SELECT EXERCISES FOR Young Proficients IN THE MATHEMATICKS.

Containing,

I. A large Variety of Algebraical Problems with their Solutions.

II. A choice Number of Geometrical Problems with their Solutions both Algebraical and Geometrical.

III. The Theory of Gunnery, independent of the Conic Sections.

IV. A new and very comprehensive Method for finding the Roots of Equations in Numbers.

V. A short Account of the Nature and first Principles of Fluxions.

VI. The Valuation of Annuities for single and joint Lives, with a Set of new Tables, far more extensive than any extant.

By THOMAS SIMPSON, F. R. S.

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TO

JOHN BACON,

Of *Newtoncap*, Esq; F. R. S.

SIR,

WHEN Gentlemen of your Station and Figure become the Patrons of Science it is a Benefit to the Publick, their Expectations of farther Improvements having then the best Foundation. And All who have the Pleasure of your Acquaintance, and know your Attachment to polite and useful Learning, in which a Knowledge of the Mathematicks may be justly included, will be sensible of my Happiness in being thus permitted to address You.

Believe

Believe me, Sir, whatever may be the Fate of theſe Sheets, I ſhall, at all Times, conſider this Uſe of your Name as a ſingular Honour to,

SIR,

Your moſt obedient, and moſt

Humble Servant,

T. SIMPSON.

Woolwich, May 1, 1752.

THE

PREFACE.

THE ensuing Work, or at least the greatest Part of it, was originally composed for my own Use in the ROYAL ACADEMY: *And it is upon a Presumption that It may also be of Service to Others, especially Those employed in a like Publick Way of Teaching, that It now appears in the World.*

The Work itself consists of six distinct Parts, or Tracts; each of which I shall here give some Account of.

The first Part contains a Number of Algebraical Problems, with their Solutions; design'd as proper Exercises for Young Beginners. In the Course of these Problems and Solutions (whereof the greater Part will appear to be new) the Art of managing Equations, and the various Methods of Substitution are taught and illustrated.

The

The second Part comprehends a Variety of Geometrical Problems with their Solutions, both by Algebra and also independent of it, from Principles purely Geometrical. In this Part the Learner will find a large Field to exercise his Industry in: He will moreover have the Opportunity of comparing the two Methods of Solution together, and from thence observing, that sometimes the One has the Advantage, and sometimes the Other; that, in some Cases They both proceed upon the very same Properties, and in others, upon quite different Ones: And it may be further remarked from hence (which will be of some Use to know) that, when Quantities are given in Magnitude only, *the Algebraic Method generally claims the Preference, in Point of Ease and Expedition, at least; whereas the Advantage is almost always on the Side of the Geometrical Effection, when the Positions of Points and Lines, and the Quantities of Angles are given.*

There is, however, one Particular, or Two, in this Part, that may be thought to stand in need of some Apology.

In the first Place, the frequent Use of Symbols, common to the Algebraic Notation, may, perhaps, be look'd upon as repugnant to the Rigour

Rigour and Strictness of Geometry. But it is not the Use of Symbols (which Some, more scrupulous than discerning, have condemned) but the Ideas annexed to them, that render the Consideration Geometrical, *or* Ungeometrical. *In pure Geometry regard is always had to the absolute Quantity of some One of the three Kinds of Extension, abstractedly considered; and, whatever Symbols are used Here, are to be considered as expressive of the Quantities themselves, and not as any Measures, or numerical Values of them. Thus by* A × B, *taken in a geometrical Sense, we have an Idea, not of the Product of two Numbers (as in the Algebraic Notation), but of a* real, *rectangular, Space comprehended under two Right-lines, represented by* A *and* B, *and two Others equal to them. So, likewise,* $\frac{B \times C}{A}$ *is not to be understood here in the Light of an Algebraic Fraction, but as a Right-line, which is Fourth Proportional to three other Right-lines, represented by* A, B, *and* C.—*These Distinctions are absolutely necessary to Those who would have an accurate Idea of the Subject.*

The second Particular, above hinted at, relates to the Quotations; wherein I have referred to my own Elements of Geometry, and

not

not to those of Euclid, so universally known and established. But for this there were two Reasons: First, those Persons, for whose Instruction these Sheets are, in a more particular manner, designed, are taught the First Principles of Geometry from other Elements than those of Euclid; and, secondly, a Number of Propositions are used Here that are only to be met with in modern Authors.

In the third Part the Theory of Gunnery, or the Motion of Projectiles, is considered, exclusive of the Conic Sections; and the practical Solutions of the several Cases depending on the Theory (as well Those where the Object is elevated or depressed as where it is situate in the Plain of the Horizon) are given, at large, by Plane Trigonometry.

The fourth Part exhibits a new, and very comprehensive Method for extracting the Roots of algebraical Equations; whereby the Number sought may be determined, to any proposed Degree of Exactness, without the Trouble of repeating the Operation, as in the common Way, by Converging Series's.

The fifth Part gives some Account of the Nature of Fluxions, together with the Investigation of the Fundamental Rules; and may be of Use, not only to Beginners, but also to

Such,

Such, who, though tolerably well versed in the Practice and Application of Fluxions, have nevertheless but an imperfect Idea of the First Principles of this difficult Branch of Science.

The sixth, and last, Part, is concerned about the Valuation of Annuities, on single, and joint Lives; wherein, besides a new Set of Tables, far more comprehensive than any yet published, are given the Solutions of upwards of Forty different Problems, on the most important and intricate Cases of the Subject: Many of which are quite new; and are, besides, such as actually occur in Business, being, most of them, taken from real Cases, proposed to the Author's Consideration, by Gentlemen in the Law, and Others.

This, sixth, Part, which the Reader will perceive is upon a different Plan from the five preceding Ones, was designed as a Supplement to my Doctrine of Annuities and Reversions, *printed in* 1742; *but, being thought too small to publish alone, it is inserted Here. This, if a Fault, will not, perhaps, be look'd upon as inexcusable: Though, as to the Performance itself, I do not in the least doubt but that it will be depreciated by Some, on Account of the Observations whereon the Calculations are grounded. I am sensible that there neither is, nor can*

can be, a Table of Observations on the Degrees of Mortality of Mankind, but what may be objected to; and that those Persons who make a very despicable Figure, when They come to Calculations, seldom fail of Displaying their Talents, and being illustrious Here; where, gratifying themselves in the Liberty which the Nature of the Subject allows them, They can boldly lanch out, without having to do with Science and Demonstration. But, though the London *Bills of Mortality, whereon I build, appear to me to be the best Foundation, at least, for this Place; yet I have no Inclination to enter the Lists with any of these Gentlemen. The Examples, given hereafter, are indeed wrought according to the* London *Bills; but the Solutions themselves are general, and may be apply'd, with equal Facility and Advantage, to any Table of Observations.*

PART

PART I.

CONTAINING

A select Number of ALGEBRAICAL PROBLEMS, *with their* SOLUTIONS.

DESIGN'D

As proper Exercises for young Beginners.

QUESTION I.

WHAT Number is that, which being doubled and 16 *added to the Product, the Sum shall be* 188?

Let x represent the required Number;
then $2x$ will denote the Double thereof;
and so $2x+16=188$, *by the Question.*
Therefore $2x=188-16=172$, *by Transposition.*

And $x=\frac{172}{2}=86$, *by Division.*

QUESTION II.

To find that Number, which being added to 56, *the Treble of the required Number shall be produced.*

If x be put for the Number sought,
then $3x$ will be the Treble thereof:
And therefore $3x=x+56$, *by the Question.*
Hence $2x=56$, *by Transposition.*

And $x=\frac{56}{2}=28$, *by Division.*

QUESTION III.

The Sum of 155 *l.* *was raiſed (for a certain Purpoſe) by three different Perſons,* A, B, *and* C; *whereof* B *advanced* 15 *l.* *more than* A; *and* C, 20 *l.* *more than* B: *How much did each contribute?*

Let x be the Number of Pounds advanced by A:
Then $x+15$, is the Number of Pounds advanced by B,
and $x+35$ the Number of Pounds advanced by C.
Therefore $3x+50=155$, *by the Queſtion.*
Whence $3x=105$,
and $x=35$.
From which it alſo appears that B contributed 50 *l.* and C 70 *l.*

QUESTION IV.

A Gentleman (by Will) left 550*l.* *to be divided among four Servants* A, B, C, *and* D; *whereof* B *was to have twice as much as* A; C *as much as* A *and* B; *and* D *as much as* C *and* B. *How much had Each?*

Let x be the Number of A's Pounds.
Then $2x$ is the Number of B's Pounds,
alſo $3x$ is the Number of C's Pounds,
and $5x$ the Number of D's Pounds:
Therefore $11x=550$, *by Queſtion.*

And, conſequently, $x=\frac{550}{11}=50$.

From which the reſt of the Shares are eaſily determined.

QUESTION V.

'Tis required to divide the Number 92 *into four ſuch Parts, that the Firſt may exceed the Second by* 10, *the Third by* 18, *and the Fourth by* 24.

Let x be the firſt Part.

Then $\left\{\begin{array}{l} x-10 \\ x-18 \\ x-24 \end{array}\right\}$ will be the other Parts.

And

And $4x-52=92$, *by the Question.*
Hence $4x=144$;
and $x=\frac{144}{4}=36$.

QUESTION VI.

A certain Sum of Money was shared among five Persons, A, B, C, D, *and* E; *whereof* B *received* 10 *l.* *less than* A; C 16 *l.* *more than* B; D 5 *l.* *less than* C; *and* E 15 *l.* *more than* D: *Moreover it appeared that* E *received as much as both* A *and* B. *What was the whole Sum shared, and how much did Each receive?*

Let x be the Share of A.

Then $\left\{\begin{array}{l} x-10 \\ x+6 \\ x+1 \\ x+16 \end{array}\right\}$ will be that of $\left\{\begin{array}{l} B, \\ C, \\ D, \\ E: \end{array}\right.$

Therefore $x+16=2x-10$, *by the Question.*
Whence $26=x$.

From which it appears that 26, 16, 32, 27, and 42 (Pounds) were the respective Shares; and that the whole Sum was 143 *l.*

QUESTION VII.

To find that Number, whereof the Double increased by 24, *shall as much exceed* 80, *as the Number itself is below* 100.

Let x be the required Number.
Then $2x+24-80=100-x$, *by the Question.*
Whence $2x+x=100-24+80$,
that is $3x=156$;
and therefore $x=\frac{156}{3}=52$.

QUESTION VIII.

What two Numbers are those, whereof the Difference is 7 and the Sum 33.

Let x be the lesser Number,
then $x+7$ will be the greater,
and $2x+7=33$.
Therefore $2x=33-7=26$,
and $x=\frac{26}{2}=13=$ the Lesser,
Whence $x+7=20=$ the Greater.

QUESTION IX.

To divide the Number 75 into two such Parts, that 3 times the greater may exceed 7 times the lesser by 15.

If x be the greater Part,
then $75-x$ will be the Lesser } *by the Question.*
and $3x=\overline{75-x}\times 7+15$ } *by the Question.*
that is, $3x=525-7x+15$;
therefore $10x=540$, and $x=54$.
From whence the lesser Part $(75-x)$ is found $=21$.

QUESTION X.

A, *after winning* 10 *Guineas of* B, *had as much Money as* B *and* 6 *Guineas more*; *and betwixt them both they had forty Guineas: What Money had Each at first?*

Let x be the Guineas that A began with;
then $40-x$, are the Guineas that B began with:
Therefore, after Play,
A had, $x+10$ Guineas;
and B, $30-x$, Guineas.
Whence $x+10=30-x+6$ *(by the Question.)*
Therefore $2x=26$.
And $x=13$.

QUESTION XI.

The Sum of 500 *l. was divided among four Persons, so that the First and Second, between them, had* 280 *l*; *the First and Third* 260 *l*; *and the First and Fourth* 220 *l. How many Pounds had Each?*

If x be the Number of Pounds the First had,

then the $\left\{\begin{array}{l}2d\\3d\\4th\end{array}\right\}$ had $\left\{\begin{array}{l}280-x\\260-x\\220-x\end{array}\right.$

The Sum of all which,
being $760-2x$, is $=500$, *by the Question.*

Whence $x=\frac{760-500}{2}=130$:

Therefore the four Shares were 130, 150, 130, and 90 Pounds, respectively.

QUESTION XII.

'Tis proposed to divide 60 *into two such Parts, that the Difference between the Greater and* 64, *may be equal to twice the Difference between the Lesser and* 38.

If x be the greater Part;
then $60-x$ will be the lesser:
Also $64-x$, will be the first mentioned Difference,
and $38-60+x$, or $x-22$, will be the Second.

Therefore $64-x=2\times\overline{x-22}$, *by the Question.*
that is, $64-x=2x-44$.
Whence $108=3x$, and $36=x$.

QUESTION XIII.

After 34 *Gallons had been drawn out of one of two, equal, Casks, and* 80 *Gallons out of the Other, there remained just twice as much Liquor in the One as in the Other: What did each Cask contain when full?*

Let x be the Number of Gallons sought;
then $x-34$ will be what remained in the first Cask,

and $x-80$, what remained in the Second.

Hence $x-34=2\times\overline{x-80}$, *by the Question.*
Or, $x-34=2x-160$.
Therefore $126=x$.

QUESTION XIV.

A Son, aſking his Father how old he was, received the following Anſwer: Your Age four Years ago, ſays the Father, was only of mine, at that Time; but now your Age is juſt of mine: What was the Age of Each?

Let x be the Age of the Son,
then $3x$ will be that of the Father:
Alſo $x-4$ will be the Age of the Son, four Years before the Time in Queſtion; and $3x-4$ will be the correſponding Age of the Father: which, *by the Queſtion*, is equal to 4 times $x-4$: Hence we have this Equation
$4x-16=3x-4$.
Therefore $x=12$, and $3x=36$; which are the two Ages required.

QUESTION XV.

What Number is that, whoſe $\frac{1}{3}$ *exceeds its* $\frac{1}{4}$ *Part by* 16?

Let x be the required Number; Then, its $\frac{1}{3}$ Part being $\frac{x}{3}$, and its $\frac{1}{4}$ Part $\frac{x}{4}$,

we have $\frac{x}{3}-\frac{x}{4}=16$, *by the Queſtion.*

Hence $4x-3x=192$, *by Reduction*;
that is, $x=192$.

QUESTION XVI.

In a Mixture of Wine and Cyder, one half of the whole + 25 Gallons was Wine; and, $\frac{1}{3}$ *Part − 5 Gallons, Cyder. How many Gallons were there of Each?*

If x be put for the Number of Gallons in the whole Mixture, the Gallons of Wine will be expreſſed by $\frac{x}{2}+25$, and thoſe of Cyder by $\frac{x}{3}-5$: Which together being equal to the *whole*, we

therefore have $\frac{x}{2}+25+\frac{x}{3}-5=x$;

Or, $20=x-\frac{x}{2}-\frac{x}{3}$:

Hence $120=6x-3x-2x$;

Or $120=x$. From which it appears that the Mixture conſiſted of 85 Gallons of Wine, and 35 of Cyder.

QUESTION XVII.

In a Lottery, conſiſting of 100000 *Tickets, half the Number of Prizes added to* $\frac{1}{3}$ *of the Number of Blanks, was* 35000. *How many Prizes were there in the Lottery?*

If x be the Number of Prizes;
then $100000-x$, will be the Number of Blanks.

And ſo, $\frac{x}{2}+\frac{100000-x}{3}=35000$.

Hence $3x+200000-2x=210000$:
And $x=10000=$ the Number ſought.

QUESTION XVIII.

To the Composition of a certain Quantity of Gunpowder, $\frac{1}{2}$ of the whole Weight + 6 lb. of Saltpetre was necessary; the Sulphur used was $\frac{1}{3}$ of the whole — 5 lb. and the Charcoal $\frac{1}{4}$ of the whole — 3 lb. How many Pounds of each of the three Ingredients were there taken?

Let x be the Number of Pounds in the Whole: Then

there were $\begin{cases} \frac{x}{2}+6, \text{ Pounds of Saltpetre.} \\ \frac{x}{3}-5, \text{ Pounds of Sulphur.} \\ \frac{x}{4}-3, \text{ Pounds of Charcoal.} \end{cases}$

And therefore $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}-2=x$, *by the Question.*

Whence $12x+8x+6x-48=24x$:

And consequently $x=\frac{48}{2}=24$.

Therefore there were taken 18 lb. of Saltpetre, 3 lb. of Sulphur, and 3 lb. of Charcoal.

QUESTION XIX.

A General, after having lost a Battle, found that he had only $\frac{1}{2}$ of his Army + 3600 left, fit for Action; $\frac{1}{8}$ of his Men + 600 being wounded, and the Rest, which were $\frac{1}{5}$ of the whole Army, either slain, taken Prisoners, or missing. What was the whole Number of the Army?

The Number sought being denoted by x, the Number of Men left unhurt will be $\frac{x}{2}+3600$.

And the Number of the Wounded $\frac{x}{8}+600$:

To which adding, $\frac{x}{5}$, the Number of the Slain and Prisoners, we have the Number of the whole Army;

Or

Or $\frac{x}{2} + 3600 + \frac{x}{8} + 600 + \frac{x}{5} = x$.

Hence $4200 \left(= x - \frac{x}{2} - \frac{x}{8} - \frac{x}{5} \right) = \frac{3x}{8} - \frac{x}{5}$

Therefore $168000\ (= 15x - 8x) = 7x$:

And $24000 = x$.

QUESTION XX.

A Prize of 2000 *l.* *was divided between two Perſons,* A *and* B; *whoſe Shares therein were in proportion as* 7 *to* 9: *What was the Share of Each?*

Let x denote the Share of A;
then $2000-x$, will be That of B.
But $x : 2000-x :: 7 : 9$, *by the Queſtion.*
From whence (as the Rectangle of the two Extremes, of any four proportional Numbers, is equal to the Rectangle of the two Means) we get this Equation,

$9 \times x = \overline{2000 - x} \times 7$;
that is, $9x = 14000 - 7x$.
Hence $16x = 14000$,
and $x = 875$.
Therefore the Share of A was 875 *l.* and that of B, 1125 *l.*

QUESTION XXI.

To divide 44 *into two ſuch Parts, that the Greater increaſed by* 5, *may be to the Leſſer increaſed by* 7, *as* 4 *is to* 3.

If x be the greater Part,
$44-x$, will be the Leſſer,
and $x + 5 : 51 - x :: 4 : 3$, *by the Queſtion.*
Therefore (by multiplying Extremes and Means) we have
$3x + 15 = 204 - 4x$:
Whence $x = \frac{204 - 15}{7} = \frac{189}{7} = 27$.

QUESTION XXII.

To find two Numbers in the proportion of 1 *to* 2; *so that,* 12 *being added to Each, the Sums shall be in proportion as* 5 *is to* 7.

Let x be the lesser Number;
then $2x$, will be the Greater:
Hence $x+12 : 2x+12 :: 5 : 7$, *by the Question.*
Therefore $\overline{2x+12}\times 5 = \overline{x+12}\times 7$;
that is $10x+60=7x+84$.
From which $3x=24$; and $x=8$.
So that the two Numbers are 8 and 16.

QUESTION XXIII.

A, *at Play, first won* 5 *Guineas of* B, *and had then as much Money as* B; *but* B, *upon winning back his own Money and* 5 *Guineas more, had* 5 *times as much Money as* A: *What Money had Each, at first?*

If x be the Number of A's Guineas, at first; then it is plain, from the Question, that $x+10$ will be the Number of B's Guineas:
Whence $x+10+5$ $(=\overline{x-5}\times 5)=5x-25$,
that is, $40=4x$:
Therefore $x=10=$ the Number of A's Guineas;
and $x+10=20=$ the Number of B's Guineas.

QUESTION XXIV.

A Grocer, with 56 *lb. of fine Tea, at* 20 *Shillings a Pound, would mix a coarser sort, of* 14 *Shillings, so as to afford the whole, together, at* 18 *Shillings, per Pound: What Quantity of the latter Sort must he take?*

Let x be the Number of Pounds required;
then 1120 is the value of finest Sort,
and $14x$ that of the Coarsest.

Moreover,

Moreover, the Number of Pounds of both Sorts together being $56+x$, it is evident that $\overline{56+x}\times 18$, or $1008+18x$, is the Value of the whole Mixture. And therefore

$1120+14x=1008+18x$:

Whence $112=4x$:

And confequently $28=x$.

QUESTION XXV.

A Farmer would mix Wheat, at 4 Shillings a Bufhel, with Rye, at 2 s. 6 d. a Bufhel; fo that the whole Mixture may confift of 90 Bufhels, and be afforded at 3 s. 4 d. a Bufhel: 'Tis required to find how many Bufhels of each Sort muft be taken.

If the Number of Bufhels of Wheat be x,

Thofe of Rye will be $90-x$.

Moreover, the Value of the Wheat will be $48x$, Pence, and the Value of the Rye $\overline{90-x}\times 30$, Pence:

Whence $48x+\overline{90-x}\times 30=90\times 38$ (*by the Queftion.*)

that is, $48x+2700-30x=3420$:

Therefore $18x=720$:

And confequently $x=\frac{720}{18}=40$.

QUESTION XXVI.

A Workman was hired for 40 Days, at 3 s. 4 d. per *Day, for every Day he worked; but with this Condition, that, for every Day he played, he was to forfeit 1 s. 4 d: And it fo happened, that, upon the whole, he had 3 l. 3 s. 4 d. to receive. The Queftion is, to find how many Days, of the 40, he work'd, and how many he play'd.*

Let x be the Number of Days he work'd;

then $40-x$ will be the Number of Days he play'd:

Moreover, as he was to receive 40 Pence, for every Day he work'd, and to forfeit 16 Pence for every Day he play'd,

we have $40x =$ the Number of Pence earn'd by Work, and $\overline{40-x} \times 16 =$ the Number of Pence forfeited by Play:
Whence $40x - \overline{40-x} \times 16 = 760$, *by the Question*;
that is, $40x - 640 + 16x = 760$:
Therefore $56x = 1400$;
and consequently $x = 25$.
From which it is evident that he work'd 25 Days, and play'd 15.

QUESTION XXVII.

A Bill of 120 *l. was paid in Guineas and Moidores, and the Number of Pieces used of both Sorts was just* 100: *How many were there of Each?*

If x be put for the Number of Guineas; then $100-x$ will be the Number of Moidores:

And so, $21x + \overline{100 - x} \times 27 = 120 \times 20$ } *by the*
Or, $21x + 2700 - 27x = 2400$ } *Question.*
Hence $300 = 27x - 21x = 6x$.

And consequently $x = \frac{300}{6} = 50$.

QUESTION XXVIII.

One bought 30 *Pounds of Sugar, of two different Sorts, and paid for the whole* 19 *Shillings; the best Sort cost* 10 *d.* per *Pound, and the worst* 7 *d. How many Pounds were there of Each?*

Let x stand for the Number of Pounds of the best Sort, and then $30-x$ will express the Pounds of the other Sort;
Therefore $x \times 10 + \overline{30-x} \times 7 = 19 \times 12$, *by the Question*;
that is, $10x + 210 - 7x = 228$:
Whence $3x = 228 - 210 = 18$,
and $x = 6$. Therefore there were 6 Pounds of the best Sort, and 24 of the Worst.

QUESTION

QUESTION XXIX.

A Lady gave a Guinea, in Charity, among a Number of Poor, consisting of Men, Women, and Children: Each Man had 12 d. each Woman 6 d. and each Child 3 d. Moreover there were twice as many Women as Men, wanting 2; and 3 times as many Children as Women, wanting 4: How many Persons were there relieved?

Let x be the Number of the Men;
then $2x-2$, will be the Number of Women,
and $6x-10$, the Number of Children.
Hence $x \times 12 + \overline{2x-2} \times 6 + \overline{6x-10} \times 3 = 21 \times 12$;
that is, $12x + 12x - 12 + 18x - 30 = 252$,
or, $42x = 294$:
Whence $x = 7$.
Therefore there were 7 Men, 12 Women, and 32 Children; in all 51 Persons.

QUESTION XXX.

To find that Number, which being divided, either into three, or four, equal Parts, the continual Product of all the Parts, in both Cases, shall be exactly the same.

Let x be the required Number;
so shall the continual Product of the three equal Parts be $\frac{x}{3} \times \frac{x}{3} \times \frac{x}{3} = \frac{x^3}{27}$; and that of the four equal Parts $\frac{x}{4} \times \frac{x}{4} \times \frac{x}{4} \times \frac{x}{4} = \frac{x^4}{256}$.
Whence $\frac{x^4}{256} = \frac{x^3}{27}$, *by the Question.*

Therefore $27x = 256$; and $x = 9\frac{13}{27}$.

QUESTION XXXI.

To find two Numbers, in the Proportion of 3 to 4, whose Sum is to the Sum of their Squares, as 7 to 50.

Let $3x$ denote the lesser Number:
Then $4x$ will express the greater.
And we shall have $3x+4x : 9x^2+16x^2 :: 7 : 50$,
or, $7x : 25x^2 :: 7 : 50$, *by the Question.*
Therefore $25x^2 \times 7 = 7x \times 50$,
or, $25x^2 = 50x$;
whence $x = \frac{50}{25} = 2$: So that 6 and 8 are the two Numbers that answer the Question.

QUESTION XXXII.

To find two Numbers in the Proportion of 9 to 7; so that the Square of their Sum, and the Cube of their Difference, shall be equal.

If $9x$ be put for the greater Number;
then $7x$ will be the lesser;
And so $\overline{16x}|^2 = \overline{2x}|^3$, *by the Question,*
that is, $256x^2 = 8x^3$.
Hence $x = \frac{256}{8} = 32$:
Therefore 288 and 224, are the two Numbers sought.

QUESTION XXXIII.

To find two Numbers whose Difference is 4 and the Difference of their Squares 120.

Let x be the lesser Number;
then $x+4$, will be the greater:
Also xx will be the Square of the lesser,
and $xx+8x+16$ that of the greater:
Whence $8x+16=120$, *by the Question:*
Therefore $8x=104$; and $x=13$:
So that 13 and 17 are the two Numbers that were to be found.

QUESTION

QUESTION XXXIV.

To divide 100 *into two ſuch Parts, that the Difference of their Squares may be* 1000.

If x be the greater Part,
$100-x$, will be the leſſer:
Therefore $xx - \overline{100-x}|^2 = 1000$;
that is, $x^2 - 10000 + 200x - x^2 = 1000$:
Whence $200x = 11000$;
and conſequently $x = \frac{11000}{200} = 55$.

QUESTION XXXV.

To divide 100 *into two Parts, ſo that the Square of their Difference may exceed the Square of twice the leſſer Part by* 2000.

The leſſer Part being denoted by x,
the greater will be expreſſed by $100-x$,
and the Difference by $100-2x$.
Therefore, by the Problem, $\overline{100-2x}|^2 = \overline{2x}|^2 + 2000$,
that is, $10000 - 400x + 4xx = 4xx + 2000$,
or, $10000 - 2000 = 400x$.
Hence $x = \frac{8000}{400} = 20$; and $100-x = 80$.

QUESTION XXXVI.

A *and* B *make a joint Stock of* 500 *l. by which they gain* 160 *l. whereof* A, *for his Share, had* 32 *l. more than* B: *What did each Perſon bring into Stock?*

If x be the Number of Pounds advanced by A;
then it will be, as 500 (the whole Stock) is to 160 (the whole Gain) ſo is x (the Stock of A) to $\frac{160x}{500}$, the Gain of A: Whence the Gain of B being $160 - \frac{16x}{50}$:
we

we have $\frac{16x}{50} = 160 - \frac{16x}{50} + 32$, *by the Question:*

Therefore $\frac{32x}{50} = 192$, or $\frac{x}{50} = 6$;

and consequently $x = 50 \times 6 = 300$.

QUESTION XXXVII.

A Sum of Money was divided between two Persons, A *and* B, *so that the Share of* A *was to That of* B, *as* 5 *to* 3, *and exceeded* $\frac{5}{9}$ *of the whole Sum by* 50 *l. What was the Share of each Person?*

Let $5x$ express the Share of A,
and $3x$ the Share of B;
then $8x$ will be the whole Sum,

and $\frac{40x}{9}$ will be $\frac{5}{9}$ thereof:

Therefore $5x - \frac{40x}{9} = 50$, *by the Question*;

that is, $\frac{5x}{9} = 50$, or $x = 90$.

Hence 450 *l.* and 270 *l.* are the two Shares required.

QUESTION XXXVIII.

A *and* B *began to play together with equal Sums of Money;* A *first won* 20 *Guineas, but afterwards lost back the Half of all he then had; and thereupon had only half as much Money as* B. *What Money did each begin with?*

Let x be the Number of Guineas required: Then A, after winning 20 Guineas, had $x + 20$; the Half of which, or $\frac{x}{2} + 10$, is therefore what he had at last: And this deducted from $2x$ (the whole Sum betwixt Both) leaves $2x - \frac{x}{2} - 10 =$ what B had at last.

Therefore

Therefore $2x - \frac{x}{2} - 10 = x + 20$;
whence $3x - 20 = 2x + 40$,
and consequently $x = 60$.

QUESTION XXXIX.

A Gentleman left his whole Estate among his four Sons; whereof the Eldest had $\frac{1}{2}$ *wanting* 800 *l. the Second* $\frac{1}{4}$ *and* 120 *l. over; the Third had half as much as the Eldest; and the Youngest* $\frac{2}{3}$ *of what the Second had. What was the whole Estate? and how much had Each?*

Let x be the whole Estate; then

The First had $\frac{x}{2} - 800$.

The Second $\frac{x}{4} + 120$.

The Third $\frac{x}{4} - 400$.

The Fourth $\frac{2x}{12} + 80$.

The Sum of all which is, consequently, equal to the Whole,

Or, $\frac{x}{2} + \frac{x}{4} + \frac{x}{4} + \frac{x}{6} - 1000 = x$.

But it is plain (without Reduction)

that $\frac{x}{2} + \frac{x}{4} + \frac{x}{4} = x$.

Hence our Equation becomes

$\frac{x}{6} - 1000 = 0$, or $\frac{x}{6} = 1000$.

Therefore the whole Estate was 6000 *l.* whereof the eldest Son had 2200 *l.* the Second 1620 *l.* the Third 1100 *l.* and the Youngest 1080 *l.*

QUESTION XL.

One being ask'd his Age, reply'd; If $\frac{2}{5}$ of my Years be multiply'd by 3, and $\frac{1}{3}$ of them be added to the Product, the Amount will be 115. What was his Age?

If x be the required Number of Years,

then $\frac{2x}{5} \times 3 + \frac{x}{3} = 115$, *by the Question*;

that is, $\frac{6x}{5} + \frac{x}{3} = 115$.

Therefore $18x + 5x = 15 \times 115$,

or $23x = 15 \times 115$:

Consequently $x = \frac{15 \times 115}{23} = 15 \times 5 = 75$.

QUESTION XLI.

A Person being ask'd the Hour of the Day, answered thus: If $\frac{1}{8}$ of the Number of Hours remaining till Midnight be multiply'd by 4, the Product will as much exceed 12 Hours, as Half the present Hour from Noon is below 4: What was the Hour after Noon?

Let x be the required Hour;

then $12 - x$, will be the Hours till Midnight,

and $\frac{36-3x}{8} \times 4 - 12 = 4 - \frac{x}{2}$, *by the Question.*

That is, $\frac{36-3x}{2} - 12 = 4 - \frac{x}{2}$,

whence $36 - 3x - 24 = 8 - x$;

and consequently $x = 2$.

QUESTION

QUESTION XLII.

A Market-woman bought in a certain Number of Eggs, at the Rate of 5 for two Pence; one half of which she sold out again at 2 a Penny, and the remaining Half at 3 a Penny; and cleared 4 Pence, by so doing: What Number of Eggs had she?

Let $2x$ be the Number sought; then, *by the Question,*

$5 : 2 :: 2x : \frac{4x}{5}$ the Number of Pence the Eggs cost:

But the Number of Pence They were sold for, again, is $\frac{x}{2}+\frac{x}{3}$: Therefore we have this Equation,

viz. $\frac{x}{2}+\frac{x}{3}-\frac{4x}{5}=4$:

From whence $15x+10x-24x=120$, or $x=120$.

QUESTION XLIII.

A certain Sum of Money, put out at Interest, amounts, in 8 Months, to 297 l. 12 s. And, in 15 Months its Amount (computed according to simple Interest) is 306 l. What is that Sum? And what the Rate of Interest?

Let x be the Number of Pounds in the required Sum: Then, the Interest thereof for 8 Months being $297,6-x$, and for 15 Months $306-x$, we have,

as $8 : 15 :: 297,6-x : 306-x$:

Whence, by multiplying Extremes and Means,

we get $2448-8x=4464-15x$.

Therefore $7x=2016$,

and consequently $x=288$ *l.* the Sum required.

For the Rate of Interest, it will be,

as 288 *l.* × 15 : 100 *l.* × 12 : : 18 *l.* (the Interest of 288 *l.* for 15 Months) to 5 *l.* the required Interest of 100 *l.* for 12 Months.

QUESTION XLIV.

A Waterman finds by Experience, that he can, with the Advantage of a common Tide, row from London to Greenwich, which is 5 Miles, in 3 Quarters of an Hour; and that, to return to London, against an equal Tide, though he rows back along-ſhore, where the Stream is only half as ſtrong as in the Middle, takes him a full Hour and Half. 'Tis required to find, from hence, at what Rate, per *Hour, the Tide runs in the Middle where it is ſtrongeſt.*

In the firſt Place, it will be

qr. qr. m m.
$3 : 4 :: 5 : 6\frac{2}{3} =$ Diſt. row'd *per* Hour with the Tide,
$6 : 4 :: 5 : 3\frac{1}{3} =$ Diſt. row'd *per* Hour againſt Tide.
If now the former of theſe two Diſtances ($6\frac{2}{3}$) be put$=a$, and the latter $(3\frac{1}{3})=b$; and x be aſſumed to expreſs the required Diſtance run, *per* Hour, by the Stream in the Middle of the River; then $a-x$ will be the *real* Effect of his Rowing, *per* Hour, in going from London, the Motion of the Tide being deducted; and $b+\frac{x}{2}$ will be the like Effect in his Return:
And ſo, theſe two Quantities being equal to each other, we have $b+\frac{x}{2}=a-x$:
Whence $2b+x=2a-2x$;
and conſequently $x=\frac{2a-2b}{3}=2\frac{2}{9}$ m.

QUESTION XLV.

To divide 36 (a) into 3 ſuch Parts, that $\frac{1}{2}$ of the Firſt, $\frac{1}{3}$ of the Second, and $\frac{1}{4}$ of the Third, may be equal to each other.

If x be put for the firſt Part,
then is $\frac{x}{2}=\frac{1}{3}$ of the ſecond Part, *(by the Queſtion.)*

And

And so $\frac{3x}{2}$ = second Part.

Moreover $\frac{x}{2}$ being $= \frac{1}{4}$ of the third Part,

therefore $\frac{4x}{2}$ (or $2x$) = third Part.

Hence $x + \frac{3x}{2} + 2x = a$,

and $2x + 3x + 4x = 2a$.

Consequently $x = \frac{2a}{9} = \frac{2 \times 36}{9} = 2 \times 4 = 8$.

From which the second Part $\left(\frac{3x}{2}\right)$ appears to be $= 12$, and the Third $(2x) = 16$.

QUESTION XLVI.

To divide the Number 90 *into* 4 *such Parts, that, if the first be increased by* 5, *the second diminished by* 4, *the third multiply'd by* 3, *and the fourth divided by* 2, *the Result, in each Case, shall be exactly the same.*

Let x be the fourth, or last, Part:

Then, three times the third Part being $= \frac{x}{2}$,

the third Part will be $\frac{x}{6}$.

Moreover, the second Part -4 being, also, $= \frac{x}{2}$,

the second Part will be $\frac{x}{2} + 4$:

And, the first Part $+ 5$ being $= \frac{x}{2}$,

the first Part, alone, will be $\frac{x}{2} - 5$.

And, by adding all the Parts thus found together, we have $x + \frac{x}{6} + \frac{x}{2} + 4 + \frac{x}{2} - 5 = 90$;

that is, $2x+\frac{x}{6}-1=90$.

Whence $13x=91\times6$; and $x=42$.
Therefore the four required Parts are 16, 25, 7, and 24, respectively.

QUESTION XLVII.

Two Workmen A *and* B *were employed together for* 50 *Days, at* 5 *shillings* per *Day, Each; during which Time* A, *by spending only Sixpence a Day less than* B, *had saved twice as much as* B, *besides the expence of* 2 *Days over: What did each Person expend a Day?*

Let x be the Pence A spent *per* Day;
then $60-x$, will be what he saved *per* Day,
and $54-x$, what B saved.
Therefore $3000-50x$ are A's whole savings,
and $2700-50x$ those of B.
Hence $3000-50x=2\times\overline{2700-50x}+2x$;
Or, $3000-50x=5400-98x$:
From which $48x=2400$, and $x=50$.

QUESTION XLVIII.

Two Persons, A *and* B, *have both the same Income;* A *lays by* $\frac{1}{5}$ *of his; but* B, *by spending* 60 *l.* per Ann. *more than* A, *at the End of three Years finds himself* 100 *l. in debt. What did Each receive, and expend,* per Annum?

Let x be the yearly Income of Each;
then $\frac{4x}{5}$ is the Sum expended by A, *per Ann.*
and $\frac{4x}{5}+60$, That expended by B.
Therefore $\frac{4x}{5}+60-x$, is what B runs in Debt.
Consequently $\overline{\frac{4x}{5}+60-x}\times3=100$,

or

or $\frac{12x}{5}+180-3x=100$;

that is, $180-\frac{3x}{5}=100$.

Whence $900-3x=500$,

and $x=\frac{400}{3}=133l.\ 6s.\ 8d.$

Therefore A expended 106*l.* 13*s.* 4*d.* and B 166*l.* 13*s.* 4*d.* *per Annum.*

QUESTION XLIX.

A Grazier bought in as many Sheep, of different Sorts, as cost him 33l. 7s. 6d. For the first Sort, which were ⅓ of the whole, he paid 9s. 6d. a-piece; for the second Sort, which were ¼ of the whole, he paid 11s. each; and for the rest, 12s. 6d. each: What Number of Sheep did he buy in all?

If x be the whole Number of Sheep;

then, the Number of the first Sort being $\frac{x}{3}$, and of the second Sort $\frac{x}{4}$, the Number of the remaining Sort (at 12*s.* 6*d.* each) must be $x-\frac{x}{3}-\frac{x}{4}=\frac{12x-4x-3x}{12}=\frac{5x}{12}$: Whence, by the conditions of the Problem, we have

$\frac{x}{3}\times 19+\frac{x}{4}\times 22+\frac{5x}{12}\times 25=1335$;

that is, $\frac{19x}{3}+\frac{22x}{4}+\frac{125x}{12}=1335$.

Let each Term of this Equation be now multiply'd by 12, and it will become

$76x+66x+125x=16020$,

or, $267x=16020$.

Therefore $x=60$.

QUESTION L.

A Draper, of a Piece of Cloth, ſtanding him in 3*s.* 2*d.* per *Yard, ſold* $\frac{1}{3}$ *Part, at* 4*s.* per *Yard*; $\frac{1}{4}$ *at* 3*s.* 8*d.* per *Yard*; $\frac{1}{5}$ *at* 3*s.* 6*d.* per *Yard*; *and the Remnant at* 3*s.* 4*d. a Yard: And his Gain upon the Whole, was* 15*s.* 2*d. How many yards did the Piece contain?*

If the Number ſought be denoted by x; then the Number of yards in the Remnant will be $x-\frac{x}{3}-\frac{x}{4}-\frac{x}{5}=\frac{60x-20x-15x-12x}{60}=\frac{13x}{60}$.
Therefore, by the Queſtion, we have

$$\frac{x}{3}\times 48+\frac{x}{4}\times 44+\frac{x}{5}\times 42+\frac{13x}{60}\times 40-38x=182,$$

Or, $16x+11x+\frac{42x}{5}+\frac{26x}{3}-38x=182$,

that is $\frac{42x}{5}+\frac{26x}{3}-11x=182$:

Whence $126x+130x-165x=2730$:

And therefore $x=\frac{2730}{91}=30$.

QUESTION LI.

A Diſtiller propoſes to mix Foreign Brandy, ſtanding him in 8 *Shillings a Gallon, with Britiſh Spirits of* 3 *Shillings* per *Gallon, in ſuch Proportion that he may gain* 30 per Cent *by ſelling out the Compound at* 9*s. a Gallon. What is that Proportion?*

Suppoſe, that, with a Gallons of Brandy, he mixes x Gallons of Spirits; then, the Brandy, ſtanding him in $8a$ (Shillings) and the Spirits in $3x$ (Shillings), the true value of the whole Mixture will be $8a+3x$:
But the Value of $a+x$ Gallons, at 9 Shillings *per* Gallon, is $9a+9x$: Therefore, by laying out $8a+3x$, he gains $a+6x$: And ſo
we have $8a+3x : a+6x :: 100 : 30$, *by the Queſtion.*

Conſequently

Consequently $100a+600x=240a+90x$;
whence $510x=140a$,
and $x=\frac{14a}{51}$.

From which it appears that, to every 51 Gallons of Brandy, there muſt be taken 14 Gallons of Spirits.

QUESTION LII.

To find two Numbers in the Proportion of 4 *to* 5, *from which two other (required) Numbers, in the proportion of* 6 *to* 7 *being, reſpectively, deducted, the Remainders ſhall be in the proportion of* 2 *to* 3, *and their Sum equal to* 20.

Let $4x$ and $5x$ be the 2 firſt Numbers, and $6y$ and $7y$, the other 2 Numbers.

Then $4x-6y : 5x-7y :: 2 : 3$ }
And $9x-13y=20$ } *by the Queſtion.*

From which Proportion, by multiplying Extremes and Means, we have $12x-18y=10x-14y$, and therefore $x=2y$; which ſubſtituted in the above Equation gives $18y-13y=20$;

whence $y=\frac{20}{5}=4$; and $x\ (=2y)=8$.

Therefore the 2 firſt Numbers are 32 and 40; and the other Two, 24 and 28.

QUESTION LIII.

A Farmer ſold, at one time, 30 *Buſhels of Wheat and* 40 *of Barley, and for the whole received* 13 *l.* 10 *s*; *and, at another time, he ſold* 50 *Buſhels of Wheat and* 30 *of Barley, at the ſame Prices as before, and for the whole received* 17 *l. The Queſtion is, to find what each Sort of Grain was ſold at* per *Buſhel.*

Let x and y expreſs the Numbers of Shillings, reſpectively, that the Wheat and Barley were ſold at *per* Buſhel; and then, from the Conditions of the Queſtion, we ſhall have the two following Equations, *viz.*

30x

$30x+40y=270,$
$50x+30y=340.$

From 4 times the Second of which Equations let 3 times the First be subtracted, and there will remain $110x=550.$

Therefore $x=\frac{55}{11}=5$:

And $y\left(=\frac{27-3x}{4}\right)=3.$

QUESTION LIV.

A Farmer, with 28 *Bushels of Barley, at* 2*s.* 4*d.* per *Bushel, would mix Rye, at* 3*s.* per *Bushel, and Wheat, at* 4*s.* per *Bushel; so that the whole Mixture may consist of* 100 *Bushels, and be worth* 3*s.* 4*d. a Bushel: How many Bushels of Rye, and how many of Wheat must he mingle with the Barley?*

Let x be the Number of Bushels of Rye, and y those of the Wheat: Then, the value of the Barley being 784 (Pence), of the Rye $36x$ (Pence), and of the Wheat $48y$ (Pence), we have

$$\left.\begin{array}{l} 784+36x+48y=4000 \\ \text{and } 28+x+y=100 \end{array}\right\} \textit{by the Question.}$$

From the first of which Equations, take 36 times the second, and there results,

$784-36\times28+12y=400,$
that is, $-224+12y=400.$
Therefore $12y=624,$

and consequently $y=\frac{624}{12}=52.$

Whence $x\ (=100-28-y)=20.$

QUESTION

QUESTION LV.

A *and* B, *working together on the ſame Work, can earn* 40 *Shillings in* 6 *Days*; A *and* C *together can earn* 54 *Shillings in* 9 *Days*; *and* B *and* C, 80 *Shillings in* 15 *Days: 'Tis required to find what each Perſon, alone, can earn* per *Day.*

Let x, y and z expreſs the Numbers of Shillings in the three required Values, reſpectively.

Then $\left\{\begin{array}{l} 6x+6y=40 \\ 9x+9z=54 \\ 15y+15z=80 \end{array}\right\}$ *by the Queſtion.*

And $\left\{\begin{array}{l} x+y=6\frac{2}{3} \\ x+z=6 \\ y+z=5\frac{1}{3} \end{array}\right\}$ *by Diviſion.*

Hence $y-z=\frac{2}{3}$, *by ſubtracting the* 2*d Equation from* 1*ſt.*
And $2y=6$, *by adding the two laſt.*
Conſequently $y=3$: From which
we have $x\ (=6\frac{2}{3}-y)=3\frac{2}{3}=3s.\ 8d.$ and $z\ (=5\frac{1}{3}-y)$ $=2\frac{1}{3}=2s.\ 4d.$

QUESTION LVI.

To find three Numbers, ſo that $\frac{1}{2}$ *the Firſt,* $\frac{1}{3}$ *of the Second, and* $\frac{1}{4}$ *of the Third, ſhall together be equal to* 62; *alſo* $\frac{1}{3}$ *of the Firſt,* $\frac{1}{4}$ *of the Second, and* $\frac{1}{5}$ *of the Third, equal to* 47; *and, laſtly,* $\frac{1}{4}$ *of the Firſt,* $\frac{1}{5}$ *of the Second, and* $\frac{1}{6}$ *of the Third, equal to* 38.

Put $a=62$, $b=47$, and $c=38$; and let the three required Numbers be denoted by x, y and z, reſpectively; then the Conditions of the Problem will be expreſſed in the three following Equations, *viz.*

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=a,$$
$$\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=b,$$
$$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=c.$$

Which,

Which, clear'd of Fractions, become

$12x + 8y + 6z = 24a$
$20x + 15y + 12z = 60b$
$30x + 24y + 20z = 120c.$

Now (in order to exterminate z) let the Second of these Equations be taken from the Double of the First, and also the Treble of the Third from the Quintuple of the Second; and there results

$4x + y = 48a - 60b$; and
$10x + 3y = 300b - 360c$:

Whence, by deducting the second of These from the Treble of the Former, and dividing by 2, there comes out

$x = 72a - 240b + 180c = 24.$

From which y $(= 48a - 60b - 4x)$ is also found $= 60$, and z $(= a - \frac{1}{2}x - \frac{1}{3}y \times 4) = 120.$

QUESTION LVII.

To divide the Number 90 (a) *into three Parts, so that, the Double of the first Part* + 40 (b); *the Treble of the Second* + 20 (c); *and the Quadruple of the Third* + 10 (d), *may be all equal to one another.*

Let x, y, and z represent the three required Parts, respectively; then, from the Conditions of the Problem, we shall have

$x + y + z = a,$
$2x + b = 3y + c,$
$2x + b = 4z + d.$

Now, in order to exterminate y and z, let 12 times the first Equation, 4 times the Second, and 3 times the Third, be added all together; and you will have

$26x + 12y + 12z + 7b = 12a + 12y + 12z + 4c + 3d.$

Therefore $26x = 12a + 4c + 3d - 7b,$

and $x = \dfrac{12a + 4c + 3d - 7b}{26} = 35.$

Whence $y\left(= \dfrac{2x + b - c}{3}\right) = 30$, and z $(= a - x - y) = 25.$

QUESTION LVIII.

To find three Numbers, so that the First with half the other Two, the Second with $\frac{1}{3}$ of the other Two, and the Third with $\frac{1}{4}$ of the other Two, may be the same, and amount to 51 in each Case.

Put $a = 51$, and let x, y and z denote the three required Numbers; then, by the Question,

$$x + \frac{y+z}{2} = a,$$

$$y + \frac{x+z}{3} = a,$$

$$z + \frac{x+y}{4} = a:$$

Which, cleared of Fractions, become

$$2x + y + z = 2a,$$

$$x + 3y + z = 3a,$$

$$x + y + 4z = 4a.$$

From whence, by taking the Second from the Third, and the First from the Double of the Second, there results $-2y + 3z = a$,
and $5y + z = 4a$.
And, by deducting the former of These, from the Treble of the latter, we have $17y = 11a$.

Therefore $y = \frac{11a}{17} = 33$,

$$z\ (= 4a - 5y) = \frac{13a}{17} = 39,$$

and $x\ (= 3a - 3y - z) = \frac{5a}{17} = 15$.

QUESTION

QUESTION LIX.

A certain Sum of Money was divided between three Persons, A, B *and* C; *so that,* A's *Share exceeded* $\frac{4}{7}$ *of the Shares of* B *and* C *by* 30 *l*; *also the Share of* B *exceeded* $\frac{3}{8}$ *of the Shares of* A *and* C *by* 30 *l*; *and the Share of* C *likewise exceeded* $\frac{2}{9}$ *of the Shares of* A *and* B, *by* 30 *l. The Question is, to find the Share of each Person.*

Let $a=30$; and let x, y, and z be assumed to express the three required Numbers; then by the Conditions of the Problem,

$$x-\frac{4y+4z}{7}=a,$$

$$y-\frac{3x+3z}{8}=a,$$

$$z-\frac{2x+2y}{9}=a.$$

Whence, by Reduction,

$$7x-4y-4z=7a,$$

$$-3x+8y-3z=8a,$$

$$-2x-2y+9z=9a.$$

Now, to get rid of y (which, because of the even Coefficients, is the easiest to be exterminated) let the Double of the first Equation and the Quadruple of the Third be, successively, added to the Second; by means whereof we have

$$11x-11z=22a,$$

$$-11x+33z=44a.$$

Moreover, by adding these two last Equations together, we have $22z=66a$.

Therefore $z=3a=90$;

whence $x\ (=2a+z)=5a=150$,

and $y\ \left(=a+\frac{3}{8}\times\overline{x+z}\right)=4a=120.$

QUESTION

QUESTION LX.

If A *and* B *together can perform a Piece of Work in* 8 *Days*; A *and* C *together in* 9 *Days*; *and* B *and* C *in* 10 *Days*: *How many Days will it take each Person, alone, to perform the same Work?*

Let the whole Work be represented by a, and let x, y, and z stand for the Parts thereof performed by A, B, and C in one Day, respectively.

Then will $\left\{\begin{array}{l} 8x+8y=a \\ 9x+9z=a \\ 10y+10x=a \end{array}\right\}$ *by the Question.*

And $\left\{\begin{array}{l} x+y=\frac{a}{8} \\ x+z=\frac{a}{9} \\ y+z=\frac{a}{10} \end{array}\right\}$ *by Division.*

Whence $y-z\left(=\frac{a}{8}-\frac{a}{9}\right)=\frac{a}{72}$,

And $2y=\frac{a}{10}+\frac{a}{72}=\frac{82a}{720}$.

Consequently $y=\frac{41a}{720}$: From which $x\left(=\frac{a}{8}-y\right)$ is found $=\frac{49a}{720}$,

and $z\left(=\frac{a}{10}-y\right)=\frac{31a}{720}$.

Now, the Part of the Work (a) performed by each Person in one single Day being thus assigned, the Number of Days it will take any one of them to do the Whole, will be found by dividing the Whole by the assigned Part.

Thus, $\frac{41a}{720}\Big)\, a = \frac{720a}{41a}=\frac{720}{41}=17\frac{23}{41}$ is the Number of Days in which B, alone, can do the Whole. And, in like

like manner, the Number of Days in which A, or C, can do the Whole, appears to be

$14\frac{34}{49}$, or $23\frac{7}{31}$, reſpectively.

QUESTION LXI.

If A, B *and* C *can, together, finiſh a Piece of Work in* 9 *Days*; A, B *and* D *together, in* 10 *Days*; A, C *and* D *together, in* 11 *Days*; *and* B, C *and* D *in* 12 *Days*: *In how long Time can they all Four, together, finiſh It?*

Here, denoting the given Numbers by a, b, c, and d, and putting u, x, y, and z, for the Parts of the whole Work (g) done by Each in one Day, reſpectively, we ſhall, by the Queſtion, have theſe Equations,

Viz. $$\left\{\begin{array}{l} a\times\overline{u+x+y}=g \\ b\times\overline{u+x+z}=g \\ c\times\overline{u+y+z}=g \\ d\times\overline{x+y+z}=g \end{array}\right\}$$

or $$\left\{\begin{array}{l} u+x+y=\frac{g}{a} \\ u+x+z=\frac{g}{b} \\ u+y+z=\frac{g}{c} \\ x+y+z=\frac{g}{d} \end{array}\right\}$$ *by Diviſion.*

The Sum of all which, divided by 3,

gives $u+x+y+z=\frac{1}{3}\times\overline{\frac{g}{a}+\frac{g}{b}+\frac{g}{c}+\frac{g}{d}}$.

Therefore, ſeeing the Work done by all the Four, in one Day, is expreſſed by $\frac{1}{3}\times\overline{\frac{g}{a}+\frac{g}{b}+\frac{g}{c}+\frac{g}{d}}$, the whole Work g, divided hereby, will conſequently give

give $\dfrac{3g}{\frac{g}{a}+\frac{g}{b}+\frac{g}{c}+\frac{g}{d}}=\dfrac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}=$

$\dfrac{3abcd}{bcd+acd+abd+abc}=7\dfrac{1797}{2289}$, for the Number of Days required.

QUESTION LXII.

To find that Number whose square Root is to its Cube Root, in the Proportion of 5 *to* 2.

Let x^6 express the required Number:
Then x^3 will be its square Root,
and x^2 its Cube Root.
Now $x^3 : x^2 :: 5 : 2$, *by the Question.*
Therefore $2x^3=5x^2$, or $2x=5$;

or, lastly, $x=\dfrac{5}{2}=2.5$.

Whence $x^6=244.140625=$ the Number sought.

QUESTION LXIII.

To find two Numbers in the Proportion of 3 *to* 5; *whereof the fifth Power of the First shall be to the third Power of the Second, as* 972 *to* 125.

If $3x$ be put for the first Number,
then $5x$ will express the Second; and we shall
have $\overline{3x}|^5 : \overline{5x}|^3 :: 972 : 125$, *by the Question,*
that is, $243x^5 : 125x^3 :: 972 : 125$:
Hence $243x^5 \times 125 = 125x^3 \times 972$,
Or, $243x^2 = 972$.

Therefore $x^2=\dfrac{972}{243}=4$;

and $x=\sqrt{4}=2$: So that 6 and 10 are the two Numbers that were to be found.

QUESTION LXIV.

To find three Numbers in the Ratio of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$; whereof the Sum of all the Squares shall be 549.

Let x denote the first Number,

then it will be $\frac{1}{2} : \frac{1}{3} : : x : \frac{2x}{3}=$ the second Number.

And $\frac{1}{2} : \frac{1}{4} : : x : \frac{x}{2}=$ the third Number.

Hence $x^2 + \overline{\frac{2x}{3}}\Big|^2 + \overline{\frac{x}{2}}\Big|^2 = 549$, *by the Question*,

that is, $x^2 + \frac{4x^2}{9} + \frac{x^2}{4} = 549$,

or, $36x^2 + 16x^2 + 9x^2 = 36 \times 549$:

From which $x^2 = \frac{36 \times 549}{61} = 36 \times 9$,

and $x = 6 \times 3 = 18$. Therefore 18, 12 and 9, are the three required Numbers.

Otherwise,

By reducing the given Fractions $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, to the same Denomination, they will appear to be in the Proportion of 6, 4 and 3. If, therefore, the first of the three Numbers sought be denoted by $6x$, the other Two will be expressed by $4x$, and $3x$, respectively: And so we shall have

$36x^2 + 16x^2 + 9x^2 = 549$.

Whence $x = 3$, and the Numbers sought, *as before*.

QUESTION LXV.

Having given the Difference of two Numbers $=6$, *and their Product* $=720$; *to find the Numbers.*

Let the Lesser of them be denoted by x; then the Greater will be $x+6$; and so, by the Question, we shall have $xx + 6x = 720$.

But in order to the Resolution of this Equation (in which both the first and second Powers of x are in-

volved)

volved) let Half the Coefficient of x, which (in this Case) is 3, be taken and squared, and let that Square be added to both Sides of the Equation: By which Means it becomes $xx + 6x + 9 = 729$.

Whereof the former Part being now a compleat Square, its Root may, therefore, be extracted; and will be expressed by the said Half Coefficient joined to x with its proper Sign; that is, by $x + 3$ (as may be very easily proved by the Multiplication of $x + 3$ into itself; whence $xx + 6x + 9$, the very Quantity above, is produced). Hence it is evident, that $x + 3 = \sqrt{729} = 27$ (for equal Quantities have equal square Roots); and consequently $x = 24$.

QUESTION LXVI.

The Sum of two Numbers being given $= 60$, *and the Sum of their Squares* $= 1872$; *to find the Numbers.*

Let x be the Greater of them;
then $60 - x$ will be the Lesser:
And therefore $x^2 + \overline{60 - x}|^2 = 1872$;
Or $x^2 + 3600 - 120x + x^2 = 1872$:
Whence $2x^2 - 120x = -1728$,
and $x^2 - 60x = -864$:
From which, by compleating the Square *(as in the last Problem)* we get $x^2 - 60x + 900$ $(= -864 + 900) = 36$:
And consequently, by taking the Root, $x - 30 = \sqrt{36} = 6$.
Therefore $x = 6 + 30 = 36$; and $60 - x = 24$: Which are the two Numbers that were to be found.

But, to solve the Problem in a more general manner (by Letters) put the Sum of the two Numbers $= a$, the Sum of their Squares $= b$, and the greater Number $= x$, *as before.*

Then will $x^2 + a^2 - 2ax + x^2 = b$, *by the Question.*
Hence $2x^2 - 2ax = b - a^2$,

and $x^2 - ax = \frac{b}{2} - \frac{a^2}{2}$.

Where, Half the Coefficient of the ſecond Term being $\frac{a}{2}$, the compleated Square will therefore be

$$x^2 - ax + \frac{a^2}{4} \left(= \frac{b}{2} - \frac{a^2}{2} + \frac{a^2}{4}\right) = \frac{b}{2} - \frac{a^2}{4}:$$

From which, by extracting the Root,

we have $x - \frac{a}{2} = \sqrt{\frac{b}{2} - \frac{aa}{4}}$;

And therefore $x = \sqrt{\frac{b}{2} - \frac{aa}{4}} + \frac{a}{2}$: Which, if a be taken $= 60$, and $b = 1872$, will come out $= 36$, *the very ſame as before.*

QUESTION LXVII.

To divide the Number 60 (a) *into two ſuch Parts, that their Product may be* 864 (b).

If x be put for the greater Part, the Leſſer will be denoted by $a - x$; and we ſhall therefore have
$ax - xx = b$; by the Conditions of the Queſtion.
This Equation, by changing the Signs of all its Terms (in order to have the higheſt Power of x affirmative) becomes $xx - ax = -b$.
Whence, by compleating the Square,

we have $xx - ax + \frac{aa}{4} = -b + \frac{aa}{4}$;

and conſequently $x - \frac{a}{2} = \sqrt{\frac{aa}{4} - b}$.

Therefore $x = \sqrt{\frac{aa}{4} - b} + \frac{a}{2} = 36$, the greater Part;

and $a - x = 24$, the Leſſer.

QUESTION

QUESTION LXVIII.

To divide the Number 60 (a) *into two Parts, ſo that the Square of the Greater multiply'd by the Leſſer, added to the Square of the Leſſer multiply'd by the Greater, may amount to* 51840 (b).

If x be the greater Part; then $a-x$ will be the Leſſer;
and $x^2 \times \overline{a-x} + \overline{a-x}|^2 \times x = b$,
that is, $ax^2 - x^3 + a^2x - 2ax^2 + x^3 = b$,
or, $a^2x - ax^2 = b$:
Whence $x^2 - ax = -\frac{b}{a}$;

and, conſequently, $x = \sqrt{\frac{a^2}{4} - \frac{b}{a}} + \frac{a}{2} = 36.$

QUESTION LXIX.

The Sum of two Numbers being given $=$ 20 (a), *and the Sum of their Cubes* $=$ 2240 (b); *to determine the Numbers.*

Let x be the Greater; then $a-x$ will be the Leſſer;
and therefore $x^3 + \overline{a-x}|^3 = b$,
that is, $x^3 + a^3 - 3a^2x + 3ax^2 - x^3 = b$:
Whence $3ax^2 - 3a^2x = b - a^3$,
and $x^2 - ax = \frac{b}{3a} - \frac{a^2}{3}$.
Therefore, by compleating the Square,

$$x^2 - ax + \frac{aa}{4}\left(= \frac{b}{3a} - \frac{a^2}{3} + \frac{a^2}{4}\right) = \frac{b}{3a} - \frac{a^2}{12}:$$

And, by extracting the Root, $x - \frac{a}{2} = \sqrt{\frac{b}{3a} - \frac{aa}{12}}$:

Conſequently $x = \frac{a}{2} + \sqrt{\frac{b}{3a} - \frac{aa}{12}} = \frac{20}{2} + \sqrt{37\frac{1}{3} - 33\frac{1}{3}} = 12.$

QUESTION LXX.

To divide the Number 240 (a) *into two ſuch Parts, that the greater Part divided by the Leſſer, may be to the leſſer Part divided by the Greater, in the Proportion of* 147 *to* 75 (*or of* m *to* n).

If the greater Part be denoted by x, the Leſſer will be expreſſed by $a-x$; and we ſhall have

$$\frac{x}{a-x} : \frac{a-x}{x} :: m : n.$$

Hence $\frac{nx}{a-x} = \frac{m \times \overline{a-x}}{x}$

and $\frac{n x^2}{m} = \overline{a-x}|^2$:

From which, by extracting the ſquare Root, on both Sides, $x\sqrt{\frac{n}{m}} = a-x$.

Whence, putting $\sqrt{\frac{n}{m}} = \frac{b}{c}$,

we have $bx = ca - cx$; and conſequently $x = \frac{ca}{b+c}$.

But, in the Caſe above propoſed, $\sqrt{\frac{n}{m}}$ being $=$ $\sqrt{\frac{75}{147}} = \sqrt{\frac{25}{49}} = \frac{5}{7}$, we have $b=5$, $c=7$; and therefore $x = \frac{7 \times 240}{12} = 7 \times 20 = 140$.

QUESTION

QUESTION LXXI.

Two Workmen A *and* B *were employ'd, by the Day, at different Rates;* A, *at the End of a certain Number of Days, had* 96 *Shillings to receive; but* B, *who play'd* 6 *of those Days, received, only,* 54 *Shillings: But, had* B *worked the whole Time, and* A *play'd* 6 *Days, They would have received exactly alike. 'Tis proposed to find the Number of Days They were employ'd; and what Each had a Day.*

Let x be the Number of Days that A work'd; then $x-6$ will be the Days that B work'd.

Moreover $\frac{96}{x}$, will be the Wages of A *per* Day;

and $\frac{54}{x-6}$, the Wages of B *per* Day.

Therefore $\frac{54}{x-6} \times x$, is what B would have earn'd, had he work'd the whole Time:

And $\frac{96}{x} \times \overline{x-6}$, what A would have earn'd had he play'd 6 Days. Which two Values being equal, by the Question, we have $\frac{54x}{x-6} = \frac{96 \times \overline{x-6}}{x}$.

Whence, by Reduction, $54x^2 = 96 \times \overline{x-6}|^2$,

Or, $\frac{9x^2}{16} = \overline{x-6}|^2$: Therefore, by extracting the square Root on both Sides, $\frac{3x}{4} = x-6$; and consequently $x = 24$. From which it is evident, that A had 4 Shillings, and B 3 Shillings, a Day.

QUESTION LXXII.

From two Places, at the Diſtance of 320 (a) *Miles, two Perſons,* A *and* B, *ſet out, at the ſame Time, in order to meet each other;* A *travelled* 8 (b) *Miles a Day more than* B; *and the Number of Days in which They met was equal to Half the Number of Miles* B *went in a Day: 'Tis required to find how far Each travelled to meet the Other.*

Let x be the Number of Days in which They met; then, by the Queſtion, $\left\{\begin{matrix} 2x \\ 2x+b \end{matrix}\right\}$ will be the Number of Miles $\left\{\begin{matrix} B \\ A \end{matrix}\right\}$ went a Day.

Therefore, by multiplying each of Theſe by (x) the Number of Days, we have $2xx$, and $2xx+bx$, for the whole Number of Miles travelled by B and A, reſpectively:

And conſequently $4xx+bx=a$.

Hence $xx+\frac{bx}{4}+\frac{bb}{64}=\frac{a}{4}+\frac{bb}{64}$,

and $x=\sqrt{\frac{a}{4}+\frac{bb}{64}}-\frac{b}{8}=8$.

Therefore $2xx=128=$ the Miles travelled by B; and $2xx+bx=192=$ Thoſe travelled by A.

QUESTION LXXIII.

Two Meſſengers, A *and* B, *were diſpatched at the ſame Time, to a Place, at the Diſtance of* 90 (a) *Miles; the Former of whom, by riding one Mile an Hour more than the Other, arrived at the End of his Journey one Hour before him: The Queſtion is, to find at what Rate Each travelled* per *Hour.*

If x be the Miles that A rode *per* Hour; then $x-1$ will be the Miles which B rode *per* Hour:

Moreover $\frac{a}{x}$ will be the Number of Hours in which A performed

performed the whole Journey; and $\frac{a}{x-1}$ will be the Number of Hours wherein B performed it.

And therefore $\frac{a}{x} = \frac{a}{x-1} - 1$, *by the Question*,

whence, by Reduction $ax - a = ax - x^2 + x$,

Or $x^2 - x = a$:

Therefore $x^2 - x + \frac{1}{4} = a + \frac{1}{4}$ (by compleating the Square)

and consequently $x = \sqrt{a + \frac{1}{4}} + \frac{1}{2} = 9\frac{1}{2} + \frac{1}{2} = 10$.

QUESTION LXXIV.

To find two Numbers, so that their Sum multiply'd by the Greater may produce 100 *times the Lesser, and being multiply'd by the Lesser may produce* 64 *times the Greater.*

Let x denote the greater Number, and y the Lesser.

Then $\overline{x+y} \times x = 100y$,

And $\overline{x+y} \times y = 64x$.

Now, the first of these Equations being multiply'd by y, and the Second by x, they become both alike;

and so we have $100y^2 = 64x^2$:

Therefore, by taking the square Root, $10y = 8x$,

and consequently $y = \frac{4x}{5}$: Which Value substituted in the second Equation, gives $\overline{x + \frac{4x}{5}} \times \frac{4x}{5} = 64x$;

Or, $\frac{9x}{5} \times \frac{4}{5} = 64$.

Whence $x = \frac{25 \times 16}{9} = 44\frac{4}{9}$; and $y = 35\frac{5}{9}$.

QUESTION

QUESTION LXXV.

To find three Numbers, so that their continual Product divided, by the Sum of each two of them, may quote given Numbers; or (which is the same Thing) to determine the Values of x, y and z, in the underwritten Equations.

$$\frac{xyz}{x+y}=200,\ \frac{xyz}{x+z}=150,\ \frac{xyz}{y+z}=120.$$

First, by Multiplication,

$xyz=200x+200y=150x+150z=120y+120z$:

Therefore, by Reduction,

$$\left.\begin{array}{l}50x+200y=150z\\ 200x+\ \ 80y=120z\end{array}\right\}\text{ or }\left\{\begin{array}{l}x+4y=3z,\\ 5x+2y=3z.\end{array}\right.$$

Hence $x+4y=5x+2y$, and therefore $y=2x$:

Which, substituted in $3z=x+4y$, gives $3z=9x$; and $z=3x$.

And, by substituting for both y and z, in the Equation $\frac{xyz}{x+y}=200$, we get $\frac{x\times 2x\times 3x}{x+2x}=200$;

that is, $2x^2=200$.

Consequently $x=10$, $y=20$, and $z=30$.

QUESTION LXXVI.

To find the Ratio of two Numbers, whose Rectangle is equal to the Square of their Difference.

Let the lesser Number be to the Greater as 1 is to x; then, if the said lesser Number be denoted by z, the Greater will be expressed by xz, and we shall have $xz\times z=\overline{xz-z}|^2$, or $xz^2=x^2z^2-2xz^2+z^2$ *(by the Question)*. Whence, dividing the whole by z^2, there results $x=x^2-2x+1$:

Therefore $x^2-3x=-1$;

and consequently $x=\frac{3+\sqrt{5}}{2}=2.618$ &c.

Hence

Hence the Ratio of any two Numbers, whose Rectangle is equal to the Square of their Difference, must be that of 2. 618 &c. to Unity.

QUESTION LXXVII.

To find two Numbers, whose Product is 300 (a) ; *so that, if* 10 (b) *be added to the Lesser, and* 8 (c) *subtracted from the Greater, the Product of the Sum and Remainder shall, also, be equal to* 300 (a).

Let the greater Number be denoted by x, and the Lesser by y;

then will $\left\{ \begin{array}{l} xy = a \\ \overline{x-c} \times \overline{y+b} = a \end{array} \right\}$ *by the Problem.*

By the last of which $xy + bx - cy - cb = a$.

From whence, the first Equation being subtracted, there rests $bx - cy - cb = 0$:

Therefore $bx = cb + cy$, and $x = \frac{cb + cy}{b}$: Which substituted in the first Equation, gives $\frac{cby + cy^2}{b} = a$.

From which we have $y^2 + by = \frac{ab}{c}$:

And, by compleating the Square, $y^2 + by + \frac{bb}{4} = \frac{ab}{c} + \frac{bb}{4}$.

Whence $y = \sqrt{\frac{ab}{c} + \frac{bb}{4}} - \frac{b}{2} = 15$;

and $x \left(= \frac{a}{y}\right) = 20$.

QUESTION LXXVIII.

To divide 100 *into two such Parts, that their Difference may be to their Sum, as their Rectangle to the Difference of their Squares.*

Let a represent half the given Number, and x half the Difference of its two Parts; then, the Greater of

them

them being expressed by $a+x$, and the Lesser by $a-x$, it will be $2x : 2a :: \overline{a+x} \times \overline{a-x} : \overline{a+x}|^2 - \overline{a-x}|^2$; that is, by Reduction, $x : a :: aa-xx : 4ax$:

Hence $4ax^2 = a \times \overline{aa-xx}$,

Or $4x^2 = a^2 - x^2$,

and therefore $x = \sqrt{\frac{aa}{5}} = 22,36$.

So that, the greater Part is 72,36; and the Lesser 27,64.

QUESTION LXXIX.

To divide the Number 60 (a) *into two such Parts, that their Product may be to the Sum of their Squares, in the Ratio of* 2 (m) *to* 5 (n).

Let x be the greater Part:

Then, the lesser Part will be $a-x$,

the Product $ax-xx$,

and the Sum of the Squares $a^2-2ax+2x^2$.

Therefore $m : n :: ax-xx : a^2-2ax+2x^2$;

and so, $2mx^2-2max+ma^2=nax-nx^2$.

Whence, $2mx^2+nx^2-2max-nax=-ma^2$;

that is, $\overline{2m+n} \times x^2 - \overline{2m+n} \times ax = -ma^2$;

Or, $x^2-ax = -\frac{maa}{2m+n}$.

Therefore $x^2-ax+\frac{aa}{4} = \left(\frac{aa}{4} - \frac{ma^2}{2m+n}\right) = \frac{aa}{4} \times \frac{n-2m}{n+2m}$,

and $x = \frac{a}{2}\sqrt{\frac{n-2m}{n+2m}} + \frac{a}{2} = 40$.

QUESTION LXXX.

To find two Numbers whose Product shall be 320 (a), *and the Difference of their Cubes to the Cube of their Difference, as* 61 (n) *is to Unity.*

Let x be the greater Number, and y the Lesser; then will $xy=a$,

and $x^3-y^3 : \overline{x-y}|^3 :: n : 1$, *by the Question:*

Which

Which, by actually involving $x-y$,
becomes $x^3-y^3 : x^3-3x^2y+3xy^2-y^3 :: n : 1$.
From whence (by ſubtracting the Conſequents from their Antecedents)
we get $3x^2y-3xy^2 : x^3-3x^2y+3xy^2-y^3 :: n-1 : 1$;
Or, which is the ſame, $3xy \times \overline{x-y} : \overline{x-y}|^3 :: n-1 : 1$.
Whence, dividing by $x-y$,
we have $3xy : \overline{x-y}|^2 :: n-1 : 1$;
Or, $3a : \overline{x-y}|^2 :: n-1 : 1$ (becauſe $xy=a$).

From whence $\overline{x-y}|^2 = \frac{3a}{n-1}$;

and conſequently $x-y = \sqrt{\frac{3a}{n-1}}$; which put $=b$:
Then from the Equations $xy=a$, and $x-y=b$, the Value of x will be found $=\frac{\sqrt{b^2+4a}+b}{2}=20$; and

That of $y=\frac{\sqrt{b^2+4a}-b}{2}=16$, *Vid. Prob.* 65.

The ſame otherwiſe.

Let z denote the Half Sum, and x the Half Difference, of the two Numbers; then the Greater will be expreſſed by $z+x$, and the Leſſer by $z-x$; and we ſhall therefore

have $\left\{\begin{array}{l} \overline{z+x} \times \overline{z-x} = a \\ \overline{z+x}|^3 - \overline{z-x}|^3 = n \times \overline{2x}|^3 \end{array}\right\}$ *by the Queſtion.*

that is $\left\{\begin{array}{l} z^2-x^2=a, \\ 6z^2x+2x^3=8nx^3. \end{array}\right.$

The laſt of which, divided by $2x$,
gives $3z^2+x^2=4nx^2$.
From whence, the Treble of the firſt being deducted, we have $4x^2=4nx^2-3a$; and conſequently $x = \sqrt{\frac{3a}{4n-4}}=2$.

Hence z $(=\sqrt{a+xx})=18$; therefore $z+x=20$, and

and $z-x=16$; which are the two Numbers that were to be found.

QUESTION LXXXI.

A Farmer received 7l. 4s. for a certain Quantity of Wheat, and an equal Sum, at a Price leſs by 1s. 6d. a Buſhel, for a Quantity of Barley, which exceeded That of the Wheat by 16 Buſhels: How many Buſhels were there of Each?

Put $a=$ the total Value of each Sort of Grain,
$b=$ the Difference of the Quantities,
$c=$ the Difference of the Prices *per* Buſhel,
and $x=$ the Number of Buſhels of the Wheat:
Then, dividing the whole Price by the Number of Buſhels, we have $\frac{a}{x}$ for the Price of the Wheat *per* Buſhel: And, in the ſame manner, the Price of the Barley *per* Buſhel will appear to be $\frac{a}{x+b}$.

Therefore $\frac{a}{x}-\frac{a}{x+b}=c$, *by the Queſtion.*

Whence (by Reduction) $ax+ab-ax=cx^2+bcx$;

Or, $\frac{ab}{c}=x^2+bx$.

Conſequently $x=\sqrt{\frac{ab}{c}+\frac{bb}{4}}-\frac{b}{2}=32$, the Number of Buſhels of Wheat; and $x+16=48$, the Number of Buſhels of Barley.

QUESTION

QUESTION LXXXII.

