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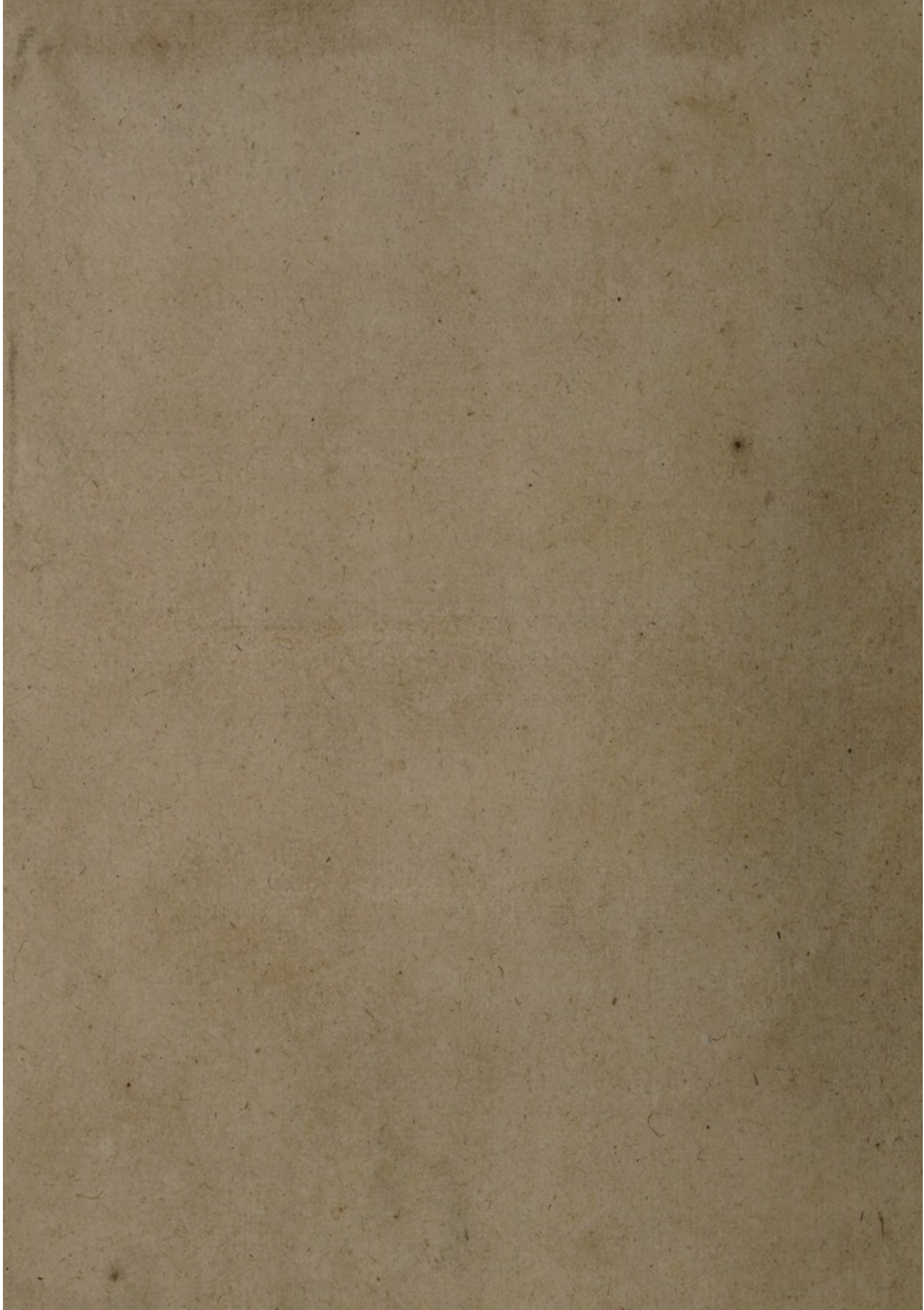


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THE
DOCTRINE
OF
CHANCES:

OR,

A METHOD of Calculating the Probabilities
of Events in PLAY.

THE THIRD EDITION,
Fuller, Clearer, and more Correct than the Former.

By A. D E M O I V R E,
*Fellow of the ROYAL SOCIETY, and Member of the ROYAL ACADEMIES
OF SCIENCES of Berlin and Paris.*



L O N D O N:

Printed for A. MILLAR, in the *Strand*.
MDCCLVI.



DOCTRINE

OF

CHANCES:

OR,

A METHOD OF CALCULATING THE PROBABILITIES
OF EVENTS IN PLAY

THE THIRD EDITION

Edited, Corrected, and with an Introduction by
J. H. W. L. and J. H. W. L.

By A. DE MOIVRE

Member of the Royal Society, and of the Academies
of Sciences of Paris, Berlin, and St. Petersburg



LONDON:

Printed for A. MILLAR, in the Strand.
MDCCLXXII.



To the RIGHT HONOURABLE the
Lord *CARPENTER*.*

MY LORD,



HERE are many People in the World who are prepossessed with an Opinion, that the Doctrine of Chances has a Tendency to promote Play; but they soon will be undeceived, if they think fit to look into the general Design of this Book: in the mean time it will not be improper to inform them, that your Lordship is pleased to espouse the Patronage of this second Edition; which your strict Probity, and the distinguished Character you bear in the World, would not have permitted, were not their Apprehensions altogether groundless.

* This Dedication was prefixed to the 2d Edition.

Your

DEDICATION.

Your Lordship does easily perceive, that this Doctrine is so far from encouraging Play, that it is rather a Guard against it, by setting in a clear Light, the Advantages and Disadvantages of those Games wherein Chance is concerned.

Besides the Endowments of the Mind which you have in common with Those whose natural Talents have been cultivated by the best Education, you have this particular Happiness, that you understand, in an eminent Degree, the Principles of Political Arithmetic, the Nature of our Funds, the National Credit, and its Influence on public Affairs.

As one Branch of this useful Knowledge extends to the Valuation of Annuities founded on the Contingencies of Life, and that I have made it my particular Care to facilitate and improve the Rules I have formerly given on that Subject; I flatter myself with a favourable Acceptance of what is now, with the greatest Deference, submitted to your Judgment, by,

MY LORD,


Your Lordship's

Most Obedient and

Most Obliged,

Humble Servant,

A. de Moivre.



P R E F A C E.*

TIS now about Seven Years, since I gave a Specimen in the Philosophical Transactions, of what I now more largely treat of in this Book. The occasion of my then undertaking this Subject was chiefly owing to the Desire and Encouragement of the Honourable † Francis Robartes Esq; who, upon occasion of a French Traët, called, L'Analyse des Jeux de Hazard, which had lately been published, was pleased to propose to me some Problems of much greater difficulty than any he had found in that Book; which having solved to his Satisfaction, he engaged me to methodize those Problems, and to lay down the Rules which had led me to their Solution. After I had proceeded thus far, it was enjoined me by the Royal Society, to communicate to them what I had discovered on this Subject: and thereupon it was ordered to be published in the Transactions, not so much as a matter relating to Play, but as containing some general Speculations not unworthy to be considered by the Lovers of Truth.

I had not at that time read any thing concerning this Subject, but Mr. Huygen's Book de Ratiociniis in Ludo Aleæ, and a little English Piece (which was properly a Translation of it) done by a very ingenious Gentleman, who, tho' capable of carrying the matter a great deal farther, was contented to follow his Original; adding only to it the computation of the Advantage of the Setter in the Play called Hazard, and some few things more. As for the French Book, I had run it over but cursorily, by reason I had observed that the Author chiefly insisted on

* This Preface was written in 1717.

† Now Earl of RADNOR.

the Method of Huygens, which I was absolutely resolved to reject, as not seeming to me to be the genuine and natural way of coming at the Solution of Problems of this kind.

I had said in my Specimen, that Mr. Huygens was the first who had published the Rules of this Calculation, intending thereby to do justice to a Man who had well deserved of the Public; but what I then said was misinterpreted, as if I had designed to wrong some Persons who had considered this matter before him: and a Passage was cited against me out of Huygen's Preface, in which he saith, Sciendum vero quod jam pridem, inter Præstantissimos totâ Galliâ Geometras, Calculus hic fuerit agitatus; ne quis indebitam mihi primæ Inventionis gloriam hac in re tribuat. But what follows immediately after, had it been minded, might have cleared me from any Suspicion of injustice. The words are these, Cæterum illi difficillimis quibusque Quæstionibus se invicem exercere soliti, methodum suam quisque occultam retinere, adeo ut a primis elementis hanc materiam evolvere mihi necesse fuerit. By which it appears, that tho' Mr. Huygens was not the first who had applied himself to those sorts of Questions, he was nevertheless the first who had published Rules for their Solution; which is all that I affirmed.

Such a Tract as this is may be useful to several ends; the first of which is, that there being in the World several inquisitive Persons, who are desirous to know what foundation they go upon, when they engage in Play, whether from a motive of Gain, or barely Diversion, they may, by the help of this or the like Tract, gratify their curiosity, either by taking the pains to understand what is here Demonstrated, or else making use of the Conclusions, and taking it for granted that the Demonstrations are right.

Another use to be made of this Doctrin of Chances is, that it may serve in Conjunction with the other parts of the Mathematicks, as a fit Introduction to the Art of Reasoning; it being known by experience that nothing can contribute more to the attaining of that Art, than the consideration of a long Train of Consequences, rightly deduced from undoubted Principles, of which this Book affords many Examples. To this may be added, that some of the Problems about Chance having a great appearance of Simplicity, the Mind is easily drawn into a belief, that their Solution may be attained by the meer Strength of natural good Sense; which generally proving otherwise, and the Mistakes occasioned thereby being not unfrequent, 'tis presumed that a Book of this Kind, which teaches to distinguish Truth from what seems so nearly to resemble it, will be looked upon as a help to good Reasoning.

Among

Among the several Mistakes that are committed about Chance, one of the most common and least suspected, is that which relates to Lotteries. Thus, supposing a Lottery wherein the proportion of the Blanks to the Prizes is as five to one; 'tis very natural to conclude, that therefore five Tickets are requisite for the Chance of a Prize; and yet it may be proved, Demonstratively, that four Tickets are more than sufficient for that purpose, which will be confirmed by often repeated Experience. In the like manner, supposing a Lottery wherein the proportion of the Blanks to the Prizes is as Thirty-nine to One, (such as was the Lottery of 1710) it may be proved, that in twenty eight Tickets, a Prize is as likely to be taken as not; which tho' it may seem to contradict the common Notions, is nevertheless grounded upon infallible Demonstration.

When the Play of the Royal Oak was in use, some Persons who lost considerably by it, had their Losses chiefly occasioned by an Argument of which they could not perceive the Fallacy. The Odds against any particular Point of the Ball were One and Thirty to One, which intitled the Adventurers, in case they were winners, to have thirty two Stakes returned, including their own; instead of which they having but Eight and Twenty, it was very plain that on the single account of the disadvantage of the Play, they lost one eighth part of all the Money they played for. But the Master of the Ball maintained that they had no reason to complain; since he would undertake that any particular point of the Ball should come up in Two and Twenty Throws; of this he would offer to lay a Wager, and actually laid it when required. The seeming contradiction between the Odds of One and Thirty to One, and Twenty-two Throws for any Chance to come up, so perplexed the Adventurers, that they begun to think the Advantage was on their side; for which reason they played on and continued to lose.

The Doctrine of Chances may likewise be a help to cure a Kind of Superstition, which has been of long standing in the World, viz. that there is in Play such a thing as Luck, good or bad. I own there are a great many judicious people, who without any other Assistance than that of their own reason, are satisfied, that the Notion of Luck is meerly Chimerical; yet I conceive that the ground they have to look upon it as such, may still be farther inforced from some of the following Considerations.

If by saying that a Man has had good Luck, nothing more was meant than that he has been generally a Gainer at play, the Expression might be allowed as very proper in a short way of speaking: But if the Word Good Luck be understood to signify a certain predominant quality, so inherent in a Man, that he must win whenever he Plays, or at least win oftner than lose, it may be denied that there is any such thing in nature.

The Asserters of Luck are very sure from their own Experience, that at some times they have been very Lucky, and that at other times they have had a prodigious Run of ill Luck against them, which whilst it continued obliged them to be very cautious in engaging with the Fortunate; but how Chance should produce those extraordinary Events, is what they cannot conceive: They would be glad, for Instance, to be Satisfied, how they could lose Fifteen Games together at Piquet, if ill Luck had not strangely prevailed against them. But if they will be pleased to consider the Rules delivered in this Book, they will see, that though the Odds against their losing so many times together be very great, viz. 32767 to 1, yet that the Possibility of it is not destroyed by the greatness of the Odds, there being One Chance in 32768 that it may so happen; from whence it follows, that it was still possible to come to pass without the Intervention of what they call Ill Luck.

Besides, This Accident of losing Fifteen times together at Piquet, is no more to be imputed to ill Luck, than the Winning with one single Ticket the highest Prize, in a Lottery of 32768 Tickets, is to be imputed to good Luck, since the Chances in both Cases are perfectly equal. But if it be said that Luck has been concerned in this latter Case, the Answer will be easy; for let us suppose Luck not existing, or at least let us suppose its Influence to be suspended, yet the highest Prize must fall into some Hand or other, not by Luck, (for by the Hypothesis that has been laid aside) but from the meer necessity of its falling somewhere.

Those who contend for Luck, may, if they please, alledge other Cases at Play, much more unlikely to happen than the Winning or Losing fifteen Games together, yet still their Opinion will never receive any Addition of Strength from such Suppositions: For, by the Rules of Chance, a time may be computed, in which those Cases may as probably happen as not; nay, not only so, but a time may be computed in which there may be any proportion of Odds for their so happening.

But supposing that Gain and Loss were so fluctuating, as always to be distributed equally, whereby Luck would certainly be annihilated; would it be reasonable in this Case to attribute the Events of Play to Chance alone? I think, on the contrary, it would be quite otherwise, for then there would be more reason to suspect that some unaccountable Fatality did rule in it: Thus, if two Persons play at Cross and Pile, and Chance alone be supposed to be concerned in regulating the fall of the Piece, is it probable that there should be an Equality of Heads and Crosses? It is Five to Three that in four times there will be an inequality; 'tis Eleven to Five in six, 93 to 35 in Eight, and about 12 to 1 in a hundred times: Wherefore Chance alone by its Nature constitutes the Inequalities of Play, and there is no need to have recourse to Luck to explain them.

Further,

Further, the same Arguments which explode the Notion of Luck, may, on the other side, be useful in some Cases to establish a due comparison between Chance and Design: We may imagine Chance and Design to be, as it were, in Competition with each other, for the production of some sorts of Events, and may calculate what Probability there is, that those Events should be rather owing to one than to the other. To give a familiar Instance of this, Let us suppose that two Packs of Piquet-Cards being sent for, it should be perceived that there is, from Top to Bottom, the same Disposition of the Cards in both Packs; let us likewise suppose that, some doubt arising about this Disposition of the Cards, it should be questioned whether it ought to be attributed to Chance, or to the Maker's Design: In this Case the Doctrine of Combinations decides the Question; since it may be proved by its Rules, that there are the Odds of above 263130830000 Millions of Millions of Millions of Millions to One, that the Cards were designedly set in the Order in which they were found.

From this last Consideration we may learn, in many Cases, how to distinguish the Events which are the effect of Chance, from those which are produced by Design: The very Doctrine that finds Chance where it really is, being able to prove by a gradual Increase of Probability, till it arrive at Demonstration, that where Uniformity, Order and Constancy reside, there also reside Choice and Design.

Lastly, One of the principal Uses to which this Doctrine of Chances may be applied, is the discovering of some Truths, which cannot fail of pleasing the Mind, by their Generality and Simplicity; the admirable Connexion of its Consequences will increase the Pleasure of the Discovery; and the seeming Paradoxes wherewith it abounds, will afford very great matter of Surprise and Entertainment to the Inquisitive. A very remarkable Instance of this nature may be seen in the prodigious Advantage which the repetition of Odds will amount to; Thus, Supposing I play with an Adversary who allows me the Odds of 43 to 40, and agrees with me to play till 100 Stakes are won or lost on either side, on condition that I give him an Equivalent for the Gain I am intitled to by the Advantage of my Odds; the Question is, what I am to give him, on supposing we play a Guinea a Stake: The Answer is 99 Guineas and above 18 Shillings*, which will seem almost incredible, considering the smallness of the Odds of 43 to 40. Now let the Odds be in any Proportion given, and let the Number of Stakes be played for be never so great, yet one general Conclusion will include all the possible Cases, and the application of it to Numbers may be wrought in less than a Minute's time.

* Guineas were then at 21^{lb}. 6^d.

I have explained, in my Introduction to the following Treatise, the chief Rules on which the whole Art of Chances depends; I have done it in the plainest manner that I could think of, to the end it might be (as much as possible) of general Use. I flatter my self that those who are acquainted with Arithmetical Operations, will, by the help of the Introduction alone, be able to solve a great Variety of Questions depending on Chance: I wish, for the sake of some Gentlemen who have been pleased to subscribe to the printing of my Book, that I could every where have been as plain as in the Introduction; but this was hardly practicable, the Invention of the greatest part of the Rules being intirely owing to Algebra; yet I have, as much as possible, endeavoured to deduce from the Algebraical Calculation several practical Rules, the Truth of which may be depended upon, and which may be very useful to those who have contented themselves to learn only common Arithmetick.

On this occasion, I must take notice to such of my Readers as are well versed in Vulgar Arithmetick, that it would not be difficult for them to make themselves Masters, not only of all the practical Rules in this Book, but also of more useful Discoveries, if they would take the small Pains of being acquainted with the bare Notation of Algebra, which might be done in the hundredth part of the Time that is spent in learning to write Short-hand.

One of the principal Methods I have made use of in the following Treatise, has been the Doctrine of Combinations, taken in a Sense somewhat more extensive, than as it is commonly understood. The Notion of Combinations being so well fitted to the Calculation of Chance, that it naturally enters the Mind whenever an Attempt is made towards the Solution of any Problem of that kind. It was this that led me in course to the Consideration of the Degrees of Skill in the Adventurers at Play, and I have made use of it in most parts of this Book, as one of the Data that enter the Question; it being so far from perplexing the Calculation, that on the contrary it is rather a Help and an Ornament to it: It is true, that this Degree of Skill is not to be known any other way than from Observation; but if the same Observation constantly recur, 'tis strongly to be presumed that a near Estimation of it may be made: However, to make the Calculation more precise, and to avoid causing any needless Scruples to those who love Geometrical Exactness, it will be easy, in the room of the word Skill, to substitute a Greater or Less Proportion of Chances among the Adventurers, so as each of them may be said to have a certain Number of Chances to win one single Game.

The general Theorem invented by Sir Isaac Newton, for raising a Binomial to any Power given, facilitates infinitely the Method of Combinations,
representing

representing in one View the Combination of all the Chances, that can happen in any given Number of Times. 'Tis by the help of that Theorem, joined with some other Methods, that I have been able to find practical Rules for the solving a great Variety of difficult Questions, and to reduce the Difficulty to a single Arithmetical Multiplication, whereof several Instances may be seen in the 46th Page of this Book.

Another Method I have made use of, is that of Infinite Series, which in many cases will solve the Problems of Chance more naturally than Combinations. To give the Reader a Notion of this, we may suppose two Men at Play throwing a Die, each in their Turns, and that he is to be reputed the Winner who shall first throw an Ace: It is plain, that the Solution of this Problem cannot so properly be reduced to Combinations, which serve chiefly to determine the proportion of Chances between the Gamesters, without any regard to the Priority of Play. 'Tis convenient therefore to have recourse to some other Method, such as the following: Let us suppose that the first Man, being willing to compound with his Adversary for the Advantage he is intitled to from his first Throw, should ask him what Consideration he would allow to yield it to him; it may naturally be supposed that the Answer would be one Sixth part of the Stake, there being but Five to One against him, and that this Allowance would be thought a just Equivalent for yielding his Throw. Let us likewise suppose the second Man to require in his Turn to have one sixth part of the remaining Stake for the Consideration of his Throw; which being granted, and the first Man's Right returning in course, he may claim again one sixth part of the Remainder, and so on alternately, till the whole Stake be exhausted: But this not being to be done till after an infinite number of Shares be thus taken on both Sides, it belongs to the Method of Infinite Series to assign to each Man what proportion of the Stake he ought to take at first, so as to answer exactly that fictitious Division of the Stake in infinitum; by means of which it will be found, that the Stake ought to be divided between the contending Parties into two parts, respectively proportional to the two Numbers 6 and 5. By the like Method it would be found that if there were Three or more Adventurers playing on the conditions above described, each Man, according to the Situation he is in with respect to Priority of Play, might take as his due such part of the Stake, as is expressible by the corresponding Term of the proportion of 6 to 5, continued to so many Terms as there are Gamesters; which in the case of Three Gamesters, for Instance, would be the Numbers 6, 5, and $4\frac{1}{6}$, or their Proportionals 36, 30, and 25.

Another Advantage of the Method of Infinite Series is, that every Term of the Series includes some particular Circumstance wherein the

Gamesters may be found, which the other Methods do not ; and that a few of its Steps are sufficient to discover the Law of its Process. The only Difficulty which attends this Method, being that of summing up so many of its Terms as are requisite for the Solution of the Problem proposed: But it will be found by Experience, that in the Series resulting from the Consideration of most Cases relating to Chance, the Terms of it will either constitute a Geometric Progression, which by the known Methods is easily summable ; or else some other sort of Progression, whose nature consists in this, that every Term of it has to a determinate number of the preceding Terms, each being taken in order, some constant relation ; in which case I have contrived some easy Theorems, not only for finding the Law of that Relation, but also for finding the Sums required ; as may be seen in several places of this Book, but particularly from page 220 to page 230.

A Third Advantage of the Method of Infinite Series is, that the Solutions derived from it have a certain Generality and Elegance, which scarce any other Method can attain to ; those Methods being always perplexed with various unknown Quantities, and the Solutions obtained by them terminating commonly in particular Cases.

There are other Sorts of Series, which tho' not properly infinite, yet are called Series, from the Regularity of the Terms whereof they are composed ; those Terms following one another with a certain uniformity, which is always to be defined. Of this nature is the Theorem given by Sir Isaac Newton, in the fifth Lemma of the third Book of his Principles, for drawing a Curve through any given number of Points ; of which the Demonstration, as well as of other things belonging to the same Subject, may be deduced from the first Proposition of his Methodus Differentialis, printed with some other of his Tracts, by the care of my Intimate Friend, and very skilful Mathematician, Mr. W. Jones. The abovementioned Theorem being very useful in summing up any number of Terms whose last Differences are equal, (such as are the Numbers called Triangular, Pyramidal, &c. the Squares, the Cubes, or other Powers of Numbers in Arithmetic Progression) I have shewn in many places of this Book how it might be applicable to these Cases.

After having dwelt some time upon various Questions depending on the general Principle of Combinations, as laid down in my Introduction, and upon some others depending on the Method of Infinite Series, I proceed to treat of the Method of Combinations properly so called, which I shew to be easily deducible from that more general Principle which had been before explained: Where it may be observed, that although the Cases it is applied to are particular, yet the Way of Reasoning, and the Consequences derived from it, are general ; that Method of
Arguing

Arguing about generals by particular Examples, being in my opinion very convenient for easing the Reader's Imagination.

Having explained the common Rules of Combinations, and given a Theorem which may be of use for the Solution of some Problems relating to that Subject, I lay down a new Theorem, which is properly a contraction of the former, whereby several Questions of Chance are resolved with wonderful ease, tho' the Solution might seem at first sight to be of insuperable difficulty.

It is by the Help of that Theorem so contracted, that I have been able to give a compleat Solution of the Problems of Pharaon and Bassette, which was never done before me: I own that some great Mathematicians had already taken the pains of calculating the advantage of the Banker, in any circumstance either of Cards remaining in his Hands, or of any number of times that the Card of the Ponte is contained in the Stock: But still the curiosity of the Inquisitive remained unsatisfied; The Chief Question, and by much the most difficult, concerning Pharaon or Bassette, being, What it is that the Banker gets per Cent. of all the Money adventured at those Games? which now I can certainly answer is very near Three per Cent. at Pharaon, and three fourths per Cent. at Bassette, as may be seen in my 33d Problem, where the precise Advantage is calculated.

In the 35th and 36th Problems, I explain a new sort of Algebra, whereby some Questions relating to Combinations are solved by so easy a Process, that their Solution is made in some measure an immediate consequence of the Method of Notation. I will not pretend to say that this new Algebra is absolutely necessary to the Solving of those Questions which I make to depend on it, since it appears that Mr. Monmort, Author of the Analyse des Jeux de Hazard, and Mr. Nicholas Bernoulli have solved, by another Method, many of the cases therein proposed: But I hope I shall not be thought guilty of too much Confidence, if I assure the Reader, that the Method I have followed has a degree of Simplicity, not to say of Generality, which will hardly be attained by any other Steps than by those I have taken.

*The 39th Problem, proposed to me, amongst some others, by the Honourable Mr. Francis Robartes, I had solved in my tract De mensura Sortis; It relates, as well as the 35th and 36th, to the Method of Combinations, as is made to depend on the same Principle. When I began for the first time to attempt its Solution, I had nothing else to guide me but the common Rules of Combinations, such as they had been delivered by Dr. Wallis and others; which when I endeavoured to apply, I was surprized to find that my Calculation swelled by degrees to an intolerable Bulk: For this reason I was forced to turn my Views another way, and to try
whether*

whether the Solution I was seeking for might not be deduced from some easier considerations; whereupon I happily fell upon the Method I have been mentioning, which as it led me to a very great Simplicity in the Solution, so I look upon it to be an Improvement made to the Method of Combinations.

The 40th Problem is the reverse of the preceding; It contains a very remarkable Method of Solution, the Artifice of which consists in changing an Arithmetic Progression of Numbers into a Geometric one; this being always to be done when the Numbers are large, and their Intervals small. I freely acknowledge that I have been indebted long ago for this useful Idea, to my much respected Friend, That Excellent Mathematician Dr. Halley, Secretary to the Royal Society, whom I have seen practise the thing on another occasion: For this and other Instructive Notions readily imparted to me, during an uninterrupted Friendship of five and Twenty years, I return him my very hearty Thanks.

The 44th and 45th Problems, having in them a Mixture of the two Methods of Combinations and Infinite Series, may be proposed for a pattern of Solution, in some of the most difficult cases that may occur in the Subject of Chance, and on this occasion I must do that Justice to Mr. Nicholas Bernoulli, to own he had sent me the Solution of those Problems before mine was Published; which I had no sooner received, but I communicated it to the Royal Society, and represented it as a Performance highly to be commended: Whereupon the Society order'd that his Solution should be Printed; which was accordingly done some time after in the Philosophical Transactions, Numb. 341, where mine was also inserted.

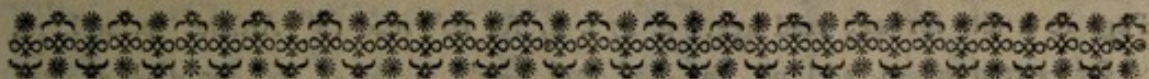
The Problems which follow relate chiefly to the Duration of Play, or to the Method of determining what number of Games may probably be played out by two Adversaries, before a certain number of Stakes agreed on between them be won or lost on either side. This Subject affording a very great Variety of Curious Questions, of which every one has a degree of Difficulty peculiar to it self, I thought it necessary to divide it into several distinct Problems, and to illustrate their Solution with proper Examples.

Tho' these Questions may at first sight seem to have a very great degree of difficulty, yet I have some reason to believe, that the Steps I have taken to come at their Solution, will easily be followed by those who have a competent skill in Algebra, and that the chief Method of proceeding therein will be understood by those who are barely acquainted with the Elements of that Art.

When I first began to attempt the general Solution of the Problem concerning the Duration of Play, there was nothing extant that could give me any light into that Subject; for altho' Mr. de Monmort, in the first Edition of his Book, gives the Solution of this Problem, as limited to three Stakes to be won or lost, and farther limited by the Supposition of an Equality

quality of Skill between the Adventurers; yet he having given no Demonstration of his Solution, and the Demonstration when discovered being of very little use towards obtaining the general Solution of the Problem, I was forced to try what my own Enquiry would lead me to which having been attended with Success, the result of what I found was afterwards published in my Specimen before mentioned.

All the Problems which in my Specimen related to the Duration of Play, have been kept entire in the following Treatise; but the Method of Solution has received some Improvements by the new Discoveries I have made concerning the Nature of those Series which result from the Consideration of the Subject; however, the Principles of that Method having been laid down in my Specimen, I had nothing now to do, but to draw the Consequences that were naturally deducible from them.



A D V E R T I S E M E N T.

THE Author of this Work, by the failure of his Eye-sight in extreme old age, was obliged to entrust the Care of a new Edition of it to one of his Friends; to whom he gave a Copy of the former, with some marginal Corrections and Additions, in his own hand writing. To these the Editor has added a few more, where they were thought necessary: and has disposed the whole in better Order; by restoring to their proper places some things that had been accidentally *misplaced*, and by putting all the Problems concerning *Annuities* together; as they stand in the late *improved* Edition of the Treatise on that Subject. An *Appendix* of several useful Articles is likewise subjoined: the whole according to a Plan concerted with the Author, above a year before his death.

P R E F A C E

quantity of 2500 between the Abacus; yet he having given no Di-
 mension of his Abacus, and the Demonstration when dis-
 cussed of my first is founded on the general Solution of the
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 which having done with this (which I thought I found was
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 sideration of the subject; however, the Principles of them being
 the same, and being in my Specimen, I had nothing new to do, but to transfer
 the Conclusions that were formerly deducible from them.

E R R A T A

Pag. 10 l. ult. for 445 read 455. p. 27 l. 4, for 10th read 12th Article. p. 29 l. 30,
 for xy read xz . p. 45 l. ult. for $\frac{1}{9}x^3$ read $\frac{1}{6}x^3$. p. 68. l. 6, for $d^5 + d^5$ read
 $d^4 + d^3$. p. 116 l. 26, for *Art. 3^d* read *Art. 4th*. p. 179 l. 8 from the bottom, for
 XV. Prob. read *Corol.* to Prob. 19. p. 181 l. 1, for 18th read 19th. p. 187 l. 25, for 38
 read 28. p. 192 l. 7. from bottom, for aab and bba read aba and bab . p. 205 l. 7, for
 $\frac{a^p}{a+b^p}$ read $\frac{b^p}{a+b^p}$. p. 238. l. 16, for $AFGz$ read AFz .



J. Mynde sculp.

THE DOCTRINE OF CHANCES.

The INTRODUCTION.

1.



HE Probability of an Event is greater or less, according to the number of Chances by which it may happen, compared with the whole number of Chances by which it may either happen or fail.

2. Wherefore, if we constitute a Fraction whereof the Numerator be the number of Chances whereby an Event may happen, and the Denominator the number of all the Chances whereby it may either happen or fail, that Fraction will be a proper designation of the Probability

bability of happening. Thus if an Event has 3 Chances to happen, and 2 to fail, the Fraction $\frac{3}{5}$ will fitly represent the Probability of its happening, and may be taken to be the measure of it.

The same thing may be said of the Probability of failing, which will likewise be measured by a Fraction whose Numerator is the number of Chances whereby it may fail, and the Denominator the whole number of Chances, both for its happening and failing; thus the Probability of the failing of that Event which has 2 Chances to fail and 3 to happen will be measured by the Fraction $\frac{2}{5}$.

3. The Fractions which represent the Probabilities of happening and failing, being added together, their Sum will always be equal to Unity, since the Sum of their Numerators will be equal to their common Denominator: now it being a certainty that an Event will either happen or fail, it follows that Certainty, which may be conceived under the notion of an infinitely great degree of Probability, is fitly represented by Unity.

These things will easily be apprehended, if it be considered, that the word Probability includes a double Idea; first, of the number of Chances whereby an Event may happen; secondly, of the number of Chances whereby it may either happen or fail.

If I say that I have three Chances to win any Sum of Money, it is impossible from that bare assertion to judge whether I am like to obtain it; but if I add that the number of Chances either to obtain it, or to miss it, is five in all, from hence will ensue a comparison between the Chances that favour me, and the whole number of Chances that are for or against me, whereby a true judgment will be formed of my Probability of success: from whence it necessarily follows, that it is the comparative magnitude of the number of Chances to happen, in respect to the whole number of Chances either to happen or to fail, which is the true measure of Probability.

4. If upon the happening of an Event, I be intitled to a Sum of Money, my Expectation of obtaining that Sum has a determinate value before the happening of the Event.

Thus, if I am to have 10 *L.* in case of the happening of an Event which has an equal Probability of happening and failing, my Expectation before the happening of the Event is worth 5 *L.*: for I am precisely in the same circumstances as he who at an equal Play ventures 5 *L.* either to have 10, or to lose his 5. Now he who ventures 5 *L.* at an equal Play, is possessor of 5 *L.* before the decision of the Play;

Play; therefore my Expectation in the case above-mentioned must also be worth 5 *L*.

5. In all cases, the Expectation of obtaining any Sum is estimated by multiplying the value of the Sum expected by the Fraction which represents the Probability of obtaining it.

Thus, if I have 3 Chances in 5 to obtain 100 *L*. I say that the present value of my Expectation is the product of 100 *L*. by the fraction $\frac{3}{5}$, and consequently that my expectation is worth 60 *L*.

For supposing that an Event may equally happen to any one of 5 different Persons, and that the Person to whom it happens should in consequence of it obtain the Sum of 100 *L*. it is plain that the right which each of them in particular has upon the Sum expected is $\frac{1}{5}$ of 100 *L*. which right is founded in this, that if the five Persons concerned in the happening of the Event, should agree not to stand the Chance of it, but to divide the Sum expected among themselves, then each of them must have $\frac{1}{5}$ of 100 *L*. for his pretension. Now whether they agree to divide that sum equally among themselves, or rather chuse to stand the Chance of the Event, no one has thereby any advantage or disadvantage, since they are all upon an equal foot, and consequently each Person's expectation is worth $\frac{1}{5}$ of 100 *L*. Let us suppose farther, that two of the five Persons concerned in the happening of the Event, should be willing to resign their Chance to one of the other three; then the Person to whom those two Chances are thus resigned has now three Chances that favour him, and consequently has now a right triple of that which he had before, and therefore his expectation is now worth $\frac{3}{5}$ of 100 *L*.

Now if we consider that the fraction $\frac{3}{5}$ expresses the Probability of obtaining the Sum of 100 *L*., and that $\frac{3}{5}$ of 100, is the same thing as $\frac{3}{5}$ multiplied by 100, we must naturally fall into this conclusion, which has been laid down as a principle, that the value of the Expectation of any Sum, is determined by multiplying the Sum expected by the Probability of obtaining it.

This manner of reasoning, tho' deduced from a particular case, will easily be perceived to be general, and applicable to any other case.

COROLLARY.

From what precedes, it necessarily follows that if the Value of an Expectation be given, as also the Value of the thing expected, then dividing the first value by the second, the quotient will express the Probability of obtaining the Sum expected: thus if I have an Expectation worth 60 *L.* and that the Sum which I may obtain be worth 100 *L.* the Probability of obtaining it will be expressed by the quotient of 60 divided by 100, that is by the fraction $\frac{60}{100}$ or $\frac{3}{5}$.

6. The Risk of losing any Sum is the reverse of Expectation; and the true measure of it is, the product of the Sum adventured multiplied by the Probability of the Loss.

7. Advantage or Disadvantage in Play, results from the combination of the several Expectations of the Gamesters, and of their several Risks.

Thus supposing that *A* and *B* play together, that *A* has deposited 5 *L.* and *B* 3 *L.* that the number of Chances which *A* has to win is 4, and the number of Chances which *B* has to win is 2, and that it were required in this circumstance to determine the advantage or disadvantage of the Adventurers, we may reason in this manner: Since the whole Sum deposited is 8, and that the Probability which *A* has of getting it is $\frac{4}{6}$, it follows that the Expectation of *A* upon the whole Sum deposited is $8 \times \frac{4}{6} = 5 \frac{1}{3}$, and for the same reason the Expectation of *B* upon that whole Sum deposited is $8 \times \frac{2}{6} = 2 \frac{2}{3}$.

Now, if from the respective Expectations which the Adventurers have upon the whole sum deposited, be subtracted the particular Sums which they deposit, that is their own Stakes, there will remain the Advantage or Disadvantage of either, according as the difference is positive or negative.

And therefore if from $5 \frac{1}{3}$, which is the Expectation of *A* upon the whole Sum deposited, 5 which is his own Stake, be subtracted, there will remain $\frac{1}{3}$ for his advantage; likewise if from $2 \frac{2}{3}$ which is the Expectation of *B*, 3 which is his own Stake be subtracted, there will remain $-\frac{1}{3}$, which being negative shews that his Disadvantage is $\frac{1}{3}$.

These conclusions may also be derived from another consideration; for if from the Expectation which either Adventurer has upon the Sum

Sum deposited by his Adversary, be subtracted the Risk of what he himself deposits, there will likewise remain his Advantage or Disadvantage, according as the difference is positive or negative.

Thus in the preceding case, the Stake of *B* being 3, and the Probability which *A* has of winning it, being $\frac{4}{6}$, the Expectation of *A* upon that Stake is $3 \times \frac{4}{6} = 2$; moreover the Stake of *A* being 5, and the Probability of losing it, being $\frac{2}{6}$, his Risk ought to be estimated by $5 \times \frac{2}{6} = 1 \frac{2}{3}$; wherefore, if from the Expectation 2, the Risk $1 \frac{2}{3}$ be subtracted, there will remain $\frac{1}{3}$ as before for the Advantage of *A*: and by the same way of proceeding, the Disadvantage of *B* will be found to be $\frac{1}{3}$.

It is very carefully to be observed, that what is here called Advantage or Disadvantage, and which may properly be called Gain or Loss, is always estimated before the Event is come to pass; and altho' it be not customary to call that Gain or Loss which is to be derived from an Event not yet determined, nevertheless in the Doctrine of Chances, that appellation is equivalent to what in common discourse is called Gain or Loss.

For in the same manner as he who ventures a Guinea in an equal Game may, before the determination of the Play, be said to be possessor of that Guinea, and may, in consideration of that Sum, resign his place to another; so he may be said to be a Gainer or Loser, who would get some Profit, or suffer some Loss, if he would sell his Expectation upon equitable terms, and secure his own Stake for a Sum equal to the Risk of losing it.

8. If the obtaining of any Sum requires the happening of several Events that are independent on each other, then the Value of the Expectation of that Sum is found by multiplying together the several Probabilities of happening, and again multiplying the product by the Value of the Sum expected.

Thus supposing that in order to obtain 90^{*L*} two Events must happen; the first whereof has 3 Chances to happen, and 2 to fail, the second has 4 Chances to happen, and 5 to fail, and I would know the value of that Expectation; I say,

The Probability of the first's happening is $\frac{3}{5}$, the Probability of the second's happening is $\frac{4}{9}$; now multiplying these two Probabilities together, the product will be $\frac{12}{45}$ or $\frac{4}{15}$; and this product being again.

again multiplied by 90, the new product will be $\frac{360}{15}$ or 24, therefore that Expectation is worth 24 *L*.

The Demonstration of this will be very easy, if it be consider'd, that supposing the first Event had happened, then that Expectation depending now intirely upon the second, would, before the determination of the second, be found to be exactly worth $\frac{4}{9} \times 90$ *L*. or 40 *L*. (by Art. 5th) We may therefore look upon the happening of the first, as a condition of obtaining an Expectation worth 40 *L*. but the Probability of the first's happening has been supposed $\frac{3}{5}$, wherefore the Expectation sought for is to be estimated by $\frac{3}{5} \times 40$, or by $\frac{3}{5} \times \frac{4}{9} \times 90$; that is, by the product of the two Probabilities of happening multiplied by the Sum expected.

And likewise, if an Expectation depends on the happening of one Event, and the failing of another, then its Value will be the product of the Probability of the first's happening by the Probability of the second's failing, and of that again by the Value of the Sum expected.

And again, if an Expectation depends on the failing of two Events, the Rule will be the same; for that Expectation will be found by multiplying together the two Probabilities of failing, and multiplying that again by the Value of the Sum expected.

And the same Rule is applicable to the happening or failing of as many Events as may be assigned.

COROLLARY.

If we make abstraction of the Value of the Sum to be obtained, the bare Probability of obtaining it, will be the product of the several Probabilities of happening, which evidently appears from this 8th Art. and from the Corollary to the 5th.

Hitherto, I have confined myself to the consideration of Events independent; but for fear that, in what is to be said afterwards, the terms independent or dependent might occasion some obscurity, it will be necessary, before I proceed any farther, to settle intirely the notion of those terms.

Two Events are independent, when they have no connexion one with the other, and that the happening of one neither forwards nor obstructs the happening of the other.

Two Events are dependent, when they are so connected together as that the Probability of either's happening is altered by the happening of the other.

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In order to illustrate this, it will not be amiss to propose the two following easy Problems.

1°. Suppose there is a heap of 13 Cards of one colour, and another heap of 13 Cards of another colour, what is the Probability that taking a Card at a venture out of each heap, I shall take the two Aces?

The Probability of taking the Ace out of the first heap is $\frac{1}{13}$: now it being very plain that the taking or not taking the Ace out of the first heap has no influence in the taking or not taking the Ace out of the second; it follows, that supposing that Ace taken out, the Probability of taking the Ace out of the second will also be $\frac{1}{13}$; and therefore, those two Events being independent, the Probability of their both happening will be $\frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$.

2°. Suppose that out of one single heap of 13 Cards of one colour, it should be undertaken to take out the Ace in the first place, and then the Deux, and that it were required to assign the Probability of doing it; we are to consider that altho' the Probability of the Ace's being in the first place be $\frac{1}{13}$, and that the Probability of the Deux's being in the second place, would also be $\frac{1}{13}$, if that second Event were considered in itself without any relation to the first; yet that the Ace being supposed as taken out at first, there will remain but 12 Cards in the heap, and therefore that upon the supposition of the Ace being taken out at first, the Probability of the Deux's being next taken will be alter'd, and become $\frac{1}{12}$; and therefore, we may conclude that those two Events are dependent, and that the Probability of their both happening will be $\frac{1}{13} \times \frac{1}{12} = \frac{1}{156}$.

From whence it may be inferred, that the Probability of the happening of two Events dependent, is the product of the Probability of the happening of one of them, by the Probability which the other will have of happening, when the first is considered as having happened; and the same Rule will extend to the happening of as many Events as may be assigned.

9. But-to determine, in the easiest manner possible, the Probability of the happening of several Events dependent, it will be convenient to distinguish by thought the order of those Events, and to suppose one of them to be the first, another to be the second, and so on: which being done, the Probability of the happening of the first may be

be looked upon as independent, the Probability of the happening of the second, is to be determined from the supposition of the first's having happened, the Probability of the third's happening, is to be determined from the supposition of the first and second having happened, and so on: then the Probability of the happening of them all will be the product of the Multiplication of the several Probabilities which have been determined in the manner prescribed.

We had seen before how to determine the Probability of the happening or failing of as many Events independent as may be assigned; we have seen likewise in the preceding Article how to determine the Probability of the happening of as many Events dependent as may be assigned: but in the case of Events dependent, how to determine the Probability of the happening of some of them, and at the same time the Probability of the failing of some others, is a disquisition of a greater degree of difficulty; which for that reason will be more conveniently transferred to another place.

10. If I have several Expectations upon several Sums, it is very evident that my Expectation upon the whole is the Sum of the Expectations I have upon the particulars.

Thus suppose two Events such, that the first may have 3 Chances to happen and 2 to fail, and the second 4 Chances to happen and 5 to fail, and that I be intitled to 90^{L.} in case the first happens, and to another like Sum of 90^{L.} in case the second happens also, and that I would know the Value of my Expectation upon the whole: I say,

The Sum expected in the first case being 90^{L.} and the Probability of obtaining it being $\frac{3}{5}$, it follows that my Expectation on that account, is worth $90 \times \frac{3}{5} = 54$; and again the Sum expected in the second case being 90, and the Probability of obtaining it being $\frac{4}{9}$, it follows that my Expectation of that second Sum is worth $90 \times \frac{4}{9} = 40$; and therefore my Expectation upon the whole is worth $54^L. + 40^L. = 94^L.$

But if I am to have 90^{L.} once for all for the happening of one or the other of the two afore-mentioned Events, the method of process in determining the value of my Expectation will be somewhat altered: for altho' my Expectation of the first Event be worth 54^{L.} as it was in the preceding Example, yet I consider that my Expectation of the second will cease upon the happening of the first, and that therefore this Expectation takes place only in case the first does happen to fail. Now the Probability of the first's failing is $\frac{2}{5}$; and
supposing

supposing it has failed, then my Expectation will be 40; wherefore $\frac{2}{5}$ being the measure of the Probability of my obtaining an Expectation worth 40^{L.}, it follows that this Expectation (to estimate it before the time of the first's being determined) will be worth $40 \times \frac{2}{5} = 16$, and therefore my Expectation upon the whole is worth $54^L. + 16^L. = 70^L.$

If that which was called the second Event be now considered as the first, and that which was called the first be now considered as the second, the conclusion will be the same as before.

In order to make the preceding Rules familiar, it will be convenient to apply them to the Solution of some easy cases, such as are the following.

C A S E Ist.

To find the Probability of throwing an Ace in two throws of one Die.

SOLUTION.

The Probability of throwing an Ace the first time is $\frac{1}{6}$; wherefore $\frac{1}{6}$ is the first part of the Probability required.

If the Ace be missed the first time, still it may be thrown on the second, but the Probability of missing it the first time is $\frac{5}{6}$, and the Probability of throwing it the second time is $\frac{1}{6}$; wherefore the Probability of missing it the first time and throwing it the second, is $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$: and this is the second part of the Probability required, and therefore the Probability required is in all $\frac{1}{6} + \frac{5}{36} = \frac{11}{36}$.

To this case is analogous a question commonly proposed about throwing with two Dice either six or seven in two throws; which will be easily solved, provided it be known that Seven has 6 Chances to come up, and Six 5 Chances, and that the whole number of Chances in two Dice is 36: for the number of Chances for throwing six or seven being 11, it follows that the Probability of throwing either Chance the first time is $\frac{11}{36}$; but if both are missed the first time, still either may be thrown the second time; now the Probability

bility of missing both the first time is $\frac{25}{36}$, and the Probability of throwing either of them the second time is $\frac{11}{36}$; wherefore the Probability of missing both of them the first time, and throwing either of them the second time, is $\frac{25}{36} \times \frac{11}{36} = \frac{275}{1296}$, and therefore the Probability required is $\frac{11}{36} + \frac{275}{1296} = \frac{671}{1296}$, and the Probability of the contrary is $\frac{625}{1296}$.

C A S E II^d.

To find the Probability of throwing an Ace in three throws.

SOLUTION.

The Probability of throwing an Ace the first time is $\frac{1}{6}$, which is the first part of the Probability required.

If the Ace be missed the first time, still it may be thrown in the two remaining throws; but the Probability of missing it the first time is $\frac{5}{6}$, and the Probability of throwing it in the two remaining times is (by Case 1st) $= \frac{11}{36}$. And therefore the Probability of missing it the first time, and throwing it in the two remaining times is $\frac{5}{6} \times \frac{11}{36} = \frac{55}{216}$, which is the second part of the Probability required; wherefore the Probability required will be $\frac{1}{6} + \frac{55}{216} = \frac{91}{216}$.

C A S E III^d.

To find the Probability of throwing an Ace in four throws.

SOLUTION.

The Probability of throwing an Ace the first time is $\frac{1}{6}$, which is the first part of the Probability required.

If the Ace be missed the first time, of which the Probability is $\frac{5}{6}$, there remains the Probability of throwing it in three times, which (by Case 2^d) is $\frac{91}{216}$; wherefore the Probability of missing the Ace the first time, and throwing it in the three remaining times, is $= \frac{5}{6} \times \frac{91}{216} = \frac{445}{1296}$, which is the second part of the Probability

bility required ; and therefore the Probability required is, in the whole,
 $\frac{1}{6} + \frac{455}{1296} = \frac{671}{1296}$, and the Probability of the contrary $\frac{625}{1296}$.

It is remarkable, that he who undertakes to throw an Ace in four throws, has just the same Advantage of his adversary, as he who undertakes with two Dice that six or seven shall come up in two throws, the odds in either case being 671 to 625 : whereupon it will not be amiss to shew how to determine easily the Gain of one Party from the Superiority of Chances he has over his adversary, upon supposition that each stake is equal, and denominated by unity. For although this is a particular case of what has been explained in the 7th Article ; yet as it is convenient to have the Rule ready at hand, and that it be easily remembered, I shall set it down here. Let therefore the odds be universally expressed by the ratio of a to b , then the respective Probabilities of winning being $\frac{a}{a+b}$, and $\frac{b}{a+b}$, the right of the first upon the Stake of the second is $\frac{a}{a+b} \times 1$, and likewise the right of the second upon the Stake of the first is $\frac{b}{a+b} \times 1$, and therefore the Gain of the first is $\frac{a-b}{a+b} \times 1$ or barely $\frac{a-b}{a+b}$: and consequently the Gain of him who undertakes that six or seven shall come up in two throws, or who undertakes to fling an Ace in four throws, is $\frac{671-625}{671+625} = \frac{46}{1296}$, that is nearly $\frac{1}{28}$ part of his adversary's Stake.

C A S E IVth.

To find the Probability of throwing two Aces in two throws.

SOLUTION.

It is plain (by the 8th Art.) that the Probability required is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

C A S E Vth.

To find the Probability of throwing two Aces in three throws.

SOLUTION.

If an Ace be thrown the first time, then it will only be required to throw it once in two throws ; but the Probability of throwing it the first time is $\frac{1}{6}$, and the Probability of throwing it once in

two throws (by the first case) is $\frac{11}{36}$: wherefore the Probability of throwing it the first time, and then throwing it once in the two remaining times is $\frac{1}{6} \times \frac{11}{36} = \frac{11}{216}$; and this is equal to the first part of the Probability required.

If the Ace be missed the first time, still there remains the Probability of throwing it twice together, but the Probability of missing it the first time is $\frac{5}{6}$, and the Probability of throwing it twice together is (by the 4th Case) $= \frac{1}{30}$; therefore the Probability of both Events is $\frac{5}{6} \times \frac{1}{30} = \frac{5}{210}$; which is the second part of the Probability required : therefore the whole Probability required is $\frac{11+5}{216} = \frac{16}{216}$.

C A S E VIth.

To find the Probability of throwing two Aces in four throws.

SOLUTION.

If an Ace be thrown the first time, no more will be required than throwing it again in three throws ; but the Probability of throwing an Ace the first time is $\frac{1}{6}$, and the Probability of throwing it in three times is $\frac{91}{216}$ (by the 2^d Case ;) wherefore the Probability of both happening is $\frac{1}{6} \times \frac{91}{210} = \frac{91}{1296} = 1^{\text{st}}$ part of the Probability required.

If the Ace be missed the first time, still there will remain the Probability of throwing two Aces in three throws ; but the Probability of missing the Ace the first time is $\frac{5}{6}$, and the Probability of throwing it twice in three throws is $\frac{16}{210}$, (by the 5th Case ;) wherefore the Probability of both together is $\frac{5}{6} \times \frac{16}{210} = \frac{80}{1296} = 2^{\text{d}}$ part of the Probability required : and therefore the Probability required $= \frac{91}{1296} + \frac{80}{1296} = \frac{171}{1296}$.

And, by the same way of reasoning, we may gradually find the Probability of throwing an Ace as many times as shall be demanded, in any given number of throws.

If, instead of employing figures in the Solutions of the foregoing Cases, we employ algebraic Characters, we shall readily perceive a most regular order in those Solutions.

11. Let therefore a be the number of Chances for the happening of an Event, and b the number of Chances for its failing; then the Probability of its happening once in any number of Trials will be expressed by the Series $\frac{a}{a+b} + \frac{ab}{(a+b)^2} + \frac{abb}{(a+b)^3} + \frac{ab^2}{(a+b)^4} + \frac{ab^3}{(a+b)^5} + \frac{ab^4}{(a+b)^6}$, &c. which Series is to be continued to so many terms as are equal to the number of Trials given: thus if $a = 1$, $b = 5$, and the number of Trials given $= 4$, then the Probability required will be expressed by $\frac{1}{6} + \frac{5}{30} + \frac{25}{210} + \frac{125}{1296} = \frac{671}{1296}$.

The same things being supposed as before, the Probability of the Event's happening twice in any given number of Trials, will be expressed by the Series $\frac{aa}{(a+b)^2} + \frac{2aab}{(a+b)^3} + \frac{2aabb}{(a+b)^4} + \frac{4aab^2}{(a+b)^5} + \frac{5aab^3}{(a+b)^6}$, &c. which is to be continued to so many terms, wanting one, as is the number of Trials given; thus let us suppose $a = 1$, $b = 5$, and the number of Trials 8, then the Probability required will be expressed by $\frac{1}{30} + \frac{10}{210} + \frac{75}{1296} + \frac{500}{7776} + \frac{3 \cdot 25}{406056} + \frac{18750}{279936} + \frac{109375}{1679616} = \frac{663991}{1679616}$.

And again, the Probability of the Event's happening three times in any given number of Trials will be expressed by the Series $\frac{a^3}{(a+b)^3} + \frac{3a^2b}{(a+b)^4} + \frac{6a^2bb}{(a+b)^5} + \frac{10a^3b^2}{(a+b)^6} + \frac{15a^3b^3}{(a+b)^7}$, &c. which is to be continued to so many terms, wanting two, as is the number of terms given.

But all these particular Series may be comprehended under a general one, which is as follows.

Let a be the number of Chances, whereby an Event may happen, b the number of Chances whereby it may fail, l the number of times that the Event is required to be produced in any given number of Trials, and let n be the number of those Trials; make $a + b = s$, then the Probability of the Event's happening l times in n Trials, will be expressed by the Series $\frac{a^l}{s^l} \times$

$$1 + \frac{lb}{s} + \frac{l \cdot l + 1 \cdot bb}{1 \cdot 2 \cdot ss} + \frac{l \cdot l + 1 \cdot l + 2 \cdot b^2}{1 \cdot 2 \cdot 3 \cdot s^3} + \frac{l \cdot l + 1 \cdot l + 2 \cdot l + 3 \cdot b^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot s^4} + \dots$$

the.

It is to be noted here, and elsewhere, that the points here made use of, stand instead of the Mark of Multiplication \times .

the common multiplicator $\frac{a^l}{s^l}$ as are denoted by the number $n - l + 1$.

And for the same reason, the Probability of the contrary, that is of the Event's not happening so often as l times, making $n - l + 1 = p$, will be expressed by the Series $\frac{b^p}{s^p} \times$

$$1 + \frac{pa}{s} + \frac{p \cdot p + 1 \cdot aa}{1 \cdot 2 \cdot ss} + \frac{p \cdot p + 1 \cdot p + 2 \cdot a^2}{1 \cdot 2 \cdot 3 \cdot s^3} + \frac{p \cdot p + 1 \cdot p + 2 \cdot p + 3 \cdot a^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot s^4},$$

which Series is to be continued to so many terms, exclusive of the common multiplicator, as are denoted by the number l .

Now the Probability of an Event's not happening being known, the Probability of its happening will likewise be known, since the Sum of those two Probabilities is always equal to Unity; and therefore the second Series, as well as the first, may be employed in determining the Probability of an Event's happening: but as the number of terms to be taken in the first is expressed by $n - l + 1$, and the number of terms to be taken in the second is expressed by l , it will be convenient to use the first Series, if $n - l + 1$ be less than l , and to use the second, if l be less than $n - l + 1$; or in other terms, to use the first or second according as l is less or greater than $\frac{n+1}{2}$.

Thus, suppose an Event has 1 Chance to happen, and 35 to fail, and that it were required to assign the Probability of its happening in 24 Trials; then because in this case $n = 24$ and $l = 1$, it is plain that 24 terms of the first Series would be requisite to answer the Question, and that one single one of the second will be sufficient: and therefore, if in the second Series we make $b = 35$, $a = 1$, and $l = 1$, the Probability of the Event's not happening once in 24 Trials, will be expressed by $\frac{35^{24}}{36^{24}} \times 1$, which by the help of Logarithms, we shall find nearly equivalent to the decimal fraction 0.50871; now this being subtracted from Unity, the remainder 0.49129 will express the Probability required; and therefore the odds against the happening of the Event will be 50 to 49 nearly.

Again, suppose it be required to assign the Probability of the preceding Event's happening twice in 60 Trials; then because $l = 2$, and $n = 60$, $n - l + 1$ will be $= 59$, which shews that 59 terms of the first Series would be required: but if we use the second, then by reason of l being $= 2$, two of its terms will be sufficient; wherefore

wherefore the two terms $\frac{35^{59}}{36^{59}} \times 1 + \frac{59}{36}$ will denote the Probability of the Event's not happening twice in 60 Trials. Now reducing this to a decimal fraction, it will be found equal to 0.5007, which being subtracted from Unity, the remainder 0.4993 will express the Probability required; and therefore the odds against the Event's happening twice in 60 times will be very little more than 500 to 499.

It is to be observed of those Series, that they are both derived from the same principle; for supposing two adversaries *A* and *B*, contending about the happening of that Event which has every time *a* chances to happen, and *b* chances to fail, that the Chances *a* are favourable to *A*, and the Chances *b* to *B*, and that *A* should lay a wager with *B*, that his Chances shall come up *l* times in *n* Trials: then by reason *B* lays a wager to the contrary, he himself undertakes that his own Chances shall, in the same number of Trials, come up $n - l + 1$ times; and therefore, if in the first Series, we change *l* into $n - l + 1$, and *vice versa*, and also write *b* for *a*, and *a* for *b*, the second Series will be formed.

It will be easy to conceive how it comes to pass, that if *A* undertakes to win *l* times in *n* Trials, his Adversary *B* necessarily undertakes in the same number of Trials to win $n - l + 1$ times, if it be considered that *A* loses his wager if he wins but $l - 1$ times; now if he wins but $l - 1$ times, then subtracting $l - 1$ from *n*, the remainder shews the number of times that *B* is to win, which therefore will be $n - l + 1$.

C A S E VIIth.

To find the Probability of throwing one Ace, and no more, in four throws.

SOLUTION.

This Case ought carefully to be distinguished from the fourth; for there it was barely demanded, without any manner of restriction, what the Probability was of throwing an Ace in 4 throws; now in this present case there is a restraint laid on that Event: for whereas in the former case, he who undertakes to throw an Ace desists from throwing when once the Ace is come up; in this he obliges himself, after it is come up, to a farther Trial which is wholly against him, excepting the last throw of the four, after which there is no Trial; and therefore we ought from the unlimited Probability of the

the Ace's being thrown once in 4 throws, to subtract the Probability of its being thrown twice in that number of throws: now the first Probability is $\frac{671}{1296}$ (by the 3^d Case, and the second Probability is $\frac{171}{1296}$ (by the 6th Case,) from which it follows that the Probability required is $\frac{500}{1296}$, and the Probability of the contrary $\frac{796}{1296}$; and therefore the Odds against throwing one Ace and no more in 4 throws are 796 to 500, or 8 to 5 nearly: and the same method may be follow'd in higher cases.

C A S E VIIIth.

If A and B play together, and that A wants but 1 Game of being up, and B wants 2; what are their respective Probabilities of winning the Set?

SOLUTION.

It is to be considered that the Set will necessarily be ended in two Games at most, for if *A* wins the first Game, there is no need of any farther Trial; but if *B* wins it, then they will want each but 1 Game of being up, and therefore the Set will be determined by the second Game: from whence it is plain that *A* wants only to win once in two Games, but that *B* wants to win twice together. Now supposing that *A* and *B* have an equal Chance to win a Game, then the Probability which *B* has of winning the first Game will be $\frac{1}{2}$, and consequently the Probability of his winning twice together will be $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; and therefore the Probability which *A* has of winning once in two Games will be $1 - \frac{1}{4} = \frac{3}{4}$, from whence it follows that the Odds of *A*'s winning are 3 to 1.

C A S E IXth.

A and B play together, A wants 1 Game of being up, and B wants 2; but the Chances whereby B may win a Game, are double to the number of Chances whereby A may win the same: 'tis required to assign the respective Probabilities of winning.

SOLUTION.

It is plain that in this, as well as in the preceding case, *B* ought to win twice together; now since *B* has 2 Chances to win a Game,
and

and A 1 Chance only for the same, the Probability which B has of winning a Game is $\frac{2}{3}$, and therefore the Probability of his winning twice together is $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, and consequently the Probability of A 's winning the Set is $1 - \frac{4}{9} = \frac{5}{9}$; from whence it follows that the Odds of A 's winning once, before B twice, are as 5 to 4.

REMARK.

Altho' the determining the precise Odds in questions of Chance requires calculation, yet sometimes by a superficial View of the question, it may be possible to find that there will be an inequality in the Play. Thus in the preceding case wherein B has in every Game twice the number of Chances of A , if it be demanded whether A and B play upon the square, it is natural to consider that he who has a double number of Chances will at long run win twice as often as his Adversary; but that the case is here otherwise, for B undertaking to win twice before A once, he thereby undertakes to win oftner than according to his proportion of Chances, since A has a right to expect to win once, and therefore it may be concluded that B has the disadvantage: however, this way of arguing in general ought to be used with the utmost caution.

12. Whatever be the number of Games which A and B respectively want of being up, the Set will be concluded at the most in so many Games wanting one, as is the sum of the Games wanted between them.

Thus suppose that A wants 3 Games of being up, and B 5; it is plain that the greatest number of Games that A can win of B before the determination of the Play will be 2, and that the greatest number which B can win of A before the determination of the Play will be 4; and therefore the greatest number of Games that can be played between them before the determination of the Play will be 6: but supposing they have played six Games, the next Game will terminate the Play; and therefore the utmost number of Games that can be played between them will be 7, that is one Game less than the Sum of the Games wanted between them.

D

CASE

CASE Xth.

Supposing that A wants 3 Games of being up, and B wants 7; but that the Chances which A and B respectively have for winning a Game are as 3 to 5, to find the respective Probabilities of winning the Set.

SOLUTION.

By reason that the Sum of the Games wanted between *A* and *B* is 10, it is plain by the preceding Paragraph that the Set will be concluded in 9 Games at most, and that therefore *A* undertakes out of 9 Games to win 3, and *B*, out of the same number, to win 7; now supposing that the first general Theorem laid down in the 11th Art. is particularly adapted to represent the Probability of *A*'s winning, then $l = 3$; and because n represents the number of Games in which the Set will be concluded, $n = 9$; but the number of terms to be used in the first Theorem being $= n - l + 1 = 7$, and the number of terms to be used in the second Theorem being $= l = 3$, it will be more convenient to use the second, which will represent the Probability of *B*'s winning. Now that second Theorem being applied to the case of n being $= 9$, $l = 3$, $a = 3$, $b = 5$, the Probability which *B* has of winning the Set will be expressed by $\frac{5^7}{8^7} \times 1 + \frac{21}{8} + \frac{252}{64} = \frac{5^7}{8^7} \times 484 = 0.28172$ nearly; and therefore subtracting this from Unity, there will remain the Probability which *A* has of winning the same, which will be $= 0.71828$; and consequently the Odds of *A*'s winning the Set will be 71828 to 28172, or very near as 23 to 9.

The same Principles explained in a different and more general way.

Altho' the principles hitherto explained are a sufficient introduction to what is to be said afterwards; yet it will not be improper to resume some of the preceding Articles, and to set them in a new light: it frequently happening that some truths, when represented to the mind under a particular Idea, may be more easily apprehended than when represented under another.

13. Let us therefore imagine a Die of a given number of equal faces, let us likewise imagine another Die of the same or any other number of equal faces; this being supposed, I say that the number of all the variations which the two Dice can undergo will be obtained by multiplying the number of faces of the one, by the number of faces of the other.

In

In order to prove this, and the better to fix the imagination, let us take a particular case: Suppose therefore that the first Die contains 8 faces, and the second 12; then supposing the first Die to stand still upon one of its faces, it is plain that in the mean time the second Die may revolve upon its 12 faces; for which reason, there will be upon that single score 12 variations: let us now suppose that the first Die stands upon another of its faces, then whilst that Die stands still, the second Die may revolve again upon its 12 faces; and so on, till the faces of the first Die have undergone all their changes: from whence it follows, that in the two Dice, there will be as many times 12 Chances as there are faces in the first Die; but the number of faces in the first Die has been supposed 8, wherefore the number of Variations or Chances of the two Dice will be 8 times 12, that is 96: and therefore it may be universally concluded, that the number of all the variations of two Dice will be the product of the multiplication of the number of faces of one Die, by the number of faces of the other.

14. Let us now imagine that the faces of each Die are distinguished into white and black, that the number of white faces upon the first is A , and the number of black faces B , and also that the number of white faces upon the second is a , and the number of black faces b ; hence it will follow by the preceding Article, that multiplying $A + B$ by $a + b$, the product $Aa + Ab + Ba + Bb$, will exhibit all the Variations of the two Dice: Now let us see what each of these four parts separately taken will represent.

1°. It is plain, that in the same manner as the product of the multiplication of the whole number of faces of the first Die, by the whole number of faces of the second, expresses all the variations of the two Dice; so likewise the multiplication of the number of the white faces of the first Die, by the number of the white faces of the second, will express the number of variations whereby the two Dice may exhibit two white faces: and therefore, that number of Chances will be represented by Aa .

2°. For the same reason, the multiplication of the number of white faces upon the first Die, by the number of black faces upon the second, will represent the number of all the Chances whereby a white face of the first may be joined with a black face of the second; which number of Chances will therefore be represented by Ab .

3°. The multiplication of the number of white faces upon the second, by the number of black faces upon the first, will express the number of all the Chances whereby a white face of the second

may be joined with a black face of the first; which number of Chances will therefore be represented by aB .

4°. The multiplication of the number of black faces upon the first, by the number of black faces upon the second, will express the number of all the Chances whereby a black face of the first may be joined with a black face of the second; which number of Chances will therefore be represented by Bb .

And therefore we have explained the proper signification and use of the several parts Aa , Ab , Ba , Bb singly taken.

But as these parts may be connected together several ways, so the Sum of two or more of any of them will answer some question of Chance: for instance, suppose it be demanded, what is the number of Chances, with the two Dice above-mentioned, for throwing a white face? it is plain that the three parts $Aa + Ab + Ba$ will answer the question; since every one of those parts comprehends a case wherein a white face is concerned.

It may perhaps be thought that the first term Aa is superfluous, it denoting the number of Variations whereby two white faces can be thrown; but it will be easy to be satisfied of the necessity of taking it in: for supposing a wager depending on the throwing of a white face, he who throws for it, is reputed a winner, whenever a white face appears, whether one alone, or two together, unless it be expressly stipulated that in case he throws two, he is to lose his wager; in which latter case the two terms $Ab + Ba$ would represent all his Chances.

If now we imagine a third Die having upon it a certain number of white faces represented by α , and likewise a certain number of black faces represented by β , then multiplying the whole variation of Chances of the two preceding Dice *viz.* $Aa + Ab + Ba + Bb$ by the whole number of faces $\alpha + \beta$ of the third Die, the product $Aa\alpha + Ab\alpha + Ba\alpha + Bb\alpha + Aa\beta + Ab\beta + Ba\beta + Bb\beta$ will exhibit the number of all the Variations which the three Dice can undergo.

Examining the several parts of this new product, we may easily perceive that the first term $Aa\alpha$ represents the number of Chances for throwing three white faces, that the second term $Ab\alpha$ represents the number of Chances whereby both the first and third Die may exhibit a white face, and the second Die a black one; and that the rest of the terms have each their particular properties, which are discovered by bare inspection.

It may also be perceived, that by joining together two or more of those terms, some question of Chance will thereby be answered: for instance,

instance, if it be demanded what is the number of Chances for throwing two white faces and a black one? it is plain that the three terms Aba , Baa , $Aa\beta$ taken together will exhibit the number of Chances required, since in every one of them there is the expression of two white faces and a black one; and therefore if there be a wager depending on the throwing two white faces and a black one, he who undertakes that two white faces and a black one shall come up, has for him the Odds of $Aba + Baa + Aa\beta$ to $Aaa + Bba + Ab\beta + Ba\beta + Bb\beta$; that is, of the three terms that include the condition of the wager, to the five terms that include it not.

When the number of Chances that was required has been found, then making that number the Numerator of a fraction, whereof the Denominator is the whole number of variations which all the Dice can undergo, that fraction will express the Probability of the Event; as has been shewn in the first Article.

Thus if it was demanded what the Probability is, of throwing three white faces with the three Dice above-mentioned, that Probability will be expressed by the fraction
$$\frac{Aaa}{Aaa + Aba + Baa + Aa\beta + Bba + Ab\beta + Ba\beta + Bb\beta}.$$

But it is to be observed, that in writing the Denominator, it will be convenient to express it by way of product, distinguishing the several multipliers whereof it is compounded; thus in the preceding case the Probability required will be best expressed as follows,

$$\frac{Aaa}{(A+B) \times (a+b) \times (\alpha+\beta)}.$$

If the preceding fraction be conceived as the product of the three fractions $\frac{A}{A+B} \times \frac{a}{a+b} \times \frac{\alpha}{\alpha+\beta}$, whereof the first expresses the Probability of throwing a white face with the first Die; the second the Probability of throwing a white face with the second Die, and the third the Probability of throwing a white face with the third Die; then will again appear the truth of what has been demonstrated in the 8th Art. and its Corollary, viz. that the Probability of the happening of several Events independent, is the product of all the particular Probabilities whereby each particular Event may be produced; for altho' the case here described be confined to three Events, it is plain that the Rule extends itself to any number of them.

Let us resume the case of two Dice, wherein we did suppose that the number of white faces upon one Die was expressed by A, and the number of black faces by B, and also that the number of white faces upon the other was expressed by a, and the number of black faces by b, which gave us all the variations $Aa + Ab + aB + Bb$; and

and let us imagine that the number of the white and black faces is respectively the same upon both Dice: wherefore $A = a$, and $B = b$, and consequently instead of $Aa + Ab + aB + Bb$, we shall have $aa + ab + ab + bb$, or $aa + 2ab + bb$; but in the first case $Ab + aB$ did express the number of variations whereby a white face and a black one might be thrown, and therefore $2ab$ which is now substituted in the room of $Ab + aB$ does express the number of variations, whereby with two Dice of the same respective number of white and black faces, a white face and a black one may be thrown.

In the same manner, if we resume the general case of three Dice, and examine the number of variations whereby two white faces and a black one may be thrown, it will easily be perceived that if the number of white and black faces upon each Die are respectively the same, then the three parts $Ab\alpha + Ba\alpha + Aa\beta$ will be changed into $aba + baa + aab$, or $3aab$, and that therefore $3aab$, which is one term of the Binomial $a + b$ raised to its Cube, will express the number of variations whereby three Dice of the same kind would exhibit two white faces and a black one.

15. From the preceding considerations, this general Rule may be laid down, *viz.* that if there be any number of Dice of the same kind, all distinguished into white and black faces, that n be the number of those Dice, a and b the respective numbers of white and black faces upon each Die, and that the Binomial $a + b$ be raised to the power n ; then 1^o, the first term of that power will express the number of Chances whereby n white faces may be thrown; 2^o, that the second term will express the number of Chances whereby $n - 1$ white faces and 1 black face may be thrown; 3^o, that the third term will express the number of Chances whereby $n - 2$ white faces and 2 black ones may be thrown; and so on for the rest of the terms.

Thus, for instance, if the Binomial $a + b$ be raised to its 6th power, which is $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$; the first term a^6 will express the number of Chances for throwing 6 white faces; the second term $6a^5b$ will express the number of Chances for throwing 5 white and 1 black; the third term $15a^4b^2$ will express the number of Chances for throwing 4 white and 2 black; the fourth $20a^3b^3$ will express the number of Chances for throwing 3 white and 3 black; the fifth $15a^2b^4$ will express the number of Chances for throwing two white and 4 black; the sixth $6ab^5$ will express the number of Chances for 2 white and 4 black; lastly, the seventh b^6 will express the number of Chances for 6 black.

And

And therefore having raised the Binomial $a + b$ to any given power, we may by bare inspection determine the property of any one term belonging to that power, by only observing the Indices wherewith the quantities a and b are affected in that term, since the respective numbers of white and black faces are represented by those Indices.

The better to compare the consequences that may be derived from the consideration of the Binomial $a + b$ raised to a power given, with the method of Solution that hath been explained before; let us resume some of the preceding questions, and see how the Binomial can be applied to them.

Suppose therefore that the Probability of throwing an Ace in four throws with a common Die of six faces be demanded.

In order to answer this, it must be considered that the throwing of one Die four times successively, is the same thing as throwing four Dice at once; for whether the same Die is used four times successively, or whether a different Die is used in each throw, the Chance remains the same; and whether there is a long or a short interval between the throwing of each of these four different Dice, the Chance remains still the same; and therefore if four Dice are thrown at once, the Chance of throwing an Ace will be the same as that of throwing it with one and the same Die in four successive throws.

This being premised, we may transfer the notion that was introduced concerning white and black faces, in the Dice, to the throwing or missing of any point or points upon those Dice; and therefore in the present case of throwing an Ace with four Dice, we may suppose that the Ace in each Die answer to one white face, and the rest of the points to five black faces, that is, we may suppose that $a = 1$, and $b = 5$; and therefore, having raised $a + b$ to its fourth power, which is $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, every one of the terms wherein a is perceived will be a part of the number of Chances whereby an Ace may be thrown. Now there are four of those parts into which a enters, *viz.* $a^4 + 4a^3b + 6a^2b^2 + 4ab^3$, and therefore having made $a = 1$, and $b = 5$, we shall have $1 + 20 + 150 + 500 = 671$ to express the number of Chances whereby an Ace may be thrown with four Dice, or an Ace thrown in four successive throws of one single Die: but the number of all the Chances is the fourth power of $a + b$, that is the fourth power of 6, which is 1296; and therefore the Probability required is measured by the fraction $\frac{671}{1296}$, which is conformable to the resolution given in the 3^d case of the questions belonging to the 10th Art.

It

It is to be observed, that the Solution would have been shorter, if instead of inquiring at first into the Probability of throwing an Ace in four throws, the Probability of its contrary, that is the Probability of missing the Ace four times successively, had been inquired into; for since this case is exactly the same as that of missing all the Aces with four Dice, and that the last term b^4 of the Binomial $a + b$ raised to its fourth power expresses the number of Chances whereby the Ace may fail in every one of the Dice; it follows, that the Probability of that failing is $\frac{b^4}{(a+b)^4} = \frac{625}{1296}$, and therefore the Probability of not failing, that is of throwing an Ace in four throws, is $1 - \frac{625}{1296} = \frac{1296 - 625}{1296} = \frac{671}{1296}$.

From hence it follows, that let the number of Dice be what it will, suppose n , then the last term of the power $(a + b)^n$, that is b^n , will always represent the number of Chances whereby the Ace may fail n times, whether the throws be considered as successive or cotemporary: Wherefore $\frac{b^n}{(a+b)^n}$ is the Probability of that failing; and consequently the Probability of throwing an Ace in a number of throws expressed by n , will be $1 - \frac{b^n}{(a+b)^n} = \frac{(a+b)^n - b^n}{(a+b)^n}$.

Again, suppose it be required to assign the Probability of throwing with one single Die two Aces in four throws, or of throwing at once two Aces with four Dice: the question will be answered by help of the Binomial $a + b$ raised to its fourth power, which being $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, the three terms $a^4 + 4a^3b + 6a^2b^2$ wherein the Indices of a equal or exceed the number of times that the Ace is to be thrown, will denote the number of Chances whereby two Aces may be thrown; wherefore having interpreted a by 1, and b by 5, the three terms above-written will become $1 + 20 + 150 = 171$, but the whole number of Chances, viz. $(a+b)^4$ is in this case $= 1296$, and therefore the Probability of throwing two Aces in four throws will be measured by the fraction $\frac{171}{1296}$.

But if we chuse to take at first the Probability of the contrary, it is plain that out of the five terms that the fourth power of $a + b$ consists of, the two terms $4ab^3 + b^4$; in the first of which a enters but once, and in the second of which it enters not, will express the number of Chances that are contrary to the throwing of two Aces; which number of Chances will be found equal to $500 + 625 = 1125$. And therefore the Probability of not throwing two Aces in four throws

throws will be $\frac{1125}{1296}$: from whence may be deduced the Probability of doing it, which therefore will be $1 - \frac{1125}{1296} = \frac{1296-1125}{1296} = \frac{171}{1296}$ as it was found in the preceding paragraph; and this agrees with the Solution of the sixth Case to be seen in the 10th Article.

Universally, the last term of any power $a + b^n$ being b^n , and the last but one being nab^{n-1} , in neither of which a^2 enters, it follows that the two last terms of that power express the number of Chances that are contrary to the throwing of two Aces, in any number of throws denominated by n ; and that the Probability of throwing two Aces will be $1 - \frac{nab^{n-1} + b^n}{a + b^n} = \frac{a + b^n - nab^{n-1} - b^n}{a + b^n}$.

And likewise, in the three last terms of the power $a + b^n$, every one of the Indices of a will be less than 3, and consequently those three last terms will shew the number of Chances that are contrary to the throwing of an Ace three times in any number of Trials denominated by n : and the same Rule will hold perpetually.

And these conclusions are in the same manner applicable to the happening or failing of any other sort of Event in any number of times, the Chances for happening and failing in any particular Trial being respectively represented by a and b .

16. Wherefore we may lay down this general Maxim; that supposing two Adversaries A and B contending about the happening of an Event, whereof A lays a wager that the Event will happen l times in n Trials, and B lays to the contrary, and that the number of Chances whereby the Event may happen in any one Trial are a , and the number of Chances whereby it may fail are b , then so many of the last terms of the power $a + b^n$ expanded, as are represented by l , will shew the number of Chances whereby B may win his wager.

Again, B laying a wager that A will not win l times, does the same thing in effect as if he laid that A will not win above $l - 1$ times; but the number of winnings and losings between A and B is n by hypothesis, they having been supposed to play n times, and therefore subtracting $l - 1$ from n , the remainder $n - l + 1$ will shew that B himself undertakes to win $n - l + 1$ times; let this remainder be called p , then it will be evident that in the same manner as the last terms of the power $a + b^n$ expanded, viz. $b^n +$

$nab^{n-1} + \frac{n}{1} \times \frac{n-1}{2} a^2b^{n-2}$, &c. the number whereof is l , do exprefs the number of Chances whereby B may be a winner, fo the first terms $a^n + na^{n-1}b + \frac{n}{1} \times \frac{n-1}{2} a^{n-2}b^2$, &c. the number whereof is p , do exprefs the number of Chances whereby A may be a winner.

17. If A and B being at play, refpectively want a certain number of Games l and p of being up, and that the refpective Chances they have for winning any one particular Game be in the proportion of a to b ; then raifing the Binomial $a + b$ to a power whole Index fhall be $l + p - 1$, the number of Chances whereby they may refpectively win the Set, will be in the fame proportion as the Sum of fo many of the first terms as are expreffed by p , to the Sum of fo many of the laft terms as are expreffed by l .

This will eafily be perceived to follow from what was faid in the preceding Article : for when A and B refpectively undertook to win l Games and p Games, we have proved that if n was the number of Games to be played between them, then p was neceffarily equal to $n - l + 1$, and therefore $l + p = n + 1$, and $n = l + p - 1$; and confequently the power to which $a + b$ is to be raifed will be $l + p - 1$.

Thus fupposing that A wants 3 Games of being up, and B 7, that their proportion of Chances for winning any one Game are refpectively as 3 to 5, and that it were required to affign the proportion of Chances whereby they may win the Set; then making $l = 3$, $p = 7$, $a = 3$, $b = 5$, and raifing $a + b$ to the power denoted by $l + p - 1$, that is in this cafe to the 9th power, the Sum of the first feven terms will be to the Sum of the three laft, in the proportion of the refpective Chances whereby they may win the Set.

Now it will be fufficient in this cafe to take the Sum of the three laft terms; for fince that Sum expreffes the number of Chances whereby B may win the Set, then it being divided by the 9th power of $a + b$, the quotient will exhibit the Probability of his winning; and this Probability being fubtracted from Unity, the remainder will exprefs the Probability of A 's winning: but the three laft terms of the Binomial $a + b$ raifed to its 9th power are $b^9 + 9ab^8 + 36a^2b^7$, which being converted into numbers make the Sum 37812500, and the 9th power of $a + b$ is 134217728, and therefore the Probability of B 's winning will be expreffed by the fraction $\frac{37812500}{134217728} = \frac{9453125}{33554432}$; let this be fubtracted from Unity, then the remainder

$$\frac{24101307}{33554432}$$

$\frac{24101307}{33554432}$ will express the Probability of A 's winning; and therefore the Odds of A 's being up before B , are in the proportion of 24101307 to 9453125, or very near as 23 to 9: which agrees with the Solution of the 10th Case included in the 10th Article.

In order to compleat the comparison between the two Methods of Solution which have been hitherto explained, it will not be improper to propose one case more.

Suppose therefore it be required to assign the Probability of throwing one Ace and no more, with four Dice thrown at once.

It is visible that if from the number of Chances whereby one Ace or more may be thrown, be subtracted the number of Chances whereby two Aces or more may be thrown, there will remain the number of Chances for throwing one Ace and no more; and therefore having raised the Binomial $a + b$ to its fourth power, which is $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, it will plainly be seen that the four first terms express the number of Chances for throwing one Ace or more, and that the three first terms express the number of Chances for throwing two Aces or more; from whence it follows that the single term $4ab^3$ does alone express the number of Chances for throwing one Ace and no more, and therefore the Probability required will be $\frac{4ab^3}{(a+b)^4} = \frac{500}{1296} = \frac{125}{324}$: which agrees with the Solution of the 7th Case given in the 10th Article.

This Conclusion might also have been obtained another way: for applying what has been said in general concerning the property of any one term of the Binomial $a + b$ raised to a power given, it will thereby appear that the term $4ab^3$ wherein the indices of a and b are respectively 1 and 3, will denote the number of Chances whereby of two contending parties A and B , the first may win once, and the other three times. Now A who undertakes that he shall win once and no more, does properly undertake that his own Chance shall come up once, and his adversary's three times; and therefore the term $4ab^3$ expresses the number of Chances for throwing one Ace and no more.

In the like manner, if it be required to assign the Chances for throwing a certain number of Aces, and it be farther required that there shall not be above that number, then one single term of the power $(a + b)^n$ will always answer the question.

But to find that term as expeditiously as possible, suppose n to be the number of Dice, and l the precise number of Aces to be thrown; then if l be less than $\frac{1}{2}n$, write as many terms of the

Series $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, $\frac{n-4}{5}$, &c. as there are Units in l ; or if l be greater than $\frac{1}{2}n$, write as many of them as there are Units in $\frac{1}{2}n - l$; then let all those terms be multiplied together, and the product be again multiplied by $a^l b^{n-l}$; and this last product will exhibit the term expressing the number of Chances required.

Thus if it be required to assign the number of Chances for throwing precisely three Aces, with ten Dice; here l will be $= 3$, and $n = 10$. Now because l is less than $\frac{1}{2}n$, let so many terms be taken of the Series $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, &c. as there are Units in 3, which terms in this particular case will be $\frac{10}{1}$, $\frac{9}{2}$, $\frac{8}{3}$; let those terms be multiplied together, the product will be 120; let this product be again multiplied by $a^l b^{n-l}$, that is (a being $= 1$, $b = 9$, $l = 3$, $n = 10$) by 6042969, and the new product will be 725156280, which consequently exhibits the number of Chances required. Now this being divided by the 10th power of $a + b$, that is, in this case, by 10000000000, the quotient 0.0725156280 will express the Probability of throwing precisely three Aces with ten Dice; and this being subtracted from Unity, the remainder 0.9274843720 will express the Probability of the contrary; and therefore the Odds against throwing three Aces precisely with ten Dice are 9274843720, to 725156280, or nearly as 64 to 5.

Although we have shewn above how to determine universally the Odds of winning, when two Adversaries being at play, respectively want certain number of Games of being up, and that they have any given proportion of Chances for winning any single Game; yet I have thought it not improper here to annex a small Table, shewing those Odds, when the number of Games wanting, does not exceed six, and that the Skill of the Contenders is equal.

Games wanting.	Odds of winning.	Games wanting.	Odds of winning.	Games wanting.	Odds of winning.
1, 2 - - -	3, 1	2, 3 - -	11, 5	3, 5 - -	99, 29
1, 3 - - -	7, 1	2, 4 - -	26, 6	3, 6 - -	219, 37
1, 4 - - -	15, 1	2, 5 - -	57, 7	4, 5 - -	163, 93
1, 5 - - -	31, 1	2, 6 - -	120, 8	4, 6 - -	382, 130
1, 6 - - -	63, 1	3, 4 - -	42, 22	5, 6 - -	638, 386

Before

Before I put an end to this Introduction, it will not be improper to shew how some operations may often be contracted by barely introducing one single Letter, instead of two or three, to denote the Probability of the happening of one Event.

18. Let therefore x denote the Probability of one Event; y , the Probability of a second Event; z , the Probability of the happening of a third Event: then it will follow, from what has been said in the beginning of this Introduction, that $1 - x$, $1 - y$, $1 - z$ will represent the respective Probabilities of their failing.

This being laid down, it will be easy to answer the Questions of Chance that may arise concerning those Events.

1°. Let it be demanded, what is the Probability of the happening of them all; it is plain by what has been demonstrated before, that the answer will be denoted by xyz .

2°. If it is inquired, what will be the Probability of their all failing; the answer will be $1 - x \times 1 - y \times 1 - z$, which being expanded by the Rules of Multiplication would be $1 - x - y - z + xy + xz + yz - xyz$; but the first expression is more easily adapted to Numbers.

3°. Let it be required to assign the Probability of some one of them or more happening; as this question is exactly equivalent to this other, what is the Probability of their not all failing? the answer will be $1 - 1 - x \times 1 - y \times 1 - z$, which being expanded will become $x + y + z - xy - xz - yz + xyz$.

4°. Let it be demanded what is the Probability of the happening of the first and second, and at the same time of the failing of the third, the Question is answered by barely writing it down algebraically; thus, $xy \times 1 - z$, or $xy - xyz$: and for the same reason the Probability of the happening of the first and third, and the failing of the second, will be $xz \times 1 - y$ or $xz - xyz$: and for the same reason again, the Probability of the happening of the second and third, and the failing of the first, will be $yz \times 1 - x$, or $yz - xyz$. And the Sum of those three Probabilities, viz. $xy + xz + yz - 3xyz$, will express the Probability of the happening of any two of them, but of no more than two.

5°. If it be demanded what is the Probability of the happening of the first, to the exclusion of the other two, the answer will be $x \times 1 - y \times 1 - z$, or $x - xy - xz + xyz$; and in the same manner, the Probability of the happening of the second to the exclusion of the other two, will be $y - xy - yz + xyz$; and again, the Probability of the happening of the third, to the exclusion of the other

other two, will be $x - xz - yz + xyz$, and the Sum of all these Probabilities together, *viz.* $x + y + z - 2xy - 2xz - 2yz + 3xyz$ will express the Probability of the happening of any one of them, and of the failing of the other two: and innumerable cases of the same nature, belonging to any number of Events, may be solved without any manner of trouble to the imagination, by the mere force of a proper Notation.

REMARK.

I. When it is required to sum up several Terms of a high Power of the Binomial $a + b$, and to divide their Sum by that Power, it will be convenient to write 1 and q for a and b ; having taken $q : 1 :: b : a$: and to use a Table of *Logarithms*.

As in the Example of *Art. 17th*, where we had to compute $\frac{b^9 + 9ab^8 + 36a^2b^7}{a + b^9}$, a being $= 3$, $b = 5$; we shall have $q = \frac{5}{3}$, and, instead of the former, we are now to compute the quantity $\frac{q^9 + 9q^8 + 36q^7}{1 + q^9} = \frac{q^7 \times q^2 + 9q + 36}{1 + q^9}$.

Now the Factor $q^2 + 9q + 36$ being $\frac{25}{9} + \frac{45}{3} + 36 = \frac{484}{9}$,
 Whose Logarithm is $L. 484 - L. 9 = - - 1.7306029$
 Add the Log. of q^7 , or $7 \times L. 5 - L. 3 = - - 1.5529409$
 And from the Sum $- - - - - 3.2835438$
 Subtract the Log. of $1 + q^9$, or $9 \times L. 8 - L. 3 = 3.8337183$
 So shall the Remainder $- - - - - 1.4498255$
 be the Logarithm of B 's chance, *viz.* 0.281725
 And the Complement of this to Unity 0.718275 is the Chance of A , in that Problem of *Art. 17th*.

An Operation of this kind will serve in most cases that occur: but if the Power is very high, and the number of terms to be summed excessively great, we must have recourse to other Rules; which shall be given hereafter.

II. When the Ratio of Chances, which we shall call that of R to S , comes out in larger numbers than we have occasion for; it may be reduced to its *least exactest* Terms, in the Method proposed by *Dr. Wallis, Huygens*, and others. As thus;

Divide the greater Term R by the lesser S ; the last Divisor by the Remainder; and so on continually, as in finding a common Divisor: and let the several Quotients, in the order they arise, be represented

represented by the Letters a, b, c, d, e , &c. Then the Ratio $\frac{S}{R}$, of the lesser Term to the greater, will be contained in this fractional Series.

$$\frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e + \text{&c.}}}}}}$$

whose Terms, from the beginning, being reduced to one Fraction, will perpetually approach to the just Value of the Ratio $\frac{S}{R}$; differing from it in *excess* and in *defect*, alternately: so that if you stop at a Denominator that stands in the 1st, 3^d, 5th, &c. place; as at a, c, e , &c. the Result of the Terms will exceed the just Value of the Ratio $\frac{S}{R}$; but if you stop at an even place, as at b, d, f , &c. it will fall short of it.

EXAMPLE I.

If it is required to reduce the Ratio just now found $\frac{281725}{718275}$, or $\frac{11269}{28731}$ ($= \frac{S}{R}$) to lower Terms; and which shall exhibit its just Quantity the nearest that is possible in Terms so low: The Quotients, found as above, will be; $a = 2, b = 1, c = 1, d = 4, e = 1, f = 1, g = 5$. And,

1°. The first Term $\frac{1}{a}$, or $\frac{1}{2}$, gives the Ratio too great; because its consequent a is too little.

2°. The Result of the two first Terms $\frac{1}{a + \frac{1}{b}} = \frac{1}{2 + \frac{1}{1}} = \frac{1}{3}$, is less than $\frac{S}{R}$, altho' it comes nearer it than $\frac{1}{2}$ did: because $\frac{1}{b} = 1$, which we added to the Denominator 2, exceeds its just Quantity $\frac{1}{b + \frac{1}{c + \text{&c.}}}$.

