio of AD to BC by the last article; therefore the ratio of C to D subtracted from the ratio of A to B leaves the ratio of AD to BC. For a further proof of this, we are to take notice, that in all subtraction whatever, the remainder and the quantity subtracted ought both together to make the quantity from whence the subtraction was made; but in our case the remainder was the ratio of AD to BC, and the quantity subtracted was the ratio of C to D, and these two added together make the ratio of ACD to BCD, or of A to B, which is the ratio from whence the subtraction was made; therefore the remainner in this case was rightly assigned.

For

For an example in numbers, let it be required to ſubtract the ratio of 4 to 5 from the ratio of 2 to 3: now the ratio of 5 to 4 added to the ratio of 2 to 3 gives the ratio of 10 to 12, or of 5 to 6, by the laſt article; therefore the ratio of 4 to 5 ſubtracted from the ratio of 2 to 3 leaves the ratio of 5 to 6, which may be confirmed thus: the ratio of 2 to 3 is the ſame with the ratio of 4 to 6, which contains the ratios of 4 to 5 and of 5 to 6; therefore, if the ratio of 4 to 5 be taken away, there will remain the other part, which is the ratio of 5 to 6.

Before I conclude this article, I ought to take notice that there is another way of conceiving the ſubtraction of ratios, which for its uſe in Phyſics and Mechanics ought not to be paſſed by in this place; it is thus: the ratio of C to D added to the ratio of A to B conſtitutes the ratio of AC to BD; therefore, *e converſo*, the ratio of C to D ſubtracted from the ratio of A to B muſt leave the ratio of $\frac{A}{C}$ to $\frac{B}{D}$, becauſe multiplication and diviſion are as much the reverſe of one another as addition and ſubtraction; but this ratio of $\frac{A}{C}$ to $\frac{B}{D}$, when reduced to integral terms, is the ſame with the ratio of AD to BC found before.

N. B. *Wherever it is ſaid that the ratio of* A *to* B *is compounded of the direct ratio of* C *to* D, *and of the inverſe or reciprocal ratio of* E *to* F, *the meaning is, that the ratio of* A *to* B *is equal to the exceſs of the ratio of* C *to* D *above the ratio of* E *to* F, *or that* A *is to* B *as* $\frac{C}{E}$ *to* $\frac{D}{F}$, *or as* CF *to* DE.

Of the multiplication and diviſion of ratios.

297. If the ratio of A to B be added to itſelf, that is, to the ratio of A to B, the ſum will be the ratio of A^2 to B^2 by the laſt article but one; and this

being added again to the ratio of *A* to *B* gives the ratio of A^3 to B^3, and so on; therefore the ratio of A^2 to B^2 is double, and the ratio of A^3 to B^3 triple, of the ratio of *A* to *B*. And universally, *The ratio of* A^n *to* B^n *is such a multiple of the ratio of* A *to* B *as is expressed by the number* n. Thus the ratio of A^4 to B^4 is four times the ratio of *A* to *B*, which I prove thus: the ratio of *A* to *B* reaches first from A^4 to A^3B, 2dly, from A^3B to A^2B^2, 3dly, from A^2B^2 to AB^3, and 4thly, from AB^3 to B^4.

To give an example in numbers, I say that five times the ratio of 2 to 3 is the ratio of the fifth power of 2 to the fifth power of 3, that is, the ratio of 32 to 243: for the ratio of 2 to 3 reaches 1st from 32 to 48, 2dly from 48 to 72, 3dly from 72 to 108, 4thly from 108 to 162, and 5thly from 162 to 243, Thus much for multiplication.

Division is the reverse of multiplication; and therefore as every ratio is doubled or trebled or quadrupled by squaring or cubing or square squaring its terms, so every ratio is bisected or trisected or quadrisected by extracting the square or cube or square-square roots of its terms. Thus half the ratio of 2 to 3 is the ratio of the square root of 2 to the square root of 3, that is (when reduced according to the first scholium in art. 179*) the ratio of 40 to 49 nearly; which is further proved thus: the ratio of 40 to 49 is half the ratio of 1600 to 2401, by what was delivered in the former part of this article; but 1600 is to 2400 as 2 to 3; therefore 1600 is to 2401 as 2 to 3 very near.

But there is no necessity of a double extraction of the root in the division of a ratio, provided the ratio proposed be reduced to an equal one whose antecedent is unity. Thus 2 is to 3 as 1 to $\frac{3}{2}$, and therefore half the ratio of 2 to 3 is the ratio of 1 to $\sqrt{\frac{3}{2}}$, or the ratio of 1 to $\sqrt{1.5}$.

From what has been said it appears that one ratio may be commensurate to another, and yet the terms of one incommensurate to the terms of the other: thus the ratio

* See the Quarto Edition, p. 283.

ratio of 2 to 3 is certainly commensurate to the ratio of the square root of 2 to the square root of 3, the former being double of the latter; and yet 2 and 3, the terms of the former ratio are incommensurate to $\sqrt{2}$ and $\sqrt{3}$ the terms of the latter.

Note. *Wherever it is said that* A *is to* B *in a sesquiplicate ratio of* C *to* D, *the meaning is, that the ratio of* A *to* B *is equal to* $\frac{3}{2}$ *of the ratio of* C *to* D: therefore, in such a case, twice the ratio of A to B will be equal to three times the ratio of C to D; but twice the ratio of A to B is the ratio of A^2 to B^2, and three times the ratio of C to D is the ratio of C^3 to D^3; therefore if A be to B in a sesquiplicate ratio of C to D, A^2 will be to B^2 as C^3 to D^3. Thus, in the revolutions of the primary planets about the Sun, and of the secondary planets about Jupiter and Saturn, their periodic times are said to be in a sesquiplicate ratio of their middle distances, that is, the squares of their periodic times are as the cubes of their middle distances.

Another way of multiplying and dividing small ratios, that is, whose terms are large in comparison of their difference.

298. Before I deliver what I have to say upon this head, I shall only observe, that *If two intermediate quantities have always the same difference, the greater the quantities are, the nearer will their ratio approach towards a ratio of equality:* thus the difference betwixt 2 and 1 is the same with the difference betwixt 100 and 99; but the ratio of 2 to 1 or of 100 to 50 is much greater than the ratio of 100 to 99. By the help of this observation, and the following theorem, I shall endeavour to shew that small ratios may sometimes be doubled, or tripled, or bisected, or trisected, by more compendious ways than those that are taught in the last article; and whenever they happen

to be so, they ought to be used, and frequently are used, rather than the other.

A THEOREM.

If there be two quantities whose difference is but small in comparison of the quantities themselves, and if so much be added to one and subtracted from the other as shall make their difference double, or triple, or half, or a third part of what it was before; I say then that the quantities after this alteration shall be in a duplicate, or a triplicate, or a subduplicate, or a subtriplicate ratio of that they were in before any such change was made, nearly.

1st, Let there be two numbers 10 and 11, whose difference is 1; then if $\frac{1}{2}$ be added to 11 and subtracted from 10, we shall have the numbers $11\frac{1}{2}$ and $9\frac{1}{2}$, whose difference is 2: I say now that $11\frac{1}{2}$ is to $9\frac{1}{2}$ in a duplicate ratio of 11 to 10 nearly. For the ratio of $11\frac{1}{2}$ to $9\frac{1}{2}$ is resolvable into these two ratios, *viz.* the ratio of $11\frac{1}{2}$ to $10\frac{1}{2}$ and the ratio of $10\frac{1}{2}$ to $9\frac{1}{2}$: now of these two ratios the former, to wit, that of $11\frac{1}{2}$ to $10\frac{1}{2}$, is somewhat less than the ratio of 11 to 10, by the observation at the beginning of this article; and the latter, to wit, that of $10\frac{1}{2}$ to $9\frac{1}{2}$, is somewhat greater than the ratio of 11 to 10, and the excess in this case is nearly equal to the defect in the former; therefore the sum of both these ratios put together, that is, the ratio of $11\frac{1}{2}$ to $9\frac{1}{2}$ will be very nearly equal to twice the ratio of 11 to 10.

2dly, As the difference betwixt 10 and 11 is 1, add 1 to 11 and subtract it from 10, and you will have the numbers 12 and 9, whose difference is 3: I say now that 12 will be to 9, or 4 to 3, in a triplicate ratio of 11 to 10 nearly. For the ratio of 12 to 9 is resolvable into these three ratios, to wit, the ratio of 12 to 11, the ratio of 11 to 10, and the ratio of 10 to 9; and of these three ratios, the first, to wit, that of 12 to 11, is somewhat less than the middle ratio of 11 to 10; and the last, to wit, that of 10 to 9, is about as much greater;

greater; therefore the first and last ratios put together will make about twice the middle ratio of 11 to 10; therefore all these three ratios put together, to wit, the ratio of 12 to 9, will make three times the ratio of 11 to 10 nearly.

3dly, And if increasing the difference increases the ratio proportionably, then diminishing the difference ought to diminish the ratio proportionably, that is, if the difference be reduced to half, or a third part of what it was at first, the ratio ought to be so reduced : now as the difference between 10 and 11 is 1, add $\frac{1}{4}$ to 10 and subtract it from 11, and you will have the numbers $10\frac{1}{4}$ and $10\frac{3}{4}$, whose difference is $\frac{1}{2}$, and $10\frac{1}{4}$ will be to $10\frac{3}{4}$ in a subduplicate ratio of 10 to 11 nearly ; but if $\frac{1}{3}$ be added to 10 and subtracted from 11, you will then have the numbers $10\frac{1}{3}$ and $10\frac{2}{3}$, whose difference is $\frac{1}{3}$; and $10\frac{1}{3}$ will be to $10\frac{2}{3}$ in a subtriplicate ratio of 10 to 11 nearly.

Let us now try how near the ratios here found approach to the truth. By the last article, the duplicate ratio of 10 to 11 is the ratio of 100 to 121, or of 1 to 1 .2100; and according to the foregoing theorem it is the ratio of $9\frac{1}{2}$ to $11\frac{1}{2}$, or of 19 to 23, or of 1 to 1 .2105.

By the last article the triplicate ratio of 10 to 11 is the ratio of 1000 to 1331, or of 1 to 1 .331 ; and according to the foregoing theorem it is the ratio of 9 to 12, or of three to 4, or of 1 to 1 .333.

By the last article the subduplicate ratio of 10 to 11 is the ratio of 1 to the square root of $\frac{11}{10}$, or of 1 to 1 .04881 ; and according to the foregoing theorem it is the ratio of $10\frac{1}{4}$ to $10\frac{3}{4}$, or of 41 to 43, or of 1 to 1 .04878.

By the last article the subtriplicate ratio of 10 to 11 is the ratio of 1 to the cube root of $\frac{11}{10}$, that is, of 1 to 1 .03228 ; and according to the foregoing theorem it is the ratio of $10\frac{1}{3}$ to $10\frac{2}{3}$, or of 31 to 32, or of 1 to 1 .03226.

By these instances we see how near these ratios come up to the truth, even when the difference is no less than a tenth or an eleventh part of the whole: but if we suppose the difference to be the hundredth or the thousandth part of the whole, they will be much more accurate; insomuch that, to multiply or divide the ratio, it will be sufficient to encrease or diminish one of the numbers only. Thus 100 is to 102 in a duplicate, and to 103 in a triplicate, ratio of 100 to 101; and 100 is to $100+\frac{1}{2}$ in a subduplicate, and to $100+\frac{1}{3}$ in a subtriplicate, ratio of 100 to 101 nearly: and universally, *If* A+z *and* A+y *be any two quantities approaching infinitely near to the quantity* A, *the ratio of* A+z *to* A *will be to the ratio of* A+y *to* A *as the infinitely small difference* z *is to the infinitely small difference* y.

I shall draw only one example out of an infinite number that might be produced to shew the use of the foregoing proposition. Suppose then I have a clock that gains one minute every day; how much must I lengthen the pendulum to set it right? Let l be the present length of the pendulum, let x be the increment to be added to its length in order to correct its motion, and let n be the number of minutes in one day; then it is plain that the pendulum l performs the same number of vibrations in the time $n-1$ that the pendulum $l+x$ is to perform in the time n. Now Monsieur *Huygens* has demonstrated that the times wherein different pendulums perform the same number of vibrations are in a subduplicate ratio of the lengths of those pedulums; therefore $n-1$ must be to n in a subduplicate ratio of l to $l+x$, or (which comes to the same thing) l must be to $l+x$ in a duplicate ratio of $n-1$ to n: but by the foregoing proposition, the duplicate ratio of $n-1$ to n is the ratio of $n-\frac{3}{2}$ to $n+\frac{1}{2}$, or of $2n-3$ to $2n+1$; therefore l is to $l+x$ as $2n-3$ is to $2n+1$, that is, the pendulum must be lengthened in the proportion of $2n-3$ to $2n+1$: but n the number of minutes in one day

is

is 1440; and therefore $2n-3$ is to $2n+1$ as 2877 is to 2881, or as 719 to 720 very near; therefore the pendulum must be lengthened in the proportion of 719 to 720. *Q. E. I.*

Had the duplicate ratio of $n-1$ to n been taken only by diminishing $n-1$ to $n-2$, without meddling with the other number n, the conclusion would still have been the same; for then l would have been to $l+x$ as $n-2$ to n, as 1438 to 1440, as 719 to 720.

Having now delivered what I intended concerning the composition and resolution of ratios, it remains that I say something further concerning the application of this doctrine, and then I shall make an end of the subject.

DEFINITION 4.

299. *If two variable quantities* Q *and* R *be of such a nature, that* R *cannot be increased or diminished in any proportion, but* Q *must necessarily be increased or diminished in the same proportion; as if R cannot be changed to any other value* r, *but* Q *must also be changed to some other value* q, *and so changed that* Q *shall always be to* q *in the same proportion as* R *to* r; *then is* Q *said to be as* R *directly, or simply as* R. Thus is the circumference of a circle said to be as the diameter; because the diameter cannot be increased or diminished in any proportion, but the circumference must necessarily be increased or diminished in the same proportion. Thus is the weight of a body said to be as the quantity of matter it contains, or proportionable to the quantity of matter; because the quantity of matter cannot be increased or diminished in any proportion, but the weight must be increased or diminished in the same proportion.

COROLLARY I.

If Q *be as* R *directly, then* e converso R *must necessarily be as* Q *directly.* For let *Q* be changed to any other value q, and at the same time let R be changed to r; then since Q is as R, Q will be to q as R to r; but if Q is to q as R is to r, then *vice versa* R will be to r as Q to q: since then Q cannot be changed to q, but R must be changed to r, and that in the same proportion, it follows by this definition that R is as Q directly.

COROLLARY 2.

If Q *be directly as* R, *and* R *be directly as* S, *then will* Q *be directly as* S. For let S be changed to s, and at the same time R to r, and Q to q; then since by the supposition R is as S, R must be to r as S to s; and since again Q is as R, Q will be to q as R to r: since then Q is to q as R to r, and R is to r as S to s, it follows that Q will be to q as S to s, and consequently that Q will be as S.

COROLLARY 3.

If Q *be as* R, *and* R *be as* S; *I say then that* Q *will be as* $R \pm S$, *and also as the square root of the product* RS. For changing Q, R, S, into q, r, s, since R is as S, we shall have R to r as S to s; whence by the twelfth and nineteenth of the fifth book of the Elements R will be to r as $R \pm S$ is to $r \pm s$; but Q is to q as R is to r, *ex hypothesi*; therefore Q is to q as $R \pm S$ is to $r \pm s$; therefore by this definition Q will be as $R \pm S$. Again, since R is as S, R^2 will be as RS, and R as $\surd RS$; but Q is as R; therefore by the last corollary Q will be as $\surd RS$.

COROLLARY 4.

If any variable quantity as Q *be multiplied by any given number as* 5; *I say then that* 5Q *will be as* Q. For it will be impossible for Q to be increased or diminished in any proportion, but $5Q$ must be increased or diminished in the same proportion: if Q in any one case be double of Q in another, then $5Q$ in the

the former case must be double of $5Q$ in the latter, and so on; therefore $5Q$ is as Q.

COROLLARY 5.

If Q *be as* R, *then* Q^2 *will be as* R^2, Q^3 *as* R^3, $\sqrt{Q}$ *as* $\sqrt{R}$, &c. For let R^2 be changed in the proportion of D to E; then will R be changed in the proportion of $\sqrt{D}$ to $\sqrt{E}$; but Q is as R; therefore Q will also be changed in the proportion of $\sqrt{D}$ to $\sqrt{E}$; therefore Q^2 will be changed in the proportion of D to E: since then R^2 cannot be changed in any proportion, suppose of D to E, but Q^2 must necessarily be changed in the same proportion, it follows from this definition that Q^2 is as R^2: and the reasoning is the same in all other cases.

COROLLARY 6.

If Q, R, *and* S, *be three variable quantities, and* Q *be as the product or rectangle* RS; *I say then, that* $\frac{Q}{R}$ *will always be as* S, *and* $\frac{Q}{S}$ *as* R, *and that* $\frac{Q}{RS}$ *will be a given quantity, or (which is chiefly meant by that phrase) that the quantity* $\frac{Q}{RS}$ *will always be the same, be the values of* Q, R, *and* S, *what they will.* For since Q is as RS, Q cannot be increased or diminished in any proportion, but RS must be increased or diminished in the same proportion; therefore $\frac{Q}{R}$ cannot be increased or diminished in any proportion, but $\frac{RS}{R}$ or S must be increased or diminished in the same proportion; therefore S is as $\frac{Q}{R}$, and $\frac{Q}{R}$ as S: and by a like proof, R will be as $\frac{Q}{S}$, and $\frac{Q}{S}$ will be as R: but if $\frac{Q}{S}$ be as R,

then

then dividing both sides by *R*, we shall have $\frac{Q}{RS}$ as 1; but 1 is a quantity that neither increases nor diminishes, but is always the same; therefore the quantity $\frac{Q}{RS}$ will always be the same: and for the same reason, *If* Q *be as any single quantity, suppose* R, $\frac{Q}{R}$ *will always be the same, let* Q *and* R *be what they will.*

COROLLARY 7.

If there be four variable quantities A, B, C, D, *all in numbers, whereof* A *is as* B, *and* C *is as* D; *I say then that the product* AC *will be as the product* BD. For since *A* is as *B*, *AC* will be as *BC*, and since *C* is as *D*, *BC* will be as *BD*; therefore by the second corollary *AC* will be as *BD*; that is, *AC* in one case will be to *AC* in any other as *BD* in the former case is to *BD* in the latter.

DEFINITION 5.

300. *If two variable quantities* Q *and* R *be of such a nature, that* R *cannot be increased in any proportion whatever, but* Q *must necessarily be diminished in a contrary proportion, or that* R *cannot be diminished in any proportion whatever, but* Q *must necessarily be increased in a contrary proportion; in a word, if* R *cannot be changed in a proportion of* D *to* E, *but* Q *must necessarily be changed in the proportion of* E *to* D; *then is* Q *said to be as* R *inversely or reciprocally.* Thus if a spherical body be viewed at any considerable distance, the apparent diameter is said to be reciprocally as the distance, because the greater the distance is, the less will be the apparent diameter, and *vice versa.* Thus if a globe be supposed to move uniformly about its *axis*, the periodical time of this motion is said to be reciprocally as the velocity with which the globe circulates (for the quicker the circulation

culation is, the ſooner it will be over); which is as much as to ſay, that the greater the velocity is with which the globe circulates, the leſs will be the periodical time of one revolution, and *vice verſa*. Thus if the numerator of a fraction continues always the ſame whilſt the denominator is ſuppoſed to vary, that fraction is ſaid to be reciprocallyas its denominator, becauſe the greater the denominator is, the leſs will be the value of the fraction, and *vice verſa*.

COROLLARY 1.

If Q *be reciprocally as* R, *then* e converſo R *will be reciprocally as* Q. For let *Q* be changed in the proportion of *D* to *E*, and at the ſame time let *R* be changed in the proportion of *A* to *B*; then ſince *Q* is reciprocally as *R*, *Q* muſt be changed in the proportion of *B* to *A*; but *Q* was changed in the proportion of *D* to *E*; therefore *B* muſt be to *A* as *D* to *E*; therefore, inverſely, *A* muſt be to *B* as *E* to *D*; but *R* was changed in the proportion of *A* to *B* by the ſuppoſition; therefore *R* was changed in the proportion of *E* to *D*. Since then *Q* cannot be changed in any proportion, ſuppoſe of *D* to *E*, but *R* muſt neceſſarily be changed in the contrary proportion of *E* to *D*, it follows from this definition that *R* muſt be reciprocally as *Q*.

