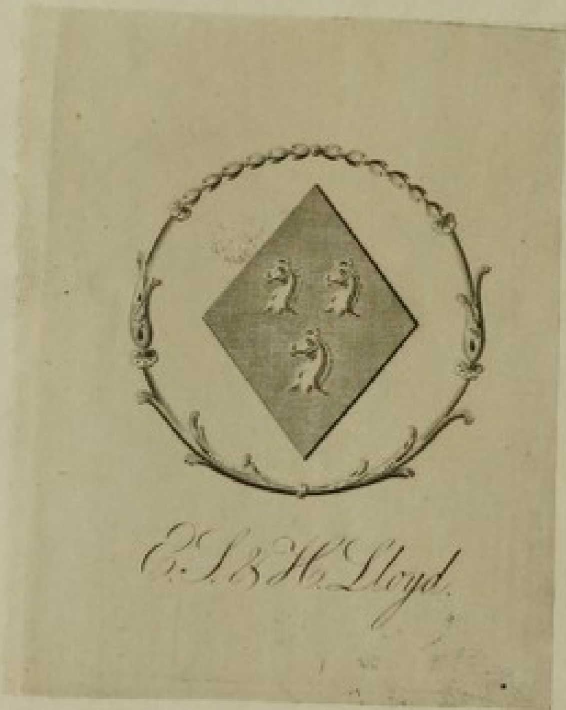
E. L. & H. Lloyd

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IOHN KERSEY BORNE AT BODICOT NEERE BANBVRY IN THE COVNTY OF OXFORD Aº. Dni. 1616.
Vox audita perit.
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THE ELEMENTS OF THAT Mathematical Art Commonly called ALGEBRA,

Expounded in Two BOOKS.

By *JOHN KERSEY.*

Nil tam difficile eſt, quod non ſolertia vincat.
Dimidium facti, qui bene cœpit, habet.

To which is added,

LECTURES read in the School of Geometry in *Oxford*,

Concerning the *Geometrical* Conſtruction of Algebraical Equations; And the *Numerical* Reſolution of the ſame by the *Compendium* of Logarithms.

By Dr. *EDMUND HALLEY*, Savilion Profeſſor of *Geometry* in the Univerſity of *Oxford*.

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A TREATISE OF THE ELEMENTS OF THE Algebraical ART.

BOOK I.

CHAP. I.

Concerning the Nature, Scope, and Kinds of ALGEBRA: *The Construction of* Cossic *Quantities, or* Powers; *with the manner of expressing them by Alphabetical Letters*: *The signification of* Characters *used in the First Book.*

I. THE Mathematical Arts or Sciences are exercis'd about *Quantity*, which is compris'd under Numbers, Lines, Superficies, and Solids: These if they be considered abstractively, and separate from all kind of Matter, are the proper Objects of *Arithmetic* and *Geometry*, which are called *Pure Mathematics*.

II. The *Method* which Mathematicians are wont to use in searching out Truth about Quantity, is twofold; *viz.* 1. *Synthetical*, or by way of Composition: 2. *Analytical*, or by way of Resolution.

III. Mathematical Composition, or the *Synthetical Method*, argues altogether with known Quantities to search out unknown; and then demonstrates that the Quantity found out will satisfie the Proposition.

IV. Mathematical Resolution, or the *Analytical Art*, commonly call'd *Algebra*, is that way of reasoning which assumes or takes the Quantity sought as if it were known or granted; and then with the help of one or more Quantities given or known, proceeds by Consequences, until at length the Quantity first only assumed or feigned to be known, is found equal to some Quantity or Quantities certainly known, and is therefore likewise known.

V. The Scope, Drift or Office of the Analytic or Algebraic Art, is to search out three kinds of Truths, *viz.*

1. *Theorems*; which are nothing else but Declarations, or Affirmations of certain Properties, Proportions, or Equalities, justly inferr'd from some Suppositions or Concessions about Quantity: Which Theorems are to be reserved in store, as ready helps to find out new, and to confirm old Truths. This kind of Resolution when it rests in a bare Invention of Truth, is called *Contemplative*, or *Notional*.

2. *Canons*, or infallible Rules, to direct how to ſolve knotty Queſtions, by the help of Quantities given or known; this kind of Reſolution is called *Problematical.*

3. *Demonſtrations*, or evident and indubitable Proofs, to manifeſt the Truth of ſuch Theorems and Canons as are Analytically found out.

VI. *Algebra* is by late Writers divided into two kinds; to wit, *Numeral* and *Literal* (or *Specious.*)

VII. *Numeral Algebra* is ſo called, becauſe in this Method of reſolving a Queſtion, the Quantity ſought or unknown is ſolely deſign'd or repreſented by ſome Alphabetical Letter, or other Character taken at Pleaſure, but all the Quantities given are expreſt by Numbers.

VIII. *Literal*, or *Specious Algebra* is ſo called, becauſe in this Method of reſolving a Queſtion, as well the given or known Quantities, as the unknown are all ſeverally expreſſed or repreſented by Alphabetical Letters. Whence it comes to paſs, that at the end of the Reſolution of a Queſtion, every Quantity appearing diſtinct under the ſame Letter or Form by which it was at firſt expreſſed, a *Canon* is diſcovered to direct how the Queſtion propos'd may be ſolved, not only by the quantities firſt given, but by any other whatſoever that are capable of ſolving the Queſtion. In this Reſpect therefore *Literal Algebra* far excels the *Numeral*; for this latter ſerves only to ſolve *Arithmetical Queſtions*, and produces not a *Canon* without much difficulty, in regard the Numbers firſt given, by reiterated Multiplications, Diviſions, and other Arithmetical Operations, will for the moſt part be ſo confounded and interwoven, that their Foot-ſteps can hardly be traced out: But *Literal* or *Specious Algebra* is applicable to the ſolving of *Geometrical Problems*, as well as *Arithmetical.*

IX. The *Doctrine* of *Algebra* is principally grounded upon the Knowlege of certain Quantities called by ſome Authors *Coſſic Quantities*, by others, *Powers*; the Conſtruction whereof is explain'd in ſix Sections next following.