3°. The three first Terms $\frac{1}{a + \frac{1}{b + \frac{1}{c}}} = \frac{1}{2 + \frac{1}{1 + \frac{1}{1}}}$; which reduced are

$\frac{1}{2 + \frac{1}{2}} = \frac{2}{5}$ exceeds the Ratio $\frac{S}{R}$: because what we added to

the

the Denominator b exceeding its just Quantity $\frac{1}{1+\frac{1}{c}}$ makes $\frac{1}{b+\frac{1}{c}}$ too little, and consequently the whole Fraction too great.

In the same manner, the following Approximation

$$\frac{1}{a+\frac{1}{b+\frac{1}{c+\frac{1}{d}}}} = \frac{1}{2+\frac{1}{1+\frac{1}{1+\frac{1}{4}}}} = \frac{9}{23}, \text{ tho' juster than the pre-}$$

ceding, errs a little in defect. And so of the rest.

But to save unnecessary trouble; and to prevent any mistake either in the Operation itself, or in distinguishing the Ratios that exceed or fall short of their just Quantity; we may use Mr. Cotes's Rule; which is to this purpose.

Write $S : R$ at the head of two Columns, under the Titles *greater*, and *less*. And place under them the two first Ratios that are found; as in our Example $1 : 2$, and $1 : 3$. Multiply the Terms of this last Ratio by the *third* Denominator c , and write the Products under the Terms of the first Ratio $1 : 2$. So shall the Sums of the Antecedents and Consequents give a juster Ratio $2 : 5$, belonging to the left-hand Column. Multiply the terms of this last by the 4th Quotient $d (= 4)$, and the Products added to $1 : 3$ give the Ratio $9 : 23$, belonging to the right-hand Column. This last multiplied by $e (= 1)$, and the Products transferred to the left hand Column, and added to the Ratio that stood last there, give the Ratio $11 : 28$. And so of the rest, as in the Scheme below.

<i>greater</i>		<i>less</i>	
$S :$	R	$S :$	R
$\frac{1}{a} = 1 :$	2	$1 :$	$3 = \frac{1}{a+\frac{1}{b}}$
$1 :$	3	$\times c = 1$	$\frac{1}{b}$
$2 :$	$5 \times d = 4$	$8 :$	20
$9 :$	23	$9 :$	$23 \times e = 1$
$11 :$	$28 \times f = 1$	$11 :$	28
$100 :$	255	$20 :$	$51 \times g = 5$
$111 :$	283		

&c.

This Method is particularly useful, when surd numbers, which have no Termination at all, enter into any Solution.

EXAMPLE

EXAMPLE II.

It will be found in the Resolution of our first Problem that the proportion of Chances there inquired into ($\frac{R}{S}$) is that of 1 to $\sqrt[3]{2} - 1$, or of 1 to 0.259921 &c. Whence our Quotients will be; $a=3$, $b=1$, $c=5$, $d=1$, $e=1$, $f=4$, &c.

And the Operation will stand as below.

<i>greater</i>	<i>left</i>
$S : R$	$S : R$
$\frac{1}{a} = 1 : 3$	$1 : 4 = \frac{1}{a + \frac{1}{b}}$
$\frac{5 : 20}{6 : 23} \times d = 1$	$\times c = 5$
$\frac{7 : 27}{13 : 50} \times f = 4$	$\frac{6 : 23}{7 : 27} \times e = 1$
&c.	$\frac{52 : 200}{59 : 227}$

End of the Introduction.

EXAMPLE II.

It will be found in the Resolution of our first Problem that the proportion of Chances that insured two ($\frac{1}{2}$) is that of 1 to $\frac{1}{2} - 1$, or of 1 to $\frac{1}{2}$ against the event. Whence our Constant will be $a = 2$, $b = 1$, $c = 1$, $d = 1$, $e = 1$, $f = 1$, $g = 1$, $h = 1$, $i = 1$, $j = 1$, $k = 1$, $l = 1$, $m = 1$, $n = 1$, $o = 1$, $p = 1$, $q = 1$, $r = 1$, $s = 1$, $t = 1$, $u = 1$, $v = 1$, $w = 1$, $x = 1$, $y = 1$, $z = 1$, $aa = 1$, $ab = 1$, $ac = 1$, $ad = 1$, $ae = 1$, $af = 1$, $ag = 1$, $ah = 1$, $ai = 1$, $aj = 1$, $ak = 1$, $al = 1$, $am = 1$, $an = 1$, $ao = 1$, $ap = 1$, $aq = 1$, $ar = 1$, $as = 1$, $at = 1$, $au = 1$, $av = 1$, $aw = 1$, $ax = 1$, $ay = 1$, $az = 1$, $ba = 1$, $bb = 1$, $bc = 1$, $bd = 1$, $be = 1$, $bf = 1$, $bg = 1$, $bh = 1$, $bi = 1$, $bj = 1$, $bk = 1$, $bl = 1$, $bm = 1$, $bn = 1$, $bo = 1$, $bp = 1$, $bq = 1$, $br = 1$, $bs = 1$, $bt = 1$, $bu = 1$, $bv = 1$, $bw = 1$, $bx = 1$, $by = 1$, $bz = 1$, $ca = 1$, $cb = 1$, $cc = 1$, $cd = 1$, $ce = 1$, $cf = 1$, $cg = 1$, $ch = 1$, $ci = 1$, $cj = 1$, $ck = 1$, $cl = 1$, $cm = 1$, $cn = 1$, $co = 1$, $cp = 1$, $cq = 1$, $cr = 1$, $cs = 1$, $ct = 1$, $cu = 1$, $cv = 1$, $cw = 1$, $cx = 1$, $cy = 1$, $cz = 1$, $da = 1$, $db = 1$, $dc = 1$, $dd = 1$, $de = 1$, $df = 1$, $dg = 1$, $dh = 1$, $di = 1$, $dj = 1$, $dk = 1$, $dl = 1$, $dm = 1$, $dn = 1$, $do = 1$, $dp = 1$, $dq = 1$, $dr = 1$, $ds = 1$, $dt = 1$, $du = 1$, $dv = 1$, $dw = 1$, $dx = 1$, $dy = 1$, $dz = 1$, $ea = 1$, $eb = 1$, $ec = 1$, $ed = 1$, $ee = 1$, $ef = 1$, $eg = 1$, $eh = 1$, $ei = 1$, $ej = 1$, $ek = 1$, $el = 1$, $em = 1$, $en = 1$, $eo = 1$, $ep = 1$, $eq = 1$, $er = 1$, $es = 1$, $et = 1$, $eu = 1$, $ev = 1$, $ew = 1$, $ex = 1$, $ey = 1$, $ez = 1$, $fa = 1$, $fb = 1$, $fc = 1$, $fd = 1$, $fe = 1$, $ff = 1$, $fg = 1$, $fh = 1$, $fi = 1$, $fj = 1$, $fk = 1$, $fl = 1$, $fm = 1$, $fn = 1$, $fo = 1$, $fp = 1$, $fq = 1$, $fr = 1$, $fs = 1$, $ft = 1$, $fu = 1$, $fv = 1$, $fw = 1$, $fx = 1$, $fy = 1$, $fz = 1$, $ga = 1$, $gb = 1$, $gc = 1$, $gd = 1$, $ge = 1$, $gf = 1$, $gg = 1$, $gh = 1$, $gi = 1$, $gj = 1$, $gk = 1$, $gl = 1$, $gm = 1$, $gn = 1$, $go = 1$, $gp = 1$, $gq = 1$, $gr = 1$, $gs = 1$, $gt = 1$, $gu = 1$, $gv = 1$, $gw = 1$, $gx = 1$, $gy = 1$, $gz = 1$, $ha = 1$, $hb = 1$, $hc = 1$, $hd = 1$, $he = 1$, $hf = 1$, $hg = 1$, $hh = 1$, $hi = 1$, $hj = 1$, $hk = 1$, $hl = 1$, $hm = 1$, $hn = 1$, $ho = 1$, $hp = 1$, $hq = 1$, $hr = 1$, $hs = 1$, $ht = 1$, $hu = 1$, $hv = 1$, $hw = 1$, $hx = 1$, $hy = 1$, $hz = 1$, $ia = 1$, $ib = 1$, $ic = 1$, $id = 1$, $ie = 1$, $if = 1$, $ig = 1$, $ih = 1$, $ii = 1$, $ij = 1$, $ik = 1$, $il = 1$, $im = 1$, $in = 1$, $io = 1$, $ip = 1$, $iq = 1$, $ir = 1$, $is = 1$, $it = 1$, $iu = 1$, $iv = 1$, $iw = 1$, $ix = 1$, $iy = 1$, $iz = 1$, $ja = 1$, $jb = 1$, $jc = 1$, $jd = 1$, $je = 1$, $jf = 1$, $jj = 1$, $jk = 1$, $jl = 1$, $jm = 1$, $jn = 1$, $jo = 1$, $jp = 1$, $jq = 1$, $jr = 1$, $js = 1$, $jt = 1$, $ju = 1$, $jv = 1$, $jw = 1$, $jx = 1$, $jy = 1$, $jz = 1$, $ka = 1$, $kb = 1$, $kc = 1$, $kd = 1$, $ke = 1$, $kf = 1$, $kg = 1$, $kh = 1$, $ki = 1$, $kj = 1$, $kk = 1$, $kl = 1$, $km = 1$, $kn = 1$, $ko = 1$, $kp = 1$, $kq = 1$, $kr = 1$, $ks = 1$, $kt = 1$, $ku = 1$, $kv = 1$, $kw = 1$, $kx = 1$, $ky = 1$, $kz = 1$, $la = 1$, $lb = 1$, $lc = 1$, $ld = 1$, $le = 1$, $lf = 1$, $lg = 1$, $lh = 1$, $li = 1$, $lj = 1$, $lk = 1$, $ll = 1$, $lm = 1$, $ln = 1$, $lo = 1$, $lp = 1$, $lq = 1$, $lr = 1$, $ls = 1$, $lt = 1$, $lu = 1$, $lv = 1$, $lw = 1$, $lx = 1$, $ly = 1$, $lz = 1$, $ma = 1$, $mb = 1$, $mc = 1$, $md = 1$, $me = 1$, $mf = 1$, $mg = 1$, $mh = 1$, $mi = 1$, $mj = 1$, $mk = 1$, $ml = 1$, $mm = 1$, $mn = 1$, $mo = 1$, $mp = 1$, $mq = 1$, $mr = 1$, $ms = 1$, $mt = 1$, $mu = 1$, $mv = 1$, $mw = 1$, $mx = 1$, $my = 1$, $mz = 1$, $na = 1$, $nb = 1$, $nc = 1$, $nd = 1$, $ne = 1$, $nf = 1$, $ng = 1$, $nh = 1$, $ni = 1$, $nj = 1$, $nk = 1$, $nl = 1$, $nm = 1$, $nn = 1$, $no = 1$, $np = 1$, $nq = 1$, $nr = 1$, $ns = 1$, $nt = 1$, $nu = 1$, $nv = 1$, $nw = 1$, $nx = 1$, $ny = 1$, $nz = 1$, $oa = 1$, $ob = 1$, $oc = 1$, $od = 1$, $oe = 1$, $of = 1$, $og = 1$, $oh = 1$, $oi = 1$, $oj = 1$, $ok = 1$, $ol = 1$, $om = 1$, $on = 1$, $oo = 1$, $op = 1$, $oq = 1$, $or = 1$, $os = 1$, $ot = 1$, $ou = 1$, $ov = 1$, $ow = 1$, $ox = 1$, $oy = 1$, $oz = 1$, $pa = 1$, $pb = 1$, $pc = 1$, $pd = 1$, $pe = 1$, $pf = 1$, $pg = 1$, $ph = 1$, $pi = 1$, $pj = 1$, $pk = 1$, $pl = 1$, $pm = 1$, $pn = 1$, $po = 1$, $pp = 1$, $pq = 1$, $pr = 1$, $ps = 1$, $pt = 1$, $pu = 1$, $pv = 1$, $pw = 1$, $px = 1$, $py = 1$, $pz = 1$, $qa = 1$, $qb = 1$, $qc = 1$, $qd = 1$, $qe = 1$, $qf = 1$, $qg = 1$, $qh = 1$, $qi = 1$, $qj = 1$, $qk = 1$, $ql = 1$, $qm = 1$, $qn = 1$, $qo = 1$, $qp = 1$, $qq = 1$, $qr = 1$, $qs = 1$, $qt = 1$, $qu = 1$, $qv = 1$, $qw = 1$, $qx = 1$, $qy = 1$, $qz = 1$, $ra = 1$, $rb = 1$, $rc = 1$, $rd = 1$, $re = 1$, $rf = 1$, $rg = 1$, $rh = 1$, $ri = 1$, $rj = 1$, $rk = 1$, $rl = 1$, $rm = 1$, $rn = 1$, $ro = 1$, $rp = 1$, $rq = 1$, $rr = 1$, $rs = 1$, $rt = 1$, $ru = 1$, $rv = 1$, $rw = 1$, $rx = 1$, $ry = 1$, $rz = 1$, $sa = 1$, $sb = 1$, $sc = 1$, $sd = 1$, $se = 1$, $sf = 1$, $sg = 1$, $sh = 1$, $si = 1$, $sj = 1$, $sk = 1$, $sl = 1$, $sm = 1$, $sn = 1$, $so = 1$, $sp = 1$, $sq = 1$, $sr = 1$, $ss = 1$, $st = 1$, $su = 1$, $sv = 1$, $sw = 1$, $sx = 1$, $sy = 1$, $sz = 1$, $ta = 1$, $tb = 1$, $tc = 1$, $td = 1$, $te = 1$, $tf = 1$, $tg = 1$, $th = 1$, $ti = 1$, $tj = 1$, $tk = 1$, $tl = 1$, $tm = 1$, $tn = 1$, $to = 1$, $tp = 1$, $tq = 1$, $tr = 1$, $ts = 1$, $tt = 1$, $tu = 1$, $tv = 1$, $tw = 1$, $tx = 1$, $ty = 1$, $tz = 1$, $ua = 1$, $ub = 1$, $uc = 1$, $ud = 1$, $ue = 1$, $uf = 1$, $ug = 1$, $uh = 1$, $ui = 1$, $uj = 1$, $uk = 1$, $ul = 1$, $um = 1$, $un = 1$, $uo = 1$, $up = 1$, $uq = 1$, $ur = 1$, $us = 1$, $ut = 1$, $uu = 1$, $uv = 1$, $uw = 1$, $ux = 1$, $uy = 1$, $uz = 1$, $va = 1$, $vb = 1$, $vc = 1$, $vd = 1$, $ve = 1$, $vf = 1$, $vg = 1$, $vh = 1$, $vi = 1$, $vj = 1$, $vk = 1$, $vl = 1$, $vm = 1$, $vn = 1$, $vo = 1$, $vp = 1$, $vq = 1$, $vr = 1$, $vs = 1$, $vt = 1$, $vu = 1$, $vv = 1$, $vw = 1$, $vx = 1$, $vy = 1$, $vz = 1$, $wa = 1$, $wb = 1$, $wc = 1$, $wd = 1$, $we = 1$, $wf = 1$, $wg = 1$, $wh = 1$, $wi = 1$, $wj = 1$, $wk = 1$, $wl = 1$, $wm = 1$, $wn = 1$, $wo = 1$, $wp = 1$, $wq = 1$, $wr = 1$, $ws = 1$, $wt = 1$, $wu = 1$, $wv = 1$, $ww = 1$, $wx = 1$, $wy = 1$, $wz = 1$, $xa = 1$, $xb = 1$, $xc = 1$, $xd = 1$, $xe = 1$, $xf = 1$, $xg = 1$, $xh = 1$, $xi = 1$, $xj = 1$, $xk = 1$, $xl = 1$, $xm = 1$, $xn = 1$, $xo = 1$, $xp = 1$, $xq = 1$, $xr = 1$, $xs = 1$, $xt = 1$, $xu = 1$, $xv = 1$, $xw = 1$, $xx = 1$, $xy = 1$, $xz = 1$, $ya = 1$, $yb = 1$, $yc = 1$, $yd = 1$, $ye = 1$, $yf = 1$, $yg = 1$, $yh = 1$, $yi = 1$, $yj = 1$, $yk = 1$, $yl = 1$, $ym = 1$, $yn = 1$, $yo = 1$, $yp = 1$, $yq = 1$, $yr = 1$, $ys = 1$, $yt = 1$, $yu = 1$, $yv = 1$, $yw = 1$, $yx = 1$, $yy = 1$, $yz = 1$, $za = 1$, $zb = 1$, $zc = 1$, $zd = 1$, $ze = 1$, $zf = 1$, $zg = 1$, $zh = 1$, $zi = 1$, $zj = 1$, $zk = 1$, $zl = 1$, $zm = 1$, $zn = 1$, $zo = 1$, $zp = 1$, $zq = 1$, $zr = 1$, $zs = 1$, $zt = 1$, $zu = 1$, $zv = 1$, $zw = 1$, $zx = 1$, $zy = 1$, $zz = 1$.

And the Operation will stand as follow.

$$\begin{array}{r}
 2 : 16 \\
 \hline
 1 : 8 \\
 \hline
 1 : 4 \\
 \hline
 1 : 2 \\
 \hline
 1 : 1
 \end{array}
 \quad
 \begin{array}{r}
 2 : 20 \\
 \hline
 1 : 10 \\
 \hline
 1 : 5 \\
 \hline
 1 : 2.5 \\
 \hline
 1 : 1.25
 \end{array}
 \quad
 \begin{array}{r}
 2 : 27 \\
 \hline
 1 : 13.5 \\
 \hline
 1 : 6.75 \\
 \hline
 1 : 3.375 \\
 \hline
 1 : 1.6875
 \end{array}
 \quad
 \begin{array}{r}
 2 : 32 \\
 \hline
 1 : 16 \\
 \hline
 1 : 8 \\
 \hline
 1 : 4 \\
 \hline
 1 : 2 \\
 \hline
 1 : 1
 \end{array}$$

End of the Introduction.

Solutions of several sorts of Problems, deduced from
the Rules laid down in the Introduction.

P R O B L E M I.

If *A* and *B* play with single Bowls, and such be the skill of *A* that he knows by Experience he can give *B* two Games out of three; what is the proportion of their skill, or what are the Odds, that *A* may get any one Game assigned?

SOLUTION.



LET the proportion of Odds be as z to 1; now since *A* can give *B* 2 Games out of 3, *A* therefore may upon an equality of Play undertake to win 3 Games together; but the probability of his winning the first time is $\frac{z}{z+1}$, and, by the 8th Article of the Introduction, the probability of his winning three times together is $\frac{z}{z+1} \times \frac{z}{z+1} \times \frac{z}{z+1}$ or $\frac{z^3}{(z+1)^3}$. Again, because *A* and *B* are supposed to play upon equal terms, the probability which *A* has of winning three times together ought to be expressed by $\frac{1}{2}$; we have therefore the Equation $\frac{z^3}{(z+1)^3} = \frac{1}{2}$, or $2z^3 = (z+1)^3$, and extracting the cube-root on both sides, $z\sqrt[3]{2} = z+1$; wherefore $z = \frac{1}{\sqrt[3]{2}-1}$, and consequently the Odds that *A* may get any one Game assigned are as $\frac{1}{\sqrt[3]{2}-1}$ to 1, or as 1 to $\sqrt[3]{2}-1$, that is in this case as 50 to 13 very near.

F 2.

COROL.

COROLLARY.

By the same process of investigation as that which has been used in this Problem, it will be found that if A can, upon an equality of Chance, undertake to win n times together, then he may justly lay the Odds of 1 to $\sqrt[n]{2} - 1$, that he wins any one Game assigned,

PROBLEM II.

If A can without advantage or disadvantage give B 1 Game out of 3; what are the Odds that A shall take any one Game assigned? Or in other terms, what is the proportion of the Chances they respectively have of winning any one Game assigned? Or what is the proportion of their skill?

SOLUTION.

Let the proportion be as z to 1: and since A can give B 1 Game out of 3; therefore A can upon an equality of play undertake to win 3 Games before B gets 2: now it appears, by the 17th Art. of the Introduction, that in this case the Binomial $z + 1$ ought to be raised to its fourth power, which will be $z^4 + 4z^3 + 6z^2 + 4z + 1$; and that the Expectation of the first will be to the Expectation of the second, as the two first terms to the three last: but these Expectations are equal by hypothesis, therefore $z^4 + 4z^3 = 6z^2 + 4z + 1$: which Equation being solved, z will be found to be 1.6 very near; wherefore the proportion required will be as 1.6 to 1, or 8 to 5.

PROBLEM III.

To find in how many Trials an Event will probably happen, or how many Trials will be necessary to make it indifferent to lay on its Happening or Failing; supposing that a is the number of Chances for its happening in any one Trial, and b the number of Chances for its failing.

SOLUTION.

Let x be the number of Trials; then by the 16th Art. of the Introd. b^x will represent the number of Chances for the Event to fail x times successively, and $a^x + b^x$ the whole number of Chances for happen-

happening or failing, and therefore $\frac{b^x}{a+b^x}$ represents the probability of the Event's failing x times together: but by supposition that Probability is equal to the probability of its happening once at least in that number of Trials; wherefore either of those two Probabilities may be expressed by the fraction $\frac{1}{2}$: we have therefore the Equation $\frac{b^x}{a+b^x} = \frac{1}{2}$, or $a+b^x = 2b^x$, from whence is deduced the Equation $x \log. a+b = x \log. b + \log. 2$; and therefore $x = \frac{\log. 2}{\log. a+b - \log. b}$.

Moreover, let us reassume the Equation $a+b^x = 2b^x$, wherein let us suppose that $a, b :: 1, q$; hence the said Equation will be changed into this $1 + \frac{1}{q^x} = 2$. Or $x \times \log. 1 + \frac{1}{q} = \log. 2$. In this Equation, if q be equal to 1, x will likewise be equal to 1; but if q differs from Unity, let us in the room of $\log. 1 + \frac{1}{q}$ write its value expressed in a Series; viz.

$$\frac{1}{q} - \frac{1}{2qq} + \frac{1}{3q^3} - \frac{1}{4q^4} + \frac{1}{5q^5} - \frac{1}{6q^6}, \&c.$$

We have therefore the Equation $\frac{x}{q} - \frac{x}{2qq}, \&c. = \log. 2$. Let us now suppose that q is infinite, or pretty large in respect to Unity, and then the first term of the Series will be sufficient; we shall therefore have the Equation $\frac{x}{q} = \log. 2$, or $x = q \log. 2$. But it is to be observed in this place that the Hyperbolic, not the Tabular, Logarithm of 2, ought to be taken, which being 0.693, &c. or 0.7 nearly, it follows that $x = 0.7q$ nearly.

Thus we have assigned the very narrow limits within which the ratio of x to q is comprehended; for it begins with unity, and terminates at last in the ratio of 7 to 10 very near.

But x soon converges to the limit $0.7q$, so that this value of x may be assumed in all cases, let the value of q be what it will.

Some uses of this Problem will appear by the following Examples.

EXAMPLE I.

Let it be proposed to find in how many throws one may undertake with an equality of Chance, to throw two Aces with two Dice.

The number of Chances upon two Dice being 36, out of which there is but one chance for two Aces, it follows that the number of Chances

Chances against it is 35; multiply therefore 35 by 0.7, and the product 24.5 will shew that the number of throws requisite to that effect will be between 24 and 25.

EXAMPLE 2.

To find in how many throws of three Dice, one may undertake to throw three Aces.

The number of all the Chances upon three Dice being 216, out of which there is but one Chance for 3 Aces, and 215 against it, it follows that 215 ought to be multiplied by 0.7; which being done, the product 150.5 will shew that the number of Throws requisite to that effect will be 150, or very near it.

EXAMPLE 3.

In a Lottery whereof the number of blanks is to the number of prizes as 39 to 1, (such as was the Lottery in 1710) to find how many Tickets one must take to make it an equal Chance for one or more Prizes.

Multiply 39 by 0.7, and the product 27.3 will shew that the number of Tickets requisite to that effect will be 27 or 28 at most.

Likewise in a Lottery whereof the number of Blanks is to the number of Prizes as 5 to 1, multiply 5 by 0.7, and the product 3.5 will shew that there is more than an equality of Chance in 4 Tickets for one or more Prizes, but less than an equality in three.

REMARK.

In a Lottery whereof the Blanks are to the Prizes as 39 to 1, if the number of Tickets in all were but 40, the proportion above-mentioned would be altered, for 20 Tickets would be a sufficient number for the just Expectation of the single Prize; it being evident that the Prize may be as well among the Tickets which are taken, as among those that are left behind.

Again if the number of Tickets in all were 80, still preserving the proportion of 39 Blanks to one Prize, and consequently supposing 78 Blanks to 2 Prizes, this proportion would still be altered; for by the Doctrine of Combinations, whereof we are to treat afterwards, it will appear that the Probability of taking one Prize or both in 20 Tickets would be but $\frac{139}{316}$, and the Probability of taking none would be $\frac{177}{316}$; wherefore the Odds against taking any Prize would be as 177 to 139, or very near as 9 to 7. And

And by the same Doctrine of Combinations, it will be found that 23 Tickets would not be quite sufficient for the Expectation of a Prize in this Lottery; but that 24 would rather be too many: so that one might with advantage lay an even Wager of taking a Prize in 24 Tickets.

If the proportion of 39 to 1 be oftner repeated, the number of Tickets requisite for the equal Chance of a Prize, will still increase with that repetition; yet let the proportion of 39 to 1 be repeated never so many times, nay an infinite number of times, the number of Tickets requisite for the equal Chance of a Prize would never exceed $\frac{7}{10}$ of 39, that is about 27 or 28.

Wherefore if the proportion of the Blanks to the Prizes is often repeated, as it usually is in Lotteries; the number of Tickets requisite for a Prize will always be found by taking $\frac{7}{10}$ of the proportion of the Blanks to the Prizes.

Now in order to have a greater variety of Examples to try this Rule by, I have thought fit here to annex a *Lemma* by me published for the first time in the year 1711, and of which the investigation for particular reasons was deferred till I gave it in my *Miscellanea Analytica* anno 1731.

LEMMA.

To find how many Chances there are upon any number of Dice, each of them of the same number of Faces, to throw any given number of points.

SOLUTION.

Let $p + 1$ be the number of points given, n the number of Dice, f the number of Faces in each Die: make $p - f = q$, $q - f = r$, $r - f = s$, $s - f = t$, &c. and the number of Chances required will be

$$\begin{aligned} &+ \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3}, \text{ \&c.} \\ &- \frac{q}{1} \times \frac{q-1}{2} \times \frac{q-2}{3}, \text{ \&c.} \times \frac{n}{1} \\ &+ \frac{r}{1} \times \frac{r-1}{2} \times \frac{r-2}{3}, \text{ \&c.} \times \frac{n}{1} \times \frac{n-1}{2} \\ &- \frac{s}{1} \times \frac{s-1}{2} \times \frac{s-2}{3}, \text{ \&c.} \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \\ &+ \text{\&c.} \end{aligned}$$

Which Series's ought to be continued till some of the Factors in each product become either $= 0$, or negative.

N. B.

N. B. So many Factors are to be taken in each of the products $\frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3}$, &c. $\frac{q}{1} \times \frac{q-1}{2} \times \frac{q-2}{3}$, &c. as there are Units in $n - 1$.

Thus for Example, let it be required to find how many Chances there are for throwing 16 Points with four Dice; then making $p + 1 = 16$, we have $p = 15$, from whence the number of Chances required will be found to be

$$\begin{aligned} & + \frac{15}{1} \times \frac{14}{2} \times \frac{13}{3} &= + 455 \\ & - \frac{9}{1} \times \frac{8}{2} \times \frac{7}{3} \times \frac{4}{1} &= - 336 \\ & + \frac{3}{1} \times \frac{2}{2} \times \frac{1}{3} \times \frac{4}{1} \times \frac{3}{2} &= + 6 \end{aligned}$$

But $455 - 336 + 6 = 125$, and therefore one hundred and twenty-five is the number of Chances required.

Again, let it be required to find the number of Chances for throwing seven and twenty Points with six Dice; the operation will be

$$\begin{aligned} & + \frac{26}{1} \times \frac{25}{2} \times \frac{24}{3} \times \frac{23}{4} \times \frac{22}{5} &= + 65780 \\ & - \frac{20}{1} \times \frac{19}{2} \times \frac{18}{3} \times \frac{17}{4} \times \frac{16}{5} \times \frac{6}{1} &= - 93024 \\ & + \frac{14}{1} \times \frac{13}{2} \times \frac{12}{3} \times \frac{11}{4} \times \frac{10}{5} \times \frac{6}{1} \times \frac{5}{2} &= + 30030 \\ & - \frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} \times \frac{5}{4} \times \frac{4}{5} \times \frac{6}{1} \times \frac{5}{2} \times \frac{4}{3} &= - 1120 \end{aligned}$$

Wherefore $65780 - 93024 + 30030 - 1120 = 1666$ is the number of Chances required.

Let it be farther required to assign the number of Chances for throwing fifteen Points with six Dice.

$$\begin{aligned} & + \frac{14}{1} \times \frac{13}{2} \times \frac{12}{3} \times \frac{11}{4} \times \frac{10}{5} &= + 2002 \\ & - \frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} \times \frac{5}{4} \times \frac{4}{5} \times \frac{6}{1} &= - 336 \end{aligned}$$

But $2002 - 336 = 1666$ which is the number required.

COROLLARY.

All the points equally distant from the Extremes, that is from the least and greatest number of Points that are upon the Dice, have the same number of Chances by which they may be produced; wherefore if the number of points given be nearer to the greater Extreme than to the lesser, let the number of points given be

be subtracted from the Sum of the Extremes, and work with the remainder; by which means the Operation will be shortened.

Thus if it be required to find the number of Chances for throwing 27 Points with 6 Dice: let 27 be subtracted from 42, Sum of the Extremes 6 and 36, and the remainder being 15, it may be concluded that the number of Chances for throwing 27 Points is the same as for throwing 15 Points.

Although, as I have said before, the Demonstration of this Lemma may be had from my *Miscellanea*; yet I have thought fit, at the desire of some Friends, to transfer it to this place.

DEMONSTRATION.

1°. Let us imagine a Die so constituted as that there shall be upon it one single Face marked 1, then as many Faces marked 11 as there are Units in r , and as many Faces marked 111 as there are Units in rr , and so on; that the geometric Progression $1 + r + rr + r^3 + r^4 + r^5 + r^6 + r^7 + r^8$, &c. continued to so many Terms as there are different Denominations in the Die, may represent all the Chances of one Die: this being supposed, it is very plain that in order to have all the Chances of two such Dice, this Progression ought to be raised to its Square, and that to have all the Chances of three Dice, the same Progression ought to be raised to its Cube; and universally, that if the number of Dice be expressed by n , that Progression ought accordingly to be raised to the Power n . Now suppose the number of Faces in each Die to be f , then the Sum of that Pro-

gression will be $\frac{1-r^f}{1-r}$; and consequently every Chance that can happen upon n Dice, will be expressed by some Term of the Series

that results from the Fraction $\frac{1-r^f}{1-r}$ raised to the power n . But as the least number of Points, that can be thrown with n Dice, is n Units, and the next greater $n + 1$, and the next $n + 2$, &c. it is plain that the first Term of the Series will represent the number of Chances for throwing n Points, and that the second Term of the Series will represent the number of Chances for throwing $n + 1$ Points, and so on. And that therefore if the number of Points to be thrown be expressed by $p + 1$, it will be but assigning that Term in the Series of which the distance from the first shall be expressed by $p + 1 - n$.

G.

But

But the Series which would result from the raising of the Fraction $\frac{1-r^f}{1-r}$ to the Power n , is the Product of two other Series, whereof one is $1 + nr + \frac{n}{1} \times \frac{n+1}{2} rr + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} r^3$, &c. the other is $1 - nr^f + \frac{n}{1} \times \frac{n-1}{2} r^{2f} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} r^{3f} + \&c.$ Wherefore, if these two Series be multiplied together, all the Terms of the product will severally answer the several numbers of Chances that are upon n Dice.

And therefore if the number of Points to be thrown be expressed by $p + 1$, it is but collecting all the Terms which are affected by the Power r^{p+1-n} , and the Sum of those Terms will answer the Question proposed.

But in order to find readily all the Terms which are affected by the Power r^{p+1-n} , let us suppose, for shortness sake, $p + 1 - n = l$; and let us suppose farther that Er^l is that Term, in the first Series, of which the distance from its first Term is l ; let also Dr^{l-f} be that Term, in the first Series, of which the distance from its first Term is $l - f$, and likewise let Cr^{l-2f} be that Term, in the first Series, of which the distance from its first Term is denoted by $l - 2f$, and so on, making perpetually a regrefs towards the first Term. This being laid down, let us write all those Terms in order, thus

$$Er^l + Dr^{l-f} + Cr^{l-2f} + Br^{l-3f}, \&c.$$

and write underneath the Terms of the second Series, in their natural order. Thus

$$Er^l + Dr^{l-f} + Cr^{l-2f} + Br^{l-3f}, \&c.$$

$$1 - nr^f + \frac{n}{1} \times \frac{n-1}{2} r^{2f} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} r^{3f}, \&c.$$

then multiplying each Term of the first Series by each corresponding Term of the second, all the Terms of the product, *viz.*

$$Er^l - nDr^l + \frac{n}{1} \times \frac{n-1}{2} Cr^l - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} Br^l, \&c.$$

will be affected with the same power r^l .

Now the Coefficient E containing so many factors $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. as there are Units in l ; it is plain, that when the Denominators of those factors are continued beyond a certain number of them, denominated by $n - 1$, then the following Denominators will be $n, n + 1, n + 2$, &c. which being the same as the first Terms of the Numerators, it follows that if from the value of the Coefficient E be rejected those Numerators and Denominators which
are

are equal, there will remain out of the Numerators, written in an inverted order, the Terms $n + l - 1$, $n + l - 2$, $n + l - 3$, &c. of which the last will be $l + 1$; and that, out of the Denominators written in their natural order, there will remain 1, 2, 3, 4, 5, &c. of which the last will be $n - 1$: all which things depend intirely on the nature of an Arithmetic Progression. Wherefore the first Term

$$Er^l \text{ is } = \frac{n+l-1}{1} \times \frac{n+l-2}{2} \times \frac{n+l-3}{3} \dots \frac{l+1}{n-1} r^l.$$

Now in the room of l , substitute its value $p + 1 - n$, then $Er^l = \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3}$, &c. $\times r^l$, and in the same manner will the second Term

$$-nDr^l \text{ be } = -\frac{p-f}{1} \times \frac{p-f-1}{2} \times \frac{p-f-2}{3} \text{ \&c. } \times nr^l, \text{ and also}$$

the third Term

$$+ \frac{n}{1} \times \frac{n-1}{2} Cr^l \text{ will be } = + \frac{p-2f}{1} \times \frac{p-2f-1}{2} \times \frac{p-3f-2}{3} \text{ \&c.}$$

$\times \frac{n}{1} \times \frac{n-1}{2} r^l$, and so on. Suppose now $r = 1$, $p - f = q$, $q - f = r$, $r - f = s$, &c. and you shall have the very Rule given in our Lemma.

Now to add one Example more to our third Problem, let it be required to find in how many throws of 6 Dice one may undertake to throw 15 Points precisely.

The number of Chances for throwing 15 Points being 1666, and the whole number of Chances upon 6 Dice being 46656, it follows that the number of Chances for failing is 44990; wherefore dividing 44990 by 1666, and the quotient being 27 nearly, multiply 27 by 0.7, and the product 18.9 will shew that the number of throws requisite to that effect will be very near 19.

PROBLEM IV.

To find how many Trials are necessary to make it equally probable that an Event will happen twice, supposing that a is the number of Chances for its happening in any one Trial, and b the number of Chances for its failing.

SOLUTION.

Let x be the number of Trials: then from what has been demonstrated in the 16th Art. of the Introd. it follows that $b^x + xab^{x-1}$ is

the number of Chances whereby the Event may fail, $\overline{a+b}^x$ comprehending the whole number of Chances whereby it may either happen or fail, and consequently the probability of its failing is $\frac{b^x + xab^{x-1}}{a+b}^x$: but, by Hypothesis, the Probabilities of happening and failing are equal; we have therefore the Equation $\frac{b^x + xab^{x-1}}{a+b}^x = \frac{1}{2}$, or $\overline{a+b}^x = 2b^x + 2xab^{x-1}$, or making $a, b :: 1, q$, $1 + \frac{1}{q}^x = 2 + \frac{2x}{q}$. Now if in this Equation we suppose $q = 1$, x will be found $= 3$, and if we suppose q infinite, and also $\frac{x}{q} = z$, we shall have the Equation $z = \log. 2 + \log. 1 + z$, in which taking the value of z , either by Trial or otherwise, it will be found $= 1.678$ nearly; and therefore the value of x will always be between the limits $3q$ and $1.678q$, but will soon converge to the last of these limits; for which reason, if q be not very small, x may in all cases be supposed $= 1.678q$; yet if there be any suspicion that the value of x thus taken is too little, substitute this value in the original Equation $1 + \frac{1}{q}^x = 2 + \frac{2x}{q}$, and note the Error. Then if it be worth taking notice of, increase a little the value of x , and substitute again this new value of x in the aforesaid Equation; and noting the new Error, the value of x may be sufficiently corrected by applying the Rule which the Arithmeticians call double false Position.

EXAMPLE 1.

To find in how many throws of three Dice one may undertake to throw three Aces twice.

The number of all the Chances upon three Dice being 216, out of which there is but 1 Chance for three Aces, and 215, against it; multiply 215 by 1.678 and the product 360.8 will shew that the number of throws requisite to that effect will be 361, or very near it.

EXAMPLE 2.

To find in how many throws of 6 Dice one may undertake to throw 15 Points twice.

The number of Chances for throwing 15 Points is 1666, the number of Chances for missing 44990; let 44990 be divided by 1666,

1666, the Quotient will be 27 very near : wherefore the Chances for throwing and missing 15 Points are as 1 to 27 respectively ; multiply therefore 27 by 1.678, and the product 45.3 will shew that the number of Chances requisite to that effect will be 45 nearly.

EXAMPLE 3.

In a Lottery whereof the number of Blanks is to the Number of Prizes as 39 to 1 : to find how many Tickets must be taken to make it as probable that two or more benefits will be taken as not.

Multiply 39 by 1.678 and the product 65.4 will shew that no less than 65 Tickets will be requisite to that effect.

PROBLEM V.

To find how many Trials are necessary to make it equally probable that an Event will happen three, four, five, &c. times ; supposing that a is the number of Chances for its happening in any one Trial, and b the number of Chances for its failing.

SOLUTION.

Let x be the number of Trials requisite, then supposing as before $a, b :: 1, q$, we shall have the Equation $1 + \frac{1}{q}^x = 2 \times$

$1 + \frac{x}{q} + \frac{x}{1} \times \frac{x-1}{2qq}$, in the case of the triple Event ; or

$1 + \frac{1}{q}^x = 2 \times 1 + \frac{x}{q} + \frac{x}{1} \times \frac{x-1}{2qq} + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3q^2}$ in

the case of the quadruple Event : and the law of the continuation of these Equations is manifest. Now in the first Equation if q be supposed $= 1$, then will x be $= 5$; if q be supposed infinite or pretty large in respect to Unity, then the aforesaid Equation, making $\frac{x}{q} = z$, will be changed into this, $z = \log. 2 + \log.$

$1 + z + \frac{1}{2} z z$; wherein z will be found nearly $= 2.675$, wherefore x will always be between $5q$ and $2.675q$.

Likewise in the second Equation, if q be supposed $= 1$, then will x be $= 7q$; but if q be supposed infinite or pretty large in respect to Unity, then $z = \log. 2 + \log. 1 + z + \frac{1}{2} z z + \frac{1}{9} z^3$;

whence

whence x will be found nearly $= 3.6719$, wherefore x will be between $7q$ and $3.6719q$.

A TABLE of the Limits.

The Value of x will always be

For a single Event, between	$1q$ and	$0.693q$
For a double Event, between	$3q$ and	$1.678q$
For a triple Event, between	$5q$ and	$2.675q$
For a quadruple Event, between	$7q$ and	$3.672q$
For a quintuple Event, between	$9q$ and	$4.670q$
For a sextuple Event, between	$11q$ and	$5.668q$
&c.		

And if the number of Events contended for, as well as the number q be pretty large in respect to Unity; the number of Trials requisite for those Events to happen n times will be $\frac{2n-1}{2}q$, or barely nq .

REMARK.

From what has been said we may plainly perceive that altho' we may, with an equality of Chance, contend about the happening of an Event once in a certain number of Trials, yet we cannot, without disadvantage, contend for its happening twice in double that number of Trials, or three times in triple that number, and so on. Thus, altho' it be an equal Chance, or rather more than an Equality, that I throw two Aces with two Dice in 25 throws, yet I cannot undertake that the two Aces shall come up twice in 50 throws, the number requisite for it being 58 or 59; much less can I undertake that they shall come up three times in 75 throws, the number requisite for it being between 93 and 94: so that the Odds against the happening of two Aces in the first throw being 35 to 1, I cannot undertake that in a very great number of Trials, the happening shall be oftner than in the proportion of 1 to 35. And therefore we may lay down this general Maxim, that Events at long run will not happen oftner than in the proportion of the Chances they have to happen in any one Trial; and that if we assign any other proportion varying never so little from that, the Odds against us will increase continually.

To this may be objected, that from the premises it would seem to follow, that if two equal Gamesters were to play together for a considerable time, they would part without Gain or Loss on either side: but the answer is easy; the longer they play the greater Probability

bability there is of an increase of absolute Gain or Loss; but at the same time, the greater Probability there is also of a decrease, in respect to the number of Games played. Thus if 100 Games produce a difference of 4 in the winnings or losings, and 200 Games produce a difference of 6, there will be a greater proportion of Equality in the second case than in the first.

P R O B L E M VI.

Three Gamesters A, B, C play together on this condition, that he shall win the Set who has soonest got a certain number of Games; the proportion of the Chances which each of them has to get any one Game assigned, or which is the same thing, the proportion of their skill, being respectively as a, b, c. Now after they have played some time, they find themselves in this circumstance, that A wants 1 Game of being up, B 2 Games, and C 3 Games; the whole Stake amongst them being supposed 1; what is the Expectation of each?

SOLUTION. I.

In the circumstance the Gamesters are in, the Set will be ended in 4 Games at most; let therefore $a + b + c$ be raised to the fourth power, which will be $a^4 + 4a^3b + 6aabb + 4ab^3 + b^4 + 4a^3c + 12aabc + 4b^3c + 6aacc + 12abcc + 6bbcc + 4ac^3 + 12acbb + 4bc^3 + c^4$.

The terms $a^4 + 4a^3b + 4a^3c + 6aacc + 12aabc + 12abcc$, wherein the dimensions of a are equal to or greater than the number of Games which A wants, wherein also the Dimensions of b and c are less than the number of Games which B and C respectively want, are intirely favourable to A , and are part of the Numerator of his Expectation.

In the same manner, the terms $b^4 + 4b^3c + 6bbcc$ are intirely favourable to B .

And likewise the terms $4bc^3 + c^4$ are intirely favourable to C .

The rest of the terms are common, as favouring partly one of the Gamesters, partly one or both of the other; wherefore these Terms are so to be divided into their parts, that the parts, respectively favouring each Gamester, may be added to his Expectation.

Take

Take therefore all the terms which are common, *viz.* $6aabb$, $4ab^3$, $12abcc$, $4ac^3$, and divide them actually into their parts; that is, 1°. $6aabb$ into $aabb$, $abab$, $abba$, $baab$, $baba$, $bbaa$. Out of these six parts, one part only, *viz.* $bbaa$ will be found to favour B , for 'tis only in this term that two Dimensions of b are placed before one single Dimension of a , and therefore the other five parts belong to A ; let therefore $5aabb$ be added to the Expectation of A , and $1aabb$ to the Expectation of B . 2°. Divide $4ab^3$, into its parts $abbb$, $babb$, $bbab$, $bbba$; of these parts there are two belonging to A , and the other two to B ; let therefore $2ab^3$ be added to the expectation of each. 3°. Divide $12abbc$ into its parts; and eight of them will belong to A , and 4 to B ; let therefore $8abbc$ be added to the Expectation of A , and $4abbc$ to the Expectation of B . 4°. Divide $4ac^3$ into its parts, three of which will be found to be favourable to A , and one to C ; add therefore $3ac^3$ to the Expectation of A , and $1ac^3$ to the Expectation of C . Hence the Numerators of the several Expectations of A , B , C , will be respectively,

1. $a^4 + 4a^3b + 4a^3c + 6aacc + 12aabc + 12abcc + 5aabb + 2ab^3 + 8abbc + 3ac^3$.
2. $b^4 + 4b^3c + 6bbcc + 1aabb + 2ab^3 + 4abcc$.
3. $4bc^3 + 1c^4 + 1ac^3$.

The common Denominator of all their Expectations being $\overline{a+b+c}^4$.

Wherefore if a , b , c , are in a proportion of equality, the Odds of winning will be respectively as 57, 18, 6, or as 19, 6, 2.

If n be the number of all the Games that are wanting, p the number of Gamesters, and a , b , c , d , &c. the proportion of the Chances which each Gamester has respectively to win any one Game assigned; let $a + b + c + d$, &c. be raised to the power $n + 1 - p$, and then proceed as before.

REMARK.

This is one general Method of Solution. But the simpler and more common Cases may be managed with very little trouble. As,

1°. Let A and B want one game each, and C two games. Then the following game will either put him in the same situation as A and B , entitling him to $\frac{1}{3}$ of the Stake; of which there is 1 Chance: or will give the whole Stake to A or B ; and of this there

are two Chances. C 's Expectation therefore is worth $\frac{1 \times \frac{1}{3} + 2 \times 0}{3}$
(*Introd.*)

(*Introd. Art. 5.*) $= \frac{1}{9}$. Take this from the Stake 1, and the Remainder $\frac{8}{9}$, to be divided equally between *A* and *B*, makes the expectations of *A*, *B*, *C*, to be 4, 4, 1, respectively; to the common Denominator 9.

2°. Let *A* want 1 Game, *B* and *C* two games each. Then the next Game will either give *A* the whole Stake; or, one of his Adversaries winning, will reduce him to the Expectation $\frac{4}{9}$, of the former Case. His present Expectation therefore is $\frac{1 \times 1 + 2 \times \frac{4}{9}}{3} = \frac{17}{27}$: and the Complement of this to Unity, *viz.* $\frac{10}{27}$, divided equally between *B* and *C*, gives the three Expectations, 17, 5, 5, the common Denominator being 27.

3°. *A* and *B* wanting each a Game, let *C* want 3. In this Case, *C* has 2 Chances for 0, and 1 Chance for the Expectation $\frac{1}{9}$, of *Casé* 1. That is, his Expectation is $\frac{1}{27}$; and those of *A* and *B* are $\frac{12}{27}$, each.

4°. Let the Games wanting to *A*, *B*, and *C*, be 1, 2, 3, respectively: then *A* winning gets the Stake 1; *B* winning, *A* is in *Casé* 3, with the Expectation $\frac{13}{27}$, or *C* winning, he has, as in *Casé* 2, the Expectation $\frac{17}{27}$. Whence his present Expectation is $\frac{1}{3} \times 1 + \frac{13}{27} + \frac{17}{27} = \frac{57}{81}$.

Again, *A* winning, *B* gets 0; himself winning, he acquires (*Casé* 3.) the Expectation $\frac{13}{27}$. And, *C* winning, he is in *Casé* 2, with the Expectation $\frac{5}{27}$. His present Expectation therefore is $\frac{1}{3} \times 0 + \frac{13}{27} + \frac{5}{27} = \frac{18}{81}$. Add this to the Expectation of *A*, which was $\frac{57}{81}$; the Sum is $\frac{75}{81}$: and the Complement of this to Unity, which is $\frac{6}{81}$, is the Expectation of *C*.

Or to find *C*'s Expectation directly: *A* winning, *C* has 0; *B* winning, he has the Expectation $\frac{1}{27}$, (*Casé* 3.) and, himself winning, he has $\frac{5}{27}$, as in *Casé* 2: In all, $\frac{1}{3} \times 0 + \frac{1}{27} + \frac{5}{27} = \frac{6}{81}$.

And thus, ascending gradually through all the inferior Cases, or by the general Rule, we may compose a Table of Odds for 3 Gamblers, supposed of equal Skill; like that for 2 Gamesters in *Art.* 17th of the Introduction.

Table for 3 Gamesters.

Games wanting.	Odds.			Games wanting.	Odds.			Games wanting.	Odds.		
A. B. C.	a.	b.	c.	A. B. C.	a.	b.	c.	A. B. C.	a.	b.	c.
1 1 2	4	4	1	1 2 3	19	6	2	2 2 4	338	338	53
1 1 3	13	13	1	1 2 4	178	58	7	2 2 5	353	353	23
1 1 4	40	40	1	1 2 5	542	179	8	2 3 3	133	55	55
1 1 5	121	121	1	1 3 4	616	82	31	2 3 4	451	195	83
1 2 2	17	5	5	1 3 5	629	87	13	2 3 5	1433	635	119
1 3 3	65	8	8	2 2 3	34	34	13	&c.	&c.		

SOLUTION II. and more General.

It having been objected to the foregoing Solution, that when there are several Gamesters, and the number of games wanting amongst them is considerable; the Operation must be tedious; and that there may be some danger of mistake, in separating and collecting the several parts of their Expectations, from the Terms of the Multinomial: I invented this other Solution, which was published in the VIIth Book of my *Miscellanea Analytica*, A. D. 1730.

The Skill of the Gamesters *A, B, C, &c.* is now supposed to be as *a, b, c, &c.* respectively: and the Games they want of the Set are *p, q, r, &c.* Then in order to find the Chance of a particular Gamester, as of *A*, or his Right in the Stake 1, we may proceed as follows.

1°. Write down Unity.

2°. Write down in order all the Letters *b, c, d, &c.* which denote the Skill of the Gamesters, excepting only the Letter which belongs to the Gamester whose Chance you are computing; as in our Example, the Letter *a* is omitted.

3°. Combine the same Letters *b, c, d, &c.* by two's, three's, four's, &c.

4°. Of these Combinations, leave out or cancel all such as make any Gamester besides *A*, the winner of the Set; that is, which give to *B, q* Games; to *C, r* Games, to *D, s* Games, &c.

5°. Multiply the whole by ap^{-1} .

6°. Prefix to each Product the Number of its *Permutations*, that is, of the different ways in which its Letters can be written*.

* Of Combinations and Permutations, See Prob. xiv. & seqq.

7°. Let

7°. Let all the Products that are of the same dimension, that is, which contain the same number of Letters, be collected into different sums.

8°. Let these several Sums, from the lowest dimension upwards, be divided by the Terms of this Series,

$f^{p-1}, f^p, f^{p+1}, f^{p+2}, \&c.$ respectively: in which Series $f = a + b + c + d + \&c.$

9°. Lastly, multiply the Sum of the Quotients by $\frac{a}{f}$, and the Product shall be the Chance or Expectation required; namely the Right of A in the Stake 1. And in the same way, the Expectations of the other Gamesters may be computed.

EXAMPLE.

Supposing $p = 2, q = 3, r = 5$; write, as directed in the Rule,

1, $b + c, bb + bc + cc, bbcc + bc^3 + c^4, bbc^3 + bc^4, bbc^4.$

Multiply each term by a^{p-1} , which in our Example is a^{2-1} , or a ; prefix to each Product the number of its Permutations, dividing at the same time the similar Sums by $f^{p-1}, f^p, f^{p+1}, \&c.$ that is by $f, f^2, f^3, \&c.$; And the whole multiplied into $\frac{a}{f}$ will give the Ex-

pectation of $A = \frac{a}{f}$ into $\frac{a}{f} + \frac{2ab+2ac}{f^2} + \frac{3abb+6abc+3acc}{f^3} + \frac{12abbc+12abcc+4ac^3}{f^4} + \frac{20abbc^2+20abc^3+5ac^4}{f^5} + \frac{60abbc^3+30abc^4}{f^6} + \frac{105abbc^4}{f^7}.$

If we now substitute for a, b, c , any numbers at pleasure, we shall have the answer that belongs to those supposed degrees of Skill. As if we make $a = 1, b = 1, c = 1$; the Expectation of A will be,

$$\frac{1}{3} \times \frac{1}{3} + \frac{4}{9} + \frac{12}{27} + \frac{28}{81} + \frac{55}{243} + \frac{90}{729} + \frac{105}{2187} = \frac{1433}{2187}.$$

And, by like Operations, those of B and C will be $\frac{635}{2187}$ and $\frac{119}{2187}$ respectively.

PROBLEM VII.

Two Gamesters A and B , each having 12 Counters, play with three Dice, on condition that if 11 Points come up, B shall give one Counter to A ; if 14 Points come up, A shall give one Counter to B ; and that he shall be the winner who shall soonest get all the Counters of his Adversary: what is the Probability that each of them has of winning?

SOLUTION.

Let the number of Counters which each of them has be $= p$; and let a and b be the number of Chances they have respectively for getting a Counter, each cast of the Dice: which being supposed, I say that the Probabilities of winning are respectively as a^p to b^p ; now because in this case $p = 12$, and that, by the preceding Lemma, $a = 27$, and $b = 15$, it follows that the Probabilities of winning are respectively as 27^{12} to 15^{12} , or as 9^{12} to 5^{12} , or as 282429536481 to 244140625: which is the proportion assigned by *Huygens* in this particular case, but without any Demonstration.

Or more generally:

Let p be the number of the Counters of A , and q the number of the Counters of B ; and let the proportion of the Chances be as a to b . I say that the Probabilities of winning will be respectively as $a^q \times \overline{a^p - b^p}$ to $b^p \times \overline{a^q - b^q}$; and consequently the Probabilities themselves will be $\frac{a^q \times \overline{a^p - b^p}}{a^{p+q} - b^{p+q}} = R$, and $\frac{b^p \times \overline{a^q - b^q}}{a^{p+q} - b^{p+q}} = S$.

DEMONSTRATION.

Let it be supposed that A has the Counters E, F, G, H, &c. whose number is p , and that B has the Counters I, K, L, &c. whose number is q : moreover, let it be supposed that the Counters are the thing played for, and that the value of each Counter is to the value of the following as a to b , in such manner as that E, F, G, H, I, K, L be in geometric Progression; this being supposed, A and B in every circumstance of their Play may lay down two such Counters as may be proportional to the number of Chances each has to get a single Counter; for in the beginning of the Play, A may lay down the Counter H, which is the lowest of his Counters, and B the Counter I, which is his highest; but $H, I :: a, b$, therefore A and B play upon equal terms. If A beats B , then A may lay down the Counter I which he has just got of his adversary, and B the Counter K; but $I, K :: a, b$, therefore A and B still play upon equal terms. But if A lose the first time, then A may lay down the Counter G, and B the Counter H, which he just now got of his adversary; but $G, H :: a, b$, and therefore they still play upon equal terms as before: So that, as long as they play together, they play without advantage or disadvantage. Now the Value of the Expectation which A has of getting all the Counters of B is the product
of

of the Sum he expects to win, and of the probability of obtaining it, and the same holds also in respect to B : but the Expectations of A and B are supposed equal, and therefore the Probabilities which they have respectively of winning, are reciprocally proportional to the Sums they expect to win, that is, are directly proportional to the Sums they are possessed of. Whence the Probability which A has of winning all the Counters of B , is to the Probability which B has of winning all the Counters of A , as the Sum of the terms, E, F, G, H , whose number is p , to the Sum of the terms I, K, L , whose number is q , that is as $a^1 \times a^p - b^p$ to $b^p \times a^1 - b^1$; as will easily appear if those terms, which are in geometric Progression, are actually summed up by the known Methods. Now the Probabilities of winning are not influenced by the Supposition here made of each Counter being to the following in the proportion of a to b ; and therefore when those Counters are supposed of equal value, or rather of no value, but serving only to mark the number of Stakes won or lost on either side, the Probabilities of winning will be the same as we have assigned.

COROLLARY 1.

If we suppose both a and b in a ratio of equality, the expressions whereby the Probabilities of winning are determined will be reduced to the proportion of p to q : which will easily appear if those expressions be both divided by $a - b$.

COROLLARY 2.

If A and B play together for a Guinea a Game, and A has but one single Guinea to lose, but B any number, let it be never so large; if A in each Game has the Chance of 2 to 1, he is more likely to win all the Stock of B than to lose his single Guinea; and just as likely, if the Stock of B were infinite

REMARK.

If p and q , or either of them be large numbers, it will be convenient to work by Logarithms.

Thus, if A and B play a Guinea a Stake, and the number of Chances which A has to win each single Stake be 43, but the number of Chances, which B has to win it, be 40, and they oblige themselves to play till such time as 100 Stakes are won or lost; (the number p being $= q = 100$, and therefore the Ratio sought being $\frac{43}{40} \bigg|^{100}$.)

From

From the logarithm of 43 = 1.6334685

Subtract the logarithm of 40 = 1.6020600

Difference = 0.0314085

Multiply this Difference by the number of Stakes to be played off, viz. 100, the product will be 3.1408500, to which answers in the Table of Logarithms 1383; therefore the Odds that *A* beats *B* are 1383 to 1.

Now in all circumstances wherein *A* and *B* venture an equal Sum, the Sum of the numbers expressing the Odds, is to their difference, as the Money played for, is to the Gain of the one, and the Loss of the other.

Wherefore, multiplying 1382 difference of the numbers expressing the Odds by 100, which is the Sum ventured by each Man, and dividing the product by 1384, Sum of the Numbers expressing the Odds, the Quotient will be, within a trifle, 99 Guineas, and 2 Shillings, supposing Guineas at 21^{sh}.

If instead of supposing the proportion of the Chances whereby *A* and *B* may respectively win a Stake to be as 43 to 40, we suppose them as 44 to 40, or as 11 to 10, the Expectation of *A* will be worth above 99 Guineas, 20 Shillings and 1 Penny.

P R O B L E M VIII.

Two Gamesters A and B lay by 24 Counters, and play with three Dice, on this condition; that if 11 Points come up, A shall take one Counter out of the heap; if 14, B shall take out one; and he shall be reputed the winner who shall soonest get 12 Counters.

This Problem differs from the preceding in this, that the Play will be at an end in 23 Casts of the Die at most; (that is, of those Casts which are favourable either to *A* or *B*) whereas in the preceding case the Counters passing continually from one hand to the other, it will often happen that *A* and *B* will be in some of the same circumstances they were in before, which will make the length of the Play unlimited.

SOLUTION.

Taking *a* and *b* in the proportion of the Chances which there are to throw 11, and 14, let $a + b$ be raised to the 23^d Power, that is,

to

to such Power as is denoted by the number of all the Counters wanting one: then shall the 12 first terms of that Power be to the 12 last in the same proportion as are the Probabilities of winning.

P R O B L E M IX.

Supposing A and B, whose proportion of skill is as a to b, to play together, till A either wins the number q of Stakes, or loses the number p of them; and that B sets at every Game the Sum G to the Sum L; it is required to find the Advantage or Disadvantage of A.

SOLUTION.

First, Let the number of Stakes to be won or lost on either side be equal, and let that number be p ; let there be also an equality of skill between the Gamesters: then I say that the Gain of A will be $pp \times \frac{G-L}{2}$, that is the square of the number of Stakes which either Gamester is to win or lose, multiplied by one half of the difference of the value of the Stakes. Thus if A and B play till such time as ten Stakes are won or lost, and B sets one and twenty Shillings to 20; then the Gain of A will be 100 times the half difference between 21 and 20 Shillings, viz. 50 sh .

Secondly, Let the number of Stakes be unequal, so that A be obliged either to win the number q of Stakes, or to lose the number p ; let there be also an equality of Chance between A and B : then I say that the Gain of A will be $pq \times \frac{G-L}{2}$; that is the Product of the two numbers of Stakes, and one half the difference of the value of the Stakes multiplied together. Thus if A and B play together till such time as either A wins eight Stakes or loses twelve, then the Gain of A will be the product of the two numbers 8 and 12, and of 6^d. half the difference of the Stakes, which product makes 2^d. 8 sh .

Thirdly, Let the number of Stakes be equal, but let the number of Chances to win a Game, or the Skill of the Gamesters be unequal, in the proportion of a to b ; then I say that the Gain of A will be

$$\frac{a^p - b^p}{a^p + b^p} \times \frac{aG - bL}{a - b}.$$

Fourthly, Let the number of Stakes be unequal, and let also the number of Chances be unequal: then I say that the Gain of A will

$$\text{be } \frac{q a^q \times a^p - b^p - p b^p \times a^q - b^q}{a^{p+q} - b^{p+q}} \text{ multiplied by } \frac{aG - bL}{a - b}.$$

DE

DEMONSTRATION.

Let R and S respectively represent the Probabilities which A and B have of winning all the Stakes of their Adversary; which Probabilities have been determined in the *viith Problem*. Let us first suppose that the Sums deposited by A and B are equal, *viz.* G , and G : now since A is either to win the Sum qG , or lose the Sum pG , it is plain that the Gain of A ought to be estimated by $RqG - SpG$; moreover since the Sums deposited are G and G , and that the proportion of the Chances to win one Game is as a to b , it follows that the Gain of A for each individual Game is $\frac{aG-bG}{a+b}$; and for the same reason the Gain of each individual Game would be $\frac{aG-bL}{a+b}$, if the Sums deposited by A and B were respectively L and G . Let us therefore now suppose that they are L and G ; then in order to find the whole Gain of A in this second circumstance, we may consider that whether A and B lay down equal Stakes or unequal Stakes, the Probabilities which either of them has of winning all the Stakes of the other, suffer not thereby any alteration, and that the Play will continue of the same length in both circumstances before it is determined in favour of either; wherefore the Gain of each individual Game in the first case, is to the Gain of each individual Game in the second, as the whole Gain of the first case, to the whole Gain of the second; and consequently the whole Gain of the second case will be $Rq - Sp \times \frac{aG-bL}{a-b}$, or restoring the values of R and S , $\frac{qa^q \times a^p - b^p - pb^p \times a^q - b^q}{a^p + q - b^p + q}$ multiplied by $\frac{aG-bL}{a-b}$.

PROBLEM X.

Three Persons A, B, C, out of a heap of 12 Counters, whereof 4 are white, and 8 black, draw blindfold one Counter at a time, in this manner; A begins to draw; B follows A; C follows B; then A begins again; and they continue to draw in the same order, till one of them who is to be reputed the winner, draws the first white. What are the respective Probabilities of their winning?

SOLU-

SOLUTION.

Let n be the number of all the Counters, a the number of white, b the number of black, and 1 the whole Stake or the Sum played for.

1°. Since A has a Chances for a white Counter, and b Chances for a black, it follows that the Probability of his winning is $\frac{a}{a+b} = \frac{a}{n}$; therefore the Expectation he has upon the Stake 1, arising from the circumstance he is in, when he begins to draw, is $\frac{a}{n} \times 1 = \frac{a}{n}$; let it therefore be agreed among the Adventurers, that A shall have no Chance for a white Counter, but that he shall be reputed to have had a black one, which shall actually be taken out of the heap, and that he shall have the Sum $\frac{a}{n}$ paid him out of the Stake, for an Equivalent. Now $\frac{a}{n}$ being taken out of the Stake there will remain $1 - \frac{a}{n} = \frac{n-a}{n} = \frac{b}{n}$.

2°. Since B has a Chances for a white Counter, and that the number of remaining Counters is $n - 1$, his Probability of winning will be $\frac{a}{n-1}$; whence his Expectation upon the remaining Stake $\frac{b}{n}$, arising from the circumstance he is now in, will be $\frac{ab}{n \cdot n-1}$. Suppose it therefore agreed that B shall have the Sum $\frac{ab}{n \cdot n-1}$ paid him out of the Stake, and that a black Counter shall also be taken out of the heap. This being done, the remaining Stake will be $\frac{b}{n} - \frac{ab}{n \cdot n-1}$ or $\frac{nb-b-ab}{n \cdot n-1}$, but $nb - ab = bb$; wherefore the remaining Stake is $\frac{b \cdot b-1}{n \cdot n-1}$.

3°. Since C has a Chances for a white Counter, and that the number of remaining Counters is $n - 2$, his Probability of winning will be $\frac{a}{n-2}$, and therefore his Expectation upon the remaining Stake arising from the circumstance he is now in, will be $\frac{b \cdot b-1 \cdot a}{n \cdot n-1 \cdot n-2}$, which we will likewise suppose to be paid him out of the Stake, still supposing a black Counter taken out of the heap.

4°. A may have out of the remainder the Sum $\frac{b \cdot b-1 \cdot b-2 \cdot a}{n \cdot n-1 \cdot n-2 \cdot n-3}$; and so of the rest till the whole Stake be exhausted.

Wherefore having written the following general Series; *viz.* $\frac{a}{n} + \frac{b}{n-1}P + \frac{b-1}{n-2}Q + \frac{b-2}{n-3}R + \frac{b-3}{n-4}S$, &c. wherein *P*, *Q*, *R*, *S*, &c. denote the preceding Terms, take as many Terms of this Series as there are Units in $b + 1$, (for since *b* represents the number of black Counters, the number of drawings cannot exceed $b + 1$.) then take for *A* the first, fourth, seventh, &c. Terms; take for *B* the second, fifth, eighth, &c. for *C* the third, sixth, &c. and the Sums of those Terms will be the respective Expectations of *A*, *B*, *C*; or because the Stake is fixed, these Sums will be proportional to the respective Probabilities of winning.

Now to apply this to the present case, make $n = 12$, $a = 4$, $b = 8$, and the general Series will become $\frac{4}{12} + \frac{8}{11}P + \frac{7}{10}Q + \frac{6}{9}R + \frac{5}{8}S + \frac{4}{7}T + \frac{3}{6}U + \frac{2}{5}X + \frac{1}{4}Y$: or multiplying the whole by 495 to take away the fractions, the Series will be $165 + 120 + 84 + 56 + 35 + 20 + 10 + 4 + 1$.

Therefore assigning to *A* $165 + 56 + 10 = 231$, to *B* $120 + 35 + 4 = 159$, to *C* $84 + 20 + 1 = 105$, the Probabilities of winning will be proportional to the numbers 231, 159, 105, or 77, 53, 35.

If there be never so many Gamesters *A*, *B*, *C*, *D*, &c. whether they take every one of them one Counter or more; or whether the same or a different number of Counters; the Probabilities of winning will be determined by the same general Series.

REMARK I.

The preceding Series may in any particular case be shortened; for if $a = 1$, then the Series will be $\frac{1}{n} \times \overline{1 + 1 + 1 + 1 + 1 + 1 + 1}$, &c.

Hence it may be observed, that if the whole number of Counters be exactly divisible by the number of Persons concerned in the Play, and that there be but one single white Counter in the whole, there will be no advantage or disadvantage to any one of them from the situation he is in, in respect to the order of drawing.

If $a = 2$, then the Series will be $\frac{2}{n \cdot n-1} \times \overline{n-1 + n-2 + n-3 + n-4 + n-5}$, &c.

If $a = 3$, then the Series will be $\frac{3}{n \cdot n-1 \cdot n-2} \times \overline{n-1 \cdot n-2 + n-2 \cdot n-3 + n-3 \cdot n-4}$, &c.

If $a = 4$, then the Series will be $\frac{4}{n \cdot n-1 \cdot n-2 \cdot n-3} \times n-1 \cdot n-2 \cdot n-3 + n-2 \cdot n-3 \cdot n-4$, &c.

Wherefore rejecting the common Multipliers; the several Terms of these Series taken in due order, will be proportional to the several Expectations of any number of Gamesters: thus in the case of this Problem where $n = 12$, and $a = 4$, the Terms of the Series will be,

For A	For B	For C
$11 \times 10 \times 9 = 990$	$10 \times 9 \times 8 = 720$	$9 \times 8 \times 7 = 504$
$8 \times 7 \times 6 = 336$	$7 \times 6 \times 5 = 210$	$6 \times 5 \times 4 = 120$
$5 \times 4 \times 3 = 60$	$4 \times 3 \times 2 = 24$	$3 \times 2 \times 1 = 6$
1386	954	630

Hence it follows that the Probabilities of winning will be respectively as 1386, 954, 630, or dividing all by 18, as 77, 53, 35, as had been before determined.

REMARK 2.

But if the Terms of the Series are many, it will be convenient to sum them up by means of the following Method, which is an immediate consequence of the fifth Lemma of Sir *Isaac Newton's Principia*, Book III; and of which the Demonstration may be deduced from his *Analysis*.

If there be a Series of Terms, A, B, C, D, E, &c. let each Term be subtracted from that which immediately follows it, and let the Remainders be called first Differences, then subtract each difference from that which immediately follows it, and let the remainders be called second differences; again, let each second difference be subtracted from that which immediately follows it, and let the remainders be called third differences, and so on. Let the first of the first Differences be called d , the first of the second d' , the first of the third d'' , &c. and let x be the interval between the first Term A, and any other Term, such as E, that is, let the number of Terms from A to E, both inclusive, be $x + 1$, then the Term

$E = A + xd + \frac{x}{1} \times \frac{x-1}{2} d' + \frac{x \times x-1 \times x-2}{1 \cdot 2 \cdot 3} d''$, &c. From hence it manifestly follows, that let the number of Terms between A and E

be never so great, if it so happen that all the differences of one of the orders are equal to one another, the following differences of all the other orders will all be $= 0$; and that therefore the last Term will be assignable by so many Terms only of the Series above-written, as are denoted by the first Difference that happens to be $= 0$.

This being premised, it will be easy to shew, how the Sums of those Terms may be taken; for if we imagine a new Series whereof the first Term shall be $= 0$; the second $= A$; the third $= A + B$; the fourth $= A + B + C$; the fifth $= A + B + C + D$, and so on; it is plain that the assigning one Term of the new Series is finding the Sum of all the Terms $A, B, C, D, \&c.$ Now since those Terms are the differences of the Sums $0, A, A + B, A + B + C, A + B + C + D, \&c.$ and that by Hypothesis some of the differences of A, B, C, D , are $= 0$, it follows that some of the differences of the

Sums will also be $= 0$; and that whereas in the Series $A + x'd + \frac{x}{1} \times \frac{x-1}{2} d''$, &c. whereby a Term was assigned, A represented

the first Term, d' the first of the first differences, d'' the first of the second differences, and that x represented the Interval between the first Term and the last, we are now to write 0 instead of A ; A in-

stead of d' ; d' instead of d'' ; d'' instead of d''' , &c. and $x + 1$ instead of x ; which being done the Series expressing the Sums will be

$$0 + \frac{x+1}{1} \times A + \frac{x+1 \cdot x}{1 \cdot 2} d' + \frac{x+1 \cdot x \cdot x-1}{1 \cdot 2 \cdot 3} d'' \text{, \&c. or } \frac{x+1}{1} \times A + \frac{x}{2} d' + \frac{x \cdot x-1}{2 \cdot 3} d'' + \frac{x \cdot x-1 \cdot x-2}{2 \cdot 3 \cdot 4} d''' \text{, \&c.}$$

where it will not perhaps be improper to take notice, that the Series by me exhibited in my first Edition, though seemingly differing from this, is the same at bottom.

But to apply this to a particular case, let us suppose that three Persons A, B, C playing on the same conditions as are expressed in this x^{th} Problem, the whole number of Counters were 100, instead of 12, still preserving the same number 4 of white Counters, and that it were required to determine the Expectations of A, B, C .

It is plain from what has been said in the first Remark, that the Expectation of A will be proportional to the sum of the numbers

$$99 \times 98 \times 97 + 96 \times 95 \times 94 + 93 \times 92 \times 91 + 90 \times 89 \times 88, \text{ \&c.}$$

that

that the Expectation of *B* will be proportional to the Sum of the numbers

$98 \times 97 \times 96 + 95 \times 94 \times 93 + 92 \times 91 \times 90 + 89 \times 88 \times 87$, &c. and lastly, that the Expectation of *C* will be proportional to the Sum of the numbers

$97 \times 96 \times 95 + 94 \times 93 \times 92 + 91 \times 90 \times 89 + 88 \times 87 \times 86$, &c. But as the number of Terms which constitute those three Series is equal to the number of black Counters increased by 1, as it has been observed before, it follows that the number of all the Terms distributed among *A*, *B*, *C*, must be 97; now dividing 97 by the number of Gamesters which in this case is 3, the quotient will be 32; and there remaining 1 after the division, it is an indication that 33 Terms enter the Expectation of *A*, that 32 Terms enter the Expectation of *B*, and 32 likewise the Expectation of *C*; from whence it follows that the last Term of those which belong to *A* will be $3 \times 2 \times 1$, the last of those which belong to *B* will be $5 \times 4 \times 3$, and the last of those which belong to *C* will be $4 \times 3 \times 2$.

And therefore if we invert the Terms, making that the first which was the last, and take the differences, according to what has been prescribed, as follows;

				<i>A</i>	<i>d</i>	<i>d</i>	<i>d</i>
3	×	2	×	1	=	6	
6	×	5	×	4	=	120	114
9	×	8	×	7	=	504	384
12	×	11	×	10	=	1320	816
15	×	14	×	13	=	2730	1410
&c.							
						270	
						432	162
						594	162

then the Expectation of *A*, as deduced from the general Theorem, will be expressed by

$$\overline{x+1} \times 6 + \frac{114x}{2} + \frac{x \cdot x-1}{2 \cdot 3} \times 270 + \frac{x \cdot x-1 \cdot x-2}{2 \cdot 3 \cdot 4} \times 162 :$$

which being contracted, then reduced into its factors, will be equivalent to

$$\frac{3}{4} \times \overline{x+1} \times \overline{x+2} \times \overline{3x+1} \times \overline{3x+4}.$$

In like manner, it will be found that the Expectation of *B* is equivalent to

$$\frac{3}{4} \times \overline{x+1} \times \overline{x+2} \times \overline{3x+5} \times \overline{3x+8}.$$

And

And that the Expectation of C is equivalent to

$$\frac{2}{4} \times x + 1 \times x + 2 \times 9xx + 27x + 16.$$

Now x in each case represents the number of Terms wanting one, which belong severally to A , B , C ; wherefore making $x + 1 = p$, the several Expectations will now be expressed by the number of Terms which were originally to be summed up, and which will be as follows.

$$\text{For } A, p \times \overline{p + 1} \times \overline{3p - 2} \times \overline{3p + 1}$$

$$\text{For } B, p \times \overline{p + 1} \times \overline{3p + 2} \times \overline{3p + 5}$$

$$\text{For } C, p \times \overline{p + 1} \times \overline{9pp + 9p - 2}$$

But still it is to be considered, that p in the first case answers to the number 33, and in the other two cases to 32; and therefore p being interpreted for the several cases as it ought to be, the several Expectations will be found proportional to the numbers 41225, 39592, 38008.

If the number of all the Counters were 500, and the number of the white still 4, then the number of all the Terms representing the Expectations of A , B , C would be 497: now this number being divided by 3, the Quotient is 165, and the remainder 2. From whence it follows that the Expectations of A and B consist of 166 terms each, and the Expectation of C only of 165, and therefore the lowest Term of all, *viz.* $3 \times 2 \times 1$ will belong to B , the last but one $4 \times 3 \times 2$ will belong to A , and the last but two will belong to C .

P R O B L E M XI.

If A, B, C throw in their turns a regular Ball having 4 white faces and eight black ones; and he be to be reputed the winner who shall first bring up one of the white faces; it is demanded, what the proportion is of their respective Probabilities of winning?

SOLUTION.

The Method of reasoning in this Problem is exactly the same as that which we have made use of in the Solution of the preceding: but whereas the different throws of the Ball do not diminish the number of its Faces; in the room of the quantities $b - 1$, $b - 2$,
 $b - 3$,

$b = 3$, &c. $n = 1$, $n = 2$, $n = 3$, &c. employed in the Solution of the aforesaid Problem, we must substitute b and n respectively, and the Series belonging to that Problem will be changed into the following, which we ought to conceive continued infinitely.

$$\frac{a}{n} + \frac{ab}{nn} + \frac{abb}{n^3} + \frac{ab^3}{n^4} + \frac{ab^4}{n^5} + \frac{ab^5}{n^6}, \text{ \&c.}$$

then taking every third Term thereof, the respective Expectations of A , B , C will be expressed by the following Series,

$$\begin{aligned} \frac{a}{n} + \frac{ab^3}{n^4} + \frac{ab^6}{n^7} + \frac{ab^9}{n^{10}} + \frac{ab^{12}}{n^{13}}, \text{ \&c.} \\ \frac{ab}{nn} + \frac{ab^4}{n^5} + \frac{ab^7}{n^8} + \frac{ab^{10}}{n^{11}} + \frac{ab^{13}}{n^{14}}, \text{ \&c.} \\ \frac{abb}{n^3} + \frac{ab^5}{n^6} + \frac{ab^8}{n^9} + \frac{ab^{11}}{n^{12}} + \frac{ab^{14}}{n^{15}}, \text{ \&c.} \end{aligned}$$

But the Terms, whereof each Series is composed, are in geometric Progreſſion, and the ratio of each Term in each Series to the following is the ſame; wherefore the Sums of theſe Series are in the ſame proportion as their firſt Terms, *viz.* as $\frac{a}{n}$, $\frac{ab}{nn}$, $\frac{abb}{n^3}$, or as nn , bn , bb ; that is, in the preſent caſe, as 144, 96, 64, or 9, 6, 4. Hence the reſpective Probabilities of winning will likewiſe be as the numbers 9, 6, 4.

COROLLARY I.

If there be any other number of Gameſters A , B , C , D , &c. playing on the ſame conditions as above, take as many Terms in the proportion of n to b , as there are Gameſters, and thoſe Terms will reſpectively denote the ſeveral Expectations of the Gameſters.

COROLLARY 2.

If there be any number of Gameſters A , B , C , D , &c. playing on the ſame conditions as above, with this difference only, that all the Faces of the Ball ſhall be marked with particular figures 1, 2, 3, 4, &c. and that a certain number p of thoſe Faces ſhall intitle A to be the winner; and that likewiſe a certain number of them, as q , r , s , t , &c. ſhall reſpectively intitle B , C , D , E , &c. to be winners: make $n - p = a$, $n - q = b$, $n - r = c$, $n - s = d$, $n - t = e$, &c. then in the following Series;

$$\frac{p}{n} + \frac{qa}{nn} + \frac{rab}{n^3} + \frac{sabc}{n^4} + \frac{tabcd}{n^5}, \text{ \&c.}$$

the Terms taken in due order will reſpectively repreſent the ſeveral Probabilities of winning.

For if the law of the Play be such, that every Man having once played in his turn, shall begin regularly again in the same manner, and that continually, till such time as one of them wins; we are to take as many Terms of the Series as there are Gamesters, and those Terms will represent the respective Probabilities of winning.

But the Reason of this Rule will best appear if we apply it to some easy Example.

Let therefore the three Gamesters A, B, C throw a Die of 12 faces in their Turns; of which 5 faces are favourable to A , 4 faces are favourable to B , and the remaining 3 give the Stake to C . Then $p = 5$, $q = 4$, $r = 3$: and there being but 3 Gamesters, the same Chances, and in the same Order A, B, C , will recur perpetually after a Round of three throws, till the Stake is won; or rather, as we suppose in the demonstration, till the Stake is totally exhausted, by each Gamester, instead of his throw, taking out of it the part to which the chance of that throw entitles him.

Now A having p Chances out of n , or 5 out of 12, to get the whole Stake at the first Throw, let him take out of it the Value of this Chance $\frac{p}{n}$; and there will remain $1 - \frac{p}{n} = \frac{n-p}{n} = \frac{a}{n}$ to be thrown for by B .

And B 's Chances for winning in his Throw being q out of n , or 4 out of 12, the Value of his present Expectation is $\frac{q}{n} \times \frac{a}{n} = \frac{qa}{n^2}$; which if he takes out of the Stake $\frac{a}{n}$ there will remain $\frac{a}{n} - \frac{qa}{n^2} = \frac{a}{n} \times 1 - \frac{q}{n} = \frac{a}{n} \times \frac{b}{n}$, to be thrown for by C .

His Chances for getting this Stake being r out of n , or 3 out of 12, the Value of his Expectation is $\frac{rab}{n^3}$; which he may take out of the Stake $\frac{ab}{n^2}$: and resign the Die to A , who begins the second Round.

But if, for the Stakes that remain after the first, second, third, &c. Rounds, we write R', R'', R''' , &c. respectively, it is manifest that the Value of a Gamester's Chance in each Round is proportional to the Stake R', R'', R''' , &c. which remained at the beginning of that Round. Thus the Value of A 's first Throw having been $\frac{p}{n} \times 1$, the Value of his second will be $\frac{p}{n} \times R'$, of his third, $\frac{p}{n} \times R''$, &c. And the Value of B 's first Throw having been $\frac{qa}{n^2} \times 1$, that of his second will $\frac{qa}{n^2} \times R'$, of his third, $\frac{qa}{n^2} \times R''$, &c. and the like for the several Expectations of C .

Put

Put $S = 1 + R' + R'' + R'''$, &c. and the Total of A 's Expectations will be $\frac{p}{n} \times S$; of B , $\frac{qa}{n^2} \times S$; of C , $\frac{rab}{n^3} \times S$: or rejecting the common Factor S , the Expectations of A , B , C , at the beginning of the Play will be as $\frac{p}{n}$, $\frac{qa}{n^2}$, $\frac{rab}{n^3}$, respectively: that is as the 3 first Terms of the Series. And the like reasoning will hold, be the Number of Gamesters, their favourable Chances, or order of Throwing, what you will.

In the present Example, $\frac{p}{n} = \frac{5}{12} = \frac{720}{1728}$; $\frac{qa}{n^2} = \frac{28}{144} = \frac{336}{1728}$; $\frac{rab}{n^3} = \frac{168}{1728}$: and the Chances of A , B , C , respectively, are as the Numerators 720, 336, 168; that is, as 30, 14, 7. or the whole Stake being 51 pieces, A can claim 30 of them, B 14, and C the remaining 7.

In making up this Stake, the Gamesters A , B , C , were, at equal play, to contribute only in proportion to their Chances of winning; that is in the proportion of p , q , r , or 5, 4, 3, respectively: and, before the Order of throwing was fixt, their Chances must have been exactly worth what they paid in to the Stake. What gives A the great advantage now is, an antecedent good luck of being the first to throw. If B had been the first; or if A , taking his first Throw, had mist of a p face, then B 's Chance had been the better of the two.

And if it were the Law of Play that every Man should play several times together, for instance twice: then taking for A the two first Terms, for B the two following, and so on; each couple of Terms will represent the respective Probabilities of winning, observing now that p and q are equal, as also r and s .

But if the Law of Play should be irregular, then you are to take for each Man as many Terms of the Series as will answer that irregularity, and continue the Series till such time as it gives a sufficient Approximation.

Yet if, at any time, the Law of the Play having been irregular, should afterwards recover its regularity, the Probabilities of winning, will (with the help of this Series) be determined by finite expressions.

Thus if it should be the Law of the Play, that two Men A and B having played irregularly for ten times together, tho' in a manner agreed on between them, they should also agree that after ten throws, they should play alternately each in his turn: distribute the ten first Terms of the Series between them, according to the order fixed upon by their convention, and having subtracted the Sum of those Terms

from Unity, divide the remainder of it between them in the proportion of the two following Terms, which add respectively to the Shares they had before; then the two parts of Unity which *A* and *B* have thus obtained, will be proportional to their respective Probabilities of winning.

PROBLEM XII.

There are any number of Gamesters, who in their Turns, which are decided by Lots, turn a Cube, having 4 of its Faces marked T, P, D, A, the other two Faces which are opposite have each a little Knob or Pivet, about which the Cube is made to turn; the Gamesters each lay down a Sum agreed upon, the first begins to turn the Cube; now if the Face T be brought up, he sweeps all the Money upon the Board, and then the Play begins anew; if any other Face is brought up, he yields his place to the next Man, but with this difference, that if the Face P comes up, he, the first Man, puts down as much Money as there was upon the Board; if the Face D comes up, he neither takes up any Money nor lays down any; if the Face A comes up, he takes up half of the Money upon the Board; when every Man has played in his Turn upon the same conditions as above, there is a recurrency of Order, whereby the Board may be very much enlarged, viz. if it so happen that the Face T is intermitted during many Trials: now the Question is this; when a Gamester comes to his Turn, supposing him afraid of laying down as much Money as there is already, which may be considerable, how must he compound for his Expectation with a Spectator willing to take his place.

SOLUTION.

Let us suppose for a little while that the number of Gamesters is infinite, and that what is upon the Board is the Sum f ; then, there

there being 1 Chance in 4 for the Face T to come up, it follows that the Expectation of the first Man, upon that score, is $\frac{1}{4}$ 2°. There being 1 Chance in 4 for the Face P to come up, whereby he would necessarily lose f , (by reason that the number of Gamesters having been supposed infinite, his Chance of playing would never return again) it follows that his Loss upon that account ought to be estimated by $\frac{1}{4}f$. 3°. There being 1 Chance in 4 for the Face D to come up, whereby he would neither win or lose any thing, we may proceed to the next Chance. 4°. There being 1 Chance in 4 for the Face A to come up, which intitles him to take up $\frac{1}{2}f$, his Expectation, upon that account, is $\frac{1}{8}f$, or supposing $8 = n$, his Expectation is $\frac{1}{n}f$; now out of the four cases above-mentioned the first and second do destroy one another, the third neither contributes to Gain or Loss, and therefore the clear Gain of the first Man is upon account of the fourth Case; let it therefore be agreed among the Adventurers, that the first Man shall not try his Chance, but that he shall take the Sum $\frac{1}{n}f$ out of the common Stake f , and that he shall yield his Turn to the next Man.

But before I proceed any farther, it is proper to prevent an Objection that may be made against what I have asserted above, *viz.* that the Face D happening to come up, the Adventurer in that case would lose nothing, because it might be said that the number of Gamesters being infinite, he would necessarily lose the Stake he has laid down at first; but the answer is easy, for since the number of particular Stakes is infinite, and that the Sum of all the Stakes is supposed only equal to f , it follows that each particular Stake is nothing in comparison to the common Stake f , and therefore that common Stake may be looked upon as a present made to the Adventurers. Now to proceed; I say that the Sum $\frac{1}{n}f$ having been taken out of the common Stake f , the remaining Stake will be $\frac{n-1}{n}f$ or $\frac{d}{f}$, supposing $n - 1 = d$: but by reason that the first Man was allowed $\frac{1}{n}$ part of the common Stake, so ought the next Man to be allowed $\frac{1}{n}$ part of the present Stake $\frac{d}{n}f$, which will make it that the Expectation of the second Man will be $\frac{d}{nn}f$; Again, the Expectation

tation of the second Man being to the Expectation of the first as $\frac{d}{n}$ to 1, the Expectation of the third must be to the Expectation of the second also as $\frac{d}{n}$ to 1, from whence it follows that the Expectation of the third Man will be $\frac{dd}{n^2}f$, and the Expectation of the fourth $\frac{d^3}{n^3}f$, and so on; which may fitly be represented by the Series f into $\frac{1}{n} + \frac{d}{nn} + \frac{dd}{n^2} + \frac{d^3}{n^3} + \frac{d^4}{n^4} + \frac{d^5}{n^5} + \frac{d^6}{n^6}$, &c. Now the Sum of that infinite Series, which is a Geometric Progression, is $\frac{f}{n-d}$, but d having been supposed $= n - 1$, then $n - d = 1$, and therefore the Sum of all the Expectations is only f , as it ought to be.

Now let us suppose that instead of an infinite number of Gamesters, there are only two; then, in this case, we may imagine that the first Man has the *first, third, fifth, seventh* Terms of that Series, and all those other Terms *in infinitum* which belong to the *odd* places, and that the second Man has all the Terms which belong to the *even* places; wherefore the Expectation of the first Man is $\frac{f}{n}$ into $1 + \frac{dd}{nn} + \frac{d^4}{n^4} + \frac{d^6}{n^6} + \frac{d^8}{n^8}$, &c. and the Expectation of the second is $\frac{df}{nn}$ into $1 + \frac{dd}{nn} + \frac{d^4}{n^4} + \frac{d^6}{n^6} + \frac{d^8}{n^8}$, &c. and therefore the Ratio of their Expectations is as $\frac{f}{n}$ to $\frac{df}{nn}$, or as 1 to $\frac{d}{n}$, that is as n to $n - 1$, or as 8 to 7; and therefore the Expectation of the first Man is $\frac{8}{15}f$, and the Expectation of the second Man is $\frac{7}{15}f$; and therefore if a Spectator has a mind to take the place of the first Man, he ought to give him $\frac{8}{15}f$.

But if the number of Gamesters be three, take a third proportional to n and d , which will be $\frac{dd}{n}$, and therefore the three Expectations will be respectively proportional to n , d , $\frac{dd}{n}$, or to nn , dn , dd , and therefore the Expectation of the first Man is $\frac{nn}{nn + dn + dd}f$ which in this case is $= \frac{64}{109}f$.

Universally, Let p be the number of Adventurers, then the Sum for which the Expectation of the first Man may be transferred to another is $\frac{n^{p-1}}{n^p - d^p}f$.

The

The Game of BASSETTE.

Rules of the Play.

The Dealer, otherwise called the *Banker*, holds a pack of 52 Cards, and having shuffled them, he turns the whole pack at once, so as to discover the last Card; after which he lays down by couples all the Cards.

The Setter, otherwise called the *Ponte*, has 13 Cards in his hand, one of every sort, from the King to the Ace, which 13 Cards are called a *Book*; out of this Book he takes one Card or more at pleasure, upon which he lays a Stake.

The *Ponte* may at his choice, either lay down his Stake before the pack is turned, or immediately after it is turned; or after any number of Couples are drawn.

The first case being particular, shall be calculated by itself; but the other two being comprehended under the same Rules, we shall begin with them.

Supposing the *Ponte* to lay down his Stake after the Pack is turned, I call 1, 2, 3, 4, 5, &c. the places of those Cards which follow the Card in view, either immediately after the pack is turned, or after any number of couples are drawn.

If the Card upon which the *Ponte* has laid a Stake comes out in any odd place, except the first, he wins a Stake equal to his own.

If the Card upon which the *Ponte* has laid a Stake comes out in any even place, except the second, he loses his Stake.

If the Card of the *Ponte* comes out in the first place, he neither wins nor loses, but takes his own Stake again.

If the Card of the *Ponte* comes out in the second place, he does not lose his whole Stake, but only a part of it, *viz.* one half, which to make the Calculation more general we shall call *y*. In this case the *Ponte* is said to be *Faced*.

When the *Ponte* chuses to come in after any number of Couples are down; if his Card happens to be but once in the Pack, and is the very last of all, there is an exception from the general Rule; for tho' it comes out in an odd place, which should intitle him to win a Stake equal to his own, yet he neither wins nor loses from that circumstance, but takes back his own Stake.