One bought two Pieces of Cloth of different Sorts, whereof the Finer coſt 4 Shillings a Yard more than the Other; ſo that, for the Fineſt, he paid 360 Shillings; whereas the Coarſeſt, which exceeded the Fineſt by 10 Yards, coſt him, only, 320 Shillings: How many Yards were there of each Piece?

Let x be the Number of Yards of the Fineſt,
and y, the Number of Shillings a Yard:
Then $x+10$, will be the Length of the coarſeſt Piece,
and $y-4$, its Price *per* Yard.

Hence $\left\{ \begin{array}{l} xy = 360 \\ \overline{x+10} \times \overline{y-4} = 320 \end{array} \right\}$ *by the Queſtion.*

By the laſt Equation $xy + 10y - 4x = 360$;
from whence deducting $xy = 360$,
we have $10y - 4x = 0$.

Therefore $10y = 4x$; and $y = \frac{2x}{5}$. Which, being ſubſtituted in the firſt Equation, we get $\frac{2x^2}{5} = 360$: Whence x comes out $= \sqrt{900} = 30$.

QUESTION LXXXIII.

There are two Numbers, whoſe Rectangle is equal to the Difference of their Squares; and the Sum of their Squares is alſo equal to the Difference of their Cubes: What are thoſe Numbers?

Let x denote the leſſer Number, and let the Greater be in Proportion thereto as y is to Unity; Or, which is the ſame Thing, let the greater Number be denoted by xy.

Then $\left\{ \begin{array}{l} x \times xy = x^2y^2 - x^2 \\ x^2y^2 + x^2 = x^3y^3 - x^3 \end{array} \right\}$ *by the Problem.*

Therefore $\left\{ \begin{array}{l} y = y^2 - 1 \\ y^2 + 1 = xy^3 - x \end{array} \right\}$ *by Diviſion.*

From

From the firſt of which Equations, $y^2 - y = 1$;

whence $y^2 - y + \frac{1}{4} = \frac{5}{4}$, and conſequently $y = \frac{1 + \sqrt{5}}{2}$.

But, by the ſecond Equation $x = \frac{yy+1}{yyy-1} = \frac{y+2}{2y}$ (becauſe $yy = y+1$) $= \frac{1}{2} + \frac{1}{y} = \frac{1}{2} + \frac{2}{\sqrt{5}+1} = \frac{1}{2} + \frac{2\sqrt{5}-2}{4}$ (by multiplying both the Numerator and Denominator by $\sqrt{5} - 1$) $= \frac{\sqrt{5}}{2}$. Therefore the two Quantities ſought are $\frac{1}{2}\sqrt{5}$, and $\frac{5+\sqrt{5}}{4}$.

QUESTION LXXXIV.

A ſets out from London for York, at the ſame time as B ſets out from York for London; and the Rate at which They travel is ſuch, that A, 9 Hours after their meeting arrives at York, and B at London, in 16 Hours after. The Queſtion is, to find in what Time each Traveller performs his Journey.

Let x denote the Number of Hours travelled by Each before the Time of their Meeting on the Road.
Then, ſince A *(by the former Part of the Queſtion)* goes over the very ſame Ground in 9 *(a)* Hours, as B travelled in x Hours, we have, therefore, as $a : x :: x : \frac{xx}{a}$, the Time wherein B travels a Diſtance equal to That gone over by A, in x Hours: But *(by the latter Part of the Queſtion)* B, in 16 *(b)* Hours, travels the very ſame Diſtance as A in x Hours. Hence it is evident that $\frac{xx}{a}$ and b are equal to each Other:

And conſequently that $x = \sqrt{ab} = 12$.

Therefore

Therefore A performs the Journey in 21 Hours, and B in 28 Hours.

QUESTION LXXXV.

The Sum of 190 *l. was divided between three Persons; whose Shares were in Geometrical Proportion, and the greatest of them exceeded the least by* 50 *l. What were all the several Shares?*

If x be put for the Least of them, then the Greatest will be $x+50$; the Sum of which two, subtracted from (190) the Whole, leaves $140-2x$ for the Mean Share: Therefore, $x : 140-2x :: 140-2x : x+50$, *by the Question.*

And consequently $\overline{140-2x}|^2 = x \times \overline{x+50}$;
that is, $19600-560x+4x^2 = x^2+50x$.
Whence $3x^2-610x = -19600$,
and $x^2 - \frac{610x}{3} = -\frac{19600}{3}$.
Hence, by compleating the Square,

$$x^2 - \frac{610x}{3} + \frac{93025}{9} \left(= -\frac{19600}{3} + \frac{93025}{9} \right) = \frac{34225}{9};$$

And, by extracting the Root, $x - \frac{305}{3} = \pm \frac{185}{3}$.

Therefore $x = \frac{305-185}{3} = 40$: And so the other two Shares are 60 and 90 Pounds.

QUESTION LXXXVI.

Two Notes, One of 120 *l. payable in* 6 *Months, and the Other of* 150 *l. payable in* 9 *Months, were discounted for* 8 *l.* 10 *s. What Rate of Interest were They discounted at?*

Let x denote the Interest of One Pound for 12 Months:

Then the Amount of 1 *l.* in 6 Months being $1+\frac{x}{2}$, and, in 9 Months $1+\frac{3x}{4}$ the *Present Value* of the Bill, due at the End of 6 Months, will therefore be $\frac{120}{1+\frac{1}{2}x}$; and *That* of the Bill, due at the End of 9 Months, $\frac{150}{1+\frac{3}{4}x}$.

Whence, we have $\frac{120}{1+\frac{1}{2}x}+\frac{150}{1+\frac{3}{4}x}\left(=120+150-8.5\right)=261,5$ *(by the Question)*
Which, by Reduction, becomes $120+90x+150+75x=261,5\times\overline{1+\frac{5}{4}x+\frac{3}{8}x^2}$;
or, $270+165x=\frac{261,5\times\overline{8+10x+3x^2}}{8}$.
Therefore $\frac{\overline{270+165x}\times 8}{261,5}=3x^2+10x+8$,
that is, $\frac{2160}{261,5}+\frac{1320x}{261,5}=3x^2+10x+8$:
From which we have
$x^2+\frac{2590}{3\times 523}\times x=\frac{136}{3\times 523}$.
Whence, by compleating the Square, *&c.*
x is found $=\sqrt{\frac{136}{1569}+\left|\frac{1295}{1569}\right|^2}-\frac{1295}{1569}=\frac{\sqrt{136\times 1569+1295\times 1295}-1295}{1569}=0,05093$. Which multiply'd by 100, gives 5,093 *l.* or *l.* 5 : 1 : $10\frac{1}{4}$, nearly, for the Rate *per Cent*, at which the Notes were discounted.

QUESTION

QUESTION LXXXVII.

A *and* B *take, in Trade,* 5940 per Annum, *each; but* A, *whose Profits are* 2 per Cent. *greater than Those of* B, *clears* 100*l.* per Annum *more than* B: *What are the Profits of Each,* per Cent? *And What do They clear* per Annum?

Let $c = 100$ (Pounds),

b $(= 5940) =$ the whole Sum taken by Each,

d $(= 100l.) =$ the given Difference of their Gains,

$x =$ What A gains *per Cent.*

$x-a$ $(= x-2) =$ What B gains *per Cent.*

Now, since A, in taking $c+x$ Pounds, gains x Pounds, it will be $c+x : x :: b : \frac{bx}{c+x}$, the whole Gain of A:

And, in the same manner, we have,

$c+x-a : x-a :: b : \frac{b \times \overline{x-a}}{c+x-a}$, the whole Gain of B:

Therefore $\frac{bx}{c+x} - \frac{b \times \overline{x-a}}{c+x-a} = d$, *by the Question.*

Hence $abc = \overline{x^2 + 2cx - ax - ac + cc} \times d$;

and consequently $x^2 + \overline{2c-a} \times x = \frac{abc}{d} + ac - cc.$

Whence, by compleating the Square,

$x^2 + \overline{2c-a} \times x + \overline{\frac{2c-a}{2}}\Big|^2 = \frac{abc}{d} + \frac{aa}{4}$;

and therefore $x = \sqrt{\frac{abc}{d} + \frac{aa}{4}} - \frac{2c-a}{2} = 10.$

From which it appears that A gained 10 *per Cent.* and cleared 540 *l. per Annum*; and that B gained 8 *per Cent.* and cleared 440 *l. per Annum.*

QUESTION LXXXVIII.

Of four Numbers in Geometrical Progreſſion, there is given the Sum of the two Leaſt $=20$ (a) *and the Sum of the two Greateſt* $=45$ (b) ; *to find the Numbers.*

Let x denote the firſt Number, and y the Third ; then the Second being expreſſed by $a-x$, and the Fourth by $b-y$, the four Terms of the Progreſſion, placed in Order, will ſtand thus, x, $a-x$, y, $b-y$,
Whence, by the Nature of Proportionals,

we have $\begin{cases} x \times \overline{b-y} = \overline{a-x} \times y, \\ xy = \overline{a-x}|^2. \end{cases}$

From the firſt of which Equations y is $= \frac{bx}{a}$: And, by ſubſtituting in the Second, we have $\frac{bx^2}{a} = \overline{a-x}|^2$:

Whence, by extracting the Square Root on both Sides,

$x\sqrt{\frac{b}{a}} = a-x$;

and conſequently $x = \frac{a}{1+\sqrt{\frac{b}{a}}} = 8.$

Therefore 8, 12, 18, and 27, are the four Numbers that were to be found.

QUESTION LXXXIX.

The Sum of 700 *l.* (a) *was divided among four Perſons* A, B, C *and* D ; *whoſe Shares were in Geometrical Progreſſion ; and the Difference between the Greateſt and Leaſt, was to the Difference between the two Means, as* 37 (m) *to* 12 (n) : *What were all the ſeveral Shares?*

Let the Share of A, or the firſt Term of the Progreſſion, be denoted by x, and let the Common Ratio be That of 1 to y:

Then

Then $x + xy + xy^2 + xy^3 = a$
and $xy^3 - x : xy^2 - xy :: m : n$ } *by the Question.*
From which Proportion, we have

$y^3 - 1 = \overline{y-1} \times \frac{my}{n}$, or $y^2 + y + 1 = \frac{my}{n}$ (by dividing the whole by $y-1$).

Hence y is found $= \frac{m-n+\sqrt{mm-2mn-3nn}}{2n} = \frac{25+7}{24}$ $= \frac{4}{3}$: Whence x $\left(= \frac{a}{1+y+y^2+y^3}\right)$ is given $= \frac{27 \times 700}{27+36+48+64} = 108$. Therefore the four Shares are 108, 144, 192, and 256 *l.*

QUESTION XC.

Of four Numbers in Arithmetical Progreſſion, the Sum of the Squares of the two Means is given = 400 (a) ; *and the Sum of the Squares of the two Extremes*=464 (b) : *To determine the Numbers.*

If x be put for the leſſer Extreme, and y for the Common Difference; then the four Numbers will be expreſſed by,

x, $x+y$, $x+2y$, and $x+3y$ reſpectively; and we ſhall therefore have $\begin{cases} \overline{x+y}|^2 + \overline{x+2y}|^2 = a \\ x^2 + \overline{x+3y}|^2 = b \end{cases}$ or

$\begin{cases} 2x^2 + 6xy + 5y^2 = a \\ 2x^2 + 6xy + 9y^2 = b \end{cases}$ *by the Question.*

Whence, by Subtraction, $4y^2 = b-a$:

And therefore $y = \frac{\sqrt{b-a}}{2} = 4$.

But, by the firſt Equation, $x^2 + 3xy = \frac{a}{2} - \frac{5yy}{2}$:

From which (looking upon y as known) we get

$x = \sqrt{\frac{a}{2} - \frac{yy}{4}} - \frac{3y}{2} = 8$.

 Therefore

Therefore 8, 12, 16, and 20, are the four Numbers that were to be found.

QUESTION XCI.

The Sum of four Numbers, in Arithmetical Progression, being given = 56 (b) *and the Sum of their Squares* = 864 (c); *to find the Numbers.*

If Half the Sum of the two middle Numbers be denoted by a, and Half their Difference by x, the Numbers themselves will be expressed by $a-x$, and $a+x$: And we shall have

$$\left.\begin{array}{l}\overline{a-3x}+\overline{a-x}+\overline{a+x}+\overline{a+3x}=b\\ \overline{a-3x}|^2+\overline{a-x}|^2+\overline{a+x}|^2+\overline{a+3x}|^2=c\end{array}\right\}$$ *by the Nature of the Question.*

Hence, by Reduction, $4a=b$, and $4aa+20xx=c$.

Therefore $a=\frac{b}{4}=14$; and $x=\sqrt{\frac{c}{20}-\frac{aa}{5}}=2$.

From which the Numbers themselves are given; being 8, 12, 16 and 20.

By the same way of proceeding the Problem may be resolved, when the Progression is supposed to consist of 6, 8, 10, or any other, even, Number (n) of Terms.

For the Sum of the Squares of the two Means $(a-x$, and $a+x)$ being $=2\times\overline{aa+xx}$; and the Sum of the Squares of the two Terms $(a-3x$ and $a+3x)$ next adjacent to them $=2\times\overline{aa+9xx}$; also the Sum of the Squares of the two Terms $(a-5x$ and $a+5x)$ next adjacent to these last being $=2\times\overline{aa+25xx}$, &c. &c. it follows that

$2\times\overline{aa+xx}+2\times\overline{aa+9xx}+2\times\overline{aa+25xx}+$ &c. $=c$.
Which Equation, by putting $1+9+25+36+$ &c. (continued to $\frac{n}{2}$ Terms) $=f$, is reduced to $naa+2fxx=c$.

Whence

Whence $x = \sqrt{\frac{c-naa}{2f}}$; from which, as a is given $= \frac{b}{n}$, the Value of x will alſo be known.

QUESTION XCII.

Having the Sum (b) *and the Sum of the Squares* (c) *of five Numbers, in Arithmetical Progreſſion; to determine the Progreſſion.*

Let a denote the middle Number, and x the Common Difference; then the five Terms of the Progreſſion will be, $a-2x$, $a-x$, a, $a+x$, and $a+2x$; whence we have $5a=b$, and $5aa+10xx=c$, by the Conditions of the Queſtion. From which a is found $=\frac{b}{5}$, and x ($= \sqrt{\frac{c}{10}-\frac{aa}{2}}$) $= \sqrt{\frac{c}{10}-\frac{bb}{50}}$.

After the ſame Manner the Problem may be inveſtigated, when any odd Number (n) of Terms is propounded: For the Sum of the Squares of the 2 Terms ($a-x$ and $a+x$) adjacent to the middle One being $= 2\times\overline{aa+xx}$; and That of the two Terms ($a-2x$ and $a+2x$) next to Theſe $= 2\times\overline{aa+4xx}$, &c. &c. it is evident therefore that

$$aa+2\times\overline{aa+xx}+2\times\overline{aa+4xx}+2\times\overline{aa+9xx}+\&c.=c.$$

Which Equation, by putting $1+4+9+16+$ &c. (to $\frac{n-1}{2}$ Terms) $=f$, becomes $naa+2fxx=c$.

Hence $x = \sqrt{\frac{c-naa}{2f}}$; from which, as a is given $= \frac{b}{n}$, all the Terms of the Progreſſion will be known.

By a like Proceſs, if, inſtead of the Sum of the Squares, the Sum of the Cubes, or Biquadrates, be given, the Problem may be reſolved.

QUESTION XCIII.

A Traveller, bound to a certain Place at the Diſtance of 140 *Miles, goes* 26 *Miles the firſt Day,* 24 *Miles the ſecond Day,* 22 *Miles the Third; and ſo on, in Arithmetical Progreſſion, decreaſing* 2 *Miles every Day. In how many Days will he arrive at the End of his Journey?*

Put $b=140$, the whole, given, Diſtance,
$c=26$, the Part thereof travelled in the firſt Day,
$d=2$, the Common Difference by which each Day's Journey decreaſes,
and $x=$ the Number of Days wherein the *whole* Journey is perform'd.

Then, from the Nature of the Queſtion, it is evident, that, the laſt Day's Journey will fall ſhort of the Firſt, by $x-1$ times the Common Difference (d); and is, therefore, truly expreſſed by $c-\overline{x-1}\times d$:
But it is well known that the Sum of any Arithmetical Progreſſion is equal to the Sum of the firſt and laſt Terms multiply'd by Half the Number of the Terms:

Hence we have $\overline{c+c-\overline{x-1}\times d}\times\frac{x}{2}$ for the whole Diſtance travelled; and conſequently this Equation,

$$\overline{2c-\overline{x-1}\times d}\times\frac{x}{2}=b.$$

Whence $2cx-dx^2+dx=2b$,

and $xx-\frac{2c+d}{d}\times x=-\frac{2b}{d}$:

From which $x=\pm\sqrt{\frac{\overline{2c+d}|^2}{4dd}-\frac{2b}{d}}+\frac{2c+d}{2d}=7.$

QUESTION

QUESTION XCIV.

After A, *who travelled at the Rate of* 4 *Miles an Hour, had been ſet out two Hours and three Quarters,* B *ſet out to overtake him; and in order thereto went* $4\frac{1}{2}$ *Miles the firſt Hour,* $4\frac{3}{4}$ *the Second,* 5 *the Third; and ſo on, gaining a Quarter of a Mile every Hour. How many Hours did it take him to come up with* A?

Put $a = 4$, the Diſtance travelled by A, *per Hour.*
$b = 11$, the Diſtance gone by A before B ſet out,
$c = 4\frac{1}{2}$, the Miles travelled by B in the firſt Hour,
$d = \frac{1}{4}$ of a Mile, the Common Exceſs,
and $x =$ the Number of Hours required.

Then it is evident (from the preceding Problem) that the Diſtance travelled by B, in the laſt Hour, will be $c + \overline{x-1} \times d$; and that $\overline{2c + \overline{x-1} \times d} \times \frac{x}{2}$ will expreſs the whole Diſtance travelled by B in x Hours.

But the Diſtance of A in x Hours being ax, the whole Diſtance travelled by A will therefore be $ax+b$; which being equal to *That* of B *(by the Queſtion)* we thence have,

$$\overline{2c + \overline{x-1} \times d} \times \frac{x}{2} = ax + b;$$

and conſequently $2cx + dx^2 - dx = 2ax + 2b$;

or $xx + \frac{2c-2a-d}{d} \times x = \frac{2b}{d}$:

Which (by making $\frac{2c-2a-d}{d} = f$) gives

$x = \sqrt{\frac{2b}{d} + \frac{ff}{4}} - \frac{f}{2} = 8$, the Number of Hours required.

QUESTION

QUESTION XCV.

To find four Numbers in Arithmetical Progression, which being increased by 2, 4, 8 and 15, respectively, the Sums shall be in Geometrical Progression.

Let x denote the least Number, and y the Common Difference. Then the four Numbers will be expressed by x, $x+y$, $x+2y$, and $x+3y$;
and the four specified Sums,
by $x+2$, $x+y+4$, $x+2y+8$, and $x+3y+15$.
Whence, by the Nature of Geometrical Proportionals,

we have $\begin{cases} \overline{x+2} \times \overline{x+2y+8} = \overline{x+y+4}|^2, \\ \overline{x+2} \times \overline{x+3y+15} = \overline{x+y+4} \times \overline{x+2y+8}, \end{cases}$

that is $\begin{cases} y^2+4y=2x \\ 2y^2+10y+2=5x \end{cases}$ *by Multiplication, and Transposition.*

Hence $5y^2+20y=4y^2+20y+4$;
therefore $y^2=4$, and $y=2$:
From which $x=6$; and so the other three Numbers are 8, 10, and 12 respectively:
For 6, 8, 10 and 12 are in Arithmetical Progression; and $6+2$, $8+4$, $10+8$, and $12+15$, are in Geometrical Progression.

QUESTION XCVI.

The Rectangle (a) *and the Sum of the Cubes* (b) *of two Numbers being given; to determine the Numbers.*

Let x denote the greater, and y the lesser Number:

Then will $\begin{cases} xy=a \\ x^3+y^3=b \end{cases}$ *by the Question.*

Whence, by Involution, $x^3y^3=a^3$,
and $x^6+2x^3y^3+y^6=b^2$.
Let the Quadruple of the former of these two Equations be subtracted from the Latter, and you will have $x^6-2x^3y^3+y^6=b^2-4a^3$;

and,

and, by extracting the ſquare Root, on both Sides, $x^3-y^3=\sqrt{b^2-4a^3}$. Which, added to the firſt Equation, gives $2x^3=b+\sqrt{b^2-4a^3}$.

Conſequently $x=\sqrt[3]{\frac{1}{2}b+\frac{1}{2}\sqrt{bb-4a^3}}$;

and $y\left(=\frac{a}{x}\right)=\frac{a}{\sqrt[3]{\frac{1}{2}b+\frac{1}{2}\sqrt{bb-4a^3}}}$.

The ſame otherwiſe.

Since $xy=a$, we have $y=\frac{a}{x}$; and therefore

$x^3+\frac{a^3}{x^3}=b$, by the Queſtion.

Whence $x^6+a^3=bx^3$,
or, $x^6-bx^3=-a^3$.
Therefore, by compleating the Square,

$$x^6-bx^3+\frac{bb}{4}\left(=\frac{bb}{4}-a^3\right)=\frac{bb-4a^3}{4};$$

and, by extracting the Root, $x^3-\frac{b}{2}=\frac{\sqrt{bb-4a^3}}{2}$.

Hence $x^3=\frac{1}{2}b+\frac{1}{2}\sqrt{bb-4a^3}$;
and conſequently the Values of x and y, *the very ſame as before.*

From either of the above Solutions, a General Theorem, for the Reſolution of Cubic Equations (according to the Manner of *Cardan*) is very eaſily deducible.

For, by putting $z=x+y$, and involving each Side to the third Power, we have $z^3=x^3+3x^2y+3xy^2+y^3=x^3+y^3+3xy\times\overline{x+y}=x^3+y^3+3axz$ (by ſubſtituting for xy and $x+y$, their Equals, a and z). From whence, by Tranſpoſition,

$z^3-3az\ (=x^3+y^3)=b$.

But it appears, from above, that $z(=x+y)$

is

is $=\sqrt[3]{\tfrac{1}{2}b+\tfrac{1}{2}\sqrt{bb-4a^3}}+\dfrac{a}{\sqrt[3]{\tfrac{1}{2}b+\tfrac{1}{2}\sqrt{bb-4a^3}}}$:
Which Value, therefore, is the true Root of the Equation $z^3-3az=b$.
From which the Root of the Equation $z^3+cz=b$ (where the fecond Term is pofitive) will alfo be given, by affuming $-3a=c$, and fubftituting for a: Whence z is found

$$=\sqrt[3]{\frac{b}{2}+\sqrt{\frac{bb}{4}+\overline{\frac{c}{3}}\Big|^3}}-\frac{\frac{1}{3}c}{\sqrt[3]{\frac{b}{2}+\sqrt{\frac{bb}{4}+\overline{\frac{c}{3}}\Big|^3}}}.$$

QUESTION XCVII.

The Difference of two Numbers being given = 4, *and the Sum of their Cubes* = 2240; *to determine the Numbers.*

Let x denote their Half Sum, and d their Half Difference, then the Greater being $x+d$, and the Leffer $x-d$, we have
$\overline{x+d}|^3+\overline{x-d}|^3=2240$,
that is $2x^3+6d^2x=2240$,
or, $x^3+3d^2x=\dfrac{2240}{2}$.
Put $c=3d^2$ $(=12)$ and $b=\dfrac{2240}{2}$ $(=1120)$
then it will become $x^3+cx=b$:
From whence by the general Theorem, in the laft Problem, $x=$

$$\sqrt[3]{\frac{b}{2}+\sqrt{\frac{bb}{4}+\overline{\frac{c}{3}}\Big|^3}}-\frac{\frac{1}{3}c}{\sqrt[3]{\frac{b}{2}+\sqrt{\frac{bb}{4}+\overline{\frac{c}{3}}\Big|^3}}}=10.$$

Therefore 12 $(=x+d)$ and 8 $(=x-d)$ are the two Numbers that were to be found.

QUESTION

QUESTION XCVIII.

'Tis proposed to divide the Number 24 ($2a$) *into two such Parts, that the Difference of their Cubes may be* 3584 ($2b$).

Let $a+x$ express the greater Part, and $a-x$ the Lesser; then will $\overline{a+x}|^3 - \overline{a-x}|^3 = 2b$;
that is, $6a^2x + 2x^3 = 2b$;
or $x^3 + 3a^2x = b$.

Put $c = 3a^2$, and $\sqrt[3]{\frac{b}{2} + \sqrt{\frac{bb}{4} + \overline{\frac{c}{3}}|^3}} = r$:

Then will $x^3 + cx = b$,

and $x = r - \frac{\frac{1}{3}c}{r} = 4$, *by the Theorem in the preceding Page.* Whence 8 and 16 are the two required Numbers.

QUESTION XCIX.

The Sum of the Squares of two Numbers being given = 208 (g) *and the Sum of their Cubes* = 2240 (h); *to find the Numbers.*

Let the greater Number be denoted by $x+y$, and the Lesser by $x-y$:

Then will $\left\{ \begin{array}{l} 2x^2 + 2y^2 = g \\ 2x^3 + 6xy^2 = h \end{array} \right\}$ *by the Question.*

From the first of these Equations, multiply'd by $3x$, let the Second be subtracted;
whence you will have $4x^3 = 3gx - h$;
and consequently $x^3 - \frac{3gx}{4} + \frac{h}{4} = 0$;
that is, in Numbers, $x^3 - 156x + 560 = 0$;
The Roots of which Equation (by Sect. 12 of my Treatise of Algebra) will be found to be 10, 4, and -14; whereof the First, only, is for our Purpose: By Means of which we get $y\ (= \sqrt{g - x^2}) = 2$.

Therefore

Therefore 12 and 8 are the two Numbers that fulfil the Conditions of the Problem.

Note. The Refolution of the above Equation, by the Theorem referred to in the preceding Examples, is impoffible; becaufe the fquare Root of a negative Quantity is to be extracted; as, upon Trial, will be found.

QUESTION C.

The Sum (a) *and the continual Product* (b) *of four Numbers, in Geometrical Progreffion, being given; to determine the Numbers.*

If the leffer of the two Means be reprefented by $x-y$, and the greater by $x+y$, we fhall have

$$\left.\begin{array}{l} \dfrac{\overline{x-y}^2}{x+y} + \overline{x-y} + \overline{x+y} + \dfrac{\overline{x+y}^2}{x-y} = a \\ \text{and } \overline{x-y}^2 \times \overline{x+y}^2 = b \end{array}\right\} \textit{by the Queftion.}$$

From the fecond of thefe Equations, by putting $c = \sqrt{b}$, and extracting the Square Root, there comes out

$$\overline{x-y} \times \overline{x+y}\ (= xx - yy) = c.$$

Moreover, from the Firft, we get

$$\overline{x-y}^3 + 2x \times \overline{x-y} \times \overline{x+y} + \overline{x+y}^3 = a \times \overline{x-y} \times \overline{x+y};$$

or, $2cx + \overline{x-y}^3 + \overline{x+y}^3 = ac$ (by Subftitution)

that is, $2cx + 2x^3 + 6xy^2 = ac$;

or $2cx + 2x^3 + 6x \times \overline{xx-c} = ac$ (becaufe $yy = xx - c$).

Hence $x^3 - \dfrac{cx}{2} = \dfrac{ac}{8}$: From which, by the Theorem in Page 60, the Value of x may be found.

QUESTION

QUESTION CI.

The Compound Interest of a certain Sum of Money, put out for 4 *Years, amounted to* 344 *l.* $\frac{81}{100}$; *but the simple Interest thereof, for the same Time, and at the same Rate, would have been only* 320 *l. What was the Sum put out? and what the Rate of Interest?*

Put $a = 344.81$, and $b = 320$; and let x denote the Interest of 1 *l.* for one Year.
Therefore, since the Simple Interest of 1 *l.* for 4 Years is $4x$, and the Compound Interest $\overline{1+x}|^4 - 1$, or $4x + 6x^2 + 4x^3 + x^4$,
we have, as $4x + 6x^2 + 4x^3 + x^4 : 4x :: a : b$, by the Nature of the Question:
and consequently $6x + 4x^2 + x^3 = \frac{4a - 4b}{b} = 0.310125$.
From the Resolution of which Equation, x will be found $= .05$: Therefore the Rate of Interest was 5 *per Cent.* and the Sum put out 1600 *l.*

QUESTION CII.

The Sum (s) *and the Product* (p) *of any two Numbers being given, to find the Sum of the Squares, Cubes, Biquadrates, &c. of those Numbers.*

Let the two Numbers be denoted by x and y:
Then $x + y = s$ and $xy = p$ } by the Question.
The former of which Equations, squared,
gives $x^2 + 2xy + y^2 = s^2$; from whence, the Double of the Latter being subtracted,
we get $x^2 + y^2 = s^2 - 2p$, *the sum of the Squares.*
Let this Equation be multiply'd by $x + y = s$,
and there arises $x^3 + xy^2 + yx^2 + y^3 = s^3 - 2sp$,
or, $x^3 + xy \times \overline{x+y} + y^3 = s^3 - 2sp$,
that is, $x^3 + p \times s + y^3 = s^3 - 2sp$ (because $xy = p$, and $x + y = s$).

Therefore

Therefore $x^3 + y^3 = s^3 - 3sp$, *the Sum of the Cubes.*
Again, multiply this laſt Equation by $x + y = s$,
and you will have $x^4 + xy \times \overline{x^2 + y^2} + y^4 = s^4 - 3s^2p$,
or $x^4 + p \times \overline{s^2 - 2p} + y^4 = s^4 - 3s^2p$ (becauſe $x^2 + y^2 = s^2 - 2p$).
Whence $x^4 + y^4 = s^4 - 4s^2p + 2p^2$, *the Sum of the fourth Powers.*
Multiply, again, by $x + y = s$, and you will have

$$x^5 + xy \times \overline{x^3 + y^3} + y^5 = s^5 - 4s^3p + 2sp^2,$$

or, $x^5 + p \times \overline{s^3 - 3sp} + y^5 = s^5 - 4s^3p + 2sp^2$,
and therefore $x^5 + y^5 = s^5 - 5s^3p + 5sp^2$, *the ſum of the fifth Powers.*
From whence the Law of Continuation is manifeſt; being ſuch, that the Sum of the next ſuperior Powers will, *always*, be obtained by multiplying the Sum of the Powers laſt found by s, and ſubtracting the Sum of the preceding Ones drawn into p, from the Product.
So that the Sum of the n^{th} Powers (expreſſed in a general Manner)

or $x^n + y^n$, will be $= s^n - ns^{n-2}p + n \times \frac{n-3}{2} s^{n-4}p^2 -$

$$n \times \frac{n-4}{2} \times \frac{n-5}{3} s^{n-6}p^3 + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} s^{n-8}p^4 - \text{\&c.}$$

QUESTION CIII.

The Sum (a) *and the Sum of the Squares* (b) *of four Numbers, in Geometrical Progreſſion being given; to find the Numbers.*

If x and y be aſſumed to repreſent the two middle Numbers, then, from the Nature of continued Proportionals, the two Extremes will be expreſſed by $\frac{xx}{y}$ and $\frac{yy}{x}$: And ſo

we ſhall have $\frac{x^2}{y} + x + y + \frac{y^2}{x} = a$,

and $\frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = b$.

Put

Put $x+y=u$, and $xy=z$:

Then, from the first Equation, $\frac{x^2}{y}+\frac{y^2}{x}=a-u$,

and therefore $x^3+y^3=xy\times\overline{a-u}$.

But, becaufe the Sum of the two Means (x and y) is expreffed by u, and their Rectangle by z, it is evident, *from Queftion* 102, that the Sum of their Squares (x^2+y^2) will be exhibited by u^2-2z, and the Sum of their Cubes (x^3+y^3) by u^3-3uz.

And, in like manner, the Sum of the two Extremes $\left(\frac{x^2}{y}+\frac{y^2}{x}\right)$ being denoted by $a-u$, and their Rectangle by z, the Sum of their Squares will be $=\overline{a-u}|^2-2z$, and the Sum of their Cubes $=\overline{a-u}|^3-\overline{3a-3u}\times z$ (but this Laft is of no Ufe in the prefent Cafe.)

Hence, by fubftituting thefe feveral Values in our fecond, and laft Equations,

we get $\overline{a-u}|^2-2z+u^2-2z=b$,

and $u^3-3uz=z\times\overline{a-u}$;

that is, by Reduction,

$2u^2-2au+a^2-b=4z$,

and $u^3=\overline{2u+a}\times z$:

And, confequently, $\overline{2u+a}\times\overline{2u^2-2au+a^2-b}=4u^3$:

Whence, by Reduction, $u^2+\frac{bu}{a}=\frac{aa-b}{2}$;

and therefore $u=\sqrt{\frac{aa-b}{2}+\frac{bb}{4aa}}-\frac{b}{2a}$:

From which the feveral Values of z, x, y will alfo become known.

QUESTION CIV.

The Sum (a) *and the Sum of the Cubes* (c) *of four Numbers in continued Geometrical Proportion being given; to determine the Numbers.*

Let the Notation of the *preceding Problem* be retained; then our two Equations *(in this Cafe)* will be $\frac{x^2}{y}+x+y+\frac{y^2}{x}=a$,

and $\frac{x^6}{y^3}+x^3+y^3+\frac{y^6}{x^3}=c$.

But, it appears, *from Thence*, that the Sum of the Cubes (x^3+y^3) of the two Means is $=u^3-3uz$; and that the Sum of the Cubes $\left(\frac{x^2}{y}+\frac{y^2}{x}\right)$ of the two Extremes is $=\overline{a-u}|^3-\overline{3a-3u}\times z$: Therefore our last Equation, by substituting these Values, becomes $u^3+\overline{a-u}|^3-3az=c$.
And, it appears, *by the last Problem*,
that the first Equation (by a like Substitution)
is reduced to $u^3=\overline{2u+a}\times z$.
Hence, by exterminating z out of the two Equations thus derived, and putting, $\frac{a^2}{3}-\frac{c}{3a}=d$, we obtain

$\overline{2u+a}\times\overline{u^2-au+d}=u^3$;

or $u^3-au^2-\overline{aa-2d}\times u+ad=0$.

From whence, the Value of u being found, the rest of the Quantities will be known.

In the same Manner the Problem may be resolved, when (instead of the Sum of the Cubes) the Sum of the 4th, 5th, or 6th, &c. Powers is given.

For the Sum of the n^{th} Powers of the two Means (or x^n+y^n) being, *universally*, $=u^n-nu^{n-2}z+n\times\frac{n-3}{2}u^{n-4}z^2$ &c. (*See Question* 102.)
and the Sum of the n^{th} Powers of the two Extremes $=\overline{a-u}|^n-n\times\overline{a+u}|^{n-2}z+n\times\frac{n-3}{2}u^{n-4}z^2$, &c. (since the Sum of the Roots is here $=a-u$);
we therefore have $u^n+\overline{a-u}|^n-nz\times\overline{u^{n-2}+\overline{a-u}|^{n-2}}+\frac{n}{1}\times\frac{n-3}{2}z^2\times\overline{u^{n-4}+\overline{a-u}|^{n-4}}-$ &c. $=c$: Which Equation, by writing $\frac{u^3}{2u+a}$ instead of its Equal z, be-

comes

comes $u^n + \overline{a-u}|^n - \frac{nu^3}{2u+a} \times \overline{u^{n-2} + a-u|^{n-2}} + \frac{n \times \overline{n-3}}{2} \times \frac{u^6}{\overline{2u+a}|^2} \times \overline{u^{n-4} + \overline{a-u}|^{n-4}} + \&c. = c.$ Whence u may be found.

QUESTION CV.

Having given the Sum (a) *and the Sum of the Squares* (b) *of five Numbers in Geometrical Progression; to determine the Progression.*

Let x, z, and y denote the three middle Numbers, taken in Order: Then $\frac{xx}{z}$ will be the first Number, and $\frac{yy}{z}$ the Last; and we shall have

$$\left.\begin{cases} \frac{xx}{z} + x + z + y + \frac{yy}{z} = a, \\ \frac{x^4}{z^2} + x^2 + z^2 + y^2 + \frac{y^4}{z^2} = b \end{cases}\right\} \text{by the Question.}$$

Put $u = x + y$; then, from the first Equation,

$$\frac{xx}{z} + \frac{yy}{z} = a - u - z.$$

Therefore, seeing the Sum of the two Extremes is expressed by $a-u-z$, and their Rectangle by z^2 (from the Nature of Proportionals) the Sum of their Squares will be $= \overline{a-u-z}|^2 - 2z^2$ *(by Question* 102*).*

Moreover, the Sum $(x+y)$ of the two Terms adjacent to the Middle one being $= u$, and their Rectangle $= z^2$, the Sum of their Squares (x^2+y^2) will therefore be $= u^2 - 2z^2$ *(by the Same).*

And so, by substituting these Values above,

we get $\overline{a-u-z}|^2 - 2z^2 + u^2 - 2z^2 + z^2 = b$,

and $\frac{u^2 - 2z^2}{z} = \overline{a-u-z}$.

 Whence

Whence $a^2-2au-2az+2u^2+2uz-2z^2=b$,
and $az-u^2-uz+z^2=0$.
The Double of which laſt Equation, added to the Former, gives $a^2-2au=b$; whence $u=\frac{a}{2}-\frac{b}{2a}$.
From which, and the Equation $az-u^2-uz+z^2=0$, the value of z will alſo become known.

QUESTION CVI.

The Sum (a) *and the Sum of the Cubes* (b) *of five Numbers, in continued Geometrical Proportion being given; to find the Numbers.*

Retaining the Notation of the laſt Problem, and proceeding in the ſame Manner, we have

$\overline{a-u-z}|^3-\overline{3a-3u-3z}\times z^2+u^3-3uz^2+z^3=b$;
and $az-u^2-uz+z^2=0$ *(as before).*

The firſt of which Equations, by Reduction, becomes

$a^3-3a^2\times\overline{u+z}+3a\times\overline{u^2+2uz}-3u^2z-3uz^2+3z^3=b$;

From whence the other Equation, multiply'd by $3z$, being ſubtracted, there remains

$a^3-3a^2\times\overline{u+z}+3a\times\overline{u^2+2uz-z^2}=b$;

therefore $u^2+2uz-z^2-a\times\overline{u+z}=\frac{b}{3a}-\frac{a^2}{3}$.

But, by the ſecond Equation, $u^2+2uz-z^2=az+uz$;

whence, by Subſtitution, $az+uz\ a\times\overline{u+z}=\frac{b}{3a}-\frac{a^2}{3}$;

that is, $uz-az=\frac{b}{3a}-\frac{a^2}{3}$;

or $az-uz=d$ $\left(\text{by putting } \frac{aa}{3}-\frac{b}{3a}=d\right)$.

From which, the ſecond Equation being ſubtracted, there reſults

$u^2-z^2=d$: Wherein let $\left(\frac{az-d}{z}\right)$ the Value of u (found from the former Equation) be now ſubſtituted, and

and we ſhall have $\frac{\overline{az-d}|^2}{z^2} - z^2 = d$; and conſequently

$z^4 - \overline{aa-d} \times z^2 + 2adz - d^2 = 0$.
Whence z will be found.

QUESTION CVII.

The Sum (a), *the Sum of the Squares* (b), *and the Sum of the Cubes* (c), *of any three Numbers being given; to determine the Numbers.*

Let them be denoted by x, y and z; then

ſince $\left\{\begin{array}{l} x+y+z=a \\ x^2+y^2+z^2=b \\ x^3+y^3+z^3=c \end{array}\right\}$ we ſhall, by Tranſpoſition, have

$x+y=a-z$
$x^2+y^2=b-z^2$
$x^3+y^3=c-z^3$.

Now, by multiplying together the two firſt of theſe Equations,
we have $x^3+x^2y+xy^3+y^3=ab-bz-az^2+z^3$.
And, by cubing of the Firſt, we alſo have
$x^3+3x^2y+3xy^2+y^3=a^3-3a^2z+3az^2-z^3$: Which, deducted from the Treble of the Former,
leaves $2x^3+2y^3=3ab-a^3+3a^2z-3bz-6az^2+4z^3$:
And, This being $=2c-2z^3$ (by the third Equation) we therefore have

$6z^3-6az^2+\overline{3a^2-3b}\times z=a^3-3ab+2c$,

and conſequently $z^3-az^2+\frac{a^2-b}{2}\times z=\frac{a^3-3ab+2c}{6}$.

From whence (when a, b and c are expreſſed in Numbers) three different Roots, or Values of z, may be found, anſwering all the Conditions of the Problem.

Thus, for Example, let $a=9$, $b=29$, and $c=99$; then our Equation will become $z^3-9z^2+26z-24=0$.

And (by either of the two firſt Methods explained in Sect. 12. of my *Treatiſe of Algebra*) the three Roots, in this Caſe, will be found to be 2, 3 and 4.

 Which

Which Numbers are, therefore, the true Values of x, y and z, in the Equations $x+y+z=9$, $x^2+y^2+z^2=29$, and $x^3+y^3+z^3=99$.

QUESTION CVIII.

The Sum (a), *the Sum of the Squares* (b), *the Sum of the Cubes* (c), *and the Sum of the Biquadrates* (d), *of any four Numbers being given; to determine the Numbers.*

Let the four Numbers be denoted by x, y, z and u; and put $A=a-u$, $B=b-u^2$, $C=c-u^3$, and $D=d-u^4$. From whence, by the Conditions of the Problem,

$$x+y+z=A,$$
$$x^2+y^2+z^2=B,$$
$$x^3+y^3+z^3=C,$$
$$x^4+y^4+z^4=D.$$

Now, if the Second of these Equations, be subtracted from the Square of the First,
we shall have $2xy+2xz+2yz=A^2-B$.
And if, in like Manner, the fourth Equation be subtracted from the Square of the Second, we shall have $2x^2y^2+2x^2z^2+2y^2z^2=B^2-D$.
Moreover, if from the Square of the former of these last Equations, the Double of the Latter be deducted, there will come out
$8x^2yz+8y^2xz+8z^2xy=A^4-2A^2B-B^2+2D$;
or, $8xyz\times\overline{x+y+z}=A^4-2A^2B-B^2+2D$;
whence $xyz=\dfrac{A^4-2A^2B-B^2+2D}{8A}$ (because $x+y+z=A$).

Again, by multiplying the first and, fifth Equations, into each other, we get $2x^2y+2x^2z+2y^2x+2y^2z+2z^2x+2z^2y+6xyz=A^3-AB$.
And, by multiplying the First and Third together, there arises $x^3+y^3+z^3+x^2y+x^2z+y^2x+y^2z+z^2x+z^2y=AB$:

The

The Double of which laſt taken from the Precedent, leaves $6xyz - 2x^3 - 2y^3 - 2z^3 = A^3 - 3AB$: And this, added to, $2x^3 + 2y^3 + 2z^3 = 2C$, gives $6xyz = A^3 - 3AB + 2C$.

Hence $\frac{A^3 - 3AB + 2C}{6} (= xyz) = \frac{A^4 - 2A^2B - B^2 + 2D}{8A}$

(*p. above*);

and conſequently, by Reduction, $A^4 - 6A^2B + 8AC + 3B^2 - 6D = 0$.

In which Equation let the ſeveral Values of A, B, C and D, be now ſubſtitued, and then (dividing the Whole by 24) we, at length, have $u^4 - au^3 + \frac{a^2 - b}{2} \times u^2 - \frac{a^3 - 3ab + 2c}{6} \times u + \frac{a^4 - 6a^2b + 8ac + 3b^2 - 6d}{24} = 0$: Whoſe four Roots (found by any of the known Methods) anſwer all the Conditions of the Problem.

QUESTION CIX.

To find the leaſt Whole Number, which being divided by 19, ſhall produce a Remainder of 7; but, being divided by 28, the Remainder ſhall be 13.

Let $19x + 7$ denote the Number ſought; where x, according to the Queſtion, muſt be a whole Number. And, by the Queſtion, it likewiſe appears that $19x + 7 - 13$, or, its Equal, $19x - 6$, muſt be diviſible by 28 (without a Remainder).

But it is plain that $28x$ is diviſible by 28: Therefore ($9x + 6$) the Difference between $19x - 6$ and $28x$, muſt alſo be diviſible by the ſame Number 28. For it is well known that, whatever Number, or Quantity, meaſures the Whole, and one Part, of Another (without a Remainder) muſt do the ſame by the remaining Part. Hence ($18x + 12$) the Double of $9x + 6$, being diviſible by 28, if the ſame be ſubtracted from $19x - 6$ (in order to get x without a Coefficient) the Remainder, $x - 18$, will, *ſtill*, be diviſible by the ſame Number; and conſequently $x-18$, either, equal to Nothing, or to ſome Multiple of 28. But, as the leaſt Value of x is requir-ed, $x-18$ muſt be $=0$: And therefore $19x+7=349$, the Number required.

QUESTION CX.

A certain Perſon bought as many Geeſe and Ducks, together, as coſt him 14 Shillings; for the Geeſe he paid 2s. 2d. a-piece; and for the Ducks 1s. 3d. What Number had he of Each?

Let x denote the Number of the Geeſe, and y That of the Ducks;
ſo ſhall $26x+15y=168$, *by the Queſtion*;
and therefore $y=\frac{168-26x}{15}=11-x-\frac{11x-3}{15}$.
Which being a whole Number, by the Nature of the Problem, $11x-3$ muſt, therefore, be (exactly) diviſible by 15.
But it is plain that $15x$ is diviſible by 15; and that its Exceſs above $11x-3$, which is $4x+3$, muſt, *likewiſe*, be diviſible by the ſame Number. Let the laſt Expreſ-ſion $(4x+3)$ be now multiply'd by 3, and the preced-ing One $(11x-3)$ ſubtracted from the Product (in order to get x without a Coefficient) whence you will have $x+12$; which being, *ſtill*, diviſible by 15, it is plain that x muſt either be 3, or 3 added to ſome Multiple of 15, as 18, 33, 48, &c. But it is apparent, from the Nature of the Queſtion, that all theſe Numbers, except the Firſt, are too large. Therefore there were 3 Geeſe, and 6 Ducks; which laſt Number (the Value of x being known) is found directly from the Equation exhibited above.

QUESTION

QUESTION CXI.

One having, at Play, won a certain Number of Guineas, not exceeding 100, *and being asked to tell the Number, made this Reply:* " *If the Number of Guineas I have* " *won be divided by* 9, *there will remain* 6; *but, if* " *the Number of Shillings contained in them be divided* " *by* 39, *there will remain* 12." *The Question is, to find what Number of Guineas he was a Winner of.*

Let $9x+6$ denote the Number sought; where, according to the Question, x must be a whole Number. Then, the Number of Shillings being $189x+54$, it also appears that $\frac{189x+126-12}{39}$, or its Equal $4x+2+\frac{11x+12}{13}$ must be a whole Number; and therefore $11x+12$, divisible by 13.

Let the Number 12 (for the sake of Brevity) be denoted by n:
Then, $13x$, and $11x+n$ being, both, divisible by 13, their Difference $2x-n$ must also be divisible by the same Number; and so, likewise, $\overline{2x-n}\times 5$, or its Equal $10x-5n$. And, if this be subtracted from $11x+n$, the Remainder $x+6n$ (or $x+72$) will, *still*, be divisible by 13. But $\frac{72}{13}$ is $=5+\frac{7}{13}$: Therefore $x+7$ must be divisible by 13; and consequently the Value of x, either, equal to 6, or 6 added to some *Multiple* of 13: But, as the Value of $9x+6$ is not to exceed 100 (by the Question), That of x cannot be greater than 6: And therefore the Number sought can be no other than 60.

QUESTION CXII.

A Perſon, in exchange for a Number of Pieces of Foreign-Gold, valued at 17 *s.* 4 *d. each, received a certain Number of Guineas (not exceeding* 50) *and one Shilling over. What was the Sum exchanged?*

If x be put for the Number of Pieces of Foreign-Gold, and y for the Number of Guineas; then it is plain, from the Queſtion, that $52x = 63y + 3$; and conſequently that

$$x = \frac{63y+3}{52} = y + \frac{11y+3}{52}.$$

Here $11y+n$ (ſuppoſing $n=3$) muſt be diviſible by 52; as muſt, alſo, its Quintuple $55y+5n$. And, if from this laſt, $52y$ be ſubtracted, and the Remainder be multiply'd by 4, we ſhall have $12y+20n$; which muſt be, ſtill, diviſible by the ſame Number; and ſo likewiſe its Exceſs $(y+19n)$ above $11y+n$. But $\frac{19n}{52} = \frac{57}{52} = 1 + \frac{5}{52}$; therefore $y+5$ is, either, equal to 52, or to ſome Multiple of it; and conſequently y equal, either, to 47, 99, 151, *&c.*
But, as the Value of y, by the Queſtion, is not greater than 50, all the Numbers, after the Firſt, are too large. Hence it appears that he received 47 Guineas and one Shilling, in exchange for 57 Foreign-Pieces; amounting in Value to 49 *l.* 8 *s.* Sterling.

QUESTION CXIII.

One laid out 10 *Shillings in* 20 *Fowls, of three different Sorts, viz. Chickens, Pigeons, and Larks: The Chickens coſt him* 12 *d, the Pigeons* 4 *d, and the Larks* 1 *d, a-piece. How many had he of Each?*

Let x, y, and z denote the Numbers of the three ſeveral Sorts, reſpectively.

Then

Then will $\left\{\begin{array}{l} x+y+z=20 \\ 12x+4y+z=120 \end{array}\right\}$ *by the Queſtion.*

And, by ſubtracting the former of theſe Equations from the Latter, we have $11x+3y=100$; and therefore $y=\frac{100-11x}{3}=33-3x-\frac{2x-1}{3}$. Now, $2x-1$ being diviſible by 3, it is evident that $(x+1)$ its Difference from $3x$, muſt likewiſe be diviſible by 3; and, conſequently, that x muſt either be 2, or 2 increaſed by ſome Multiple of 3; that is, equal to ſome one of the Numbers 2, 5, 8, 11, 14. &c. But, as neither y nor z can be greater than 18 (by the Queſtion) ſo all the foregoing Numbers, below and above 8, give the Value of y either too great, or too ſmall.

But, when x is taken 8, y will come out $=4$, and $z=8$; which are the three Numbers required.

QUESTION CXIV.

To determine all the ſeveral Ways whereby it is poſſible to pay 60 l. in Guineas and Moidores, only.

Let x denote the Number of Guineas, and y the Number of Moidores.

Then will $21x+27y=1200$; or $7x+9y=400$, *by the Problem*; and therefore $x=\frac{400-9y}{7}=57-y-\frac{2y-1}{7}$. From whence, as $2y-1$ is diviſible by 7, it will appear, by Reaſoning as in the preceding Examples, that $y+3$ muſt be diviſible by the ſame Number; and conſequently that the leaſt Value of y is $=4$,

and the correſponding Value of x $\left(=57-y-\frac{2y-1}{7}\right)=52$.

Now, having found the leaſt Value of y, and the Greateſt of x, the reſt of the Anſwers will be obtained, by adding 7 (the Coefficient of x in the above Equation) to the laſt Value of y, *continually*, and ſubtracting 9 (the Coefficient of y) from the laſt Value of x. By

means

means of which we get the 6 following Solutions, being all the Queſtion admits of.

viz. $\begin{cases} x = 52,\ 43,\ 34,\ 25,\ 16,\ \text{or},\ 7, \\ y = 4,\ 11,\ 18,\ 25,\ 32,\ \text{or},\ 39. \end{cases}$

QUESTION CXV.

To find how many Ways it is poſſible to pay 20 l. in Half-Guineas, and Half-Crowns, without any other Sort of Coin.

If x be the Number of Half-Guineas, and y the Number of Half-Crowns; we ſhall have $21x + 5y = 800$; and therefore $y = \frac{800 - 21x}{5} = 160 - 4x - \frac{x}{5}$. Whence it appears, at one View, that x is a Multiple of 5: And therefore, the ſeveral, required, Values of x being expreſſed by 5, 10, 15, 20, 25, 30 and 35, Thoſe of y, anſwering thereto, muſt be 139, 118, 97, 76, 55, 34, and 13, reſpectively. So that there are 7 Anſwers in this Caſe.

QUESTION CXVI.

A Reckoning of 20 Shillings was ſpent by a Company of twenty Perſons, conſiſting of Officers, Sailors, and Marines: Each Officer paid 2 s. 6 d. each Sailor 12 d. and each Marine 8 d. How many Perſons were there of each Denomination.

Let the three required Numbers be denoted by x, y, and z, reſpectively;

ſo ſhall $\begin{cases} x + y + z = 20 \\ 30x + 12y + 8z = 240 \end{cases}$ *by the Queſtion.*

And, by ſubtracting 8 times the former of theſe Equations from the Latter, we ſhall have $22x + 4y = 80$; and therefore $y = 20 - \frac{11x}{2}$.

But, y being a whole Number, it is plain that x muſt be an even Number, and, alſo, leſs than 4; and therefore

fore can be no other than 2. From whence y is given $= 9$, and $z = 9$, likewise.

QUESTION CXVII.

To find a Number, which, being divided by 28, *shall produce a Remainder of* 19; *but, being divided by* 19, *the Remainder shall be* 15; *and, being divided by* 15, *the Remainder shall be* 11.

Let $28x + 19$ denote the Number sought; where x, according to the first Condition of the Problem, must be a whole Number. And, by the second Condition, it appears that $28x + 19 - 15$ must be divisible by 19. Whence (following the Method observed in the preceding Examples) the least Value of x is found $= 8$: And so $19z + 8$ (where z denotes any whole Number) is a general Value of x, answering the two first Conditions.

Let this be, therefore, substituted instead of x; and our assumed Expression will become $532z + 243$. From whence, as $532z + 232$ is divisible by 15, the least Value of z will be found $= 14$. And $15n + 14$, will be a general Value of z: Which, substituted in $532z + 243$, gives $7980n + 7691$ for a general Answer to the Problem; where n may be, either, equal to Nothing, or any whole Number.

QUESTION CXVIII.

To find three Numbers, in the proportion of 5, 7, *and* 9; *which being, severally, divided by* 11, 13, *and* 15, *there shall remain* 1, 2, *and* 3, *respectively.*

Let $5x$, $7x$, and $9x$ denote the three required Numbers: Then, by the Question, $5x-1$, $7x-2$, and $9x-3$, must be, respectively, divisible by 11, 13, and 15 (without leaving any Remainder).

But it will be found (by proceeding as in the foregoing Problems) that the least Value of x, to answer the first of these Conditions, will be $= 9$: Therefore $9 + 11z$ (where

(where z denotes any whole Number) will be a general Value of x, anſwering the ſame Condition.

Let this Value be, therefore, ſubſtituted in the ſecond and third Expreſſions; which, by that means, will become $77z+61$, and $99z+78$. And then, as the former of Theſe is diviſible by 13, the leaſt Value of z, to fulfil this Condition, will (alſo) be found $=9$.

Let, therefore, $9+13u$ (which is a general Value for z) be ſubſtituted in the laſt of the three Expreſſions, and it will become $\overline{13u+9}\times 99+78$. Which being diviſible by 15, the $\frac{1}{3}$ Part thereof, or $\overline{13u+9}\times 33+26$ $(=429u+323)$ muſt, conſequently, be diviſible by 5. Whence u is found $=3$: Therefore z $(=13u+9)=48$, and x $(=11z+9)=537$. So that the three leaſt Numbers, anſwering the Conditions of the Problem, are 2685, 3759, and 4833.

QUESTION CXIX.

Suppoſing $6x+7y+8z=100$; *'tis required to find all the poſſible Values of* x, y, *and* z, *in whole Numbers.*

In Queſtions of this Kind, where you have three, or more, indeterminate Quantities, and but one Equation, it will be proper, firſt of all, to find the Limits of thoſe Quantities. Thus, becauſe $x=\frac{100-7y-8z}{6}=16-y-z-\frac{y+2z-4}{6}$, it appears that x cannot be greater than 14. And, in the ſame Manner, it will appear that y cannot be greater than 12; nor z, greater than 10.

Now, as x is a whole Number, by the Queſtion, $y+2z-4$ muſt therefore be diviſible by 6: And, as $2z$ and 4 are even Numbers, it is plain that y muſt alſo be an even Number (ſince an odd One cannot be divided by an even One, without a Remainder). Let y be, therefore, firſt expounded by the leaſt even Number (2), ſo will $y+2z-4$ become $=2z-2$; which, being diviſible by 6, it is plain that $z-1$ (the Half Thereof) is

diviſible

divisible by 3; and consequently that the several Values of z (when $y=2$) are 1, 4, 7, and 10.
Whence the corresponding Values of x, by substituting above, will appear to be 13, 9, 5, and 1.

Let y be now taken $=4$; then $y+2z-4$ will be $=2z$: And so, z, being divisible by 3, the several Values of z, in this Case, will be 3, 6, 9. But the two first of These, only, are for our Purpose, the Last giving $x=0$.

By taking $y=6$, and proceding in the same Manner, we shall get two other Answers; wherein z will be 2, and 5; and x, 7 and 3. And, by taking $y=8$, two more Answers will be found (making 10 in the whole) which are all the Question admits of; and which, being placed in Order, will stand as below.

y	z	x
2	1.4.7.10	13.9.5.1
4	3.6.	8.4.
6	2.5.	7.3.
8	1.4.	6.2.

QUESTION CXX.

If $17x+19y+21z=400$; *'tis proposed to find all the possible Values of* x, y *and* z, *in whole positive Numbers.*

When the Coefficients of the indeterminate Quantities x, y and z are nearly equal, as in this Example, it will be convenient to substitute for the Sum of those Quantities: Thus, let $x+y+z=m$; then, by substracting 17 times this last Equation from the preceding One, we shall have $2y+4z=400-17m$; and by substracting the given Equation from 21 times the assumed One $x+y+z=m$, there will remain $4x+2y=21m-400$. Therefore, since y and z can have no Values less than Unity, it is plain, from the first of these two Equations, that $400-17m$ cannot be less than 6, and therefore m not

not greater than $\frac{400-6}{17}$ or 23: Alſo, becauſe by the ſecond of the two laſt Equations, $21m-400$ cannot be leſs than 6, it is obvious that m cannot be leſs than $\frac{400+6}{21}$ or 19: Therefore 19 and 23 are the Limits of m in this Caſe. Theſe being determined, let $4x$ be tranſpoſed in the laſt Equation, and the Whole divided by 2, and we ſhall have $y = 10m - 200 - 2x + \frac{m}{2}$; which being a whole Number, by the Queſtion, it is evident that $\frac{m}{2}$ muſt likewiſe be a whole Number, and conſequently m equal to an even Number; which, as the Limits of m are 19 and 23, can only be 20, or 22: Let, therefore, m be firſt taken $=20$, then y will become $=10-2x$ and $z\,(m-x-y)\;10+x$; wherein x being taken equal to 1, 2, 3 and 4 ſucceſſively, we ſhall have y equal to 8, 6, 4, 2 and z equal to 11, 12, 13, 14 reſpectively; which are four of the Anſwers required. Again, let m be taken $=22$, then will $y=31-2x$ and $z=x-9$, in which let x be taken equal to 10, 11, 12, 13, 14 and 15 ſucceſſively, and y will come out $=11$, 9, 7, 5, 3 and 1, and $z=1$, 2, 3, 4, 5 and 6, reſpectively. Therefore we have the ten following Anſwers in whole Numbers; which are all the Queſtion admits of.

$x =$	1	2	3	4	10	11	12	13	14	15
$y =$	8	6	4	2	11	9	7	5	3	1
$z =$	11	12	13	14	1	2	3	4	5	6.

QUESTION

QUESTION CXXI.

To find two Whole Numbers, whereof the Difference of the Squares shall be 77.

Let the leſſer Number be x, and the Greater $x+m$; and ſuppoſe the Number given to be repreſented by a:

So ſhall $\overline{x+m}|^2-x^2=a$,
that is, $2mx+mm=a$;
and conſequently $x=\frac{a-mm}{2m}=\frac{a}{2m}-\frac{m}{2}$.

Whence we alſo have $x+m=\frac{a}{2m}+\frac{m}{2}\left(=\frac{a+mm}{2m}\right)$.

But, in the Caſe propoſed, a being $=77$,
x becomes $=\frac{77}{2m}-\frac{m}{2}$, and $x+m=\frac{a}{2m}+\frac{m}{2}$:
Which being both required in Whole Numbers, it is evident, in the firſt Place, that m muſt be ſome Diviſor of 77; and, ſecondly, that $\frac{77}{m}$ muſt be greater than m; and conſequently m leſs than 9.

But the Diviſors of 77, below 9, are 1 and 7: Which Numbers being wrote ſucceſſively, in the Room of m, the correſponding Values of x will come out 33, and 2; and Thoſe of $x+m$, 34 and 9, reſpectively: So that the Queſtion, in the Caſe propoſed, admits of two Anſwers, and no more.

QUESTION CXXII.

To find a Whole Number, to which 12 *and* 25 *being, ſucceſſively, added, both the Sums ſhall be ſquare Numbers.*

Let z be the Number ſought; and aſſume x and $x+m$ for the Roots of the two Squares:

Then will $\left\{\begin{array}{l} z+12=x^2 \\ z+25=\overline{x+m}|^2 \end{array}\right\}$ *by the Queſtion.*

 Hence,

Hence, by ſubtracting the former Equation from the Latter, we get $13 = \overline{x+m}|^2 - x^2 = 2mx + mm$;

and therefore $x = \frac{13}{2m} - \frac{m}{2}$. Which being a Whole Number, m muſt be $= 1$: Whence $x = 6$; and z $(= x^2 - 12) = 24$.

QUESTION CXXIII.

To find three Whole Numbers, ſo that the Sum of the Squares of the Two leaſt of them ſhall be equal to the Square of the Greateſt.

It appears, from the Problem preceding the Laſt, that the Difference of the Squares of $\frac{a+mm}{2m}$, and $\frac{a-mm}{2m}$ is, univerſally, equal to a,

or $\frac{\overline{a+mm}|^2}{4mm} - \frac{\overline{a-mm}|^2}{4mm} = a$; let a and m be what they will.

Whence it is alſo plain that $\overline{a+mm}|^2 = \overline{a-mm}|^2 + 4mma$.

But, ſince it is required to have $4mma$, a ſquare Number (as well as the other Two) a muſt therefore be a ſquare Number; let it be n^2, and then our Equation will become $\overline{nn+mm}|^2 = \overline{nn-mm}|^2 = \overline{2mn}|^2$: Where m and n may be expounded by any Whole Numbers, at Pleaſure.

Thus, for Example, ſuppoſe $m=1$, and $n=2$; then there will come out $5^2 = 3^2 + 4^2$. Again, let $m = 2$, and $n = 3$, and there ariſes $\overline{13}|^2 = \overline{5}|^2 + \overline{12}|^2$. Laſtly, let $m=2$ and $n=5$, and you will get $\overline{29}|^2 = \overline{21}|^2 + \overline{20}|^2$.

QUESTION CXXIV.

To find three Whole Numbers, whose Squares are in Arithmetical Progression.

Let x, $x+m$, and $x+n$ express three such Numbers. So shall $\overline{x+m}|^2 - x^2 = \overline{x+n}|^2 - \overline{x+m}|^2$, by the Nature of the Problem.

Whence x is found $= \frac{nn-2mm}{4m-2n}$.

Put $4m-2n = a$, and $n^2-2m^2 = b$; then $x = \frac{b}{a}$, $x+m = \frac{b+am}{a}$, and $x+n = \frac{b+an}{a}$. Now, as the Squares of $\frac{b}{a}$, $\frac{b+am}{a}$, and $\frac{b+an}{a}$ are in Arithmetical Progression, it is plain that the Squares of their *Equimultiples*, b, $b+am$, and $b+an$, must be in Arithmetical Progression likewise. From whence, by expounding m and n by different Whole Numbers, successively, as many particular Answers as you please, may be exhibited.

Thus, if $m=2$ and $n=3$; then, a being $=2$, and $b=1$, there will come out 1, 5, and 7. But, if $m=3$ and $n=5$, we shall get 7, 13, and 17, for another Answer.

QUESTION CXXV.

Supposing $x^2 = z^2 + az + b$ *(where* a *and* b *denote given Numbers); 'tis required to find the Values of* x *and* z *(if possible) in Whole Numbers.*

Put $x = z+m$; then, by Substitution,
$z^2+2mz+m^2 = z^2+az+b$;

and consequently $z = \frac{m^2-b}{a-2m}$. Which Value, by putting $a-2m = n$ $\left(\text{or } m = \frac{a-n}{2}\right)$ becomes $= \frac{a^2-2an+n^2-4b}{4n} =$

 $\frac{1}{4}$ of

$\frac{1}{4}$ of $\sqrt{\frac{aa-4b}{n}} - 2a+n$. From which it is evident, that, to have the Value of z a Whole Number, n muſt be ſome Diviſor of the given Quantity $aa-4b$, and therefore even or odd, according as a is even or odd.

Example. Suppoſe the given Equation to become $x^2 = z^2+20z$; in which Caſe, a being $=20$, and $b=0$, we have $z = \frac{1}{4} \times \sqrt{\frac{400}{n}} - 40+n$; where the, even, Diviſors of 400 are 2, 4, 8, 10, *&c.* whereof the Second will be found to anſwer; the Values of z and x coming out 16 and 24, reſpectively.

Again, ſuppoſe the given Equation to become $x^2 = z^2+100z+1000$: Here we have $z = \frac{1}{4} \times \sqrt{\frac{6000}{n}} - 200+n$: And the, even, Diviſors of 6000, are 2, 4, 6, 8, 10, 12, 16, 20, *&c.* Whereof 4, 12, and 20 ſucceed: By the laſt of Theſe (which determines the leaſt Values) z comes out $=30$, and $x=70$.

QUESTION CXXVI.

Having given $x^2 = a^2 + bz + cz^2$, *wherein* a, b *and* c *denote given Whole Numbers; 'tis required to find the Values of* x *and* z *(if poſſible) in Whole Numbers.*

Put $x = a+mz$; then will $\overline{a+mz}|^2 = a^2+bz+cz^2$; that is, $a^2+2amz+m^2z^2 = a^2+bz+cz^2$.

Whence z comes out $= \frac{b-2am}{mm-c}$:

Where it is evident, that, in order to have a poſitive Value, m muſt be taken equal to ſome Number between $\sqrt{c}$, and $\frac{b}{2a}$.

Thus, ſuppoſing the given Equation to become $x^2 = 64-12z+5z^2$, the Value of m, in this Caſe, muſt be

leſs

leſs than $\sqrt{5}$, and greater than $-\frac{12}{16}$. Let it therefore be expounded by 2 and 1, ſucceſſively;

Whence $\frac{-12-16m}{mm-5}$, or its Equal $\frac{12+16m}{5-mm}$ (which is, here, the Value of z) will come out 44 and 7 reſpectively; and the correſponding Values of x $(a+mz)$ are found to be 96 and 15: Both which anſwer the Conditions of the Problem.

PART II.

CONTAINING

A Variety of GEOMETRICAL PROBLEMS, *with their* SOLUTIONS: *Both by* ALGEBRA, *and independent of it, from Principles purely* GEOMETRICAL.

ALTHOUGH the chief Defign of this Work confifts not in Exhibiting a Number of Rules and Precepts, but in illuftrating Thofe already known, on the Subject, by a Set of proper and ufeful Examples; yet, as we are in this Part to treat of the Inveftigation of Geometrical Problems, without calling in the Affiftance of Algebra (a Thing hitherto very little confidered by Authors, tho' in itfelf very interefting and ufeful) it may not be amifs, before we proceed to particular Cafes, to premife a few General Obfervations on this Head.

1. In the firft Place, then, it is neceffary, in order to the Conftruction of Geometrical Problems, that fomething of the *Geometric Loci* fhould be underftood— Thus it will be of Ufe to know that the Place of the Vertex of a Triangle, whereof the Bafe and oppofite Angle are fuppofed to remain conftant, while the other Sides and Angles vary, will *always* fall in the Circumference of a Circle paffing thro' the Extremities of the Bafe. This appears from *Euclid B.* 3. *Prop.* 21; and is alfo demonftrated in my *Elements of plane Geometry, B.* 3, *Prop.* 9.

It ought moreover to be known, that, the Locus of the Vertex, when the Ratio of the two Sides and the Bafe

Base of the Triangle are given, or continue invariable, will, *also*, be the Circumference of a Circle, dividing the Base in the given Ratio (*For the Domonstration of which, see Elem. plane Geometry B.* 4, *Prop.* 15.).

If the Sum, or the Difference, of the Squares of the two Sides, together with the Base, be supposed given, the Locus of the Vertex will, *still*, be the Circumference of a Circle, in the former Case; and a Right-line, in the Latter (*Vid. B.* 2, *Prop.* 12).

But, if the Sum, or the Difference, of the Sides, themselves, is given; then the Locus will, either, be the Arch of an Ellipsis or an Hyperbola (as is well known to Those who have touch'd upon Conic-Sections). But these two Last are not admitted in the Construction of linear and plane Problems; of which, only, we purpose to treat ——The Problems, of the following Collection, wherein the Use of the *Geometric-Loci*, above specified, is particularly exemplified, are the 11, 12, 19, 34, 35, 48, and 51st.

2°. When, in the Figure to be constructed, the Sum, or the Difference, of two adjacent Sides happens to be given; it will be proper, first, to form a Triangle, so that the said Sum, or Difference, may be one of its Sides; and, then, to consider, what other Sides, or Angles, will be given, or become known, in Consequence Thereof. This Rule is illustrated in the 1, 2, 18, 26, 27th, and some other of the succeeding Problems.

3°. It often happens that the Ratio of two, or more, Lines is given, from the Nature of the Figure, or by Hypothesis, though the Lines themselves are absolutely unknown: In all such Cases we must endeavour, by drawing Parallels (or some other Way) to obtain other Lines in the same given Ratio; so that, one of them being given, or known from the Nature of the Figure or Problem, the other may also become known— The Use of this Rule, which is very extensive, will particularly appear in the Solutions to the 3, 13, 16, 22, 26, 33, 37, and 56th. Problems.

4°. But, if the Lines, whereof the Ratio is given, ſhould happen to lye remote of each other; then, by the Help of Thoſe, we muſt endeavour to determine the Ratio of Others, lying nearer together; and ſo on, till we obtain (if poſſible) the Ratio of two Lines, that are both Sides of the ſame Triangle; wherein one Angle and the remaining Side (or ſome other, two, Parts) are given — For the better underſtanding this Rule, conſult the Solutions to the 49th and 57th Problems, in particular.

5°. Laſtly, when the Rectangle under two unknown Lines is given, either, a mean Proportional muſt be found, or elſe, two other Lines muſt be aſſigned, by forming ſimilar Triangles (or ſome other Way) ſo as to comprehend an equal Rectangle; and ſo that, one of them being given by the Nature of the Figure, the Other may alſo become known. This Rule is exemplified in the 4, 5, 6, 9, 10, 21, 38, 40, 41, 44, and 53^{d}. Problems.

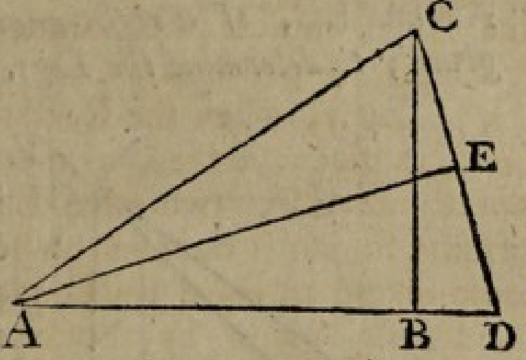
Beſides the above, other Obſervations might be here laid down; but Thoſe already delivered being the moſt General that have occur'd to me, I ſhall now proceed on in the Reſolution of Problems; the proper Buſineſs of this Work.

PROBLEM

PROBLEM I.

One Leg BC, *and the Difference between the other Leg* AB *and the Hypothenuſe* AC, *of a right-angled Triangle* ABC, *being given; to find both* AB *and* BC.

Put $BC = a$, $AB = x$, and $AC = x + b$; (b being the given Difference) Then,



$\overline{AC}|^2$ being $= \overline{AB}|^2 + \overline{BC}|^2$ (*Elem.* 7. 2. *) we have $xx + 2bx + bb = xx + aa$: Whence $2bx = aa - bb$; and conſequently $x = \frac{aa-bb}{2b} = \frac{aa}{2b} - \frac{b}{2}$. From which AC $(x+b)$ is given $= \frac{aa}{2b} + \frac{b}{2}$.

Geometrically.

If, in AB produced, there be taken BD equal to the given Difference of AC and AB, and DC be drawn (according to the ſecond General Obſervation) it is evident that AD will be equal to AC; and the Angle ACD, alſo, equal to the Angle D.

Therefore, having taken BD as above ſpecified, and made BC perpendicular thereto, and of the given Length, and joined D, C; let CA be ſo drawn as to make the Angle DCA=D; or, inſtead thereof, let a perpendicular EA be erected on the middle of CD; then the Interſection of either of theſe Lines with DB, produced, determines the Triangle.

From this Conſtruction we have the very ſame Theorem, for the numerical Solution, as is derived above, from the Algebraical Proceſs: For the Triangles ADE and CDB, having each a Right-Angle, and D

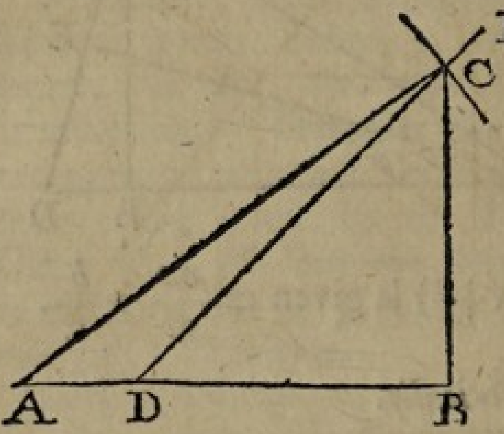
* *Note. The Quotations, in this, and the ſucceeding Problems, refer to my Elements of Plane Geometry; printed for* J. NOURSE.

common,

common, are ſimilar. Therefore DB : DC : : DE ($\frac{1}{2}$DC) : AD (AC) $= \frac{DC^2}{2DB} = \frac{BC^2+BD^2}{2BD} = \frac{BC^2}{2BD} + \frac{BD}{2}$; *as before.*

PROBLEM II.

The Hypothenuſe AC, *and the Difference of the two Legs* AB *and* BC, *of a right-angled Triangle* ABC, *being given; to determine the Legs.*



Put AC $= a$, AB $= x$, and BC $= x - b$: So ſhall $xx + \overline{x-b}|^2 = aa$ (*Elem.* 7. 2.). that is, $2xx - 2bx + bb = aa$. Whence $xx - bx = \frac{aa}{2} - \frac{bb}{2}$: And conſequently $x = \sqrt{\frac{1}{2}aa - \frac{1}{4}bb} + \frac{1}{2}b$.

Geometrically.

If, in AB there be taken AD equal to the given Difference, and CD be drawn; then, DB being = BC, the Angle BDC will alſo be = BCD = Half a Right-Angle.