COROLLARY 2.

If Q *be directly as* R, *and* R *be reciprocally as* S, *then* Q *muſt be reciprocally as* S. For let *S* be changed in the proportion of *D* to *E*; then ſince *R* is reciprocally as *S*, *R* muſt be changed in the proportion of *E* to *D*; but *Q* is directly as *R* by the ſuppoſition; therefore *Q* muſt alſo be changed in the proportion of *E* to *D*. Since then *S* cannot be changed in the proportion of *D* to *E*, but *Q* muſt neceſſarily be changed in the proportion of *E* to *D*, it follows from this definition that *Q* is reciprocally as *S*.

COROL-

Corollary 3.

By a like way of reasoning, if Q *be reciprocally as* R, *and* R *be reciprocally as* S, Q *will be directly as* S.

Corollary 4.

If two variable quantities Q *and* R *be of such a nature that their product or rectangle* QR *is always the same; I say then that* Q *will be reciprocally as* R. For since *QR* is always the same, it will be as the number 1, which neither increases nor diminishes; but if *QR* be as one, then *Q* will be as the fraction $\frac{1}{R}$ by the sixth corollary to the fourth definition. Since then *Q* is directly as the fraction $\frac{1}{R}$, and the fraction $\frac{1}{R}$ is reciprocally as its denominator *R* by this definition, it follows from the second corollary that *Q* will be reciprocally as *R*.

Corollary 5.

Every fraction is reciprocally as the same fraction inverted. Thus the fraction $\frac{R}{S}$ is reciprocally as the fraction $\frac{S}{R}$. This is evident from the last corollary; for if the fractions $\frac{R}{S}$ and $\frac{S}{R}$ be multiplied together, their product will always be unity, let *R* and *S* be what they will.

Corollary 6.

If Q *be reciprocally as* R, *or reciprocally as* $\frac{R}{1}$

then

then Q *will be directly as* $\frac{1}{R}$. For since Q is reciprocally as $\frac{R}{1}$, and $\frac{R}{1}$ is reciprocally as $\frac{1}{R}$ by the last corollary, it follows from the third corollary that Q will be directly as $\frac{1}{R}$. For the same reason, *If* Q *be reciprocally as* $\frac{1}{R}$, *it will be directly as* R.

DEFINITION 6.

301. *If any quantity as* Q *depends upon several others as* R, S, T, V, X, *all independent of one another, so that any one of them may be changed singly without affecting the rest; and if none of the quantities* R, S, T, *can be changed singly, but* Q *must be changed in the same proportion, nor any of the quantities* V, X, *but* Q *must be changed in a contrary proportion; then is* Q *said to be as* R *and* S *and* T *directly, and as* V *and* X *reciprocally or inversely*. Thus the fraction $\frac{RST}{VX}$ is said to be as R and S and T directly, and as V and X inversely, because none of the factors belonging to the numerator can be changed, but the value of the fraction must be changed in the same proportion, and none of the factors belonging to the denominator can be changed, but the value of the fraction must be changed in a contrary proportion.

N. B. *If* Q *be as* R *and* S *and* T *directly, without any reciprocals, then it is said to be as* R *and* S *and* T conjunctim, *jointly*.

A THEOREM.

302. *If* Q *be as* R *and* S *and* T *directly, and as* V *and* X *reciprocally; and if the quantities* R, S, T, V, X, *be changed into* r, s, t, v, x, *and so* Q *into* q; *I say then that the ratio of* Q *to* q *will be equal to the excess*

excess of all the direct ratios taken together above all the reciprocal ones taken together: as if the ratios of R *to* r, *of* S *to* s, *and of* T *to* t *(which I call direct ratios) when added together make the ratio of* A *to* B; *and if the ratios of* V *to* v, *and of* X *to* x *(which I call reciprocal ratios) when added together make the ratio of* C *to* D; *I say then that the ratio of* Q *to* q *will be equal to the excess of the ratio of* A *to* B *above the ratio of* C *to* D.

For suppoſing all but *R* to continue the ſame, let *R* be changed into *r*; then will *Q* be changed from its firſt value in the ratio of *R* to *r* by the *hypotheſis*: let now *r*, *T*, *V*, *X* continue, and let *S* be changed into *s*: then will *Q* be changed from its laſt value in the ratio of *S* to *s*: in like manner if *T* be changed into *t*, *cæteris paribus*, *Q* will be changed from its laſt value in the ratio of *T* to *t*: therefore if *R*, *S*, *T* be changed into *r*, *s*, *t*, *Q* will be changed from one value to another in a ratio compounded of all the direct ratios of *R* to *r*, of *S* to *s*, and of *T* to *t*; that is, *Q* will be changed in the ratio of *A* to *B*. This being ſo, let us now imagine *V* to be changed, *cæteris paribus*, into *v*; then will *Q* be further changed in the ratio of *v* to *V*; and if after this we imagine *X* to be changed into *x*, *Q* will be changed in the proportion of *x* to *X*, and will now be arrived at its laſt value *q*: therefore if to the ratio of *A* to *B* you add the ratios of *v* to *V* and of *x* to *X*, you will have the ratio of *Q* to *q*: but to add the ratio of *v* to *V* is the ſame thing as to ſubtract the ratio of *V* to *v* by art. 296; and ſo again, to add the ratio of *x* to *X* is the ſame as to ſubtract the ratio of *X* to *x*; therefore if from the ratio of *A* to *B* you ſubtract the ratios of *V* to *v* and of *X* to *x*, you will have the ratio of *Q* to *q*; but the ratios of *V* to *v* and of *X* to *x*, when added together, make the ratio of *C* to *D*, *ex hypotheſi*; therefore if from the ratio of *A* to *B* you ſubtract the ratio of *C* to *D*, there will remain the ratio of *Q* to *q*; therefore the ratio of

Q to

Q to *q* is the excess of the ratio of *A* to *B* above the ratio of *C* to *D*; or (which is the same thing) *Q* is to *q* in a ratio compounded of the ratio of *A* to *B* directly, and of the ratio of *C* to *D* inversely. See art. 296.

This is upon a supposition that the quantities *R*, *S*, *T*, *V*, *X* were changed into *r*, *s*, *t*, *v*, *x* one after another in time: but since the ratio of *Q* to *q* does not depend upon the intervals of time between the several changes, but will be the same whether those intervals be greater or less, it follows that the ratio of *Q* to *q* will be the same as if all these changes had been made at once. *Q. E. D.*

COROLLARY I.

If the quantities *R*, *S*, *T*, *V*, *X*, and consequently *A*, *B*, *C*, *D*, be expressed by numbers, as they must be before they can be of use in any computation; then the ratio of *A* to *B* will be the ratio of *RST* to *rst*, and the ratio of *C* to *D* will be the ratio of *VX* to *vx*; and the excess of the ratio of *A* to *B* above the ratio of *C* to *D* will be the ratio of $\frac{RST}{VX}$ to $\frac{rst}{vx}$; (see the second way of subtracting ratios in art. 296) therefore, in this case, *Q* will be to *q* as the fraction $\frac{RST}{VX}$ is to the fraction $\frac{rst}{vx}$. *Since then the fraction* $\frac{\mathrm{RST}}{\mathrm{VX}}$ *cannot be changed into* $\frac{\mathrm{rst}}{\mathrm{vx}}$ *but at the same time* Q *must be changed into* q, *and so changed that* Q *will be to* q *as* $\frac{\mathrm{RST}}{\mathrm{VX}}$ *is to* $\frac{\mathrm{rst}}{\mathrm{vx}}$, *it follows from the fourth definition that* Q *will be as the fraction* $\frac{\mathrm{RST}}{\mathrm{VX}}$; *and consequently that* Q *in any one case will be to* Q *in any other as the fraction* $\frac{\mathrm{RST}}{\mathrm{VX}}$

in the former case is to the fraction $\frac{RST}{VX}$ *in the latter.*

Corollary 2.

If there be no reciprocals then Q *will be as the product of all the direct terms, that is, as the product* RS *if there be two of them, or as the product* RST *if there be three of them*, &c.

Scholium.

In the demonstration of the foregoing proposition as well as in the sixth definition it was supposed, that the quantities *R*, *S*, *T*, *V*, *X*, upon which *Q* depended, were themselves entirely independent of one another, so as that any of them might be changed singly without affecting the rest; and in such a case, if *Q* be as *R* and *S* directly, it may be concluded to be as the product *RS*. But this conclusion must not be carried farther than can be justified by the demonstration: for if in any case the quantities *R* and *S* should not be independent, if neither of them can be changed whilst the other continues the same, then though no change can be made either in *R* or *S* but what will make a proportionable change in *Q*, yet here *Q* must not be said to be as the product *RS*. As for example, let *Q* be an arc of a circle subtending at the distance *R* an angle whose quantity is represented by *S*; then it is plain that neither *R* nor *S* can be changed singly, but *Q* must be changed proportionably; it is plain also that either *R* or *S* may be changed singly whilst the other remains the same; and therefore in this case it is lawful to conclude that *Q* is as the product *RS*. But let us now suppose *Q* to be the circumference of a circle whose *radius* is *R*, and let *S* be the side of a regular polygon of any given sort inscribed in that circle; as for instance, let *S* be the side of an inscribed square: here then it is plain that neither *R* nor *S* can be changed, but *Q* must be changed

changed proportionably; and yet if we should conclude in this case that *Q* is as *RS*, the illation would be false, because *R* and *S* have here as much dependence upon one another as *Q* upon both; for every one knows that the *radius* of a circle cannot be increased or diminished in any proportion, but the side of a square inscribed in that circle must be increased or diminished in the same proportion: in this case it may be concluded that *Q* is as $R+S$, or as $R-S$, or as the square root of *RS* by the third corollary in art. 299, but it must by no means be allowed that *Q* is as *RS*; for should *Q* be as *RS*, since in this case *S* is as *R*, and consequently *RS* as R^2, *Q* would be as R^2 by the second corollary in art. 299, which contradicts the supposition that *Q* is as *R*.

Examples to illustrate the foregoing theorem, where direct ratios are only concerned.

303. Ex. 1. *If a body moves for any time with any uniform velocity through any space, that space will be as the time and velocity jointly.* For if we suppose the velocity to be the same in all cases, but the time to differ, then the space described will be greater or less in proportion as the time is so, and therefore will be as the time: on the other hand, if we suppose the time to be the same in all cases, and the velocity to differ, then the space described in these equal times will be greater or less as the velocity is so, and consequently will be as the velocity: lastly, let us suppose both the time and velocity to vary; then the space will vary upon both these accounts, and therefore will vary in a ratio equal to the ratio wherein the time varies, and the ratio wherein the velocity varies put together; that is, the space in any one case will be to the space in any other in a ratio compounded of the ratio of the time in the former case to the time in the latter, and of the velocity in the former case to the velocity in the latter. This is universal; but if we

suppose the time and velocity to be expressed by numbers, we must then say that the space described is as the product of the number representing the time multiplied into the number representing the velocity, by the second corollary in the last article; or that the space described in any one case is to the space described in any other as the product of the time and velocity in the former case is to a like product in the latter.

Ex. 2. *The quantity of matter in any body depends upon two things,* viz. *its magnitude and density (where by density I mean the compactness or closeness of its matter).* For if two bodies of equal densities but of unequal magnitudes be compared, one body must have more matter than the other, or less, according as its solid content is greater or less, that is, according as its magnitude is greater or less; therefore in this case the quantities of matter in any two bodies thus compared will be as their magnitudes: on the other hand, if two bodies of the same magnitude but of different densities be compared, their quantities of matter will be as their densities, because the closer the parts of a body are, so much more matter will be crowded into the same space; therefore, if the bodies be different both in magnitude and density, the quantity of matter in one body will be to the quantity of matter in the other in a ratio compounded of the ratio of the magnitude of one body to the magnitude of the other, and of the ratio of the density of the former body to the density of the latter; and therefore, if these quantities be represented by numbers, the quantity of matter in any body will be as its magnitude and density multiplied together. Thus if D and d be the diameters of two globes whose densities are as E to e, the quantity of matter in the former globe will be to the quantity of matter in the latter as $D^3 \times E$ is to $d^3 \times e$; for the solid contents of all globes are as the cubes of their diameters.

Ex. 3. *The* momentum, *or force, or* impetus *with which a body moves, and with which it will strike any obstacle*

obstacle that lies in its way to oppose or stop it, is as the velocity of the motion and the quantity of matter in the body jointly. For the same quantity of matter moving with different velocities will strike an obstacle with forces proportionable to the velocities: on the other hand, different quantities of matter moving with the same velocity will strike with forces proportionable to their matter; a double body will strike with a double force, &c.; therefore, in the case where the velocity is the same, the *momentum* of a body is as the quantity of matter it contains; and in the case where the quantity of matter is the same, the *momentum* is as the velocity; therefore, if neither the velocity nor the matter be the same, the *momentum* will be as the matter and velocity jointly; and, in numbers, as the product of the number expressing the matter multiplied into the number expressing the velocity.

Ex. 4. *If a heavy body be suspended perpendicularly upon a lever (by which I mean an inflexible rod moving about a fixt point in the middle), the* momentum *or efficacy of that body to turn the lever about its center is,* cæteris paribus, *as the weight of the body and as the distance of the point of suspension from the center of the lever jointly.* For if we suppose this distance to be the same, the *momentum* of the body to turn the lever must be greater or less according as its weight is so, from whence that *momentum* arises: on the other hand, if we suppose the weight to be always the same, but to be removed, sometimes farther from, and sometimes nearer to the center, the *momentum* of the body to turn the lever will be greater or less in proportion to the distance of the point of suspension from the center of the lever, as is demonstrated in Mechanics, and may easily be tried by experience: therefore universally, the *momentum* of the body will be as this distance and the weight of the body jointly; and in numbers is as the product of the weight multiplied into the distance.

To illustrate this, I shall put the following question. Let a body weighing five pounds be suspended at the distance of six inches from the center of a lever, and let another body of seven pounds be suspended on the same side of the center at the distance of eight inches; then let a third body of nine pound weight be suspended on the other side of the center at the distance of ten inches: *Quære*, whether will these bodies sustain each other *in æquilibrio* or not; and if not, on which side will the lever dip, and with what *momentum?*

To resolve this, since we are at liberty to represent any one of these *momenta* by what numbers we please, provided the rest be represented proportionably, let us represent the *momentum* of the nine pound body by the product of its weight and distance multiplied together, that is, by 9×10 or 90; then must the other *momenta* be represented by like products, or they would not be represented by numbers proportionable to them: therefore the *momentum* of the five pound body will be 5×6 or 30, and that of the seven pound body 7×8 or 56; and therefore the sum of the *momenta* on this side the center acting the same way will be 86: whence now it plainly appears that the lever will dip on the side of the nine pound body, because 90, the *momentum* on that side, is greater than 86, the sum of the *momenta* on the other side: and since the excess of 90 above 86 is 4, it follows that 4 will be the difference of the *momenta* on one side and the other; insomuch that if any one sustains this lever immoveable, he will sustain the same force as if all the weights now upon the lever were taken away, and a single pound weight was suspended at the distance of four inches from the center of the lever: therefore when all the weights were upon the lever, if a single pound weight had been suspended at four inches distance, and on the same side of the center with the other two bodies whose weights were five and seven pounds, the whole system would then have consisted *in æquilibrio.*

Upon

Upon this theorem, that the force of a body upon a lever is as its weight and diftance from the center multiplied together, is founded the method of finding the centers of gravity of bodies, or the center of gravity of any fyftem of bodies, let their places or pofitions be what they will: but I muft not carry this matter any farther.

Ex. 5. *If a globe be made to move uniformly in an uniform fluid, the refiftance it will meet with in any given time by impinging againft the particles of the fluid, will be as the denfity of the fluid, and as the fquare of the diameter of the globe, and as the fquare of the velocity it moves with jointly.*

To determine rightly in this cafe, we muft here do what we all along have done, and what we always muft do in like cafes; that is, we muft take the whole to pieces, examine every particular circumftance by itfelf, *cæteris paribus*, and then put them all together. Firft then let us fuppofe the fame globe to move with the fame velocity, but fometimes in a denfer fluid, and fometimes in a rarer; then it is plain that the denfer the fluid is, the more particles of it the body will be likely to meet with in any given time, and confequently the greater refiftance it will fuffer from them; therefore the refiftance of the body, *cæteris paribus*, will be as the denfity of the fluid. In the next place let us fuppofe different globes to move in the fame fluid, and with the fame velocity; then, fince the refiftance of thefe globes arifes only from their furfaces, or rather from half their furfaces, and fince the furfaces of all globes are as the fquares of their diameters, it follows that the refiftance thefe globes meet with will be as the fquares of their diameters. Laftly, let us fuppofe the fame globe to move in the fame fluid with different velocities; then it is plain that a globe which moves with a double velocity will ftrike twice as many particles of the fluid in any given time, as it would if it was to move with a fingle velocity:

but if the body ſtrikes twice as many particles, then twice as many particles will ſtrike it, whence ariſes the reſiſtance; therefore the reſiſtance of a body moving with a double velocity is upon this account double of what it would have been in the caſe of a ſingle velocity: but this is not all; for it will not only ſtrike twice as many particles, but it will ſtrike every particle with twice the force in this caſe of what it would in the caſe of a ſingle velocity; and therefore, ſince action and reaction are always equal, and ſince it is the reaction of the medium that creates the reſiſtance, it follows that a body moving with a double velocity meets with four times the reſiſtance of what it would meet with when moving with a ſingle velocity. In like manner, a body that moves with a triple velocity will act three times as ſtrong upon three times the number of particles, and therefore will ſuffer nine times the reſiſtance of what it would ſuffer with a ſingle velocity; therefore the ſame globe moving in the ſame medium with different velocities will meet with a reſiſtance proportionable to the ſquare of the velocity it moves with. Put now all theſe conſiderations together, and the reſiſtance of a globe moving uniformly in an uniform fluid (I mean that reſiſtance which ariſes from the globe's impinging againſt the particles of the medium) will be as the denſity of the medium, as the ſquare of the diameter of the globe, and as the ſquare of the velocity it moves with jointly. Thus if two globes whoſe diameters are D and d move with velocities which are to one another as V to v in two fluids whoſe denſities are as E to e, the reſiſtance of the former will be to the reſiſtance of the latter as $V^2 \times D^2 \times E$ is to $v^2 \times d^2 \times e$.

Other

Other examples, wherein direct and reciprocal ratios are mixt together.

304. Ex. 6. *If a body be put into motion by any force directly applied, whether this force be a single impulse acting at once, or whether it be divided into several impulses acting successively; I say that the last velocity of this motion will be as the moving force directly, and as the quantity of matter in motion reciprocally.* For if different forces be applied to the same quantity of matter, the greater the force is, the greater will be the velocity, and *vice versa*; therefore in this case the velocity will be as the *vis motrix*: but if we suppose the same force to be applied to different quantities of matter, then the greater the quantity of matter is, the less will be the velocity, and *vice versa*, which I thus demonstrate. Suppose the moving force M, when applied to a certain quantity of matter as Q, will produce the velocity V; I say then that the same force M, applied to a quantity of matter equal to $2Q$, will only produce a velocity equal to $\frac{1}{2}V$: for M acting upon $2Q$ will produce the same velocity as $\frac{1}{2}M$ acting upon $1Q$; but $\frac{1}{2}M$ acting upon Q will produce a velocity equal to $\frac{1}{2}V$, because by the supposition M acting upon Q will produce the velocity V; therefore M acting upon $2Q$ will produce a velocity equal to $\frac{1}{2}V$; and for the same reason, M acting upon $3Q$ will produce a velocity equal to $\frac{1}{3}V$, &c.; therefore, if the *vis motrix* be the same, the velocity of the motion produced will be reciprocally as the quantity of matter: therefore universally, the velocity will be as the *vis motrix* directly, and as the quantity of matter inversely. As if M be changed into m, Q into q, and so V into v, the ratio of V to v will be equal to the excess of the ratio of M to m above the ratio of Q to q. In numbers thus; V will be to

v as $\frac{M}{Q}$ is to $\frac{m}{q}$; see the first corollary in art. 302. Otherwise thus; the *momentum* or *impetus* with which a body moves, is the force with which it will strike an object that lies in its way to stop it; therefore since action and reaction are equal, the force necessary to destroy any motion must be equal to the *momentum* with which the body moves: but the force necessary to destroy any motion is equal to the force that produced it, which we call the *vis motrix*; therefore in all motion whatever, the *vis motrix* must be equal to the *momentum*, and must be as the quantity of matter in the body moved multiplied into the velocity of the motion, because the *momentum* is so; see the last article, example the 3d: therefore M will always be as $V \times Q$ and V as $\frac{M}{Q}$.