X. Numbers are ſaid to be *in Geometrical Proportion continued*, when as the firſt is to the ſecond, ſo is the ſecond to the third, and ſo is the third to the fourth, &c. As, for Example, theſe Numbers, 1, 2, 4, 8, 16, 32, &c. are Continual Proportionals; for, as the firſt Term 1, is the half of the ſecond Term 2; ſo is the ſecond Term 2, the half of the third Term 4; and ſo is 4 the half of 8, &c. Likewiſe theſe Numbers, 3, 9, 27, 81, 243, &c. are in Geometrical Proportion continued; For as the firſt Term 3 is a third part of the ſecond Term 9, ſo is the ſecond Term 9 a third part of the third Term 27; and ſo is 27 one third of 81, &c. Alſo, theſe numbers are continual Proportionals, to wit, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. for as the firſt Term 1, is the double of the ſecond Term $\frac{1}{2}$, ſo is $\frac{1}{2}$ the double of $\frac{1}{4}$, and $\frac{1}{4}$ the double of $\frac{1}{8}$, &c.

XI. In any ſeries or rank of Numbers proceeding from Unity in a continued Geometrical proportion, whether aſcending or deſcending, all the Numbers or Terms except the firſt, which is ſuppoſed to be 1, (to wit, Unity,) are called *Coſſic Numbers*, or *Powers*; *viz.* the ſecond Term or Proportional is called the *Root*, or firſt Power; the third Proportional is called the *Square*, or ſecond Power; the fourth Proportional is called the *Cube*, or third Power; the fifth Proportional is called the *Biquadrate*, or fourth Power, the ſixth Proportional, the fifth Power, &c. As for Example, in this rank of Continual Poportionals, 1, 2, 4, 8, 16, 32, &c. the ſecond Term 2 is the Root; the third Term 4 is the ſecond Power, or the Square of the Root 2; the fourth Term 8 is the third Power, or the Cube of the Root 2; the fifth Term 16 is the Biquadrate or fourth Power of the ſame Root 2, &c.

In like manner in this rank of continual Proportionals deſcending from 1, to wit, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c. the ſecond Term $\frac{1}{2}$ is the Root; the third Term $\frac{1}{4}$ is the ſecond Power; the fourth Term $\frac{1}{8}$ is the third Power, &c. The like is to be underſtood of any other Rank of Numbers in a continued Geometrical Proportion, whoſe firſt Term or Proportional is Unity.

XII. From the two laſt preceding Sections, (which are grounded upon 10. Prop. 8. Elem. *Euclid.*) it is evident that any Number whatſoever being propoſed for a *Root*, the ſecond Power, or the *Square*, is produced by the Multiplication of the Root by it ſelf; the third Power, or the *Cube*, is produced by the Multiplication of the ſecond Power by the Root; the fourth Power is produced by the Multiplication of the third Power by the Root, &c.

As, for Example, if 2 be given for the *Root*, this 2 multiplied by it ſelf, produces 4 for the ſecond Power, to wit, the Square of the Root 2: Again, 4 the ſecond Power

being

being multiplied by the Root 2 gives 8 the third Power, or the Cube; which third Power multiplied by the Root 2, produces the fourth Power 16, &c.

In like manner, if this Fraction $\frac{2}{3}$ be prescribed for a Root, by multiplying $\frac{2}{3}$ by it self, there comes forth $\frac{4}{9}$ for the second Power, or the Square of the Root $\frac{2}{3}$; Again, the second Power $\frac{4}{9}$ multiplied by the Root $\frac{2}{3}$ produces the third Power $\frac{8}{27}$, or the Cube of the Root $\frac{2}{3}$; and the third Power $\frac{8}{27}$ multiplied by the Root $\frac{2}{3}$ gives the fourth Power $\frac{16}{81}$, &c.

But when the *Root* is 1, to wit, Unity, every one of its Powers will also be 1; for multiplication by 1 makes no alteration. All which will be further illustrated by the Scales of Cossic numbers or Powers in the following Table, which shews that if the Root be 5, the Square is 25, the Cube 125, the Biquadrate or fourth Power 625, the fifth Power 3125, &c.

A Table of Powers *in Numbers.*

The *Root* or first Power.	1	2	3	4	5
The *Square* or second Power.	1	4	9	16	25
The *Cube* or third Power.	1	8	27	64	125
The *Biquadrate* or fourth Power.	1	16	81	256	625
The fifth Power.	1	32	243	1024	3125
The sixth Power.	1	64	729	4096	15625
The seventh Power.	1	128	2187	16384	78125
The eighth Power, &c.	1	256	6561	65536	390625

XIII. The *Root* or first Power being given, the third, fifth, eighth, or any other Power may be found out without respect to the intermediate Power or Powers, in this manner; *viz.* Suppose the number 3 be prescribed for the Root, and that the fifth Power be desired; first write down the Root 3 five times thus, 3, 3, 3, 3, 3; then multiply these five equal numbers one into another according to the Rule of continual Multiplication, so the last Product 243 shall be the desired fifth Power raised from the Root 3.

In like manner, if the eighth Power of the Root 2 be desired, you may write the Root 2 eight times thus, 2, 2, 2, 2, 2, 2, 2, 2, these multiplied continually produce 256, which is the eighth Power of the Root 2. After the same manner you may find out any other Power from a number given for the Root.

XIV. If over or under any Series or Rank of Cossic numbers or Algebraic Powers, constituted according to the three last foregoing Sections, there be placed a rank of Numbers beginning with Unity, and proceeding according to the natural order of numbers, as 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. these numbers so placed are usually called the *Indices*, or *Exponents* of those Powers, as well because they shew the order, seat, or place of each Power, as also its number of Degrees or Dimensions; that is, how many times the Root is involved or multiplied in producing each Power respectively: As for Example, let there be a Rank or Scale of Algebraic powers raised from the root 3, as 3, 9, 27, 81, 243, 729, 2187, &c. and over them let there be so many numbers placed in an Arithmetical progression, beginning with 1, and proceeding according to the natural order of Numbers, as here you see:

INDICES.	1	2	3	4	5	6	7	8	&c.
POWERS.	3	9	27	81	243	729	2187	6561	&c.