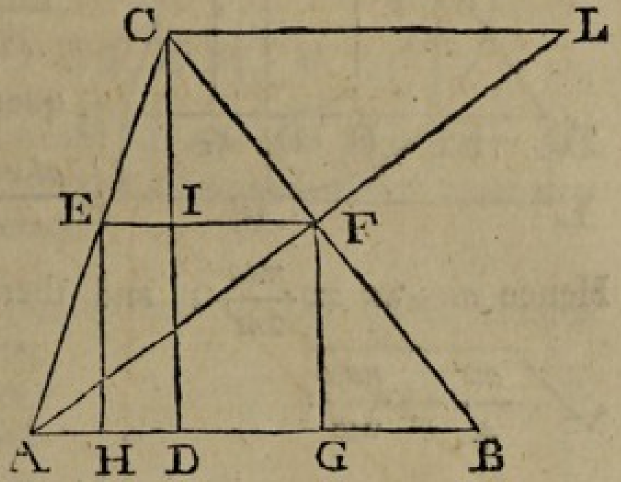
Therefore, having laid down AD, and drawn an indefinite Line DCE to make the Angle BDE = $\frac{1}{2}$ Right-Angle; upon the Center A, with the given Interval AC, let an Arch be deſcribed, interſecting DE in C; from which Point, upon AD produced, let fall the Perpendicular CB; ſo ſhall ABC be the Triangle required.

The numerical Solution, according to this Conſtruction, is very eaſy, by the Help of Trigonometry: For, two Sides and one Angle of the Triangle ADC being given, the other Angles may from thence be found; and then, all the Angles and one Side (AC) of the propoſed Triangle being known, the other Sides AB and BC may alſo be determined.

PROBLEM

PROBLEM III.

The Baſe AB *and the Perpendicular* CD *of any Triangle* ABC *being given*; *to find the Side* EF, *or* EH, *of the inſcribed Square* EFGH.



Put CD$=a$, AB$=b$, and DI ($=$EF)$=x$: Then will CI $=a-x$; and, by Reaſon of the ſimilar Triangles A B C and ECF, we ſhall have $a : b :: a-x : x$ ($=$ EF). Therefore $ax = ab-bx$; and conſequently $x = \frac{ab}{a+b}$.

Geometrically.

The Ratio of EH to EA being given, as CD to CA, EF muſt therefore be to EA in the ſame given Ratio: And, if CL be drawn parallel to EF, meeting AF produced in L *(agreeable to the* 3[d] *General Obſervation)* it is evident, becauſe of the ſimilar Triangles, that the Line CL, ſo drawn, will be to CA, *ſtill*, in the ſame given Ratio; that is, CL : CA : : CD : CA; and conſequently CL $=$ CD. Whence the Method of Conſtruction is manifeſt.

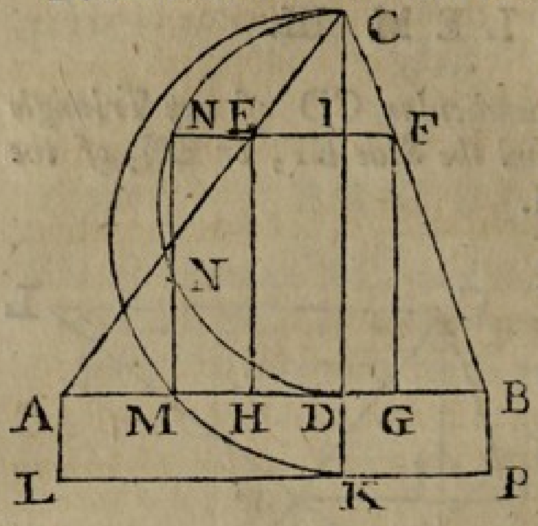
PROBLEM IV.

To determine the Sides of a Rectangle, EFGH, *inſcribed in a given Triangle* ABC, *whoſe Area ſhall be to That of the Triangle in a given Ratio.*

Put the Perpendicular CD$=a$, the Baſe AB$=b$, and the Altitude EH of the Rectangle $=x$; and let the given Ratio of ABC to EFGH be that of m to n.

Becauſe

Becauſe of the ſimilar Triangles ABC and EFC, it



will be, CD (a) : AB (b) : : CI $(a-x)$: EF $= \frac{ba-bx}{a}$ (*Elem.* 12. 4.) Therefore EF × EH $= \frac{abx - bx^2}{a}$; and conſequently $m : n :: \frac{ab}{2} : \frac{abx-bx^2}{a}$ (*by the Queſtion*). Hence $ax-xx = \frac{naa}{2m}$; and therefore $x = \frac{a}{2} \pm \sqrt{\frac{aa}{4} - \frac{naa}{2m}}$.

Geometrically.

The Rectangle HF being to the Triangle ABC in a given Ratio, and the latter of Theſe being actually given, the Magnitude of the Former is alſo given; and therefore may be expreſſed by a given Rectangle ABPL, on the Baſe AB; whoſe Altitude KD is to Half That of the Triangle in the aforeſaid given Ratio.

But it appears that the Rectangle DI × EF is to the Rectangle DI × IC in the given Ratio of EF to IC, or of AB to CD (*Elem.* 1 *and* 12. 4); and that the Rectangle DK × AB is alſo to DK × CD in the ſame given Ratio. Therefore, the Antecedents being equal, the Conſequents muſt likewiſe be equal, or DI × IC = DK × CD. Whence this Conſtruction.

Deſcribe, upon CD and CK, two Semi-circles; and, from the Point M wherein the Circumference of the latter cuts AB, let MN be drawn, parallel to DC interſecting the Former in N; ſo ſhall MN be the required Altitude of the Rectangle. Since DI × CI $= \overline{IN}|^2 = \overline{DM}|^2 =$ DK × CD (*Elem.* 11. 4), *as above.*

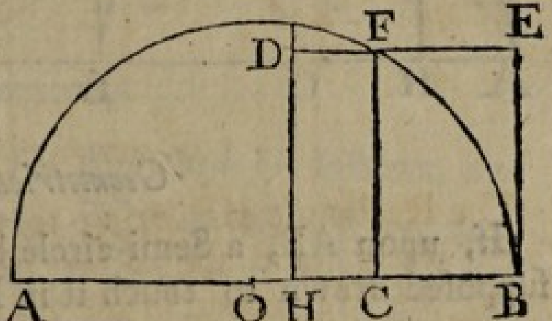
This

This Problem, it is obſervable, becomes impoſſible when MN paſſes intirely without the leſſer Semi-circle, that is, when the given Rectangle is ſuppoſed greater than half the Triangle. The ſame Thing appears alſo from the Algebraic Solution, in which Caſe the Quantity $\left(\frac{aa}{4} - \frac{naa}{2m}\right)$ under the Radical-Sign, becomes negative.

PROBLEM V.

To divide a given Right-line AB *into two ſuch Parts* AC *and* BC, *that the Rectangle contained under them may be of a given Magnitude.*

Put AB$=a$, and AC$=x$, and let the given Magnitude, or Content, of the propoſed Rectangle be repreſented by the Square BD whoſe Side BE, or ED, let be denoted by b: Then will $x \times \overline{a-x} = bb$; or $xx - ax = -bb$.



Whence $x = \pm\sqrt{\tfrac{1}{4}aa - bb} + \frac{a}{2}$.

Geometrically.

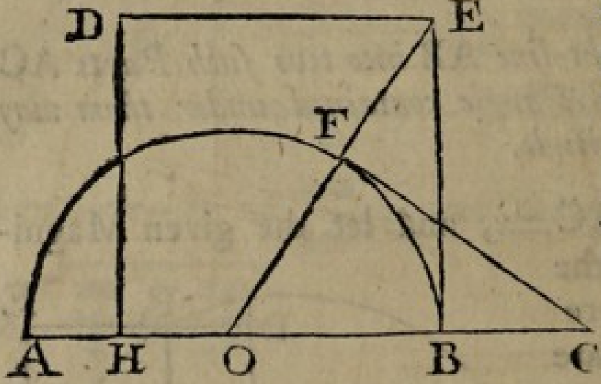
If upon AB, as a Diameter, a Semi-circle AFB be deſcribed, it is evident (*by Elem.* 11. 4.) that a Perpendicular FC, drawn from the Point wherein the Circumference interſects DE, will cut AB in the Point required—It is plain, from both theſe Solutions, that the given Rectangle muſt not be greater than the Square of half the given Line, to be divided.

PROBLEM

PROBLEM VI.

To a given Line AB, *it is required to add another Line* BC, *so that the Rectangle under the whole, compounded, Line* AC, *and the Part added, may be of a given Magnitude.*

Let AB $=a$, BC $=x$, and the Side of the given Square BEDH, expressing the Magnitude of the proposed Rectangle, $=b$. Then we shall have $\overline{a+x}\times x=bb$; and consequently $x=\sqrt{bb+\frac{1}{4}aa}-\frac{1}{2}a$.



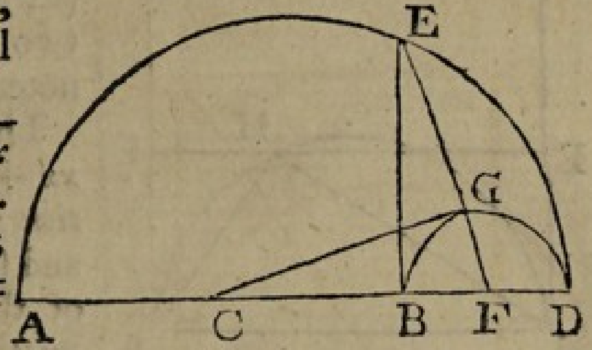
Geometrically.

If, upon AB, a Semi-circle be described, and CF be supposed drawn to touch it in D, it is plain, *from Elem. Corol. to* 17. 3, that $\overline{CF}|^2$ is $=$ AC $\times$ BC $=\overline{BE}|^2$ *(by Hypothesis)*; and consequently CF $=$ BE: Therefore, OF being $=$ OB, it follows that OE and OC are likewise equal. Hence, if to the Middle of AB, we draw EO, and take OC $=$ EO, *the Thing is done.*

PROBLEM

PROBLEM VII.

To divide a given Right-line AB *into two ſuch Parts, that, the Rectangle under one of them* AC *and another, given, Line* BD, *may be equal to the Square of the remaining Part* BC.



Put AB$=a$, BD$=b$, and BC$=x$; then will AC $=a-x$, and therefore $xx=\overline{a-x}\times b$, *by the Queſtion.* Hence $xx+bx=ab$; and, conſequently $x=\sqrt{ab+\frac{1}{4}bb}-\frac{1}{2}b$.

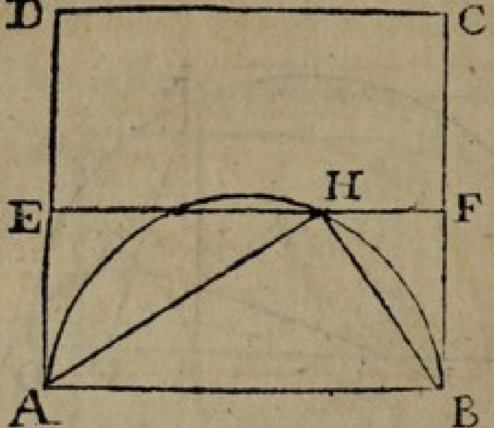
Geometrically.

Since $\overline{BC}|^2=AC\times BD$ *(by Hypotheſis)* it follows, by adding BC×BD to Each, that $\overline{BC}|^2+BC\times BD=AC\times BD+BC\times BD$; or that, $BC\times CD=BD\times AB=\overline{BE}|^2$, taking BE a Mean Proportional between BD and AB *(by Elem.* 14. 5.) But the Rectangle BC×CD, if a Semi-circle be deſcribed upon the Diameter BD, is known to be equal to the Square of the Tangent CG *(Elem. Cor. to* 17. 3). Hence $\overline{CG}|^2=\overline{BE}|^2$; and conſequently CG$=$BE: Therefore, FG being alſo$=$FB, it follows that FC is equal to FE; whence the Method of Conſtruction is manifeſt.

PROBLEM

PROBLEM VIII.

To determine two Lines, whereof the Rectangle ſhall be equal to a given Rectangle ABFE, *and the Sum of their Squares equal to a given Square* ABCD.



Put AB (=BC) $= a$, BF (= AE) $= b$, and let the two required Lines be denoted by x and y.

Then will $xy = ab$, and $xx + yy = aa$, *by the Queſtion:* Whence by adding, and ſubtracting, the Double of the former of theſe Equations from the Latter,

we have $\begin{cases} xx+2xy+yy\ (=\overline{x+y}|^2) = aa+2ab, \\ xx-2xy+yy\ (=\overline{x-y}|^2) = aa-2ab. \end{cases}$

Therefore $\begin{cases} x+y = \sqrt{aa+2ab} \\ x-y = \sqrt{aa-2ab} \end{cases}$

From which $\begin{cases} 2x = \sqrt{aa+2ab} + \sqrt{aa-2ab} \\ 2y = \sqrt{aa+2ab} - \sqrt{aa-2ab}. \end{cases}$

Geometrically.

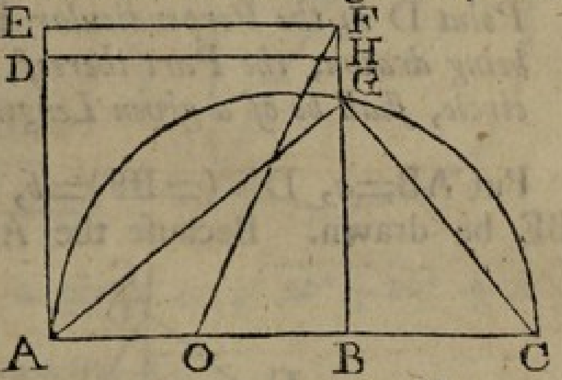
If upon AB a Semi-circle be deſcribed, interſecting EF in H, then the Lines joining A, H, and B, H will anſwer the Conditions of the Problem. For, the Angle AHB being a Right-one *(Elem.* 11. 3.) thence is $\overline{AH}|^2 + \overline{BH}|^2 = \overline{AB}|^2 =$ ABCD *(Elem.* 7. 2.); and AH × BH (= twice the Triangle ABH) = ABFE *(Elem. Corol. to* 2. 2).

PROBLEM

PROBLEM IX.

To determine two Lines, whereof the Rectangle shall be equal to a given Rectangle ABFE, *and the Difference of their Squares equal to a given Square* ABHD.

Put AB$=a$, BF$=b$; and let x be the greater, and y the lesser, of the two Lines required: So shall $xy=ab$; and $xx-yy=aa$, *by the Question*. From the former of which Equations we have $y=\frac{ab}{x}$; which



Value, substituted in the Latter, gives $xx-\frac{aabb}{xx}=aa$.

Hence $x^4-a^2x^2=a^2b^2$;

and consequently $x=\surd:\frac{1}{2}aa+a\sqrt{bb+\frac{1}{4}aa}:$ Whence y will also be known.

Geometrically.

It is evident, in the first Place, that the two Lines to be determined will be the Hypothenuse and one Leg of a right-angled Triangle (ABG) whose remaining Leg is the given Line AB. And, since the Rectangle under these Lines is supposed given, another Triangle ACG, similar to ABG, must therefore be assumed; so that CG, in the Former, and BG, in the Latter, may be homologous Sides (*according to the* 5[th] *General Observation*; *vid.* *p*. 88.)

Hence we have CG×AB=BG×AG (*Elem.* 18. 3)= AB×BF (*by Hyp.*) and consequently CG = BF:

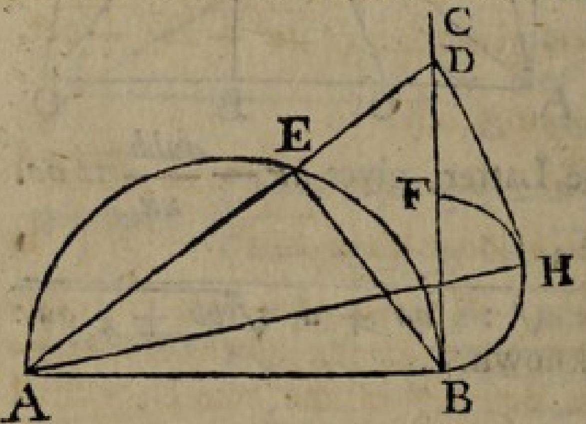
And so, AC×BC ($=\overline{CG}|^2$, *Elem. Cor.* 11. 4) being given $=\overline{BF}|^2$, the Case under Consideration is now reduced to our 6[th] Problem: Whence we have the following Construction.

To the Middle of AB, let FO be drawn; and, having taken OC = OF, let a Semi-circle be described upon AC, cutting BF in G; ſo ſhall BG be the leſſer, and AG the greater of the two Lines required.

PROBLEM X.

The Diameter AB *of a Semi-circle being given, to find a Point* D *in the Perpendicular* BC, *from whence* DA *being drawn, the Part thereof,* ED, *without the Semi-circle, ſhall be of a given Length* (BF).

Put AB=a, DE (=BF)=b, and AD=x; alſo let BE be drawn. Becauſe the Angle AEB is a Right-one, the Triangles ADB and ABE are ſimilar. And therefore AD (x) : AB (b) : : AB(b) : AE($x-a$). Whence $xx-ax=bb$; and conſequently $x=\sqrt{bb+\frac{1}{4}aa}+\frac{1}{2}a$.



Geometrically.

Since DA×EA is = AB^2 *(Elem.* 18. 3.) where the Part DE of DA is given (=BF) the Caſe is therefore, reduced to our 6[th] Problem.

From whence it will appear, that, if upon BF a Semi-circle be deſcribed, and through the Center thereof, AH be drawn, meeting the Periphery in H, an Arch deſcribed from the Center A, with the Radius AH, will cut BC in the Point required.

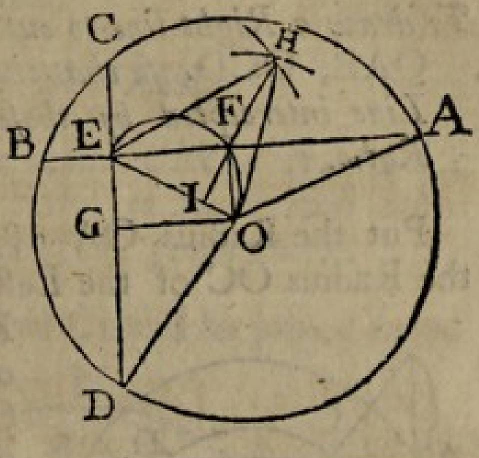
PROBLEM XI.

Having the Length of two Chords AB *and* CD, *cutting each other at Right Angles, together with the Diſtance* OE *of the Point of their Interſection from the Center; to determine the Diameter of the Circle.*

Upon the given Chords, from the Center O, let fall

the

the Perpendiculars OF and OG; and draw the Radii OA and OD: Alſo put AF ($=\frac{1}{2}$ AB) $= a$, DG ($=\frac{1}{2}$ CD) $= b$, OE $= c$, and AO (= DO) $= x$.



Then will $\overline{OF}^2 = x^2 - a^2$, and $\overline{OG}^2 = x^2 - b^2$; but $\overline{OF}^2 + \overline{OG}^2$ ($= \overline{OF}^2 + \overline{FE}^2$) is $= \overline{OE}^2$; that is, $2x^2 - a^2 - b^2 = c^2$; and conſequently $x = \sqrt{\frac{aa+bb+cc}{2}}$.

Whence the Diameter is given $= \sqrt{2a^2 + 2b^2 + 2c^2}$.

Geometrically.

Since $\overline{FE}^2 + \overline{DG}^2$ is ($= \overline{OD}^2 = \overline{OA}^2$) $= \overline{OF}^2 + \overline{AF}^2$, it is evident that $\overline{FE}^2 - \overline{OF}^2$ is given $= \overline{AF}^2 - \overline{DG}^2$. We are therefore to conſtruct a right-angled Triangle upon the given Hypothenuſe OE, whereof the Squares of the two Legs ſhall have the ſame Difference as the two given Squares $\overline{AF}^2$ and $\overline{DG}^2$.

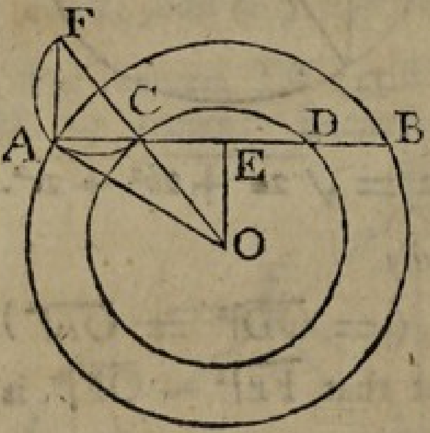
In order to which, upon OE, let a Semi-circle be deſcribed; alſo, from the Centers O and E, with Radii equal to DG and AF, reſpectively, let two Arcs be deſcribed, interſecting each other in H; from which Point, upon OE, let fall the Perpendicular HG; which will interſect the Semi-circle in (F) the Vertex of the required Triangle: Since it is evident that $\overline{FE}^2 - \overline{OF}^2 = \overline{EI}^2 - \overline{OI}^2 = \overline{EH}^2 - \overline{OH}^2 = \overline{AF}^2 - \overline{DG}^2$ (*by Conſtruction.*)

Therefore, if in EF produced, FA be taken of the given Length, OA (when drawn) will be the Radius of the required Circle.

PROBLEM XII.

To draw a Right line to cut two given concentric Circles, OAB, OCD, ſo that the Chords, or Parts of the ſaid Line intercepted by thoſe Circles, may obtain a given Ratio.

Put the Radius OA of the greater Circle $=a$, and the Radius OC of the Leſſer $=b$; and let the given Ratio of AB to CD be That of m to n: Then, denoting OE, the Diſtance of AB from the Center, by x, we have $AE^2 = aa-xx$, and $CE^2 = bb-xx$. Therefore, AE being $= \frac{1}{2}$ AB, and CE $= \frac{1}{2}$ CD, it follows, that, $aa-xx : bb-xx :: m^2 : n^2$; and conſequently $n^2a^2 - n^2x^2 = m^2b^2 - m^2x^2$: Whence $x = \sqrt{\frac{mm\,bb - nn\,aa}{mm-nn}}$. By Means of which AB may be drawn.



Geometrically.

Since the Ratio of AE to CE is given, let OC be produced to F, ſo that OF may be to OC in the ſame, given Ratio, *(agreeable to the 3^d General Obſervation)* then, A, F being joined, the Triangle CAF will be ſimilar to the Triangle CEO; and conſequently the Angle CAF a Right one. Hence the following Conſtruction.

Having drawn the Radius OC, and in it, produced, taken OF in proportion thereto, as AB is to CD *(as above ſpecified)* let a Semi-Circle, upon CF be deſcribed, interſecting the greater of the two given Circles in A; from which Point, through C, draw AB, *and the Thing is done.*

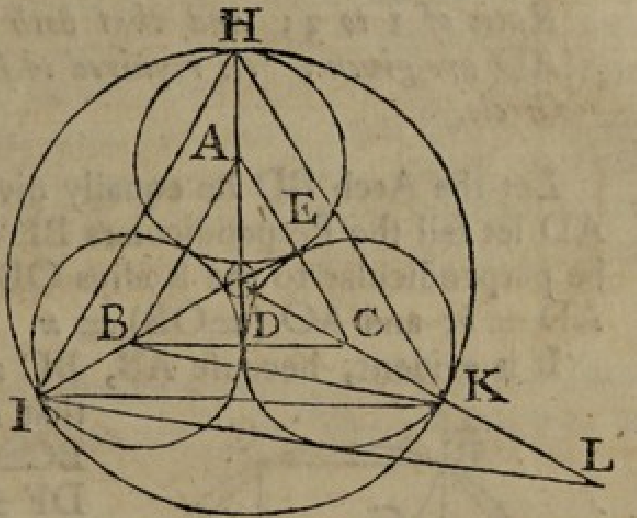
It is manifeſt, both from This, and the Algebraic Solution, that the Ratio of m to n (or of AB to CD) cannot

cannot be given lefs than That of OA to OC, without rendering the Problem impoffible.

PROBLEM XIII.

To determine the Radii of three equal Circles A, B, C, *defcribed in a given Circle* HIK *to touch each other and likewife the Circumference of the given Circle.*

Let the Centers of the feveral Circles be joined; and let AO and BO be produced to bifect BC and AC in D and E: Alfo let the Radius (OI) of the given Circle be denoted by a, and That of each of the required Ones by x. Now the Triangles BCE and BOD being fimilar, and CE $= \frac{1}{2}$ BC, it appears that OD is alfo $= \frac{1}{2}$ OB. But $\overline{OB}|^2 - \overline{OD}|^2$ is $= \overline{BD}|^2$; that is, in Species, $\overline{a-x}|^2 - \frac{\overline{a-x}|^2}{4} = xx$. Which, folved, gives $x = \sqrt{12aa} - 3a = a \times \overline{2\sqrt{3} - 3}$.



Geometrically.

It is evident that the Right-line IK, joining the Points of Contact I and K, is the Side of an Equilateral Triangle infcribed in the given Circle: And, that, if in OK produced there be taken KL $= \frac{1}{2}$ IK, a Line drawn from I to L, will be parallel to another Line drawn from B to K; becaufe the Triangles IKL and BCK (having IKL $=$ BCK, and IK : KL (: : 2 : 1) : : BC : CK) are equiangular.

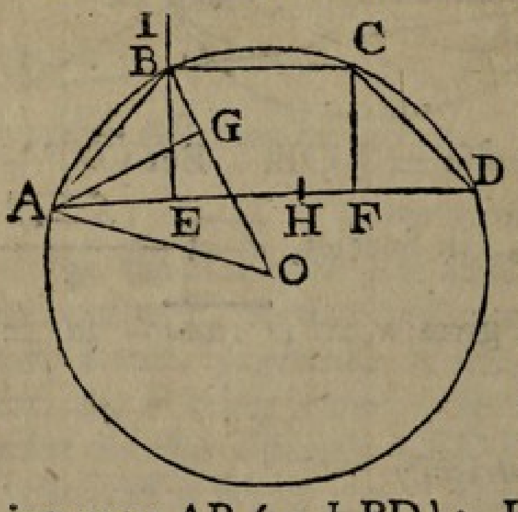
Therefore, in order to the Geometrical Conftruction, having firft drawn the Radii OH, OI, and OK, to divide the Circumference in three equal Parts, and taken KL, in OK produced, equal to $\frac{1}{2}$ IK; draw LI, and KB, parallel thereto, meeting OI in B; make OA and OC each = OB; and upon the Centers A, B, and C, through A, I, K let the three required Circles be defcribed.

PROBLEM XIV.

Suppofing AB *and* ABD *to be two Arcs of a Circle, in the Ratio of* 1 *to* 3; *and that both their Chords* AB *and* AD *are given; 'tis required to find the Radius of the Circle.*

Let the Arch BD be equally divided in C, and upon AD let fall the Perpendiculars BE and CF; alfo let AG be perpendicular to the Radius OB: And put AB $= a$, AD $= b$, and AO ($=$OB) $= x$.

It is evident, becaufe AB, BC and CD are all equal, that AE=DF, and EF= BC$=a$: Therefore AE+DF $= b-a$, and AE $= \frac{b-a}{2}$. It alfo appears that the Triangles ABE and AOG are equiangular, becaufe the Angle BAD, ftanding upon the Arch BD, is equal to the Angle O, at the Center, ftanding upon AB ($= \frac{1}{2}$ BD): Hence we have, AB (a) : AE $\left(\frac{b-a}{2}\right)$: : AO (x) : OG $= \frac{\overline{b-a} \times x}{2a}$. But



$\overline{AB}|^2 = \overline{OA}|^2 + \overline{OB}|^2 - 2$ OB $\times$ OG; that is, in Species, $a^2 = x^2 + x^2 - \frac{\overline{b-a} \times x^2}{a}$; or $a^3 = 3ax^2 - bx^2$.

Therefore $x = \sqrt{\frac{a^3}{3a-b}} = a\sqrt{\frac{a}{3a-b}}$.

From

From the ſame Equation, if the Radius AO *(x)* and the Chord *(b)* of an Arch ABD be ſuppoſed given, the Chord AB *(a)* of the Sub-triple of that Arch, may be determined: But this, by-the-bye.

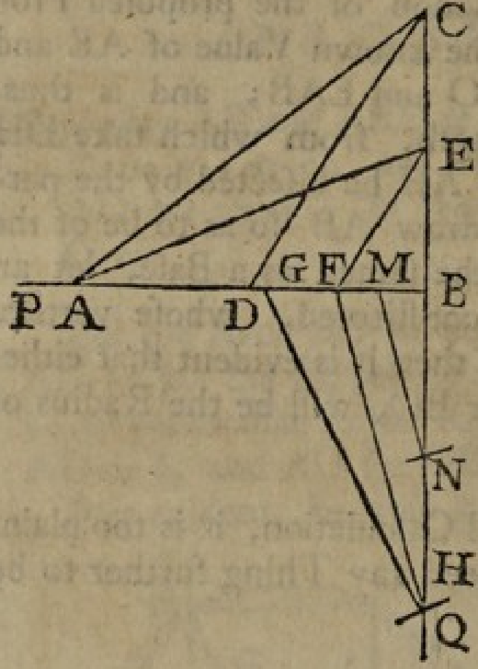
Geometrically.

The Geometrical Conſtruction of the propoſed Problem is alſo obvious from the known Value of AE and the Equality of the Angles O and EAB; and is thus. Draw AD of the given Length, from which take DH = AB; let the Remainder AH be biſected by the perpendicular EI; to which draw AB ſo as to be of the given Length; and upon the ſame, as a Baſe, let an Iſoſceles Triangle AOB be conſtituted, whoſe vertical Angle O ſhall be = EAB; then it is evident that either of the equal Sides AO, or BO, will be the Radius of the Circle.

As to the Trigonometrical Calculation, it is too plain, from the Conſtruction, to need any Thing further to be ſaid about it.

PROBLEM XV.

Having the Lengths of two Lines AE *and* CD *drawn from the acute Angles of a right-angled Triangle* ABC, *to bisect, and terminate in, the opposite Sides; to determine the Triangle.*



Call AE, a; CD, b; and BD (or AD), x; Then will $\overline{CB}^2 = b^2 - x^2$; whence $\overline{BE}^2 \left(= \frac{1}{4}\overline{CB}^2\right) = \frac{bb-xx}{4}$; and therefore $a^2 \left(= \overline{AE}^2 = \overline{BE}^2 + \overline{BA}^2\right) = \frac{bb-xx}{4} + 4xx$.

Hence $15xx = 4aa - bb$; and $x = \sqrt{\frac{4aa-bb}{15}}$.

Geometrically.

If EF be supposed parallel to CD, meeting AB in F, the Length thereof, being half That of CD, will consequently be given: Whence $\overline{AB}^2 - \overline{BF}^2 \left(= \overline{AE}^2 - \overline{EF}^2\right)$ is also given.

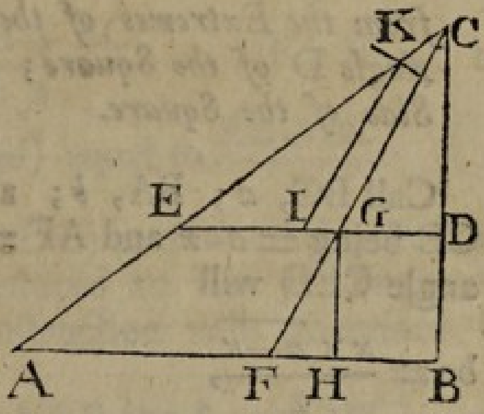
Therefore the Problem is reduced to this; To determine two Lines AB and BF, in the Ratio of 4 to 1, so that the Difference of their Squares may be equal to the Difference of the Squares of two given Lines AE and EF.

Hence, having drawn two indefinite Lines BP and BQ at right-Angles to each other, take BG, in the Former, equal to EF; and from the Point G, to BQ, draw

draw GH=AE: So shall $\overline{AB}|^2 - \overline{BF}|^2 (=\overline{AE}|^2 - \overline{EF}|^2) = BH^2$ (*Elem. Cor.* 7. 2). Therefore, if from any Point M in PB, to BQ there be drawn MN = 4 MB; it is evident that a Line, HF, drawn from H parallel to NM, will cut off BF as required——This Problem becomes impossible when either of the two given Lines is greater than the Double of the Other.

PROBLEM XVI.

The Length and Position of a Right-line DE, *drawn parallel to the Base of a given right-angled Triangle* ABC *being known; 'tis proposed to draw another Right-line* CF *from the Vertex of the Triangle, so that the Part thereof* (FG) *intercepted by* AB *and* ED, *may be equal to* (EG) *the Part of* ED *intercepted by* CA *and* CF.



Upon AB let fall the Perpendicular GH: And put ED=a, CD=b, DB=c, and EG=x: Then, from the Similarity of the Triangles CDG and GHF, it will be, CD (b) : DG ($a-x$) : : GH (c) : HF $= \frac{c \times \overline{a-x}}{b}$:

And therefore $\frac{c^2 \times \overline{a-x}|^2}{b^2} + c^2$ ($HF^2 + GH^2$) $= x^2 (GF^2)$

Whence, by Reduction, $b^2x^2 - c^2x^2 + 2c^2ax = c^2a^2 + c^2b^2$;

and $x^2 + \frac{2ac^2}{bb-cc} \times x = \frac{\overline{a^2+b^2} \times c^2}{bb-cc}$.

From which x is found $= \frac{bc\sqrt{aa+bb-cc} - ac^2}{bb-cc}$.

Geometrically.

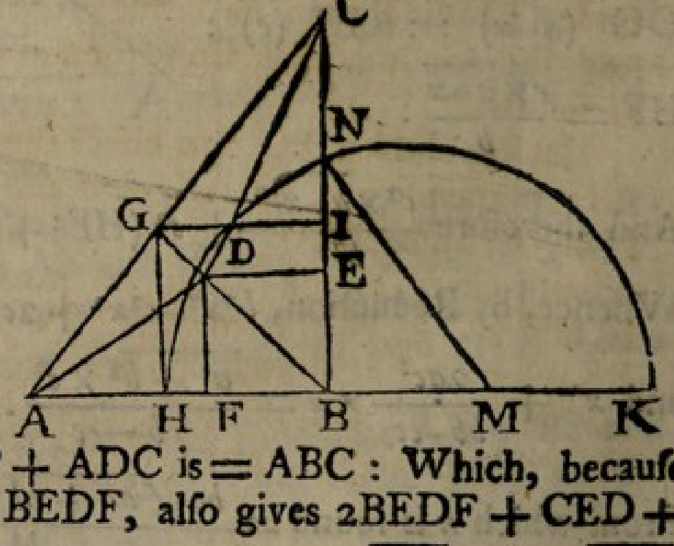
Geometrically.

Since GF is to GC every-where in the given Ratio of DB to DC, GE, in the required Position, must therefore be to GC, in the same, given, Ratio. Hence, if, in ED, there be taken EI = DB, and from the Center I, at the Distance of CD, an Arch be described, intersecting EC in K; then a Line CGF, drawn parallel to the Radius KI, will determine both the Length and Position of GF: For it is evident that CG : EG : : EK : EI : : CD : DB : : CG : GF; and consequently that EG = GF——This Problem appears to be impossible when BD is greater than CE.

PROBLEM XVII.

Supposing the Area of a Square BE DF, *form'd within a given right-angled Triangle* ABC, *to be equal to the Area of the Triangle* ADC, *made by drawing Lines from the Extremes of the Hypothenuse to the adjacent Angle* D *of the Square; 'tis proposed to determine the Side of the Square.*

Call BC, a; BA, b; and BE (or BF), x: Then, CE being $= a-x$ and AF $= b-x$, the Area of the Triangle CED will be $= \frac{x \times \overline{a-x}}{2}$, and That of the Triangle AFD $= \frac{x \times \overline{b-x}}{2}$. But it is evident that BEDF



+ CED + AFD + ADC is = ABC: Which, because ADC is equal to BEDF, also gives 2BEDF + CED + AFD = ABC, that is, $2x^2 + \frac{x \times \overline{a-x}}{2} + \frac{x \times \overline{b-x}}{2}$

=

$=\frac{ab}{2}$: Hence $x^2 + \frac{a+b}{2} \times x = \frac{ab}{2}$; and $x = \sqrt{\frac{ab}{2} + \overline{\tfrac{1}{4}a + \tfrac{1}{4}b}|^2} - \frac{a+b}{4}$. Q. E. I.

Geometrically.

If BD be produced to meet AC in G, and GH be drawn perpendicular to AB, it is evident that the Triangle ADC will be to the Triangle ABC $\left(\frac{AB \times BC}{2}\right)$ as GD : GB, or as HF to HB *(Elem.* 2. *and* 5. *of the* 4): And it alſo appears *(by Elem.* 14. 4.) that $HB \times \overline{AB+BC}$ is $= AB \times BC$. Therefore it follows that Triang. ADC : $\frac{HB \times \overline{AB+BC}}{2}$: : HF : HB : : $HF \times \frac{AB+BC}{2}$: $HB \times \frac{AB+BC}{2}$; and conſequently that the Triangle $ADC = HF \times \frac{AB+BC}{2} = HF \times BK$; by taking BK (in AB produced) equal to $\frac{AB+BC}{2}$.

Hence, $\overline{BF}|^2$ being $(=ADC) = HF \times BK$, the Caſe under Conſideration is reduced to our 7th Problem; and the Geometrical Conſtruction will, therefore, be as follows.

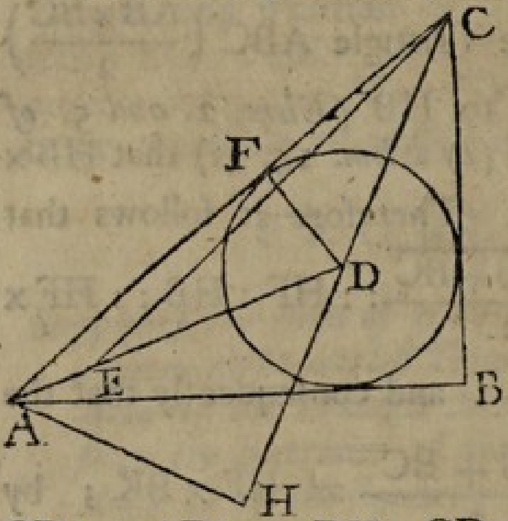
Having drawn BG (to biſect the Angle ABC), and GH perpendicular to AB, and alſo taken BK equal to Half the Sum of AB and BC *(as above intimated)*, let a Semi-circle, upon HK, be next deſcribed, interſecting BC in N; from which Point, to the Middle of BK, let NM be drawn; then make MF = MN, and BF will be the Side of the Square; as is manifeſt from the Problem above quoted.

PROBLEM

PROBLEM XVIII.

Having given the Hypothenuse AC *of a right-angled Triangle* ABC, *and the Difference of two Lines* AD *and* CD, *drawn from the Extremes thereof to the Center* D *of the inscribed Circle; to determine the remaining Sides* AB *and* BC, *of the Triangle.*

Upon CD, produced, let fall the Perpendicular AH; and make $AC = a$, $AD = x$, $DC = y$, and the given Difference, $x - y$, $= b$.



It is evident that the Angle ADH is $=$ DAC $+$ DCA $= \frac{1}{2}$ BAC $+ \frac{1}{2}$ BCA $= \frac{1}{2}$ a Right Angle; and therefore AH $=$ HD $= \frac{AD}{\sqrt{2}} = \frac{x}{\sqrt{2}}$. But $CD^2 + AD^2 + 2DH \times CD = AC^2$; that is, in Species, $y^2 + x^2 + xy\sqrt{2} = a^2$. Which Equation, by substituting $x - b$ instead of y its Equal, and denoting $\sqrt{2}$ by c, becomes $x^2 - 2bx + b^2 + x^2 + cx^2 - cbx = a^2$; that is $\overline{2+c} \times x^2 - \overline{2+c} \times bx + b^2 = a^2$: Whence $x^2 - bx = \frac{aa - bb}{2+c}$, and $x = \sqrt{\frac{aa-bb}{2+c} + \frac{1}{4}bb} + \frac{b}{2}$.

Geometrically.

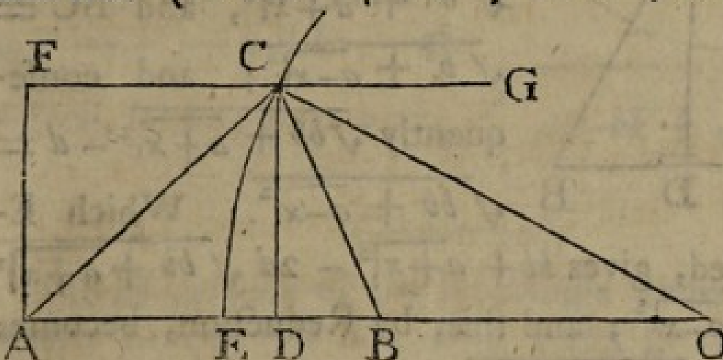
The Geometrical Construction of this Problem, as the Angle ADH is given ($= \frac{1}{2}$ a Right Angle) is exceeding obvious: For, if DE be supposed $=$ DC, so that AE may express the given Difference of AD and CD, the Angle DEC (supposing CE drawn) will be given $= \frac{1}{2}$ ADH. Therefore the Triangle AEC, by Means of the given Angle AEC, and the two given Sides AE and AC, may be constructed. And then, by producing

producing AE, and making the Angle ECD = DEC, the Point D will likewise be determined; and consequently the Radius of the Circle, by letting fall a Perpendicular DF, upon AC: Whence the Circle itself may be described; and two Lines may be drawn from A and C to touch the same; and thereby form the Triangle ABC, *as required.*

PROBLEM XIX.

Having the Base AB, *the Perpendicular* CD, *and the Ratio of the two Sides* AC, BC, *of a Triangle* ABC; *to find the Sides.*

Call AB, a; CD, b; and AD, x; and let the given Ratio of AC to BC be expounded by That of m to n. Hence BD $= a-x$; $\overline{AC}|^2 \ (= \overline{CD}|^2 + \overline{AD}|^2) = bb + xx$ and $\overline{BC}|^2 \ (= \overline{CD}|^2 + \overline{BD}|^2) = bb + aa - 2ax + xx$: And therefore $mm : nn :: bb + xx : bb + aa - 2ax + xx$. From which, by multiplying Extremes and Means,



we have $m^2b^2 + m^2a^2 - 2m^2ax + m^2x^2 = n^2b^2 + n^2x^2$: Whence $\overline{mm-nn} \times xx - 2mmax = \overline{nn-mm} \times bb - m^2a^2$; and $xx - \frac{2mma}{mm-nn} \times x = -bb - \frac{m^2a^2}{mm-nn}$: Which, solved, gives $x = \frac{mma}{mm-nn} \pm \sqrt{\overline{\frac{mma}{mm-nn}}|^2 - b^2}$.

Geometrically.

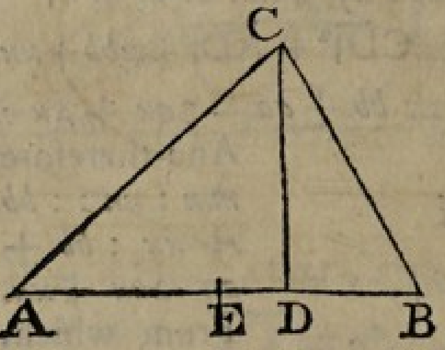
The Geometrical Construction of this Problem is given *by Elem.* 15. 4. For, if the Base AB be divided at E in the given Ratio of AC to BC, and, in AB produced, there be taken, EO, a Fourth Proportional

portional to AE–BE, AE, and BE, *it is there demonstrated*, that two Lines drawn from A and B to meet any-where in the Circumference of a Circle described thro' E, from the Center O, will be in the same given Ratio of AE to BE. Whence it is evident that the Intersection of the said Circumference with a Right-line FG, drawn parallel to AB, at the given Distance DC, will determine the Vertex of the Triangle.

PROBLEM XX.

Having the Base AB, *the Perpendicular* CD, *and the Difference of the two Sides,* AC *and* BC *of a Triangle* ABC; *to find the Sides.*

Making AE $= \frac{1}{2}$ AB $= a$, CD $= b$, AC–BC $= d$, and ED $= x$, we have AD $= a+x$, BD $= a-x$, AC $= \sqrt{b^2 + \overline{a+x}^2}$, and BC $= \sqrt{b^2 + \overline{a-x}^2}$; and consequently $\sqrt{bb + \overline{a+x}^2} - d = \sqrt{bb + \overline{a-x}^2}$. Which Equation, squared, gives $bb + \overline{a+x}^2 - 2d\sqrt{bb + \overline{a+x}^2} + dd = bb + \overline{a-x}^2$; and this, by Reduction, becomes $4ax + dd = 2d\sqrt{bb + \overline{a+x}^2}$. And this, again squared, produces $16a^2x^2 + 8addx + d^4 = 4dd \times \overline{bb + aa + 2ax + xx}$; or, $16a^2x^2 + d^4 = 4dd \times \overline{aa + bb} + 4ddxx$. Whence $x =$



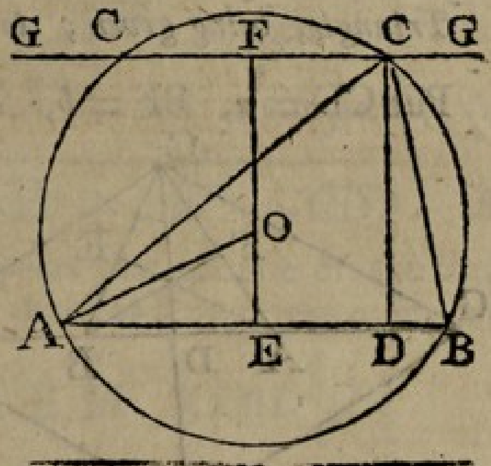
$$\sqrt{\frac{4dd \times \overline{aa+bb} - d^4}{16aa - 4dd}}.$$

The Geometrical Construction of this Problem being only a particular Case of a more general One, given at large hereafter (*Problem* 49) I shall not insert it here: But observe, with respect to the Algebraical Solution, that, if d be supposed to denote the Sum, instead of the Difference, of the Sides, the Value of x (or DE) will be be given by the very Equation above exhibited; as is manifest from the Process.

PROBLEM XXI.

Having the Base AB, *the Perpendicular* CD, *and the Rectangle of the two Sides* AC *and* BC, *of a Triangle* ABC; *to determine the Triangle.*

By retaining the Notation of the preceding Problem, and making the given Rectangle $= c^2$, we have $\sqrt{bb+\overline{a+x}|^2} \times \sqrt{bb+\overline{a-x}|^2} = c^2$; and therefore $\overline{bb+\overline{a+x}|^2} \times \overline{bb+\overline{a-x}|^2} = c^4$; that is $b^4 + b^2 \times \overline{2a^2+2x^2} + a^4 - 2a^2x^2 + x^4 = c^4$. Whence $x^4 + \overline{2bb-2aa} \times x^2 = c^4 - b^4 - 2a^2b^2 - a^4$.



Which, solved, gives $x = \sqrt{aa-bb \pm \sqrt{c^4-4a^2b^2}}$.

Geometrically.

The Magnitude of the Rectangle under the two unknown Lines, AC and BC, being given, two other Lines must, therefore, be assigned, containing an equal Rectangle; whereof One being given, the Other will also become known. (*Vid. Observation* 5, *P.* 88. But it is known that the Rectangle under the, given, Perpendicular CD and the Diameter of a Circle circumscribing the Triangle, is equal to the Rectangle under the said, unknown, Sides of the Triangle (*Elem.* 19. 3): Hence the Diameter of the circumscribing Circle is given; and from thence the following Construction.

Find a Third-Proportional to CD and the Side of the Square, expressing the Magnitude of the proposed Rectangle; and with the Half Thereof, from the Point A (or B) describe an Arch, cutting EF, perpendicular to

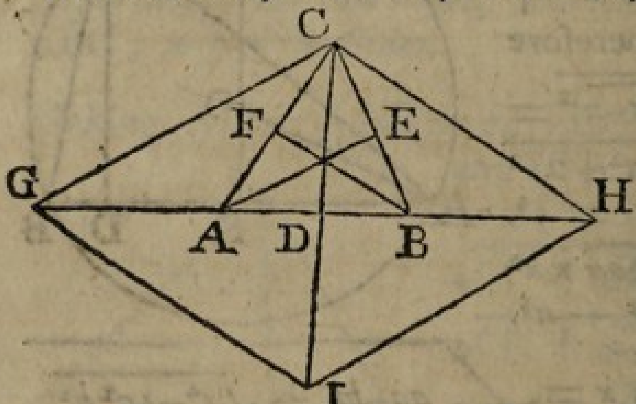
AB,

AB, in O; from which Point, as a Center, with the ſame Radius, let a Circle ACB be deſcribed; then a Right-line, GFG, drawn parallel to AB, at the given Diſtance DC, will interſect the ſaid Circle in the Vertex of the Triangle.

PROBLEM XXII.

The Lengths of three Lines AE, BF *and* CD, *drawn from the Angles to the Middle of the oppoſite Sides of a Triangle, being given; to find the Sides.*

Put CD $= a$, BF $= b$, AE $= c$, AB $= x$, AC $= y$ and BC $= z$. Since, by a known Property of Triangles, $\overline{AC}^2 + \overline{BC}^2 = 2\overline{CD}^2 + 2\overline{AD}^2$ (*Elem.* 12. 2) we have $y^2 + z^2 = 2a^2 + \frac{1}{2}x^2$, and therefore $y^2 + z^2 - \frac{1}{2}x^2 = 2a^2$. In the ſame manner



$x^2 + z^2 - \frac{1}{2}y^2 = 2b^2$.

$x^2 + y^2 - \frac{1}{2}z^2 = 2c^2$.

From whence (by taking the former of theſe Equations from twice the Sum of the two Latter) there comes out $4x^2 + \frac{1}{2}x^2 = 2 \times \overline{2b^2 + 2c^2 - a^2}$: And conſequently $x = \frac{2}{3}\sqrt{2b^2 + 2c^2 - a^2}$. By the ſame Argument, $y = \frac{2}{3}\sqrt{2a^2 + 2c^2 - b^2}$; and $z = \frac{2}{3}\sqrt{2a^2 + 2b^2 - c^2}$.

Geometrically.

If CG and CH be drawn parallel to AE and BF, meeting AB, produced, in G and H; it is plain, becauſe CE=BE, and CF=AF, that AG=AB=BH; and alſo that CG=2AE, and CH=2BF. Therefore, the two Sides CG, CH, and the Line CD, biſecting the Baſe of the Triangle GCH being given, the Diagonal CI (=2CD) of the Parallelogram GCHI (as well

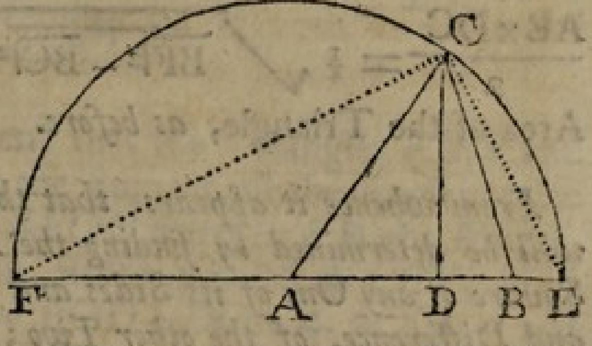
well as the Sides) will be given (*Elem.* 13. 2). Hence, in order to the Construction, let a Triangle CGI, whose three Sides are equal to the Doubles of the three given Lines, be constituted, and draw GDH to bisect CI in D; also set off DA and DB, each equal to $\frac{1}{3}$ of GD; join A, C, and B, C, *and the Thing is done.*

From this Construction we have the very same Numerical Solution as from the algebraic Process: For, since, $2\overline{GD}|^2 + 2\overline{CD}|^2 = \overline{CG}|^2 + \overline{CH}|^2$ (*Elem.* 12. 2.) $= 4\overline{AE}|^2 + 4\overline{BF}|^2$, thence is $GD = \sqrt{2\overline{AE}|^2 + 2\overline{BF}|^2 - \overline{CD}|^2}$, and consequently AB $(= \frac{2}{3}GD) = \frac{2}{3}\sqrt{2\overline{AE}|^2 + 2\overline{BF}|^2 - \overline{CD}|^2}$. By the Construction it also appears that no one of the three given Lines must be greater than the Sum of the other two.

PROBLEM XXIII.

All the Sides of a Triangle ABC *being given; to find the Perpendicular* CD, *the Segments of the Base* AD *and* BD, *together with the Area of the Triangle.*

Put $AC=a$, $AB=b$, $BC=c$, and $AD=x$: Then $BD = b-x$; and $c^2 - \overline{b-x}|^2 (=CD^2) = a^2 - x^2$; that is, $c^2 - b^2 + 2bx - x^2 = a^2 - x^2$. Whence $2bx = a^2 + b^2 - c^2$, and $x = \frac{aa+bb-cc}{2b}$.



Now $\overline{CD}|^2 = \overline{AC}|^2 - \overline{AD}|^2 = \overline{AC+AD} \times \overline{AC-AD}$
$= \overline{a + \frac{aa+bb-cc}{2b}} \times \overline{a - \frac{aa+bb-cc}{2b}} = \frac{aa+2ab+bb-cc}{2b} \times \frac{-aa+2ab-bb+cc}{2b} = \frac{\overline{b+a}|^2 - c^2}{2b} \times \frac{c^2 - \overline{b-a}|^2}{2b}$.

Hence $CD = \frac{1}{2b}\sqrt{\overline{\overline{b+a}^2 - c^2} \times \overline{c^2 - \overline{b-a}^2}}$; and the Area $\left(\frac{CD \times AB}{2}\right) = \frac{1}{4}\sqrt{\overline{\overline{b+a}^2 - c^2} \times \overline{c^2 - \overline{b-a}^2}}$.

Geometrically.

From the Center A, with the Radius AC, let a Semicircle ECF be deſcribed, cutting AB produced in E and F; ſo that BF may be the Sum, and BE the Difference, of the Sides AC and AB: Alſo let EC and FC be drawn.

Then will $\overline{FC}^2 = FE \times FD$ $(= 2AF \times FD)$; and $\overline{EC}^2 = FE \times ED$ $(= 2AE \times ED)$, *by Elem. Corol. to* 12. 4.

Alſo $\overline{BC}^2 = \overline{BF}^2 + \overline{FC}^2$ $(2AF \times FD) - 2FB \times FD$ $= \overline{BF}^2 - 2AB \times FD$ *(Elem.* 8. 2): And likewiſe $\overline{BC}^2 = \overline{BE}^2 + \overline{EC}^2$ $(2AE \times ED) - 2EB \times ED = BE^2 + 2AB \times ED$.

Hence it appears that $2AB \times FD$ is $= \overline{BF}^2 - \overline{BC}^2$; and $2AB \times ED = \overline{BC}^2 - \overline{BE}^2$: And, conſequently, that $\overline{\overline{BF}^2 - \overline{BC}^2} \times \overline{\overline{BC}^2 - \overline{BE}^2} = 4\overline{AB}^2 \times FD \times ED$ $= 4\overline{AB}^2 \times \overline{DC}^2$ *(by Elem. Cor. to* 12. 4). Therefore $\frac{AB \times DC}{2} = \frac{1}{4}\sqrt{\overline{\overline{BF}^2 - \overline{BC}^2} \times \overline{\overline{BC}^2 - \overline{BE}^2}}$ = the Area of the Triangle, *as before.*

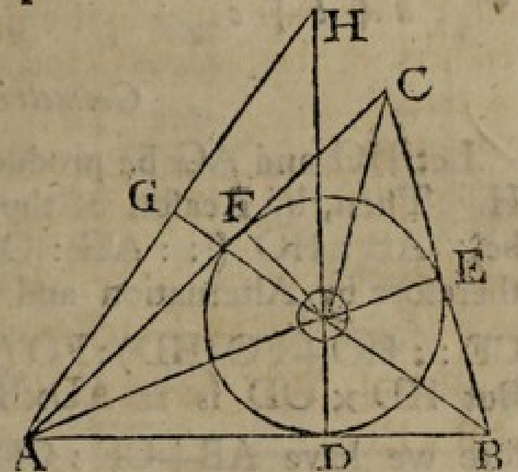
From whence it appears, that the Area of any Triangle will be determined by finding the Differences between the Square of any One of its Sides and the Squares of the Sum, and Difference, of the other Two; and then taking $\frac{1}{4}$ *of the ſquare Root of the Product ariſing by the Multiplication of the ſaid Differences into each other.*

PROBLEM

PROBLEM XXIV.

Having all the Sides of a Triangle ABC; *to find the Radius of its inſcribed Circle* DEF.

From the Center O, to the angular Points and the Points of Contact, let Lines be drawn; and, upon BO produced, let fall a perpendicular AG.



It is plain, in the firſt Place (becauſe OD=OE =OF) that AD=AF, BD=BE, and CF=CE: Therefore, by Addition, BD+CF (=BE+CE) =BC: Take each of theſe equal Quantities from AB+AC, and there will remain AD+AF= AB+AC−BC; From whence (AD being = AF) we get AD (or AF) = $\frac{AB+AC-BC}{2}$. By ſubtracting of which from AB, and from AC, we alſo have BD $=\frac{AB+BC-AC}{2}$, and CF $=\frac{AC+BC-AB}{2}$.

Moreover, it is evident that the Triangles AOG and COF are ſimilar: For the Sum of all the Angles at the Center, DOE+DOF+FOE being =4 Right Angles, the Sum of their Halves, BOD+DOA+COF, muſt be =2 Right Angles = BOD+DOA+AOG; and conſequently COF=AOG.

Now let the Values of AD, BD, and CF (found above) be denoted by a, b and c, reſpectively; and put OD (OE=OF)=x: Then, it will be BO ($\sqrt{bb+xx}$) : OD (x) : : AB ($a+b$) : AG $=\frac{ax+bx}{\sqrt{bb+xx}}$; and BO:

BD : : AB : BG $= \frac{ab + bb}{\sqrt{bb + xx}}$. Therefore OG (BG − BO) $= \frac{ab + bb}{\sqrt{bb + xx}} - \sqrt{bb + xx} = \frac{ab - xx}{\sqrt{bb + xx}}$. But, AG : OG : : CF : OF; Or, $ax + bx : ab - xx :: c : x$ whence $ax^2 + bx^2 = abc - cx^2$; and confequently $x = \sqrt{\frac{abc}{a + b + c}}$.

Geometrically.

Let DO and AG be produced to meet each other in H. Then, by Reafon of the fimilar Triangles, it will be, AB : HO (: : AG : OG) : : CF : FO; And therefore by Alternation and Compofition, AB+CF : CF : : HO+FO (HD) : FO (OD) : : HD×OD : $\overline{OD}|^2$. But HD × OD is = AD×BD (*Elem.* 18. 3): Therefore we have AB+CF : CF : : AD × BD : $\overline{OD}|^2 = \frac{AD \times BD \times CF}{AD + BD + CF}$, the very fame as before.

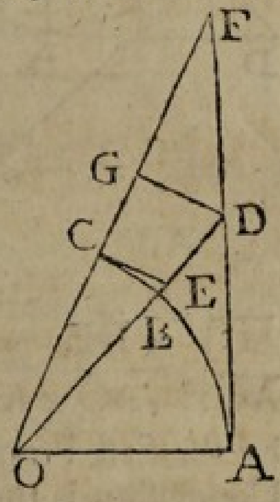
From this Conclufion, the Rule in common Practice, for finding the Area of a Triangle, having the three Sides given, is eafily deduced: For it is evident that the Area of the Triangle ABC is equal to the Radius (OD) drawn into the Half Sum of the Sides (AD+BD+CF); that is $= \sqrt{\overline{AD + BD + CF} \times AD \times BD \times CF}$. Where AD, BD, and CF, are the Differences between the Half Sum and each particular Side.

PROBLEM

PROBLEM XXV.

The Radius of a Circle and the Tangents of two Arcs thereof being given, to determine the Tangent of the Sum of thoſe Arcs.

Let AB and BC be the propoſed Arcs, whereof the given Tangents are AD and CE; and let the former of Theſe be continued out to meet the Radius OC, produced, in F; ſo ſhall AF be the Tangent of AC, the Sum of the ſaid Arcs.



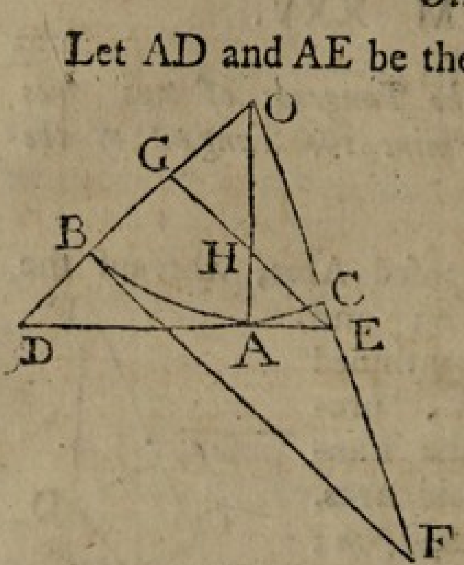
Now, calling AO, r; AD, m; CE, n; AF, x; and FO, y; and making DG perpendicular to FO; we have *(by ſim. Triangles)* OF (y) : AO (r) : : DF $(x-m)$: DG $= \frac{rx-rm}{y}$.

Alſo OF (y) : AF (x) : : DF $(x-m)$: FG $= \frac{xx-mx}{y}$: From which laſt we have OG $(=$ OF $-$ FG$) = y - \frac{xx-mx}{y} = \frac{yy-xx+mx}{y} = \frac{rr+mx}{y}$ (becauſe $yy-xx=rr$).

But OG $\left(\frac{rr+mx}{y}\right)$: DG $\left(\frac{rx-rm}{y}\right)$: : OC (r) : CE (n) and conſequently $r^2x-r^2m=r^2n+mnx$. Whence $x = \frac{rr\times\overline{m+n}}{rr-mn}$; or AF $= \frac{\overline{AO}|^2\times\overline{AD+CE}}{\overline{AO}|^2-AD\times CE}$.

Otherwise.

Let AD and AE be the Tangents of the two Arcs AB and AC, and BF That of their Sum BC; alſo let EG be drawn perpendicular to OD, interſecting the Radius OA in H. Then, by Reaſon of the ſimilar Triangles, it will be AO : AD : : AE : AH. And, OH (AO–AH) : DE (: : OG : GE) : : OB (AO) : BF $= \frac{AO \times DE}{AO - AH} =$



$\frac{\overline{AO}|^2 \times DE}{\overline{AO}|^2 - AO \times AH} = \frac{\overline{AO}|^2 \times \overline{AD + AE}}{\overline{AO}|^2 - AD \times AE}$; becauſe, by the firſt Proportion $AO \times AH = AD \times AE$. Which Concluſion is the very ſame with That above.

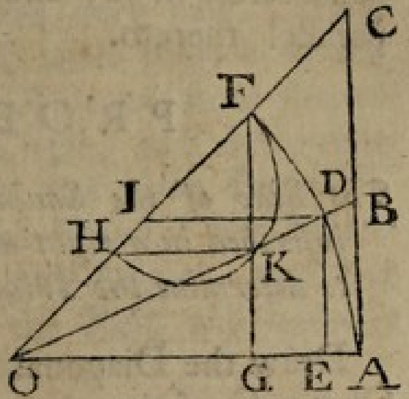
If the Tangents of the Arcs AC and AB (*Fig.* 1.) were to be given, in order to find the Tangent of their Difference BC; then, by the Proportion $\frac{rr+mx}{y}$: $\frac{rx-rm}{y}$: : r : n (above derived) we ſhould have $n = \frac{rr \times \overline{x-m}}{rr+mx}$; or $CE = \frac{\overline{AO}|^2 \times \overline{AF-AD}}{AO^2 + AF \times AD}$.

PROBLEM XXVI.

The Ratio of the Sines DE, FG *of two Arcs* AD, AF, *of a given Circle, together with That of their Tangents* AB, AC *being given; to find both the Sines and the Tangents.*

Put the Radius AO $= a$; and let the given Ratio of AB to AC be That of m to n; moreover let DE be to

FG as p to q; and call AB, x; and AC, y: Then (*by similar Triangles*) $\overline{OB}^2$ (a^2+x^2) : $\overline{AB}^2$ (x^2) : : $\overline{OD}^2$ (a^2) : $\overline{DE}^2 = \frac{a^2x^2}{aa+xx}$; and $\overline{OC}^2$ (a^2+y^2) : $\overline{AC}^2$ (y^2) : : $\overline{OF}^2$ (a^2) : $\overline{FG}^2 = \frac{a^2y^2}{aa+yy}$. Therefore, *by the Question*, p^2 : q^2 : : $\frac{a^2x^2}{aa+xx}$: $\frac{a^2y^2}{aa+yy}$, and



consequently $p^2y^2 \times \overline{a^2+x^2} = q^2x^2 \times \overline{a^2+y^2}$. But, by the Question, we also have, $m : n : : x : y, = \frac{nx}{m}$; which Value, substitued in the preceding Equation, gives $\frac{n^2p^2x^2}{mm} \times \overline{a^2+x^2} = q^2x^2 \times \overline{a^2+\frac{nnxx}{mm}}$: Whence $n^2p^2 \times \overline{a^2+x^2} = q^2 \times \overline{a^2m^2+n^2x^2}$; and $x = \frac{a}{n} \times \sqrt{\frac{nnpp-mmqq}{qq-pp}}$. From which AC, DE and FG are also given.

Geometrically.

Since the Ratio of AB to AC is given, as m to n; GK (supposing K to be the Intersection of FG and OB) will be to GF, in the same given Ratio: And, if (agreeable to the 3[d] General Observation) KH be drawn parallel to AO, meeting OF in H, OH will be to OF *still* in the same given Ratio.

Again, if DI be drawn parallel to AO, meeting OF in I; then OK : OH (: : OD (OF) : OI : : FG : DE) : : q : p; Whence OK is also given; and from thence the following Construction.

In any Radius OF of the given Circle, take OH to OF in the given Ratio of m to n; and upon HF let a Semi-circle be described: Take also a Fourth pro-

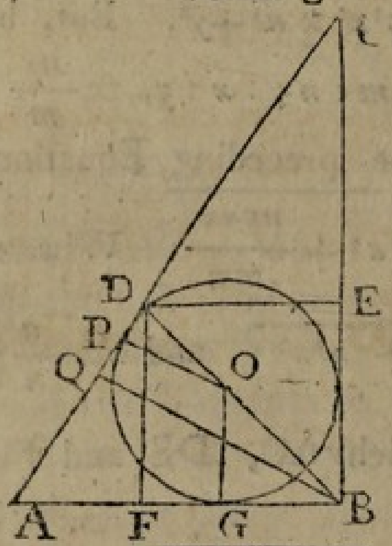
 portional

portional to p, q, and OH; with which, as a Radius, from the Center O, describe an Arch, cutting the Semi-circle in K; and, having drawn HK, make OA parallel thereto.

PROBLEM XXVII.

The Side of the Square, and the Radius of the Circle, inscribed in a right angled Triangle ABC, *being given; to determine the Triangle.*

Draw the Diagonal BD of the Square; and from the Center O of the given Circle, to the Points of contact,



draw the Radii OG and OP; and upon the Hypothenuse AC let fall the Perpendicular BQ: Calling the Side of the Square, a; the Radius of the Circle, b; and AQ, x: Then, because of the parallel Lines, we shall have FG $(a-b)$: BF (a) :: (OD : BD ::) OP (b) : BQ $= \frac{ab}{a-b}$:

Whence DQ $(=\sqrt{BD^2-BQ^2})$ $=\sqrt{2a^2 - \frac{aabb}{\overline{a-b}|^2}}$, is also given.

Let it, for Brevity-sake, be denoted by c; and let BQ $\left(\frac{ab}{a-b}\right) = d$: Then we shall have, AQ$(x)$: BQ(d) :: BQ (d) : CQ $= \frac{dd}{x}$; and also (*by Elem.* 10. 4.) AD$(x+c)$: CD $\left(\frac{dd}{x}-c\right)$ :: AB : BC :: AQ(x) : BQ(d) whence, by multiplying Extremes and Means, we get $dx+cd = dd-cx$; and, from thence, $x = \frac{d \times \overline{d-c}}{d+c}$. By Means whereof every thing else is readily found.

Geometrically,

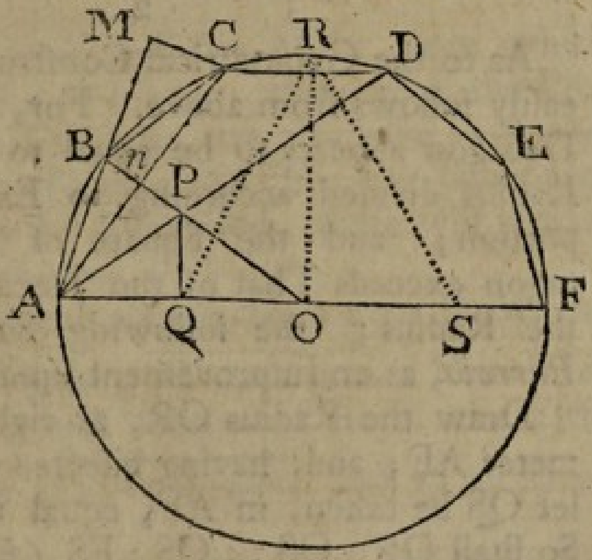
Geometrically.

The Geometrical Conſtruction, and the Trigonometrical Solution of this Problem are very eaſy. For, ſince the Poſition of the Point D with reſpect to the Circle, is given; a Right-line DPA drawn from thence (*by Elem.* 16. 5.) to touch the Circle, will determine the Triangle: And then, from the given Lines OP and OD, the Angle D may be found: By Means whereof and ABD ($=\frac{1}{2}$ a Right Angle) together with the given Side BD, all the Reſt will become known.

PROBLEM XXVIII.

To determine the Sides of a regular Pentagon and Decagon, inſcribed in a given Circle.

Let AB, BC, CD, *&c.* be Sides of the Decagon, and AC a Side of the Pentagon: And let AD be drawn, interſecting the Radius OB in P. It is evident in the firſt Place, that the Angles BAP and OAP, ſtanding on the equal Arches BD and BF, are equal to one another, and alſo equal, each of them, to the Angle AOP, inſiſting on the Arch AB (*Elem.* 9. 3.) And, ſecondly, that the Triangle BAP (as well as APO) is an



iſoſceles One, becauſe the Perpendicular A*n*C, makes equal Angles BAC, DAC with the two Sides AB, AP of the Triangle. Hence it appears very plain that all the three Lines AB, AP, and OP are equal among themſelves; and likewiſe that AO (OB) : AB (OP) : : OP (AB) : BP (*by Elem.* 10. 4.) ſeeing the Angle BAO is bisected by AP. Moreover, by letting fall the two Perpendiculars PQ and CM, upon AO and ABM, the Triangles BCM and APQ (as BC is = AB=AP, and the Angle MBC = MAD = PAQ) will appear to be equal in all respects; and so, BM being (= AQ) = $\frac{1}{2}$ AO, we have $\overline{AC}|^2$ (= $\overline{BC}|^2 + \overline{AB}|^2 + 2BM \times AB$, *Elem.* II. 2.) = $\overline{BC}|^2 + \overline{AB}|^2 + AO \times AB$. But, by the above Proportion, $\overline{AB}|^2$ is = AO × BP: Therefore, by writing AO × BP in the Room of $\overline{AB}|^2$, we get $\overline{AC}|^2 = \overline{BC}|^2 + AO \times AB + AO \times BP = \overline{BC}|^2 + \overline{AO}|^2$: Whence (BC being first found) AC will also become known.—If AO be now denoted by a, and AB by x; then from the Equality of $\overline{AB}|^2$ and AO × BP, you will have $x^2 = a \times \overline{a-x}$; from which x will be found $= \sqrt{\frac{5aa}{4}} - \frac{a}{2}$; And from thence AC (= $\sqrt{xx+aa}$) $= a\sqrt{\frac{5-\sqrt{5}}{2}}$.

As to the Geometrical Construction, it likewise very easily follows from above. For, since the Side of the Decagon appears to be equal to the greater Part of the Radius divided according to Extreme-and-Mean Proportion; and the Square of the Side of the Pentagon exceeds That of the Decagon by the Square of the Radius; the following Solution (given by Dr. *Barrow*, as an Improvement upon *Euclid's*) is manifest.

Draw the Radius OR, at right-Angles to the Diameter AF; and, having bisected the Radius AO in Q, let QS be taken, in AF, equal to the Distance QR: So shall OF : OS : : OS : FS (*Elem.* 22. 5.) and consequently OS = AB the Side of the Decagon.

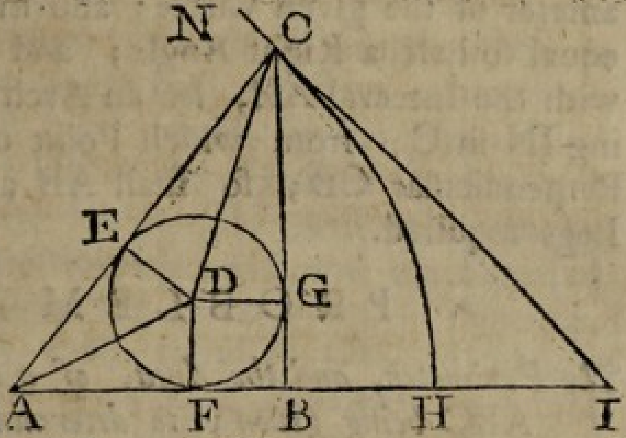
And, because $\overline{RS}|^2$ (supposing RS drawn) is $= \overline{OS}|^2 + \overline{OR}|^2$, it is plain also that RS will be equal to the Side AC of the Pentagon.

PROBLEM

PROBLEM XXIX.

Having the Hypothenuse AC *of a right-angled Triangle* ABC, *and also the Radius of the inscribed Circle* DEFG ; *to find the two Legs* AB *and* BC.

From the Center D of the given Circle, to the Points of Contact, let DE, DF, and DG be drawn ; also draw AD and CD: And put DE ($=$ DG $=$ DF) $= a$, AC $= b$, AB $= x$, and BC $= y$.



It is evident, that CE must be $=$ CG $= y - a$; because the right-angled Triangles CDE and CDG, having DE $=$ DG, and CD common, are equal in all respects. In the very same manner is AE $=$ AF $= x - a$.

Therefore $y - a + x - a = b$ ($=$ AC) ; from which Equation we have $x + y = b + 2a$. But, from the Property of right-angled Triangles, we also have $xx + yy = bb$. And, if from the Double of This, the Square of the former Equation be subtracted, there will remain $xx - 2xy + yy = bb - 4ab - 4aa$.

From whence, by extracting the square Root, on both Sides, we get $x - y = \sqrt{bb - 4ab - 4aa}$. Which last Equation, added to, and subtracted from, the First, gives $2x = 2a + b + \sqrt{bb - 4ab - 4aa}$, and $2y = 2a + b - \sqrt{bb - 4ab - 4aa}$.

Geometrically.