If M be as Q, then $\frac{M}{Q}$ will be a standing quantity, and therefore the velocity V in this case will always be the same. Thus if the weights of all bodies be proportionable to the quantities of matter they contain, they will be equally accelerated in equal times; and *vice versa*, if all bodies, how different soever in the kinds and quantities of matter, be equally accelerated in equal times (as by undoubted experiments upon pendulums we find they are, setting aside the resistance of the air), it follows that the weights of bodies are proportionable to their quantities of matter only, without depending upon their forms, constitutions, or any thing else.

Ex. 7. *The velocity of a planet moving uniformly in a circle round the Sun is as its distance from the centre of the Sun directly, and as its periodical time inversely.* For if two planets at different distances from the Sun perform their revolutions in the same time, that planet must move with the greatest velocity that has the greatest circumference to describe; therefore in in this case, where the periodical time is given or always the same, the velocity of the planet must be as the circumference of the circle to be described: but the circumference of every circle is as its diameter or semidiameter; therefore, if the periodical time be given, the velocity of a planet must be as its distance from the Sun directly. Let us now suppose two planets revolving at the same distance from the Sun, but in different periodical times; then it is plain that the swifter planet will perform its revolution in less time, and *vice versa*; and therefore, if the distance be given, the velocity will be reciprocally as the periodical time. Put both these cases together, and the velocity of a planet moving uniformly round the Sun will be as its distance from the center of the Sun directly, and as its periodical time inversely. Thus the Earth's distance from the Sun is to that of Jupiter as 10 to 52 nearly; and the Earth's periodical time is to that of Jupiter as 1 year to 12 years nearly, or as 1 to 12; therefore the Earth's velocity is to Jupiter's velocity as $\frac{10}{1}$ is to $\frac{52}{12}$, or as 120 to 52, or as 30 to 13.

This way of reasoning is applicable to all bodies moving uniformly in circles, let the law of their motions be what it will. But if (as that accurate Astronomer *Kepler* has demonstrated) the planetary motions be so tempered that their periodical times are in a sesquiplicate ratio of their distances, or (which is the same thing by art. 297) that the squares of their periodical times are as the cubes of their distances, we shall then have a more simple way of expressing the velocity of a planet thus: let V be the velocity, and D the distance of any planet from the Sun, and let T be the periodical time; then since, from what has been said, V is as $\frac{D}{T}$, we shall have V^2 as $\frac{D^2}{T^2}$; but, according to *Kep-*

ler's

ler's propofition, T^2 is as D^3, and $\frac{D^2}{T^2}$, as $\frac{D^2}{D^3}$, or as $\frac{1}{D}$; therefore V^2 is as $\frac{1}{D}$, and V as $\frac{1}{\sqrt{D}}$; that is, in this cafe, the velocity of a planet is reciprocally in a fubduplicate ratio of its diftance from the fun. So the velocity of a planet whofe diftance is D is to the velocity of a planet whofe diftance is d as $\sqrt{d}$ is to $\sqrt{D}$, or as 1 is to $\sqrt{\frac{D}{d}}$.

Ex. 8. *If a wheel turns uniformly about its* axis, *the time of one round will be as the diameter of the wheel directly, and as the abfolute velocity of every point in the circumference of the wheel inverfely.* For if the circumference of a great wheel moves with the fame velocity as the circumference of a fmall one, the periodical time of the former wheel will be as much greater in proportion than the periodical time of the latter as the circumference of the former wheel is greater than the circumference of the latter, or as the diameter of the former is greater than the diameter of the latter; therefore if the velocity of the wheel's circumference be given, the periodical time will be as the diameter of the wheel directly: let us now fuppofe the velocity of the circumference of the fame wheel to be in any cafe increafed; then will the periodical time be diminifhed in a contrary proportion, and *vice verfa*; therefore if the diameter of a wheel be given, the periodical time will be reciprocally as the velocity of the circumference; therefore if neither the diameter nor the velocity of the circumference be given, the periodical time will be as the diameter of the wheel directly, and as the abfolute velocity of every point in the circumference inverfely. In numbers the periodical time will be as $\frac{D}{V}$.

Ex.

Ex. 9. *The relative gravity of any ſpecies of bodies is as the abſolute weight of any body of that ſpecies directly, and as its magnitude inverſely*; where by the magnitude or bulk of a body is meant the quantity of ſpace it takes up, and not the quantity of matter it contains.

All bodies of the ſame kind are ſuppoſed to weigh in proportion to their magnitudes; and therefore if a body of any one kind be compared with a body of the ſame magnitude of another kind, the proportion of their weights will always be the ſame, let their common magnitude be what it will; and hence ariſes the compariſon in general of the weight of one ſpecies of bodies with the weight of another: if a cubic inch of gold be 19 times as heavy as a cubic inch of water, then a cubic foot of gold will be 19 times as heavy as a cubic foot of water, *&c.*; and ſo we pronounce in general that gold is 19 times as heavy as water, though we mean bulk for bulk. In this ſenſe therefore may any one ſpecies of bodies be ſaid to be heavier or lighter than another, in proportion as any one body of the former ſpecies is heavier or lighter than a body of the ſame magnitude of the latter, which is the ſame in effect with the firſt part of my aſſertion. Let us now compare bodies of the ſame weight, but of different magnitudes; and then it will appear that the ſpecific gravities of theſe bodies, that is, of the ſeveral ſpecies to which they belong, will be reciprocally as the magnitudes of the bodies compared: thus if a cubic inch of gold be as heavy as 19 cubic inches of water, then the ſpecific gravity of gold will be to the ſpecific gravity of water, not as 1 to 19, but as 19 to 1; for if one cubic inch of gold be as heavy as 19 cubic inches of water, then 1 cubic inch of gold will be 19 times as heavy as 1 cubic inch of water; and therefore, from what has been ſaid in the former caſe, the ſpecific gravity of gold will be to the ſpecific gravity of water as 19 to 1. Put both theſe caſes together, and the relative gravity of any ſpecies of bodies will be as the abſolute weight of any one body

body of that species directly, and as its magnitude inversely. Thus if in numbers P and p be the weights of two globes whose diameters are D and d, the specific gravities of the metals out of which these two globes were formed are as $\frac{P}{D^3}$ to $\frac{p}{d^3}$.

EX. 10. *If a body as* A *gravitates toward the center of a planet as* B *at the distance* D; *I say then that the weight of* A *will be as the quantity of matter in* A *directly, and as the quantity of matter in* B *directly, and as the square of the distance* D *inversely.* For the weight of the whole body *A* towards *B* arises, *cæteris paribus*, from the weight of all its parts; and therefore in such a case will be as the quantity of matter in *A*. Again, the weight of *A* towards the whole planet *B* arises, *cæteris paribus*, from the weight of *A* to all the parts of *B*; and therefore in such a case will be as the quantity of matter in *B*. Lastly, if the quantities of matter in *A* and *B* continue the same, and the distance *D* be supposed to vary, the great *Newton* has demonstrated that the weight of *A* towards *B* will be reciprocally as the square of the distance *D*. Therefore if neither the quantities of matter in *A* and *B*, nor the distance *D* be the same, the weight of *A* towards *B* will be as the quantity of matter in *A* directly, and as the quantity of matter in *B* directly, and as the square of the distance *D* inversely. Thus if *A* and *B* be numbers representing the quantities of matter in the bodies *A* and *B* respectively, the weight of *A* towards *B* at the distance *D* will be as $\frac{AB}{D^2}$, that is, the weight of *A* towards *B* at the distance *D* will be to the weight of *a* towards *b* at the distance *d* as the fraction $\frac{AB}{D^2}$ is to the fraction $\frac{ab}{d^2}$.

Hence

Hence the weight of A towards B will be equal to the weight of B towards A, since both will be represented by the same quantity $\frac{AB}{D^2}$.

Another way of treating the examples in the two last articles.

305. *If there be ever so many quantities, and these all heterogeneous to one another, we are at liberty to represent them by what number we please, or even all by unity itself, provided we take care to represent all other quantities of like kinds by proportionable numbers.* Thus I am at liberty to call any quantity of time I please 1, or any degree of velocity 1, or any quantity of space 1; but then I must take care to call a double time, or a double velocity, or a double space, by the number 2, and so on. This consideration suggests to us another way of treating the examples in the two last articles, somewhat different from the former; which, as it may be explained by a bare instance or two, I shall give the learner as follows:

In the first example we were taught that the space described by a body moving uniformly for any time, and with any velocity, is in numbers as the time and velocity multiplied together; which may also be demonstrated thus: suppose that a body moving uniformly in some known time called 1, and with some velocity called 1, shall describe a space which we will also call 1; then if in the time 1, and with the velocity 1, there be described the space 1, it is plain that in the time T, and with the velocity 1, there will be described the space T; but if in the time T, and with the velocity 1, there be described the space T, then in the time T, and with the velocity V, there will be described the space VT, and that, let the quantities V and T be what they will; and therefore, in all cases, the space will be as $T \times V$.

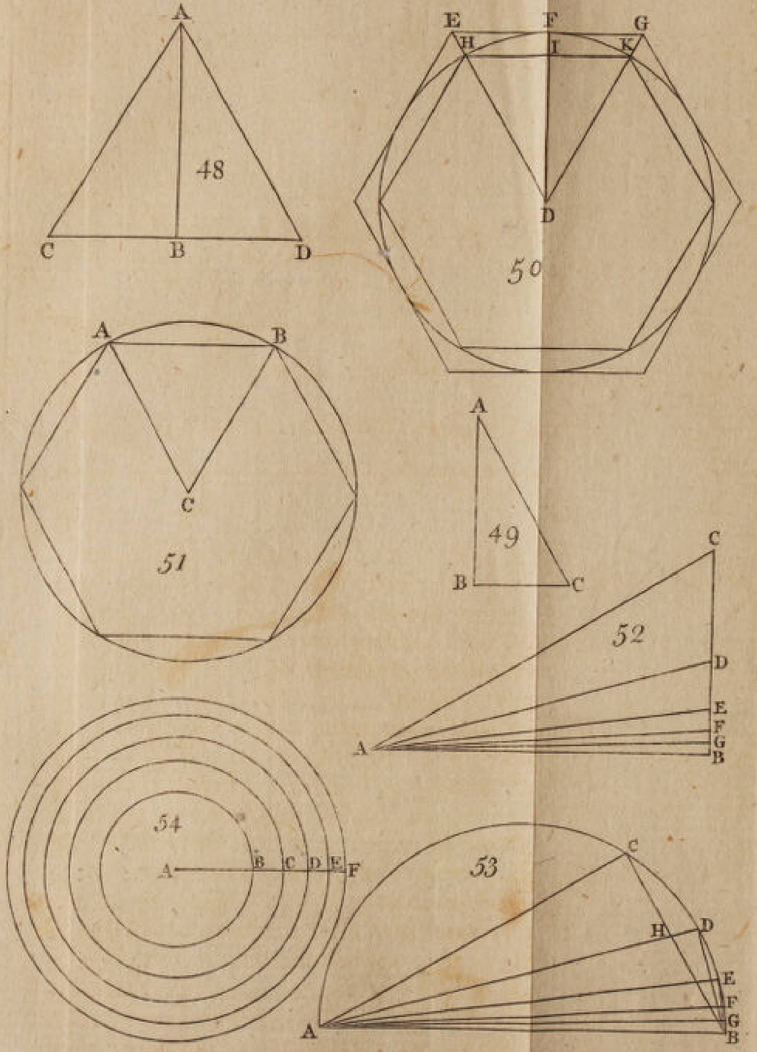
Again

Again, in the ſixth example it was ſhewn that if any moving force as M be directly applied to any body whoſe quantity of matter is Q, the velocity thereby produced will be as $\frac{M}{Q}$: for a future demonſtration whereof, let us ſuppoſe that ſome known force called 1, when applied to ſome quantity of matter called 1, will produce the velocity 1; then will the force 2 applied to the ſame quantity of matter 1 produce the velocity 2; but if the force 2 when applied to the quantity of matter 1 produce the velocity 2, then the ſame force 2 applied to a quantity of matter as 3 will produce a velocity equal to a third part of the former, to wit $\frac{2}{3}$; and for the ſame reaſon the force M applied to a quantity of matter as Q will produce the velocity $\frac{M}{Q}$; and therefore this velocity will always be as $\frac{M}{Q}$.

It is not impoſſible but that ſome of my leſs judicious readers may be inclined to think I have ſpun out this ſubject to too great a length: but I eaſily perſuade myſelf that there are none who have thoroughly conſidered the very great uſefulneſs and importance of this doctrine, eſpecially in Mechanical and Natural Philoſophy, but will readily acquit me of this charge; and the more ſo, becauſe none that I know of have digeſted theſe matters into a ſyſtem, or have written ſo diſtinctly upon them as the importance of the ſubject requires.

to face pa. 351.

Pl. I.



THE

ELEMENTS OF ALGEBRA.

BOOK VIII. PART II.

Of Prisms, Cylinders, Pyramids, Cones, and Spheres.

MANY of the following articles concerning the circle, ſphere, and cylinder, are taken out of *Archimedes*, but demonſtrated another way: and though they have no immediate relation to Algebra, yet as there are not many of them, and as they are a ſort of ſupplement to *Euclid's* Geometry, I have been prevailed upon to inſert them here, for the ſake of thoſe who cannot read *Archimedes*, and for the eaſe of thoſe who can. Moreover, as *Euclid's* doctrine of ſolids is ſomewhat hard of digeſtion as it is delivered in the Elements, I have not ſcrupled to transfer ſome of the chief properties of cones and pyramids into this book, and to demonſtrate them after a more eaſy and ſimple manner. And laſtly, as the menſuration of the circle is abſolutely neceſſary to the menſuration of the cylinder, cone, and ſphere, I ſhall, before I enter upon the reſt, explain what *Archimedes* has delivered upon that head.

A Lemma.

340. *If in a right-angled triangle one of the acute angles be thirty degrees, or a third part of a right one, the oppoſite ſide will be equal to half the hypotenuſe.* (Fig. 48.)

Let *ABC* be a right-angled triangle, right-angled at *B*, and let the angle *BAC* be 30 degrees; I ſay then that the oppoſite ſide *BC* will be half the hypotenuſe *AC*.

For producing *CB* beyond *B* to *D*, ſo that *BD* may be equal to *BC*, and drawing *AD*, the two triangles *ABC* and *ABD* will be ſimilar and equal; therefore the angle *CAD* will be 60 degrees, and the lines *AC* and *AD* will be equal; therefore the other two angles at *C* and *D* will be 60 degrees each, and the triangle *ACD* will be equilateral; therefore the line *BC*, which is the half of *CD*, will alſo be the half of *AC*. *Q. E. D.*

A Lemma. (*Fig.* 49, 50).

341. *Let* ABC *be a right-angled triangle, right-angled at* B; *and ſuppoſing two ſimilar and equilateral polygons, one to be circumſcribed about a circle, and the other to be inſcribed in it, let the angle* BAC *be equal to half the angle at the center ſubtended by a ſide of either polygon: I ſay then that* AB *will be to* BC *as the diameter of the circle to the ſide of the circumſcribea polygon; and that* AC *will be to* BC *as the diameter of the circle is to the ſide of the inſcribed polygon.*

Let *D* be the center of the circle, let *EFG* be a ſide of the circumſcribed polygon, touching the circle in the point *F*, and let *HIK* be the ſide of a like polygon inſcribed, and let *HK* and *EG* be ſuppoſed parallel, ſo as to ſubtend the ſame angle at the center. Draw the lines *DHE*, *DIF*, *DKG*; then will the three triangles *ABC*, *DEF* and *DHI* be ſimilar, having the angles at *B*, *F*, and *I* right, and the angle *BAC* being equal to the angle *EDF* by the ſuppoſition; therefore

AB

AB will be to *BC* as *DF* to *EF*, or as 2*DF* to *EG*, that is, as the diameter of the circle is to the fide of the circumfcribed polygon; and *AC* will be to *BC* as *DH* to *HI*, or as 2*DH* to *HK*, that is, as the diameter of the circle is to the fide of the infcribed polygon. *Q. E. D.*

If the angle BAC *be a 48th part of a right one*, AB *will be to* BC *as the diameter of any circle is to the fide of a regular polygon of* 96 *fides circumfcribed about it, and* AC *will be to* BC *as the diameter is to the fide of a like polygon infcribed.* For if the line *HIK* be the fide of an infcribed regular polygon of 96 fides, the arc *HFK* will be a 96th part of the whole circumference, or a 24th part of a quadrant, and the arc *HF* a 48th part of a quadrant; whence the angle *EDF* or *HDI* will be a 48th part of a right angle.

A THEOREM.

342. *The circumference of every circle is fomewhat more than three diameters.* (Fig. 51.)

Let *AB* be the fide of a regular hexagon infcribed in a circle whofe center is *C*, and draw *AC* and *BC*: then will the angle at *C* in the triangle *ABC* be 60 degrees, as containing a fixth part of the whole circumference; therefore fince *AC* and *BC* are equal, the other two angles at *A* and *B* will be 60 degrees each; therefore the triangle *ABC* will be equiangular, and confequently equilateral; therefore *AB* will be equal to *AC*, and 6*AB* to 6*AC*; but 6*AB* is equal to the perimeter of the infcribed hexagon, and 6*AC* is equal to three diameters; therefore the perimeter of a regular hexagon infcribed in a circle is equal to three times the diameter of that circle: whence it follows that the circumference of the circle itfelf will be fomewhat more than three diameters. *Q. E. D.*

A THEOREM.

343. *If the diameter of a circle be called* 1, *the circumference will be fomewhat lefs than* $3\frac{10}{70}$ *and fomewhat greater than* $3\frac{10}{71}$.

The demonftration of the firft part. (Fig. 52.)

Let *ABC* be a right angle, in which infcribe the lines *AC*, *AD*, *AE*, *AF*, *AG* in the manner following: make the angle *BAC* a third part of a right one, *BAD* a 6th part, *BAE* a 12th part, *BAF* a 24th part, and *BAG* a 48th part: then will *AC* be double of *BC* by the 340th article, and *AB* will be to *BG* as the diameter of any circle is to the fide of a regular polygon of 96 fides circumfcribed about it by the 341ft article. Moreover, as the line *AD* bifects the angle *BAC*, we fhall have as *AB* to *AC* fo *BD* to *DC* by the third of the fixth book of the Elements; and by art. 330*, *AB*+*AC* is to *AB* as *BC* is to *BD*; and by permutation, *AB*+*AC* is to *BC* as *AB* is to *BD*: therefore if *BC* be divided into any number of equal parts, how many foever of thefe parts are contained in the fum of the lines *AB* and *AC*, the fame number of like parts of *BD* will be contained in the line *AB* alone; as if *BC* be divided into 10000 equal parts, and the fum *AB*+*AC* contains 37320 of thofe parts, then if the line *BD* be divided into 10000 equal parts, the line *AB* alone will contain 37320 of them. After the fame manner it may be demonftrated, that whatever parts of *BD* are contained in the fum of the lines *AB*, *AD*, the fame number of like parts of *BE* will be contained in *AB* alone, and fo on: whence we have the following procefs.

1ft. Let *BC* be divided into 10000 equal parts, or (which is the fame thing) let *BC* be called 10000;

* See the Quarto Edition, p. 539.

then

then will AC be 20000, and conſequently AB will be greater than 17320, and $AB+AC$ will be greater than 37320.

2dly, Therefore if $BD=10000$, AB will be greater than 37320, AD greater than 38636, and $AB+AD$ greater than 75956.

3dly, Therefore if $BE=10000$, AB will be greater than 75956, AE greater than 76611, and $AB+AE$ greater than 152567.

4thly, Therefore if $BF=10000$, AB will be greater than 152567, AF greater than 152894, and $AB+AF$ greater than 305461.