P R O B L E M XIII.

To estimate at Bassette the Loss of the Ponte under any circumstance of Cards remaining in the Stock, when he lays his Stake; and of any number of times that his Card is repeated in the Stock.

The Solution of this Problem containing four cases, viz. of the Ponte's Card being *once, twice, three or four* times in the Stock; we shall give the Solution of all these cases severally.

SOLUTION of the *first* Case.

The Ponte has the following chances to win or lose, according to the place his Card is in.

1	1	Chance for winning	0
2	1	Chance for losing	y
3	1	Chance for winning	1
4	1	Chance for losing	1
5	1	Chance for winning	1
6	1	Chance for losing	1
*	1	Chance for winning	0

It appears by this Scheme, that he has as many Chances to win 1 as to lose 1, and that there are two Chances for neither winning or losing, viz. the first and the last, and therefore that his only Loss is upon account of his being *Faced*: from which it is plain that the number of Cards covered by that which is in view being called n , his Loss will be $\frac{y}{n}$, or $\frac{1}{2n}$, supposing $y = \frac{1}{2}$.

SOLUTION of the *second* Case.

By the first Remark belonging to the x^{th} Problem, it appears † that the Chances which the Ponte has to win or lose are proportional to the numbers, $n-1$, $n-2$, $n-3$, &c. Wherefore his Chances for winning and losing may be expressed by the following Scheme.

† Namely, by calling the Ponte's two Cards two white Counters, drawn for alternately by A and B ; and supposing all A 's Chances to belong to the Banker's right hand, and those of B to his left. And the like for the Cases of the Ponte's Card being in the Stock 3 or 4 times.

1	$n-1$	Chances for winning	0
2	$n-2$	Chances for losing	y
3	$n-3$	Chances for winning	1
4	$n-4$	Chances for losing	1
5	$n-5$	Chances for winning	1
6	$n-6$	Chances for losing	1
7	$n-7$	Chances for winning	1
8	$n-8$	Chances for losing	1
9	$n-9$	Chances for winning	1
*	1	Chance for losing	1

Now setting aside the first and second number of Chances, it will be found that the difference between the 3^d and 4th is $= 1$, that the difference between the 5th and 6th is also $= 1$, and that the difference between the 7th and 8th is also $= 1$, and so on. But the number of differences is $\frac{n-3}{2}$, and the Sum of all the Chances is $\frac{n}{1} \times \frac{n-1}{2}$: wherefore the Gain of the Ponte is $\frac{n-3}{n \times n-1}$. But his Loss upon account of the Face is $n-2 \times y$ divided by $\frac{n \times n-1}{1 \times 2}$ that is $\frac{2n-4 \times y}{n \times n-1}$: hence it is to be concluded that his Loss upon the whole is $\frac{2n-4 \times y-n-3}{n \times n-1}$ or $\frac{1}{n \times n-1}$ supposing $y = \frac{1}{2}$.

That the number of differences is $\frac{n-3}{2}$ will be made evident from two considerations.

First, the Series $n-3, n-4, n-5, \&c.$ decreases in Arithmetic Progression, the difference of its terms being Unity, and the last Term also Unity, therefore the number of its Terms is equal to the first Term $n-3$: but the number of differences is one half of the number of Terms; therefore the number of differences is $\frac{n-3}{2}$.

Secondly, it appears, by the xth Problem, that the number of all the Terms including the two first is always $b+1$, but a in this case is $= 2$, therefore the number of all the Terms is $n-1$; from which excluding the two first, the number of remaining Terms will be $n-3$, and consequently the number of differences $\frac{n-3}{2}$.

That the Sum of all the Terms is $\frac{n}{1} \times \frac{n-1}{2}$, is evident also from two different considerations.

First

First in any Arithmetic Progression whereof the first Term is $n - 1$, the difference Unity, and the last Term also Unity, the Sum of the Progression will be $\frac{n}{1} \times \frac{n-1}{2}$.

Secondly, the Series $\frac{2}{n \times n-1} \times n-1 + n-2 + n-3, \&c.$ mentioned in the first Remark upon the tenth Problem, expresses the Sum of the Probabilities of winning which belong to the several Gamblers in the case of two white Counters, when the number of all the Counters is n . It therefore expresses likewise the Sum of the Probabilities of winning which belong to the Ponte and Banker in the present case: but this Sum must always be equal to Unity, it being a certainty that the Ponte or Banker must win; supposing therefore that $n-1 + n-2 + n-3, \&c. = S$, we shall have the Equation $\frac{2S}{n \times n-1} = 1$, and therefore $S = \frac{n}{1} \times \frac{n-1}{2}$.

SOLUTION of the *third* Case.

By the first Remark of the tenth Problem, it appears that the Chances which the Ponte has to win and lose, may be expressed by the following Scheme.

1	$n-1 \times n-2$	for winning	0
2	$n-2 \times n-3$	for losing	y
3	$n-3 \times n-4$	for winning	1
4	$n-4 \times n-5$	for losing	1
5	$n-5 \times n-6$	for winning	1
6	$n-6 \times n-7$	for losing	1
7	$n-7 \times n-8$	for winning	1
8	$n-8 \times n-9$	for losing	1
*	2	\times 1	for winning 1

Setting aside the first, second, and last number of Chances, it will be found that the difference between the 3^d and 4th is $2n-8$; the difference between the 5th and 6th, $2n-12$; the difference between the 7th and 8th, $2n-16$, &c. Now these differences constitute an Arithmetic Progression, whereof the first Term is $2n-8$, the common difference 4, and the last Term 6, being the difference between 4×3 and 3×2 . Wherefore the Sum of this Progression is $\frac{n-1}{1} \times \frac{n-5}{2}$, to which adding the last Term 2×1 , which is favourable to the Ponte, the Sum total will be $\frac{n-1}{1} \times \frac{n-3}{2}$: but the

the Sum of all the Chances is $\frac{n}{1} \times \frac{n-1}{1} \times \frac{n-2}{3}$, as may be collected from the first Remark of the x^{th} Problem, and the last Paragraph of the second case of this Problem: therefore the Gain of the Ponte is $\frac{3 \cdot n-3 \cdot n-3}{2 \cdot n \cdot n-1 \cdot n-2}$. But his Loss upon account of the Face is $\frac{3 \cdot n-2 \cdot n-3 \cdot y}{n \cdot n-1 \cdot n-2}$ or $\frac{3y \cdot n-3}{n \cdot n-1}$, therefore his Loss upon the whole is $\frac{3y \cdot n-3}{n \cdot n-1} - \frac{3 \cdot n-3 \cdot n-3}{2 \cdot n \cdot n-1 \cdot n-2}$; or $\frac{3n-9}{2 \cdot n \cdot n-1 \cdot n-2}$ supposing $y = \frac{1}{2}$.

SOLUTION of the *fourth* Case.

The Chances of the Ponte may be expressed by the following Scheme.

1	$n-1 \times n-2 \times n-3$	for winning	0
2	$n-2 \times n-3 \times n-4$	for losing	y
3	$n-3 \times n-4 \times n-5$	for winning	1
4	$n-4 \times n-5 \times n-6$	for losing	1
5	$n-5 \times n-6 \times n-7$	for winning	1
6	$n-6 \times n-7 \times n-8$	for losing	1
7	$n-7 \times n-8 \times n-9$	for winning	1
*	3 x 2 x 1	for losing	1

Setting aside the first and second numbers of Chances, and taking the differences between the 3^d and 4th, 5th and 6th, 7th and 8th, the last of these differences will be found to be 18. Now if the number of those differences be p , and we begin from the last 18, their Sum, from the second Remark of the x^{th} Problem, will be found to be $p \times \overline{p+1} \div 2$, but p in this case is $= \frac{n-5}{2}$, and therefore the Sum of these differences will easily appear to be $\frac{n-5}{2} \times \frac{n-3}{2} \times \frac{2n-5}{1}$, but the Sum of all the Chances is $\frac{n}{1} \times \frac{n-1}{1} \times \frac{n-2}{1} \times \frac{n-3}{4}$; wherefore the Gain of the Ponte is $\frac{n-5 \cdot n-3 \cdot 2n-5}{n \cdot n-1 \cdot n-2 \cdot n-3}$: now his Loss upon account of the Face is $\frac{n-2 \cdot n-3 \cdot n-4 \cdot 4y}{n \cdot n-1 \cdot n-2 \cdot n-3}$, and therefore his Loss upon the whole will be $\frac{n-4 \cdot 4y}{n \cdot n-1} - \frac{n-5 \cdot 2n-5}{n \cdot n-1 \cdot n-2}$ or $\frac{3n-9}{n \cdot n-1 \cdot n-2}$, supposing $y = \frac{1}{2}$.

There still remains one single case to be considered, *viz.* what the Loss of the Ponte is, when he lays a Stake before the Pack is turned up : but there will be no difficulty in it, after what we have said ; the difference between this case and the rest being only, that he is liable to be faced by the first Card discovered, which will make his Loss to be $\frac{3n-6}{n \cdot n-1 \cdot n-3}$, that is, interpreting n by the number of all the Cards in the Pack, *viz.* 52, about $\frac{1}{866}$ part of his Stake.

From what has been said, a Table may easily be composed, shewing the several Losses of the Ponte in whatever circumstance he may happen to be.

0	for winning
1	for losing
2	for winning
3	for losing
4	for winning
5	for losing
6	for winning
7	for losing
8	for winning
9	for losing
10	for winning
11	for losing
12	for winning
13	for losing
14	for winning
15	for losing
16	for winning
17	for losing
18	for winning
19	for losing
20	for winning
21	for losing
22	for winning
23	for losing
24	for winning
25	for losing
26	for winning
27	for losing
28	for winning
29	for losing
30	for winning
31	for losing
32	for winning
33	for losing
34	for winning
35	for losing
36	for winning
37	for losing
38	for winning
39	for losing
40	for winning
41	for losing
42	for winning
43	for losing
44	for winning
45	for losing
46	for winning
47	for losing
48	for winning
49	for losing
50	for winning
51	for losing
52	for winning

A Table for BASSETTE.

N	1	2	3	4
52	***	***	***	866
51	***	***	1735	867
49	98	2352	1602	801
47	94	2162	1474	737
45	90	1980	1351	675
43	86	1806	1234	617
41	82	1640	1122	561
39	78	1482	1015	507
37	74	1332	914	457
35	70	1190	818	409
33	66	1056	727	363
31	62	930	642	321
29	58	812	562	281
27	54	702	487	243
25	50	600	418	209
23	46	506	354	177
21	42	420	295	147
19	38	342	242	121
17	34	272	194	97
15	30	210	151	75
13	26	156	114	57
11	22	110	82	41
9	18	72	56	28
7	14	42	35	17

The use of this Table will be best explained by some Examples.

EXAMPLE 1.

Let it be proposed to find the Loss of the Ponte, when there are 26 Cards remaining in the Stock, and his Card is twice in it.

In the Column N find the number 25, which is less by 1 than the number of Cards remaining in the Stock: over-against it, and under the number 2, which is at the head of the second Column, you will find 600; which is the Denominator of a fraction whose Numerator is Unity, and which shews that his Loss in that circumstance is one part in six hundred of his Stake.

EXAMPLE 2.

To find the Loss of the Ponte when there are eight Cards remaining in the Stock, and his Card is three times in it.

In the Column N find the number 7, less by one than the number of Cards remaining in the Stock: over-against 7, and under the number 3, written on the top of one of the Columns, you will find 35, which denotes that his Loss is one part in thirty-five of his Stake.

COROLLARY 1.

'Tis plain from the construction of the Table, that the fewer Cards are in the Stock, the greater is the Loss of the Ponte.

COROLLARY 2.

The least Loss of the Ponte, under the same circumstances of Cards remaining in the Stock, is when his Card is but twice in it; the next greater but three times; still greater when four times; and the greatest when but once. If the Loss upon the Face were varied, 'tis plain that in all the like circumstances, the Loss of the Ponte would vary accordingly; but it would be easy to compose other Tables to answer that variation; since the quantity y , which has been assumed to represent that Loss, having been preserved in the general expression of the Losses, if it be interpreted by $\frac{2}{3}$ instead of $\frac{1}{2}$, the Loss, in that case, would be as easily determined as in the other: thus supposing that 8 Cards are remaining in the Stock, and that the Card of the Ponte is twice in it, and also that y should be interpreted

interpreted by $\frac{2}{3}$, the Loss of the Ponte would be found to be $\frac{4}{63}$ instead of $\frac{1}{42}$.

The Game of PHARAON.

The Calculation for *Pharaon* is much like the preceding, the reasonings about it being the same; it will therefore be sufficient to lay down the Rules of the Play, and the Scheme of Calculation.

Rules of the Play.

First, the Banker holds a Pack of 52 Cards.

Secondly, he draws the Cards one after the other, and lays them down at his right and left-hand alternately.

Thirdly, the Ponte may at his choice set one or more Stakes upon one or more Cards, either before the Banker has begun to draw the Cards, or after he has drawn any number of couples.

Fourthly, the Banker wins the Stake of the Ponte, when the Card of the Ponte comes out in an odd place on his right-hand; but loses as much to the Ponte when it comes out in an even place on his left-hand.

Fifthly, the Banker wins half the Ponte's Stake, when it happens to be twice in one couple.

Sixthly, when the Card of the Ponte being but once in the Stock, happens to be the last, the Ponte neither wins nor loses.

Seventhly, the Card of the Ponte being but twice in the Stock, and the last couple containing his Card twice, he then loses his whole Stake.

P R O B L E M XIV.

To find at Pharaon the Gain of the Banker in any circumstance of Cards remaining in the Stock, and of the number of times that the Ponte's Cards is contained in it.

This Problem having four Cases, that is, when the Ponte's Card is *once, twice, three, or four* times in the Stock; we shall give the Solution of these four cases severally.

S O L U -

SOLUTION of the *first* Case.

The Banker has the following number of Chances for winning and losing.

1	1	Chance for winning	1
2	1	Chance for losing	1
3	1	Chance for winning	1
4	1	Chance for losing	1
5	1	Chance for winning	1
*	1	Chance for losing	0

Wherefore, the Gain of the Banker is $\frac{1}{n}$, supposing n to be the number of Cards in the Stock.

SOLUTION of the *second* Case.

The Banker has the following Chances for winning and losing.

1	{	$n-2$	Chances for winning	1
		1	Chance for winning	y
2		$n-2$	Chances for losing	1
3	{	$n-4$	Chances for winning	1
		1	Chance for winning	y
4		$n-4$	Chances for losing	1
5	{	$n-6$	Chances for winning	1
		1	Chance for winning	y
6		$n-6$	Chances for losing	1
7	{	$n-8$	Chances for winning	1
		1	Chance for winning	y
8		$n-8$	Chances for losing	1
*			1 Chance for winning	1

The Gain of the Banker is therefore $\frac{\overline{n-2} \cdot y}{n \cdot n-1} + \frac{2}{n \cdot n-1}$, or $\frac{\frac{1}{2}n+1}{n \cdot n-1}$ supposing $y = \frac{1}{2}$.

The only thing that deserves to be explained here, is this; how it comes to pass, that whereas at *Bassette*, the first number of Chances for winning was represented by $n-1$, here 'tis represented by $n-2$; to answer this, it must be remembered, that according to the Law
of

of this Play, if the Ponte's Cards come out in an odd place, the Banker is not thereby entitled to the Ponte's whole Stake: for if it so happens that his Card comes out again immediately after, the Banker wins but one half of it; therefore the number $n - 1$ is divided into two parts, $n - 2$ and 1 , whereof the first is proportional to the Probability which the Banker has for winning the whole Stake of the Ponte, and the second is proportional to the Probability of winning the half of it.

SOLUTION of the *third* Case.

The number of Chances which the Banker has for winning and losing, are as follow :

1	{	$n - 2 \times n - 3$	Chances for winning	1
		$2 \times n - 2$	Chances for winning	y
2		$n - 2 \times n - 3$	Chances for losing	1
3	{	$n - 4 \times n - 5$	Chances for winning	1
		$2 \times n - 4$	Chances for winning	y
4		$n - 4 \times n - 5$	Chances for losing	1
5	{	$n - 6 \times n - 7$	Chances for winning	1
		$2 \times n - 6$	Chances for winning	y
6		$n - 6 \times n - 7$	Chances for losing	1
7	{	$n - 8 \times n - 9$	Chances for winning	1
		$2 \times n - 8$	Chances for winning	y
*		2×1	Chances for losing	1

Wherefore the Gain of the Banker is $\frac{3y}{2 \cdot n - 1}$, or $\frac{3}{4 \cdot n - 1}$ supposing $y = \frac{1}{2}$.

The number of Chances for the Banker to win, is divided into two parts, whereof the first expresses the number of Chances he has for winning the whole Stake of the Ponte, and the second for winning the half of it.

Now for determining exactly those two parts, it is to be considered, that in the first couple of Cards that are laid down by the Banker, the number of Chances for the first Card to be the Ponte's is $n - 1 \times n - 2$; also, that the number of Chances for the second to be the Ponte's, but not the first, is $n - 2 \times n - 3$: wherefore the number of Chances for the first to be the Ponte's, but not the second, is likewise $n - 2 \times n - 3$. Hence it follows, that if from
the

the number of Chances for the first Card to be the Ponte's, *viz.* from $n-1 \times n-2$, there be subtracted the number of Chances for the first to be the Ponte's, and not the second, *viz.* $n-2 \times n-3$, there will remain the number of Chances for both first and second Cards to be the Ponte's, *viz.* $2 \times n-2$, and so for the rest.

SOLUTION of the *fourth* Case.

The number of Chances which the Banker has for winning and losing, are as follow :

1	{	$n-2 \times n-3 \times n-4$	for winning	1
		$3 \times n-2 \times n-3$	for winning	y
2		$n-2 \times n-3 \times n-4$	for losing	1
3	{	$n-4 \times n-5 \times n-6$	for winning	1
		$3 \times n-4 \times n-5$	for winning	y
4		$n-4 \times n-5 \times n-6$	for losing	1
5	{	$n-6 \times n-7 \times n-8$	for winning	1
		$3 \times n-6 \times n-7$	for winning	y
6		$n-6 \times n-7 \times n-8$	for losing	1
7	{	$n-8 \times n-9 \times n-10$	for winning	1
		$3 \times n-8 \times n-9$	for winning	y
8		$n-8 \times n-9 \times n-10$	for losing	1
*	{	$2 \times 1 \times 0$	for winning	1
		$3 \times 2 \times 1$	for winning	y
		$2 \times 1 \times 0$	for losing	1

Wherefore the Gain of the Banker, or the Loss of the Ponte, is $\frac{2n-5}{n-1 \cdot n-3} y$ or $\frac{2n-5}{2 \times n-1 \cdot n-3}$ supposing y to be $= \frac{1}{2}$.

It will be easy, from the general expressions of the Losses, to compare the disadvantage of the Ponte at *Bassette* and *Pharaon*, under the same circumstances of Cards remaining in the hands of the Banker, and of the number of times that the Ponte's Card is contained in the Stock; but to save that trouble, I have thought fit here to annex a Table of the Gain of the Banker, or Loss of the Ponte, for any particular circumstance of the Play, as it was done for *Bassette*.

A Ta-

A Table for PHARAON.

N	1	2	3	4
52	***	***	***	*50
50	***	94	65	48
48	48	90	62	46
46	46	86	60	44
44	44	82	57	42
42	42	78	54	40
40	40	74	52	38
38	38	70	49	36
36	36	66	46	34
34	34	62	44	32
32	32	58	41	30
30	30	54	38	28
28	28	52	36	26
26	26	46	33	24
24	24	42	30	22
22	22	38	28	20
20	20	34	25	18
18	18	30	22	16
16	16	26	20	14
14	14	22	17	12
12	12	18	14	10
10	10	14	12	8
8	8	11	9	6

The numbers of the foregoing Table, as well as those of the Table for *Bassette*, are sufficiently exact to give at first view an idea of the advantage of the Banker in all circumstances, and the Method of using it is the same as that which was given for *Bassette*. It is to be observed at this Play, that the least disadvantage of the Ponte, under the same circumstances of Cards remaining in the Stock, is when the Card of the Ponte is but twice in it, the next greater when three times, the next when once, and the greatest when four times.

Of PERMUTATIONS and COMBINATIONS.

Permutations are the Changes which several things can receive in the different orders in which they may be placed, being considered as taken two and two, three and three, four and four, &c.

Combinations are the various Conjunctions which several things may receive without any respect to order, being taken two and two, three and three, four and four.

The Solution of the Problems that relate to Permutations and Combinations depending entirely upon what has been said in the 8th and 9th Articles of the Introduction, if the Reader will be pleased to consult those Articles with attention, he will easily apprehend the reason of the Steps that are taken in the Solution of those Problems.

P R O B L E M XV.

Any number of things a, b, c, d, e, f, being given, out of which two are taken as it happens: to find the Probability that any of them, as a, shall be the first taken, and any other, as b, the second.

SOLUTION.

The number of Things in this Example being six, it follows that the Probability of taking *a* in the first place is $\frac{1}{6}$: let *a* be considered as taken, then the Probability of taking *b* will be $\frac{1}{5}$; wherefore the Probability of taking *a*, and then *b*, is $\frac{1}{6} \times \frac{1}{5} = \frac{1}{30}$:

COROLLARY.

Since the taking a in the first place, and b in the second, is but one single Case of those by which six Things may change their order, being taken two and two; it follows that the number of Changes or Permutations of six Things, taken two and two, must be 30.

Universally; let n be the number of Things; then the Probability of taking a in the first place, and b in the second will be $\frac{1}{n} \times \frac{1}{n-1}$; and the number of Permutations of those Things, taken two and two, will be $n \times n - 1$.

PROBLEM XVI.

Any number of Things a, b, c, d, e, f , being given, out of which three are taken as it happens; to find the Probability that a shall be the first taken, b the second, and c the third.

SOLUTION.

The Probability of taking a in the first place is $\frac{1}{6}$: let a be considered as taken, then the Probability of taking b will be $\frac{1}{5}$: suppose now both a and b taken, then the Probability of taking c will be $\frac{1}{4}$: wherefore the Probability of taking first a , then b , and thirdly c , will be $\frac{1}{6} \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{120}$.

COROLLARY.

Since the taking a in the first place, b in the second, and c in the third, is but one single Case of those by which six Things may change their Order, being taken three and three; it follows, that the number of Changes or Permutations of six Things taken three and three, must be $6 \times 5 \times 4 = 120$.

Universally, if n be the number of Things; the Probability of taking a in the first place, b in the second, and c in the third, will be $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}$; and the number of Permutations of n Things taken three and three, will be $n \times n - 1 \times n - 2$.

GENERAL COROLLARY.

The number of Permutations of n things, out of which as many are taken together as there are Units in p , will be $n \times n - 1 \times n - 2 \times n - 3$, &c. continued to so many Terms as there are Units in p .

Thus the number of Permutations of six Things taken four and four, will be $6 \times 5 \times 4 \times 3 = 360$, likewise the number of Permutations of six Things taken all together will be $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

PROBLEM XVII.

To find the Probability that any number of things, whereof some are repeated, shall all be taken in any order proposed: for instance, that aabbbccccc shall be taken in the order wherein they are written.

SOLUTION.

The probability of taking a in the first place is $\frac{2}{9}$; suppose one a to be taken, the Probability of taking the other is $\frac{1}{8}$. Let now the two first Letters be supposed taken, the Probability of taking b will be $\frac{3}{7}$: let this be also supposed taken, the Probability of taking another b will be $\frac{2}{6}$: let this be supposed taken, the Probability of taking the third b will be $\frac{1}{5}$; after which there remaining nothing but the Letter c , the Probability of taking it becomes a certainty, and consequently is expressed by Unity. Wherefore the Probability of taking all those Letters in the order given is $\frac{2}{9} \times \frac{1}{8} \times \frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} \times 1 = \frac{1}{1260}$.

COROLLARY 1.

The number of Permutations which the Letters $aabbbccccc$ may receive being taken all together will be $\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 1260$.

COROLLARY 2.

The same Letters remaining, the Probability of the Letters being taken in any other given Order will be just the same as before: thus

thus the Probability of those Letters being taken in the order *cabaccbb* will be $\frac{1}{1260}$.

GENERAL COROLLARY.

The number of Permutations which any number n of Things may receive being taken all together, whereof the first Sort is repeated p times, the second q times, the third r times, the fourth s times, &c. will be the Series $n \times n - 1 \times n - 2 \times n - 3 \times n - 4$, &c. continued to so many Terms as there are Units in $p + q + r$ or $n - s$ divided by the product of the following Series; viz. $p \times p - 1 \times p - 2$, &c. $q \times q - 1 \times q - 2$, &c. $r \times r - 1 \times r - 2$, &c. whereof the first must be continued to so many Terms as there are Units in p , the second to so many Terms as there are Units in q , the third to so many as there are Units in r , &c.

PROBLEM XVIII.

Any number of Things a, b, c, d, e, f, being given: to find the Probability that in taking two of them as it may happen, both a and b shall be taken, without any regard to order.

SOLUTION.

The Probability of taking a or b in the first place will be $\frac{2}{6}$; suppose one of them taken, as for instance a , then the Probability of taking b will be $\frac{1}{5}$. Wherefore the Probability of taking both a and b will be $\frac{2}{6} \times \frac{1}{5}$.

COROLLARY.

The taking of both a and b is but one single Case of all those by which six Things may be combined two and two; wherefore the number of Combinations of six Things taken two and two will be $\frac{6}{1} \times \frac{5}{2}$.

Universally. The number of Combinations of n Things taken two and two will be $\frac{n}{1} \times \frac{n-1}{2}$.

PRO-

P R O B L E M XIX.

Any number of things a, b, c, d, e, f being given, to find the Probability that in taking three of them as they happen, they shall be any three proposed, as a, b, c, no respect being had to order.

SOLUTION.

The Probability of taking either *a*, or *b*, or *c*, in the first place, will be $\frac{3}{6}$; suppose one of them as *a* to be taken, then the Probability of taking either *b* or *c* in the second place will be $\frac{2}{5}$; again, let either of them be taken, suppose *b*, then the Probability of taking *c* in the third place will be $\frac{1}{4}$; wherefore the Probability of taking the three things proposed, viz. *a, b, c*, will be $\frac{3}{6} \times \frac{2}{5} \times \frac{1}{4}$.

COROLLARY.

The taking of *a, b, c*, is but one single case of all those by which six Things may be combined three and three; wherefore the number of Combinations of six Things taken three and three will be $\frac{6}{1} \times \frac{5}{2} \times \frac{4}{3} = 20$.

Universally. The number of Combinations of *n* things combined according to the number *p*, will be the fraction $\frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ &c. both Numerator and Denominator being continued to so many Terms as there are Units in *p*.

P R O B L E M XX.

To find what Probability there is, that in taking at random seven Counters out of twelve, whereof four are white and eight black, three of them shall be white ones.

SOLUTION.

First, Find the number of Chances for taking three white out of four, which will be $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} = 4$.

Secondly,

Secondly, Find the number of Chances for taking four black out of eight: these Chances will be found to be $\frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} \times \frac{5}{4} = 70$.

Thirdly, Because every one of the first Chances may be joined with every one of the latter, it follows that the number of Chances for taking three white, and four black, will be $4 \times 70 = 280$.

Fourthly, Altho' the case of taking four white and three black, be not mentioned in the Problem, yet it is to be understood to be implied in it; for according to the Law of Play, he who does more than he undertakes, is still reputed a winner, unless the contrary be expressly stipulated; let therefore the case of taking four white out of four be calculated, and it will be found $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} \times \frac{1}{4} = 1$.

Fifthly, Find the Chances for taking three black Counters out of eight, which will be found to be $\frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} = 56$.

Sixthly, Multiply the two last numbers of Chances together, and the Product 56 will denote the number of Chances for taking four white and three black.

And therefore the whole number of Chances, which answer to the conditions of the Problem, are $280 + 56 = 336$.

There remains now to find the whole number of Chances for taking seven Counters out of twelve, which will be $\frac{12}{1} \times \frac{11}{2} \times \frac{10}{3} \times \frac{9}{4} \times \frac{8}{5} \times \frac{7}{6} \times \frac{6}{7} = 792$.

Lastly, Divide therefore 336 by 792, and the Quotient $\frac{336}{792}$ or $\frac{14}{33}$ will express the Probability required; and this Fraction being subtracted from Unity, the remainder will be $\frac{19}{33}$, and therefore the Odds against taking three white Counters are 19 to 14.

COROLLARY.

Let a be the number of white Counters, b the number of black, n the whole number of Counters $= a + b$, c the number of Counters to be taken out of the number n ; let also p represent the number of white Counters to be found precisely in c , then the number of Chances for taking none of the white, or one single white, or two

two white and no more, or three white and no more, or four white and no more, &c. will be expressed as follows ;

$$1 \times \frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} \times \frac{a-3}{4}, \text{ \&c. } \times \frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3}, \text{ \&c.}$$

The number of Terms in which a enters being equal to the number p , and the number of Terms in which b enters being equal to the number $c-p$.

And the number of all the Chances for taking a certain number c of Counters out of the number n , is expressed by the Series $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}, \text{ \&c.}$ to be continued to as many Terms as there are Units in c , for a Denominator.

EXAMPLES.

Suppose as in the last problem ; only that of the 7 Counters drawn, there shall not be one white. In this Case, since $p=0$, and $c-p=7=b-1$: we are to take 1 of the first Series, and 7 (or 1) Terms of the second ; which gives the number of Chances 1×8 ; the Ratio of which to all the 7's that can be taken out of 12, is $\frac{8}{792} = \frac{1}{99}$. So that there is the Odds of 98 to 1, that there shall be one or more white Counters among the 7 that are drawn.

Again, if there is to be 1 white Counter and no more, we are now to take the Terms $1 \times \frac{a}{1} \dots \times \frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} \times \frac{b-3}{4} \times \frac{b-4}{5} \times \frac{b-5}{6} = 4 \dots \times \frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} \times \frac{5}{4} \times \frac{4}{5} \times \frac{3}{6} = 4 \times \frac{8 \times 7}{2} = \frac{112}{1}$: Which gives the probability $\frac{112}{792} = \frac{14}{99}$; or the odds 85 to 14 ; that there shall be more than 1 white Counter, or that all the 7 shall be black.

Lastly, If it is undertaken to draw all the 4 white among the seven, the Number of Chances will be $1 \times \frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} = 56$. And the Probability $\frac{56}{792} = \frac{7}{99}$; that is, the Odds of 92 to 7 that there shall be, of the 7 drawn, fewer than 4 white Counters, or none at all.

REMARK.

If the numbers n and c were large, such as $n=40000$ and $c=8000$, the foregoing Method would seem impracticable, by reason of the vast number of Terms to be taken in both Series, whereof the first is to be divided by the second : tho' if those Terms were

were actually set down, a great many of them being common Divisors might be expunged out of both Series; for which reason it will be convenient to use the following Theorem, which is a contraction of that Method, and which will be chiefly of use when the white Counters are but few.

Let therefore n be the number of all the Counters; a the number of white; c the number of Counters to be taken out of the number n ; p the number of the white that are to be taken precisely in the number c ; then making $n - c = d$. The Probability of taking precisely the number p of white Counters, will be

$$\frac{c \cdot c-1 \cdot c-2, \&c. \times d \cdot d-1 \cdot d-2, \&c. \times \frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3}, \&c.}{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4 \cdot n-5 \cdot n-6 \cdot n-7 \cdot n-8, \&c.}$$

Here it is to be observed, that the Numerator consists of three Series which are to be multiplied together; whereof the first contains as many Terms as there are Units in p ; the second as many as there are Units in $a-p$; the third as many as there are Units in p ; and the Denominator as many as there are Units in a .

P R O B L E M XXI.

In a Lottery consisting of 40000 Tickets, among which are three particular Benefits, what is the Probability that taking 8000 of them, one or more of the particular Benefits shall be amongst them.

SOLUTION.

First, In the Theorem belonging to the Remark of the foregoing Problem, having substituted respectively 8000, 40000, 32000, 3 and 1, in the room of c, n, d, a , and p ; it will appear that the Probability of taking one precisely of the three particular Benefits, will be

$$\frac{8000 \cdot 32000 \cdot 31999 \cdot 3}{40000 \cdot 39999 \cdot 39998} = \frac{48}{125} \text{ nearly.}$$

Secondly, c, n, d, a being interpreted as before, let us suppose $p = 2$: hence the Probability of taking precisely two of the particular Benefits will be found to be

$$\frac{8000 \cdot 7000 \cdot 32000 \cdot 3}{40000 \cdot 39999 \cdot 39998} = \frac{12}{125} \text{ nearly.}$$

N

Thirdly,

Thirdly, making $p = 3$, the Probability of taking all the three particular Benefits will be found to be $\frac{8000 \cdot 7999 \cdot 7998}{40000 \cdot 39999 \cdot 39998} = \frac{1}{125}$.

Wherefore the Probability of taking one or more of the three particular Benefits will be $\frac{48+12+1}{125}$ or $\frac{61}{125}$ very near.

It is to be observed, that those three Operations might have been contracted into one, by inquiring the Probability of not taking any of the three particular Benefits, which will be found to be $\frac{32000 \cdot 31999 \cdot 31998}{40000 \cdot 39999 \cdot 39998} = \frac{64}{125}$ nearly, which being subtracted from Unity, the remainder $1 - \frac{64}{125}$ or $\frac{61}{125}$ will shew the Probability required, and therefore the Odds against taking any of three particular Benefits will be 64 to 61 nearly.

PROBLEM XXII.

To find how many Tickets ought to be taken in a Lottery consisting of 40000, among which are Three particular Benefits, to make it as probable that one or more of those Three may be taken as not.

SOLUTION.

Let the number of Tickets requisite to be taken be $= x$; it will follow therefore from the Remark belonging to the xxth Problem, that the Probability of not taking any of the particular Benefits will be $\frac{n-x}{n} \times \frac{n-x-1}{n-1} \times \frac{n-x-2}{n-2}$; but this Probability is equal to $\frac{1}{2}$, since by Hypothesis the Probability of taking one or more of them is equal to $\frac{1}{2}$, from whence we shall have the Equation $\frac{n-x}{n} \times \frac{n-x-1}{n-1} \times \frac{n-x-2}{n-2} = \frac{1}{2}$, which Equation being solved, the Value of x will be found to be nearly 8252.

N. B. The Factors whereof both the Numerator and Denominator are composed, being but few, and in arithmetic progression; and besides, the difference being very small in respect of n ; those Terms may be considered as being in geometric Progression: wherefore the Cube of the middle Term $\frac{n-x-1}{n-1}$, may be supposed equal to the product of the Multiplication of those Terms; from whence

whence will arise the Equation $\frac{(n-x-1)^3}{n-1} = \frac{1}{2}$; or, neglecting Unity in both Numerator and Denominator, $\frac{(n-x)^3}{n^3} = \frac{1}{2}$ and consequently x will be found to be $= n \times 1 - \sqrt[3]{\frac{1}{2}}$ or $n \times 1 - \frac{1}{2}\sqrt[3]{4}$, but $n = 40000$, and $1 - \frac{1}{2}\sqrt[3]{4} = 0.2063$; wherefore $x = 8252$.

In the Remark belonging to the second Problem, a Rule was given for finding the number of Tickets that were to be taken to make it as probable, that one or more of the Benefits would be taken as not; but in that Rule it was supposed, that the proportion of the Blanks to the Prizes was often repeated, as it usually is in Lotteries: now in the case of the present Problem, the particular Benefits being but three in all, the remaining Tickets are to be considered as Blanks in respect of them; from whence it follows, that the proportion of the number of Blanks to one Prize being very near as 13332 to 1, and that proportion being repeated but three times in the whole number of Tickets, the Rule there given would not have been sufficiently exact, for which reason it was thought necessary to give another Rule in this place.

PROBLEM XXIII.

Supposing a Lottery of 100000 Tickets, whereof 90000 are Blanks, and 10000 are Benefits, to determine accurately what the odds are of taking or not taking a Benefit, in any number of Tickets assigned.

SOLUTION.

Suppose the number of Tickets to be 6; then let us inquire into the Probability of taking no Prize in 6 Tickets, which to find let us make use of the Theorem set down in the Corollary of the xxth Problem, wherein it will appear that the number of Chances for taking no Prize in 6 Tickets, making $a = 10000$, $b = 90000$, $c = 6$, $p = 0$, $n = 100000$, will be

$$\frac{90000}{1} \times \frac{89999}{2} \times \frac{89998}{3} \times \frac{89997}{4} \times \frac{89996}{5} \times \frac{89995}{6},$$

and that the whole number of Chances will be

$$\frac{100000}{1} \times \frac{99999}{2} \times \frac{99998}{3} \times \frac{99997}{4} \times \frac{99996}{5} \times \frac{99995}{6}; \text{ then}$$

N 2

dividing

dividing the first number of Chances by the second, which may easily be done by Logarithms, the Quotient will be 0.53143, and this shews the Probability of taking no Prize in 6 Tickets: and this decimal fraction being subtracted from Unity, the Remainder 0.46857 shews the Probability of taking one Prize or more in 6 Tickets; wherefore the Odds against taking any Prize in 6 Tickets, will be 53143 to 46857.

If we suppose now that the number of Tickets taken is 7, then carrying each number of Chances above-written one step farther, we shall find that the Probability of taking no Prize in 7 Tickets is 0.47828, which fraction being subtracted from Unity, the remainder will be 0.52172, which shews the Odds of taking one Prize or more in 7 Tickets to be 52172 to 47828.

REMARK.

When the number of Tickets taken bear a very inconsiderable proportion to the whole number of Tickets, as it happens in the case of this Problem, the Question may be resolved as a Problem depending on the Cast of a Die: we may therefore suppose a Die of 10 Faces having one of its Faces such as the Ace representing a Benefit, and all the other nine representing Blanks, and inquire into the Probability of missing the Ace 6 times together, which by the Rules given in the Introduction, will be found to be $\frac{9^6}{10^6} = 0.53144$ differing from what we had found before but one Unit in the fifth place of Decimals. And if we inquire into the Probability of missing the Ace 7 times, we shall find it 0.47829 differing also but one Unit in the fifth of Decimals, from what had been found before, and therefore in such cases as this we may use both Methods indifferently; but the first will be exact if we actually multiply the numbers together, the second is only an approximation.

But both Methods confirm the truth of the practical Rule given in our third Problem, about finding what number of Tickets are necessary for the equal Chance of a Prize; for multiplying as it is there directed, the number 9 representing the Blanks by 0.7, the Product 6.3 will shew that the number requisite is between 6 and 7.

PROBLEM XXIV.

The same things being given as in the preceding Problem, suppose the price of each Ticket to be 10 L. and that after the Lottery is drawn, 7 L. — 10 s. be returned

to the Blanks, to find in this Lottery the value of the Chance of a Prize.

SOLUTION.

There being 90000 Blanks, to every one of which 7 *L.* — 10 *ſb.* is returned, the total Value of the Blanks is 675000 *L.* and consequently the total Value of the Benefits is 325000 *L.* which being divided by 10000, the number of the Benefits, the Quotient is 32 *L.* — 10 *ſb.*; and therefore one might for the Sum of 32 *L.* — 10 *ſb.* be intitled to have a Benefit certain, taken at random out of the whole number of Benefits: the Purchaser of a Chance has therefore 1 Chance in 10 for the Sum of 32 *L.* — 10 *ſb.* and 9 Chances in 10 for losing his Money; from whence it follows, that the value of his Chance is the 10th part of 32 *L.* — 10 *ſb.* that is 3 *L.* — 5 *ſb.* And therefore the Purchaser of a Chance, by giving the Seller 3 *L.* — 5 *ſb.* is intitled to the Chance of a Benefit, and ought not to return any thing to the Seller, altho' he should have a Prize; for the Seller having 3 *L.* — 5 *ſb.* sure, and 9 Chances in 10 for 7 *L.* — 10 *ſb.* the Value of which Chances is 6 *L.* — 15 *ſb.*; it follows that he has his 10 *L.*

P R O B L E M XXV.

Supposing still the same Lottery as has been mentioned in the two preceding Problems, let A engage to furnish B with a Chance, on condition that whenever the Ticket on which the Chance depends, shall happen to be drawn, whether it proves a Blank or a Prize, A shall furnish B with a new Chance, and so on, as often as there is occasion, till the whole Lottery be drawn; to find what consideration B ought to give A before the Lottery begins to be drawn, for the Chance or Chances of one or more Prizes, admitting that the Lottery will be 40 days a drawing.

SOLUTION.

Let 3 *L.* — 5 *ſb.*, which is the absolute Value of a Chance, be called *s.*

1°. *A* who is the Seller ought to consider, that the first Day, he furnishes necessarily a Chance whose Value is s .

2°. That the second day, he does not necessarily furnish a Chance, but conditionally, *viz.* if it so happen that the Ticket on which the Chance depends, should be drawn on the first day; but the Probability of its being drawn on the first day is $\frac{1}{40}$; and therefore he ought to take $\frac{1}{40}s$ for the consideration of the second day.

3°. That in the same manner, he does not necessarily furnish a Chance on the third day, but conditionally, in case the only Ticket depending (for there can be but one) should happen to be drawn on the second day; of which the Probability being $\frac{1}{39}$, by reason of the remaining 39 days from the second inclusive to the last, it follows, that the Value of that Chance is $\frac{1}{39}s$.

4°. And for the same reason, the Value of the next is $\frac{1}{38}s$, and so on.

The Purchaser ought therefore to give the Seller

$1 + \frac{1}{40} + \frac{1}{39} + \frac{1}{38} + \frac{1}{37} \dots \dots \dots + \frac{1}{2}$, the whole multiplied by s , or

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \dots \dots \dots + \frac{1}{40}$, the whole

multiplied by s . Now it being pretty laborious to sum up those 40 Terms, I have here made use of a Rule which I have given in the Supplement to my *Miscellanea Analytica* *, whereby one may in a very short time sum up as many of those Terms as one pleases, tho' they were 10000 or more; and by that Rule, the Sum of those 40 Terms will be found to be 4.2785 very near, which being multiplied by s which in this case is 3.25, the product 13.9 will shew that the Purchaser ought to give the Seller about 13 *L.* — 18 *lb.*

COROLLARY.

The Value of the Chance s for one single day that shall be fixed upon, is the Value of that Chance divided by the number of Days intercepted between that Day inclusive, and the number of Days remaining to the end of the Lottery: which however must be understood with this restriction, that the Day fixed upon must be chose before the Lottery begins; or if it be done on any other Day, the State
of

of the Lottery must be known, and a new Calculation made accordingly for the Value of s .

* SCHOLIUM.

If there is a Series of Fractions of this Form $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} \dots + \frac{1}{a-1}$; the first of which is $\frac{1}{n}$, and the last $\frac{1}{a-1}$; their Sum will be,

$$\log. \frac{a}{n} + \frac{1}{2n} + \frac{1}{2n^2} A + \frac{1}{4n^4} B + \frac{1}{6n^6} C + \frac{1}{8n^8} D + \&c. \quad \text{Where it is}$$

$$- \frac{1}{2a} + \frac{1}{2a^2} A + \frac{1}{4a^4} B + \frac{1}{6a^6} C + \frac{1}{8a^8} D + \&c.$$

to be observed,

1°. That the mark (*log.*) denoting *Neper's*, or the *Natural*, Logarithm, affects only the first Term $\frac{a}{n}$.

2°. That the Values of the Capital Letters are, $A = \frac{1}{6}$, $B = -\frac{1}{30}$, $C = +\frac{1}{42}$, $D = -\frac{1}{30}$, $E = +\frac{5}{66}$, &c. being the numbers of Mr. *James Bernoulli* in his excellent Theorem for the Summing of Powers; which are formed from each other as follows;

$$A = \frac{1}{2} - \frac{1}{3}$$

$$B = \frac{1}{2} - \frac{1}{5} - \frac{4}{2} A.$$

$$C = \frac{1}{2} - \frac{1}{7} - \frac{6}{2} A - \frac{6 \cdot 5 \cdot 4}{2 \cdot 3 \cdot 4} B.$$

$$D = \frac{1}{2} - \frac{1}{9} - \frac{8}{2} A - \frac{8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 4} B - \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} C.$$

$$E = \frac{1}{2} - \frac{1}{11} - \frac{10}{2} A - \frac{10 \cdot 9 \cdot 8}{2 \cdot 3 \cdot 4} B - \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} C - \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} D.$$

&c.

3°. In working by this Rule, it will be convenient to sum a few of the first terms, in the common way; that the powers of $\frac{1}{n}$ may the sooner converge.

4°. The same Rule furnishes an easy Computation of the Logarithm of any ratio $\frac{a}{n}$, the difference of whose terms is not very great.

P R O B L E M XXVI.

To find the Probability of taking four Hearts, three Diamonds, two Spades, and one Club in ten Cards out of a Stock containing thirty-two.

SOLUTION.

First, The number of Chances for taking four Hearts out of the whole number of Hearts that are in the Stock, that is out of Eight, will be $\frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 70$.

Secondly, The number of Chances for taking three Diamonds out of Eight, will be $\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$.

Thirdly, The number of Chances for taking two Spades out of Eight, will be $\frac{8 \cdot 7}{1 \cdot 2} = 28$.

Fourthly, The number of Chances for taking one Club out of Eight, will be $\frac{8}{1} = 8$.

And therefore multiplying all those particular Chances together, the product $70 \times 56 \times 28 \times 8 = 878080$ will denote the whole number of Chances for taking four Hearts, three Diamonds, two Spades, and one Club.

Fifthly, The whole number of Chances for taking any ten Cards out of thirty-two is

$$\frac{32 \cdot 31 \cdot 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} = 64512240.$$

And therefore dividing the first Product by the second, the quotient $\frac{878080}{64512240}$ or $\frac{1}{75}$ nearly, will express the Probability required; from which it follows that the Odds against taking four Hearts, three Diamonds, two Spades, and one Club in ten Cards, out of a Stock containing thirty-two, are very near 74 to 1.

R E M A R K.

But if the numbers in this Problem had not been restricted each to a particular suit of Cards; that is, if it had been undertaken only that in drawing the ten Cards, 4 of them should be of one suit, 3 of another, 2 of another, and one of the fourth; then writing for the four suits, the Letters *A . B . C . D*; and under them the Numbers

4 . 3 . 2 . 1; since this
is

is but one Position out of 24, which the numbers can have with respect to the Letters (by the general Corollary to Prob. xvi) we must now multiply the number of Chances before found, which was 878080, by 24; and the probability required will be $\frac{2107392}{6451224}$; that is, it is the Odds of about 2 to 1, or very nearly of 68 to 33, that of 10 Cards drawn out of a *Piquet* pack *four, three, two, and one*, shall not be of different suits.

Of the Game of QUADRILLE.

PROBLEM XXVII.

The Player having 3 Matadors and three other Trumps by the lowest Cards in black or red, what is the Probability of his forcing all the Trumps?

SOLUTION.

In order to solve this Problem, it is to be considered, that the Player whom I call *A* forces the Trumps necessarily, if none of the other Players whom I call *B, C, D*, has more than three Trumps; and therefore, if we calculate the Probability of any one of them having more than three Trumps, which case is wholly against *A*, we may from thence deduce what will be favourable to him; but let us first suppose that he plays in black.

Since the number of Trumps in black is 11, and that *A* by supposition has 6 of them, then the number of Trumps remaining amongst *B, C, D* is 5; and again, since the number of all the other remaining Cards, which we may call Blanks, is 29, whereof *A* has 4, it follows that there are 25 Blanks amongst *B, C, D*; and therefore the number of Chances for *B* in his 10 Cards to have 4 Trumps and 6 Blanks, is by the Corollary of the xxth Problem.

$$\frac{5 \times 4 \times 3 \times 2}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

And likewise the number of Chances for his having 5 Trumps and 5 Blanks, is by the same Corollary.

$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

And therefore the number of all the Chances of *B* against *A* is $106 \times 5 \times 7 \times 11 \times 23$: but the number of Chances whereby any 10 Cards may be taken out of 30 is $\frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}$ which being reduced to $5 \times 7 \times 9 \times 11 \times 13 \times 23 \times 29$, it follows that the Probability of *B*'s having more than three Trumps is $\frac{106 \times 5 \cdot 7 \cdot 11 \cdot 23}{3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 29} = \frac{106}{9 \times 13 \cdot 29}$: but this Probability falls as well upon *C* and *D* as upon *B*, and therefore it ought to be multiplied by 3, which will make it $\frac{106}{3 \times 13 \times 29} = \frac{106}{1131}$; and this being subtracted from Unity, the remainder $\frac{1025}{1131}$ will express the Probability of *A*'s forcing all the Trumps; and therefore the Odds of his forcing the Trumps are 1025 against 106, that is 29 to 3 nearly.

But if *A* plays the same Game in red, his advantage will be considerably less than before; for there being 12 Trumps in red, whereof he has 6, *B* may have 4, or 5, or 6 of them, so that the number of the Chances which *B* has for more than three Trumps will be respectively as follows:

$$\begin{array}{l} \frac{6 \times 5 \times 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \\ \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \times \frac{24 \cdot 23 \cdot 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4} \end{array}$$

Now the Sum of all those Chances being $215 \times 23 \times 22 \times 21$, and the Sum of all the Chances for taking any 10 Cards out of 30, being $5 \times 7 \times 9 \times 11 \times 13 \times 23 \times 29$, as appears by the preceding case, it follows, that the Probability of *B*'s having more than three Trumps is $\frac{215 \cdot 23 \cdot 22 \cdot 21}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 23 \cdot 29} = \frac{86}{3 \cdot 13 \cdot 29}$; but this Probability falling as well upon *C* and *D*, as upon *B*, ought to be multiplied by 3, which will make it $\frac{86}{13 \cdot 29} = \frac{86}{377}$; and this being subtracted from Unity, the remainder $\frac{291}{377}$, will express the Probability of *A*'s forcing all the Trumps; and therefore the Odds of his forcing all the Trumps is in this case 291 to 86, that is nearly 10 to 3.

PROBLEM XXVIII.

The Player A having Spadille, Manille, King, Queen, and two small Trumps in black, to find the Probability of his forcing all the Trumps.

SOLUTION.

A forces the Trumps necessarily, if *Baste* accompanied with two other Trumps be not in one of the Hands of B, C, D, and as *Baste* ought to be in some Hand, it is indifferent where we place it; let it therefore be supposed that B has it, in consequence of which let us consider the number of Chances for his having besides *Baste*,

- 1°. 2 Trumps and 7 Blanks.
- 2°. 3 Trumps and 6 Blanks.
- 3°. 4 Trumps and 5 Blanks.

Now the Blanks being in all 29, whereof A has 4, it follows that the number of remaining Blanks is 25; and the number of Trumps being in all 11, whereof A has 6 by Hypothesis, and B has 1, viz. *Baste*, it follows that the number of remaining Trumps is 4; and therefore the Chances which B has against the Player are respectively as follows:

$$\begin{array}{l} \frac{4 \cdot 3}{1 \cdot 2} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \\ \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \\ \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \end{array}$$

The Sum of all which is $1441 \times 5 \times 23 \times 22$; but the Sum of all the Chances whereby B may join any 9 Cards to the *Baste* which he has already is $\frac{29 \times 28 \times 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \times 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} = 29 \times 7 \times 3 \times 13 \times 5 \times 23 \times 11$. and therefore the Probability of *Baste* being in one Hand, accompanied with two Trumps at least, is expressed by the Fraction $\frac{1441 \cdot 5 \cdot 23 \cdot 22}{29 \cdot 7 \cdot 3 \cdot 13 \cdot 5 \cdot 23 \cdot 11} = \frac{131 \cdot 22}{29 \cdot 7 \cdot 3 \cdot 13} = \frac{2882}{7917}$ and this being subtracted from Unity, the remainder will be $\frac{5035}{7917}$, and therefore the Odds of A's forcing the Trumps are 5035 to 2882, which are very near 7 to 4.

But if it be in red, *A* has the small disadvantage of 19703 against 19882, or nearly 110 against 111.

It is to be noted in this Proposition, that it is not now necessary to multiply by 3; by reason that *B* represents indeterminately any one of the three *B, C, D*: else if the case of having *Baste* was determined to *B* in particular, his probability of having it would only be $\frac{1}{3}$: so that the Chances afterwards being multiplied by 3, the Solution would be the same.

PROBLEM XXIX.

The Player having Spadille, Manille, and 5 other Trumps more by the lowest in red, what is the Probability, by playing Spadille and Manille, of his forcing 4 Trumps?

SOLUTION.

The 5 remaining Trumps being between *B, C, D*, their various dispositions are the following:

<i>B,</i>	<i>C,</i>	<i>D</i>
1,	2,	2
2,	3,	0
3,	1,	1
4,	1,	0
5,	0,	0

Which must be understood in such manner, that what is here assigned to *B* may as well belong to *C* or *D*.

Now it is plain, that out of those five dispositions there are only the two first that are favourable to *A*; let us therefore see what is the Probability of the first disposition.

The number of Chances of *B* to have 1 Trump and 9 Blanks are $\frac{5}{1} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} = 5 \times 5 \times 5 \times 11 \times 17 \times 19 \times 23$, but the number of all the Chances whereby he may take any 10 Cards out of 30, is $5 \times 7 \times 9 \times 11 \times 13 \times 23 \times 29$ as has been seen already in one of the preceding Problems; and therefore the Probability of *B*'s having one Trump and nine Blanks is $\frac{5 \cdot 5 \cdot 5 \cdot 11 \cdot 17 \cdot 19 \cdot 23}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 23 \cdot 29} = \frac{25 \cdot 19 \cdot 17}{7 \cdot 9 \cdot 13 \cdot 29}$.

Now

Now in order to find the number of Chances for *C* to have 2 Trumps and 8 Blanks, it must be considered that *A* having 7 Trumps, and *B* 1, the number of remaining Trumps is 4; and likewise that *A* having 3 Blanks, and *B* 9, the number of remaining Blanks is 16, and therefore that the number of Chances for *C* to have 2 Trumps and 8 Blanks is

$$\frac{4 \cdot 3}{1 \cdot 2} \times \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} = 6 \times 9 \times 10 \times 11 \times 13.$$

But the number of all the Chances whereby *C* may take any 10 Cards out of 20 remaining between him and *D*, is

$$\frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} = 4 \times 11 \times 13 \times 17 \times 19,$$

and therefore the Probability of *C*'s having 2 Trumps and 8 Blanks is

$$\frac{6 \cdot 9 \cdot 10 \cdot 11 \cdot 13}{4 \cdot 11 \cdot 13 \cdot 17 \cdot 19} = \frac{6 \cdot 9 \cdot 10}{4 \cdot 17 \cdot 19} = \frac{9 \cdot 15}{17 \cdot 19}$$

Now *A* being supposed to have had 7 Trumps, *B* 1, and *C* 2, *D* must have 2 necessarily, and therefore no new Calculation ought to be made on account of *D*. It follows therefore that the Probability of the disposition 1, 2, 2, belonging respectively to *B*, *C*, *D*, ought to be expressed by $\frac{25 \cdot 19 \cdot 17}{7 \cdot 9 \cdot 13 \cdot 29} \times \frac{9 \cdot 15}{17 \cdot 19} = \frac{15 \cdot 25}{7 \cdot 13 \cdot 29}$.

Now three things, whereof two are alike, being to be permuted 3 different ways, it follows that the Probability of the Disposition 1, 2, 2, as it may happen in any order, will be $\frac{3 \cdot 15 \cdot 25}{7 \cdot 13 \cdot 29} = \frac{1125}{2639}$.

It will be found in the same manner, that the Probability of the Disposition 2, 3, 0 as it belongs respectively to *B*, *C*, *D*, is $\frac{2 \cdot 5 \cdot 10}{7 \cdot 13 \cdot 29}$; but the number of Permutations of three things which are all unlike being 6, it follows that the said Probability ought to be multiplied by 6, which will make it $\frac{6 \cdot 2 \cdot 5 \cdot 10}{7 \cdot 13 \cdot 29} = \frac{600}{2639}$.

From all which it follows, that the Probability of *A*'s forcing 4 Trumps is $\frac{1125 + 600}{2639} = \frac{1725}{2639}$; which fraction being subtracted from Unity, the remainder will be $\frac{914}{2639}$, and therefore the Odds of *A*'s forcing 4 Trumps are 1725 to 914, that is very near 17 to 9.

P R O B L E M XXX.

A the Player having 4 Matadors, in Diamonds, with the two black Kings, each accompanied with two small Cards of their own suit; what is the Probability that no one of the others B, C, D, has more than 4 Trumps, or in case he has more, that he has also of the suit of both his Kings; in which cases A wins necessarily?

SOLUTION.

The Chances that are against *A* are as follows; it being possible that *B* may have

Diamonds,	Hearts,	Number of Chances.
5,	5	14112
6,	4	5880
7,	3	960
8,	2	45
		Sum 20997

Diamonds	Spades,	Hearts,	Number of Chances.
5,	1	4	70560
5,	2	3	100800
5,	3	2	50400
5,	4	1	8400
5,	5	0	336
6,	1	3	20160
6,	2	2	18900
6,	3	1	5600
6,	4	0	420
7,	1	2	2160
7,	2	1	1200
7,	3	0	160
8,	1	1	60
8,	2	0	15
			279171

Now

Now by reason that among *B, C, D*, there are as many Clubs as Spades, viz. 6 of each sort, it follows that the Clubs may be substituted in the room of the Spades, which will double this last number of Chances, and make it 558342; and therefore, adding together the first and second number of Chances, the Sum will be 579339, which will be the whole number of Chances, whereby *B* may withstand the Expectation of *A*; but the number of all the Chances which *B* has for taking any 10 Cards out of 30, is $5 \times 7 \times 9 \times 11 \times 13 \times 23 \times 29 = 30045015$; from which it follows that the Probability of *B*'s withstanding the Expectation of *A* is $\frac{579339}{30045015}$; but as this may fall as well upon *C* and *D* as upon *B*, it follows that this Probability ought to be multiplied by 3, then the Product $\frac{1738017}{30045015}$ will express the Probability of *A*'s losing; and this being subtracted from Unity, the remainder will express the Probability of *A*'s winning; and therefore the Odds of *A*'s winning will be little more than $16 \frac{1}{4}$ to 1.

PROBLEM XXXI.

A having Spadille, Manille, King, Knave, and two other small Trumps in black, what is the Probability that Baste accompanied with two other Trumps, or the Queen accompanied with three other, shall not be in the same hand; in which case *A* wins necessarily?

SOLUTION.

The Probability of Baste being in one hand, accompanied with two other Trumps, has been found, in Problem xxviii, to be $\frac{2882}{7917}$.

The number of Chances for him who has the Queen, to have also three other Trumps, excluding Baste, is

$$\frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 5 \times 7 \times 11 \times 20 \times 23$$

but the number of Chances for joining any 9 Cards to the Queen is $3 \times 5 \times 7 \times 11 \times 13 \times 23 \times 29$, and therefore the Probability of the Queen's being in one hand, accompanied with three other Trumps, is

$$\frac{5 \cdot 7 \cdot 11 \cdot 20 \cdot 23}{3 \times 5 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 29} = \frac{20}{3 \cdot 13 \cdot 29} = \frac{20}{1131} = \frac{140}{7917},$$

now

now this Probability being added to the former, the Sum will be $\frac{3022}{7917}$; and therefore the Odds of *A*'s not being withstood either from *Baſte* being accompanied with two other Trumps, or from the Queen accompanied with three, are 4895 to 3022, nearly as 13 to 8.

It may be obſerved, that the reaſon of *Baſte* being excluded from among the Trumps that accompany the Queen is this; if the Queen be accompanied with *Baſte* and two other Trumps, the *Baſte* itſelf is accompanied with three Trumps, which caſe had been taken in already in the firſt part of the Solution.

P R O B L E M. XXXII.

A having three Matadors in Spades with the Kings of Hearts, Diamonds, and Clubs, two ſmall Hearts, and two ſmall Diamonds; to find the Probability that not above three Spades ſhall be in one hand, or that, if there be above three, there ſhall be alſo of the ſuits of the three Kings; in which caſe *A* wins neceſſarily.

SOLUTION.

The Probability of not above [three Trumps being in one hand] = 0.332141.

The Probability that one of the Oppoſers ſhall have 4 Trumps, and at the ſame time Hearts, Diamonds, and Clubs, and that no other ſhall have 4 Trumps, is = 0.393501.

The Probability that two of the Oppoſers ſhall have 4 Trumps, and at the ſame time Hearts, Diamonds, and Clubs, is = 0.013836.

The Probability that one of the Oppoſers ſhall have 5 Trumps, and at the ſame time Hearts, Diamonds, and Clubs, is = 0.103019.

The Probability that one of the Oppoſers ſhall have 6 Trumps, Hearts, Diamonds, and Clubs, is = 0.001041.

The Probability of one of the Oppoſers having 7 Trumps, and at the ſame time Hearts, Diamonds, and Clubs, is = 0.000313.

Now the Sum of all theſe Probabilities is 0.843851, which being ſubtracted from Unity, the remainder is 0.156149; and therefore the Odds of the Player's winning are as 843851 to 156149, that is very near as 27 to 5.

PROBLEM XXXIII.

To find at Pharaon, how much it is that the Banker gets per Cent. of all the Money that is adventured.

HYPOTHESIS.

I suppose *first*, that there is but one single Ponte; *Secondly*, that he lays his Money upon one single Card at a time; *Thirdly*, that he begins to take a Card in the beginning of the Game; *Fourthly*, that he continues to take a new Card after the laying down of every couple: *Fifthly*, that when there remain but six Cards in the Stock, he ceases to take a Card.

SOLUTION.

When at any time the Ponte lays a new Stake upon a Card taken as it happens out of his Book, let the number of Cards already laid down by the Banker be supposed equal to x , and the whole number of Cards equal to n .

Now in this circumstance the Card taken by the Ponte has past four times, or three times, or twice, or once, or not at all.

First, If it has passed four times, he can be no loser upon that account.

Secondly, If it has passed three times, then his Card is once in the Stock: now the number of Cards remaining in the Stock being $n - x$, it follows by the first case of the xiiith Problem, that the Loss of the Ponte will be $\frac{1}{n-x}$: but by the Remark belonging to the xxth Problem, the Probability of his Card's having passed three times precisely in x Cards is $\frac{x \cdot x-1 \cdot x-2 \cdot n-x+4}{n \cdot n-1 \cdot n-2 \cdot n-3}$: now supposing the Denominator equal to S , multiply the Loss he will suffer, if he has that Chance, by the Probability of having it, and the product $\frac{x \cdot x-1 \cdot x-2 \cdot 4}{S}$ will be his absolute Loss in that circumstance.

Thirdly, If it has passed twice, his absolute Loss will, by the same way of reasoning, be found to be $\frac{x \cdot x-1 \cdot n-x+2 \cdot 6}{20}$

Fourthly, If it has passed once, his absolute Loss will be found to be $\frac{x \cdot n-x \cdot n-x-2 \cdot 3}{S}$

Fifthly, If it has not yet passed, his absolute Loss will be

$$\frac{n - x \cdot n - x - 2 \cdot 2n - 2x - 5}{28}$$

Now the Sum of all these Losses of the Ponte will be

$$\frac{n^3 - \frac{9}{2}nn + 5n - 3x - \frac{3}{2}xx + 3x^2}{8},$$

and this is the Loss he suffers by venturing a new Stake after any number of Cards x are passed.

But the number of Couples which at any time are laid down, is always one half of the number of Cards that are passed; wherefore calling t the number of those Couples, the Loss of the Ponte

may be expressed thus

$$\frac{n^3 - \frac{9}{2}nn + 5n - 6t - 6tt + 24t^3}{8}$$

Let now p be the number of Stakes which the Ponte adventures; let also the Loss of the Ponte be divided into two parts, *viz.*

$$\frac{n^3 - \frac{9}{2}nn + 5n}{8}, \text{ and } \frac{-6t - 6tt + 24t^3}{8}.$$

And since he adventures a Stake p times; it follows that the first part of his loss will be

$$\frac{pn^3 - \frac{9}{2}pnn + 5pn}{8}.$$

In order to find the second part, let t be interpreted successively by 0, 1, 2, 3, &c. to the last term $p - 1$; then in the room of $6t$ we shall have a Sum of Numbers in Arithmetic Progression to be multiplied by 6; in the room of $6tt$ we shall have a Sum of Squares, whose Roots are in Arithmetic Progression, to be multiplied by 6; and in the room of $24t^3$ we shall have a Sum of Cubes, whose Roots are in Arithmetic Progression, to be multiplied by 24.

These several Sums being collected according to the Rule given in the second Remark on the x^{th} Problem, will be found to be

$$\frac{6p^4 - 14p^3 + 6pp + 2p}{8}$$

and therefore the whole Loss of the Ponte will be

$$\frac{pn^3 - \frac{9}{2}pnn + 5pn + 6p^4 - 14p^3 + 6pp + 2p}{8}.$$

Now this being the Loss which the Ponte sustains by adventuring the Sum p , each Stake being supposed equal to Unity, it follows that the Loss *per Cent.* of the Ponte, is the quantity above-written multiplied by 100, and divided by p , which considering that S has been supposed equal to $n \times n - 1 \times n - 2 \times n - 3$, will make it to be

$$\frac{2n - 5}{2 \cdot n - 1 \cdot n - 3} \times 100 + \frac{p - 1 \times 6pp - 8p - 2}{n \cdot n - 1 \times n - 2 \cdot n - 3} \times 100;$$

let

let now n be interpreted by 52, and p by 23; and the Loss *per Cent.* of the Ponte, or Gain *per Cent.* of the Banker, will be found to be 2.99251; that is 2^{L.} — 19^{sb.} — 10^{d.} *per Cent.*

By the same Method of process, it will be found that the Gain *per Cent.* of the Banker at Bassette will be $\frac{3^{n-9}}{n \cdot n-1 \cdot n-2} \times 100 +$

$\frac{4p \cdot p-1 \cdot p-2}{n \cdot n-1 \cdot n-2 \cdot n-3} \times 100$. Let n be interpreted by 51, and p by 23;

and the foregoing expression will become 0.790582 or 15^{sb.} — 9 $\frac{1}{2}$ ^{d.}

The consideration of the first Stake which is adventured before the Pack is turned being here omitted, as being out of the general Rule; but if that case be taken in, the Gain of the Banker will be diminished, and be only 0.76245, that is 15^{sb.} — 3^{d.} very near; and this is to be estimated as the Gain *per Cent.* of the Banker, when he takes but half Face.

Now whether the Ponte takes one Card at a time, or several Cards, the Gain *per Cent.* of the Banker continues the same: whether the Ponte keeps constantly to the same Stake, or some time doubles or triples it, the Gain *per Cent.* is still the same: whether there be one single Ponte or several, his Gain *per Cent.* is not thereby altered. Wherefore the Gain *per Cent.* of the Banker, upon all the Money that is adventured at Pharaon, is 2^{L.} — 19^{sb.} — 10^{d.} and at Bassette 15^{sb.} — 3^{d.}

PROBLEM XXXIV.

Supposing A and B to play together, that the Chances they have respectively to win are as a to b , and that B obliges himself to set to A so long as A wins without interruption: what is the advantage that A gets by his hand?

SOLUTION.

First, If A and B stake 1 each, the Gain of A on the first Game is $\frac{a-b}{a+b}$.

Secondly, His Gain on the second Game will also be $\frac{a-b}{a+b}$, provided he should happen to win the first: but the Probability of A's winning the first Game is $\frac{a}{a+b}$. Wherefore his Gain on the second Game will be $\frac{a}{a+b} \times \frac{a-b}{a+b}$.

P 2

Thirdly,

Thirdly, His Gain on the third Game, after winning the two first, will be likewise $\frac{a-b}{a+b}$: but the Probability of his winning the two first Games is $\frac{aa}{(a+b)^2}$; wherefore his Gain on the third Game is $\frac{aa}{(a+b)^2} \times \frac{a-b}{a+b}$.

Fourthly, Wherefore the Gain of the Hand of A is an infinite Series; viz. $1 + \frac{a}{a+b} + \frac{aa}{(a+b)^2} + \frac{a^3}{(a+b)^3} + \frac{a^4}{(a+b)^4} + \dots$ to be multiplied by $\frac{a-b}{a+b}$: but the Sum of that infinite Series is $\frac{a+b}{b}$; wherefore the Gain of the Hand of A is $\frac{a+b}{b} \times \frac{a-b}{a+b} = \frac{a-b}{b}$.

COROLLARY 1.

If A has the advantage of the Odds, and B sets his hand out, the Gain of A is the difference of the numbers expressing the Odds, divided by the lesser. Thus if A has the Odds of 5 to 3, then his Gain will be $\frac{5-3}{3} = \frac{2}{3}$.

COROLLARY 2.

If B has the disadvantage of the Odds, and A sets his hand out, the Loss of B will be the difference of the numbers expressing the Odds divided by the greater: thus if B has but 3 to 5, his Loss will be $\frac{2}{5}$.

COROLLARY 3.

If A and B do mutually engage to set to one another, as long as either of them wins without interruption, the Gain of A will be found to be $\frac{aa-bb}{ab}$; that is the Sum of the numbers expressing the Odds multiplied by their difference, the Product of that Multiplication being divided by the Product of the numbers expressing the Odds. Thus if the Odds were as 5 to 3, the Sum of 5 and 3 being 8, and the difference being 2, multiply 8 by 2, and divide the product 16, by the product of the numbers expressing the Odds, which is 15, and the Quotient will be $\frac{16}{15}$, or $1 \frac{1}{15}$, which therefore will be the Gain of A .

COROLLARY 4.

But if he be only to be set to, who wins the first time, and that he be to be set to as long as he wins without interruption; then the Gain

Gain of A will be $\frac{a^3 - b^3}{ab \times a + b}$: thus if a be 5, and b 3, the Gain of A will be $\frac{98}{120} = \frac{49}{60}$.

PROBLEM XXXV.

Any number of Letters a, b, c, d, e, f, &c. all of them different, being taken promiscuously as it happens : to find the Probability that some of them shall be found in their places according to the rank they obtain in the Alphabet ; and that others of them shall at the same time be displaced.

SOLUTION.

Let the number of all the Letters be $= n$; let the number of those that are to be in their places be $= p$, and the number of those that are to be out of their places $= q$. Suppose for brevity's sake $\frac{1}{n} = r$, $\frac{1}{n \cdot n - 1} = s$, $\frac{1}{n \cdot n - 1 \cdot n - 2} = t$, $\frac{1}{n \cdot n - 1 \cdot n - 2 \cdot n - 3} = v$, &c. then let all the quantities $1, r, s, t, v$, &c. be written down with Signs alternately positive and negative, beginning at 1, if p be $= 0$; at r , if p be $= 1$: at s , if p be $= 2$, &c. Prefix to these Quantities the Coefficients of a Binomial Power, whose index is $= q$; this being done, those Quantities taken all together will express the Probability required. Thus the Probability that in 6 Letters taken promiscuously, two of them, viz. a and b shall be in their places, and three of them, viz. c, d, e , out of their places, will be

$$\frac{1}{6 \cdot 5} - \frac{3}{6 \cdot 5 \cdot 4} + \frac{3}{6 \cdot 5 \cdot 4 \cdot 3} - \frac{1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = \frac{11}{720}$$

And the Probability that a shall be in its place, and b, c, d, e , out of their places, will be

$$\frac{1}{6} - \frac{4}{6 \cdot 5} + \frac{6}{6 \cdot 5 \cdot 4} - \frac{4}{6 \cdot 5 \cdot 4 \cdot 3} + \frac{1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = \frac{53}{720}$$

The Probability that a shall be in its place, and b, c, d, e, f , out of their places, will be

$$\frac{1}{6} - \frac{5}{6 \cdot 5} + \frac{10}{6 \cdot 5 \cdot 4} - \frac{10}{6 \cdot 5 \cdot 4 \cdot 3} + \frac{5}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} - \frac{1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{44}{720} = \frac{11}{180}$$

The Probability that a, b, c, d, e, f shall all be displaced is

$$1 - \frac{6}{6} + \frac{15}{6 \cdot 5} - \frac{20}{6 \cdot 5 \cdot 4} + \frac{15}{6 \cdot 5 \cdot 4 \cdot 3} - \frac{6}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} + \frac{1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \text{ or } 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \\ = \frac{265}{720} = \frac{53}{144}.$$

Hence it may be concluded, that the Probability that one of them at least shall be in its place, is $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144}$, and that the Odds that one of them at least shall be so found, are as 91 to 53.

It must be observed, that the foregoing Expression may serve for any number of Letters, by continuing it to so many Terms as there are Letters: thus if the number of Letters had been seven, the Probability required would have been $\frac{177}{280}$.

DEMONSTRATION.

The number of Chances for the Letter a to be in the first place, contains the number of Chances by which a being in the first place, b may be in the second, or out of it: This is an Axiom of common Sense of the same degree of evidence, as that the whole is equal to all its parts.

From this it follows, that if from the number of Chances that there are for a to be in the first place, there be subtracted the number of Chances that there are for a to be in the first place, and b at the same time in the second, there will remain the number of Chances by which a being in the first place, b may be excluded the second.

For the same reason it follows, that if from the number of Chances for a and b to be respectively in the first and second places, there be subtracted the number of Chances by which a, b , and c may be respectively in the first, second, and third places; there will remain the number of Chances by which a being in the first, and b in the second, c may be excluded the third place: and so of the rest.

Let $+a$ denote the Probability that a shall be in the first place, and let $-a$ denote the Probability of its being out of it. Likewise let the Probabilities that b shall be in the second place, or out of it, be respectively express'd by $+b$ and $-b$.

Let

Let the Probability that a being in the first place, b shall be in the second, be expressed by $a + b$: Likewise let the Probability that a being in the first place, b shall be excluded the second, be expressed by $a - b$.

Universally. Let the Probability there is that as many as are to be in their proper places, shall be so, and that as many others as are at the same time to be out of their proper places shall be so found, be denoted by the particular Probabilities of their being in their proper places, or out of them, written all together: So that, for instance $a + b + c - d - e$, may denote the Probability that a, b , and c shall be in their proper places, and that at the same time both d and e shall be excluded their proper places.

Now to be able to derive proper Conclusions by virtue of this Notation, it is to be observed, that of the Quantities which are here considered, those from which the Subtraction is to be made are indifferently composed of any number of Terms connected by $+$ and $-$; and that the Quantities which are to be subtracted do exceed by one Term those from which the Subtraction is to be made; the rest of the Terms being alike, and their Signs alike; then nothing more is requisite to have the remainder, than to preserve the Quantities that are alike, with their proper Signs, and to change the Sign of the Quantity exceeding.

It having been demonstrated in what we have said of Permutations and Combinations, that $a = \frac{1}{n}$; $a + b = \frac{1}{n \cdot n-1}$; $a + b + c = \frac{1}{n \cdot n-1 \cdot n-2}$, &c. let $\frac{1}{n}$, $\frac{1}{n \cdot n-1}$, &c. be respectively called r, s, t, v , &c. this being supposed, we may come to the following Conclusions.

$$\begin{array}{rcl} b & = & r \\ b + a & = & s \\ \text{then } 1^{\circ}. & b - a & = r - s \end{array}$$

$$c + b = s \text{ for the same reason that } a + b = s.$$

$$c + b + a = t$$

$$2^{\circ}. c + b - a = s - r$$

$$e - a = r - s \text{ by the first Conclusion.}$$

$$c - a + b = s - r \text{ by the second.}$$

$$3^{\circ}. e - a - b = r - 2s + t$$

$$d + c + b = t$$

$$d + c + b + a = v$$

$$4^{\circ}. d + c + b - a = t - r$$

$$d + c$$

$$d + c - a = s - t \quad \text{by the second Conclusion.}$$

$$d + c - a + b = t - v \quad \text{by the fourth.}$$

$$5^{\circ}. \quad d + c - a - b = s - 2t + v$$

$$d - b - a = r - 2s + t \quad \text{by the third Conclusion.}$$

$$d - b - a + c = s - 2t + v \quad \text{by the fifth.}$$

$$6^{\circ}. \quad d - b - a - c = r - 3s + 3t - v.$$

By the same process, if no letter be particularly assigned to be in its place the Probability that such of them as are assigned may be out of their places, will likewise be found thus.

$$-a = 1 - r, \text{ for } +a \text{ and } -a \text{ together make Unity.}$$

$$-a + b = r - s \quad \text{by the first Conclusion.}$$

$$7^{\circ}. \quad -a - b = 1 - 2r + s$$

$$-a - b = 1 - 2r + s \quad \text{by the seventh Conclusion.}$$

$$-a - b + c = r - 2s + t \quad \text{by the third.}$$

$$8^{\circ}. \quad -a - b - c = 1 - 3r + 3s - t.$$

Now examining carefully all the foregoing Conclusions, it will be perceived, that when a Question runs barely upon the displacing any given number of Letters, without requiring that any other should be in its place, but leaving it wholly indifferent; then the Vulgar Algebraic Quantities which lie at the right-hand of the Equations, begin constantly with Unity: it will also be perceived, that when one single Letter is assigned to be in its place, then those Quantities begin with r , and that when two Letters are assigned to be in their places, they begin with s , and so on: moreover 'tis obvious, that these Quantities change their Signs alternately, and that the numerical Coefficients, which are prefixed to them are those of a Binomial Power, whose Index is equal to the number of Letters which are to be displaced.

P R O B L E M XXXVI.

Any given number of Letters a, b, c, d, e, f, &c. being each repeated a certain number of times, and taken promiscuously as it happens: To find the Probability that of some of those sorts, some one Letter of each may be found in its place, and at the same time, that of some other sorts, no one Letter be found in its place.

SOLU-

SOLUTION.

Suppose n be the number of all the Letters, l the number of times that each Letter is repeated, and consequently $\frac{n}{l}$ the whole number of Sorts: suppose also that p be the number of Sorts of which some one Letter is to be found in its place, and q the number of Sorts of which no one Letter is to be found in its place. Let now the prescriptions given in the preceding Problem be followed in all respects, saving that r must here be made $= \frac{l}{n}$, $s = \frac{ll}{n \cdot n-1}$, $t = \frac{l^3}{n \cdot n-1 \cdot n-2}$, &c. and the Solution of any particular case of the Problem will be obtained.

Thus if it were required to find the Probability that no Letter of any sort shall be in its place, the Probability thereof would be expressed by the Series

$$1 - qr + \frac{q \cdot q-1}{1 \cdot 2} s - \frac{q \cdot q-1 \cdot q-2}{1 \cdot 2 \cdot 3} + \frac{n \cdot q-1 \cdot q-2 \cdot q-3}{1 \cdot 2 \cdot 3 \cdot 4} v, \text{ \&c.}$$

of which the number of Terms is equal to $q + 1$.

But in this particular case q would be equal to $\frac{n}{l}$, and therefore, the foregoing Series might be changed into this, *viz.*

$$\frac{1}{2} \times \frac{n-l}{n-1} - \frac{1}{6} \times \frac{n-l \cdot n-2l}{n-1 \cdot n-2} + \frac{1}{24} \times \frac{n-l \cdot n-2l \cdot n-3l}{n-1 \cdot n-2 \cdot n-3}, \text{ \&c.}$$

of which the number of Terms is equal to $\frac{n-l}{l}$.

COROLLARY 1.

From hence it follows, that the Probability of one or more Letters, indeterminately taken, being in their places, will be expressed as follows:

$$1 - \frac{1}{2} \times \frac{n-l}{n-1} + \frac{1}{6} \times \frac{n-l \cdot n-2l}{n-1 \cdot n-2} - \frac{1}{24} \times \frac{n-l \cdot n-2l \cdot n-3l}{n-1 \cdot n-2 \cdot n-3}, \text{ \&c.}$$

COROLLARY 2.

The Probability of two or more Letters indeterminately taken, being in their places, will be

$$\frac{1}{2} \times \frac{n-l}{n-1} - \frac{2}{1 \cdot 3} \times \frac{n-2l}{n-2} A + \frac{3}{2 \cdot 4} \times \frac{n-3l}{n-3} B - \frac{4}{3 \cdot 5} \times \frac{n-4l}{n-4} C + \frac{5}{4 \cdot 6} \times \frac{n-5l}{n-5} D, \text{ \&c.}$$

wherein it is necessary to observe, that the Capitals A, B, C, D , &c. denote the preceding Terms.

Q

Altho'

Altho' the formation of this last Series flows naturally from what we have already established, yet that nothing may be wanting to clear up this matter, it is to be observed, that if one Species is to have some one of its Letters in its proper place, and the rest of the Species to be excluded, then the Series whereby the Problem is determined being to begin at r , according to the Precepts given in the preceding Problem, becomes

$$r - qs + \frac{q \cdot q-1}{1 \cdot 2} t - \frac{q \cdot q-1 \cdot q-2}{1 \cdot 2 \cdot 3} v, \&c.$$

but then the number of Species being $\frac{n}{l}$, and all but one being to be excluded, it follows that q in this case is $= \frac{n}{l} - 1 = \frac{n-l}{l}$ wherefore the preceding Series would become, after the proper Substitutions,

$$\frac{l}{n} - \frac{n-l \cdot l}{n \cdot n-1} + \frac{1}{2} \times \frac{n-l \cdot n-2l \cdot l}{n \cdot n-1 \cdot n-2} - \frac{1}{6} \times \frac{n-l \cdot n-2l \cdot n-3l \cdot l}{n \cdot n-1 \cdot n-2 \cdot n-3}, \&c.$$

And this is the Probability that some one of the Letters of the Species particularly given, may obtain its place, and the rest of the Species be excluded; but the number of Species being $\frac{n}{l}$, it follows that this Series ought to be multiplied by $\frac{n}{l}$, which will make it

$$1 - \frac{n-l}{n-1} + \frac{1}{2} \times \frac{n-l \cdot n-2l}{n-1 \cdot n-2} - \frac{1}{6} \times \frac{n-l \cdot n-2l \cdot n-3l}{n-1 \cdot n-2 \cdot n-3}, \&c.$$

And this is the Probability that some one Species indeterminately taken, and no more than one, may have some one of its Letters in its proper place.

Now if from the Probability of one or more being in their places, be subtracted the Probability of one and no more being in its place, there will remain the Probability of two or more indeterminately taken being in their places, which consequently will be the difference between the following Series, *viz.*

$$1 - \frac{1}{2} \times \frac{n-l}{n-1} + \frac{1}{6} \times \frac{n-l \cdot n-2l}{n-1 \cdot n-2} - \frac{1}{24} \times \frac{n-l \cdot n-2l \cdot n-3l}{n-1 \cdot n-2 \cdot n-3}, \&c.$$

and $1 - \frac{n-l}{n-1} + \frac{1}{2} \times \frac{n-l \cdot n-2l}{n-1 \cdot n-2} - \frac{1}{6} \times \frac{n-l \cdot n-2l \cdot n-3l}{n-1 \cdot n-2 \cdot n-3}, \&c.$

which difference therefore will be

$$\frac{1}{2} \times \frac{n-1}{n-1} - \frac{1}{3} \times \frac{n-1 \cdot n-2}{n-1 \cdot n-2} + \frac{1}{8} \times \frac{n-1 \cdot n-2 \cdot n-3}{n-1 \cdot n-2 \cdot n-3}, \&c.$$

$$\text{or } \frac{1}{2} \times \frac{n-1}{n-1} - \frac{2}{1 \cdot 3} \times \frac{n-2}{n-2} A + \frac{3}{2 \cdot 4} \times \frac{n-3}{n-3} B - \frac{4}{3 \cdot 5} \times \frac{n-4}{n-4} C, \&c.$$

as we had expressed it before: and from the same way of reasoning, the other following Corollaries may be deduced.

COROLLARY 3.

The Probability that three or more Letters indeterminately taken may be in their places, will be expressed by the Series

$$\frac{1}{6} \times \frac{n-1 \cdot n-2}{n-1 \cdot n-2} - \frac{3}{1 \cdot 4} \times \frac{n-3}{n-3} A + \frac{4}{2 \cdot 5} \times \frac{n-4}{n-4} B$$

$$- \frac{5}{3 \cdot 6} \times \frac{n-5}{n-5} C + \frac{6}{4 \cdot 7} \times \frac{n-6}{n-6} D, \&c.$$

COROLLARY 4.

The Probability that four or more Letters indeterminately taken may be in their places will be thus expressed

$$\frac{1}{24} \times \frac{n-1 \cdot n-2 \cdot n-3}{n-1 \cdot n-2 \cdot n-3} - \frac{4}{1 \cdot 5} \times \frac{n-4}{n-4} A + \frac{5}{2 \cdot 6} \times \frac{n-5}{n-5} B$$

$$- \frac{6}{3 \cdot 7} \times \frac{n-6}{n-6} C, \&c.$$

The Law of the continuation of these Series being manifest, it will always be easy to assign one that shall fit any case proposed.

From what we have said it follows, that in a common Pack of 52 Cards, the Probability that one of the four Aces may be in the first place, or one of the four Deuces in the second, or one of the four Trays in the third; or that some of any other sort may be in its place (making 13 different places in all) will be expressed by the Series exhibited in the first Corollary.

It follows likewise, that if there be two Packs of Cards, and that the order of the Cards in one of the Packs be the Rule whereby to estimate the rank which the Cards of the same Suit and Name are to obtain in the other; the Probability that one Card or more in one of the Packs may be found in the same position as the like Card in the other Pack, will be expressed by the Series belonging to the first Corollary, making $n = 52$, and $l = 1$. Which Series will in this case be $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720}, \&c.$ whereof 52 Terms are to be taken,

If the Terms of the foregoing Series are joined by Couples, the Series will become

$$\frac{1}{2} + \frac{1}{2 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 6} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 8} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 10}$$

&c. of which 26 Terms ought to be taken.

But by reason of the great Convergency of the said Series, a few of its Terms will give a sufficient approximation, in all cases; as appears by the following Operation

$$\begin{array}{r} \frac{1}{2} = 0.500000 \\ \frac{1}{2 \cdot 4} = 0.125000 \\ \frac{1}{2 \cdot 3 \cdot 4 \cdot 6} = 0.006944 + \\ \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 8} = 0.000174 + \\ \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 10} = 0.000002 + \\ \hline \text{Sum} = 0.632120 + \end{array}$$

Wherefore the Probability that one or more like Cards in two different Packs may obtain the same position, is very nearly 0.632, and the Odds that this will happen once at least as 632 to 368, or 12 to 7 very near.

But the Odds that two or more like Cards in two different Packs will not obtain the same position are very nearly as 736 to 264, or 14 to 5.

REMARK.

It is known that $1 + y + \frac{1}{2} y y + \frac{1}{6} y^3 + \frac{1}{24} y^4$ &c. is the number whose hyperbolic Logarithm is y , and therefore $1 - y + \frac{1}{2} y y - \frac{1}{6} y^3 + \frac{1}{24} y^4$ &c. is the Number whose hyperbolic Logarithm is $-y$. Let N be $= y - \frac{1}{2} y y + \frac{1}{6} y^3 - \frac{1}{24} y^4$ &c. then $1 - N$ is the Number whose hyperbolic Logarithm is $-y$. Let now y be $= 1$, therefore $1 - N$ is the number whose hyperbolic Logarithm is -1 ; but the number whose hyperbolic Logarithm is -1 , is the reciprocal of that whose hyperbolic Logarithm is 1 , or whose *Briggian* Logarithm is 0.4342944. Therefore 9.5657056 is the *Briggian* Logarithm answering to the hyperbolic Logarithm -1 , but the number answering to it is 0.36788. Therefore $1 - N = 0.36788$; and $N = 1 - 0.36788 = 0.63212$; and therefore

fore the Series $y - \frac{1}{2} y y + \frac{1}{6} y^3 - \frac{1}{24} y^4 \&c.$ in infinitum, when $y = 1$, that is $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} \&c. = 0.63212.$

COROLLARY 5.

If A and B each holding a Pack of Cards, pull them out at the same time one after another, on condition that every time two like Cards are pulled out, A shall give B a Guinea; and it were required to find what consideration B ought to give A to play on those Terms: the Answer will be one Guinea, let the number of Cards be what it will.

Altho' this be a Corollary from the preceding Solutions, yet it may more easily be made out thus; one of the Packs being the Rule whereby to estimate the order of the Cards in the second, the Probability that the two first Cards are alike is $\frac{1}{52}$, the Probability that the two second are alike is also $\frac{1}{52}$, and therefore there being 52 such alike combinations, it follows that the Value of the whole is $\frac{52}{52} = 1.$

COROLLARY 6.

If the number of Packs be given, the Probability that any given number of Circumstances may happen in any number of Packs, will easily be found by our Method: thus if the number of Packs be k , the Probability that one Card or more of the same Suit and Name in every one of the Packs may be in the same position, will be expressed as follows,

$$\frac{1}{n^{k-2}} - \frac{1}{2 \cdot n \cdot n-1} \frac{1}{n^{k-2}} + \frac{1}{6 \cdot n \cdot n-1 \cdot n-2} \frac{1}{n^{k-2}} - \frac{1}{24 \cdot n \cdot n-1 \cdot n-2 \cdot n-3} \frac{1}{n^{k-2}}, \&c.$$

PROBLEM XXXVII.

If A and B play together each with a certain number of Bowls = n : what are the respective Probabilities of winning, supposing that each of them want a certain number of Games of being up?

SOLU-

SOLUTION.

First, The Probability that some Bowl of B may be nearer the Jack than any Bowl of A is $\frac{1}{2}$; A and B in this Problem being supposed equal Players.

Secondly, Supposing one of his Bowls nearer the Jack than any Bowl of A , the number of his remaining Bowls is $n - 1$, and the number of all the Bowls remaining between them is $2n - 1$: wherefore the Probability that some other of his Bowls may be nearer the Jack than any Bowl of A will be $\frac{n-1}{2n-1}$, from whence it follows, that the Probability of his winning two Bowls or more is $\frac{1}{2} \times \frac{n-1}{2n-1}$.

Thirdly, Supposing two of his Bowls nearer the Jack than any Bowl of A , the Probability that some other of his Bowls may be nearer the Jack than any Bowl of A , will be $\frac{n-2}{2n-2}$; wherefore the Probability of his winning three Bowls or more is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2}$: the continuation of which process is manifest.

Fourthly, The Probability that one single Bowl of B shall be nearer the Jack than any Bowl of A is $\frac{1}{2} - \frac{1}{2} \times \frac{n-1}{2n-1}$ or $\frac{1}{2} \times \frac{n}{2n-1}$; for if from the Probability that one or more of his Bowls may be nearer the Jack than any Bowl of A , there be subtracted the Probability that two or more may be nearer, there remains the probability of one single Bowl of B being nearer: in this case B is said to win one Bowl at an end.

Fifthly, The Probability that two Bowls of B , and not more, may be nearer the Jack than any Bowl of A , will be found to be $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$, in which case, B is said to win two Bowls at an end.

Sixthly, The Probability that B may win three Bowls at an end will be found to be $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2} \times \frac{n}{2n-3}$; the process whereof is manifest.