I say

I say the Index 4 in the Arithmetical progression, shews that the fourth Power 81, which stands under 4, is produced by the multiplication of the Root 3 four times into it self, *viz.* these four numbers 3, 3, 3, 3, multiplied continually will produce 81; likewise the Index 7 in the Arithmetical progression shews, that the seventh Power 2187, which stands under 7, is produced by the multiplication of the Root 3 seven times into it self; *viz.* these seven equal numbers 3, 3, 3, 3, 3, 3, 3, multiplied continually produce 2187. And so of others.

To that use of *Indices*, this may be added; *viz.* If any two or more Indices be added together, the sum will be an Index shewing what power will be produced by the multiplication of those Powers one into another which answer to the Indices that were added together: As for Example, if the Indices 3 and 5 be added together, the sum is the Index 8; which shews, that if the third and fifth Powers be multiplied one by the other, the eighth Power will be produced: As in the rank of Powers in the preceding Tabulet, if the third power 27 be multiplied by the fifth Power 243, the Product will give the eighth Power 6561. In like manner, for as much as the Indices 2 and 6 added together make the Index 8; therefore the second Power 9 multiplied by the sixth Power 729 will also produce the eighth Power 6561: Again because the Indices 1, 2, and 5 added together make the Index 8; therefore the first, second and fifth Powers, to wit, 3, 9, and 243 multiplied continually will likewise produce the eighth Power 6561. And as the Index 3 added to it self makes the Index 6, so the third Power 27 multiplyed by it self, or squared, will produce the sixth Power 729.

And as the Addition of Indices answers to the Multiplication of their correspondent Powers, so the subtraction of Indices answers to the division of their correspondent Powers: As, for Example, because the Index 8 lessened by the Index 5, leaves for a Remainder the Index 3; therefore the eighth Power 6561 divided by the fifth Power 243 gives in the Quotient the third Power 27. Likewise, as the Index 7 lessened by the Index 3 leaves the Index 4; so the seventh Power 2187 divided by the third Power 27, gives the fourth Power 81.

XV. From the premisses it is evident, that upon an Arithmetical foundation, a Scale or Rank of Algebraic Powers may be raised and continued as far as you please; the three first of which have an affinity with, and may be expounded by Geometrical dimensions: For first, we may conceive any terminated Right-line, to be divided into a number of equal parts at pleasure, suppose 12; then this number 12, or that Right-line, may be esteemed as a Root: Secondly, the said 12 multiplied by it self produces 144 the second Power, which is equal to the Area of a square Superficies whose side is 12: Thirdly, the said second Power 144 multiplied by the Root 12 produces the third Power 1728, which is equal to the Solid content of a Cube, (to wit, a Solid in the form of a Dye) whose side is 12.

But none of the rest of the Algebraic powers can properly be explain'd by any Geometrical quantity, in regard there are but three dimensions in Geometry, to wit, Length, Breadth, and Depth (or Thickness.)

XVI. In searching out the solution of a Question by the Algebraic Art, the number or line sought is usually called a *Root*, which so long as it remains unknown cannot be really exprest, and therefore it must be design'd or represented by some Symbol or Character, at the will of the Artist; also the Powers which may be imagined to proceed from the said Root in such manner as has before bin declared are likewise to be represented by Symbols or Characters; concerning which there is much diversity among *Algebraical* Writers, every one pleasing his fancy in the choice of Characters: But in this matter I shall imitate Mr. *Thomas Harriot* in his *Ars Analytica*, and *Renates des Cartes* in his *Geometry*, but chiefly the former; whose method of expressing Quantities by Alphabetical Letters, I conceive to be the plainest for Learners, *viz.*

To design or represent the Root sought, whether it be a number or a Line in a Question proposed, we may assume any Letter of the Alphabet, as *a*, *b*, or *c*, &c. but for the better distinguishing of known quantities from unknown, some *Analysts* are wont to assume one of the five Vowels, as, *a*, or *e*, &c. to represent the quantity sought; and Consonants, as, *b*, *c*, *d*, &c. to represent quantities known or given: Now if the letter *a* be assumed to represent the Root sought, then (according to Mr. *Harriot*) the second Power, or the Square raised from that Root, may be represented by *aa*; the third Power, or the Cube, by *aaa*; the fourth Power by *aaaa*; the fifth Power by *aaaaa*; and after the

the ſame manner any higher Power of the Root or number a may be repreſented: For ſo many Dimenſions or Degrees as are in the Power, ſo many times the Letter which at firſt was aſſumed for the Root is to be repeated.

Or after the manner of *Renates des Cartes*, if the letter a be aſſumed to repreſent the Root, the Square may be deſigned thus, a^2. the Cube thus, a^3. the fourth Power thus, a^4. the fifth Power thus, a^5. And ſo any other power may be expreſt by writing the Index or Exponent of the Power in a ſmall figure next after, and near the head of the letter aſſumed to repreſent the Root. Both which ways will be further illuſtrated by the following Table.

A Table ſhewing two ways (now moſt in uſe) to expreſs ſimple Powers *by Alphabetical Letters.*

The *Root* or firſt Power,	a.	a
The *Square* or ſecond Power,	$a\,a$.	a^2
The *Cube* or third Power,	$a\,a\,a$.	a^3
The fourth Power,	$a\,a\,a\,a$.	a^4
The fifth Power,	$a\,a\,a\,a\,a$.	a^5
The ſixth Power,	$a\,a\,a\,a\,a\,a$.	a^6
The ſeventh Power,	$a\,a\,a\,a\,a\,a\,a$.	a^7
The eighth Power,	$a\,a\,a\,a\,a\,a\,a\,a$.	a^8

After the ſame manner, known Quantities and their Powers may be repreſented by Conſonants; as, b may be put for any known number in a Queſtion, and then its Square may be ſignified by bb, the Cube by bbb, the fourth Power by $bbbb$, the fifth Power by $bbbbb$, the ſixth by $bbbbbb$, and ſo forwards: Or the Square of the Root b may be expreſt thus, b^2. the Cube thus, b^3. the fourth Power thus, b^4. the fifth Power thus, b^5. the ſixth Power thus, b^6. and ſo forward.