5thly, Therefore if $BG=10000$, AB will be greater than 305461: therefore *e converſo*, if AB be ſuppoſed equal to 305461, BG will be leſs than 10000: but it was ſhewn before that AB is to BG as the diameter of any circle is to the ſide of a regular polygon of 96 ſides circumſcribed about that circle; therefore if the diameter of any circle be called 305461, the ſide of ſuch a polygon will be leſs than 10000, and the whole perimeter leſs than 960000; therefore the perimeter of ſuch a polygon will be leſs than the product of the diameter multiplied into $3\frac{10}{70}$ or $\frac{22}{7}$: for $305461\times\frac{22}{7}=960020\frac{2}{7}$: therefore if the diameter of any circle be called 1, the perimeter of a regular polygon of 96 ſides circumſcribed about it will be leſs than $3\frac{10}{70}$; but the circumference of every circle is leſs than the perimeter of any polygon circumſcribed about it; therefore the circumference of the circle will ſtill be leſs than $3\frac{10}{70}$. *Q. E. D.*

The demonſtration of the ſecond part. (Fig. 53.)

Let *ACDEFGB* be a ſemicircle whoſe diameter is *AB*, and in this ſemicircle let the lines *AC*, *AD*, *AE*, *AF*, *AG* be inſcribed in the manner following: make the angle *BAC* a third part of a right one, *BAD* a ſixth part, *BAE* a 12th part, *BAF* a 24th part, and *BAG* a 48th part, and join *BC*, *BD*, *BE*, *BF*, *BG*; then will *AB* be double of *BC*, and *AB* will be to *BG* as the diameter of any circle is to the ſide of a regular polygon of 96 ſides inſcribed. Let *AD* cut *BC* in *H*; and by the demonſtration of the firſt part of this theorem, *AC*+*AB* will be to *CB* as *AC* to *CH*, ſince by the conſtruction the line *AH* biſects the angle *BAC*: but the triangles *ACH* and *ADB* are ſimilar, having the angles at *C* and *D* right, as being in a ſemicircle, and the angle *CAH* being equal to the angle *DAB*; therefore *AC* will be to *HC* as *AD* to *BD*: but it was before demonſtrated, that as *AC* is to *HC* ſo is *AB*+*AC* to *BC*; therefore as *AB*+*AC* is to *BC* ſo is *AD* to *BD*; and whatever parts of *BC* are contained in the ſum of the lines *AB*, *AC*, the ſame number of like parts of *BD* will be contained in the line *AD* alone: whence the following proceſs.

1ſt, Let *BC*=10000; then will *AB*=20000, *AC* will be leſs than 17321, and *AB*+*AC* will be leſs than 37321.

2dly, Therefore if *BD*=10000, *AD* will be leſs than 37321, *AB* will be leſs than 38638, and *AB*+*AD* will be leſs than 75959.

3dly, Therefore if *BE*=10000, *AE* will be leſs than 75959, *AB* will be leſs than 76615, and *AB*+*AE* will be leſs than 152574.

4thly, Therefore if *BF*=10000, *AF* will be leſs than 152574, *AB* will be leſs than 152902, and *AB*+*AF* will be leſs than 305476.

5thly, Therefore if *BG*=10000, *AG* will be leſs than 305476, and *AB* will be leſs than 305640;

therefore

therefore *e converſo*, if *AB* be equal to 305640, *BG* will be greater than 10000: but *AB* is to *BG* as the diameter of any circle is to the ſide of a regular polygon of 96 ſides inſcribed in it; therefore if the diameter of any circle be 305640, the ſide of ſuch an inſcribed polygon will be greater than 10000, and its perimeter greater than 960000; therefore the perimeter of ſuch a polygon will be greater than the product of the diameter multiplied into $3\frac{10}{71}$ or $\frac{223}{71}$: for $305640 \times \frac{223}{71} = 959968 - \frac{8}{71}$: therefore if the diameter of a circle be called 1, the perimeter of a regular hexagon of 96 ſides inſcribed in it will be greater than $3\frac{10}{71}$: but the circumference of every circle is greater than the perimeter of any inſcribed polygon; therefore the circumference of this circle will be greater ſtill than $3\frac{10}{71}$. *Q. E. D.*

Thus then if the diameter of a circle be called 1, the circumference muſt lie between theſe two very narrow limits, to wit, $3\frac{10}{70}$ and $3\frac{10}{71}$: the whole difference of theſe limits is but $\frac{1}{497}$, and therefore, by this method, the circumference of a circle is determined to a 497th part of the diameter.

The moſt compendious way of obtaining the numbers in the laſt article.

344. If any one has a mind to examine the foregoing calculations, it may not be amiſs to let him know, that the hypotenuſes of the triangles *ABD*, *ABE*, *ABF* and *ABG* (*Fig.* 52, 53) may be computed without either ſquaring the greater leg, or ex-

tracting the more confiderable part of the fquare root. As if *AD* (*Fig.* 52.) the hypotenufe of the triangle *ABD* in the firft part be required, having given *AB* 37320 and *BD* 10000, the method I ufe is as follows:

1ft, Whatever the hypotenufe *AD* may be, this is certain, that the greater leg *AB* will be equal to a confiderable part of it; and therefore if *AD* be to be found by a feries, as is ufual in extracting the fquare root, it will be proper to make *AB* the firft term; and hence it is that I call 37320=*AB* my firft root. Again, as the fquare of *AD* is to exceed the fquare of *AB* by the fquare of *BD*, that is, by 100000000; this number I call my firft refolvend, and then doubling my firft root, the product 74640 I call my firft divifor, and fo am prepared for the following operation.

2dly, Thus prepared, I divide my firft refolvend by my firft divifor, and the firft figure of the quotient (for I am concerned for no more at prefent) I find to be 1, which, as it comes out of the place of thoufands, fignifies 1000; this number therefore 1000 I add to my firft root, and fo have 38320 for a more correct or fecond root. The fame number 1000 I add alfo to my firft divifor, and then multiplying the fum 75640 by 1000, the number that was added, I fubtract the product 75640000 from my firft refolvend, and there remains 24360000; this I call my fecond refolvend, and the double of my fecond root, to wit 76640, I call my fecond divifor, and fo proceed to the next operation.

3dly, Now I divide my fecond refolvend by my fecond divifor, and the firft figure of the quotient is 3, which, as it comes out of the place of hundreds, fignifies 300; therefore I add 300 to my fecond root, and fo have 38620 for my third root: the fame number 300 I alfo add to my fecond divifor, and the fum 76940 I multiply by 300, and the product is 23082000, which, being fubtracted from my fecond refolvend, leaves me 1278000 for a third refolvend, and

and the double of my third root, to wit 77240, I have for my third diviſor.

4thly, I divide my third reſolvend by my third diviſor, and the firſt figure of the quotient is 1, which ſignifies 10; therefore I add 10 to my third root, and ſo have a fourth root 38630: moreover adding 10 to my third diviſor, the ſum is 77250, which being multiplied by 10, and the product 772500 being ſubtracted from the third reſolvend, leaves 505500 for the fourth reſolvend, and the double of my fourth root, to wit 77260, makes a fourth diviſor.

5thly and laſtly, I divide my fourth reſolvend by my fourth diviſor, and the neareſt quotient too little is *6*; therefore I add *6* to my fourth root, and ſo have a fifth root, to wit, 38636, which is the neareſt root leſs than the true that can be expreſſed in whole numbers: therefore the hypotenuſe *AD* is greater than 38636.

The reaſon of theſe operations will not be difficult to any one who thoroughly underſtands the foundation of the common method of extracting the ſquare root.

Van Ceulen's *numbers expreſſing the circumference of a circle whoſe diameter is* 1.

345. This method of *Archimedes* is capable of being purſued to any degree of exactneſs required: nay *Ludolf Van Ceulen* has computed the circumference of a circle to no fewer than 36 places, upon a ſuppoſition that the diameter is unity. His numbers expreſſing this circumference are,

3.1415 9265 3589 7932 3846 2643 3832 7950 288 +.

But ſince the invention of fluxions by its great author Sir *Iſaac Newton*, he (Sir *Iſaac*) has from this method drawn serieſes almoſt infinitely more expeditious than the biſections of *Archimedes* or *Van Ceulen*, whereby the circumference of a circle may be computed to 12 or 13 places in little more than half an hour's time,

 as

as Doctor *Halley* from his own experience assures us.

Note, that *Metius's* proportion of the diameter of a circle to the circumference is as 113 to 355, the most accurate of any in such small numbers. (See Schol. 1. in art. 179*).

Why the circle cannot be squared geometrically.

346. If, having given the diameter or semidiameter of any circle, a right line could be found exactly equal to the circumference, whether such a line could be expressed by numbers or not, the circle might be squared as well as any right-lined figure whatever, that is, a square might be constructed whose area would be equal to that of the circle, which I thus demonstrate.

Let $2r$ represent the diameter of any circle, and $2c$ the circumference; then will rc, the product of the *radius* into the semicircumference, be its area, by cor. 4 in art. 311 †. Let now x be the side of a square whose area is equal to that of the circle, and we shall have $xx=rc$; whence x will be a mean proportional between r and c, and may be found by the 13th of the sixth book of the Elements. But it is impossible upon *Euclid's postulata*, having given the diameter or semidiameter of any circle, to find a right line exactly equal to the circumference; and therefore the circle cannot be squared upon the same foundation on which we are taught to square all right-lined figures; and hence it is that we say, the circle cannot be squared geometrically. But as the three *postulata* of *Euclid*, how simple soever they may appear in theory, are never a one of them capable of being perfectly executed, but that errors must necessarily arise in drawing and producing lines, in taking the distances of points, *&c.*; and as from these errors others must necessarily arise in subsequent

* See the Quarto Edition, p. 282. † Ibid. p. 504.

operations;

operations; and laſtly, as the circumference of a circle may be had from the diameter in numbers, to any aſſignable degree of exactneſs, it follows that in practice, a circle is as capable of being ſquared as any other figure whatever that is not a ſquare.

Corollaries drawn from art. 343.

347. From the 343d article may be deduced ſeveral corollaries, ſome of the moſt uſeful whereof are inſerted here as follows:

1ſt, *The diameter of every circle is to the circumference as 7 to 22 nearly:* for 1 is to $3\frac{10}{70}$ or $\frac{22}{7}$ as 7 to 22.

2d, *If* d *be the diameter of any circle, its area will be* $\frac{11dd}{14}$: for as 7 is to 22, ſo is d the diameter to $\frac{22d}{7}$ the circumference; and if $\frac{d}{2}$ the *radius* be multiplied into $\frac{11d}{7}$ the ſemicircumference, the product $\frac{11dd}{14}$ will be the area, by corollary 4 in art. 311*.

3d, *Hence we have a ready way, having the diameter of any circle given to find its area, and* vice verſa, *without the mediation of the circumference, by ſaying, as* 14 *is to* 11, *ſo is the ſquare of the given diameter to the area ſought. But if the area be given, in order to find the diameter, the proportion muſt be reverſed, by ſaying as* 11 *is to* 14, *ſo is the given area to a fourth, which fourth number will be the ſquare of the diameter, and its ſquare root the diameter itſelf.*

4th, Arguing as in the two laſt corollaries, *If the diameter of a circle be to the circumference as* a *to* b, *then the ſquare of the diameter of any circle will be to its area as* 4a *to* b; *and* vice verſa, *the area will be to the ſquare of the diameter as* b *to* 4a.

* See the Quarto Edition, p. 504.

5th, *The circumferences of all circles are as their diameters or semidiameters, and their areas as the squares of the diameters or semidiameters.* For if d and e be the diameters of two circles, their circumferences will be $\frac{22d}{7}$ and $\frac{22e}{7}$; and $\frac{22d}{7}$ is to $\frac{22e}{7}$ (dropping the common denominator 7, and the common factor 22) as d to e. Again, the area of the circle whose diameter is d is $\frac{11dd}{14}$ as in the second corollary; and for the same reason, the area of the other circle whose diameter is e is $\frac{11ee}{14}$; and $\frac{11dd}{14}$ is to $\frac{11ee}{14}$ as dd to ee; therefore the circumferences of all circles are as their diameters, and their areas as the squares of their diameters. And since the halves of all quantities are as the wholes, and the squares of the halves as the squares of the wholes, it follows also that the circumferences of circles are as their semidiameters, and their areas as the squares of the semidiameters.

6th, *If there be three circles such, that the sum of the squares of the semidiameters of two of them is equal to the square of the semidiameter of the third; I say then that the areas of the two first circles put together will be equal to the area of the third.* For let a, b, c represent the semidiameters of the three circles, and let $a^2+b^2=c^2$: since then the semidiameter of the first circle is a, the diameter will be $2a$, and the square of the diameter $4aa$: but as 14 is to 11 so is $4aa$ to $\frac{44a^2}{14}$ or $\frac{22a^2}{7}$; therefore the area of the first circle will be $\frac{22a^2}{7}$ by the third corollary; and for the same reason, the areas of the other two circles will

will be $\frac{22b^2}{7}$ and $\frac{22c^2}{7}$: but $a^2+b^2=c^2$ *ex hypothesi*: therefore $\frac{22a^2}{7}+\frac{22b^2}{7}=\frac{22c^2}{7}$.

N. B. This last corollary is not demonstrated in the 31st of the sixth book of the Elements, as some may imagine, that demonstration not reaching farther than right-lined figures.

The following easy problems may serve to shew some uses of the following corollaries.

PROBLEM I.

348. *To find the proportion between the diameter of any circle and the side of an equal square.*

Call this diameter 1, and by the second corollary in the foregoing article, the area of this circle will be $\frac{11}{14}$ nearly; and the side of a square whose area is $\frac{11}{14}$ will be $\sqrt{\frac{11}{14}}$: therefore the diameter of any circle is to the side of an equal square as 1 to $\sqrt{\frac{11}{14}}$. But because this fraction $\frac{11}{14}$, though it serves well enough for common purposes, is not accurately true, and because its square root cannot be accurately expressed in numbers neither, to reduce the error (for there must be an error) to a more simple point, let c be the circumference of a circle whose diameter is 1; and the area of such a circle, that is, the product of the *radius* into the semi-circumference, will be $\frac{1}{2}\times\frac{c}{2}=\frac{c}{4}$; and the side of an equal square will be $\sqrt{\frac{c}{4}}$: but, according to *Van Ceulen*, $c=3.1415926536$, and $\frac{c}{4}=.7853981634$,

and

and $\sqrt{\frac{c}{4}} = .88623$; therefore the diameter of a circle is to the ſide of an equal ſquare as 1 to .88623. or as 100000 is to 88623: ſuppoſe the proportion to be that of 100000 to 88625, which makes but an error of 2 in the fifth place, and then multiplying both terms by 8, the proportion will be that of 800000 to 709000, or of 800 to 709; therefore *As* 800 *is to* 709, *ſo is the diameter of any circle to the ſide of an equal ſquare, nearly true to five places.*

N. B. If the diameter of a circle be 9 yards, the ſide of an equal ſquare found as above will not err an hundredth part of an inch.

PROBLEM 2.

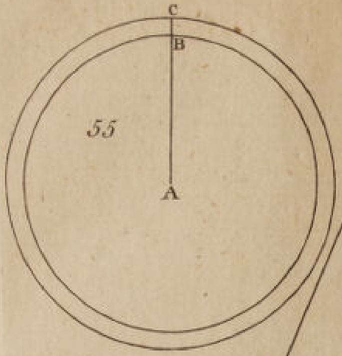
349. *To find the ſemidiameter of a circle that will comprehend within its circumference the quantity of an acre of land.*

An acre of land contains 4840 ſquare yards, and therefore this muſt be the area of our circle. Say then, as 11 to 14, ſo 4840 to 6160; and this laſt number will be the ſquare of the diameter, by the third corollary in art. 347; whence the diameter itſelf will be 78 .486 yards, and the ſemidiameter 39 .243 yards, that is 39 yards $8\frac{1}{7}$ inches nearly.

PROBLEM 3.

350. *Let a ſtring of a given length be diſpoſed into the form of a circle: It is required to find the area of this circle.*

Let c be the length of the ſtring, and conſequently the circumference of the circle made by it, and the diameter of the circle will be $\frac{7c}{22}$, the ſemidiameter $\frac{7c}{44}$, and the area $\frac{7cc}{88}$. Suppoſe now this ſtring to be diſpoſed into the form of a ſquare, and the ſide of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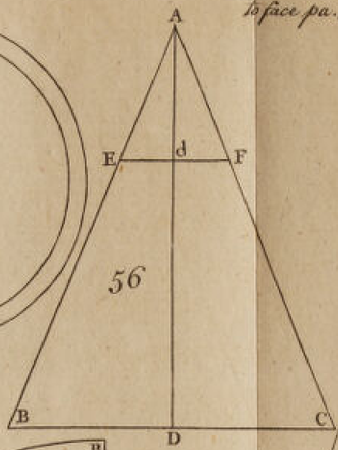
A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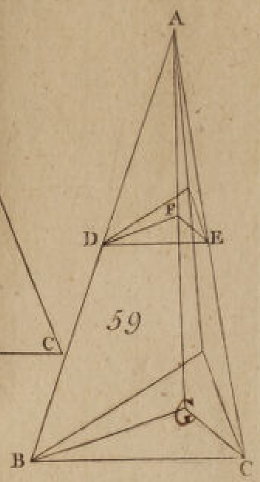
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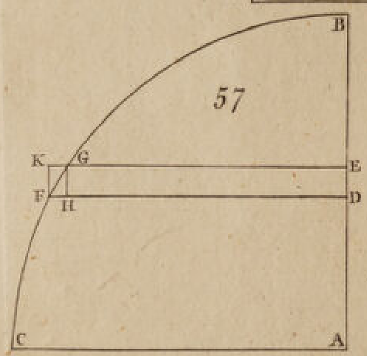
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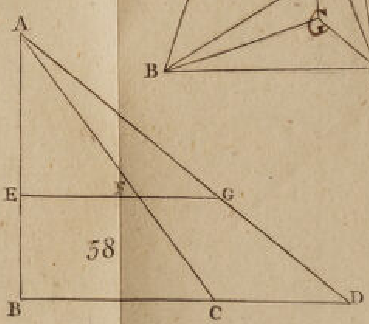
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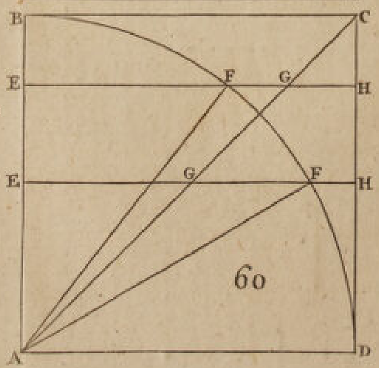
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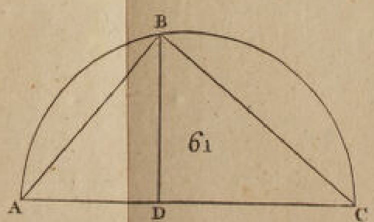
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of this ſquare would be $\frac{c}{4}$, and its area $\frac{cc}{16}$, and the area of the ſquare would be to the area of the circle as $\frac{cc}{16}$ is to $\frac{7cc}{88}$, that is, as $\frac{1}{16}$ is to $\frac{7}{88}$, or as 11 to 14: therefore, *As* 11 *is to* 14, *ſo is the area comprehended by the ſtring when in form of a ſquare, to the area comprehended by the ſame ſtring when in form of a circle.* Q. E. I.

N. B. By the anſwer here given it appears, that if c be the circumference of any circle, its area will be $\frac{7cc}{88}$; and conſequently that *As* 88 *is to* 7, *ſo is the ſquare of the circumference of any circle to its area nearly.*

PROBLEM 4.

351. *It is required to divide a given circle into any number of equal parts by means of concentric circles drawn within it.* (Fig. 54.)

Let A be the center, and AF be the ſemidiameter of the circle given, and let it be required to divide the area of this circle into five equal parts by means of four concentric circles deſcribed within the former, and cutting the line AF in the points B, C, D, E, that is, let the area of the circle AB, and the areas of the *annuli* BC, CD, DE, and EF be ſuppoſed all equal; then the circle AB will be $\frac{1}{5}$ of the whole, the circle AC $\frac{2}{5}$, the circle AD $\frac{3}{5}$, *&c.*; and the area of the circle AF will be to the area of the circle AB as 5 to 1: but the area of the circle AF is to the area of the circle AB as the ſquare of the ſemidiameter AF is to the ſquare of the ſemidiameter AB, by cor. 5. in art. 347; therefore AF^2 is to AB^2 as 5 to 1: but AF^2 is given by the ſuppoſition; therefore AB^2, and conſequently AB the ſemidiameter of the inmoſt circle is given. In like manner AF^2 is to AC^2 as 5 to 2; whence *AC* the semidiameter of the next concentric circle is given; and so of the rest. *Q. E. I.*

Problem 5.