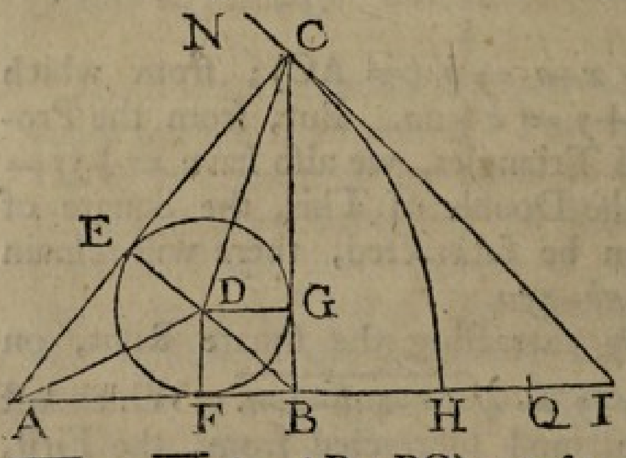
Seeing the Difference between each Leg of the Triangle and the adjacent Segment of the Hypothenuse, is equal to the Radius of the Circle, it is plain that the Sum of the two Legs (AB+BC) will exceed the Sum of of the two Segments (or the whole Hypothenuse) by twice the Radius; and will therefore be given = AC + 2BF. Whence, if BI be supposed equal to BC, AI will likewise be given = AC + 2BF, and the Angle BIC = half a Right-One. Hence the following Construction.

Draw an indefinite Line in which take AH equal to the given Hypothenuse, and HI equal to the Diameter of the given Circle; also make the Angle AIN equal to half a Right Angle; and from the Center A, with the Interval AH, let an Arch be described, meeting IN in C; from which Point upon AP let fall the Perpendicular CB; so shall AB and BC be the two Legs required.

PROBLEM XXX.

The Perimeter, and the Area, of a right-angled Triangle ABC *being given; to determine the Triangle.*

Put the given Perimeter (AB+BC+AC) $= p$, the Area ($\frac{1}{2}$ AB × BC) $= a^2$; and let half the Sum of the Legs AB and BC be denoted by x and half their Difference by y: Then, AB being $= x+y$, BC $= x-y$, and AC $= p-2x$, we shall have



$\overline{x+y} \times \overline{x-y}$ (= AB × BC) $= 2a^2$; and $\overline{x+y}|^2 + \overline{x-y}|^2$ (= AC^2) $= \overline{p-2x}|^2$; that is, by Reduction, $xx-yy=2a^2$; aud $2xx+2yy = pp-4px+4xx$.

Now, by the Addition of the latter of these Equations to the Double of the former, there arises $4xx = pp-4px + 4xx+4a^2$: Whence x comes out $= \frac{pp+4aa}{4p} = \frac{1}{4}p + \frac{aa}{p}$. From which Value that of y $= \sqrt{x^2-2a^2}$, will likewise be given.

Geometrically,

Geometrically.

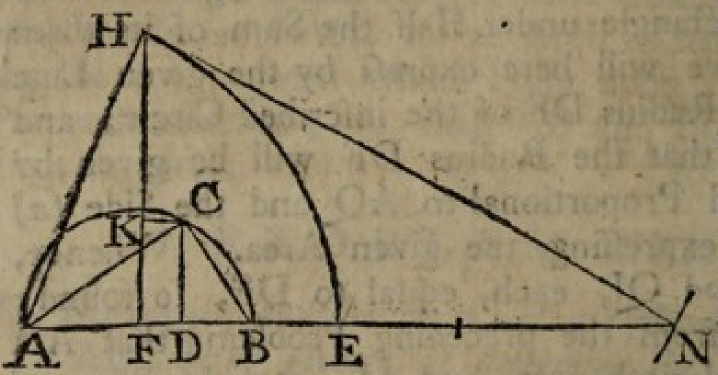
If from the Center D, of a Circle inſcribed in the Triangle, Lines be ſuppoſed drawn to the angular Points, the propoſed Triangle will, by that Means, be divided into three Others ADB, BDC, and ADC; whoſe Baſes are the three Sides of the firſt Triangle, and their Perpendiculars all Radii of the ſaid Circle. From which it is evident that the Triangle ABC is equal to a Rectangle under Half the Sum of its three Sides (which we will here expreſs by the given Line AQ) and the Radius DF of the inſcribed Circle; and conſequently that the Radius DF will be given by taking a Third Proportional to AQ and the Side (a) of the Square expreſſing the given Area. Whence, making QH aud QI, each, equal to DF, ſo found; it will appear from the preceding Problem that AH will be = the Hypothenuſe, and AI = to the Sum of the two Legs, of the propoſed Triangle: Which Quantities being both given, the Method of Conſtruction is manifeſt *from the laſt Problem.*

PROBLEM

PROBLEM XXXI.

The Sum, or Difference of the two Legs AC *and* BC *of a right-angled Triangle* ABC *being given, together with the Sum, or Difference of the Hypothenuse* AB *and a Perpendicular* CD *falling thereon from the Right-Angle; to find all the Sides of the Triangle.*

Put $AC+BC=s$, $AC-BC=d$, $AB+CD=p$, $AB-CD=q$: Then will $AC=\frac{s+d}{2}$, $BC=\frac{s-d}{2}$, $AB=\frac{p+q}{2}$, $CD=\frac{p-q}{2}$: But $\overline{AB}|^2=\overline{AC}|^2+\overline{BC}|^2$; and $AB\times CD$ ($=2$ Area ABC) $=AC\times BC$: Which, in Species, give $p^2+2pq+q^2=2s^2+2d^2$, and $p^2-q^2=s^2-d^2$.



By adding, and subtracting the Double of the last of these Equations from the former, we have these two other Equations,

viz. $3p^2+2pq-q^2=4s^2$;
and, $-p^2+2pq+3q^2=4d^2$.

From which, when any two of the Quantities, s, d, p, q, are given, the other two will, easily, be determined.

Thus, let s and p be given; then, from the former Equation, we have $q^2-2pq=3p^2-4s^2$; whence $q^2-2pq+p^2=4p^2-4s^2$, and $p-q=2\sqrt{p^2-s^2}$: Therefore CD $\left(=\frac{p-q}{2}\right)=\sqrt{p^2-s^2}$, and $AB=p-\sqrt{p^2-s^2}$.

If d and q be given; we shall have $CD=\sqrt{q^2-d^2}$; because $p^2-q^2=s^2-d^2$, or $p^2-s^2=q^2-d^2$ (p. above.) But,

But, if s and q be given, then will $3p^2+2pq = 4s^2+q^2$; which, ſolv'd, gives $p = \frac{2}{3}\sqrt{3ss+qq} - \frac{1}{3}q$.

Laſtly, if d and p be given, we ſhall have $3q^2+2pq = 4d^2+p^2$, and conſequently $q = \frac{2}{3}\sqrt{3dd+pp} - \frac{1}{3}p$.

Geometrically.

The very ſame Properties whereby the algebraical Solution is above brought out, lead us alſo to Geometrical Conſtructions of the ſeveral Caſes of the Problem under Conſideration. But it will be ſufficient, here, to exhibit That of the Caſe, wherein the Sum of the Legs (AE) and the Difference of the Hypothenuſe and Perpendicular (AF) are ſuppoſed given, being the moſt difficult.

Thus, becauſe $\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$,
and $2AC \times BC$ $(= 2AB \times CD) = 2AB \times BF$;
we ſhall, by adding theſe equal Quantities,
have $\overline{AC}^2 + \overline{BC}^2 + 2AC \times BC$ (or $\overline{AE}^2$, *Elem.* 5. 2.) $= \overline{AB}^2 + \overline{2AB} \times BF = \overline{AB}^2 + 2AB \times \overline{AB-AF} = 3\overline{AB}^2 - 2AB \times AF$.

Therefore, if an Arch, from the Center A, with the Radius AE, be deſcribed; and, from its Interſection (H) with FH perpendicular to AE, another Arch be alſo deſcribed, with the Radius 2AE, cutting AE produced in N; then the Hypothenuſe AB of the required Triangle, will be $\frac{1}{3}$ of the Line AN thus determined.

For $\overline{HN}^2$ $(4\overline{AE}^2)$ being $= \overline{AN}^2 + \overline{AH}^2$ $(\overline{AE}^2)$ $- 2AN \times AF$ (*Elem.* 9. 2), and therefore $3\overline{AE}^2 = \overline{AN}^2 - 2AN \times AF$; it is plain, if AB be taken $= \frac{1}{3}$ AN, that $3\overline{AE}^2 = 3AB \times 3AB - 6AB \times AF$: And conſequently that $\overline{AE}^2 = 3\overline{AB}^2 - 2AB \times AF$, *the very ſame as above.*

From the Value of AB, thus given, what yet remains to be done, will be effected with great Facility. For, if in FH there be taken FK = FB, and KC be

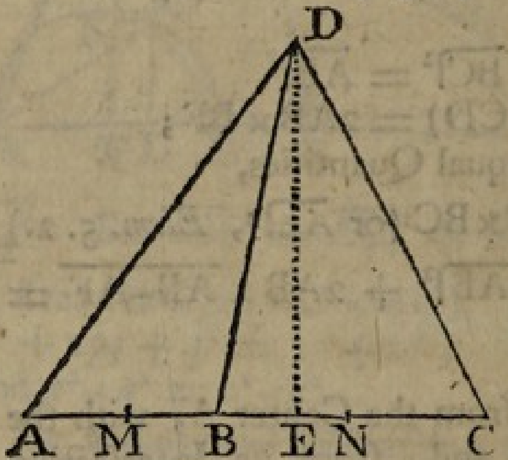
drawn

drawn parallel to AN, interſecting a Semicircle, deſcribed upon AE, in the Point C, that Point will, it is evident, be the Vertex of the Triangle.

L E M M A.

If a Line be drawn from the Vertex to any Point in the Baſe of a Triangle, the Sum of the two Solids under the Squares of the two Sides and the alternate Segments of the Baſe, will be equal to the Solid under the whole Baſe and its two Segments, together with the Solid under the ſame Baſe and the Square of the dividing Line.

Let ACD be any propoſed Triangle, and BD the dividing Line; then, I ſay, that $\overline{AD}^2 \times BC + \overline{CD}^2 \times AB = AC \times AB \times BC + AC \times \overline{BD}^2$.



For, if AB and CB be biſected in M and N, and a Perpendicular DE be let fall upon AC, it is known

that $\left\{\begin{array}{l}\overline{AD}^2 - \overline{BD}^2 = AB \times 2ME \\ \overline{CD}^2 - \overline{BD}^2 = BC \times 2NE\end{array}\right\}$ *Elem.* 8. *and* 10. *of* 2.

Whence it follows,

that $\overline{AD}^2 \times BC - \overline{BD}^2 \times BC = AB \times BC \times 2ME$;

and $\overline{CD}^2 \times AB - \overline{BD}^2 \times AB = AB \times BC \times 2NE$.

Let theſe equal Quantities be added together, and the Sums will alſo be equal;

that is, $\overline{AD}^2 \times BC + \overline{CD}^2 \times AB - AC \times \overline{BD}^2$ ($= AB \times BC \times 2MN$) $= AB \times BC \times AC$; and conſequently $\overline{AD}^2 \times BC + \overline{CD}^2 \times AB = AC \times AB \times BC + AC \times \overline{BD}^2$. *Q. E. D.*

COROL. 1. Hence, if $AB = BC$, then will $\overline{AD}^2 + \overline{CD}^2 = 2AB^2 + 2\overline{BD}^2$.

COROL.

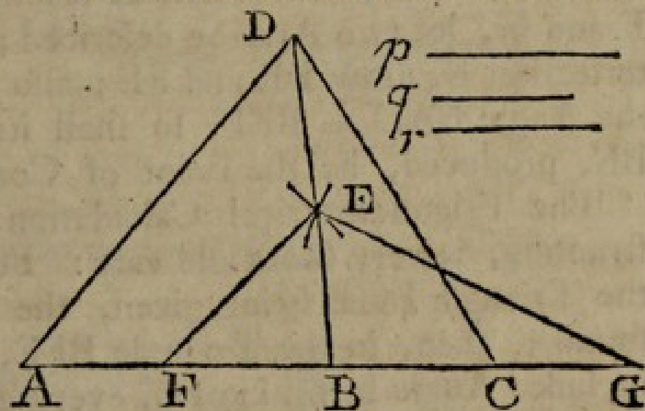
COROL. 2. But, if $AD = DC$, then we ſhall have $\overline{AD}^2 \times BC + \overline{CD}^2 \times AB = \overline{AD}^2 \times BC + \overline{AD}^2 \times AB = \overline{AD}^2 \times AC$: Whence $\overline{AD}^2 \times AC = AB \times BC \times AC + \overline{BD}^2 \times AC$; and conſequently $\overline{AD}^2 = AB \times BC + \overline{BD}^2$.

COROL. 3. Laſtly, if the Angle $ADB =$ the Angle CDB (or $AD : CD :: AB : BC$, *Elem.* 10. 4.) then $AD \times BC$ being $= CD \times AB$, it follows that $\overline{AD}^2 \times BC = AD \times CD \times AB$; and that $\overline{CD}^2 \times AB = AD \times CD \times BC$. Hence $\overline{AD}^2 \times BC + \overline{CD}^2 \times AB = AD \times CD \times \overline{AB + BC} = AD \times CD \times AC = AB \times BC \times AC + \overline{BD}^2 \times AC$ *(p. above)*; and conſequently $AD \times CD = AB \times BC + \overline{BD}^2$.

PROBLEM XXXII.

From three given Points, A, B, C, *in the ſame Right-line, to draw as many Lines, to meet in a fourth Point* D, *ſo as to obtain a given Ratio among themſelves.*

Call AB, a; BC, b; AC, c; and AD, x; and let AD, BD, and CD be, in Proportion to one another, as p, q, and r, reſpectively. Then, BD being $= \frac{qx}{p}$, and $CD = \frac{rx}{p}$, we ſhall, *by the preceding Lemma*, have $x^2 \times b + \frac{r^2x^2}{pp} \times a = abc + c \times \frac{q^2x^2}{pp}$.



Whence $\overline{bp^2 + ar^2 - cq^2} \times x^2 = abcp^2$;

And conſequently $x = p\sqrt{\frac{abc}{bpp+arr-cqq}}$.
After the ſame manner the Problem may be reſolved, when, inſtead of the Ratio, the Sums, the Differences, or the Rectangles of the three Lines, are given.

Geometrically.

If from any Point E, in BD, two Lines EF and EG be ſuppoſed drawn, ſo as to form the Angles BEF and BEG, reſpectively, equal to BAD and BCD, it is evident, from the Similarity of the Triangles BEF, BDA, and BEG, BDC,

that $\begin{cases} BE : BF :: BA : BD \\ BE : BG :: BC : BD. \end{cases}$

And, conſequently, that BF : BG : : BC : BA. Therefore, if BF be taken = BC, BG will be = AB. Moreover, from the abovementioned, ſimilar, Triangles,

we have $\begin{cases} BD : AD\ (:: q : p) :: BF\ (BC) : FE \\ BD : CD\ (:: q : r) :: BG\ (AB) : GE. \end{cases}$

Whence FE and GE are given; and from thence the following Conſtruction.

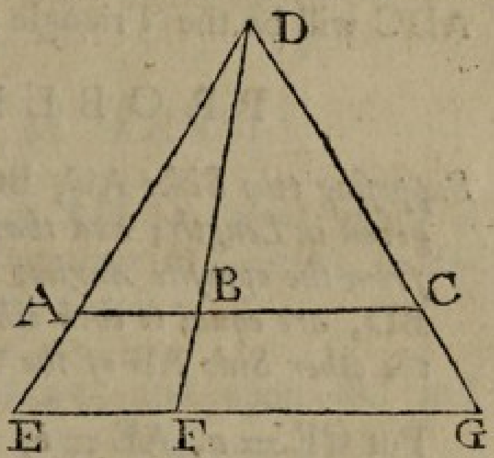
Take BF = BC and BG = AB; alſo take a Fourth Proportional to $q : p$, and BC, and Another to q, r, and AB; And, with Theſe as Radii, from the Centers F and G, let two Arcs be deſcribed; and, to their Interſection E, draw BE and FE; alſo draw AD, making the Angle BAD = BEF, ſo ſhall its Interſection with BE, produced, be the Point of Concourſe required.

The Trigonometrical Calculation, from this Conſtruction, is very ſhort and eaſy: For, all the Sides of the Triangle FGE being given, the Angle F may be found; then, in the Triangle BFE, two Sides and the included Angle being known, every Thing elſe is readily determined.——It may be obſerved that there is another Conſtruction of this Problem; by Means of the Interſection of two Circles, ſo deſcribed that Lines drawn from the given Points to meet in the Peripheries Thereof, may obtain the given Ratios (*See Elem.* 15. 4.) but the Method given above I look upon as preferable.—As to to the Limitations, it is plain the Problem becomes impossible when the two Circles, described from C and F, do not meet each other; that is, when q is given either, less than the Difference, or greater than the Sum of $\frac{pb}{c}$ and $\frac{ra}{c}$.

PROBLEM XXXIII.

The two Sides AD, CD, *of a Triangle being given in Length, together with the Length of a Line* DB *dividing the Base* AC *in a given Ratio; to determine the Base, or Line so divided.*

Call AD, a; BD, b; CD, c; and AB, x; and let the given Ratio of AB to BC be that of m to n. Hence BC $= \frac{nx}{m}$, and AC $\left(= x + \frac{nx}{m}\right) = \frac{\overline{m+n} \times x}{m}$. And therefore, *by the Lemma*, $a^2 \times \frac{nx}{m} + c^2 \times x = \frac{\overline{m+n} \times x}{m} \times \frac{nx}{m} \times x + \frac{\overline{m+n} \times x}{m} \times b^2$.



Whence, by Reduction, $mna^2 + m^2c^2 - m \times \overline{m+n} \times b^2 = \overline{m+n} \times nx^2$.

And $x = \sqrt{\frac{mnaa + mmcc}{\overline{m+n} \times n} - \frac{mbb}{n}}$.

If DB be supposed to bisect the Base; then, m and n being equal, x becomes $= \sqrt{\frac{aa+cc}{2} - bb}$.

But, if DB be supposed to bisect the vertical Angle, we shall have $m : n$ (: : AD : CD) : : $a : c$ (*Elem.* 10. 4).

 Whence,

Whence, by writing a and c instead of m and n, our Equation becomes $x \left(= \sqrt{\frac{a^3c+a^2c^2}{\overline{a+c} \times c} - \frac{ab^2}{c}}\right) = \sqrt{a^2 - \frac{ab^2}{c}}$.

Geometrically.

In any Right-line EG, at pleasure, let there be taken EF and FG in the given Ratio of AB to BC; and, from the Points E, F and G, let three Lines be drawn to meet in a Point D (*by the last Problem*) so as to obtain the given Ratio of AD, BD, and CD, respectively: Then, if DA be taken of the given Length, and ABC be drawn parallel to EFG, it is manifest that ADC will be the Triangle required.

PROBLEM XXXIV.

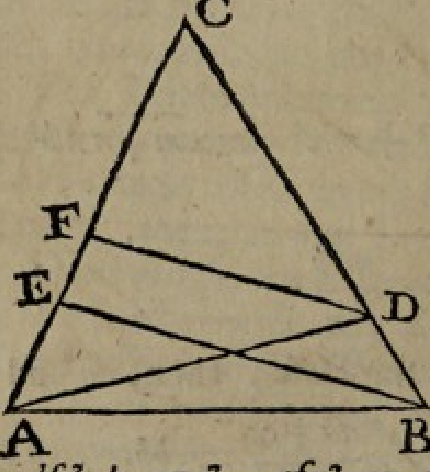
Supposing two Sides AC, BC, *of a Triangle* ABC, *to be given in Length; and that two Lines* BE, AD, *drawn from the opposite Angles, to cut off given Segments* AE, BD, *are equal to each other; 'tis proposed to determine the other Side* AB *of the Triangle.*

Put CE $= a$, AE $= b$, CD $= c$, BD $= d$, CA $= f$, CB $= g$, AB $= x$, and AD (or BE) $= z$. Then, *by the Lemma*, we have the two following Equations.

$g^2 \times b + x^2 \times a = fab + f \times z^2$,

$f^2 \times d + x^2 \times c = gcd + g \times z^2$.

From the former of which, multiply'd by g, let the latter, multiply'd by f, be subtracted, and there will arise $bg^3 - df^3 + agx^2 - cfx^2 = abfg - cdfg$.



Therefore $x = \sqrt{\frac{\overline{ab-cd} \times fg + df^3 - bg^3}{ag-cf}}$.

Geometrically.

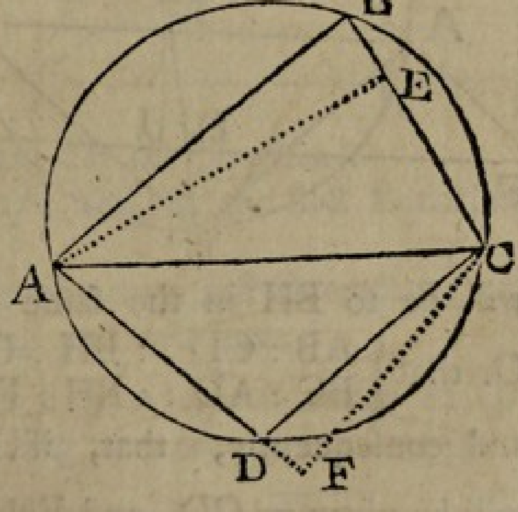
Geometrically.

If DF be drawn parallel to BE, the Ratio Thereof to BE (or AD) will be given, as CD to CB; and CF will likewiſe be to CE in the ſame given Ratio, and therefore will be given in Length. Hence it is evident that the Poſition of the Point D with reſpect to the Side AC (firſt laid down) will be determined by the Interſection of two Circles; One deſcribed from the Center C, with the given Interval CD; and the Other, *by Elem.* 15. 4, ſo that Lines drawn from F and A, to meet any-where in the Periphery Thereof, may obtain the ſaid given Ratio of CD to CB. Through which Point, ſo determined, the other Side CB of the Triangle muſt be drawn; and then, AB being joined, the Thing is done.

PROBLEM XXXV.

All the Sides of a Trapezium ABCD, *about which a Circle may be inſcribed, being given in Length; to determine the Diameter of the Circle.*

Let the Diagonal AC be drawn, and upon BC and AD let fall the Perpendiculars AE and CF. And put AB $=a$, BC $=b$, CD $=c$, AD $=d$, and BE $=x$.



Now the external Angle CDF being equal to the internal, oppoſite, Angle B (*Elem. Corol. to* 13. 3.) the Triangles ABE and CDF are ſimilar: And therefore AB(a) : BE (x) :: CD (c) : DF $=\frac{cx}{a}$. But $\overline{AB}|^2+\overline{BC}|^2-2BC\times BE$

($=\overline{AC}|^2$)$=\overline{AD}|^2+\overline{CD}|^2+2AD\times DF$; that is, in Species, $aa+bb-2bx=cc+dd+\frac{2cdx}{a}$.

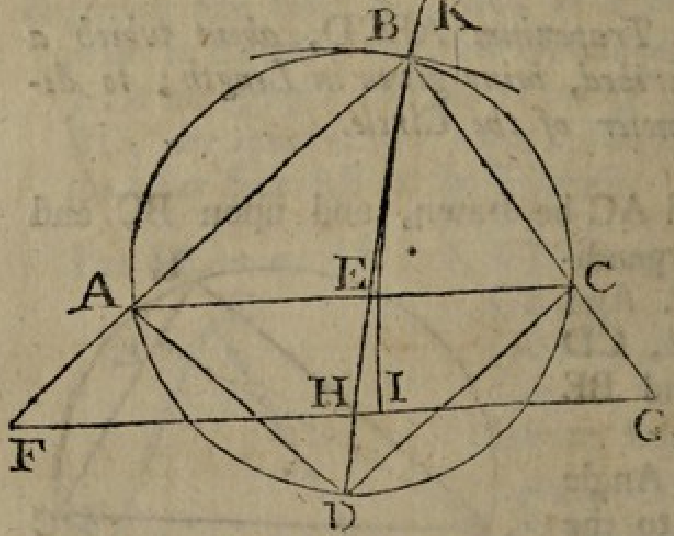
From which Equation x is found $=\frac{aa+bb-cc-dd}{2b+\frac{2cd}{a}}$:

Whence AE ($=\sqrt{aa-xx}$) and AC ($=\sqrt{aa+bb-2bx}$) will alſo be given: And then it will be AE : AC :: AB : the Diameter ſought (*Elem.* 19. 3).

Geometrically.

If the two Diagonals of the Trapezium be drawn, interſecting each other in E, the Triangles ABE and CED, as well as CEB and AED, will be equiangular (*Elem. Corol. to* 9. 3).

Whence $\begin{cases} AB : CD :: BE : CE \\ BC : AD :: BE : AE. \end{cases}$



And, ſince the Ratios of CE and AE to BE are thus given, it follows, that, if any Line FG be drawn parallel to AC, the Parts thereof GH and FH, intercepted by BC, BE, and BA (produced) will be to BH in the ſame given Ratios,

Or that $\begin{cases} AB : CD :: BH : GH \\ BC : AD :: BH : FH; \end{cases}$

and conſequently, that, if BH be taken $=$ AB, GH will be given $=$ CD, and FH$=\frac{AD\times AB}{BC}$. Whence the following Conſtruction.

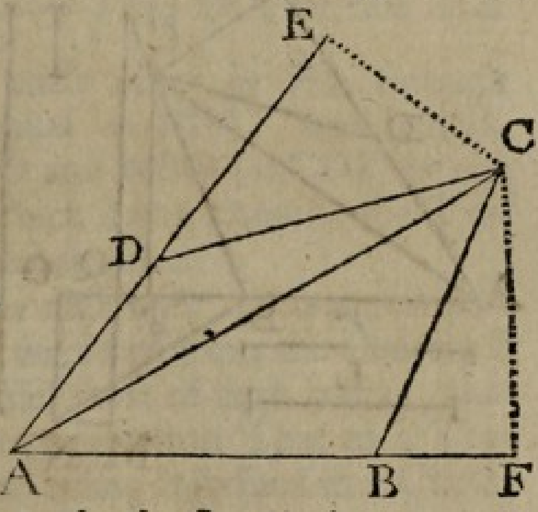
Make

Make FH a Fourth-Proportional to BC, AB, and AD (*by Elem.* 13. 5); and, in the ſame Line produced, take HG = CD: Then (*by Elem.* 15. 4.) let a Circle IK be deſcribed, ſo that two Lines drawn from F and G, to meet any-where in the Periphery Thereof, may obtain the given Ratio of AB to CB. And from H, as a Center, with the Radius AB, let another Arch be deſcribed, interſecting the Former in B; draw BF and BG, in which ſet off BA and BC of the given Lengths; then, thro' the three Points A, B, and C, let a Circle be deſcribed, and the Thing is done.

PROBLEM XXXVI.

All the Sides and, the Area of a Trapezium ABCD *being given; to determine the Trapezium.*

Suppoſe the Diagonal AC to be drawn; ſuppoſe alſo CE and CF to be perpendicular to AD and AB: And make AD $=a$, DC $=b$, BC $=c$, AB $=d$, the given Area ABCD $=r^2$, DE $=x$, and BF $=y$.



Then will $aa+bb+2ax$ ($=\overline{AC}|^2$) $= cc+dd$ $2dy$ (*Elem.* 11. 2.) and $a\sqrt{bb-xx}+d\sqrt{cc-yy}$ (=2ADC+2ABC) $= 2r^2$ (*by the Queſtion*).

Put $c^2+d^2-a^2-b^2=2f$; and, by the firſt Equation, you will have $ax-dy=f$.

Moreover, by ſquaring the two laſt Equations, and then adding them together, you will have

$a^2b^2+c^2d^2+2ad\sqrt{bb-xx}\times\sqrt{cc-yy}-2adxy=4r^4+f^2$. Which, by dividing by $2ad$, and making $g=\frac{4r^4+f^2}{2ad}-\frac{ab^2}{2d}-\frac{c^2d}{2a}$, is reduced to $\sqrt{bb-xx}\times\sqrt{cc-yy}=g+xy$.

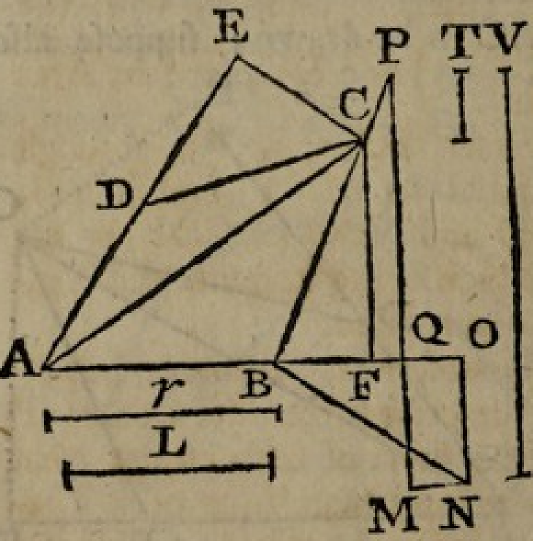
This, ſquared, gives $b^2c^2-b^2y^2-c^2x^2=g^2+2gxy$:

 Which,

Which, by writing $\frac{dy+f}{a}$ in the Room of, its Equal, x, will become $b^2c^2 - b^2y^2 - \frac{c^2 \times \overline{dy+f}|^2}{aa} = g^2 + 2gy \times \frac{dy+f}{a}$: Whence, putting $b^2c^2 - \frac{c^2f^2}{a^2} - g^2 = h$, $\frac{2c^2df}{a^2} + \frac{2fg}{a} = k$, and $b^2 + \frac{c^2d^2}{a^2} + \frac{2dg}{a} = l$, we get $h = ky + ly^2$; and therefore $y = \sqrt{\frac{h}{l} + \frac{kk}{4ll}} - \frac{k}{2l}$.

Geometrically.

Because $\overline{AD}|^2 + \overline{DC}|^2 + 2AD \times DE$ $(= \overline{AC}|^2) = \overline{AB}|^2 + \overline{BC}|^2 + 2AB \times BF$ (*Elem.* 11. 2.) it is plain that $2AD \times DE - 2AB \times BF$ is given $= \overline{AB}|^2 + \overline{BC}|^2 - \overline{AD}|^2 - \overline{CD}|^2$.



Find (*by Elem.* 2 *and* 3. 6.) a Line L whose square shall be $= \overline{AB}|^2 + \overline{BC}|^2 - \overline{AD}|^2 - \overline{CD}|^2$; so shall $2AD \times DE - 2AB \times BF = L^2$, and consequently $DE - \frac{AB \times BF}{AD} = \frac{L^2}{2AD}$.

But $\frac{AB \times BF}{AD}$, supposing PQ perpendicular to AB, and BP a Fourth Proportionol to AD, AB, and BC, appears to be = BQ: And AB × CF (since AD : AB : : BC : BP : : CF : PQ) will also be equal to AD × PQ (*Elem.* 3. 4.)

Hence

Hence we have $DE - BQ = \frac{L^2}{2AD}$, and $AD \times EC + AB \times CF$ ($= 2r^2$) $= AD \times EC + AD \times PQ$: Whence, by taking a Third Proportional (T) to 2AD, and L, and Another (V) to $\frac{1}{2}$AD and r, there reſults $DE - BQ = T$; and $EC + PQ = V$.

Therefore the Problem is reduced to this; *to find two Angles* CDE, PBQ, *ſo that the Sum of their Sines* CE, PQ, *and the Difference of their Co-ſines* DE, BQ *(anſwering to given, but unequal, Radii* DC, BP) *may be both given Quantities.*

And the Conſtruction thereof (which is exceeding evident) will be as follows.

Draw two Lines cutting each other at Right-Angles in M, in which take $MP = V$, the given Sum of the Sines; and $MN = T$, the given Difference of the Co-ſines: Then, from the Centers P and N, with the given Radii PB$\left(= \frac{AB \times BC}{AD}\right)$ and DC, let two Arcs be deſcribed, interſecting each other in B; through which Point draw OA parallel to MN; and join B, P, and B, N; ſo ſhall PBO and NBO (=CDE) be the two Angles required. Which being known, the Trapezium itſelf is very eaſily conſtructed.

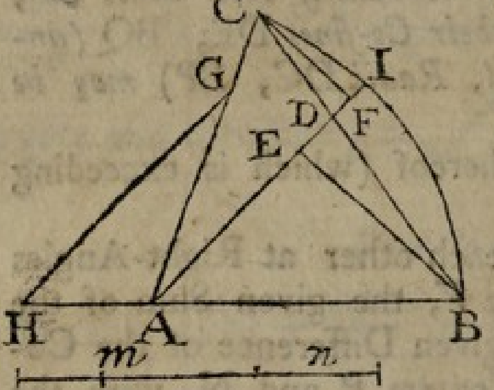
This Problem, it may be obſerved, becomes impoſſible when the two Arcs, deſcribed from the Centers P and N, do not meet, but fall ſhort of each other; that is, when the given Area is greater than That of a Trapezium of the ſame given Sides, inſcribed in a Circle, determined by the preceding Problem.

But, beſides the above, there is another Limit, for the leaſt Value of the Area (except in one particular Caſe) and the Problem will be impoſſible when one of the two Circles falls wholly within the other, as well as when it falls wholly without it: But this Laſt depends upon the particular Order of joining the given Lines; whereas the Caſe is otherwiſe with reſpect to the firſt, or greateſt Limit.

PROBLEM XXXVII.

To divide a given Angle BAC *into two Parts* BAI *and* CAI, *so that their Sines* BE *and* CF *may obtain a given Ratio; suppose that of* m *to* n.

Put the Chord BC of the given Angle $= a$, and the Part thereof BD, intercepted by AI, $= x$: Then, because of the similar Triangles BDE and CDF, it will be, BD (x) : CD $(a-x)$: : BE : CF : : $m : n$, *by Hypothesis.*



Therefore $nx = m \times \overline{a-x}$; and consequently $x = \frac{ma}{m+n}$. From whence, and the given Angle ABD, the Angle BAD will be found.

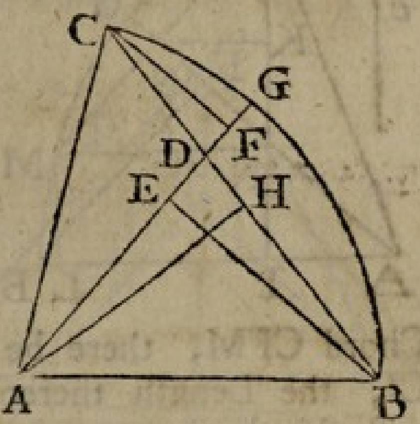
The Geometrical Construction (like the Algebraical Operation) may be performed by dividing the Subtense BC in the given Ratio of m to n: But the following Method is preferable.

In AC take AG $= m$; and, in BA produced, take AH $= n$; then a Line AE drawn parallel to That joining the Points G and H, will divide the Angle as required. For, *by Trigonometry*, AG (m) : AH (n) : : Sine AHG (BAE) : Sine AGH (CAE): From this Construction the numerical Solution is exceeding easy; both the Sides AG and AH, and the included Angle HAG being given.

PROBLEM XXXVIII.

To divide a given Angle BAC *into two ſuch Parts* BAG, CAG, *that the Rectangle under their Sines*, BE *and* CF *(to a given Radius) may be of a given Magnitude.*

Let AH be perpendicular to the Chord BC; alſo let BH $(=CH)=a$, AH $=b$, BE $\times$ CF $=c^2$, and HD $=x$; ſuppoſing AG to interſect BC in D.



Becauſe of the ſimilar Triangles DAH, DBE, and DCF.

we have $\begin{cases} AD\,(\sqrt{bb+xx}) : AH\,(b) :: BD\,(a+x) : BE \\ AD\,(\sqrt{bb+xx}) : AH\,(b) :: CD\,(a-x) : CF. \end{cases}$

Whence, by compounding the two Proportions,

$$bb+xx : bb :: \overline{a+x} \times \overline{a-x} : BE \times CF = \frac{bb \times \overline{aa-xx}}{bb+xx} = c^2.$$

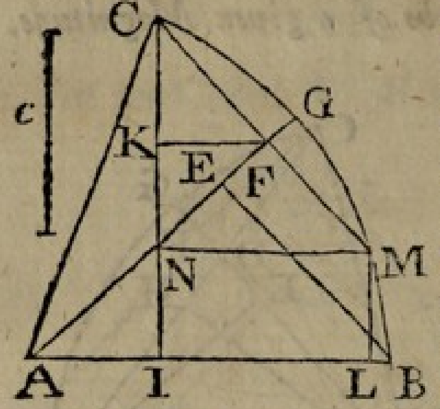
From which Equation x is found $= b\sqrt{\frac{aa-cc}{bb+cc}}$.

By means of which and the foregoing Proportions both BE and CF will become known.

Geometrically.

Geometrically.

If CI be ſuppoſed perpendicular to AB, and FK to CI; then, the oppoſite Angles CNF and ANI being equal, their Complements NCF and NAI will likewiſe be equal: And therefore, the Triangles CFK and ABE being equiangular, we have AB : BE : : CF : FK, or AB × FK = BE × CF = c^2 (*according to Obſervation* 5.) whence FK is given. Therefore, if, from the Extremity of the Chord CFM, there be drawn MN perpendicular to CI, the Length thereof, being twice That of FK, will alſo be given; and, from thence, the following Conſtruction.



Take IL a Third Proportional to $\frac{1}{2}$ AB and the Side (*c*) of the given Square, expreſſing the Magnitude of the propoſed Rectangle: Draw LM perpendicular to AB, meeting the Arch BC in M; then a Line AG drawn to biſect MC will divide the Angle BAC as required.

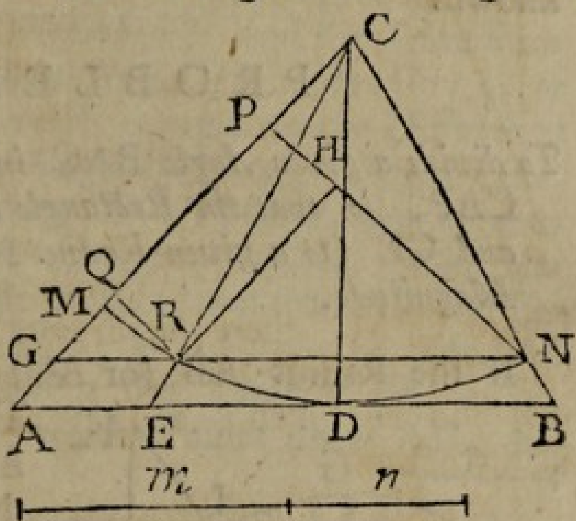
The Numerical Solution, from hence, is very conciſe and eaſy: For, having found the Value of IL (by dividing the Meaſure of the given Rectangle by Half the Radius) let the Co-ſine AI of the whole, given, Angle be added thereto; then the Sum AL will be the Co-ſine of (BM) the Difference of the two, required, Parts.— This Problem becomes impoſſible when IL is given greater than IB; that is, when the Rectangle propoſed is greater than Half the Rectangle under AB and BI.

PROBLEM

PROBLEM XXXIX.

To divide a given Angle MCN *into two ſuch Parts* MCD, NCD *that their Tangents* AD, BD *may obtain a given Ratio*; *ſuppoſe That of* m *to* n.

Put the Radius CD $=a$, the Tangent of the given Angle MCN$=b$, and That (AD) of the Part MCD$=x$. Then it will appear, from Problem 25, that the Tangent BD of the remaining Part NCD is truly expreſſed by $\frac{a^2\times\overline{b-x}}{aa+bx}$.



Hence we have $m : n :: x : \frac{a^2\times\overline{b-x}}{aa+bx}$: And therefore $nx\times\overline{aa+bx}=ma^2\times\overline{b-x}$. From which x is found $=a\sqrt{\frac{m}{n}+\left|\frac{\overline{m+n}\times a}{2nb}\right|^2}-\frac{\overline{n+m}\times a^2}{2nb}$.

Geometrically.

The Ratio of AD to BD being given, as m to n, the Ratio of their Sum AB to their Difference AE (ſuppoſing DE $=$ DB) will be given, as $m+n$ to $m-n$. And, if NG be drawn parallel to BA, meeting CE and CA in R and G, the whole Line NG will be to the Part GR in the ſame Ratio of $m+n$ to $m-n$: And it is evident, that, if two Perpendiculars NP and RQ be let fall from the Points N and R upon AC, they will likewiſe be in that Ratio. Whence the following Conſtruction.

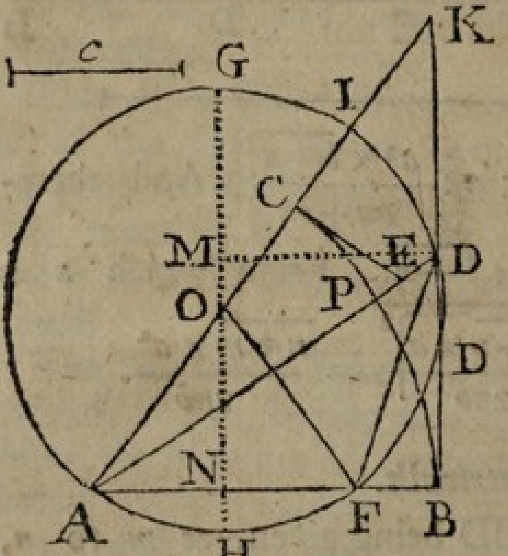
In NP, perpendicular to MC, take PH a Fourth Proportional to $m+n$, $m-n$, and PN; draw HR parallel

rallel to CM interfecting the Arch MN in R, and draw CD to bifect NR, *and the Thing is done.*

For the Trigonometrical Calculation, it will be $m+n : m-n :: NP : RQ$;
that is, as $m+n$ is to $m-n$, fo is the Sine of the whole, given, Angle to the Sine of the Difference of its two, required, Parts; whence the Parts themfelves will be known.

PROBLEM XL.

To divide a given Angle BAC *into two Parts* BAP *and* CAP, *fo that the Rectangle under their Tangents* BD *and* CE *(to a given Radius* AB) *may be of a given Magnitude.*



If the Radius AB (or AC) be denoted by a; the Tangent BK of the given Angle BAC, by b; and the Tangent BD of the Part BAP, by x; then the Tangent CE of the remaining Part CAP will be reprefented by $\frac{a^2 \times \overline{b-x}}{aa+bx}$ *(See the Note to Prob.* 25). And we fhall, therefore, have $\frac{a^2 \times \overline{b-x} \times x}{aa+bx}$ ($=$ BD $\times$ CE) $= c^2$. Whence $bx - xx = \frac{cc \times \overline{aa+bx}}{aa} = cc + \frac{bc^2x}{aa}$: Which, by making $d = b - \frac{bcc}{aa}$, becomes $dx - xx = cc$; whence $x = \frac{1}{2}d \pm \sqrt{\frac{1}{4}dd - cc}$.

Geometrically.

Geometrically.

If a Line DF be drawn to make the Angle BDF equal to CAE, the Triangles BDF and CAE will be ſimilar; and we ſhall, therefore, have AC × BF = BD × CE = c^2 (*according to Obſervation* 5. *p.* 88) whence BF, and conſequently AF, will be given. Moreover the Sum of the Angles BAD and BDF being equal to the given Angle BAC (*by Hypotheſis*) and the ſame Sum + ADF equal to a Right-Angle (*by Elem. Cor.* 4. *to* 10. 1.) it is evident that ADF is equal to the Difference between the ſaid given Angle BAC and a Right-One.

Therefore, having taken BF a Third Proportional to AB and the Side (c) of the given Square, make the Angle AFO = the Angle FAC; and from (O) the Interſection of AC and FO, let a Circle be deſcribed thro' A and F, interſecting the Tangent BK in D, and D; from either of which Points draw AD, *and the Thing is done.*

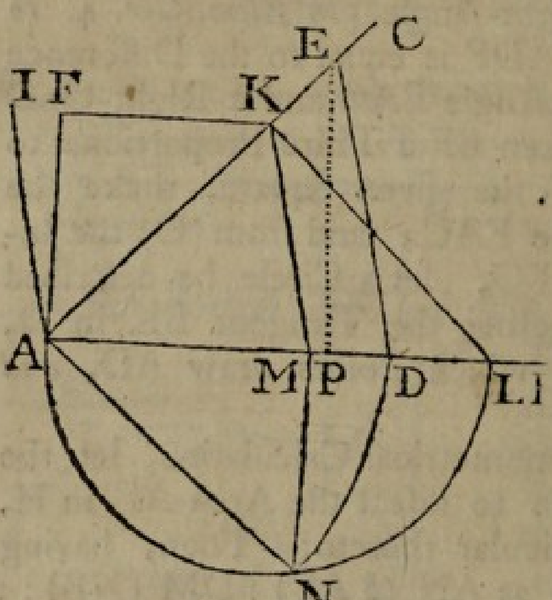
In order to the Trigonometrical Calculation, let the Diameter GH be drawn to biſect the Arch AF in H, and let DM be perpendicular thereto: Then, having found AF, it will be, as AN ($\frac{1}{2}$ AF) : DM (NB) : : Sine AH (Co-ſine NAO) : Sine GD (= Co-ſine $\frac{1}{2}$ DD) = Co-ſine of the Difference of the two required Angles BAD and DAI; whence, as their Sum is given, the Angles themſelves will be known.

This Problem becomes impoſſible, when the Circle OAFG neither cuts, nor touches, the Line BK; that is, when the given Rectangle is greater than the Square of the Tangent of Half the propoſed Angle, and the Angle itſelf is acute; or, when the ſaid Rectangle is leſs than the Square of half the Tangent of the propoſed Angle, and the Angle itſelf is obtuſe.

PROBLEM XLI.

To draw a Line DE *parallel to a given Line* AI, *so as to interſect two other Lines* AB, AC, *given by Poſition, and thereby form a Triangle* ADE *of a given Magnitude.*

Let the given Area of the Triangle be denoted by a^2; and let the Sines of the given Angles DAF, ADE, and AED (to the Radius r) be expreſſed by m, n, and p, reſpectively: Then, making EP perpendicular to AD, and calling AD, x; we have (*by plane Trigonometry*) As $p : m :: x : \text{DE} = \frac{mx}{p}$; And $r : n ::$ DE : EP $= \frac{mnx}{rp}$.



Hence $\frac{mnx}{rp} \times \frac{x}{2} = a^2$:

And therefore $x = \sqrt{\frac{2rpaa}{mn}} = a\sqrt{\frac{2rp}{mn}}$.

Geometrically.

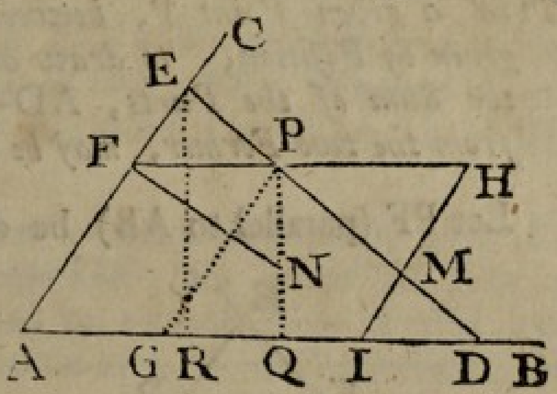
Let AF, perpendicular to AB, be the Side of a Square equal to the Triangle ADE; then, if AL be taken = 2AF, and FK be drawn parallel to AB, &c. it is evident that the Triangle ALK, being $= \overline{\text{AF}}|^2$, will alſo be equal to the Triangle ADE. Moreover, by making KM parallel to AI, it will be $\overline{\text{AD}}|^2 : \overline{\text{AM}}|^2 ::$ ADE (ALK) : AMK (*Elem.* 17.4.) :: AL : AM (*Elem.* 1.4.) :: AL × AM : $\overline{\text{AM}}|^2$. And conſequently $\overline{\text{AD}}|^2$ = AL × AM

$\times$AM. Therefore, AD being a Mean Proportional between AL and AM, upon the former of These let a Semi-Circle ANL be described, intersecting MN, perpendicular to AB, in N; make AD = the Chord AN, and draw DE parallel to AI, and the Thing is done (*Elem.* 11. 4).

PROBLEM XLII.

Thro' a given Point P, *between two Lines* AB *and* AC, *given by Position, to draw another Line* DE, *so that the Triangle* ADE, *formed thereby, may be of a given Magnitude.*

Let PG and PF be parallel to AC and AB, and PQ and ER perpendicular to AB: Also let AG and PQ (which are given by the Position of P) be denoted by a and b, respectively. Then, calling AD, x, and denoting the given Area by c^2, we



shall have, DG $(x-a)$: PQ (b) :: AD (x) : ER $= \frac{bx}{x-a}$:

And therefore $\frac{bx}{x-a} \times \frac{x}{2}$ $(= \text{ER} \times \frac{1}{2}\text{AD}) = c^2$.

Whence $xx - \frac{2c^2x}{b} = -\frac{2ac^2}{b}$; and consequently $x = \frac{c^2 \pm c\sqrt{c^2-2ab}}{b}$.

Geometrically.

If upon AF, a Parallelogram AFHI be constituted, to contain the given Area, it will appear, by taking away AFPMI from each of the equal Quantities AH and ADE, that the Remainders PHM and PFE + IDM will likewise be equal: And so, these three Triangles

 being

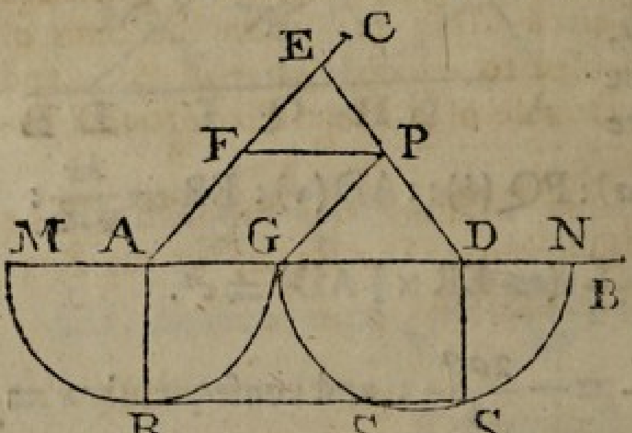
being all ſimilar to one another, it follows that $\overline{PH}^2$ will be $= \overline{PF}^2 + \overline{ID}^2$ (*Elem.* 17. 4). Whence this Conſtruction.

From the Center F, with the Interval PH, let an Arch be deſcribed, cutting PQ perpendicular to AB, in N; make ID = PN, and draw DPE, and the Thing is done. For it is evident that $\overline{PH}^2$ ($\overline{FN}^2$) is $= \overline{PF}^2 + \overline{DI}^2$ ($\overline{PN}^2$).——Hence it alſo appears that the Problem will be impoſſible when PH is leſs than PF; or when the Triangle propoſed to be conſtructed is leſs than twice the Parallelogram AFPG.

PROBLEM XLIII.

Thro' a given Point P, *between two Lines* AB, AC *given by Poſition, to draw another Line* BC, *ſo that the Sum of the Parts*, AD+AE, *cut off thereby, from the two Former, may be a given Quantity.*

Let PF (parallel to AB) be denoted by a, and PG (parallel to AC) by b; alſo let AD + AE = c, and AD = x: Then, by Reaſon of the parallel Lines, we ſhall have DG ($x - a$) : PG (b) : : AD (x) : AE $= \frac{bx}{x-a}$.



And therefore $\frac{bx}{x-a} + x$ (= AE + AD) = c.

From which, by Reduction, $xx - \overline{c+a-b}\,xx = -ac$; and conſequently $x = \frac{c+a-b}{2} \pm \sqrt{\frac{\overline{c+a-b}^2}{4} - ac}$.

Geometrically,

Geometrically.

The Triangles DGP and PFE being ſimilar, the Rectangle under DG and FE will therefore be equal to the given Rectangle under AF and AG $= \overline{AR}|^2$; by taking AM $=$ AF, and making AR a Mean Proportional between AM and AG. Moreover, if MN be taken $=$ the given Sum of AD and AE, it is manifeſt that the Sum of DG and FE (whoſe Rectangle is above given) will, alſo, be given $=$ GN.

Therefore, having deſcribed a Semi-circle upon GN, let RS be drawn parallel to AB, interſecting the Periphery thereof in S; from which Point, upon AB, let fall the Perpendicular SD, and thro' P draw DPE, and the Thing is done. For $DG \times DN = \overline{DS}|^2 = \overline{AR}|^2 = AM \times AG = AF \times AG = DG \times EF$: Whence $EF = DN$; and conſequently $AD + AE$ $(= AD + AF + DN)$ $= MN$.

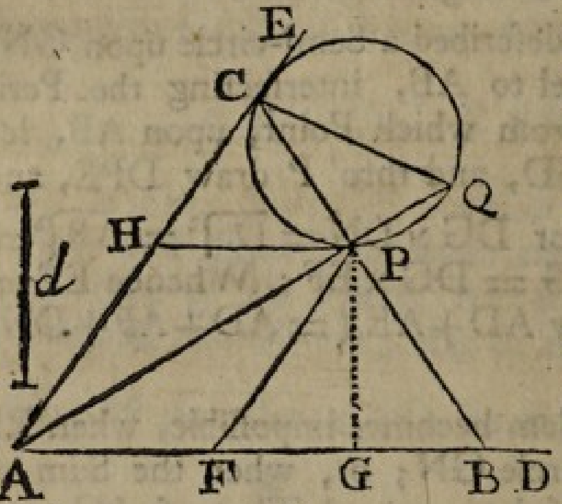
It is plain, this Problem becomes impoſſible, when RS falls below the Semi-circle GN; or, when the Sum of AD and AE is ſuppoſed to exceed That of AG and AF by leſs than the Double of a Mean Proportional between AG and AF.

Much after the ſame Manner the Problem will be reſolved, when the Difference, the Ratio, or the Rectangle of AD and AE is given.

PROBLEM XLIV.

Through a given Point, P, *betwixt two Right-Lines* AD, AE, *given by Position, so to draw a Right-line* BPC *that the Rectangle* (BP × CP) *under the Parts Thereof, intercepted by that Point and those Lines, may be of a given Magnitude.*

Let PF and PH be parallel to AC and AB, and PG perpendicular to AB: Then, calling PF, a; PH, b; FG, c; and BF, x; we have $\overline{BP}|^2 = aa + xx - 2cx$ (*Elem.* 9. 2).



And, by similar Triangles, BF (x) : BP :: PH (b) : CP $= \frac{b}{x} \times$BP. Consequently BP × CP $= \frac{b}{x} \times BP^2 = \frac{b}{x} \times \overline{aa + xx - 2cx}$. Hence, if the given Value of BP × CP be denoted by d^2, then will $\frac{d^2x}{b} = aa + xx - 2cx$: Which Equation, solved, gives

$$x = \frac{dd}{2b} + c \pm \sqrt{\overline{\frac{dd}{2b} + c}\Big|^2 - a^2}.$$

Geometrically.

The Rectangle under two unknown Lines being given, another Line must therefore be found, or assumed, under which and some given Line in the Figure, an equal Rectangle may be contained. (*Vid. Obser.* 5. *p.* 88.)

As,

As, in the prefent Cafe, the Line AP is given, both in Length and Pofition, let the Rectangle under it, and a Part, PQ, of the fame Line produced, be therefore affumed $= BP \times CP$; then the Confequence will be, *that*, befides obtaining $PQ = \frac{d^2}{AP}$, the Triangles PQC and PBA (fuppofing QC drawn) will alfo be fimilar (becaufe AP : BP :: CP : PQ) And fo, the Angle PCQ being = the given Angle PAB, it is evident that a Segment of a Circle defcribed upon PQ *(by Elem.* 17. 5.) capable of containing the faid given Angle, will interfect AE in the Point (or Points) required.

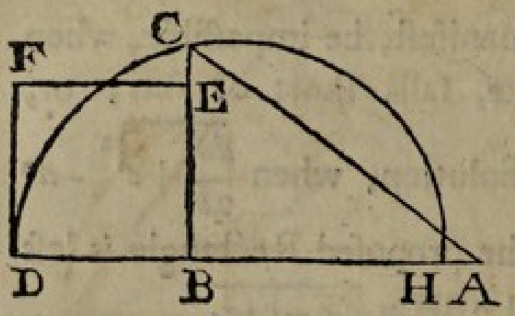
This Problem will, it is manifeft, be impoffible, when the Circle, defcribed as above, falls fhort of AE; or, according to the Algebraic Solution, when $\overline{\frac{dd}{2b}+c}\,^2 - a^2$ is negative; that is, when the propofed Rectangle is lefs than $2b \times \overline{a-c}$, or, its Equal $2PH \times \overline{PF-FG}$.

PROBLEM XLV.

The Area (a^2) *of a right-angled Triangle* ABC, *whose Sides are in Arithmetical Progression, being given, to determine the Triangle.*

Put the greater Leg AB $= x$, and the Common Difference $= y$; so shall BC $= x-y$, and AC $= x+y$ And therefore

$$\left.\begin{array}{l} \overline{x+y}|^2 = \overline{x-y}|^2 + x^2 \\ \frac{1}{2} x \times \overline{x-y} = a^2 \end{array}\right\} \text{by the Question.}$$



From the former of which Equations we have $xx + 2xy + yy = 2xx - 2xy + yy$; and consequently $4y = x$: Whence, by substituting for x in the second Equation, we get $2y \times 3y = a^2$: From which y is given $= \sqrt{\frac{a^2}{6}}$; and $x\ (=4y) = 4\sqrt{\frac{a^2}{6}}$. Therefore BC $= 3\sqrt{\frac{a^2}{6}}$, AB $= 4\sqrt{\frac{a^2}{6}}$, and AC $= 5\sqrt{\frac{a^2}{6}}$.

Geometrically.

It is well known that $\overline{AC+BC} \times \overline{AC-BC}$ ($= \overline{AC}|^2 - \overline{BC}|^2$) is $= \overline{AB}|^2$ *(Elem. Corol. to* 7. 2). And, by the Question, AC + BC is $=$ 2AB (because AB is an Arithmetical Mean between AC and BC) Therefore $2AB \times \overline{AC-BC} = \overline{AB}|^2$; and consequently AC$-$BC $= \frac{1}{2}$ AB: Take these equal Quantities from the equal Quantities AC+BC and 2AB, and the Remainders, 2BC and $1\frac{1}{2}$AB, will be equal; and consequently 3AB $=$ 4BC. But, $\frac{1}{2}$ AB $\times$ BC ($= \frac{2}{3}$ BC $\times$ BC) $= \overline{BD}|^2$ ($=a^2$); and therefore $\overline{BC}|^2 = \frac{3}{2}\overline{BD}|^2$.

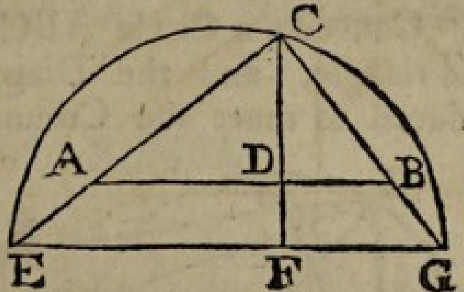
Hence

Hence, if BH be taken to BD (a) in the Proportion of 3 to 2, a Mean Proportional BC between DB and HB, will be the leſſer Leg of the Triangle required; whoſe greater Leg BA, being in Proportion thereto, as 4 to 3, is alſo given from hence.

PROBLEM XLVI.

The Area (a^2) *of a right-angled Triangle* ABC, *whoſe Sides are in Geometrical Proportion, being given; to determine the Triangle.*

Make AC $= x$; then BC $= \frac{2aa}{x}$, and AB $=$ $\sqrt{x^2 + \frac{4a^4}{xx}}$.



Therefore, $\frac{2a^2}{x} : x :: x : \sqrt{x^2 + \frac{4a^4}{xx}}$, *by the Queſt.*

Hence $x^2 = \frac{2a^2}{x}\sqrt{x^2 + \frac{4a^4}{xx}}$; $x^4 = \frac{4a^4}{xx} \times \frac{x^4 + 4a^4}{xx}$; $x^8 - 4a^4x^4 = 16a^8$; $x^4 - 2a^4 = a^4\sqrt{20}$; and $x = a \times \sqrt[4]{2 + \sqrt{20}}$.

Geometrically.

Since, *by Hypotheſis*, AB : AC :: AC : BC, therefore is $\overline{AB}^2 : \overline{AC}|^2 :: \overline{AC}|^2 : \overline{BC}|^2$ (*Elem. Cor.* 1. *to* 9.4.)

But $\overline{AC}^2$, ſuppoſing CD perpendicular to AB, is equal to AB $\times$ AD; and $\overline{BC}|^2$ equal to AB $\times$ BD (*Elem. Cor. to* 11. 4).

Therefore $\overline{AB}|^2$: AB $\times$ AD :: AB $\times$ AD : AB $\times$ BD; or, AB : AD :: AD : BD.

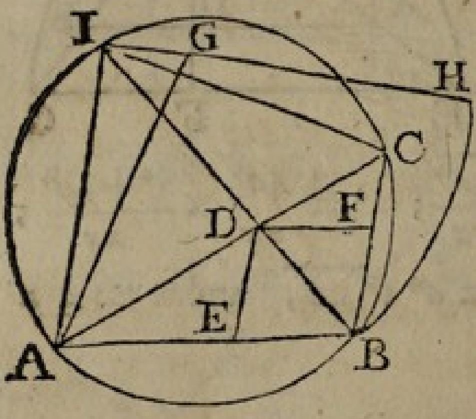
Whence the following Conſtruction.

Draw any Line EG, at pleaſure; which divide at F (*by Elem.* 22. 5.) according to Extreme-and-Mean Proportion (ſo that EG : EF :: EF : GF); erect the perpendicular FC, and upon EG let a Semi-circle be deſcribed interſecting it in C; join E, C, and G, C; and let AB *(by Prob. 41.)* be drawn, parallel to EG, to cut off the given Area ABC $(=a^2)$, *and the Thing is done.* For it is manifeſt that AB is divided, by CDF, in the ſame Proportion with EG; or that, AB : AD : : AD : BD, *as above.*

PROBLEM XLVII.

Suppoſing one Side AC, *and the oppoſite Angle Angle* ABC *of a Triangle to be given, together with the Side* DE, *or* DF, *of the inſcribed Rhombus* EF; *to find from thence the other two Sides of the Triangle.*

Conceive a Circle ABCI to be deſcribed about the Triangle, and the Diagonal of the Rhombus produced to meet the Circumference thereof in I: Alſo let AI and CI be drawn; which will be equal to each other *(Elem. 10. 3.)* as being the Subtenſes of the equal Angles IBA and IBC, form'd by the Diagonal and Sides of the Rhombus: And, ſince the Arcs AI and CI, as well as the Chords, are equal, the Angles IAD and IBA, inſiſting upon Them, muſt likewiſe be equal; and conſequently the Triangle IAD ſimilar to the Triangle IBA.



But the vertical Angle AIC of the Iſoſceles Triangle ACI (as well as the Baſe AC) is given, being = 2 Right-angles − ABC *(Elem. 13. 3.)* whence IA will be given *(by plane Trigonometry).*

Put, therefore, $IA=a$, $BD=b$, and $IB=x$; then, from the Similarity of the Triangle above ſpecified, we ſhall have, $x-b$ (ID) : a (IA) : : a (IA) : x (IB) whence $xx-bx=aa$; and conſequently $x=\sqrt{aa+\frac{1}{4}bb}+\frac{1}{2}b$. From which, and the known Values of IA and the Angle ABI, the Value of AB, &c. will alſo be known.

Geometrically.

Geometrically.

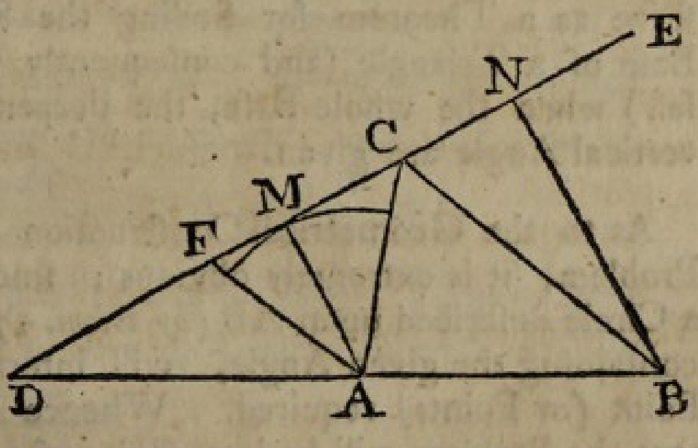
The Geometrical Conſtruction hereof, as the Rectangle BI×DI (by the foregoing Proportion) is given = $\overline{AI}|^2$, is obvious from Problem 6; and is thus,

Having, upon the given Side AC, deſcribed a Segment of a Circle ABC capable of containing the given Angle ABC (*Elem.* 17. 5); and, in the other Segment, AIC, conſtituted the Iſoſceles Triangle AIC, make IG perpendicular to IA, and equal to Half (BD) the given Diagonal of the Rhombus; alſo, in IG produced, take GH equal to the Diſtance GA; and, through H, from the Center I, let an Arch be deſcribed interſecting the Circle ABC in B; then draw BA and BC, and the Thing is done.

PROBLEM XLVIII.

From two given Points A *and* B, *to draw two Lines* AC *and* BC, *to meet in a Right-line* DE *given by Poſition, and form an Angle* ACB *of a given Magnitude.*

Make AM and BN perpendicular to DE; and let AF be ſuppoſed parallel to BC: Then, calling AM, a; BN, b; MN, c; and MC, x; it will be as $b : c-x$ (NC) $:: a :$ MF $= \frac{ac-ax}{b}$:



But MF and MC are Tangents of the Angles MAF and MAC; which Angles together (becauſe of the parallel Lines AF and BC) make an Angle CAF equal to the given Angle ACB: Therefore, if the Tangent of the ſaid given Angle, to the Radius AM, be denoted

noted by t, it will appear, *from Problem* 25, that $\frac{\overline{MC+MF}\times AM^2}{AM^2-MC\times MF}=t$.

This, in Species, gives $\frac{\overline{x+\frac{ac-ax}{b}}\times a^2}{a^2-\frac{acx-axx}{b}}=\frac{\overline{bx-ax+ac}\times a}{ab-cx+xx}=t$.

Which, by Reduction, becomes $\frac{\overline{b-a}\times ax}{t}+\frac{aac}{t}=xx-cx+ab$. Whence, making $c+\frac{\overline{b-a}\times a}{t}=d$, and $a\times\overline{\frac{ac}{t}-b}=f$, we have $f=xx-dx$; and conſequently $x=\frac{1}{2}d\pm\sqrt{f+\frac{1}{4}dd}$.

When the Line MN is parallel to That joining the given Points A, B; then, BN (b) becoming $=$ AM (a) we have, in this Caſe, $d=c$, and $f=\frac{aac}{t}-a^2$; and therefore $x=\frac{c}{2}\pm\sqrt{\frac{aac}{t}-aa+\frac{1}{4}cc}$. Which may ſerve as a Theorem for finding the Segments of the Baſe of a Triangle (and conſequently the Triangle itſelf) when the whole Baſe, the Perpendicular, and the vertical Angle are given.

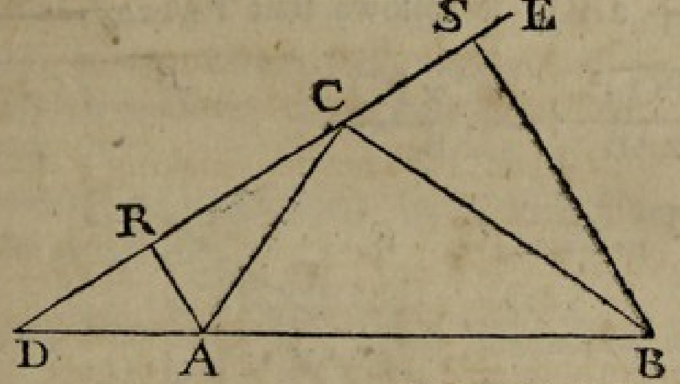
As to the Geometrical Conſtruction of the General Problem, it is extremely obvious; ſince a Segment of a Circle deſcribed upon AB *(by Elem.* 17. 5.*)* capable of containing the given Angle, will interſect DE in the Point (or Points) required. Whence it alſo appears that the Problem will be impoſſible when the Circle falls ſhort of the Line FE; and, conſequently, that the Angle ACB will be the greateſt poſſible when the Circle touches the ſaid Line; or, when DC is a Mean Proportional between DA and DB *(Elem. Cor. to* 17. 3*)*.

PROBLEM

PROBLEM XLIX.

To find a Point C *in a Right-line* DE, *given by Position; so that two Lines* CA, CB *being drawn from thence to two given Points* A *and* B, *the One* BC *shall exceed the Other* AC *by a given Difference* d.

Let AR and BS be perpendicular to DE; and let these Perpendiculars, together with RS (which are all known from the Position of DE) be denoted by a, b, and c, respectively: Then,



putting AC $= x$, we have RC $\left(= \sqrt{\overline{AC}|^2 - \overline{AR}|^2}\right) = \sqrt{xx-aa}$; and SC $\left(= \sqrt{\overline{BC}|^2 - \overline{BS}|^2}\right) = \sqrt{xx+2xd+dd-bb} = c - \sqrt{xx-aa}$.

From which Equation, by squaring both Sides thereof, we get $xx+2dx+dd-bb = cc-2c\sqrt{xx-aa}+xx-aa$.

This contracted becomes $2c\sqrt{xx-aa} = bb-aa-dd+cc-2dx$, or, $\sqrt{xx-aa} = m-nx$; by dividing by $2c$, and putting $\frac{bb-aa-dd+cc}{2c} = m$, and $\frac{d}{c} = n$.

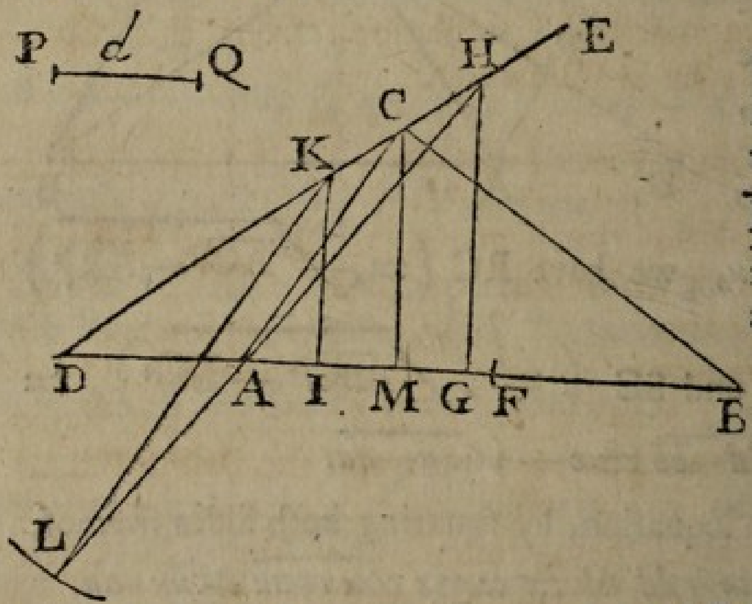
Therefore, by squaring again, $xx-aa = mm-2mnx+nnxx$; whence $\overline{1-nn} \times xx + 2mnx = aa+mm$, or $xx + \frac{2mnx}{1-nn} = \frac{aa+mm}{1-nn}$.

And consequently $x = \sqrt{\frac{aa+mm}{1-nn} + \frac{m^2n^2}{\overline{1-nn}|^2}} - \frac{mn}{1-nn}$.

Geometrically,

Geometrically.

If AB be drawn, and biſected in F, and CM be ſuppoſed perpendicular thereto, it is well known that FM $= \frac{\overline{BC+AC} \times \overline{BC-AC}}{2AB}$ (*Elem.* 8. 2.): Therefore, ſince $BC - AC = PQ$, and, conſequently, $BC + AC = PQ + 2AC$, it follows that $FM = \frac{\overline{PQ+2AC} \times PQ}{2AB} = \frac{\overline{PQ}|^2}{2AB} + \frac{AC \times PQ}{AB}$.



But, ſince both AB and PQ are given, the Value of $\frac{\overline{PQ}|^2}{2AB}$, (being a Third Proportional to 2AB and PQ) is alſo given. Let this, therefore, be expreſſed by FG; ſo that FM may be $= FG + \frac{AC \times PQ}{AB}$: Then it is plain that $GM = \frac{AC \times PQ}{AB}$; or that, GM is to AC in the given Ratio of PQ to AB.

But the Ratio of GM to HC (ſuppoſing GH perpendicular to AB) is alſo given by the Poſition of the Line DE; whence the Ratio of HC to AC (where Both belong to the ſame Triangle AHC) will be exhibited: For, taking $GI = PQ$, and making IK perpendicular to AB, it will be $\frac{HC}{HK} = \frac{GM}{GI} \left(= \frac{GM}{PQ} \right) = \frac{AC}{AB}$, (*p. above*) and conſequently HC : AC : : HK : AB.

But,

But, if KL be ſuppoſed parallel to AC, meeting HA produced (if need be) in L, it is manifeſt that HC : AC : : HK : KL. Therefore KL = AB; whence the following Conſtruction.

Having taken $FG = \frac{\overline{PQ}|^2}{2AB}$, GI = PQ, and erected the Perpendiculars GH and IK, and alſo drawn HA *(as above ſpecified)*, from the Center K, with the Interval AB, let an Arch be deſcribed, cutting HA produced in L; and, having drawn LK, make AC parallel thereto; which will cut the given Line DE in the Point required.

The Trigonometrical Calculation, from this Conſtruction, may be as follows.

Having computed FG $\left(= \frac{\overline{PQ}|^2}{2AB}\right)$ and ſubtracted it from FD and FA, the Remainders GD and GA will be given; and it will then be as GD : GA : : *Tang.* DHG (or *Co-tang.* D) : *Tang.* AHG; whence the Angle LHK is likewiſe known. And, ſince *Sin.* DHG (or *Co-ſin.* D) : *Rad.* : : IG (PQ) : HK, the Value of HK (as well as Thoſe of KL and the Angle HKL) will be known: From which, the other two Angles of the Triangle HKL being found, the Angle DAC (= ECA−D = HKL−D) will alſo be obtained.

After the very ſame Manner the Problem may be reſolved, when the Sum, inſtead of the Difference, of the Lines AC and BC is given.

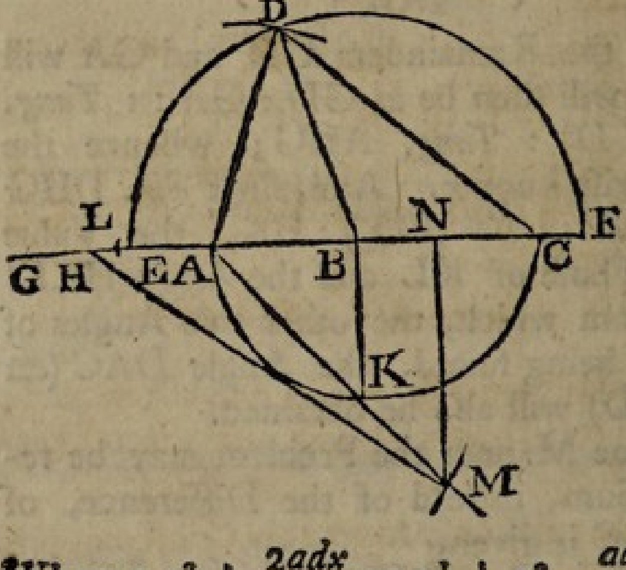
As to the Reſtrictions of the laſt Problem, it is evident that the given Difference muſt never exceed the Diſtance AB: But when the Line DE paſſeth between the Points A and B, the Limit will be ſtill leſs; but is eaſily determined, in any Caſe, from the given Poſition of DE. It is a little remarkable, that the above Solution fails in that particular Caſe, only, wherein the General Problem becomes moſt ſimple; that is, when DE is perpendicular to AB. But here the Operation will, alſo, become more ſimple and expeditious: For the

the Position of MC (with which DE is supposed to coincide) being actually given, the Length of AC $= \frac{GM \times AB}{PQ}$ (*p. above*) will also be known. And, if with this, as a Radius, an Arch be described, from the Center A, it will intersect the Perpendicular MC (or DC) in the Point required.