It may be observed, that the foregoing Expressions might be reduced to fewer Terms; but leaving them unreduced, the Law of the Process is thereby made more conspicuous.

It is carefully to be observed, when we mention henceforth the probability of winning two Bowls, that the Sense of it ought to be extended to two Bowls or more; and that when we mention

tion the winning two Bowls at an end, it ought to be taken in the common acceptation of two Bowls only: the like being to be observed in other cafes.

This preparation being made; suppose that *A* wants one Game of being up and *B* two; and it be required in that circumstance to determine their probabilities of winning.

Let the whole Stake between them be supposed = 1. Then either *A* may win a Bowl, or *B* win one Bowl at an end, or *B* may win two Bowls.

In the first case he loses his Expectation.

In the second case *B* becomes intitled to $\frac{1}{2}$ of the Stake, but the probability of this case is $\frac{1}{2} \times \frac{n}{2n-1}$. Wherefore his Expectation arising from that part of the Stake he will be intitled to, if this Case should happen, and from the probability of its happening, will be $\frac{1}{4} \times \frac{n}{2n-1}$.

In the third case, *B* wins the whole Stake 1, but the probability of this Case, is $\frac{1}{2} \times \frac{n-1}{2n-1}$.

From this it follows, that the whole Expectation of *B* is equal to $\frac{1}{4} \times \frac{n}{2n-1} + \frac{1}{2} \times \frac{n-1}{2n-1}$, or $\frac{3n-2}{8n-4}$. Which being subtracted from Unity, the remainder will be the Expectation of *A*, viz. $\frac{5n-2}{8n-4}$. It may therefore be concluded that the Probabilities which *A* and *B* have of winning are respectively as $5n-2$ to $3n-2$.

'Tis remarkable, that the fewer the Bowls are the greater is the proportion of the Odds; for if *A* and *B* play with single Bowls, the proportion will be as 3 to 1; if they play with two Bowls each, the proportion will be as 2 to 1; if they play with three Bowls each, the proportion will be as 13 to 6; yet let the number of Bowls be never so great, that proportion will not descend so low as 5 to 3.

Let us now suppose that *A* wants one Game of being up, and *B* three; then either *A* may win a Bowl, or *B* one Bowl at an end, or two Bowls at an end, or three Bowls.

In the first Case, *B* loses his Expectation.

If the second Case happen, then *B* will be in the circumstance of wanting but two to *A*'s one; in which case his Expectation will be $\frac{3n-2}{8n-4}$, as it has been before determined: but the probability

that this Case may happen is $\frac{1}{2} \times \frac{n}{2n-1}$; wherefore the Expectation

tion of B arising from the prospect of this Case will be equal to

$$\frac{1}{2} \times \frac{n}{2n-1} \times \frac{3n-2}{8n-4}.$$

If the third Case happen, then B will be intitled to one half of the Stake: but the Probability of its happening is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$; wherefore the Expectation of B arising from the prospect of this case is $\frac{1}{4} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$ or $\frac{1}{8} \times \frac{n}{2n-1}$.

If the fourth Case happen, then B wins the whole Stake 1: but the Probability of its happening is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$ or $\frac{1}{4} \times \frac{n-2}{2n-1}$.

From this it follows, that the whole Expectation of B will be $\frac{9nn-13n+4}{8 \times 2n-1}^2$; which being subtracted from Unity, the remainder will be the Expectation of A , viz. $\frac{23nn-19n+4}{8 \times 2n-1}^2$. It may therefore be concluded that the Probabilities which A and B have of winning will be as $23nn-19n+4$ to $9nn-13n+4$.

N. B. If A and B play only with one Bowl each, the Expectation of B as deduced from the foregoing Theorem would be found $= 0$, which we know from other principles ought to be $= \frac{1}{8}$. The reason of which is, that the case of winning two Bowls at an end, and the case of winning three Bowls enter the general conclusion, which cases do not belong to the Supposition of playing with single Bowls; wherefore excluding those two Cases, the Expectation of B will be found to be $\frac{1}{2} \times \frac{n}{2n-1} \times \frac{3n-2}{8n-4} = \frac{1}{8}$, which will appear if n be made $= 1$. But the Expectation of B in the case of two Bowls would be rightly determined from the general Solution: the reason of which is, that the Probability of winning three Bowls being universally $\frac{1}{4} \times \frac{n-2}{2n-1}$, that Expression becomes $= 0$, when n is interpreted by 2; which makes it that the general Expression is applicable to this Case.

After what has been said, it will be easy to extend this way of reasoning to any circumstance of Games wanting between A and B ; by making the Solution of each simpler Case subservient to the Solution of that which is next more compounded.

Having given formerly the Solution of this Problem, proposed to me by the Honourable *Francis Robartes* Esq;, in the *Philosophical Transactions* Number 329; I there said, by way of Corollary, that
if

if the proportion of Skill in the Gamesters were given, the Problem might also be solved: since which time M. *de Moivre*, in the second Edition of a Book by him published upon the Subject of Chance, has solved this Problem as it is extended to the consideration of the Skill, and to carry his Solution to a great number of Cases, giving also a Method whereby it might be carried farther: But altho' his Solution is good, as he has made a right use of the Doctrine of Combinations, yet I think mine has a greater degree of Simplicity, it being deduced from the original Principle whereby I have demonstrated the Doctrine of Permutations and Combinations: wherefore to make it as familiar as possible, and to shew its vast extent, I shall now apply it to the general Solution of this Problem, by taking in the consideration of the Skill of the Gamesters.

But before I proceed, it is necessary to define what I call *Skill*: viz. that it is the proportion of Chances which the Gamesters may be supposed to have for winning a single Game with one Bowl each.

P R O B L E M XXXVIII.

If A and B, whose proportion of skill is as a to b, play together each with a certain number of Bowls: what are their respective Probabilities of winning, supposing each of them to want a certain number of Games of being up?

SOLUTION.

First, The Chance of *B* for winning one single Bowl being *b*, and the number of his Bowls being *n*, it follows that the Sum of all his Chances is nb ; and for the same reason, the Sum of all the Chances of *A* is na : wherefore the Sum of all the Chances for winning one Bowl or more is $na + nb$; which for brevity's sake we may call f . From whence it follows, that the Probability which *B* has of winning one Bowl is $\frac{nb}{f}$.

Secondly, Supposing one of his Bowls nearer the Jack than any of the Bowls of *A*, the number of his remaining Chances is $n-1 \times b$; and the number of Chances remaining between them is $s-b$: wherefore the Probability that some other of his Bowls may be nearer the Jack than any Bowl of *A* will be $\frac{n-1 \times b}{f-b}$; from whence it follows that the Probability of his winning two Bowls or more is $\frac{nb}{f} \times \frac{n-1 \times b}{f-b}$.

Thirdly, Supposing two of his Bowls nearer the Jack than any of the Bowls of A , the number of his remaining Chances is $n-2 \times b$; and the number of Chances remaining between them is $s-2b$; wherefore the Probability that some other of his Bowls may be nearer the Jack than any Bowl of A will be $\frac{n-2 \times b}{s-2b}$. From whence it follows that the Probability of his winning three Bowls or more is $\frac{nb}{f} \times \frac{n-1 \times b}{f-b} \times \frac{n-2 \times b}{f-2b}$; the continuation of which process is manifest.

Fourthly, If from the Probability which B has of winning one Bowl or more, there be subtracted the Probability which he has of winning two or more, there will remain the Probability of his winning one Bowl at an end: which therefore will be found to be

$$\frac{nb}{f} - \frac{nb}{f} \times \frac{n-1 \times b}{f-b}, \text{ or } \frac{nb}{f} \times \frac{s-nb}{f-b}, \text{ or } \frac{nb}{f} \times \frac{an}{f-b}.$$

Fifthly, For the same reasons as above, the Probability which B has of winning two Bowls at an end, will be $\frac{nb}{f} \times \frac{n-1 \times b}{f-b} \times \frac{an}{f-2b}$.

Sixthly, And for the same reason likewise, the Probability which B has of winning three Bowls at an end will be found to be $\frac{nb}{f} \times \frac{n-1 \times b}{f-b} \times \frac{n-2 \times b}{f-2b} \times \frac{an}{f-3b}$; The continuation of which process is manifest.

N. B. The same Expectations which denote the Probability of any circumstance of B will denote likewise the Probability of the like circumstance of A , only changing b into a , and a into b .

These things being premised, suppose *first*, that each wants one Game of being up; 'tis plain, that the Expectations of A and B are respectively $\frac{an}{f}$ and $\frac{bn}{f}$. Let this Expectation of B be called P .

Secondly, Suppose A wants one Game of being up, and B two, and let the Expectation of B be required: then either A may win a Bowl, or B win one Bowl at an end, or B win two Bowls.

If the first Case happens, B loses his Expectation.

If the second happens, he gets the Expectation P ; but the Probability of this Case is $\frac{nb}{f} \times \frac{an}{f-b}$: wherefore the Expectation of B arising from the possibility that it may so happen is $\frac{nb}{f} \times \frac{an}{f-b} \times P$.

If the third Case happens, he gets the whole Stake 1; but the Probability of this Case is $\frac{nb}{f} \times \frac{nb-b}{f-b}$; wherefore the Expectation of *B* arising from the Probability of this Case is $\frac{nb}{f} \times \frac{nb-b}{f-b} \times 1$.

From which it follows, that the whole Expectation of *B* will be $\frac{nb}{f} \times \frac{an}{f-b} \times P + \frac{nb}{f} \times \frac{nb-b}{f-b}$. Let this Expectation be called *Q*.

Thirdly, Suppose *A* to want one Game of being up, and *B* three: then either *B* may win one Bowl at an end, in which Case he gets the Expectation *Q*; or two Bowls at an end, in which Case he gets the Expectation *P*; or three Bowls, in which Case he gets the whole Stake 1. Wherefore the Expectation of *B* will be found to be

$$\frac{nb}{f} \times \frac{an}{f-b} \times Q + \frac{nb}{f} \times \frac{n-1 \times b}{f-b} \times \frac{an}{f-2b} \times P + \frac{nb}{f} \times \frac{n-1 \times b}{f-b} \times \frac{n-2 \times b}{f-2b}$$

An infinite number of these Theorems may be formed in the same manner, which may be continued by inspection, having well observed how each of them is deduced from the preceding.

If the number of Bowls were unequal, so that *A* had *m* Bowls, and *B*, *n* Bowls; then supposing $ma + nb = s$, other Theorems might be found to answer that inequality: and if that inequality should not be constant, but vary at pleasure; other Theorems might also be found to answer that Variation of inequality, by following the same way of arguing. And if three or more Gamesters were to play together under any circumstance of Games wanting, and of any given proportion of Skill, their Probabilities of winning might be determined in the same manner.

P R O B L E M XXXIX.

To find the Expectation of A, when with a Die of any given number of Faces, he undertakes to fling any number of them in any given number of Casts.

SOLUTION.

Let $p + 1$ be the number of all the Faces in the Die, *n* the number of Casts, *f* the number of Faces which he undertakes to fling.

The number of Chances for the *Ace* to come up once or more in any number of Casts n , is $\overline{p+1}^n - p^n$: as has been proved in the Introduction.

Let the *Deux*, by thought, be expunged out of the Die, and thereby the number of its Faces reduced to p , then the number of Chances for the *Ace* to come up will at the same time be reduced to $p^n - \overline{p-1}^n$. Let now the *Deux* be restored, and the number of Chances for the *Ace* to come up without the *Deux*, will be the same as if the *Deux* were expunged: But if from the number of Chances for the *Ace* to come up with or without the *Deux*, viz. from $\overline{p+1}^n - p^n$ be subtracted the number of Chances for the *Ace* to come up without the *Deux*, viz. $p^n - \overline{p-1}^n$, there will remain the number of Chances for the *Ace* and the *Deux* to come up once or more in the given number of Casts, which number of Chances consequently will be $\overline{p+1}^n - 2p^n + \overline{p-1}^n$.

By the same way of arguing it will be proved, that the number of Chances, for the *Ace* and *Deux* to come up without the *Tray*, will be $p^n - 2 \times \overline{p-1}^n + \overline{p-2}^n$, and consequently that the number of Chances for the *Ace*, the *Deux*, and *Tray* to come up once or more, will be the difference between $\overline{p+1}^n - 2p^n + \overline{p-1}^n$, and $p^n - 2 \times \overline{p-1}^n + \overline{p-2}^n$, which therefore will be $\overline{p+1}^n - 3 \times p^n + 3 \times \overline{p-1}^n - \overline{p-2}^n$.

Again, it may be proved that the number of Chances for the *Ace*, the *Deux*, the *Tray*, and the *Quatre* to come up is $\overline{p+1}^n - 4 \times p^n + 6 \times \overline{p-1}^n - 4 \times \overline{p-2}^n + \overline{p-3}^n$; the continuation of which process is manifest.

Wherefore if all the Powers $\overline{p+1}^n$, p^n , $\overline{p-1}^n$, $\overline{p-2}^n$, $\overline{p-3}^n$, &c. with Signs alternately positive and negative be written in order, and to those Powers there be prefixed the respective Coefficients of a Binomial raised to the Power f , expressing the number of Faces required to come up; the Sum of all those Terms will be the Numerator of the Expectation of A , of which the Denominator will be $\overline{p+1}^n$.

EXAMPLE I.

Let Six be the number of Faces in the Die, and let A undertake in eight Casts to fling both an *Ace* and a *Deux*, without any regard to order: then his Expectation will be $\frac{6^8 - 2 \times 5^8 + 4^8}{6^8}$

$$= \frac{964502}{1680216} = \frac{4}{7} \text{ nearly.}$$

EXAM-

EXAMPLE 2.

Let A undertake with a common Die to fling all the Faces in 12 Casts, then his Expectation will be found to be

$$\frac{6^{12} - 6 \times 5^{12} + 15 \times 4^{12} - 20 \times 3^{12} + 15 \times 2^{12} - 6 \times 1^{12} + 1 \times 0^{12}}{6^{12}} = \frac{10}{23}$$

nearly.

EXAMPLE 3.

If A with a Die of 36 Faces undertake to fling two given Faces in 43 Casts; or which is the same thing, if with two common Dice he undertake in 43 Casts to fling two Aces at one time, and two Sixes at another time; his Expectation will be

$$\frac{36^{43} - 2 \times 35^{43} + 34^{43}}{30^{43}} = \frac{49}{100} \text{ nearly.}$$

N. B. The parts which compose these Expectations are easily obtained by the help of a Table of Logarithms.

PROBLEM XL.

To find in how many Trials it will be probable that A with a Die of any given number of Faces shall throw any proposed number of them.

SOLUTION.

Let $p + 1$ be the number of Faces in the Die, and f the number of Faces which are to be thrown: Divide the Logarithm of $\frac{1}{1 - \sqrt{\frac{1}{2}}}$ by the Logarithm of $\frac{p+1}{p}$, and the Quotient will ex-

press the number of Trials requisite to make it as probable that the proposed Faces may be thrown as not.

DEMONSTRATION.

Suppose Six to be the number of Faces that are to be thrown, and n the number of Trials, then by what has been demonstrated in the preceding Problem the Expectation of A will be

$$\frac{(p+1)^n - 6 \times p^n + 15 \times (p-1)^n + 20 \times (p-2)^n - 15 \times (p-3)^n + 6 \times (p-4)^n + (p-5)^n}{p+1}$$

Let

Let it be supposed that the Terms, $p+1$, p , $p-1$, $p-2$, &c. are in geometric Progression, (which Supposition will very little err from the truth, especially if the proportion of p to 1, be not very small.) Let now r be written instead of $\frac{p+1}{p}$, and then the Expectation of A will be changed into $1 - \frac{6}{r^n} + \frac{15}{r^{2n}} - \frac{20}{r^{3n}} + \frac{15}{r^{4n}} - \frac{6}{r^{5n}} + \frac{1}{r^{6n}}$, or $1 - \frac{1}{r^n} \sqrt[6]{6}$. But this Expectation of A ought to be made equal to $\frac{1}{2}$, since by Supposition he has an equal Chance to win or lose, hence will arise the Equation $1 - \frac{1}{r^n} \sqrt[6]{6} = \frac{1}{2}$ or $r^n = \frac{1}{1 - \sqrt[6]{\frac{1}{2}}}$, from which it may be concluded that $n \text{ Log. } r$, or $n \times \text{Log. } \frac{p+1}{p} = \text{log. } \frac{1}{1 - \sqrt[6]{\frac{1}{2}}}$, and consequently that n is equal to the Logarithm of $\frac{1}{1 - \sqrt[6]{\frac{1}{2}}}$, divided by the Logarithm of $\frac{p+1}{p}$. And the same demonstration will hold in any other Case.

EXAMPLE 1.

To find in how many Trials A may with equal Chance undertake to throw all the Faces of a common Die.

The Logarithm of $\frac{1}{1 - \sqrt[6]{\frac{1}{2}}} = 0.9621753$; the Logarithm of $\frac{p+1}{p}$ or $\frac{6}{5} = 0.0791812$: wherefore $n = \frac{0.9621753}{0.0791812} = 12 +$. From hence it may be concluded, that in 12 Casts A has the worst of the Lay, and in 13 the best of it.

EXAMPLE 2.

To find in how many Trials A may with equal Chance with a Die of thirty-six Faces undertake to throw six determinate Faces; or, in how many Trials he may with a pair of common Dice undertake to throw all the Doublets.

The Logarithm of $\frac{1}{1 - \sqrt[6]{\frac{1}{2}}}$ being 0.9621753, and the Logarithm of $\frac{p+1}{p}$ or $\frac{36}{35}$ being 0.0122345; it follows that the number of Casts requisite to that effect is $\frac{0.9621753}{0.0122345}$, or 79 nearly.

But

But if it were the Law of the Play, that the Doublets must be thrown in a given order, and that any Doublet happening to be thrown out of its Turn should go for nothing; then the throwing of the six Doublets would be like the throwing of the two Aces six times: to produce which, the number of Casts requisite would be found by multiplying 35 by 5.668, as appears from the Table annexed to our vth Problem; and consequently would be about 198.

N. B. The Fraction $\frac{1}{1 - \sqrt{\frac{1}{2}}}$, may be reduced to another form

viz. $\frac{f \sqrt{2}}{f \sqrt{2} - 1}$; which will facilitate the taking of its Logarithm.

PROBLEM XLI.

Supposing a regular Prism having a Faces marked I, b Faces marked II, c Faces marked III, d Faces marked IV, &c. what is the Probability that in a certain number of throws n, some of the Faces marked I will be thrown, as also some of the Faces marked II?

SOLUTION.

Make $a + b + c + d, \&c. = s$, then the Probability required will be expressed by $\frac{s^n - \overbrace{s - a}^n + \overbrace{s - a - b}^n}{s^n}$; the Demonstration

of which flowing naturally from the Method of arguing employed in the xxxixth Problem, there can be no difficulty about it.

EXAMPLE.

Suppose it be required to find the Probability of throwing in 8 throws the two Chances v and vi, with a pair of common Dice.

The number of all the Chances upon two Dice being 36, whereof 4 belong to the Chance v, and 5 to the Chance vi; it follows that s ought to be interpreted by 36, a by 4, and b by 5: which being done, the Probability required will be expressed by

$\frac{36^8 - 4 \cdot 36^7 + 5 \cdot 36^6}{36^8}$

$36^8 - 32^8 + 27^8$, which by help of a Table of Log. will be
 $\frac{36^8 - 32^8 + 27^8}{36^8}$ found thus:

$\frac{36^8}{36^8} = 1$, $\frac{32^8}{36^8} = 0.38975$, $\frac{31^8}{36^8} = 0.30176$, $\frac{27^8}{36^8} = 0.10012$,
 but $1 - 0.38975 - 0.30176 + 0.10012 = 0.40861$, and this
 being subtracted from Unity, there remains 0.59139 , and therefore
 the Odds against throwing v and vi in 8 throws are 59139 to 40861 ,
 that is about 13 to 9 .

But if it be required, that some of the Faces marked I , some
 of the Faces marked II , and some of the Faces marked III , be thrown,
 the Probability of throwing those Chances in a given number of throws
 n will be expressed by $s^n - \frac{s^n - a^n}{s^n} + \frac{s^n - a^n - b^n}{s^n} - \frac{s^n - a^n - b^n - c^n}{s^n}$

$$\frac{s^n - a^n + s^n - a^n - b^n - s^n - a^n - b^n - c^n}{s^n}$$

s^n

And if the Faces marked IV are farther required to be thrown, the
 Probability of it will be expressed by

$$\frac{s^n - s^n - a^n + s^n - a^n - b^n - s^n - a^n - b^n - c^n + s^n - a^n - b^n - c^n - d^n}{s^n}$$

$$\frac{-s^n - b^n + s^n - a^n - c^n - s^n - a^n - b^n - d^n}{s^n}$$

$$\frac{-s^n - c^n + s^n - a^n - d^n - s^n - a^n - c^n - d^n}{s^n}$$

$$\frac{-s^n - d^n + s^n - b^n - c^n - s^n - b^n - c^n - d^n}{s^n}$$

$$\frac{+s^n - b^n - d^n}{s^n}$$

$$\frac{+s^n - c^n - d^n}{s^n}$$

Now the order of the preceding Solutions being manifest, it will
 be easy by bare inspection to continue them as far as there is occasion.

P R O B L E M XLII.

*If A obliges himself in a certain number of throws n with
 a pair of common Dice not only to throw the Chances
 v and vi , but v before vi ; with this restriction, that if
 he happens to throw vi before v , he does not indeed
 lose his wager, but is to proceed as if nothing had been
 done, still deducting so many throws as have been vain
 from the number of throws which he had at first given
 him; to find the Probability of his winning.*

SOLU-

SOLUTION.

Let the number of Chances which there are for throwing v be called a , the number of Chances for throwing vi , b ; the number of all the Chances upon two Dice f , and the number of throws that A takes $= n$. This being supposed,

1°. If A throws v the first throw, of which the Probability is $\frac{a}{f}$, he has nothing more to do than to throw vi in $n-1$ times,

of which the Probability is $1 - \frac{(f-b)^{n-1}}{f^{n-1}}$, and therefore the Probability of throwing v the first time, and throwing afterwards vi in $n-1$ times is $\frac{a}{f} \times 1 - \frac{(f-b)^{n-1}}{f^{n-1}}$.

2°. If A misses v the first time, and throws it the second, of which the Probability is $\frac{f-a}{f} \times \frac{a}{f}$, then he is afterwards to throw

vi in $n-2$ times, of which the Probability being $1 - \frac{(f-b)^{n-2}}{f^{n-2}}$ it follows that the Probability of missing v the first time, throwing it the second, and afterwards throwing vi , will be $\frac{f-a}{f} \times \frac{a}{f} \times 1 - \frac{(f-b)^{n-2}}{f^{n-2}}$.

3°. If A misses v the two first times, and throws it the third, then he is afterwards to throw vi in $n-3$ times, the Probability of all which is $\frac{(f-a)^2}{f^2} \times \frac{a}{f} \times 1 - \frac{(f-b)^{n-3}}{f^{n-3}}$; and so on. Now all this added together constitutes two geometric Progressions, the number of whose Terms in each is $n-1$.

Wherefore the Sum of the whole will be

$$\frac{f^{n-1} - (f-a)^{n-1}}{f^{n-1}} - \frac{af-ab}{a-b} \times \frac{(f-b)^{n-1} - (f-a)^{n-1}}{f^n} : \text{ and}$$

if a and b are equal, then the second part will be reduced to $-\frac{1}{n-1} \times a \times (f-a)^{n-1} \times \frac{1}{f^n}$.

Now for the application of this to numbers; a in the Case proposed is $= 4$, $b = 5$, $f = 36$. Let n be $= 12$, and the Probability required will be found to be 0.44834, which being subtracted from unity the remainder will be 0.55166, and therefore the Odds against A are 55166 to 44834, that is nearly as 21 to 17.

But if the conditions of the Play were that A in 12 times should throw both v and vi , and that vi should come up before v , the Odds against A would not be so great; being only 54038 to 45962, that is nearly as 20 to 17.

It would not be difficult after what we have said, tho' perhaps a little laborious, to extend these kinds of Solutions to any number of Chances given.

P R O B L E M XLIII.

Any number of Chances being given, to find the Probability of their being produced in a given order, without any limitation of the number of times in which they are to be produced.

SOLUTION.

1°. Let the Chances be a and b , and let it be required to produce them in the order a, b .

The Probability of producing a before b is $\frac{a}{a+b}$, which being supposed to have happened, b must be produced of necessity; and therefore the Probability of producing the Chances a and b in the given order a, b , is $\frac{a}{a+b}$.

2°. Let the Chances given be a, b, c , and let it be required to produce them in the order in which they are written; then the Probability of producing a before b or c is $\frac{a}{a+b+c}$; which being supposed, the Probability of producing b before c is by the preceding case $\frac{b}{b+c}$; after which c must necessarily be produced, and therefore the Probability of this case is $\frac{a}{a+b+c} \times \frac{b}{b+c}$.

3°. Let the Chances be a, b, c, d , and let it be required to produce them in the order in which they are written; then the Probability of producing a before all the rest is $\frac{a}{a+b+c+d}$; which being supposed, the Probability of producing b before all the remaining is $\frac{b}{b+c+d}$; which being supposed, the Probability of producing c before d is $\frac{c}{c+d}$. And therefore the Probability of the whole is $\frac{a}{a+b+c+d} \times \frac{b}{b+c+d} \times \frac{c}{c+d}$; and in the same manner may these Theorems be continued in infinitum.

And

And therefore if it was proposed to find the Probability of throwing with a pair of common Dice the Chances IV, V, VI, VIII, IX, X before VII; let the Chances be called respectively a, b, c, d, e, f , and m , then the Probability of throwing them in the order they are written will be

$$\frac{a}{a+b+c+d+e+f+m} \times \frac{b}{b+c+d+e+f+m} \times \frac{c}{c+d+e+f+m} \times \frac{d}{d+e+f+m} \times \frac{e}{e+f+m} \times \frac{f}{f+m}.$$

But as the order in which they may be thrown is not the thing particularly required here, except that the Chances m are to be thrown the last; so it is plain that there will be as many different parts like the preceding as the position of the 6 Letters a, b, c, d, e, f , may be varied, which being 720 different ways, it follows, that in order to have a compleat Solution of this Question, there must be 720 different parts like the preceding to be added together.

However the Chances IV and X, V and IX, VI and VIII being respectively the same, those 720 might be reduced to 90, which being added together, and the Sum multiplied by 8, we should have the Probability required.

Still those Operations would be laborious, for which reason it will be sufficient to have an approximation, by supposing that all the Chances a, b, c, d, e, f , that is, 3, 4, 5, 5, 4, 3 are equal to the mean Chance 4, which will make it that the Probability required will be expressed by

$$\frac{6b}{6b+m} \times \frac{5b}{5b+m} \times \frac{4b}{4b+m} \times \frac{3b}{3b+m} \times \frac{2b}{2b+m} \times \frac{b}{b+m} \text{ or } \frac{24}{30} \times \frac{20}{26} \times \frac{16}{22} \times \frac{12}{18} \times \frac{8}{14} \times \frac{4}{10} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13} = \frac{1024}{15015};$$

and therefore the Odds against throwing the Chances IV, V, VI, VIII, IX, X before VII are about 13991 to 1024, or nearly 41 to 3.

But the Solution might be made still more exact, if instead of taking 4 for the mean Chance, we find the several Probabilities of throwing all the Chances before VII, and take the sixth part of the Sum for the mean Probability; thus because the several Probabilities of throwing all the Chances before VII are respectively $\frac{3}{9}$,

$\frac{4}{10}$, $\frac{5}{165}$, $\frac{5}{11}$, $\frac{4}{10}$, $\frac{3}{9}$, the Sum of all which is $\frac{392}{165}$, if we divide the whole by 6, the Quotient will be $\frac{392}{990}$ or $\frac{59}{149}$

nearly, and this being supposed $= \frac{z}{z+5}$ wherein z represents the mean Chance, we shall find $z = 3 \frac{14}{15}$. And therefore the Probability of throwing all the Chances before VII, will be found to be $\frac{354}{444} \times \frac{295}{385} \times \frac{236}{326} \times \frac{177}{267} \times \frac{118}{208} \times \frac{59}{149} = 0.065708$ nearly, which being subtracted from Unity, the remaining is 0.934292, and therefore the Odds against throwing all the Chances before VII are 934292 to 65708, that is about $14 \frac{1}{5}$ to 1.

But if it was farther required not only to throw all the Chances before VII, but also to do it in a certain number of times assigned, the Problem might easily be solved by imagining a mean Chance.

P R O B L E M XLIV.

If A, B, C play together on the following conditions; First that they shall each of them stake 1^L. Secondly that A and B shall begin the Play; Thirdly, that the Loser shall yield his place to the third Man, which is constantly to be observed afterwards; Fourthly, that the Loser shall be fined a certain Sum p, which is to serve to increase the common Stock; Lastly, that he shall have the whole Sum deposited at first, and increased by the several Fines, who shall first beat the other two successively: 'Tis demanded what is the Advantage or Disadvantage of A and B, whom we suppose to begin the Play.

SOLUTION.

Let BA signify that B beats A, and AC that A beats C, and so let always the first Letter denote the Winner, and the second the Loser.

Let us suppose that B beats A the first time; then let us inquire what the Probability is that the Set shall be ended in any number of Games, and also what is the Probability which each Gamester has of winning the Set in that given number of Games.

First, If the Set be ended in two Games, B must necessarily be the winner, for by Hypothesis he wins the first time; which may be expressed by BA, BC.

Secondly,

Secondly, If the Set be ended in three Games, *C* must be the winner, as appears by the following Scheme, *viz.* BA, CB, CA.

Thirdly, If the Set be ended in four Games, *A* must be the winner, as appears by the Scheme BA, CB, AC, AB.

Fourthly, If the Set be ended in five Games, *B* must be the winner, which is thus expressed, BA, CB, AC, BA, BC.

Fifthly, If the Set be ended in six Games, *C* must be the winner, as appears still by the following process, thus, BA, CB, AC, BA, CB, CA.

And this process recurring continually in the same order needs not be prosecuted any farther.

Now the Probability that the first Scheme shall take place is $\frac{1}{2}$, in consequence of the Supposition made that *B* beats *A* the first time; it being an equal Chance whether *B* beat *C*, or *C* beat *B*.

And the Probability that the second Scheme shall take place is $\frac{1}{4}$: for the Probability of *C*'s beating *B* is $\frac{1}{2}$, and that being supposed, the Probability of his beating *A* will also be $\frac{1}{2}$; wherefore the Probability of *C*'s beating *B*, and then *A*, will be $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

And from the same consideration, the Probability that the third Scheme shall take place is $\frac{1}{8}$: and so on.

Hence it will be easy to compose a Table of the Probabilities which *B*, *C*, *A* have of winning the Set in any given number of Games; and also of their Expectations: which Expectations are the Probabilities of winning multiplied by the common Stock deposited at first, and increased successively by the several Fines.

TABLE of the Probabilities, &c.

	B	C	A
2	$\frac{1}{2} \times 3 + 2p$	-----	-----
3	-----	$\frac{1}{4} \times 3 + 3p$	-----
4	-----	-----	$\frac{1}{8} \times 3 + 4p$
5	$\frac{1}{16} \times 3 + 5p$	-----	-----
6	-----	$\frac{1}{32} \times 3 + 6p$	-----
7	-----	-----	$\frac{1}{64} \times 3 + 7p$
8	$\frac{1}{128} \times 3 + 8p$	-----	-----
9	-----	$\frac{1}{256} \times 3 + 9p$	-----
10	-----	-----	$\frac{1}{512} \times 3 + 10p$
&c.			

Now the several Expectations of B , C , A may be summed up by the following Lemma.

LEMMA.

$\frac{n}{b} + \frac{n+d}{bb} + \frac{n+2d}{b^3} + \frac{n+3d}{b^4} + \frac{n+4d}{b^5}$, &c. in infinitum, is equal to $\frac{n}{b-1} + \frac{d}{(b-1)^2}$.

Let the Expectations of B be divided into two Series, viz.

$$\begin{aligned} & \frac{3}{2} + \frac{3}{16} + \frac{3}{128} + \frac{3}{1024}, \text{ \&c.} \\ & + \frac{2p}{2} + \frac{5p}{16} + \frac{8p}{128} + \frac{11p}{1024}, \text{ \&c.} \end{aligned}$$

The first Series constituting a Geometric Progression continually decreasing, its Sum by the known Rules will be found to be $\frac{12}{7}$.

The second Series may be reduced to the form of the Series in our Lemma, and may be thus expressed

$\frac{p}{2} \times \frac{2}{1} + \frac{5}{8} + \frac{8}{8^2} + \frac{11}{8^3} + \frac{14}{8^4}$, &c. wherefore dividing the whole by $\frac{p}{2}$, and laying aside the first term 2, we shall have the Series $\frac{5}{8} + \frac{8}{8^2} + \frac{11}{8^3} + \frac{14}{8^4}$, &c. which has the same form as the Series of the Lemma, and may be compared with it: let therefore n be made $= 5$, $d = 3$, and $b = 8$, and the Sum of the Series will be $\frac{5}{7} + \frac{3}{49}$ or $\frac{38}{49}$; to this adding the first Term 2 which had been laid aside, the new Sum will be $\frac{136}{49}$, and that being multiplied by $\frac{p}{2}$ whereby it had been divided, the product will be $\frac{68}{49}p$, which is the Sum of the second Series expressing the Expectation of B : from whence it may be concluded that all the Expectations of B contained in both the abovementioned Series will be equal to $\frac{12}{7} + \frac{68}{49}p$.

And by the help of the same Lemma, it will be found that all the Expectations of C will be equal to $\frac{6}{7} + \frac{48}{49}p$.

It will be also found that all the Expectations of A will be equal to $\frac{3}{7} + \frac{31}{49}p$.

We have hitherto determined the several Expectations of the Gamblers upon the Sum by them deposited at first, and also upon the Fines by which the common Stock is increased: it now remains to estimate the several Risks of their being fined; that is to say, the Sum of the Probabilities of their being fined multiplied by the respective Values of the Fines.

Now after the Supposition made of A 's being beat the first time, by which he is obliged to lay down his Fine p , B and C have an equal Chance of being fined after the second Game; which makes the Risk of each to be $= \frac{1}{2}p$, as appears by the following Scheme.

$$\frac{BA}{CB} \text{ or } \frac{BA}{BC}$$

In like manner, it will be found, that C and A have one Chance in four, for their being fined after the third Game, and consequently that the Risk of each is $\frac{1}{4}p$, according to the following Scheme.

BA

$$\frac{BA}{CB} \text{ or } \frac{BA}{CA}$$

And by the like Process, it will be found that the Risk of *B* and *C* after the fourth Game is $\frac{1}{8}p$.

Hence it will be easy to compose the following Table, which expresses the Risks of each Gamester.

TABLE of Risks.

	<i>B</i>	<i>C</i>	<i>A</i>
2	$\frac{1}{2}p$	$\frac{1}{2}p$	-----
3	-----	$\frac{1}{4}p$	$\frac{1}{4}p$
4	$\frac{1}{8}p$	-----	$\frac{1}{8}p$
5	$\frac{1}{16}p$	$\frac{1}{16}p$	-----
6	-----	$\frac{1}{32}p$	$\frac{1}{32}p$
7	$\frac{1}{64}p$	-----	$\frac{1}{64}p$
8	$\frac{1}{128}p$	$\frac{1}{128}p$	-----
9	-----	$\frac{1}{256}p$	$\frac{1}{256}p$
&c.			

In the Column belonging to *B*, if the vacant places were filled up by interpolating the Terms $\frac{1}{4}p$, $\frac{1}{32}p$, $\frac{1}{256}p$, &c. the Sum of the Risks of *B* would compose one uninterrupted geometric Progression, the Sum of whose Terms would be $= p$; but the Terms interpolated constitute a geometric Progression whose Sum is $= \frac{2}{7}p$: wherefore if from p there be subtracted $\frac{2}{7}p$, there will remain $\frac{5}{7}p$ for the Sum of the Risks of *B*.

In like manner it will be found that the Sum of the Risks of *C* will be $= \frac{6}{7}p$.

And

And the Sum of the Risks of A , after his being fined the first time, will be $\frac{3}{7}p$.

Now if from the several Expectations of the Gamesters, there be subtracted each Man's Stake, and also the Sum of his Risks, there will remain the clear Gain or Loss of each of them.

Wherefore, from the Expectations of $B = \frac{12}{7} + \frac{68}{49}p$.

Subtracting *first* his Stake $= 1$

Then the Sum of his Risks $= \frac{5}{7}p$.

There remains the clear Gain of $B = \frac{5}{7} + \frac{33}{49}p$.

Likewise from the Expectations of $C = \frac{6}{7} + \frac{48}{49}p$.

Subtracting *first* his Stake $= 1$

Then the Sum of his Risks $= \frac{6}{7}p$.

There remains the clear Gain of $C = -\frac{1}{7} + \frac{6}{49}p$.

In like manner, from the Expectations of $A = \frac{3}{7} + \frac{31}{49}p$.

Subtracting, *first* his Stake $= 1$

Secondly, the Sum of his Risks $= \frac{3}{7}p$.

Lastly, the Fine p due to
the Stock by the Loss of
the first Game $\} = p$.

There remains the clear Gain of $A = -\frac{4}{7} - \frac{39}{49}p$.

But we had supposed, that in the beginning of the Play A was beaten; whereas A had the same Chance to beat B , as B had to beat him: wherefore dividing the Sum of the Gains of B and A into two equal parts, each Part will be $\frac{1}{14} - \frac{3}{49}p$, which consequently must be reputed to be the Gain of each of them.

COROLLARY I.

The Gain of C being $-\frac{1}{7} + \frac{6}{49}p$, let that be made $= 0$, then p will be found to be $= \frac{7}{6}$. If therefore the Fine has the same proportion to each Man's Stake as 7 has to 6, the Gamesters play all upon equal terms: But if the Fine bears a less proportion

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to

to the Stake than 7 to 6, *C* has the disadvantage: thus supposing $p = 1$, his Loss would be $\frac{1}{49}$, but if it bears a greater proportion to the Stake than 7 to 6, *C* has the advantage.

COROLLARY 2.

If the common Stake were constant, that is if there were no Fines, then the Probabilities of winning would be proportional to the Expectations; wherefore supposing $p = 0$, the Expectations after the first Game would be $\frac{12}{7}$, $\frac{6}{7}$, $\frac{3}{7}$, whereof the first belongs to *B*, the second to *C*, and the third to *A*: and therefore dividing the Sum of the Probabilities belonging to *B* and *A* into two equal parts, it will follow that the Probabilities of winning would be proportional to the numbers 5, 4, 5, and therefore it is five to two before the Play begins that either *A* or *B* win the Set, or five to four that one of them that shall be fixed upon wins it.

COROLLARY 3.

Hence likewise if three Gamesters *A*, *B*, *C*, are engaged in a *Poule*, and have not time to play it out; but agree to divide (*S*) the Sum of the Stake and Fines, in proportion to their respective Chances: $\frac{4}{7}$ *S* will be the Share of *B*, whom we suppose to have got one Game; $\frac{2}{7}$ *S* that of *C*, who should next come in; and $\frac{1}{7}$ *S* the Share of *A* who was last beat. For, as they agree to give over playing, all consideration of the subsequent Fines p is now set aside, and the Case comes to that of the first part of *Corol.* 2.

Or the same thing may be shortly demonstrated as follows.

Put $S = 1$, and the Share of *A* = z . Then *B* playing with *C* has an equal Chance for the whole Stake *S*, and for being reduced to the present Expectation of *A*; that is, *B*'s Expectation is $\frac{1+z}{2}$. *C* has an equal chance for 0, and for *B*'s present Expectation; that is,

C's Expectation is $\frac{0 + \frac{1}{2} \times 1 + z}{2} = \frac{1+z}{4}$. But the Sum of the three Expectations $z + \frac{1}{2} \times 1 + z + \frac{1}{4} \times 1 + z = S = 1$; or $z + \frac{3}{4} z (= \frac{7}{4} z) = \frac{1}{4}$; and $z = \frac{1}{7}$, which is *A*'s Share; those of *B* and *C* being $\frac{1}{2} \times 1 + \frac{1}{7}$, and $\frac{1}{4} \times 1 + \frac{1}{7}$; or $\frac{4}{7}$ and $\frac{2}{7}$, respectively.

P R O-

PROBLEM XLV.

If four Gamesters play on the conditions of the foregoing Problem, and he be to be reputed the Winner who beats the other three successively, what is the Advantage of A and B whom we suppose to begin the Play?

SOLUTION.

Let BA denote as in the preceding Problem that B beats A, and AC that A beats C; and universally, let the first Letter always denote the Winner, and the second the Loser.

Let it be also supposed that B beats A the first time: then let it be inquired what is the Probability that the Play shall be ended in any number of Games; as also what is the Probability which each Gamester has of winning the Set in that given number of Games.

First, If the Set be ended in three Games, B must necessarily be the Winner; since by hypothesis he beats A the first Game, which is expressed as follows:

1	BA
2	\overline{BC}
3	BD

Secondly, If the Set be ended in four Games, C must be the winner; as it thus appears.

1	BA
2	\overline{CB}
3	CD
4	CA

Thirdly, If the Set be ended in five Games, D will be the Winner; for which he has two Chances, as it appears by the following Scheme.

1	BA	BA
2	\overline{CB}	\overline{BC}
3	DC	DB
4	DA	DA
5	DB	DC

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Fourthly,

Fourthly, If the Set be ended in six Games, *A* will be the Winner; and he has three Chances for it, which are thus collected.

1	BA	BA	BA
2	\overline{CB}	\overline{CB}	\overline{BC}
3	DC	CD	DB
4	AD	AC	AD
5	AB	AB	AC
6	AC	AD	AB

Fifthly, If the Set be ended in seven Games, then *B* will have three Chances to be the Winner, and *C* will have two, thus;

1	BA	BA	BA	BA	BA
2	\overline{CB}	\overline{CB}	\overline{CB}	\overline{BC}	\overline{BC}
3	DC	DC	CD	DB	DB
4	AD	DA	AC	AD	DA
5	BA	BD	BA	CA	CD
6	BC	BC	BD	CB	CB
7	BD	BA	BC	CD	CA

Sixthly, If the Set be divided in eight Games, then *D* will have two Chances to be the Winner, *C* will have three, and *B* also three, thus;

1	BA	BA	BA	BA	BA	BA	BA	BA
2	\overline{CB}	\overline{CB}	\overline{CB}	\overline{CB}	\overline{CB}	\overline{BC}	\overline{BC}	\overline{BC}
3	DC	DC	DC	CD	CD	DB	DB	DB
4	AD	AD	DA	AC	AC	AD	AD	DA
5	BA	AB	BD	BA	AB	CA	AC	CD
6	CB	CA	CB	DB	DA	BC	BA	BC
7	CD	CD	CA	DC	DC	BD	BD	BA
8	CA	CB	CD	DA	DB	BA	BC	BD

Let now the Letters by which the Winners are denoted be written in order, prefixing to them the numbers which express their several Chances for winning; in this manner.

3	1B
4	1C
5	2D
6	3A
7	3B + 2C
8	3C + 2D + 3B
9	3D + 2A + 3C + 3D + 2A
10	3A + 2B + 3D + 3A + 2B + 2C + 3A + 3D
&c.	

Then

Then carrying this Table a littler farther, and examining the Formation of these Letters, it will appear; *First*, that the Letter B is always found so many times in any Rank, as the Letter A is found in the two preceding Ranks: *Secondly*, that C is found so many times in any Rank as B is found in the preceding Rank, and D in the Rank before that. *Thirdly*, that D is found so many times in any Rank, as C is found in the preceding, and B in the Rank before that: And, *Fourthly*, that A is found so many times in any Rank as D is found in the preceding Rank, and C in the Rank before that.

From all which it may be concluded, that the Probability which the Gamester B has of winning the Set in any number of Games, is $\frac{1}{2}$ of the Probability which A has of winning it one Game sooner, together with $\frac{1}{4}$ of the Probability which A has of winning it two Games sooner.

The Probability which C has of winning the Set in any given number of Games, is $\frac{1}{2}$ of the Probability which B has of winning it one Game sooner, together with $\frac{1}{4}$ of the Probability which D has of winning it two Games sooner.

The Probability which D has of winning the Set in any number of Games is $\frac{1}{2}$ the Probability which C has of winning it one Game sooner, and also $\frac{1}{4}$ of the Probability which B has of winning it two Games sooner.

The Probability which A has of winning the Set in any number of Games is $\frac{1}{2}$ of the Probability which D has of winning it one Game sooner, and also $\frac{1}{4}$ of the Probability which C has of winning it two Games sooner.

These things being observed, it will be easy to compose a Table of the Probabilities which B, C, D, A have of winning the Set in any number of Games, as also of their Expectations, which will be as follows:

		B	C	D	A
I	3	$\frac{1}{4} \times 4 + 3p$	-----	-----	-----
II	4	-----	$\frac{1}{8} \times 4 + 4p$	-----	-----
III	5	-----	-----	$\frac{2}{16} \times 4 + 5p$	-----
IV	6	-----	-----	-----	$\frac{3}{32} \times 4 + 6p$
V	7	$\frac{3}{64} \times 4 + 7p$	$\frac{2}{64} \times 4 + 7p$	-----	-----
VI	8	$\frac{3}{128} \times 4 + 8p$	$\frac{3}{128} \times 4 + 8p$	$\frac{2}{128} \times 4 + 8p$	-----
VII	9	-----	$\frac{3}{256} \times 4 + 9p$	$\frac{6}{256} \times 4 + 9p$	$\frac{4}{256} \times 4 + 9p$
VIII	10	$\frac{4}{512} \times 4 + 10p$	$\frac{2}{512} \times 4 + 10p$	$\frac{6}{512} \times 4 + 10p$	$\frac{9}{512} \times 4 + 10p$
IX	11	$\frac{13}{1024} \times 4 + 11p$	$\frac{10}{1024} \times 4 + 11p$	$\frac{2}{1024} \times 4 + 11p$	$\frac{9}{1024} \times 4 + 11p$
X	12	$\frac{18}{2048} \times 4 + 12p$	$\frac{19}{2048} \times 4 + 12p$	$\frac{14}{2048} \times 4 + 12p$	$\frac{4}{2048} \times 4 + 12p$
&c.	&c.				

The Terms whereof each Column of this Table is composed, being not easily fummable by any of the known Methods, it will be convenient, in order to find their Sums to use the following *Analysis*.

Let $B' + B'' + B''' + B^{IV} + B^V + B^{VI}$, &c. represent the respective Probabilities which *B* has of winning the Set, in any number of Games answering to 3, 4, 5, 6, 7, 8, &c. and let the Sum of these Probabilities *in infinitum* be supposed $= y$.

In the same manner, let $C' + C'' + C''' + C^{IV} + C^V + C^{VI}$, &c. represent the Probabilities which *C* has of winning, which suppose $= z$.

Let the Probabilities which *D* has of winning be represented by $D' + D'' + D''' + D^{IV} + D^V + D^{VI}$, &c. which suppose $= v$.

Lastly, Let the Probabilities which *A* has of winning be represented by $A' + A'' + A''' + A^{IV} + A^V + A^{VI}$, &c. which suppose $= x$.

Now from the Observations set down before in the Table of Probabilities, it will follow, that

B'

$$B' = B'$$

$$B'' = B''$$

$$B''' = \frac{1}{2}A'' + \frac{1}{4}A'$$

$$B^{IV} = \frac{1}{2}A''' + \frac{1}{4}A''$$

$$B^V = \frac{1}{2}A^{IV} + \frac{1}{4}A'''$$

$$B^{VI} = \frac{1}{2}A^V + \frac{1}{4}A^{IV}$$

&c.

From which Scheme we may deduce the Equation following, $y = \frac{1}{4} + \frac{3}{4}x$: for the Sum of the Terms in the first Column is equal to the Sum of the Terms in the other two. But the Sum of the Terms in the first Column is $= y$ by Hypothesis; wherefore y ought to be made equal to the Sum of the Terms in the other two Columns.

In order to find the Sum of the Terms of the second Column, I argue thus,

$$A' + A'' + A''' + A^{IV} + A^V + A^{VI}, \&c. = x \text{ by Hypoth.}$$

$$\text{Theref. } A'' + A''' + A^{IV} + A^V + A^{VI}, \&c. = x - A'$$

$$\text{and } \frac{1}{2}A'' + \frac{1}{2}A''' + \frac{1}{2}A^{IV} + \frac{1}{2}A^V + \frac{1}{2}A^{VI}, \&c. = \frac{1}{2}x - \frac{1}{2}A'$$

Then adding $B' + B''$ on both Sides of the last Equation, we shall have

$$B' + B'' + \frac{1}{2}A'' + \frac{1}{2}A''' + \frac{1}{2}A^{IV} + \frac{1}{2}A^V + \frac{1}{2}A^{VI}, \&c.$$

$$= \frac{1}{2}x - \frac{1}{2}A' + B' + B''.$$

But $A' = 0$, $B' = \frac{1}{4}$, $B'' = 0$, as appears from the Table: wherefore the Sum of the Terms of the second Column is $= \frac{1}{2}x + \frac{1}{4}$.

The Sum of the Terms of the third Column is $\frac{1}{4}x$ by Hypothesis; and consequently the Sum of the Terms in the second and third Columns is $= \frac{3}{4}x + \frac{1}{4}$, from whence it follows that the Equation $y = \frac{1}{4} + \frac{3}{4}x$ had been rightly determined.

And

And by a reasoning like the preceding, we shall find $z = \frac{1}{2}y + \frac{1}{4}v$, and also $v = \frac{1}{2}z + \frac{1}{4}y$, and lastly $x = \frac{1}{2}v + \frac{1}{4}z$.

Now these four Equations being resolved, it will be found that

$$\begin{aligned} B' + B'' + B''' + B^{IV} + B^V + B^{VI}, \&c. = y = \frac{56}{149} \\ C' + C'' + C''' + C^{IV} + C^V + C^{VI}, \&c. = z = \frac{36}{149} \\ D' + D'' + D''' + D^{IV} + D^V + D^{VI}, \&c. = v = \frac{32}{149} \\ A' + A'' + A''' + A^{IV} + A^V + A^{VI}, \&c. = x = \frac{25}{149} \end{aligned}$$

Hitherto we have determined the Probabilities of winning: but in order to find the several Expectations of the Gamesters, each Term of the Series expressing those Probabilities is to be multiplied by the respective Terms of the following Series,

$$4 + 3p, 4 + 4p, 4 + 5p, 4 + 6p, 4 + 7p, 4 + 8p, \&c.$$

The first part of each product being no more than a Multiplication by 4, the Sums of all the first parts of those Products are only the Sums of the Probabilities multiplied by 4; and consequently are $4y, 4z, 4v, 4x$, or $\frac{224}{149}, \frac{144}{149}, \frac{128}{149}, \frac{100}{149}$, respectively.

But to find the Sums of the other parts,

$$\begin{aligned} \text{Let } 3B'p + 4B''p + 5B'''p + 6B^{IV}p, \&c. \text{ be } = pt. \\ 3C'p + 4C''p + 5C'''p + 6C^{IV}p, \&c. \text{ be } = ps. \\ 3D'p + 4D''p + 5D'''p + 6D^{IV}p, \&c. \text{ be } = pr. \\ 3A'p + 4A''p + 5A'''p + 6A^{IV}p, \&c. \text{ be } = pq. \end{aligned}$$

$$\text{Now since } \begin{aligned} 3B' &= 3B' \\ 4B'' &= 4B'' \end{aligned}$$

$$5B''' = \frac{5}{2}A'' + \frac{5}{4}A'$$

$$6B^{IV} = \frac{6}{2}A''' + \frac{6}{4}A''$$

$$7B^V = \frac{7}{2}A^{IV} + \frac{7}{4}A'''$$

$$8B^{VI} = \frac{8}{2}A^V + \frac{8}{4}A^{IV}$$

it follows that $t = \frac{3}{4} + \frac{3}{4}q + x$; for 1°, the first Column is $= t$ by *Hypothesis*.

2°, $3A' + 4A'' + 5A''' + 6A^{IV} + 7A^V, \&c. = q$ by *Hypothesis*.

3°, $A' + A'' + A''' + A^{IV} + A^V, \&c.$ has been found $= \frac{25}{49}$
 $=$ to the value of x . Where

Wherefore adding these two Equations together, we shall have

$$4A' + 5A'' + 6A''' + 7A^{IV} + 8A^V, \&c. = q + x,$$

$$\text{or } \frac{4}{2}A' + \frac{5}{2}A'' + \frac{6}{2}A''' + \frac{7}{2}A^{IV} + \frac{8}{2}A^V, \&c. = \frac{1}{2}q + \frac{1}{2}x.$$

But $A' = 0$, therefore there remains still

$$\frac{5}{2}A'' + \frac{6}{2}A''' + \frac{7}{2}A^{IV} + \frac{8}{2}A^V, \&c. = \frac{1}{2}q + \frac{1}{2}x.$$

Now the Terms of this last Series, together with $3B' + 4B''$, compose the second Column: but $3B' = \frac{3}{4}$ and $4B'' = 0$, as appears from the Table; consequently the Sum of the Terms of the second Column is $= \frac{3}{4} + \frac{1}{2}q + \frac{1}{2}x$.

By the same Method of proceeding, it will be found that the Sum of the Terms of the third Column is $= \frac{1}{4}q + \frac{1}{2}x$.

From whence it follows, that $y = \frac{3}{4} + \frac{1}{2}q + \frac{1}{2}x + \frac{1}{4}q + \frac{1}{2}x$
or $t = \frac{3}{4} + \frac{3}{4}q + x$.

And by the same way of reasoning, we shall find

$$s = \frac{1}{2}t + \frac{1}{2}y + \frac{1}{4}r + \frac{1}{2}v, \text{ and also}$$

$$r = \frac{1}{2}s + \frac{1}{2}z + \frac{1}{4}t + \frac{1}{2}y, \text{ and lastly}$$

$$q = \frac{1}{2}r + \frac{1}{2}v + \frac{1}{4}s + \frac{1}{2}z.$$

But for avoiding confusion, it will be proper to restore the values of x, y, z, v , which being done, the Equations will stand as follows.

$$t = \frac{3}{4} + \frac{3}{4}q + \frac{25}{149} \text{ or } t = \frac{547}{596} + \frac{3}{4}q.$$

$$s = \frac{44}{149} + \frac{1}{2}t + \frac{1}{4}r.$$

$$r = \frac{46}{149} + \frac{1}{2}s + \frac{1}{4}t.$$

$$q = \frac{34}{149} + \frac{1}{2}r + \frac{1}{4}s.$$

Now the foregoing Equations being solved, it will be found that
 $t = \frac{45536}{22201}$, $s = \frac{38724}{22201}$, $r = \frac{37600}{22201}$, $q = \frac{33547}{22201}$.

From which we may conclude that the several Expectations of $B, C, D, A, \&c.$ are respectively,

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First,

First, $4 \times \frac{56}{149} + \frac{45536}{22201} p$; *Secondly*, $4 \times \frac{36}{149} + \frac{38724}{22201} p$.
Thirdly, $4 \times \frac{32}{149} + \frac{37600}{22201} p$; *Fourthly*, $4 \times \frac{25}{149} + \frac{33547}{22201} p$.

The Expectations of the Gamesters being found, it will be necessary to find the Risks of their being fined, or otherwise what Sum each of them ought justly to give to have their Fines insured. In order to which, let us form so many Schemes as are sufficient to find the Law of their Process.

And *First*, we may observe, that upon the Supposition of *B* beating *A* the first Game, in consequence of which *A* is to be fined, *B* and *C* have one Chance each for being fined the second Game, as it thus appears :

1	BA	BA
2	CB	BC

Secondly, that *C* has one Chance in four for being fined the third Game, *B* one Chance likewise, and *D* two; according to the following Scheme.

1	BA	BA	BA	BA
2	CB	CB	BC	BC
3	DC	CD	DB	BD

Thirdly, that *D* has two Chances in eight for being fined the fourth Game, that *A* has three, and *C* one according to the following Scheme.

1	BA	BA	BA	BA	BA	BA
2	CB	CB	CB	CB	BC	BC
3	DC	DC	CD	CD	DB	DB
4	AD	DA	AC	CA	AD	DA

N. B. The two Combinations BA, BC, BD, AB, and BA, BC, BD, BA, are omitted in this Scheme as being superfluous; their disposition shewing that the Set must have been ended in three Games, and consequently not affecting the Gamesters as to the Probability of their being fined the fourth Game; yet the number of all the Chances must be reckoned as being eight; since the Probability of any one Circumstance is but $\frac{1}{8}$.

These Schemes being continued, it will easily be perceived that the circumstances under which the Gamesters find themselves, in respect of their Risks of being fined, stand related to one another in the same manner as did their Probabilities of winning; from which

which consideration a Table of the Risks may easily be composed as follows.

A TABLE of Risks.

		B	C	D	A
I	2	$\frac{1}{2}p$	$\frac{1}{2}p$	-----	-----
II	3	$\frac{1}{4}p$	$\frac{1}{4}p$	$\frac{2}{4}p$	-----
III	4	-----	$\frac{1}{8}p$	$\frac{2}{8}p$	$\frac{3}{8}p$
IV	5	$\frac{3}{16}p$	$\frac{2}{16}p$	$\frac{2}{16}p$	$\frac{3}{16}p$
V	6	$\frac{6}{32}p$	$\frac{5}{32}p$	$\frac{2}{32}p$	$\frac{3}{32}p$
VI	7	$\frac{6}{64}p$	$\frac{8}{64}p$	$\frac{8}{64}p$	$\frac{4}{64}p$
VII	8	$\frac{7}{128}p$	$\frac{8}{128}p$	$\frac{14}{128}p$	$\frac{13}{128}p$
VIII	9	$\frac{17}{256}p$	$\frac{15}{256}p$	$\frac{14}{256}p$	$\frac{22}{256}p$
&c.					

Wherefore supposing $B' + B'' + B'''$, &c. $C' + C'' + C'''$, &c. $D' + D'' + D'''$, &c. $A' + A'' + A'''$, &c. to represent the several Probabilities; and supposing that the several Sums of these Probabilities are respectively y, x, z, v , we shall have the following Equations $y = \frac{3}{4} + \frac{3}{4}x$; $z = \frac{1}{2} + \frac{1}{2}y + \frac{1}{4}v$; $v = \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}y$; $x = \frac{1}{2}v + \frac{1}{4}z$. Which Equations being solved we shall have $y = \frac{243}{149}$, $z = \frac{252}{149}$, $v = \frac{224}{149}$, $x = \frac{175}{149}$.

Let now every one of those Fractions be multiplied by p , and the Products $\frac{243}{149}p$, $\frac{252}{149}p$, $\frac{224}{149}p$, $\frac{175}{149}p$ will express the respective Risks of B, C, D, A , or the Sums they might justly give to have their Fines insured.

But if from the several Expectations of the Gamesters there be subtracted, *First*, the Sums by them deposited in the beginning of the Play, and *Secondly*, the Risks of their Fines, there will remain the clear Gain or Loss of each. Wherefore

$$\text{From the Expectations of } B = \frac{224}{149} + \frac{45536}{22201}p.$$

$$\text{Subtracting his own Stake} = 1$$

$$\text{And also the Sum of his Risks} = \frac{243}{149}p.$$

$$\text{There remains his clear Gain} = \frac{75}{149} + \frac{9329}{22201}p.$$

$$\text{From the Expectations of } C = \frac{144}{149} + \frac{38724}{22201}p.$$

$$\text{Subtracting his own Stake} = 1$$

$$\text{And also the Sum of his Risks} = \frac{252}{149}p.$$

$$\text{There remains his clear Gain} = -\frac{5}{149} + \frac{1176}{22201}p.$$

$$\text{From the Expectations of } D = \frac{128}{149} + \frac{37600}{22201}p.$$

$$\text{Subtracting his own Stake} = 1$$

$$\text{And also the Sum of his Risks} = \frac{224}{149}p.$$

$$\text{There remains his clear Gain} = -\frac{21}{149} + \frac{4224}{22201}p.$$

$$\text{From the Expectations of } A = \frac{100}{149} + \frac{33547}{22201}p.$$

$$\text{Subtracting his own Stake} = 1$$

$$\text{And also the Sum of his Risks} = \frac{175}{149}p.$$

$$\text{Lastly, the Fine due to the } \left. \begin{array}{l} \text{Stock by the Loss of the} \\ \text{first Game} \end{array} \right\} = p.$$

$$\text{There remains his clear Gain} = -\frac{49}{149} + \frac{14729}{22201}p.$$

The foregoing Calculation being made upon the Supposition of *B* beating *A* in the beginning of the Play, which Supposition neither affects *C* nor *D*, it follows that the Sum of the Gains between *B* and *A* ought to be divided equally; then their several Gains will stand as follows:

Gain

$$\begin{array}{rcl} \text{Gain of } \left\{ \begin{array}{l} A = \frac{13}{149} - \frac{2700}{22201}p \\ B = \frac{13}{149} - \frac{2700}{22201}p \\ C = -\frac{5}{149} + \frac{1176}{22201}p \\ D = -\frac{21}{149} + \frac{4224}{22201}p \end{array} \right. & & \\ \text{Sum of the Gains} = & 0 & 0 \end{array}$$

If $\frac{13}{149} - \frac{2700}{22201}p$, which is the Gain of A or B be made $= 0$; then p will be found $= \frac{1937}{2700}$; from which it follows, that if each Man's Stake be to the Fine in the proportion of 2700 to 1937, then A and B are in this case neither Winners nor Losers ; but C wins $\frac{1}{225}$ which D loses.

And in the like manner may be found what the proportion between the Stake and the Fine ought to be, to make C or D play without advantage or disadvantage ; and also what this proportion ought to be, to make them play with any advantage or disadvantage given.

COROLLARY 1.

A Spectator S might at first, in consideration of the Sum $4 + 7p$ paid him in hand, undertake to furnish the four Gamesters with Stakes, and pay all their Fines.

COROLLARY 2.

If the Stock is considerably increased, and the Gamesters agree either to pay no more Fines, or to give over playing, then

1°. If we suppose B to have got the last Game, by beating out A , and call the Stock *Unity* ; the Expectations, or Shares, belonging to B, C, D, A , respectively, will be $\frac{56}{149}, \frac{36}{149}, \frac{32}{149}, \frac{25}{149}$.

2°. If B has got 2 Games, by beating D and A successively, the Shares of B, C, D, A , are $\frac{87}{149}, \frac{28}{149}, \frac{18}{149}, \frac{16}{149}$. For B has now an equal Chance for the whole Stake, or for the lowest Chance of the former Case : that is, his Expectation is worth

$$\frac{1}{2}$$

$\frac{1}{2} \times 1 + \frac{25}{149} = \frac{87}{149}$. C has an equal Chance for 0, and for $\frac{56}{149}$; that is, his Expectation is $\frac{28}{149}$, and in the same way the Numerators of the Expectations of D and A are found.

COROLLARY 3.

If the proportion of Skill between the Gamesters be given, then their Gain or Loss may be determined by the Method used in this and the preceding Problem.

COROLLARY 4.

If there be never so many Gamesters playing on the conditions of this Problem, and the proportion of Skill between them all be supposed to be equal, then the Probabilities of winning or of being fined may be determined as follows.

Let $\overline{B}^I, \overline{C}^I, \overline{D}^I, \overline{E}^I, \overline{F}^I, \overline{A}^I$, denote the Probabilities which B, C, D, E, F, A have of winning the Set, or of being fined in any number of Games; and let the Probabilities of winning or of being fined in any number of Games less by one than the preceding, be denoted by $\overline{B}^{II}, \overline{C}^{II}, \overline{D}^{II}, \overline{E}^{II}, \overline{F}^{II}, \overline{A}^{II}$: and so on; then I say that

$$\begin{aligned}\overline{B}^I &= \frac{1}{2}\overline{A}^{II} + \frac{1}{4}\overline{A}^{III} + \frac{1}{8}\overline{A}^{IV} + \frac{1}{16}\overline{A}^V \\ \overline{C}^I &= \frac{1}{2}\overline{B}^{II} + \frac{1}{4}\overline{B}^{III} + \frac{1}{8}\overline{B}^{IV} + \frac{1}{16}\overline{B}^V \\ \overline{D}^I &= \frac{1}{2}\overline{C}^{II} + \frac{1}{4}\overline{C}^{III} + \frac{1}{8}\overline{C}^{IV} + \frac{1}{16}\overline{C}^V \\ \overline{E}^I &= \frac{1}{2}\overline{D}^{II} + \frac{1}{4}\overline{D}^{III} + \frac{1}{8}\overline{D}^{IV} + \frac{1}{16}\overline{D}^V \\ \overline{F}^I &= \frac{1}{2}\overline{E}^{II} + \frac{1}{4}\overline{E}^{III} + \frac{1}{8}\overline{E}^{IV} + \frac{1}{16}\overline{E}^V \\ \overline{A}^I &= \frac{1}{2}\overline{F}^{II} + \frac{1}{4}\overline{F}^{III} + \frac{1}{8}\overline{F}^{IV} + \frac{1}{16}\overline{F}^V\end{aligned}$$

Now the Law of these relations being visible, it will be easy to extend it to any other number of Gamesters.

COROLLARY 5.

If there be several Series so related to one another, that each Term of one Series may have a certain given proportion to some one assigned Term in each of the other Series, and that the order of these proportions be constant and uniform, then will all those Series be exactly summable.

REMARK

REMARK.

As the Application of the Doctrine contained in these Solutions and Corollaries may appear difficult when the Gamesters are many, and when it is required to put an end to the play by a fair distribution of the money in the *Poule*; which I look upon as the most useful Question concerning this Game: I shall explain this Subject a little more particularly.

1. Let us then Suppose any number of Gamesters, $n + 1$ (as, in our Scheme, 6) and having written down so many Letters

Number of Games won by B	$n-5$	0	$A \}$ $B \}$	C	D	E	F	
			b	c	d	e	f	
	$n-4$	I	B	C	D	E	F	A
			b^I	c^I	d^I	e^I	f^I	a^I
	$n-3$	II	b^{II}	c^{II}	d^{II}	e^{II}	f^{II}	a^{II}
	$n-2$	III	b^{III}	c^{III}	d^{III}	e^{III}	f^{III}	a^{III}
	$n-1$	IV	b^{IV}	c^{IV}	d^{IV}	e^{IV}	f^{IV}	a^{IV}
	n	V	1	0	0	0	0	0

as there are Gamesters in the Order they are to succeed one another, place under them their respective small Letters, to denote the Probabilities which the several Gamesters have of winning the *Poule*, immediately after their Order of Succession

is fixt, and before the play is begun. Where note that the Letter b signifies ambiguously the Expectation of A or of B : and this Case being particular, not to occur again in the same *Poule*, may be separated from the others by a line.

We shall always suppose B to be the Winner of the first Game; and that A takes the lowest place in the second Row of Capitals. Under these repeat $n-1$ Rows of the small Letters which, with the small strokes or dots affixed to them mark the Expectations of the several Gamesters, when any one Gamester has got as many Games as is the Number of dots, or that which is marked in Roman Characters to the right of the Row. For it is to be observed, that, after the first Game, the small Letters thus marked do not, unless by accident, signify the Expectations of the particular Gamesters at first denoted by their Capitals; but the Expectations which belong to the Rank and Column where any Letter stands. For Example, b^{III} does not denote the Expectations of him who was supposed to get the first Game, unless perhaps he has got two more successively; but indefinitely, those of whatever Gamester has got 3 Games following. And the other Letters of the same Row, as c^{III} , d^{III} , e^{III} , signify the simultaneous Expectations of the three Gamesters that follow him in the Order of playing.

2. This

2. This preparation being made, it will be obvious in what manner the Expectations are varied by the Event of every Game; and how they are always reducible to known Numbers.

For if we suppose *B*, the Gamester who is in Play, to have got 3 Games, for instance, and to want two more of the *Poule*; then his present Expectation being b''' , if he wins the next Game which he is to play with *C*, the Consequence will be; 1°. His own Expectations will be changed into b'' ; having now got 4 Games. 2°. All the other Expectations in the same Row, will likewise be transferred to the next inferior (IV.) but marked each by the preceding Letter of the Alphabet: that is, d''' becomes c'' , e''' becomes d'' , &c. excepting only c''' , the Expectation of him who lost the Game, which is thereby reduced to the lowest Expectation a'' . And if *B* had already gained $(n-1=)$ 4 Games, and consequently wanted but one; if he gains this, all the Expectations c'' , d'' , e'' , &c. will vanish together, while b'' becomes $=1$, the Exponent of Certainty.

But if *B* loses his Game with *C*, all the Expectations, of whatever Rank, are transferred to the Rank I, and their Ratios are restored as when the first Game was won: only the Letters are changed into the next preceding. As b''' becomes a' , c''' becomes b' , d''' becomes c' , and so on.

3. Now there being supposed an equal Chance of *B*'s winning and losing a Game; any Expectation of his, as when he has got 3 Games, will be thus expressed; $b''' = \frac{b'' + a'}{2}$; in which, substituting for b'' its equal in our Example $\frac{1+a'}{2}$, we shall have $b''' = \frac{1+3 \times a'}{4}$. The same way, $b'' = \frac{b' + a'}{2} = \frac{1+7a'}{8}$; and $b' = \frac{1+15a'}{16}$. In general; when the number of Games that *B* wants of gaining the *Poule* is m , then shall $\frac{1+2^m-1 \times a'}{2^m}$ be the value of his Expectations.

4. The other Expectations are collected nearly in the same manner. As $c''' = \frac{a''' + b'}{2}$, in which substituting for a''' its equal (in our example) $\frac{0+f'}{2}$, we have $c''' = \frac{f' + 2b'}{4} = \frac{1}{2} b' + \frac{1}{4} f'$. The same way, $c'' = \frac{1}{2} b' + \frac{1}{4} f' + \frac{1}{8} e'$; and $c' = \frac{1}{2} b' + \frac{1}{4} f' + \frac{1}{8} e' + \frac{1}{16} d'$; the number of Terms added to the Games won being always $=n$, and the Letter a' always omitted.

From

From all which it appears, that the Expectation of a Gamester, in any State of the Play, is expressed by the Expectations a' , b' , c' , &c. after one Game is won: and that these, therefore, are first to be computed.

5. In order to which, I say, that the Letters b , c , d , e , &c. expressing, as above, the Chances of the Gamesters, B , C , D , E , &c. immediately after their Order of playing is fixt by lot, or otherwise; these Chances are in the geometrical Progreffion of $1 + 2^n$ to 2^n .

For either of the Gamesters (as A) who play the first Game, has 1 out of 2^n Chances of beating all his Adversaries in one Round. And therefore he may, in consideration of the Sum $\frac{1}{2^n} \times \overline{b+c+d+e}$

give up his expectations arising from the Probability of that Event, and take the lowest place with the Expectation e ; the Gamester C succeeding to his place, D to that of C ; and so on. But B having, on the score of Priority, the same demand upon A , as A has upon B ; that is, neither having any demand upon the other, the Term $\frac{1}{2^n} \times b$ is to be cancelled; and the Value of A 's place, with

respect to the other Gamesters, reduced to $\frac{1}{2^n} \times c + \frac{1}{2^n} \times d +$
&c. And now each of the Gamesters C , D , E , &c. being raised to the next higher Expectation b , c , d , &c. for which he has paid $\frac{1}{2^n}$ of his former Expectation; it follows that $b = 1 + \frac{1}{2^n} \times c$,

$c = 1 + \frac{1}{2^n} \times d$, &c. and that, before the play is begun, every

Expectation is to the next below it as $1 + \frac{1}{2^n}$ to 1, or as $1 + 2^n$ to 2^n . Which coincides with Theor. I. of Mr. *Nicolas Bernoulli* in Phil. Transf. N. 341.

Thus if the Gamesters are 3, (A) B , C ; their first Expectations are (5) 5, 4, with the common Denominator 14. If they are 4, (A) B , C , D , their Expectations are (81) 81, 72, 64, with the Denominator 298. If there are 5 Gamesters, their Expectations are (17³) 17³, 17² × 16, 17 × 16², 16³, with their Sum for a Denominator; that is, (4913), 4913, 4624, 4352, 4096, with the Denominator 22898. And the like for any number of Gamesters.

6. It is plain likewise that the Expectations of all the Gamesters, excepting A and B , remain the same after one Game is plaid, as they were at first; $c' = c$, $d' = d$, $e' = e$, &c. because the contest

X

in

in the first Game concerns A and B alone; its Event making no alteration in the Expectations of the others: but only raising B 's first expectation, which was b , to the Value b' , and diminishing the equal Expectation of A by the same quantity: so that $a' + b' = 2b$.

And therefore, to find all the Expectations after the first Game is played, we have now only to compute the first and last of that Rank, b' and a' .

But it was found already that if m represents the number of Games that the last Winner B wants to gain the *Poule*, his Expectations in that Circumstance will be equal to $\frac{1 + 2^m - 1 \times a'}{2^m}$. From which, putting

$m = n - 1$, which is the Case when B has got one Game, and the Expectation b' ; and substituting for b' its equal $2b - a'$, we shall get $a' = \frac{2^n b - 1}{2^n - 1}$.

As when there are 3 Gamesters, $n = 2$, $b = \frac{5}{14}$ and $a' = \frac{\frac{20}{14} - 1}{3} = \frac{6}{42} = \frac{1}{7}$. And $b' = 2b - a' = \frac{10}{14} - \frac{2}{14} = \frac{4}{7}$.

If there are 4 Gamesters, $n = 3$, $b = \frac{81}{298}$; and therefore $a' = 8 \times \frac{81}{298} - 1 \times \frac{1}{7} = \frac{350}{298} \times \frac{1}{7} = \frac{50}{298} = \frac{25}{149}$. And $b' = \frac{81}{149} - \frac{25}{149} = \frac{56}{149}$.

If there are 5 Gamesters, $n = 4$, $b = \frac{4913}{22898}$; whence $a' = 16 \times \frac{4913}{22898} - 1 \times \frac{1}{15} = \frac{55710}{22898} \times \frac{1}{15} = \frac{3714}{22898} = \frac{1857}{11449}$.

And $b' = 2b - a' = \frac{4913}{11449} - \frac{1857}{11449} = \frac{3056}{11449}$. So that the Expectations of the Gamesters, B having got one Game, will stand thus:

B	C	D	E	A
b'	c'	d'	e'	a'

3056, 2312, 2176, 2048, 1857; these numbers expressing the Ratios of the Expectations; and with the Denominator 11449 subscribed, their absolute quantity; or the Shares of the whole Stake due to each Gamester, if they were to give over playing.

7. And thus the Probabilities which the several Gamesters have of gaining the *Poule* may in all Cases be computed, and disposed into Tables. But the 6 following, will, 'tis thought, be more than sufficient for any Case that happens in play.

TABLE

TABLE I. For a Poule of Three.

Games won	$\left. \begin{matrix} A \\ B \end{matrix} \right\}$	C	Denom.
0	5	4	$5 + 5 + 4 = 14.$

I.	B	C	A	Denom.
	4	2	1	7.

Tab. II. For a Poule of Four.

	$\left. \begin{matrix} A \\ B \end{matrix} \right\}$	C	D	Denom.
0	81	72	64	298.

	B	C	D	A	Denom.
I.	56	36	32	25	149.
II.	87	28	18	16	

Tab. III. For a Poule of Five.

	$\left. \begin{matrix} A \\ B \end{matrix} \right\}$	C	D	E	Denom.
0	4913	4624	4352	4096	22898.

	B	C	D	E	A	Denom.
I.	3056	2312	2176	2048	1857	11449.
II.	4255	2040	1920	1666	1568	
III.	6653	1528	1156	1088	1024	

Tab. IV. For a Poule of Six.

	$\left. \begin{matrix} A \\ B \end{matrix} \right\}$	C	D	E	F	Denom.
0	1185921	1149984	1115136	1081344	1048576	6766882.

	B	C	D	E	F	A
I.	682976	574992	557568	540672	524288	502945
II.	863007	540144	523776	507904	481602	467008
III.	1223069	472560	458240	422532	409728	397312
IV.	1943193	341488	287496	278784	270336	262144

X 2

Denom. 3383441.
One

One Example will shew the Use of the Tables: Suppose 5 Gamesters engaged in a *Poule*, with this condition, that if it is not ended when a certain number of Games are played, they shall give over, and divide the Money in proportion to the Chances they shall then have of winning the *Poule*. That number of Games being played, suppose the *Poule* risen to 30 Guineas, and that a Gamester (*B*) has got two Games: *Qu.* how the 30 Guineas are to be shared?

Divide $31 \frac{1}{2} l.$ into Shares proportional to the numbers 4255, 2040, &c. (in Tab. III.) which stand in the Row of *Games won* II. and those Shares will be as follow: $31 \frac{1}{2} l. \times \frac{4255}{11449} =$ the Share of

	<i>l.</i>	<i>s.</i>	<i>d.</i>
<i>B</i>	11	14	2
<i>C</i>	5	12	3
<i>D</i>	5	5	8
<i>E</i>	4	11	8
<i>A</i>	4	6	3
<i>L.</i>	31	10	0

Note, the pricked Line which is drawn in each of the Tables separates the Chances of the Gamesters who are *necessarily* to come into the play before the *Poule* is won, from the Chances of those who *may possibly* not come in again; which lie below that line. And, setting aside the Column *B*, all the Chances in any Row above the line are in the continued Ratio of $1 + 2^n$ to 2^n . As in Tab. III. $d'' = \frac{16}{17} \times c''$, or $1920 = 1 - \frac{1}{17} \times 2040$.

The same is true of the Terms of any Row that lie both below the line. But if one lies above and the other below it, their Relation is different, and is to be found by *Art.* 3. of this Remark.

It remains to compute the Profit and Loss upon the Fines *p*: as follows.

1. The present Expectations of a Gamester who is *entering*, or to enter, into play, that he shall be the Winner, are made up of his several present Expectations, upon the Events of his coming in *once*, *twice*, *thrice*, &c. as is manifest. And as, immediately after the Order of playing is fixt, it was shewn that those *total* Expectations are in the geometrical Progression of $1 + 2^n$ to 2^n , the number of Gamesters being $n + 1$; so, in any other State of the *Poule*, their Ratio is always given.

But

But every time that a Gamester enters, his Chance of *winning* in that *Turn*, is to his Chance of paying a *Fine*, as 1 to $2^n - 1$: and therefore, *componendo*, the Sum of a Gamester's several Expectations of *winning*, is to the Sum of his several Risks of paying a *Fine*, in the same Ratio; the whole Stake, and also each *Fine* p , being put $= 1$. And the whole *Risks* of the several Gamesters are in the same Ratios as their *Expectations*.

Thus in the Case of *Three* Gamesters, whose Expectations are $\frac{5}{14}$, $\frac{5}{14}$, $\frac{4}{14}$, their Chances of paying the *Fine* p will be the same Fractions multiplied into 3 ($= 2^n - 1$); that is, they will be $\frac{15}{14}$, $\frac{15}{14}$, $\frac{12}{14}$.

And the first Expectations of *Four* Gamesters being 81, 81, 72, 64, to the Denominator 298; their Chances of being *Fined* will be the same Numerators multiplied into 7 ($= 2^n - 1$), that is, $\frac{567}{298}$, $\frac{567}{298}$, $\frac{504}{298}$, $\frac{448}{298}$; respectively.

Hence again it appears, that the Total of the *Fines*, or the Sum for which they may be furnished throughout the *Poule*, is $2^n - 1 \times p$. For the Sum of the Expectations upon the Stake 1, is 1; and these are to the Number of *Fines* as 1 to $2^n - 1$.

2. Suppose now that one of the first Players of *Three*, as *A*, is beat out, and his *Fine* paid, as must always necessarily happen; and thence, the Expectations of getting the *Poule* reduced to

$\frac{4}{7}$	$\frac{2}{7}$	$\frac{1}{7}$
<i>C</i>	<i>B</i>	<i>A</i>

: then the Risks of *C* and *A* will be $\frac{6}{7}$, $\frac{3}{7}$, respectively: whose Sum $\frac{9}{7}$ taken from 2 ($= 2^n - 2$) leaves $\frac{5}{7}$ for the *Fines* of *B*.

In like manner, the Expectations of *Four* Gamesters, after one Game is won, being 56, 36, 32, 25, with the Denominator 149; the Numerators of the Risks of the Three last Gamesters *C*, *D*, *A*, will be 36, 32, 25, multiplied by 7 ($= 2^n - 1$) to the same Denominator; and their Sum taken from the *Fines* to be paid after one Game is won, which are $6 = 2^n - 2$, leaves for the Risks of *B*, $\frac{243}{149}$: those of *C*, *D*, *A*, being $\frac{252}{149}$, $\frac{224}{149}$, $\frac{175}{149}$, respectively.

3. If *B* has got more than one Game, the Sums for which a Spectator *R* may furnish all the subsequent *Fines*, will be found as follows.

Let

Let the Number of *Fines* which *R* risks to pay, when *B* has got 1, 2, 3, 4, &c. Games, be x, y, z, v , &c. respectively; then $\frac{x+1+y+1}{2} = x$; or $y = x - 2$, $\frac{x+1+z+1}{2} = y = x - 2$; or $z = x - 2^2 + 2$. And the same way $v = x - 2^3 + 2^2 + 2$, &c.; in an obvious Progression.

Because when *B* has got 1 Game, there is an equal Chance of his winning or losing the next; in the former Case, *R* pays the *Fine* $1 \times p$ for *C*, and comes to have the Risk y ; but if *C* wins, *R* pays $1 \times p$ for *B*, and his Risk x is the same as before: and so of the rest. So that the number of pieces p for which *R* may engage to furnish the subsequent *Fines*, when *B* has got 2, 3, 4, &c. Games, is had by the continual Subtraction of 2 and its Powers from $2^n - 2$. As in a *Poule* of four, when *B* has got 2 Games, the Sum of the Risks is $6 - 2 = 4$. In a *Poule* of five, $x = 2^n - 2 = 14$, $y = 12$, $z = 8$, $v = 0$.

And from these numbers subtracting the Risks of the other Gamesters *C, D, E*, &c. found as above, there will remain the Risks of *B* the Gamester who continues in play.

4. The Expectations of the several Gamesters upon the *Fines* may likewise be determined by an obvious, but more troublesome, Operation.

Under the Capitals, *B C D E A*, write their small Letters thus:

I. $b^I \quad c^I \quad d^I \quad e^I \quad a^I$
 II. $b^{II} \quad c^{II} \quad d^{II} \quad e^{II} \quad a^{II}$
 III. $b^{III} \quad c^{III} \quad d^{III} \quad e^{III} \quad a^{III}$

IV. $b^I \quad 0 \quad 0 \quad 0 \quad 0$ Signifying, respectively, the Number of *Fines* which a Gamester, winning the *Poule*, may expect to find in it, *B* having already got so many Games as the *Dots* affixed to the Letter: and to these Letters prefix their fractional Coefficients taken from the *Tables of Probabilities*. Then, by the law of the Game, there will be formed a Series of Equations determining the Expectations sought.

As in the Case of 3 Gamesters, write,

$$\left. \begin{array}{l} B \quad C \quad A \\ \text{I. } \frac{4}{7}b^I \quad \frac{2}{7}c^I \quad \frac{1}{7}a^I \\ \text{II. } 1 \times b^{II} = 2 \quad 0 \quad 0 \end{array} \right\} \text{and the Equations } \left\{ \begin{array}{l} 1^\circ. \frac{4}{7}b^I = \frac{1}{2} \times 2 + \frac{1}{7} \times a^I + 1 \\ 2^\circ. \frac{2}{7}c^I = \frac{1}{2} \times \frac{4}{7} \times b^I + 1 + 0 \\ 3^\circ. \frac{1}{7}a^I = \frac{1}{2} \times \frac{2}{7} \times c^I + 1 + 0 \end{array} \right.$$

Which

Which being reduced give $B' (= \frac{4}{7}b') = \frac{68}{49}$, $C' (= \frac{2}{7}c') = \frac{48}{49}$, $A' (= \frac{1}{7}a') = \frac{31}{49}$. From which subtracting their respective Risks; for B , $\frac{35}{49}$; for C , $\frac{42}{49}$; and for A , $\frac{70}{49} (= \frac{3}{7} + 1$, his *Fines*, and the *Fine* already paid in), remain the Gains $+\frac{33}{49}$, $+\frac{6}{49}$, $-\frac{39}{49}$, multiplied into p .

If it is a *Poule* of four, the Expectations on the *Fines* will stand thus:

$$\text{I. } \frac{56}{149}b' \quad \frac{36}{149}c' \quad \frac{32}{149}d' \quad \frac{25}{149}a'$$

$$\text{II. } \frac{87}{149}b'' \quad \frac{28}{149}c'' \quad \frac{18}{149}d'' \quad \frac{16}{149}a''$$

$$\text{III. } 1 \times b''' = 3 \quad 0 \quad 0 \quad 0 \quad \text{and setting aside the}$$

common Denominator 149, the Equations will be;

$$1^\circ. 56b' = \frac{1}{2} \times 25 \cdot \overline{a' + 1} + 87b'' \quad 5^\circ. 28c'' = \frac{1}{2} \times 56 \cdot \overline{b' + 2} + 0 \text{ i. e. } c'' = b' + 2$$

$$2^\circ. 36c' = \frac{1}{2} \times 56 \cdot \overline{b' + 1} + 16a'' \quad 6^\circ. \quad \quad \quad d'' = c' + 2$$

$$3^\circ. 32d' = \frac{1}{2} \times 36 \cdot \overline{c' + 1} + 28c'' \quad 7^\circ. \quad \quad \quad a'' = d' + 2$$

$$4^\circ. 25a' = \frac{1}{2} \times 32 \cdot \overline{d' + 1} + 18d'' \quad 8^\circ. \frac{87}{149}b'' = \frac{1}{2} \times 3 + \frac{25}{149} \cdot \overline{a' + 1}.$$

Whence will be found

$$\left. \begin{aligned} b' &= \frac{813 \frac{1}{7}}{149}; \text{ and } \left(\frac{56}{149} b' \right) B' = \frac{45536}{22201} \\ c' &= \frac{1075 \frac{2}{3}}{149}; \text{ and } C' = \frac{38724}{22201} \\ d' &= \frac{1175}{149}; \text{ and } D' = \frac{37600}{22201} \\ a' &= \frac{1341 \frac{22}{25}}{149}; \text{ and } A' = \frac{33547}{22201} \end{aligned} \right\} \left\{ \begin{aligned} b'' &= \frac{618 \frac{34}{87}}{149}; \text{ and } B'' = \frac{53800}{22201} \\ c'' &= \frac{1111 \frac{1}{7}}{149}; \text{ and } C'' = \frac{31112}{22201} \\ d'' &= \frac{1373 \frac{2}{3}}{149}; \text{ and } D'' = \frac{24726}{22201} \\ a'' &= \frac{1473}{149}; \text{ and } A'' = \frac{23568}{22201} \end{aligned} \right.$$

And the like Computations may be made for the superior *Poules*; the Composition of the Equations to be reduced being regular and obvious.

P R O B L E M XLVI.

Of HAZARD.

To find at Hazard the Advantage of the Setter upon all Suppositions of Main and Chance.

SOLUTION.

Let the whole Money played for be considered as a common Stake, upon which both the Caster and the Setter have their several Expectations; then let those Expectations be determined in the following manner.

First, Let it be supposed that the Main is VII: then if the Chance of the Caster be VI or VIII, it is plain that the Setter having 6 Chances to win, and 5 to lose, his Expectation will be $\frac{6}{11}$ of the Stake: but there being 10 Chances out of 36 for the Chance to be VI, or VIII, it follows, that the Expectation of the Setter resulting from the Probability of the Chance being VI or VIII, will be $\frac{10}{36}$ multiplied by $\frac{6}{11}$ or $\frac{60}{11}$ divided by 36.

Secondly, If the Main being VII, the Chance should happen to be V or IX, the Expectation of the Setter would be $\frac{24}{5}$ divided by 36.

Thirdly, If the Main being VII, the Chance should happen to be IV or X, it follows that the Expectation of the Setter would be 4 divided by 36.

Fourthly, If the Main being VII, the Caster should happen to throw II, III, or XII, then the Setter would necessarily win, by the Law of the Game; but there being 4 Chances in 36 for throwing II, III, or XII, it follows that before the Chance of the Caster is thrown, the Expectation of the Setter resulting from the Probability of the Caster's Chance being II, III, or XII, will be 4 divided by 36.

Lastly, If the Main being VII, the Caster should happen to throw VII, or XI, the Setter loses his Expectation.

From the Solution of the foregoing particular Cases it follows, that the Main being VII, the Expectation of the Setter will be expressed

pressed by the following Quantities, *viz.* $\frac{60}{11} + \frac{24}{5} + \frac{4}{1} + \frac{4}{1}$

$$\frac{60}{11} + \frac{24}{5} + \frac{4}{1} + \frac{4}{1} = \frac{60 \cdot 5 + 24 \cdot 11 + 4 \cdot 55 + 4 \cdot 55}{55} = \frac{300 + 264 + 220 + 220}{55} = \frac{1004}{55}$$

which may be reduced to $\frac{251}{495}$; now this fraction being subtracted from Unity, to which the whole Stake is supposed equal, there will remain the Expectation of the Caster, *viz.* $\frac{244}{495}$.

But the Probabilities of winning being always proportional to the Expectations, on Supposition of the Stake being fixt, it follows that the Probabilities of winning for the Setter and Caster are respectively proportional to the two numbers 251 and 244, which properly denote the Odds of winning.

Now if we suppose each Stake to be 1, or the whole Stake to be 2, the Gain of the Setter will be expressed by the fraction $\frac{7}{495}$, it being the difference of the numbers expressing the Odds, divided by their Sum, which supposing each Stake to be a Guinea of 21 Shillings will be about $3^d. - 2 \frac{1}{4}f$.

By the same Method of Process, it will be found that the Main being VI or VIII, the Gain of the Setter will be $\frac{167}{7128}$ which is about $5^d. - 3 \frac{1}{2}f$ in a Guinea.

It will be also found that the Main being V or IX, the Gain of the Setter will be $\frac{43}{2835}$, which is about $3^d. - 3 \frac{1}{3}f$ in a Guinea.

COROLLARY I.

If each particular Gain made by the Setter, in the Case of any Main, be respectively multiplied by the number of Chances which there are for that Main to come up, and the Sum of the Products be divided by the number of all those Chances, the Quotient will express the Gain of the Setter before a Main is thrown: from whence it follows that the Gain of the Setter, if he be resolved to set upon the first Main that may happen to be thrown, is to be estimated by $\frac{42}{495} + \frac{1670}{7128} + \frac{344}{2835}$, the whole to be divided by 24, which being reduced will be $\frac{37}{2010}$, or about $4^d. - 2 \frac{1}{2}f$ in a Guinea.

COROLLARY 2.