XVII. Numbers ſet before, that is, on the left hand of quantities expreſt by letters are called Numbers prefixt; but if no number be prefixt to the letter, then 1 or unity muſt be imagined to be prefixt: As, in theſe quantities a, (or $1\,a$,) $2\,a$, $3\,a$, $\frac{1}{2}a$, $\frac{2}{3}a$, $5bbb$ (or $5b^3$) the numbers prefixt are (as you ſee) 1, 2, 3, $\frac{1}{2}$, $\frac{2}{3}$, and 5, every one of which numbers (and the like ſo prefixt) ſhews how often the quantity repreſented by the letter or letters immediately following the number is taken; ſo a or $1\,a$ ſignifies ſome number or line once taken, alſo $2\,a$ repreſents the double, $\frac{1}{2}a$ the half, and $\frac{2}{3}a$ two third parts of the number or line repreſented by a. In like manner $5bbb$, or $5b^3$, ſignifies that the Cube of the number or line repreſented by b is taken five times.

XVIII. All numbers expreſt by figures and cyphers (as in vulgar Arithmetic) not having any letter or letters annexed to them, are for diſtinction ſake called Abſolute numbers; as theſe numbers, 5, 20, 105, $\frac{1}{2}$, $\frac{2}{3}$, and all others when they be not prefixt or annext to any letter or letters are called abſolute numbers.

XIX. All *Algebraical Operations* are perform'd in an Arithmetical manner, partly in the vulgar way by numbers, and partly by Alphabetical letters in all the parts of Arithmetic, to wit, Addition, Subtraction, Multiplication, Diviſion, and the Extraction of Roots: But ſince letters cannot be diſpoſed like numbers to perform thoſe operations, ſome Characters muſt of neceſſity be uſed to ſignifie ſuch operations. The Characters uſed in this firſt Book are explained in the following Sections.

XX. This Character $+$ is a ſign of *Affirmation*, as alſo of *Addition*, and always belongs to the quantity that follows the ſign; as, $+a$ affirms the quantity denoted by a to be real, or greater than nothing; the like may be ſaid of $+b$, and $+2c$, &c.

When no ſign is prefixt before a quantity, the ſign $+$ is always to be underſtood, and muſt be imagined to be prefixt; ſo a implies $+a$, likewiſe $2b$ ſignifies the ſame thing with $+2b$; the like of others.

But when the ſign $+$ is placed between two quantities, it imports as much as the word *plus*, or *more*, and ſignifies that thoſe quantities are added or to be added together

gether: As $3+4$ (or 3 more 4) ſignifies the ſum of 3 and 4; or it hints that 4 is to be added to 3. In like manner $a+b$ ſignifies the ſum of numbers or quantities repreſented by a and b; and $a+b+c$ ſignifies the ſum of quantities denoted by a, b, and c.

XXI. This Character — is a ſign of *Negation*, as alſo of *Subtraction*, and always belongs to the following quantity; as for Example, -5 is a fictitious number leſs than nothing by 5; *viz.* as $+5$ *l.* may repreſent five pounds in money, or the Eſtate of ſome perſon who is clearly worth five pounds; ſo -5 *l.* may repreſent a Debt of five pounds owing by ſome perſon who is worſe than nothing by five pounds.

But when the ſign — is placed between two quantities, it imports as much as the word *minus*, or *leſs*; and intimates that the number or quantity following that ſign is ſubtracted or to be ſubtracted from the number or quantity that ſtands next before the ſame ſign: As $8-3$ (or 8 leſs 3) ſignifies that 3 is ſubtracted or to be ſubtracted from 8; or $8-3$ denotes the exceſs of 8 above 3, to wit, 5.

In like manner $a-b$ (or a leſs b) ſignifies that the quantity denoted by b is ſubtracted or to be ſubtracted from the quantity a; or $a-b$ may ſignifie the exceſs of the quantity a above the quantity b.

XXII. This Character $\backsim$ ſignifies the *Difference* of two quantities, to wit, the exceſs of the greater above the leſs, when 'tis not determin'd or known in which of thoſe quantities the exceſs lyes; ſo $a \backsim b$ ſignifies the difference of two quantities repreſented by a and b when 'tis not known whether a be greater or leſs than b.

XXIII. This Character $\times$ is a ſign of *Multiplication*, and is put for the word *into*, or *by*; *viz.* when 'tis ſet between two quantities it ſignifies that they are multiplied, or to be multiplied mutually one by the other: As, 6×3 (or 6 into or by 3) imports the Product of the multiplication of 6 by 3, to wit, 18.

In like manner $a\times b$ ſignifies that the quantity repreſented by a is multiplied or to be multiplied by the quantity b: alſo $a\times b\times c$ ſignifies the Product made by the continual multiplication of the quantities a, b, and c, one into another.

But for the moſt part the Multiplication of quantities denoted by letters is ſignified by the joyning of letters together, like letters in a word; as ab ſignifies the Product of the multiplication of the quantity a by the quantity b. Alſo abc ſignifies the Product of the continual multiplication of the quantities a, b and c one into another: All which will be further illuſtrated in Chap. 4.

XXIV. Quantities deſign'd or repreſented by letters are either Simple or Compound.

XXV. A Simple quantity is deſigned or expreſſed either by a ſingle letter or by two or more letters joyned together like letters in a word: As a (or $+a$) is a ſimple quantity; likewiſe $2aa$, $3abc$, and $dddd$ are ſimple quantities.

XXVI. A Compound quantity conſiſts of two or more ſimple quantities connected or joyned one to another by $+$ or $-$; ſo $a+b$ is a compound quantity, likewiſe $a-c$, alſo $a+b+c$, and $a+b-c$ are compound quantities.

XXVII. Every one of theſe four Characters, to wit, $+$, $-$, $\backsim$, and $\times$, (before defined in Sect. 20, 21, 22, and 23.) may ſometimes have reference to ſuch a Compound quantity as follows the ſign, and has a line drawn over every member of it. As, for Example, by $a+\overline{b \backsim c}$, you are to underſtand that the difference of the quantities b and c (whether the Exceſs be in b or in c) is added or to be added to the quantity a.

In like manner, $a-\overline{b+c}$ ſhews that the Compound quantity $b+c$ is ſubtracted or to be ſubtracted from the quantity a; where in regard of the line drawn over $b+c$, the ſign — hath reference to the ſubtraction of c as well as b from the quantity a. But if that line were omitted, then the ſign — would only refer to the next following ſimple quantity: As, $a-b+c$, (or $a+c-b$) ſignifies the ſubtraction of b only from $a+c$.