PROBLEM L.

From two given Points A, C, *in the Diameter* EF *of a given Semi-circle, to draw two Lines to meet in the Circumference, so that One of them* CD *may exceed the Other* AD *by a given Difference, not greater than the Distance of the two given Points.*

The Radius BD being supposed drawn; put $AB=a$, $BC=b$, $AC=c$, $BD=r$, $AD=x$, and $CD=x+d$; d being the given Difference.



Then, by the Lemma at p. 128, we shall have $x^2 \times b + \overline{x+d}|^2 \times a = abc + cr^2$; or, $bx^2 + ax^2 + 2adx + ad^2 = abc + cr^2$.

Whence $x^2 + \frac{2adx}{c} = ab + r^2 - \frac{add}{c}$

And consequently $x = \sqrt{ab + r^2 - \frac{abdd}{cc}} - \frac{ad}{c}$:

Geometrically.

The Geometrical Construction is also deducible from the *Lemma* above specified. For it is evident from thence, that $\overline{AD}|^2 \times \frac{BC}{AB} + \overline{CD}|^2$ is $= AC \times BC +$

A

$\frac{AC \times \overline{BD}|^2}{AB} = AC \times BC + AC \times BG$ (by taking $BG = \frac{\overline{BD}|^2}{AB}$) $= AC \times \overline{BC + BG} = AC \times CG$.

Therefore, having taken BG a Third Proportional to AB and BD; and CL a Mean-Proportional between AC and CG; draw BK perpendicular to GF, meeting the Circumference of a Semi-circle, deſcribed upon AC, in K; and, having drawn AKM, and taken AH equal to the given Difference of AD and CD, upon H as a Center, with the Radius LC, let an Arch be deſcribed, interſecting AK in M; from which Point upon GF let fall a Perpendicular MN; then, if from A, with the Radius AN, another Arch be deſcribed, it will interſect the Arch of the given Semi circle in the Point, D, required.

For $AB : BC :: \overline{AB}|^2 : AB \times BC$ ($= \overline{BK}|^2$, *Elem.* 11. 4) $:: \overline{AN}|^2$ ($\overline{AD}|^2$) $: \overline{NM}|^2 = \overline{AD}|^2 \times \frac{BC}{AB}$. And $\overline{NM}|^2$ $\left(\overline{AD}|^2 \times \frac{BC}{AB}\right) + \overline{HN}|^2 = \overline{HM}|^2 = AC \times CG$ (*by Conſtr.*) $= \overline{AC}|^2 \times \frac{BC}{AB} + \overline{CD}|^2$ (*p. above*):

Therefore $CD = HN = AN + AH = AD + AH$; and conſequently $CD - AD = AH =$ the given Difference, *by Conſtruction.*

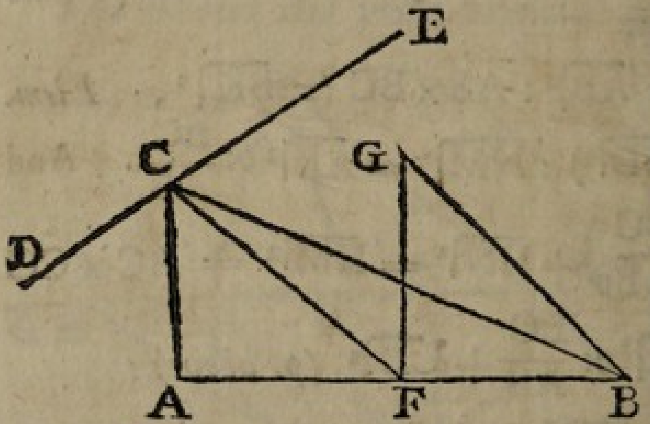
The Method of Calculation, from this Conſtruction, is ſufficiently eaſy: For, having computed $BG \left(= \frac{\overline{BD}|^2}{AB}\right)$ and HM $(= \sqrt{CA \times \overline{CB + BG}})$ and alſo the Angle BAK (which is had by the Proportion AB : BC :: Squ. Rad. : Squ. Tang. BAK); you will then have, in the Triangle HAM, two Sides and one Angle; whence every-thing elſe is readily determined.

PROBLEM

PROBLEM LI.

From the Middle F, *and the two Extremes* A *and* B, *of a given Right-line* AB, *to draw three Lines to meet in a Point* C, *in a Right-line* DE *given by Position, so as to be in Geometrical Proportion, or so that* AC : FC : : FC : BC.

Because $\overline{BC}|^2 + \overline{AC}|^2 = 2\overline{FC}|^2 + 2\overline{BF}|^2$ (*Elem.* 12. 2). And BC x AC = $\overline{FC}|^2$ (*by Hyp. and Elem.* 3. 4.) it is evident that $\overline{BC}|^2 - 2BC \times AC + \overline{AC}|^2 = 2\overline{BF}|^2$, or $\overline{BC-AC}|^2 = 2\overline{BF}|^2 = \overline{BG}|^2$, by taking BG as the Diagonal of the Square whose Side is BF.



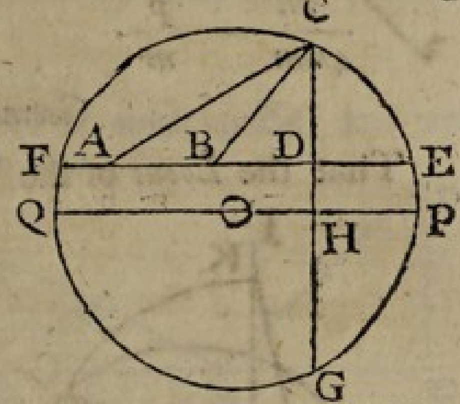
Hence BC-AC is given = BG: And so the Case in Question is reduced to Problem 49; to which I shall therefore refer for the remaining Part of the Solution.—It appears from hence that a Point in the Circumference of a given Circle, whose Center is in the Line AB, may be so determined, *by the last Problem*, that three Lines drawn from thence to the three given Points A, F, B, shall be in Geometrical Proportion.

PROBLEM

PROBLEM LII.

From two given Points A, B, *within a given Circle, to draw two Lines* AC, BC, *to meet in the Periphery Thereof, so that the Sum of their Squares may be a given Quantity.*

Through the given Points let EF be drawn, meeting the Circumference of the Circle in E and F; parallel to which, draw the Diameter PQ; and let the Chord CG be drawn to cut EF and PQ, at Right-angles, in D and H.



Put EF $=a$, EA $=b$, EB $=c$, DH $=d$, ED $=x$, and DC $=y$; and let the given Quantity, $\overline{AC}|^2+\overline{BC}|^2$, be denoted by e^2: Then will DF $=a-x$, DA $=b-x$, DB $=c-x$, DG $=y+2d$.

But ED × DF = CD × DG (*Elem.* 16. 3).

And $\overline{DA}|^2+\overline{DC}|^2+\overline{DB}|^2+\overline{DC}|^2=\overline{AC}|^2+\overline{BC}|^2$ (*Elem.* 7. 2). Which, in Species,

give $\begin{cases} x\times\overline{a-x}=y\times\overline{y+2d} \\ \overline{b-x}|^2+y^2+\overline{c-x}|^2+y^2=e^2. \end{cases}$

or $\begin{cases} ax-xx=yy+2dy \\ 2xx+2yy-2bx-2cx=ee-bb-cc. \end{cases}$

Whence, by adding the double of the former Equation to the Latter, we get $2ax-2bx-2cx=ee-bb-cc+4dy$; and consequently $y=\frac{bb+cc-ee}{4d}+\frac{a-b-c}{2d}\times x=f+gx$, by making $\frac{bb+cc-ee}{4d}=f$, and $\frac{a-b-c}{2d}=g$.

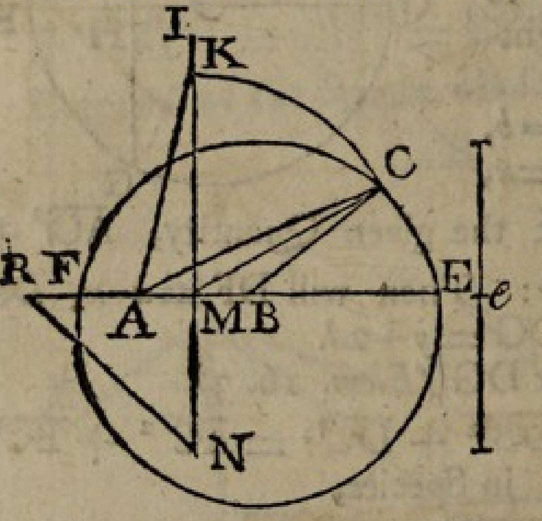
Now, the Value of *y* thus found being ſubſtituted in the firſt Equation, there ariſes $ax-xx=ff+2fgx+g^2x^2+2df+2dgx$;

or, $\overline{1+gg}\times xx+\overline{2dg+2fg-a}\times x=-ff-2df$.

From which, by making $m=1+gg$, $n=a-2dg-2fg$, and $p=f\times\overline{f+2d}$, the Value of x is found $=\frac{n}{2m}\pm\sqrt{\frac{nn}{4mm}-\frac{p}{m}}$.

Geometrically.

That the *Locus* of the Vertex of a Triangle, whereof the Baſe and Sum of the Squares of its 2 Sides are given, is the Circumference of a Circle, deſcribed from the Middle of the Baſe as a Center, is evident, *from Elem.* 12. 2; becauſe the Line drawn from the Vertex to the Middle of the Baſe, is an invariable Quantity.



Therefore, having biſected AB with the Perpendicular IMN, take MN and MR equal, each, to Half the Side *(e)* of the given Square; draw NR, and, from A to the Perpendicular MI, draw AK equal to NR; and upon the Center M, at the Diſtance MK, let an Arch be deſcribed; which will meet the Circumference of the given Circle in the Point C, required. For AC, BC, and MC being drawn, it is evident that $\overline{AC}|^2+\overline{BC}|^2=2\overline{AM}|^2+2\overline{CM}|^2=2\overline{AM}|^2+2\overline{MK}|^2=2\overline{AK}|^2=2\overline{RN}|^2=4\overline{RM}|^2=e^2$. In this Problem it is requiſite that MC $(=\sqrt{\frac{1}{2}e^2-\overline{AM}|^2})$ ſhould be greater than the Difference, and leſs than the Sum, of the

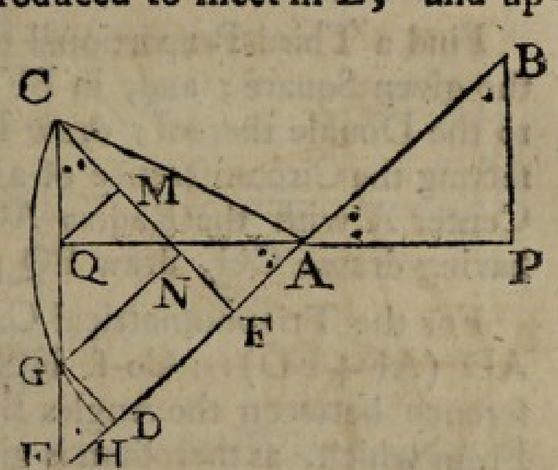
Radius

Radius of the given Circle and the Diſtance of the Point M from its Center.

PROBLEM LIII.

To draw a Line PQ *thro' a given Point* A, *ſo that the Rectangle of two Perpendiculars, falling thereon from two other given Points,* B *and* C, *may be of a given Magnitude.*

Let BA and CQ be produced to meet in E; and upon BE let fall the Perpendicular CF; which, and the Segment AF, are both given, by the Poſition of the propoſed Points.



Put, therefore, CF $=a$, AF$=b$, AB$=c$, and EF$=x$: Then will CE$=\sqrt{aa+xx}$: And it will be (*by ſimilar Triangles*),

as CE : EF : : AB : BP : $=\frac{cx}{\sqrt{aa+xx}}$;

and CE : EF : : AE : EQ $=\frac{x\times\overline{b+x}}{\sqrt{aa+xx}}$:

Whence CQ (= CE−EQ) $=\sqrt{aa+xx}-\frac{bx+xx}{\sqrt{aa+xx}}=\frac{aa-bx}{\sqrt{aa+xx}}$:

And, conſequently, BP × CQ $=\frac{cx\times\overline{aa-bx}}{aa+xx}=g^2$; ſuppoſing g^2 to denote the given Magnitude of the propoſed Rectangle.

From this Equation, by making $d=\frac{aac}{bc+gg}$, and $f=$

$\frac{aag}{bc+gg}$, the Value of x is found $= \frac{1}{2} d \pm \sqrt{\frac{1}{4} dd - fg}$. Whence every Thing elſe is readily determined.

Geometrically.

If QM be ſuppoſed parallel to AB, the Triangle CQM will, it is plain, be ſimilar to ABP; and conſequently $AB \times QM = BP \times CQ$ (*Elem.* 18. 3.) $= g^2$: Whence QM is given; and, from thence, the following Conſtruction.

Find a Third-Proportional to AB and the Side (g) of the given Square; and, in BF produced, take FD equal to the Double thereof; draw DG parallel to FC, interſecting the Circumference of a Circle, deſcribed from the Center A with the Radius AC, in the Point G; and, having drawn CG, draw PQ perpendicular thereto.

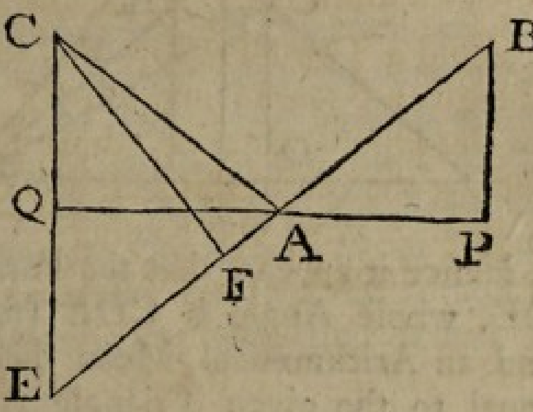
For the Trigonometrical Calculation, it will be AF : AD (AF+FD) : : Co-ſ. HGC : Co-ſ. HG, the Difference between the Angles BAP (HAQ) and CAQ: From which, as their Sum is given, the Angles themſelves will be known.—This Problem becomes impoſſible when FD is greater than FH; that is, when the propoſed Rectangle is greater than Half the Rectangle under AB and FH.

PROBLEM

PROBLEM LIV.

Two Lines AB, AC, *drawn from the ſame Point* A, *being given both in Poſition and Length*; *to draw another Line* PQ *thro' that Point*, *ſo that two Perpendiculars* BP, CQ, *falling thereon from the Extremes of the two given Lines*, *may form two Triangles* ABP, ACQ, *equal to each other.*

Let BA and CQ be produced to meet in E; and upon BE let fall the Perpendicular CF: And put $CF = a$, $AF = b$, $AB = c$, and $EF = x$. The Areas of ſimilar Triangles being, in Proportion, as the Squares of their homologous Sides; and both the Triangles AEQ, ABP being ſimilar to ECF; we have $\overline{CE}|^2$ $(aa+xx)$: $\frac{1}{2}CF \times EF$ $(\frac{1}{2}ax)$: : $\overline{AB}|^2$ (c^2) : Area $ABP = \frac{\frac{1}{2}ac^2x}{aa+xx}$.

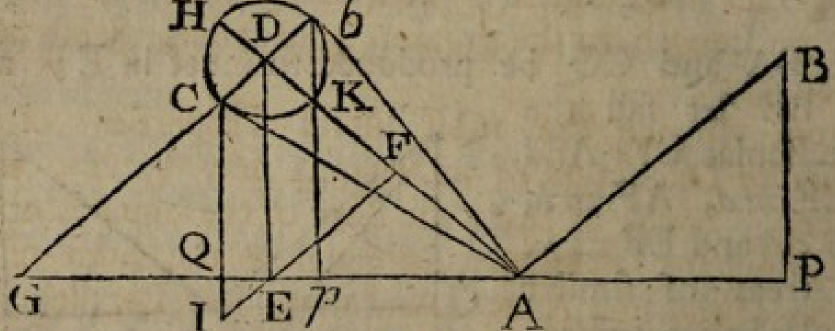


And, $aa+xx : \frac{1}{2}ax : : \overline{x+b}|^2$ $(=\overline{AE}|^2)$: Area $AEQ = \frac{\frac{1}{2}ax \times \overline{x+b}|^2}{aa+xx}$. This, taken from the Area AEC $(= \frac{1}{2}a \times \overline{x+b})$ leaves the Area $ACQ = \frac{\frac{1}{2}a \times \overline{x+b} \times \overline{aa-bx}}{aa+xx}$.

Which being equal to ABP, by Hypotheſis, we therefore have $\overline{x+b} \times \overline{aa-bx} = c^2x$: From whence, making $d = \frac{cc-aa}{b} + b$, x is found $= \sqrt{aa + \frac{1}{4}dd} - \frac{1}{2}d$.

Geometrically.

If A*b* be ſuppoſed perpendicular, and equal, to AB, and *bp* perpendicular to PQ, the Triangle A*bp*, being ſimilar to ABP, will alſo be equal to it; and conſequently equal to ACQ: And, if theſe equal Triangles be, ſucceſſively, taken from the Trapezium AQC*b*, the Remainders *pb* CQ, and AC*b* will likewiſe be equal.



Hence it appears that the Parallelogram under C*b* and DE, whoſe Angle is CDE (ſuppoſing DE parallel to, and an Arithmetical Mean between, CQ and *bp*) is equal to the given Triangle AC*b*; and conſequently the Altitude Thereof equal to half That of the Triangle. Whence it is evident that the Point E muſt fall, ſomewhere, in a Line FI drawn thro' the Middle of AD (or AC) parallel to *b*C (*Elem.* 2. 2). But the Point E, ſince the Angle ADE is a Right-one, will likewiſe fall in the Circumference of a Semi-circle deſcribed upon the Diameter AD (*Elem.* 11. 3). And therefore FE, being a Radius, muſt be equal to AF; and conſequently DG = AD; ſuppoſing DC produced to meet AQ in G.

Therefore, in order to the Geometrical Conſtruction, having made A*b* perpendicular, and equal to, AB, and drawn AD to the middle of C*b* (*as above intimated*) let DG, in DC produced, be taken equal to AD; and from G, through A, draw GP, *and the Thing is done.*

It often happens that the Demonſtration of a Geometrical Conſtruction, to be the moſt neat and elegant, proceeds upon Principles very different from Thoſe whereby we firſt arrived at ſuch Conſtruction. The

Caſe

Case above is an Instance of it: Where, from the similar Triangles, it is manifest that GQ (Ap) : QC : : Gp (AQ) : pb; and therefore $\frac{1}{2}$ AQ × QC = $\frac{1}{2}$ Ap × pb = $\frac{1}{2}$ BP × AP. *Q. E. D.*

Note. If AB and AC be two Semi-conjugate Diameters of an Ellipsis, then the Line PAQ, determined as above, will be the Position of the Greater Axis: And, if, upon the Diameter Cb, a Circle be described intersecting AD in H and K; then AH, and AK will be equal in Length to the greater, and lesser, Semi-Axis, respectively.

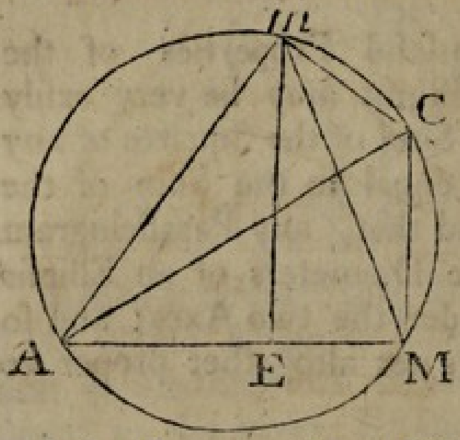
From whence the most useful Properties of the Conjugate Diameters of an Ellipsis may be very easily deduced. Such, as that, the Sum of the Squares of any two Conjugate Diameters, is equal to the Sum of the Squares of the two Axes: And that, any Parallelogram described about the Conjugate Diameters of an Ellipsis is equal to the Rectangle under the two Axes; and so forth. But these are Matters not altogether proper to be insisted on in this Place.

However it will not be improper to observe, that the last Problem is always possible, except in those two Cases, wherein the given Lines are perpendicular, and form one continued Line.

PROBLEM LV.

If, at the Extremes, M *and* m, *of two given Right-lines* AM *and* Am, *making a given Angle* MAm, *two Perpendiculars* MC *and* mC *be erected; 'tis proposed to find the Distance of their Intersection,* C, *from* A *the given angular Point.*

Since the Angles AMC and A*m*C are both Right Ones, the Circumference of a Circle, described upon the Diameter AC, will pass thro' M and *m*. Therefore, if M*m* be drawn, and a Perpendicular *m*E be let fall upon AM, the Triangles A*m*C and *m*EM will appear to be equiangular; because the Angles AC*m* and EM*m*, insisting on the same Arch A*m*, are equal; and A*m*C is equal to *m*EM, being both Right Angles.



If now the Ratio of *m*E to A*m* (which is given because the Angle EA*m* is given) be denoted by That of *s* to *r*; and the Ratio of AE to A*m* by That of *c* to *r*; we shall have AE $= \frac{c}{r} \times$ A*m*; and therefore $\overline{Mm}^2$ ($= \overline{AM}^2 + \overline{Am}^2 - 2AM \times AE$) $= \overline{AM}^2 + \overline{Am}^2 - \frac{2c}{r} \times AM \times Am$: Whence, by Reason of the similar Triangles above specified, it will be, $\overline{mE}^2 : \overline{mA}^2$ ($:: s^2 : r^2$) $:: \overline{Mm}^2 : \overline{AC}^2 = \frac{r^2}{s^2} \times \overline{AM}^2 + \overline{Am}^2 - \frac{2rc}{ss} \times AM \times Am$; whence AC is given.

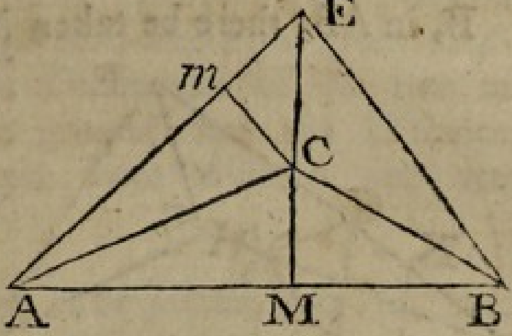
As to the Geometrical Construction, it is indicated by the Conditions of the Problem, without any sort of Argumentation.

PROBLEM

PROBLEM LVI.

To find a Point C, *from whence three Right-lines drawn to ſo many given Points* A, B, *and* E, *ſhall obtain the Ratio of three given Quantities* a, b, *and* e, *reſpectively.*

The given Points being joined; make AB $= f$, AE $= g$, and AC $= x$: Then, BC being $= \frac{bx}{a}$, and EC $= \frac{ex}{a}$ *(by Hypotheſis)* we alſo have AM $\left(= \frac{1}{2} \text{AB} + \frac{\text{AC}^2 - \text{BC}^2}{2\text{AB}}\right) = \frac{f}{2} + \frac{\overline{aa-bb} \times x^2}{2aaf}$ *(Elem.* 8. 2.); and Am $\left(= \frac{1}{2} \text{AE} + \frac{\text{AC}^2 - \text{EC}^2}{2\text{AE}}\right) = \frac{g}{2} + \frac{\overline{aa-ee} \times x^2}{2aag}$; ſuppoſing CM and Cm to be perpendicular to AB and AE, reſpectively.



Hence, putting $\frac{aaf}{aa-bb} = h$, $\frac{aag}{aa-ee} = k$, and $z = \frac{f}{2} + \frac{\overline{aa-bb} \times x^2}{2aaf}$ $(= \text{AM}) = \frac{f}{2} + \frac{xx}{2h}$, we get $xx = 2hz - fh$; and therefore Am $\left(= \frac{g}{2} + \frac{xx}{2k}\right) = \frac{g}{2} + \frac{2hz - fh}{2k} = l + \frac{hz}{k}$; by making $l = \frac{g}{2} - \frac{fh}{2k}$.

Now, ſince *(by the laſt Problem)* $\overline{\text{AM}}|^2 + \overline{\text{A}m}|^2 - \frac{2c}{r} \times \text{AM} \times \text{A}m = \frac{ss}{rr} \times \overline{\text{AC}}|^2$ (where *s* and *c* denote the Sine and Co-ſine of the given Angle MA*m*, to the Radius *r*) we ſhall, from hence, by ſubſtituting the above

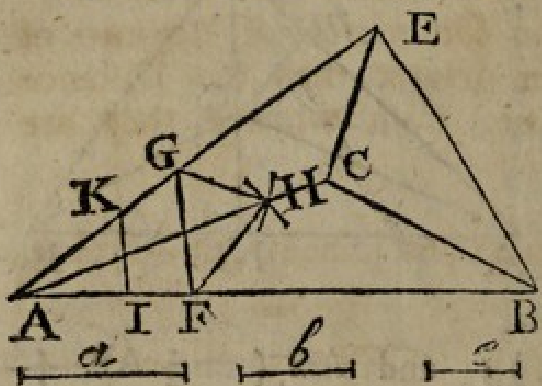
above Values of AM, A*m*, and AC, obtain the following Equation,

$$z^2 + \overline{l + \frac{hz}{k}}\Big|^2 - \frac{2cz}{r} \times \overline{l + \frac{hz}{k}} = \frac{ss}{rr} \times \overline{2hz - fh}.$$

From the Reſolution of which the Value of z, and from thence the Poſition of the Point D, will be determined.

Geometrically.

If, in AB, there be taken AF$=a$, and FH be drawn, to make the Angle AFH $=$ACB and meet AC in H; then, the Ratio of FH to AF (by Reaſon of the ſimilar Triangles AFH, ACB) being the ſame with That of BC to AC, it is evident that FH is given $=b$.



Moreover, if HG be drawn, making the Angle AHG $=$AEC, it will likewiſe appear, that both HG and AG are given:

For, ſince AG : AH : : AC : AE,
and AF : AH : : AC : AB;
it follows that AG : AF : : AB : AE; whence AG is given: And then it will be $a : e$ (: : AC : EC) : : AG : HG. Whence HG is given; and, from thence, the following

Conſtruction.

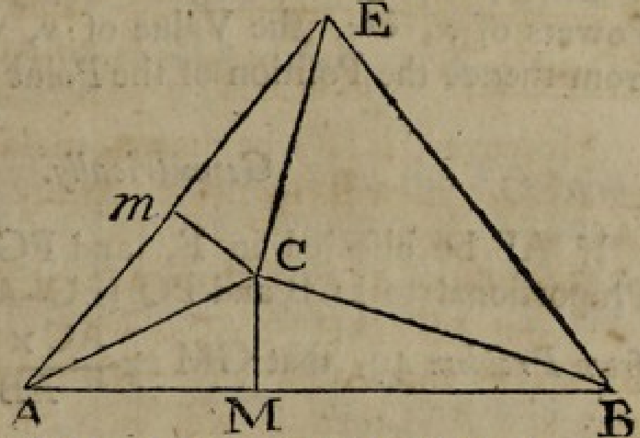
Take AF$=a$, and make the Angle AFG $=$ AEB; alſo take AI $= e$, and draw IK parallel to FG; moreover, from the Centers F and G, with the Intervals b and AK, deſcribe two Arcs, and from the Point H of their Interſection draw HF and HA, then a Line, BC, drawn to make the Angle ABC $=$ AHF, will cut AH, produced, in the Point required.

As to the Trigonometrical Solution, it is too obvious, from the Conſtruction, to need an Explanation. But it it will be proper to take Notice that the Problem itſelf becomes impoſſible, when the two Arcs, deſcribed from the Centers F and G, neither cut, nor touch, each other; that is, when the Diſtance FG is, either, greater than the Sum, or leſs than the Difference, of b and AK.

PROBLEM LVII.

Three Points, A, B *and* E, *being given, to determine a fourth Point* C, *ſo that Lines* (AC, BC, EC) *drawn from thence to the three former, may have given Differences.* (Provided the Difference of no two of the ſaid Lines be given greater than the Diſtance of the two given Points from whence they are drawn.)

Suppoſing the given Points to be joined, put $AB = a$, $AE = b$, $AC = x$, $BC = x + p$, and $EC = x + q$ (where p and q repreſent the given Differences). Then, if upon AB and AE, the Perpendiculars CM and Cm be let fall, it will be (by a known Property of Triangles) AB (a) : BC + AC $(2x+p)$: : BC — AC (p) : BM — AM $= \frac{2px+pp}{a}$: Whence AM $= \frac{1}{2}a - \frac{2px+pp}{2a}$: And, by the very ſame Argument, Am $= \frac{1}{2}b - \frac{2qx+qq}{2b}$. Which two Values, by putting $\frac{1}{2}a - \frac{pp}{2a} = f$, and $\frac{1}{2}b - \frac{qq}{2b} = g$ (for the ſake of Brevity) will become $f - \frac{px}{a}$, and $g - \frac{qx}{b}$.



Moreover,

Moreover, if the Sine, and the Co-ſine of the given Angle MAm, to the Radius r, be denoted by s and c, reſpectively, it will appear, from the Problem preceding the Laſt, that $\overline{AM}|^2 + \overline{Am}|^2 - \frac{2c}{r} \times AM \times Am = \frac{ss}{rr} \times \overline{AC}|^2$; or, (in Species) $\overline{f - \frac{px}{a}}|^2 + \overline{g - \frac{qx}{b}}|^2 - \frac{2c}{r} \times \overline{f - \frac{px}{a}} \times \overline{g - \frac{qx}{b}} = \frac{s^2x^2}{r^2}$. Which Equation may be reduced to $\overline{\frac{pp}{aa} + \frac{qq}{bb} - \frac{2cpq}{rab} - \frac{ss}{rr}} \times x^2 - \overline{\frac{fp}{a} + \frac{gq}{b} - \frac{cgp}{ra} - \frac{cfq}{rb}} \times 2x = -f^2 - g^2 - \frac{2cfg}{r}$: Whence, by ſubſtituting for the Coefficients of the Powers of x, &c. the Value of x, will be found, and from thence the Poſition of the Point C.

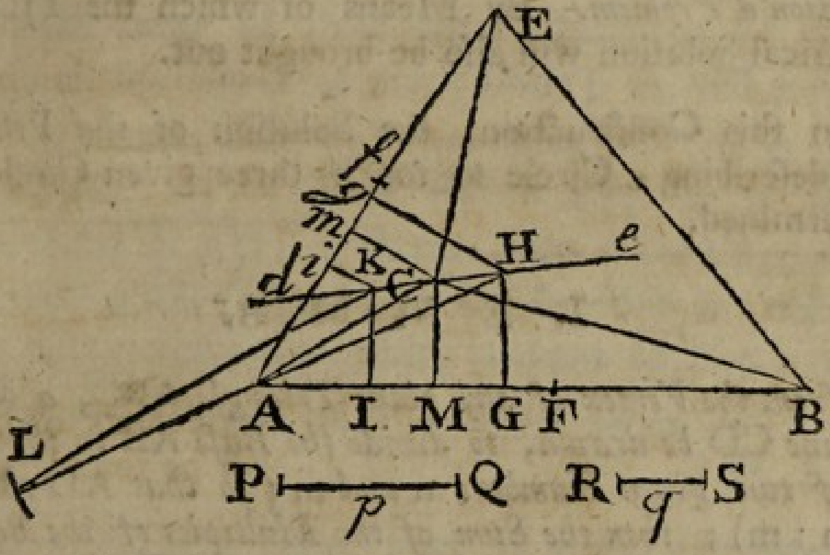
Geometrically.

If AB be biſected in F, and FG be taken a Third Proportional to 2AB and PQ (BC–AC), it is evident, *from Problem* 49, that $GM = \frac{AC \times PQ}{AB}$.

And, for the very ſame Reaſons, if AE be biſected in f, and fg be taken a Third Proportional to 2AE and RS (EC—AC), we ſhall alſo have $gm = \frac{AC \times RS}{AE}$. Whence it appears that GM is to gm, in the given Ratio of $\frac{PQ}{AB}$ to $\frac{RS}{AE}$, or of PQ to $\frac{AB \times RS}{AE}$, or, laſtly, of GI to gi; by taking GI = PQ, and $gi = \frac{AB \times RS}{AE}$.

If

If therefore GH and *g*H, and alfo IK and *i*K, be drawn, interfecting in H and K, it is manifeft that the required Point C muft fall, fomewhere, in the Right-line *de* paffing thro' H and K; fince, in this Cafe (and no other) it will be GM : GI (: : HC : HK) : : *gm* : *gi*; or, alternately, GM : *gm* : : GI : *gi*. By



Reafoning in the very fame Manner, from the Triangles ACB and BEC, the Pofition of another Right-line, wherein the Point C falls, may alfo be determined; whofe Interfection with *de* will confequently be the Point required.

But, inafmuch as the Cafe, from the Pofition of *de* thus given, is now reduced to our 49th *Problem*, the remaining Part of the Solution is likewife given, *from thence*, by a different Method: According to which, and what is above demonftrated, we have the following

Conftruction.

Having taken *fg* a Third Proportional to 2AE and RS; and *gi* a Fourth Proportional to AE, AB, and RS; and having alfo taken FG a Third Proportional to 2AB and PQ, and fet off GI = PQ, and drawn the Right-line *de* through the Interfections of the Perpendiculars

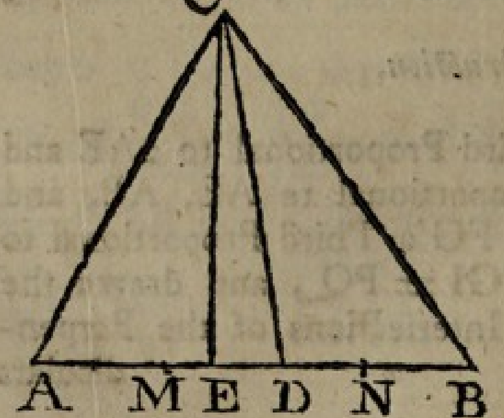
diculars GH, *g*H; IK, *i*K, *as above ſpecified*; let an Arch of a Circle, from the Center K, at the Diſtance of AB be next deſcribed, and through A let HN be drawn, meeting it in L; then a Right-line AC drawn parallel to That joining the Points L and K, will cut *de* in the Point required; *as is evident from the above-mention'd Problem.* By Means of which the Trigonometrical Solution will alſo be brought out.

In this Conſtruction, the Solution of the Problem for deſcribing a Circle to touch three given Circles, is determined.

L E M M A.

If from the Vertex of any plane Triangle ABC, *a Right-line* CD *be drawn, to divide the Baſe* AB *in the Ratio of two given Numbers* n *and* m (*ſo that* AD : BD : : n : m); *then the Sum of the Multiples of the Squares of the two Sides of the Triangle, whoſe Factors are the ſaid given Numbers, taken alternately, will be equal to the Sum of the Rectangle of the two Parts of the Baſe and the Square of the dividing Line, repeated as often as there are Units in the two propoſed Numbers* (*that is,* m *times* $\overline{AC}|^2 +$ n *times* $\overline{BC}|^2 = \overline{m+n}$ *times* $\overline{AD \times BD + \overline{DC}|^2}$)

For, let AD and BD be biſected in M and N, and upon AB let fall the Perpendicular CE; then will $\overline{AC}|^2 - \overline{DC}|^2 = AD \times 2ME$ (*Elem.* 8. 2). Whence it is evident that *m* times $\overline{AC}|^2 -$ *m* times $\overline{DC}|^2 =$ *m* times AD × 2ME.



And, by the very ſame Argument, it appears that *n* times $\overline{BC}|^2 -$ *n* times $\overline{DC}|^2$ is $=$ *n* times BD × 2NE.

Therefore,

Therefore, by adding these equal Quantities together, m times $\overline{AC}^2 + n$ times $\overline{BC}^2 - m+n$ times $\overline{DC}^2 =$ m times $AD \times 2ME + n$ times $BD \times 2NE$. But $m : n :: BD : AD$ (*by Hyp.*) $:: BD \times 2NE : AD \times 2NE$ (*Elem.* 1. 4); and therefore n times $BD \times 2NE = m$ times $AD \times 2NE$.

Let the latter of these equal Quantities be wrote above, in the Room of the Former; so will m times $\overline{AC}^2 + n$ times $\overline{BC}^2 - m+n$ times $\overline{DC}^2 = m$ times $AD \times 2ME + m$ times $AD \times 2NE = m$ times $\overline{AD \times 2ME + AD \times 2NE} = m$ times $AD \times AB = m+n$ times $AD \times BD$ (because, by Construction $m : m+n ::$ $BD : AB$); and consequently m times $\overline{AC}^2 + n$ times $\overline{BC}^2 = m+n$ times $\overline{DC}^2 + m+n$ times $AD \times BD$

Q. E. D

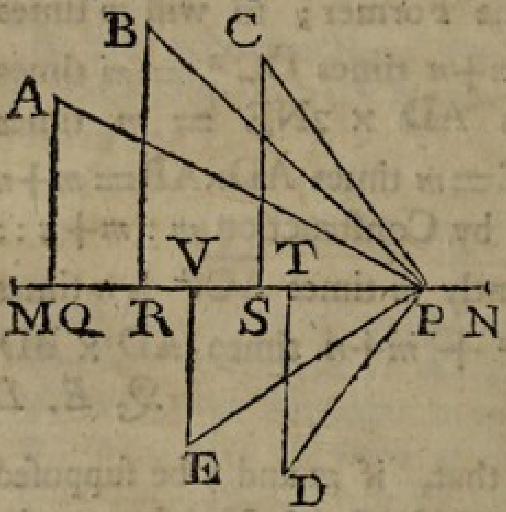
It is plain from hence that, if m and n be supposed to denote two given Lines, instead of Numbers, the Sum of the Solids $m \times \overline{AC}^2$ and $n \times \overline{BC}^2$ will be equal to the Sum of the Solids $\overline{m+n} \times \overline{DC}^2$ and $\overline{m+n} \times AD \times BD$.

PROBLEM

PROBLEM LVIII.

From any Number of given Points, A, B, C, D, &c. *to draw as many Lines* AP, BP, CP, &c. *to meet in a Right-line* MN *given by Position, so that the Sum of all their Squares may be a given Quantity.*

Upon the given Line, from all the given Points, let fall Perpendiculars, AQ, BR, CS, *&c.* which Perpendiculars, as well as the Distances between Them, are all given from the Position of MN: Put, therefore, $AQ=a$, $BR=b$, $CS=c$, $DT=d$, $EV=e$, $QR=r$, $QS=s$, $QT=t$, *&c.* Also put $QP=x$, and the given Sum of all the proposed Squares $=k^2$.



Then, RP being $=x-r$, $SP=x-s$, *&c.*

We shall have, *by Elem.* 7. 2.
$$\left.\begin{cases}\overline{AP}^2 = aa + xx \\ \overline{BP}^2 = bb + xx - 2rx + rr \\ \overline{CP}^2 = cc + xx - 2sx + ss \\ \overline{DP}^2 = dd + xx - 2tx + tt \\ \quad \&c. \qquad\qquad \&c.\end{cases}\right\} = k^2$$

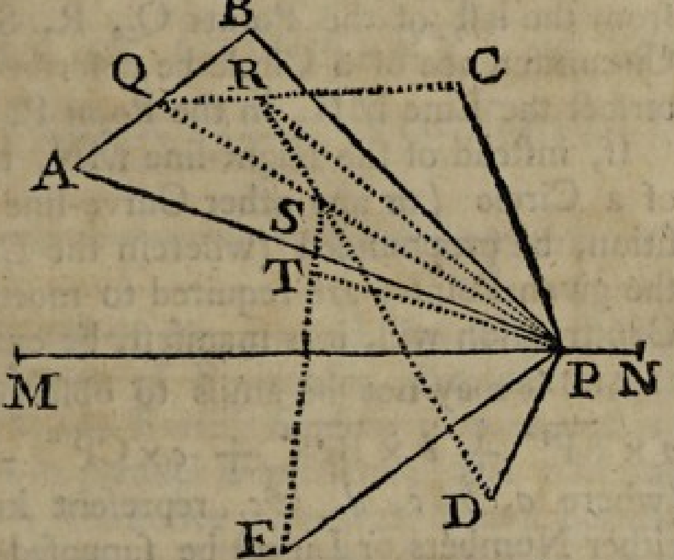
Put $aa+bb+cc+dd$ *&c.* $=\alpha^2$, $r+s+t$ *&c.* $=\beta$, and $rr+ss+tt$ *&c.* $=\gamma^2$; and let the Number of the given Points be denoted by n:
Then our Equation will be reduced to
$\alpha^2+nx^2-2\beta x+\gamma^2=k^2$:
From whence x is found $=\frac{\beta}{n}\pm\sqrt{\frac{k^2-\alpha^2-\gamma^2}{n}+\frac{\beta^2}{nn}}$.

Geometrically.

If PQ be drawn to bisect the Distance AB, then will $\overline{AP}^2+\overline{BP}^2=2AQ\times BQ+2\overline{QP}^2$, *by the Lemma.*

And,

And, if QC be drawn, and QR be taken equal to $\frac{1}{3}$ Thereof; then will $2\overline{QP}^2 + \overline{CP}^2 = 3QR \times CR$ $3\overline{RP}^2$, *by the Same.* Whence, by adding theſe equal Quantities together, and taking $2QP^2$ (common) away, we have $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 = 2AQ \times BQ + 3QR \times CR + 3\overline{RP}^2$.



Again, by drawing RD, and taking $RS = \frac{1}{4}$ Thereof we have $3\overline{RP}^2 + \overline{DP}^2 = 4RS \times DS + 4\overline{SP}^2$, *by the Lemma.*

And therefore $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 = 2AB \times BQ + 3QR \times CR + 4RS \times DS + 4\overline{SP}^2$, by the Addition of equal Quantities.

In the ſame manner, if SE be drawn, and ST be taken $= \frac{1}{5}$ Thereof, we ſhall have $4\overline{SP}^2 + \overline{EP}^2 = 5ST \times ET + 5\overline{TP}^2$. Whence, again, by the Addition of equal Quantities, $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 + \overline{EP}^2 = 2AQ \times BQ + 3QR \times CR + 4RS \times DS + 5ST \times ET + 5\,\overline{TP}^2$.

Hence it is evident (without proceeding further) that, let the Number of the given Points be what it will, the Square of the laſt of the Lines QP, RP, SP, TP, *&c.* drawn as above, will always be a *given* Quantity; becauſe the Sum of all the Rectangles $2AQ \times BQ$, $3QR \times CR$, $4RS \times DS$, *&c.* (or That of their Equals $AB \times QB$, $QC \times RC$, $RD \times SD$, *&c.*) is given, by the Poſition of the Points A, B, C, D, *&c.*

In order, therefore, to the Geometrical Conſtruction let a Rectangle be conſtituted *(by Elem.* 7. 6.) equal to the Exceſs of the given Sum of the Squares above the Sum of the ſaid Rectangles: Then find a Mean Proportional between the Length of the Rectangle, ſo determined, and that Part of its Breadth defined by the Number of the given Points; with which, as a Radius, from the laſt of the Points Q, R, S, T, *&c.* let the Circumference of a Circle be deſcribed; which will interſect the Line MN, in the Point P, required.

If, inſtead of the Right-line MN, the Circumference of a Circle (or any other Curve-line) given by Poſition, be propounded (wherein the Lines, drawn from the given Points, are required to meet), the Method of Conſtruction will, it is manifeſt, be exactly the ſame.

And it may not be amiſs to obſerve farther, that if $a \times \overline{AP}^2 + b \times \overline{BP}^2 + c \times \overline{CP}^2 + d \times \overline{DP}^2$, *&c.* (where a, b, c, d, *&c.* repreſent known Quantities, either Numbers or Lines) be ſuppoſed given, the *Locus* of the Point P will, *ſtill*, be the Circumference of a Circle, determined after the ſame manner, from the premiſed *Lemma*.

For, by making in this Caſe
$$\begin{cases} AQ : BQ :: b : a \\ QR : RC :: c : a+b \\ RS : DS :: d : a+b+c \\ ST : ET :: e : a+b+c+d, \end{cases}$$

and then proceeding as above, it will appear that $a \times \overline{AP}^2 + b \times \overline{BP}^2 + c \times \overline{CP}^2 +$ *&c.* is equal to $\overline{a+b} \times AQ \times QB + \overline{a+b+c} \times QR \times RC + \overline{a+b+c+d} \times RS \times SD + \overline{a+b+c+d+e} \times ST \times TE + \overline{a+b+c+d+e} \times \overline{TP}^2$: Whence it is evident that TP is a given Quantity.

PART

PART III.

WHEREIN

The THEORY of GUNNERY, or the MOTION of PROJECTILES, is confidered.

IT is, ufually, taken for granted, by *Thofe* who treat of the Motion of Projectiles, that the Force of Gravity near the Earth's Surface is every-where the fame, and acts in parallel Directions; and that the Effect of the Air's Refiftance upon very heavy Bodies, fuch as Bombs and Cannon Balls, is too fmall to be taken into Confideration.

That the Error arifing from the Suppofition of Gravity acting uniformly, and in parallel Lines, muft be exceeding fmall, is very obvious; becaufe, even, the greateft Diftance of a Projectile above the Surface of the Earth, is inconfiderable in Comparifon of its Diftance from the Center, to which the Gravitation tends: But then, on the other hand, it is very certain, that the Refiftance of the Air, to very fwift Motions, is much greater than it has been commonly reprefented.

Neverthelefs, if the Amplitude of the Projection, anfwering to one given Elevation, be firft found by Experiment (which our Method fuppofes to be done) the Amplitudes in all other Cafes, where the Elevations and Velocities do not very much differ from the Firft, may be determined, to a fufficient Degree of Exactnefs, from the foregoing Hypothefes: Becaufe, in all fuch Cafes, the Effects of the Refiftance will be nearly as the Amplitudes *themfelves*; and, were They *accurately* fo, the Proportions of the

N 2 Amplitudes,

Amplitudes, at different Elevations, would then be the very ſame as *in vacuo*.

For this Reaſon, and to avoid having Recourſe to Principles and Calculations no-ways adequate to the Experience and Underſtanding of Beginners, for whoſe Uſe this little Tract is chiefly intended, I ſhall, in what follows, conform to the Method of other Writers, ſo far, as to take no Notice of the Air's Reſiſtance; but conſider the Motions as performed *in vacuo*.

Now, in order to form a clear Idea of the Subject here propoſed, the Path of every Projectile is to be conſidered as depending on two different Forces; that is to ſay, on the impellent Force, whereby the Motion is firſt began (and would be continued, in a Right-line), and on the Force of Gravity, by which the Projectile, during the whole Time of its Flight, is continually urged downwards, and made to deviate more and more from its firſt Direction.

As whatever relates to the Track and Flight of a Ball (neglecting the Reſiſtance of the Air) is to be determined from the Action of theſe two Forces, it will be proper, before we proceed to conſider their joint Effect, to premiſe ſomething concerning the Nature of the Motion produced by Each, when ſuppoſed to act alone, independent of the Other; to which End the two firſt, of the four following *Lemmas*, are premiſed.

LEMMA

LEMMA I.

Every Body, after the impreſs'd Force, whereby it is put in Motion, ceaſes to act, continues to move uniformly in a Right-line; unleſs it be interrupted by ſome other Force or Impediment.

This is a Law of Nature, and has its Demonſtration from Experience and Matter-of-Fact.

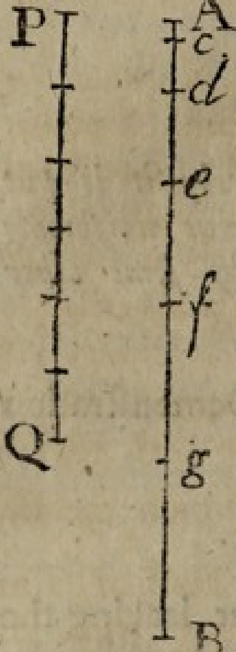
COROLLARY.

It follows from hence that a Ball, after leaving the Mouth of the Piece, would continue to move along the Line of its firſt Direction, and deſcribe Spaces therein proportional to the Times of their Deſcription, were it not for the Action of Gravity; whereby the Direction is changed, and the Motion interrupted.

LEMMA II.

The Motion, or Velocity, acquired by a Ball, in freely deſcending from Reſt, by the Force of an uniform Gravity, is as the Time of the Deſcent; and the Space fallen thro', as the Square of that Time.

The firſt Part of the *Lemma* is extremely obvious: For, ſince every Motion is proportional to the Force whereby it is generated, That generated by the Force of an uniform Gravity muſt be as the Time of the Deſcent; becauſe the *whole Effort* of ſuch a Force is proportional to the Time of its Action; that is, as the Time of the Deſcent.



To demonſtrate that the Diſtances deſcended are proportional to the Squares of the Times, let the Time of falling thro' any propoſed Diſtance AB be repreſented by the Right-line PQ; which conceive to be divided into an indefinite Number of very ſmall, equal, Particles, repreſented, each, by the *Symbol m*; and let the Diſtance deſcended in the Firſt of them be A*c*; in the Second *cd*; in the Third *de*; and ſo on.

Then the Velocity acquired being always as the Time from the Beginning of the Deſcent, it will at the Middle of the firſt of the ſaid Particles be repreſented by $\frac{1}{2}m$; at the Middle of the Second, by $1\frac{1}{2}m$; at the Middle of the Third, by $2\frac{1}{2}m$, &c. Which Values conſtitute the Series $\frac{m}{2}$, $\frac{3m}{2}$, $\frac{5m}{2}$, $\frac{7m}{2}$, $\frac{9m}{2}$, &c.

But, ſince the Velocity, at the Middle of any One of the ſaid Particles of Time, is an exact Mean between the Velocities at the two Extremes thereof, the correſponding Particle of the Diſtance AB may be therefore conſidered as deſcribed with that mean Velocity: And ſo, the Spaces A*c*, *cd*, *de*, *ef*, &c. being, reſpectively, equal to the abovementioned Quantities $\frac{m}{2}$, $\frac{3m}{2}$, $\frac{5m}{2}$, $\frac{7m}{2}$, &c. it follows, by the continual Addition of Theſe, that the Spaces A*c*, A*d*, A*e*, A*f*, &c. fallen thro' from the Beginning, will be expreſſed by $\frac{m}{2}$, $\frac{4m}{2}$, $\frac{9m}{2}$, $\frac{16m}{2}$, $\frac{25m}{2}$, &c. Which are, evidently, to one another in proportion, as 1, 4, 9, 16, 25, &c. that is, as the Squares of the Times. *Q. E. D.*

COROL.

COROLLARY.

Seeing the Velocity acquired in any Number *(n)* of the aforesaid, equal, Particles of Time (measured by the Space that would be described in one single Particle) is represented by *n* times *m*, or *n m*, it will therefore be, as 1 Particle of Time, is to *n* such Particles, so is *nm*, the said Distance answering to the former Time, to the Distance, n^2m, corresponding to the Latter, with the same Celerity, acquired at the End of the said *n* Particles. Whence it appears that the Space $\frac{n^2m}{2}$ *(found above)* thro' which the Ball falls, in any given Time *n*, is just the Half of That (n^2m) which might be uniformly described with the last, or greatest, Celerity, in the same Time.

SCHOLIUM.

It is found, by Experiment, that any heavy Body, near the Earth's Surface (where the Force of Gravity may be consider'd as uniform) descends about 16 Feet, from Rest, in the first Second of Time.

Therefore, as the Distances fallen thro' are proved above to be, in Proportion, as the Squares of the Times; it follows that, as the Square of 1 Second, is to the Square of any given Number of Seconds, so is 16 Feet, to the Number of Feet a heavy Body will freely descend in the said given Number of Seconds. Whence the Number of Feet descended in any given Time will be found, by multiplying the Square of the Number of Seconds by 16.

Thus the Distance descended in 2, 3, 4, 5, *&c.* Seconds, will appear to be 64, 144, 256, 400 F, *&c.* respectively.

Moreover, from hence, the Time of the Descent thro' any given Distance will be obtained, by dividing the said Distance, in Feet, by 16, and extracting the square Root of the Quotient; or, which comes to the ſame Thing, by extracting the ſquare Root of the whole Diſtance, and then taking $\frac{1}{4}$ of that Root for the Number of Seconds required. Thus, if the Diſtance be ſuppoſed 2640 Feet; then, by either of the two Ways, the Time of the Deſcent will come out 12, 84 Seconds, or 12″ : 50‴.

It appears alſo (*from the Corol.*) that the Velocity *per* Second (in Feet), at the End of the Fall, will be determined, by multiplying the Number of Seconds in the Fall by 32: Thus it is found that a Ball, at the End of 10 Seconds, has acquired with a Velocity of 320 Feet *per* Second.——After the ſame Manner, by having any two of the four following Quantities, *viz.* the *Force*, the *Time*, the *Velocity*, and *Diſtance*, the other Two may be determined: But this not being abſolutely neceſſary in what follows (though equally uſeful in other Diſquiſitions) I ſhall put down the ſeveral Rules, or Equations below, in a Note *, to be taken, or omitted, at Pleaſure.

LEMMA

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* *Let the Space freely deſcended by a Ball, in the firſt Second of Time (which is as the accelerating Force) be denoted by* f; *alſo let* t *denote the Number of Seconds wherein any Diſtance,* d, *is deſcended; and let* v *be the Velocity,* per *Second, at the End of the Deſcent: Then will*

$$v = 2ft = 2\sqrt{fd} = \frac{2d}{t}$$

$$t = \sqrt{\frac{d}{f}} = \frac{v}{2f} = \frac{2d}{v}$$

$$d = ftt = \frac{vv}{4f} = \frac{tv}{2}$$

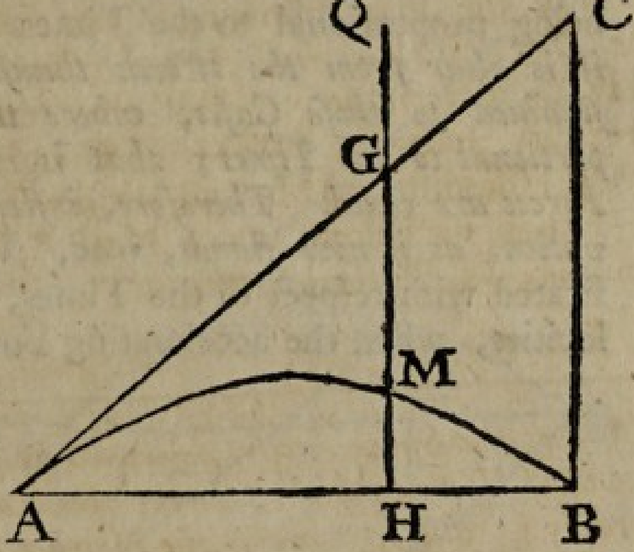
$$f = \frac{d}{tt} = \frac{v}{2t} = \frac{vv}{4d}.$$

All which Equations are very eaſily deduced from the two original Ones, $d = ftt$, *and* $v = 2ft$, *already demonſtrated; the Former in the Propoſition itſelf; and the Latter, in the Corollary to it; by which it appears that the Meaſure*

L E M M A III.

A Ball, projected in the Direction of a Right-line AC *making an Angle with the Horizon* AB, *is, by Means of its Gravity, deflected continually, more and more, from its first Direction; but the Celerity with which it approaches any Perpendicular* (BC) *to the Horizon, is neither increased nor decreased by the Action of Gravity.*

For, let a Line HMQ, perpendicular to the Horizon, be conceived to move parallel to itself, towards BC, along with the Ball: Then, as the Gravity always acts in this Line, it can have no Effect in making it, either, move faster or slower towards BC; but is wholly employ'd in drawing down the Projectile along the same, from its first Direction AC; and thereby causing it to describe a Curve-line AMB.



COROL.

Measure of the Velocity at the End of the first Second is 2f; *whence the Velocity* (v) *at the End of* t *Seconds must consequently be expressed by* 2f × t *or* 2ft.

Having proceeded thus far, I shall here take the Opportunity to point out, and rectify an inadvertent Expression, at p. 230 *of my Book of Fluxions, relating to this Subject.* — *It is there said, by way of Remark,* that, whatever Ratio the Times have with Respect to the Distances descended, *&c.* the same also will the Velocities have, being

COROLLARY.

Hence it appears that the Projectile, at the End of a given Time, is in the very same vertical Line HG, as it would be in, if Gravity was not to act; and that the horizontal Distance AH, as well as the Hypothenuse AG, is proportional to the Time wherein the Projectile actually moves through the Arch AM, corresponding to the said Distance.

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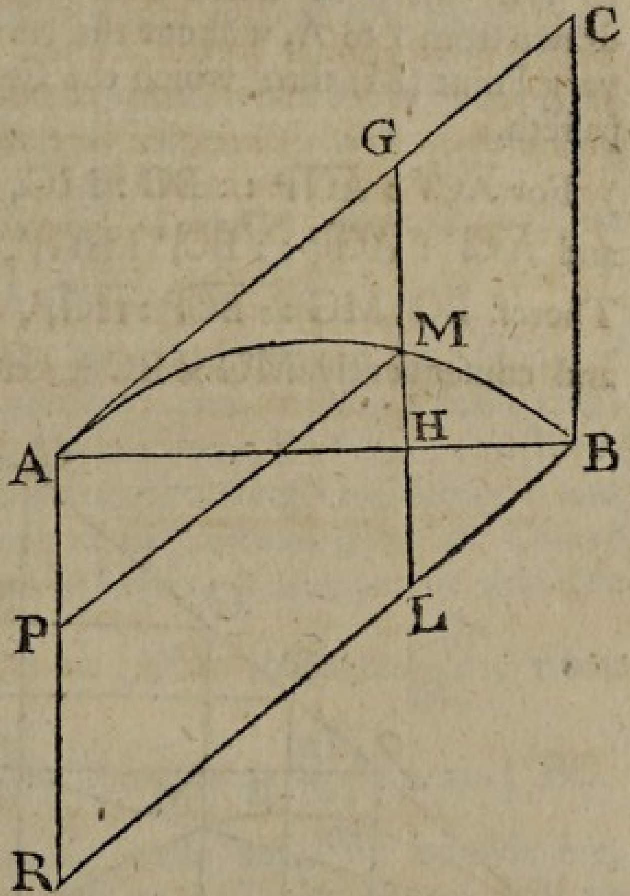
being proportional to the Times. *Which Observation, it is clear from the Words themselves, ought to be restrained to those Cases, where the Velocities are proportional to the Times; that is, where the accelerating Forces are equal. Therefore, instead of the said Observation, as it now stands, read,* What is above demonstrated with respect to the Times, holds also in the Velocities, when the accelerating Forces are equal.

LEMMA

LEMMA IV.

The Deflections arising from the Action of Gravity, or the vertical Distances, MG, BC, *intercepted by the Path of the Ball and the Line of Direction, are in proportion, to one another, as the Squares of the corresponding Parts* AG, *and* AC, *of the said Line of Direction.*

Conceive a Line GML to be carried along, parallel to BC, so that its extreme Point G may trace out the Line of Direction AC, in the very same manner as the Projectile itself, would describe it, were it not to be deflected therefrom by the Action of Gravity: Then, since, by the preceding Lemma, the Projectile is always in the Line GML, and the Force of Gravity is, wholly, employ'd in urging it downwards, along that Line; the Effect produced by the said Force, or the Distance MG of the



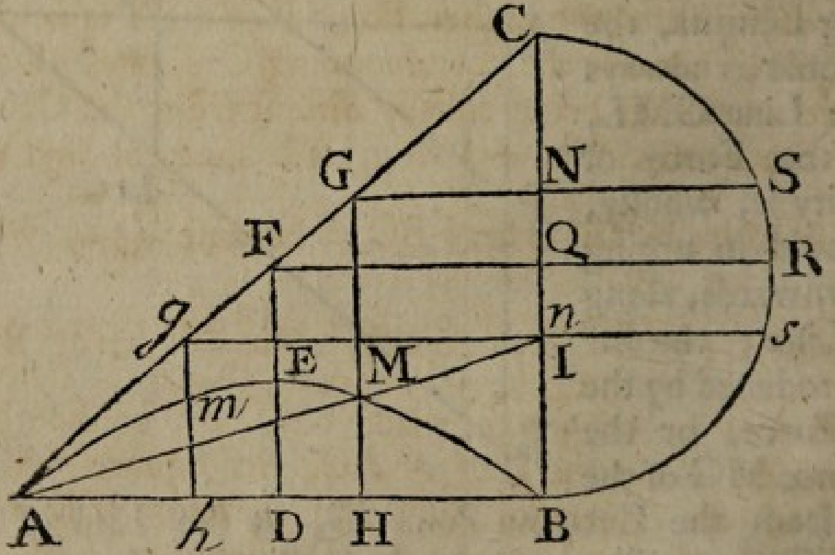
Ball from the Extreme Point G, at the End of any given Time, will consequently be the very same, as if the Line GML (instead of being carry'd uniformly towards CB) was to have continued in its first Position AR, and the Ball suffered to descend from Rest along that Line; the Force employ'd being the same in both Cases. But it is proved, *in Lemma* 2, that the Spaces AP and AR that would be described in descending, freely, from

from Reſt, are as the Squares of the Times; therefore their Equals GM and CB are likewiſe as the Squares of the ſame Times; that is, of the Times wherein AM and AMB are deſcribed: Which are to each other as $\overline{AG}|^2$ to $\overline{AC}|^2$ *(by Corol. to Lem. 3.)*

COROL. I.

If B be taken as the Point where the Projectile impinges on the Horizon AB, and in BC (perpendicular to AB) there be taken CI = GH; then a Right-line, drawn from I to A, will cut the vertical Line HG in the very Point (M) thro' which the Center of the Projectile paſſeth.

For $\overline{AC}|^2 : \overline{AG}|^2 :: BC : MG$, *as above.*
and $\overline{AC}|^2 : \overline{AG}|^2 :: \overline{BC}|^2 : \overline{HG}|^2$, *by ſim. Triangles.*
Theref. $BC : MG :: \overline{BC}|^2 : \overline{HG}|^2$, *by Equality of Ratios:*
and conſequently $MG \times BC = \overline{HG}|^2$.



This, turn'd into an Analogy, gives BC : HG :: HG : MG; and ſo likewiſe is BC to CI (becauſe of the parallel Lines). Whence it is evident that HG and CI are equal to each other, in all Poſitions of HG: By Means whereof, as many Points in the Curve, as you pleaſe, may be determined.

COROL.

COROL. II.

If GNS be drawn, parallel to AB, interſecting BC in N, and meeting the Circumferenec of a Semicircle, deſcribed upon BC, in S; it will further appear, that the Height (HM) of the Ball is everywhere a Third-proportional to BC and NS. For, ſince BC : HG : : HG : MG, it follows, by Diviſion, that, BC : CN : : HG (BN) : HM $= \frac{CN \times BN}{BC} = \frac{\overline{NS}|^2}{BC}$ (*Elem.* 16. 3). Hence it is evident that the Projectile will be at its greateſt Heighth, when it has performed, juſt, the Half of its Flight; ſince it is well known that NS (and conſequently $\frac{\overline{NS}|^2}{BC}$) is the greateſt poſſible when it coincides with the Radius QR.

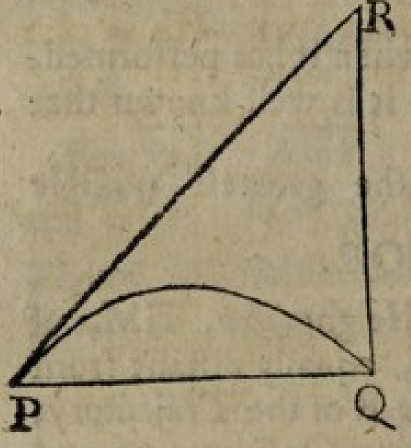
It appears moreover that the Heights *hm*, HM, of the Ball, in any two vertical Lines, equally diſtant from That paſſing thro' the higheſt Point E of the Trajectory, are equal; becauſe the correſponding Ordinates *ns* and NS are equal, as being equally diſtant from the Center Q of the Semi-circle. —Laſtly, it is apparent that the greateſt Altitude DE is $= \frac{1}{4}$ BC; becauſe $\frac{\overline{NS}|^2}{BC}$, when NS coincides with QR, becomes $= \frac{\frac{1}{4}\overline{BC}|^2}{BC} = \frac{1}{4}$ BC. Which may be otherwiſe made out, by conſidering, that, as AF is but the Half of AC, its Square will be only $\frac{1}{4}$ of the Square of BC; and therefore FE only $\frac{1}{4}$ of BC (*by the Lem*). And ſo, DF being $= \frac{1}{2}$ BC (*by ſim. Triang.*) DE, as well as FE, muſt be $= \frac{1}{4}$ BC.

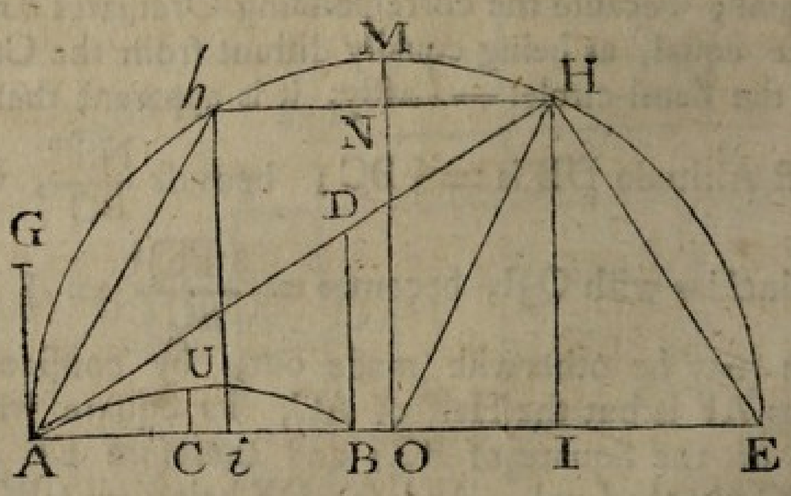
PROPOSITION

PROPOSITION I.

The Amplitude, or horizontal Range of a Piece, with a given Charge of Powder, at an Elevation of 45 Degrees, being known, from Experiment; to determine the Elevation ſo as to hit an Object, at a given Diſtance, on the plane of the Horizon; the Quantity of Power remaining the ſame.

Let PQ be the given Amplitude, at an Elevation (QPR) of 45 Degrees; let AB be the given Diſtance of the Object, and BAD the required Elevation: In AB, produced, take AO=PQ, with which as a Radius, from the Center O, let a Semi-circle AME be deſcribed, and let AD, produced, meet the Periphery thereof in H; join E, H, and make DB and HI perpendicular to AE.





Since the Charge of Powder, or the Velocity at both the Elevations QPR and BAD, is ſuppoſed to be the ſame, the Times of Flight, during which the Diſtances PR and AD would be uniformly deſcribed with that Velocity, will therefore be to each other, directly as the ſaid Diſtances; and conſequently $\overline{PR}^2 : \overline{AD}^2 ::$

RQ

RQ : DB (*by Principles already explained. vid. Lem.* 3 *and* 4).

But $\overline{PR}|^2$ ($= \overline{PQ}|^2 + \overline{RQ}|^2$) $= 2\overline{RQ}|^2$ (becauſe the Angle P ($= 45°$) $=$ R). Hence the above Proportion becomes $2\overline{RQ}|^2 : \overline{AD}|^2 :: RQ : DB$; from which we have $\overline{AD}|^2 = 2RQ \times DB = AE \times DB$, *by Conſtruction.*

Moreover, from the ſimilar Triangles ABD and AHE, we have AE : EH : : AD : DB : Whence $EH \times AD = AE \times DB = \overline{AD}|^2$ (*p. above*) and conſequently EH $=$ AD : And ſo, the Triangles EHI and ADB being equiangular, it is plain that HI is alſo equal to AB. Whence follows this eaſy

Conſtruction.

With an Interval equal to the given Amplitude, at the Elevation of 45°, let a Semi-circle AME be deſcribed; make OM perpendicular to the Diameter thereof, in which take ON equal to the given Diſtance AB, and thro' N, parallel to AE, draw HN*h*, interſecting the Circumference in H and *h*; then either of the Directions AH, or A*h*, will anſwer the Conditions of the Problem. *Q. E. I.*

CoROL. I.

If OH be drawn, the Angle EOH will be $=$ 2EAH (*Elem.* 9. 3.) and it will be, as OH (PQ) : HI (AB) : : *Radius : Sine* EOH; that is, in Words, as the given Amplitude, at 45° Elevation, is to any other propoſed Amplitude, ſo is the *Radius*, to the *Sine* of twice the Elevation correſponding to the Latter. From whence it is evident, that the horizontal Amplitudes, at different Elevations, are to one another as the *Sines* of the Doubles of the ſaid Elevations; and that, the Amplitude of the Projection at an Elevation of 45 Degrees (when HI coincides with MO) is the greateſt poſſible.

CoROL.

COROL. II.

Since it is found above that $AE \times DB$ ($= \overline{AD}|^2$) $=$ $\overline{EH}|^2$, it follows, that $\overline{AE}|^2 : \overline{EH}|^2 :: \overline{AE}|^2 : AE \times DB :: AE : DB :: \frac{1}{2} AO$ ($\frac{1}{4}$ AE) : CV ($\frac{1}{4}$ DB. *vid. Corol.* 2. *to Lem.* 4.) that is, as the Square of the *Radius*, is to the Square of the *Sine* of the Angle of Elevation, ſo is Half the greateſt Horizontal Amplitude, to the greateſt Altitude of the Projectile.

Hence it appears, that the Diſtance which the Ball would aſcend, if projected in a vertical Direction *(uſually call'd the Impetus)* is juſt one Half of the greateſt Amplitude; ſince, in this Caſe, the *Sine* of the Elevation becomes equal to the *Radius*.

Therefore, as a Body *(in vacuo)* aſcends and deſcends with the ſame Celerity; and ſeeing the Diſtance AG, expreſſing the perpendicular Aſcent, is as the Square of the Celerity at A, *(p. Lem.* 2) ; it follows that the greateſt horizontal Amplitude AO, being $=2AG$, is alſo as the Square of the ſame Celerity.

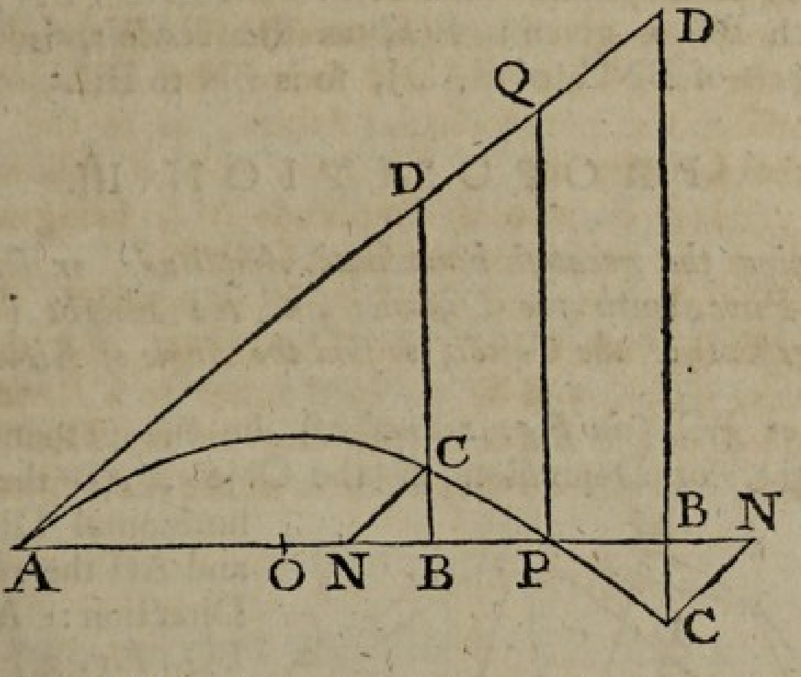
From whence, and *Corol.* 1, it is manifeſt, that the Amplitudes, when both the Elevations, and the Velocities, differ, will be to each other in a *Ratio* compounded of the *Ratio* of the *Sines* of the Double Elevations, and the *Duplicate Ratio* of the Velocities.

PROPOSITION

PROPOSITION II.

The greatest horizontal Amplitude, and the Angle of Elevation being given; to find at what Distance the Piece ought to be planted, to hit an Object, whose Height above, or Depression below, the Level of the Piece is also given.

Let BC be the perpendicular Height, or Depression of the proposed Object, and AB the required Distance; let BC, produced, meet the Line of Direction AD in D, and let P be the Place where the Path of the Projectile (produced) meets the Level of the Piece: Make PQ perpendicular to AP, and CN parallel to AD.



By the last Proposition it will be, as *Radius* : *Sin.* 2A : : the greatest horizontal Amplitude, to the Distance AP; which, therefore, is known.

Moreover, it appears from the fourth Proportion, *in Corol.* 1, *to Lem.* 4, that PQ : BD : : BD : CD.

But $\left\{\begin{array}{l} PQ : BD : : AP : AB \\ BD : CD : : AB : AN \end{array}\right\}$ *by similar Triangles.*

Therefore, *by Equality of Ratios*,
AP : AB : : AB : AN;

 Whence

Whence AP : PB : AB : BN *(by Division)* and consequently $AP \times BN = AB \times PB$.

Let AP be now bisected, in O; then $AB \times BP$ (or its Equal $AP \times BN$) being $= \overline{AO}^2 \backsim \overline{OB}^2$ *(Elem.* 6. 2). we therefore have $\overline{OB}^2 = \overline{AO}^2 \mp AP \times BN = AO \times \overline{AO \pm 2BN}$; whence AB is likewise known. *Q. E. I.*

COROLLARY.

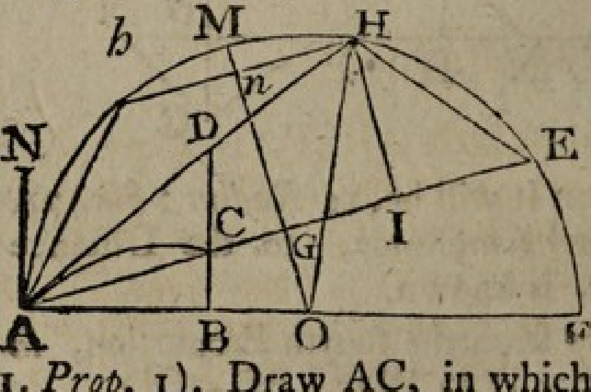
If the Amplitude AP, the Elevation PAQ, and the Distance AB of any perpendicular BCD from the Place of Projection A, be supposed given; then the Height, or Depression of the Ball in that Perpendicular, may, from hence, be found.

For it is proved that AP : BP : : AB : BN; from which BN is given: But, as the *Radius*, is to the *Tangent* of BNC (or BAD), so is BN to BC.