352. *Whoever makes a tour round the earth, must necessarily take a larger compass with his head than with his feet: The question is, how much larger?*

Let *A* (*Fig.* 55.) represent the center of the earth, *AB* its semidiameter, *BC* the traveller's height, *AC* the semidiameter of the circle described by his head: let also *b* represent the circumference of the circle whose semidiameter is *AB*, and *c* the circumference of the circle whose semidiameter is *AC*, and *c*—*b* will be the difference we are now enquiring into, which may be thus determined.

By the fifth corollary in art. 347, *AC* is to *AB* as *c* is to *b*; and by division of proportion, *BC* is to *AB* as *c*—*b* is to *b*; and alternately, *BC* is to *c*—*b* as *AB* is to *b*. Let *d* be the circumference of a circle whose semidiameter is *BC*, and *BC* will be to *d* also as *AB* to *b*; therefore *BC* is to *d* as *BC* is to *c*—*b*; therefore *c*—*b*=*d*; that is, *The traveller's head will pass through more space than his feet by the circumference of a circle whose semidiameter is his own length*: as if the man be 6 feet high, his head will travel farther than his heels by 37 feet $8\frac{4}{7}$ inches nearly, and that whether the semidiameter *AB* be greater or less, or nothing at all.

Problem 6.

353. *It is required, having given the depth and the diameter of the base of any cylindrical vessel, to find its content in ale gallons.*

Here it must be observed, that in the mensuration of a solid, all its dimensions must be taken in the same kind of measure: as, if any one dimension be taken in inches, the rest must be taken so too, and then the number representing the content of any solid

solid will be the number of cubic inches to which that solid is equivalent.

Let then a be the given altitude of the cylindrical vessel to be measured, d the diameter of its base, and by the second corollary in art. 347, $\frac{11dd}{14}$ will give a number of square inches equal to the base, and this area multiplied into the altitude a, will give $\frac{11\,add}{14}$, a number of cubic inches equal to the solid content of the vessel: but 282 cubic inches constitute an ale gallon; and therefore if $\frac{11\,add}{14}$, the solid content of the vessel, be divided by 282, the quotient, to wit, $\frac{11\,add}{3948}$, will be the number of gallons therein contained. Instead of 3948 put 3949, which will make no considerable difference in so great a denominator, and the fraction $\frac{11\,add}{3949}$ (dividing both the numerator and denominator by 11) will be reduced to $\frac{add}{359}$: whence the following canon:

Having taken both the depth, and the diameter of the base in inches, multiply the square of the diameter into the depth of the vessel, and the product divided by 359 will give the number of gallons required.

N. B. This substitution of 3949 instead of 3948 corrects about $\frac{7}{10}$ of the error that would otherwise have been committed in supposing the square of the diameter of the base to be to its area as 14 to 11.

PROBLEM 7.

354. *To measure a frustum of a cone, whose perpendicular altitude and the diameters of the two bases are given.*

3 *N. B.*

N. B. By a frustum of a cone I mean a cone having its top cut off by a plane parallel to the base.

Let the isosceles triangle *ABC* (*Fig* 56.) represent the section of a cone made through its *axis* *AD*, and let *EF* be any line parallel to the base *BC*, cutting *AB* in *E*, *AC* in *F*, and the *axis* *AD* in *d*; then will the trapezium *BEFC* be the section of a frustum of this cone whose perpendicular altitude is *Dd*. Call *BC*, the diameter of the greater base, g; *EF*, the diameter of the lesser base, l; and *Dd*, the height of the frustum, b: call also *AD*, the unknown altitude of the whole cone, x; and consequently *Ad*, the altitude of the cone to be cut off, $x-b$; and from the similar triangles *ABC*, *AEF* we have this proportion; *AD* is to *Ad* as *BC* is to *EF*, that is, according to our notation, x is to $x-b$ as g to l; whence, by multiplying extremes and means, we have $gx-gb=lx$, and x, or the altitude of the cone, equal to $\frac{gb}{g-l}$. In like manner if from the value of x we subtract b, the altitude of the frustum, we shall find *Ad*, or the height of the cone to be cut off, equal to $\frac{lb}{g-l}$. Now the solid content of every cone is found by multiplying the base into a third part of its altitude; therefore since the base of the cone *ABC* is $\frac{11\,gg}{14}$, and its altitude $\frac{gb}{g-l}$, its solid content will be $\frac{g^3}{g-l} \times \frac{b}{3} \times \frac{11}{14}$: in like manner the solid content of the cone *AEF* will be $\frac{l^3}{g-l} \times \frac{b}{3} \times \frac{11}{14}$: subtract the latter cone from the former, and there will remain the solid content of the frustum *BEFC*

BEFC equal to $\frac{g^3-l^3}{g-l} \times \frac{b}{3} \times \frac{11}{14}$: but the fraction $\frac{g^3-l^3}{g-l}$ may be reduced to an integer by dividing the numerator by the denominator, and the quotient will be $gg+gl+ll$; therefore the solid content of the frustum *BEFC* will now be expressed thus, $\overline{gg+gl+ll} \times \frac{b}{3} \times \frac{11}{14}$. Whence we have the following canon:

Add the squares and the rectangle of the two given diameters together; multiply the sum into a third part of the given altitude, and the product will be the frustum of a pyramid of the same height having square bases whose sides are equal to the two diameters given; and as 14 *is to* 11 *so will this frustum be to the frustum sought.* Q. E. I.

N. B. 1st. Since a cone differs nothing from a frustum of a cone whose lesser base is equal to nothing, if l be made equal to nothing in the foregoing canon, it ought to give the solid content of a cone whose height is b, and the diameter of whose base is g: and so it will; for then $\overline{gg+gl+ll} \times \frac{b}{3} \times \frac{11}{14}$ becomes $\frac{11gg}{14} \times \frac{b}{3}$.

2dly, Since a cylinder may be considered as a frustum of a cone whose bases are equal, if l be made equal to g in the foregoing canon, it ought to give the solid content of a cylinder whose height is b, and the diameter of whose base is g: and so we find it will; for $\overline{gg+gl+ll} \times \frac{b}{3} \times \frac{11}{14}$ in this case becomes $3gg \times \frac{b}{3} \times \frac{11}{14} = \frac{11}{14}ggb$.

3dly, If the leſſer baſe of the fruſtum be ſuppoſed to be increaſed till it becomes equal to the greater; and if, on the other hand, the greater baſe be ſuppoſed to be diminiſhed till it becomes equal to that which was the leſſer baſe before, the ſolid content of the fruſtum will be the ſame as at the firſt; and therefore, if the foregoing canon be juſt, it ought not to be altered by changing the quantities g and l one for the other: which is true; for $gg+gl+ll$ by this means only becomes $ll+lg+gg$, which is the ſame quantity.

In ſolving this laſt problem it is taken for granted that every cone is the third part of a cylinder having the ſame baſe and height; which may ſafely be done, ſince *Euclid* has demonſtrated it in the 10th of the twelfth book of the Elements: but becauſe *Euclid's* doctrine of ſolids is not ſo eaſy to the imaginations of young beginners, I ſhall (in another place) give another demonſtration of this propoſition, independently of any thing that has here been ſaid.

Lemma. (*Fig.* 57.)

355. *Let* ABC *be any curvilinear ſpace comprehended between two ſtraight lines* AB *and* AC *at right angles to each other, and a curve as* BC *concave towards* AB; *and from any two points* D *and* E *in the line* AB *let the lines* DF *and* EG *be drawn parallel to the baſe* AC, *and terminating in the curve at the points* F *and* G, *and compleat the parallelogram* DEGH: *then it is plain that the curvilinear ſpace* DEGF *will be greater than the parallelogram* DEGH. *But what I here propoſe to demonſtrate is, that if the line* EG *be ſuppoſed to move towards* DF *in a poſition always parallel to itſelf, and at laſt to coincide with* DF, *the nearer* EG *approaches to* DF, *the nearer will the ratio of the curvilinear ſpace* DEGF *to the parallelogram* DEGH *approach towards a ratio of equality, and that it will at laſt terminate in a ratio of equality when* EG *coincides with* DF.

For,

For, completing the parallelogram *GHFK*, the parallelogram *DEKF* will be to the parallelogram *DEGH* upon the ſame baſe as *DF* is to *EG*; therefore the curvilinear ſpace *DEGF* will be to the parallelogram *DEGH* in a leſs ratio than that of *DF* to *EG*; that is, though that ſpace be greater than this parallelogram, yet the ratio of the former to the latter is a leſs ratio than that of *DF* to *EG*: but the nearer the line *EG* approaches towards *DF*, the nearer will the ratio of *DF* to *EG* approach towards a ratio of equality, and it will at laſt become a ratio of equality when *EG* coincides with *DF*; therefore *a fortiori*, the ultimate ratio of the curvilinear ſpace *DEGF* to the parallelogram *DEGH* will be a ratio of equality.

Hence it follows, that if we ſuppoſe the line AB *to be divided into a certain number of parts, ſuch as* DE, *and theſe parts to be made the baſes of ſo many inſcribed parallelograms, ſuch as is the parallelogram* DG, *the more there are of theſe parallelograms, the nearer will the ſum of all the curvilinear ſpaces* DEGF, *that is, the whole curvilinear ſpace* ABC, *approach towards the ſum of all the inſcribed parallelograms; and if we ſuppoſe the baſes of theſe parallelograms to be diminiſhed, and ſo their number to be augmented* ad infinitum, *they will make up the whole curvilinear ſpace* ABC; *ſo that the ſpace* ABC *will be to the ſum of all the inſcribed parallelograms ultimately in a ratio of equality.* For, letting aſide the parts infinitely near the point of interſection *B*, which will be of no conſequence in the account, let the parallelogram *DEGH* be that which, of all the reſt, differs moſt from its correſpondent curvilinear ſpace *DEGF*; and the conſequence will be that the curvilinear ſpace *ABC* will be to the ſum of all the inſcribed parallelograms in a leſs ratio than that of the ſpace *DEGF* to the ſpace *DEGH*: but even this ratio becomes at laſt a ratio of equality, when *DE* is infinitely ſmall, by this lemma: whence it follows *a fortiori*, that the ultimate ratio of the curvilinear ſpace *ABC* to the ſum of all the inſcribed parallelograms will be a ratio of equality.

I thought myſelf obliged to demonſtrate this propoſition, becauſe I have known it objected, that though the difference between any particular parallelogram and its correſpondent curvilinear ſpace be allowed to be infinitely ſmall when the common baſe is ſo, yet how do we know but that an infinite number of theſe differences may amount to a finite quantity? and if ſo, then the whole curvilinear ſpace cannot be ſaid to be to the ſum of all the inſcribed parallelograms in a ratio of equality. To this I anſwer, that it muſt be the buſineſs of Geometry to determine whether an infinite number of theſe infinitely ſmall differences amount to a finite quantity or not; and by the demonſtration here given it appears that they do not, but that the ſum of all theſe differences actually diminiſhes as their number increaſes, and at laſt comes to nothing when their number is infinite.

A Lemma. (*Fig.* 57.)

356. *Suppoſing the line* AB *ſtill to keep its place, let us imagine the whole ſpace* ABC *to turn round it, ſo as to deſcribe or generate a ſolid whoſe* axis *is* AB, *and the ſemidiameter of whoſe baſe is* AC, *and the inſcribed parallelograms will at the ſame time by their common motion deſcribe ſo many thin cylindric* laminæ, *which, taken all together, will be leſs than the ſolid generated by the ſpace* ABC; *but, the more they are in number, the nearer they will approach to it, and their circular edges will at laſt terminate in the curve ſurface of the ſolid when their number is infinite.*

For, completing the parallelogram *GHFK* as before, the *lamina* generated by the parallelogram *DK* will be to the *lamina* generated by the parallelogram *DG* as the ſquare of *DF* is to the ſquare of *EG*, all circles being as the ſquares of their ſemidiameters; therefore the *lamina* generated by the curvilinear ſpace *DEGF* will be to the *lamina* generated by the parallelogram

lelogram DG in a leſs ratio than that of DF^2 to EG^2: but when D and E coincide, DF will be equal to EG, and the ſquare of DF to the ſquare of EG; therefore, in this caſe, every particular cylindric *lamina* will be the ſame with a correſpondent *lamina* of the ſolid; and *componendo*, all the cylindric *laminæ* will conſtitute the ſolid itſelf. This may alſo be further evident by applying the demonſtration in the laſt lemma to this caſe. Therefore we need not ſcruple to ſuppoſe round ſolids, generated after the ſame manner as this is, to be made up of an infinite number of infinitely thin cylindric *laminæ*: *Nay even the cone itſelf may be conſidered as being ſo conſtituted: for if we ſuppoſe* BC *to be a ſtraight line inſtead of a curve, the reaſoning of the laſt article and this will equally ſucceed; in which caſe, the ſpace* ABC *will be a triangle, and the figure generated a cone.*

If a ſolid be made up of a finite number of priſmatic or cylindric laminæ, *decreaſing in their diameters as they are farther and farther diſtant from the baſe, the ſurface of ſuch a ſolid muſt neceſſarily aſcend by ſteps: but the thinner theſe elementary* laminæ *are (ſuppoſing their thinneſs to be compenſated by a greater number), the narrower and the ſhallower theſe ſteps will be, ſo as to terminate at laſt in a regular ſurface when their number is infinite.*

A THEOREM,

357. *All iſoſceles cones of equal heights are as their baſes; that is, the ſolid content of any one iſoſceles cone is to the ſolid content of any other of an equal height, as the baſe of the former cone is to the baſe of the latter.*

Note, that by an iſoſceles cone I mean a cone whoſe baſe is a circle, and whoſe vertex is every-where equally diſtant from the circumference of the baſe; and by an iſoſceles pyramid is meant a pyramid having a regular polygon for its baſe, and whoſe vertex is equally diſtant from all the angles of that polygon: laſtly, by iſoſceles *prisms and cylinders are meant such as have equal and regular polygons and circles for their bases, and are so constituted, that a right line joining the centers of their two bases is perpendicular to them.*

Let *ABC* (*Fig.* 58.) be a right-angled triangle at *B*, and producing the base *BC* out to *D*, join *AD*; let also the line *EFG* be drawn any where within the triangle parallel to the base *BCD*, so as to cut *AB* in *E*, *AC* in *F*, and *AD* in *G*: then will *EF* be to *BC* as *EG* is to *BD*, since both are as *AE* to *AB* by similar triangles: therefore, if they be taken alternately, *EF* will be to *EG* as *BC* to *BD*, and EF^2 to EG^2 as BC^2 to BD^2. This being allowed, let the triangle *ABD* be supposed to turn round the fixed side *AB*, so as to generate a cone whose *axis* is *AB*; then will the triangle *ABC* generate another cone having the same common altitude *AB*. Let both these cones be considered as constituted of an infinite number of infinitely thin cylindric *laminæ*, and let *EF* represent indifferently the semidiameter of any one of these *laminæ* belonging to the cone *ABC*; then will *EG* be the semidiameter of a correspondent *lamina* belonging to the cone *ABD*; and the *lamina* whose semidiameter is *EF* will be to the *lamina* whose semidiameter is *EG*, having the same thickness, as EF^2 is to EG^2, or (according to what is already demonstrated) as BC^2 is to BD^2; that is, every particular *lamina* of the cone *ABC* will be to a like *lamina* of the cone *ABD* as the base of the former cone is to the base of the latter; therefore *componendo*, the whole cone *ABC* will be to the whole cone *ABD* as the base of the former is to the base of the latter: but the planes *ABC* and *ABD* may be so constituted as to generate any two isosceles cones whatever that have equal heights; therefore if the heights of two isosceles cones be equal, these cones will be to each other as their bases. *Q. E. D.*

A The-

A THEOREM.

358. *Every isosceles pyramid is equal to an isosceles cone of an equal base and height.*

Let *P* represent any isosceles pyramid, and *C* an isosceles cone of an equal base and height: I say then that the pyramid *P* will be equal to the cone *C*.

To demonstrate this, imagine the pyramid *P* to have a cone, as *c*, inscribed in it, so as to touch every side of the pyramid in so many lines of contact, and imagine the circumscribing pyramid, and consequently the inscribed cone, to be constituted of an infinite number of infinitely thin *laminæ*; and every *lamina* of the circumscribing pyramid will be to a correspondent *lamina* of the inscribed cone as the base of the pyramid is to the base of the cone. For the plane of every *lamina* that constitutes the pyramid will be a polygon similar to the base, and the plane of every correspondent *lamina* that constitutes the inscribed cone will be a circle inscribed in such a polygon: therefore *componendo*, all the *laminæ* constituting the pyramid *P* will be to all those that constitute the cone *c*, that is, the whole pyramid *P* will be to the whole cone *c* as the base of *P* is to the base of *c*: but the cone *c* is to the cone *C* of an equal height, as the base of *c* is to the base of *C*. Since then *P* is to *c* as the base of *P* is to the base of *c*, and *c* is to *C* as the base of *c* is to the base of *C*, it follows *ex æquo* that *P* is to *C* as the base of *P* is to the base of *C*: but the base of *P* is equal to the base of *C* by the supposition; therefore the pyramid *P* is equal to the cone *C*, having an equal base and altitude. *Q. E. D.*

COROLLARY.

Hence it follows, that whether cones be compared with cones, or cones with pyramids, or pyramids with pyramids, all such as have equal heights will be to one another as their bases. For cones are so by the last article, and pyramids are equal to cones having equal bafes and heights by this: I mean ifofceles pyramids and ifofceles cones.

Scholium.

As the folid content of a prifm or cylinder of a given perpendicular altitude depends upon the quantity, and not upon the figure of the bafe, fo by the demonftration of this propofition it appears, that the folid content of an ifofceles pyramid or cone of a given perpendicular altitude depends upon the quantity, and not upon the figure of the bafe: let the perpendicular altitude and the area of the bafe be the fame, and the quantity of the folid will continue the fame, whether that bafe be in the form of a triangle, or a fquare, or a polygon, or a circle. Other pyramids and cones will be confidered in another place.

A Lemma.

359. *If from the center of any cube ftraight lines be imagined to be drawn to all its angles, the cube will by this means be diftinguifhed into as many equal ifofceles pyramids as it has fides, to wit 6, whofe bafes will be in the fides of the cube, and whofe common vertex will be in the center.*

For if from a point above any right-lined plain figure lines be drawn to all its angles, and then the interftices of thefe lines be imagined to be filled up with triangular planes, the folid figure thus inclofed will be a pyramid; therefore, by the lines above defcribed, the cube will be diftinguifhed into as many pyramids as it hath fides. And that thefe pyramids will be all equal and ifofceles, is evident: for firft, their bafes will be all equal from the nature of the cube; and in the next place, their heights will be all equal from the nature of the center, which is fuppofed to be equally diftant from all the fides of the cube; and laftly, as this center muft alfo be equally diftant from all its angles, it follows that thefe pyramids muft be all ifofceles pyramids. *Q. E. D.*

Corol-

COROLLARY.

Hence every one of these pyramids will be the sixth part of the whole cube.

A THEOREM.

360. *Every isosceles pyramid or cone is a third part of an isosceles prism or cylinder having an equal base, and an equal perpendicular height.*

Let a be the perpendicular altitude of any isosceles pyramid or cone, and let b be the area of its base, both taken according to the directions given in art. 353: imagine also a cube whose side, that is the side of whose square base, is $2a$; then will $4a^2$ be the area of the base, and $8a^3$ the solid content of this cube: and if, from the center of the cube, lines be imagined to be drawn to the four angles of the base, they will be the outlines of an isosceles pyramid whose base is the same with the base of the cube, to wit, $4a^2$, and whose perpendicular altitude is a; whence the solid content of this pyramid will be $\frac{8a^3}{6}$ or $\frac{4a^3}{3}$, by the corollary in the last article: but as this pyramid has the same height a with the pyramid proposed, these two pyramids will be to one another as their bases, by the corollary in the last article but one: hence the solid content of the pyramid proposed will easily be had by saying, as $4a^2$, the base of the pyramid within the cube, is to b the base of the pyramid proposed, so is $\frac{4a^3}{3}$ the solid content of the former, to a fourth, to wit $\frac{ab}{3}$, which will be the solid content of the latter; therefore the solid content of an isosceles pyramid or cone whose base is b, and whose altitude is a, is found to be $\frac{ab}{3}$: but the solid content of an isosceles prism or cylinder having an an equal base and height is ab; therefore every isosceles pyramid or cone is a third part of an isosceles prism or cylinder having an equal base and an equal perpendicular altitude. *Q. E. D.*

Corollary 1.