Moreover, $a \backsim \overline{b+c}$ ſignifies the difference between the ſimple quantity a, and the compound quantity $b+c$.

And $a\times\overline{b-c}$ ſignifies that the quantity a is multiplied or to be multiplied by the exceſs of the quantity b above the quantity c.

XXVIII. This Character $\surd$ is called a radical ſign, and ſignifies that the Square root of the number or quantity that ſtands next after the ſaid ſign $\surd$, is extracted, or to be extracted; as $\surd 25$ ſignifies the ſquare root of 25, to wit, 5; and $\surd 36$ ſignifies the ſquare root of 36, to wit, 6.

Like-

Likewiſe $\sqrt{ab}$ ſignifies the ſquare root of the quantity ab. So that when a number or quantity immediately follows the ſaid radical ſign $\sqrt{}$, the ſquare root of that number or quantity is thereby denoted.

But to deſign or repreſent the Root of a Power higher than a Square, ſome *Algebraical* Writers (whom in this matter I ſhall follow) are wont to write the Index of the Power within a Circle next after the ſign $\sqrt{}$; As for Example, $\sqrt{(3)}27$ ſignifies the Cubic root of 27, to wit, 3. Likewiſe, $\sqrt{(4)}16$ denotes the Biquadrate root of 16, to wit, 2; that is, the root from whence 16 conſidered as the fourth Power is produced. Again, $\sqrt{(5)}243$ ſignifies the root from whence 243 conſider'd as the fifth Power is raiſed, which Root is 3. And if you pleaſe you may write $\sqrt{(2)}81$ to denote the ſquare root of 81, to wit, 9.

Likewiſe $\sqrt{(3)}a$ ſignifies the Cubic root of ſome number or quantity repreſented by a. Alſo $\sqrt{(4)}bc$ ſignifies the Biquadrate root of the Quantity bc.

Sometimes the Radical Sign belongs to as many of the following Quantities as have a Line drawn over them; as $\sqrt{}:\overline{b+c}:$ or, $\sqrt{(2)}:\overline{b+c}:$ ſignifies the Square root of the ſum of the Quantities b and c. Likewiſe $\sqrt{}:\overline{bb-c}:$ imports the Square root of the Remainder when the quantity c is ſubtracted from the Square of the quantity b. Which Roots, and ſuch like, are called *Univerſal Roots*.

Again, $d+\sqrt{}:\overline{bb-c}:$ ſignifies that the Quantity c is firſt to be ſubtracted from the Square bb, and then the Square root of the Remainder is to be added to the quantity d. But that the Learner may the better perceive my meaning in the three laſt Examples concerning Univerſal Roots, let b ſignifie 4; bb, 16; c, 12; and d, 23. Then $\sqrt{}:\overline{b+c}:$ ſignifies $\sqrt{}:\overline{4+12}:$ that is, $\sqrt{16}$, to wit, 4. Alſo $\sqrt{}:\overline{bb-c}:$ ſignifies $\sqrt{}:\overline{16-12}:$ that is, $\sqrt{4}$, to wit, 2. And $d+\sqrt{}:\overline{bb-c}:$ ſignifies $23+2$, that is, 25. After the ſame manner the Univerſal Square root of $d+\sqrt{}:\overline{bb-c}:$ may be expreſt thus;

$\sqrt{}:\overline{d+\sqrt{bb-c}}:$ that is, 5.

XXIX. Four points ſet in this form :: are always in the middle of four Geometrical Proportionals, as, for Example, theſe four Numbers 2 . 4 :: 6 . 12 are Geometrical Proportionals, and to be read thus; As 2 is to 4, ſo is 6 to 12; or, (in the Phraſe of *The Rule of Three*) If 2 give 4, then 6 will give 12.

In like manner theſe four Quantities, $b\,.\,d :: c\,.\,a$ are to be read thus; As b is to d, ſo c to a, that is, look what proportion b has to d, the ſame proportion has c to a.

Alſo theſe four Quantities, $b+c\,.\,d-a :: f\,.\,g$ do intimate that the ſum of b and c has ſuch proportion to the Exceſs of d above a, as f has to g. The like is be underſtood of others.

XXX. This Character $\div$ ſet at the end of three or more Quantities, imports that they are Continual Proportionals Geometrical; ſo by 2 . 4 . 8 . 16 . 32 $\div$ it is ſignified that ſuch proportion as 2 has to 4, the ſame has 4 to 8, 8 to 16, and 16 to 32.

Likewiſe by theſe $a\,.\,b\,.\,c \div$ you are to underſtand that the quantity a has the ſame proportion to the quantity b, as b to c.

XXXI. This Character $=$ is the ſign of an Equation or Equality, and imports as much as the Word *Equal*; as $8+4=7+5$ ſignifies that the ſum of 8 and 4 is equal to the ſum of 7 and 5. Likewiſe $8=12-4$ that 8 is equal to 12 leſs 4, to wit, the exceſs of 12 above 4.

Again, $8\times 3=4\times 6$ denotes the Product of 8 multiplied by 3 to be equal to the Product of 4 into 6.

So alſo $a+b=c+d$ ſignifies that the ſum of the quantities a and b is equal to the ſum of the quantities c and d. This will be farther explained in the XI. Chapter.

XXXII. This Character $\sqsubset$ ſtands for the Word *Greater*, *viz.* it ſignifies that the Quantity which ſtands before, that is, on the left hand of the ſaid Character is greater than the quantity following the ſame; ſo $5\sqsubset 4$ muſt be read thus, 5 is greater than 4. Likewiſe $a+b\sqsubset c$ ſignifies that the Compound quantity $a+b$ is greater than the Simple quantity c. And $d\sqsubset a+c$ ſignifies that the quantity d is greater than $a+c$.

XXXIII. This Character $\sqsupset$ ſignifies that the quantity ſtanding before the Character is leſs than the quantity following the ſame; as $4\sqsupset 5$ muſt be read thus, 4 is leſs than 5. Likewiſe, $a+b\sqsupset c+d$ ſignifies that the compound quantity $a+b$ is leſs than the compound quantity $c+d$.