The Probability of no Main, is to the Probability of a Main as $109 + 2$ to $109 - 2$, or as 111 to 107.

COROLLARY 3.

If it be agreed between the Caster and Setter, that the Main shall always be VII, and it be farther agreed, that the next Chance happening to be Ames-ace, the Caster shall lose but half his Stake, then the Caster's Loss is only $\frac{1}{3960}$ of his Stake, that is about $\frac{1}{4}f$ in a Guinea.

COROLLARY 4.

The Main being VI or VIII, and the Caster has $\frac{3}{4}$ of his money returned in case he throws Ames-ace, what is his Loss? And if the Main being V or IX, and he has $\frac{1}{2}$ of his Money returned in case he throws Ames-ace, what is his Loss? In answer to the first, the Gain of the Setter or Loss of the Caster is $\frac{1}{385 \frac{11}{37}}$.

In answer to the second the Loss of the Caster would be but $\frac{1}{782 \frac{2}{29}}$.

COROLLARY 5.

If it be made a standing Rule, that whatever the Main may happen to be, if the Caster throws Ames-ace immediately after the Main, or in other words, if the Chance be Ames-ace, the Caster shall only lose $\frac{1}{3}$ of his own Stake, then the Play will be brought so near an Equality, that it will hardly be distinguishable from it; the Gain of the Caster being upon the whole but $\frac{1}{6048}$ of his own Stake, or $\frac{1}{6}$ of a farthing in a Guinea.

The Demonstration of this is easily deduced from what we have said before *viz.* that the Loss of the Caster is $\frac{37}{2016}$; now let us consider what part of his own Stake should be returned him in case he throws Ames-ace next after the Main; Let x be that part, but the Probability of throwing Ames-ace next after the Main is $\frac{1}{36}$, therefore, the real Value of what is returned him is $\frac{1}{36}x$, and since the Play is supposed to be reduced to an Equality, then what is returned him must equal his Loss; for which reason, we have the Equation $\frac{x}{36} = \frac{37}{2016}$, or $x = \frac{37}{56}$ which being very near

near $\frac{2}{3}$, it follows that $\frac{2}{3}$ of his own Stake ought to be returned him.

Or thus; if the Caster has returned him $\frac{37}{50}$ when that happens, he loses nothing; but there being but 1 Chance in 36 for that Case to happen; the real Value of what is returned is but $\frac{37}{50 \times 36}$; and in the same manner if $\frac{2}{3}$ is returned, the real Value is $\frac{2}{3 \times 36}$: and so, the Difference $\frac{2}{3 \times 36} - \frac{37}{50 \times 36} = \frac{1}{6048}$ is the Gain of the Caster.

P R O B L E M XLVII.

To find at Hazard the Gain of the Box for any number of Games divisible by 3.

SOLUTION.

Let a and b respectively represent the Chances for winning a Main or for losing it, which is usually called a Main and no Main; then,
1°, It is very visible that when the four last Mains are $baaa$, otherwise that when a Main has been lost, if the three following Mains are won successively, then the Box must be paid.

2°, That the last 7 Mains being $baaaaaa$, there is also a Box to be paid.

3°, That the last 10 Mains being $baaaaaaaaaa$, the Box is to be paid, and so on.

Now the Probability of the 4 last Mains being $baaa$ is $\frac{ba^3}{(a+b)^4}$, and consequently, if the number of Mains thrown from the beginning is represented by n , the Gain of the Box upon this account will be $\frac{n - 3 \times ba^3}{(a+b)^4}$.

But to obviate a difficulty which may perhaps arise concerning the foregoing Expression which one would naturally think must be $\frac{nba^3}{(a+b)^4}$, it must be remembered that the Termination $baaa$ belongs to 4 Games at least, and that therefore the three first Games are to be excluded from this Case, tho' they shall be taken notice of afterwards.

Again the Probability of the 7 last Mains terminating thus *baaaaaa*, will be $\frac{ba^6}{(a+b)^7}$, but this Case does not belong to the 6 first Mains, therefore the Gain of the Box upon this account will be $\frac{n-6 \times ba^6}{(a+b)^7}$; and so on.

And therefore the first part of the Expectation of the Box is expressed by the Series

$$\frac{n-3 \times ba^3}{(a+b)^4} + \frac{n-6 \times ba^6}{(a+b)^7} + \frac{n-9 \times ba^9}{(a+b)^{10}} + \frac{n-12 \times ba^{12}}{(a+b)^{13}}, \&c.$$

of which the number of Terms is $\frac{n-3}{3}$.

The second part of the Expectation of the Box arises from all the Mains being won successively without any interruption of a no Main, and this belongs particularly to the three first Mains, as well as to all those which are divisible by 3, and therefore the second part of the Expectation of the Box will be expressed by the Series $\frac{a^3}{(a+b)^3} + \frac{a^6}{(a+b)^6} + \frac{a^9}{(a+b)^9} + \frac{a^{12}}{(a+b)^{12}}, \&c.$ of which the number of Terms is $\frac{n}{3}$.

Those who will think it worth their while to sum up these Series, may without much difficulty do it, if they please to consult my *Miscellanea*, wherein such sorts of Series, and others more compound, are largely treated of.

In the mean time, I shall here give the Result of what they may see there demonstrated.

If the first Series be distinguished into two others, the first positive, the other negative, we shall now have three Series, the Sums of which will be, supposing $\frac{a}{a+b} = r$.

$$1^o, \frac{nb}{a+b} \times \frac{r^3 - r^n}{1 - r^3}$$

$$2^o, - \frac{3b}{a+b} \times \frac{r^3 \frac{1}{3} nr^n + \frac{1}{3} \times n-1 \times r^{n+3}}{1 - r^3}$$

$$3^o, \frac{r^3 - r^{n+3}}{1 - r^3}$$

the sum of all which will be reduced to the Expression $\frac{n}{14} - \frac{5}{49}$ $+\frac{5}{49 \times 2^n}$, when *a* and *b* are in a Ratio of Equality.

COROLLARY I.

If *n* be an infinite number, the Gain of the Box will be universally expressed by $\frac{nb}{a+b} \times \frac{a^3}{(a+b)^3 - a^3}$; but when *a* and *b* are in a Ratio of Equality by $\frac{n}{14}$.

CORO-

COROLLARY 2.

The Gain of the Box being such as has been determined for an infinite number of Mains, it follows that, one with another, the Gain of the Box for one single Main ought to be estimated by

$$\frac{b}{a+b} \times \frac{a^3}{a+b)^3 - a^3}, \text{ or } \frac{1}{14} \text{ if } a \text{ and } b \text{ are equal.}$$

COROLLARY 3.

And consequently, it follows that in so many Mains as are expressed by $\frac{a+b \times a+b)^3 - a^3}{a^3 b}$, or in 14 Mains if a and b are equal, the Expectation of the Box is 1, calling 1 whatever is stipulated to belong to the Box, which usually is 1 Half-Guinea.

COROLLARY 4.

Now supposing that a and b are respectively as 107 to 111, a Box is payed one with another in about 14.7 Mains.

After I had solved the foregoing Problem, which is about 12 years ago, I spoke of my Solution to Mr. *Henry Stuart Stevens*, but without communicating to him the manner of it: As he is a Gentleman who, besides other uncommon Qualifications, has a particular Sagacity in reducing intricate Questions to simple ones, he brought me, a few days after, his Investigation of the Conclusion set down in my third Corollary; and as I have had occasion to cite him before, in another Work, so I here renew with pleasure the Expression of the Esteem which I have for his extraordinary Talents: Now his Investigation was as follows.

Let a and b respectively represent the number of Chances for a Main and no Main; Let also 1 be the Sum which the Box must receive upon Supposition of three Mains being won successively; now the Probability of winning a Main is $\frac{a}{a+b}$, and the Probability of winning three Mains is $\frac{a^3}{a+b)^3}$, and therefore the Box-keeper might without advantage or disadvantage to himself receive from the Caster at a certainty, the Sum $\frac{a^3}{a+b)^3} \times 1$, which would be an Equivalent for the uncertain sum 1, payable after three Mains.

Let

Let it therefore be agreed between them, that the Caster shall pay but the Sum $\frac{a^3}{a+b} \times 1$ for his three Mains; now let us see what consideration the Box-keeper gives to the Caster in return of that Sum. 1^o, he allows him one Main sure, 2^o, he allows him a second Main conditionally, which is provided he wins the first, of which the Probability being $\frac{a}{a+b}$, it follows that the Box allows him only, if one may say so, the portion $\frac{a}{a+b}$ of a second Main, and for the same reason the portion $\frac{aa}{(a+b)^2}$ of a third Main, and therefore the Box allows in all to the Caster $1 + \frac{a}{a+b} + \frac{aa}{(a+b)^2}$ Mains, or $\frac{3aa+3ab+bb}{(a+b)^2}$; and therefore if for the Sum received $\frac{a^3}{a+b} \times 1$, there be the allowance of $\frac{3aa+3ab+bb}{(a+b)^2}$ Mains, how many are allowed for the Sum 1? and the Term required will be $\frac{3aa+3ab+bb \times a+b}{a^3}$, or $\frac{(a+b)^4}{ab^3} - \frac{a+b}{b}$: and therefore in so many Mains as are denoted by the foregoing Expression, the Box gets the Sum 1; which Expression is reduced to 14 if a and b are equal.

P R O B L E M XLVIII.

Of RAFFLING.

If any number of Gamesters A, B, C, D, &c. play at Raffles, what is the Probability that the first of them having thrown his Chance, and before the other Chances are thrown, wins the Money of the Play?

SOLUTION.

In order to solve this Problem, it is necessary to have a Table ready composed of all the Chances which there are in three Raffles, which Table is the following.

A TABLE of all the Chances which are in three Raffles.

Points.		Chances to win or lose.	Chances to win or lose.	Equality of Chance.	
LIV	}	IX	884735	0	1
LIII		X	884726	1	9
LII		XI	884681	10	45
LI		XII	884534	55	147
L		XIII	884165	202	369
XLIX		XIV	883400	571	765
XLVIII		XV	881954	1336	1446
XLVII		XVI	879470	2782	2484
XLVI		XVII	875501	5266	3969
XLV		XVIII	869632	9235	5869
XLIV	} or {	XIX	861199	15104	8433
XLIII		XX	849706	23537	11493
XLII		XXI	834679	35030	15027
XLI		XXII	815392	50057	19287
XL		XXIII	791506	69344	23886
XXXIX		XXIV	762838	93230	28668
XXXVIII		XXV	728971	121898	33867
XXXVII		XXVI	690100	155765	38871
XXXVI		XXVII	646929	194636	43171
XXXV		XXVIII	599472	237807	47457
XXXIV	}	XXIX	548865	285264	50607
XXXIII		XXX	496314	335871	52551
XXXII		XXXI	442368	388422	53946

The Sum of all the numbers expressing the Equality of Chance being 442368, if that Sum be doubled it will make 884736, which is equal to the Cube of 96.

The first Column contains any number of Points which *A* may be supposed to have thrown in three Raffles.

The second Column contains the number of Chances which *A* has for winning, if his Points be above xxxi, or the number of Chances he has for losing, if his Points be either xxxi or below it.

The third Column contains the number of Chances which *A* has for losing, if his Points be above xxxi, or for winning, if they be either xxxi or below it.

The fourth Column, which is the principal, and out of which the other two are formed, contains the number of Chances whereby any number of Points from IX to LIV can be produced in three Raffles; and consequently contains the number of Chances which any of the Gamesters *B, C, D, &c.* may have for coming to an equality of Chance with *A*.

The Construction of the fourth Column depends chiefly on the number of Chances which there are for producing one single Raffle, whereof XVIII or III have 1 Chance

XVII or IV have 3 Chances

XVI or V have 6 Chances

XV or VI have 4 Chances

XIV or VII have 9 Chances

XIII or VIII have 9 Chances

XII or IX have 7 Chances

XI or X have 9 Chances

Which number of Chances being duly combined, will afford all the Chances of three Raffles.

But it will be convenient to illustrate this by one Instance; let it therefore be required to find the number of Chances for producing XII Points in three Raffles.

1°, It may plainly be perceived that those Points may be produced by the following single Raffles III, III, VI, or III, IV, V, or IV, IV, IV; then considering the first Case, and knowing from the Table of single Raffles, that the Raffles III, III, VI, have respectively 1, 1, 4 Chances to come up, it follows from the Doctrine of Combinations that those three numbers ought to be multiplied together, which in the present Case makes the product to be barely 4, but as the disposition, III, III, VI, may be varied twice; *viz.* by III, VI, III, and VI, III, III, which will make in all three dispositions, it follows that the number 4, which expresses the Chances of one disposition, ought to be multiplied by 3, which being done, the product 12 must be set apart.

2°, The Disposition III, IV, V, has for its Chances the product of the numbers 1, 3, 6, which makes 18; but this being capable of 6 permutations, the number 18 ought to be multiplied by 6, which being done, the product 108 must likewise be set apart.

3°, The Disposition IV, IV, IV has for its Chances the product of 3, 3, 3, which makes 27; but this not being capable of any variation, we barely write 27, which must be set apart.

4°, Adding together those numbers that were severally set apart, the Sum will be found to be 147, which therefore expresses the number of Chances for producing XII Points in three Raffles: and in the same manner may all the other numbers belonging to the Table of three Raffles be calculated.

This being laid down, let us suppose that *A* has thrown the Points XL in three Raffles, that there are four Gamesters besides himself, and that under that circumstance of *A*, it be required to find the Probability of his beating the other four.

Let *m* universally represent the number of Chances which any other Gamester has of coming to an equality with *A*, which number of Chances in this particular Case is 23886; Let *a* universally represent the number of Chances which *A* has for beating any one of his Adversaries, which number of Chances is found in the Table to be 791506; Let *f* represent the number of all the Chances that there are in three different Raffles, which number is the Cube of 96, by reason that there are no more than 96 single Raffles in three Dice, and therefore *f* constantly stands for the number 884736; Let *p* universally represent the number of Gamesters in all, which in this Case will be 5; then the Probability which *A* has of beating the

other four will be $\frac{a + m \cdot p - a^p}{m \cdot p \times f^{p-1}}$; and therefore if each of the Gamesters stake 1, the Expectation of *A* upon the whole Stake *p*,

will be expressed by $\frac{a + m \cdot p - a^p}{m \cdot f^{p-1}}$; and consequently his Gain, or what he might clearly get from his Adversaries by an equitable composition with them for the Value of his Chance, will be

$$\frac{a + m \cdot p - a^p}{m \cdot f^{p-1}} - 1.$$

Now the Logarithm of $a + m = 5.9113665$, Log. *a* = 5.8984542, Log. *m* = 4.3781434, Log. *f* = 5.9468136; and therefore Log. $a + m \cdot p =$ or Log. $a + m \cdot 5 = 29.5568325$, Log. $a^p = 29.4922710$, Log. $m \cdot f^{p-1} = 28.1653978$; from which Logarithms it will be convenient to reject the least index 28, and treat those Logarithms as if they were respectively 1.5568325, 1.4922710, 0.1653978: but the numbers belonging to the two first are 36.044 and 31.065, whose difference is 4.979 from the Logarithm of which, viz. 0.6971421, if the Log. 0.1653978 be subtracted, there will remain the Log. 0.5317433, of which the corresponding number being 3.402, it follows that the Gain of *A* ought to be estimated by 2.402.

DEMONSTRATION.

1°, When A has thrown his Chance, the Probability of B 's having a worse Chance will be $\frac{a}{f}$; wherefore the Probability which A has of beating all his Adversaries whose number is $p - 1$, will be $\frac{a^{p-1}}{f^{p-1}}$.

2°, The Probability which B has in particular of coming to an Equality with A is $\frac{m}{f}$, which being supposed, the Probability which A has of beating the rest of his Adversaries whose number is $p - 2$, is $\frac{a^{p-2}}{f^{p-2}}$; which being again supposed, the Probability which A now has of beating B , with whom he must renew the Play, is $\frac{1}{2}$; wherefore the Probability of the happening of all these

things is $= \frac{m}{f} \times \frac{a^{p-2}}{f^{p-2}} \times \frac{1}{2} = \frac{\frac{1}{2}ma^{p-2}}{f^{p-1}}$: but because C , or D or E , &c. might as well have come to an equality with A as B himself, it follows that the preceding Fraction ought to be multiplied by $p - 1$, which will make it, that the Probability which A has of beating all his Adversaries except one, who comes to an equality with him, and then of his beating him afterwards, will be $\frac{\frac{p-1}{2}ma^{p-2}}{f^{p-1}}$.

3°, The Probability which both B and C have of coming to an equality with A is $\frac{mm}{ff}$; which being supposed, the Probability which A has of beating the rest of his Adversaries whose number is $p - 3$, is $\frac{a^{p-3}}{f^{p-3}}$; which being again supposed, the Probability which A now has of beating B and C with whom he must renew the Play, (every one of them being now obliged to throw for a new Chance) is $\frac{1}{3}$; wherefore the Probability of the happening of all these things will be $= \frac{mm}{ff} \times \frac{a^{p-3}}{f^{p-3}} \times \frac{1}{3} = \frac{\frac{1}{3}mma^{p-3}}{f^{p-1}}$: but the number of the Adversaries of A being $p - 1$,

and

and the different Variations which that number can undergo by elections made two and two being $\frac{p-1}{1} \times \frac{p-2}{2}$, as appears from the Doctrine of Combinations, it follows that the Probability which any two, and no more, of the Adversaries of A have of coming to an Equality with him, that A shall beat all the rest, and that he shall beat afterwards those two that were come to an Equality, is $\frac{\frac{p-1}{2} \times \frac{p-2}{3} mma^{p-3}}{f^{p-1}}$ and so of the rest.

From hence it follows that the Probability which A has of beating all his Adversaries, will be expressed by the following Series,

$$\frac{a^{p-1} + \frac{p-1}{2} ma^{p-2} + \frac{p-1}{2} \times \frac{p-2}{3} mma^{p-3} + \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} m^2 a^{p-4}, \&c.}{f^{p-1}}$$

the Terms of whose Numerator are continued till such time as their number be $= p$; now to those who understand how to raise a Binomial to a Power given, by means of a Series, it will plainly appear that the foregoing Expression is equivalent to this other $\frac{(a+m)^p - a^p}{mp \times f^{p-1}}$; which consequently denotes the Probability required.

P R O B L E M XLIX.

The same things being given as in the preceding Problem, to find how many Gamesters there ought to be in all, to make the Chance of A, after he has thrown the Point XL, to be the most advantageous that is possible.

SOLUTION.

It is very easily perceived that the more Adversaries A has, the more his Probability of winning will decrease; but he has a Compensation, which is, that if he beats them all, his Gain will be greater than if he had had fewer Competitors: for which reason, there being a balance between the Gain that he may make on one side, and the decrease of the Probability of winning on the other, there is a certain number of Gamesters, which till it be attained, the Gain will be more prevalent than the decrease of Probability; but which being exceeded, the decrease of Probability will prevail over the Gain; so that what was advantage, till a certain time, may gradually turn to equality, and even to disadvantage. This Problem is therefore proposed in order to determine those Circumstances.

Let $\text{Log. } f - \text{Log. } a$ be made $= g$, let also $\text{Log. } f - \text{Log. } \overline{a + m}$ be made $= f$, which being done, then the number of Gamesters requisite to make the Advantage the greatest possible will be expressed by the fraction $\frac{\log. g - \log. f}{\log. a + m - \log. a}$, so that supposing as in the preceding Problem that $a = 791506$, $m = 23886$, and consequently $a + m = 815392$, as also $f = 884736$, and $\text{Log. } f = 5.9468136$, $\text{Log. } a = 5.8984542$, $\text{Log. } m = 4.3781434$, $\text{Log. } a + m = 5.9113665$, then g will be $= 0.0483594$, and f will be $= 0.354471$. Theref. $\text{Log. } g - \text{Log. } f = 0.1349014$, and $\text{Log. } \overline{a + m} - \text{Log. } a = 0.0129123$ and therefore the number of Gamesters will be $\frac{1349014}{129123} = 10.4$ nearly, which shews that the number required will be about ten or eleven.

As the Demonstration of this last Operation depends upon principles that are a little too remote from the Doctrine of Chances, I have thought fit to omit it in this place; however if the Reader will be pleased to consult my *Miscellanea Analytica*, therein he will find it, pag. 223 and 224.

It is proper to observe, that the method of Solution of this last Problem, as well as of the preceding, may be applied to an infinite variety of other Problems, which may happen to be so much easier than these, as they may not require Tables of Chances ready calculated.

P R O B L E M L.

Of WHISK.

If four Gamesters play at Whisk, to find the Odds that any two of the Partners, that are pitched upon, have not the four Honours.

SOLUTION.

First, Suppose those two Partners to have the Deal, and the last Card which is turned up to be an Honour.

From the Supposition of these two Cases, we are only to find what Probability the Dealers have of taking three set Cards in twenty-five, out of a Stock containing fifty-one. To resolve this the shortest way, recourse must be had to the Theorem given in the Remark belonging to our xxth Problem, in which making the Quantities n ,

c, d, p, a , respectively equal to the numbers 51, 25, 26, 3, 3, the Probability required will be found to be $\frac{25 \times 24 \times 23}{51 \times 50 \times 49}$ or $\frac{92}{833}$.

Secondly, If the Card which is turned up be not an Honour, then we are to find what Probability the Dealers have of taking four given Cards in twenty-five out of a Stock containing fifty-one; which by the aforesaid Theorem will be found to be $\frac{25 \times 24 \times 23 \times 22}{51 \times 50 \times 49 \times 48}$ or $\frac{253}{4998}$.

But the Probability of taking the four Honours being to be estimated before the last Card is turned up; and there being sixteen Chances in fifty-two, or four in thirteen for an Honour to turn up, and nine in thirteen against it, it follows that the Probability of the first Case ought to be multiplied by 4; that the fraction expressing the Probability of the second ought to be multiplied by 9; and that the Sum of those Products ought to be divided by 13, which being done, the Quotient $\frac{115}{1666}$ or $\frac{2}{29}$ nearly, will express the Probability required.

And by the same Method of proceeding it will be found, that the Probability which the two Eldest have of taking four Honours is $\frac{69}{1666}$, that the Probability which the Dealers have of taking three Honours is $\frac{468}{1666}$, and that the Probability which the Eldest have of taking three Honours is $\frac{364}{1666}$. Moreover, that the Probability that there are no Honours on either side will be $\frac{650}{1666}$.

Hence it may be concluded, 1^o, that it is 27 to 2 nearly that the Dealers have not the four Honours.

That it is 23 to 1 nearly that the Eldest have not the four Honours.

That it is 8 to 1 nearly that neither one side nor the other have the four Honours.

That is 13 to 7 nearly that the two Dealers do not reckon Honours.

That it is 20 to 7 nearly that the two Eldest do not reckon Honours.

And that it is 25 to 16 nearly that either one side or the other do reckon Honours, or that the Honours are not equally divided.

COROLLARY I.

From what we have said, it will not be difficult to solve this Case at Whisk; viz. which side has the best, of those who have VIII of the Game, or of those who at the same time have IX? In

In order to which it will be necessary to premise the following Principle.

1°, That there is but 1 Chance in 8192 to get VII by Triks.

2°, That there are 13 Chances in 8192 to get VI.

3°, That there are 78 Chances in 8192 to get V.

4°, That there are 286 Chances in 8192 to get IV.

5°, That there are 715 Chances in 8192 to get III.

6°, That there are 1287 Chances in 8192 to get II.

7°, That there are 1716 Chances in 8192 to get I.

All this will appear evident to those who can raise the Binomial $a + b$ to its thirteenth power.

But it must carefully be observed that the foregoing Chances express the Probability of getting so many Points by Triks, and neither more nor less.

For if it was required, for Instance, to assign the Probability of getting one or more by Triks, it is plain that the Numerator of the Fraction expressing that Probability would be the Sum of all the Chances which have been written, *viz.* 4096, and consequently that this Probability would be $\frac{4096}{8192}$ or $\frac{1}{2}$.

2°, That the Probability of getting two or more by Triks would be $\frac{2380}{8192}$, or $\frac{1190}{4096}$.

3°, That the Probability of getting three or more by Triks would be $\frac{1093}{8192}$.

4°, That the Probability of getting IV or more by Triks would be $\frac{378}{8192}$.

5°, That the Probability of getting V or more by Triks would be $\frac{92}{8192}$.

6°, That the Probability of getting VI or more would be $\frac{14}{8192}$.

7°, That the Probability of getting VII would be $\frac{1}{8192}$.

This being laid down, I proceed thus.

1°, If those that have VII of the Game are Dealers, their Probability of getting II by Honours is $\frac{583}{1666}$: for the Dealers will get II by Honours if they have either 3 of the 4 Honours, or all the 4 Honours, but the Probability of taking three Honours is $\frac{468}{1666}$, and the Probability they have of taking the four Honours is $\frac{115}{1666}$, and the Sum of this is $\frac{583}{1666}$.

The

The Probability which they have of getting them by Triks is $\frac{2380}{8192}$ or $\frac{1100}{4096}$.

And therefore adding these two Probabilities together, the Sum will be $\frac{4370508}{6823936}$.

Now subtracting from this, the Probability of both circumstances happening together, *viz.* $\frac{693770}{6823936}$ the remainder will be $\frac{3676738}{6823936}$; and this expresses their Expectation upon the common Stake which we suppose to be = 1.

But they have a farther Expectation, which is that of getting one single Game by Triks, which is $\frac{1716}{8192}$ or $\frac{429}{2048}$; and their Probability of not getting by Honours is $\frac{1083}{1666}$ ($= 1 - \frac{583}{1666}$); and therefore their Probability of getting one single Game by Triks independently from Honours is $\frac{464607}{3411958}$; but then if this happen they will be but equal with their Adversaries, and therefore this Chance entitles them to no more than half of the common Stake; therefore taking the half of the foregoing fraction, it will be $\frac{464607}{6823936}$; and therefore the whole Expectation of the Dealers is $\frac{3676738 + 464607}{6823936} = \frac{4141345}{6823936}$; whence there remains for those who have 1X of the Game $\frac{2682591}{6823936}$; which will make that the Odds for the VIII against the 1X will be 4141345 to 2682591, which is about 3 to 2, or something more, *viz.* 17 to 11.

2°. But if those who have VIII of the Game are Eldest, then their Probability of having three of the four Honours is $\frac{364}{1666}$, and their Probability of having the four Honours is $\frac{69}{1666}$, and therefore their Probability of getting their two Games by Honours is $\frac{364 + 69}{1666} = \frac{433}{1666}$. The Probability of getting them by Triks is as before $\frac{1190}{4096}$, now adding these two Probabilities together, the Sum will be $\frac{3756108}{6823936}$, from which subtracting, the Probability of both circumstances happening together, *viz.* $\frac{515270}{6823936}$, there will remain $\frac{3240838}{6823936}$, and this expresses the Expectation arising from the Prospect of their winning at once either by Honours or by Triks.

But

But their Expectation arising from the Prospect of getting one single Game, and then being upon an equal foot with their Adversaries, found the same way as it was in the Supposition of their being Dealers, is $\frac{528957}{6823936}$. For the Probability of the Eldest taking 4 Honours is $\frac{69}{1006}$, and of their taking 3 Honours, $\frac{364}{1006}$; whose Sum taken from Unity, leaves $\frac{1233}{1006}$, for the Probability of their not getting by Honours; and this multiplied by $\frac{429}{2048}$ the Probability of their getting one Game by Triks, gives $\frac{528957}{3411968}$; the half of which is $\frac{528957}{6823936}$. And therefore their Expectation upon the whole is $\frac{3240338 + 528957}{6823936} = \frac{3769795}{6823936}$, and consequently there remains for the IX, $\frac{3054141}{6823936}$, and therefore the Odds of the VIII against the IX are now 3769795 to 3054141, which is nearly as 95 to 77.

From whence it follows that without considering whether the VIII are Dealers or Eldest, there is one time with another the Odds of somewhat less than 7 to 5; and very nearly that of 25 to 18.

COROLLARY 2.

It is a Question likewise belonging to this Game, what the Probability is that a Player has a given number of Trumps dealt him: particularly, it has been often taken as an equal Wager that the Dealer has at least 4 Trumps.

Now altho' the Solution of all such Questions is included in our xxth Problem; yet as this Game is much in use, I have, for the Reader's ease, computed the following Tables; shewing, for the Dealer as well as the other Gamesters, what the Probability is of taking *precisely* any assigned number of Trumps in one deal.

And thence by a continual addition of the numbers, or of such part of them as is necessary, it is easily found what the Probability is of taking *at least* that number.

Chances of the Dealer to have besides the Card turned up.		Trumps	Chances of any other Gamester to have precisely.	
Tab. I.	3910797436	0	8122425444	Tab. II.
	20112672528	I.	46929569232	
	41959196136	II.	110619698904	
	46621329040	III.	139863987120	
	30454255260	IV.	104897990340	
	12181702104	V.	48726808416	
	3014663652	VI.	14211985788	
	455999544	VII.	2583997416	
	40714245	VIII.	284999715	
	2010580	IX.	18095220	
	48906	X.	603174	
	468	XI.	8892	
	1	XII.	39	
Sum = 158753389900 is the common Denominator; being the Combinations of 12 Cards in 51.			476260169700 = Sum, is the common Denominator; being the Combinations of 13 in 51.	

By the help of these Tables several useful Questions may be resolved; as 1°. If it is asked, what is the Probability that the Dealer has precisely III Trumps, besides the Trump Card? The Answer, by *Tab. I.* is $\frac{4662}{15875}$; and the Probability of his having some other number of Trumps is $\frac{11213}{15875}$. But if the Question had been, What is the Probability that some other Gamester, the eldest hand for instance, has precisely IV Trumps? The answer, by *Tab. II.* is $\frac{104898}{476260}$.

2°. To find the Chance of the Dealer's not having fewer than IV Trumps: add his Chances to take 0, I, II, which are 39108, 201127, 419592; and their Sum 659827 taken from the Denominator 1587534, and the Remainder made its Numerator, the Probability of the Dealer having IV or more Trumps will be $\frac{927707}{1587534} = \frac{329}{503}$, a little above $\frac{7}{12}$. The Wager therefore that the Dealer has not IV Trumps is so far from equal, that whoever lays it throws away above $\frac{1}{6}$ of his Stake.

But if the Wager is that the Dealer has not V Trumps, then 466213 (the Chances of his having III. besides the Trump Card) is to be added to the Chances for 0, I, II; which will make the Chance of him who lays this Wager to be nearly $\frac{317}{455}$; and that of his Adversary $\frac{138}{455}$.

And hence, if Wagers are laid that the Dealer has not IV Trumps, and has not V Trumps, *alternately*; the advantage of him who lays in this manner will be nearly $11 \frac{1}{4}$ per Cent. of his Stakes.

3°. To find the Odds of laying that the eldest hand has at least III, and at least IV Trumps, *alternately*; the Numerator of the one Expectation is (by Tab. II.) 31501119, and of the other 17514720, to the Denominator 47626017; whence the advantage of the Bet will be $\frac{15}{514}$, or 3 per Cent. nearly.

Again, if it is laid that the Trumps in the Dealer's hand shall be either I, II, III or VI; the disadvantage of this Bet will be only $15^{\text{lb.}} 4^{\text{d}}$, or about $\frac{3}{4}$, per Cent.

In like manner, the Odds of any proposed Bet of this kind may be computed: And from the Numbers in the Tables, and their Combinations, different Bets may be found which shall approach to the Ratio of Equality; or if they differ from it, other Bets may be assigned, which, repeated a certain Number of Times, shall ballance that difference.

4°. And if the Bet includes any other Condition besides the number of Trumps, such as the Quality of one or more of them; then proper Regard is to be had to that restriction.

Let the Wager be that the Eldest has IV Trumps dealt him; and that two of them shall be the Ace and King. The Probability of his having IV Trumps precisely is, by Tab II. $\frac{104808}{476260}$; and the different fours in 12 Cards are $\frac{12}{1} \times \frac{11}{2} \times \frac{10}{3} \times \frac{9}{4}$. But because 2 out of the 12 Trumps are specified, all the Combinations of 4 in 12 that are favourable to the Wager are reduced to the different two's that are found in the remaining 10 Cards, which are $\frac{10}{3} \times \frac{9}{2}$. And this number is to the former as 1 to 11: the Probability therefore is reduced by this restriction to $\frac{1}{11}$, of what else it had been: that is, it is reduced from near $\frac{1}{5}$ to about $\frac{1}{52}$.

Note;

Note; these Tables and others of a like kind, which different Games may require, are best computed and examined by beginning with the lowest number, and observing the Law by which the others are formed successively. As in Tab. I, putting $A = 1$; and the Letters B, C, D , &c. standing for the other Terms regularly ascending; we shall have $B = \frac{39}{1} \times \frac{12}{1} \times A$, $C = \frac{38}{2} \times \frac{11}{2} \times B$, $D = \frac{37}{3} \times \frac{10}{3} \times C$, &c. till we arrive at the Term $N = \frac{28}{12} \times \frac{1}{12} \times M$.

And if the corresponding Terms in Tab. II. are marked by the same Letters dotted, then is $A' = \frac{39}{1} \times A$, $B' = \frac{38}{2} \times B$, $C' = \frac{37}{3} \times C$, $D' = \frac{36}{4} \times D$, &c. up to $N' = \frac{27}{13} \times N$.

PROBLEM LI.

Of PIQUET.

To find at Piquet the Probability which the Dealer has for taking one Ace or more in three Cards, he having none in his Hand.

SOLUTION.

From the number of all the Cards which are thirty-two, subtracting twelve which are in the Dealer's Hands, there remain twenty, among which are the four Aces.

From which it follows that the number of all the Chances for taking any three Cards in the bottom, is the number of Combinations which twenty Cards may afford being taken three and three; which by the Rule given in our xv Problem is $\frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3}$ or 1140.

The number of all the Chances being thus obtained, find the number of Chances for taking one Ace precisely with two other Cards; find next the number of Chances for taking two Aces precisely with any other Card; lastly, find the number of Chances for taking three Aces; then these Chances being added together, and their Sum divided by the whole number of Chances, the Quotient will express the Probability required.

But the number of Chances for taking one Ace are 4, and the number of Chances for taking any two other Cards, are $\frac{16}{1} \cdot \frac{15}{2}$, and therefore the number of Chances for taking one Ace and two other Cards are $\frac{4}{1} \times \frac{16 \cdot 15}{1 \cdot 2} = 480$, as appears from what we have said in the Doctrine of Combinations.

If there remains any difficulty in knowing why the number of Chances for joining any two other Cards with the Ace already taken is $\frac{16 \cdot 15}{1 \cdot 2}$, it will be easily resolved if we consider that there being in the whole Pack but 4 Aces and 28 other Cards, out of which other Cards, the Dealer has 12 in his Hands, there remain only 16, out of which he has a Choice, and therefore the number of Chances for taking two other Cards is what we have determined.

In like manner it will appear that the number of Chances for taking two Aces precisely are $\frac{4 \cdot 3}{1 \cdot 2}$ or 6, and that the number of Chances for taking any other Card are $\frac{16}{1}$ or 16; from whence it follows that the number of Chances for taking two Aces with another Card are 6×16 or 96.

Lastly, it appears that the number of Chances for taking three Aces is equal to $\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} = 4$.

Wherefore the Probability required will be found to be $\frac{480+96+4}{1140}$ or $\frac{580}{1140}$ or $\frac{29}{57}$, which fraction being subtracted from Unity, the remainder will be $\frac{28}{57}$.

From whence it may be concluded that it is 29 to 28 that the Dealer takes one Ace or more in three Cards, he having none in his Hand.

The preceding Solution may be contracted by inquiring at first what the Probability is of not taking any Ace in three Cards, which may be done thus.

The number of Cards in which the four Aces are contained being twenty, and consequently the number of Cards out of which the four Aces are excluded being sixteen, it follows that the number of Chances which there are for the taking of three Cards, among which no Ace shall be found, is the number of Combinations which sixteen Cards may afford being taken three and three, which number

ber of Chances by our 18th Problem will be found to be $\frac{16 \cdot 15 \cdot 14}{1 \cdot 2 \cdot 3}$ or 560.

But the number of all the Chances for taking any three Cards in twenty has been found to be 1140; from whence it follows that the Probability of not taking any Ace in three Cards, is $\frac{560}{1140}$ or $\frac{28}{57}$, and therefore the Probability of the contrary, that is of taking one Ace or more in three Cards is $\frac{29}{57}$ as we had found it before.

P R O B L E M LII.

To find at Piquet the Probability which the Eldest has of taking an Ace in five Cards, he having no Ace in his Hand.

SOLUTION.

First, Find the number of Chances for taking one Ace and four other Cards, which will be 7280.

Secondly, The number of Chances for taking two Aces and three other Cards, which will be found to be 3360.

Thirdly, The number of Chances for taking three Aces and two other Cards, which will be found to be 480.

Fourthly, The number of Chances for taking four Aces and any other Card, which will be found to be 16.

Lastly, The number of Chances for taking any five Cards in twenty, which will be found to be 15504.

Let the Sum of all the particular Chances, *viz.* 7280 + 3360 + 480 + 16, be divided by the Sum of all the Chances, *viz.* by 15504, and the Quotient will be $\frac{11136}{15504}$ or $\frac{232}{323}$ which being subtracted from Unity, the remainder will be $\frac{91}{323}$; and therefore the Odds of the Eldest hand taking an Ace or more in five Cards are as 232 to 91, or 5 to 2 nearly.

But if the Probability of not taking an Ace in five Cards be inquired into, the work will be considerably shortened; for this Probability will be found to be expressed by $\frac{16 \times 15 \times 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ or 4368

4368 to be divided by the whole number of Chances, viz. by 15504, or 91 by 323; which makes the Probability of taking one or more Aces $\frac{232}{323}$ as before.

P R O B L E M LIII.

To find at Piquet the Probability which the Eldest has of taking both an Ace and a King in five Cards, he having none in his Hand.

SOLUTION.

Let the following Chances be found; viz.

- 1°, For one Ace, one King, and three other Cards.
- 2°, For one Ace, two Kings, and two other Cards.
- 3°, For one Ace, three Kings, and any other Card.
- 4°, For one Ace, and four Kings.
- 5°, For two Aces, one King, and two other Cards.
- 6°, For two Aces, two Kings, and any other Card.
- 7°, For two Aces, and three Kings.
- 8°, For three Aces, one King, and any other Card.
- 9°, For three Aces, and two Kings.
- 10°, For four Aces, and one King.

Among these Cases, there being four pairs that are alike, viz. the second and fifth, the third and eighth, the fourth and tenth, the seventh and ninth; it follows that there are only six Cases to be calculated, whereof the first and sixth are to be taken singly, but the second, third, fourth and seventh to be doubled; now the Operation is as follows.

The *first* Case has $\frac{4}{1} \times \frac{4}{1} \times \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3}$ or 3520 Chances.

The *second* $\frac{4}{1} \times \frac{4 \cdot 3}{1 \cdot 2} \times \frac{12 \cdot 11}{1 \cdot 2}$ or 1584, the double of which is 3168.

The *third* $\frac{4}{1} \times \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \times \frac{12}{1}$ or 192, the double of which is 384 Chances.

The

The *fourth* $\frac{4}{1} \times \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}$ or 4, the double of which is 8 Chances.

The *sixth* $\frac{4 \cdot 3}{1 \cdot 2} \times \frac{4 \cdot 3}{1 \cdot 2} \times \frac{12}{1}$ or 432 Chances.

The *seventh* $\frac{4 \cdot 3}{1 \cdot 2} \times \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \times$ or 24, the double of which is 48 Chances.

Now the Sum of all those Chances being 7560, and the whole number of Chances for taking any five Cards out of 20 being $\frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ or 15504, it follows that the Probability required will be $\frac{7560}{15504}$ or $\frac{315}{646}$, and therefore the Probability of the contrary will be $\frac{331}{646}$, from whence it follows that the Odds against the Eldest hand taking an Ace and a King are 331 to 315, or 21 to 20 nearly.

P R O B L E M LIV.

To find at Piquet the Probability of having twelve Cards dealt to, without King, Queen or Knave, which Case is commonly called Cartes Blanches.

SOLUTION.

Altho' this may be derived from what has been said in the xxth Problem, yet I shall here prescribe a Method which will be somewhat more easy, and which may be followed in many other Instances.

Let us therefore imagine that the twelve Cards dealt to are taken up one after another, and let us consider, 1^o, the Probability of the first's being a Blank; now there being 20 Blanks in the whole Pack, and 32 Cards in all, it is plain that the Probability of it is $\frac{20}{32}$. 2^o, Let us consider the Probability of the second's being a Blank, which by reason the first Card is accounted for, and because, there remain now but 19 Blanks and 31 Cards in all, will be found to be $\frac{19}{31}$; and in like manner the Probability of the third Card's being a Blank will be $\frac{18}{30}$, and so on; and therefore the

Proba-

Probability of the whole will be expressed by the Fraction

$$\frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{32 \cdot 31 \cdot 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}$$
 the number of Mul-
 tiplicators in both Numerator and Denominator being equal to twelve.

Now that Fraction being shortened will be reduced to $\frac{323}{578950}$ or
 $\frac{1}{1792}$ nearly, and therefore the Odds against *Cartes Blanches* are
 1791 to 1 nearly.

P R O B L E M LV.

*To find how many different Sets, essentially different from
 one another, one may have at Piquet before taking in.*

SOLUTION.

Let the Suits be disposed in order, and let the various dispositions
 of the Cards be written underneath, together with the number of
 Chances that each disposition will afford, and the Sum of all those
 Chances will be the thing required.

Let also the Letters D, H, S, C respectively represent Diamonds,
 Hearts, Spades, and Clubs.

	D,	H,	S,	C	Chances.
1	0,	0,	4,	8 =	70
2	0,	0,	5,	7 =	448
3	0,	0,	6,	6 =	748
4	0,	1,	3,	8 =	448
5	0,	1,	4,	7 =	4480
6	0,	1,	5,	6 =	12544
7	0,	2,	2,	8 =	784
8	0,	2,	3,	7 =	812544
9	0,	2,	4,	6 =	54880
10	0,	2,	5,	5 =	87808
11	0,	3,	3,	6 =	87808
12	0,	3,	4,	5 =	219520
13	0,	4,	4,	4 =	343000
14	1,	1,	2,	8 =	1792
15	1,	1,	3,	7 =	28672
16	1,	1,	4,	6 =	125440
17	1,	1,	5,	5 =	200704
18	1,	2,	2,	7 =	50176
19	1,	2,	3,	6 =	351232
20	1,	2,	4,	5 =	878080
21	1,	3,	3,	5 =	1404928
22	1,	3,	4,	4 =	2195200
23	2,	2,	2,	6 =	614656
24	2,	2,	3,	5 =	2458624
25	2,	2,	4,	4 =	3851600
26	2,	3,	3,	4 =	6146560
27	3,	3,	3,	3 =	9834496
Sum					28,967,278

Which Sum would seem incredibly great, if Calculation did not prove it to be so.

But it will not be inconvenient to shew by one Example how the numbers expressing the Chances have been found, for which we must have recourse to our xxth and xxith Problems, and there examine the Method of Solution, the same being to be observed in this place. Let it therefore be required to assign the 19th Case, which is for taking 1 Diamond, 2 Hearts, 3 Spades and 6 Clubs. Then it will easily be seen that the variations for taking 1 Diamond are 8, that the variations for taking 2 Hearts are $\frac{8 \cdot 7}{1 \cdot 2} = 28$, and that

the variations for taking 3 Spades are $\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$, and that the variations for taking 6 Clubs are $\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 28$. And therefore that the number of Chances for the 19th Case is the product of the several numbers 8, 28, 56, 28, which will be found 351232.

There is one thing worth observing, which is, that when the number of Cards of any one Suit being to be combined together, exceed one half the number of Cards of that Suit, then it will be sufficient to combine only the difference between that number and the whole number of Cards in the Suit, which will make the operation shorter; thus being to combine the 8 Clubs by six and six, I take the difference between eight and six, which being 2, I combine the Cards only two and two, it being evident that as often as I take 6 Cards of one Suit, I leave 2 behind of the same Suit, and that therefore I cannot take them oftner six and six, than I can take them two and two.

It may perhaps seem strange that the number of Sets which we have determined, notwithstanding its largeness, yet should not come up to the number of different Combinations whereby twelve Cards might be taken out of thirty-two, that number being 225792840; but it ought to be considered, that in that number several sets of the same import, but differing in Suit might be taken, which would not introduce an essential difference among the Sets.

REMARK.

It may easily be perceived from the Solution of the preceding Problem, that the number of variations which there are in twelve Cards make it next to impossible to calculate some of the Probabilities relating to Piquet, such as that which results from the priority of Hand, or the Probabilities of a Pic, Repic or Lurch; however notwithstanding that difficulty, one may from observations often repeated, nearly estimate what those Probabilities are in themselves, as will be proved in its place when we come to treat of the reasonable conjectures which may be deduced from Experiments; for which reason I shall set down some Observations of a Gentleman who has a very great degree of Skill and Experience in that Game, after which I shall make an application of them.

HYPOTHESES.

1°, That 'tis 5 to 4 that the Eldest hand wins a Game,

2°,

2°, That is 2 to 1, that the Eldest wins rather without lurching than by lurching.

3°, That it is 4 to 1, that the Youngest Hand wins rather without lurching than by lurching.

But it must carefully be observed that these Odds are restrained to the beginning of a Game.

From whence, to avoid Fractions, we may suppose that the Eldest has 75 Chances to win one Game, and the Youngest 60.

That out of these 75 Chances of the Eldest, he has 50 to win without Lurch, and 25 with a Lurch.

That of the 60 Chances of the Youngest, he has 48 to win without a Lurch, and 12 with a Lurch.

This being laid down, I shall proceed to determine the Probabilities of winning the Set, under all the circumstances in which *A* and *B* may find themselves.

1°, When *A* and *B* begin, he who gets the Hand has the Odds of 6478643 to 3362857 or 23 to 20 nearly that he wins the Set.

2°, If *A* has 1 Game and *B* none.

Before they cut for the Hand, the Odds in favour of *A* are 682459 to 309067 or 38 to 23 nearly.

If *A* has the Hand, the Odds are 4627 to 1448, or 16 to 5 nearly.

If *B* has the Hand, the Odds in favour of *A* are 511058 to 309067, or 38 to 23 nearly.

3°, If *A* has 1 Game, and *B* 1 Game.

He who gets the Hand has the Odds of 10039 to 8186 or 27 to 22 nearly.

4°, If *A* has 2 Games and *B* none.

Before they cut for the Hands the Odds are 59477 to 13423, or 31 to 7 nearly.

If *A* has the Hand, the Odds are 5117 to 958, or 16 to 3 nearly.

If *B* has the Hand, the Odds in favour of *A* are 1151 to 307, or 25 to 7 nearly.

5°, If *A* has 2 Games and *B* 1.

Before they cut for the Hand, the Odds are 92 to 43, or 15 to 7 nearly.

If *A* has the Hand, the Odds are 11 to 4. *

If

* In this Case *B* has 12 Chances for 1, and 48 for $\frac{1}{2}$, but the number of all the Chances

If B has the Hand, the Odds in favour of A are 17 to 10.

6°, If A has 2 Games and B 2 Games, he who gets the Hand has 5 to 4 in his favour.

I hope the Reader will easily excuse my not giving the Demonstration of the foregoing Calculation, it being so easily deduced from the Rules given before, that this would seem entirely superfluous.

P R O B L E M LVI.

Of SAVING CLAUSES.

A has 2 Chances to beat B , and B has 1 Chance to beat A ; but there is one Chance which intitles them both to withdraw their own Stake, which we suppose equal to f ; to find the Gain of A .

SOLUTION.

This Question tho' easy in itself, yet is brought in to caution Beginners against a Mistake which they might commit by imagining that the Case, which intitles each Man to recover his own Stake, needs not be regarded, and that it is the same thing as if it did not exist: This I mention so much more readily, that some people who have pretended great skill in these Speculations of Chance have themselves fallen into that error. Now there being 4 Chances in all, whereof A has 2 to gain f , 'tis evident that the Expectation of that Gain is worth $\frac{2}{4}f$; but A having 1 Chance in 4 to lose f , the Risk of that is a Loss which must be estimated by $\frac{1}{4}f$, and therefore the absolute Gain of A is $\frac{2}{4}f - \frac{1}{4}f$, or $\frac{1}{4}f$. But supposing the saving Clause not considered, A would have 2 Chances in 3 to win f , and 1 Chance in 3 to lose f , and therefore the Expectation of his Gain

Chances between A and B are 135, therefore B has $\frac{12+24}{135} = \frac{36}{135} = \frac{4}{15}$, Odds 11 to 4. If B has the Hand, then he has 25 for 1, 50 for $\frac{1}{2} = \frac{25+25}{135} = \frac{50}{135} = \frac{10}{27}$, Odds 17 to 10. But before they cut for the Hand B has $\frac{4}{15} + \frac{10}{27} + \frac{1}{2} = \frac{43}{135}$, Odds 92 to 43.

would

would be worth $\frac{2}{3}f$, and the Risk of his Loss would be estimated by $\frac{1}{3}f$; which would make his Gain to be $\frac{2}{3}f - \frac{1}{3}f = \frac{1}{3}f$. From whence it may evidently be seen that the condition of drawing Stakes is to be considered; and indeed in this last Case, there are the Odds of 2 to 1 that A beats B , whereas in the former it cannot be said but very improperly that A has 2 to 1 the best of the Game; for if A undertakes without any limitation to beat B , then he must lose if the saving Clause happens, and therefore he has but an equality of Chance to beat or not to beat; however it may be said with some propriety of Expression, that it is 2 to 1 that A rather beats B than that A beats him.

But to make the Question more general, let A and B each depofite the Sum f ; let a represent the Chances which A has to beat B , and b the Chances which B has to beat A ; let there be also a certain number m of Chances which may be called common, by the happening of which A shall be entitled to take up such part of the common Stake $2f$ as may be denominated by the fraction $\frac{p}{r}$, and B shall be entitled to take the remainder of it.

Then 1°, it appears that the number of all the Chances being $a + b + m$, whereof there are the number a which intitle A to gain f ; thence his Gain upon that score is $\frac{a}{a+b+m} \times f$.

2°, It appears that the number of Chances whereby A may lose, being b , his Loss upon that account is $\frac{b}{a+b+m} \times f$.

3°, It appears that if the Chances m should happen, then A would take up the part $\frac{p}{r}$ of the common Stake $2f$, and thereby gain $\frac{2p}{r}f - f$, or $\frac{2p-r}{r} \times f$. But the Probability of the happening of this is $\frac{m}{a+b+m}$; and therefore his Gain arising from the Probability of this circumstance is $\frac{m}{a+b+m} \times \frac{2p-r}{r} \times f$.

From all which it appears that his absolute Gain is $\frac{a}{a+b+m} \times f - \frac{b}{a+b+m} \times f + \frac{m}{a+b+m} \times \frac{2p-r}{r} \times f$.

Now suppose there had been no common Chances, the Gain of A would have been $\frac{a-b}{a+b} \times f$.

Let it therefore be farther required to assign what the proportion of p to r ought to be, to make the Gain of A to be the same in both Cases.

This

This will be easily done by the Equation $\frac{a-b}{a+b+m} + \frac{2pm-rm}{r \times a+b+m} = \frac{a-b}{a+b}$; wherein multiplying all the Terms by $a+b+m$ we shall have the new Equation $a-b + \frac{2pm-rm}{r} = \frac{aa+am-bb-bm}{a+b}$ or $\frac{2pm-rm}{r} = \frac{am-bm}{a+b}$, or $\frac{2p-r}{r} = \frac{a-b}{a+b}$, or $2pa - ra + 2bp - br = ra - br$, or $2pa + 2bp = 2ra$, and therefore $pa + bp = ra$, and $\frac{p}{r} = \frac{a}{a+b}$. From which we may conclude, that if the two parts of the common Stake $2f$ which A and B are respectively to take up, upon the happening of the Chances m , are respectively in the proportion of a to b , then the common Chances give no advantage to A above what he would have had if they had not existed.

P R O B L E M LVII.

Odds of Chance and Odds of Money compared.

A and B playing together deposit £. apiece; A has 2 Chances to win £, and B 1 Chance to win £, whereupon A tells B that he will play with him upon an equality of Chance, if he B will set him 2£ to 1£, to which B assents: to find whether A has any advantage or disadvantage by that Bargain.

SOLUTION.

In the first circumstance, A having 2 Chances to win f , and 1 Chance to lose f , his Gain, as may be deduced from the Introduction, is $\frac{2f-f}{3} = \frac{1}{3}f$.

In the second circumstance, A having 1 Chance to win $2f$, and 1 Chance to lose f , his Gain is $\frac{2f-f}{2} = \frac{1}{2}f$, and therefore he gets $\frac{1}{6}f$ by that Bargain.

But if B , after the Bargain proposed, should answer, let us play upon an equality of Chance, and you shall stake but $\frac{1}{2}f$, and I shall stake f , and so I shall have set 2 to 1, and that A should assent: then he has 1 Chance to win f , and 1 Chance to lose $\frac{1}{2}f$, and therefore his

his Gain is $\frac{f - \frac{1}{2}f}{2} = \frac{1}{4}f$, and therefore he is worse by $\frac{1}{12}f$ than he was in the first circumstance.

But if A , after this proposal of B , answers; let us preserve the quantity of the whole Stake $2f$, but do you stake $\frac{4}{3}f$, and I shall stake $\frac{2}{3}f$, whereby the proportion of 2 to 1 will remain, and that B assents; then A has 1 Chance to win $\frac{4}{3}f$ and 1 Chance to lose $\frac{2}{3}f$, which

makes his Gain to be $\frac{\frac{4}{3}f - \frac{2}{3}f}{2} = \frac{2}{3}f - \frac{1}{3}f = \frac{1}{3}f$, which is the same as in the first circumstance.

And universally, A having a Chances to win f , and B having b Chances to win f , if they should agree afterwards to play upon an equality of Chance, and set to each other the respective Stakes $\frac{2b}{a+b}f$ and $\frac{2a}{a+b}f$, then the Gain of A would thereby receive no alteration, it being in both Cases $\frac{a-b}{a+b}f$.

P R O B L E M LVIII.

OF THE DURATION OF PLAY.

Two Gamesters A and B whose proportion of skill is as a to b, each having a certain number of Pieces, play together on condition that as often as A wins a Game, B shall give him one Piece; and that as often as B wins a Game, A shall give him one Piece; and that they cease not to play till such time as either one or the other has got all the Pieces of his Adversary: now let us suppose two Spectators R and S concerning themselves about the ending of the Play, the first of them laying that the Play will be ended in a certain number of Games which he assigns, the other laying to the contrary. To find the Probability that S has of winning his wager.

SOLU-

SOLUTION.

This Problem having some difficulty, and it having given me occasion to inquire into the nature of some Series naturally resulting from its Solution, whereby I have made some improvements in the Method of summing up Series, I think it necessary to begin with the simplest Cases of this Problem, in order to bring the Reader by degrees to a general Solution of it.

CASE I.

Let 2 be the number of Pieces, which each Gamester has; let also 2 be the number of Games about which the Wager is laid: now because 2 is the number of Games contended for, let $a + b$ be raised to its Square, *viz.* $aa + 2ab + bb$; then it is plain that the Term $2ab$ favours S, and that the other two are against him; and consequently that the Probability he has of winning is $\frac{2ab}{a+b}^2$.

COROLLARY

If a and b are equal, neither R or S have any Advantage or Disadvantage; but if a and b are unequal, R has the Advantage.

CASE II.

Let 2 be the number of Pieces of each Gamester, as before, but let 3 be the number of Games about which the Wager is laid: then $a + b$ being raised to its Cube, *viz.* $a^3 + 3aab + 3abb + b^3$, it will be seen that the two Terms a^3 and b^3 are contrary to S, they denoting the number of Chances for winning three times together; it will also be seen that the other two Terms $3aab$ and $3abb$ are partly for him, partly against him. Let therefore those two Terms be divided into their proper parts, *viz.* $3aab$ into $aab + aba + baa$, and $3abb$ into $abb + bab + bba$, and it will plainly be perceived that out of those six parts there are four which are favourable to S, *viz.* aab, baa, abb, bba or $2aab + 2abb$; from whence it follows that the Probability which S has of winning his Wager will be $\frac{2aab+2abb}{a+b}^3$, or dividing both Numerator and Denominator by $a + b$, it will be found to be $\frac{2ab}{a+b}^2$, which is the same as in the preceding Case. The reason of which is, that the winning of a certain number of even Pieces in an odd number of Games is impossible, unless it was done in the even number of Games immediately preceding the

the odd number, no more than an odd number of Pieces can be won in an even number of Games, unless it was done in the odd number immediately preceding it; but still the Problem of winning an even number of Pieces in an odd number of Games is rightly proposed; for Instance, the Probability of winning either of one side or the other, 8 Pieces in 63 Games; for, provided it be done either before or at the Expiration of 62 Games, he who undertakes that it shall be done in 63 wins his Wager.

CASE III.

Let 2 be the number of Pieces of each Gamester, and 4 the number of Games upon which the Wager is laid: let therefore $a + b$ be raised to the fourth Power, which is $a^4 + 4a^3b + 6aabb + 4ab^3 + b^4$; which being done, it is plain that the Terms $a^4 + 4a^3b + 4ab^3 + b^4$ are wholly against S , and that the only Term $6aabb$ is partly for him, and partly against him, for which reason, let this Term be divided into its parts, viz. $aabb$, $abab$, $abba$, $baab$, $baba$, $baaa$, and 4 of these parts, viz. $abab$, $abba$, $baab$, $baba$, or $4aabb$ will be found to favour S ; from which it follows that his Probability of winning will be $\frac{4aabb}{a+b)^4}$.

CASE IV.

If 2 be the number of Pieces of each Gamester, and 5 the number of Games about which the Wager is laid, the Probability which S has of winning his wager will be the same as in the preceding Case, viz.

$$\frac{4aabb}{a+b)^4}$$

Universally, Let 2 be the number of Pieces of each Gamester, and $2 + d$ the number of Games upon which the Wager is laid;

and the Probability which S has of winning will be $\frac{\overline{2ab}^1 1 + \frac{1}{2}d}{a+b)^2 + d}$

if d be an even number; or $\frac{\overline{2ab}^{\frac{1+d}{2}}}{a+b)^{1+d}}$ if d be odd, writing $d-1$ instead of d .

CASE V.

If 3 be the number of Pieces of each Gamester, and $3 + d$ the number of Games upon which the Wager is laid, then the Probabi-

lity which S has of winning will be $\frac{\overline{3ab} \sqrt{1 + \frac{1}{2}d}}{a + b \sqrt{2 + d}}$ if d be an even number, or $\frac{\overline{3ab} \sqrt{\frac{1+d}{2}}}{a + b \sqrt{1+d}}$ if it be odd.

CASE VI.

If the number of Pieces of each Gamester be more than 3, the Expectation of S , or the Probability there is that the Play shall not be ended in a given number of Games, may be determined in the following manner.

A General Rule for determining what Probability there is that the Play shall not be determined in a given number of Games.

Let n be the number of Pieces of each Gamester. Let also $n + d$ be the number of Games given; raise $a + b$ to the Power n , then cut off the two extream Terms, and multiply the remainder by $aa + 2ab + bb$: then cut off again the two Extreams, and multiply again the remainder by $aa + 2ab + bb$, still rejecting the two Extreams; and so on, making as many Multiplications as there are Units in $\frac{1}{2}d$; make the last Product the Numerator of a Fraction whose Denominator let be $\overline{a + b}^{n+d}$, and that Fraction will express the Probability required, or the Expectation of S upon a common Stake 1, supposed to be laid between R and S ; still observing that if d be an odd number, you write $d - 1$ in its room.

EXAMPLE I.

Let 4 be the number of Pieces of each Gamester, and 10 the number of Games given: in this Case $n = 4$, $n + d = 10$; wherefore $d = 6$, and $\frac{1}{2}d = 3$. Let therefore $a + b$ be raised to the fourth Power, and rejecting continually the extreams, let three Multiplications be made by $aa + 2ab + bb$. Thus,

$$\begin{array}{r}
 a^4 + 4a^3b + 6aabb + 4ab^3 + b^4 \\
 \underline{aa + 2ab + bb} \\
 4a^5b + 6a^4bb + 4a^3b^3 \\
 + 8a^4bb + 12a^3b^3 + 8aabb^4 \\
 + 4a^3b^3 + 6aabb^4 + 4ab^5
 \end{array}$$

$$\begin{array}{r}
 14a^4bb + 20a^3b^3 + 14a^2b^4 \\
 aa + 2ab + bb \\
 \hline
 14^6abb| + 20a^5b^3 + 14a^4b^4 \\
 + 28a^5b^3 + 40a^4b^4 + 28a^3b^5 \\
 + 14a^4b^4 + 20a^3b^5 + 14a^2b^6 \\
 \hline
 48a^5b^3 + 68a^4b^4 + 48a^3b^5 \\
 aa + 2ab + bb \\
 \hline
 48a^7b^3| + 68a^6b^4 + 48a^5b^5 \\
 + 96a^6b^4 + 136a^5b^5 + 96a^4b^6 \\
 + 48a^5b^5 + 68a^4b^6 + 48a^3b^7 \\
 \hline
 164a^6b^4 + 232a^5b^5 + 164a^4b^6
 \end{array}$$

Wherefore the Probability that the Play will not be ended in 10 Games will be $\frac{164a^6b^4 + 232a^5b^5 + 164a^4b^6}{a+b}^{10}$, which Expression will be reduced to $\frac{560}{1024}$, if there be an equality of Skill between the Gamesters; now this Fraction $\frac{560}{1024}$ or $\frac{35}{64}$ being subtracted from Unity, the remainder will be $\frac{29}{64}$, which will express the Probability of the Play's ending in 10 Games, and consequently it is 35 to 29 that, if two equal Gamesters play together, there will not be four Stakes lost on either side, in 10 Games.

N. B. The foregoing operation may be very much contracted by omitting the Letters a and b , and restoring them after the last Multiplication; which may be done in this manner. Make $n + \frac{1}{2}d - 1 = p$, and $\frac{1}{2}d + 1 = q$; then annex to the respective Terms resulting from the last Multiplication the literal Products $a^p b^q$, $a^{p-1} b^{q+1}$, $a^{p-2} b^{q+2}$, &c.

Thus in the foregoing Example, instead of the first Multiplicand $4a^3b + 6aabb + 4ab^3$, we might have taken only $4 + 6 + 4$, and instead of multiplying three times by $aa + 2ab + bb$, we might have multiplied only by $1 + 2 + 1$, which would have made the last Terms to have been $164 + 232 + 164$. Now since that $n = 4$ and $d = 6$, p will be $= 6$ and $q = 4$, and consequently the literal Products to be annexed respectively to the Terms $164 + 232 + 164$ will be a^6b^4 , a^5b^5 , a^4b^6 , which will make the Terms resulting from the last Multiplication to be $164a^6b^4 + 232a^5b^5 + 164a^4b^6$, as they had been found before.

EXAMPLE II.

Let 5 be the number of Pieces of each Gamester, and 10 the number of Games given: let also the proportion of Skill between A and B be as 2 to 1.

Since $n = 5$, and $n + d = 10$, it follows that $d = 5$. Now d being an odd number must be supposed $= 4$, so that $\frac{1}{2}d = 2$: let therefore $1 + 1$ be raised to the fifth Power, and always rejecting the Extreams, multiply twice by $1 + 2 + 1$, thus

$$\begin{array}{r}
 1|+5+10+10+5|+1 \\
 \hline
 1+2+1 \\
 \hline
 5|+10+10+5 \\
 \quad +10+20+20+10 \\
 \quad \quad +5+10+10+5 \\
 \hline
 20+35+35+20
 \end{array}$$

$$\begin{array}{r}
 20+35+35+20 \\
 \hline
 1+2+1 \\
 \hline
 20|+35+35+20 \\
 \quad +40+70+70+40 \\
 \quad \quad +20+35+35+20 \\
 \hline
 75+125+125+75
 \end{array}$$

Now to supply the literal Products that are wanting, let $n + \frac{1}{2}d - 1$ be made $= p$, and $\frac{1}{2}d + 1 = q$, and the Products that are to be annexed to the numerical quantities will be $a^p b^q$, $a^{p-1} b^{q+1}$, $a^{p-2} b^{q+2}$, $a^{p-3} b^{q+3}$, &c. wherefore n , in this Case, being $= 5$, and $d = 4$, then p will be $= 6$, and $q = 3$, it follows that the Products to be annexed in this Case be $a^5 b^3$, $a^5 b^4$, $a^4 b^5$, $a^3 b^6$, and consequently the Expectation of S will be found to be

$$\frac{75a^6b^3 + 125a^5b^4 + 125a^4b^5 + 75a^3b^6}{(a+b)^9}$$

N. B. When n is an odd number, as it is in this Case, the Expectation of S will always be divisible by $a + b$. Wherefore dividing both Numerator and Denominator by $a + b$, the foregoing Expression will be reduced to

$$\frac{75a^5b^3 + 50a^4b^4 + 75a^3b^5}{(a+b)^8} \text{ or } 25a^3b^3 \times \frac{3aa+2ab+3bb}{(a+b)^8}$$

Let now a be interpreted by 2, and b by 1, and the Expectation of S will become $\frac{3800}{6561}$.

PROBLEM LIX.

The same things being given as in the preceding Problem, to find the Expectation of R, or otherwise the Probability that the Play will be ended in a given number of Games.

SOLUTION.

First, It is plain that if the Expectation of S obtained by the preceding Problem be subtracted from Unity, there will remain the Expectation of R.

Secondly, Since the Expectation of S decreases continually, as the number of Games increases, and that the Terms we rejected in the former Problem being divided by $aa + 2ab + bb$ are the Decrement of his Expectation; it follows that if those rejected Terms be divided continually by $aa + 2ab + bb$ or $a + b$, they will be the Increment of the Expectation of R. Wherefore the Expectation of R may be expressed by means of those rejected Terms. Thus in the second Example of the preceding Problem, the Expectation of R expressed by means of the rejected Terms will be found to be

$$\frac{a^5 + b^5}{a + b} \times 1 + \frac{5a^4b + 5ab^4}{a + b} + \frac{20a^3b^2 + 20a^2b^3}{a + b} \text{, or}$$

$$\frac{a^5 + b^5}{a + b} \times 1 + \frac{5ab}{a + b} + \frac{20a^2bb}{a + b}$$

In like manner, if 6 were the number of the Pieces of each Gamester, and the number of Games were 14, it would be found that the Expectation of R would be

$$\frac{a^6 + b^6}{a + b} \times 1 + \frac{6aa}{a + b} + \frac{27aabb}{a + b} + \frac{110a^3b^3}{a + b} + \frac{429}{a + b} a^4b^4.$$

And if 7 were the number of Pieces of each Gamester, and the number of Games were 15, then the Expectation of R would be found to be

$$\frac{a^7 + b^7}{a + b} \times 1 + \frac{7ab}{a + b} + \frac{35aabb}{a + b} + \frac{154a^3b^3}{a + b} + \frac{637a^4b^4}{a + b}.$$

N. B. The number of Terms of these Series will always be equal to $\frac{1}{2}d + 1$, if d be an even number, or to $\frac{d+1}{2}$, if it be odd.

Thirdly,

Thirdly, All the Terms of these Series have to one another certain Relations, which being once discovered, each Term of any Series resulting from any Case of this Problem, may be easily generated from the preceding ones.

Thus in the first of the two last foregoing Series, the numerical Coefficient belonging to the Numerator of each Term may be derived from the preceding, in the following manner. Let K, L, M be the three last Coefficients, and let N be the Coefficient of the next Term required; then it will be found that N in that Series will constantly be equal to $6M - 9L + 2K$. Wherefore if the Term which would follow $\frac{429a^4b^4}{(a+b)^4}$ in the Case of 16 Games given, were desired; then make $M = 429$, $L = 110$, $K = 27$, and the following Coefficient will be found 1638. From whence it appears that the Term itself would be $\frac{1638a^5b^5}{(a+b)^{10}}$.

Likewise, in the second of the two foregoing Series, if the Law by which each Term is related to the preceding were demanded, it might thus be found. Let K, L, M be the Coefficients of the three last Terms, and N the Coefficient of the Term desired; then N will in that Series constantly be equal to $7M - 14L + 7K$, or $M - 2L + K \times 7$. Now this Coefficient being obtained, the Term to which it belongs is formed immediately.

But if the universal Law by which each Coefficient is generated from the preceding be demanded, it will be expressed as follows.

Let n be the number of Pieces of each Gamester: then each Coefficient contains

$$\begin{aligned} & n \text{ times the last} \\ & - n \times \frac{n-3}{2} \text{ times the last but one} \\ & + n \times \frac{n-4}{2} \times \frac{n-5}{3} \text{ times the last but two} \\ & - n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} \text{ times the last but three} \\ & + n \times \frac{n-6}{2} \times \frac{n-7}{3} \times \frac{n-8}{4} \times \frac{n-9}{5} \text{ times the last but four.} \\ & \text{\&c.} \end{aligned}$$

Thus the number of Pieces of each Gamester being 6, the first Term n would be $= 6$, the second Term $n \times \frac{n-3}{2}$ would be $= 9$, the third Term $n \times \frac{n-4}{2} \times \frac{n-5}{3}$ would be $= 2$. The rest of the Terms vanishing in this Case. Wherefore if K, L, M are the three last

last Coefficients, the Coefficient of the following Term will be $6M - 9L + 2K$.

Fourthly, The Coefficient of any Term of these Series may be found independently from any relation they may have to the preceding: in order to which, it is to be observed that each Term of these Series is proportional to the Probability of the Play's ending in a certain number of Games precisely: thus in the Series which expresses the Expectation of R , when each Gamester is supposed to have 6 Pieces; *viz.*

$$\frac{a^6+b^6}{a+b)^6} \times 1 + \frac{6ab}{a+b)^7} + \frac{27a^2b^2}{a+b)^8} + \frac{110a^3b^3}{a+b)^9} + \frac{420a^4b^4}{a+b)^{10}}$$

the last Term being multiplied by the common Multiplier $\frac{a^6+b^6}{a+b)^6}$ set down before the Series, the Product $\frac{420a^4b^4 \times a^6+b^6}{a+b)^{14}}$

will denote the Probability of the Play's ending in 14 Games precisely. Wherefore if that Term were desired which expresses the Probability of the Play's ending in 20 Games precisely, or in any number of Games denoted by $n + d$, I say that the Coefficient of that Term will be

$\frac{n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}$, &c. continued to so many Terms as there are Units in $\frac{1}{2}d$.

$-\frac{3n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}$, &c. continued to so many Terms as there are Units in $\frac{1}{2}d-n$.

$+\frac{5n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}$, &c. continued to so many Terms as there are Units in $\frac{1}{2}d-2n$.

$-\frac{7n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}$, &c. continued to so many Terms as there are Units in $\frac{1}{2}d-3n$.

&c.

Let now $n + d$ be supposed = 20, n being already supposed = 6, then the Coefficient demanded will be found from the general Rule to be

$$\begin{aligned} &\frac{6}{1} \times \frac{19}{2} \times \frac{18}{3} \times \frac{17}{4} \times \frac{16}{5} \times \frac{15}{6} \times \frac{14}{7} = 23256 \\ &-\frac{18}{1} = -18 \end{aligned}$$

Where-

Wherefore the Coefficient demanded will be $23256 - 18 = 23238$, and then the Term itself to which this Coefficient does belong, will be $\frac{23238a^7b^7}{(a+b)^{14}}$, and consequently the Probability of the Play's ending in 20 Games precisely will be $\frac{a^6+b^6}{(a+b)^6} \times \frac{23238a^7b^7}{(a+b)^{14}}$.

But some things are to be observed about this formation of the Coefficients, which are,

First, that whenever it happens that $\frac{1}{2}d$, or $\frac{1}{2}d - n$, or $\frac{1}{2}d - 2n$, or $\frac{1}{2}d - 3n$, &c. expressing respectively the number of Multipliers to be taken in each Line, are $= 0$, then 1 ought to be taken to supply that Line.

Secondly, That whenever it happens that those quantities $\frac{1}{2}d$, or $\frac{1}{2}d - n$, or $\frac{1}{2}d - 2n$, or $\frac{1}{2}d - 3n$, &c. are less than nothing, otherwise that they are negative, then the Line to which they belong, as well as all the following, ought to be cancelled.

P R O B L E M LX.

Supposing A and B to play together till such time as four Stakes are won or lost on either side; what must be their proportion of Skill, otherwise what must be their proportion of Chances for winning any one Game assigned, to make it as probable that the Play will be ended in four Games as not?

SOLUTION.

The Probability of the Play's ending in four Games is by the preceding Problem $\frac{a^4+b^4}{(a+b)^4} \times 1$: now because, by Hypothesis, it is to be an equal Chance whether the Play ends or ends not in four Games; let this Expression of the Probability be made $= \frac{1}{2}$, then we shall have the Equation $\frac{a^4+b^4}{(a+b)^4} = \frac{1}{2}$: which, making $b, a :: 1, z$, is reduced to $\frac{z^4+1}{z+1^4} = \frac{1}{2}$, or $z^4 - 4z^3 - 6zz - 4z + 1 = 0$. Let $12zz$ be added on both sides of the Equation, then will $z^4 - 4z^3 + 6zz - 4z + 1$ be $= 12zz$, and extracting the Square-

Square-root on both sides, it will be reduced to this *quadratic* Equation, $zz - 2z + 1 = z\sqrt{12}$, of which the two Roots are $z = 5.274$ and $z = \frac{1}{5.274}$. Wherefore whether the Skill of *A* be to that of *B*, as 5.274 to 1, or as 1 to 5.274, there will be an Equality of Chance for the Play to be ended or not ended in four Games.

PROBLEM LXI.

Supposing that A and B play till such time as four Stakes are won or lost: What must be their proportion of Skill to make it a Wager of three to one, that the Play will be ended in four Games?

SOLUTION.

The Probability of the Play's ending in four Games arising from the number of Games 4, from the number of Stakes 4, and from the proportion of Skill, *viz.* of *a* to *b*, is $\frac{a^4+b^4}{a^4+b^4}$; the same Probability arising from the Odds of three to one, is $\frac{3}{4}$: Wherefore $\frac{a^4+b^4}{a^4+b^4} = \frac{3}{4}$, and supposing *b*, *a* :: 1, *z*, that Equation will be changed into $\frac{z^4+1}{z+1^4} = \frac{3}{4}$ or $z^4 - 12z^3 + 38zz - 12z + 1 = 56zz$, and extracting the Square Root on both sides, $zz - 6z + 1 = z\sqrt{56}$, the Roots of which Equation will be found to be 13.407 and $\frac{1}{13.407}$: Wherefore if the Skill of either be to that of the other as 13.407 to 1, 'tis a Wager of three to one, that the Play will be ended in 4 Games.

PROBLEM LXII.

Supposing that A and B play till such time as four Stakes are won or lost; What must be their proportion of Skill to make it an equal Wager that the Play will be ended in six Games?

SOLUTION.

The Probability of the Play's ending in six Games, arising from the given number of Games 6, from the number of Stakes 4, and

from the proportion of Skill a to b , is $\frac{a^4+b^4}{(a+b)^4} \times \frac{1+ab}{(a+b)^2}$; the same Probability arising from an equality of Chance, is $= \frac{1}{2}$, from whence results the Equation $\frac{a^4+b^4}{(a+b)^4} \times \frac{1+ab}{(a+b)^2} = \frac{1}{2}$, which making $b, a :: 1, z$ must be changed into the following $z^6 + 6z^5 - 13z^4 - 20z^3 - 13zz + 6z + 1 = 0$.

In this Equation, the Coefficients of the Terms equally distant from the Extreams, being the same, let it be supposed that the Equation is generated from the Multiplication of two other Equations of the same nature, viz. $zz - yz + 1 = 0$, and $z^4 + pz^3 + qzz + pz + 1 = 0$. Now the Equation resulting from the Multiplication of those two will be

$$\begin{aligned} z^6 - yz^5 + 1z^4 + 2pz^3 + pz + 1 = 0. \\ + pz^5 - pyz^4 - qyz^3 - yz \\ + qz^4 \end{aligned}$$

which being compared with the first Equation, we shall have $p - y = 6$, $1 - py + q = -13$, $2p - qy = -20$, from whence will be deduced a new Equation, viz. $y^3 + 6yy - 16y - 32 = 0$, of which one of the Roots will be 2.9644, and this being substituted in the Equation $zz - yz + 1 = 0$, we shall at last come to the Equation $zz - 2.9644z + 1 = 0$, of which the two Roots will be 2.576 and $\frac{1}{2.576}$; it follows therefore that if the Skill of either Gamester be to that of the other as 2.576 to 1, there will be an equal Chance for four Stakes to be lost or not to be lost, in six Games.

COROLLARY

If the Coefficients of the extream Terms of an Equation, and likewise the Coefficients of the other Terms equally distant from the Extreams be the same, that Equation will be reducible to another, in which the Dimensions of the highest Term will not exceed half the Dimensions of the highest Term in the former.

PROBLEM LXIII.

Supposing A and B whose proportion of Skill is as a to b, to play together till such time as A either wins a certain number q of Stakes, or B some other number p of them: what is the Probability that the Play will not be ended in a given number of Games (n)?

SOLV-

SOLUTION.

Multiply the Binomial $a + b$ so many times by it self as there are Units in $n - 1$, always observing after every Multiplication to reject those Terms in which the Dimensions of the Quantity a exceed the Dimensions of the Quantity b , by q ; as also those Terms in which the Dimensions of the Quantity b exceed the Dimensions of the Quantity a , by p ; then shall the last Product be the Numerator of a Fraction expressing the Probability required, of which Fraction the Denominator must be the Binomial $a + b$ raised to that Power which is denoted by n .

EXAMPLE.

Let p be $= 3$, $q = 2$, and let the given number of Games be $= 7$. Let now the following Operation be made according to the foregoing Directions.

$$\begin{array}{r}
 a+b \\
 a+b \\
 \hline
 aa|+2ab+bb \\
 \quad a+b \\
 \hline
 \quad 2aab+3abb|+b^3 \\
 \quad \quad a+b \\
 \hline
 \quad \quad 2a^3b|+5aabb+3ab^3 \\
 \quad \quad \quad a+b \\
 \hline
 \quad \quad \quad 5a^3bb|+8aabb^3|+3ab^4 \\
 \quad \quad \quad \quad a+b \\
 \hline
 \quad \quad \quad \quad 5a^4bb|+13a^3b^3+8aabb^4 \\
 \quad \quad \quad \quad \quad a+b \\
 \hline
 \quad \quad \quad \quad \quad 13a^4b^3+21a^3b^4|+8ab^5
 \end{array}$$

From this Operation we may conclude, that the Probability of the Play's not ending in 7 Games is equal to $\frac{13a^4b^3+21a^3b^4}{(a+b)^7}$. Now if an equality of Skill be supposed between A and B , the Expression of this Probability will be reduced to $\frac{13+21}{128}$ or $\frac{17}{64}$: Wherefore the Probability of the Play's ending in 7 Games will be $\frac{47}{64}$; from which it follows that it is 47 to 17 that, in seven Games, either A wins two Stakes of B , or B wins three Stakes of A .

P R O B L E M LXIV.

The same things being supposed as in the preceding Problem, to find the Probability of the Play's ending in a given number of Games.

SOLUTION.

First, If the Probability of the Play's not ending in the given number of Games, which we may obtain from the preceding Problem, be subtracted from Unity, there will remain the Probability of its ending in the same number of Games.

Secondly, This Probability may be expressed by means of the Terms rejected in the Operation belonging to the preceding Problem: Thus if the number of Stakes be 3 and 2, the Probability of the Play's ending in 7 Games may be expressed as follows.

$$\frac{aa}{(a+b)^2} \times 1 + \frac{2ab}{(a+b)^2} + \frac{5aabb}{(a+b)^4}$$

$$\frac{b^3}{(a+b)^3} \times 1 + \frac{3ab}{(a+b)^2} + \frac{8aabb}{(a+b)^4}$$

Supposing both a and b equal to Unity, the Sum of the first Series will be $= \frac{29}{64}$, and the Sum of the second will be $\frac{18}{64}$; which two Sums being added together, the aggregate $\frac{47}{64}$ expresses the Probability that, in seven Games, either A shall win two Stakes of B , or B three Stakes of A .

Thirdly, The Probability of the Play's ending in a certain number of Games is always composed of a double Series, when the Stakes are unequal: which double Series is reduced to a single one, in the Case of an Equality of Stakes

The first Series always expresses the Probability there is that A , in a given number of Games, or sooner, may win of B the number q of Stakes, excluding the Probability there is that B before that time may have been in a circumstance of winning the number p of Stakes; both which Probabilities are not inconsistent together: for A , in fifteen Games for Instance or sooner, may win two Stakes of B , though B before that time may have been in a circumstance of winning three Stakes of A .

The second Series always expresses the Probability there is that B , in that given number of Games, may win of A a certain number

ber p of Stakes, excluding the Probability there is that A , before that time, may win of B the number q of Stakes.

The first Terms of each Series may be represented respectively by the following Terms.

$$\begin{aligned} & \frac{a^q}{a+b)^q} \times 1 + \frac{qab}{a+b)^2} + \frac{q \cdot q+3 \cdot aabb}{1 \cdot 2 \cdot a+b)^4} + \frac{q \cdot q+1 \cdot q+5 \cdot a^3b^3}{1 \cdot 2 \cdot 3 \cdot a+b)^6} \\ & + \frac{q \cdot q+5 \cdot q+6 \cdot q+7 \cdot a^4b^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot a+b)^8}, \&c. \\ & \frac{a^p}{a+b)^p} \times 1 + \frac{pab}{a+b)^2} + \frac{p \cdot p+3 \cdot aabb}{1 \cdot 2 \cdot a+b)^4} + \frac{p \cdot p+4 \cdot p+5 \cdot a^3b^3}{1 \cdot 2 \cdot 3 \cdot a+b)^6} \\ & + \frac{p \cdot p+5 \cdot p+6 \cdot p+7 \cdot a^4b^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot a+b)^8}, \&c. \end{aligned}$$

Each of these Series continuing in that regularity till such time as there be a number p of Terms taken in the first, and a number q of Terms taken in the second; after which the Law of the continuation breaks off.

Now in order to find any of the Terms following in either of these Series, proceed thus: let $p + q - 2$ be called l ; let the Coefficient of the Term desired be T ; let also the Coefficients of the preceding Terms taken in an inverted order, be $S, R, Q, P, \&c.$ then will T be equal to $lS - \frac{l-1}{1} \times \frac{l-2}{2} R + \frac{l-2}{1} \times \frac{l-3}{2} \times \frac{l-4}{3} Q - \frac{l-3}{1} \times \frac{l-4}{2} \times \frac{l-5}{3} \times \frac{l-6}{4} P, \&c.$ Thus if p be $= 3$ and $q = 2$. then l will be $3 + 2 - 2 = 3$, wherefore $lS - \frac{l-1}{1} \times \frac{l-2}{2} R$ would in this Case be equal to $3S - R$, which shews that the Coefficient of any Term desired would be three times the last, *minus* once the last but one.

To apply this, let it be required to find what Probability there is that in fifteen Games or sooner, either A shall win two Stakes of B , or B three Stakes of A ; or which is all one, to find what Probability there is that the Play shall end in fifteen Games at farthest; A and B resolving to play till such time as A either wins two Stakes or B three.

Let 2 and 3, in the two foregoing Series, be substituted respectively in the room of q and p , the three first Terms of the first Series will be, setting aside the common Multiplicator, $1 + \frac{2ab}{a+b)^2} + \frac{5aabb}{a+b)^4}$: likewise the two first Terms of the second will be $1 + \frac{3ab}{a+b)^2}$. Now because the Coefficient of any Term desired in each

each Series is respectively three times the last, *minus* once the last but one, it follows that the next Coefficient in the first Series will be found to be 13, and by the same Rule the next to it 34, and so on. In the same manner, the next Coefficient in the second Series will be found to be 8, and the next to it 21, and so on. Wherefore restoring the common Multipliers the two Series will be

$$\begin{array}{l} \frac{a^2}{a+b} \times 1 + \frac{2ab}{a+b} + \frac{5abb}{a+b} + \frac{13a^3b^3}{a+b} + \frac{34a^4b^4}{a+b} \\ + \frac{80a^5b^5}{a+b} + \frac{233a^6b^6}{a+b} . \\ \frac{b^3}{a+b} \times 1 + \frac{3ab}{a+b} + \frac{8aabb}{a+b} + \frac{21a^3b^3}{a+b} + \frac{55a^4b^4}{a+b} \\ + \frac{134a^5b^5}{a+b} + \frac{377a^6b^6}{a+b} . \end{array}$$

If we suppose an equality of Skill between *A* and *B*, the Sum of the first Series will be $\frac{18778}{32768}$, the Sum of the second will be $\frac{12393}{32768}$, and the Aggregate of those two Sums will be $\frac{31171}{32768}$, which will express the Probability of the Play's ending in fifteen Games or sooner. This last Fraction being subtracted from Unity, there will remain $\frac{1597}{32768}$, which expresses the Probability of the Play's continuing beyond fifteen Games: Wherefore 'tis 31171 to 1597, or 39 to 2 nearly that one of the two equal Gamesters that shall be pitched upon, shall in fifteen Games at farthest, either win two Stakes of his Adversary, or lose three to him.

N. B. The Index of the Denominator in the last Term of each Series, and the Index of the common Multiplier prefixed to it being added together, must either equal the number of Games given, or be less than it by Unity. Thus in the first Series, the Index 12 of the Denominator of the last Term, and the Index 2 of the common Multiplier being added together, the Sum is 14, which is less by Unity than the number of Games given. So likewise in the second Series, the Index 12 of the Denominator of the last Term, and the Index 3 of the common Multiplier being added together, the Sum is 15, which precisely equals the number of Games given.

It is carefully to be observed that those two Series taken together express the Expectation of one and the same person, and not of two different persons; that is properly of a Spectator, who lays a wager that

that the Play will be ended in a given number of Games. Yet in one Case, they may express the Expectations of two different persons: for Instance, of the Gamesters themselves, provided that both Series be continued infinitely; for in that Case, the first Series infinitely continued will express the Probability that the Gamester *A* may sooner win two Stakes of *B*, than that he may lose three to him: likewise the second Series infinitely continued will express the Probability that the Gamester *B* may sooner win three Stakes of *A*, than lose two to him. And it will be found, (when I come to treat of the Method of summing up this sort of Series, whose Terms have a perpetual recurrency of relation to a fixed number of preceding Terms) that the first Series infinitely continued is to the second infinitely continued, in the proportion of $aa \times aa + ab + bb$ to $b^3 \times a + b$; that is in the Case of an Equality of Skill as 3 to 2, which is conformable to what I have said in the 19th Problem.

Fourthly, Any Term of these Series may be found independently from any of the preceding: for if a Wager be laid that *A* shall either win a certain number of Stakes denominated by *q*, or that *B* shall win a certain number of them denominated by *p*, and that the number of Games be expressed by $q + d$; then I say that the Coefficient of any Term in the first Series answering to that number of Games will be

$+ \frac{q}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to so many Multipliers as there are Units in $\frac{1}{2}d$.

$- \frac{q+2p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to so many Terms as there are Units in $\frac{1}{2}d - p$.

$+ \frac{3q+2p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to so many Terms as there are Units in $\frac{1}{2}d - p - q$.

$- \frac{3q+4p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to so many Terms as there are Units in $\frac{1}{2}d - 2p - q$.

$+ \frac{5q+4p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to so many Terms as there are Units in $\frac{1}{2}d - 2p - 2q$.

$- \frac{5q+6p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to so many Terms as there are Units in $\frac{1}{2}d - 3p - 2q$.

$+ \frac{7q+6p}{1}$

$\frac{7q+6p}{1} \times \frac{q+d-1}{2} \times \frac{q+d+2}{3} \times \frac{q+d-3}{4}$, &c. continued to so many Terms as there are Units in $\frac{1}{2}d-3p-3q$.
And so on.

And the same Law will hold for the other Series, calling $p+\delta$ the number of Games given, and changing q into p , and p into q , as also d into δ , still remembring that when d is an odd number, $d-1$ ought to be taken in the room of it, and the like for δ .

And the same observation must be made here as was made at the end of the LIXth Problem, viz. that if $\frac{1}{2}d$, or $\frac{1}{2}d-p$, or $\frac{1}{2}d-p-q$, or $\frac{1}{2}d-2p-q$, or $\frac{1}{2}d-2p-2q$, &c. expressing respectively the number of Multipliers to be taken in each Line, are $= 0$, then 1 ought to be taken for that Line, and also, that if $\frac{1}{2}d$, or $\frac{1}{2}d-p$, or $\frac{1}{2}d-p-q$, &c. are less than nothing, otherwise negative, then the Line to which they belong as well as all the following ought to be cancelled.

P R O B L E M LXV.

If A and B, whose proportion of skill is supposed as a to b, play together: What is the Probability that one of them, suppose A, may in a number of Games not exceeding a number given, win of B a certain number of Stakes? leaving it wholly indifferent whether B, before the expiration of those Games, may or may not have been in a circumstance of winning the same, or any other number of Stakes of A.

SOLUTION.

Supposing n to be the number of Stakes which A is to win of B , and $n+d$ the number of Games; let $a+b$ be raised to the Power whose Index is $n+d$; then if d be an odd number, take so many Terms of that Power as there are Units in $\frac{d+1}{2}$; take also so many of the Terms next following as have been taken already, but prefix to them in an inverted order, the Coefficients of the preceding Terms. But if d be an even number, take so many Terms of the
said

faid Power as there are Units in $\frac{1}{2}d + 1$; then take as many of the Terms next following as there are Units in $\frac{1}{2}d$, and prefix to them in an inverted order the Coefficients of the preceding Terms, omitting the last of them; and those Terms taken all together will compose the Numerator of a Fraction expressing the Probability required, the Denominator of which Fraction ought to be $a + b$ raised to the power $n + d$.

EXAMPLE I.

Supposing the number of Stakes, which *A* is to win, to be *Three*, and the given number of Games to be *Ten*; let $a + b$ be raised to the tenth power, viz. $a^{10} + 10a^9b + 45a^8bb + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45aab^8 + 10ab^9 + b^{10}$. Then by reason that $n = 3$, and $n + d = 10$, it follows that $d = 7$, and $\frac{d+1}{2} = 4$. Wherefore let the Four first Terms of the said Power be taken, viz. $a^{10} + 10a^9b + 45a^8bb + 120a^7b^3$, and let the four Terms next following be taken likewise without regard to their Coefficients, then prefix to them in an inverted order, the Coefficients of the preceding Terms: thus the four Terms following with their new Coefficients will be $120a^6b^4 + 45a^5b^5 + 10a^4b^6 + 1a^3b^7$. Then the Probability which *A* has of winning three Stakes of *B* in ten Games or sooner, will be expressed by the following Fraction

$$\frac{a^{10} + 10a^9b + 45a^8bb + 120a^7b^3 + 120a^6b^4 + 45a^5b^5 + 10a^4b^6 + a^3b^7}{(a + b)^{10}}$$

which in the Case of an Equality of Skill between *A* and *B* will be reduced to $\frac{352}{1024}$ or $\frac{11}{32}$.