Hence the solid content of an isosceles pyramid or cone is found by multiplying the area of the base into a third part of the perpendicular altitude, or else by multiplying the area of the base into the whole altitude, and then taking a third part of the product.

Corollary 2.

Hence also it follows that all isosceles pyramids and cones upon equal bases are to one another as their heights.

A Lemma.

361. *If a pyramid of any kind be cut by a plane parallel to its base, the quantity of the section, or (which is all one) the quantity of the base of the pyramid cut off, will always be the same, let the figure of the pyramid be what it will, so long as the base and perpendicular altitude of the whole pyramid and the perpendicular altitude of the pyramid cut off continue the same: in which case, the perpendicular distance of the plane of the section from the plane of the base will also be the same.* (See *Fig.* 59.)

Let A be the vertex of the pyramid, and let BC be any one side of the base; let the lines AB and AC be cut by the plane of the section in the points D and E respectively, and let AFG be the perpendicular altitude of the whole pyramid, cutting the plane of the section in F, and the plane of the base in G, both produced if need be: join FD, FE, GB, GC: then since the base of the pyramid cut off will always be similar to the base of the whole pyramid, whereof DE and BC are correspondent sides; and since all similar plain figures are to each other as the squares of their cor-
respondent

respondent sides by the 20th of the sixth book of the Elements, it follows that the base whose side is DE will be to the base whose side is BC as DE^2 to BC^2, that is, by similar triangles, as AD^2 is to AB^2, or as AF^2 is to AG^2. Since then as AG^2 is to AF^2 so is the base of the whole pyramid to the base of the pyramid cut off; so long as the three first continue the same, the last must also continue the same. *Q. E D.*

COROLLARY.

Since the number of sides of the pyramid is not concerned in the demonstration of this proposition, which will be equally true, be the number of sides what it will, it must also be true of the cone, which is nothing else but a pyramid of an infinite number of sides, let the shape of the cone be what it will; that is, whether AG *the perpendicular altitude of the cone passes through the center of the base or not.*

A THEOREM.

362. *If a prism or cylinder of any kind be described by the motion of a plain figure ascending uniformly in a horizontal position to any given height, the quantity of the solid thus generated will be the same, whether the describing plane ascends directly or obliquely to the same height; and consequently all prisms and cylinders of what kind soever, that have equal bases and equal perpendicular heights, are equal, whether they stand upon those bases erect or reclining.*

For the better conceiving of this, let the describing plane be made, not to ascend all the way, but sometimes to ascend perpendicularly, and sometimes to move laterally or edgeway, and that by turns: then it is plain that the quantity of solid space, or rather the sum of all the solid spaces thus described, will amount to no more than if the describing plane had ascended all the way perpendicularly to the same height. Let the times of these alternate motions wherein they are performed be diminiſhed and their number be increaſed *ad infinitum*, and they will terminate at laſt in an uniform oblique motion, and the ſolid generated by this motion will be equal to a ſolid generated by a perpendicular motion of the ſame plane to the ſame height. *Q. E. D.*

N. B. What has here been demonſtrated concerning priſms and cylinders, may be further illuſtrated by ſliding a pack of cards, or a pile of halfpence, out of an erect into an oblique poſture; whereby it may eaſily be ſeen, that neither the baſe or the perpendicular altitude, nor the quantity of the ſolid, can be affected by this change of poſture; but the finer, that is the thinner, theſe conſtituent *laminæ* are, the nearer they will repreſent an oblique ſolid.

A Theorem.

363. *All pyramids and cones of what kind ſoever, that have equal baſes and equal perpendicular heights, are equal.*

To evince this, let us imagine two plain figures (whether ſimilar or diſſimilar to each other it matters not) to riſe together from the ſame level, one directly, and the other obliquely, but both in a horizontal poſition, and always upon the ſame level; and let theſe planes be imagined not to retain all along their firſt magnitude (as was ſuppoſed in the laſt article) but to leſſen by degrees as they riſe, ſo as by their motion to deſcribe tapering figures, and at laſt to vaniſh each in a point: then it is eaſy to ſee, that if the tapering figures thus deſcribed be pyramids or cones having equal baſes and equal perpendicular heights, theſe deſcribing planes muſt not only be equal to each other at firſt, and vaniſh at equal heights, but they muſt leſſen ſo together as to be equal to each other at all other equal altitudes whatever: this is evident from the laſt article but one: and therefore the ſolids deſcribed by them muſt neceſſarily be equal. *Q. E. D.*

COROLLARY.

Hence it follows, that whatever we have demonstrated concerning isosceles pyramids, cones, prisms, and cylinders, with respect to their proportion one to another, will be equally true of all others, whatever shape or posture they may be in: as, that all pyramids and cones of the same height are to each other as their bases, that all pyramids and cones upon equal bases are as their heights, and that every pyramid or cone is a third part of a prism or cylinder having an equal base, and an equal perpendicular altitude.

A LEMMA. (*Fig.* 60.)

364. *Let* ABCD *be a square whose base is* AD, *and whose diagonal is* AC; *and upon the center* A, *and with the* radius AB, *describe the quadrant or quarter of a circle* BAD: *draw also the line* EFGH *or* EGFH *any where within the square, parallel to the base* AD, *cutting the side* AB *in* E, *the quadrant* BD *in* F, *the diagonal* AC *in* G, *and the opposite side* CD *in* H, *and join* AF: *I say then that the square of* EF *and the square of* EG *put together will always be equal to the square of* EH.

For the triangles ABC and AEG are similar, as having one angle at A in common, and the angles at B and E right; therefore EG will be to EA as BC is to BA; but BC is equal to BA, from the nature of a square; therefore EG will be equal to EA, and EG^2 to EA^2, and EF^2+EG^2 to $EF^2+EA^2=AF^2=AD^2=EH^2$, that is, $EF^2+EG^2=EH^2$. *Q. E. D.*

A THEOREM.

365. *Every sphere is two thirds of a circumscribing cylinder, that is, a cylinder that will just contain it.*

For supposing all things as in the last article, (see *Fig.* 60.) let the square $ABCD$ be now supposed

to

to turn round its fixed ſide *AB*, ſo that the ſquare may generate a cylinder, the quadrant a hemiſphere, and the triangle *ABC* an inverted cone; and let this cylinder, and conſequently the cone and hemiſphere be conſidered as conſiſting of an infinite number of infinitely thin cylindric *laminæ*: then if *EH* repreſents the ſemidiameter of any one of theſe *laminæ* belonging to the cylinder, *EG* will be the ſemidiameter of ſo much of this *lamina* as lies within the cone, and *EF* the ſemidiameter of ſo much as lies within the hemiſphere: and becauſe (by the laſt article) the ſquare of *EF* and the ſquare of *EG* are both together equal to the ſquare of *EH*, it follows from corollary 6 in art. 347, that the two circles, whoſe ſemidiameters are *EF* and *EG*, are both together equal to the circle whoſe ſemidiameter is *EH*; which is as much as to ſay in other words, ſince the line *EH* was taken indifferently, that every particular *lamina* of the cylinder is equal to two correſpondent *laminæ* at the ſame height, whereof one belongs to the cone, and the other to the hemiſphere; therefore *componendo*, the whole cylinder is equal to the cone and the hemiſphere put together: but the cone has been proved already to be a third part of the cylinder, as having the ſame baſe and height, (ſee art. 360); therefore the hemiſphere muſt be the remaining two thirds of it; that is, every hemiſphere is two thirds of a cylinder of the ſame baſe and height.

Let us now imagine two ſuch hemiſpheres, and two ſuch cylinders to be put together in one common baſe, and the two hemiſpheres will conſtitute a ſphere, and the two cylinders a cylinder circumſcribed about that ſphere, and the ſphere will now be two thirds of the circumſcribing cylinder, *Q. E. D.*

A The-

A THEOREM.

366. *Every ſphere is equal to a cone or pyramid whoſe baſe is the ſurface of the ſphere and whoſe perpendicular altitude is its ſemidiameter.*

To demonſtrate this, let a body terminated by plain ſides, regular or irregular, be ſo conſtituted as to admit of a ſphere to be inſcribed in it, touching every ſide in ſome point or other, as the cube and an infinite number of other bodies will; and from the center of the inſcribed ſphere imagine lines to be drawn to every angle of the circumſcribing body: then will this body be diſtinguiſhed into as many pyramids as it hath ſides, whoſe baſes will be the ſeveral ſides of the body, whoſe common vertex will be in the center of the ſphere, and whoſe perpendicular heights will be *radii* drawn to the ſeveral points of contact, and conſequently will be equal: for as when a circle is touched by right lines, all *radii* drawn to the ſeveral points of contact will be perpendicular to the reſpective tangent lines; ſo when a ſphere is touched by planes, all *radii* drawn to the ſeveral points of contact will be perpendicular to the reſpective tangent planes.

Let then r be the *radius* of the ſphere, and let a, b, c, d repreſent the quantities or areas of the ſeveral ſides of the circumſcribing body; and the ſolid contents of the pyramids whereof that body is compoſed will be $\frac{ar}{3}, \frac{br}{3}, \frac{cr}{3}, \frac{dr}{3}$, and the ſum of all theſe pyramids, or the ſolid content of the body, will be $\frac{ar+br+cr+dr}{3}$. Let $a+b+c+d=s$, that is, let s be the whole ſurface of the circumſcribing body, and its ſolid content will be $\frac{rs}{3}$: but $\frac{rs}{3}$ is the content of a pyramid whoſe baſe is s, and whoſe perpendicular altitude is r; therefore every body circumſcribed

circumſcribed about a ſphere is equal to a pyramid whoſe baſe is the ſurface of the body, and whoſe perpendicular altitude is the ſemidiameter of the inſcribed ſphere.

Let us now imagine the ſeveral angles of this circumſcribing body to be pared off cloſe by the ſurface of the ſphere, ſo as to create more ſides and more angles, and we ſhall ſtill have a body circumſcribed about the ſphere, though in another ſhape; and therefore the proportion already advanced will be as true in relation to this body as to the former: whence it follows, that if we ſuppoſe the angles of the circumſcribing body to be pared off *ad infinitum*, that is, till it differs nothing from the inſcribed ſphere, the propoſition will be as true of the ſphere itſelf as it was before of the body circumſcribed about it, to wit, that every ſphere is equal to a cone or pyramid whoſe baſe is the ſurface of the ſphere, and whoſe perpendicular altitude is its ſemidiameter. *Q. E. D.*

A Theorem.

367. *The ſurface of every ſphere is equal to four great circles of the ſame ſphere.*

Where note, that by a great circle of a ſphere, I mean any circle ariſing from a ſection of a ſphere into two equal hemiſpheres by a plane paſſing through its center, in contradiſtinction to a leſſer circle ariſing from a ſection of a ſphere into unequal parts: or a great circle of a ſphere may be defined to be a circle whoſe diameter is the ſame with that of the ſphere.

Let s be the ſurface of any ſphere, d the diameter and b the area of a great circle of that ſphere; then will b be the baſe of a circumſcribing cylinder, d its altitude, and $b\,d$ its ſolid content; therefore, by the laſt article but one, $\frac{2bd}{3}$ will be the ſolid content of the ſphere: but by the laſt article, this ſphere is equal to a cone or pyramid whoſe baſe is s the ſurface of of the ſphere, and whoſe perpendicular altitude is $\frac{d}{2}$ its ſemidiameter, a third part whereof is $\frac{d}{6}$; therefore $\frac{sd}{6}$ will alſo repreſent the ſolid content of the ſphere: whence we have the following equation, to wit, $\frac{sd}{6} = \frac{2bd}{3}$, which being reduced gives $s=4b$. *Q. E. D.*

From the three laſt articles may be deduced the following corollaries:

COROLLARIES.

368. 1ſt, *As the diameter of a circle is to the circumference, that is, as 7 to 22 nearly, ſo is the ſquare of the diameter of any ſphere to its ſurface.* For ſuppoſing the diameter of a circle to be to the circumference as 1 to c, and putting d for the diameter of any ſphere, cd will be the circumference of a great circle of that ſphere, ſince as 1 is to c, ſo is d to cd; multiply then $\frac{cd}{2}$ the ſemicircumference, into $\frac{d}{2}$ the *radius*, and you will have $\frac{cdd}{4}$ the area of a great circle; therefore four great circles, or the ſurface of the ſphere, will be cdd: but as 1 is to c, ſo is dd to cdd; therefore, &c.

2d, Whence it follows, that *The ſurface of every ſphere is equal to the product of the circumference of a great circle multiplied into the diameter of the ſphere.* For, retaining the notation of the laſt article, cdd the ſurface of the ſphere is equal to cd the circumference of a great circle multiplied into d the diameter.

3d, *The ſurface of every ſphere is equal to the convex ſurface of a circumſcribed cylinder.* For if a concave cylinder without its two baſes be ſlit, and then opened into a plane, the figure of that plane will be a parallelogram, whose base will be that line which before was the circumference of the base of the cylinder, and whose height will be the same with that of the cylinder; therefore, as the area of a parallelogram is found by multiplying the base into the height, the surface of every cylinder must be found by multiplying the circumference of the base into the height of the cylinder: but the circumference of a cylinder circumscribed about a sphere is equal to the circumference of a great circle of the sphere, and the height of such a cylinder is equal to the diameter of the sphere; therefore the convex surface of the cylinder will be equal to the circumference of a great circle of the sphere multiplied into the diameter, which by the last corollary is the surface of the inscribed sphere.

4th, *The solid content of every sphere is equal to the product of its surface multiplied into a third part of the* radius, *or the* radius *into a third part of the surface.* This is evident from art. 366.

5th, *As six times the diameter of a circle is to the circumference, that is, as 42 is to 22 or 21 to 11 nearly, so is the cube of the diameter of any sphere to its solid content.* For if we suppose the diameter of a circle to be to the circumference as 1 to c, the surface of a sphere whose diameter is d will be cdd by the first corollary; and this surface multiplied into a third part of the *radius*, or into a third part of $\frac{d}{2}$, which is $\frac{d}{6}$, gives $\frac{cd^3}{6}$ the solid content of the sphere: but as 6 is to c, so is d^3 to $\frac{cd^3}{6}$; therefore as six times the diameter of a circle is to the circumference so is the cube of the diameter of any sphere to its solid content.

6th, *The surfaces of all spheres are as the squares, and the solid contents as the cubes, of their diameters or semidiameters.* For supposing the diameter of any circle

circle to be to the circumference as 1 to c, and ſuppoſing d and e to be the diameters of two ſpheres, the ſurfaces will be cd^2 and ce^2 by the firſt corollary, and the ſolid contents will be $\frac{cd^3}{6}$ and $\frac{ce^3}{6}$ by the laſt: but cd^2 is to ce^2 as d^2 is to e^2, or as $\frac{d^2}{4}$ is to $\frac{e^2}{4}$; and $\frac{cd^3}{6}$ is to $\frac{ce^3}{6}$ as d^3 is to e^3, or as $\frac{d^3}{8}$ is to $\frac{e^3}{8}$.

To ſhew the uſe of the properties of the ſphere above deſcribed, I ſhall add the following problems:

PROBLEM I.

369. *To find how many acres the ſurface of the whole earth contains.*

Let the diameter of a circle be to the circumference as d to c, and let e be the circumference of the earth; then will $\frac{de}{c}$ be its diameter, and $\frac{de^2}{c}$ its ſurface by the ſecond corollary in the laſt article. Now the circumference of the earth is 131630573 Engliſh feet, or 24930 Engliſh miles nearly, allowing 5280 feet to a mile: therefore if we make $e=24930$, we ſhall have $e^2=621504900$. Now the numbers 7 and 22 are ſcarce exact enough to expreſs the proportion of the diameter of a circle to the circumference in company with ſo large a number as e^2; let us therefore uſe that of 113 to 355, which we have elſewhere ſhewn (ſchol. 1. in art. 179*) to be much more exact; that is, let $d=113$ and $c=355$, and $\frac{de^2}{c}$ or the ſurface of the earth will be 197831137 ſquare miles: but every ſquare mile contains 640 acres; therefore, if the foregoing number of ſquare miles be multiplied by 640, the product 126611927680 will be the number of acres required.

* See the Quarto Edition, p. 282.

N. B. To be more exact in any computation than the *data* on which it is founded, can be to little or no purpose.

PROBLEM 2.

370. *What must be the diameter of a concave sphere that will just hold an English gallon?*

By the fifth corollary in art. 368, as 11 is to 21, so is the solid content of any sphere to the cube of its diameter: but the solid content of our sphere is 282 cubic inches or an English gallon by the supposition: therefore the cube of its diameter will be $538\frac{4}{11}$, the cube root whereof 8.135 will be the diameter itself.

N. B. The extraction of the cube root is taught in most books of Arithmetic, and depends on the nature of a binomial, as doth the extraction of the square root; and therefore whoever sees the reason of the latter, may (without much difficulty) reason himself into the former: but the extraction of the roots of all simple powers will best be performed by the help of logarithms, as will be shewn hereafter when we come to treat of the nature and properties of those numbers.

Of the Spheroid.

373. *If a sphere be resolved into an infinite number of infinitely thin cylindric* laminæ, *and then these* laminæ, *retaining their circular figure, be all increased or all diminished in the same proportion, they will constitute a figure called a spheroid; and it is said to be prolate or oblong, according as these constituent* laminæ *are increased or diminished.* This a learner, who is unacquainted with the nature of the *ellipsis*, may (if he pleases) take for the definition of a spheroid.

From the definition here given it follows,

1st, that *Every spheroid is to a sphere upon the same* axis, *as any one* lamina *in the former is to a like* lamina *in the latter from whence it was derived; or as any*

number

number of laminæ *in the former is to the ſame number of like* laminæ *in the latter, that is, as any portion of the former comprehended between two parallel planes perpendiuclar to its* axis, *is to a like portion of the latter.*

2dly, it follows, that *Every ſpheroid, as well as every ſphere, is two thirds of a circumſcribing cylinder.* For though a ſpheroid be greater or leſs than a ſphere upon the ſame *axis*, the cylinder circumſcribed about the ſpheroid will be proportionably greater or leſs than the cylinder circumſcribed about the ſphere: for, having the ſame length, they will be as their baſes; therefore the ſpheroid will have the ſame proportion to a cylinder circumſcribed about it, as the ſphere hath to a cylinder circumſcribed about the ſphere.

A Lemma.

374. *The chord of any circular arc is a mean proportional between the verſed ſine of that arc and the diameter.*

Let *ABC* (*Fig.* 61.) be a ſemicircle whoſe diameter is *AC*, and aſſuming any arc as *AB*, draw the ſtraight line *AB*, which is its chord; draw alſo *BD* perpendicular to the diameter *AC* in *D*, and the intercepted line *AD* is called the verſed ſine of the arc *AB*. What we are then to demonſtrate is, that the chord *AB* is a mean proportional between the verſed ſine *AD* and the whole diameter *AC*: and this is eaſily done by drawing the other chord *BC*; for then the triangle *ABC* will be right-angled at *B*, as being in a ſemicircle, and conſequently will be ſimilar to the right-angled triangle *ADB*; whence *AD* will be to *AB* as *AB* to *AC*. *Q. E. D.*

Problem 5.

375. *To find the ſolid content of a fruſtum of a hemiſphere or hemiſpheroid comprehended between a great circle perpendicular to its* axis *and any other leſſer circle parallel to it, having theſe two oppoſite baſes and the height of the fruſtum given.*

N. B. As □ *AD* is ſometimes uſed for the ſquare of *AD*, or a ſquare whoſe ſide is *AD*, ſo in our notation in this and ſome of the following articles, we ſhall not ſcruple to uſe ⊙ *AD* for the area of a circle whoſe ſemidiameter is *AD*, 2 ⊙ *AD* for two ſuch circles, &c.