XXXIV. Quan-

XXXIV. Quantities, whether they be Simple or Compound, which are expreſt either wholly by Letters, or partly by Letters and partly by Numbers written upon one Line, are called Algebraical Integers, or whole Quantities; as theſe, $a, ab, cd+ff, a+3$, &c. But theſe quantities, $\frac{b}{c}, \frac{aa+bb}{a+c}, \frac{a+3}{b}$, and others ſo written, are called Algebraical Fractions, becauſe each of them like a Fraction in vulgar *Arithmetic* conſiſts of a Numerator placed above a Line, and a Denominator underneath.

CHAP. II.

Addition of Algebraical Integers.

I. ALgebraical *Addition* finds out the Sum or Aggregate of two or more Quantities expreſt either wholly by Letters, or partly by Letters and partly by Numbers.

II. The Operations in Algebraic Addition depend principally upon a diligent obſervation of three things, *viz.*

Firſt, You muſt obſerve whether the Quantities to be added be Like or Unlike.

Like Quantities are thoſe which are expreſt by the ſame Letters equally repeated in every one of the Quantities; ſuch are theſe, $a, 5a, -2a$, each of which is expreſt by the ſingle letter a. Alſo theſe are like quantities, $3aa, aa, -2aa$, each of which is expreſt by a double a, to wit, aa. Likewiſe theſe, $2ab, 3ab, -ab$ are called Like quantities becauſe every one of them is expreſt by the ſame Letters, to wit, ab.

Unlike Quantities are thoſe which are expreſt by different Letters, or elſe by the ſame letters unequally repeated; as, for Example, b and c are unlike quantities, becauſe they are expreſt by different letters; alſo $2abc$ and $2ab$ are unlike quantities, becauſe the letter c is in the one, but not in the other. Again, a and aa are unlike quantities, in regard the letter a is not equally repeated in both. The like is to be underſtood of others.

Secondly, You muſt obſerve whether the Signs (to wit, + and —) belonging to like quantities given to be added be Like or Unlike: As, for Example, theſe quantities $+2a$ and $+3a$ have like ſigns, the ſame ſign + being prefixt before each quantity. Alſo theſe quantities, $-2a$ and $-3a$ have like ſigns, the ſame ſign — being prefixt to each quantity; but theſe quantities $+2a$ and $-3a$ have unlike or different ſigns prefixt.

Thirdly, The Numbers prefixed before the Letters muſt be diligently obſerved, for their ſum or difference will be concern'd in Algebraical Addition, as will be manifeſt by the following Rules.

III. When two or more ſimple Algebraical Integers (or whole quantities) propos'd to be added or collected into one Sum are like, and have like ſigns, Firſt collect the numbers prefixt into one Sum; then to that Sum annex the letter or letters by which any one of the quantities propos'd is expreſt; laſtly, prefix the given ſign whether it be + or —, ſo ſhall this new quantity be the Sum deſired. As, for Example, if it be deſired to add a to a, or $+1a$ to $+1a$, the Sum will be $2a$ or $+2a$; for (according to the Rule) the Sum of the prefixed Numbers 1 and 1 is 2, to which I annex a and prefix + (or imagine it to be prefixed,) ſo $2a$ or $+2a$ is the Sum deſired.

$$\begin{array}{lc|c} \text{Add} & a & +1a \\ & a & +1a \\ \hline \text{Sum} & 2a & +2a \end{array}$$

In like manner, if to $-2b$ you would add $-b$, the Sum will be $-3b$. For the numbers prefixt are 2 and 1, which added together make 3, to which annexing b, and prefixing the given ſign —, there ariſes $-3b$, the Sum deſired.

$$\begin{array}{lc} \text{Add} & -2b \\ & -b \\ \hline \text{Sum} & -3b \end{array}$$

More

More Examples of the Rule of Addition in the foregoing Sect. III.

To be added,	$5a$ $3a$	$-5aa$ $-2aa$	$+7ab$ $+13ab$
The Sum,	$8a$	$-7aa$	$+20ab$

To be added,	ac $2ac$ $3ac$	$-3bcd$ $-bcd$ $-6bcd$	$+3a^3$ $+2a^3$ $+7a^3$
The Sum,	$6ac$	$-10bcd$	$+12a^3$

IV. When two simple Quantities propos'd to be added together be like, and have equal Numbers prefix'd, but unlike or contrary Signs, the Sum will be o, or nothing; for the affirmative Quantity will destroy or extinguish the Negative: As for Example, if it be required to add c, or $+c$, to $-c$, the Sum will be o; to wit, nothing. For supposing $-c$, or $-1c$ to be a Debt of one Crown that I owe; and $+c$, or $+1c$ to be one Crown in my Purse, it is evident that one Crown in ready Money will discharge or strike off a Debt of one Crown; and so that Debt and Credit being added or compared together, the Sum amounts to o.

$$\text{Add,}\ \begin{cases} -c \\ +c \end{cases}$$
$$\text{Sum,}\quad 0$$

In like manner, if it be desired to add $-6\,l.$ to $+6\,l.$ the Sum will be o; for if my whole Estate be worth but 6 Pounds, and I owe a Debt of 6 Pounds, it is manifest that my clear Estate is worth or amounts to just nothing.

$$\text{Add,}\ \begin{cases} +6\,l. \\ -6\,l. \end{cases}$$
$$\text{Sum,}\quad 0$$

More Examples of the Rule of Addition in the preceding Sect. IV.

To be added,	$+3a$ $-3a$	$-5abc$ $+5abc$	$+7ddd$ $-7ddd$
The Sum,	o	o	o

V. When two simple Quantities propos'd to be added together be like, but their Signs unlike, and the prefixed Numbers unequal between themselves; first subtract the lesser Number prefixed from the greater, then to the Remainder annex the Letter or Letters by which either of the Quantities proposed is exprest; lastly, before the said Remainder set the Sign which stands before the greater Number prefix'd, so shall this new Quantity be the Sum desired.