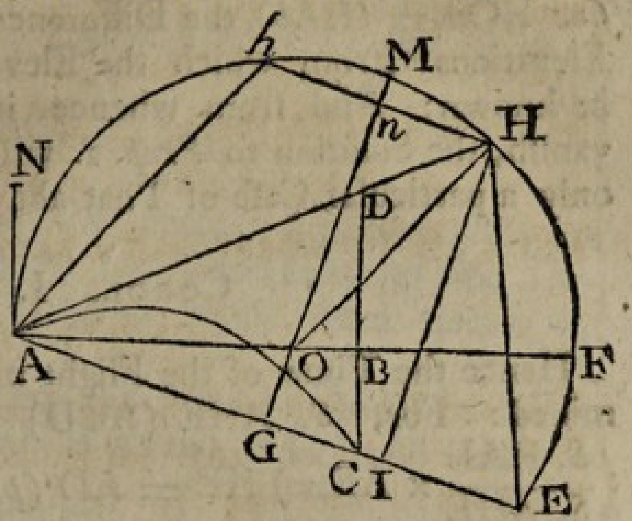
PROPOSITION III.

Having the greatest horizontal Amplitude, or Range of a Piece, with the Distance and the Height (or Depression) of the Object, to find the Angle of Elevation.

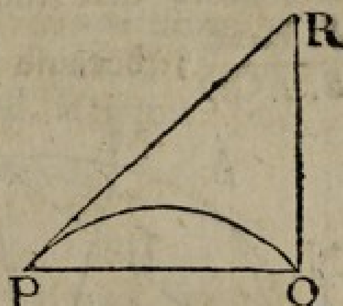
Let BC (*in Fig.* 1 *and* 2) be the perpendicular Height, or Depression of the Object, AB the given horizontal Distance, and AH the required Direction: Also let PQ (*Fig.* 3.) be the greatest horizontal Amplitude, answering to 45° of Elevation, (*vid. Corol.* 1. *Prop.* 1). Draw AC, in which produced (if need be) take AG = PQ; make MGO perpendicular to AG, meeting AB produced (if necessary) in O; and from the Center O, with the Interval OA, let the Circumference of a Circle be described, intersecting AG produced,



duced, in E; join E, H, and let HI, AN, and QR, be perpendicular to AE, AO, and PQ reſpectively, and let BC be produced to meet AH in D.



It appears, from what has been already delivered, that $\overline{AD}^2 : \overline{PR}^2 :: DC : RQ$. Whence, $\overline{PR}^2$ being $= 2\overline{PQ}^2 = 2\overline{AG}^2 = \frac{1}{2}\,\overline{AE}^2$, and $RQ = PQ = \frac{1}{2}\,AE$ *(by Conſtruction)* we have $\overline{AD}^2 : \frac{1}{2}\,\overline{AE}^2 :: DC : \frac{1}{2}\,AE$; and conſequently $\overline{AD}^2 = AE \times DC$.



But, the Triangles ADC and AEH being equiangular (becauſe $ADC = DAN = E$, and DAC common) we likewiſe have $AD : DC :: AE : EH$; and therefore $AE \times DC = AD \times EH = \overline{AD}^2$ *(p. above)* whence $EH = AD$; and ſo, the Triangles ADB and EHI being equiangular, we likewiſe have $HI = AB$; and from thence this eaſy

Conſtruction.

Having deſcribed the Circle AHEF, as above directed, and drawn MG perpendicular to AE, take therein G*n* equal to the given horizontal Diſtance AB, and thro' *n*, parallel to AE, draw H*h*, cutting the Circumference of the Circle in H and *h*; then either of the Directions AH, or A*h*, will anſwer the Conditions of the Problem.

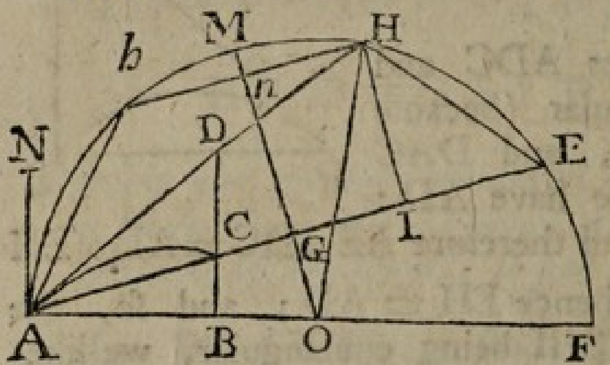
From this Conſtruction we have the following Calculation, *viz.* As $AB : BC :: AG\ (PQ) : GO$; which, added to, or ſubtracted from G*n* (AB), gives O*n*.

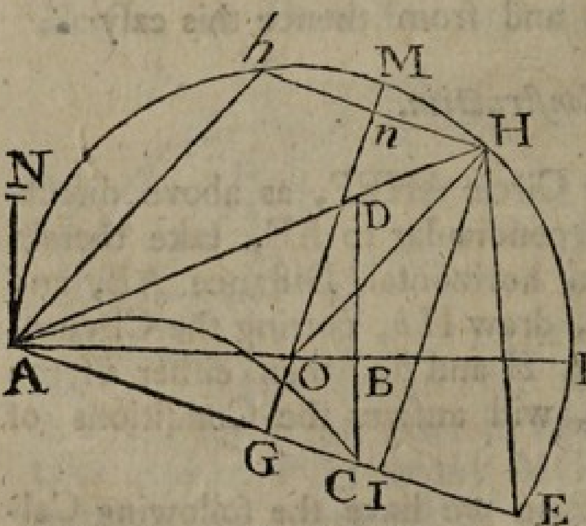
O 2 Then,

Then, as AG (PQ) : O*n* : : *Co-ſine* of OAG : *Co-ſine* HO*n* $=$ (HA*h*) the Difference of the two required Elevations: From which the Elevations *themſelves* will be known: And from whence, if BC be ſuppoſed to vaniſh, the Solution to *Prop.* I. will be obtained; being only a particular Caſe of That above. *Q. E. I.*

COROL. I.

Hence the Time of the Flight may likewiſe be determined: For, *S.* AHE (ACD) : *S.* HAE : : AE : $\left(\frac{S.\ HAE}{S.\ AHE} \times AE =\right)$ HE $=$ AD (*p. above*) Which being proportional to the Time of the Flight (*vid. Lem.* 3.) it follows that the ſaid Time will, always, be as $\frac{S.\ HAE}{S.\ ACD}$; becauſe AE is ſuppoſed conſtant. Therefore, if the Time of the perpendicular Aſcent, or Deſcent, thro' the Impetus AN, be found, and denoted by *T*, it is evident that the Time ſought will be truly repreſented by $\frac{S.\ HAE}{S.\ ACD} \times 2T$; that is, the *Co-ſine* of the Object's Elevation above, or Depreſſion below the Level of the Piece, will be to the *Sine* of the Elevation of the Piece above the Object, as twice the Time of the perpendicular Aſcent, or Deſcent, is to the true Time of the Flight.





COROL.

COROL. II.

If the Elevation of the Piece, together with the Diſtance, and the Height (or Depreſſion) of the Object, be given, the *Impetus* may, from hence, be alſo found.

For, firſt, it will be AB : BC : : *Radius* : *Tang.* BAC; whence (as BAD is given) EAH will likewiſe be known.

Then, *S.* EAH (CAD) : *S.* AHE (ACD) : : HE (AD) : AE. Alſo *S.* ADC : *Radius* : : AB : AD.

Therefore, by compounding theſe Proportions, we have *S.* CAD × *S.* ADC : *Rad.* × *S.* ACD : : AB : AE; which is 4 times the required Impetus *(vid. Corol.* 3. *to Prop.* 1).

COROL. III.

Moreover, if the Elevation, and the Impetus be given, the Amplitude of the Projection on an aſcending, or deſcending Plane ACE, whoſe Inclination is given, may from hence be derived.

For, *S.* AHE (ACD) : *S.* EAH (CAD) : : AE : EH (AD). And *S.* ACD : *S.* ADC : : AD : AC.

By the Compoſition of which Proportions we have □ *S.* ACD : *S.* CAD × *S.* ADC : : AE : AC; whence AC is given.

COROL. IV.

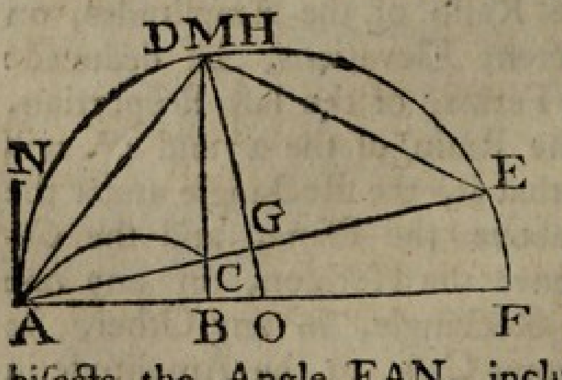
Hence, alſo, may the Ratio of the Amplitudes, on the ſame Plane, at different Elevations, be deduced: For the firſt and third Terms, of the laſt Proportion, continuing invariable, the Ratio of the 2^{d} and 4th. will likewiſe be invariable; that is, the Rectangle under the *Sine* of the Elevation above the Plane, and the *Coſine* of the Elevation above the Horizon, in any one Caſe, will be to the like Rectangle, in any Other, as the Amplitude in the former Caſe, to the Amplitude in the Latter.

COROL. V.

But, if the Elevation be ſuppoſed conſtant, and the Plane's Inclination to vary; then, ſince, by the above Proportion, AC is *univerſally* $= \frac{S.\ CAD \times S.\ ADC}{\square\ Sine\ ACD} \times$ AE (where AE and the Angle ADC are ſuppoſed conſtant) it follows that the Amplitude will be, barely, as $\frac{S.\ CAD}{\square\ S.\ ACD}$; that is, *inverſely*, as the ſquare of the *Coſine* of the Inclination of the Plane, apply'd to the *Sine* of the Elevation above the Plane.—If both the Inclinations, and the Elevations, differ, it will appear, from the ſame Equation, that, the Amplitude will be, *univerſally*, as the Rectangle of the *Sine* of the Elevation above the Plane, and the *Co-ſine* of the Elevation above the Horizon, apply'd to the Square of the *Co-ſine* of the Plane's Inclination.

COROL. VI.

Since it is proved that HI is, always, equal to AB, it is evident that, when the Former coincides with MG, and thereby becomes a *Maximum*, the Latter will alſo be a *Maximum:* In which Circumſtance AC will likewiſe be a *Maximum*; and the Point D will then coincide with M and H *(as in the annexed Figures)* becauſe AD and EH are always equal to each other.

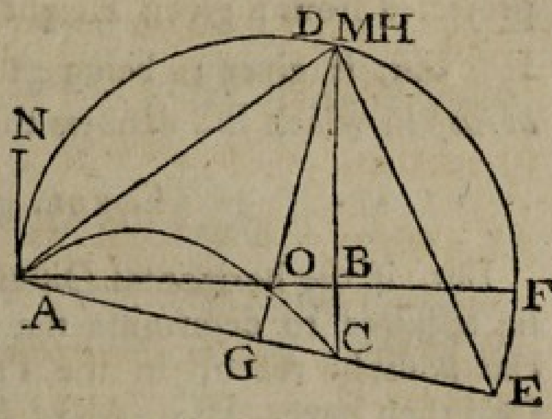


Therefore, ſince, in this Caſe, the Angle HAE is (= E) = NAH, it appears that the Amplitude, on any inclin'd Plane, will be the greateſt poſſible when the Line of Direction AH biſects the Angle EAN, included between the Plane and the Zenith.

COROL.

COROL. VII.

Hence the greateſt Amplitude, on any inclin'd Plane, with a given Impetus, may be determined : For the right-angled Triangles AOG and HOB, having AO = HO, and the Angle O, common, are equal in all reſpects : Therefore it will be, as the *Tangent* of AHG (or BAH, the Piece's Elevation) is to the *Tang.* of CHG (or CAB, the Plane's Inclination) ſo is AG (twice the given Impetus) to the Difference CG between the ſaid Double Impetus and the Amplitude ſought.



COROL. VIII.

Hence, alſo, if the greateſt Amplitude, on an inclin'd Plane be given, the Impetus may be found : For, it will be as the *Radius*, is to the *Sine* of the Plane's Inclination BAC, ſo is the given Amplitude AC to the Difference (BC or CG) betwixt it and twice the Impetus. *Vid. Corol.* 2. *to Prop.* 1.

COROL. IX.

But if, inſtead of the Plane's Inclination, the perpendicular Height, or Depreſſion of the Object above, or below the Level of the Piece, be given ; then, AC being $= AG \mp BC$, and $\overline{AB}|^2$ $(= \overline{AC}|^2 - \overline{BC}|^2) =$ $\overline{AG}|^2 \mp 2AG \times BC$, the greateſt Diſtance AB, at which the Ball can poſſibly hit the Object, will therefore be $= \sqrt{AG \times \overline{AG \mp 2BC}}$. From which, as all the Sides of the Triangle ABC are given, the Angle

BAC will likewiſe be known; and, from thence, the Elevation, BAH, *by Corol.* 6.

From the latter of the two Caſes here conſider'd (where the Object is ſuppoſed below the Level of the Piece, *as in Fig.* 2.) the greateſt Amplitude of a Ball, projected from a given Height above the Plane of the Horizon, is given; being $= \sqrt{AG \times \overline{AG + 2BC}}$, *(as above)* in which BC denotes the given Height.

COROL. X.

But, if the horizontal Diſtance AB is given, and it be required to determine the greateſt Height the Ball can poſſibly reach, in the Perpendicular BCD; we ſhall then have, HG (AB) : AG : : *Radius : Tang.* of the Elevation (BAH or AHG); and, as *Radius : Tang.* BAC ($= 2BAH \backsim 90$) : : AB : BC; which therefore is known. But, becauſe $AC = AG \mp BC$, and $\overline{BC}|^2 = \overline{AC}|^2 - \overline{AB}|^2$, the Value of BC will, *alſo*, be truly expreſſed by $\frac{AG^2 \backsim AB^2}{2AG}$.

COROL. XI.

Laſtly, if there be given the perpendicular Height, or Depreſſion, of the Object, and its horizontal Diſtance, in order to determine the Elevation, and the *leaſt* Impetus, to hit the Object: Then it will be as AB : BC : : *Radius : Tang.* BAC; whence the Elevation BAH is alſo known, *by Corol.* 6: And, as *Radius : Tang.* AHG (BAH) : : HG (AB) : AG; the Half of which is the Impetus, *by Prop.* 1. *Corol.* 2.

Here

Here follow the practical Solutions of the several Cases depending on the foregoing THEORY.

I. *Of Projections made on the Plane of the Horizon.*

PROBLEM I.

The greatest Amplitude of a Piece being known (from Experiment) to find the Amplitude, at any proposed Elevation.

SOLUTION.

As the Radius, is to the Sine of double the proposed Elevation, so is the given, to the required, Amplitude *(by Prop.* 1. *Cor.* 1.)

Ex. Let the gr. Amp. be 8000 Feet, and the given Elev. 30° : 16′.

Then, as Radius ————	10. 0000
to Sine 60° : 32′ ————	9. 9398
so is —— 8000 ————	3. 9030
Amp. req. 6965 F. ————	3. 8428

PROBLEM II.

The Impetus, or the greatest Amplitude (which is the Double thereof) being known, to find the Elevation, to strike an Object at a given Distance.

SOLUTION.

As the greatest Amplitude, is to the given Distance; so is the Radius, to the Sine of the Double Elevation *(by Prop.* 1. *Corol.* 1).

Ex. Let the gr. Amp. be 7500 F. and the given Distance 5620 F.

Then,

Then, as 7500 ————	3. 8750
is to 5620 ————	3. 7497
so is Radius ————	10. 0000
to the sine of 48° : 32′, or 131° : 28′ —	9. 8747

Therefore 24° : 16′ is the lower, and 65° : 44′ the higher Elevation, required.

PROBLEM III.

The Angle of Elevation, and the Distance of an Object, being given, to find the Impetus, so as to strike the Object.

SOLUTION.

As the Sine of twice the Elevation, is to the Radius, so is the Distance of the Object to twice the Impetus (*by Prop.* 1).

Ex. Let the Elev. be 32° : 12′ and the given Dist. 6500. F.

Then, as Sine 64° : 24′ ————	9. 9551
is to Radius ————	10. 0000
so is – 6500 ————	3. 8129
to – – 7208 ————	3. 8578

Whence 3604 is the Impetus required.

PROBLEM IV.

The Amplitude, at any one known Elevation being given, to find the Amplitude at any other known Elevation.

SOLUTION.

As the Sine of Double the first Elevation, is to the Sine of Double the Second, so is the Amplitude at the Former, to that at the Latter, *(by Prop.* 1).

Ex. Let the first Elev^{n}. be 25° : 12′, the 2^{d}. 36° : 15′, and the given Amp. 5250. F.

Then,

Then, as Sine 50° : 24′ —	Co. Ar.	0. 1132
to Sine — 72° : 30′	————	9. 9794
so is — — 5250 F.	————	3. 7201
to the Amp. req. 6498 F.	————	1|3. 8127

PROBLEM V.

The Amplitude of the Projection, with a given Quantity of Powder, being known; to find the Requisite of Powder, so as to strike an Object at a given Distance; the Elevation remaining the same.

SOLUTION.

As the given Amplitude, is to the proposed Distance, so is the given Weight, or Quantity of Powder, to the Quantity sought, nearly *.

Ex. Suppose the Requisite of Powder to throw a Shot 4000 Feet, at 45° Elevation, to be 16 lb. What Quantity is necessary, to strike an Object at the Distance of 5000 Feet.

Here it will be, as 4000 : 5000 : : 16 : 20 lb. the Quantity sought.

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* *In this Solution, the Velocity communicated to the Ball, is supposed to be in the Subduplicate Ratio of the Quantity of Powder; which is not strictly true, especially in large Charges; since a considerable Part of the Powder, in such Cases, is blown out, unfired: There are, besides, other Reasons to be assigned, why, the Velocity cannot be exactly in the Proportion above specified.*

PROBLEM

PROBLEM VI.

The Distance of the Object, and the Elevation of the Piece, being given; to determine the Time of the Flight.

SOLUTION.

As the Radius is to the Tangent of the Elevation, so is the given Distance of the Object, *in Feet*, to the Square of 4 times the Number of *Seconds*, required *(vid. p.* 184 *and* 187).

Ex. Let the Elevation be 32°, and the Distance of the Object 5280 Feet.

Then, Radius ———————	10. 0000
Tang. 32° ———————	9. 7958
Dist. 5280 ———————	3. 7226
	1|3. 5184
57. 44 ———————	1. 7592

¼ of which, or 14^s, 36, is the Time required.

But, when the Elevation is 45 Degrees (which is commonly the Case, in Throwing Bombs, &c.) then ¼ of the square Root of the Distance, *in Feet*, will give the Number of *Seconds* taken up in the Flight. The Knowing of which will be of Use in adjusting the Fuse.

II. *Of Projections, when the Object fired at, is above, or below, the Level of the Piece.*

PROBLEM VII.

The horizontal Distance, and the Angle of Elevation, or Depression of an Object, being given, with the Elevation of the Piece; to find the Impetus, so as to hit the Object.

SOLUTION.

As the Rectangle of the Sine of the Elevation of the Piece above the Object, and the Co-sine of its Elevation

vation above the Horizon, is to the Rectangle under the Radius and the Co-ſine of the Object's Elevation, or Depreſſion; ſo is $\frac{1}{4}$ of the given horizontal Diſtance of the Object, to the Impetus required (*by Prop.* 3. *Corol.* 2).

Ex. Let the horizontal Diſtance of the Object be 5600 Feet, and its Elevation 8° : 15′; and let the Elevation of the Piece be 32° : 30′: Then,

24° : 15′, Co-ar. of its Sine ——	0. 3865	to be added
32° : 30′, Co-ar. of its Co-ſ. ——	0. 0740	
Radius ——————	10. 0000	
8° : 15′, its Co-ſine ————	9. 9955	
1400 F. ——————	3. 1461	
4000 F. the Imp. req^d. ————	2|3. 6021	

PROBLEM VIII.

The horizontal Diſtance, and the Angle of Elevation, or Depreſſion of an Object, being given, together with the Impetus; to find the Elevation of the Piece, to hit the Object.

SOLUTION.

As the Radius is to the Tangent of the Object's Elevation, or Depreſſion, ſo is twice the Impetus to a fourth Number; which add to, or ſubtract from, the given horizontal Diſtance, according as the Object is elevated, or depreſſed: Then ſay,

As twice the Impetus is to the Sum, or Remainder, ſo is the Co-ſine of the given Elevation, or Depreſſion, to the Co-ſine of an Angle; which added to, and ſubtracted from, the Angle included between the Object and the Zenith (or vertical Point), gives the Double of the Complements of two different Elevations, whereby the Ball may hit the Object (*See Prop.* 3).

Ex. Let the horizontal Diſtance of the Object be 5600 Feet, and its Elevation 8° : 15′; and let the given Impetus be 4000 F.

Then,

Then, as Radius ————	10. 0000
is to Tang. 8° : 15′ ————	9. 1613
so is 8000 ————	3. 9031
to — 1160 ————	1|3. 0644

This added to 5600 gives 6760:

Therefore, as 8000 ————	3. 9031
is to —— 6760 ————	3. 8299
so is Co-s. 8° : 15′ ————	9. 9955
to the Co-s. 33 : 15 ————	9. 9223

Which, added to, and subtracted from 81° : 45′, gives 115° : 00′, and 48° : 30′, respectively: The Halves of which are 57° : 30′, and 24° : 15′; whose Complements 32° : 30, and 65° : 45′, are the two Elevations required.

PROBLEM IX.

The Impetus, and the Angle of Elevation, being given; to find at what Distance the Piece ought to be planted, to hit an Object, whose Distance above, or below, the Level of the Piece, is also given.

SOLUTION.

As the Radius, is to the Sine of twice the given Elevation, so is the Impetus, to half the horizontal Amplitude, at that Elevation *(by Prop.* 1).

And, as the Radius, is to the Co-tangent of the Elevation, so is twice the perpendicular Height, or Depression, of the Object, to a Fourth-Proportional; which take from, or add to, half the horizontal Amplitude, according as the Object is elevated, or depressed; then find a Mean Proportion between the Half-Amplitude, and the Sum, or Remainder; which, added to the said Half-Amplitude, gives the Distance sought *(by Prop.* 2).

Ex. Let the Impetus be 3000 Feet, the Elevation 40°, and the Heighth of the Object 200 Feet.

Then,

Then, as Radius ————	10. 0000
to Sine – 80° ————	9. 9933
so is — 3000 ————	3. 4771
to —— 2954 ————	1|3. 4704

And, as Radius ————	10. 0000
is to Co-t. 40° ————	10. 0762
so is —— 400 ————	2. 6021
to —— 477 ————	1|2. 6783

Now, the Difference between the two Numbers above found is 2477; whose Log. being added to that of 2954, and the Sum divided by 2, the Quotient will be the Log. of 2705, the required Mean Proportional: Whence the Distance sought comes out 5659 Feet.

III. *Of Projections on Planes, inclin'd to that of the Horizon.*

PROBLEM X.

The Inclination of the Plane, and the Elevation and Impetus of the Piece, being known; to find the Amplitude of the Projection.

SOLUTION.

As the Square of the Co-sine of the Plane's Inclination to the Horizon, is to the Rectangle of the Sine of the Elevation above the Plane and the Co-sine of the Elevation above the Horizon, so is 4 times the Impetus, to the Amplitude of the Projection *(by Prop.* 3. *Cor.* 3).

Ex. 1. Let the Impetus be 4000 Feet, the Elevation 32° : 30′, and the Ascent of the Plane 8° : 15′. Then the Elevation above the Plane will be 24° : 15′; and the Operation as follows.

8° : 15′, Co-Arith. of its Co-ſ. —	0. 0045	to be added
The ſame repeated ———	0. 0045	
24° : 15′ — its Sine ———	9. 6135	
32° : 30′ — its Co-ſ. ———	9. 9260	
16000 F. — its Log. ———	4. 2041	
5658 Feet the Amp. req^d^. ——	2|3. 7526	

Ex. 2. Let the Elevation and Impetus be the ſame as in the laſt Example; but let the Plane in this Caſe have a Deſcent of 8° : 15′ (inſtead of an equal Aſcent).

Then the Operation will ſtand thus,

8° : 25′, Co-Ar. of its Co-ſ. ———	0. 0045	to be added
The ſame repeated ———	0. 0045	
40° : 45′, its Sine ———	9. 8147	
32° : 20′, its Co-ſine ———	9. 9260	
16000 F. its Log. ———	4. 2041	
8992 F. Amp. req^d^. ———	2|3. 9538	

PROBLEM XI.

The Inclination of the Plane, the Elevation of the Piece, and the Amplitude of the Projection, being given; to find the Impetus.

SOLUTION.

As the Rectangle of the Sine of the Elevation above the Plane and the Co-ſine of the Elevation above the Horizon, is to the Square of the Co-ſine of the Plane's Inclination, ſo is the given Amplitude, to 4 times the required Impetus *(by Prop.* 3. *Cor.* 3).

Ex. Suppoſe the Plane to have an Aſcent of 8° : 15′, and that the Amplitude thereon, at an Elevation of 32° : 30′, is 5658 Feet. Then, the Elevation above the Plane being 24° : 15′, we ſhall have,

24° : 15′ Co-Ar. of its Sine	0. 3865	to be added
32° : 30′ Co-Ar. of its Co ſ.	0. 0740	
8° : 15 its Co-ſ.	9. 9955	
The ſame	9. 9955	
5658 F. its Log.	3. 7526	
16000 F.	2|4. 2041	

¼ of which, or 4000 Feet, is the Impetus, req[d].

PROBLEM XII.

The Inclination of the Plane, the Impetus, and the Amplitude being given, to determine the Elevation.

SOLUTION.

As the Radius is to the Co-ſine of the Plane's Inclination, ſo is the given Diſtance, on the Plane, to the horizontal Diſtance correſponding: From which, the Impetus, and the Plane's (or Object's) Elevation or Depreſſion, the Elevation of the Piece may be found, *by Prob.* 8.

Ex. Let the Plane have an Aſcent of 8° : 15′, and let the given Amplitude thereon be 5658 Feet, ſuppoſing the *Impetus* to be 4000 Feet.

Then, as the Radius	10. 0000
is to the Co-ſine of 8° : 15′	9. 9955
ſo is 5658 Feet	3. 7526
to – 5600 Feet	1|3. 7481

the horizontal Diſtance of the Place where the Ball impinges: As for the reſt of the Operation, it is exactly the ſame as in the Example to *Prob.* 8; for which Reaſon it will be needleſs to repeat it here.

PROBLEM XIII.

Having the Angle of Elevation, or Depreſſion of the Object, together with the Elevation, and Impetus of the Piece; to determine the Time of the Flight.

SOLUTION.

As the Co-ſine of the Elevation, or Depreſſion of the Object, is to the Sine of the Elevation of the Piece above the Object, ſo is Half the ſquare Root of the Number of Feet in the *Impetus* given, to the requir'd Number of Seconds in the Flight.

Ex. Suppoſe the Elevation of the Object to be $8^\circ : 30'$; That of the Piece 45°; and the *Impetus* 3600 Feet.

Then, the ſquare Root of 3600 being 60, we have

As the Co-ſine $8^\circ : 30'$ — Co-Ar.	0. 0048
is to the Sine $36^\circ : 30$ ———	9. 7744
ſo is 30 ———	1. 4771
to 18,04 Seconds req^{d}. ———	1|1. 2563

IV. *Of the* MAXIMA *and* MINIMA, *in the Motion of Projectiles.*

PROBLEM XIV.

The greateſt horizontal Amplitude being given, to find the greateſt Amplitude on a Plane whoſe Inclination to the Horizon is alſo given.

SOLUTION.

Take half the Angle included between the Plane and the Zenith; the Complement of which is the required Elevation *(by Prop.* 3. *Cor.* 6). Then ſay, as the Tangent of the ſaid Elevation, is to the Tangent of the Plane's Inclination, ſo is the given Amplitude, to the Difference between it and the Amplitude ſought *(by Prop.* 3. *Corol.* 7).

Ex.

Ex. Let the greateſt horizontal Range be 8000 Feet, and the Inclination of the Plane 12° : 30, deſcending.

Here the Angle included between the Plane and the Zenith, or vertical Point, being 102° : 30′, the Half thereof will be 51° : 15′, and the Elevation 38° : 45′.

Therf. as Tang. 38° : 45′ ———	9. 9045
is to Tang. – 12° : 30′ ———	9. 3457
ſo is – – – – 8000 F. ———	3. 9031
to ——— — 2210 F. ———	3. 3443

Which, added to 8000 (becauſe the Plane deſcends), gives 10210 Feet, for the true Anſwer, in this Caſe.

PROBLEM XV.

The greateſt Amplitude, on an inclin'd Plane, being given, to find the greateſt Amplitude, on the Plane of the Horizon.

SOLUTION.

As the Radius, is to the Sine of the Plane's Inclination, ſo is the given Amplitude, to the Difference between *it* and the required Amplitude *(by Prop.* 3, *Corol.* 8).

Ex. Let the Inclination of the Plane be 12° : 30′, deſcending, and the given Amplitude 10210 Feet.

Then, as Radius ———————	10. 0000
is to *S.* 12° : 30′ ———————	9. 3353
ſo is – – 10210 ———————	4. 0090
to – — – 2210 ———————	3. 3443

Which taken from 10210, gives 8000 Feet, for the true Diſtance ſought.

PROBLEM XVI.

The Impetus, and the perpendicular Height, or Depreſſion, of an Object being given, to find the greateſt horizontal Diſtance at which the Ball can, poſſibly, hit the Object, and alſo the Elevation anſwering thereto.

SOLUTION.

Take the Difference of the two given Quantities, if the Object be elevated; but otherwiſe, their Sum; then find a Mean Proportional between the Impetus and ſaid Difference, or Sum; the Double of which will be the Diſtance ſought (*by Prop.* 3. *Cor.* 9).

For the Elevation, it will be, as the Diſtance thus found is to the Height, or Depreſſion of the Object, ſo is the Radius to the Tangent of an Angle; which added to, or ſubtracted from, 90 Degrees, reſpectively, gives the Double of the required Elevation (*by Prop.* 3. *Cor.* 9).

Ex. Let the Impetus be 4000 Feet, and the Depreſſion of the Object, below the Level of the Piece, 2210 Feet.

Here we are to take a Mean Proportional between 4000 and 6210; which is $= \sqrt{4000 \times 6210} = 4984$; whoſe Double 9968 is the required Diſtance.

Moreover, we have, as	9968	3. 9986
is to	2210	3. 3443
ſo is Radius		10. 0000
to Tang.	12° : 30	9. 3457

whence the Elevation appears to be 38° : 45.

From this Problem, the greateſt Amplitude of a Ball, projected from a given Height above the Level of the Horizon, is given.

PROBLEM XVII.

Having the horizontal Diſtance of the Object, together with its perpendicular Height, or Depreſſion, above or below the Level of the Piece; to determine the leaſt Impetus, poſſible, whereby the Ball may reach the Object; and alſo the Elevation correſponding.

SOLUTION.

As the horizontal Diſtance of the Object is to its perpendicular Height or Depreſſion, ſo is the Radius to the Tangent of an Angle; which, added to, or ſubtracted from 90 Degrees, gives the Double of the Elevation (*by Prop.* 3. *Corol.* 6).

And, as the Radius is to the Tangent of the Elevation, ſo is the given horizontal Diſtance, to twice the Impetus required (*by Prop.* 3. *Corol.* 11).

Ex. Let the horizontal Diſtance be 9968 Feet, and the Diſtance of the Object below the Level of the Piece 2210 Feet.

Then, as -- 9968	3. 9986
is to — — 2210	3. 3443
ſo is Rad.	10. 0000
to Tang. 12° : 30′	9. 3457

Therefore the Elevation is 38° : 45′; and it will be

As Radius	10. 0000
is to Tang. 38° : 45′	9. 9945
ſo is — 9968	3. 9986
to — 8000	3. 9031

The Half of which, or 4000 Feet, is the Impetus ſought.

PROBLEM XVIII.

To find the greatest Height a Ball can, possibly, reach in a Perpendicular to the Horizon; the Impetus, and the Distance of the Piece from the said Perpendicular, being given.

SOLUTION.

Find a Third Proportional to the Impetus and half the given Distance, which subtract from the Impetus, and the Remainder will be the Answer (*by Prop.* 3. *Corol.* 10).

Thus, if the Impetus be 4000 F. and the given Distance 3000 Feet, then it will be, as 4000 : 3000 : : 3000 : 2250; which taken from 4000, leaves 1750 Feet, for the greatest Height the Ball can possibly reach in the proposed Perpendicular, with that Impetus.

PART

PART IV.

EXHIBITING

A new, and very comprehensive Method for extracting the Roots of Equations in Numbers; by increasing the Dimensions of the unknown Quantity.

IN this Method (as in the common Method by converging Series's) the Value sought is, first of all, to be nearly estimated (either from the Equation itself, or from the Nature of the Problem whence it is derived); and some unknown Quantity (as z) must be assumed to express the Difference between that Value (which we will denote by r) and the true Value required: And then, by substituting $r \pm z$ instead of its Equal, in the given Equation, a new Equation will emerge, affected only with z and known Quantities.

The Equation being thus prepared for a Solution, multiply it into two, or more succeeding Terms of the Series $1 + Az + Bz^2 + Cz^3 + Dz^4$ &c. (according to the Degree of Exactness Necessary) and let the Coefficients of the homologous Terms of the new Equation hence arising, wherein the Square, and the next, higher Powers of z are concerned, be made equal to Nothing: By which Means the Value of A, &c. will be determined, and as many Terms of the Equation destroyed, at the same time, as there are Terms in the assumed Multiplicator *minus* one. And the Terms involving the superior Powers, which yet remain, being of very small Value, may also be rejected: Whence the Equation will be reduced to a Simple One: From which the Value of z will be found.

Ex. 1. Let the Equation propounded be $x^2 = 50$.

Here, x being something greater than 7, put $7 + z = x$; then the given Equation will become $49 + 14z + z^2 = 50$;

or, $-1 + 14z + z^2 = 0$. This, multiply'd by $1 + Az + Bz^2$,

$$\text{gives}\left\{\begin{array}{cccccc} -1 & +14z & +z^2 & * & * \\ * & -Az & +14Az^2 & +Az^3 & * \\ * & * & -Bz^2 & +14Bz^3 & +Bz^4 \end{array}\right\} = 0.$$

Whence, by equating the Coefficients of the like Terms, we have $1 + 14A - B = 0$, and $A + 14B = 0$:

From which Equations A is found $= -\frac{14}{197}$; and B $\left(= -\frac{A}{14}\right) = \frac{1}{197}$:

And, by substituting these Values above,

we have $-1 + \overline{14 + \frac{14}{197}} \times z + \frac{z^4}{197} = 0$.

Which, by rejecting the exceeding small Quantity $\frac{z^4}{197}$,

becomes $-1 + \overline{14 + \frac{14}{197}} \times z = 0$.

Hence $-197 + 2772z = 0$;

and $z = \frac{197}{2772} = .0710678$, nearly: Which Value is true to the last Decimal Place: And, if more Terms of the Series $1 + Az + Bz^2 + Cz^3$ &c. had been taken, the Conclusion would have been still exacter in Proportion.

Ex. 2. Suppose the given Equation (when prepared for a Solution) to be $-2 + 5z - z^3 = 0$.

Here, if four Terms of the general Series $1 + Az + Bz^2 + Cz^3$ &c. be taken, and multiply'd by $-2 + 5z - z^3$, our Equation will be chang'd into the following One,

viz.

$$viz. \left\{\begin{array}{l} -2-2Az-2Bz^2-2Cz^3 \quad * \\ \;\;*+5z+5Az^2+5Bz^3+5Cz^4 \\ \;\;* \quad * \quad * \quad -z^3-Az^4 \; \&c. \end{array}\right\}=0.$$

Where, by comparing the homologous Terms, we have $2B=5A$, $2C=5B-1$, and $5C=A$:

Hence, $25B-5\ (=10C)=2A$; but, by the firſt of theſe Equations, $B=\frac{5A}{2}$; therefore $\frac{125A}{2}-5=2A$, and conſequently $A=\frac{10}{121}$.

Let this Value be now wrote inſtead of A; by which means our laſt Equation for the Value of z (neglecting the Terms $-Bz^5-Cz^6$) is reduced to $-2+\overline{5-\frac{20}{121}}\times z=0$.

Whence z comes out $=\frac{242}{585}=0.414$ nearly.

Let there be now given the general Equation,
$$-p+az+bz^2+cz^3+dz^4 \; \&c.=0.$$

Then, multiplying by $(1+Az)$ the two firſt Terms of the Series, only,

$$\text{we get} \left\{\begin{array}{l} -p+az+bz^2 \; \&c. \\ \;\;*-pAz+aAz^2 \; \&c. \end{array}\right\}=0.$$

Here, by making $b+aA=0$, A is found $=-\frac{b}{a}$;

and our Equation becomes $-p+\overline{a+\frac{bp}{a}}\times z=0$:

Whence z is given $=\frac{p}{a+\frac{bp}{a}}=\frac{ap}{aa+bp}$.

The Value of z, here determined, taking in two Terms of the given Series, $az+bz^2+cz^3$ &c. I call an Approximation of the ſecond Degree (as the common Method of Converging Series's, which takes in the firſt Term

Term only, may, by the ſame Rule, be call'd an Approximation of the Firſt Degree.)

But to obtain an Approximation of the Third Degree, or ſuch an One as ſhall include three Terms of the original Series, let the given Equation, $-p+az+bz^2+cz^3$ &c. $=0$, be now multiply'd by three Terms of the aſſumed Series $1+Az+Bz^2$ &c: Whence there ariſes this

$$\text{new Equation}\left\{\begin{array}{l} -p+az+bz^2+cz^3 \text{ \&c.} \\ \ast-Apz+Aaz^2+Abz^3 \text{ \&c.} \\ \ast\quad\ast\quad -Bpz^2+Baz^3 \text{ \&c.}\end{array}\right\}=0.$$

Where, by equating the homologous Terms, we have $b+Aa-Bp=0$, and $c+Ab+Ba=0$.

Let the former of theſe Equations be multiply'd by a, and the Latter by p; and then add the two Products together; ſo ſhall $ab+Aa^2+pc+Abp=0$; and conſequently $A=-\dfrac{ab+cp}{aa+bp}$. Whence $z\left(=\dfrac{p}{a-pA}\right)$ is likewiſe given.

To have an Approximation of the Fourth Degree, four Terms of the Series $1+Az+Bz^2$ &c. muſt be taken, for a Multiplicator: By which means the Equation given will be

$$\text{transform'd to}\left\{\begin{array}{l} -p+az+bz^2+cz^3+dz^4. \text{\&c.} \\ \ast-Apz+Aaz^2+Abz^3+Acz^4. \text{\&c.} \\ \ast\quad\ast\quad -Bpz^2+Baz^3+Bbz^4. \text{\&c.} \\ \ast\quad\ast\quad\ast\quad -Cpz^3+Caz^4. \text{\&c.}\end{array}\right\}=0.$$

Here we have

$B=\dfrac{Aa}{p}+\dfrac{b}{p}$,

$C=\dfrac{Ba}{p}+\dfrac{Ab}{p}+\dfrac{c}{p}$,

$0=Ca+Bb+Ac+d$,

From whence, exterminating C, there ariſes

$\dfrac{Ba}{p}+\dfrac{Ab}{p}+\dfrac{c}{p}+\dfrac{Bb}{a}+\dfrac{Ac}{a}+\dfrac{d}{a}=0$, or $\overline{\frac{a}{p}+\frac{b}{a}}\times B+\overline{\frac{b}{p}+\frac{c}{a}}\times A+\frac{c}{p}+\frac{d}{a}=0.$

Which, by ſubſtituting the Value of B,

becomes $\overline{\frac{a}{p}+\frac{b}{a}}\times\overline{\frac{Aa}{p}+\frac{b}{p}}+\overline{\frac{b}{p}+\frac{c}{a}}\times A+\frac{c}{p}+\frac{d}{a}=0;$

that is, $\overline{\frac{aa}{pp}+\frac{2b}{p}+\frac{c}{a}}\times A+\frac{ab}{pp}+\frac{bb}{ap}+\frac{c}{p}+\frac{d}{a}=0.$

Multiply the Whole by ap^2; then will $\overline{a^3+2abp+cpp}$ $\times A+a^2b+b^2p+acp+dp^2=0:$

Whence A is found $=-\frac{aab+\overline{ac+bb}\times p+dpp}{a^3+2abp+cpp};$

And $z=\frac{p}{a-pA}$, *as in the preceding Caſe.*

By the ſame Method, an Approximation of any higher Degree, to take in as many Terms of the propoſed Series as you pleaſe, may be derived.

For, it is evident, from above, that in all Caſes whatever,

B will be $=\frac{Aa}{p}+\frac{b}{p}$,

$C=\frac{Ba}{p}+\frac{Ab}{p}+\frac{c}{p}$,

$D=\frac{Ca}{p}+\frac{Bb}{p}+\frac{Ac}{p}+\frac{d}{p}$,

$E=\frac{Da}{p}+\frac{Cb}{p}+\frac{Bc}{p}+\frac{Ad}{p}+\frac{e}{p}$,

&c. &c.

where the laſt Value is, always, equal to Nothing.

Therefore

Therefore, by making

$$\begin{cases} Q=\frac{a}{p} \\ R=\frac{aQ+b}{p} \\ S=\frac{aR+bQ+c}{p} \\ T=\frac{aS+bR+cQ+d}{p} \\ V=\frac{aT+bS+cR+dQ+e}{p} \\ \text{\&c.} \end{cases}$$

and

$$\begin{cases} q=\frac{b}{p} \\ r=\frac{aq+c}{p} \\ s=\frac{ar+bq+d}{p} \\ t=\frac{as+br+cq+e}{p} \\ v=\frac{at+bs+cr+dq+f}{p} \\ \text{\&c.} \end{cases}$$

the Value of *A* will be determined, by dividing the laſt of theſe Quantities q, r, s, t, &c. by the correſponding Quantity of the upper Series Q, R, S, &c. and changing the Sign of the Quotient.

Whence we get $\frac{p}{a+p\times\frac{q}{Q}}$, $\frac{p}{a+p\times\frac{r}{R}}$, $\frac{p}{a+p\times\frac{s}{S}}$, &c.

or, $\frac{1}{\frac{a}{p}+\frac{q}{Q}}$, $\frac{1}{\frac{a}{p}+\frac{r}{R}}$, $\frac{1}{\frac{a}{p}+\frac{s}{S}}$, &c. for ſo many different Values of z; whereof each is more exact than the preceding One.

Theſe

Theſe Equations are eaſily derived from Thoſe above, expreſſing the Relation of the Quantities A, B, C, D, &c.

For, by writing Q and q, inſtead of their Equals, in the firſt of thoſe Equations $\left(B = \frac{Aa}{p} + \frac{b}{p}\right)$ it becomes $B = QA + q$.

Which Value of B, wrote in the 2d Equation, $C = \frac{Ba}{p} + \frac{Ab}{p} + \frac{c}{p}$, gives $C = \frac{aQA}{p} + \frac{aq}{p} + \frac{Ab}{p} + \frac{c}{p} = RA + r$; by ſubſtituting R and r in the Room of their Equals.

Moreover the third Equation, by ſubſtituting for C and B, becomes $D = \frac{aRA}{p} + \frac{ra}{p} + \frac{bQA}{p} + \frac{bq}{p} + \frac{Ac}{p} + \frac{d}{p} = SA + s$.

After the very ſame manner $E = TA + t$, $F = VA + v$, &c. And, by putting all theſe ſeveral Values, ſucceſſively, equal to Nothing, we have $-\frac{q}{Q}$, $-\frac{r}{R}$, $-\frac{s}{S}$, &c. for ſo many different Approximations of the Value A. Which being ſubſtituted in the general Equation $z = \frac{p}{a - pA}$, the very Expreſſions before given, for the Value of z, are obtained.

It will be proper, now, to ſhew the Uſe of the ſeveral Approximations, or Theorems, derived above, by a few Examples.

In

In the first Place, then, let the Equation given be $x^3 = 10$.

This Equation, by making $2 + z = x$, will be transformed to $-2 + 12z + 6z^2 + z^3 = 0$: Which being compared with the general Equation
$$-p + az + bz^2 + cz^3 + dz^4 \text{ \&c.} = 0,$$
we have $p = 2$, $a = 12$, $b = 6$, $c = 1$, $d = 0$, &c.

Therefore (by the first of the three Approximations, at p. 217) $-A\left(= \frac{b}{a}\right)$ is found $= \frac{1}{2}$.

Whence $z \left(= \frac{p}{a - Ap}\right)$ comes out $= \frac{1}{13} = 0.1538$, nearly.

But, according to the second Approximation, or Theorem, the Value of $-A\left(= \frac{ab + cp}{aa + bp}\right)$ will be $= \frac{72 + 2}{144 + 12} = \frac{37}{78}$: And consequently that of $z \left(= \frac{p}{a - Ap}\right) = \frac{78}{505} = 0.15445$.

Lastly, the Value of $-A$, according to the third Approximation being $= \frac{a^2b + \overline{ac + bb} \times p + dp^2}{a^3 + 2abp + cpp} = \frac{144 \times 6 + 96}{144 \times 12 + 144 \times 2 + 4} = \frac{36 \times 6 + 24}{36 \times 14 + 1} = \frac{48}{101}$, the corresponding Value of z will therefore come out $= \frac{101}{654} = 0.154434$: Which Number is true in all its Places.

The very same Conclusions will likewise be brought out, from the general Solution.

For,

For, p being here $= 2$, $a = 12$, $b = 6$, $c = 1$, $d = 0$. &c. (*as before*).

we have $Q\left(=\frac{a}{p}\right) = 6$,

$$R\left(\frac{aQ+b}{p}\right) = \frac{12\times 6+6}{2} = 39,$$

$$S\left(\frac{aR+bQ+c}{p}\right) = \frac{505}{2},$$

&c.

Alſo, $q\left(\frac{b}{p}\right) = 3$,

$$r\left(\frac{aq+c}{p}\right) = \frac{37}{2},$$

$$s\left(\frac{ar+bq+d}{p}\right) = 120,$$

&c.

Therefore $\frac{q}{Q} = \frac{1}{2}$, $\frac{r}{R} = \frac{37}{78}$, $\frac{s}{S} = \frac{240}{505} = \frac{48}{101}$; and conſequently $\frac{1}{\frac{a}{p}+\frac{q}{Q}} = \frac{1}{6+\frac{1}{2}} = \frac{2}{13}$, &c. *as before*.

For a ſecond Example, ſuppoſe $10z - z^3 = 2$, or $-2 + 10z - z^3 = 0$.

In this Caſe $p = 2$, $a = 10$, $b = 0$, $c = -1$, $d = 0$, &c. And therefore, for an Approximation of the fourth Degree, we have $-A\left(\frac{aab+\overline{ac+bb}\times p+dpp}{a^3+2abp+cpp}\right) = \frac{-20}{1000-4} = \frac{-5}{249}$; and conſequently $z\left(=\frac{p}{a-pA}\right) = \frac{249}{1240} = 0.2008045$; which Value is true to the laſt Fignre.

For

For a third Example, let there be given the Equation

$$x^3 + 36x^2 + 432x = 2272.$$

Here the Value of x appears to be about 4; let therefore $4+z$ be wrote for x; whence our Equation is reduced to

$96 + 768z + 48z^2 + z^3 = 0.$

Which being compared with

$-p + az + bz^2 + cz^3 + dz^4$ &c. $= 0$,

we have, in this Cafe,

$p = -96,\ a = 768,\ b = 48,\ c = 1,\ d = 0$, &c.

Or, $\frac{a}{p} = -8, \frac{b}{p} = -\frac{1}{2}, \frac{c}{p} = -\frac{1}{96}$, &c.

Hence $Q\left(\frac{a}{p}\right) = -8$,

$$R\left(\frac{aQ}{p} + \frac{b}{p}\right) = 64 - \tfrac{1}{2} = \frac{127}{2},$$

$$S\left(\frac{aR}{p} + \frac{bQ}{p} + \frac{c}{p}\right) = -\frac{48385}{96};$$

&c.

Alfo $q\left(\frac{b}{p}\right) = -\frac{1}{2}$,

$$r\left(\frac{aq}{p} + \frac{c}{p}\right) = 4 - \frac{1}{96} = \frac{383}{96},$$

$$s\left(\frac{ar}{p} + \frac{bq}{p} + \frac{d}{p}\right) = -\frac{380}{12},$$

&c.

Therefore $\frac{q}{Q} = \frac{1}{16}, \frac{r}{R} = \frac{383}{127 \times 48} = \frac{383}{6096}, \quad \frac{s}{S} = \frac{380 \times 8}{48385} = \frac{3040}{48385}$;

From whence $z = -\frac{16}{127} = -0.12598$, nearly;

or, $z = -\frac{6096}{48385} = -0.1259894$, *more* nearly,

or, $z = -\frac{9677}{76808} = -0.1259894802$, *ftill* nearer.

Laftly,

Laftly, let there be given the Equation

$$\frac{x^2}{2} - \frac{x^4}{2.3.4} + \frac{x^6}{2.3.4.5.6.} - \frac{x^8}{2.3.4.5.6.7.8.} \text{ \&c.} = \frac{1}{2}.$$

Then, making $z = x^2$, &c. we have

$$-1 + z - \frac{z^2}{12} + \frac{z^3}{12 \times 30} - \frac{z^4}{12 \times 30 \times 56} \text{ \&c.} = 0.$$

Here $p = 1$, $a = 1$, $b = -\frac{1}{12}$, $c = \frac{1}{12 \times 30}$, $d = -\frac{1}{12 \times 30 \times 56}$: &c.

And, by fubftituting thefe Values in the third General Approximation (*vid. p.* 219), we have — A

$$\left(= \frac{aab + \overline{ac + bb} \times p + dpp}{aaa + 2abp + cpp}\right)$$

$$= \frac{-\frac{1}{12} + \frac{1}{12 \times 30} + \frac{1}{12 \times 12} - \frac{1}{12 \times 30 \times 56}}{1 - \frac{2}{12} + \frac{1}{12 \times 30}} = -$$

$$\frac{30 \times 56 - 56 - 140 + 1}{10 \times 30 \times 56 + 56} = -\frac{1485}{16856}.$$

Therefore $z \left(= \frac{p}{a - pA}\right) = \frac{16856}{15371} = 1.09661$;

and $x \; (= \sqrt{z}) = 1.04719.$

After the fame manner the Roots of other Equations may be approximated: But I fhall here fhew, how the general Theorems themfelves may be rendered more commodious, for certain particular Cafes, by Means of a proper Transformation.

It is known, if two Quantities be, refpectively, increafed, or decreafed by two other, fmall, Quantities, nearly in the fame Ratio with the two Firft, that the Sums, or Differences will ftill be in the fame Ratio with the two firft Quantities, very near.

Wherefore, feeing the Numerator of the Fraction $\frac{ab+cp}{aa+bp}$, expreffing the 2^d. General Value of – A. (*vid.* *p.* 218) is in proportion to the Denominator, nearly, as b to a, or cp to $\frac{acp}{b}$; let cp be therefore taken from the Numerator, and $\frac{acp}{b}$ from the Denominator; agreeable to the above Obfervation: By which Means the Fraction itfelf will be transformed to $\frac{ab}{aa+bp-\frac{acp}{b}} =$

$$\frac{b}{a+\overline{\frac{b}{a}-\frac{c}{b}}\times p} = \frac{b}{a+rp}; \text{ fuppofing } r = \frac{b}{a} - \frac{c}{b}.$$

And, in the very fame Manner the Fraction $\frac{aab+\overline{ac+bb}\times p+dpp}{aaa+2abp+cpp}$, expreffing the third Value of – A, is changed to $\frac{aab+\overline{ac+bb}\times p}{aaa+2abp+cpp-\frac{adpp}{b}} =$

$$\frac{aab+\overline{ac+bb}\times p}{a^3+2abp+sp^2} \left(\text{by putting } s = c - \frac{ad}{b}\right).$$

But this laft Value is ftill capable of a further Reduction: For, the Ratio of the Numerator to the Denominator being That of $\frac{bsp}{r}$ to $\frac{asp}{r}+sp^2$, nearly (as appears from the preceding Cafe) let, therefore, the former of thefe Quantities be fubtracted from the Numerator, and the latter from the Denominator: Whence the Fraction itfelf becomes

$$\frac{a^2b+\overline{ac+bb}\times p-\frac{bsp}{r}}{a^3+2abp-\frac{asp}{r}} = \frac{ab+\overline{c+\frac{bb}{a}-\frac{bs}{ra}}\times p}{a^2+\overline{2b-\frac{s}{r}}\times p} =$$

$= \frac{ab + \overline{c + bw} \times p}{aa + \overline{b + aw} \times p}$; by putting $w = \frac{b}{a} - \frac{s}{ar}$ $\left(= \frac{b}{a} - \frac{cb - ad}{bb - ac}\right)$

Now, to exemplify the Uſe of the Theorems thus transformed, let the Equation $96 + 768z + 48z^2 + z^3 = 0$ (given at p. 224) be here reſumed:

Then, in this Caſe, p being $= -96$, $a = 768$, $b = 48$, $c = 1$, $d = 0$, *&c.* we have $r \left(= \frac{b}{a} - \frac{c}{b}\right) = \frac{1}{24}$, $s \left(= c - \frac{ad}{b}\right) = 1$, and $w \left(= \frac{b}{a} - \frac{s}{ar}\right) = \frac{1}{32}$.

Whence, according to the firſt Approximation,

$-A \left(= \frac{b}{a + rp}\right) = \frac{48}{768 - 4} = \frac{12}{191}$:

And, according to the Second,

$-A \left(= \frac{ab + \overline{c + bw} \times p}{aa + \overline{b + aw} \times p}\right) = \frac{768 \times 48 + \overline{1 + \frac{3}{2}} \times -96}{768 \times 768 + \overline{48 + 24} \times -96}$

$= \frac{768 - 5}{768 \times 16 - 72 \times 2}$ (by dividing every Term by 48)

$= \frac{763}{12144}$.

Hence the Value of $z \left(= \frac{1}{\frac{a}{p} - A}\right)$ comes out $= -\frac{191}{1516} = -0,1259894$, *nearly*; or equal to $-\frac{12144}{96389}$ $= -0.1259894802$, *more nearly.*

Theſe Concluſions agree with Thoſe before given, by the former Method. But the laſt general Approximations, containing the feweſt Dimenſions of the Quantity p, will commonly be found to have the Advantage, in point of Expedition, when the Value of that Quantity conſiſts of ſeveral Decimal

 Places,

Places, and alſo when the Coefficients a, b, c, d, of the Powers of the unknown Quantity z, are related to each other according to ſome known Law.

Of this Kind are the Coefficients of ſuch Series's as ariſe in extracting the Roots of pure Powers; and in theſe Caſes the general Theorems, or Equation themſelves, are capable of being rendered ſtill more commodious.

Let there be aſſumed the Equation $x^n = k$ (which includes all the Caſes of pure, or ſimple Powers, according to the Value of the Exponent n).

Then, by aſſuming r nearly equal to x, and making $r \times \overline{1+z} = x$, we ſhall have $r^n \times \overline{1+z}|^n = k$;

and therefore $-\frac{k}{r^n} + \overline{1+z}|^n = 0$; that is

$$-\frac{k}{r^n} + 1 + nz + \frac{n}{1} \times \frac{n-1}{2} z^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} z^3$$

&c. $= 0$; or laſtly $-p + z + \frac{n-1}{2} z^2 + \frac{n-1}{2} \times \frac{n-2}{3} z^3 +$ &c. $= 0$; by dividing the Whole by n, and putting $p = \frac{k - r^n}{nr^n}$.

Here (by a Compariſon with the general Equation $-p + az + bz^2$ &c. $= 0$) we have $a = 1$, $b = \frac{n-1}{1}$, $c = \frac{n-1}{2} \times \frac{n-2}{3}$, $d = \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c.

Theref. $r\left(\frac{b}{a} - \frac{c}{b}\right) = \frac{n-1}{2} - \frac{n-2}{3} = \frac{n+1}{6}$,

$$s\left(c - \frac{ad}{b}\right) = \frac{n-2}{3} \times \overline{\frac{n-1}{2} - \frac{n-3}{4}} = \frac{n-2}{3} \times \frac{n+1}{4},$$

$$w\left(\frac{b}{a} - \frac{s}{ar}\right) = \frac{n-1}{2} - \frac{n-2}{2} = \frac{1}{2},$$

Whence,

Whence, for an Approximation of the third Degree,

$$-A\left(=\frac{b}{a+rp}\right)=\frac{\frac{1}{2}\times\overline{n-1}}{1+\overline{n+1}\times\frac{1}{6}p};$$

and z

$$\left(=\frac{p}{a-Ap}\right)=\frac{p+\overline{n+1}\times\frac{1}{6}p^2}{1+\overline{2n-1}\times\frac{1}{3}p}.$$

But, for an Approximation of the fourth Degree,

$$-A\left(=\frac{ab+\overline{c+bw}\times p}{aa+\overline{b+aw}\times p}\right)=\frac{\frac{1}{2}\times\overline{n-1}+\overline{n-1}\times\overline{2n-1}\times\frac{1}{12}p}{1+\frac{1}{2}np};$$

and

$$z=\frac{p+\frac{1}{2}npp}{1+\frac{\overline{2n-1}}{2}\times p+\frac{\overline{2n-1}\times\overline{n-1}}{12}\times pp}.$$

Hence it is manifest that the Root x, of the proposed Equation $x^n=k$, is equal to

$$r+\frac{rp\times\overline{1+\overline{n-1}\times\frac{1}{6}p}}{1+\overline{2n-1}\times\frac{1}{3}p}, \text{ *nearly*;}$$

Or, equal to,

$$r+\frac{rp\times\overline{1+\frac{1}{2}np}}{1+\frac{\overline{2n-1}}{2}\times p+\frac{\overline{2n-1}\times\overline{n-1}}{12}\times p^2},$$

more nearly.

But both these Expressions, in Cases where p is a proper Fraction, will be better adapted to practice by making $\frac{nr^n}{k-r^n}=v\left(=\frac{1}{p}\right)$, and substituting $\frac{1}{v}$ for p, its Equal: Whence (after proper Reduction)

$$x=r+\frac{r}{v}\times\frac{6v+n+1}{6v+4n-2}, \text{ *nearly*;}$$

$$\text{Or, } x=r+\frac{r\times\overline{2v+n}}{v\times\overline{2v+2n-1}+\frac{1}{6}\times\overline{2n-1}\times\overline{n-1}},$$

more nearly.

To shew now the Use and great Exactness of these last Approximations, by an Example, let it be proposed to extract the square Root of 441. Here the general Equation $x^n = k$, becoming $x^2 = 441$, we have $n = 2$, $k = 441$, and $v\left(= \frac{nr^n}{k-r^n}\right) = \frac{800}{41}$; supposing r to be assumed $= 20$.

Therefore, by the first Approximation,
$x = 20 + \frac{41}{40} \times \frac{1641}{1682} = 20 + \frac{67281}{67280} = 21 + \frac{1}{67280}$, nearly.

And, by the Second, $x = 20 + \frac{2758480}{2758481} = 21 - \frac{1}{2758481}$, more nearly.

Again, let there be given the Equation $x^3 = 500$: Then, assuming $r = 8$, we have $v\left(\frac{nr^n}{k-r^n}\right) = \frac{3 \times 512}{-12} = -128$.

Hence, by the first Approximation,
$x = 8 - \frac{1}{16} \times \frac{-768+4}{-768+10} = 8 - \frac{191}{3032} = 7,93700527$, nearly:

And, by the Second, $x = 8 - \frac{8 \times 253}{128 \times 251 + \frac{5}{3}} = 8 - \frac{6072}{96389} = 7 \cdot 9370052599$, more nearly.

All the different Approximations hitherto delivered were, originally, derived by multiplying the given Equation into a certain Number of Terms of the assumed Series $1 + Az + Bz^2 + Cz^3$ &c. But there are other Methods (though, perhaps, none so general) by which the second, third, &c. Dimensions of the unknown Quantity may, in like sort, be destroyed (without assum-
ing

ing any Series) and from thence Value of that Quantity approximated, to what Degree of Exactneſs you pleaſe.

Let there, for Inſtance, be given the Equation,
$2z + z^2 = 1$, or $z^2 = 1 - 2z$:

Then, by ſquaring both Sides thereof, there ariſes $z^4 = 1 - 4z + 4z^2$.

And if, inſtead of the laſt Term $4z^2$, its Equal, $1 - 2z$, be ſubſtituted, you will have $z^4 = 5 - 12z$: This, ſquared, gives $z^8 = 25 - 120z + 144z^2 = 169 - 408z$, by ſubſtituting for z^2, as before.

Here, rejecting z^8 (on Account of its Smallneſs in Compariſon of the other Terms) we have $408z = 169$, and therefore $z = \frac{169}{408} = 0{,}414213$, nearly; which is true to the laſt Figure, incluſive. But, if you would have the Anſwer ſtill nearer the Truth, let the above Equation $z^8 = 169 - 408z$ be, either, multiply'd, again, by itſelf, or into ſome one of the preceding Ones. Thus, if it be multiplied into $z^4 = 5 - 12z$, you will have
$z^{12} = 845 - 4068z + 4896z^2 = 5741 - 13860z$:
Where, z^{12} being rejected, z is found $= \frac{5741}{13860}$ $= 0.41421356$, &c.

Again, if there be given the Equation $z^3 = 3z - p$; then, by ſquaring both Sides thereof, we have
$z^6 = 9z^2 - 6pz + p^2$:
And therefore $z^7 = 9z^3 - 6pz^2 + p^2z = -6pz^2 + \overline{pp + 27} \times z - 9p$; by writing $27z - 9p$ inſtead of its Equal $9z^3$.

Now, to the triple of this laſt Equation, let the 2^d. Equation, multiply'd by $2p$, be added:

By which Means z^2 will be exterminated, and you then will have $3z^7 + 2pz^6 = \overline{81 - 9pp} \times z - 27p + 2p^3$. Whence (rejecting $3z^7 + 2pz^6$) the Value of z is found $= \frac{\overline{27 - 2pp} \times p}{\overline{9 - pp} \times 9}$, nearly.

Various other Expedients might be uſed, to exterminate the 2^d, 3^d, *&c.* Powers of the unknown Quantity; but what is already delivered may ſuffice.

PART

PART V.

GIVING

Some Account of the Nature of FLUXIONS; *together with the Investigation of the fundamental Rules.*

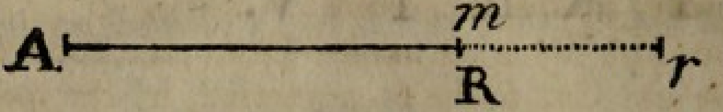
1. IN the Doctrine of Fluxions all Kinds of Magnitudes are considered as generated by the continual Motion of some of their Bounds or Extremes; as a Line by the Motion of a Point; a Surface by the Motion of a Line; and a Solid by the Motion of a Surface. So likewise Time may be consider'd as represented by a Line, increasing uniformly by the Motion of a Point: And, as Quantities of all Kinds whatever are capable of Increase and Decrease, They may be represented, in like manner, by Lines, Surfaces, or Solids, conceived to be generated by Motion.

2. *Every Quantity thus generated is call'd a Fluent, or Flowing Quantity: And the Magnitude by which any Flowing Quantity would be uniformly increased, in a given Time, with the generating Celerity at any proposed Position, or Instant* * *(was it from thence to continue invariable) is the Fluxion of the said Quantity at that Position, or Instant.*

3. Thus,

* *Some Authors define the generating Celerity itself (and not the Magnitude it would produce) to be the Fluxion; but use that Magnitude as the Measure of the said Celerity or Fluxion: Which is, in effect, coming to the same Thing.*

3. Thus, let a Point *m* be conceived to move from A, and thereby generate a Right-line A*m*, with a Motion any how regulated ; and ſuppoſe the Celerity



thereof, at any propoſed Poſition R, to be ſuch, *as would*, (was it to continue invariable from that Poſition) be ſufficient to deſcribe, or paſs uniformly over the Diſtance R*r*, in the given Time allowed for the Fluxion : Then will the ſaid Diſtance R*r* truly expreſs the required Fluxion of the Flowing Line A*m*, in that Poſition.

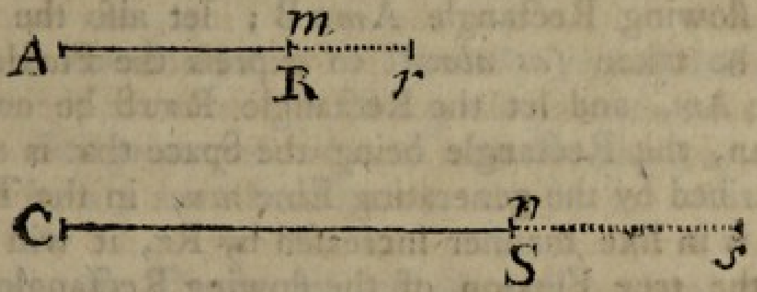
4. It appears from hence, that, when the generating Motion is uniform, the Fluxion, and the Increment *actually* deſcribed in the given Time, are one and the ſame Thing: But, if the Velocity continually increaſes, or decreaſes, the Fluxion muſt then be either leſs, or greater than the ſaid Increment, or the Space *actually* deſcribed: Since an Increaſe of the Velocity muſt neceſſarily cauſe an Increaſe in the Diſtance gone over, and *vice verſâ*.

Although, in Forming a juſt and diſtinct Conception of the Nature, and Quantity of a Fluxion, the Conſideration of Time is, abſolutely, neceſſary (on which, even, our Ideas of Velocity depend), yet in the Buſineſs and Application of Fluxions, it is not always requiſite, that ſome, vulgar (or common) Meaſure of Time (as a Second, Minute, Hour, &c.) ſhould be propounded for, the Production of Fluxions of the Quantities under Conſideration. A Line generated by the uniform Motion of a Point, it is obſerved above, may be taken as a proper Repreſentative, or Meaſure of Time : And that Interval of Time (be it what it will) wherein the Line, ſo generated, is augmented by any Length, or Fluxion, aſſigned, may be taken as the Time underſtood in the Definition, allowed for the Production of the Fluxions of all other Quantities that have any Relation to, or Dependence upon, the ſaid uniformly-generated Line. And theſe Fluxions themſelves, by means of the ſaid Relation and the given Length, or Fluxion, may be alſo truly exhibited, independant of any particular, known Meaſure of Time ; as will, hereafter, be fully made to appear.

5. If

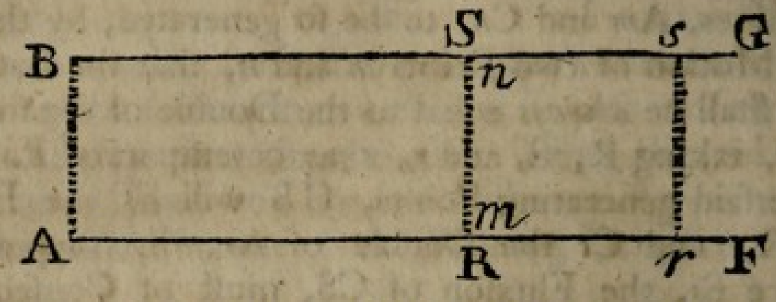
5. It appears moreover, from tne above Definition, that Quantities, which flow, or increaſe together, ſo as to continue, *always*, in a conſtant Ratio, have their Fluxions, likewiſe, in *the ſame* conſtant Ratio.

To illuſtrate This by a particular Example, ſuppoſe two Lines, A*m* and C*n*, to be ſo generated, by the unifrom Motion of two Points *m* and *n*, that the Latter of them ſhall be *always* equal to the Double of the former: Then, taking R, S, and *r*, *s* as cotemporary Poſitions of the ſaid generating Points, CS will be the Double of AR, and C*s* the Double of A*r*, by ſuppoſition; whence S*s*, the Fluxion of CS, muſt of Conſequence be the Double of R*r*, the Fluxion of AR.



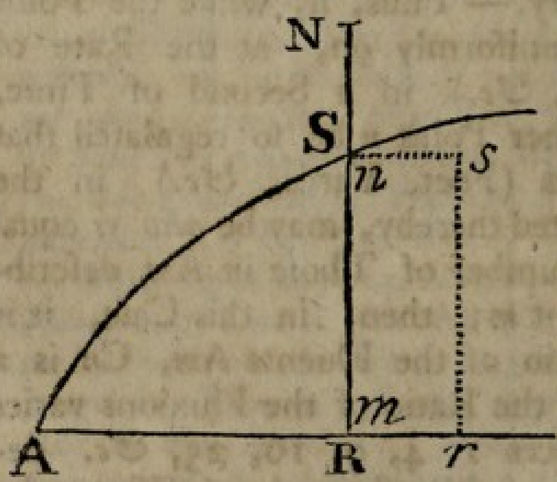
It is equally plain, on the other hand, that if the Ratio of the Fluents A*m*, C*n* is variable, That of the Fluxions muſt alſo vary.—Thus, if, while the Point *m* continues to move uniformly on, at the Rate of one Inch (Foot, Yard, &c.) in a Second of Time, the Motion of the other Point *n* be ſo regulated that the Number of Inches (Feet, Yards, &c.) in the flowing Line C*n* generated thereby, may be *always* equal to the Square of the Number of Thoſe in A*m* deſcribed by the former Point *m*; then, in this Caſe, it is manifeſt, that the Ratio of the Fluents A*m*, C*n* is a variable One; and that the Ratio of the Fluxions varies alſo; ſeeing the Diſtances 1, 4, 9, 16, 25, &c. deſcribed in 1, 2, 3, 4, 5, &c. Seconds of Time, by the Point *n*, increaſe much faſter in Proportion than 1, 2, 3, 4, 5, &c. the correſponding Diſtances gone over by the other Point *m*, moving uniformly.

6. Hitherto Regard has been had to the Fluxions of Lines: But the Fluxions of Superficies and Solids are confidered in the fame manner, and are comprehended with equal Facility. Let a given Right-line *mn*



be conceived to move parallel to itfelf, with an equable Motion, from the Pofition AB, and thereby generate the flowing Rectangle A*mn*B; let alfo the Diftance R*r* be taken *(as above)* to exprefs the Fluxion of the Bafe A*m*, and let the Rectangle R*rs*S be completed: Then, this Rectangle being the Space that is uniformly defcribed by the generating Line *mn*, in the Time that A*m* is in like manner increafed by R*r*, it will therefore be the true Fluxion of the flowing Rectangle A*n*, by the Definition. *Art.* 2.

7. The Generation, and the Fluxion of any triangular, or curvilineal, Space ASR, are conceived in like manner; by fuppofing a Right-line *m*N to be carry'd along, continually parallel to itfelf, fo that the intercepted Part Thereof *mn* (which is itfelf a variable Quantity) may pafs over, and thereby generate, the Space ARS propounded. And the Fluxion



of the Space thus generated, if R*r* be taken as the Fluxion of the Bafe (or Abfciffa) A*m*, will be truly expreffed by the Rectangle (R*s*) under R*r* and RS; as we fhall have Occafion to fhew more at large hereafter.

8. From

8. From what has been thus far delivered, it will not be difficult to form a juſt Idea of the Fluxion of a Solid: But it is time we now come to ſhew the manner of determining the Fluxions of Algebraic Quantities; by means whereof all Others, of what Kind ſoever, are explicable: In order to which it will be requiſite, firſt of all, to premiſe the following Obſervations.

1. *That, the final Letters u, w, x, y, z, of the Alphabet are uſually put for variable Quantities; and the initial Letters a, b, c, d, &c. for invariable Ones:* Thus, the *variable* Baſe Am of the flowing Rectangle AmnB *(in Art.* 6.) may be repreſented by x, and the *invariable* Altitude mn, by a.

2. *That, the Fluxion of a Quantity repreſented by a ſingle Letter is commonly expreſſed by the ſame Letter with a Dot, or Full-point, over it:* Thus the Fluxion of x is denoted by $\dot{x}$; and the Fluxion of y by $\dot{y}$.

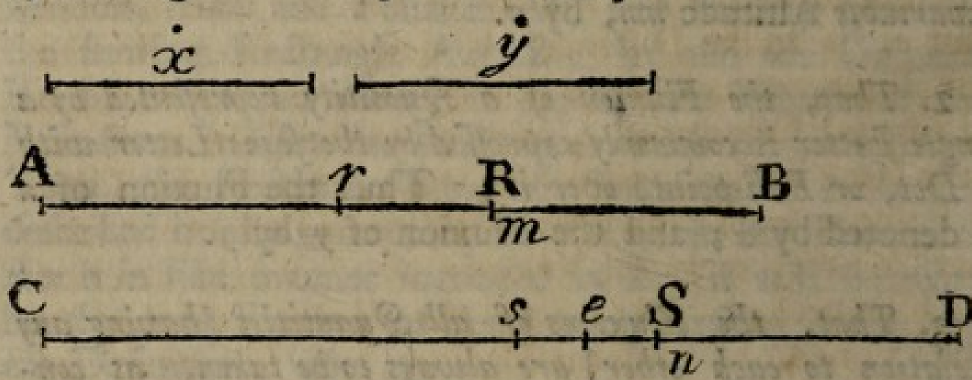
3. *That, the Fluxions of all Quantities (having any Relation to each other) are always to be taken, as contemporaneous, or ſuch as may be generated together, with their reſpective Celerities, in one and the ſame Time.*

PROPOSITION

PROPOSITION I.

9. *The Fluxion of any variable Algebraic Quantity being given, 'tis proposed to find the Fluxion of the Square of that Quantity.*

Let two Points m and n be supposed to move, at the same Time, from two other, fix'd, Points A and C, along the Right-lines AB and CD, in such a manner, that the Measure of the Distance Cn, described by the Latter, may be, *always*, equal to the Square of the cotemporary Distance Am described by the former Point m, moving with an equable Celerity.



Moreover, let R and S be any two contemporary Positions of the said Points; and, supposing the described Distances AR and CS to be denoted by x and y, let the Lines $\dot{x}$ and $\dot{y}$ be taken to represent the Spaces that *would be* uniformly passed over, in the same given Time, with the Celerities of the said Points at R and S: So shall those Lines express the Fluxions of the variable Quantities Am and Cn, when the generating Points m and n arrive at the foresaid cotemporary Positions R and S (*by the Definition*, *Art.* 2).

Furthermore, if r and s be considered as any other corresponding Places of the proposed Points, and the Interval rR be denoted by v; then, AR being $= x$, and A$r = x - v$, we shall have CS $(= y) = x^2$, and C$s = \overline{x-v}|^2$, *by Hypothesis*; and consequently Ss $(=$ CS $-$ C$s) = 2xv - vv$.

From

From which it appears, that, while the former Point *m* moves, uniformly, over the Distance v, the other Point *n* passeth over a Space expressed by $2xv - vv$.

But this last Distance, since the Velocity of the generating Point *n* increases continually (*see Art.* 5.) is less than the Space that would be uniformly described, in the same Time, with the Velocity at S; and greater than That which would be described with the Velocity at *s*; and, therefore, is equal to, and may be taken to express, the Space which *might be* uniformly gone over by the Celerity at some intermediate Point *e*, between *s* and S, in the same Time.

Therefore, seeing the Distance ($2xv - vv$) that might be described with the Celerity at the said intermediate Point *e*, is to the Distance (v) described by *m*, in the same Time, as $2x - v$ to Unity, it is evident that the said (mean) Celerity at *e*, must be to the Celerity of *m*, in the same Ratio; and consequently, that, in the Time the Point *m* would move over the given Distance $\dot{x}$, the other Point *n*, with its Velocity at *e*, would describe the Distance $2x\dot{x} - v\dot{x}$: Since the Spaces described in equal Times, by uniform Motions, are known to be as the Velocities of the said Motions.