Let *ABCD* (*Fig.* 60.) be a ſquare whoſe baſe is *AD* and diagonal *AC*; and upon the center *A* and with the *radius AB* deſcribe the quadrant *BAD*; draw alſo the line *EFGH* any where within the ſquare parallel to *AD*, cutting *AB* in *E*, the quadrant in *F*, the diagonal in *G*, and the oppoſite ſide *CD* in *H*. This done, imagine the whole figure to turn round its fixed ſide *AB*: then will the ſquare generate a cylinder, the quadrant a hemiſphere, the triangle *ABC* an inverted cone, and the curvilinear ſpace *AEFD* ſuch a fruſtum of an hemiſphere as we are to find the ſolid content of, having given *AD* and *EF* the ſemidiameters of the two oppoſite baſes, and *AE* the height of the fruſtum.

In the 365th article, by the help of this conſtruction, it was demonſtrated, that the hemiſphere generated by the quadrant *ABD* and the cone generated by the triangle *ABC* were together equal to the cylinder generated by the ſquare *ABCD*; and the reaſons there given for ſuch an equality, equally prove that the fruſtum generated by the ſpace *AEFD* and the cone generated by the triangle *AEG* will both together be equal to the cylinder generated by the parallelogram *AEHD*: but the cone generated by the triangle *AEG* is equal to $\odot\, EG \times \frac{AE}{3}$; and the cylinder generated by the parallelogram *AEHD* is equal to $\odot AD \times AE = 3 \odot AD \times \frac{AE}{3} = \overline{2\odot AD + \odot EH} \times \frac{AE}{3}$, therefore, if *ſ* be put for the ſolid content of the fruſtum, we ſhall have the following equation,

ſ+

$f + \odot EG \times \frac{AE}{3} = \overline{2\odot AD + \odot EH} \times \frac{AE}{3}$; transpose $\odot EG \times \frac{AE}{3}$, and then we shall have $f = \overline{2\odot AD + \odot EH - \odot EG} \times \frac{AE}{3}$: but by the 364th article, and the sixth corollary in the 347th, $\odot EH = \odot EF + \odot EG$; therefore $\odot EH - \odot EG = \odot EF$: substitute $\odot EF$ instead of $\odot EH - \odot EG$ in the foregoing equation ($f = \overline{2\odot AD + \odot EH - \odot EG} \times \frac{AE}{3}$) and you will have $f = \overline{2\odot AD + \odot EF} \times \frac{AE}{3}$: this is upon a supposition that the solid proposed is a frustum of a hemisphere. Let us now suppose the solid f to consist of an infinite number of infinitely thin cylindric *laminæ* parallel to its base, and then that these *laminæ*, retaining their circular figure, be all diminished in some given proportion, suppose in the proportion of r to s; then it is plain that the solid f will degenerate into a frustum of an hemispheroid, and that it will be diminished in the proportion of r to s; but then the quantity $\overline{2\odot AD + \odot EF} \times \frac{AE}{3}$ will also be diminished in the same proportion; and therefore f will still be equal to $\overline{2\odot AD + \odot EF} \times \frac{AE}{3}$; whence we have the following theorem for finding the solid content of the frustum proposed, whether it be a frustum of a hemisphere or hemispheroid.

To twice the area of the greater base add the area of the less; multiply the sum by a third part of the altitude of the frustum, and the product will be its solid content. Q. E. I.

Problem 6.

376. *To find the convex surface of any segment of a sphere whose base and height are given.* (Fig. 60.)

Retaining the construction of the last article, and supposing what was there proved, if from the hemisphere generated by the space *ABD* be subtracted the frustum generated by the space *AEFD*, there will remain a segment of the sphere generated by the space *BEF*; and if to this segment again be added the cone generated by the triangle *AEF*, they will both together constitute a sector of the sphere generated by the space *ABF*; and lastly, if the solid content of this spherical sector be applied to, or divided by a third part of the *radius AD*, the plane or quotient thence arising will be equal to the convex surface generated by the arc *BF*, which is here proposed to be determined. For as every sphere is equal to a cone whose base is its surface and whose altitude is its *radius*, (see art. 366) so (and for the same reason) must every sector of a sphere be equal to a cone whose base is the spherical part of its surface, and whose altitude is the *radius*. Now the hemisphere generated by the space *ABD* being two thirds of a cylinder of the same base and height, as was demonstrated in art. 365, its solid content will be expressed by $2\odot AD\times\frac{AB}{3}=2\odot AD\times\frac{AE}{3}+2\odot AD\times\frac{EB}{3}$; and the solid content of the frustum generated by the space *AEFD* was $2\odot AD\times\frac{AE}{3}+\odot EF\times\frac{AE}{3}$; subtract the latter from the former, and there will remain the segment generated by the space *BEF* equal to $2\odot AD\times\frac{EB}{3}-\odot EF\times\frac{AE}{3}$; add to this the cone generated by the triangle *AEF*, whose content is $\odot EF\times\frac{AE}{3}$, and you will have the spherical sector generated by the

space

ſpace ABF equal to $2 \odot AD \times \frac{EB}{3}$. Let the diameter of a circle be to the circumference as 1 to c, and $2AD \times c$ will be the circumference of a great circle, whoſe half $AD \times c$ multiplied into AD the *radius*, will give $AD^2 \times c$ for the area of a great circle; therefore $\odot AD = AD^2 \times c$, and $2 \odot AD \times \frac{EB}{3}$, or the content of the ſector, will be $2AD^2 \times c \times \frac{EB}{3}$: but EB is the verſed ſine of the arc BF; and therefore if we put l for the chord of that arc, we ſhall have $2AD \times EB = l^2$ by the laſt article but one; and the ſolid content of the ſector will now be $l^2 \times c \times \frac{AD}{3}$; divide by $\frac{AD}{3}$, and you will have the ſurface generated by the arc BF equal to $l^2 \times c$: but as $AD^2 \times c$ was equal to $\odot AD$, ſo will $l^2 \times c$ be equal to $\odot l$, that is, to a circle whoſe *radius* is the chord of the arc BF: therefore *the ſurface of every ſegment of a ſphere is equal to a circle whoſe* radius *is the diſtance of the pole, or vertical point of the ſegment, from the circumference of its baſe.*

What has here been determined concerning the convex ſurface of a ſegment of a ſphere agrees entirely with what was determined in art. 367 concerning the ſurface of a whole ſphere. For if we ſuppoſe the arc BF to be a ſemicircle, its chord will then be a diameter, and the ſurface generated by this arc will be the ſurface of the whole ſphere; and therefore the ſurface of this ſphere will be equal to a circle whoſe *radius* is the diameter of the ſphere, that is $2AD$: but a circle whoſe *radius* is $2AD$, is quadruple of a circle whoſe *radius* is AD, becauſe all circles are as the ſquares of their ſemidiameters; therefore the ſurface of every ſphere is equal to four great circles of the ſame, as was there demonſtrated.

THE

THE ELEMENTS OF ALGEBRA.

BOOK IX. PART I.

Of powers and t eir indexes.

378. THE indexes of powers have been already considered, so far as they serve for a sort of short-hand writing in Algebra: but the incomparable *Newton* has very much enlarged our views with respect to these indexes or exponents, insomuch that it is by their means chiefly, that so many excellent, useful, and comprehensive theorems have been discovered both in Algebra and Geometry, and more particularly in the doctrine of Fluxions. This sort of notation therefore I shall now endeavour further to explain in my observations upon the following small table:

Powers without their indexes.

$$xxxxx.\quad xxxx.\quad xxx.\quad xx.\quad x.\quad 1.\quad \frac{1}{x}.\quad \frac{1}{xx}.\quad \frac{1}{xxx},\quad \frac{1}{xxxx}.\quad \frac{1}{xxxxx}.$$

Powers with their indexes.

$$x^5.\quad x^4.\quad x^3.\quad x^2.\quad x^1.\quad x^0,\quad x^{-1}.\quad x^{-2}.\quad x^{-3}.\quad x^{-4}.\quad x^{-5}.$$

This table consists of two rows, whereof the upper is a series of powers expressed without their indexes, the common root or fundamental quantity being x; the lower expresses the same powers by the help of their indexes.

OBSER-

OBSERVATIONS.

379. 1ſt, *By this table it appears that every ſubſequent power is the quotient of the next before it divided by the common root* x, *and that every ſubſequent index is generated by ſubtracting unity from the next before it.* Thus x^1 divided by x gives x, x divided by x gives 1, 1 divided by x gives $\frac{1}{x}$, $\frac{1}{x}$ divided by x gives $\frac{1}{xx}$, *&c.*: thus again, $2-1=1$, $1-1=0$, $0-1=-1$, $-1-1=-2$ *&c.* Since then each row exhibits a regular ſeries, it follows that the negative indexes have the ſame right to expreſs the powers they belong to as the affirmative ones, and that x^{-2} repreſents $\frac{1}{xx}$ upon the ſame foundation that x^2 repreſents xx.

2dly, *Therefore whatever number is the index of any power, its negative will be the index of the reciprocal of that power, or of unity divided by that power.* Thus if 2 be the index of xx, -2 will be the index of $\frac{1}{xx}$; if 1 be the index of x, -1 will be the index of $\frac{1}{x}$; and ſo of the reſt.

3dly, *In all caſes whatever, the addition of indexes anſwers to the multiplication of the powers to which they belong; that is, if any two powers of the ſame quantity be multiplied together, the index of the multiplicator added to the index of the multiplicand will give the index of the product.* Thus x^2 multiplied into x^3 gives x^5, as $xx \times xxx$ gives $xxxxx$: thus $x^2 \times x^{-3}$ gives x^{-1}, as $xx \times \frac{1}{xxx}$ gives $\frac{1}{x}$: thus $x^{-2} \times x^{-3}$ gives x^{-5}, as $\frac{1}{xx} \times \frac{1}{xxx}$ gives $\frac{1}{xxxxx}$: thus $x^2 \times x^{-2}$ gives x^0, as $xx \times \frac{1}{xx}$ gives 1: $x^3 \times x^0$ gives x^3, as $xxx \times 1$ gives xxx.

4thly,

4thly, *In like manner the ſubtraction of indexes anſwers to the diviſion of powers; that is, if any power of any quantity be divided by a power of the ſame quantity, the index of the diviſor ſubtracted from the index of the dividend leaves the index of the quotient.* Thus x^3 divided by x^2 quotes x^1, as xxx divided by xx quotes x: thus x^2 divided by x^{-3} quotes x^5, as xx divided by $\frac{1}{xxx}$ quotes $xxxxx$: thus x^{-2} divided by x^3 quotes x^{-5}, as $\frac{1}{xx}$ divided by xxx quotes $\frac{1}{xxxxx}$: thus x^{-2} divided by x^{-3} gives x^1, as $\frac{1}{xx}$ divided by $\frac{1}{xxx}$ gives x: thus x^0 divided by x^{-2} gives x^2, as 1 divided by $\frac{1}{xx}$ gives xx: laſtly, x^2 divided by x^2 gives x^0, as xx divided by xx gives 1.

5thly, *If the index of any power be multiplied by* 2, 3, 4, &c. *the product will be the index of the ſquare, cube, ſquare-ſquare,* &c. *of that power: and therefore if the index of any power be divided by* 2, 3, 4, &c. *the quotient will be the index of the ſquare root, cube root, ſquare-ſquare root,* &c. *of that power.* Thus the ſquare of x^2 is x^4, its cube x^6, its ſquare-ſquare x^8: thus again, the ſquare root of x^{12} is x^6, its cube root x^4: its ſquare-ſquare root x^3, *&c.*: thus the ſquare root of x or x^1 is $x^{\frac{1}{2}}$, its cube root $x^{\frac{1}{3}}$, its ſquare-ſquare root $x^{\frac{1}{4}}$ *&c.*: thus the ſquare root of $\frac{1}{x}$ or x^{-1} is $x^{-\frac{1}{2}}$, its cube root $x^{-\frac{1}{3}}$, its ſquare-ſquare root $x^{-\frac{1}{4}}$ *&c.*: thus $x^{\frac{2}{3}}$ ſignifies the cube root of x^2, $x^{\frac{3}{4}}$ the ſquare-ſquare root of x^3. And univerſally, $x^{\frac{m}{n}}$ ſignifies that root of x^m whoſe index is n; as if $y^n = x^m$, then y is ſaid to be that root of x^m whoſe index

index is n, and muſt be expreſſed by $x^{\frac{m}{n}}$; and therefore if in any caſe $x^m = y^n$, it will be a good inference to ſay that y is equal to $x^{\frac{m}{n}}$, or that x is equal to $y^{\frac{n}{m}}$.

6thly, *Powers are reducible to more ſimple powers, as often as their fractional indexes are reducible to more ſimple fractions.* Thus the ſquare-ſquare root of x^2 is the ſame with the ſquare root of x, becauſe $x^{\frac{2}{4}} = x^{\frac{1}{2}}$.

7thly, *If the index of any power be an improper fraction, and that fraction be reduced into a whole number and a fraction, the power will hereby be reſolved into two factors, whereof one will have the whole number for its index, and the other the fractional part.* Thus $\frac{5}{2} = 2 + \frac{1}{2}$, and therefore $x^{\frac{5}{2}} = x^2 \times x^{\frac{1}{2}}$; that is, the ſquare root of x^5 is equal to xx multiplied into the ſquare root of x.

8thly, *Surd powers may be reduced to the ſame root by a reduction of their fractional indexes to the ſame denomination, and that, whether they be powers of the ſame quantity or not.* Thus $x^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$ are the ſame as $x^{\frac{3}{6}}$ and $y^{\frac{2}{6}}$; that is, the ſquare root of x and the cube root of y are the ſame as the ſixth root of x^3 and the ſixth root of y^2, and thus may ſurds of different roots be compared together without any extraction of thoſe roots. As for inſtance, if any one ſhould aſk me, which of theſe two quantities is the greater, the ſquare root of 2, or the cube root of 3? I ſhould anſwer, the cube root of 3; for the ſquare root of 2 or $2^{\frac{1}{2}}$ or $2^{\frac{3}{6}}$ is equal to $8^{\frac{1}{6}}$; but the cube root of 3, or $3^{\frac{1}{3}}$, or $3^{\frac{2}{6}}$, is equal to $9^{\frac{1}{6}}$; and $9^{\frac{1}{6}}$ is greater than $8^{\frac{1}{6}}$.

9thly, *That the addition and subtraction of indexes answers to the multiplication and division of the powers to which they belong, holds equally true in fractional indexes, as in integral ones.* Thus $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, and $x^{\frac{1}{2}}\times x^{\frac{1}{3}}=x^{\frac{5}{6}}$, which I thus demonstrate. Let $y^6=x$; then by the fifth observation we shall have $y=x^{\frac{1}{6}}$, $y^3=x^{\frac{3}{6}}$, or $x^{\frac{1}{2}}$, $y^2=x^{\frac{2}{6}}$, or $x^{\frac{1}{3}}$, and $y^5=x^{\frac{5}{6}}$: but $y^3\times y^2$ is equal to y^5 by the third observation; therefore $x^{\frac{1}{2}}$ multiplied into $x^{\frac{1}{3}}$ gives $x^{\frac{5}{6}}$. After the same manner, since $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$, it may be demonstrated that $x^{\frac{1}{2}}$ divided by $x^{\frac{1}{3}}$ will give $x^{\frac{1}{6}}$; for y^3 divided by y^2 gives y, which is equal to $x^{\frac{1}{6}}$; and the demonstrations will be the same in all other cases.

PART II.

Of logarithms, and their uses.

The definition of logarithms, and consectaries drawn from it.

Art. 390. LOGARITHMS *are a set of artificial numbers placed over-against the natural ones, usually from* 1 *to* 100000, *and so contrived that their addition answers to the multiplication of the natural numbers to which they belong; that is, if any two numbers be multiplied together, and so produce a third, their logarithms being added together will constitute the logarithm of that third.*

Thus 0 .3010300, the common logarithm of 2, added to

0 .4771213, the logarithm of 3, gives

0 .7781513, the logarithm of 6, because 6 is the product of 2 and 3 multiplied together.

From this definition it follows, first, *That in any system or table of logarithms whatever, the logarithm of unity or* 1 *will be nothing*: for as 1 neither increases nor diminishes the number multiplied by it, so neither will its logarithm either increase or diminish the logarithm to which it is added; and therefore the logarithm of 1 must be nothing.

2dly, *For a like reason, the logarithm of a proper fraction will always be negative*: for such a fraction always diminishes the number multiplied by it, and therefore its logarithm will always diminish the logarithm to which it is added.

3dly, *This property of logarithms, whereby they are defined as above, affords us no small compendium in multiplication*: for whenever one number is to be multiplied by another, it is but taking out their logarithms, and adding them together, and their sum will be a third logarithm whose natural number being taken out of the tables will be the product required.

4thly, *The subtraction of logarithms answers to the division of the natural numbers to which they belong*; that is, whenever one number is to be divided by another, it is but subtracting the logarithm of the divisor from the logarithm of the dividend, and the remainder will be the logarithm of the quotient: and thus by the help of logarithms may the operation of division be performed by mere subtraction as that of multiplication was by addition. *Hence as every fraction is nothing else but the quotient of the numerator divided by the denominator, its logarithm will be found by subtracting the logarithm of the denominator from the logarithm of the numerator.* To demonstrate this, to wit, that the logarithm of the divisor subtracted from the logarithm of the dividend will leave the logarithm of the quotient, let the number A be divided by the number B, and let the quotient be the number C, and let the logarithms of the numbers A, B, and C, be a, b, and c respectively; I say then that $a-b$ will be equal to c: for since by the supposition fition $\frac{A}{B}=C$, we fhall have $A=BC$, $a=b+c$ by the definition; whence $a-b=c$.

5thly, *As every fourth proportional is found by multiplying the fecond and third numbers together, and dividing the product by the firft, fo the logarithm of every fuch fourth proportional will be found by adding the logarithms of the fecond and third numbers together, and fubtracting from the fum the logarithm of the firft.* This renders all operations by the rule of proportion very compendious and eafy; efpecially after the practitioner has pretty well inured himfelf to take out of the table logarithms to his numbers, and numbers to his logarithms: but this compendium is chiefly ufeful in Trigonometry, both plain and fpherical, where every thing he wants is put down ready to his hands.

6thly, *If* A *be any number whofe logarithm is* a, *then the logarithm of* A^2 *will be* 2a, *that of* A^3, 3a, *&c. that of* $\frac{1}{A}$, —a, *that of* $\frac{1}{A^2}$, —2a, *&c. And univerfally, the logarithm of* A^m *will be* $a \times m$, *and that, whether the index* m *be integral or fractional, affirmative or negative: on the other hand, if* q *be the logarithm of any power of* A, *as of* A^m, *then* $\frac{q}{m}$ *will be the logarithm of* A. The reafon of all this is plain; for as A^2 is the product of A multiplied into itfelf, fo its logarithm will be the logarithm of A added to itfelf or doubled, that is $2a$; and fo of the higher powers. Again, as $\frac{1}{A}$ is the quotient of unity divided by A, its logarithm will be found by fubtracting a, the logarithm of A, from 0, the logarithm of 1, which gives $-a$; and fo of the lower powers. Laftly, as $\sqrt{A}$, when multiplied into itfelf, produces A, fo its logarithm, when added to itfelf, ought to make a; therefore the logarithm of $\sqrt{A}$ will be $\frac{1}{2}a$; and fo of all the other fractional powers. Here then again we have another inſtance of the very great uſefulneſs of a good table of logarithms, to wit, in raiſing a number to any given power, or in extracting any given root out of it, all which is performed with equal facility, only by multiplying its logarithm by the index of the given power, or dividing it by the index of the given root; as doubling it for the ſquare, tripling it for the cube, *&c.*; halving it for the ſquare root, triſecting it for the cube root *&c.*: this, I ſay, cannot but be very uſeful in a great many caſes, and more eſpecially in Annatociſm, where we have ſometimes occaſion to extract even the three hundred ſixty-fifth root of a number, as at other times to raiſe it to the three hundred ſixty-fifth power, ſcarce poſſible to be performed any other way; to ſay nothing of the innumerable miſtakes that in ſo long and laborious a calculation would be almoſt unavoidable, all which are prevented by the uſe of logarithms. It cannot indeed be expected that entire powers, and much leſs entire roots, ſhould be gained this way; but it will be eaſy in moſt caſes to obtain as many terms as can be of any uſe to us.