EXAMPLE II.

Supposing the number of Stakes which *A* has to win to be *Four*, and the given number of Games to be *Ten*; let $a + b$ be raised to the tenth Power, and by reason that $n = 4$, and $n + d = 10$, it follows that $d = 6$, and $\frac{1}{2}d + 1 = 4$; wherefore let the four first Terms of the said Power be taken, viz. $a^{10} + 10a^9b + 45a^8bb + 120a^7b^3$; take also three of the Terms following, but prefix to them, in an inverted order, the Coefficients of the Terms already taken, omitting the last of them; hence the three Terms following with their new Coefficients will be $45a^6b^4 + 10a^5b^5 + 1a^4b^6$. Then

E e

the

the Probability which *A* has of winning four Stakes of *B* in ten Games, or sooner, will be expressed by the following Fraction

$$\frac{a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 45a^6b^4 + 10a^5b^5 + 1a^4b^6}{a+b}^{10}$$

which in the Case of an Equality of Skill between *A* and *B* will be reduced to $\frac{232}{1024}$ or $\frac{29}{128}$.

Another SOLUTION.

Supposing as before that *n* be the number of Stakes which *A* is to win, and that the number of Games be $n + d$, the Probability which *A* has of winning will be expressed by the following Series

$$\frac{a^n}{a+b} \times 1 + \frac{nab}{a+b} + \frac{n \cdot n+3 \cdot aabb}{1 \cdot 2 \cdot a+b} + \frac{n \cdot n+4 \cdot n+5 \cdot a^3b^2}{1 \cdot 2 \cdot 3 \cdot a+b} + \frac{n \cdot n+5 \cdot n+6 \cdot n+7 \cdot a^4b^3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot a+b} + \&c.$$

which Series ought to be continued to so many Terms as there are Units in $\frac{1}{2}d + 1$; always observing to substitute $d - 1$ in the room of d in Case d be an odd number, or which is the same thing, taking so many Terms as there are Units in $\frac{d+1}{2}$.

Now supposing, as in the first Example of the preceding Solution, that Three is the number of Stakes, and Ten the given number of Games, and also that there is an equality of Skill between *A* and *B*, the foregoing Series will become $\frac{1}{8} \times 1 + \frac{3}{4} + \frac{9}{16} + \frac{28}{64} = \frac{11}{32}$, as before.

REMARK.

In the first attempt that I had ever made towards solving the general Problem of the Duration of Play, which was in the Year 1708, I began with the Solution of this LXVth Problem, well knowing that it might be a Foundation for what I farther wanted, since which time, by a due repetition of it, I solved the main Problem: but as I found afterwards a nearer way to it, I barely published in my first Essay on those matters, what seemed to me most simple and elegant, still preserving this Problem by me in order to be published when I should think it proper. Now in the year 1713 Mr. *de Monmort* printed a Solution of it in a Book by him published upon Chance, in which was also inserted a Solution of the same by Mr. *Nicolas Bernoulli*; and as those two Solutions seemed to

to me, at first sight, to have some affinity with what I had found before, I considered them with very great attention; but the Solution of Mr. *Nicolas Bernoulli* being very much crouded with Symbols, and the verbal Explication of them too scanty, I own I did not understand it thoroughly, which obliged me to consider Mr. *de Monmort's* Solution with very great attention: I found indeed that he was very plain, but to my great surprize I found him very erroneous; still in my Doctrine of Chances I printed that Solution, but rectified and ascribed it to Mr. *de Monmort*, without the least intimation of any alterations made by me; but as I had no thanks for so doing, I resume my right, and now print it as my own: but to come to the Solution.

Let it be proposed to find the number of Chances there are for *A* to win two Stakes of *B*, or for *B* to win three Stakes of *A*, in fifteen Games.

The number of Chances required is expressed by two Branches of Series; all the Series of the first Branch taken together express the number of Chances there are for *A* to win two Stakes of *B*, exclusive of the number of Chances there are for *B* before that time, to win three Stakes of *A*. All the Series of the second Branch taken together express the number of Chances there are for *B* to win three Stakes of *A*, exclusive of the number of Chances there are for *A* before that time to win two Stakes of *B*.

First Branch of Series.

$$\begin{array}{cccccccccccccccc}
 a^{15} & a^{14}b & a^{13}b^2 & a^{12}b^3 & a^{11}b^4 & a^{10}b^5 & a^9b^6 & a^8b^7 & a^7b^8 & a^6b^9 & a^5b^{10} & a^4b^{11} & a^3b^{12} & a^2b^{13} \\
 1 + 15 + 105 + 455 + 1365 + 3003 + 5005 + 5005 + 3003 + 1365 + 455 + 105 + 15 + 1 \\
 - 1 - 15 - 105 - 455 - 455 - 105 - 15 - 1 \\
 + 1 + 15 + 15 + 1
 \end{array}$$

Second Branch of Series.

$$\begin{array}{cccccccccccccccc}
 b^{15} & b^{14}a & b^{13}a^2 & b^{12}a^3 & b^{11}a^4 & b^{10}a^5 & b^9a^6 & b^8a^7 & b^7a^8 & b^6a^9 & b^5a^{10} & b^4a^{11} & b^3a^{12} & b^2a^{13} \\
 1 + 15 + 105 + 455 + 1365 + 3003 + 5005 + 3003 + 1365 + 455 + 105 + 15 + 1 \\
 - 1 - 15 - 105 - 455 - 1365 - 455 - 105 - 15 - 1 \\
 + 1 + 15 + 1
 \end{array}$$

The literal Quantities which are commonly annexed to the numerical ones, are here written on the top of them; which is done, to the end that each Series being contained in one Line, the dependency they have upon one another, may thereby be made more conspicuous.

The first Series of the first Branch expresses the number of Chances there are for A to win two Stakes of B , including the number of Chances there are for B before, or at the Expiration of the fifteen Games, to be in a Circumstance of winning three Stakes of A ; which number of Chances may be deduced from the LXVth Problem.

The second Series of the first Branch is a part of the first, and expresses the number of Chances there are for B to win three Stakes of A , out of the number of Chances there are for A , in the first Series to win two Stakes of B . It is to be observed about this Series, *First*, that the Chances of B expressed by it are not restrained to happen in any order, that is, either before or after A has won two Stakes of B . *Secondly*, that the literal products belonging to it are the same with those of the corresponding Terms of the first Series. *Thirdly*, that it begins and ends at an Interval from the first and last Terms of the first Series equal to the number of Stakes which B is to win. *Fourthly*, that the numbers belonging to it are the numbers of the first Series repeated in order, and continued to one half of its Terms; after which those numbers return in an inverted order to the end of that Series: which is to be understood in case the number of its Terms should happen to be even; for if it should happen to be odd, then that order is to be continued to the greatest half, after which the return is made by omitting the last number. *Fifthly*, that all the Terms of it are affected with the sign *minus*.

The Third Series is part of the second, and expresses the number of Chances there are for A to win two Stakes of B , out of the number of Chances there are in the second Series for B to win three Stakes of A ; with this difference, that it begins and ends at an Interval from the first and last Terms of the second Series, equal to the number of Stakes which A is to win; and that the Terms of it are all positive.

It is to be observed, that let the number of those Series be what it will, the Interval between the beginning of the first and the beginning of the second, is to be equal to the number of Stakes which B is to win; and that the Interval between the beginning of the second and the beginning of the third, is to be equal to the number of Stakes which A is to win; and that these Intervals recur alternately in the same order. It is to be observed likewise that all these Series are alternately positive and negative.

All the Observations made upon the first Branch of Series belonging also to the second, it would be needless to say any thing more of them.

Now

Now the Sum of all the Series of the first Branch, being added to the Sum of all the Series of the second, the Aggregate of these Sums will be the Numerator of a Fraction expressing the Probability of the Play's terminating in the given number of Games; of which the Denominator is the Binomial $a + b$ raised to a Power whose Index is equal to that number of Games. Thus supposing that in the Case of this Problem both a and b are equal to Unity, the Sum of the Series in the first Branch will be 18778, the Sum of the Series in the second will be 12393, and the Aggregate of both 31171; and the Fifteenth Power of 2 being 32768, it follows that the Probability of the Play's terminating in Fifteen Games will be $\frac{31171}{32768}$, which being subtracted from Unity, the remainder will be $\frac{1597}{32768}$: From whence we may conclude that it is a Wager of 31171 to 1597, that either A in Fifteen Games shall win two Stakes of B , or B win three Stakes of A : which is conformable to what was found in the LXIVth Problem.

P R O B L E M LXVI.

To find what Probability there is that in a given number of Games A may be winner of a certain number q of Stakes, and at some other time B may likewise be winner of the number p of Stakes, so that both circumstances may happen.

SOLUTION.

Find by our LXVth Problem the Probability which A has of winning, without any limitation, the number q of Stakes: find also by the LXIII^d Problem the Probability which A has of winning that number of Stakes before B may happen to win the number p ; then from the first Probability subtracting the second, the remainder will express the Probability there is that both A and B may be in a circumstance of winning, but B before A . In the like manner, from the Probability which B has of winning without limitation, subtracting the Probability which he has of winning before A , the remainder will express the Probability there is that both A and B may be in a circumstance of winning, but A before B : wherefore adding these two remainders together, their Sum will express the Probability required.

Thus

Thus if it were required to find what Probability there is, that in Ten Games A may win Two Stakes of B , and that at some other time B may win Three:

The first Series will be found to be

$$\frac{aa}{a+b} \times 1 + \frac{2ab}{a+b} + \frac{5aabb}{a+b} + \frac{14a^3b}{a+b} + \frac{17a^4b^4}{a+b}$$

The second Series will be

$$\frac{aa}{a+b} \times 1 + \frac{2ab}{a+b} + \frac{5aabb}{a+b} + \frac{13a^3b^3}{a+b} + \frac{34a^4b^4}{a+b}$$

The difference of these Series being $\frac{aa}{a+b} \times \frac{a^3b^3}{a+b} + \frac{8a^4b^4}{a+b}$ expresses the first part of the Probability required, which in the Case of an equality of Skill between the Gamesters would be reduced to $\frac{3}{256}$.

The third Series is as follows,

$$\frac{b^3}{a+b} \times 1 + \frac{3ab}{a+b} + \frac{9aabb}{a+b} + \frac{28a^3b^4}{a+b}$$

The fourth Series is

$$\frac{b^3}{a+b} \times 1 + \frac{3ab}{a+b} + \frac{8aabb}{a+b} + \frac{21a^3b^3}{a+b}$$

The difference of these two Series being $\frac{b^3}{a+b} \times \frac{aabb}{a+b} + \frac{7a^3b^3}{a+b}$ expresses the second part of the Probability required, which in the Case of an equality of Skill would be reduced to $\frac{11}{512}$. Wherefore the Probability required would in this Case be $\frac{3}{256} + \frac{11}{512} = \frac{17}{512}$. Whence it follows, that it is a Wager of 495 to 17, or 29 to 1 very near, that in Ten Games A and B will not both be in a circumstance of winning, viz. A the number q and B the number p of Stakes. But if by the conditions of the Problem, it were left indifferent whether A or B should win the two Stakes or the three, then the Probability required would be increased, and become as follows; viz.

$$\frac{aa+bb}{a+b} \times \frac{a^3b^3}{a+b} + \frac{8a^4b^4}{a+b} + \frac{a^3+b^3}{a+b} \times \frac{aabb}{a+b} + \frac{7a^3b^3}{a+b}$$

which

which, in the Case of an equality of Skill between the Gamesters, would be double to what it was before.

P R O B L E M LXVII.

To find what Probability there is, that in a given number of Games A may win the number q of Stakes; with this farther condition, that B during that whole number of Games may never have been winner of the number p of Stakes.

SOLUTION.

From the Probability which A has of winning without any limitation the number q of Stakes, subtract the Probability there is that both A and B may be winners, viz. A of the number q , and B of the number p of Stakes, and there will remain the Probability required.

But if the conditions of the Problem were extended to this alternative, viz. that either A should win the number q of Stakes, and B be excluded the winning of the number p ; or that B should win the number p of Stakes, and A be excluded the winning of the number q , the Probability that either the one or the other of these two Cases may happen, will easily be deduced from what we have said.

The Rules hitherto given for the Solution of Problems relating to the Duration of Play are easily practicable, if the number of Games given is but small; but if that number is large, the work will be very tedious, and sometimes swell to that degree as to be in some manner impracticable: to remedy which inconveniency, I shall here give an Extract of a paper by me produced before the Royal Society, wherein was contained a Method of solving very expeditiously the chief Problems relating to that matter, by the help of a Table of Sines, of which I had before given a hint in the first Edition of my *Doctrine of Chances*, pag. 149, and 150.

P R O B L E M LXVIII.

To solve by a Method different from any of the preceding, the Problem LIX, when a is to b in a ratio of Equality.

SOLU-

SOLUTION.

Let n be the number of Games given, and p the number of Stakes; let Q represent 90 degrees of a Circle whose Radius is equal to Unity; let $C, D, E, F, \&c.$ be the Sines of the Arcs $\frac{Q}{p}, \frac{3Q}{p}, \frac{5Q}{p}, \frac{7Q}{p}, \&c.$ till the Quadrant be exhausted; let also, $c, d, e, f, \&c.$ be the Co-sines of those Arcs: then if the difference between n and p be an even number, the Probability of the Play's not ending in the given number of Games will be represented by the Series

$$\frac{2}{p} \times \frac{c^{n+1}}{C} - \frac{d^{n+1}}{D} + \frac{e^{n+1}}{E} - \frac{f^{n+1}}{F}, \&c.$$

of which Series very few Terms will be sufficient for a very near approximation. But if the difference between n and p be odd, then the Probability required will be $\frac{2}{p} \times \frac{c^n}{C} - \frac{d^n}{D} + \frac{e^n}{E} - \frac{f^n}{F} \&c.$

In working by Logarithms, you are perpetually to subtract, from the Logarithm of every Term, the Product of 10 into the number n , in case the number $n - p$ be even; but in case it be odd, you are to subtract the Product of 10 into $n - 1$, and if the Subtraction cannot be made without making the remainder negative, add 10, 20, or 30, &c. and make such proper allowances for those additions as those who are conversant with Logarithms know how to make.

To apply this to some particular cases, let it be required to find the Probability of Twelve Stakes being not lost in 108 Games.

Here because the difference between 108 and 12 is 96, I take the first form, thus

The Arcs $\frac{Q}{p}, \frac{3Q}{p}, \frac{5Q}{p}, \frac{7Q}{p}, \frac{9Q}{p}, \frac{11Q}{p}, \frac{13Q}{p}, \&c.$ being respectively $7^d - 30'$, $22^d - 30'$, $37^d - 30'$, $52^d - 30'$, $67^d - 30'$, $82^d - 30'$, $97^d - 30'$, &c. I take only the six first, as not exceeding 90^d .

Now the Logarithm of the Co-sine of $7^d - 30'$ being 9.9962686, I multiply it by $n + 1$, that is in this Case by 109, and the product will be 1089.5932774, which is the Logarithm of the Numerator of the first Fraction $\frac{c^{n+1}}{C}$.

From

From that Logarithm, I subtract the Logarithm of the Sine of $7^d - 30'$ here represented by C, which being 9.1156977, the remainder will be 1080.4775797, out of which rejecting 1080 product of 10 by the given number of Games 108, and taking only 0.4775797 the number answering will be 3.00327, which being multiplied by the common Multiplicator $\frac{2}{p}$, that is in this Case by $\frac{2}{12}$ or $\frac{1}{6}$, the product will be 0.50053, which Term alone determines nearly the Probability required.

For if we intend to make a Correction by means of the second Term $\frac{d^{n+1}}{D}$, we shall find the Logarithm of $\frac{d^{n+1}}{D}$ to be 1076.6692280 to which adding 10, and afterwards subtracting 1080, the remainder will be 6.6692280, to which answers 0.0004669, of which the 6th part is 0.0000778, which being almost nothing may be safely rejected. And whenever it happens that n is a large number in respect to p , the first Term alone of these Series will exceeding near determine the Probability required.

Let it now be required to find the Probability of 45 Stakes being not lost on either side in 1519 Games.

The Arcs $\frac{Q}{p}$, $\frac{3Q}{p}$, $\frac{5Q}{p}$, &c. being respectively 2^d , 6^d , 10^d , &c. I take, 1^o, the Logarithm of the Co-sine of 2^d which is 9.9997354, which being multiplied by $n + 1$, that is in this Case by 1520, the product will be 15199.5988080, out of which subtracting the Logarithm of the Sine of 2^d , viz. 8.5428192, the remainder will be 15191.0559888, out of which rejecting 15190, the number answering will be 11.3759, which being multiplied by $\frac{2}{p}$, that is, in this Case by $\frac{2}{45}$, the product will be .50559 which nearly determines the Probability required.

Now if we want a Correction by means of the second Term, we shall find $\frac{d^{n+1}}{D} = .00002081$, which Term being so very inconsiderable may be entirely rejected, and much more all the following.

Considering therefore that when the Arc $\frac{Q}{p}$ is small, the first Term alone is sufficient for a near approximation, it will not be amiss to inquire what must be the number of Games that shall make it an equal Probability of the Play's being ended in that number of Games; which to do,

Suppose $\frac{c^{n+1}}{C} \times \frac{2}{p} = \frac{1}{2}$, hence $4c^{n+1} = Cp$, then supposing p a large number, whereby the number n must be still much larger, we may barely take for our Equation $4c^n = pC$, then taking the Logarithms, we shall have $\text{Log. } 4 + n \text{ Log. } c = \text{Log. } C + \text{Log. } p$, let the magnitude of the Arc $\frac{Q}{p}$ be supposed $= z$; now since the number p has been supposed very large, it follows that the Arc z must be very small; wherefore the Sine of that Arc will also be nearly $= z$, and its Co-sine $1 - \frac{1}{2}zz$ nearly, of which Co-sine the Logarithm will be $-\frac{1}{2}zz$ nearly; we have therefore the Equation $\text{Log. } 4 - \frac{1}{2}nzz = \text{Log. } p + \text{Log. } z$; let now the Magnitude of an Arc of 90° , to a Radius equal to Unity, be $= M$, hence we shall have $\frac{M}{p} = z$, and $\text{Log. } z = \text{Log. } M - \text{Log. } p$, wherefore the Equation will at last be changed into this, $\text{Log. } 4$

$-\frac{1}{2}nMM = \text{Log. } M$, and therefore $n = \frac{2 \log. 4 - 2 \log. M}{MM} \times pp$, but $\frac{2 \log. 4 - 2 \log. M}{MM} = 0.756$ nearly, and therefore $n = 0.756pp$.

N. B. The Logarithms here made use of are supposed to be Hyperbolic Logarithms, of which I hear a Table will soon be published.

Mr. *de Monmort* in the second Edition of his Tract, *Des jeux de Hazard*, tells us that he found that if p denoted an odd number of Stakes to be won or lost, making $\frac{p+1}{2} = f$, that then the Quantity $3ff - 3f + 1$ would denote a number of Games wherein there would be more than an equal Probability of the Play's being ended; but at the same time he owns, that he has not been able to find a Rule like it for an even number of Stakes.

Whereupon I shall observe, *first*, that his Expression may be reduced to $\frac{3}{4}pp + \frac{1}{4}$. Which tho' near the Truth in small numbers, yet is very defective in large ones, for it may be proved that the number of Games found by his Expression, far from being above what is requisite, is really below it. *Secondly*, that his Rule does not err more in an even number of Stakes than in an odd one; but that Rule being founded upon an induction gathered from the Solution of some of the simplest Cases of this Problem, it is no wonder that he restrained it to the odd Cases, he happening to be mistaken in determining

mining the number of Games requisite to make it an even Wager that twelve Stakes would be won or lost before or at the expiration of those Games, which he finds by a very laborious calculation to have been 122; in which however he was afterwards rectified by Mr. *Nicolas Bernoulli*, who informed him that he had found by his own Calculation that the number of Games requisite for that purpose was above 108, and below 110; and this is exactly conformable to our Rule, for multiplying $pp = 144$ by 0.756, the Product will be 108.864.

For a Proof that his Rule falls short of the Truth, let us suppose $p = 45$, then f will be $= 23$, and $3ff - 3f + 1$ will be $= 1519$, let us therefore find the Probability of the Play's terminating in that number of Games; but we have found by this LXVIIIth Problem, that the Probability of the Play's not terminating in that number of Games is 0.50559; and therefore the Probability of its terminating within them is 0.49441; which being less than $\frac{1}{2}$, shews 'tis not more than an equal Wager that the Play would be terminated in 1519 Games.

But farther, let us see what number of Games would be necessary for the equal wager, then multiplying 2025 square of 45 by 0.756, the Product will be 1530.9; which shews that about 1531 Games are requisite for it.

P R O B L E M LXIX.

The same things being given as in the preceding Problem, except that now the ratio of a to b is supposed of inequality, to solve the same by the Sines of Arcs.

SOLUTION.

Let n represent the number of Games given, p the number of Stakes to be won or lost on either side, let also A be the Semi-circumference of a Circle whose Radius is equal to Unity: let C, D, E, F, &c. be the Sines of the Arcs $\frac{A}{p}$, $\frac{3A}{p}$, $\frac{5A}{p}$, $\frac{7A}{p}$, &c. till the Semi-circumference be exhausted; let also $c, d, e, f, \&c.$ be the respective versed Sines of those Arcs; let $\frac{a^n + b^n}{a + b}$ be made $= L$,

$\frac{a - b}{a + b} = t$, $\frac{ab}{a + b} = r$; let $c, 2r :: CC, m$; $d, 2r :: DD, q$;

$e, 2r :: EE, f, \&c.$ then the Probability of the Play not ending in n Games will be expreffed by the following Series

$$\frac{C}{2rc+t} \times m^{\frac{1}{2}n} - \frac{D}{2rd+t} \times q^{\frac{1}{2}n} + \frac{E}{2re+t} \times f^{\frac{1}{2}n}, \&c.$$

the whole to be multiplied by $\frac{2L}{p \times r^{\frac{1}{2}p-1}}$

As there are but few Tables of Sines, wherein the Logarithms of the verfed Sines are to be found, it will be eafy to remedy that inconveniency, by adding the Logarithm of 2 to the excefs of twice the tabular Logarithm of the Sine of half the given Arc above 10; for that Sum will give the Logarithm of the verfed Sine of the whole Arc.

It will be eafily perceived that inftead of referring the Arcs to the Divifion of the Semi-circumference, we might have referred them to the Divifion of the Quadrant, as in the Cafe of the preceding Problem.

Of the Summation of recurring Series.

The Reader may have perceived that the Solution of feveral Problems relating to Chance depends upon the Summation of Series; I have, as occafion has offered, given the Method of fummung them up; but as there are others that may occur, I think it neceffary to give a fummery View of what is moft requifite to be known in this matter; defiring the Reader to excufe me, if I do not give the Demonftrations, which would fwell this Tract too much; efpecially confidering that I have already given them in my *Mifcellanea Analytica*.

I call that a *recurring* Series which is fo conftituted, that having taken at pleafure any number of its Terms, each following Term fhall be related to the fame number of preceding Terms, according to a conftant law of Relation, fuch as the following Series

$$\begin{array}{cccccc} A & B & C & D & E & F \\ 1 & + 2x & + 3xx & + 10x^3 & + 34x^4 & + 97x^5, \&c. \end{array}$$

in which the Terms being refpectively represented by the Capitals A, B, C, D, &c. we fhall have

$$\begin{array}{l} D = 3Cx - 2Bxx + 5Ax^3 \\ E = 3Dx - 2Cxx + 5Bx^3 \\ F = 3Ex - 2Dxx + 5Cx^3 \\ \&c. \end{array}$$

Now

Now the Quantities $3x - 2xx + 5x^3$, taken together and connected with their proper Signs, is what I call the Index, or the *Scale of Relation*; and sometimes the bare Coefficients $3 - 2 + 5$ are called the Scale of Relation.

PROPOSITION I.

If there be a recurring Series $a + bx + cxx + dx^3 + ex^4$, &c. of which the Scale of Relation be $fx - gxx$; the Sum of that Series continued *in infinitum* will be

$$\begin{array}{r} a + bx \\ - fax \\ \hline 1 - fx + gxx \end{array}$$

PROPOSITION II.

Supposing that in the Series $a + bx + cxx + dx^3 + ex^4$, &c. the Law of Relation be $fx - gxx + bx^3$; the Sum of that Series continued *in infinitum* will be

$$\begin{array}{r} a + bx + cxx \\ - fax - fbxx \\ + gaxx \\ \hline 1 - fx + gxx - bx^3 \end{array}$$

PROPOSITION III.

Supposing that in the Series, $a + bx + cxx$, &c. the Law of Relation be $fx - gxx + bx^3 - kx^4$, the Sum of the Series will be

$$\begin{array}{r} a + bx + cxx + dx^3 \\ - fax - fbxx - fcx^3 \\ + gaxx + gbxx^3 \\ - bax^3 \\ \hline 1 - fx + gxx - bx^3 + kx^4 \end{array}$$

As the Regularity of those Sums is conspicuous, it would be needless to carry them any farther.

Still it is convenient to know that the Relation being given, it will be easy to obtain the Sum by observing this general Rule.

1°, Take as many Terms of the Series as there are parts in the Scale of Relation.

2°, Subtract the Scale of Relation from Unity, and let the remainder be called the *Differential Scale*.

3°, Multi-

3°, Multiply those Terms which have been taken in the Series by the Differential Scale, beginning at Unity, and so proceeding orderly, remembering to leave out what would naturally be extended beyond the last of the Terms taken.

Then the Product will be the Numerator of a Fraction expressing the Sum, of which the Denominator will be the Differential Scale.

Thus to form the preceding Theorem,

$$\begin{array}{r} \text{Multiply } a + bx + cxx + dx^3 \\ \text{by } 1 - fx + gxx - bx^3 \dots \\ \text{and beginning from Unity, we shall have} \\ a + bx + cxx + dx^3 \\ - fax - fbxx - fcx^3 \dots \\ + gaxx + gbxx^3 \dots \\ - bax^3 \dots \end{array}$$

omitting the superfluous Terms, and thus will the Numerator be formed; but the Denominator will be the Differential Scale, viz.
 $1 - fx + gxx - bx^3 + kx^4.$

COROLLARY.

If the first Terms of the Series are not taken at pleasure, but begin from the second Term to follow the Law of Relation, in so much that

$$\begin{array}{lcl} b \text{ shall be} & = & fa \\ c & = & fb - ga \\ d & = & fc - gb + ba \\ \&c. & & \end{array}$$

then the Fraction expressing the Sum of the Series will have barely the first Term of the Series for its Numerator.

PROPOSITION IV.

If a Series is so constituted, as that the last Differences of the Coefficients of the Terms whereof it is composed be all equal to nothing, the Law of the Relation will be found in the Binomial $\sqrt[n]{1-x}$, n denoting the rank of those last Differences; thus supposing the Series

$$\begin{array}{ccccccc} A & B & C & D & E & F & G \\ 1 & + 4x & + 10xx & + 20x^3 & + 35x^4 & + 56x^5 & + 84x^6, \&c. \end{array}$$

whereof the Coefficients are,

$$1 + 4$$

	1	+	4	+	10	+	20	+	35	+	56	+	84
1 st Differences	-	-	3	+	6	+	10	+	15	+	21	+	28
2 ^d Differences	-	-	-	-	3	+	4	+	5	+	6	+	7
3 ^d Differences	-	-	-	-	-	-	1	+	1	+	1	+	1
4 th Differences	-	-	-	-	-	-	-	-	0	+	0	+	0

I say that the Relation of the Terms will be found in the Binomial $1 - x^4$, which being expanded will be $1 - 4x + 6xx - 4x^3 + x^4$ and is the Differential Scale, and therefore the Scale properly so called will be $4x - 6xx + 4x^3 - x^4$; thus, in the foregoing Series, the Term

$$G = 4Fx - 6Exx + 4Dx^3 - 1Cx^4.$$

COROLLARY.

The Sums of those infinite Series which begin at Unity, and have their Coefficients the figurate numbers of any order, are always expressible by the Fraction $\frac{1}{1-x^p}$, wherein p denotes the rank or order which those figurative numbers obtain; for Instance if we take the Series

$1 + 1x + 1xx + 1x^3 + 1x^4 + 1x^5 + 1x^6$, &c. which is a geometric Progression, and whose Coefficients are the numbers of the first order, the Sum will be $\frac{1}{1-x}$, and if we take the Series $1 + 2x + 3xx + 4x^3 + 5x^4 + 6x^5 + 7x^6$, &c. whose Coefficients compose the numbers of the second order, the Sum will be $\frac{1}{1-x^2}$; and again, if we take the Series $1 + 3x + 6xx + 10x^3 + 15x^4$, &c. whose Coefficients are the numbers of the third order, otherwise called Triangular numbers, the Sum will be $\frac{1}{1-x^3}$.

PROPOSITION V.

The Sum of any finite number of Terms of a recurring Series $a + bx + cxx + dx^3 + ex^4$, &c. is always to be obtained.

Thus supposing the Scale of Relation to be $fx - gxx$; n the number of Terms whose Sum is required; and $ax^n + \beta x^{n+1}$ the two Terms which would next follow the last of the given Terms, if the Series was continued; then the Sum will be

$$\frac{a + bx - x^n \times a + \beta x - fax - fax}{1 - fx + gxx}$$

But

But if the Scale of Relation be $fx - gxx + bx^3$, n the number of Terms given, and $ax^n + \beta x^{n+1} + \gamma x^{n+2}$, the three Terms that would next follow the last of the given Terms, then the Sum will be

$$\begin{array}{r} a + bx + cxx - x^n \times a + \beta x + \gamma xx \\ - fax - fbxx \qquad - fax - f\beta x \\ + gaxx \qquad \qquad \qquad + gaxx \\ \hline 1 - fx + gxx - bx^3 \end{array}$$

The continuation of which being obvious, those Theorems need not be carried any farther.

But as there is a particular elegancy for the Sums of a finite number of Terms in those Series whose Coefficients are figurate numbers beginning at Unity, I shall set down the *Canon* for those Sums.

Let n denote the number of Terms whose Sum is to be found, and p the rank or order which those figurate numbers obtain, then the Sum will be

$$\begin{aligned} & \frac{1 - x^n}{1 - x^p} - \frac{nx^n}{1 - x^{p-1}} - \frac{n \cdot n+1 \cdot x^n}{1 \cdot 2 \cdot 1 - x^{p-2}} \\ & - \frac{n \cdot n+1 \cdot n+2 \cdot x^n}{1 \cdot 2 \cdot 3 \cdot 1 - x^{p-3}} - \frac{n \cdot n+1 \cdot n+2 \cdot n+3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 - x^{p-4}}, \text{ \&c.} \end{aligned}$$

which is to be continued till the number of Terms be $= p$.

Thus supposing that the Sum of twelve Terms of the Series, $1 + 3x + 6xx + 10x^3 + 15x^4$, &c. were demanded, that Sum will be

$$\frac{1 - x^{12}}{1 - x^3} - \frac{12x^{12}}{1 \cdot 2 \cdot 1 - x^2} - \frac{12 \cdot 13x^{12}}{1 \cdot 2 \cdot 3 \cdot 1 - x}.$$

PROPOSITION VI.

In a recurring Series, any Term may be obtained whose place is assigned.

It is very plain, from what we have said, that after having taken so many Terms of the Series as there is in the Scale of Relation, the Series may be protracted till it reach the place assigned; however if that place be very distant from the beginning of the Series, the continuation of those Terms may prove laborious, especially if there be many parts in the Scale.

But there being frequent Cases wherein that inconveniency may be avoided, it will be proper to shew by what Rule this may be known; and then to shew how we are to proceed.

The Rule will be to take the Differential Scale, and to suppose it $= 0$, then if the roots of that supposed Equation be all real, and unequal, the thing may be effected as follows. Let the Series be represented by

$$a + br$$

$$a + br + crr + dr^3 + er^4, \&c.$$

and 1^o if $fr - grr$ be the Scale of Relation, and consequently $1 - fr + grr$ the differential Scale, then having made $1 - fr + grr = 0$; multiply the Terms of that Scale respectively by xx , x , 1 , so as to have $xx - frx + grr = 0$, let m and p be the two roots of that Equation, then having made $A = \frac{br - pa}{m - p}$ and $B = \frac{br - ma}{p - m}$, and supposing l to be the interval between the first Term and the place assigned, that Term will be $Am^l + Bp^l$.

Secondly, If the Scale of Relation be $fr - grr + br^3$, make $1 - fr + grr - br^3 = 0$, the Terms of which Equation being multiplied respectively by x^3 , xx , x , 1 , we shall have the new Equation $x^3 - frxx + grrx - br^3 = 0$, let m , p , q be the roots of that Equation, then having made $A = \frac{crr - p + q \times br + pqa}{m - p \times m - q}$,

$$B = \frac{crr - \overline{m + q} \times br + mqa}{p - m \times p - q}, C = \frac{crr - \overline{p + m} \times br + mqa}{q - m \times q - p};$$

And supposing as before l to be the Interval between the first Term and the Term whose place is assigned, that Term will be $Am^l + Bp^l + Cq^l$.

Thirdly, If the Scale of Relation be $fr - grr + br^3 - kr^4$ make $1 - fr + grr - br^3 + kr^4 = 0$, and multiply its Terms respectively by x^4 , x^3 , xx , x , 1 , so as to have the new Equation $x^4 - frx^3 + grrx^2 - br^3x + kr^4 = 0$, let m , p , q , f , be roots of that Equation, then having made

$$A = \frac{dr^3 - \overline{p + q + f} \times crr + \overline{pq + pf + qf} \times br - pqf \times a}{m - p \times m - q \times m - f}$$

$$B = \frac{dr^3 - \overline{q + f + m} \times crr + \overline{qf + qm + fm} \times br - qfm \times a}{p - q \times p - f \times p - m}$$

$$C = \frac{dr^3 - \overline{f + m + p} \times crr + \overline{fm + fp + mp} \times br - fmp \times a}{q - f \times q - m \times q - p}$$

$$D = \frac{dr^3 - \overline{m + p + q} \times crr + \overline{mp + mq + pq} \times br - mpq \times a}{f - m \times f - p \times f - q}$$

then, still supposing l to be the Interval between the first Term and the Term whose place is assigned, that Term will be $Am^l + Bp^l + Cq^l + Df^l$.

Altho' one may by a narrow inspection perceive the Order of those Theorems, it will not be amiss to express them in words at length.

GENERAL RULE.

Let the Roots m , p , q , f , &c. determined as above, be called respectively,

G g

spectively, first, second, third, fourth Root, &c. let there be taken as many Terms of the Series beginning from the first, as there are parts in the Scale of Relation : then multiply in an inverted order, 1°, the last of these Terms by Unity ; 2°, the last but one by the Sum of the Roots wanting the first ; 3°, the last but two, by the Sum of the Products of the Roots taken two and two, excluding that product wherein the first Root is concerned ; 4°, the last but three, by the Sum of the Products of the Roots taken three and three, still excluding that Product in which the first Root is concerned, and so on ; then all the several parts which are thus generated by Multiplication being connected together by Signs alternately positive and negative, will compose the Numerator of that Fraction to which *A* is equal ; now the Numerator of that Fraction to which *B* is equal will be formed in the same manner, excluding the second Root instead of the first, and so on

As for the Denominators, they are formed in this manner : From the first Root subtract severally all the others, and let all the remainders be multiplied together, and the Product will constitute the Denominator of the Fraction to which *A* is equal ; and in the same manner, from the second Root subtracting all the others, let all the remainders be multiplied together, and the Product will constitute the Denominator of the Fraction to which *B* is equal, and so on for the Rest.

COROLLARY I.

If the Series in which a Term is required to be assigned, be the Quotient of Unity divided by the differential Scale $1 - fr + grr - br^3 + kr^4$, multiply the Terms of that Scale respectively by $x^4, x^3, x^2, x, 1$, so as to make the first Index of x equal to the last of r , then make the Product $x^4 - frx^3 + grrxx - br^3x + kr^4$ to be $= 0$. Let as before m, p, q, f , be the Roots of that Equation, let also z be the number of those Roots, and l the Interval between the first Term, and the Term required, then make

$$A = \frac{m^{z-1}}{m - p \times m - q \times m - f}, B = \frac{p^{z-1}}{p - m \times p - q \times p - f}$$

$$C = \frac{q^{z-1}}{q - m \times q - p \times q - f}, D = \frac{f^{z-1}}{f - m \times f - p \times f - q}$$

and the Term required will be $Am^l + Bp^l + Cq^l + Df^l$; and the Sum of the Terms will be

$$A \times \frac{1-m^{l+1}}{1-m} + B \times \frac{1-p^{l+1}}{1-p} + C \times \frac{1-q^{l+1}}{1-q} + D \times \frac{1-f^{l+1}}{1-f}$$

It is to be observed, that the Interval between the first Term and the Term required is always measured by the number of Terms wanting one, so that having for Instance the Terms, *a, b, c, d, e, f*, whereof *a* is the first and *f* the Term required, the Interval between *a* and *f* is 5, and the Number of all the Terms 6.

COROLLARY 2.

If in the recurring Series $a + br + crr + dr^3 + er^4$, &c. whereof the Differential Scale is supposed to be $1 - fr + grr - br^3 + kr^4$, we make $x^4 - fxx^3 + grrxx - br^3x + kr^4 = 0$, and that the Roots of that Equation be *m, p, q, f*, and that it so happen that so many Terms of the Series $a + br + crr + dr^3 + er^4$, &c. as there are Roots, be every one of them equal to Unity, then any Term of the Series may be obtained thus; let *l* be the Interval between the first Term and the Term required, make

$$A = \frac{1-p \times 1-q \times 1-f}{m-m \times p \times m-q \times m-f}, \quad B = \frac{1-q \times 1-f \times 1-m}{p-p \times q \times p-f \times p-m}$$

$$C = \frac{1-f \times 1-m \times 1-p}{q-q \times f \times q-m \times q-p}, \quad D = \frac{1-m \times 1-p \times 1-q}{f-m \times f-p \times f-q}$$

and the Term required will be $Am^l + Bp^l + Cq^l + Df^l$.

PROPOSITION VII.

If there be given a recurring Series whose Scale of Relation is $fr - grr$, and out of that Series be composed two other Series, whereof the first shall contain all the Terms of the Series given which are posited in an odd place, and the second shall contain all the Terms that are posited in even place; then the Scale of Relation in each of these two new Series may be obtained as follows:

Take the differential Scale $1 - fr + grr$, out of which compose the Equation $xx - frx + grr = 0$; then making $xx = z$, expunge the Quantity *x*, whereby the Equation will become $z - fr \sqrt{z} + grr = 0$, or $z + grr = fr \sqrt{z}$; and squaring both parts, to take away the Radicality, we shall have the new Equation $zz + 2grrz - ffrz = 0$, or $zz + 2grrz - ffrz = 0$; and dividing its

Terms respectively by $zz, z, 1$, we shall have a new differential Scale for each of the two new Series into which the Series given was divided, which will be $1 + 2grr - ffr + ggr^4$: and this being ob-

tained, it is plain from our *first Proposition*, that each of the two new Series may be summed up.

But if the Scale of Relation be extended to three Terms, such as the Scale $fr - grr + br^3$, then the differential Scale for each of the two Series into which the Series given may be supposed to be divided, will be $1 - ffr - 2fbr^4 - bbr^6$, whereby it appears that each of the two new Series may be summed up.

If instead of dividing the Series given into two Series, we divide it into three, whereof the first shall be composed of the

1st, 4th, 7th, 10th, &c. Terms; the second of the

2^d, 5th, 8th, 11th, &c. Terms; the third of the

3^d, 6th, 9th, 12th, &c. Terms; and that the Scale of Relation be supposed $fr - grr$; then taking the differential Scale

$1 - fr + grr$, and having out of it formed the Equation $xx - frx + grr = 0$, suppose $x^3 = z$; let now x be expunged, and the Equation will be changed into this $zz + 3fgr^3z + g^3r^6 = 0$,
 $- f^3r^3z$

of which the Terms being divided respectively by zz , z , 1 , we shall have a differential Scale $1 - f^3r^3 + g^3r^6$, which will serve
 $+ 3fgr^3$

for every one of the three Series into which the Series given is divided; and therefore every one of those three Series may be summed up, by help of the two first Terms of each.

If the Scale of Relation be composed of never so many parts, still if the Series given be to be divided into three other Series; from the supposition of x^3 being made $= z$, will be derived a Scale of Relation for the three parts into which the Series given is to be divided.

But if the Series given was to be divided into 4, 5, 6, 7, &c. Series given, suppose accordingly $x^4 = z$, $x^5 = z$, $x^6 = z$, $x^7 = z$, &c. and x being expunged by the common Rules of Algebra, the Scale of Relation will be obtained for every one of the Series into which the Series given is to be divided.

PROPOSITION VIII.

If there be given two Series, each having a particular Scale of Relation, and that the corresponding Terms of both Series be added together, so as to compose a third Series, the differential Scale for this third Series will be obtained as follows.

Let

Let $1 - fr + grr$ be the differential Scale of the first, and $1 - mr + prr$, the differential Scale of the second; let those two Scales be multiplied together, and the Product $1 - m + f \times r + p + g + mf \times rr - mg + pf \times r^3 + pg \times r^4$, will express the differential Scale of the Series resulting from the addition of the other two.

And the same Rule will hold, if one Series be subtracted from the other.

PROPOSITION IX.

If there be given two recurring Series, and that the corresponding Terms of those two Series be multiplied together, the differential Scale of the Series resulting from the Multiplication of the other two may be found as follows.

Suppose $1 - fr + grr$ to be the differential Scale of the first, and $1 - ma + paa$ the differential Scale of the second, so that the first Series shall proceed by the powers of r , and the second by the powers of a ; imagine those two differential Scales to be Equations equal to nothing, and both r and a to be indeterminate quantities; make $ar = z$, and now by means of the three Equations, $1 - fr + grr = 0$, $1 - ma + paa = 0$, $ar = z$, let both a and r be expunged, and the Equation resulting from that Operation will be

$$1 - fmz + ffpzz - fgmpz^3 + ggppz^4 = 0$$

$$+ mmgz^2$$

$$- 2gpzz$$

$$\text{or } 1 - fmar + fspa^2r^2 - fgmpa^3r^3 + ggppa^4r^4 = 0$$

$$+ mmga^2r^2$$

$$- 2gpa^2r^2$$

by substituting ar in the room of z ; and the Terms of that Equation, without any regard to their being made $= 0$, which was purely a fiction, will express the differential Scale required: and in the same manner may we proceed in all other more compound Cases.

But it is very observable, that if one of the differential Scales be the Binomial $1 - a$ raised to any Power, it will be sufficient to raise the other differential Scale to that Power, only substituting ar for r , or leaving the Powers of r as they are, if a be restrained to Unity; and that Power of the other differential Scale will constitute the differential Scale required.

Some Uses of the foregoing Propositions.

We have seen in our LVIIIth Problem, that if two Adversaries, whose proportion of Skill be as a to b , play together till such time as either of them wins a certain number of Stakes, such as 4 for instance, the Probability of the Play's not ending in any given number of Games will be determined by

$$\frac{4a^3b + 6aabb + 4ab^3}{a+b} \text{ for 4 Games.}$$

$$\frac{14a^4bb + 20a^3b^2 + 14a^2b^3}{a+b} \text{ for 6 Games.}$$

$$\frac{48a^5b^3 + 68a^4b^4 + 48a^3b^5}{a+b} \text{ for 8 Games.}$$

$$\frac{164a^6b^4 + 232a^5b^5 + 164a^4b^6}{a+b} \text{ for 10 Games.}$$

$$\frac{560a^7b^5 + 732a^6b^6 + 560a^5b^7}{a+b} \text{ for 12 Games.}$$

&c.

Wherein it is evident that each Term in each of the three Columns written above is referred to the two preceding by a constant Scale of Relation, so that if the Terms of the first Column which are

$\frac{4a^3b}{a+b}$, $\frac{14a^4bb}{a+b}$, $\frac{48a^5b^3}{a+b}$, $\frac{164a^6b^4}{a+b}$, $\frac{560a^7b^5}{a+b}$, &c. be respectively called E, F, G, H, K, &c. and that for shortness sake we suppose

$\frac{ab}{a+b} = r$, we shall find $G = 4rF - 2rrE$, $H = 4rG - 2rrF$, and so on; and therefore considering the Sum of every three Terms whereby each Probability is expressed as one single Term, and denoting those Sums respectively by S, T, U, X, &c. we shall find $U = 4rT - 2rrS$, $X = 4rU - 2rrT$, and so on; from which it follows that the Method of determining the Probability of the Play's not ending in any number of Games given, is no more than the finding of a Term in a recurring Series.

Let it therefore be required to find the Probability of 4 Stakes not being lost in 60 Games, to answer this, let it be imagined that the Probabilities of not ending in

0, 2, 4, 6, 8, 10 - - - - - 60 Games,
are expressed by C, D, E, F, G, H, - - - - - K respectively;
then calling l the number of Games given, it is evident that the Term K is distant from the Term C by an Interval $= \frac{1}{2}l$, in this Case $= 30$, the odd numbers being omitted, by reason it is impossible

ble an even number of Stakes should be won or lost exactly in an odd number of Games: moreover it being a certainty that the Set of 4 Stakes to be won or lost can neither be concluded before the Play begins, nor when no more than two Games are played off, it follows that the two Terms C, and D, are each of them equal to Unity; for which reason, if out of the Scale of Relation $4r - 2rr$, or rather out of the differential Scale $1 - 4r + 2rr$, we form the Equation, $xx - 4rx + 2rr = 0$, and that the roots of that Equation be m and p , and then make $A = \frac{1-p}{m-p}$, $B = \frac{1-m}{p-m}$, the

two Terms alone $Am^{\frac{1}{2}l} + Bp^{\frac{1}{2}l}$ will determine the Probability required. This being conformable to Corollary 2^d of our VIth Proposition, it will be proper to consult it.

But because in higher Cases, that is when the number of Stakes to be won or lost is larger, it would sometimes be infinitely laborious to extract the Roots of those Equations, it will be proper to shew how those Roots are actually to be found in a Table of Sines. Of which to give one Instance, let it be proposed to find the Probability of the Play's not ending in any number of Games l , when the number of Stakes to be won or lost is 6; then arguing in the same manner as in the preceding Case, let the Probabilities of the Play's not being concluded in 0, 2, 4, 6, 8, 10 - - - l Games be respectively

D, E, F, G, H, K - - - z ; then

we may conclude that the three Terms D, E, F standing respectively over-against the number of Games 0, 2, 4, are each of them equal to Unity, it being a certainty that the Play cannot be concluded in that number of Games. Wherefore having taken the differential Scale $1 - 6r + 9rr - 2r^3$, which belongs to that number of Stakes 6, and formed out of it the Equation $x^3 - 6rxx + 9rrx - 2r^3 = 0$, let the Roots of that Equation be denoted by m, p, q ; then making

$A = \frac{1-p \times 1-q}{m-p \times m-q}$, $B = \frac{1-q \times 1-m}{p-q \times p-m}$, $C = \frac{1-m \times 1-p}{q-m \times q-p}$, the

Probability required will be $Am^{\frac{1}{2}l} + Bp^{\frac{1}{2}l} + Cq^{\frac{1}{2}l}$.

Now I say that the Roots m, p, q of the Equation above written, may be derived from a Table of Sines; for if the Semi-circumference of a Circle whose Radius is $2r$, be divided into 6 equal parts, and we take the Co-versed Sines of the Arcs that are $\frac{1}{6}$, $\frac{3}{6}$, $\frac{5}{6}$ of the Semi-circumference, so that the Numerators of those Fractions be all the odd numbers contained in 6, those Co-versed Sines will

will be the Values of m , p , q , and the Rule is general and extends to all Cafes; still it is observable that when the number of Stakes is odd, for Instance 9, we ought to take only $\frac{1}{9}$, $\frac{3}{9}$, $\frac{5}{9}$, $\frac{7}{9}$ of the Semi-circumference, and reject the last Term $\frac{9}{9}$ expressing the whole Semi-circumference.

But what ought chiefly to recommend this Method is, that supposing m to be the greatest Co-versed Sine, the first Term alone

$Am^{\frac{1}{2}l}$ will give a sufficient approximation to the Probability required, especially if l be a large number in itself, and it be also large in respect to the number of Stakes.

Still these Rules would not be easily practicable by reason of the great number of Factors which might happen to be both in the Numerator and Denominator to which A is supposed equal, if I had not, from a thorough inspection into the nature of the Equations which determine the Values of m , p , q , &c. deduced the following Theorems.

1^o, If n represents the number of Stakes to be won or lost, whether that number be even or odd, then the Numerator of the Fraction to which A is equal, viz. $1 - p \times 1 - q \times 1 - f \times 1 - t$, &c. will always be equal to the Fraction $\frac{a^n + b^n}{a + b \times 1 - m}$; and in the same manner that the Numerator of the Fraction to which B is equal, viz. $1 - q \times 1 - f \times 1 - t$, &c. will always be equal to the Fraction $\frac{a^n + b^n}{a + b \times 1 - p}$, and so on.

2^o, If n be an even number, and that m' be the right Sine corresponding to the Co-versed Sine m ; then the Denominator of the Fraction to which A is equal, viz. $m - p \times m - q \times m - f \times m - t$,

&c. will always be equal to the Fraction $\frac{nr^{\frac{1}{2}n}}{m'}$; and in the same

manner if p' represent the right Sine belonging to the Co-versed Sine p , then the Denominator of the Fraction to which B is equal, viz. $p - q \times p - f \times p - t$, &c. will always be equal to the Fraction

$\frac{nr^{\frac{1}{2}n}}{p'}$, and so on.

3^o, If n be an odd number, and that m' be, as before, the right Sine corresponding to the Co-versed Sine m ; then the Denominator

nator of the Fraction to which A is equal will be $\frac{nr^{\frac{1}{2}}}{m'\sqrt{m}}$, and the Denominator of the Fraction to which B is equal will be $\frac{nr^{\frac{1}{2}}}{p'\sqrt{p}}$.

COROLLARY

From all which it follows, that the Method of determining the Probability of a certain number n of Stakes not being lost in a given number l of Games, may be thus expressed.

Let L be supposed $= \frac{a^n + b^n}{a+b}$, and $r = \frac{ab}{a+b}$, then that Probability will be

$$\frac{L}{nr^{\frac{1}{2}}} \text{ into } \frac{m'}{1-m} \times m^{\frac{1}{2}l} - \frac{p'}{1-p} \times p^{\frac{1}{2}l} + \frac{q'}{1-q} \times q^{\frac{1}{2}l} - \frac{s'}{1-s} \times s^{\frac{1}{2}l}$$

&c. when n is an even number, or

$$\frac{L}{nr^{\frac{1}{2}}} \text{ into } \frac{m'\sqrt{m}}{1-m} \times m^{\frac{l-1}{2}} - \frac{p'\sqrt{p}}{1-p} \times p^{\frac{l-1}{2}} + \frac{q'\sqrt{q}}{1-q} \times q^{\frac{l-1}{2}} - \frac{s'\sqrt{s}}{1-s} \times s^{\frac{l-1}{2}}$$

&c. when n is an odd number.

But because $m^{\frac{l-1}{2}} \times \sqrt{m}$, $p^{\frac{l-1}{2}} \times \sqrt{p}$, &c. are the same as $m^{\frac{1}{2}l}$, $p^{\frac{1}{2}l}$ respectively, it is plain that both Cases are reduced to one and the same Rule.

It was upon this foundation that I prescribed the Rule to be seen in my LXIXth Problem, wherein I did not distinguish the odd Cases from the even.

But altho' the Rule there given seems somewhat different from what it is here, yet at bottom there is no difference; it consisting barely in this, that whereas $2r$ in this place is the Radius of the Circle to which the Calculation is adapted, there it is *Unity*, and that there the Co-versed Sines were expressed by their Equivalents in right Sines; there was also this little difference, that the Denominators $1-m$, $1-p$, &c. were expressed by means of the versed Sines of those Arcs, to which m and p are co-versed Sines.

Other Variations might be introduced, such for instance as might arise from the consideration of \sqrt{mr} , \sqrt{pr} , &c. being the right Sines

Sines of $\frac{1}{2}$ the Complements to a Quadrant of the Arcs originally taken.

But to shew the farther use of these Series, it will be convenient to propose a Problem or two more relating to that Subject.

P R O B L E M LXX.

M and N, whose proportion of Chances to win one Game are respectively as a to b, resolve to play together till one or the other has lost 4 Stakes: two Standers by, R and S, concern themselves in the Play, R takes the side of M, and S of N, and agree betwixt them, that R shall set to S, the Sum L to the Sum G on the first Game, 2L to 2G on the second, 3L to 3G on the third, 4L to 4G on the fourth, and in case the Play be not then concluded, 5L to 5G on the fifth, and so increasing perpetually in Arithmetic Progression the Sums which they are to set to one another, as long as M and N play; yet with this farther condition, that the Sums, set down by them R and S, shall at the end of each Game be taken up by the Winner, and not left upon the Table to be taken up at once upon the Conclusion of the Play: it is demanded how the Gain of R is to be estimated before the Play begins.

SOLUTION.

Let there be supposed a time wherein the number p of Games has been played; then R having the number a of Chances to win the Sum $p + 1 \times G$ in the next Game; and S having the number b of Chances to win the Sum $p + 1 \times L$, it is plain that the Gain of R in that circumstance ought to be estimated by the quantity $\frac{aG - bL}{a + b}$; but this Gain being to be estimated before the Play begins, it follows that it ought to be estimated by the quantity $p + 1 \times \frac{aG - bL}{a + b}$ multiplied by the respective Probability there

there is that the Play will not then be ended; and therefore the whole Gain of R is the Sum of the Probabilities of the Play's not ending in 0, 1, 2, 3, 4, 5, 6, &c. *Games in infinitum*, multiplied by the respective Values of the quantity $p + 1 \times \frac{aG - bL}{a+b}$, p being interpreted successively by the Terms of the Arithmetic Progression, 0, 1, 2, 3, 4, 5, 6, &c. Now, let these Probabilities of the Play's not ending be respectively represented by A, B, C, D, E, F, G, I , &c. let also the Quantity $\frac{aG - bL}{a+b}$ be called S , and then it will follow that the Gain of R will be expressed by the Series $AS + 2BS + 3CS + 4DS + 5ES + 6FS + 7GS$, &c. but in this Problem, altho' the Probabilities of the Play's not ending decrease continually, yet the number of Stakes being even, the Probability of the Play's not ending in an odd number of Games is not less than the Probability of not ending in the even number that immediately precedes the odd; and therefore $B = A, D = C, F = E, I = G$, &c. from whence it follows that the Gain of R will be expressed by the product of S into $3A + 7C + 11E + 15G + 19I$, &c. but the differential Scale for the Series $A + C + E + G$, &c. is $1 - 4r + 2rr$, wherein r is supposed $= \frac{ab}{a+b\sqrt{2}}$, and the differential Scale for the Series $3 + 7 + 11 + 15 + 19$, &c. is $1 - 3a + 3aa - a^3$, wherein $a = 1$. And therefore the differential Scale for the Series $3A + 7C + 11E$, &c. consisting of the products of the Terms of one Series by the corresponding Terms of the other, will be $1 - 4r + 2rr\sqrt{2}$, or $1 - 8r + 20rr - 16r^3 + 4r^4$; and therefore having written down the four first Terms of the Series to be summed up, viz. as many Terms wanting one as there are in the differential Scale, multiply them in order by the differential Scale according to the prescription given in the Remark belonging to our third Proposition, and the Product will be the Numerator of the Fraction expressing the Sum, of which Fraction the Denominator will be $1 - 4r + 2rr\sqrt{2}$; But to make this the plainer, here follows the Operation,

$$\begin{array}{r}
 3A + 7C + 11E + 15G \\
 1 - 8r + 20rr - 16r^3 \dots\dots \\
 \hline
 3A + 7C + 11E + 15G \\
 - 24rA - 56rC - 88rE \dots\dots \\
 + 60rrA + 140rrC \dots\dots \\
 - 48r^3A
 \end{array}$$

And thus is the Numerator obtained: but $A = 1$, it being a certainty that the Play cannot be ended before it is begun, and C is likewise $= 1$, it being a certainty that 4 Stakes cannot be lost neither before nor at the expiration of 2 Games; but by the law of Relation of the Terms of the Series, $E = 4rC - 2rrA$, and $G = 4rE - 2rrC$, and therefore the proper Substitutions being made, the Sum of the Series will be found to be S into $\frac{10 - 36r + 36rr + 8r^3}{1 - 4r + 2rr^2}$ and now in the room of S and r substituting their respective Values

$$\frac{aG - bL}{a + b} \text{ and } \frac{ab}{(a+b)^2} \text{ the Sum } \frac{aG - bL}{a + b} \text{ into}$$

$$\frac{10 + 2aa^5b + 42a^4bb + 64a^3b^3 + 42aab^4 + 24ab^5 + 10b^6 \times \overline{a+b}^2}{a^4 + b^4}^2$$

will express the Gain of R .

COROLLARY 1.

If the Stake L be greater than the Stake G , in the same proportion as a is greater than b , there can be no advantage on either side.

COROLLARY 2.

If a and b are equal, the Gain of R will be 216 times the half difference between the Stakes G and L : thus if G stands for a Guinea of 21^s . and L for 20^s . the Gain of R will be 216 Sixpences, that is, $5^L. - 8^s$.

COROLLARY 3.

If a be greater than b , the Gain of R , according to that inequality, will vary an infinite number of ways, yet not be greatest when the proportion of a to b is greatest; so that for Instance, if the proportion of a to b is 2 to 1, and G and L are equal, the Gain of R will be about $29 \frac{1}{4}G$; but if a is to b as 3 to 1, the Gain of R will be no more than about $22 \frac{1}{4}G$; and if the proportion of a to b be infinitely great, which would make R win infallibly, the Gain of R will be only $10G$. But altho' this may seem at first a very strange Paradox, yet the reason of it will easily be apprehended from this consideration, that the greater the proportion is of a to b , so much the sooner is the Play likely to be concluded; and therefore if that proportion were infinite, the Play would necessarily be terminated in 4 Games, which would make the Gain of R to be $1 + 2 + 3 + 4 = 10$.

But

But if it was required what must be the proportion of a to b which will afford to R the greatest advantage possible, the answer will be very near 2 to 1, as may be found easily upon Trial; and may be found accurately by the Method which the Geometricians call *de Maximis & Minimis*.

P R O B L E M LXXI.

If M and N, whose number of Chances to win one Game are respectively as a to b , play together till four Stakes are won or lost on either side; and that at the same time, R and S whose number of Chances to win one Game are respectively as c to d , play also together till five Stakes are won or lost on either side; what is the Probability that the Play between M and N will be ended in fewer Games, than the Play between R and S.

SOLUTION.

The Probability of the first Play's being ended in any number of Games before the second, is compounded of the Probability of the first Play's being ended in that number of Games, and of the second's not being ended with the Game immediately preceding: from whence it follows, that the Probability of the first Play's ending in an indeterminate number of Games before the second, is the Sum of all the Probabilities *in infinitum* of the first Play's ending, multiplied by the respective Probabilities of the second's not being ended with the Game immediately preceding.

Let A, B, C, D, E, &c. represent the Probabilities of the first Play's ending in 4, 6, 8, 10, 12, &c. Games respectively; let also F, G, H, K, L, &c. represent the Probabilities of the second's not being ended in 3, 5, 7, 9, 11, &c. Games respectively: hence, by what we have laid down before, the Probability of the first Play's ending before the second will be represented by the infinite Series $AF + BG + CH + DK + EL$, &c. Now to find the Law of Relation in this third Series, we must fix the Law of Relation in the first and second, which will be done by our LXth Problem, it being for the first $4r - 2rr$, wherein r is supposed $= \frac{ab}{(a+b)^2}$; and because, as we have observed before, the Law of Relation in those Series

Series which expresse the Probability of not ending, is the same as the Law of Relation in the respective Series which expresse the Probability of ending; it will also be found by the directions given in our LXth Problem, that if we suppose $\frac{cd}{c+a} = m$, the Law of Relation for the second Series will be $5m - 5mm$, and therefore the Laws of Relation in the first and second Series will respectively be $1 - 4r + 2rr$, $1 - 5m + 5mm$. And now having supposed those two differential Scales as Equations $= 0$, and supposed also $rm = z$, we shall find by the Rules delivered in our IXth Proposition, that the Scale of Relation for the third Series will be $1 - 20z + 110z^2 - 200z^3 + 100z^4$; and therefore having taken the four first Terms of the third Series, and multiplied them by the differential Scale, according to the proper Limitations prescribed in our III^d Proposition, we shall find the Sum of the third Series to be

$$\begin{array}{r}
 AF + BG \quad +CH \quad +DK \\
 -20AFGz - 20BGz \quad -20CHz \\
 +110AFz^2 + 110BGz^2 \\
 -200AFz^3 \\
 \hline
 1 - 20z + 110z^2 - 200z^3 + 100z^4
 \end{array}$$

Now supposing S to represent the Fraction $\frac{a^4 + b^4}{a + b}$, the four Terms A, B, C, D will be found to be $1S + 4rS + 14rrS + 48r^3S$; but the four Terms F, G, H, K wherein S is not concerned will be found to be $1, 5m - 5mm, 20mm - 25m^3, 75m^3 - 100m^4$; and therefore the proper Substitutions being made in the Sum above written, we shall have that Sum reduced to its proper *Data*; and that Sum thus reduced will exhibit the Probability required. But because those *Data* are many, it cannot be expected that the Solution should have so great a degree of Simplicity as if we had restrained a and b to a ratio of Equality, which if we had, the Probability required would have been expressed by the Fraction $\frac{2z - 10zz + z^3}{1 - 20z + 110zz - 200z^3 + 100z^4}$; but because r has been supposed $= \frac{ab}{a+b}$, it follows that r in this Case is $= \frac{1}{4}$: and again, because m has been supposed $= \frac{cd}{c+a}$, then m is also $= \frac{1}{4}$, for which reason rm or $z = \frac{1}{16}$, for which reason substituting $\frac{1}{16}$ instead of z , the Probability required will be expressed by the Fraction

Fraction $\frac{476}{723}$: Now subtracting this Fraction from Unity, the remainder will be the Fraction $\frac{247}{723}$, and therefore the Odds of the first Play's ending before the second will be 476 to 247, or 27 to 14 nearly.

P R O B L E M LXXII.

A and B playing together, and having an equal number of Chances to win one Game, engage to a Spectator S that after an even number of Games n is over, the Winner shall give him as many Pieces as he wins Games over and above one half the number of Games played, it is demanded how the Expectation of S is to be determined.

SOLUTION.

Let E denote the middle Term of the Binomial $a + b$ raised to the Power n , then $\frac{\frac{1}{2}nE}{2^n}$ will express the number of Pieces which the Spectator has a right to expect.

Thus supposing that A and B were to play 6 Games, then raising $a + b$ to the 6th Power, all the following Terms will be found in it, *viz.* $a^6 + 6a^5b + 15a^4bb + 20a^3b^3 + 15aab^4 + 6ab^5 + b^6$.

But because the Chances which A and B have to win one Game have been supposed equal, then a and b may both be made $= 1$, which will make it that the middle Term E will be 20; therefore this number being multiplied by $\frac{1}{2}n$, that is in this Case by 3, the Product will be 60, which being divided by 2^n or 2^6 , that is by 64, the Quotient will be $\frac{60}{64}$ or $\frac{15}{16}$, and therefore the Expectation of S is as good to him as if he had $\frac{15}{16}$ of a Piece given him, and for that Sum he might transfer his Right to another.

It will be easy by Trial to be satisfied of the Truth of this Conclusion, for resuming the 6th Power of $a + b$, and considering the first Term a^6 , which shews the number of Chances for A to win 6 times; in which Case S would have 3 Pieces given him, then the Expecta-

Expectation of S arising from that prospect is $\frac{3a^6}{a+b}^6$, that is $\frac{3}{64}$; considering next the Term $6a^5b$ which denotes the number of Chances for A to win 5 times and losing once, whereby he would get two Games above 3, and consequently S get 2 Pieces, then the Expectation of S arising from that prospect would be $\frac{2 \times 6a^5b}{a+b}^6$ or $\frac{12}{64}$; lastly considering the third Term $15a^4b^2$ which shews the number of Chances for A to get 4 Games out of 6, and consequently for S to get 1 Piece, the Expectation of S arising from that prospect would be $\frac{1 \times 15a^4b^2}{a+b}^6$ or $\frac{15}{64}$, the fourth Term $20a^3b^3$ would afford nothing to S , it denoting the number of Chances for A to win no more than 3 Games; and therefore that part of the Expectation of S , which is founded on the Engagement of A to him, would be $\frac{3+12+15}{64} = \frac{30}{64}$; but he expects as much from B , and therefore his whole Expectation is $\frac{60}{64} = \frac{15}{16}$ as had been before determined.

And in the same manner, if A and B were to play 12 Games the Expectation of S would be $\frac{5544}{4096}$, which indeed is greater than in the preceding Case, but less than in the proportion of the number of Games played, his Expectation in this Case being to the former as 5544 to 3840, which is very little more than in the proportion of 3 to 2, but very far from the proportion of 12 to 6, or 2 to 1.

And if we suppose still a greater number of Games to be played between A and B , the Expectation of S would still increase, but in a less proportion than before; for instance, if A and B were to play 100 Games, the Expectation of S would be 3.9795; if 200, 5.6338; if 300, 6.9041; if 400, 7.9738; if 500, 8.9161; if 700, 800, 900, 10.552, 11.280, 11.965 respectively, so that in 100 Games the Expectation of S would be in respect to that number of Games about $\frac{1}{25}$, and in 900 Games that Expectation would not be above $\frac{1}{75}$. Now how to find the middle Terms of those high Powers will be shewn afterwards.

COROLLARY.

From the foregoing considerations, it follows, that if after taking a great number of Experiments, it should be observed that the happenings

penings or failings of an Event have been very near a ratio of Equality, it may safely be concluded, that the Probabilities of its happening or failing at any one time assigned are very near equal.

PROBLEM LXXIII.

A and B playing together, and having a different number of Chances to win one Game, which number of Chances I suppose to be respectively as a to b, engage themselves to a Spectator S, that after a certain number of Games is over, A shall give him as many Pieces as he wins Games, over and above $\frac{a}{a+b}n$, and B as many as he wins Games, over and above the number $\frac{b}{a+b}n$; to find the Expectation of S.

SOLUTION.

Let E be that Term of the Binomial $a+b$ raised to the Power n , in which the Indices of the Powers of a and b shall be in the same ratio to one another as a is to b ; let also p and q denote respectively those Indices, then will the Expectation of S from A and B together be

$\frac{2pq}{n \times a+b} E$, or $\frac{pq}{n \times a+b} E$ from either of them in particular.

Thus supposing the number of Games n to be 6, and that the ratio of a to b is as 2 to 1; then that Term E of the Binomial $a+b$ raised to its 6th Power, wherein the Indices have the same ratio to one another as 2 to 1, is $15a^4b^2$, and therefore $p=4$, and $q=2$; and because, a, b, p, q, n are respectively 2, 1, 4, 2, 6, thence the Expectation $\frac{2pq}{n \times a+b} \times E$ will be in this particular Case $\frac{16}{4374} \times 240$, or $\frac{640}{729} = \frac{9}{10}$ nearly.

But supposing that A and B resolve to play 12 Games, then that Term of the Binomial $a+b$ raised to its 12th Power, wherein the Indices p and q have the same ratio as 2 to 1, is $495a^8b^4$; and because the Quantities a, b, p, q, n , are respectively 2, 1, 8, 4, 12, the Expectation of S will be $\frac{675840}{531441}$ or $\frac{14}{11}$ nearly.

And again, if A and B play still a greater number of Games, the Expectation of S will perpetually increase, but in a less proportion than of the number of Games played.

COROLLARY.

From this it follows, that if after taking a great number of Experiments, it should be perceived that the happenings and failings have been nearly in a certain proportion, such as of 2 to 1, it may safely be concluded that the Probabilities of happening or failing at any one time assigned will be very near in that proportion, and that the greater the number of Experiments has been, so much nearer the Truth will the conjectures be that are derived from them.

But suppose it should be said, that notwithstanding the reasonableness of building Conjectures upon Observations, still considering the great Power of Chance, Events might at long run fall out in a different proportion from the real Bent which they have to happen one way or the other; and that supposing for Instance that an Event might as easily happen as not happen, whether after three thousand Experiments it may not be possible it should have happened two thousand times and failed a thousand; and that therefore the Odds against so great a variation from Equality should be assigned, whereby the Mind would be the better disposed in the Conclusions derived from the Experiments.

In answer to this, I'll take the liberty to say, that this is the hardest Problem that can be proposed on the Subject of Chance, for which reason I have reserved it for the last, but I hope to be forgiven if my Solution is not fitted to the capacity of all Readers; however I shall derive from it some Conclusions that may be of use to every body: in order thereto, I shall here translate a Paper of mine which was printed *November 12, 1733*, and communicated to some Friends, but never yet made public, reserving to myself the right of enlarging my own Thoughts, as occasion shall require.

Novemb. 12, 1733.

A Method of approximating the Sum of the Terms of the Binomial $a + b^n$ expanded into a Series, from whence are deduced some practical Rules to estimate the Degree of Assent which is to be given to Experiments.

ALTHO' the Solution of Problems of Chance often requires that several Terms of the Binomial $a + b^n$ be added together, nevertheless in very high Powers the thing appears so laborious, and of so great difficulty, that few people have undertaken that Task; for besides *James* and *Nicolas Bernoulli*, two great Mathematicians, I know of no body that has attempted it; in which, tho' they have shewn very great skill, and have the praise which is due to their Industry, yet some things were farther required; for what they have done is not so much an Approximation as the determining very wide limits, within which they demonstrated that the Sum of the Terms was contained. Now the Method which they have followed has been briefly described in my *Miscellanea Analytica*, which the Reader may consult if he pleases, unless they rather chuse, which perhaps would be the best, to consult what they themselves have writ upon that subject: for my part, what made me apply myself to that Inquiry was not out of opinion that I should excel others, in which however I might have been forgiven; but what I did was in compliance to the desire of a very worthy Gentleman, and good Mathematician, who encouraged me to it: I now add some new thoughts to the former; but in order to make their connexion the clearer, it is necessary for me to resume some few things that have been delivered by me a pretty while ago.

I. It is now a dozen years or more since I had found what follows; If the Binomial $1 + 1$ be raised to a very high Power denoted by n , the ratio which the middle Term has to the Sum of all the Terms, that is, to 2^n , may be expressed by the Fraction

$\frac{2A \times n - 1}{n^2 \times \sqrt{n-1}}$, wherein A represents the number of which the Hyperbolic Logarithm is

$\frac{1}{12} - \frac{1}{360} + \frac{1}{1260} - \frac{1}{1680}$, &c. But because

cause the Quantity $\frac{n-1}{n^n}$ or $1 - \frac{1}{n}$ is very nearly given when n is a high Power, which is not difficult to prove, it follows that, in an infinite Power, that Quantity will be absolutely given, and represent the number of which the Hyperbolic Logarithm is -1 ; from whence it follows, that if B denotes the Number of which the Hyperbolic Logarithm is $-1 + \frac{1}{12} - \frac{1}{360} + \frac{1}{1260} - \frac{1}{1680}$, &c. the Expression above-written will become $\frac{2B}{\sqrt{n-1}}$ or barely $\frac{2B}{\sqrt{n}}$: and that therefore if we change the Signs of that Series, and now suppose that B represents the Number of which the Hyperbolic Logarithm is $1 - \frac{1}{12} + \frac{1}{360} - \frac{1}{1260} + \frac{1}{1680}$, &c. that Expression will be changed into $\frac{2}{B\sqrt{n}}$.

When I first began that inquiry, I contented myself to determine at large the Value of B , which was done by the addition of some Terms of the above-written Series; but as I perceived that it converged but slowly, and seeing at the same time that what I had done answered my purpose tolerably well, I desisted from proceeding farther till my worthy and learned Friend Mr. *James Stirling*, who had applied himself after me to that inquiry, found that the Quantity B did denote the Square-root of the Circumference of a Circle whose Radius is Unity, so that if that Circumference be called c , the Ratio of the middle Term to the Sum of all the Terms will be expressed by $\frac{2}{\sqrt{nc}}$.

But altho' it be not necessary to know what relation the number B may have to the Circumference of the Circle, provided its value be attained, either by pursuing the Logarithmic Series before mentioned, or any other way; yet I own with pleasure that this discovery, besides that it has saved trouble, has spread a singular Elegancy on the Solution.

II. I also found that the Logarithm of the Ratio which the middle Term of a high Power has to any Term distant from it by an Interval denoted by l , would be denoted by a very near approximation, (supposing $m = \frac{1}{2}n$) by the Quantities $m + l - \frac{1}{2} \times \text{Log. } m + l - 1 + m - l + \frac{1}{2} \times \text{Log. } m - l + 1 - 2m \times \text{Log. } m + \text{Log. } \frac{m+l}{m}$.

COROLLARY 1.

This being admitted, I conclude, that if m or $\frac{1}{2}n$ be a Quantity infinitely great, then the Logarithm of the Ratio, which a Term distant from the middle by the Interval l , has to the middle Term, is $-\frac{2ll}{n}$.

COROLLARY 2.

The Number, which answers to the Hyperbolic Logarithm $-\frac{2ll}{n}$, being

$$1 - \frac{2ll}{n} + \frac{4l^4}{2nn} - \frac{8l^6}{6n^3} + \frac{16l^8}{24n^4} - \frac{32l^{10}}{120n^5} + \frac{64l^{12}}{720n^6}, \&c.$$

it follows, that the Sum of the Terms intercepted between the Middle, and that whose distance from it is denoted by l , will be

$$\frac{2}{\sqrt{nc}} \text{ into } l - \frac{2l^3}{1 \times 3n} + \frac{4l^5}{2 \times 5nn} - \frac{8l^7}{6 \times 7n^3} + \frac{16l^9}{24 \times 9n^4} - \frac{32l^{11}}{120 \times 11n^5}, \&c.$$

Let now l be supposed $= s\sqrt{n}$, then the said Sum will be expressed by the Series

$$\frac{2}{\sqrt{c}} \text{ into } s - \frac{2s^3}{3} + \frac{4s^5}{2 \times 5} - \frac{8s^7}{6 \times 7} + \frac{16s^9}{24 \times 9} - \frac{32s^{11}}{120 \times 11}, \&c.$$

Moreover, if s be interpreted by $\frac{1}{2}$, then the Series will become

$$\frac{2}{\sqrt{c}} \text{ into } \frac{1}{2} - \frac{1}{3 \times 4} + \frac{1}{2 \times 5 \times 8} - \frac{1}{6 \times 7 \times 10} + \frac{1}{24 \times 9 \times 32} - \frac{1}{120 \times 11 \times 64}, \&c.$$

which converges so fast, that by help of no more than seven or eight Terms, the Sum required may be carried to six or seven places of Decimals: Now that Sum will be found to be 0.427812, independently from the common Multiplicator $\frac{2}{\sqrt{c}}$, and therefore to the Tabular Logarithm of 0.427812, which is 9.6312529, adding the Logarithm of $\frac{2}{\sqrt{c}}$, viz. 9.9019400, the Sum will be 19.5331929, to which answers the number 0.341344.

LEMMA.

If an Event be so dependent on Chance, as that the Probabilities of its happening or failing be equal, and that a certain given number n of Experiments be taken to observe how often it happens and fails, and also that l be another given number, less than $\frac{1}{2}n$, then the Probability of its neither happening more frequently than $\frac{1}{2}n + l$ times,

times, nor more rarely than $\frac{1}{2}n - l$ times, may be found as follows.

Let L and L be two Terms equally distant on both sides of the middle Term of the Binomial $1 + 1^n$ expanded, by an Interval equal to l ; let also f be the Sum of the Terms included between L and L together with the Extrems, then the Probability required will be rightly expressed by the Fraction $\frac{f}{2^n}$; which being founded on the common Principles of the Doctrine of Chances, requires no Demonstration in this place.

COROLLARY 3.

And therefore, if it was possible to take an infinite number of Experiments, the Probability that an Event which has an equal number of Chances to happen or fail, shall neither appear more frequently than $\frac{1}{2}n + \frac{1}{2}\sqrt{n}$ times, nor more rarely than $\frac{1}{2}n - \frac{1}{2}\sqrt{n}$ times, will be expressed by the double Sum of the number exhibited in the second Corollary, that is, by 0.682688, and consequently the Probability of the contrary, which is that of happening more frequently or more rarely than in the proportion above assigned will be 0.317312, those two Probabilities together completing Unity, which is the measure of Certainty: Now the Ratio of those Probabilities is in small Terms 28 to 13 very near.

COROLLARY 4.

But altho' the taking an infinite number of Experiments be not practicable, yet the preceding Conclusions may very well be applied to finite numbers, provided they be great: for Instance, if 3600 Experiments be taken, make $n = 3600$, hence $\frac{1}{2}n$ will be $= 1800$, and $\frac{1}{2}\sqrt{n} = 30$, then the Probability of the Event's neither appearing oftner than 1830 times, nor more rarely than 1770, will be 0.682688.

COROLLARY 5.

And therefore we may lay this down for a fundamental Maxim, that in high Powers, the Ratio, which the Sum of the Terms included between two Extrems distant on both sides from the middle Term by an Interval equal to $\frac{1}{2}\sqrt{n}$, bears to the Sum of all
the

the Terms, will be rightly expressed by the Decimal 0.682688, that is $\frac{28}{41}$ nearly.

Still, it is not to be imagined that there is any necessity that the number n should be immensely great; for supposing it not to reach beyond the 900th Power, nay not even beyond the 100th, the Rule here given will be tolerably accurate, which I have had confirmed by Trials.

But it is worth while to observe, that such a small part as is $\frac{1}{2}\sqrt{n}$ in respect to n , and so much the less in respect to n as n increases, does very soon give the Probability $\frac{28}{41}$ or the Odds of 28 to 13; from whence we may naturally be led to enquire, what are the Bounds within which the proportion of Equality is contained? I answer, that these Bounds will be set at such a distance from the middle Term, as will be expressed by $\frac{1}{4}\sqrt{2n}$ very near; so in the Case above mentioned, wherein n was supposed = 3600, $\frac{1}{4}\sqrt{2n}$ will be about 21.2 nearly, which in respect to 3600, is not above $\frac{1}{169}$ th part: so that it is an equal Chance nearly, or rather something more, that in 3600 Experiments, in each of which an Event may as well happen as fail, the Excess of the happenings or failings above 1800 times will be no more than about 21.

COROLLARY 6.

If l be interpreted by \sqrt{n} , the Series will not converge so fast as it did in the former Case when l was interpreted by $\frac{1}{2}\sqrt{n}$, for here no less than 12 or 13 Terms of the Series will afford a tolerable approximation, and it would still require more Terms, according as l bears a greater proportion to \sqrt{n} : for which reason I make use in this Case of the Artifice of Mechanic Quadratures, first invented by Sir *Isaac Newton*, and since prosecuted by Mr. *Cotes*, Mr. *James Stirling*, myself, and perhaps others; it consists in determining the Area of a Curve nearly, from knowing a certain number of its Ordinates A, B, C, D, E, F, &c. placed at equal Intervals, the more Ordinates there are, the more exact will the Quadrature be; but here I confine myself to four, as being sufficient for my purpose: let us therefore suppose that the four Ordinates are A, B, C, D, and that the Distance between the first and last is denoted by l , then

l , then the Area contained between the first and the last will be $\frac{1 \times A + D + 3 \times B + C}{8} \times l$; now let us take the Distances $0 \sqrt{n}$, $\frac{1}{6} \sqrt{n}$, $\frac{2}{6} \sqrt{n}$, $\frac{3}{6} \sqrt{n}$, $\frac{4}{6} \sqrt{n}$, $\frac{5}{6} \sqrt{n}$, $\frac{6}{6} \sqrt{n}$, of which every one exceeds the preceding by $\frac{1}{6} \sqrt{n}$, and of which the last is \sqrt{n} ; of these let us take the four last, *viz.* $\frac{3}{6} \sqrt{n}$, $\frac{4}{6} \sqrt{n}$, $\frac{5}{6} \sqrt{n}$, $\frac{6}{6} \sqrt{n}$, then taking their Squares, doubling each of them, dividing them all by n , and prefixing to them all the Sign —, we shall have $-\frac{1}{2}$, $-\frac{8}{9}$, $-\frac{25}{18}$, $-\frac{2}{1}$, which must be looked upon as Hyperbolic Logarithms, of which consequently the corresponding numbers, *viz.* 0.60653, 0.41111, 0.24935, 0.13534 will stand for the four Ordinates A, B, C, D. Now having interpreted l by $\frac{1}{2} \sqrt{n}$, the Area will be found to be $= 0.170203 \times \sqrt{n}$, the double of which being multiplied by $\frac{2}{\sqrt{n}}$, the product will be 0.27160; let therefore this be added to the Area found before, that is, to 0.682688, and the Sum 0.95428 will shew what, after a number of Trials denoted by n , the Probability will be of the Event's neither happening oftner than $\frac{1}{2}n + \sqrt{n}$ times, nor more rarely than $\frac{1}{2}n - \sqrt{n}$, and therefore the Probability of the contrary will be 0.04572: which shews that the Odds of the Event's neither happening oftner nor more rarely than within the Limits assigned are 21 to 1 nearly.

And by the same way of reasoning, it will be found that the Probability of the Event's neither appearing oftner than $\frac{1}{2}n + \frac{3}{2} \sqrt{n}$, nor more rarely than $\frac{1}{2}n - \frac{3}{2} \sqrt{n}$ will be 0.99874, which will make it that the Odds in this Case will be 369 to 1 nearly.

To apply this to particular Examples, it will be necessary to estimate the frequency of an Event's happening or failing by the Square-root of the number which denotes how many Experiments have been, or are designed to be taken; and this Square-root, according as it has been already hinted at in the fourth Corollary, will be as it were the *Modulus* by which we are to regulate our Estimation; and therefore suppose the number of Experiments to be taken is 3600, and that it were required to assign the Probability of the Event's neither happening oftner than 2850 times, nor more rarely than 1750, which two numbers may be varied at pleasure, provided they be equally distant from the middle Sum 1800, then make the half difference

difference between the two numbers 1850 and 1750, that is, in this Case, $50 = f\sqrt{n}$; now having supposed $3600 = n$, then \sqrt{n} will be $= 60$, which will make it that 50 will be $= 60f$, and consequently $f = \frac{50}{60} = \frac{5}{6}$; and therefore if we take the proportion, which in an infinite power, the double Sum of the Terms corresponding to the Interval $\frac{5}{6}\sqrt{n}$, bears to the Sum of all the Terms, we shall have the Probability required exceeding near.

LEMMA 2.

In any Power $(a + b)^n$ expanded, the greatest Term is that in which the Indices of the Powers of a and b , have the same proportion to one another as the Quantities themselves a and b ; thus taking the 10th Power of $a + b$, which is $a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$; and supposing that the proportion of a to b is as 3 to 2, then the Term $210a^6b^4$ will be the greatest, by reason that the Indices of the Powers of a and b , which are in that Term, are in the proportion of 3 to 2; but supposing the proportion of a to b had been as 4 to 1, then the Term $45a^8b^2$ had been the greatest.

LEMMA 3.

If an Event so depends on Chance, as that the Probabilities of its happening or failing be in any assigned proportion, such as may be supposed of a to b , and a certain number of Experiments be designed to be taken, in order to observe how often the Event will happen or fail; then the Probability that it shall neither happen more frequently than so many times as are denoted by $\frac{an}{a+b} + l$, nor more rarely than so many times as are denoted by $\frac{an}{a+b} - l$, will be found as follows:

Let L and R be equally distant by the Interval l from the greatest Term; let also S be the Sum of the Terms included between L and R, together with those Extreams, then the Probability required will be rightly expressed by $\frac{S}{(a+b)^n}$.

COROLLARY 8.

The Ratio which, in an infinite Power denoted by n , the greatest Term bears to the Sum of all the rest, will be rightly expressed by

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the Fraction $\frac{a+b}{\sqrt{abnc}}$, wherein c denotes, as before, the Circumference of a Circle for a Radius equal to Unity.

COROLLARY 9.

If, in an infinite Power, any Term be distant from the Greatest by the Interval l , then the Hyperbolic Logarithm of the Ratio which that Term bears to the Greatest will be expressed by the Fraction $-\frac{(a+b)^2}{2abn} \times ll$; provided the Ratio of l to n be not a finite Ratio, but such a one as may be conceived between any given number p and \sqrt{n} , so that l be expressible by $p\sqrt{n}$, in which Case the two Terms L and R will be equal.

COROLLARY 10.

If the Probabilities of happening and failing be in any given Ratio of inequality, the Problems relating to the Sum of the Terms of the Binomial $a + b^n$ will be solved with the same facility as those in which the Probabilities of happening and failing are in a Ratio of Equality.

REMARK I.

From what has been said, it follows, that Chance very little disturbs the Events which in their natural Institution were designed to happen or fail, according to some determinate Law; for if in order to help our conception, we imagine a round piece of Metal, with two polished opposite faces, differing in nothing but their colour, whereof one may be supposed to be white, and the other black; it is plain that we may say, that this piece may with equal facility exhibit a white or black face, and we may even suppose that it was framed with that particular view of shewing sometimes one face, sometimes the other, and that consequently if it be tossed up Chance shall decide the appearance; But we have seen in our LXXII^d Problem, that altho' Chance may produce an inequality of appearance, and still a greater inequality according to the length of time in which it may exert itself, yet the appearances, either one way or the other, will perpetually tend to a proportion of Equality: But besides, we have seen in the present Problem, that in a great number of Experiments, such as 3600, it would be the Odds of above 2 to 1, that one of the Faces, suppose the white, shall not appear more frequently than 1830 times, nor more rarely than 1770, or in other Terms,

that

that it shall not be above or under the perfect Equality by more than $\frac{1}{120}$ part of the whole number of appearances; and by the same Rule, that if the number of Trials had been 14400 instead of 3600, then still it would be above the Odds of 2 to 1, that the appearances either one way or other would not deviate from perfect Equality by more than $\frac{1}{260}$ part of the whole: and in 1000000 Trials it would be the Odds of above 2 to 1, that the deviation from perfect Equality would not be more than by $\frac{1}{2000}$ part of the whole. But the Odds would increase at a prodigious rate, if instead of taking such narrow limits on both sides the Term of Equality, as are represented by $\frac{1}{2} \sqrt{n}$, we double those Limits or triple them; for in the first Case the Odds would become 21 to 1, and in the second 369 to 1, and still be vastly greater if we were to quadruple them, and at last be infinitely great; and yet whether we double, triple or quadruple them, &c. the Extension of those Limits will bear but an inconsiderable proportion to the whole, and none at all, if the whole be infinite; of which the reason will easily be perceived by Mathematicians, who know, that the Square-root of any Power bears so much a less proportion to that Power, as the Index of it is great.

What we have said is also applicable to a Ratio of Inequality, as appears from our 9th Corollary. And thus in all Cases it will be found, that *altho' Chance produces Irregularities, still the Odds will be infinitely great, that in process of Time, those Irregularities will bear no proportion to the recurrency of that Order which naturally results from ORIGINAL DESIGN.*

REMARK II.

As, upon the Supposition of a certain determinate Law according to which any Event is to happen, we demonstrate that the Ratio of Happenings will continually approach to that Law, as the Experiments or Observations are multiplied: so, *conversely*, if from numberless Observations we find the Ratio of the Events to converge to a determinate quantity, as to the Ratio of P to Q; then we conclude that this Ratio expresses the determinate Law according to which the Event is to happen.

For let that Law be expressed not by the Ratio P : Q, but by some other, as R : S; then would the Ratio of the Events converge to this last, not to the former: which contradicts our *Hypothesis*. And the like, or greater, Absurdity follows, if we should suppose the

Event not to happen according to any Law, but in a manner altogether desultory and uncertain; for then the Events would converge to no fixt Ratio at all.

Again, as it is thus demonstrable that there are, in the constitution of things, certain Laws according to which Events happen, it is no less evident from Observation, that those Laws serve to wise, useful and beneficent purposes; to preserve the stedfast Order of the Universe, to propagate the several Species of Beings, and furnish to the sentient Kind such degrees of happiness as are suited to their State.

But such Laws, as well as the original Design and Purpose of their Establishment, must all be *from without*; the *Inertia* of matter, and the nature of all created Beings, rendering it impossible that any thing should modify its own essence, or give to itself, or to any thing else, an original determination or propensity. And hence, if we blind not ourselves with metaphysical dust, we shall be led, by a short and obvious way, to the acknowledgment of the great MAKER and GOVERNOUR of all; *Himself all-wise, all-powerful and good.*

Mr. *Nicolas Bernoulli* *, a very learned and good Man, by not connecting the latter part of our reasoning with the first, was led to discard and even to vilify this Argument from *final Causes*, so much insisted on by our best Writers; particularly in the Instance of the nearly equal numbers of *male* and *female* Births, adduced by that excellent Person the late Dr. *Arbuthnot*, in *Phil. Trans.* N°. 328.

Mr. *Bernoulli* collects from Tables of Observations continued for 82 years, that is from *A. D.* 1629 to 1711, that the number of Births in *London* was, at a *medium*, about 14000 yearly: and likewise, that the number of *Males* to that of *Females*, or the facility of their production, is nearly as 18 to 17. But he thinks it the greatest weakness to draw any Argument from this against the Influence of *Chance* in the production of the two Sexes. For, says he,

“ Let 14000 Dice, each having 35 faces, 18 white and 17 black, be thrown up, and it is great Odds that the numbers of white and black faces shall come as near, or nearer, to each other, as the numbers of Boys and Girls do in the Tables.”

To which the short answer is this: Dr. *Arbuthnot* never said, “ that supposing the facility of the production of a Male to that

* See his two Letters to Mr. *de Monmort*, one dated at *London*, 11 Oct. 1712, the other from *Paris*, 23 Jan. 1713, in the Appendix to the *Analyse des Jeux de hazard*, 2d Edit.

“ of the production of a female to be already *fixt* to nearly the Ratio
“ of equality, or to that of 18 to 17 ; he was *amazed* that the Ratio
“ of the numbers of Males and Females born should, for many years,
“ keep within such narrow bounds :” the only Proposition against
which Mr. *Bernoulli*’s reasoning has any force.