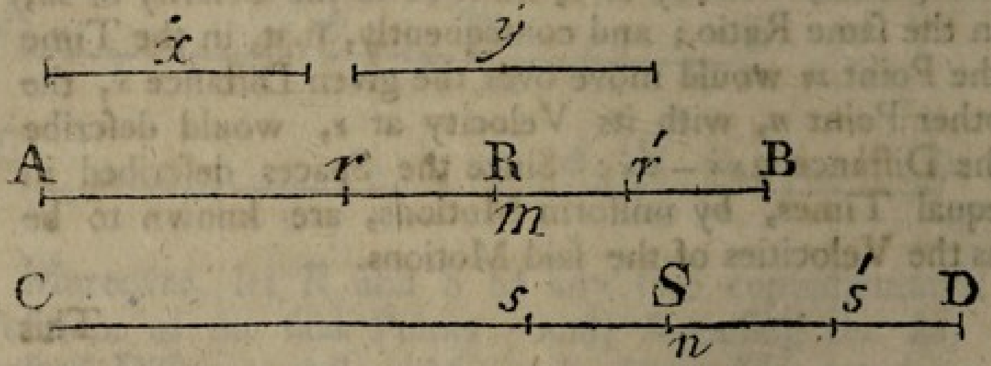
This

The above Method of Investigation hath been represented, as bearing a near Affinity to the Method of Prime and Ultimate Ratios: *Against which so many Objections have been started, by the celebrated Author of the Analyst, as to embarrass and stagger a great Number of Persons; who are not well apprized how far the said Objections are justifiable, nor wherein their main Force consists. It is not my Design to take Part in a Dispute, on which enough hath been already said by Others: Though, that the Method itself is perfectly Scientific, I believe no One, that understands it, will deny; but whether the great Inventor has therein expressed himself with all the Caution and Accuracy he was capable of, is another Question. Had he call'd That a* LIMITING-RATIO *which he names an* Ultimate-One, *the ingenious Author above mention'd might not, perhaps, have found Room for* his Ghosts of departed Quantities. *However, be this as it will, those*

This being determined, let r be now ſuppoſed to coincide with R, and s with S, by means of the Arrival of the generating Points at R and S; then e, being always between s and S, will likewiſe coincide with S; and the foreſaid Diſtance $2x\dot{x} - v\dot{x}$, that might be uniformly deſcribed with the Velocity at e (now at S) will become, barely, $2x\dot{x}$; which (*by Art.* 2) is equal to ($\dot{y}$) the Fluxion of Cn or x^2. From whence it appears, that the Fluxion of the Square of any variable, or flowing Quantity is found by multiplying twice the Root, or Quantity itſelf, into its Fluxion.

The ſame otherwiſe.

10. Let the Diſtance Ar be denoted by u, and let other Things remain as before: Then, CS being =



xx, and Cs = uu, *by Suppoſition*, the Diſtance sS, deſcribed in the ſame Time with rR ($= x - u$) will therefore be truly expreſſed by $xx - uu$, or its Equal $\overline{x+u} \times \overline{x-u}$. Which Diſtance, as the Velocity of n continually increaſes, muſt be greater than That which would be uniformly deſcribed, with the Celerity at s, in the ſame Time. Whence it is evident, that the Velocity

thoſe Objections have nothing at all to do with our Method of Inveſtigation given above. The preſſing (and only) Difficulty, in the Buſineſs of ultimate Ratios, conſiſts, it is known, in conſidering the Values of the Quantities compared, in their ultimate State, wherein their Ratio is ſuppoſed to be taken: For

Velocity at s is to the, uniform, Velocity of m, in a leſs Ratio than That of $\overline{x+u}\times\overline{x-u}$ to $x-u$, or of $\overline{x+u}$ to *Unity*, or laſtly, of $\overline{x+u}\times\dot{x}$ to $\dot{x}$ (becauſe the Velocities of uniform Motions are as the Spaces deſcribed in equal Times). Therefore, ſince the uniform Velocity of m (meaſured by the Diſtance moved over in a given Time) is defined by $\dot{x}$ it is plain that the Meaſure of the Velocity of the other Point at s (or the Diſtance that *might be* uniformly deſcribed in the ſame given Time) will be leſs than $\overline{x+u}\times\dot{x}$. Which Quantity being, itſelf, leſs than $2x\times\dot{x}$ (becauſe u is leſs than x) the Celerity at s muſt conſequently be leſs than $2x\dot{x}$, take the Point s where you will on This-ſide of S.

Let now r' and s' be any other cotemporary Poſitions of the two Points, on the other Side of R and S, and let Ar' be denoted by w: So ſhall the Diſtance Ss, deſcribed in the ſame Time with Rr' $(w-x)$, be truly expreſſed by $ww-xx$, or its Equal $\overline{w+x}\times\overline{w-x}$ (*by Hypotheſis*) Which Diſtance, as the Velocity of the de-

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For, if we look upon them as real Magnitudes, it is objected, that their Ratio will not ſtrictly agree with the Ultimate Ratio aſſigned: And if, on the other hand, they be taken as mere Nothings, we then loſe the very Idea of Proportion. But our Inveſtigation, as is already obſerved, is not embarraſſed with any ſuch Difficulty: For, though the Diſtance Ss *grows, indeed, leſs and leſs, continually, and even vaniſhes when the generating Point* n *arrives at* S; *yet the Velocity of that Point, which is the Quantity in Queſtion, neither vaniſhes, nor aſſumes a new Law; but ſtill continues to increaſe in the ſame manner as before. — It may, poſſibly, be objected, that, as the Meaſure of the ſaid Velocity is, originally, derived by Means of the Diſtance* nS, *we cannot retain a clear Idea of it, when that Diſtance is vaniſhed out of the Equation. But, with equal Reaſon, it might be urged, that, we can have no juſt Conception of the Dimenſions and true Proportion of a Building, after the Scaffolding by Means of which it was raiſed, is taken away.*

fcribing Point n continually increafes, muft, evidently, be lefs than That which would uniformly arife from the Celerity at s', in the fame Time. Whence it is alfo plain that the faid Velocity at s', is to the uniform Velocity of m, in a greater Ratio than That of $\overline{w+x} \times \overline{w-x}$ to $\overline{w-x}$, or of $\overline{w+x} \times \dot{x}$ to $\dot{x}$. But the Quantity $\overline{w+x} \times \dot{x}$ is, itfelf, greater than $2x \times \dot{x}$, becaufe w is greater than x; and fo the Meafure of the faid Vecocity at s' muft confequently be greater than $2x\dot{x}$.

Therefore, fince the Velocity increafes continually, from C to D; and feeing the Meafure Thereof, before the Arrival of the generating Point at S, is every-where lefs, and afterwards every-where greater, than $2x\dot{x}$; it is manifeft, that, at S, it can be neither leffer nor greater, but muft have, or pafs thro', the very Value, or Degree, expreffed by $2x\dot{x}$. Q. E. I.

If the Line Am (x) be fuppofed to be generated with an accelerated, or a retarded Motion, inftead of an uniform One, it will readily appear, from the firft of the foregoing Methods, that the required Fluxion of x^2, fuppofing $\dot{x}$ to denote the Meafure of the Velocity at R, will, *ftill*, be expounded by $2x\dot{x}$.

For the Spaces rR (v) and sS ($2xv - vv$) defcribed in the fame Time, being to each other, in the Ratio of $\dot{x}$ to $2x\dot{x}—v\dot{x}$, the Mean Celerities of the generating Motions, at certain intermediate Points between the extreme Ones r, R, and s, S, muft be likewife, in that Ratio: Which Ratio, when v becomes $=$ o, and the Points coincide, will become That of $\dot{x}$ to $2x\dot{x}$.

PROPO-

PROPOSITION II.

11. *The Fluxions, $\dot{x}$ and $\dot{y}$, of two flowing Quantities, x and y, being given; 'tis proposed to determine the Fluxion of the Rectangle, or Product, xy, of the said Quantities.*

Let z be, *always*, equal to the Sum of the two proposed Quantities x and y: Then, the Fluxions of equal Quantities being also equal, $\dot{z}$ must be $= \dot{x} + \dot{y}$. Moreover, since z is $= x + y$, we shall have $zz = xx + 2xy + yy$; and therefore $xy = \frac{1}{2}zz - \frac{1}{2}xx - \frac{1}{2}yy$. But the Fluxion of $\frac{1}{2}zz - \frac{1}{2}xx - \frac{1}{2}yy$ (and consequently That of its Equal xy) appears, from the preceding Proposition, to be $z\dot{z} - x\dot{x} - y\dot{y}$: Which, by writing $x+y$, and $\dot{x}+\dot{y}$, in the Room of their Equals z and $\dot{z}$, will become $(\overline{x+y} \times \overline{\dot{x}+\dot{y}} - x\dot{x} - y\dot{y}) = y\dot{x} + x\dot{y}$, the required Fluxion of xy. Hence it is apparent that the Fluxion of the Product, or Rectangle, of any two flowing Quantities, is expressed by the Sum of the Products arising from the Multiplication of each Quantity into the Fluxion of the Other.

12. From the Fluxion of a Rectangle, above determined, the Fluxion of a Fraction, $\frac{z}{y}$, is very easily deduced.

For, by putting $x = \frac{z}{y}$ (the proposed Fraction) and multiplying both Sides of the Equation by y we have $xy = z$; and therefore $x\dot{y} + y\dot{x} = \dot{z}$, *as above*, (the Fluxions of equal Quantities being, necessarily, equal). From this Equation, by transposing $x\dot{y}$, and dividing the Whole by y, we get $\dot{x} = \frac{\dot{z}}{y} - \frac{x\dot{y}}{y}$: This, by writing $\frac{z}{y}$ in the Room of its Equal x, becomes $\dot{x} = \frac{\dot{z}}{y} -$

$\frac{z\dot{y}}{yy} = \frac{y\dot{z} - z\dot{y}}{yy}$: Which Value is therefore the true Fluxion of x, or, its Equal, $\frac{z}{y}$, the Fraction proposed.

13. Moreover, from the Fluxion of a Rectangle, the Fluxion of the continual Product of three, four, five, or any other Number, of flowing Quantities, may be determined.

Thus, let the Fluxion of yzu, where the Number of Factors is 3, be first required: Then, by putting $x = zu$, our given Expression will be reduced to yx; and its Fluxion will be $y\dot{x} + x\dot{y}$ (*by Prop.* 2). But, x being $= zv$, and therefore $\dot{x} = z\dot{v} + v\dot{z}$ (*by the same*), if these Values be substituted in $y\dot{x} + x\dot{y}$, it will become $y \times \overline{z\dot{v} + v\dot{z}} + zv\dot{y} = yz\dot{v} + y\dot{z}v + \dot{y}zv$, the true Fluxion of yzu, required.

Again, if the Fluxion of $yzvw$, where the Number of Factors is four, was to be demanded; then, by making $x = zvw$, the Quantity proposed will be reduced to yx; and its Fluxion will therefore be expressed by $y\dot{x} + x\dot{y}$; which, because x is $= zvw$, and $\dot{x} = zv\dot{w} + z\dot{v}w + \dot{z}vw$ (*as appears from above*) will be likewise expressed by $y \times \overline{zv\dot{w} + z\dot{v}w + \dot{z}vw} + zvw\dot{y}$, or $yzv\dot{w} + yz\dot{v}w + y\dot{z}vw + \dot{y}zvw$.

In the same manner the Fluxion of $yzvws$, will appear to be $yzvw\dot{s} + yzv\dot{w}s + yz\dot{v}ws + y\dot{z}vws + \dot{y}zvws$; and so of others.

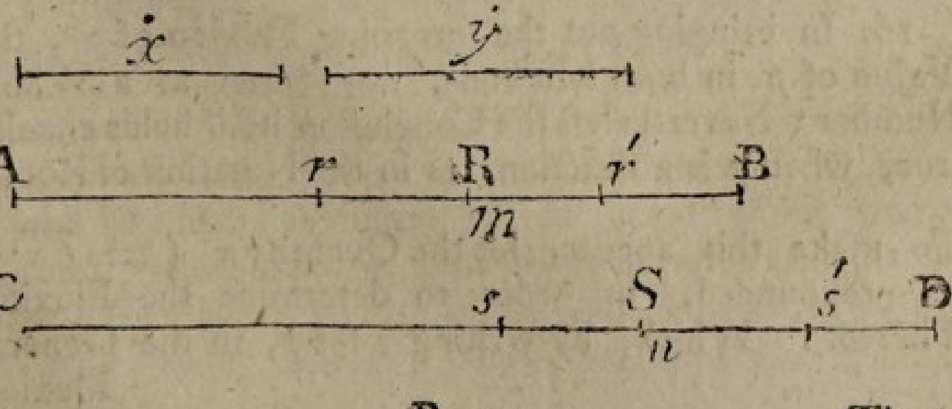
14. From the Fluxions thus determined, the Fluxion of any Power of a variable Quantity (whose Exponent is a whole positive Number) is very readily obtained; nothing more being herein required, than to consider all the Factors as equal among themselves. Thus, the

the Fluxion of yvw being found $yv\dot{w}+y\dot{v}w+\dot{y}vw$, it is plain, if both v and w be ſuppoſed equal to y, that the Fluent, or Quantity propounded will become y^3, and its Fluxion $yy\dot{y}+y\dot{y}y+\dot{y}yy=3y^2\dot{y}$.

Thus, alſo, becauſe the Fluxion of $yzvw$ is $yzv\dot{w}+yz\dot{v}w+y\dot{z}vw+\dot{y}zvw$, it appears that the Fluxion of y^4 will be truly expreſſed by $4y^3\dot{y}$. And, from the given Fluxion of $yzvws$, That of y^5 will in like manner appear to be $5y^4\dot{y}$. From whence the Law of Continuation is manifeſt; the Fluxion of y^p being univerſally expounded by $py^{p-1}\dot{y}$.

15. This laſt General Concluſion, which is of very great Importance in the Buſineſs of Fluxions, being the Reſult of ſeveral Deductions, whereby its Truth and Evidence may, perhaps, loſe a Part of their Force, a more direct Inveſtigation of the *ſame* may not here be amiſs; though the former Method will, doubtleſs, appear the moſt eaſy and proper for Beginners, to whom, the manner of working by General Indices, is not plain and familiar.

Conceive two Points m and n to move, at the ſame Time, from two other Points A and C, along the Right-lines AB and CD; and let every Thing be ſuppoſed as in Prop. 1; *only*, let the Meaſure of the Diſtance deſcribed by the Point n be, *always*, equal to the p Power, (inſtead of the Square) of That deſcribed by the other Point m, moving uniformly.



 Then,

Then, ſince by Hypotheſis, the Value of CS is here $= x^p$, and That of C*s* $= u^p$, *(See the ſecond Solution to the foreſaid Prop.)* it is evident that the Diſtance *s*S, deſcribed in the ſame Time with *r*R ($= x-u$), will be truly defined by $x^p - u^p$, or its Equal $\overline{x-u} \times \overline{x^{p-1} + x^{p-2}u + x^{p-3}u^2 + x^{p-4}u^3 + --- + u^{p-1}}$. Whence, ſuppoſing $\dot{x}$ to denote the Meaſure of the uniform Velocity of *m*, it will appear, by Reaſoning as in the ſaid Propoſition, that the Meaſure of the Velocity of *n*, at any Place *s*, on This-ſide of S, muſt be leſs than $\dot{x} \times \overline{x^{p-1} + x^{p-2}u + x^{p-3}u^2 --- + u^{p-1}}$, and conſequently leſs than $\dot{x} \times p x^{p-1}$; ſeeing each of the (p) Terms of the ſaid Series (the Firſt only excepted) is leſs than x^{p-1}, u being leſs than x.

Again, by conſidering r' and s' as two other, cotemporary Poſitions, beyond R and S, and Reaſoning in the ſame manner, the Meaſure of the Velocity at s' will appear to be greater, now, than the aboveſaid Quantity $\dot{x} \times p x^{p-1}$.

Therefore, as the Velocity increaſes continually, and ſeeing the Value thereof, before the Point arrives at S, is every-where leſs; and afterwards, every-where greater, than $\dot{x} \times p x^{p-1}$ (or $p x^{p-1} \dot{x}$) it is evident, that, at S, it muſt be neither leſſer nor greater, but exactly equal to $p x^{p-1} \dot{x}$: Which Quantity is therefore the true Fluxion of x^p; and agrees exactly with That determined above.

16. In bringing out the foregoing Fluxion of x^p, the Value of p, in both Methods, was taken as a Whole Number: Nevertheleſs the Concluſion itſelf holds equally true, when p is a Fraction, as in the Notation of Roots.

To make this appear, let the Quantity $x^{\frac{3}{2}}$ ($= \sqrt{x^3}$) be propounded, in order to determine the Fluxion Thereof: Which, by writing $\frac{3}{2}$ for p, in the General Fluxion

Fluxion $px^{p-1}\dot{x}$, comes out $\frac{3}{2}x^{\frac{1}{2}}\dot{x}$. Now, to prove that this is the true Fluxion, put $y = x^{\frac{3}{2}}$, the Quantity given; and then, by ſquaring both Sides, you will have $y^2 = x^3$; which, in Fluxions, gives $2y\dot{y} = 3x^2\dot{x}$ (as has been already ſhewn). This, by ſubſtituting for y, becomes $2x^{\frac{3}{2}}\dot{y} = 3x^2\dot{x}$. Whence, by Diviſion, $\dot{y} = \frac{3}{2}x^{\frac{1}{2}}\dot{x}$, *the very ſame as before.*

But, to demonſtrate the ſame Thing in a General Manner, ſuppoſe the Fluxion of $x^{\frac{m}{n}}$ to be required (m and n being any whole Numbers, whatever) Put $y = x^{\frac{m}{n}}$; and then, by raiſing both Sides to the Power n, you will have $y^n = x^m$: Which, in Fluxions, gives $ny^{n-1}\dot{y} = mx^{m-1}\dot{x}$; and, conſequently, $\dot{y} = \frac{m}{n} \times \frac{x^{m-1}\dot{x}}{y^{n-1}} = \frac{m}{n} \times$

$$\frac{yx^{m-1}\dot{x}}{y^n} = \frac{m}{n} \times \frac{yx^{m-1}\dot{x}}{x^m} = \frac{m}{n} \times \frac{x^{\frac{m}{n}}x^{m-1}\dot{x}}{x^m} = \frac{m}{n} \times x^{\frac{m}{n}-1}\dot{x}.$$

Which Value, of the Fluxion of $x^{\frac{m}{n}}$, is evidently the very ſame, as That ariſing by expounding p in the General Fluxion ($px^{p-1}\dot{x}$) by $\frac{m}{n}$; *which was to be proved.*

17. Now, from what has been thus far delivered, the following practical Rules, for determing the Fluxions of Algebraic Quantities, are obtained.

RULE I.

To find the Fluxion of any given Power of a flowing Quantity.

Multiply the Fluxion of the Root by the Exponent of the given Power, and the Product into that Power of the ſame

Root which arises by subtracting Unity from the given Exponent.

The Reason of this Rule is seen above; the Rule itself being nothing more than $p\dot{x}x^{p-1}$ (the Fluxion of x^p) expressed in Words.

RULE II.

To find the Fluxion of the Product of several variable Quantities, multiplied together.

Multiply the Fluxion of each, by the Product of the rest of the Quantities; so shall the Sum of all the Products thus arising be the true Fluxion required.

The Reason of which is, likewise, evident from what has been already delivered. *See Art.* 13.

RULE III.

To find the Fluxion of a Fraction, arising from the Division of one variable Quantity by another.

From the Fluxion of the Numerator, drawn into the Denominator, subtract the Fluxion of the Denominator, drawn into the Numerator; and divide the Remainder by the Square of the Denominator.

This appears from $\frac{y\dot{z}-z\dot{y}}{yy}$, the Fluxion of $\frac{z}{y}$, determined in *Art.* 12.

Though I might here, with Propriety enough, put an End to this Part, as my professed Design therein extends no farther than Giving the Young Beginner some Account of the Nature, and First Principles, of Fluxions, together with the Investigation of the Fundamental Rules exhibited above; nevertheless, as different Ways of bringing out the same Truths have often a very good Effect, and seeing the Fluxions of all Quantities whatever (whether Powers, Fractions, &c.) are deducible from the Fluxion of a Rectangle, I shall, therefore, subjoin a different Method, whereby the said

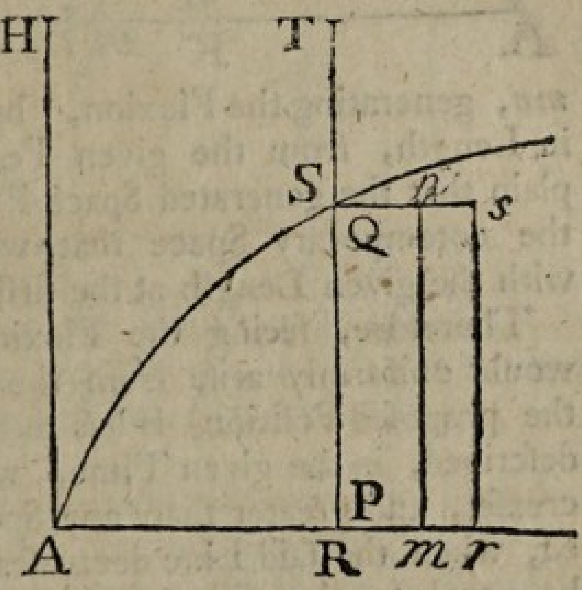
Fluxion

Fluxion of a Rectangle *(given by Prop. 2.)* may be investigated: In order to which it will be requisite, first of all, to premise the following

LEMMA.

18. *The Fluxion of a curvilineal Space* ARS, *generated by the Ordinate* RS *(or the intercepted Part of a Right-line* RT *(moving parallel to itself, is equal to the Rectangle* (Rs) *under the said Ordinate, and the Fluxion* (Rr) *of the Abscissa* AR.

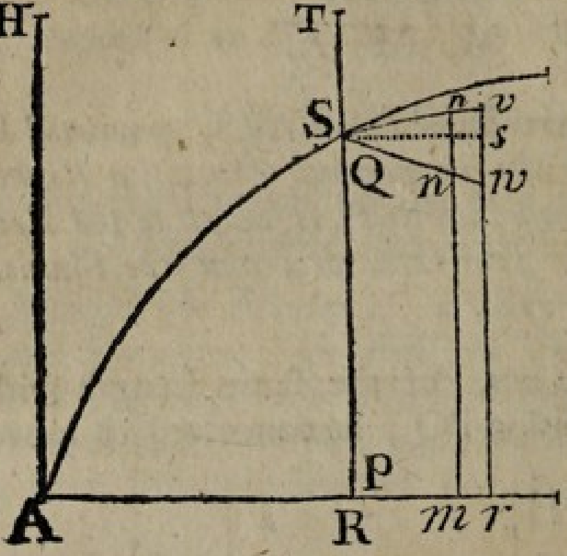
For, let a Right-line *mn*, of the same Length with RS in the proposed Position PQ, be conceived to move from thence, parallel to itself, with the same Celerity that the generating Line itself has in that Position: By which Means the Rectangle P*rs*Q will be *uniformly* generated, with the very Celerity by which it begins to be generated, or, by which the Space ARS is increased in the proposed Position PQ; since both the Length, and Velocity of *mn*, are the same as Those of RS in the said Position. Hence the Rectangle so generated must, consequently, be the true Fluxion of the Space ARS, *by the Definition.*



But the same Thing may be otherwise made to appear, from a different, and more logical Method of Arguing; by proving that the required Fluxion can neither be greater, nor less, than the said Rectangle.

Thus,

Thus, if the Line *mn* (while it moves uniformly on towards *rs*) be ſuppoſed to increaſe in Length, the Area P*mn*Q, generated thereby, will evidently be greater than That which would uniformly ariſe in the ſame Time, with the given Length at the firſt Poſition PQ; ſince the new Parts, produced each ſucceeding Moment (as the generating Line continues to lengthen) are greater and greater.



And, in the ſame manner, if the Line *mn*, generating the Fluxion, be ſuppoſed to decreaſe, in Length, from the given Poſition PQ, it is equally plain that the generated Space P*mn*Q will be leſs than the cotemporary Space that would, uniformly, ariſe with the given Length at the firſt Poſition PQ.

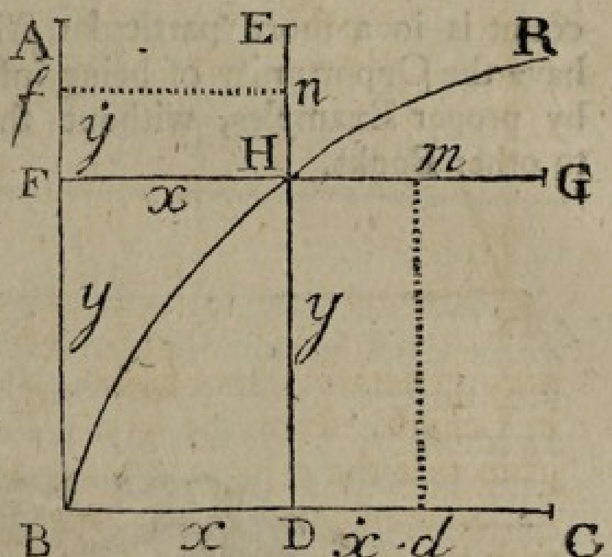
Therefore, ſeeing the Fluxion (or the Space that would uniformly ariſe from the generating Celerity at the propoſed Poſition) is leſs than any Space that can be deſcribed, in the given Time, when the Line *mn* increaſes, and greater than any Space that can be deſcribed, when the ſaid Line decreaſes; it muſt conſequently be equal to that Space which will ariſe, when the Length of the ſaid Line, from the given Poſition, is ſuppoſed neither to increaſe nor decreaſe; that is, when the generated Space P*mn*Q is a Rectangle, as in the preceding Figure.

PRO.

PROPOSITION.

19. *To determine the Fluxion of the Product, or Rectangle, of two variable Quantities* (x *and* y).

Let two Right-lines DE and FG, perpendicular to each other, be conceived to move from two other Right-lines BA and BC, continually parallel to themselves, and thereby generate the variable Rectangle DF: Let the Path of their Intersection, or the Place of the Angle H, be the Line BHR, dividing the generated Rectangle DF in two Parts, BDH and BHF: Moreover let



D*d* ($\dot{x}$) and F*f* ($\dot{y}$) be the Fluxions of the Sides BD (*x*) and BF (*y*); and suppose *dm* and *fn* to be drawn parallel, and equal, to DH and FH, respectively.

Then, since, *by the preceding Lemma*, the Fluxion of the Space or Area BDH, is truly expressed by the Rectangle D*m* ($=y\dot{x}$) and That of the Space or Area BFH, by the Rectangle F*n* ($= x\dot{y}$), it follows, because equal Quantities have equal Fluxions, that the Fluxion of the proposed Rectangle xy ($=$ BDH $+$ BFH) is truly expressed by $x\dot{y} + y\dot{x}$, the very Expression before determined: *See Art.* 11.

It

It may, perhaps, be expected, that I ſhould now give ſome Inſtances of the Uſe and Application of the Theory hitherto explained, in the Reſolution of Problems: But, having inſiſted very largely on this Head in my *Doctrine and Application of Fluxions*, I ſhall take the Liberty to recommend that Work, to the Peruſal of Such as are deſirous of farther Information in the Matter. Thoſe, for whoſe Uſe the above Account is in a more particular Manner deſigned, may have the Opportunity of being inſtructed in the Practice, by proper Examples, without the Trouble of turning to other Books.

PART VI.

TREATING OF

The VALUATION *of* ANNUITIES, *for single and joint* LIVES.

THE following Tract being designed for *real Use*, and adapted to the Understanding of Such whose Acquisitions, in the Mathematical Way, extend not beyond Vulgar, or Decimal Arithmetick; it therefore seem'd proper to omit the Investigation of several Particulars therein delivered, depending upon higher Principles. This, I hope, my Mathematical Readers will have the Candour to excuse; when They consider the Importance of the Subject to a Multitude of Persons, who cannot be expected to see into the Nature and Usefulness of an Algebraical Process.—Without further Apology I shall therefore proceed now to my Purpose; which is to exhibit, in a plain, easy manner, by Means of proper Tables, the practical Solutions of the most useful and necessary Problems on the Subject; without any Intermixture of *Analytical Operations* (whereof the bare Appearance, to Those unacquainted therewith, would seem to cast a Darkness over the *Whole*) However, for the sake of Those that are Judges, I shall, in an annex'd Scholium, give the Reasons of what is most material, and necessary to be explained.

TABLE

TABLE I.

Shewing the Probability of the Duration of Life, from Observations on the Bills of Mortality of the City of London.

Years.	N°. Persons.	Years.	N°. Persons.	Years.	N°. Persons.	Years.	N°. Persons.
0	1000	20	360	40	229	60	102
1	680	21	355	41	222	61	97
2	547	22	350	42	214	62	92
3	496	23	345	43	206	63	87
4	469	24	339	44	199	64	82
5	452	25	333	45	192	65	77
6	440	26	327	46	185	66	72
7	430	27	321	47	178	67	67
8	422	28	315	48	171	68	62
9	415	29	308	49	165	69	58
10	410	30	301	50	159	70	54
11	405	31	294	51	153	71	50
12	400	32	287	52	147	72	46
13	395	33	280	53	141	73	42
14	390	34	273	54	135	74	39
15	385	35	266	55	129	75	36
16	380	36	259	56	123	76	33
17	375	37	252	57	117	77	30
18	370	38	245	58	112	78	27
19	365	39	237	59	107	79	25

TABLE II.

Exhibiting the Number of Years of Life, which a Person, of a given Age, may, upon an Equality of Chance, expect to enjoy; according to the aforesaid Observations.

Age.	Exp.	Age.	Exp.	Age.	Exp.	Age.	Exp.
1	27.0	21	28.3	41	19.2	61	12.0
2	32.0	22	27.7	42	18.8	62	11.6
3	34.0	23	27.2	43	18.5	63	11.2
4	35.6	24	26.6	44	18.1	64	10.8
5	36.0	25	26.1	45	17.8	65	10.5
6	36.0	26	25.6	46	17.4	66	10.1
7	35.8	27	25.1	47	17.0	67	9.8
8	35.6	28	24.6	48	16.7	68	9.4
9	35.2	29	24.1	49	16.3	69	9.1
10	34.8	30	23.6	50	16.0	70	8.8
11	34.3	31	23.1	51	15.6	71	8.4
12	33.7	32	22.7	52	15.2	72	8.1
13	33.1	33	22.3	53	14.9	73	7.8
14	32.5	34	21.9	54	14.5	74	7.5
15	31.9	35	21.5	55	14.2	75	7.2
16	31.3	36	21.1	56	13.8	76	6.8
17	30.7	37	20.7	57	13,4	77	6.4
18	30.1	38	20.3	58	13.1	87	6.0
19	29.5	39	19.9	59	12.7	79	5.5
20	28.9	40	19.6	60	12.4	80	5.0

TABLE III.

A TABLE *shewing the present Value of one Pound, to be received at the end of any number of Years, not exceeding* 90, *discounting at the Rates of* 5, 4, *and* 3 per Cent. *compound Interest.*

Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*	Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*
1	.9524	.9615	.9709	21	.3589	.4388	.5375
2	.9070	.9245	.9426	22	.3418	.4219	.5219
3	.8638	.8890	.9151	23	.3255	.4057	.5067
4	.8227	.8548	.8885	24	.3100	.3901	.4919
5	.7835	.8219	.8626	25	.2953	.3757	.4776
6	.7462	.7903	.8375	26	.2812	.3607	.4637
7	.7107	.7599	.8131	27	.2678	.3468	.4502
8	.6768	.7307	.7894	28	.2551	.3335	.4371
9	.6446	.7026	.7664	29	.2429	.3206	.4243
10	.6139	.6756	.7441	30	.2314	.3083	.4120
11	.5847	.6496	.7224	31	.2204	.2965	.4000
12	.5568	.6246	.7014	32	.2099	.2851	.3883
13	.5303	.6006	.6809	33	.1999	.2741	.3770
14	.5051	.5775	.6611	34	.1903	.2636	.3660
15	.4810	.5553	.6419	35	.1813	.2534	.3554
16	.4581	.5339	.6232	36	.1726	.2437	.3450
17	.4363	.5134	.6050	37	.1644	.2343	.3350
18	.4155	.4936	.5874	38	.1566	.2253	.3252
19	.3957	.4746	.5703	39	.1491	.2166	.3158
20	.3769	.4564	.5537	40	.1420	.2083	.3066

Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*	Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*
41	.1353	.2003	.2977	66	.0399	.0751	.1421
42	.1288	.1926	.2890	67	.0380	.0722	.1380
43	.1227	.1852	.2805	68	.0362	.0695	.1340
44	.1169	.1780	.2724	69	.0345	.0668	.1301
45	.1113	.1712	.2644	70	.0329	.0642	.1263
46	.1060	.1646	.2567	71	.0313	.0617	.1226
47	.1010	.1583	.2493	72	.0298	.0594	.1190
48	.0961	.1522	.2420	73	.0284	.0571	.1156
49	.0916	.1463	.2349	74	.0270	.0549	.1122
50	.0872	.1407	.2281	75	.0257	.0528	.1089
51	.0830	.1353	.2215	76	.0245	.0508	.1058
52	.0791	.1301	.2150	77	.0233	.0488	.1027
53	.0753	.1251	.2087	78	.0222	.0469	.0997
54	.0717	.1203	.2027	79	.0212	.0451	.0968
55	.0683	.1156	.1968	80	.0202	.0434	.0940
56	.0651	.1112	.1910	81	.0192	.0417	.0912
57	.0620	.1069	.1855	82	.0183	.0401	.0886
58	.0590	.1028	.1801	83	.0174	.0386	.0860
59	.0562	.0989	.1748	84	.0166	.0371	.0835
60	.0535	.0951	.1697	85	.0158	.0357	.0811
61	.0510	.0914	.1648	86	.0151	.0343	.0787
62	.0485	.0879	.1600	87	.0143	.0330	.0764
63	.0462	.0845	.1553	88	.0136	.0317	.0742
64	.0440	.0813	.1508	89	.0130	.0305	.0720
65	.0419	.0781	.1464	90	.0124	.0292	.0690

TABLE IV.

A TABLE *shewing the present Value of an Annuity of one Pound for any number of Years, not exceeding* 90, *when Interest is at* 5, 4, *and* 3 per Cent.

Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*	Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*
1	0.952	0.961	0.971	21	12.821	14.029	15.415
2	1.859	1.886	1.913	22	13.163	14.451	15.939
3	2.723	2.775	2.829	23	13.488	14.857	16.444
4	3.546	3.630	3.717	24	13.799	15.247	16.936
5	4.329	4.452	4.580	25	14.094	15.622	17.413
6	5.076	5.242	5.497	26	14.375	15.983	17.877
7	5.786	6.002	6.230	27	14.643	16.329	18.327
8	6.463	6.733	7.020	28	14.898	16.663	18.764
9	7.108	7.435	7.786	29	15.141	16.984	19.188
10	7.721	8.111	8.530	30	15.372	17.292	19.600
11	8.306	8.760	9.253	31	15.593	17.588	20.000
12	8 863	9.385	9.954	32	15.803	17.873	20.389
13	9.393	9.985	10.635	33	16.002	18.148	20.766
14	9.899	10.563	11.296	34	16.193	18.411	21.132
15	10.380	11.118	11.938	35	16.374	18.665	21.487
16	10.838	11.652	12.561	36	16.547	18.908	21.832
17	11.274	12.166	13.166	37	16.711	19.142	22.167
18	11.689	12.659	13.753	38	16.868	19.368	22.492
19	12.085	13.134	14.324	39	17.017	19.584	22.808
20	12.462	13.590	14.877	40	17.159	19.793	23.115

Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*	Years.	Value at 5 *per cent.*	Value at 4 *per cent.*	Value at 3 *per cent.*
41	17.294	19.993	23.412	66	19.201	23.122	28.595
42	17.423	20.186	23.701	67	19.239	23.194	28.733
43	17.546	20.371	23.982	68	19.275	23.263	28.867
44	17.663	20.549	24.254	69	19.310	23.330	28.997
45	17.774	20.720	24.519	70	19.343	23.394	29.123
46	17.880	20.885	24.775	71	19.374	23.456	29.246
47	17.981	21.043	25.025	72	19.404	23.516	29.365
48	18.077	21.195	25.267	73	19.432	23.573	29.481
49	18.169	21.341	25.502	74	19.459	23.628	29.593
50	18.256	21.482	25.730	75	19.485	23.680	29.702
51	18.339	21.617	25.951	76	19.509	23.731	29.808
52	18.418	21.747	26.166	77	19.533	23.780	29.910
53	18.493	21.873	26.375	78	19.555	23.827	30.010
54	18.565	21.993	26.578	79	19.576	23.872	30.108
55	18.633	22.109	26.774	80	19.596	23.915	30.201
56	18.698	22.220	26.965	81	19.615	23.957	30.292
57	18.760	22.327	27.151	82	19.634	23.997	30.381
58	18.819	22.430	27.331	83	19.652	24.036	30.467
59	18.876	22.528	27.506	84	19.668	24.073	30.550
60	18.929	22.623	27.676	85	19.684	24.108	30.631
61	18.980	22.715	27.840	86	19.699	24.143	30.710
62	19.029	22.803	28.000	87	19.713	24.176	30.786
63	19.075	22.887	28.156	88	19.727	24.207	30.860
64	19.119	22.968	28.306	89	19.740	24.238	30.932
65	19.161	23.046	28.453	90	19.752	24.267	31.002

TABLE V.

For the Valuation of Annuities upon one LIFE.

Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*
6	14.1	16.2	18.8	21	12.9	14.7	17.0
7	14.2	16.3	18.9	22	12.7	14.5	16.8
8	14.3	16.4	19.0	23	12.6	14.3	16.5
9	14.3	16.4	19.0	24	12.4	14.1	16.3
10	14.3	16.4	19.0	25	12.3	14.0	16.1
11	14.3	16.4	19.0	26	12.1	13.8	15.9
12	14.2	16.3	18.9	27	12.0	13.6	15.6
13	14.1	16.2	18.7	28	11.8	13.4	15.4
14	14.0	16.0	18.5	29	11.7	13.2	15.2
15	13.9	15.8	18.3	30	11.6	13.1	15.0
16	13.7	15.6	18.1	31	11.4	12.9	14.8
17	13.5	15.4	17.9	32	11.3	12.7	14.6
18	13.4	15.2	17.6	33	11.2	12.6	14.4
19	13.2	15.0	17.4	34	11.0	12.4	14.2
20	13.0	14.8	17.2	35	10.9	12.3	14.1

Age.	Year's Purch. at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch. at 3 *per cent.*	Age.	Year's Purch at 5 *per cent.*	Year's Purch. at 4 *per cent.*	Year's Purch at 3 *per cent.*
36	10.8	12.1	13.9	56	8.4	9.1	10.1
37	10.6	11.9	13.7	57	8.2	8.9	9.9
38	10.5	11.8	13.5	58	8.1	8.7	9.6
39	10.4	11.6	13.3	59	8.0	8.6	9.4
40	10.3	11.5	13.2	60	7.9	8.4	9.2
41	10.2	11.4	13.0	61	7.7	8.2	8.9
42	10.1	11.2	12.8	62	7.6	8.1	8.7
43	10.0	11.1	12.6	63	7.4	7.9	8.5
44	9.9	11.0	12.5	64	7.3	7.7	8.3
45	9.8	10.8	12.3	65	7.1	7.5	8.0
46	9.7	10.7	12.1	66	6.9	7.3	7.8
47	9.5	10.5	11.9	67	6.7	7.1	7.6
48	9.4	10.4	11.8	68	6.6	6.9	7.4
49	9.3	10.2	11.6	69	6.4	6.7	7.1
50	9.2	10.1	11.4	70	6.2	6.5	6.9
51	9.0	9.9	11.2	71	6.0	6.3	6.7
52	8.9	9.8	11.0	72	5.8	6.1	6.5
53	8.8	9.6	10.7	73	5.6	5.9	6.2
54	8.6	9.4	10.5	74	5.4	5.6	5.9
55	8.5	9.3	10.3	75	5.2	5.4	5.6

TABLE VI.

Exhibiting the Value of an Annuity for a given Term of Years, on the Contingency of its ceasing upon the Extinction of an assigned Life.

Given Age.	N. Years.	Val. at 5 per Cent.	Val. at 4 p. Cent.	Val. at 3 per Cent.	Given Age.	N. Years	Val. at 5 per Cent.	Val. at 4 per Cent.	Val at 3 per Cent.
5	5	4.0	4.1	4.2	15	5	4.1	4.2	4.4
	10	7.0	7.3	7.7		10	7.1	7.5	7.9
	15	9.1	9.8	10.5		15	9.3	10.0	10.7
	20	10.7	11.7	12.8		20	10.8	11.8	12.9
	25	11.8	13.0	14.5		25	11.8	13.1	14.5
	30	12.6	14.1	15.8		30	12.5	14.0	15.7
	35	13.2	14.9	16.8		35	13.0	14.6	16.6
	40	13.6	15.4	17.5		40	13.3	15.0	17.2
	45	13.8	15.7	18.0		45	13.5	15.3	17.6
	50	13.9	15.9	18.3		50	13.6	15.5	17.9
10	5	4.1	4.2	4.3	20	5	4.1	4.2	4.3
	10	7.2	7.6	8.0		10	7.0	7.4	7.8
	15	9.4	10.1	10.8		15	9.1	9.7	10.5
	20	11.0	12.0	13.1		20	10.5	11.4	12.5
	25	12.1	13.4	14.8		25	11.4	12.6	13.9
	30	12.8	14.4	16.2		30	12.0	13.4	15.0
	35	13.3	15.1	17.1		35	12.5	13.9	15.7
	40	13.7	15.5	17.7		40	12.7	14.3	16.3
	45	13.9	15.8	18.2		45	12.8	14.5	16.6
	50	14.0	16.0	18.5		50	12.9	14.6	16.8

Given Age.	N. Years.	Val. at 5 *per Cent.*	Val. at 4 *per Cent.*	Val. at 3 *per Cent.*
25	5	4.1	4.2	4.3
	10	6.9	7.3	7.6
	15	8.9	9.5	10.1
	20	10.2	11.0	12.0
	25	11.0	12.1	13.4
	30	11.5	12.8	14.3
	35	11.8	13.2	15.0
	40	12.0	13.5	15.4
	45	12.1	13.7	15.6
30	5	4.0	4.1	4.2
	10	6.8	7.1	7.5
	15	8.6	9.2	9.8
	20	9.8	10.6	11.5
	25	10.6	11.6	12.7
	30	11.0	12.2	13.5
	35	11.2	12.6	14.0
	40	11.4	12.8	14.4
35	5	4.0	4.1	4.2
	10	6.6	6.9	7.3
	15	8.3	8.9	9.5
	20	9.4	10.2	11.1
	25	10.1	11.1	12.2
	30	10.5	11.6	13.0
	35	10.7	11.9	13.5
40	5	3.9	4.0	4.1
	10	6.4	6.8	7.1
	15	8.0	8.6	9.2
	20	9.1	9.8	10.6
	25	9.7	10.6	11.6
	30	10.0	11.0	12.2

Given Age.	N. Years.	Val. at 5 *per Cent.*	Val. at 4 *per Cent.*	Val. at 3 *per Cent.*
45	5	3.9	4.0	4.1
	10	6.3	6.7	7.0
	15	7.9	8.4	9.0
	20	8.8	9.5	10.3
	25	9.4	10.2	11.2
	30	9.6	10.6	11.7
50	5	3.8	3.9	4.0
	10	6.2	6.5	6.8
	15	7.6	8.2	8.8
	20	8.5	9.1	9.9
	25	8.9	9.7	10.6
55	5	3.8	3.9	4.0
	10	6.1	6.4	6.7
	15	7.4	7.9	8.4
	20	8.0	8.7	9.4
	25	8.3	9.0	9.9
60	5	3.7	3.8	3.9
	10	5.8	6.1	6.4
	15	7.0	7.4	7.9
	20	7.5	8.0	8.7
65	5	3.6	3.7	3.8
	10	5.4	5.7	6.0
	15	6.4	6.8	7.2
70		3.4	3.5	3.6
	10	5.0	5.2	5.4

TABLE VII.

Serving as a Supplement to TABLE VI.

Diff.	1	2	3	4
3.4	0.8	1.6	2.3	2.9
3.3	0.8	1.5	2.2	2.8
3.2	0.8	1.5	2.1	2.7
3.1	0.7	1.4	2.0	2.6
3.0	0.7	1.3	1.9	2.5
2.9	0.7	1.3	1.9	2.4
2.8	0.7	1.3	1.8	2.3
2.7	0.6	1.2	1.7	2.2
2.6	0.6	1.2	1.7	2.2
2.5	0.6	1.1	1.6	2.1
2.4	0.6	1.1	1.6	2.0
2.3	0.5	1.0	1.5	1.9
2.2	0.5	1.0	1.4	1.8
2.1	0.5	0.9	1.3	1.7
2.0	0.5	0.9	1.3	1.7
1.9	0.4	0.8	1.2	1.6
1.8	0.4	0.8	1.1	1.5
1.7	0.4	0.7	1.1	1.4
1.6	0.4	0.7	1.0	1.3
1.5	0.4	0.7	1.0	1.3
1.4	0.3	0.6	0.9	1.2
1.3	0.3	0.6	0.9	1.1
1.2	0.3	0.6	0.8	1.0
1.1	0.3	0.5	0.7	0.9
1.0	0.2	0.4	0.6	0.8

TABLE VIII.

Shewing the Value of an Annuity for two joint Lives (i. e. *for as long as They exist together*).

Age of the Youngest.	Age of the Eldest.	Val. at 5 *per Cent.*	Val. at 4 *per Cent.*	Val. at 3 *per Cent.*
10	10	11.6	13.0	14.7
	15	11.3	12.7	14.3
	20	10.8	12.2	13.8
	25	10.2	11.6	13.1
	30	9.7	10.9	12.3
	35	9.1	10.2	11.5
	40	8.6	9.6	10.7
	45	8.1	9.0	10.0
	50	7.6	8.4	9.3
	55	7.1	7.8	8.6
	60	6.6	7.2	7.8
	65	6.1	6.5	6.9
	70	5.5	5.8	6.1
	75	4.9	5.1	5.3
15	15	11.0	12.3	13.9
	20	10.5	11.8	13.3
	25	10.1	11.2	12.6
	30	9.5	10.6	11.9
	35	9.0	10.0	11.2
	40	8.5	9.4	10.4
	45	8.0	8.8	9.6
	50	7.5	8.2	8.9
	55	7.0	7.6	8.2
	60	6.5	7.0	7.5
	65	6.0	6.4	6.8
	70	5.4	5.7	6.0
	75	4.8	5.0	5.2
20	20	10.1	11.3	12.8
	25	9.7	10.8	12.2
	30	9.2	10.3	11.6
	35	8.8	9.8	10.9
	40	8.4	9.2	10.2
	45	7.9	8.6	9.5
	50	7.4	8.0	8.8
	55	6.9	7.5	8.1
	60	6.4	6.9	7.4
	65	5.9	6.3	6.7
	70	5.4	5.7	6.0
	75	4.8	5.0	5.2
25	25	9.4	10.5	11.8
	30	9.0	10.1	11.3
	35	8.6	9.6	10.7
	40	8.2	9.1	10.0
	45	7.8	8.5	9.4
	50	7.3	7.9	8.7
	55	6.8	7.4	8.0
	60	6.3	6.8	7.3
	65	5.8	6.2	6.6
	70	5.3	5.6	5.9
	75	4.7	4.9	5.1
30	30	8.6	9.6	10.8
	35	8.3	9.2	10.3
	40	8.0	8.8	9.7

Age of the Youngest.	Age of the Eldest.	Val. at 5 *per Cent.*	Val. at 4 *per Cent.*	Val. at 3 *per Cent.*
30	45	7.6	8.3	9.1
	50	7.2	7.8	8.5
	55	6.7	7.3	7.9
	60	6.2	6.7	7.2
	65	5.7	6.1	6.5
	70	5.2	5.5	5.8
	75	4.7	4.9	5.1
35	35	8.0	8.8	9.9
	40	7.7	8.5	9.4
	45	7.4	8.1	8.9
	50	7.0	7.6	8.3
	55	6.6	7.1	7.7
	60	6.1	6.5	7.1
	65	5.6	6.0	6.4
	70	5.1	5.4	5.7
	75	4.6	4.8	5.0
40	40	7.3	8.1	9.1
	45	7.1	7.8	8.7
	50	6.8	7.4	8.2
	55	6.4	6.9	7.6
	60	6.0	6.4	7.0
	65	5.5	5.9	6.4
	70	5.1	5.4	5.7
	75	4.6	4.8	5.0
45	45	6.7	7.4	8.3
	50	6.5	7.1	7.9
	55	6.2	6.7	7.4
	60	5.8	6.3	6.8

Age of the Youngest.	Age of the Eldest.	Val. at 5 *per Cent.*	Val. at 4 *per Cent.*	Val. at 3 *per Cent.*
45	65	5.4	5.8	6.3
	70	5.0	5.3	5.6
	75	4.5	4.7	4.9
50	50	6.2	6.8	7.6
	55	6.0	6.5	7.2
	60	5.7	6.1	6.7
	65	5.3	5.7	6.2
	70	4.9	5.2	5.5
	75	4.4	4.6	4.8
55	55	5.7	6.2	6.9
	60	5.5	5.9	6.5
	65	5.2	5.6	6.0
	70	4.8	5.1	5.4
	75	4.3	4.5	4.7
60	60	5.2	5.6	6.1
	65	4.9	5.3	5.7
	70	4.6	4.9	5.2
	75	4.2	4.4	4.6
65	65	4.7	5.0	5.4
	70	4.4	4.6	4.9
	75	4.0	4.2	4.4
70	70	4.2	4.4	4.6
	75	3.9	4.0	4.2
75	75	3.6	3.7	3.8

TABLE IX.

For the Value of an Annuity upon the longest of two given Lives.

Age of the Youngest.	Age of the Eldest.	Val. at 5 per Cent.	Val. at 4 per Cent.	Val. at 3 per Cent.
10	10	17.1	19.9	23.4
	15	16.8	19.5	22.9
	20	16.6	19.1	22.5
	25	16.4	18.8	22.2
	30	16.2	18.6	21.9
	35	16.1	18.4	21.6
	40	16.0	18.3	21.4
	45	15.9	18.2	21.2
	50	15.8	18.0	20.9
	55	15.7	17.8	20.7
	60	15.5	17.6	20.4
	65	15.3	17.4	20.1
	70	15.1	17.2	19.8
	75	14.8	16.9	19.5
15	15	16.7	19.3	22.8
	20	16.4	18.9	22.3
	25	19.2	18.6	21.9
	30	16.0	18.3	21.6
	35	15.9	18.1	21.3
	40	15.7	17.9	21.1
	45	15.6	17.8	·0 9
	50	15.4	17.6	20.7
	55	15.3	17.4	20.4
	60	15.2	17.2	20.1
	65	15.0	16.9	19.8
	70	14.7	16.6	19.4
	75	14.4	16.3	18.9
20	20	15.8	18.3	21.6
	25	15.5	17.9	21.1
	30	15.3	17.6	20.7
	35	15.1	17.4	20.4
	40	15.0	17.2	20.1
	45	14.9	17.0	19.9
	50	14.7	16.8	19.6
	55	14.5	16.6	19.4
	60	14.3	16.3	19.1
	65	14.1	16.0	18.7
	70	13.8	15.7	18.2
	75	13.5	15.3	17.7
25	25	15.1	17.4	20.3
	30	14.9	17.0	19.8
	35	14.7	16.7	19.4
	40	14.5	16.5	19.2
	45	14.3	16.3	18.9
	50	14.2	16.1	18.7
	55	14.0	15.9	18.4
	60	13.8	15.6	18.0
	65	13.6	15.3	17.6
	70	13.3	15.0	17.2
	75	12.9	14.6	16.7
30	30	14.5	16.6	19.3
	35	14.2	16.2	18.8
	40	14.0	15.9	18.4
	45	13.8	15.6	18.1

Age of the Youngest.	Age of the Eldest.	Val. at 5 *per Cent.*	Val. at 4 *per Cent.*	Val. at 3 *per Cent.*
30	50	13.6	15.4	17.8
	55	13.4	15.1	17.4
	60	13.2	14.8	17.0
	65	12.9	14.5	16.6
	70	12.6	14.1	16.1
	75	12.2	13.7	15.6
35	35	13.8	15.8	18.3
	40	13.5	15.4	17.8
	45	13.3	15.1	17.4
	50	13.1	14.8	17.1
	55	12.9	14.5	16.7
	60	12.7	14.2	16.3
	65	12.4	13.8	15.8
	70	12.0	13.4	15.3
	75	11.6	13.0	14.8
40	40	13.3	15.0	17.3
	45	13.0	14.6	16.8
	50	12.7	14.2	16.3
	55	12.4	13.9	15.9
	60	12.1	13.5	15.4
	65	11.8	13.1	14.9
	70	11.4	12.7	14.5
	75	11.0	12.3	14.0
45	45	12.8	14.2	16.2
	50	12.5	13.8	15.7
	55	12.1	13.4	15.2
	60	11.7	12.9	14.7

Age of the Youngest.	Age of the Eldest.	Val. at 5 *per Cent.*	Val. at 4 *per Cent.*	Val. at 3 *per Cent.*
45	65	11.4	12.5	14.1
	70	11.0	12.0	13.6
	75	10.6	11.6	13.1
50	50	12.1	13.3	15.0
	55	11.7	12.9	14.5
	60	11.3	12.4	13.9
	65	10.9	12.0	13.3
	70	10.5	11.5	12.8
	75	10.1	11.0	12.3
55	55	11.3	12.4	13.6
	60	10.9	11.9	13.0
	65	10.5	11.3	12.4
	70	10.0	10.8	11.8
	75	9.5	10.3	11.3
60	60	10.5	11.2	12.2
	65	10.0	10.6	11.5
	70	9.5	10.1	10.9
	75	9.0	9.5	10.3
65	65	9.4	10.0	10.7
	70	8.9	9.4	10.0
	75	8.3	8.7	9.3
70	70	8.2	8.6	9.2
	75	7.6	7.9	8.4
75	75	6.9	7.2	7.6

TABLE

X

TABLE X.

For finding the Value of an Annuity upon One, or more Lives, at the Rates of $3\frac{1}{4}$, $3\frac{1}{2}$, $3\frac{3}{4}$, $4\frac{1}{4}$, $4\frac{1}{2}$, *and* $4\frac{3}{4}$, per Cent. *Interest; supposing the Value, at the Rate of* 4 per Cent. *to be known, from the preceding Tables.*

Val. at 4 p. cent	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	$4\frac{1}{4}$	$4\frac{1}{2}$	$4\frac{3}{4}$
	add	add	add	ſub.	ſub.	ſub.
6	0.3	0.2	0.1	0.1	0.2	0.3
7	0.4	0.2	0.1	0.1	0.2	0.3
8	0.5	0.3	0.1	0.1	0.3	0.4
9	0.7	0.4	0.2	0.2	0.4	0.6
10	0.9	0.6	0.3	0.3	0.5	0.7
11	1.1	0.7	0.3	0.3	0.6	0.9
12	1.3	0.8	0.4	0.4	0.7	1.0
13	1.4	0.9	0.4	0.4	0.8	1.2
14	1.6	1.0	0.5	0.5	0.9	1.3
15	1.7	1.1	0.5	0.5	1.0	1.4
16	1.9	1.3	0.6	0.6	1.1	1.6
17	2.1	1.4	0.7	0.6	1.2	1.8
18	2.3	1.5	0.7	0.7	1.3	1.9
19	2.5	1.6	0.8	0.7	1.4	2.0
20	2.7	1.7	0.8	0.8	1.5	2.2

PROBLEM

PROBLEM I.

To find the present Value of any Sum of Money, to be received at the End of a given Term of Years; discounting at the Rate of 3, 4, *or* 5 per Cent. *compound Interest.*

SOLUTION.

Find, *by Tab.* III. the present Value of 1*l.* to be received at the End of the given Term; which multiply by the Number of Pounds proposed (cutting off 4 Figures from the Product, on Account of the Decimals) then the Result will be the Value sought.

EXAMPLE. Let the Sum proposed be 800*l.* the given Term 7 Years, and the Rate of Interest, 4 *per Cent.* Then the Answer will appear to be .7599 multiply'd by 800, or 607.9200*l.* that is, 607*l*: 18*s*: 5*d*, nearly.

PROBLEM II.

To find the present Value of an Annuity, certain, for a given Number of Years; according to any of the Rates of Interest specified in the preceding Problem.

SOLUTION.

Seek, *in Tab.* IV. the Number of Year's-Purchase answering to the given Term of Years; which, multiply'd by the proposed Annuity, gives the Answer.

EXAMPLE. Suppose the Annuity to be 100*l.* the Number of Years 7, and the Rate of Interest 4 *per Cent.* Then the Value sought will be 6.002 multiply'd by 800; or 600*l*: 4*s*.

PROBLEM III.

To find (according to Observations on the Bills of Mortality of the City of London) the Probability, or Proportion of Chance, that a Person, of a given Age, continues in Being a given Number of Years.

SOLUTION.

Let the given Age be 40, and the Number of Years proposed 15.

Look, *in Tab.* I. against 40 Years, and also against 55 Years, the Age to which the Person must arrive, if he lives to the End of the given Term; corresponding to which you will find the Numbers 229 and 129, respectively; shewing that, of 229 Persons who attain to the Age of Forty, only 129 reach the Age of Fifty-five: Now the Excess of 229 above 129 being 100, it is evident that the Odds, or the Ratio of the Chances, for and against, surviving the proposed Term, will be as 129 to 100, or as 9 to 7, nearly: And, in the same manner, the Answer will be found in any other Case.

Note. Though This, and the following Problem, are not immediately concern'd about the Business of Annuities, yet they are the Foundation whereon the *whole* is grounded; and therefore do not improperly fill up the Place here allotted them.

PROBLEM IV.

To find (according to the foresaid Observations) the Number of Years of Life, which a Person, of a given Age, may, upon an Equality of Chance, expect to enjoy.

SOLUTION.

Seek the given Age, *in Table* II. and against it you will have the Answer, in Years and Decimal-Parts.

Thus it will appear that, a Person, 30 Years old, may, upon an Equality of Chance, expect 23.6 Years more, for his Share of Life *.

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* *By the Expectation, or Share, of Life, is not here to be understood, that particular Period, which a Person hath an equal Chance of surviving; this last being a different, and more simple Consideration. The Expectation of a Life (to put it in the most familiar Light) may be taken as the Number of Years at which the Purchase of an Annuity, granted thereon, without Discount of Money, ought to be valued. Which Number of Years will differ more or less from the Period above mention'd, according to the different Degrees of Mortality to which the several Stages of Life are incident.—Thus, it is much more than an equal Chance (according to the Table of the Probability of the Duration of Life, p.* 254) *that an Infant, just come into the World, arrives not to the Age of* 10 *Years; yet the Expectation, or Share of Life, due to It, upon an Average, is near* 20 *Years. The Reason of which wide Difference, is, the great Excess of the Probability of Mortality in the first, tender Years of Life, above That respecting the more mature, and stronger Ages.— If the Numbers that die at every Age were to be the same, the two Quantities above specified would also be equal; but when the said Numbers become continually less and less, the Expectation must, consequently, be the greater of the Two.*

PROBLEM V.

To find the Value of an Annuity for an assigned Life.

SOLUTION.

This Problem is resolved from *Tab.* V. by looking against the given Age, under the proposed Rate of Interest; corresponding to which you will have the Number of Years-Purchase required.

Ex. Let the given Age be Thirty-six, and the Rate of Interest 4 *per Cent.* and let the proposed Annuity be 250*l.* Then the Value thereof will appear to be 12.1 Years-Purchase, or 12.1 times 250*l.* Therefore, multiplying 250 by 12.1 (and cutting off one Figure, upon account of the Decimal) the Answer comes out 3025*l.* After the same manner the Answer will be found in any other Case, falling within the Limits of the Table. But, as there may be Occasion, sometimes, to know the Values of Lives, computed at higher Rates of Interest than any There exhibited, the two following, practical, Rules are subjoined; by which the Problem is resolved, independent of Tables.

RULE I.

If the given Age is not less than 45 Years (nor greater than 85) subtract it from 92; multiply the Remainder by the *Perpetuity* *, and divide the Product by the said Remainder added to $2\frac{1}{2}$ times the *Perpetuity*; then the Quotient will be the Number of *Years Purchase* required.

* *It may, perhaps, be of Use to some of my Readers, to be informed here, that, by* the Perpetuity *of an Estate, or Annuity, is understood the Number of Years-Purchase of the* Fee Simple; *found by dividing* 100 *by the Rate* per Cent. *at which Interest is reckoned.*

EXAMPLE.

EXAMPLE.

Let the given Age be 50 Years, and the Rate of Interest 10 *per Cent.* Then, subtracting 50 from 92, there remains 42; which, multiply'd by 10, the *Perpetuity*, gives 420; this divided by 67, the Remainder increased by $2\frac{1}{2}$ times (10) the Perpetuity, quotes 6.3 nearly: Therefore, supposing the Annuity to be 100 *l.* its Value, in present Money, will be 630 *l.*

RULE II.

If the given Age is less than 45 *Years (but not less than* 10) *take* $\frac{8}{10}$ *of what it wants of* 45; *which divide by the Rate* per Cent. *increased by* 1.2; *then, if the Quotient be added to the Value of a Life of* 45 *Years, found by the preceding Rule, you will have the required Number of Years-Purchase in this Case.*

EXAMPLE.

Let the proposed Age be 20 Years, and the Rate of Interest 5 *per Cent.* Here, taking 20 from 45, there remains 25; $\frac{8}{10}$ whereof is 20; which, divided by 6.2, quotes 3.2; and this added to 9.8, the Value of a Life of 45 *(found by Rule* I.) gives 13.0, for the Number of Years-Purchase that a Life of Twenty ought to be valued at.

It will be needless, I presume, to offer any thing farther by way of Example to the preceding Rules; which bring out the Conclusions so near the true Values, computed from *real* Observations, as seldom to differ therefrom by more than about $\frac{1}{10}$ or $\frac{2}{10}$ of an Year's-Purchase.

The Observations here understood are Those whereon the foregoing Tables are grounded (as the most proper Foundation for this *Place)* But, if any Person is desirous of seeing a similar Method of Solution accommodated to the *Breslau* Observations (publish'd by Dr. *Halley*, in N° 196 of the Philosophical Transactions, which are considerably different from Those abovementioned, deduced from the Bills of Mortality of the City of *London)* what follows may, perhaps, anſwer His Expectation. The Rule is thus, *Multiply the Difference between the given Age and* 85 *Years by the Perpetuity, and divide the Product by* $\frac{8}{10}$ *of the ſaid Difference increaſed by twice the Perpetuity; then the Quotient will be the Anſwer.* Which, from the Age of Eight, to Eighty, will, for the general Part, come within leſs than $\frac{1}{8}$ of an Year's-Purchaſe of the Truth.

PROBLEM VI.

To determine the Value of an Annuity, granted for a given Term of Years, upon the Contingency of its ceaſing on the Failing of a propoſed Life, if this ſhould happen before the Expiration of the ſaid given Term.

Solution.

Find, *in Tab.* VI. *Column* 1. the Age of the aſſigned Life (or That neareſt it), and find, *in Column* 2. the propoſed Term of Years; againſt which laſt you will have the Anſwer.

Example I.

Let the Age be 10, the Number of Years 15, and the Rate of Intereſt 5 *per Cent.* Then it appears, at one View, that the Value ſought will be 9.4 Years Purchaſe.

Example II.

Suppoſe the given Age to be 5 Years, the propoſed Term 16 Years, and Intereſt as above: In which Caſe it appearing that the Value of a Term of 15 Years is 9.1, and That of a Term of 20 Years, 10.7, it is evident that the true Anſwer here muſt be about 9.5 — But, that there may be no Difficulty in allowing for the odd Years (which is the harder to do, as the Differences are unequal) the Table at *p.* 264 is annexed, as a Supplement to that preceding it: To comprehend the Use of which the Difference of the Values, answering to the two nearest Tabular Numbers to the given Term (*in Tab.* VI) must be taken; which, in this Case, is 1.6: And then, by entering *Tab.* VII. *Column* 1. with this Difference, you will find against it, under the Excess (1) of the given Term above the next inferior Tabular Number, the Value 0.4 to be added to That answering to the said inferior Number in order to have the true Conclusion. — After the same manner the Value corresponding to the same Age, and a Term of 19 Years, will be found 10.4. But it may be proper to observe, that, to avoid Trouble, it will be sufficient, in most Cases, when the Age given cannot be exactly found in the Table, to take the Tabular Number that comes nearest to it, whether greater or lesser.

PROBLEM VII.

To find the Value of an Annuity, for two assigned joint Lives, *that is, for as long as they both continue in Being together.*

SOLUTION.

Seek, *in Tab.* VIII, *Column* 1. the Age of the youngest Life (or That nearest to it), and find, *in Column* 2, the Age of the Elder; against which last you will have the Number of Years-Purchase required.

EXAMPLE I.

Let the two Ages be 20, and 35, Years, and the Rate of Interest 4 *per Cent.* Then it appears, at one View, that the Value sought will be 9.8.

Example II.

Let the propoſed Ages be 25, and 37, Years, and Intereſt as before.

Here, if the Age of the Elder was to be $\left\{ \begin{matrix} 35 \\ 40 \end{matrix} \right\}$ the Value ſought would be $\left\{ \begin{matrix} 9.6 \\ 9.1 \end{matrix} \right\}$

Therefore, when the Age is 37, it muſt be 9.4.

Example III.

Suppoſe the given Ages to be 32 and 57, and the Intereſt of Money at 3 *per Cent.* In this Caſe, the Value correſponding to the Ages $\left\{ \begin{matrix} 30 \\ 35 \end{matrix} \right\}$ and $\left\{ \begin{matrix} 57 \\ 57 \end{matrix} \right\}$, appears to be $\left\{ \begin{matrix} 7.6 \\ 7.4 \end{matrix} \right\}$ Whence That anſwering to the given Ages 32 and 57, muſt be about 7.5.

But, in order to avoid Trouble, you may, upon Occaſion, add an Year, or two, to one of the propoſed Ages, and ſubtract as much from the Other, when they are nearly equal: But, if One of them much exceeds the Other, it will then be ſufficient to take the neareſt Number in the Table for the Leſſer.

PROBLEM VIII.

To find the Value of an Annuity, for the longeſt of two Lives, that is, for as long as either of them continues in Being.

Solution.

Find, *in Tab.* IX, *Column* 1, the Age of the youngeſt Life (or That neareſt it), and *in Column* 2, find

Note. *The Solutions to the ſeveral Problems inſerted in the Courſe of this Work are, not improperly, divided into two different* find the Age of the Elder; against which last you will have the Number of Years-Purchase required.

Example I.

Let the given Ages be 15, and 40, Years; and let the Rate of Interest be 4 *per Cent.* Then it is apparent that the Value of an Annuity upon the longest of two such Lives will be 17.9, or nearly 18, Years-Purchase.

Example II.

Let the Ages propounded be 25 and 62, and the Rate of Interest as before: In which Case we have 15.5 for the Value of the Annuity.

PROBLEM IX.

To find the Value of an Annuity for three joint *Lives,* A, B, *and* C.

Solution.

† Let A be the youngest, and C the oldest, of the three proposed Lives: Take the Value of the two *joint Lives* B and C *(by Tab.* VIII.) and find the Age of a single Life D, of the same Value *(by Tab.* V.) then find the Value of the *joint Lives* A and D; which will be the Answer.

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*different Classes: The first whereof, hereafter distinguish'd thus *, are general, and strictly true, according to any Table of Observations, or Probability of Life what-so-ever: But the second Sort, marked thus †, are near Approximations only, yet such as are applicable, likewise, to any Table of Observation.*

EXAMPLE I.

Let the three given Ages be 20, 35 and 44 Years; and let the Rate of Intereſt be 3 *per Cent.* Then, the Value of the two oldeſt joint Lives B and C, will be found 9.0; anſwering to a ſingle Life (D) of 61 Years: And the Value of the joint Lives A and D, which is 7.3 Years-Purchaſe, will be the Value ſought.

EXAMPLE II.

Suppoſe the three Ages to be 17, 23 and 38, and the Rate of Intereſt 5 *per Cent.* Here the Value of the joint Lives B and C, will be 8.5; agreeing with That of a ſingle Life (D) of 55 Years; whence the Value ſought, or That of the joint Lives A and D, is in this Caſe 7.0, or, juſt 7 Years-Purchaſe.

PROBLEM X.

To find the Value of an Annuity for the longeſt of three Lives, A, B, *and* C.

SOLUTION.

† Let A be the youngeſt, and C the oldeſt, of the three propoſed Lives. Find the Value of the joint Lives B and C *(by Tab.* VIII), and find *(by Tab.* V) the Age of a ſingle Life, D, of the ſame Value: Moreover find *(by Tab.* IX) the Value of the longeſt of the Lives A and B, alſo That of the longeſt of the Lives A and C, and likewiſe That of the longeſt of the Lives A and D; then the laſt of theſe three Values, ſubtracted from the Sum of the two former, leaves the Value ſought,

EXAMPLE I.

Let the three Ages be 20, 40 and 66 Years, and let the Rate of Intereſt be 4 *per Cent.* Then the Value

Value of the joint Lives B and C will be found 5.8; answering to a single Life (D) of 73 Years: Moreover the Value of the longest of the Lives A and B will be 17.2; that of the longest of the Lives A and C 16.0; and that of the longest of the Lives A and D 15.5. Therefore the Value sought is 17.7 Years Purchase.

EXAMPLE II.

Let the given Ages be 15, 28 and 37 Years, and the Rate of Interest as before: Here (proceeding as in *Prob.* 7. *Ex.* 3.) the Value of the two oldest joint Lives will be had 9.2, equal to That of a single Life (D) of 55½ Years *(by Tab.* V). Hence, the three Values specified in the latter Part of the Rule will, in this Case, be 18.4, 18.0, and 17.4, respectively, and consequently the Value sought, just, 19 Years Purchase.

PROBLEM XI.

To find the Value of an Annuity granted upon three Lives, A, B, C, *on Condition of its ceasing as soon as any two of them become extinct.*

SOLUTION.

† Find, *(by Tab.* VIII.) the Value of each Pair of *joint* Lives, *viz.* of A and B, of A and C, and of B and C; then, from the Sum of those three Values, let twice the Value of the three *joint Lives* A, B, and C, *(found by Prob.* IX) be deducted, and the Residue will be the Answer.

EXAMPLE.

Example.

Suppoſe the given Ages of A, B and C to be 20, 35 and 44 Years, reſpectively; and let the Rate of Intereſt be 3 *per Cent.*

Here the Value of the *joint Lives*

$$\left\{\begin{array}{l}\text{A and B}\\ \text{A and C}\\ \text{B and C}\end{array}\right\} \text{ will be } \left\{\begin{array}{r}10.9\\ 9.6\\ 9.0\end{array}\right\}.$$

The Sum of which three Numbers is 29.5; moreover the Value of the three *joint Lives*, A, B and C is, in this Caſe, 7.3. (*See Prob.* 9. *Ex.* 1.) Therefore 14.9 is the Value ſought.

PROBLEM XII.

The Value of an Annuity upon One, two, or three Lives, at the Rate of 4 per Cent. *being known (from the preceding Problems); to find the Value of the ſame Life, or Lives, at the Rate of* $3\frac{1}{4}$, $3\frac{1}{2}$, $3\frac{3}{4}$, $4\frac{1}{4}$, $4\frac{1}{2}$, *or* $4\frac{3}{4}$, *per Cent.*

Solution.

† This Problem is ſolved by Means of Tab. X; from whence the Value that ought to be added to, or ſubtracted from, the given Value, at the Rate of 4 *per Cent.* is had by bare Inſpection.

Thus it will appear that an Annuity, upon One, or more Lives, which at the Rate of 4 *per Cent.* is worth 16 Years Purchaſe, will, at the Rate of $3\frac{1}{2}$ *per Cent.* be worth 17 Years Purchaſe and three Tenths. This Method, though exact only, when apply'd to a ſingle Life, is ſufficiently near in the Caſe of two, or more, Lives.

Remark.

REMARK.

As it is cuſtomary for an *Annuitant* to receive his Money half-yearly in equal Portions (inſtead of yearly) he hath, in this, a double Advantage; for, beſides the Uſe of, each, firſt-half-yearly Payment for 6 Months, he alſo hath a Chance of receiving one Half-year's Value more, than if he was to be paid yearly. Now the Value of both theſe Conſiderations put together (let the Rate of Intereſt and the Number of Lives on which the Annuity depends be what they will) will always amount to $\frac{1}{4}$ of a Year's-Purchaſe. Therefore $\frac{1}{4}$ of a Year's Purchaſe muſt be always added to the Values found by the foregoing Problems, in order to have the true Anſwer, when the Payments are made Half-yearly. And, if the Payments are to be made Quarterly, then $\frac{3}{8}$ of a Year's Purchaſe ought to be added to the Value found by the preceding Calculations.

OF

Of REVERSIONS.

PROBLEM XIII.

To find the Value of the Reversion of an assigned Life after a given Term of Years.

SOLUTION.

* From the Value of the proposed Life, subtract the Value of an Annuity for the given Term of Years, on the Contingency of its ceasing upon the Extinction of the foresaid Life; the Remainder will be the Answer.

EXAMPLE.

A, aged 15 Years, expects to enter upon an Estate of 500 *l. per Annum*, after the Expiration of 10 Years; which he is to hold thence-forward, during Life; what is the present Value of his Expectation, reckoning Interest of Money at 4 *per Cent.*

Here the Answer will be 8.3 Years-Purchase, or 4150*l.* For the Value of the whole Life A being 15.8 *(by Tab.* V), and That of the first 10 Years 7.5 *(by Tab.* VI), the Difference will be 8.3, *as above.*

PROBLEM XIV.

To find the Value of the Reversion of an Annuity, for the Remainder of a given Term of Years, after an assigned Life.

SOLUTION.

* From the Value of an Annuity certain for the given Term, subtract the Value of the Annuity for the said Term, on the Contingency of its ceasing upon the Failing of the proposed Life; the Remainder will be the Value of the Reversion.

EXAMPLE.

EXAMPLE.

A, aged Twenty-five, who has the Right of an Annuity for 31 Years certain, makes over the Reversion thereof to B and his Heirs, to enjoy the same after his Decease, for the Remainder of the said Term. Now, in order to find the Value of B's Expectation, the Value of an Annuity *certain*, for 31 Years, is to be found; which, at the Rate of 4 *per Cent.* will be 17.58 *(by Tab.* IV). Moreover the Value of an Annuity for the same Term, on the Contingency of its Failing on the Extinction of A, will appear to be 12.9 *(by Tab.* VI). Therefore the Value sought, in this Case, is 4.68 Years Purchase.

PROBLEM XV.

To find the Value of the Reversion of one Life after another.

SOLUTION.

* From the Value of the Life in Expectation subtract the Value of the two *joint Lives*; the Remainder will be the required Value of the Reversion.