As for Example, if it be desired to add $-2a$ to $+3a$, the Sum will be a. For first Subtracting 2 from 3 the Remainder is 1, to which annexing a and prefixing $+$ (because $+$ belongs to that Quantity which has the greater Number prefix'd) there arises $+1a$, or $+a$ for the Sum sought.

$$\text{Add,}\ \begin{cases} +3a \\ -2a \end{cases}$$
$$\text{Sum,}\quad +1a,\ \text{or},\ +a$$

Again, to add $+b$ to $-3b$, I subtract 1 the lesser Number prefix'd, from 3 the greater, and to the Remainder 2 annexing b and prefixing $-$, (because $-$ belongs to $3b$ whose prefix'd Number 3 is greater than that of $+b$ or $+1b$) I find $-2b$ for the Sum desired.

$$\text{Add,}\ \begin{cases} -3b \\ +b \end{cases}$$
$$\text{Sum,}\quad -2b$$

Thus you see that this last Rule of Addition is performed by Subtraction, and may easily be understood under the Notion of discharging or paying off a Debt, or at least part of a Debt by so much ready Money or Credit, and then observing what Debt remains unpaid,

 or

or what Money or Credit remains as an overplus: So in the first of the two last Examples, you may conceive $+3a$ to be three Pounds in ready Cash, and $-2a$ to be a Debt of two Pounds; then comparing the said ready Money and Debt together, you will find by Subtraction that the clear Money remaining after the Debt is pay'd, will be one Pound, to wit, $+1a$ or a which is the Sum of the Quantities $+3a$ and $-2a$. Likewise in the latter Example, if $-3b$ be conceived to represent a Debt of three Pounds, and $+b$ or $+1b$ one Pound in ready Money; 'tis evident that this will strike off one Pound of that Debt, and so the Debt remaining will be two Pounds, to wit, $-2b$, which is the Sum of $-3b$ and $+b$.

More Examples of the Rule of Addition in the preceding Sect. V.

To be added,	$+5aa$ $-7aa$	$+6abcd$ $-4abcd$	$-8f^4$ $+3f^4$
The Sum,	$-2aa$	$+2abcd$	$-5f^4$

VI. When three or more simple Quantities propos'd to be added be like, but have unlike Signs; First, (by the Rule in *Sect.* III. of this *Chap.*) collect the Affirmative quantities into one Sum, and the Negative quantities into another; then (by *Sect.* IV. or V.) add those two Sums into one, so this last Sum shall be that which is sought.

As, for Example, If the Sum of these four Quantities, $7a$, $2a$, $-3a$, $-5a$ be desired; First, (by *Sect.* III.) the Sum of $7a$ and $2a$ is $+9a$; also the Sum of $-3a$ and $-5a$ is $-8a$; lastly (by *Sect.* V.) $+9a$ added to $-8a$ makes $+a$, that is, a, which is the Sum desired.

More Examples of the Rule of Addition in Sect. VI.

To be added,	$+5a$ $+3a$ $-8a$	$-2bc$ $+3ba$ $-4bc$	$+4d^5$ $+3d^5$ $-5d^5$
The Sum,	0	$-3bc$	$+2d^5$

To be added,	$+5ee$ $+2ee$ $-ee$ $-4ee$	$-4fff$ $-3fff$ $-2fff$ $+8fff$	$+4ggbb$ $-3ggbb$ $+2ggbb$ $-ggbb$
The Sum,	$+2ee$	$-fff$	$+2ggbb$

VII. When two or more Simple quantities given to be added be unlike, write them down one after another without altering their Signs; as, if the Number (or Line) a be to be added to the Number (or Line) b; I write $a+b$, or, $b+a$ for the Sum.

In like manner the Sum of these Quantities, a, b, c, may be written thus, $a+b+c$; or thus, $a+c+b$; or thus, $b+a+c$.

More Examples of the Rule of Addition in Sect. VII.

To be added,	$+3a$ $+2d$	$+aa$ $-bb$
The Sum,	$3a+2d$	$+aa-bb$

Again,

Again,

To be added,	$+ab$ $-ac$ $+ad$	$+5ddd$ $-3dd$ $-4d$
The Sum,	$+ab-ac+ad$	$+5ddd-3dd-4d$

Addition of Compound Algebraical Integers.

VIII. The Addition of Compound whole Quantities may easily be dispatch'd by the help of the Rules in the preceding *Sections* of this *Chapter*, as will appear by the following Examples.

First then, If this Compound quantity $a+b$ be to be added to $a+2b$, their Sum is $a+b+a+2b$, that is $2a+3b$; for $a+a$ makes $2a$; and $+b+2b$ makes $+3b$.

Again, The Sum of these two Compound quantities $3b+5a$ and $2b-2a$ is $3b+5a+2b-2a$, that is, $5b+3a$; for $3b+2b$ makes $5b$; and (by *Sect.* V.) $+5a-2a$ makes $+3a$.

Likewise, The Sum of these two Compound quantities $5ee+3f-8$ and $3ee-2f+6$ will be found $8ee+f-2$: For $5ee$ added to $3ee$ makes $8ee$; also $+3f$ added to $-2f$ gives $+f$, and -8 added to $+6$ makes -2.

After the same manner, $3a-8$ added to $10-a$ makes $2a+2$; (for $+3a$ added to $-a$ makes $+2a$, and -8 added to $+10$ gives $+2$.)

Again, The Sum of these two Compound quantities $a+b$ and $c-d$ is $a+b+c-d$, which Sum admits of no Contraction, in regard all the Simple quantities are unlike.

More Examples of the Addition of Compound whole Quantities.

To be added,	$a+b$ $a-b$	$aa+2a-3$ $aa+a-6$
The Sum,	$2a$	$2aa+3a-9$

To be added,	$aa-2ab$ $aa+ab$	$4c-d+3$ $-4c+2d-2$
The Sum,	$2aa-ab$	$d+1$

To be added,	$2ee+3ef-ff$ $-3ee+5ef$	$a^3-abc+6$ $+3abc-6$
The Sum,	$-ee+8ef-ff$	a^3+2abc

To be added,	$-aaa+2bba$ $8aaa+4bba$ $6aaa-6bba$	$aa-5a+24$ $aa+a-17$ $-2aa+2a+12$
The Sum,	$13aaa$	$-2a+19$

To be added,	$a+b$ $c-d$ $e+f$	$5b^3+24$ $-2b^3+40$ $6b^3-64$
The Sum,	$a+b+c-d+e+f$	$9b^3$, or, $9bbb$

CHAP. III.

Subtraction in Algebraic Integers.