7thly, *If any ſet of numbers, as* A, B, C, D *be in continual geometrical proportion, their logarithms, which we ſhall call* a, b, c, d, *will be in arithmetical progreſſion:* for ſince by the ſuppoſition A is to B as B is to C as C is to D, that is, ſince $\frac{B}{A}=\frac{C}{B}=\frac{D}{C}$, we ſhall have $b-a=c-b=d-c$ by the fourth conſectary; therefore a, b, c, d are in arithmetical progreſſion. *Q. E. D.*

8thly, *From this laſt conſectary it will be eaſy, having two numbers given, to find as many mean proportionals as we pleaſe between them.* Let the given numbers be A and F, and let it be required to find four mean proportionals between them, which we ſhall call B, C, D, E, ſo that A, B, C, D, E, F, may be in continual geometrical proportion. Here then it is evident from the laſt conſectary, that, as theſe numbers are in continual geometrical proportion, their logarithms, which we shall call a, b, c, d, e, f, will be in arithmetical progression, whereof the extremes a and f are known, as being the logarithms of the known numbers A and F, and the intermediates may be found thus. Put x for the common difference of this arithmetic progression; then will $a+x=b$, $a+2x=c$, $a+3x=d$, $a+4x=e$, $a+5x=f$; whence $x=\frac{f-a}{5}$; whence $a+x$ or $b=a+\frac{f-a}{5}=\frac{4a+f}{5}$, $a+2x$ or $c=\frac{3a+2f}{5}$, $a+3x$ or $d=\frac{2a+3f}{5}$, $a+4x$ or $e=\frac{a+4f}{5}$; so that the logarithms of the four mean proportionals sought are $\frac{4a+f}{5}$, $\frac{3a+2f}{5}$, $\frac{2a+3f}{5}$, $\frac{a+4f}{5}$: take then the natural numbers B, C, D, E of these logarithms, and they will be the mean proportionals required. *Q. E. I.*

Logarithms the measures of ratios.

391. *Logarithms are so called from their being the arithmetical or numeral exponents of ratios:* for if unity be made the common consequent of all ratios, or the common standard to which all other numbers are to be referred, then every logarithm will be the numeral exponent of the ratio of its natural number to unity. As for instance, the ratio of 81 to 1 actually contains within itself these four ratios, to wit, the ratio of 81 to 27, that of 27 to 9, that of 9 to 3, and that of 3 to 1 (see art. 293); all which ratios are equal to one another, and to the ratio of 3 to 1; therefore the ratio of 81 to 1 is said to be four times as big as the ratio of 3 to 1 (see art. 294): and hence it is that the logarithm of 81 is four times as big as the logarithm of 3. Again, the ratio of 24 to 1 contains, and may be resolved into these three

ratios, to wit, the ratio of 24 to 12, that of 12 to 4, and that of 4 to 1; the firſt of theſe ratios, to wit, the ratio of 24 to 12, is the ſame with that of 2 to 1; the ſecond, to wit, the ratio of 12 to 4, is the ſame with that of 3 to 1; and therefore the ratio of 24 to 1 is equal to the ratios of 2 to 1, 3 to 1, and 4 to 1, put together; and hence it is that the logarithm of 24 is equal to the logarithms of 2, 3 and 4 put together: *And univerſally, the magnitude of the ratio of* A *to* 1 *is to the magnitude of the ratio of* B *to* 1 *as the logarithm of* A *is to the logarithm of* B. *And hence we have a way of meaſuring all ratios whatever, let their conſequents be what they will:* as for example, the ratio of *A* to *B* is the exceſs of the ratio of *A* to 1 above the ratio of *B* to 1 (ſee art. 296); therefore the numeral exponent of the ratio of *A* to *B* will be the exceſs of the numeral exponent of the ratio of *A* to 1 above the numeral exponent of the ratio of *B* to 1, that is, the exceſs of the logarithm of *A* above the logarithm of *B*; therefore *The magnitude of the ratio of* A *to* B *is to the magnitude of the ratio of* C *to* D *as the exceſs of the logarithm of* A *above the logarithm of* B, *which is the meaſure of the former ratio, is to the exceſs of the logarithm of* C *above the logarithm of* D, *which is the meaſure of the latter ratio:* and thus we ſee that logarithms are as true and as proper meaſures of ratios as circular arcs are of angles.

I might have defined logarithms from the idea here given of them, and thence have deduced all the other properties above deſcribed: but, as it is not every one that hath a juſt and diſtinct notion of the nature and compoſition of ratios, I thought it more adviſeable to treat of them in a way more familiar to the learner.

Of Briggs's *Logarithms.*

392. *From the definition given in art.* 390, *it may eaſily be ſeen, that, if any one ſyſtem of logarithms be once obtained, an infinite number of others may be derived*

from them by increasing or diminishing the logarithms of that system in some given proportion. As for instance, in the system given let a, b, c, be the logarithms of three numbers, A, B, and C, whereof the third is the product of the other two multiplied together; then will $a+b=c$, by the definition. Let us now imagine all the logarithms of this given system to be doubled; then will a, b, and c be changed into $2a$, $2b$ and $2c$; but as $a+b$ was equal to c in the former system, so now will $2a+2b$ be equal to $2c$ in the latter; that is, all the numbers of this new system will still retain the property of logarithms. *But though all these different systems be equally perfect, if computed to the same degree of accuracy, yet they will not all be equally convenient for use: for of all systems or tables of logarithms, that is certainly best accommodated for practice which is now in use, and is commonly known by the name of* Briggs's *logarithms.* The Lord *Napeir*, a Scotch nobleman, was the first inventor of logarithms; but our countryman Mr. *Briggs*, Professor of Geometry in *Gresham* College, was undoubtedly the first who thought of this system; and, proposing it to the noble inventor the Lord *Napeir*, he afterwards published it with that Lord's consent and approbation.

The distinguishing mark of this system is, that herein the logarithm of 10 *is* 1, *and consequently that of* 100, 2, *that of* 1000, 3, *that of* 10000, 4, &c.; *that of* 1, 0, *that of* $\frac{1}{10}$ *or of* 0.1, —1. *that of* $\frac{1}{100}$ *or of* 0.01, —2, &c. *In this system the integral parts of the logarithms are always distinguished from the rest, and called the indexes or characteristics of the logarithms whereof they are parts:* thus the logarithm of 20 is 1 .3010300, where the characteristic is 1; that of 2 is 0.3010300, where the characteristic is 0; that of $\frac{2}{10}$ or 0 .2 is —1+.3010300, where—1 is the characteristic, *&c.*

Some

Some advantages of this ſyſtem.

393. Some of the chief advantages of this ſyſtem, beyond all others, will appear from the following conſiderations.

1ſt, Whereas we have frequent occaſion to multiply and divide by 10, 100, 1000, *&c.* this in this ſyſtem is very readily performed, only by adding to or ſubtracting from the characteriſtic the numbers 1, 2, 3, *&c.*; and as theſe are whole numbers, they can only influence the index or characteriſtic of a logarithm, without affecting the decimal part.

2dly, So long as the digits that compoſe any number are the ſame, and in the ſame order, whatever be their places with reſpect to the place of units, the decimal parts of the logarithm of ſuch a number will always be the ſame. As for inſtance, let $4+l$ be the logarithm of this number 34567 .89, where 4 is the characteriſtic, and l repreſents the ſum of all the decimal parts; then will $5+l$ be the logarithm of 345678 .9, $6+l$ that of 3456789, $7+l$ that of 34567890, *&c.* On the other hand, $3+l$ will be the logarithm of 3456 .789, $2+l$ that of 345 .6789, $1+l$ that of 34 .56789, $0+l$ that of 3 .456789, $-1+l$ that of 0.3456789, $-2+l$ that of 0.03456789, *&c.*: the reaſon of this is plain; for if the number 34567 .89 be multiplied by 10, the product will be 345678 .9; therefore if to $4+l$, the logarithm of the former number be added 1, the logarithm of 10, the ſum $5+l$ will be the logarithm of the latter. Again, if the number 34567 .89 be divided by 10, the quotient will be 3456 .789; therefore if from $4+l$, the logarithm of the former number be ſubtracted 1, the logarithm of 10, the remainder $3+l$ will be the logarithm of the latter. Here then we ſee the reaſon why in *Briggs's* tables the decimal part of every logarithm is affirmative, whether the whole logarithm taken together be ſo or not; for, in the logarithm of all numbers greater than unity, both the integral and decimal parts are affirmative; and therefore the decimal parts muſt always be ſo, ſince theſe are not changed by changing the natural number, ſo long as the digits that compoſe it are the ſame, and in the ſame order: thus $\frac{-3}{10}$ or—. 3 may be a logarithm; but it is never expreſſed ſo, but rather thus,—1+.7, the negation being thrown wholly upon the characteriſtic.

3dly, By this means in *Briggs's* ſyſtem the characteriſtic of the logarithm of any number is eaſily known thus: ſuppoſe I was aſked, what is the characteriſtic of the logarithm of this number 34567 .89? Here I conſider that this number lies between 10000 and 100000; therefore its logarithm muſt be ſome number between 4 and 5; therefore it muſt be 4 with ſome decimal parts annexed, that is, the characteriſtic muſt be 4. And again, ſuppoſe it was required to aſſign the characteriſtic of the logarithm of this number, 0 .03456789: here I conſider that this number lies between $\frac{1}{10}$ and $\frac{1}{100}$, that is, between 0 .1 and 0 .01, and therefore its logarithm muſt lie between —1 and —2, that is, its logarithm muſt be —2 with ſome affirmative decimal parts annexed, to leſſen the negation; therefore the characteriſtic will be —2.

To find the characteriſtic of Briggs's *logarithm of any number.*

394. Hence may be drawn a ſhort and eaſy rule for determining the index or characteriſtic of the logarithm of any number given, thus. *If the number given be a whole number, or a mixt number conſiſting of integral and decimal parts, then ſo many removes as is the place of units to the right hand of the firſt figure, of ſo many units will the characteriſtic conſiſt; but if the number propoſed be a pure decimal, then ſo many*

many removes as is the place of units to the left hand of the first significant figure, of so many negative units will the characteristic consist. Thus the index or characteristic of the logarithm of this number 34567 .89 is 4, because 7 in the place of units is four removes to the right hand of the first figure 3: thus again, the characteristic of the logarithm of this number 0 .03456789 is —2, because 0 in the place of units is two removes to the left hand of the first significant figure 3.

These rules are the more to be observed, because in some tables the integral parts of all logarithms are omited, being left to be supplied by the operator himself, as occasion requires: by this means, the logarithms become of much more general use than if, by having their characteristics prefixed, they were tied down to particular numbers.

Another idea of Logarithms.

395. *In the system here described, every natural number is, or may be, considered as some power of* 10, *and its logarithm as the index of that power:* for let a be the logarithm of any natural number as A; then since *Briggs's* logarithm of 10 is 1, his logarithm of 10^a will be a; this is evident from art. 390 consect. 6; therefore A must be equal to 10^a, since they have both the same logarithm; that is, the natural number A is such a power of 10 as is expressed by its logarithm a. This consideration gives us a new idea of logarithms, and to one acquainted with the nature of powers and their indexes, it will be no wonder that the addition, subtraction, multiplication, and division, of these logarithms answer to the multiplication, division, involution and evolution of their natural numbers.

Precautions to be used in working by Briggs's *logarithms.*

396. Though these logarithms (as I observed before) are preferable to all others, on account of their simplicity

ſimplicity and facility in practice, yet in uſing them ſome precautions are to be obſerved, which (to prevent miſtakes) I ſhall here juſt point out to the learner; as

1ſt, *In the addition of logarithms, whatever is carried over from the decimal to the integral parts muſt be conſidered as affirmative, and as ſuch muſt be added to thoſe integral parts, whether they be affirmative or negative.* Thus —3+.7000000 being added to —4 +.8000000, the ſum will be —6+.5000000; for though the ſum of the characteriſtics —3 and —4 be —7, the affirmative unit drawn from the decimals reduces it to —6.

2dly, *Whenever a ſubtraction is to be made in logarithms, it muſt be performed in the decimal parts as uſual; but if the characteriſtic of the ſubtrahend, or of the number from whence the ſubtraction is to be made, or of both, be negative, they muſt be treated in the ſubtraction as the nature of ſuch quantities requires.* Thus —3+.8900000 ſubtracted from —1+.7600000 leaves 1.8700000: for if +1, on account of the decimals, be added to —3, the characteriſtic of the ſubtrahend, it will be reduced to —2, which being ſubtracted from —1 as above, leaves +1. Nay, the learner muſt not be diſcouraged if he ſometimes finds himſelf obliged to ſubtract a greater logarithm from a leſs, as will always be the caſe where the logarithm of a proper fraction is required: as for example, let it be required to find the logarithm of $\frac{1}{2}$: here ſubtracting 0.3010300, the logarithm of 2, from 0.0000000, the logarithm of 1, there will remain —1+.6989700, the logarithm of $\frac{1}{2}$; for in this ſubtraction, +1, on account of the decimals being added to the characteriſtic of the ſubtrahend, gives 1, which ſubtracted from 0 above, leaves —1.

Note, *The logarithm of a vulgar fraction may alſo be obtained by throwing it into a decimal.* Thus the logarithm of $\frac{2}{3}$ may be obtained, either by ſubtracting the logarithm of 3 from that of 2, or elſe by taking out the logarithm of this decimal fraction .6666667, which is the ſame as the logarithm of the whole number 6666667, except that the characteriſtic of the former logarithm is —1, and that of the latter +6.

3dly, *in the multiplication of logarithms the ſame care muſt be taken as in addition.* Thus if it be required to multiply this logarithm —3+ .7000000 by 9, the product will be —21+ .3000000; for though the product of —3×9 be —27, yet the +6 drawn from the decimals reduces it to —21.

4thly, *Whenever a logarithm is to be divided by* 2, 3, 4, &c. *in order to obtain the ſquare, cube, biquadrate,* &c. *root of its natural number, if the characteriſtic be negative, and will not be divided without a fraction, my way is to reſolve it into two parts, to wit, into a negative part which will be divided, and an affirmative part which will incorporate with the decimals annexed.* Thus if I was to take the half of this logarithm —1+ .7000000, I cannot join the —1 to the decimals annexed, becauſe they are quantities of different kinds; therefore I reſolve the characteriſtic —1 into two parts, to wit, —2+1, and then taking the half of —2, which is —1, I join the affirmative part +1 to the decimals annexed, and ſo take the half of +17, which is +8 *&c.*; therefore the half of the aforeſaid logarithm is —1+ .8500000: had the characteriſtic been —3, I ſhould have reſolved it into —4+1. Had $\frac{1}{3}$ of the foreſaid logarithm been required, I ſhould have reſolved the characteriſtic —1 into —3+2, and ſo ſhould have taken, firſt, the third part of —3, which is —1 and then of +27, which is +9: had the characteriſtic been —2, I ſhould have reſolved it into —3+1; had it been —4, I ſhould have reſolved it into —6+2, and ſo on.

N. B. Of all the tables hitherto in uſe whoſe logarithms do not run to above ſeven decimal places, I take thoſe publiſhed by Doctor *Sherwin* to be the beſt upon many accounts, and particularly in the diſpoſition of the logarithms: theſe therefore I ſhall not

ſcruple

ſcruple to recommend to my readers, whom I ſhall alſo refer to the directions there given for finding the logarithms of all abſolute numbers from 1 to 10000000, and *vice verſa*. But I muſt own I cannot with equal juſtice recommend the method there taken to avoid negative indexes by creating new ones, and by uſing arithmetical complements. It is not to be denied but that this ſort of practice may be abſolutely neceſſary to ſuch as know nothing of the nature and uſe of negative quantities; but thoſe who do, I believe, will find the rules here laid down more natural and convenient; and as they carry their own reaſons along with them, I doubt not but that the learner will find them eaſier to be remembered, and leſs liable to be miſunderſtood.

397. In the tables above recommended, after the logarithms on every page, are two columns, one called a column of differences, and ſigned *D*; the other called a column of proportional parts, and ſigned *Pts* above, and *Pro* below: theſe two columns, as well as the reſt, have been explained by the author; but, leſt they ſhould not be thoroughly underſtood by what is there ſaid of them, I ſhall take the liberty, by a ſingle inſtance, to explain more at large the reaſon and uſe of theſe columns: I ſhall take my example from the author himſelf. Let it then be required to find by the tables the logarithm of this number of ſeven places, to wit, 5423758: to do this, I firſt put down 6, the characteriſtic of the logarithm ſought, according to the directions given in art. 394; then I conſider in the next place, that though by the help of the tables we can find the logarithm of any number under 10000000, yet that the abſolute numbers there do not, properly ſpeaking, run to above five places; therefore I lower the abſolute number given, to wit, 5423758, to this, 54237.58, which will not affect the decimal part of the logarithm ſought; then ſetting aſide the characteriſtic, I take out of the tables the logarithm of the five integral places 54237

according

according to the directions there given, and find it to be 7342957; this I ſubtract from the logarithm of 54238, that is, from 7343037, and find the difference to be 80. But the deſign of the column of differences is on purpoſe to avoid this ſubtraction: for, had I taken out of that column the number oppoſite to 54237, the integral part of the abſolute number propoſed, or if no ſuch oppoſite number was to be found, had I taken the neareſt number above, (not below), I ſhould have found the number 80.1, that is, in a whole number, 80, without any ſubtractraction. Thus then the caſe ſtands: as the abſolute number propoſed 54237.58 lies between the two neareſt tabular numbers 54237 and 54238, whoſe difference is 1, ſo muſt the logarithm ſought lie between the logarithms of the tabular numbers above mentioned, whoſe difference is 80; therefore I ſay by the golden rule, as 1, the difference of the two tabular numbers, between which mine lies, is to 80, the difference of the two tabular logarithms between which the logarithm ſought lies, ſo is .58, the difference be twixt my number and the neareſt leſs tabular number, to 46, the difference betwixt the logarithm ſought and the neareſt leſs tabular logarithm; therefore adding this difference 46 to the neareſt leſs tabular logarithm, to wit, 7342957, I have 7343003, which being joined as decimal parts to the characteriſtic 6, gives 6.7343003 for the logarithm ſought. This number 46, which was the fourth proportional above found, is called the proportional part, becauſe it is the ſame proportional part of 80, the difference of the two neareſt tabular logarithms, that .58, the decimal part of the number propoſed, is of 1, the difference of the two neareſt tabular numbers. Whoever attends to the foregoing operation will eaſily perceive, that this proportional part 46 was gained from multiplying 80, the common difference, by .58, the decimal part of the abſolute number propoſed; and the ſame would have been obtained if the common difference

80

80 had first been multiplied by .5 and then by .08, and the products been taken into one sum: now it is to save these two multiplications that the column of proportional parts was contrived; for whoever looks there for the common difference 80 will find all the products of the said common difference multiplied by .1, .2, .3, .4, .5, &c. to .9 inclusively; and looking for the number over against .5, he will find the number 40, which shews that the number 40 is $\frac{5}{10}$ of the common difference 80; so also over against 8 he will find the number 64, which shews that the number 64 is $\frac{8}{10}$ of the common difference; but we do not want $\frac{8}{10}$ of it, but 8 hundredth parts; therefore he must not take the number 64, but a tenth part of that number, to wit, 6 .4 or 6, which being added to 40, the proportional part before found, gives 46, to be added to the nearest less tabular logarithm in order to obtain the logarithm sought.

But when all possible exactness is required, and no errors are intended to be committed, but such as unavoidably arise from the imperfection of the logarithms themselves; I would advise the reader to compute the proportional parts himself, as above, rather than trust to the table for them, though he will rarely find any considerable difference. My reason for this advice is, because in the table of proportional parts, no notice is taken of decimals; whereas those decimals ought not in all cases to be neglected, at least not till the operation is over, and the artist sees what it is he throws away or takes into his account, to lessen the error as much as he can.

FINIS.

[illegible]

[illegible] which shews that the number [illegible] of the common difference [illegible] is also [illegible]

[illegible] he will find the number [illegible] which shews that [illegible] of the common difference; but we do not want [illegible] of it, but [illegible] hundredth parts [illegible]

[illegible]

But when [illegible]

[illegible] pute the proportional parts [illegible] than omit in the table for them, though [illegible] find any considerable difference. [illegible] reason [illegible] advice is, because in the [illegible] proportional parts [illegible] taken of [illegible] [illegible] arrows away [illegible]

[illegible] as much as he can.

F I N I S.