EXAMPLE.

Let the Age of the Life in Possession be 55 Years, that of the Life in Expectation 20 Years, the Rate of Interest 5 *per Cent.* and the proposed Annuity 100 *l.* Then, *by Tab.* VIII, the Value of the two *joint Lives* will be 6.9; which, subtracted from 13.0, the Value of the Life in Expectation *(found by Tab.* V), leaves 6.1 Years-Purchase, for the Value of the Reversion. Which, multiply'd by the proposed Annuity, gives 610 *l.* its Value in present Money.

Note. In the Resolution of the preceding Problem, the Value that ought to be paid for putting in, or joining, a new Life to One already in Possession, is likewise determined

determined. Thus, for Example, if A and his Heirs hold an Annuity of 100 *l.* upon a single Life, of Fifty-five, and They would put in another Life, of Twenty, to hold the Annuity as long as either of the two Lives continue in Being; then the Sum which ought to be paid, as an Equivalent for that Consideration, is 610 *l.* the very Value above exhibited.

PROBLEM XVI.

To find the Value of the Reversion of two Lives after One.

SOLUTION.

* From the Value of the three Lives subtract the Value of the Life in Possession, the Remainder will be the Value of the Reversion.

EXAMPLE.

Let the Age of the Life in Possession be 40 Years, and the Ages of the two Lives in Expectation 20 and 66 Years; and let Interest be supposed at 4 *per Cent.* In which Case, the Value of the three Lives being 17.7 (*by Prob.* 10, *Ex.* 1), and That of the Life in Possession 11.5 (*by Tab.* V), the Answer comes out 6.2 Years-Purchase: So that, if the Annuity was to be 500 *l.* the Value of the Reversion would be 3100 *l.* Which Sum also expresses the Consideration that ought to be allow'd for putting in two Lives, of 20 and 66 Years, to a Life of Forty, already in Possession.

PROBLEM

PROBLEM XVII.

To find the Value of the Reversion of one Life after two joint *Lives.*

Solution.

* From the Value of the Life in Expectation, subtract the Value of the three *joint Lives*, there will remain the Value of the Reversion.

Example.

Suppose the Age of the Life in Expectation to be 20 Years, and the Ages of the two Lives in Possession 35 and 44 Years; and suppose Interest of Money to be at 3 *per Cent.*

Here the Value of the three *joint Lives* will be found 7.3 (*by Prob.* 9, *Ex.* 1.) which being deducted from 17.2, there rests 9.9 Years-Purchase, for the Value of the Reversion.

PROBLEM XVIII.

To find the Value of the Reversion of One Life after Two.

Solution.

* From the Value of the three Lives, subtract the Value of the two Lives in Possession, there will remain the Value of the Reversion.

Example.

Let 40 and 66 be the Ages of the two Lives in Possession, and 20 That of the Life in Expectation, and let the Rate of Interest be 4 *per Cent.*

Then (*by Prob.* 10, *Ex.* 1.) the Value of the three Lives will be found 17.7; from which taking, 13.0, the Value of the two Lives in Possession (*found by Tab.* IX.) the

to add one Life to the two

the Remainder, 4.7, will be the Number of Years-Purchase at which the Value sought is to be estimated. Which, therefore, also expresses the Value that ought to be paid for putting in, or joining, a Life of 20 Years to two Others of 40 and 66 Years.

PROBLEM XIX.

To find the Value of the Reversion of two joint Lives *after One.*

SOLUTION.

* From the Value of the two joint Lives in Expectation, subtract the Value of the three joint Lives, there will rest the Value of the Reversion.

EXAMPLE.

Let the Ages of the two Lives in Expectation be 20 and 44 Years, and That of the Life in Possession 35 Years; and let the Rate of Interest be 3 *per Cent.* Then, the Value of the three *joint* Lives being 7.3 (*by Prob.* 9, *Ex.* 1.) and That of the two joint Lives in Expectation 9.6 (*by Tab.* VIII) the Answer in this Case will be 2.3 Years-Purchase.

PROBLEM XX.

Q *and his Heirs, as soon as any two of three given Lives* A, B, C, *become extinct, are to enter upon an Annuity, in order to enjoy the same, during the Life of the Survivor: To determine the present Value of the Expectation.*

SOLUTION.

* To the Sum of the Values of all the single Lives, add three times the Value of the three *joint Lives*; and from this Sum deduct twice the Sum of the Values of each

each Pair of *joint Lives (viz.* of A and B; A and C; and B and C) the Remainder will be the Anſwer.

EXAMPLE.

Let the three given Ages be 20, 35, and 44 Years, and the Rate of Intereſt 3 *per Cent.* Then the Values of the three *ſingle Lives, by Tab.* V, will appear to be 17.2, 14.1, and 12.5. And, *by Prob.* IX. *Ex.* 1, the Value of the 3 *joint Lives* is found 7.3: The treble of which Value, added to the Sum of the three Former, gives 65.7. But the Values of the three different Pairs of *joint Lives* are found, *by Tab.* VIII, to be 10.9, 9.6, and 9.0: The Double of all which, taken from 65.7, gives 6.7 for the Number of Years-Purchaſe required.

PROBLEM XXI.

A *and* B *enjoy an Annuity, equally, betwixt them; which, after the Deceaſe of either of them, is to belong intirely to the Survivor for Life; To find the Value of the Right of Each in that Annuity.*

SOLUTION.

* From the Value of the Life A, or B, ſubtract Half the Value of the two joint Lives; the Remainder will be the Value of the Right of A, or B, accordingly.

EXAMPLE.

Let the Age of A be 25 Years, that of B 40 Years, and the Rate of Intereſt 5 *per Cent.* Then the Value of the two *joint* Lives will be 8.2 (*by Tab.* VIII.) whereof the Half is 4.1; this taken from 12.3, the Value of the Life A, leaves 8.2 for the Expectation of A; but, being taken from 10.3, the Remainder 6.2, will be the Value of B's Expectation.

PROBLEM XXII.

A given Annuity, after the Decease of A, *is to be divided equally between* B *and* C, *during their* joint Lives; *and then is to go intirely to the last Survivor for Life; 'tis proposed to find the Value of* B's *Expectation.*

Solution.

* From the Value of the Reversion of the Life B after the Life A (*found by Prob.* 15) subtract half the Value of the Reversion of the *joint Lives* B and C after the Life A (*found by Prob.* 19), the Remainder will be the true Value of B's Expectation.

Example.

Let the given Ages of A, B and C be 35, 20, and 44 Years, respectively; and let the Rate of Interest be 3 *per Cent.* Then the Value of the Life B after the Life A will appear to be 6.3; moreover the Value of the joint Lives B and C after the Life A will come out 2.3 (*Vid. Ex. to Prob.* 19). Therefore the Answer, in this Case, is 5.15 Years-Purchase.

PROBLEM XXIII.

A, B, *and* C *share an Annuity equally amongst them; which, upon the Decease of any One of the Three, is to be divided equally between the two Survivors during their joint Continuance, and then is to go intirely to the last Survivor for Life: To find the Value of the Right of* A *in the said Annuity.*

Solution.

* From the Value of the Life A, take half the Sum of the Values of the *joint Lives* A B, and the *joint Lives* A C; then to the Remainder add ⅓ of the Value of

of the three *joint Lives* A B C, and the Sum resulting will be the true Answer.

EXAMPLE.

Suppose the three Ages to be 17, 23, and 38 Years, respectively, and the Rate of Interest 5 *per Cent.*

Here the Value of the Life A will be 13.5; also that of the *joint Lives* A and B will be 10.1 (*by Prob.* 7) that of the *joint Lives* A and C, 8.6, and that of the three *joint Lives* A, B and C, 7.0 (*by Prob.* 9, *Ex.* 2). Therefore the Answer comes out 6.5 Years-Purchase.

Of successive Lives, and the Renewing of Leases.

PROBLEM XXIV.

Supposing A *to enjoy an Annuity for Life, and, at his Decease, to have the Nomination of a Successor, who is likewise to enjoy the Annuity for his Life; 'tis proposed to find the present Value of the two successive Lives.*

SOLUTION.

* Multiply the Value of the Life A by the Value of the Life put in at his Decease, and divide the Product by the Perpetuity; then let the Quotient be subtracted from the Sum of the said Values, and the Remainder will be the Answer.

EXAMPLE.

Suppose the Value of the Life A, at 4 *per Cent.* to be 12 Years-Purchase, and that he hath the Liberty, at his Decease, to name a Life to succeed him of the greatest Value possible; which Value, according to *Tab.* V, is 16.4, answering to an Age of 10 Years. Therefore, in this Case, the Product of the two Values will be 196.8; which divided by 25, the Perpetuity, quotes 7.872; this, deducted from 28.4, the Sum of the said Values, leaves 20.528 Years-Purchase for the true Value of the two successive Lives.

PROBLEM XXV.

To find the present Value of any Number of Lives in Succession.

Solution.

* Multiply the Value of each Life (considered independent of the Rest, without Regard to Time, or Order of Succession) by the Rate of Interest, *per Cent.* dividing the Product by 100; subtract each Quotient from Unity, and multiply all the Remainders continually together, subtracting their Product, also, from Unity; then the last Remainder, multiply'd by the Perpetuity, will be the true Value of all the successive Lives.

Otherwise, thus.

* Find, *in Tab.* IV. the Number of Years answering to the Value of each of the given Lives; then find, *in the same Table*, the Value of an Annuity *certain* for the Whole of all those Years added together; which Value will be the Answer.

Example.

Suppose the Number of Lives to be Three; whereof the Values, at 4 *per Cent.* are 8, 10 and 15 Years-Purchase. These being multiply'd, each, by 4, and the Products divided by 100, we have 0.32, 0.4, and 0.6, respectively: Which, subtracted severally, from Unity, leave 0.68, 0.6, and 0.4; whereof the continual Product is 0.1632; This being also taken from Unity, there rests 0.8368; which, multiply'd by 25, the Perpetuity, gives 20.92 for the true Answer.

Otherwise, by the second Method.

It appears that the Number of Years corresponding, *in Tab.* IV,

$$\text{to}\left\{\begin{matrix}8\\10\\15\end{matrix}\right\}\text{Years Purch. is about}\left\{\begin{matrix}9.8\\13.0\\23.4\end{matrix}\right\}\text{whereof the Sum is 46.2;}$$

against which we have 20.9, *the same as before.*

PROBLEM

PROBLEM XXVI.

A given Sum of Money, is to be received (as a Legacy), on the Decease of B, *who is now of a given Age: What is the Value thereof in present Money?*

SOLUTION.

* Subtract the Value of the Life B from the *Perpetuity*; then it will be, as the *Perpetuity* is to the Remainder, so is the proposed Sum, to its Value in present Money.

EXAMPLE.

Let the Age of B be 32 Years, the Rate of Interest 4 *per Cent.* and the given Sum 500 *l.* Then the Value of the Life B being 12.7, and the *Perpetuity* 25.0, it will be, as 25.0 : 12.3 : : 500 *l.* : 246 *l.* the Value sought.

PROBLEM XXVII.

A given Sum is to be received on the Extinction of the first, second, or third, of three assigned Lives; to find the present Value of the Expectation.

SOLUTION.

* Find *(by Prob.* 9, 10, or 11) the Value of an Annuity granted upon the assigned Lives, on Condition of its ceasing on the Failing of the first, second, or third of those Lives, according to the Case proposed; which Value subtract from the *Perpetuity*, and then proceed as in the last Problem.

EXAMPLE.

Let the given Sum be 1000 *l.* the Rate of Interest 3 *per Cent.* and the Ages of the three Lives 20, 35 and 44 Years.

Here the Value of an Annuity to ceaſe upon Failing

$$\text{of the}\left\{\begin{matrix}1\text{ſt}\\2\text{d}\\3\text{d}\end{matrix}\right\}\text{Life will be}\left\{\begin{matrix}7.3\\14.9\\21.3\end{matrix}\right\}\text{by Prob.}\left\{\begin{matrix}9\\11\\10\end{matrix}\right\};\text{ which being}$$

ſubtracted, ſeverally, from the *Perpetuity* 33.3, there reſts 26.0, 18.4, and 12.0; whence the Anſwer comes out 780 *l.* 552 *l.* or 360 *l.* according to the three foreſaid Caſes, reſpectively.

PROBLEM XXVIII.

A *purchaſes a Leaſe-hold Eſtate, upon One, two, or three aſſigned Lives, for a given Sum, on Condition that his Heirs fill up the Leaſe,* continually, *whenever all the Lives become vacant, paying a ſtated Fine: The Queſtion is, to find the preſent Value of the whole Purchaſe allowed for the Eſtate, with the Annuity, or Rack-rent, anſwering thereto; ſuppoſing the Lives put in at each Renewal to be of the ſame, given, Ages.*

Solution.

* Find *(by Prob.* 5, 8, or 10.) the Value of the Life, or Lives, firſt propoſed; find alſo *(by the ſame)* the Value of the Life, or Lives, with which the Leaſe is to be, conſtantly, renewed; ſubtract the former of Theſe from the *Perpetuity*; then it will be as the Latter is to the Remainder, ſo is the Fine propoſed to the preſent Value of all the Fines: Which added to the Sum, paid at entering, gives the Value allowed for the whole Purchaſe. This being multiply'd by the Rate of Intereſt, the Product, divided by 100, will be the Annuity, or Rack-rent correſponding.

Example.

Let the Number of Lives be Three; whereof the Ages are 15, 28, and 37 Years; let the given Sum be 2000 *l.* the Fine 1000, and the Rate of Intereſt 4 *per Cent.* Then the Value of the three Lives will appear to be 19.0 *(Vid.*

(*Vid. Ex.* 2, *Prob.* 10), which, subtracted from 25.0, the Perpetuity, leaves 6.0: Therefore, if the Value of the Lives (found in like manner) with which the Lease is to be renewed, be taken at 17.7 Years-Purchase, we shall have 17.7 : 6 : : 1000*l.* 339*l.* Hence the whole Value allowed for the Purchase is 2339*l.* and the Annuity corresponding appears to be 93.56*l.* or 93*l.* 11*s.* 2*d.*

PROBLEM XXIX.

Supposing A *to purchase an Estate, in Copy-hold, upon any Number of assigned Lives, for a given Sum, on Condition that He and his Heirs fill up the Lease, continually, whenever a Life becomes vacant, by paying a proposed Fine: To find the present Value of the whole Purchase allowed for the Estate, with the Annuity, or Rack-rent, corresponding; supposing the Life put in, at each Renewal, to be of the same, given, Value.*

SOLUTION.

* Subtract the Sum of the Values of all the, single, Lives, upon which the Lease is first granted, from the *Perpetuity* multiply'd by the Number of those Lives; then it will be, as the given Value of the Life to be put in at each Renewal, is to the Remainder, found above, so is the Fine proposed, to the present Value of all the Sums paid, for ever, for Renewing; which, added to the Value paid at first entering, gives the total Value of the Purchase. This being multiply'd by the Rate of Interest, the Product, divided by 100, will give the Annuity answering thereto.

EXAMPLE.

Let the Number of Lives be Three, and their Values (at 4 *per Cent.*) 10, 12, and 15 Years-Purchase; also let the Sum paid upon entering be 1600*l.* and the proposed Fine 400*l.* and suppose the Purchaser to have the Liberty of renewing with Lives of what Ages he thinks proper, or most to his own Advantage; which Ages, according to our *Table at p.* 260, appear to be between 7 and 12 Years; answering to 16.4 Years Purchase. Therefore, the Sum of the Values of the three first Lives being 37, and the Perpetuity 25, we have here 16.4 : 38 : : 400 : 926.8, or 926 *l.* 16*s.* the present Value of all the Sums paid for renewing. Hence the whole Value allowed for the Purchase is 2526*l.* 16*s.* and the corresponding Annuity 101 *l.* 1 *s.* 5*d.*

PROBLEM XXX.

The same being supposed as in the last Problem; to find how much the Rent-Roll of the first Proprietor's Estate ought to be increased, on account of the Fines paid at Renewing.

SOLUTION.

* Multiply the present Value of all the Sums paid for Renewing *(found as in the preceding Problem)* by the Rate *per Cent.* assigned; then the Product, divided by 100, will be the Answer. Which, in the Case, there proposed, will be found 37*l.* 1 *s.* 5 *d.*

PROBLEM XXXI.

There is an Estate, which, if A *(who is a Minor) happens to die in a given Time, or before he attains to a certain given Age (suppose Twenty-one) is, after his Decease, to go to* B *and his Heirs for ever; to find the Value of* B's *Expectation.*

SOLUTION.

* From the *Perpetuity*, subtract the Value of a Life of that Age to which the Expectation of B is limited; multiply the Remainder by the present Value of 1 *l.* to be received at the End of the given Term *(found by Tab.* III). And let this last Product be, again, multiply'd

ply'd by the Number of Persons, *in Tab.* 1, arriving to the Age above mentioned, dividing the Product thence arising by the Number, in the same Table, answering to A's present Age; then to the Quotient add the Value of the Life A, and subtract the Sum from the Perpetuity; the Remainder will be the true Value required.

EXAMPLE.

Let the Age of A be 8 Years, the proposed Estate 500 *l. per Annum*, and Interest at 5 *per Cent.* Here the Value of a Life of Twenty-one (the Age to which B's Expectation is limited) being 12.9, and the *Perpetuity* 20, the Difference will be 7.1; which multiply'd by 0.5303, the present Value of 1 *l.* to be received 13 Years hence (when A arrives to the Age of 21) gives 3.76513, or 3.76, nearly: This being multiply'd by 355, the Product, divided by 422, gives 3.16; to this adding 14.3, the Value of the Life A, and from the Sum deducting 20, the Perpetuity, there results 2.54, or $2\frac{1}{2}$ Years-Purchase, nearly, for the required Value of B's Expectation.

Of Reversions depending upon the Probability of one particular Life, in Possession, surviving the Rest.

PROBLEM XXXII.

B, *who is of a given Age, will, if he lives 'till the Decease of* A, *whose Age is also given, become possessed of an Estate of a given Value; to find the Worth of his Expectation in present Money.*

SOLUTION.

† Find *(by Tab.* VIII.) the Value of an Annuity on two equal *joint Lives*, whereof the common Age is equal to the Age of the older of the two proposed Lives and B; which Value subtract from the *Perpetuity*, and take Half the Remainder; then say, as the Expectation of

of the Duration of the younger of the two Lives A and B, *found against the Age corresponding, in Tab.* II, is to That of the Elder, so is the said Half-Remainder, to a fourth Proportional; which will be the Number of Years-Purchase required, when the Life B, in Expectation is the older of the Two. But, if B be the younger, then add the Value so found to That of the *joint Lives* A and B (*found by Tab.* VIII), and let the Sum be subtracted from the *Perpetuity*, and you will, also, have the Answer, in this Case.

Example I.

Suppose the Age of A to be Thirty, that of B Forty, the Rate of Interest 4 *per Cent.* and the given Legacy 5000 *l.* or 200 *l. per Annum.* Then the Value of two equal *joint Lives*, of Forty, being 8.1, and the *Perpetuity* 25, the Remainder, or Difference, will here be 16.9; whereof the Half is 8.45: Therefore it will be, as 23.6 : 19.6 : : 8.45 : 7.02, Years-Purchase, or 1404 *l.* the required Value of B's Expectation.

Example II.

Let the Age of A be Forty, and that of B Thirty, and the Rest as in the preceding Example. Here the Value of the *joint Lives* A and B will be 8.8; which added to 7.02 (found above), the Sum will be 15.82; whence the Answer, in this Case, is 9.18 Years-Purchase, or 1836 *l.*

PROBLEM XXXIII.

C *and his Heirs are intitled to an Estate of a given Value, upon the Decease of* B, *provided* B *survives* A; *to find the Value of Their Expectation* in present *Money.*

Solution.

† Find (*by Tab.* IX.) the Value of an Annuity upon the longest of two equal Lives, whereof the common

Age

Age is That of the older of the Lives A and B; which Value subtract from the *Perpetuity*, and take Half the Remainder; then it will be, as the Expectation of Duration of the younger of the Lives A and B (*found by Tab.* II.) is to That of the older, so is the said Half-remainder, to the Number of Years-Purchase required, when the Life B *is the older of the Two:* But, *if* B *be the younger*; then, to the Number thus found, add the Value of an Annuity on the longest of the Lives A and B, and subtract the Sum from the Perpetuity, for the Answer in this Case.

EXAMPLE.

Let the Age of A be Thirty, That of B Forty, the Rate proposed 4 *per Cent.* and the given Estate 200 *l. per Annum.*

Here, the Value of the longest of two Lives, aged Forty each, will be found 15.0; which, taken from 25, leaves 10, for the Remainder. Therefore, it will be 23.6 : 19.6 :: 5 : 4.15, the Number of Years-Purchase required; answering to 830 *l.*

But, if A had been Forty, and B Thirty, the Value sought would have been 4.95 × 200, or 990*l.* Because the Value of the longest of the Lives A and B is 15.9 Years-Purchase.

PROBLEM XXXIV.

C is to enter upon a given Annuity, for Life, on the Decease of B, *in case* B *survives* A: *To determine the Value of his Expectation in present Money.*

SOLUTION.

† Case 1°. *If the Life* C, *in Expectation, be older than either of the other Two:* Find, *by Prob.* 18, the Value of the Reversion of the Life C after the longest of the Lives A and B; One Half of which will be the Answer.

Case 2°. *If the Life* C *be younger than one, or both, of the Other:* Then find the Reversion of the Life C after the

the longest of two equal Lives, of the same Age with the oldest of A or B; which multiply by the Expectation of Duration of the said oldest Life *(given in Tab.* II) and divide the Product by twice That of the Younger; so shall the Quotient be the Answer, *in Case* B *is older than* A: But, if B be younger than A, let the said Quotient be subtracted from the Value of the Reversion of the Life C after the longest of the Lives A and B, and the Remainder will be the Answer.

EXAMPLE I.

Let the Age of A be 30; That of B, 40; and That of C, 50 Years; and suppose the Rate of Interest to be 3 *per Cent.*

Then, *according to Case* 1, the Value of the longest of the three Lives will be found to be 19.6 (*by Prob.* 10). From which deducting 18.4 the Value of the longest of the two Lives A and B, the Remainder 1.2 will be the Value of the Reversion of the Life C after A and B; the Half of which, or $\frac{6}{10}$ of a Years-Purchase, is therefore the Value required.

EXAMPLE II.

Suppose the Ages of A, B, and C to be 66, 40, and 20, respectively, and the Rate of Interest 4 *per Cent.*

Here, *according to Case* 2, we are first to find the Value of the Reversion of a Life of Twenty after the longest of two equal Lives of 66; which *(by Prob.* 18) will come out 7.0: This being multiply'd by 10.1, and the Product divided by 39.2 *(vid. Tab.* II) there results 1.8 nearly. Which, as B is here younger than A, must be subtracted from (4.7) the Value of the Reversion of the Life C after the longest of the Lives A and B *(found by Prob.* 18) whence we get 2.9 for the true Answer in this Case.

PROBLEM

PROBLEM XXXV.

C *is to enter upon a given Annuity, for Life, on the Decease of* B, *provided the Latter is survived by* A: *To find the present Value of his Expectation.*

SOLUTION.

† Find, *by the last Problem*, the Value of the Reversion of the Life C after the Life B, on the Contingency of B's surviving A; which, subtract from the absolute Value of the Reversion of the Life C after the Life B *(found by Prob.* 15); the Remainder will be the Answer.

EXAMPLE.

Let every Thing be supposed, as in the second Example of the last Problem: Then the former of the [above mentioned] Reversions will appear to be 2.9: And the Latter (by taking 9 2 the Value of the *joint Lives* B and C from 14.8, that of the Life C) will be found 5.6. Therefore 2.7 Years-Purchase is the Value sought.

PROBLEM XXXVI.

A *and* B *are possessed of an Annuity between them; which, if* B *survives* A, *is afterwards to be divided equally between* B *and* C, *during their joint Existence; and then is to go intirely to the last Survivor for Life: To find the Value of* C'*s Expectation.*

SOLUTION.

† Find, *by Prob.* 34, the Value of the Reversion of the Life C after the Life B, on the Contingency of B's surviving A; to which add Half the Value of the Reversion of the *joint Lives* B and C after the Life A *(found by Prob.* 19); the Sum will be the Value sought.

EXAMPLE.

Example.

Suppose the given Ages of A, B, and C, to be 66, 40, and 20 Years, respectively, and the Rate of Interest 4 *per Cent.*

Then the Value of the former of the two Reversions above mentioned will be 2.9 Years-Purchase *(vid. Ex.* 2, *to Prob.* 34), and That of the Latter (by taking 5.2, the Value of the *joint Lives* A, B, and C from 9.2, the Value of the joint Lives B and C) will come out 4.0: Therefore 4.9 Years-Purchase is the required Value of C's Expectation.

PROBLEM XXXVII.

C *engages to pay to* A *a given Annuity, till such Time (if they both live so long) as the Latter comes to the Possession of an Estate, which he is intitled to on the Decease of* B: *In consideration whereof,* A, *on his Part, obliges himself, if he lives to enter upon the said Estate, to cause a given Annuity to be paid back to* C *for Life (but if either* C, *or* A, *happens to die before* B, *then the Money advanced by* C *is to be wholly lost to his Heirs). The Question is, to find the Advantage, or Disadvantage of either Party, in such a Contract.*

Solution.

† Find, *by Prob.* 9, the Number of Years-Purchase of the three joint Lives A, B and C; which multiply by the Annuity that A is to receive, the Product will be the Value of A's whole Expectation from C.

Find also, *by Prob.* 34, the Value of the Reversion of the Life C after the Life B, on the Contingency of B's surviving A; which subtract from the Value of the Reversion of the Life C after the Life B *(found by Prob.* 15); then the Remainder, multiply'd by the Annuity paid back to C, will give the Whole of C's Expectation from A: And the Difference of the two Values thus found will consequently be the Answer.

Example.

EXAMPLE.

A young Gentleman (A) aged Twenty-five, having greatly diſobliged his Father (B) a Gentleman of Sixty, to whom he is ſole Heir, is forced to contract with a certain Perſon (C) aged 35, for an Annuity of 200 *l.* (in order to a Support) which he engages to repay, after his Father's Deceaſe, with another of 300 *l.* to continue during the Life of C; *according to the Conditions ſpecified in the Problem.*

Now, we are firſt to find the Value of the three joint Lives; which, at the Rate of 4 *per Cent.* comes out 5.6 Years-Purchaſe; and this, multiply'd by 200, gives 1120*l.* for the whole Value of A's Expectation.

Moreover, the Value of the Reverſion of the Life C after the Life B, on the Contingency of B's ſurviving A, will come out one Year's Purchaſe, very near: And the Value of the Reverſion of the Life C after the Life B (without Reſtriction) will appear to be 5.8 Years-Purchaſe *(by Prob. 7).* Hence 4.8, multiply'd by 300, which is 1440 *l.* will be the total Value of C's Expectation: Who, therefore, gains 320 *l.* by the Contract.

PROBLEM XXXVIII.

C, *if he lives till the Deceaſe of* B, *is to receive a given Legacy, in caſe* A *is then extinct; to determine the Value of his Expectation in preſent Money.*

SOLUTION.

† Caſe 1°. *If the Life* C *be the oldeſt of the Three:* From the Value of an Annuity on the Life C, take the Value of the two *joint Lives* B and C; multiply the Remainder by the given Sum, or Legacy, and divide the Product by twice A's *Expectation of Duration (found in Tab.* II); the Reſult will be the Value ſought.

Caſe 2°. *If the Life* B *be the oldeſt of the Three:* Then, from the Value of an Annuity, for as many Years

Years of C's Life as are expreffed by the Double of B's Expectation of Duration *(found by Tab.* II and VI), fubtract the Value of the two *joint Lives* B and C; multiply the Remainder by the given Sum, or Legacy, and divide the Product by twice the Expectation of A's Duration, *as in the preceding Cafe.*

Cafe 3°. *If the Life* A *be the oldeft of the Three:* Then find the Value of the Life C, if older than B; otherwife, find (*by Tab.* VI.) the Value of as many Years Thereof, as are expreffed by the Double of B's *Expectation of Duration:* And, from the Value thus found, let the Value of the *joint Lives* A and C be fubtracted; multipy the Remainder by the given Legacy, then the Product, divided by twice B's *Expectation of Duration*, will be the Anfwer, in this Cafe.

Example I.

Let the Age of C be 15 Years, that of B 70, and that of A 45; and let the propfed Legacy be 1000 *l.* and the Rate of Intereft 3 *per Cent.*

This Example, it is plain, belongs to Cafe 2°. According to which we muft firft find B's Expectation of Duration *(by Tab.* II.) which appears to be 8.8, and the Double of it 17.6: Now (*by Tab.* VI.) the Value of 17.6 Years of C's Life is found 11.9; and (*by Tab.* VIII.) the Value of the joint Lives B and C appears to be 6.0; which laft Value, fubtracted from the Former, leaves 5.9; this multiply'd by 1000 (the given Legacy) produces 5900; which being divided by 35.6, the Double of A's Expectation of Duration, there refults 165 *l.* for the required Value of C's Expectation.

Example II.

Suppofe the Age of C to be 15 Years, that of B 45, and that of A 70; fuppofe alfo the given Legacy to be 1000 *l.* and the Rate of Intereft 3 *per Cent.*

Here, according to Cafe 3°, we muft firft feek, *in Tab.* II. for B's Expectation of Duration; this appears

to

to be 17.8, and the Double thereof 35.6: Now the Value of 35.6 Years of the Life C (*by Tab.* VI.) will be found 16.7; from which subtracting (6.0) the Value of the Joint Lives A and C (*found by Tab.* VIII.) there remains 10.7; this, multiply'd by 1000, gives 10700; which, divided by 35.6, quotes 300 *l.* for the required Value of C's Expectation, in present Money.

PROBLEM XXXIX.

Q, *in case he lives till* R *and* S *are both extinct, is to receive a Legacy of a given Value; to determine the Worth of his Expectation in present Money.*

SOLUTION.

† Find, by the last Problem, the Expectation of Q, on the Contingency of R's surviving S; and also his Expectation, on the Contingency of S's surviving R: The Sum of which Values will, consequently, be the whole of his Expectation, in the present Case.

EXAMPLE.

Let the Age of Q be 15 Years, that of R 45 Years, and that of S 70; also suppose the given Legacy to be 1000 *l.* and the Rate of Interest 3 *per Cent.*

Here the first Part of Q's Expectation, depending on R's surviving S, appears, from Ex. 2. of the preceding Problem (by substituting Q for C, R for B, and S for A) to be 300 *l.*

And the latter Part of his Expectation, depending on S's surviving R, appears, from *Ex.* 1 (by substituting Q for C, S for B, and R for A) to be 165 *l.* Therefore the Answer is 465 *l.*

PROBLEM XL.

Q *and* R, *if they live till the Decease of* S, *are to receive, each of Them a given Legacy; but, in Case one of Them dies before Him, then the Whole is to go to the Survivor: To determine the Value of* Q's *Expectation.*

SOLUTION.

† Find, *by Prob.* 32, the Value of Q's Expectation on the Sum assigned to himself (depending on the Chance of his surviving S, without further Restriction): Then find, *by Prob* 38, the remaining Part of his Expectation, on the Sum assigned to R (depending on the Contingency of S's surviving R:) The Sum of the two Values so found will consequently be the Answer.

EXAMPLE.

Suppose Q to be 15 Years of Age, R 45, and S, 70; suppose also the Legacy of Q to be 600 *l.* That of R 1000 *l.* and the Rate of Interest 3 *per Cent.*

In the first Place we are to find the present Value of 600 *l.* (or an Annuity of 18 *l.*) to be received (or enter'd upon) by Q, at the Decease of S: Which Value comes out 23.37 Years-Purchase, or 420 *l.* Moreover, *by Ex.* 1, *to Prob.* 38, the latter Part of Q's Expectation, depending upon the Chance of his Receiving the 1000 *l.* assigned to B, will appear to be 165 *l.* So that the whole is 585 *l.*

PROBLEM

PROBLEM XLI.

Q, R, *and* S *share an Annuity equally among them; which, upon the Decease of any One of the Three, is to be divided equally between the two Survivors, during their joint Continuance; and then is to go intirely to the last Survivor and his Heirs, for ever: To determine the Value of the Right of* Q *in the said Annuity.*

SOLUTION.

† First find, *by Prob.* 39, that Part of Q's Expectation which depends on his becoming possessed of the whole *Annuity*, on being the last Survivor: To which Value add half the Sum of the Values of the Joint Lives Q R, and Q S; and from the Aggregate let ⅔ of the Value of the three Joint Lives Q R S be subtracted; the Remainder will be the Value sought.

EXAMPLE.

Suppose Q to be 15 Years of Age, R 45, and S 70; supppose moreover the Annuity to be 300 *l.* and the Rate of Interest 3 *per Cent.* according to which the Value of the Perpetuity is 10000 *l.* Whence, by the Example to the Problem above quoted, the former Part of Q's Expectation will appear to be 4650 *l.* We are next to find the Values of the Joint Lives Q R, and Q S; These, *by Tab.* VIII, appear to be 9.6, and 6.0, Years-Purchase, respectively: Whereof the Half Sum is 7.8 Years-Purchase, or 2340 *l.* Moreover, *by Prob.* 9, the Value of the three Joint Lives Q R S, will be found 5.2 Years-Purchase, or 1560 *l.* and ⅔ thereof 1040 *l.* And so, by adding together 4650 and 2340, and taking 1040 from the Sum, the Answer comes out 5950 *l.*

SCHOLIUM.

Wherein the Reasons of what is most material and difficult in the preceding Solutions, are explained.

The Solutions to the several Problems proposed in the foregoing Pages being deliver'd in a practical manner, without their Investigation (in order to render the Work as useful as possible, and acceptable to Those, to whom the Sight of an Algebraical Process would, at best, afford no satisfactory, or pleasing Idea) and there being, in the Number of the said Solutions, some few that, even, the Mathematical Reader (whom I should be loth to leave dissatisfy'd) may not readily see into the Reasons of; I shall here, according to my Promise, put down the Invention of what seems most material, and necessary to be explained. It is not, indeed, my Design to insist, in this Place, on the first Principles of the Laws of Chance, and the general Methods of Computation whereon the Subject is grounded; but, to *such* as want Information therein, shall take the Liberty to recommend my *Doctrine of Annuities*, printed in 1742. Neverthelefs, as the Merit of a Work of this Nature, when apply'd to the common Uses of Life, consists in its Conformity with Truth and *real* Observations, it is but reasonable that, for the general Satisfaction of Those into whose Hands this Tract may happen to fall, something should be, first of all, said relative to the Observations whereon the foregoing Tables have their Foundation.

The first of these Tables (shewing the Probability of the Duration of Life) on which the Rest depend, was computed from 10 Years Observations on the Bills of Mortality of the City of *London*; and is the very same, in effect, with that printed in my Book of Annuities above mentioned; tho' adapted to a different Radix. Both these Tables agree with That first of all published by Mr. *Smart*, whom I have followed, except in very low Ages; where it seem'd necessary to make some Alterations.

Alterations. I am senſible, indeed, that theſe Alterations have been condemned, as arbitrary and without Foundation: But that ſome ſuch Allowance ought to be made, on account of the great, and unequal, Afflux of People, of different Ages, to Town, is, I think, ſufficiently evident, from the following Reaſons.

Firſt, it is to be obſerved, agreeable to what I have elſewhere advanced, that the ſaid continual Reſort of Strangers to Town, would no way influence the Values of Annuities deduced from Obſervations on the Bills of Mortality, if Thoſe arriving and ſettling *There*, at the ſeveral different Ages of Life, were to be in the ſame Proportion as the whole Number of the Living of the ſame Ages; ſince it is, not on the Greatneſs of the Numbers that die at each particular Age, but on their Ratio that the whole of the Buſineſs depends.

To render this as obvious as may be, by a familiar Example; ſuppoſe A and B to be two equal Cities, with regard to the Number of their Inhabitants, and let Half the Inhabitants of the Former, who are above the Age of Twenty, be ſuppoſed to remove to the Latter; then it is plain that the Bills of Mortality of B, for all Ages above Twenty, will be immediately increaſed by One Half; and ſo the ſame Proportion ſtill preſerved: For the Equimultiples of Quantities are to one another as the Quantities themſelves.

But, with reſpect to Ages below Twenty, the Bills ſtill continuing the ſame, the Numbers here cannot be compared with Thoſe above, in order to obtain the true Ratio of the Probabilities of Mortality at the different Ages of Life; ſeeing Theſe laſt are too little by One Half.

It is evident from hence, that, if there be any Part of Life wherein the Number of Thoſe that remove to Town falls ſhort of the Proportion above ſpecified, the Bills of Mortality, for that Interval, will not truly exhibit the Probability of Mortality without ſome Allowance or Correction.

Now it is certain, from manifold Experience, that very few Perſons come to live in Town under the Age of Fifteen, in compariſon of the Numbers that arrive There, between the Ages of Fifteen and Thirty; though the Number of all the Living comprehended in the former Period is much greater than That in the Latter.

Add to this, that the Bills of Mortality, with Reſpect to ſmall Ages, are alſo lower than they would otherwiſe be, on account of a great Number of Youth, of the Better Sort, who are ſent into the Country for the Benefit of Air and Education. Theſe, at their Return, together with the Arrival of a Multitude of Working People (who, after having ſerved an Apprenticeſhip in the Country, are willing to learn Experience, and try their Fortune in Town) very much increaſe the Body of Inhabitants: And it is chiefly to this Conſideration that the great Increaſe in the Bills of Mortality after the Age of Twenty, like a Rivulet ſwoln by a ſudden Rain, is to be aſcribed: For I dare affirm, that there is Nothing in Man's Conſtitution whereby ſo great a Diſproportion during ſo ſhort an Interval, can be, phyſically, accounted for: Nor does it ſeem, in the leaſt, Reaſonable to ſuppoſe that a Perſon, of twenty-five Years of Age, has more than 4 times a greater Chance to die within the Compaſs of an Year, than One of fifteen; and yet this is the Caſe according to the Table, I have been condemned for altering — As to the Reaſon of my Beginning the Table with the Number 1280 (inſtead of 1000), it was with no other Intent than to avoid Trouble in making, what I judged, the neceſſary Alterations. Nor does it at all ſignify what Number is made the Radix of a Table of this Kind, provided the ſame Proportions are preſerved throughout. However, in conformity to Cuſtom, I have here been at the Pains to reduce the ſaid Table to the common Radix.

Amongſt Thoſe who have been ſevere on the above Alterations, there is a certain Foreign Gentleman, *M. Parcieux*, Member of the Royal Academy of Sciences at *Paris*, whom I muſt not paſs by without a more particular Notice. This Gentleman, in a Book intitled

intitled *Essai sur les Probabilités de la vie humaine*, (as appears by an Account given of it in the *Memoirs* of the said Academy, 1746) accuses me of *Having assigned much too great a Probability of Mortality to the youngest Ages*; and further affirms that *this Error* (for such He calls it) *arises from omitting to include, in the Account, those Persons who, having escaped the Weakness of Infancy, go out of the Bills of Mortality, and die elsewhere.*

Whether, or no, the Fact be as it is here represented, this Writer does not appear to be a sufficient Judge: He does not seem to be appriz'd, that there is not, perhaps, a City in the World so fatal to the Infant State as the City of *London* (owing, too much, to the Intemperance of Parents and profuse, irregular manner of Living). Besides, were the Bills of Mortality of a Place to be ever so regular, and well kept, it ought to be remembered that there can be no Arriving at an absolute Certainty, in Matters of this Nature. Now, were it for these Reasons alone, I could not help looking upon this Gentleman's Assertions (to view Them in the most favourable Light) as a little too peremptory; especially where He tells the World that, *I was not aware, that, by avoiding* Sir W. Petty*'s Fault, I my self fell into the contrary Mistake.* I declare, I am not conscious of any such Mistake; nay I can positively affirm, that the Reason he assigns for it, is alogether chimerical: Nor is there any thing I have said in my Preface, or elsewhere, that can justify such a Construction. But, what need have I to insist upon this! Every body Here knows that the Number of Those, who are born in Town, and die elsewhere, is inconsiderable in comparison of the Number of Those who, on the contrary, leave the Country and die in Town: So that, had they been, *actually*, disregarded, no such Error as He speaks of could possibly have arisen from thence.

The second Table, in this Work, *shewing the Number of Years that a Person of any Age, may, upon an Equality of Chance, expect to enjoy*, is deduced from the First; by a Method of Computation no way different from That, for finding the Value of an Annuity upon a single Life, when Money is supposed to bear no Interest; only the Value of one Half-Year, extraordinary, is to be added *here*; because an *Annuitant*, who receives his Money yearly, has an equal Chance of living Half a Year after the Time of the last Payment.

As to the Manner of computing the Tables, for the Values of single, and joint, Lives; this I have shewn (with proper Expedients to facilitate the Trouble of such a Work) in my Book of Annuities aforesaid, to which I must beg Leave to refer. — The first eight Problems, relating only to the Use of these Tables, need no other Explanation than is already given them.

The 9th Problem, *determining the Value of three joint Lives*, is the first that seems necessary to be taken Notice of *here:* Whereof the Solution is not, indeed, strictly conformable to the Table of Observations, being only an Approximation, but such an One as answers very near the Truth; and which may be apply'd with Advantage to, almost, any Hypothesis, or Table of Observations. The Reasonableness of the Method of proceeding is evident from the Nature of the Subject, without calling in the Assistance of any Kind of Computation. And, in a Number of Examples, respecting Lives of different Ages, I scarce ever found the Error to exceed $\frac{1}{8}$ of an Years-Purchase.

The Solution to our 10th Problem, *for determining the Value of the longest of three Lives*, is equally exact, and built upon the same Foundation, with the preceding One. For, the Value of the longest of three Lives being truly, and universally, expressed by $A + B + C - (AB) - (AC) - (BC) + (ABC)$; (where (AB), (AC), and (BC) denote the Values of each Pair of *joint Lives*, &c. *Vid. Doct. Ann. p.* 23); it follows that the said Value, if (according to *Prob.* 9) a single Life D be taken equal in Value to the two *joint Lives* (BC), will also be expressed by $A + B + C - (AB) - (AC) - D + (AD)$

or

or by its Equal $\overline{A+B-(AB)} + \overline{A+C-(AC)} - \overline{A+D-(AD)}$: But $\overline{A+B-(AB)}$ is the Value of the longeſt of the two Lives A and B, and $\overline{A+C-(AC)}$, that of the longeſt of the Lives A and C, *&c.* Whence the Whole is manifeſt.

The Solution to the next Problem, and Thoſe that follow afterwards on Reverſions, and ſucceſſive Lives, are either ſuch as are evident from plain Reaſoning, excluſive of Mathematical Principles, or ſuch, whoſe Demonſtrations I have already given elſewhere: For which Reaſons it ſeems needleſs to ſay any Thing further about them in this Place.

But, in thoſe Kinds of Reverſions conſider'd in the Thirty-ſecond, and the following Problems, depending upon the Probability of one particular Life, in Poſſeſſion, ſurviving the Reſt, the Reader muſt, I apprehend, have met with ſome Difficulty, the Rules there laid down being derived by Virtue of a certain Hypotheſis; which, however, anſwers very near the Truth.

In this Hypotheſis the Numbers of the Living at each ſeveral Age, differing by an, equal, Interval of one Year, are ſuppoſed to form an Arithmetical Progreſſion (or a Series whoſe Terms decreaſe continually by a common Difference). This Aſſumption was firſt adopted by Mr. *De Moivre*, and apply'd by him, to good Purpoſe, in calculating the Values of Lives, according to the *Breſlaw* Obſervations *(mentioned in p.* 275), with which it, indeed, much better agrees than with Thoſe deduced from the Bills of Mortality of the City of *London*.

But, though the Hypotheſis aforeſaid cannot be apply'd with equal Advantage Here, without ſome farther Conſideration, ſeeing the Differences of the Numbers *in Tab.* I, are far from being the ſame throughout; yet, by taking the Mean of the ſaid Differences, the Value of a *real* Life may be converted into That of an equal *imaginary* One, conformable to the ſaid Hypotheſis.

This is actually done, *in Tab.* II. in which you have the Number of Years that a Perſon of any Age, may upon an Equality of Chance, be expected to enjoy: The Double whereof is the *Complement* of an equal, *imaginary* Life, according to the ſaid Hypotheſis; where, by the *Complement*, is to be underſtood the Number of Years that *ſuch* a Life has a Chance of continuing in Being.

Thus it will appear, that the Value of a *real* Life, of Fifty, is equal to That of an *imaginary* One, whoſe *Complement* is 32 Years. This indeed is exact, only, when Money is ſuppoſed to bear no Intereſt; in all other Caſes, the Latter is only an Approximation, but ſo near as to come within a ſufficient Degree of Exactneſs, as will be ſeen from the annexed Table, wherein are exhibited the Values of Annuities for every tenth Year of Life, both according to real Obſervation, and to the ſaid Hypotheſis.

Age.	Value at 5 *per cent.* by Obſ.	Value at 5 *per cent.* by Hyp.	Value at 4 *per cent.* by Obſ.	Value at 4 *per cent.* by Hyp.	Value at 3 *per cent.* by Obſ.	Value at 3 *per cent.* by Hyp.
10	14.3	14.2	16.4	16.3	19.0	19.1
20	13.0	13.1	14.8	15.0	17.2	17.2
30	11.6	11.9	13.1	13.4	15.0	15.1
40	10.3	10.7	11.5	11.9	13.2	13.4
50	9.2	9.6	10.1	10.4	11.4	11.5
60	7.9	8.1	8.4	8.6	9.2	9.4
70	6.2	6.3	6.5	6.6	6.9	7.1

By inſpecting of which Table it will appear, that the Error ſeldom exceeds $\frac{1}{40}$ Part of the whole Value, and is the leaſt of all, where the Intereſt of Money is loweſt, agreeable to what was before intimated.

But before I proceed to the Uſe of the foregoing Hypotheſis in the Reſolution of the ſeveral Problems above

above mentioned, I shall put down the Investigation of a known Theorem, for finding the Value of any proposed Life, according to the said Hypothesis: Which (being done without the Summation of Series's) may perhaps appear more obvious than That delivered, at *p.* 15, of my former Tract.

Let, therefore, a be taken to denote the *Complement* of the proposed Life A *(as above defined)*, also let P denote the Value of the Perpetuity, and r the Amount of 1*l.* in one Year: And, in order to render the Operation as easy as may be, suppose another Person (Q) and his Heirs to be intitled to the Reversion of the Annuity, for ever, after the Demise of A: Then, if the Value of the said Reversion be computed, and subtracted from the Perpetuity, the Remainder will, evidently, be the required Value of the Life A. But, to find the true Value of this Reversion, we must determine the several Parts Thereof, which depend on the Contingency of A's becoming extinct in the first, second, third, and each succeeding Year.

Thus, if the Life A fails the first Year, whereof the Probability *(according to the Hypothesis)* is $\frac{1}{a}$, then, as Q and his Heirs become possessed of the whole Estate, or Perpetuity (without any Deduction) it is plain that the true Value of their Expectation, on the Contingency of A's becoming extinct the first Year, will be $\frac{1}{a} \times P$.

Again, if the Life A fails the second Year, whereof the Probability is also $\frac{1}{a}$, the Value of Q's Expectation will be the Value of the Perpetuity, discounted for one Year: And so the Expectation on this Event is $\frac{1}{a} \times \frac{P}{r}$.

And, in the same manner, it will appear, that the Expectation, on the Contingency of A's dying in the 3d, 4th, 5th, *&c.* Year, will be represented by $\frac{1}{a} \times$

$\frac{P}{r^2}$, $\frac{1}{a} \times \frac{P}{r^3}$, $\frac{1}{a} \times \frac{P}{r^4}$, *&c.* respectively; and consequently, that the Sum of all These, or its Equal $\frac{rP}{a}$ into $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + ---- + \frac{1}{r^a}$, will be the total Value of the Reversion. But the Series $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} ----- + \frac{1}{r^a}$ is known to express the Value of an Annuity, *certain*, for a Years: Which let be denoted by M; then, the Value of the Reversion being $= \frac{rP}{a} \times M = \frac{\overline{P+1} \times M}{a}$ $\left(\text{because } P = \frac{1}{r-1}\right)$ the Value of the proposed Life A, in Possession, will consequently be $P - \frac{\overline{P+1} \times M}{a}$. *Q. E. I.*

I come now to the Investigation of the Solutions of the Thirty-second, and the succeeding Problems.

In order to which let a and b be taken to represent the Complements of two imaginary Lives, of the same Values with the two real Ones A and B *(Vid. Prob.* 32). Moreover, let S denote the given Sum, or Legacy, to be received on the Decease of A, if B be then living; and let r represent the proposed Rate, or 1 *l.* increased by its Interest for one Year.

Now, to obtain an Answer to the said Problem, we must find the Value of B's Expectation on each particular Year, according to the Probability he has of receiving the Legacy (S) at the End of that Year: And each of these Values must be, again, consider'd in two Parts, according to the Chance of both the Lives Failing, or not Failing, within the Compass of the same Year.

The Probability that both A and B are extinct at the End of the first Year being $\frac{1}{a} \times \frac{1}{b}$, the Expectation

thereon

thereon would be $\frac{1}{a} \times \frac{1}{b} \times \frac{S}{r}$ $\left(\text{or } \frac{S}{abr}\right)$ was it a Certainty that A would drop first; but as there is an equal Chance for the Contrary (under the above Restrictions), the true Expectation can therefore be only $\frac{S}{2abr}$. In the same manner, the Expectation, on the Chance of both the Lives failing in the 2d, 3d, 4th, &c. Years, will be $\frac{S}{2abr^2}$, $\frac{S}{2abr^3}$, $\frac{S}{2abr^4}$ &c. And the Sum of all These $\frac{S}{2ab} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \text{\&c.}}$ will consequently be the whole of B's Expectation, on the Contingency of both the Lives dropping in one, and the same, Year: Where the Series is to be continued to as many Terms as are expressed by the *Complement* b of the oldest Life.

But now, with regard to the other Part of B's Expectation, depending on the Probability of his surviving the Year wherein A expires (which is, by far, the most considerable), it is evident that the Part thereof which depends upon the Chance of his coming into Possession at the End of any proposed Year, is compounded of the Probability of his being then alive, and That of A's dying within the Compass of the same Year (allowing also for the Discompt of Money to that Time).

Hence the Expectation on the proposed Sum S, on the Contigency of receiving it at the End of 1, 2, 3, 4, &c. Years, will be expressed by $\frac{b-1}{b} \times \frac{1}{a} \times \frac{S}{r}$, $\frac{b-2}{b} \times \frac{1}{a} \times \frac{S}{r^2}$, $\frac{b-3}{b} \times \frac{1}{a} \times \frac{S}{r^3}$, &c. respectively. The Aggregate of all which Values, added to the former Part of B's Expectation (found above) gives $\frac{S}{a} \times \overline{\frac{2b-1}{2br} + \frac{2b-3}{2br^2} + \frac{2b-5}{2br^3} + \frac{2b-7}{2br^4}}$ &c. for the whole

whole Value required. Where the Series is to be continued to as many Terms as there are Units in b, the Complement of the oldeſt Life.

Now, ſeeing the Concluſion, here brought out, is not otherwiſe affected by a th n its being a general Diviſor to the *whole*, it is evident that the true Value of the Expectation, ſuppoſing b to remain the ſame, will be reciprocally as the Complement (a) of the younger Life, and therefore, alſo, as the Expectation of its Duration, which is the Half therof. But the ſaid Value, when $a=b$, is in the ſame Proportion to the given Sum S, as Half the Reverſion of an Annuity, for ever, after two ſuch equal *joint Lives*, is *to the Perpetuity* (*Vid. Doct. Ann. p.* 76.) Whence the Reaſon of the Solution to *Prob.* 32 is manifeſt. And, by proceeding according to the very ſame Method, the 33[d] will alſo appear conſpicuous.

But, as to the 34[th], the Exegeſis will be ſomething more complex; ſince we muſt, firſt of all, determine *the Probability that both* A *and* B *will be extinct at the End of any Number of Years, on the Contingency of* A's *dying firſt.*

In order to which, let a and b be taken as above ſpecified; and ſuppoſe that a Sum of Money S is to be received on the Deceaſe of B (whenever that happens) provided A is then extinct.

Now (as before) the Probability of receiving the Sum S, within the Compaſs of any propoſed Year, muſt be conſidered in two Parts, according to the Chance of One, or both Lives, failing in that Year.

Thus the Probability of receiving it the 2[d], 3[d], or 4[th], &c. Year, on the Contingency of A's being dead before the Commencement of that Year, will be defined by $\frac{1}{a} \times \frac{1}{b}$, $\frac{2}{a} \times \frac{1}{b}$, or $\frac{3}{a} \times \frac{1}{b}$, &c. reſpectively.

And the Probability of receiving it, any Year, on the Contingency of both the Lives failing within that Year, will, conſtantly, be expreſſed by $\frac{1}{2ab}$,

for

for as long as They both have a Chance of continuing together.

Hence the whole Probability, depending on the 1^{st}, 2^{d}, 3^{d}, 4^{th}, *&c.* Year (by adding $\frac{1}{2ab}$ to each of the Values first found) appears to be $\frac{1}{2ab}$, $\frac{3}{2ab}$, $\frac{5}{2ab}$, $\frac{7}{2ab}$, *&c.* respectively: Therefore the Sum of all These, continued to as many Terms (n) as there are Years proposed, will, evidently, be the true Value of the Probability required. But the Sum of n Terms of the Series, 1, 3, 5, 7, *&c.* is $= n^2$: Therefore the required Probability is also truly expressed by $\frac{n^2}{2ab}$.

But here we must not omit to take Notice, that this Conclusion holds, only, when (n) the Number of Years is less than the Time that A and B have a Chance of living together: For, if n be greater than a, and the Age of A, at the same time, exceeds That of B; then, besides the Probability of Receiving the Sum S, in the Time (a) during which A hath a Chance of living, there is a further Chance, or Probability, depending on the Contingency of receiving it, or of B's dying, after the Expiration of the said Time: Which (as the Life A is then out of the Question) will consequently be expressed by $\frac{n-a}{b}$. And this added to $\frac{aa}{2ab}$, found above (by writing a for n), gives $\frac{2n-a}{2b}$, for the true Measure of the required Probability in this Case, where n is greater than a.

Now, to apply these Conclusions to the Problem in Question, let c denote the Complement of the Life C: Then, as the Probability of his receiving the Rent of any proposed Year, is compounded of the Probability of his living till then, and of the Probability that both A and B are extinct, with the Restriction of A's dying first *(as above determined)*; it is manifest that the present

Value

Value of his Expectation depending on the 1, 2, 3, 4th, *&c.* Years Income, will be expressed by $\frac{1^2 \times \overline{c-1}}{2abcr}$, $\frac{2^2 \times \overline{c-2}}{2abcr^2}$, $\frac{3^2 \times \overline{c-3}}{2abcr^3}$, $\frac{4^2 \times \overline{c-4}}{2abcr^4}$, *&c.* respectively. And so the whole Expectation, or the Value sought, will be truly exhibited by the following Series $\frac{1}{2abc} \times \overline{\frac{1^2 \times \overline{c-1}}{r} + \frac{2^2 \times \overline{c-2}}{r^2} + \frac{3^2 \times \overline{c-3}}{r^3} + \text{-----} \frac{c^2 \times \overline{c-c}}{r^c}}$, when C (*according to Case* 1 *of the Solution*) is older than either A or B; and by $\frac{1}{2abc} \times \overline{\frac{1^2 \times \overline{c-1}}{r} + \frac{2^2 \times \overline{c-2}}{r^2} + \text{----} \frac{b^2 \times \overline{c-b}}{r^b} + \frac{b^2 \times \overline{c-b-1}}{r^{b+1}} + \text{---} \frac{b^2 \times \overline{c-c}}{r^c}}$, when C is younger than B, and older than A (*according to the former Part of Case* 2).

But the first of these two Values is known to express Half the Value of the Reversion of the Life C after the longest of the Lives A and B.

And, with regard to the Second, it is evident (because $\frac{1}{bbc} \times \overline{\frac{1^2 \times \overline{c-1}}{r} + \frac{2^2 \times \overline{c-2}}{r^2} + \text{---} \frac{b^2 \times \overline{c-b}}{r^b} + \frac{b^2 \times \overline{c-b-1}}{r^{b+1}} + \text{----} \frac{b^2 \times \overline{c-c}}{r^c}}$ is the Value of the Reversion of the Life C after the longest of two equal Lives B, B) that the true Value Thereof must be equal to That of the said Reversion, multiply'd by $\frac{b}{2a}$: *Hence the former Part of Case* 2° is also manifest. — As to *the latter Part*, it is derived *from thence*, by plain Reasoning. For, since the absolute Value of the Reversion of the Life C after the two given Lives A and B, may be considered as composed of two Parts, depending on the

Chance

Chance of the Older of the two Lives surviving the Younger, and on the Chance of the Younger's surviving the Older; it is very easy to perceive that, if the former of these Parts be taken from the Whole, the other Part will remain.

Thus much concerning the Solution of our 34th Problem; on which the three succeeding Ones, almost, intirely depend; and therefore it will be unnecessary to dwell upon them here.

The Solution to the 38th may, however, need some sort of Explanation: In regard to which it is evident, that the Probability of the Expectant's receiving the proposed Legacy (*S*) at the End of any particular Year, is compounded of the Probability of his continuing alive till then, and That of B's dying within the Compass of the said Year, on the Contingency of A's being extinct before. Now the last of These is already found to be $\frac{1}{2ab}$, $\frac{3}{2ab}$, $\frac{5}{2ab}$, &c. according as the 1st, 2d, 3d, &c. Year is assigned; whence it is plain that the Expectation, on the Contingency of coming into Possession at the End of the first, second, third, &c. Year, will be $\frac{1\times\overline{c-1}\times S}{2abcr}$, $\frac{3\times\overline{c-2}\times S}{2abcr^2}$, $\frac{5\times\overline{c-3}\times S}{2abcr^3}$, &c. respectively;

and, consequently, that the Sum of all These, or $\frac{S}{2bac}\times$

$$\overline{\frac{1\times\overline{c-1}}{r}+\frac{3\times\overline{c-2}}{r^2}+\frac{5\times\overline{c-3}}{r^3}+\frac{7\times\overline{c-4}}{r^4}+\&c.}$$

will be the, whole, Value required.

But this Series may be divided into two Others, *viz.*

$$\frac{S}{abc}\times\overline{\frac{1\times\overline{c-1}}{r}+\frac{2\times\overline{c-2}}{r^2}+\frac{3\times\overline{c-3}}{r^3}+\frac{4\times\overline{c-4}}{r^4}+\&c.}$$

and

$$\frac{-S}{2abc}\times\overline{\frac{c-1}{r}+\frac{c-2}{r^2}+\frac{c-3}{r^3}+\frac{c-4}{r^4}+\frac{c-5}{r^5}}+\&c.$$

 And

And the former of These may be, again, resolved into $\frac{S}{a} \times \overline{\frac{c-1}{cr} + \frac{c-2}{cr^2} + \frac{c-3}{cr^3} + \frac{c-4}{cr^4} + \frac{c-5}{cr^5} \&c.}$

and $\frac{-S}{a} \times \frac{\overline{b-1} \times \overline{c-1}}{bcr} + \frac{\overline{b-2} \times \overline{c-2}}{bcr^2} + \frac{\overline{b-3} \times \overline{c-3}}{bcr^3} \&c.$ whereof the last (exclusive of the General Multiplicator) is known to express the Value of the joint Lives B and C: And the preceding Part (or Series) is likewise known to represent, either the Value of the *whole* Life C, or else the Value of as many Years of it as are expressed by the Complement of the oldest of the Lives A and B, according to the different Cases specified in the Solution of the Problem: Whence the Reason of the Whole is apparent.—As to the Series $\frac{-S}{2abc}$ $\times \overline{\frac{c-1}{r} + \frac{c-2}{r^2} + \frac{c-3}{r^3} \&c.}$ whereof no Notice is taken in the Solution, it is so small, in Comparison of the other Two, that to have embarrass'd the Solution with it would not only have been unnecessary, but quite improper, considering the Nature of the Subject; where an absolute Exactness (for want of knowing the precise Law of the Decrements of Life) is impossible.

There remain yet three other Solutions to speak of; but as These depend intirely on the 38th, and some of the preceding Ones, it will be needless to say any thing further about Them in this Place. I have, already, been very particular on these Kinds of Problems, and the more so, as there is no Method yet published (that I know of) by which They can be rightly determined. 'Tis true the Manner of proceeding by first finding the Probability of Survivorship (which Method is used in my former Work, and which a celebrated Author on this Subject has largely insisted on, in three successive Editions) may be applied to good Advantage, when the given Ages are nearly equal: But then

then, it is certain, that this is not a genuine Way of going to Work; and that the Conclusions hence derived are, at the best, but near Approximations. The Rate of Interest that Money bears must be compounded with the Probability of Survivorship, and the Expectation on each particular Year must be determined, in order to have a true Solution.

I shall conclude what I have to say on this Subject of Annuities with the Solutions of two, or three, of the most useful Problems, according to a Method very different from That whereby They are usually investigated.

PROBLEM I.

To determine the present Value of an Estate in Land, for a single Life A, *of a given Age, according to the Hypothesis of an uniform decrease of the Probability of Life.*

Let r denote the Rate, p the Perpetuity, and a the Complement of the proposed Life. Moreover let x (considered as flowing uniformly) be any Time elapsed from his entering into Possession: Then the present Value of the Expectation on the next succeeding Moment $\dot{x}$, will, evidently, be expressed by $\frac{a-x}{a} \times \frac{\dot{x}}{r^x}$, or, its Equal, $\dot{x}q^x - \frac{x\dot{x}q^x}{a}$ (by putting $q = \frac{1}{r}$) whereof the Fluent, when $x = a$, will consequently be the true Value required.

But the Fluent of the first Term $(\dot{x}q^x)$ is found $= \frac{q^x}{m}$; and That of the Second $\left(\frac{x\dot{x}q^x}{a}\right) = \frac{q^x}{ma} \times \overline{x - \frac{1}{m}}$; *by Prob.* 1, *Sect.* 6, *Vol.* 2, *of my Doctrine and Application of Fluxions*: Where m denotes the Hyperbolical Logarithm of q.

 These

These Fluents, corrected by their constant Quantities, become $\frac{q^x - 1}{m}$, and $\frac{q^x}{am} \times \overline{x - \frac{1}{m}} + \frac{1}{am^2}$: Which, when x is $= a$, are expressed by $\frac{q^a - 1}{m}$, and $\frac{q^a}{m} + \frac{1 - q^a}{am^2}$, respectively. The Difference of which, or $-\frac{1}{m} - \frac{1 - q^a}{am^2}$, is consequently the true Value of the proposed Life.

But $\frac{q^x - 1}{m}$ (the Fluent of $\dot{x} q^x$), when x becomes infinite, will be equal to the Perpetuity, that is $-\frac{1}{m}$ $\left(= \frac{1}{\text{Hyp. Log. } r}\right) = p$: Whence, $-m$ being $= \frac{1}{p}$, the Value found above will, also, be expressed by $p - \overline{1 - q^a} \times \frac{p^2}{a}$, or, its Equal, $p - \frac{pM}{a}$; supposing M to be put for $(\overline{1 - q^a \times p})$ the Value, *certain*, for the Time a.

The Value here brought, it may be observed, exceeds $\left(p - \frac{\overline{p + 1} \times M}{a}\right)$ That given in p. 316, by $\frac{M}{a}$: Which is owing to the Advantage of enjoying an Estate in Land (by an actual Possession to the last Moments of Life) preferable to an Annuity, where no Regard is had to the Time elapsed from the last Yearly-Payment to the Dropping of the proposed Life.

PROBLEM

PROBLEM II.

To determine the Value of an Estate in Land, to continue duringthe joint Existence of any Number of assigned Lives, A, B, C, D, *&c. according to the aforesaid Hypothesis,*

Let the Complements of the proposed Lives A, B, C, D, *&c.* be denoted by a, b, c, d, &c. respectively: Then, supposing other Things to remain as in the preceding Problem, the Value of the Expectation on the Moment $\dot{x}$ will, here, be expressed by $\frac{a-x}{a}\times\frac{b-x}{b}\times\frac{c-x}{c}\times\frac{d-x}{d}$ *&c.* $\times\frac{\dot{x}}{r^x}$, or its equal, $\overline{1-\frac{x}{a}}\times\overline{1-\frac{x}{b}}\times\overline{1-\frac{x}{c}}$ *&c.* $q^x\dot{x}$. Which, if the Sum of all the Quantities $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, *&c.* be put $=\alpha$; the Sum of all their Rectangles $=\beta$; the Sum of all their Solids $=\gamma$, *&c.* will be reduced to $\dot{x}q^x-\alpha x\dot{x}q^x+\beta x^2\dot{x}q^x-\gamma x^3\dot{x}q^x+\delta x^4\dot{x}q^x$, *&c.*

Whereof the Fluent *(by the Problem above quoted)* will be found $=\frac{q^x}{m}$ into $\overline{1-\alpha\times\overline{x-\frac{1}{m}}+\beta\times\overline{x^2-\frac{2x}{m}+\frac{2}{m^2}}-\gamma\times\overline{x^3-\frac{3x^2}{m}+\frac{6x}{m^2}-\frac{6}{m^3}}+}$ *&c.* which, when x is $=0$, becomes $=\frac{1}{m}+\frac{\alpha}{m^2}+\frac{2\beta}{m^3}+\frac{6\gamma}{m^4}+\frac{24\delta}{m^5}$ *&c.*

Hence the corrected Fluent (by subtracting this last Expression, and writing p, instead of, its Equal, $-\frac{1}{m}$) will come out $p-\alpha p^2+2\beta p^3-6\gamma p^4+24\delta p^5$, *&c.* $-q^xp$ into $\overline{1-\alpha\times\overline{x+p}+\beta\times\overline{x^2+2xp+2p^2}-\gamma\times\overline{x^3+3x^2p+6xp^2+6p^3}}$, *&c.* which, by taking x equal

equal to the Complement of the oldest Life, gives the true Value required.

When all the Lives are equal, the Conclusion will become much more simple: For, the Fluxion above given, by making $z = a - x$ ($= b - x = c - x$, &c.) will be here transformed to $-\frac{z^n \dot{z} r^z}{a^n r^a}$. Whose Fluent, supposing m to denote the Hyperbolical Logarithm of r, will appear to be $-\frac{r^z}{ma^n r^a} \times \overline{z^n - \frac{nz^{n-1}}{m} + \frac{n.\overline{n-1}.z^{n-2}}{m^2}}$ &c.

Which, when $x = 0$, or $z = a$, becomes $-\frac{1}{ma^n} \times \overline{a^n - \frac{na^{n-1}}{m} + \frac{n.\overline{n-1}.a^{n-2}}{mm}}$ &c. But when $x = a$, or $z = 0$, it becomes $-\frac{1}{ma^n r^a}$ into $\pm \frac{1.2.3.4 \ldots n}{m^n}$:

The Difference of which Values, or its Equal,

$$p - \frac{np^2}{a} + \frac{n.\overline{n-1}.p^3}{a^2} - \frac{n.\overline{n-1}.\overline{n-2}.p^4}{a^3} \text{ \&c.}$$

$\pm \frac{1.2.3.4 \ldots n}{a^n r^a} \times p^{n+1}$, is consequently the true Value of all the Joint Lives: where the Sign of the last Term must be negative, or positive, according as (n) the Number of Lives is even or odd.

PROBLEM III.

Suppose that a given Sum (S) *is to be received, as a Legacy, on the Decease of* B, *in case* A *is then extinct: To find its Value in present Money, allowing for that Contingency.*

If a, b, q, &c. be supposed as in the preceding Problems, it is easy to perceive that the Expectation on the Moment $\dot{x}$, will here be defined by $\frac{x}{a} \times \frac{\dot{x}}{b} \times$

Sq^x

Sq^x, or its Equal, $\frac{S}{a} \times \frac{x\dot{x}q^x}{b}$. But the Fluent of $\frac{x\dot{x}q^x}{b}$ (when $x = b$) is known to expreſs the Difference between the Value of the Life B, and the Value (M) of an Annuity *certain*, for b Years (*Vid. Prob.* 1). Whence it plainly follows, that $\frac{S}{a} \times \overline{M-B}$, will be the true Value ſought; provided B is older than A.

But, if B be younger than A; then, beſides the Expectation depending on the Chance of his dying within the Term (a) to the End of which A has a Poſſibility of living, there is a farther Expectation ariſing from the Probability of B's not dying till after the Expiration of the ſaid Term. Which Expectation, with regard to any particular Moment $\dot{x}$ (as A is here out of the Queſtion) will be repreſented by $\frac{\dot{x}}{b} \times Sq^x$. Whoſe Fluent (generated while x is increaſed from a to b) is expreſſed by $\frac{S}{b}$ drawn into the preſent Value (V) of an Annuity *certain*, for the Time ($b-a$) which B has a Chance of living after the greateſt Limit of A's Duration. This, therefore, added to $\frac{S}{b} \times \overline{N-A}$ (found as above, by changing a for b, A for B, and N for M) gives $\frac{S}{b} \times \overline{N+V-A}$, for the whole Value of the Expectation, in this Caſe. But (N) the Value of an Annuity certain for the Time a, with (V) the Value of the Reverſion Thereof for the Time $b-a$, after the Time a, is conſequently equal to It's whole Value (M) for the Time b. Therefore our laſt Expreſſion is reduced to $\frac{S}{b} \times \overline{M-A}$. From which and the other Value found above, we have the following General Rule.

From

From the Value of an Annuity in Land, *certain* for as many Years as are expreſſed by the Complement of B's Age, ſubtract the Value of the oldeſt Life (be it which it will) then ſay, as the Complement of the youngeſt Life, is to the Remainder, ſo is the propoſed Sum to its required Value in preſent Money.

PROBLEM IV.

Q and his Heirs are intitled to an Eſtate of a given Value (S) on the Deceaſe of C, *in Caſe* B *ſurvives* A, *and is himſelf ſurvived by* C: *To determine the preſent Value of Q's Expectation.*

Here the Expectation on the Moment $\dot{x}$ depends on the Probability of C's dying in that Moment, and on the Probability that both A and B are before extinct, with the further Reſtriction of A's dropping firſt.

In order to determine the latter of Theſe, it will be proper to obſerve, from the laſt Problem, that, as $\frac{x\dot{x}}{ab}$ denotes the Probability of the Expectant's receiving the Sum S in the Moment $\dot{x}$, ſo $\left(\frac{x^2}{2ab}\right)$ the Fluent thereof muſt conſequently expreſs the Probability of receiving it during the Interval x; which is, evidently, the Probability here required. Hence it follows that the Expectation on the Moment $\dot{x}$, in the preſent Caſe, is truly defined by $\frac{\dot{x}}{c} \times \frac{x^2}{2ab}\ Sq^x$. Whoſe Fluent, $\frac{Sq^x}{2abcm} \times \overline{x^2 - \frac{2x}{m} + \frac{2}{mm}}$, properly corrected, will conſequently be the true Value of Q's Expectation.

After the ſame manner the Value ſought, in more complicated Caſes, may be determined. But I ſhall here put down a general Theorem for finding the Value of an Annuity, granted upon any Number (n) of aſſigned Lives, A, B, C, D, *&c.* but ſo, as to continue,

tinue, only, as long as a given Number *(m)* of Them are in Being.

Let *Q* be the Value of all the Joint Lives A, B, C, *&c.* that is, the Value of an Annuity, for as long as They all continue in Being together; also let *R* be the Sum of the Values of all the Joint Lives that can arise, by combining A, B, C, *&c.* so as to leave out one Life at each Combination; and *S* the Sum of all the Joint Lives that can arise by combining the same, so as to leave out two Lives at each Combination, *&c. &c.*

Then will the Value of the Purchase be truly expressed by $\pm \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3}\ (n-m) \times Q \mp \frac{n-2}{1} \times \frac{n-3}{2} \times \frac{n-4}{3}\ (n-m-1) \times R \pm \frac{n-3}{1} \times \frac{n-4}{2} \times \frac{n-5}{3}\ (n-m-2) \times S \mp \frac{n-4}{1} \times \frac{n-5}{2} \times \frac{n-6}{3} (n-m-3) \times T$, *&c.*

Where the upper, or the lower, Signs obtain, according as $n-m$ is an even, or an odd, Number; and where the Quantities between the Hooks, express the Number of the preceding Factors to be taken; with regard to which it is to be observed, that in the last Term, where the Number of Factors becomes $=0$, an Unit must be taken.

This Theorem (which is strictly true according to any Law of the Decrements of Life) is the same, in effect, with that given at *p.* 26, of my Doctrine of Annuities, but rather more commodious.—Nevertheless, as the Trouble of computing the Values Q, R, S, *&c.* will still be very great, when several Lives are concerned, the following Approximation for the Value of the longest Life (as the Annuity is usually held thereon) will also be found of Use.

Let *a*, *b*, *c*, *d*, *e*, *f*, *&c.* express the Values of any Number of single Lives; of which Values let *a* be the greatest, *b* the next, and so on; also let *m* denote the Ineterst and eight Tenths of the Interest of one Pound for one Year:

 Then

Then will the Value of all the Lives be nearly $= a + \frac{\overline{2-mb} \times b^2}{4a} + \frac{\overline{2-mc} \times c^3}{9ab} + \frac{\overline{2-md} \times d^4}{16abc} + \frac{\overline{2-me} \times e^5}{25abcd} +$ &c. Which Theorem, in that Case where all the Lives are equal, will become much more simple, being then reducible to

$a + \overline{2-ma} \times a \times \overline{\tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \tfrac{1}{25}}$, &c. Where as many Terms of the Series $\frac{1}{4} + \frac{1}{9} + \frac{1}{16}$ &c. are to be taken, as are expressed by the Number of the Lives, less one.

The E N D.

ERRATA.

Page 11, l. 11, for $4d$ read $2d$; p. 61, l. last, for g r. $\frac{1}{2}g$; p. 73, l. 23, after *either*, r. *be*; p. 81, l. 11, for *a* r. 77; p. 82, l. 18, for $= \overline{2mn}|^2$ r. $+ \overline{2mn}|^2$; p. 87, l. 19, for 51, r. 52; l. 27, for 26 and 27, r. 29 and 30; p. 89, l. 4, for BC r. AC; p. 98, l. 10, for *b* r. *a*, and for *a* r. *b*; p. 99, l. 24, for HG r. HI; p. 106, l. 10, for EK r. IK; p. 109, l. 27, for *mma*, under the Radical Sign, r. *mna*; p. 126, l. 18, r. *to, and from*; p. 133, l. 18, for *inscribed* r. *described*; p. 157, l. 20, for HKL r. KHL; p. 166, l. 16, for ADE r. AED; p. 174, l. 3, for HN r. HL; p. 184, l. 11, dele *with*; p. 180, l. 7, for *Power* r. *Powder*; p. 214, l. 12, for 3000 r. 6000; p. 228, l. 6, r. *Equations*; l. 20, for $\frac{n-1}{1}$ r. $\frac{n-1}{2}$; p. 231, l. 8, for $1-2z$ r. $4 \times \overline{1-2z}$; p. 234, l. 28, dele the *Comma*, and before *Fluxions* r. *the*; p. 311, l. 14, before *profuse* r. *a*.