But he might have said, and we do still insist, that “ as, from
“ the Observations, we can, with Mr. *Bernoulli*, infer the facili-
“ ties of production of the two Sexes to be nearly in a Ratio of
“ equality ; so from this Ratio once discovered, and *manifestly serv-*
“ *ing to a wise purpose*, we conclude the Ratio itself, or if you will
“ the *Form of the Die*, to be an Effect of *Intelligence* and *Design*.”
As if we were shewn a number of Dice, each with 18 white and 17
black faces, which is Mr. *Bernoulli*’s supposition, we should not
doubt but that those Dice had been made by some Artist ; and that
their form was not owing to *Chance*, but was adapted to the particu-
lar purpose he had in View.

Thus much was necessary to take off any impression that the
authority of so great a name might make to the prejudice of our ar-
gument. Which, after all, being level to the lowest understanding,
and falling in with the common sense of mankind, needed no formal
Demonstration, but for the scholastic subtleties with which it may be
perplexed ; and for the abuse of certain words and phrases ; which
sometimes are imagined to have a meaning merely because they are
often uttered.

Chance, as we understand it, supposes the *Existence* of things, and
their general known *Properties* : that a number of Dice, for instance,
being thrown, each of them shall settle upon one or other of its
Bases. After which, the *Probability* of an assigned Chance, that is
of some particular disposition of the Dice, becomes as proper a sub-
ject of Investigation as any other quantity or Ratio can be.

But *Chance*, in atheistical writings or discourse, is a sound ut-
terly insignificant : It imports no determination to any *mode of Ex-*
istence ; nor indeed to *Existence* itself, more than to *non-existence* ;
it can neither be defined nor understood : nor can any Proposition
concerning it be either affirmed or denied, excepting this one, “ That
“ it is a mere word.”

The like may be said of some other words in frequent use ; as
fate, *necessity*, *nature*, a *course of nature* in contradistinction to
the *Divine energy* : all which, as used on certain occasions, are mere
sounds : and yet, by artful management, they serve to found spe-
cious conclusions : which however, as soon as the latent fallacy of the
Term is detected, appear to be no less absurd in themselves, than they
commonly are hurtful to society. I shall

I shall only add, That this method of Reasoning may be usefully applied in some other very interesting Enquiries; if not to force the Assent of others by a strict Demonstration, at least to the Satisfaction of the Enquirer himself: and shall conclude this Remark with a passage from the *Ars Conjectandi* of Mr. James Bernoulli, Part IV. Cap. 4. where that acute and judicious Writer thus introduceth his Solution of the Problem for *Assigning the Limits within which, by the repetition of Experiments, the Probability of an Event may approach indefinitely to a Probability given*, “Hoc igitur est illud Problema &c.” This, says he, is the Problem which I am now to impart to the Publick, after having kept it by me for twenty years: new it is, and difficult; but of such excellent use, that it gives a high value and dignity to every other Branch of this Doctrine. Yet there are Writers, of a Class indeed very different from that of James Bernoulli, who insinuate as if the Doctrine of Probabilities could have no place in any serious Enquiry; and that Studies of this kind, trivial and easy as they be, rather disqualify a man for reasoning on every other subject. Let the Reader chuse.

P R O B L E M LXXIV.

To find the Probability of throwing a Chance assigned a given number of times without intermission, in any given number of Trials.

SOLUTION.

Let the Probability of throwing the Chance in any one Trial be represented by $\frac{a}{a+b}$, and the Probability of the contrary by $\frac{b}{a+b}$: Suppose n to represent the number of Trials given, and p the number of times that the Chance is to come up without intermission; then supposing $\frac{b}{a+b} = x$, take the quotient of Unity divided by $1 - x - axx - aax^3 - a^3x^4 - a^4x^5 - \dots - a^{p-1}x^p$, and having taken as many Terms of the Series resulting from that division, as there are Units in $n - p + 1$, multiply the Sum of the whole by $\frac{a^p x^p}{b^p}$, or by $\frac{a^p}{(a+b)^p}$, and that Product will express the Probability required.

EXAMPLE I.

Let it be required to throw the Chance assigned three times together, in 10 trials, when a and b are in a ratio of Equality, otherwise when each of them is equal to Unity; then having divided 1 by $1 - x - xx - x^3$, the Quotient continued to so many Terms as there are Units in $n - p + 1$, that is, in this Case to $10 - 3 + 1 = 8$, will be $1 + x + 2xx + 4x^3 + 7x^4 + 13x^5 + 24x^6 + 44x^7$. Where x being interpreted by $\frac{b}{a+b}$, that is in this Case by $\frac{1}{2}$, the Series will become $1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{7}{16} + \frac{13}{32} + \frac{24}{64} + \frac{44}{128}$, of which the Sum is $\frac{520}{128} = \frac{65}{16}$, and this being multiplied by $\frac{a^p x^p}{b^p}$, that is, in this Case by $\frac{1}{8}$, the Product will be $\frac{65}{128}$, and therefore 'tis something more than an equal Chance, that the Chance assigned will be thrown three times together some time in 10 Trials, the Odds for it being 65 to 63.

N. B. The continuation of the Terms of those Series is very easy; for in the Case of the present Problem, the Coefficient of any Term is the Sum of 3 of the preceding; and in all Cases, 'tis the Sum of so many of the preceding Coefficients as are denoted by the number p .

But if, in the foregoing Example, the ratio of a to b was of inequality, such as, for instance 2 to 1, then according to the prescription given before, divide Unity by $1 - x - 2xx - 4x^3$, and the Quotient will be $1 + x + 3xx + 9x^3 + 19x^4 + 49x^5 + 123x^6 + 297x^7$, in which the quantity x , which has universally been supposed $= \frac{b}{a+b}$, will in this Case be $= \frac{1}{3}$; wherefore in the preceding Series having interpreted x by $\frac{1}{3}$, we shall find the Sum of 8 of its Terms will be $= \frac{5904}{2187} = \frac{74}{27}$, and this being multiplied by $\frac{a^p x^p}{b^p}$ which in this Case is $\frac{8}{27}$, the Product $\frac{592}{729}$ will express the Probability required, so that there are the Odds of 592 to 137, that the Chance assigned will happen three times together in 10 Trials or before; and only the Odds of 41 to 40 that it does not happen three times together in 5.

After having given the general Rule, it is proper to consider of Expedients to make the Calculation more easy; but before we proceed, it is proper to take a new Case of this Problem: Suppose therefore it be required to find the Probability of throwing the Chance assigned 4 times together in 21 Trials. And first let us suppose the Chance assigned to be of Equality, then we should begin to divide Unity by $1 - x - xx - x^3 - x^4$; but if we consider that the Terms $x + xx + x^3 + x^4$ are in geometric Progression, and that the Sum of that Progression is $\frac{x - x^5}{1 - x}$, if we subtract that from 1, the remainder $\frac{1 - 2x + x^5}{1 - x}$ will be equivalent to $1 - x - xx - x^3 - x^4$, and consequently $\frac{1 - x}{1 - 2x + x^5}$ will be equivalent to the Quotient of Unity divided by $1 - x - xx - x^3 - x^4$; and therefore by that expedient, the most complex Case of this Problem will be reduced to the contemplation of a Trinomial; let us therefore begin to take so many Terms of the Series resulting from the Division of Unity by the Trinomial $1 - 2x + x^5$ as there are Units in $n - p + 1$, that is in $21 - 4 + 1$, or 18, and those Terms will be $1 + 2x + 4x^2 + 8x^3 + 16x^4 + 31x^5 + 60x^6 + 116x^7 + 224x^8 + 432x^9 + 833x^{10} + 1606x^{11} + 3096x^{12} + 5968x^{13} + 11494x^{14} + 22155x^{15} + 42704x^{16} + 82312x^{17}$. Now although these Terms may seem at first sight to be acquired by very great labour, yet if we consider what has been explained before concerning the nature of a recurring Series, we shall find that each Coefficient of the Series is generated from the double of the last, subtracting once the Coefficient of that Term which stands 5 places from the last inclusive; so that for instance if we wanted one Term more, considering that the last Coefficient is 82312, and that the Coefficient of that Term which stands five places from the last inclusive is 5968, then the Coefficient required will be twice 82312, wanting once 5968, which will make it 158656, so that the Term following the last will be $158656x^{18}$.

But to make this more conspicuous, if we take the Binomial $2x - x^5$, and raise it successively to the Powers, whose Indices are 0, 1, 2, 3, 4, 5, 6, &c. and add all those powers together, and write against one another all the Terms which have the same power of x , we shall have a very clear view of the quotient of 1 divided by $1 - 2x + x^5$. Now it will stand thus, supposing that a stands for 2.

$$\begin{array}{rcl}
 & 1 & \\
 + ax & & \\
 + a^2xx & & \\
 + a^3x^3 & & \\
 + a^4x^4 & & \\
 + a^5x^5 & - & x^5 \\
 + a^6x^6 & - & 2ax^6 \\
 + a^7x^7 & - & 3aax^7 \\
 + a^8x^8 & - & 4a^2x^8 \\
 + a^9x^9 & - & 5a^3x^9 \\
 + a^{10}x^{10} & - & 6a^4x^{10} + 1x^{10} \\
 + a^{11}x^{11} & - & 7a^5x^{11} + 3ax^{11} \\
 + a^{12}x^{12} & - & 8a^6x^{12} + 6aax^{12} \\
 + a^{13}x^{13} & - & 9a^7x^{13} + 10a^3x^{13} \\
 + a^{14}x^{14} & - & 10a^8x^{14} + 15a^4x^{14} \\
 + a^{15}x^{15} & - & 11a^{10}x^{15} + 21a^5x^{15} - 1x^{15} \\
 + a^{16}x^{16} & - & 12a^{11}x^{16} + 28a^6x^{16} - 4ax^{16} \\
 + a^{17}x^{17} & - & 13a^{12}x^{17} + 36a^7x^{17} - 10aax^{17}
 \end{array}$$

When the Terms have been disposed in that manner, it will be easy to sum them up by the help of a Theorem which may be seen pag. 224. Now a being $= 2$, and $x = \frac{1}{2}$, every one of the Terms of the first Column will be equal to 1, and therefore the Sum of the first Column is so many Units as there are Terms, which Sum consequently will be 18; but the Terms of the second Column being reduced to their proper Value, will constitute the Series

$$\frac{1}{32} + \frac{2}{32} + \frac{3}{32} + \frac{4}{32} + \frac{5}{32} + \frac{6}{32} + \frac{7}{32} + \frac{8}{32} + \frac{9}{32} + \frac{10}{32} + \frac{11}{32} + \frac{12}{32} + \frac{13}{32} \text{ of which the Sum will be } \frac{91}{32};$$

the Terms of the third Column will constitute the Series $\frac{1}{1024} + \frac{3}{1024} + \frac{6}{1024} + \frac{10}{1024} + \frac{15}{1024} + \frac{21}{1024} + \frac{28}{1024} + \frac{36}{1024}$ of which the Sum is $\frac{120}{1024}$; the Terms of the fourth Column

added together are $\frac{15}{32768}$, and therefore the Sum of all Terms may be expressed by $18 - \frac{91}{32} + \frac{120}{1024} - \frac{15}{32768} = \frac{500465}{32768}$.
L But

But this Sum ought to be multiplied by $1 - x$, that is, by $1 - \frac{1}{2} = \frac{1}{2}$, which will make the Product to be $\frac{500465}{65530}$.

Nevertheless, this Multiplication by $1 - x$, takes off too much from the true Sum, by one half of the lowest Term of each Column, therefore that half must be added to the foregoing Sum; now all the lowest Terms of each Column put together will be $1 - \frac{13}{32} + \frac{36}{1024} - \frac{10}{32768} = \frac{20598}{32768}$, of which the half $\frac{10299}{32768}$ ought to be added to the Sum $\frac{500465}{65536}$, which will make the true Sum to be $\frac{521063}{65536}$; but this is farther to be multiplied by $\frac{a^p x^p}{i^p}$, which by reason that a and b are in a ratio of equality will be reduced to $x^p = \frac{1}{16}$; and therefore the Sum $\frac{521063}{65536}$ ought to be divided by 16, which will make it to be $\frac{521063}{1048576}$: and this last Fraction will denote the Probability of producing the Chance assigned 4 times successively some time in 21 Trials, the Odds against it being 527513 to 521063, which is about 82 to 81.

But what is remarkable in this Problem is this, that the oftner the Chance assigned is to be produced successively, the fewer Columns will be necessary to be used to have a sufficient Approximation, and in all high Cases, it will be sufficient to use only the first and second, or three at most, whereof the first is a geometric Progression, of which a very great number of Terms will be as easily summed up, as a very small number; and the second Column by what we have said concerning the nature of a recurring Series, as easily as the first, and in short all the Columns.

But now 'tis time to consider the Case wherein a to b has a ratio of inequality; we had said before that in this Case we ought to divide Unity by $1 - x - axx - aax^3 - a^3x^4 - \dots - a^{p-1}x^p$, but all the Terms after the first which is 1, constitute a geometric Progression, of which the first is x , and the last $a^{p-1}x^p$, and therefore the Sum of that Progression is $\frac{x - a^p x^{p+1}}{1 - ax}$, and this being subtracted from Unity, the remainder will be $1 - ax + a^p x^{p+1}$, and

$$\frac{x - a^p x^{p+1}}{1 - ax}$$

therefore Unity being divided by the Series above-written will be $\frac{1}{1 - ax}$

$\frac{1 - ax}{1 - ax + a^p x^{p+1}}$, and if $a + 1$ be supposed $= m$, this Fraction
 $= x$

will be $\frac{1 - ax}{1 - mx - a^p x^{p+1}}$, and therefore if we raise successively $mx - a^p x^{p+1}$ to the several Powers denoted by 0, 1, 2, 3, 4, 5, 6, &c. and rank all those Powers in several Columns, and write against one another all the Terms that have the same power of x , we shall be able to sum up every Column extended to the number of Terms denoted by $n - p + 1$, which being done, the whole must be multiplied by $1 - ax$, and to the Sum is to be added the Sum of the lowest Term of each Column multiplied by ax .

But if it be required to assign what number of Games are necessary, in all Cases, to make it an equal Chance whether or not p Games will be won without intermission, it may be done by approximation, thus; let $\frac{a + b^p - a^p}{a^p}$ be supposed $= q$, and let

$\frac{a + b}{b} - \frac{a^p}{b \times a + b^{p-1}}$ be supposed $= r$, then the number

of Games required will be expressed by $\frac{7}{10}qr$; thus supposing $a = 1$, $b = 1$, $p = 6$, then the number of Games would be found between 86 and 87; but if a be supposed $= 1$, and $b = 2$, still supposing $p = 6$, the number of Games requisite to that effect would be found to be between 763 and 764; but it is to be observed, that the greater the number p is, so much the more exact will the Solution prove.

The first part of the paper is devoted to a discussion of the general principles of the theory of the function $f(x)$ and its properties. It is shown that the function $f(x)$ is continuous and differentiable at every point of its domain.

It is also shown that the function $f(x)$ is bounded on every closed interval of its domain. This is proved by the Weierstrass theorem. The function $f(x)$ is also shown to be uniformly continuous on every closed interval of its domain. This is proved by the Heine theorem. The function $f(x)$ is also shown to be differentiable at every point of its domain. This is proved by the definition of the derivative.

The second part of the paper is devoted to a discussion of the properties of the function $f(x)$ and its derivatives. It is shown that the function $f(x)$ is continuous and differentiable at every point of its domain. This is proved by the definition of the derivative. The function $f(x)$ is also shown to be bounded on every closed interval of its domain. This is proved by the Weierstrass theorem.

The third part of the paper is devoted to a discussion of the properties of the function $f(x)$ and its derivatives. It is shown that the function $f(x)$ is continuous and differentiable at every point of its domain. This is proved by the definition of the derivative. The function $f(x)$ is also shown to be bounded on every closed interval of its domain. This is proved by the Weierstrass theorem.

A
T R E A T I S E
O F
A N N U I T I E S
O N
L I V E S:

Dedicated to

The RIGHT HONOURABLE
GEORGE Earl of *MACCLESFIELD*
PRESIDENT of the ROYAL SOCIETY.

P R E F A C E
T O T H E
Second E D I T I O N.

DR. Halley published in the Philosophical Transactions, No. 196. an Essay concerning the Valuation of Lives; it was partly built upon five Years Observation of the Bills of Mortality taken at Breslaw, the Capital of Silesia, and partly on his own Calculation.

Altho

Altho' he had thereby confirmed the great Opinion which the World entertained of his Skill and Sagacity, yet he was sensible, that his Tables and Calculations were susceptible of farther Improvements; of this he expressed his Sense in the following Words; Were this Calculus founded on the Experience of a very great Number of Years, it would very well be worth the while, to think of Methods to facilitate the Computation of two, three or more Lives.

From whence it appears, that the Table of Observations being only the Result of a few Years Experience, it was not so entirely to be depended upon, as to make it the Foundation of a fixed and unalterable Valuation of Annuities on Lives; and that even admitting such a Table could be obtained, as might be grounded on the Experience of a great Number of Years, still the Method of applying it to the Valuation of several Lives, would be extremely laborious, considering the vast Number of Operations, that would be requisite to combine every Year of each Life with every Year of all the other Lives.

The Subject of Annuities on Lives, had been long neglected by me, partly prevented by other Studies, partly wanting the necessary means to treat of it as it deserved: But two or three Years after the Publication of the first Edition of my Doctrine of Chances, I took the Subject into Consideration; and consulting Dr. Halley's Table of Observations, I found that the Decrements of Life, for considerable Intervals of Time, were in Arithmetic Progression; for Instance, out of 646 Persons of twelve Years of Age, there remain 640 after one Year; 634 after two Years; 628, 622, 616, 610, 604, 598, 592, 586, after 3, 4, 5, 6, 7, 8, 9, 10 Years respectively, the common Difference of those Numbers being 6.

Examining afterwards other Cases, I found that the Decrements of Life for several Years were still in Arithmetic Progression; which may be observed from the Age of 54, to the Age of 71, where the Difference for 17 Years together, is constantly 10.

After having thoroughly examined the Tables of Observation, and discovered that Property of the Decrements of Life, I was inclined to compose a Table of the Values of Annuities on Lives, by keeping close to the Tables of Observation; which would have been done with Ease, by taking in the whole Extent of Life, several Intervals whether equal or unequal: However, before I undertook the Task, I tried what
would

would be the Result, of supposing those Decrements uniform from the Age of Twelve; being satisfied that the Excesses arising on one side, would be nearly compensated by the Defects on the other; then comparing my Calculation with that of Dr. Halley, I found the Conclusion so very little different, that I thought it superfluous to join together several different Rules, in order to compose a single one: I need not take notice, that from the Time of Birth to the Age of Twelve, the Probabilities of Life increase, rather than decrease, which is a Reason of the apparent Irregularity of the Tables in the beginning.

Another thing was necessary to my Calculation, which was, to suppose the Extent of Life confined to a certain Period of Time, which I suppose to be at 86: What induced me to assume that Supposition was 1st, That Dr. Halley terminates his Tables of Observations at the 84th Year; for altho' out of 1000 Children of one Year of Age, there are twenty, who, according to Dr. Halley's Tables, attain to the Age of 84 Years, this Number of 20 is inconsiderable, and would still have been reduced, if the Observations had been carried two Years farther. 2^o. It appears from the Tables of Graunt, who printed the first Edition of his Book above 80 Years ago, that out of 100 new-born Children, there remained not one after 86 Years; this was deduced from the Observations of several Years, both in the City and the Country, at a Time when the City being less populous, there was a greater Facility of coming at the Truth, than at present. 3^o. I was farther confirmed in my Hypothesis, by Tables of Observation made in Switzerland, about the Beginning of this Century, wherein the Limit of Life is placed at 86: As for what is alledged, that by some Observations of late Years, it appears, that Life is carried to 90, 95, and even to 100 Years; I am no more moved by it, than by the Examples of Parr or Jenkins, the first of whom lived 152 Years, and the other 167. To this may be added, that the Age for purchasing Annuities for Life, seldom exceeds 70, at which Term, Dr. Halley ends his Tables of the Valuation of Lives.

The greatest Difficulty that occurred to me in this Speculation, was to invent practical Rules that might easily be applied to the Valuation of several Lives; which, however, was happily overcome, the Rules being so easy, that by the Help of them, more can be performed in a Quarter of an Hour, than by any Method before extant, in a Quarter of a Year.

Since

Since the Publication of my first Edition, which was in 1724, I made some Improvements to it, as may be seen in the second Edition of my Doctrine of Chances; but this Edition of the Annuities has many Advantages over the former, and that in respect to the Disposition of the Precepts, the Conciseness of the Rules, the Multiplicity of Problems, and Usefulness of the Tables I have invented.

Before I make an End of this Preface, I think it proper to observe, that altho' I have given Rules for finding the Value of Annuities for any Rate of Interest, yet I have confined myself in my Tables, to the several Rates of 4, 5 and 6 per Cent. which may be interpreted, as if I thought it reasonable, that when Land scarce produces three and a half per Cent. and South-Sea Annuities barely that Interest, yet the Purchaser of an Annuity should make 4 per Cent. or above; but those Cases can hardly admit of Comparison, it being well known, that Land in Fee-simple procures to the Proprietor Credit, Honour, Reputation, and other Advantages, in consideration of which, he is contented with a smaller Income. As to the Value of South-Sea Annuities, it has its Foundation on the Punctuality of Payments, and on a Parliamentary Security; but Annuities on Lives, have not the former Security, and seldom the latter.

It was found necessary, however, in a subsequent Edition, to add the Tables of 3 and $3\frac{1}{2}$ per Cent. Interest.

OF ANNUITIES on LIVES.

Part I. containing the Rules and Examples.

BEFORE I come to the Solution of Questions on Lives, it will be necessary to explain the Meaning of some Words which I shall often have occasion to mention.

1°. Supposing the *Probabilities of Life* to decrease in Arithmetic Progression in such manner, as that supposing, for Instance, 36 Persons each of the Age of 50, if after one Year expired there remain but 35, after two 34, after three 33, and so on; it is very plain that such Lives would necessarily be extinct in 36 Years, and that therefore the *Probabilities of living* 1, 2, 3, 4, 5, &c. Years from this Age of 50 would fitly be represented by the Fractions $\frac{35}{36}$, $\frac{34}{36}$, $\frac{33}{36}$, $\frac{32}{36}$, $\frac{31}{36}$, &c. which decrease in Arithmetic Progression.

I will not say that the Decrements of Life are precisely in that Proportion; still comparing that Hypothesis with the Table of Dr. Halley, from the Observations made at *Breslaw*, they will be found to be exceedingly approaching.

2°. I call that the *Complement of Life*, which remains from the Age given, to the Time of the Extinction of Life, which will be at 86, according to our Hypothesis. Thus supposing an Age of 50, because the Difference between 50 and 86 is 36, I call 36 the *Complement of Life*.

3°. I call that the *Rate of Interest* which is properly the Amount of one Pound, put out at Interest for one Year; otherwise one Pound joined with the Interest it produces in one Year: thus supposing Interest at 5 per Cent the Interest of 1*l.* would be 0.05, which being joined to the Principal 1, produces 1.05; which is what I call the *Rate of Interest*.

PROBLEM I.

Supposing the Probabilities of Life to decrease in Arithmetic Progression, to find the Value of an Annuity upon a Life of an Age given.

SOLUTION.

Let the Rent or Annuity be supposed = i , the Rate of Interest = r , the Complement of Life = n , the Value of an Annuity certain to continue

M m during

during n Years $= P$, then will the Value of the Life be $\frac{1 - \frac{r}{n}P}{r-1}$, which is thus expressed in Words at length ;

Take the Value of an Annuity certain for so many Years, as are denoted by the Complement of Life ; multiply this Value by the Rate of Interest, and divide the Product by the Complement of Life, then let the Quotient be subtracted from Unity, and let the Remainder be divided by the Interest of 1 l. then this last Quotient will express the Value of an Annuity for the Age given.

Thus suppose it were required to find the present Value of an Annuity of 1 l. for an Age of 50, Interest being at 5 per Cent.

The Complement of Life being 36, let the Value of an Annuity certain, according to the given Rate of Interest, be taken out of the Tables annexed to this Book, this Value will be found to be 16.5468.

Let this Value be multiplied by the Rate of Interest 1.05, the Product will be 17.3741.

Let this Product be divided by the Complement of Life, viz. by 36, the Quotient will be 0.4826.

Subtract this Quotient from Unity, the Remainder will be 0.5174.

Lastly, divide this Quotient by the Interest of 1 l. viz. by 0.05, and the new Quotient will be 10.35 ; which will express the Value of an Annuity of 1 l. or how many Years Purchase the said Life of 50 is worth.

And in the same manner, if Interest of Money was at 6 per Cent. an Annuity upon an Age of 50, would be found worth 9.49 Years Purchase.

But as I have annexed to this Treatise the Values of Annuities for an Interest of 3, $3\frac{1}{2}$, 4, 5, and 6 per Cent. it will not be necessary to calculate those Cases, but such only as require a Rate of Interest higher or lower, or intermediate ; which will seldom happen, but in case it does, the Rule may easily be applied.

P R O B L E M II.

The Values of two single Lives being given, to find the Value of an Annuity granted for the Time of their joint continuance.

SOLV-

SOLUTION.

Let M be the Value of one Life, P the Value of the other, r the Rate of Interest; then the Value of an Annuity upon the two joint Lives will be $\frac{MP}{M+P-r-1MP}$, in Words thus;

Multiply together the Values of the two Lives, and reserve the Product.

Let that Product be again multiplied by the Interest of 1 l. and let that new Product be subtracted from the Sum of the Values of the Lives, and reserve the Remainder.

Divide the first Quantity reserved by the second, and the Quotient will express the Value of the two joint Lives.

Thus, supposing one Life of 40 Years of Age, the other of 50, and Interest at 5 per Cent. The Value of the first Life will be found in the Tables to be 11.83, the Value of the second 10.35, the Product will be 122.4405, which Product must be reserved.

Multiply this again by the Interest of 1 l. viz. by 0.05, and this new Product will be 6.122025.

This new Product being subtracted from the Sum of the Lives which is 22.18, the Remainder will be 16.057975, and this is the second Quantity reserved.

Now dividing the first Quantity reserved by the second, the Quotient will be 7.62 nearly; and this expresses the Values of the two joint Lives.

If the Lives are equal, the Canon for the Value of the joint Lives will be shortened and be reduced to $\frac{M}{2-r-1 \times M}$, which in words may be thus expressed;

Take the Value of one Life, and reserve that Value.

Multiply this Value by the Interest of 1 l. and then subtract the Product from the Number 2, and reserve the Remainder.

Divide the first Quantity reserved by the second, and the Quotient will express the Value of the two equal joint Lives.

Thus, supposing each Life to be 45 Years of Age, and Interest at 5 per Cent.

The Value of one Life will be found to be 11.14, the first Quantity reserved.

This being multiplied by 0.05 the Interest of 1 l. the Product will be 0.557.

This Product being subtracted from the Number 2, the Remainder will be 1.443, the second Quantity reserved.

M m 2

Divide

Divide the first Quantity reserved *viz.* 11.14; by the second, *viz.* 1.443, and the Quotient 7.72 will be the Value of the two joint Lives, each of 45 Years of Age.

P R O B L E M III.

The Values of three single Lives being given, to find the Value of an Annuity for the Time of their joint continuance.

SOLUTION.

Let M, P, Q , be the respective Values of the single Lives, then the Value of the three joint Lives will be $\frac{MPQ}{MP+MQ+PQ-2dMPQ}$, supposing d to represent the Interest of 1 *l.* in words thus;

Multiply the Values of the single Lives together, and reserve the Product.

*Let that Product be multiplied again by the Interest of 1 *l.* and let the Double of that new Product be subtracted from the Sum of the several Products of the Lives taken two and two, and reserve the Remainder.*

Divide the first Quantity reserved by the second, and the Quotient will be the Value of the three joint Lives.

Thus, supposing one Life to be worth 13 Years Purchase, the second 14, the third 15, and Interest at 4 *per Cent.* the Product of the three Lives will be 2730, which being multiplied by the Interest of 1 *l.* *viz.* by 0.04, the new Product will be 109.20, whereof the double is 218.40: Now the Product of the first Life by the second is 182; the Product of the first Life by the third is 195; and the Product of the second Life by the third is 210, the Sum of all which is 587; from which subtracting the Number 218.40 found above, the Remainder will be 368.60, by which the Product of the three Lives, *viz.* 2730 being divided, the Quotient 7.41 will be the Value of the three joint Lives.

But if the three Lives were equal, the general Expression of the Value of the joint Lives will be much shorter: for let M represent the Value of one Life, d the Interest of 1 *l.* then the Value of the three joint Lives will be $\frac{M}{3-2dM}$, in Words thus;

Take

Take the Value of one Life, and reserve it, multiply this Value by the Interest of 1 l. and double the Product.

Subtract this double Product from the number 3, and reserve the Remainder.

Divide the first Quantity reserved by the second, and the Quotient will be the Value of the three joint Lives.

Thus, supposing three equal Lives each worth 14 Years Purchase, reserve the Number 14.

Multiply this by 0.04, Interest of 1 l. the Product will be 0.56, which being doubled, will be 1.12.

This being subtracted from the Number 3, the Remainder will be 1.88, which is the second Quantity to be reserved.

Divide 14, the first Quantity reserved by the second 1.88, and the Quotient 7.44 will be the Value of the three joint Lives.

From the two last Examples it appears, that in estimating the Values of joint Lives, it would be an Error to suppose that they might be reduced to an Equality, by taking a Mean Life betwixt the longest and shortest, for altho' 14 is a Medium betwixt 13 and 15, yet an Annuity upon those three joint Lives was found to be 7.41, whereas supposing them to be each 14 Years Purchase, the Value is 7.44; it is true that the Difference is so small, that it might be neglected, yet this arises meerly from a near Equality in the Lives; for if there had been a greater Inequality, the Conclusion would have considerably varied.

Before I come to the fourth Problem, I think it proper to explain the Meaning of some Notations which I make use of, in order to be as clear and concise as I can.

I denote the Value of an Annuity upon two joint Lives, whose single Values are M and P by \overline{MP} , which ought carefully to be distinguished from the Notation MP ; this last denoting barely the Product of one Value multiplied by the other, whereas \overline{MP} stands for what was denoted in our second Problem by $\frac{MP}{M+P-r-rMP}$.

In the same manner, the Value of an Annuity upon the three joint Lives whose single Values are M , P , Q , is denoted by \overline{MPQ} , which is equivalent to what has been expressed in the third Problem by

$$\frac{MPQ}{MP+MQ+PQ-2dMPQ}.$$

This being premised, I proceed to the fourth Problem.

P R O B L E M IV.

The Values of two single Lives being given, to find the Value of an Annuity upon the longest of them, that is, to continue so long as either of them is in being.

SOLUTION.

Let M be the Value of one Life, P the Value of the other, \overline{MP} the Value of the two joint Lives, then the Value of the longest of the two Lives will be $M + P - \overline{MP}$. In Words thus ;

From the Sum of the Values of the single Lives, subtract the Value of the joint Lives, and the Remainder will be the Value of the longest.

Let us suppose two Lives, one worth 13 Years Purchase, the other 14, and Interest at 4 per Cent. The Sum of the Values of the Lives is 27, the Value of the two joint Lives by the Rules before given, will be found 9.23. Now, subtracting 9.23 from 27, the Remainder 17.77 is the Value of the longest of the two Lives.

If the two Lives are equal, the Operation will be something shorter.

But it is proper to observe in this place, that if several equal Lives are concerned in an Annuity, I commonly denote one single Life by M' , two joint Lives by M'' , three joint Lives by M''' , and so on ; so that the Rule for an Annuity to be granted till such Time as either of the equal Lives is in being may be expressed by $2 M' - M''$.

P R O B L E M V.

The Values of three single Lives being given, to find the Value of an Annuity upon the longest of them.

SOLUTION.

Let M, P, Q , be the Values of the single Lives, $\overline{MP}, \overline{MQ}, \overline{PQ}$, the Values of all the joint Lives combined two and two, \overline{MPQ} the Value of three joint Lives, then the Value of an Annuity upon the longest of them is $M + P + Q - \overline{MP} - \overline{MQ} - \overline{PQ} + \overline{MPQ}$, in Words thus ;

Take the Sum of the three single Lives, from which Sum subtract the Sum of all the joint Lives combined two and two, then to the Remainder add the Value of the three joint Lives, and the Result will be the Value of the longest of the three Lives.

Thus,

Thus, Supposing the single Lives to be 13, 14, and 15 Years Purchase, the Sum of the Values will be 42; the Values of the first and second joint Lives is 9.24, of the first and third 9.05, of the second and third 10.18, the Sum of all which is 29.06 which being subtracted from the Sum of the Lives found before, *viz.* 42, the Remainder will be 12.94, to which adding the Value of the three joint Lives 7.41, the Sum 20.35 will be the Value of the longest of the three joint Lives.

But if the three Lives are equal, the Rule for the Value of the Life that remains last is $3M' - 3M'' + M'''$.

OF REVERSIONS.

PROBLEM VI.

Suppose A is in Possession of an Annuity, and that B after the Decease of A is to have the Annuity for him, and his Heirs for ever, to find the present Value of the Reversion.

SOLUTION.

Let M be the Value of the Life in Possession, r the Rate of Interest, then the present Value will be $\frac{1}{r-1} - M$, that is, from the Value of the Perpetuity, subtract the Value of the Life in Possession, and the Remainder will be the Value of the Reversion.

Thus, Supposing that A is 50 Years of Age, an Annuity upon his Life, Interest at 5 per Cent. would be 8.39, which being subtracted from the Perpetuity 20, the Remainder will be 11.61, which is the present Value of the Expectation of B .

In the same manner, supposing that C were to have an Annuity for him and his Heirs for ever, after the Lives of A and B , then from the Perpetuity subtracting the Value of the longest of the two Lives of A and B , the Remainder will express the Value of C 's Expectation.

Thus, Supposing the Ages of A and B be 40 and 50, the Value of an Annuity upon the longest of these two Lives would be found by the 4th Problem to be 14.56; and this being subtracted from the Perpetuity 20, the Remainder is 5.44, which is the Value of C 's Expectation, and the Rule will be the same in any other Case that may be proposed.

P R O B L E M VII.

Supposing that A is in Possession of an Annuity for his Life, and that B after the Life of A, should have an Annuity for his Life only; to find the Value of the Life of B after the Life of A.

This Case ought carefully to be distinguished from the Case of the 6th Problem; for in that Problem, altho' the Expectant *B* should die before *A*, still the Heirs of *B* have the Reversion; but in the Case of the present Problem, if *B* dies before *A*, the Heirs of *B* have no Expectation.

SOLUTION.

Let *M* be the Value of the Life of the present Possessor, *P* the Value of the Life of the Expectant, then the Value of his Expectation is $P - \overline{MP}$. In Words thus;

From the present Value of the Life of B, subtract the present Value of the joint Lives of B and A, and the Remainder will be the Value of B's Expectation.

The Reason of which Operation is very plain, for if *B* were now to begin to receive the Annuity, it would be worth to him the Sum *P* in present Value; but as he is to receive nothing during the joint Lives of himself and *A*, the present Value of their two joint Lives ought to be subtracted from the Value of his own Life.

P R O B L E M VIII.

To find the Value of one Life after two.

Thus, Suppose A in Possession of an Annuity for his Life, that B is to have his Life in it after A, and that C is likewise to have his Life in it after B, but so that B dying before A, C succeeds A immediately; to find the Value of C's Expectation.

SOLUTION.

Let *M*, *P*, *Q*, be the respective Values of the Lives of *A*, *B*, *C*,
then

then the Value of C's Expectation is $\frac{Q - \frac{MQ}{P} + \frac{MPQ}{PQ}}{-PQ}$, which in

Words is thus expressed;

From the present Value of the Life of C, subtract the Sum of the joint Lives of himself and A, and of himself and B, and to the Remainder add the Sum of the three joint Lives, and the Result of these Operations will express the present Value of the Expectation of C.

P R O B L E M IX.

If A, B, C agree among themselves to buy an Annuity to be by them equally divided, whilst they live together, then after the Decease of one of them, to be equally divided between the two Survivors, then to belong entirely to the last Survivor for his Life; to find what each of them ought to contribute towards the Purchase.

SOLUTION.

Let M, P, Q , be the respective Values of the Lives of A, B, C , then what A is to contribute, is

$$M - \frac{1}{2} \frac{MP}{P} + \frac{1}{3} \frac{MPQ}{PQ} - \frac{1}{2} \frac{MQ}{P}$$

What B is to contribute, is

$$P - \frac{1}{2} \frac{PM}{M} + \frac{1}{3} \frac{MPQ}{PQ} - \frac{1}{2} \frac{PQ}{M}$$

What C is to contribute, is

$$Q - \frac{1}{2} \frac{QM}{M} + \frac{1}{3} \frac{MPQ}{PQ} - \frac{1}{2} \frac{QP}{M}$$

In Words thus;

From the Value of the Life of A, subtract the half Sum of the Values of the joint Lives of himself and B, and of himself and C, and to the Remainder add $\frac{1}{3}$ of the Value of the three joint Lives, and the Sum will be what A is to contribute towards the Purchase.

In like manner, from the Value of B's Life subtract the half Sum of the Values of the joint Lives of himself and A, and of himself and C, and to the Remainder add $\frac{1}{3}$ of the Value of the three joint Lives, and the Sum will be what B is to contribute.

And again, from the Value of the Life of *C*, subtract the half Sum of the Values of himself and *A*, and of himself and *B*; then to the Remainder add $\frac{1}{3}$ of the Values of the three joint Lives, and this last Operation will shew what *C* is to contribute.

PROBLEM X.

Supposing three equal Lives of any Age given, for Instance 30, and that upon the Failing of any one of them, that Life shall be immediately replaced, and I then receive a Sum I agreed upon, and that to Perpetuity for me and my Heirs; what is the present Value of that Expectation, and at what Intervals of Time, one with another, may I expect to receive the said Sum?

SOLUTION.

Imagine that there is an Annuity of 1 *l.* to be received as long as the three Lives are in being, and that the Present Value is M''' , which Symbol we make use of to represent the Present Value of an Annuity upon three equal joint Lives; now, since each Life is supposed to be 30 Years of Age, and that the Rate of Interest is 5 *per Cent.* we shall find, by following the Directions given in Prob. III. that the Present Value of the three joint Lives is $7.64 = M'''$; this being fixed, the Present Value of all the Payments to be made to Eternity at equal Intervals of Time, will be $\frac{1-dM'''}{dM'''} \times s$, where the Quantity d signifies the Interest of 1 *l.* In words thus;

*Multiply the Present Value of the three joint Lives, viz. 7.64, by the Interest of 1 *l.* which in this Case is 0.05, and that Product, which is 0.382, must be reserved.*

*Subtract this Quantity from Unity, and the Remainder, viz. 0.618 being divided by the Quantity reserved, the Quotient will be 1.62, and this being multiplied by the Sum *s*, which we may suppose 100 *l.* the Product will be 162 *l.* and this is the present Value of all the Payments that will be made to Eternity, at equal Intervals of Time upon the failing of a Life, which is to be immediately replaced.*

As for the Intervals of Time after which those Replacements will be made, they may be found thus;

Look in the seventh of our Tables for the Number 7.64, which is the Value of the three joint Lives, and over against it will be found the Number answering, which is between 9 and 10; and so it may be said that the Replacements will be made at every Interval of about 9 or 10 Years.

But that Interval may be determined a little more accurately, by help of a Table of Logarithms, by taking the Logarithm of the Quantity $\frac{1}{1-dM''}$ and dividing it by the Logarithm of r .

The Logarithm of $\frac{1}{1-dM''}$ is 0.2090115; the Logarithm of r is 0.0211893; and the first being divided by the second, the Quotient is 9.86, which shews that the Replacements will be made at Intervals a little more than $9\frac{1}{4}$ Years.

PROBLEM XI.

Supposing, as before, three equal Lives of 30, and that the Lives are not to be renewed, till after the failing of any two of them, and that a Sum p is then to be received, and that perpetually, after the failing of two Lives, what is the present Value of that Expectation?

SOLUTION.

Make $3M'' - 2M''' = A$, let the Interest of 1 $l.$ be $= d$, then the present Value of that Expectation will be $\frac{1-dA}{dA} \times p$.

But to know the Intervals of Time after which the Lives will be filled up, take the Logarithm of the Quantity $\frac{1}{1-dA}$, and divide it by the Logarithm of r .

The Value of a single Life of 30, Interest at 5 per Cent. is found in our Tables to be 13 Years Purchase, the Value M'' of two joint Lives by Problem II. is 9.63; and the Value M''' of three joint Lives by Problem III. is 7.64; then $3M'' - 2M'''$, or the Difference between the Triple of two joint Lives, and the Double of three joint Lives will be $13.59 = A$, then $\frac{1-dA}{dA} \times p$ will be found to be 0.473 p , and the Intervals of Time will be 23.32, that is, nearly $23\frac{1}{3}$ Years.

P R O B L E M XII.

Supposing still the Lives to be 30, and that they are not to be renewed till after the Extinction of all three, and that a Sum q is then to be received, and that perpetually after every Renewal, what is the present Value of that Expectation?

SOLUTION.

Make $3M' - 3M'' + M''' = B$, then the present Value of that Expectation will be $\frac{1-dB}{dB} \times q$; here B will be found to be 17.76, and consequently $\frac{1-dB}{dB} \times q$ will be $= 0.121 \times q$.

And the Intervals of Time will be the Logarithm of the Quantity $\frac{1}{1-dB}$ divided by the Logarithm of r , which in this Case would be 44.87, that is, nearly 45 Years.

COROLLARY.

Hence it will be easy for the Proprietor of the Lives, to find which is most advantageous to him, to fill up a Life as soon as it is vacant, or not to fill up before the Vacancy of two, or to let them all drop before the Renewal.

REMARK.

It is not to be imagined that if Interest of Money was higher or lower than 5 *per Cent.* the Intervals of Time after which the Renewals are made, would be the same as they are now, for it will be found, that as Interest is higher, the Intervals will be shorter; and as it is lower, so the Intervals will be longer; yet one might make it an Objection to our Rules, that the length of Life would thereby seem to depend upon the Rate of Interest. To answer this Difficulty, it must be observed, that the calculating of Time imports no more, than that considering the Circumstances of the Purchaser and the Proprietor of the Lives, in respect to the Rate of Interest agreed upon, and the Sum to be given upon the Renewal of a Life, or Lives, the Proprietor makes the same Advantage of his Money, as if he had agreed with the Purchaser, that he should pay him a certain Sum of Money at equal Intervals of Time, for redeeming the

Risque

Risque which he the Purchaser runs of paying that Sum when the Life or Lives drop : but the real Intervals of Time will be shewn afterwards.

Altho' it seldom happens that in Contracts about Lives, any more than three are concerned, yet I hope it will not be displeasing to our Readers to have this Speculation carried a little farther.

But as general Rules are best inculcated by particular Examples, I shall take the Case of five Lives, and express the several Circumstances of them in such manner, as that they may be a sure Guide in all other Cases of the same kind, let the Number of Lives be what it will ; let therefore the following Expressions be written,

$$\begin{array}{r}
 M'''' \\
 5M'''' - 4M'''' \\
 10M''' - 15M'''' + 6M'''' \\
 10M'' - 20M''' + 15M'''' - 4M'''' \\
 5M' - 10M'' + 10M''' - 5M'''' + 1M''''
 \end{array}$$

The first Term M'''' represents properly the present Value of an Annuity upon five equal joint Lives, but from thence may be deduced the Time of their joint continuance, or the Time in which it may be expected that one of them will fail, it being as I have said before, the Logarithm of $\frac{1}{1-dM''''}$ divided by the Logarithm of r : however, for shortness sake, I call for the present that Expression the Time.

The two next Terms, $5M'''' - 4M''''$, represent the Time in which two of the Lives will fail.

The three next Terms, $10M''' - 15M'''' + 6M''''$, represent the Time in which three out of the five Lives will fail.

The four next, $10M'' - 20M''' + 15M'''' - 4M''''$, represent the Time in which four out of the five Lives will fail.

The five next, $5M' - 10M'' + 10M''' - 5M'''' + 1M''''$, represent the Time in which all the five Lives will be extinct.

Now the Law of the Generation of the Co-efficients is thus.

1°. Take all the Terms which are affected with the Mark M'''' , beginning from the uppermost, with the Co-efficients $1 - 4 + 6 - 4 + 1$, which are the Terms of the Binomial $1 - 1$, raised to the fourth Power, which is less by one than the Number of Lives concerned.

2°. Take the Terms which are affected with the Mark M''' , and prefix to them in order, the product of the Number 5 by the Co-efficients $1 - 3 + 3 - 1$, which are the Terms of the Binomial $1 - 1$ raised to its Cube, that is, to a Power less by two than the Number of Lives concerned.

3°. Take all the Terms which are affected with the Mark M''' , and prefix to them in order, the Product of the Number 10, multiplied by the Co-efficients $1-2+1$, which are the Terms of the Binomial $1-1$ raised to its Square, that is, to a Power less by three than the Number of Lives concerned.

4°. Take all the Terms which are affected with the Mark M'' , and prefix to them the product of the Number 10, multiplied by the Terms of the Binomial $1-1$, raised to the Power whose Index is 1, that is to a Power less by four than the Number of Lives concerned.

5°. Take all the Terms which are affected with the Mark M' , and prefix to them the Product of the Number 5, multiplied by the Binomial $1-1$, raised to a Power less by 5 than the Number of Lives concerned; which in this Case happening to be nothing, or 0, degenerates barely into Unity.

As for the Multipliers, conceiving that the Multiplier of the first Term M'''' is 1, all the Multipliers will be 1, 5, 10, 10, 5, which are all, except the last, the Coefficients of the Binomial $1+1$, raised to its fifth Power, that is, to a Power equalling the Number of all the Lives.

N. B. The Exception here given, does not fall upon the Number 5, but upon the last Term of the fifth Power, $1+5+10+10+5+1$, which last 1 is rejected.

OF SUCCESSIVE LIVES.

PROBLEM XIII.

If A enjoys an Annuity for his Life, and at his Decease has the Nomination of a Successor B, who is also to enjoy the Annuity for his Life, to find the present Value of the two successive Lives.

SOLUTION.

Let the Values of the Lives be M and P ; let d be the Interest of 1 l. then the Value of the two successive Lives will be $M+P-dMP$.

But if the Successor B was himself to have the Nomination of a Life Q ; then the Value of the three successive Lives would be $M+P+Q-d \times MP+MQ+PQ+dd \times MPQ$.

But

But before I proceed, it is proper to observe that the Expressions MP , $M\mathcal{Q}$, $P\mathcal{Q}$, and $MP\mathcal{Q}$, signify barely Products, which is conformable to the usual Algebraic Notation; this I take notice of, for fear those Expressions should be confounded with others that I have made use of before, viz. \overline{MP} , $\overline{M\mathcal{Q}}$, $\overline{P\mathcal{Q}}$, and $\overline{MP\mathcal{Q}}$, which denoted joint Lives.

But to comprise under one general Rule all the possible Cases that may happen about any Number of successive Lives, it will be proper to express it in Words at length, thus;

From the Sum of all the Lives, subtract the Sum of the Products of all the Lives combined two and two, which Sum of Products before they are subtracted, ought to be multiplied by the Interest of 1 l.

To this add the Sum of the Products of all the Lives taken three and three, but multiplied again by the Square of the Interest of 1 l.

From this subtract the Sum of the Products of all the Lives taken four and four, but multiplied again by the Cube of the Interest of 1 l. and so on by alternate Additions and Subtractions still observing that if there was occasion to take the Lives five and five, six and six, &c. the Interest of 1 l. ought to be raised to the 4th Power, and to the 5th, and so on.

But all those Operations would be very much contracted, if the Lives to be nominated were always of the same Age, for Instance 30: for suppose M to be the Value of an Annuity on an Age of 30, and d to be the Interest of 1 l. then the present Value of all the successive Lives, of which the Number is n , would be $\frac{1 - 1 - d M n^r}{d}$.

In Words thus;

Multiply the Value of one Life by the Interest of 1 l. let the Product be subtracted from Unity, and let the Remainder be raised to that Power which answers to the Number of Lives; then this Power being again subtracted from Unity, let the Remainder be divided by the Interest of 1 l. and the Quotient will be the present Value of all the successive equal Lives.

And again, if the Number of those Lives were infinite, the Sum would barely be $\frac{1}{d}$.

P R O B L E M XIV.

Of a Perpetual Advowson.

1°. I suppose that at the Time of the Demise of the Incumbent, the Patron would receive the Sum f , for alienating his Right of the
next

next Presentation, if the Law did not forbid the Alienation in that Circumstance of Time.

2°. I suppose that when this Right is transferred, the Age of the Incumbent is such, that an Annuity upon his Life would be worth M Years Purchase, when the Interest of 1 $l.$ is d .

This being supposed, the Right of the next Presentation is worth $\frac{1 - dM}{d} \times f$, and the Right of Patronage, or perpetual Recurrency of the like Circumstances to Eternity, would be worth $\frac{1 - dM}{d} \times f$. In words thus;

Take the present Value of the Life of the Incumbent, and multiply it by the Interest of 1 $l.$ and reserve the Product.

Subtract this Product from Unity, and let the Remainder be multiplied by the Sum expected f , and the new Product will shew the Right of the next Presentation; let also this be reserved.

Then divide the second Quantity reserved by the first, and the Quotient will shew the present Value of the Right of Patronage, or perpetual Recurrency.

Thus, supposing the Life of the Incumbent worth 8 Years Purchase, the Rate of Interest 5 *per Cent.* and the Sum f to be 100 $l.$ the Right of the next Presentation would be worth 60 $l.$ and the Right of perpetual Recurrency 150 $l.$

P R O B L E M XV.

Of a Copy-hold.

Supposing that every Copy-hold Tenant pays to the Lord of the Manor a certain Fine on Admittance, and that every Successor does the like; to find the Value of the Copy-hold computed from the Time of a Fine being paid, independently from the Fine that may be given on Alienation.

SOLUTION.

I suppose that the Value of the Life of the present Tenant, and the Life of every future Successor when he comes to Possession is the same; this being admitted, let M be the Value of a Life, d the Interest of 1 $l.$ and f the Fine to be paid, then the present Value of the Copy-

Copy-hold will be $\frac{1-d^M}{dM} \times f$: and this Expression being exactly the same as that whereby the Right of Patronage has been determined, needs no Explanation in Words.

Only it is necessary to observe, that the Sum f paid in Hand being added to this, will make the Canon shorter, and will be reduced to $\frac{f}{dM}$, which may be expressed thus in Words.

Divide the Fine by the Product of the Life, multiplied by the Interest of 1 l.

Thus, if the Life of a Tenant is worth 12 Years Purchase, and the Fine to be paid on Admittance 56 l. and also the Rate of Interest 5 per Cent. then the present Value of the Copy-hold is $93 \frac{1}{3}$ l.

P R O B L E M X V I .

A borrows a certain Sum of Money, and gives Security that it shall be repaid at his Decease with the Interests; to fix the Sum which is then to be paid.

S O L U T I O N .

Let the Sum borrowed be f , the Life of the Borrower M Years purchase, d the Interest of 1 l. then the Sum to be paid at A 's Decease will be $\frac{f}{1-d^M}$; thus, supposing $f = 800$, $M = 11.83$, $d = 0.05$, then $\frac{f}{1-d^M}$ would be found $= 1958$ l: in the same manner, if the Sum to be paid at A 's Decease, was to be an Equivalent for his Life, unpaid at the Time of the Purchase, that Sum would be $\frac{M}{1-d^M} = 2895$ l. Supposing the Annuity received to be 100 l. as also the Life of A 11.83 Years Purchase.

P R O B L E M X V I I .

A borrows a Sum f , payable at his Decease, but with this Condition, that if he dies before B, then the whole Sum is to be lost to the Lender; to find what A ought to pay at his Decease in case he survives B.

SOLUTION.

Let us suppose, as before, that A is 40 Years of Age, that the Sum borrowed is 800 $l.$ and that Interest of Money is 5 *per Cent.* Farther, let it be supposed that B is 70 Years of Age, then, 1°. determine what A should pay at his Decease, if the Life of B was not concerned; by the Solution of the preceding Problem, we find the Sum to be 1958 $l.$ But we ought to consider that the Lender having a Chance to lose his Money, there ought to be a Compensation for the Risque he runs, which is founded on the possibility of a Man of seventy outliving a Man of forty. Now, by the Rules to be delivered in the next Problem, we shall find that the Probability of that Contingency is measured by the Fraction $\frac{4}{23}$, and therefore the Probability of the youngest Life's surviving the oldest is $\frac{19}{23}$. Now this being the Measure of the Probability which the Lender has of being repaid, the Sum 1958 ought to be increased in the proportion of 23 to 19, which will make it to be 2370 $l.$ nearly.

Of the Probabilities of Survivorship.

P R O B L E M XVIII.

Any Number of Lives being given, to find their Probability of Survivorship.

SOLUTION.

Let $A, B, C, D, \&c.$ be the Lives, whereof A is supposed to be the youngest, B the next to it, C the next, $\&c.$ and so the last the oldest.

Let $n, p, q, s, t, \&c.$ be the respective Intervals intercepted between the Ages of those Lives, and the Extremity of old Age supposed at 86; then the Probabilities of any one of those Lives surviving all the rest, will be

for A	1	-	$\frac{p}{2n}$	-	$\frac{qq}{6np}$	-	$\frac{s^3}{12npq}$	-	$\frac{t^4}{20npqs}$
B	+	$\frac{p}{2n}$	-	$\frac{qq}{6np}$	-	$\frac{s^3}{12npq}$	-	$\frac{t^4}{20npqs}$	
C			+	$\frac{qq}{3np}$	-	$\frac{s^3}{12npq}$	-	$\frac{t^4}{20npqs}$	
D					+	$\frac{s^3}{4npq}$	-	$\frac{t^4}{20npqs}$	
E							+	$\frac{t^4}{5npqs}$	
&c.									

Here some few things may be observed.

1°. That the Probability of the youngest Life surviving all the rest, always begins with Unity, and that it is expressed by so many Terms as there are Lives concerned.

2°. That the Probabilities of the other Lives surviving all the rest, are always expressed each by one Term less than the preceding.

3°. That each first Term of those whereby each Probability is expressed, is always the Sum of all the other Terms standing above it.

4°. That the Numbers 2, 6, 12, 20, 30, &c. made use of in the Denominators of the Fractions are generated by the Multiplication of the following Numbers, 1×2 , 2×3 , 3×4 , 4×5 , &c. It would take up too much room to explain this general Rule in Words at length, for which Reason I shall content my self with explaining only the Cases of two and three Lives, which are the most necessary.

And, First, if there be two Lives of a given Age, such as 40 and 70, take their Complements of Life, which as I have explained before, are the Differences between 86 and the respective Ages, those Complements therefore are 46 and 16.

Divide the shortest Complement by the Double of the Longest, and the Quotient will express the Probability of the oldest Life surviving the youngest.

Thus in the present Case, the shortest Complement being 16, and the double of the longest being 92, I divide 16 by 92, and the Quotient $\frac{16}{92}$ or $\frac{4}{23}$ will express the Probability required.

Subtract this Fraction from Unity, and the Remainder $\frac{19}{23}$ will express the Probability of the youngest Life surviving the oldest.

So that the Odds of the youngest Life surviving the oldest, are 19 to 4.

The Case of three Lives is thus: Suppose there are three Lives of a given Age, such as 40, 45, and 60; take their respective Complements of Life, which are 46, 41, 26, then divide the Square of the

O o 2

shortest

shortest Complement by 3 times the Product of the other two, and the Quotient will express the Probability of the oldest Life surviving the other two.

Divide the middlemost Complement by the Double of the greatest, and from the Quotient subtract the Square of the least divided by 6 Times the Product of the other two, and the Remainder will express the Probability of the middlemost Life surviving the other two.

Subtract the Sum of the two foregoing Probabilities from Unity, and the Remainder will express the Probability of the youngest Life surviving the other two.

Thus in the Case proposed, the Probability of the oldest Life surviving the other two, will be found $\frac{676}{5658} = \frac{3}{25}$ nearly.

The Probability of the middlemost Life surviving the other two will be $\frac{4367}{11316} = \frac{5}{13}$ nearly.

The Probability of the youngest Life surviving the other two will be $\frac{3}{5}$ nearly.

P R O B L E M XIX.

Any Number of Lives being given, to find the Probability of the Order of their Survivorship.

SOLUTION.

Suppose the three Lives to be those of *A*, *B*, *C*, and that it be required to assign the Probability of Survivorship as limited to the Order in which they are written, so that *A* shall both survive *B* and *C*, and *B* also survive *C*. This being supposed, let *n*, *p*, *q*, represent the respective Complements of Life, of the youngest, middlemost, and oldest, then the Probabilities of the six different Orders that there are in three things, will be as follows;

<i>A</i> , <i>B</i> , <i>C</i>	$1 - \frac{p}{2n} - \frac{q}{2p} + \frac{qq}{6np}$
<i>A</i> , <i>C</i> , <i>B</i>	$\frac{q}{2p} - \frac{qq}{3np}$
<i>B</i> , <i>A</i> , <i>C</i>	$\frac{p}{2n} - \frac{q}{2n} + \frac{qq}{6np}$
<i>B</i> , <i>C</i> , <i>A</i>	$\frac{q}{2n} - \frac{qq}{3np}$
<i>C</i> , <i>A</i> , <i>B</i>	$\frac{qq}{6np}$
<i>C</i> , <i>B</i> , <i>A</i>	$\frac{qq}{6np}$

In Words thus;

1°. Divide

1°. Divide the middlemost Complement by the double of the greatest, and let the Quotient be subtracted from Unity.

2°. From that Remainder subtract again the Quotient of the shortest Complement divided by the Double of the Middlemost.

3°. To that new Remainder add the Quotient arising from the Square of the shortest Complement divided by six times the Product of the greatest and middlemost multiplied together, and this last Sum will express the Probability of the first Order.

The probability of the Second will be found thus;

1°. Divide the shortest Complement by the double of the middlemost, and reserve the Quotient.

2°. Divide the Square of the shortest by three times the Product of the longest Complement, multiplied by the Middlemost, and reserve the new Quotient.

3°. Let the second Quotient be subtracted from the first, and the Remainder will express the Probability of the happening of the second Order.

The Probability of the third Order will be found as follows.

1°. Divide the middlemost Complement by the Double of the Greatest, and reserve the Quotient.

2°. Divide the shortest Complement by the Double of the longest, and reserve the Quotient.

3°. Divide the Square of the shortest Complement by six times the Product of the longest and middlemost multiplied together, and reserve the Quotient.

4°. From the first Quotient reserved, subtract the second; then to the Remainder add the Third, and the Result of these Operations will express the Probability of the third Order.

The Probability of the fourth Order will be found thus.

1°. Divide the shortest Complement by the Double of the longest, and reserve the Quotient.

2°. Divide the Square of the shortest Complement by three Times the Product of the longest and middlemost, and reserve the new Quotient.

3°. From the first Quotient reserved, subtract the second, and the Remainder will express the Probability of the fourth Order.

The fifth Order will be found as follows.

Divide the Square of the shortest Complement by six times the Product of the longest and middlemost, multiplied together, and the Quotient will express the Probability required.

The Probability of the last Order is the same as that of the fifth.

P R O B L E M XX.

D, whilst in Health, makes a Will, whereby he bequeaths 500 l. to E, and 300 l. to F. with this Condition, that if either of them dies before him, the whole is to go to the Survivor of the two; what are the Values of the Expectations of E and F, estimated from the time that the Will was writ?

SOLUTION.

Suppose *D* to be 70 Years of Age, *E* 36, and *F* 45; suppose also that *d* represents the Interest of 1 *l.* when Interest is at 5 per Cent.

An Annuity upon the Life of *D* is worth 5.77, as appears from our Tables, which Value we may call *M*.

Wherefore if it was sure that *D* would die before either of them, the Expectation of *E* upon that Account, would be worth in present Value $1 - dM \times 500$, and the Expectation of *F*, $1 - dM \times 300$; which being reduced to Numbers, are respectively 355 *l.* 15 *s.* and 213 *l.* 9 *s.*

But as this depends on the Probability of *D*'s dying first, we are to look for that Probability, which is composed of two Parts, that is, when the Order of Survivorship is either *E, F, D*, or *F, E, D*; now the Order *E, F, D*, is the same as *A, B, C*, in the preceding Problem, whereof the Probability is $1 - \frac{p}{2n} - \frac{q}{2p} + \frac{qq}{6np}$, and the Order *F, E, D*, is the same as *B, A, C*, whereof the Probability is $\frac{p}{2n} - \frac{q}{2n} + \frac{qq}{6np}$, and the Sum of those Probabilities, viz. $1 - \frac{q}{2p} - \frac{q}{2n} + \frac{qq}{3np}$, will express the Probability of *D*'s dying before them both.

Now the Ages being given, their Complements of Life will also be given, so that *n* will be found = 50, *p* = 41, *q* = 16; for which reason the Probability just now set down being expressed in Numbers, will be 0.6865, and this being multiplied by the Expectations before found, viz. 355 *l.* 15 *s.* and 213 *l.* 9 *s.* will produce 244 *l.* 3 *s.* 5 *d.* and 146 *l.* 10 *s.* 8 *d.* and these Sums express the present Expectations of *E* and *F*, arising from the Prospect of *D*'s dying before either of them.

But

But both *E* and *F* have a farther Expectation; which, in respect to *E*, is, that he shall survive *D*, and that *D* shall survive *F*, in which Case he obtains 800 *l.* but this not being to be obtained before the Decease of *D*, is reduced in present Value to 569 *l.* 4 *s.* Now the Probability of obtaining this answers to the Order, *A, C, B*, in the preceding Problem, which is expressed by $\frac{q}{2p} - \frac{qq}{3^{np}} = 0.1535$; and therefore multiplying the Sum 569 *l.* 4 *s.* by 0.1535, the Product will be 87 *l.* 7 *s.* 5 *d.* and this will be the second Part of *E*'s Expectation, which being joined with the first Part found before, viz. 244 *l.* 3 *s.* 5 *d.* the Sum will be 331 *l.* 10 *s.* 10 *d.* which is the total Expectation of *E*, or the present Sum he might justly expect, if he would sell his Right to another.

In the same manner the total Expectation of *F* will be found to be 213 *l.* 18 *s.* 6 *d.*

Otherwise, and more exactly, thus;

1. Let the Value of an Annuity of 40 *l.* for *D*'s Life, be taken off; which reduces the Sum to *l.* 569.2. as above.

2. The Heirs of *D* have likewise a demand upon this last Sum, for the Contingency of his outliving both the Legatees; which is implied tho' not expressed in the Question. Subtract therefore from the Value of the longest of the 3 Lives *D, E, F*, which, by Prob. V, is 15.477, the Value of the longest of the two Lives *E, F*; which, by Prob. IV, is 15.197; and the Remainder 0.28, *D*'s Survivorship due to the Heirs, taken from *l.* 569.2, considered as 20 Years Purchase, or the Perpetuity, reduces it to *l.* 561.23.

3. This Sum, now clear of all demands, might be paid down immediately to *E* and *F*, in the proportions of 5 and 3, according to the Will; were their Ages equal. And altho' they are not, we shall suppose that *D*, or his Executor named, pays it them in that manner; the Share of *E* being *l.* 350.77, and that of *F*, *l.* 210.46. leaving them to adjust their Pretensions, on account of Age, between themselves.

4. In order to which; the Sums which *E* and *F* have received being called *G* and *L*, respectively; let the Value of *E*'s Survivorship after *D* and *F*, found as above, be denoted by *e*, and that of *F* after *D* and *E* by *f*: Then those Values, *e* and *f*, will represent the Chances, or Claims, which *E* and *F* have upon each other's Sums *L* and *G*. And therefore the Ballance of their Claims is $\frac{eL - fG}{e + f}$; due by *F* or *E* as the Sign is positive or negative.

As in our Example, e and f being 3.26, 2.269 respectively, E must refund to F ($\frac{eL-fG}{e+f} = -$) $l. 19.857$; and the just Values of their Legacies will be $l. 330. 18 s.$ and $l. 230. 6 s.$

This last Computation is to be used when the Testator D is not very old, or the Ages of E and F are considerably different; or when both these Conditions obtain: For in those Cases, the Ratio of the *Probabilities* of Survivorship will differ sensibly from that of the *Values of the Probabilities* reckoned in Years purchase. And the like caution is to be observed in all similar Cases.

Of the Expectations of Life.

I call that the Expectation of Life, the Time which a Person of a given Age may justly expect to continue in being.

I have found by a Calculation deduced from the Method of Fluxions, that upon Supposition of an equable Decrement of Life, the Expectation of Life would be expressed by $\frac{1}{2}n$, supposing n to denote its Complement.

However, if that Interval be once attained, there arises a new Expectation of $\frac{1}{4}n$, and afterwards of $\frac{1}{8}n$, and so on. This being laid down, I shall proceed farther.

P R O B L E M XXI.

To find the Expectation of two joint Lives, that is, the Time which two Lives may expect to continue together in being.

SOLUTION.

Let the Complements of the Lives be n and p , whereof n be the longest and p the shortest, then the Expectation of the two joint Lives, will be $\frac{1}{2}p - \frac{pp}{6n}$, in Words thus.

From $\frac{1}{2}$ the shortest Complement, subtract the 6th Part of its Square, divided by the greatest, the Remainder will express the Number of Years sought.

Thus, supposing a Life of 40, and another of 50, the shortest Complement will be 36, the greatest 46, $\frac{1}{2}$ of the shortest will be

18, the Square of 36 is 1296, whereof the sixth Part is 216, which being divided by 46, the Quotient will be $\frac{216}{46} = 4.69$ nearly; and this being subtracted from 18, the Remainder 13.31 will express the Number of Years due to the two joint Lives.

COROLLARY.

If the two Lives be equal, the Expectation of the two joint Lives will be $\frac{1}{3}$ part of their common Complement.

P R O B L E M XXII.

Any Number of Lives being given, whether equal or unequal, to find how many Years they may be expected to continue together.

SOLUTION.

1°. Take $\frac{1}{2}$ of the shortest Complement.

2°. Take $\frac{1}{6}$ part of the Square of the shortest, which divide successively by all the other Complements, then add all the Quotients together.

3°. Take $\frac{1}{12}$ part of the Cube of the shortest Complement, which divide successively by the Product of all the other Complements, taken two and two.

4°. Then take $\frac{1}{20}$ part of the Biquadrate of the shortest Complement, which divide successively by the Products of all the other Complements, taken three and three, and so on.

5°. Then from the Result of the first Operation, subtract the Result of the second, to the Remainder add the Result of the third, from the Sum subtract the Result of the fourth, and so on.

6°. The last Quantity remaining after these alternate Subtractions and Additions, will be the thing required.

N. B. The Divisors 2, 6, 12, 20, &c. are the Products of 1 by 2, of 2 by 3, of 3 by 4, of 4 by 5, &c.

COROLLARY.

If all the Lives be equal, add Unity to the Number of Lives, and divide their common Complement by that Number thus increased by

P p

Unity,

Unity, and the Quotient will always express the Time due to their joint Continuance.

P R O B L E M XXIII.

Two Lives being given, to find the Number of Years due to the Longest.

SOLUTION.

From the Sum of the Years due to each Life, subtract the Number of Years due to their Joint Continuance, the Remainder will be the Number of Years due to the Longest, or Survivor of them both.

Thus, supposing a Life of 40, and another of 50, the Number of Years due to the Life of 40, is 23; the Number of Years due to the Life of 50, is 18; from the Sum of 23 and 18, *viz.* 41, subtract 13.31 due to their joint Continuance, the Remainder 27.69 will be the Time due to the longest.

COROLLARY.

If the Lives be equal, then $\frac{2}{3}$ of their common Complement will be the Number of Years due to the Survivor.

Thus, supposing two Lives of 50, then their Complement will be 36; whereof two thirds will be 24; which is the Time due to the Survivor of the two.

P R O B L E M XXIV.

Any Number of Lives being given, to find the Number of Years due to the Longest.

SOLUTION.

Let the Years due to each Life be respectively denoted by $M, P, Q, S, \&c.$ then let the joint Lives, taken two and two, be denoted by $\overline{MP}, \overline{MQ}, \overline{MS}, \overline{PQ}, \&c.$ let also the joint Lives, taken three and three be denoted by $\overline{MPQ}, \overline{MPS}, \overline{MQS}, \overline{PQS}, \&c.$ Moreover, let the joint Lives, taken four and four, be denoted by $\overline{MPQS}, \&c.$ then if there be three Lives, the Time due to the longest will be

$$\begin{aligned} &M - \overline{MP} + \overline{MPQ} \\ &+ P - \overline{MQ} \\ &+ Q - \overline{PQ} \end{aligned}$$

But

But if all the Lives be equal, let n be their common Complement, then the Time due to the longest, will be $\frac{3}{4}n$.

If there be four Lives, the Time due to the longest will be

$$\begin{aligned} & M - \overline{MP} + \overline{MPQ} - \overline{MPQS} \\ & + P - \overline{MQ} + \overline{MPS} \\ & + Q - \overline{MS} + \overline{MQS} \\ & + S - \overline{PQ} + \overline{PQS} \\ & \quad - \overline{PS} \\ & \quad - \overline{QS} \end{aligned}$$

But if all the Lives be equal, the Time due to the longest will be expressed by $\frac{4}{5}$ of their common Complement.

Universally, if the common Complement of equal Lives be n , and the Number of Lives p , the Number of Years due to the Longest of them will be $\frac{p}{p+1} \times n$.

PROBLEM XXV.

Any Number of equal Lives being given, to find the Time in which one, or two, or three, &c. of them will fail.

SOLUTION.

Let n be their common Complement, p the Number of all the Lives, q the Number of those which are to fail, then $\frac{q}{p+1} \times n$ will express the Time required. In words thus;

Multiply the common Complement of the Lives by the Number of the Lives that are to drop, and divide the Product by the Number of all the Lives increased by Unity.

Thus, supposing 100 Lives, each of 40 Years of Age, it will be found that 5 of them will drop in about two Years and a Quarter.

But if we put t for the Time given, we shall have the four following Equations;

$$\begin{aligned} 1^\circ. \quad t &= \frac{qn}{p+1} \\ 2^\circ. \quad q &= \frac{p+1 \times t}{n} \\ 3^\circ. \quad p &= \frac{qn-t}{t} \\ 4^\circ. \quad n &= \frac{p+1 \times t}{q} \end{aligned}$$

In which any three of the four Quantities n, p, q, t , being given, the fourth will be known.

This Speculation might be carried to any Number of unequal Lives: but my Design not being to perplex the Reader with too great Difficulties, I shall forbear at present to prosecute the thing any farther.

P R O B L E M XXVI.

A, who is 30 Years of Age, buys an Annuity of 1 l. for a limited Time of his Life, suppose 10 Years, on Condition that if he dies before the Expiration of that Time, the Purchase Money is wholly to be lost to his Heirs; to find the present Value of the Purchase, supposing Interest at 5 per Cent.

SOLUTION.

Let n be the Complement of A 's Life, m the limited Number of Years, p the Difference of n and m ; Q the Value of an Annuity of 1 l. certain for m Years, and V the Value of the Perpetuity: then the present Value of the Purchase will be $\frac{mV + p - V + 1 \times Q}{n}$. In Words thus;

1°. Multiply V , the Value of a Perpetuity, at the given Rate of Interest, by m the limited Number of Years, and reserve the Product.

2°. To the same V add Unity, and take the Difference between their Sum and p , which is the Excess of the Complement of A 's Age above the limited Number of Years: multiply this Difference by Q , an Annuity certain for m Years, to get the second Product.

3°. Let the Sum of these Products, if p is greater than $V + 1$; and their Difference, if it is lesser, be divided by n , the Complement of A 's Age; and the Quotient shall be the Value of the Purchase.

As, in the Question proposed, where $n = 56$, $m = 10$, $p = 46$, $Q = 7.7212$, and $V = 20$; the first Product (mV) is that of 20 by 10, or 200. And $p - V + 1$ being $46 - 21 = 25$, the second Product is 25×7.7212 , that is 193.0302. The two Products added (p being greater than $V + 1$) make 393.0302: which divided by 56 quotes, for the Answer, 7.0184 Years Purchase.

Note, 1. When it happens that p is equal to $V + 1$; as, Interest being at 5 per Cent, if the Difference of n and m is 21; the second Product $p - V + 1 \times Q$ vanishing, the Answer is simply $\frac{mV}{n}$.

2. If

2. If $m=n$, or $p=0$, seeing V is equal to $\frac{1}{r-1}$, the Expression will be changed into $\frac{1}{r-1} - \frac{r^2}{n \times r-1}$; which coincides with the Solution of *Prob. I*: Q representing now the same Thing as P did in that Problem.

3. By this Proposition, some useful Questions concerning *Insurances* may be resolved.

Suppose A , at 30 Years of Age, assigns over to B an Annuity of 1000 *l* a Year, limited to 10 Years, and depending likewise upon A 's Life: then, by the foregoing Solution, A ought to receive for it only 7018 *l*. 8 *s*. Interest being at 5 per Cent. But if B wants that the Annuity should stand clear of all Risques, he must pay for it the Value certain, which is 7721 *l*. 4 *s*. and A ought to have his Life insured for 702 *l*. 16 *s*. the just Price of such an Insurance being the Difference of the Values of the Annuity certain, and of the same Annuity subject to the Contingency of the Annuitant's Life failing.

The same 702 *l*. 16 *s*. is likewise the Value of the Reversion of this Annuity to a Person and his Heirs, who should succeed to the Remainder of the 10 Years, upon A 's Decease. See *Prob. XXVIII*.

It is evident by the foregoing Process, that altho' the Question there proposed is particular, yet the Solution is general; which Method, often practised in my *Doctrine of Chances*, is of singular Use to fix the Reader's Imagination.

P R O B L E M XXVII.

A pays an Annuity of 100 *l*. during the Lives of B and C , each 34 Years of Age; to find what A ought to give in present Money to buy off the Life of B , supposing Interest at 4 per Cent.

SOLUTION.

It will be found by our Tables that an Annuity upon a Life of 34 is worth 14.12 Years Purchase; and, by the Rules before delivered, that an Annuity upon the longest of the two Lives of B and C is worth 18.40: hence it is very plain, that, to buy off the Life of B , A must pay the Difference between 18.40 and 14.12, which being 4.28, it follows that A ought to pay 428 *l*.

In the same manner, if A were to pay an Annuity during the three Lives of B , C , D , whether of the same or different Ages, it would

be

be easy to determine what *A* ought to pay to buy off one of the Lives of *B*, *C*, *D*, or any two of them, or to redeem the whole.

For, 1°. if the Life of *D* is to be bought off, then from the Value of the three Lives, subtract the Value of the two Lives of *B* and *C*, and the Remainder is what is to be given to buy off the Life of *D*.

2°. If the two Lives of *C* and *D* were to be bought off, then from the Value of the three Lives, subtract the Life of *B*, and the Remainder is what is to be given to buy off those two Lives.

Lastly, It is plain that to redeem the whole, the Value of the three Lives ought to be paid.

P R O B L E M XXVIII.

A, whose Life is worth 14 Years Purchase, supposing Interest at 4 per Cent. is to enjoy an Annuity of 100 l. during the Term of 31 Years; *B* and his Heirs have the Reversion of it after the Decease of *A* for the Term remaining; to find the Value of *B*'s Expectation.

SOLUTION.

Since the Life of *A* is supposed to be worth 14 Years Purchase when Interest is at 4 per Cent. it follows from the Tables that *A* must be about 35 Years of Age, therefore find, by the twenty-sixth Proposition, the Value of an Annuity on a Life of 35, to continue the limited Time of 31 Years; let that Value be subtracted from the Value of an Annuity certain, to continue 31 Years; and the Remainder will be the Value of the Reversion.

P R O B L E M XXIX.

A is to have an Annuity of 100 l. for him and his Heirs after the failing of any one of the Lives *M*, *P*, *Q*, the first of which is worth 13 Years Purchase, the second 14, and the third 15; to find the present Value of his Expectation, Interest of Money being supposed at 4 per Cent.

SOLUTION.

By the Example to Prob. III. it appears, that an Annuity upon the above 3 joint Lives is worth 7.41 Years Purchase; let this be supposed

posed $= R$, and let f represent the present Value of a Perpetuity of 100 l . which in this Case is 2500 l . then the present Expectation of A will be worth $1 - dR \times f$. In Words thus;

Multiply the Value of the three joint Lives by the Interest of 1 l . then subtracting that Product from Unity, let the Remainder be multiplied by the Value of the Perpetuity, and the Product will be the Expectation required.

In this Case 7.41, multiplied by 0.04, produces 0.2964, and this Product subtracted from Unity, leaves 0.7036; now this Remainder being multiplied by 2500, produces 1759 l . the Expectation of A .

But if the Problem had been, that A should not have the Annuity before the Failing of any two of those Lives; from the Sum of all the joint Lives combined two and two, subtract the double Value of the three joint Lives, and let the Remainder be called T , then the Expectation of A will be worth $1 - dT \times f$; now, by the Rules before delivered, we shall find that the Sum of all the joint Lives combined two and two, is 29.06, from which subtracting the double of the three joint Lives, viz. 14.82, the Remainder is 14.24. Hence supposing $T = 14.24$, then $1 - dT \times f$ will be found to be 1076 l . and this is the Value of A 's Expectation.

Lastly, If A was not to have the Annuity before the Extinction of the three Lives, suppose the Value of the three Lives $= V$, then the Expectation of A would be worth $1 - dV \times f$, which in this Case is 465 l .

P R O B L E M XXX.

To determine the Fines to be paid for renewing any Number of Years in a College-Lease of twenty; and also what Rate of Interest is made by a Purchaser, who may happen to give an advanced Price for the same, upon Supposition that the Contractor is allowed 8 per Cent. of his Money.

Altho' the Problem here proposed does not seem to relate to the Subject of this Book, yet as some useful Conclusions may be derived from the Solution of it, I have thought fit to insert it in this Place.

Table of Fines.

1	0.2146	8	2.2821	15	5.8254
2	0.4463	9	2.6792	16	6.5060
3	0.6965	10	3.1081	17	7.2411
4	0.9666	11	3.5713	18	8.0349
5	1.2587	12	4.0715	19	8.9922
6	1.4133	13	4.6118	20	9.8181
7	1.9144	14	5.1953		

If a Purchaser gives the Original Contractor 11 Years Purchase for his Lease of 20, he makes above $6\frac{1}{2}$ per Cent. of his Money.

If he gives 12 Years Purchase for the same, he makes above 5 l. 8 s. per Cent. of his Money.

If he gives 13 Years Purchase, he makes $4\frac{1}{2}$ per Cent. of his Money.

PROBLEM XXXI.

To determine the Fines to be paid for renewing any Number of Years in a College-Lease of One and Twenty; as also what Rate of Interest is made by a Purchaser who may happen to give an advanced Price for the same, upon Supposition that the Contractor is allowed 8 per Cent. of his Money.

Table of Fines.

1	0.1987	8	2.1131	15	5.3940
2	0.4133	9	2.2808	16	6.0241
3	0.6450	10	2.8779	17	6.7047
4	0.8952	11	3.3068	18	7.4398
5	1.1653	12	3.7700	19	8.2336
6	1.4574	13	4.2702	20	9.0909
7	1.6120	14	4.8105	21	10.0168

He that gives 11 Years Purchase, instead of 10.0168 for renewing his Lease for 21 Years, makes 6 l. 16 s. per Cent. of his Money.

He who gives 12 Years Purchase for the same, makes very near 5 l. 16 s. per Cent. of his Money.

He who gives 13 Years Purchase for the same, makes a little more than 4 l. 16 s. per Cent. of his Money.

The Values of Annuities for Lives having been calculated, in this Book, upon a supposition that the Payments are made Yearly, and there being some Occasions wherein it is stipulated that the Payments should be made Half-Yearly, I have thought fit to add the two following Problems; whereby, 1°. It is shewn what the Half-Yearly Payments ought to be, if the Price of the Purchase is preserved. 2°. How the Price of the Purchase ought to be increased, if the Half-Yearly Payments are required to be the Half of the Yearly Payments.

P R O B L E M XXXII.

An Annuity being given, to find what Half-Yearly Payments will be equivalent to it, when Interest of Money is 4, 5, or 6 per Cent

SOLUTION.

Take Half of the Annuity, and from that Half subtract its 100th, or 80th, or 68th Part, according as the Interest is 4, 5, or 6 per Cent. and the Remainder will be the Value of the Half-Yearly Payments required; thus, if the Annuity was 100 *l.* the Half-Yearly Payments would respectively be 49 *l.* 10 *s.* 49 *l.* 7 *s.* 6 *d.* 49 *l.* 5 *s.* 3 *d.* nearly.

P R O B L E M XXXIII.

The present Value of an Annuity being given, to find how much this present Value ought to be increased, when it is required that the Payments shall be Half-Yearly, and also one Half of the Yearly Payments, when Interest is at 4, 5, or 6 per Cent.

SOLUTION.

To the present Value of the Annuity add respectively its 99th, 79th, or 67th, and the Sums will be the Values increased.

As there are some Persons who may be desirous to see a general Solution of the two last Problems, I have thought fit to add what follows.

In the first of the two last Problems, let *A* be the Yearly Payments agreed on, and *B* the Half-Yearly Payments required, *r* the Yearly Rate of Interest, then $B = \frac{r^{\frac{1}{2}} - 1}{r - 1} \times A$. In the second, let *M* be the present Value of the Yearly Payments, *P* the present Value of those that are to be Half-Yearly, then $P = \frac{\frac{1}{2} \times r - 1}{r^{\frac{1}{2}} - 1} \times M$.

TABLE I.

The present Value of an annuity of one pound, for any Number of Years not exceeding 100, Interest at 3 per Cent.

Years.	Value.	Years.	Value.	Years.	Value.	Years.	Value.
1	0.9709	26	17.8768	51	25.9512	76	29.8076
2	1.9135	27	18.3270	52	26.1662	77	29.9103
3	2.8286	28	18.7641	53	26.3750	78	30.0100
4	3.7170	29	19.1884	54	26.5777	79	30.1068
5	4.5797	30	19.6004	55	26.7744	80	30.2008
6	5.4172	31	20.0004	56	26.9655	81	30.2920
7	6.2303	32	20.3887	57	27.1509	82	30.3806
8	7.0197	33	20.7658	58	27.3310	83	30.4666
9	7.7861	34	21.1318	59	27.5058	84	30.5501
10	8.5302	35	21.4872	60	27.6756	85	30.6311
11	9.2526	36	21.8323	61	27.8404	86	30.7099
12	9.9540	37	22.1672	62	28.0003	87	30.7863
13	10.6350	38	22.4925	63	28.1557	88	30.8605
14	11.2961	39	22.8082	64	28.3065	89	30.9325
15	11.9379	40	23.1148	65	28.4529	90	31.0024
16	12.5611	41	23.4124	66	28.5950	91	31.0703
17	13.1611	42	23.7014	67	28.7330	92	31.1362
18	13.7535	43	23.9819	68	28.8670	93	31.2001
19	14.3238	44	24.2543	69	28.9971	94	31.2622
20	14.8775	45	24.5187	70	29.1234	95	31.3224
21	15.4150	46	24.7754	71	29.2460	96	31.3809
22	15.9369	47	25.0247	72	29.3651	97	31.4377
23	16.4436	48	25.2667	73	29.4807	98	31.4928
24	16.9355	49	25.5017	74	29.5929	99	31.5463
25	17.4131	50	25.7298	75	29.7018	100	31.5984

The Value of the Perpetuity is $33\frac{1}{3}$ Years Purchase.

TABLE II.

The present Value of an Annuity of one Pound, to continue so long as a Life of a given Age is in being, Interest being estimated at 3 per Cent.

Age.	Value	Age.	Value	Age.	Value	Age.	Value
1	15.05	26	17.50	51	12.26	76	4.05
2	16.62	27	17.33	52	12.00	77	3.63
3	17.83	28	17.16	53	11.73	78	3.21
4	18.46	29	16.98	54	11.46	79	2.78
5	18.90	30	16.80	55	11.18	80	2.34
6	19.33	31	16.62	56	10.90	81	1.89
7	19.60	32	16.44	57	10.61	82	1.43
8	19.74	33	16.25	58	10.32	83	0.96
9	19.87	34	16.06	59	10.03	84	0.49
10	19.87	35	15.86	60	9.73	85	0.00
11	19.74	36	15.67	61	9.42	86	0.00
12	19.60	37	15.46	62	9.11		
13	19.47	38	15.26	63	8.79		
14	19.33	39	15.05	64	8.46		
15	19.19	40	14.84	65	8.13		
16	19.05	41	14.63	66	7.79		
17	18.90	42	14.41	67	7.45		
18	18.76	43	14.19	68	7.10		
19	18.61	44	13.96	69	6.75		
20	18.46	45	13.73	70	6.38		
21	18.30	46	13.49	71	6.01		
22	18.15	47	13.25	72	5.63		
23	17.99	48	13.01	73	5.25		
24	17.83	49	12.76	74	4.85		
25	17.66	50	12.51	75	4.45		

TABLE III.

The present Value of an Annuity of one Pound, for any Number of Years not exceeding 100, Interest at $3\frac{1}{2}$ per Cent.

Years.	Value.	Years.	Value.	Years.	Value.	Years.	Value.
1	0.9662	26	16.8904	51	23.6286	76	26.4799
2	1.8997	27	17.2854	52	23.7958	77	26.5506
3	2.8016	28	17.6670	53	23.9573	78	26.6190
4	3.6731	29	18.0358	54	24.1133	79	26.6850
5	4.5151	30	18.3920	55	24.2641	80	26.7488
6	5.3286	31	18.7363	56	24.4097	81	26.8104
7	6.1145	32	19.0689	57	24.5504	82	26.8700
8	6.8740	33	19.3902	58	24.6864	83	26.9275
9	7.6077	34	19.7007	59	24.8178	84	26.9831
10	8.3166	35	20.0007	60	24.9447	85	27.0368
11	9.0015	36	20.2905	61	25.0674	86	27.0887
12	9.6633	37	20.5705	62	25.1859	87	27.1388
13	10.3027	38	20.8411	63	25.3004	88	27.1873
14	10.9205	39	21.1025	64	25.4110	89	27.2341
15	11.5174	40	21.3551	65	25.5178	90	27.2793
16	12.0941	41	21.5991	66	25.6211	91	27.3230
17	12.6513	42	21.8349	67	25.7209	92	27.3652
18	13.1897	43	22.0627	68	25.8173	93	27.4060
19	13.7098	44	22.2828	69	25.9104	94	27.4454
20	14.2124	45	22.4955	70	26.0004	95	27.4835
21	14.6980	46	22.7009	71	26.0873	96	27.5203
22	15.1671	47	22.8994	72	26.1713	97	27.5558
23	15.6204	48	23.0912	73	26.2525	98	27.5902
24	16.0584	49	23.2766	74	26.3309	99	27.6234
25	16.4815	50	23.4556	75	26.4067	100	27.6554

The Value of the Perpetuity is $28\frac{1}{2}$ Years Purchase.

TABLE IV.

The present Value of an Annuity of one Pound, so long as a Life of a given Age is in being, Interest being estimated at $3\frac{1}{2}$ per Cent.

Age.	Value	Age.	Value	Age.	Value	Age.	Value
1	14.16	26	16.28	51	11.69	76	3.98
2	15.53	27	16.13	52	11.45	77	3.57
3	16.56	28	15.98	53	11.20	78	3.16
4	17.09	29	15.83	54	10.95	79	2.74
5	17.46	30	15.68	55	10.69	80	2.31
6	17.82	31	15.53	56	10.44	81	1.87
7	18.05	32	15.37	57	10.18	82	1.42
8	18.16	33	15.21	58	9.91	83	0.95
9	18.27	34	15.05	59	9.64	84	0.48
10	18.27	35	14.89	60	9.36	85	0.00
11	18.16	36	14.71	61	9.08	86	0.00
12	18.05	37	14.52	62	8.79		
13	17.94	38	14.34	63	8.49		
14	17.82	39	14.16	64	8.19		
15	17.71	40	13.98	65	7.88		
16	17.59	41	13.79	66	7.56		
17	17.46	42	13.59	67	7.24		
18	17.33	43	13.40	68	6.91		
19	17.21	44	13.20	69	6.57		
20	17.09	45	12.99	70	6.22		
21	16.96	46	12.78	71	5.87		
22	16.83	47	12.57	72	5.51		
23	16.69	48	12.36	73	5.14		
24	16.56	49	12.14	74	4.77		
25	16.42	50	11.92	75	4.38		

TABLE V.

The present Value of an Annuity of one Pound, for any Number of Years not exceeding 100, Interest at 4 per Cent.

Years.	Value.	Years.	Value.	Years.	Value.	Years.	Value.
1	0.9615	26	15.9827	51	21.6714	76	23.7311
2	1.8860	27	16.3295	52	21.7475	77	23.7799
3	2.7750	28	16.6630	53	21.8726	78	23.8268
4	3.6298	29	16.9837	54	21.9929	79	23.8720
5	4.4518	30	17.2920	55	22.1086	80	23.9153
6	5.2421	31	17.5884	56	22.2198	81	23.9571
7	6.0020	32	17.8735	57	22.3267	82	23.9972
8	6.7327	33	18.1476	58	22.4295	83	24.0357
9	7.4353	34	18.4111	59	22.5284	84	24.0728
10	8.1108	35	18.6646	60	22.6234	85	24.1085
11	8.7604	36	18.9082	61	22.7148	86	24.1428
12	9.3850	37	19.1425	62	22.8027	87	24.1757
13	9.9856	38	19.3678	63	22.8872	88	24.2074
14	10.5631	39	19.5844	64	22.9685	89	24.2379
15	11.1183	40	19.7927	65	23.0466	90	24.2672
16	11.6522	41	19.9930	66	23.1218	91	24.2954
17	12.1656	42	20.1856	67	23.1940	92	24.3225
18	12.6592	43	20.3707	68	23.2635	93	24.3486
19	13.1339	44	20.5488	69	23.3302	94	24.3736
20	13.5903	45	20.7200	70	23.3945	95	24.3977
21	14.0291	46	20.8846	71	23.4562	96	24.4209
22	14.4511	47	21.0429	72	23.5156	97	24.4431
23	14.8568	48	21.1951	73	23.5727	98	24.4646
24	15.2469	49	21.3414	74	23.6276	99	24.4851
25	15.6220	50	21.4821	75	23.6804	100	24.5049

TABLE VI.

The present Value of an Annuity of one Pound, to continue so long as a Life of a given Age is in being, Interest being estimated at 4 per Cent.

Age.	Value	Age.	Value	Age.	Value	Age.	Value
1	13.36	26	15.19	51	11.13	76	3.91
2	14.54	27	15.06	52	10.92	77	3.52
3	15.43	28	14.94	53	10.70	78	3.11
4	15.89	29	14.81	54	10.47	79	2.70
5	16.21	30	14.68	55	10.24	80	2.28
6	16.50	31	14.54	56	10.01	81	1.85
7	16.64	32	14.41	57	9.77	82	1.40
8	16.79	33	14.27	58	9.52	83	0.95
9	16.88	34	14.12	59	9.27	84	0.48
10	16.88	35	13.98	60	9.01	85	0.00
11	16.79	36	13.82	61	8.75	86	0.00
12	16.64	37	13.67	62	8.48		
13	16.60	38	13.52	63	8.20		
14	16.50	39	13.36	64	7.92		
15	16.41	40	13.20	65	7.63		
16	16.31	41	13.02	66	7.33		
17	16.21	42	12.85	67	7.02		
18	16.10	43	12.68	68	6.71		
19	15.99	44	12.50	69	6.39		
20	15.89	45	12.32	70	6.06		
21	15.78	46	12.13	71	5.72		
22	15.67	47	11.94	72	5.38		
23	15.55	48	11.74	73	5.02		
24	15.43	49	11.54	74	4.66		
25	15.31	50	11.34	75	4.29		

TABLE VII.

The present Value of an Annuity of one Pound, for any number of Years not exceeding 100, Interest at 5 per Cent.

Years.	Value.	Years.	Value.	Years.	Value.	Years.	Value.
1	0.9523	26	14.3751	51	18.3389	76	19.5094
2	1.8594	27	14.6430	52	18.4180	77	19.5328
3	2.7232	28	14.8981	53	18.4934	78	19.5550
4	3.5459	29	15.1410	54	18.5651	79	19.5762
5	4.3294	30	15.3724	55	18.6334	80	19.5964
6	5.0756	31	15.5928	56	18.6985	81	19.6156
7	5.7863	32	15.8026	57	18.7605	82	19.6339
8	6.4632	33	16.0025	58	18.8195	83	19.6514
9	7.1078	34	16.1929	59	18.8757	84	19.6680
10	7.7212	35	16.3741	60	18.9292	85	19.6838
11	8.3064	36	16.5468	61	18.9802	86	19.6988
12	8.8632	37	16.7112	62	19.0288	87	19.7132
13	9.3935	38	16.8678	63	19.0750	88	19.7268
14	9.8986	39	17.0170	64	19.1191	89	19.7398
15	10.3796	40	17.1590	65	19.1610	90	19.7522
16	10.8377	41	17.2943	66	19.2010	91	19.7640
17	11.2740	42	17.4232	67	19.2390	92	19.7752
18	11.6895	43	17.5459	68	19.2753	93	19.7859
19	12.0853	44	17.6627	69	19.3098	94	19.7961
20	12.4622	45	17.7740	70	19.3426	95	19.8058
21	12.8211	46	17.8800	71	19.3739	96	19.8151
22	13.1630	47	17.9810	72	19.4037	97	19.8239
23	13.4885	48	18.0771	73	19.4321	98	19.8323
24	13.7986	49	18.1687	74	19.4592	99	19.8403
25	14.0939	50	18.2559	75	19.4849	100	19.8479

TABLE VIII.

The present Value of an Annuity of one Pound, to continue so long as a Life of a given Age is in being, Interest at 5 per Cent.

Age.	Value	Age.	Value	Age.	Value	Age.	Value
1	11.96	26	13.37	51	10.17	76	3.78
2	12.88	27	13.28	52	9.99	77	3.41
3	13.55	28	13.18	53	9.82	78	3.03
4	13.89	29	13.09	54	9.63	79	2.64
5	14.12	30	12.99	55	9.44	80	2.23
6	14.34	31	12.88	56	9.24	81	1.81
7	14.47	32	12.78	57	9.04	82	1.38
8	14.53	33	12.67	58	8.83	83	0.94
9	14.60	34	12.56	59	8.61	84	0.47
10	14.60	35	12.45	60	8.39	85	0.00
11	14.53	36	12.33	61	8.16	86	0.00
12	14.47	37	12.21	62	7.93		
13	14.41	38	12.09	63	7.68		
14	14.34	39	11.96	64	7.43		
15	14.27	40	11.83	65	7.18		
16	14.20	41	11.70	66	6.91		
17	14.12	42	11.57	67	6.64		
18	14.05	43	11.43	68	6.36		
19	13.97	44	11.29	69	6.07		
20	13.89	45	11.14	70	5.77		
21	13.81	46	10.99	71	5.47		
22	13.72	47	10.84	72	5.15		
23	13.64	48	10.68	73	4.82		
24	13.55	49	10.51	74	4.49		
25	13.46	50	10.35	75	4.14		

TABLE IX.

The present Value of an Annuity of one Pound, for any Number of Years not exceeding 100, Interest at 6 per Cent.

Years.	Value.	Years.	Value.	Years.	Value.	Years.	Value.
1	0.9433	26	13.0031	51	15.8130	76	16.4677
2	1.8333	27	13.2105	52	15.8613	77	16.4790
3	2.6730	28	13.4061	53	15.9069	78	16.4896
4	3.4651	29	13.5907	54	15.9499	79	16.4996
5	4.2123	30	13.7648	55	15.9905	80	16.5091
6	4.9173	31	13.9290	56	16.0288	81	16.5180
7	5.5823	32	14.0840	57	16.0649	82	16.5264
8	6.2097	33	14.2302	58	16.0989	83	16.5343
9	6.8016	34	14.3681	59	16.1311	84	16.5418
10	7.3600	35	14.4982	60	16.1614	85	16.5489
11	7.8868	36	14.6209	61	16.1900	86	16.5556
12	8.3838	37	14.7367	62	16.2170	87	16.5618
13	8.8526	38	14.8460	63	16.2424	88	16.5678
14	9.2949	39	14.9490	64	16.2664	89	16.5734
15	9.7122	40	15.0462	65	16.2891	90	16.5786
16	10.1058	41	15.1380	66	16.3104	91	16.5836
17	10.4772	42	15.2245	67	16.3306	92	16.5883
18	10.8276	43	15.3061	68	16.3496	93	16.5928
19	11.1581	44	15.3831	69	16.3676	94	16.5969
20	11.4699	45	15.4558	70	16.3845	95	16.6009
21	11.7640	46	15.5243	71	16.4005	96	16.6046
22	12.0415	47	15.5890	72	16.4155	97	16.6081
23	12.3033	48	15.6500	73	16.4297	98	16.6114
24	12.5503	49	15.7075	74	16.4431	99	16.6145
25	12.7833	50	15.7618	75	16.4558	100	16.6175

TABLE X.

The present Value of an Annuity of one Pound, to continue so long as a Life of a given Age is in being, Interest being estimated at 6 per Cent.

Age.	Value	Age.	Value	Age.	Value	Age.	Value
1	10.80	26	11.90	51	9.34	76	3.66
2	11.53	27	11.83	52	9.20	77	3.31
3	12.04	28	11.76	53	9.04	78	2.95
4	12.30	29	11.68	54	8.90	79	2.57
5	12.47	30	11.61	55	8.72	80	2.18
6	12.63	31	11.53	56	8.56	81	1.78
7	12.74	32	11.45	57	8.38	82	1.36
8	12.79	33	11.36	58	8.20	83	0.92
9	12.84	34	11.60	59	8.02	84	0.77
10	12.84	35	11.18	60	7.83	85	0.00
11	12.79	36	11.09	61	7.63	86	0.00
12	12.74	37	11.00	62	7.42		
13	12.69	38	10.90	63	7.21		
14	12.63	39	10.80	64	7.00		
15	12.58	40	10.70	65	6.77		
16	12.53	41	10.60	66	6.53		
17	12.47	42	10.50	67	6.22		
18	12.41	43	10.37	68	6.03		
19	12.36	44	10.26	69	5.77		
20	12.30	45	10.14	70	5.50		
21	12.23	46	10.02	71	5.22		
22	12.17	47	9.90	72	4.93		
23	12.11	48	9.76	73	4.63		
24	12.04	49	9.63	74	4.32		
25	11.97	50	9.49	75	4.00		

Note; The 1st, 3^d, 5th, 7th and 9th Tables serve likewise to resolve the Questions concerning *Compound Interest*: as

I.

To find the present Value of 1000 *l.* payable 7 Years hence, at $3\frac{1}{2}$ per Cent. From the present Value of an Annuity of 1 *l.* certain for 7 Years, which, in *Tab. III.* is 6.1145, I subtract the like Value for 6 Years, which is 5.3286; and the Remainder .7859 is the Value of the 7th Year's Rent, or of 1 *l.* payable after 7 Years; which multiplied by 1000 gives the Answer 785 *l.* 18*sh.*

II.

If it is asked, *what will be the Amount of the Sum S in 7 Years at $3\frac{1}{2}$ per Cent?* Having found .7859 as above, 'tis plain the Amount will be $\frac{S}{.7859}$.

III.

If the Question is, *In what time a Sum S will be doubled, tripled, or increased in any given Ratio at 3, $3\frac{1}{2}$, &c. per Cent.* I take, in the proper Table, two contiguous Numbers whose Difference is nearest the Reciprocal of the Ratio given, as $\frac{1}{2}$, $\frac{1}{3}$, &c. And the Year against the higher number is the Answer.

Thus in *Tab. I.* against the Years 22, 23, stand the Numbers 15.9369 and 16.4436; whose Difference .5067 being a little more than .5, or $\frac{1}{2}$, shews that in 23 Years, a Sum *S* will be a little less than doubled, at 3 per Cent. Compound Interest. And against the Years 36 and 37 are 21.8323, and 22.1672; the Difference whereof being .3349, nearly $\frac{1}{3}$, shews that in 14 Years more it will be almost tripled.

If more exactness is required; take the adjoining Difference whose Error is contrary to that of the Difference found; and thence compute the proportional part to be added or subtracted thus, in the last of these Examples, the Difference between the Years 37 and 38 is .3252, which wants .0081 of .3333 ($=\frac{1}{3}$), as the other Difference .3349 exceeded it by .0016. The 38th Year is therefore to be divided in the Ratio of 16 to 81; that is $\frac{16}{97}$ of a Year, or about 2 Months, is to be added to the 37 Years.

IV. To

IV.

To find at what Rate of Interest I ought to lay out a Sum S , so as it may encrease $\frac{1}{3}$ for Instance, or become $\frac{4}{3} S$ in 7 Years. Here the Fraction I am to look for among the Differences is $\frac{3}{4}$, or the Decimal .75; which is not to be found in *Tab. I.* or *III.* till after the limited Time of 7 Years. But in *Tab. V.* the Numbers against 6 and 7 Years give the Difference .7599; and the Rate is 4 *per Cent.* nearly.

To find how nearly; we may proceed as under the foregoing Rule. Take the Difference between 6 and 7 Years in *Tab. VII.* for 5 *per Cent.*; which being .7107, wanting .0393 of .75, as .7599 exceeded it by .0099; divide Unity in the Ratio of 99 to 393, that is of 33 to 131, and the lesser Part added to 4 *per Cent.* gives the Rate sought, $4 \frac{33}{131}$, or $4 \frac{1}{5}$.

PART II.

Containing the Demonstrations of some of the principal Propositions in the foregoing Treatise.

CHAPTER I.

I Observed formerly, that upon Supposition that the Decrements of Life were in Arithmetic Progression, the Conclusions derived from thence would very little vary from those, that could be deduced from the Table of Observations made at *Breslaw*, concerning the Mortality of Mankind; which Table was about fifty Years ago inserted by Dr. *Halley* in the *Philosophical Transactions*, together with some Calculations concerning the Values of Lives according to a given Age.

Upon the foregoing Principle, I supposed that if n represented the Complement of Life, the Probabilities of living 1, 2, 3, 4, 5, &c. Years, would be expressed by the following Series, $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, $\frac{n-4}{n}$, $\frac{n-5}{n}$, &c. and consequently that the Value of a Life, whose Complement is n , would be expressed by the Series

$\frac{n-1}{nr} + \frac{n-2}{nrr} + \frac{n-3}{nrr^2} + \frac{n-4}{nrr^3} + \frac{n-5}{nrr^4}$, &c. the Sum of which I have

asserted in *Problem I.* to be $\frac{1 - \frac{r}{n}P}{r-1}$, where the Signification of the Quantities P and r is explained.

As the Reasonings that led me to that general Expression, require something more than an ordinary Skill in the Doctrine of Series, I shall forbear to mention them in this Place; and content myself with pointing out to the Reader a Method, whereby he may satisfy himself of the Truth of that Theorem, provided he understand so much of a Series, as to be able to sum up a Geometric Progression.

DEMON-

DEMONSTRATION.

$$P = \frac{1}{r} + \frac{1}{rr} + \frac{1}{r^3} + \frac{1}{r^4} + \frac{1}{r^5} \dots \frac{1}{r^n}.$$

Therefore,

$$rP = 1 + \frac{1}{r} + \frac{1}{rr} + \frac{1}{r^3} + \frac{1}{r^4} \dots + \frac{1}{r^{n-1}}.$$

And

$$\frac{rP}{n} = \frac{1}{n} + \frac{1}{nr} + \frac{1}{nrr} + \frac{1}{nr^3} + \frac{1}{nr^4} \dots + \frac{1}{nr^{n-1}}.$$

Therefore,

$$1 - \frac{rP}{n} = \frac{n-1}{n} - \frac{1}{nr} - \frac{1}{nrr} - \frac{1}{nr^3} - \frac{1}{nr^4} - \dots - \frac{1}{nr^{n-1}}.$$

But this is to be divided by $r-1$, or multiplied by

$$\frac{1}{r-1} = \frac{1}{r} + \frac{1}{rr} + \frac{1}{r^3} + \frac{1}{r^4} + \frac{1}{r^5} + \frac{1}{r^6} \dots \text{&c.}$$

Then multiplying actually those two Series's together, the Product will be found to be

$$\begin{aligned} & \frac{n-1}{nr} - \frac{1}{nrr} - \frac{1}{nr^3} - \frac{1}{nr^4} - \frac{1}{nr^5} - \frac{1}{nr^6} \text{&c.} \\ & + \frac{n-1}{nrr} - \frac{1}{nr^3} - \frac{1}{nr^4} - \frac{1}{nr^5} - \frac{1}{nr^6} \text{&c.} \\ & + \frac{n-1}{nr^3} - \frac{1}{nr^4} - \frac{1}{nr^5} - \frac{1}{nr^6} \text{&c.} \\ & + \frac{n-1}{nr^4} - \frac{1}{nr^5} - \frac{1}{nr^6} \text{&c.} \\ & + \frac{n-1}{nr^5} - \frac{1}{nr^6} \text{&c.} \\ & + \frac{n-1}{nr^6} \text{&c.} \end{aligned}$$

And adding the Terms of the perpendicular Columns together, we shall have $\frac{n-1}{nr} + \frac{n-2}{nrr} + \frac{n-3}{nr^3} + \frac{n-4}{nr^4} + \frac{n-5}{nr^5} + \frac{n-6}{nr^6} \text{&c.}$

which consequently is equal to $\frac{1 - \frac{r^n}{n}}{r-1}$: which was to be demonstrated.

If it be required that upon the Failing of a Life, such Part of the Annuity should be paid, as may be proportional to the Time elapsed from the Beginning of the last Year, to the Time of the Life's failing, then the Value of the Life will be $\frac{1}{r-1} - \frac{1}{an}P$, wherein a represents the hyperbolic Logarithm of the Rate of Interest.

But

But because there are no Tables printed of hyperbolic Logarithms, and that the Reduction of a common Logarithm to an hyperbolic is somewhat laborious, it will be sufficient here to set down the hyperbolic Logarithms of 1.03, 1.035, 1.04, 1.05, 1.06, which are respectively, 0.02956, 0.0344, 0.03922, 0.04879, 0.05825, or $\frac{1}{35}$, $\frac{1}{31}$, $\frac{2}{51}$, $\frac{2}{41}$, $\frac{6}{103}$ nearly.

CHAPTER II.

Explaining the Rules of combined Lives.

Supposing a fictitious Life, whose Number of Chances to continue in being from Year to Year, are constantly equal to a , and the Number of Chances for failing are constantly equal to b , so that the Odds of its continuing during the Space of any one Year, be to its failing in the same Interval of Time constantly as a to b , the Value of an Annuity upon such a Life would be easily found.

For, if we make $a+b=s$, the Probabilities of living 1, 2, 3, 4, 5, &c. Years would be represented by the Series $\frac{a}{s}$, $\frac{aa}{ss}$, $\frac{a^3}{s^3}$, $\frac{a^4}{s^4}$, $\frac{a^5}{s^5}$, &c. continued to Eternity; and consequently the Value of an Annuity upon such Life would be expressed by this new Series $\frac{a}{sr} + \frac{aa}{ssrr} + \frac{a^3}{s^3r^3} + \frac{a^4}{s^4r^4}$ &c. which being a geometric Progression perpetually decreasing, the Sum of it will be found to be $\frac{a}{sr-a}$: thus, if a stands for 21, and b for 1, and also r for 1.05, the Value of such Life would be ten Years Purchase.

From these Premises the following Corollaries may be drawn:

COROLLARY I.

An Annuity upon a fictitious Life being given, the Probability of its continuing one Year in being is also given; for let the Value be $=M$, then $\frac{a}{f} = \frac{Mr}{M+1}$.

COROLLARY II.

If a Life, whose Value is deduced from our Tables is found to be worth 10 Years Purchase, then such Life is equivalent to a fictitious Life, whose Number of Chances for continuing one Year, is to the Number of Chances for its failing in that Year, as 21 to 1.

COROL-

COROLLARY III.

Wherefore having taken the Value of a Life from our Tables, or calculated it according to the Rules prescribed; we may transfer the Value of that Life to that of a fictitious Life, and find the Number of Chances it would have for continuing or failing Yearly.

COROLLARY IV.

And the Combination of two or more *real Lives* will be very near the same as the Combination of so many corresponding *fictitious Lives*; and therefore an Annuity granted upon one or more real Lives, is nearly of the same Value as an Annuity upon a fictitious Life.

These things being premised, it will not be difficult to determine the Value of an Annuity upon two or three, or as many joint Lives as may be assigned.

For let x represent the Probability of one Life's continuing from Year to Year, and y the Probability of another Life's continuing the same Time; then according to the Principles of the Doctrine of Chances, the Terms

$$xy, xxyy, x^3y^3, x^4y^4, x^5y^5, \&c.$$

will respectively represent the Probabilities of continuing together, 1, 2, 3, 4, 5, &c. Years; and the Value of an Annuity upon the two joint Lives, will be $\frac{xy}{r} + \frac{xxyy}{rr} + \frac{x^3y^3}{r^3} + \frac{x^4y^4}{r^4} + \frac{x^5y^5}{r^5} \&c.$

which being a Geometrical Progression perpetually decreasing, the Sum of it will be found to be $\frac{xy}{r-xy}$: let now M be put for the Value of the first Life, and P for the Value of the second, then by our

first Corollary it appears that $x = \frac{Mr}{M+1}$, and $y = \frac{Pr}{P+1}$; and there-

fore having written these Values of x and y in the Expression $\frac{xy}{r-xy}$, which is the Value of the two joint Lives, it will be changed into

$\frac{MPr}{M+1 \times P+1 - MPr}$: which is the same Theorem that I had given in my first Edition.

It is true that in the Solution of *Prob. II.* I have given a Theorem which seems very different from this; making the Value of the joint Lives to be $\frac{MP}{M+P-dMP}$, wherein d represents the Interest of 1 *l.* and yet I may assure the Reader, that this last Expression is originally derived from the first; and that whether one or the

other is used, the Conclusions will very little differ: but the first Theorem is better adapted to Annuities paid in Money, it being customary that the last Payment, whether it be Yearly or Half-Yearly, is lost to the Purchaser; whereas the second Theorem is better fitted to Annuities paid by a Grant of Lands, whereby the Purchaser makes Interest of his Money to the last Moment of his Life: for which Reason I have chose to use the last Expression in my Book.

By following the same Method of Investigation, we shall find that if M, P, Q , denote three single Lives, an Annuity upon those joint Lives will be $\frac{MPQrr}{M+1 \times P+1 \times Q+1 - MPQrr}$, in the Case of Annuities payable in Money; or $\frac{MPQ}{MP+MQ+PQ-2dMQ}$, in the Case of Annuities paid by a Grant of Lands.

CHAPTER III.

Containing the Demonstration of the Rules given in Problems 4th and 5th, for determining the Value of longest Life.

Let x and y represent the respective Probabilities which two Lives have of continuing one Year in being, therefore $1-x$ is the Probability of the first Life's failing in one Year, and $1-y$ the Probability of the second Life's failing in one Year: Therefore multiplying these two Probabilities together, the Product $1-x-y+xy$ will represent the Probability of the two Lives failing in one Year; and if this be subtracted from Unity, the Remainder $x+y-xy$ will express the Probability of one at least of the two Lives outliving one Year: which is sufficient for establishing the first Year's Rent.

And, for the same Reason $xx+yy-xyy$ will express the Probability of one at least of the two Lives outliving two Years: which is sufficient to establish the second Year's Rent.

From the two Steps we have taken, it plainly appears that the longest of two Lives is expressible by the three following Series;

$$\left. \begin{array}{l} \frac{x}{r} + \frac{xx}{rr} + \frac{x^3}{r^3} + \frac{x^4}{r^4} + \frac{x^5}{r^5} \\ + \frac{y}{r} + \frac{yy}{rr} + \frac{y^3}{r^3} + \frac{y^4}{r^4} + \frac{y^5}{r^5} \\ - \frac{xy}{r} - \frac{xyy}{rr} - \frac{x^3y^3}{r^3} - \frac{x^4y^4}{r^4} - \frac{x^5y^5}{r^5} \end{array} \right\} \&c.$$

Whereof

Whereof the first represents an Annuity upon the first Life, the second an Annuity upon the second Life, and the third an Annuity upon the two joint Lives; and therefore we may conclude that an Annuity upon the longest of two Lives, is the Difference between the Sum of the Values of the single Lives, and the Value of the joint Lives: which have been expressed in *Problem IV.* by the Symbols $M+P-\overline{MP}$.

In the same manner it will be found that if x, y, z , represent the respective Probabilities of three Lives continuing one Year, then the Probability of their not failing all three in one Year will be expressed by $x+y+z-xy-xz-yz+xyz$; which is sufficient to ground this Conclusion, that an Annuity upon the longest of three Lives, is the Sum of the single Lives, *minus* the Sum of the joint Lives, *plus* the three joint Lives: which has been expressed by me, by the Symbols $M+P+Q-\overline{MP}-\overline{MQ}-\overline{PQ}+\overline{MPQ}$.

From the foregoing Conclusions, it is easily perceived how the Value of the longest of any Number of Lives ought to be determined; *viz.* by the Sum of the Values of the single Lives, *minus* the Sum of the Values of all the joint Lives taken two and two, *plus* the Sum of all the joint Lives taken three and three, *minus* the Sum of all the joint Lives taken four and four, and so on by alternate Additions and Subtractions.

CHAPTER IV.

Containing the Demonstrations of what has been said concerning Reversions, and the Value of one Life after one or more Lives.

1°. It plainly appears that the present Value of a Reversion after one Life, is the Difference between the Perpetuity, and the Value of the Life in Possession: Thus, if the Life in Possession be worth 14 Years Purchase, and that I have the Reversion after that Life, and have a mind to sell it, I must have for it 11 Years purchase, which is the Difference between the Perpetuity 25, and 14 the Value of the Life, when Money is rated at 4 *per Cent.*

2°. It is evident that the Reversion after two, three, or more Lives, is the Difference between the Perpetuity, and the longest of all the Lives.

But the Value of a Life after one or more Lives not being so obvious, I think it is proper to insist upon it more largely: let x therefore represent the Probability of the Expectant's Life continuing one Year in being, and y the Probability of the second Life's continuing also one Year in being, and therefore $1-y$ is the Probability of that second Life's failing in that Year; from which it follows, according to the Doctrine of Chances, that the Probability of the first Life's continuing one Year, and of the second's failing in that Year, is $x \times 1-y$, or $x-xy$; which is a sufficient foundation for drawing the following Conclusion, *viz.* that the Value of the first Life after the second is the Value of that first Life *minus* the Value of the two joint Lives: which I have expressed by the Symbols $M-\overline{MP}$.

In the same manner, if x, y, z , represent the respective Probabilities of three Lives continuing one Year, then $x \times 1-y \times 1-z$, will represent the Probability of the first Life's continuing one Year, and of the other two Lives failing in that Year; but the foregoing Expression is brought, by actual Multiplication, to its Equivalent $x-xy-xz+xyz$; from whence can be deduced by meer Inspection the Rule given in *Prob. VIII.* *viz.* that the present Value of the first Life's Expectation after the Failing of the other two, is $M-\overline{MP}-\overline{MQ}+\overline{MPQ}$.

CHAPTER V.

Containing the Demonstration of what has been asserted in the Solution of the 10th and 29th Problems.

In the Solution of the 10th Problem, M''' denoting the present Value of an Annuity to continue so long as three Lives of the same Age subsist together, let us suppose that n denotes the Number of Years during which the Annuity will continue; then supposing r to express the Rate of Interest, it is well known that the present Value of that

Annuity will be $\frac{1-\frac{1}{r^n}}{r-1}$, wherefore we have the Equation $M''' = \frac{1-\frac{1}{r^n}}{r-1}$, or making $r-1=d$, $M''' = \frac{1-\frac{1}{r^n}}{d}$, from whence will be deduced $\frac{1}{r^n} = 1-dM'''$, and consequently $r^n = \frac{1}{1-dM'''}$.

Now let us suppose that a Sum f is to be received to eternity at the equal

equal Intervals of Time, denoted by n , and that we want to find the present Value of it; it is plain to those who have made some Proficiency in Algebra, that $\frac{f}{r^n - 1}$ is the present Value of it, let us therefore in the room of r^n substitute its Value found before, *viz.* $\frac{1}{1 - dM''}$, and then $r^n - 1$ will be found equal to $\frac{dM''}{1 - dM''}$, and consequently $\frac{f}{r^n - 1} = \frac{1 - dM''}{dM''} \times f$: as in the Solution of *Prob. X.*

Now it will be easy to find n ; for let us suppose $\frac{1}{1 - dM''} = T$, then $r^n = T$, and therefore $n = \frac{\log. T}{\log. r}$.

The 29th Problem has some Affinity with the 10th; in the former it was required to know the present Value of a Sum f , payable at the Failing of any one of three equal Lives, but in the latter the three Lives are supposed unequal; but besides, it is extended to two other Cases, *viz.* to the present Value of a Sum f to be paid after the Failing of any two of the Lives, as also to the present Value of a Sum f to be paid after the Failing of the three Lives.

For in the first Case, let us imagine an Annuity to be paid as long as the three Lives are in being; or, which is the same thing, till one of the Lives fails; and let us suppose that R represents the Value of the three joint Lives; let us also suppose that n is the Number of Years after which this will happen, and that d is the Interest of 1 *l.* therefore $\frac{f}{r^n}$ is the present Value of the Sum f to be then paid; but

$$R = \frac{1 - \frac{1}{r^n}}{d}, \text{ therefore } \frac{1}{r^n} = 1 - dR, \text{ and therefore } \frac{f}{r^n} = \frac{f}{1 - dR} \times f.$$

But the second Case has something more of Difficulty, and therefore I shall enlarge a little more upon it: let us imagine now that there is an Annuity to continue not only as long as the three equal Lives are in being, but as long as any two of the said Lives are in being; now in order to find the present Value of the said Annuity, let us suppose that x, y, z , represent the respective Probabilities of the said Lives continuing one Year. Therefore.

1°. $x y z$ represents the Probability of their all outliving the Year.

2°. $xy \times 1 - z$, or $xy - xyz$ represents the Probability of the two first outliving the Year, and of the third failing in that Year.

3°. $xz \times \overline{1-y}$ or $xz - xyz$ represents the Probability of the first and third's outliving the Year, and of the second's failing in that Year.

4°. $yx \times \overline{1-z}$, or $yz - xyz$ represents the Probability of the second and third's outliving the Year, and of the first's failing in that Year.

Then adding those several Products together, their Sum will be found equal to $xy + xz + yz - 2xyz$, which is an Indication that the present Value of an Annuity to continue as long as two of the said Lives are in being is $\overline{MP} + \overline{MQ} + \overline{PQ} - 2\overline{MPQ}$, which we may suppose $= T$.

Let us now compare this with an Annuity certain to continue n Years, the Rate of Interest being supposed $= r$, and $r - 1 = d$,

then we shall have the Equation $\frac{1 - \frac{1}{r^n}}{d} = T$, from whence we shall find $\frac{1}{r^n} = 1 - dT$, and consequently $\frac{f}{r^n}$, which is the present Value of the Expectation required, is $= \overline{1 - dT} \times f$.

By the same Method of Process, we may find the present Value of an Annuity to continue so long as any one of the three Lives in question is subsisting; for let x, y, z , represent the same things as before.

1°. xyz represents the Probability of the three Lives outliving the first Year.

2°. $xy + xz + yz - 3xyz$ represents the Probability of two of them outliving the Year, and of the third's failing in that Year.

3°. $x \times \overline{1-y} \times \overline{1-z}$, or $x - xy - xz + xyz$ represents the Probability of the first Life's outliving the Year, and of the other two failing in that Year.

4°. $y \times \overline{1-x} \times \overline{1-z}$, or $y - xy - yz + xyz$ represents the Probability of the second Life's outliving the Year, and of the other two failing in that Year.

5°. $z \times \overline{1-x} \times \overline{1-y}$, or $z - xz - yz + xyz$ represents the Probability of the third Life's outliving the Year, and of the other two failing in that Year.

Now the Sum of all this is $x + y + z - xy - xz - yz + xyz$; which is an Indication that the Value of an Annuity to continue as long as any one of three Lives is in being ought to be expressed by $\overline{M} + \overline{P} + \overline{Q} - \overline{MP} - \overline{MQ} - \overline{PQ} + \overline{MPQ}$: and this last Case may be looked upon as a Confirmation of the Rule given in our 5th Problem.

CHAPTER VI.

Containing the Demonstration of what has been said concerning successive Lives in the Solution of Prob. XIII.

What has been there said amounts to this; The present Values of Annuities certain for any particular Number of Years being given, to find the present Value of an Annuity to continue as long as the Sum of those Years.

Let us suppose that M represents the present Value of an Annuity to continue n Years, and that P represents the present Value of an Annuity to continue p Years; the first Question is, how from these *Data* to find the present Value of an Annuity to continue $n + p$ Years, the Investigation of which is as follows: let r be the Rate of Interest, and suppose $r - 1$ which denotes the Interest of 1 $l. = d$;

then, 1°. $M = \frac{1 - \frac{1}{r^n}}{d}$, therefore $\frac{1}{r^n} = 1 - dM$; and for the same Reason $\frac{1}{r^p} = 1 - dP$. Therefore $\frac{1}{r^{n+p}} = \frac{1}{1 - dM} \times \frac{1}{1 - dP} = \frac{1}{1 - dM - dP + ddMP}$. Let now f be supposed to be the Value of the Annuity which is to continue $n + p$ Years, then $\frac{1}{r^{n+p}} = 1 - df$. Therefore $1 - df = 1 - dM - dP + ddMP$; then subtracting Unity on both Sides, dividing all by d , and changing the Signs, we shall have $f = M + P - dMP$.

2°. By the same Method of Process, it will be easy to find that if M, P, Q , represent Annuities to continue for the respective Number of Years n, p, q , then the Value of an Annuity to continue $n + p + q$ Years will be $M + P + Q - dMP - dMQ - dPQ + ddMPQ$: the Continuation of which is obvious.

Let us now suppose that the Intervals n, p, q , are equal, then the Values M, P, Q , are also equal; in which Case, the foregoing Canon will be changed into this, $3M - 3dMM + d^2M^2$, or $\frac{3dM - 3ddMM + d^2M^2}{d}$; but if this Numerator be subtracted from Unity, the Remainder will be $1 - 3dM + 3ddMM - d^2M^2 = 1 - dM^3$; and subtracting this again from Unity, the original Numerator will be restored, and will be equivalent to $1 - 1 - dM^3$, and consequently, if M represents the Value of an Annuity to continue

tinue a certain Number of Years, then $\frac{1 - \sqrt[n]{1 - dM^3}}{d}$ will represent the Value of an Annuity to continue three times as long.

And universally, if M stands for the Value of an Annuity to continue a certain Number of Years, then $\frac{1 - \sqrt[n]{1 - dM^n}}{d}$ will represent the Value of an Annuity to continue n times as long.

And if n were infinite, I say that $1 - \sqrt[n]{1 - dM^n}$ would be $= 0$; from whence the Value would be $= \frac{1}{d}$ or $\frac{1}{r-1}$, which represents the Value of the Perpetuity.

But that there may remain no scruple about what we have asserted above, that in the Case of n being infinite, $1 - \sqrt[n]{1 - dM^n}$ would vanish; I prove it thus, $\frac{1}{d} > M$, therefore $1 > dM$, therefore $1 - dM$ is a Fraction less than Unity: now it is well known that a Fraction less than Unity being raised to an infinite Power, is nothing, and was therefore safely neglected.

CHAPTER VII.

Containing the Demonstration of what has been asserted in the 32d and 33d Problems concerning half-yearly Payments; as also the Investigation of some Theorems relating to that Subject.

It is well known that if an Annuity A is to continue n Years, the

present Value of it is $A - \frac{A}{r^n}$; supposing r to represent the Rate of Interest; now to make a proper Application of this Theorem to half-yearly Payments, I look upon n as representing indifferently the Number of Payments and the Number of Years; let us now suppose a half-yearly Rent B of the same present Value as the former, and to continue as long, then the Number of Payments in this Case will be $2n$, but the Rate of Interest, instead of being r , is now $r^{\frac{1}{2}}$, which being raised to the Power $2n$, will be r^n as before; for which

Reason the present Value of the half-yearly Payments is $B - \frac{B}{r^{\frac{n}{2}} - 1}$:

but by Hypothesis, the present Values of the yearly and half-yearly Pay-

Payments are the same; therefore $\frac{A - \frac{A}{r^n}}{r-1} = \frac{B - \frac{B}{r^{\frac{n}{2}}}}{r^{\frac{1}{2}}-1}$, and dividing both sides of the Equation by $1 - \frac{1}{r^n}$, we shall have $\frac{A}{r-1} = \frac{B}{r^{\frac{1}{2}}-1}$, from whence will be deduced $B = \frac{r^{\frac{n}{2}}-1}{r-1} \times A$: and in the same manner, if the Payments were to be made quarterly, then B would be $= \frac{r^{\frac{n}{4}}-1}{r-1} \times A$; and so on.

But if we suppose that a Rent shall be paid half-yearly, and that it shall be also one half of what would be given for an annual Rent, and that the two Rents shall be of the same Duration; then the present Values of the yearly and half-yearly Rents will be different: for let M and P be the present Values of the yearly and half-yearly

Rents, then $M = \frac{A - \frac{A}{r^n}}{r-1}$, and $P = \frac{\frac{1}{2}A - \frac{\frac{1}{2}A}{r^{\frac{n}{2}}}}{r^{\frac{1}{2}}-1}$, and dividing both Values by $A - \frac{A}{r^n}$, we shall have $M, P :: \frac{1}{r-1}, \frac{\frac{1}{2}}{r^{\frac{1}{2}}-1}$; and consequently $P = \frac{\frac{1}{2} \times r-1}{r^{\frac{1}{2}}-1} \times M$.

The two last Problems bring to my Mind an Assertion which was maintained, about six Years ago, in a Pamphlet then published; which was that it would be of great Advantage to a Person who pays an Annuity, to discharge it by half-yearly Payments, each of one half the Annuity in Question: the Reason of which was, that then the time of paying off the Principal would be considerably shortened. I had not the Curiosity to read the Author's Calculation, because I thought it too long; since which Time I thought fit to examine the thing, and found that indeed the Time would be shortened, but not so considerably as the Author imagined: which to prove, I supposed a Principal of 2000 *l.* an Annuity of 100 *l.* and the Rate of Interest 1.04: in consequence of which, I found that the Principal would be discharged in 41 Years; this being founded on the general Theorem

$\frac{A - \frac{A}{r^n}}{r-1} = P$, in which A represents the Annuity, P the Principal, r the Rate of Interest, and n the Number of Years: now to apply this to the Case of half-yearly Payments, let us suppose that p denotes the Number of Years in which the Principal will be discharged; therefore $2p$ will be the Number of Payments, $\frac{1}{2}A$ the Annuity, and $r^{\frac{1}{2}}$ the Rate of Interest: which being respectively substituted in the Room of

n, A, r , we shall have now $\frac{\frac{1}{2}A - \frac{\frac{1}{2}A}{r^p}}{r^{\frac{1}{2}} - 1} = P$, but $r^{\frac{1}{2}} - 1 = 0.019804$, which being supposed $= m$, we shall have $\frac{1}{2}A - \frac{\frac{1}{2}A}{r^p} = m P$, and $\frac{\frac{1}{2}A}{r^p} = \frac{1}{2}A - m P$, or $\frac{50}{r^p} = 10.392$; therefore $\frac{r^p}{50} = \frac{1}{10.392}$, or $r^p = \frac{50}{10.392}$, and $p \log. r = \log. 50 - \log. 10.392 = 0.6822709$, therefore $p = \frac{0.6822709}{\log. r}$; again, $\log. r = 0.0170333$, therefore $p = \frac{0.6822709}{0.0170333} = 40.05$: and therefore the Advantage of paying half-yearly would amount to no more than gaining one Year in 41.

Quarterly Payments, or half-quarterly, nay even Payments made at every Instant of Time, would not much accelerate the Discharge of the Principal. Which to prove, let us resume once more our ge-

neral Theorem $\frac{A - \frac{A}{r^n}}{r - 1} = P$; let us now imagine that the Number of Instants in the Year is $= t$, let us further suppose that s is the Number of Years in which the Principal will be discharged, then in the room of A , writing $\frac{1}{t}A$; in the room of r , writing $r^{\frac{1}{t}}$; and in the room of n , writing $s t$, we shall have $\frac{\frac{1}{t}A - \frac{\frac{1}{t}A}{r^{\frac{1}{t}}}}{r^{\frac{1}{t}} - 1} = P$. But it is known, that if t represents an infinite Number, such as is the Number of Instants in one Year, then $r^{\frac{1}{t}} - 1 = \frac{1}{t} \log. r$, we have

therefore $\frac{\frac{1}{t}A - \frac{\frac{1}{t}A}{r^{\frac{1}{t}}}}{\frac{1}{t} \log. r} = P$, or $\frac{A - \frac{A}{r^{\frac{1}{t}}}}{\log. r} = P$; let the Logarithm of r

be supposed $= a$, therefore $A - \frac{A}{r^{\frac{1}{t}}} = a P$, and $\frac{A}{r^{\frac{1}{t}}} = A - a P$, and $r^{\frac{1}{t}} = \frac{A}{A - a P}$, which suppose $= Q$, then $s = \frac{\log. Q}{\log. r}$: But it is to be noted, that a represents the hyperbolic Logarithm of r , which is, as we have seen before, 0.0392207 when r stands for 1.04; this being supposed, the Logarithm of Q will be found to be 0.6663794, which being divided by the Logarithm of r viz. 0.0170333, the

Quotient

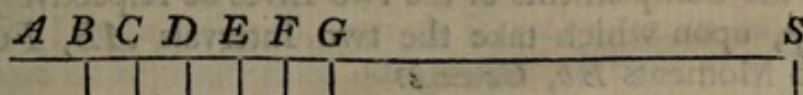
Quotient will be 39,1 Years; but in this last Operation the Logarithms of Q and r , may be taken out of a common Table.

CHAPTER VIII.

Containing the Demonstration of what has been said concerning the Probabilities of Survivorship.

What I call Complement of Life having been defined before *pag.* 265. I shall proceed to make use of that Word as often as occasion shall require.

HYPOTHESIS.



Let it be supposed that the Complement of Life AS being divided into an infinite Number of equal Parts representing Moments, the Probabilities of living from A to B , from A to C , from A to D , &c. are respectively proportional to the several Complements SB , SC , SD , in so much that these Probabilities may respectively be represented by the Fractions $\frac{SB}{SA}$, $\frac{SC}{SA}$, $\frac{SD}{SA}$, &c. This Hypothesis being admitted the following Corollaries may be deduced from it.

COROLLARY I.

The Probability of Life's failing in any Interval of Time AF is measured by the Fraction $\frac{FA}{SA}$.

COROLLARY II.

When the Interval AF is once past, the Probability of Life's continuing from F to G is $\frac{SG}{SF}$, for at F , the Complement of Life is SF , and the Probability of its failing is $\frac{FG}{SF}$.

COROLLARY III.

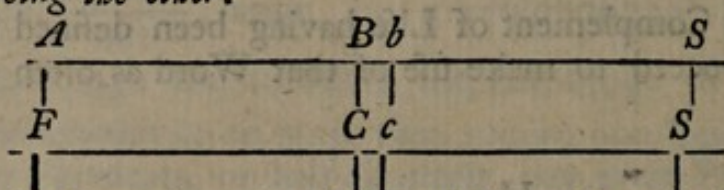
The Probability of Life's continuing from A to F , and then failing from F to G , is $\frac{SF}{SA} \times \frac{FG}{SF} = \frac{FG}{SA}$.

COROLLARY IV.

The Probability of Life's failing in any two or more equal Intervals of Time assigned between A and S are exactly the same, the Estimation being made at A considered as the present Time.

These things premised, it will not be difficult to solve the following Problem.

Two Lives being given, to find the Probability of one of them fixed upon, surviving the other.



For, let the Complements of the two Lives be respectively $AS=n$ and $FS=p$, upon which take the two Intervals AB , $FC=z$, as also the two Moments Bb , $Cc=z$.

The Probability of the first Life's continuing from A to B , or beyond it, is $\frac{n-z}{n}$; the Probability of the second's continuing from F to C , and then failing in the Interval Cc , is by the third *Corollary* $\frac{z}{p}$; therefore the Probability of the first Life's continuing during the time AB or beyond it, and of the second's failing just at the end of that Time, is measured by $\frac{n-z}{n} \times \frac{z}{p} = \frac{nz-zz}{np}$, whose Fluent $\frac{nz - \frac{1}{2}zz}{np}$ will express the Probability of the first Life's continuing during any Interval of Time or beyond it, and of the second's failing any time before or precisely at the end of that Interval.

Let now p be written instead of z , and then the Probability of the first Life's surviving the second, will be $\frac{np - \frac{1}{2}pp}{np} = 1 - \frac{\frac{1}{2}p}{n}$.

From the foregoing Conclusion we may immediately infer that the Probability of the second Life's surviving the first is $\frac{\frac{1}{2}p}{n}$.

By the same method of arguing, we may proceed to the finding the Probability of any one of any Number of given Lives surviving all the rest, and thereby verifying what we have said in *Prob. XVIII.* and *XIX.*

CHAPTER

CHAPTER IX.

Serving to render the Solutions in this Treatise more general, and more correct.

I.

Altho', in treating this subject of Annuities, I have made use only of Dr. *Halley's* Table, founded upon the *Breslaw* Bills of mortality; from which I deduced the *Hypothesis* of an equable Decrement of Life: Yet are my Rules easily applicable to any other Table of Observations; by *Prob. II.* of my Letter to Mr. *Jones* in *Phil. Trans.* N^o. 473, which the Reader may see below, in the *Appendix*.

Or instead of the Theorem there given, he may use that by which *Prob. XXVI.* was resolved, which is rather more independent of *Tables*: And its application to our present purpose may be explained as follows.

As in all *Tables* of Observations deduced from Bills of mortality, or if we should combine several of them into one, it will be found that, for certain Intervals at least, *the Decrements of Life continue nearly the same*; if we conceive the whole Extent of Life to be represented by a right Line *AZ*, in which there are taken distances *PQ*, *QR*, *RS*, &c. proportional to those Intervals, and at the points *P*, *Q*, *R*, *S*, &c. there be erected perpendiculars proportional to the Numbers of the Living at the beginning of the respective Intervals, and their Extremities are connected by right Lines; then there will be formed a *Polygon* Figure on the Base *AZ*, whose Ordinates will every where represent the Numbers of that *Table* from which the Figure was constructed; and the Inclinations of the Sides of the *Polygon* to its Base will express the Convergencies of Life to its End, or the Degrees of Mortality belonging to the respective Intervals.

Say therefore, as the difference of the Ordinates at *P* and *Q*, is to the Ordinate at *P*: so is the Interval *PQ*, to a fourth *PZ'*; and *PZ'* shall be the *Complement* of Life at the age *P*; and the Point *Z'* in the Base shall be that from which the Complements are to be reckoned throughout the Interval *PQ*.

Let *PZ'*, thus found, be substituted for *n* in the *Canon* of *Prob. XXVI.* and the Interval *PQ* for *m*, so shall the *Value* of that Interval be known: and in like manner the subsequent Values of *QR*, *RS*, &c. giving to each Interval its proper Complement *QZ''*, *RZ'''*, &c. And

And lastly, these Values being severally discounted, *First*, in the Ratio of their respective Ordinates at P , Q , R , &c. to some preceding Ordinate as at N , at the Age 12, for instance; and *Secondly*, by the present Value of 1*l*. payable after the Years denoted by NP , NQ , NR , &c. their Sum will be the *Value* of the Life at N , according to the given Table of Observations. After which, the younger Lives must be computed from Year to Year: as those after 70, or when an Interval contains but one Year, ought likewise to be computed.

If it is proposed, for Example, to find how nearly my *Hypothesis* agrees with Dr. *Halley's* Table for the Interval of 8 Years between 33 and 41, it's Value, at 5 *per Cent.* computed by *Prob. XXVI.* will, to an Annuitant 33 Years old, be 5.9456, according to the *Hypothesis*. But the Numbers of the Living at those Ages being, in the Table, 507 and 436, if we compute immediately from it, we must take $n = \frac{507}{71} \times 8 = 57.14$; and the same Rule will give the Value 5.9831. Discount now the Values found as belonging to a Life of 12 Years; that is multiply the first by $\frac{53}{74}$, and the other by $\frac{507}{646}$; and the Products 4.2583 and 4.6957 discounted the *second* time, that is, multiplied by .3589, the present Value of 1*l*. payable after 21 ($= 33 - 12$) Years gives the Values 1.5283 and 1.6853; the difference being 0.157, near $\frac{1}{6}$ of a Year's purchase.

In general, the *Hypothesis* will be found to give the Value of a *single* Life, or of an assigned Interval, somewhat below what the *Table* makes it: but then, as both the young and the middle aged are observed to die off faster in *England* than at *Breslaw*, my Rules may very well be preferable, for the Purchases and Contracts that are made upon single Lives in this Country.

In the same manner may any other *Tables* be compared with the *Hypothesis*, and with one another. And if we give the preference to any particular Table, and would at the same time retain the *Hypothesis* of equal Decrement we may, by the *differential Method*, easily find that *mean Termination* of Life, Z , which shall best correspond to the *Table*.

II.

To preserve somewhat of Elegance and Uniformity in my Solutions, as well as to avoid an inconvenient multiplicity of *Canons* and *Symbols*, I did transfer the Decrement of Life from an *Arithmetical* to a *Geometrical*

metrical Series: which however, in many Questions concerning *Combined Lives*, creates an error too considerable to be neglected. This hath not escaped the Observation of my Friends, no more than it had my own: but the same Persons might have observed likewise, that such Errors may, when it is thought necessary, be corrected by my own Rules; particularly upon this obvious principle, That, *if money is supposed to bear no Interest, the Values of Lives will coincide with what I call their Expectations.*

But as the Computation of such Corrections might seem tedious; and because practical Rules ought to be of ready Use, as well as sufficiently exact; I chuse rather to give another Rule for *joint Lives*, which will answer both these Purposes; at the same time that it is general, and easily retained in the Memory.

General Rule for the Valuation of joint Lives.

The given Ages being each increased by unity, find, by Problem XXI. or XXII. the Number of Years due to their joint Continuance; and the Complement of twice this Number to 86, taken as a single Life, will, in the proper Table, give nearly the Value required.

EXAMPLE I.

The Value of two joint Lives of 40 and 50, at 5 per Cent. was, in Prob. II. found to be 7.62. But if they are made 41 and 51, their joint *Expectation*, by Prob. XXI. will be 13 Years, these doubled and taken from 86 leave 60, against which in Table VIII, stands 8.39 Years purchase, nearly the Value sought.

EXAMPLE 2.

The 3 joint Lives whose single Values, at 4 per Cent. are 13, 14, 15 Years purchase, are in Prob. II. worth 7.41. But by Table VI, the Ages to which these Values belong, increased by Unity, are 42, 36, 28; whose Complements to 86 substituted for p, n, q , in $\frac{1}{2}p - \frac{p^2 \times n + q}{6nq} + \frac{p^3}{12nq}$, the Canon for the *Expectation* of 3 joint Lives, gives 12.43. And $86 - 2 \times 12.43$ is nearly 61; at which Age a single Life, in Table VI, is worth 8.75 Years purchase.

It is needless to add any thing concerning *longest Lives*, *Survivorships*, *Reversions* and *Insurances*; the Computation of their Values being

being only the combining those of *single* and *joint* Lives, by Addition and Subtraction: which being performed according to the Rules of this Treatise, the Answer may be depended upon as sufficiently exact, in all useful Questions that can occur. For we do not here aim at an Accuracy beyond what the determination of our main *Data*, the Probabilities of human Life, and the conformity of our Hypothesis to nature, can bear; nor do we give our Conclusions for perfectly exact, as is required in such as are *purely* arithmetical, but only as very near Approximations; upon which business may be transacted, without considerable Loss to any party concerned.

III.

The same Rule serves for the Case of an Annuity secured, upon *joint* Lives, by a Grant of Lands; or when the fractional part of the last Year is to be accounted for. Only, in this Case, 1°. *The Addition of Unity to each Life is to be omitted.* 2°. The single Life is not

now to be taken out of our Tables, or computed from $\frac{1 - \frac{r}{n} P}{r - 1}$ the Canon of *Prob. I*, but from $\frac{1}{r - 1} - \frac{1}{an} P$, a being *Neper's* Logarithm of r : as in *Phil. Trans.* N°. 473, and in *Chap. I.* foregoing.

According to which, if the Ages and Interest are as in Example 1; the *Expectation* of joint Life will be 13.3 Years; and thence $n = 26.6$; $P = 14.5358$; $a = .04879$: And the Value of the Annuity $20 - 11.2 = 8.8$; exceeding what it would have been upon yearly Payments by about $\frac{4}{10}$ of a Year's purchase.

And if the Payments are half yearly or quarterly, the skillful Computist cannot be at a loss after what has been said of those Cases in *Chap. VII* *.

* See, on the Subject of Annuities, *Mathem. Repository*, Vol. II. and III. by the ingenious Mr. *James Dodson*, F. R. S.

F I N I S.

A P P E N D I X.

Nº. I.

Dedication of the First Edition of this Work (1718.)

T O

Sir ISAAC NEWTON, Kt. President of the *Royal Society*.

S I R,

THE greatest Help I have received in writing upon this Subject having been from your incomparable Works, especially your Method of Series; I think it my Duty publickly to acknowledge, that the Improvements I have made, in the matter here treated of, are principally derived from yourself. The great benefit which has accrued to me in this respect, requires my share in the general Tribute of Thanks due to you from the learned World: But one Advantage which is more particularly my own, is the Honour I have frequently had of being admitted to your private Conversation; wherein the Doubts I have had upon any Subject relating to *Mathematics*, have been resolved by you with the greatest Humanity and Condescension. Those marks of your Favour are the more valuable to me, because I had no other pretence to them but the earnest desire of understanding your sublime and universally useful Speculations. I should think my self very happy, if having given my Readers a Method of calculating the Effects of Chance, as they are the result of Play, and thereby fixing certain Rules, for estimating how far some sort of Events may rather be owing to Design than Chance, I could by this small Essay excite in others a desire of prosecuting these Studies, and of learning from your Philosophy how to collect, by a just Calculation, the Evidences of exquisite Wisdom and Design, which appear in the *Phænomena* of Nature throughout the Universe. I am, with the utmost Respect,

Sir,

Your most humble,

and obedient Servant

U u

A. de MOIVRE.

Note upon Coroll. I. Prob. VII; and upon Prob. IX.

In that Corollary, it was found that the Probabilities of winning all each others Stakes being as $a^q \times \overline{a^p - b^p}$ and $b^p \times \overline{a^q - b^q}$; If we divide by $a - b$, and suppose the Chances for one Game to be equal, or $a = b$; then the Probabilities will be as the Number of pieces, or, in the Ratio of p to q .

But when we have to divide such Expressions continually, that is by some Power of $a - b$, as $a - b^2$, $a - b^4$, &c. it will be more convenient to use a *General Rule* for determining the Value of a Ratio whose Terms vanish by the contrariety of Signs. The Rule is this;

For the difference of the Quantities that destroy each other in any Case proposed, write an indeterminate Quantity x ; in the Result reject all those Terms that vanish when x becomes less than any finite Quantity: so shall the remaining homogeneous Terms, divided by their greatest common Measure, express the Ratio sought.

As in our example, if we make $a - b = x$, or $a = b + x$, and for a^p , a^q , write their equals $\overline{b + x}^p$, $\overline{b + x}^q$, expanded by the Binomial Theorem; the Ratio of R to S , in Prob. VII, will be reduced to that of $pb^{p+q-1} \times x + p \cdot \frac{p-1}{2} \times pq \times b^{p+q-2} \times x^2 + \&c.$ to $qb^{p+q-1} \times x + q \cdot \frac{q-1}{2} \times b^{p+q-2} \times x^2 + \&c.$ Of which retaining only the two Terms that involve x , and dividing them by $b^{p+q-1} \times x$, we get $\frac{R}{S} = \frac{p}{q}$.

The Solution of Prob. IX. gives for the Gain of A the Product $\frac{qa^q \times \overline{a^p - b^p} - pb^p \times \overline{a^q - b^q}}{a^{p+q} - b^{p+q}}$ by $\frac{aG - bL}{a - b}$: and when $a = b$, if we substitute as before, the Terms involving x vanish in the Numerator of the first of these Factors; reducing it to $* * + \frac{pq}{2} \times \overline{p + q} \times b^{p+q-2} \times x^2 + \&c.$ and the Denominator is $* \overline{p + q} \times b^{p+q-1} \times x + \&c.$ The other Factor is $\frac{b \times \overline{G - L} + xG}{x}$, or when x vanishes with respect to b , $\frac{b \times \overline{G - L}}{x}$; and the Product of the two is $pq \times \frac{G - L}{2}$;

as

as in *Case 2*. *Case 1* follows immediately from this; and the 3^d has as little difficulty.

Another Example of our Rule may be; *To find, from the Canon of Prob. I. of the Treatise on Annuities, the Expectation of a Life whose Complement is n; that is, the present Value of a Rent or Annuity upon that Life, money bearing no Interest.* Now that Canon being

$$\frac{1 - \frac{r}{n} P}{r - 1}, \text{ or } \frac{n - rP}{n \times r - 1}, \text{ if for } P \text{ we write its equal } \frac{1 - r^{-n}}{r - 1}, \text{ and } 1 + x$$

for r , the Value sought will be $\frac{n - n - r + r}{n \times r - 1} = ** +$

$$\frac{1 - n \times - \frac{n}{2} \times x^2 + \&c.}{n x^2} = \frac{n - 1}{2}.$$

This Value wants half a year of $\frac{n}{2}$, its quantity according to the Rule given above, *pag.* 288: because there the Probabilities of Life were supposed to decrease as the Ordinates of a Triangle; whereas, in the Hypothesis of yearly payments in *Prob. I.*, they decrease *per saltum*, like a Series of parallelograms inscribed in a Triangle.

The Reader will likewise observe that our general Rule for computing the Value of a Fraction whose form becomes $\frac{0}{0}$, is in effect the same as that given by the Marquis *de l'Hospital* in his *Analyse des infinimens petits*. And that, from the Number of Terms that vanish in the Operation, and from the *Sign* of the Term which determines the Ratio, the *Species* of algebraical Curve Lines, and the *Position* of their Branches, are discovered. See *Mac Laurin's Fluxions*, Book I. Chap. 9. and Book II. Chap. 5.

N^o. III.

Note to Prob. XLV. from Mr. Nicolas Bernoulli, Phil. Transf. 341.

To find the Probability that a *Poule* shall be ended in a given Number of Games: a Series of Fractions beginning with $\frac{1}{2^n - 1}$, whose Denominators increase in a double proportion, and the Numerator of each Fraction is the Sum of as many next preceding Numerators as there are Units in $n - 1$, will give the successive Probabilities that the *Poule* shall be ended *precisely* in $n, n + 1, n + 2, n + 3, \&c.$ Games; and consequently if as many Terms of this Series are added together, as there are units in $p + 1$, their Sum will express the Probability that the *Poule* shall be ended at least in $n + p$ Games. For Example, if there are 4 Players, and thence $n = 3$, we shall have this Series $\frac{1}{4}, \frac{1}{8}, \frac{2}{16}, \frac{3}{32}, \frac{5}{64}, \frac{8}{128}, \frac{13}{256}, \frac{21}{512}, \&c.$ Out of which if we form this other $\frac{1}{4}, \frac{3}{8}, \frac{8}{16}, \frac{19}{32}, \frac{43}{64}, \frac{94}{128}, \frac{201}{256}, \&c.$ whose Terms are the Sums of the Terms of former Series, these last will shew the Probability of the *Poule* ending in 3, 4, 5, 6, &c. Games, at least.

N°. IV.

A correct Table of the Sums of Logarithms, from the Author's Supplement to his Miscellanea Analytica.

10.	6.55976.30328.7678.	460.	1026.82368.84245.7267.
20.	18.38612.46168.7770.	470.	1053.50280.26009.6230.
30.	32.42366.00749.2572.	480.	1080.27422.85779.2496.
40.	47.91164.50681.5991.	490.	1107.13604.49151.6763.
50.	64.48307.48724.7209.	500.	1134.08640.85351.3508.
60.	81.92017.48493.9024.	510.	1161.12355.00246.5923.
70.	100.07840.50356.8004.	520.	1188.24576.93048.6770.
80.	118.85472.77224.9966.	530.	1215.45143.16339.6251.
90.	138.17193.57900.1086.	540.	1242.73896.39114.8380.
100.	157.97000.36547.1585.	550.	1270.10685.12561.5931.
110.	178.20091.76448.7008.	560.	1297.55363.38324.8209.
120.	198.82539.38472.1977.	570.	1325.07790.39038.2121.
130.	219.81069.31561.4815.	580.	1352.67830.30922.0491.
140.	241.12910.99886.9689.	590.	1380.35351.98269.6983.
150.	262.75689.34109.2616.	600.	1408.10228.69662.7898.
160.	284.67345.62406.8298.	610.	1435.92337.95771.1124.
170.	306.86078.19948.2847.	620.	1463.81561.28607.3923.
180.	329.30297.14247.9393.	630.	1491.77784.02119.6951.
190.	351.98588.98339.3535.	640.	1519.80895.14015.3428.
200.	374.89688.86400.4044.	650.	1547.90787.08720.1888.
210.	398.02458.26149.3624.	660.	1576.07355.61385.9540.
220.	421.35866.95421.3259.	670.	1604.30499.62866.2770.
230.	444.88978.26514.6048.	680.	1632.60121.05589.2142.
240.	468.60936.87056.4794.	690.	1660.96124.70260.3147.
250.	492.50958.63954.6190.	700.	1689.38418.13336.1091.
260.	516.58322.09826.1269.	710.	1717.86911.55213.0134.
270.	540.82361.20667.5295.	720.	1746.41517.69081.2925.
280.	565.22459.20470.1654.	730.	1775.02151.70397.9157.
290.	589.78043.33690.9860.	740.	1803.68731.06935.9463.
300.	614.48580.30437.7387.	750.	1832.41175.49371.5144.
310.	639.33572.32255.0106.	760.	1861.19406.82372.5655.
320.	664.32553.68741.5328.	770.	1890.03348.96156.3791.
330.	689.45087.77060.3828.	780.	1918.92927.78485.4396.
340.	714.70764.37846.5691.	790.	1947.88071.07073.5663.
350.	740.09197.42162.3279.	800.	1976.88708.42376.3542.
360.	765.60022.85067.1998.	810.	2005.94771.20741.9152.
370.	791.22896.82108.4658.	820.	2035.06192.47899.6883.
380.	816.97493.05636.3600.	830.	2064.22906.92766.7182.
390.	842.83506.38337.0506.	840.	2093.44850.81552.2793.
400.	868.80641.41777.2588.	850.	2122.71961.92143.1027.
410.	894.88621.38085.1630.	860.	2152.04179.48752.7013.
420.	921.07182.03166.5465.	870.	2181.41444.16819.4477.
430.	947.36071.70083.7526.	880.	2210.83697.98139.1145.
440.	973.75050.41416.4285.	890.	2240.30884.26218.5633.
450.	1000.23889.09583.9930.	900.	2269.82947.61838.1577.

If we would examine these Numbers, or continue the Table farther on, we have that excellent Rule communicated to the Author by Mr. *James Stirling*; published in his Supplement to the *Miscellanea Analytica*, and by Mr. *Stirling* himself in his *Methodus Differentialis*, Prop. XXVIII.

“ Let $x = \frac{1}{2}$ be the last Term of any Series of the natural Numbers 1, 2, 3, 4, 5 $x = \frac{1}{2}$; $a = .43429448190325$ the reciprocal of *Neper's* Logarithm of 10: Then three or four Terms of this Series $x \text{ Log. } x = ax = \frac{a}{2.122} + \frac{7a}{8.360x^3} - \frac{31a}{32.1260x^5} + \frac{127a}{128.1680x^7} - \&c.$ added to 0.399089934179, &c. which is half the Logarithm of a Circumference whose Radius is Unity, will be the Sum of the Logarithms of the given Series; or the Logarithm of the Product $1 \times 2 \times 3 \times 4 \times 5 - - - - \times x = \frac{1}{2}.$ ”

The Coefficients of all the Terms after the first two being formed as follows.

$$\text{Put } -\frac{1}{3.4} = A$$

$$-\frac{1}{5.8} = A + 3B$$

$$-\frac{1}{7.12} = A + 10B + 5C$$

$$-\frac{1}{9.16} = A + 21B + 35C + 7D$$

$$-\frac{1}{11.20} = A + 36B + 126C + 84D + 9E.$$

&c.

In which the Numbers 1, 1, 1, &c. 3, 10, 21, 36, &c. 5, 35, 126, &c. that multiply *A*, *B*, *C*, &c. are the alternate *Unciæ* of the odd Powers of a Binomial. Then the Coefficients of the several Terms will be $\frac{1}{2} \times A = -\frac{1}{2.12}$, $\left(\frac{1}{2}\right)^3 \times B = \frac{7}{8.360}$, $\left(\frac{1}{2}\right)^5 \times C = \frac{31}{32.1260}$, &c. See the general Theorem and Demonstration in Mr. *Stirling's* Proposition quoted above.

Nº. V.

Some Useful Cautions.

One of the most frequent occasions of Error in managing Problems of Chance, being to allow more or fewer Chances than really there are; but more especially in the first Case, for the fault lies commonly that way, I have in the Introduction taken great care to settle the Rules of proceeding cautiously in this matter; however it will not be amiss to point out more particularly the danger of being mistaken.

Suppose

Suppose therefore I have this Question proposed; There are two Parcels of three Cards, the first containing King, Queen, and Knave of Hearts, the second the King, Queen, and Knave of Diamonds, and that I were promised the Sum S , in case that in taking a Card out of each Parcel, I should take out either the King of Hearts, or the King of Diamonds, and that it were required I should determine the value of my Expectation.

If I reason in this manner; the Probability of taking out the King of Hearts is $\frac{1}{3}$, therefore $\frac{1}{3}f$ is my due upon that account; the Probability of taking out the King of Diamonds is also $\frac{1}{3}$, and therefore that part of my Expectation is $\frac{1}{3}f$ as the other was, and consequently my whole Expectation is $\frac{2}{3}f$; this would not be a legitimate way of reasoning: for I was not promised that in case I should take out both Kings, I should have the Sum $2f$, but barely the Sum f . Therefore we must argue thus; the Probability of taking out the King of Hearts is $\frac{1}{3}$, the probability of missing the King of Diamonds is $\frac{2}{3}$, and therefore the probability of taking out the King of Hearts, and missing the King of Diamonds is $\frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$, for which reason that part of my Expectation which arises from the probability of taking out the King of Hearts, and missing the King of Diamonds is $\frac{2}{9}f$; for the same reason that part of my Expectation which arises from the probability of taking the King of Diamonds and missing the King of Hearts is $\frac{2}{9}f$, but I ought not to be deprived of the Chance of taking out the two Kings of which the probability is $\frac{1}{9}$, and therefore the value of that Chance is $\frac{1}{9}f$; for which reason, the value of my whole Expectation is $\frac{2}{9}f + \frac{2}{9}f + \frac{1}{9}f = \frac{5}{9}f$ which is less by $\frac{1}{9}f$ than $\frac{2}{3}f$.

But suppose I were proposed to have $2f$ given me in case I took out both Kings, then this last Expectation would be $\frac{2}{9}f$, which would make the whole value of my Expectation to be $\frac{2}{9}f + \frac{2}{9}f + \frac{2}{9}f = \frac{6}{9}f = \frac{2}{3}f$.

One:

One may perceive by this single instance, that when two Events are such, that on the happening of either of them I am to have a Sum f , the probability of that Chance ought to be estimated by the Sum of the Probabilities of the happening of each, wanting the probability of their both happening.

But not to argue from particulars to generals. Let x be the probability of the happening of the first, and y the probability of the happening of the second, then $x \times 1 - y$ or $x - xy$ will represent the probability of the happening of the first and failing of the second, and $y \times 1 - x$ or $y - xy$ will represent the probability of the happening of the second and failing of the first, but xy represents the happening of both; and therefore $x - xy + y - xy + xy$ or $x + y - xy$ will represent the probability of the happening of either.

This conclusion may be confirmed thus; $1 - x$ being the probability of the first's failing, and $1 - y$ the probability of the second's failing, then the Product $1 - x \times 1 - y$ or $1 - x - y + xy$ will represent the probability of their both failing; and this being subtracted from Unity, the remainder, viz. $x + y - xy$ will represent the probability of their not both failing, that is of the happening of either.

And if there be three Events concerned, of which the Probabilities of happening are respectively x, y, z , then multiplying $1 - x$ by $1 - y$ and that again by $1 - z$, and subtracting the Product from Unity, the remainder will express the probability of the happening of one at least of them, which consequently will be $x + y + z - xy - xz - yz + xyz$; and this may be pursued as far as one pleases.

A difficulty almost of the same nature as that which I have explained is contained in the two following Questions: the first is this;

A Man throwing a Die six times is promised the Sum f every time he throws the Ace, to find the value of his Expectation.

The second is this; a Man is promised the Sum f if at any time in six trials he throws the Ace, to find the value of his Expectation.

In the first Question every throw independently from any other is entitled to an Expectation of the Sum f , which makes the value of the Expectation to be $\frac{1}{6}f + \frac{1}{6}f + \frac{1}{6}f + \frac{1}{6}f + \frac{1}{6}f + \frac{1}{6}f = f$; but in the second, none but the first throw is independent, for the second has no right but in case the first has failed, nor has the

the third any right but in case the two first have failed, and so on; and therefore the value of the Expectation being the Sum expected, multiplied by the Sum of the Probabilities of the Ace's being thrown at any time, exclusive of the Probabilities of its having been thrown before, will be $\frac{1}{6}f + \frac{5}{36}f + \frac{25}{216}f + \frac{125}{1296}f + \frac{625}{7776}f + \frac{3125}{46656}f = \frac{31031}{46656}$ that is nearly $\frac{2}{3}f$.

We may also proceed thus; the probability of the Ace's being missed six times together is $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{15625}{46656}$, and therefore the probability of its not being missed six times, that is of its happening some time or other in 6 throws is $1 - \frac{15625}{46656} = \frac{31031}{46656}$, and consequently the value of the Expectation is $\frac{31031}{46656}$ as it was found before.

Another Instance may be, the computing the Odds of the Bet, *That one of the 4 Players at Whist shall have above 4 Trumps.* The Solution one might think was by adding all the Chances (in the Tables pag. 177) which the 4 Gamesters have for 5 or more Trumps; and this would be true, were every Gamester to lay for himself in particular. But as it may happen that *two* of the Gamesters have above 4 Trumps, and yet, as the Bet is commonly laid, only one Stake is paid, half the Number of these last Chances (computed by *Prob. XX.*) is to be subtracted: which reduces the Wager nearly to an equality.

Nº. VI.

A short method of calculating the value of Annuities on Lives, from Tables of Observations, In a Letter to W. Jones Esq; Phil. Trans. Nº. 473.

Although it has been an established custom, in the payment of Annuities on Lives, that the last rent is lost to the heirs of the late possessor of an annuity, if the person happens to die before the expiration of the term agreed on for payment, whether yearly, half-yearly, or quarterly: nevertheless, in this Paper I have supposed, that such a part of the rent should be paid to the heirs of the late possessor, as may be exactly proportioned to the time elapsed between that of the last payment, and the very moment of the Life's expiring; and this by a proper, accurate, and geometrical calculation.

I have been induced to take this method, for the following reasons; first, by this supposition, the value of Lives would receive but

an inconsiderable increase; secondly, by this means, the several intervals of life, which, in the Tables of Observations, are found to have uniform decrements, may be the better connected together. It is with this view that I have framed the two following Problems, with their Solutions.

PROBLEM I.

To find the value of an Annuity, so circumstantiated, that it shall be on a Life of a given age; and that upon the failing of that life, such a part of the rent shall be paid to the heirs of the late possessor of an Annuity, as may be exactly proportioned to the time intercepted between that of the last payment, and the very moment of the life's failing.

SOLUTION.

Let n represent the complement of life, that is, the interval of time between the given age, and the extremity of old-age, supposed at 86.

r the amount of 1 $l.$ for one year.

α the Logarithm of r .

P the present value of an Annuity of 1 $l.$ for the given time.

\mathcal{Q} the value of the life sought.

$$\text{Then } \frac{1}{r-1} - \frac{P}{\alpha n} = \mathcal{Q}.$$

DEMONSTRATION.

For, let z represent any indeterminate portion of n . Now the Probability of the life's attaining the end of the interval z , and then failing, is to be expressed by $\frac{z}{n}$, (as shewn in my book of Annuities upon Lives) upon the supposition of a perpetual and uniform decrement of life.

But it is well known, that if an Annuity certain of 1 $l.$ be paid during the time z , its present value will be $P = \frac{1-r^{\frac{1}{z}}}{r-1}$ or $\frac{1}{r-1} - \frac{1}{r-1 \times r^{\frac{1}{z}}}$.

And, by the laws of the Doctrine of Chances, the Expectation of such a life, upon the precise interval z , will be expressed by $\frac{z}{n \times r-1} - \frac{z}{nr^z \times r-1}$; which may be taken for the ordinate of a curve, whose area is as the value of the life required.

In

In order to find the area of this curve, let $p = n \times \overline{r - 1}$; and then the ordinate will become $\frac{z}{p} - \frac{z}{p r^z}$, a much more commodious expression.

Now it is plain, that the fluent of the first part is $\frac{z}{p}$: but as the fluent of the second part is not so readily discovered, it will not be improper, in this place, to shew by what artifice I found it; for I do not know, whether the same method has been made use of by others: all that I can say, is, that I never had occasion for it, but in the particular circumstance of this Problem.

Let, therefore, $r^z = x$; hence $z \text{ Log. } r = \text{Log. } x$; therefore $z \text{ Log. } r = (\text{Fluxion of the Log. } x) = \frac{\dot{x}}{x}$, or $\alpha z = \frac{\dot{x}}{x}$; consequently $z = \frac{\dot{x}}{\alpha x}$, and $\frac{z}{r^z} = \frac{\dot{x}}{\alpha x x}$: but the fluent of $\frac{\dot{x}}{\alpha x x}$ is $(-\frac{1}{\alpha x} =) -\frac{1}{\alpha r^z}$; and therefore the fluent of $-\frac{z}{p r^z}$ will be $+\frac{1}{p \alpha r^z}$.

The sum of the two fluents will be $\frac{z}{p} + \frac{1}{p \alpha r^z}$; but, when $z = 0$, the whole fluent should be $= 0$; let therefore the whole fluent be $\frac{z}{p} + \frac{1}{p \alpha r^z} + q = 0$.

Now, when $z = 0$, then $\frac{z}{p} = 0$, and $\frac{1}{\alpha p r^z}$ becomes $\frac{1}{\alpha p}$ (for $r^z = 1$), consequently $\frac{1}{\alpha p} + q = 0$; and $q = -\frac{1}{\alpha p}$: therefore the area of a curve, whose ordinate is $\frac{z}{p} - \frac{z}{p r^z}$ will be $(\frac{z}{p} - \frac{1}{\alpha p} + \frac{1}{\alpha p r^z} =) \frac{z}{p} - \frac{1}{\alpha p} + \frac{1}{\alpha p r^z}$.

But $P = \frac{1}{r-1} - \frac{1}{r-1 \times r^z}$; therefore $1 - \frac{1}{r^z} = \overline{r-1} \times P$, and the expression for the area becomes $\frac{z}{n \times r-1} - \frac{P}{\alpha n}$: And putting n instead of z , that area, or the value of the life, will be expressed by $\frac{1}{r-1} - \frac{P}{\alpha n}$. *Q. E. D.*

Those who are well versed in the nature of Logarithms, I mean those that can deduce them from the Doctrine of Fluxions and infinite Series, will easily apprehend, that the quantity here called α , is that which some call the hyperbolic Logarithm; others, the natural Logarithm: it is what Mr. Cotes calls the Logarithm whose modulus is 1: lastly, it is by some called *Neper's* Logarithm. And, to save the reader some trouble in the practice of this last theorem, the most necessary natural Logarithms, to be made use of in the present disquisition about Lives, are the following: X x 2 If

If $r = 1.04$, then will $\alpha = 0.0392207$.
 $r = 1.05$, - - - $\alpha = 0.0487901$.
 $r = 1.06$, - - - $\alpha = 0.0582589$.

It is to be observed, that the Theorem here found makes the Values of Lives a little bigger, than what the Theorem found in the first Problem of my book of Annuities on Lives, does; for, in the present case, there is one payment more to be made, than in the other; however, the difference is very inconsiderable.

But, although it be indifferent which of them is used, on the supposition of an equal decrement of life to the extremity of old-age; yet, if it ever happens, that we should have Tables of Observations, concerning the mortality of mankind, intirely to be depended upon, then it would be convenient to divide the whole interval of life into such smaller intervals, as, during which, the decrements of life have been observed to be uniform, notwithstanding the decrements in some of those intervals should be quicker, or slower, than others; for then the Theorem here found would be preferable to the other; as will be shewn hereafter.

That there are such intervals, Dr. *Halley's* Tables of Observations sufficiently shew; for instance; out of 302 persons of 54 years of age, there remain, after 16 years (that is, of the age of 70) but 142; the decrements from year to year having been constantly 10; and the same thing happens in other intervals; and it is to be presumed, that the like would happen in any other good Tables of Observations.

But, in order to shew, in some measure, the use of the preceding Theorem, it is necessary to add another Problem; which, though its Solution is to be met with in the first edition of my book of Annuities on Lives, yet it is convenient to have it inserted here, on account of the connexion that the application of the preceding Problem has with it.

In the mean time, it will be proper to know, *What part of the yearly rent should be paid to the heirs of the late possessor of an Annuity, as may be exactly proportioned to the time elapsed between that of the last payment, and the very moment of the life's expiring.* To determine

this, put A for the yearly rent; $\frac{1}{m}$ for the part of the year intercepted between the time of the last payment, and the instant of the life's fail-

ing; r the amount of 1 $l.$ at the year's end: then will $\frac{r^{\frac{1}{m}} - 1}{r - 1} A$ be the sum to be paid.

P R O B L E M II.

To find the Value of an Annuity for a limited interval of life, during which the decrements of life may be considered as equal.

SOLUTION.

Let a and b represent the number of people living in the beginning and end of the given interval of years.

s represent that interval.

P the Value of an Annuity certain for that interval.

\mathcal{Q} the Value of an Annuity for life supposed to be necessarily extinct in the time s ; or (which is the same thing) the Value of an Annuity for a life, of which the complement is s .

Then $\mathcal{Q} + \frac{b}{a} \times \overline{P - \mathcal{Q}}$ will express the Value required.

DEMONSTRATION.

For, let the whole interval between a and b be filled up with arithmetical mean proportionals; therefore the number of people living in the beginning and end of each year of the given interval s will be represented by the following Series; viz.

$$a, \frac{sa-a+b}{s}, \frac{sa-2a+2b}{s}, \frac{sa-3a+3b}{s}, \frac{sa-4a+4b}{s}, \&c. \text{ to } b.$$

Consequently, the Probabilities of the life's continuing during 1, 2, 3, 4, 5, &c. years will be expressed by the Series,

$$\frac{sa-a+b}{sa}, \frac{sa-2a+2b}{sa}, \frac{sa-3a+3b}{sa}, \frac{sa-4a+4b}{sa}, \&c. \text{ to } \frac{b}{a}.$$

Wherefore, the Value of an Annuity of 1 $l.$ granted for the time s , will be expressed by the Series

$$\frac{sa-a+b}{sar} + \frac{sa-2a+2b}{sar^2} + \frac{sa-3a+3b}{sar^3} + \frac{sa-4a+4b}{sar^4}, \&c. \text{ to } + \frac{b}{ar^s};$$

this Series is divisible into two other Series's, viz.

$$1st. \frac{s-1}{sr} + \frac{s-2}{sr^2} + \frac{s-3}{sr^3} + \frac{s-4}{sr^4}, \&c. \text{ to } + \frac{s-s}{sr^s}.$$

$$2d. \frac{b}{a} \times \frac{1}{sr} + \frac{2}{sr^2} + \frac{3}{sr^3} + \frac{4}{sr^4}, \&c. \text{ to } \frac{s}{sr^s}.$$

Now, since the first of these Series's begins with a Term whose Numerator is $s-1$, and the subsequent Numerators each decrease by unity; it follows, that the last Term will be $= 0$; and consequently, that Series expresses the Value of a life necessarily to be extinct in the time s . The sum of which Series may be esteemed as a given quantity; and is what I have expressed by the symbol \mathcal{Q} in Problem I.

The

The second Series is the difference between the two following Series's,

$$\frac{b}{a} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \&c. \text{ to } \frac{1}{r^s}.$$

$$\frac{b}{a} \times \frac{s-1}{sr} + \frac{s-2}{sr^2} + \frac{s-3}{sr^3} + \frac{s-4}{sr^4} \&c. \text{ to } + \frac{s-s}{sr^s}.$$

Where, neglecting the common multiplier $\frac{b}{a}$, the first Series is the Value of an Annuity certain to continue s years; which every mathematician knows how to calculate, or is had from Tables already composed for that purpose: this Value is what I have called P ; and the second Series is \mathcal{Q} .

Therefore $\mathcal{Q} + \frac{b}{a} \times \overline{P - \mathcal{Q}}$ will be the Value of an Annuity on a life for the limited time. \mathcal{Q} E. D.

It is obvious, that the Series denoted by \mathcal{Q} , must of necessity have one Term less than is the number of equal intervals contained in s ; and therefore, if the whole extent of life, beginning from an age given, be divided into several intervals, each having its own particular uniform decrements, there will be, in each of these intervals, the defect of one payment; which to remedy, the Series \mathcal{Q} must be calculated by Problem 1.

EXAMPLE.

To find the Value of an Annuity for an age of 54, to continue 16 years, and no longer.

It is found, in Dr. Halley's Tables of Observations, that a is 302, and b 172: now $n = s = 16$; and, by the Tables of the Values of Annuities certain, $P = 10.8377$; also (by Problem 1.) $\mathcal{Q} = (\frac{1}{r-1} - \frac{P}{ar}) = 6.1168$. Hence it follows (by this Problem), that the Value of an Annuity for an age of 54, to continue during the limited time of 16 years, supposing interest at 5 per cent. per annum, will be worth $(\mathcal{Q} + \frac{b}{a} \times \overline{P - \mathcal{Q}}) = 8.3365$ years purchase.

From Dr. Halley's Tables of Observations, we find, that from the age of 49 to 54 inclusive, the number of persons, existing at those several ages, are, 357, 346, 335, 324, 313, 302, which comprehends a space of five years; and, following the precepts before laid down, we shall find, that an Annuity for a life of 49, to continue for the limited time of 5 years, interest being at 5 per cent. per annum, is worth 4.0374 years purchase.

And,

And, in the same manner, we shall find, that the Value of an Annuity on a life, for the limited time comprehended between the ages of 42 and 49, is worth 5.3492 years purchase.

Now, if it were required to determine the Value of an Annuity on life, to continue from the age of 42 to 70, we must proceed thus:

It has been proved, that an Annuity on life, reaching from the age of 54 to 70, is worth 8.3365 years purchase; but this Value, being estimated from the age of 49, ought to be diminished on two accounts: First, because of the Probability of the life's reaching from 49 to 54, which Probability is to be deduced from the Table of Observations, and is proportional to the number of people living at the end and beginning of that interval, which, in this case, will be found 302 and 357: The second diminution proceeds from a discount that ought to be made, because the Annuity, which reaches from 54 to 70, is estimated 5 years sooner, *viz.* from the age of 49, and therefore that diminution ought to be expressed by $\frac{1}{r^5}$; so that the total diminution of the Annuity of 16 years will be expressed by the fraction $\frac{302}{357r^5}$, which will reduce it from 8.3365 years purchase to 5.5259; this being added to the Value of the Annuity to continue from 49 to 54, *viz.* 4.0374, will give 9.5633, the Value of an Annuity to continue from the age of 49 to 70. For the same reason, the Value 9.5633, estimated from the age of 42, ought to be reduced, both upon account of the Probability of living from 42 to 49, and of the discount of money for 7 years, at 5 *per cent. per annum*, amounting together to 3.8554, which will bring it down to 5.7079; to this adding the Value of an Annuity on a life to continue from the age of 42 to 49, found before to be 5.3492, the sum will be 11.0571 years purchase, the Value of an Annuity to continue from the age of 42 to 70.

In the same manner, for the last 16 years of life, reaching from 70 to 86, when properly discounted, and also diminished upon the account of the Probability of living from 42 to 70, the Value of those last 16 years will be reduced to 0.8; this being added to 11.0571 (the Value of an Annuity to continue from the age of 42 to 70, found before), the sum will be 11.8571 years purchase, the Value of an Annuity to continue from the age of 42 to 86; that is, the Value of an Annuity on a life of 42; which, in my Tables, is but 11.57, upon the supposition of an uniform decrement of life, from an age given to the extremity of old-age, supposed at 86.

It is to be observed, that the two diminutions, above-mentioned, are conformable to what I have said in the Corollary to the second Problem of the first edition, printed in the year 1724.

Those who have sufficient leisure and skill to calculate the Value of joint Lives, whether taken two and two, or three and three, in the same manner as I have done the first Problem of this tract, will be greatly assisted by means of the two following Theorems:

If the ordinate of a curve be $\frac{z}{r^x}$; its area will be $\frac{1}{a^2} - \frac{1}{a^2 r^2} - \frac{z}{a r^2}$.

If the ordinate of a curve be $\frac{z^2}{r^x}$; its area will be $\frac{2}{a^3} - \frac{2}{a^3 r^2} - \frac{2z}{a^2 r^2} - \frac{z^2}{a r^2}$.

The Probabilities of human Life, according to different Authors.

Table I, by Dr. Halley.

Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.
1	1000	16	622	31	523	46	387	61	232	76	78
2	855	17	616	32	515	47	377	62	222	77	68
3	798	18	610	33	507	48	367	63	212	78	58
4	760	19	604	34	499	49	357	64	202	79	* 49
5	732	20	598	35	* 490	50	* 346	65	192	80	41
6	710	21	592	36	481	51	335	66	182	81	34
7	692	22	586	37	472	52	324	67	172	2	28
8	680	23	580	38	463	53	313	68	162	83	23
9	670	24	574	39	454	54	302	69	152	84	19
10	661	25	* 567	40	445	55	* 292	70	142	*	*
11	653	26	560	41	436	56	282	71	* 131		
12	646	27	553	42	427	57	272	72	120		
13	* 640	28	546	43	* 417	58	261	73	109		
14	634	29	539	44	407	59	252	74	98		
15	628	30	* 531	45	397	60	242	75	* 88		

Table II. by M. Kerffboom.

Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.
0	1400										
1	1125	16	849	31	699	46	550	61	369	76	160
2	1075	17	842	32	687	47	540	62	356	77	145
3	1030	18	835	33	675	48	530	63	343	78	130
4	993	19	826	34	665	49	518	64	329	79	115
5	964	20	817	35	655	50	507	65	315	80	100
6	947	21	808	36	645	51	495	66	301	81	87
7	930	22	800	37	635	52	482	67	287	82	75
8	913	23	792	38	625	53	470	68	273	83	64
9	904	24	783	39	615	54	458	69	259	84	55
10	895	25	772	40	605	55	446	70	245	85	45
11	886	26	760	41	596	56	434	71	231	86	36
12	878	27	747	42	587	57	421	72	217	87	28
13	870	28	735	43	578	58	408	73	203	88	21
14	863	29	723	44	569	59	395	74	189	89	15
15	8 6	30	711	45	560	60	382	75	175	90	10

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Table III.

A P P E N D I X.

Table III. by M. de *Parcieux*.

Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.
1	****	21	806	41	650	61	450	81	101
2	****	22	798	42	643	62	437	82	85
3	1000	23	790	43	636	63	423	83	71
4	970	24	782	44	629	64	409	84	59
5	948	25	774	45	622	65	395	85	48
6	930	26	766	46	615	66	380	86	38
7	915	27	758	47	607	67	364	87	29
8	902	28	750	48	599	68	347	88	22
9	890	29	742	49	590	69	329	89	16
10	880	30	734	50	581	70	310	90	11
11	872	31	726	51	571	71	291	91	7
12	866	32	718	52	560	72	271	92	4
13	860	33	710	53	549	73	251	93	2
14	854	34	702	54	538	74	231	94	1
15	848	35	694	55	526	75	211	95	0
16	842	36	686	56	514	76	192	96	*
17	835	37	678	57	502	77	173	97	*
18	828	38	671	58	489	78	154	98	
19	821	39	664	59	476	79	136	99	
20	814	40	657	60	463	80	118	100	

Table IV. by Messieurs *Smart* and *Simpson*.

Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.
1	1280 } 870 }	17	480	33	358	49	212	65	99
2	700	18	474	34	349	50	204	66	93
3	635	19	468	35	340	51	196	67	87
4	600	20	462	36	331	52	188	68	81
5	580	21	455	37	322	53	180	69	75
6	564	22	448	38	313	54	172	70	69
7	551	23	441	39	304	55	165	71	64
8	541	24	434	40	294	56	158	72	59
9	532	25	426	41	284	57	151	73	54
10	524	26	418	42	274	58	144	74	49
11	517	27	410	43	264	59	137	75	45
12	510	28	402	44	255	60	130	76	41
13	504	29	394	45	246	61	123	77	38
14	498	30	385	46	237	62	117	78	35
15	492	31	376	47	228	63	111	79	32
16	486	32	367	48	220	64	105	80	29

Remarks.

Remarks on the foregoing Tables.

The first Table is that of Dr. *Halley*, composed from the Bills of Mortality of the City of *Breslaw*; the best, perhaps, as well as the first of its kind; and which will always do honour to the judgment and sagacity of its excellent Author.

Next follows a Table of the ingenious Mr. *Kerſſeboom*, founded chiefly upon Registers of the *Dutch* Annuitants, carefully examined and compared, for more than a century backward. And *Monsieur de Parcieux* by a like use of the Lists of the *French Tontines*, or long *Annuities*, has furnished us Table III; whose numbers were likewise verified upon the *Necrologies* or mortuary Registers of several religious houses of both Sexes.

To these is added the Table of *Messieurs Smart and Simpson*, adapted particularly to the City of *London*; whose inhabitants, for reasons too well known, are shorter lived than the rest of mankind.

Each of these Tables may have its particular use: The *Second* or *Third* in valuing the *better* sort of Lives, upon which one would chuse to hold an Annuity; the *Fourth* may serve for *London*, or for Lives such as those of its Inhabitants are supposed to be: while Dr. *Halley's* numbers, falling between the two Extremes, seem to approach nearer to the general course of nature. And in Cases of *combined* Lives, two or more of the Tables may perhaps be usefully employed.

Besides these, the celebrated *Monsieur de Buffon*† has lately given us a new Table, from the actual Observations of *Monsieur du Pré de S. Maur* of the *French Academy*. This Gentleman, in order to strike a just *mean*, takes three populous parishes in the City of *Paris*, and so many country Villages as furnish him nearly an equal number of Lives: and his care and accuracy in that performance have been such as to merit the high approbation of the learned Editor. It was therefore proposed to add this Table to the rest; after having purged its numbers of the inequalities that necessarily happen in fortuitous things, as well as of those arising from the careless manner in which *Ages* are given in to the parish Clerks; by which the years that are multiples of 10 are generally overloaded.

But this having been done with all due care, and the whole reduced to Dr. *Halley's* Denomination of 1000 Infants of a year old; there resulted only a mutual confirmation of the two Tables; Mr. *du Pré's* Table making the Lives somewhat better as far as 39 years, and thence a small matter worse than they are by Dr. *Halley's*.

We may therefore retain this last as no bad standard for mankind in *general*; till a better *Police*, in this and other nations, shall furnish

† *Histoire Naturelle*, Tome II. the

the proper *Data* for correcting it, and for expressing the Decrements of Life more accurately, and in larger numbers.

For which purpose, the parish Registers ought to be kept in a better manner, according to one or other of the Forms that have been proposed by Authors. Or, if we suppose the numbers annually born to have been nearly the same for an age past, the thing may be done at once, by taking the numbers of the living, with their ages, throughout every Parish in the Kingdom: as was in part ordered some time ago by the Right Reverend the Bishops: but their Order was not universally obeyed; for what reason we pretend not to guess. Certain it is, that a *Census* of this kind once established, and repeated at proper intervals, would furnish to our Governours, and to ourselves, much important instruction of which we are now in a great measure destitute: Especially if the whole was distributed into the proper *Classes* of *married* and *unmarried*, *industrious* and *chargeable* Poor, *Artificers* of every kind, *Manufacturers*, &c. and if this was done in each County, City, and Borough, separately; that particular useful conclusions might thence be readily deduced; as well as the general state of the Nation discovered; and the Rate according to which *human Life* is wasting from year to year. See, on this subject, the judicious Observations of Mr. Corbyn Morris, addressed to Thomas Potter Esq; in the year 1751.

F I N I S.