I. *Algebraical Subtraction* takes one Quantity, whether it be expreſs'd by a Letter or Letters, or partly by Letters and partly by Number, out of, or from an-ther, in ſuch manner, that if the Remainder be added (according to the Rules of Algebraic Addition) to the Quantity ſubtracted, the Sum will be always equal to the ſaid other Quantity.

II. A general Rule to find out the Remainder in all caſes of Algebraical Subtraction is this: Firſt, joyn both the given Quantities together, by writing one after the other; but with this caution, that every Sign of the Quantity given to be ſubtracted, be ever changed into the contrary Sign, *viz.* + into — and — into +; then ſhall the Sum of both Quantities ſo connected be the Remainder ſought, which is to be contracted (when it may be done) into the feweſt and ſmalleſt Terms, by the Rules of Algebraical Addition.

As for Example, If from $5a$ it be deſired to ſubtract $3a$, firſt, I write down $5a$, then next after the ſame I write $-3a$; (where obſerve, that according to the Rule above given I change +, the Sign belonging to $3a$ the Quantity given to be ſubtracted, into —,) ſo there ariſes $5a-3a$, which being contracted (by the Rule of Addition in *Sect.* V. *Chap.* II.) makes $2a$ the Remainder ſought.

Out of	$5a$
Subtract	$3a$
Remainder,	$5a-3a$
Remainder contracted,	$2a$

Likewiſe, if from $3b$ it be deſired to ſubtract $-2b$, I firſt write down $3b$, and next after the ſame I write $+2b$; ſo $3b+2b$, that is, $5b$ is the Remainder ſought; where obſerve (as before) that I change the Sign —, which belongs to $2b$ the Quantity propos'd, to be taken out of $3b$, into the contrary Sign +. But that the ſaid $5b$ is a true Remainder, we may prove by Addition; for $+5b$ added to $-2b$ the Quantity ſubtracted, makes $+3b$, which is the Quantity out of which the ſaid $-2b$ was ſubtracted.

Out of	$+3b$
Subtract	$-2b$
Remainder,	$3b+2b$
Remainder contracted,	$5b$

Moreover, if a be to be ſubtracted from a, the Remainder will be $a-a$, that is, 0 or nothing. And if from $2b$ there be ſubtracted $-4b$, the Remainder will be $2b+4b$, that is, $6b$.

Likewiſe, if from $-2m$ it be required to ſubtract $-m$, the Remainder will be found $-2m+m$, that is, $-m$. In every one of which Examples you may obſerve that the Sign of the Quantity propos'd to be ſubtracted is changed into the contrary Sign.

Again, if from $2bc$, it be deſired to ſubtract $2ab$, the Remainder will be $2bc-2ab$ which, becauſe it conſiſts of unlike Quantities, cannot be contracted into fewer or leſſer Terms, by any of the Rules of Algebraical Addition. But according to the definition of Subtraction, the ſaid $2bc-2ab$ is a true Remainder, for if it be added to $2ab$ the Quantity ſubtracted, the Sum is $2bc$, which is the Quantity out of which the ſaid $2ab$ was ſubtracted.

Out of	$2bc$
Subtract	$2ab$
Remainder,	$2bc-2ab$

More Examples of Subtraction in Simple Algebraic Integers.

Out of	$2b$	$+3c$	$-2n$
Subtract	b	$-c$	$-n$
Remainder,	$2b-b$	$+3c+c$	$-2n+n$
Remainder contracted,	b	$+4c$	$-n$

Again,

Again,

Out of	$3a$	$-8d$	$-a$
Subtract	$5a$	$-10d$	$+a$
Remainder,	$3a-5a$	$-8d+10d$	$-a-a$
Remainder contracted,	$-2a$	$+2d$	$-2a$

Out of	$-bcd$	$-4rs$	$+4abc$
Subtract	$-bcd$	$+9rs$	$-abc$
Remainder,	$-bcd+bcd$	$-4rs-9rs$	$+4abc+abc$
Remainder contracted,	0	$-13rs$	$+5abc$

From	d	$-2b$	$+a^3$
Subtract	e	$-3a$	$-3a$
Remainder,	$d-e$	$-2b+3a$	$+a^3+3a$

From	$8bbd$	$+3abcd$
Subtract	$7bbb$	$-7aa$
Remainder,	$8bbd-7bbb$	$+3abcd+7aa$

Nor will the Operation be otherwise in the Subtraction of Compound Algebraic Integers; as for Example, if from this Compound quantity $3a+2b$, it be desired to subtract $a+3b$. First I write down $3a+2b$, then next after the same I write $-a-3b$, where observe, that the Sign $+$ which belongs to a, and also to $3b$, in the Quantity propos'd to be subtracted, is changed into the contrary Sign $-$ (according to the Rule of Subtraction before given; (so the Remainder sought is $3a+2b-a-3b$, that is, $2a-b$, (by *Sect.* V. *Chap.* II.)

From	$3a+2b$
Subtract	$a+3b$
Remainder,	$3a+2b-a-3b$
Remainder contracted,	$2a-b$

Again, If from $2a+b$, it be desired to subtract $5a-6b$, the Remainder will be $2a+b-5a+6b$, that is, $7b-3a$ for (according to the Rule of Algebraical Subtraction) I joyn together the two given Quantities, changing only the Signs of $+5a-6b$ (the Quantity to be Subtracted) into the contrary Signs, so there arises $2a+b-5a+6b$ which contracted (by the Rules of Addition in *Sect.* III. and V. of *Chap.* II.) make $7b-3a$, which is the Remainder sought, as will easily appear by the Proof.

Out of	$2a+b$
Subtract	$5a-6b$
Remainder,	$2a+b-5a+6b$
Remainder contracted,	$7b-3a$

Likewise, to subtract $c-d$ from $a+b$, I change the Signs of $c-d$ into the contrary Signs; *viz.* instead of $c-d$, I take $-c+d$, which added to $a+b$ makes $a+b-c+d$, which because it consists altogether of unlike Quantities, cannot be contracted into fewer Terms, and therefore the said $a+b-c+d$ is the Remainder sought, to wit, that which arises by subtracting $c-d$ from $a+b$.

From	$a+b$
Subtract	$c-d$
Remainder,	$a+b-c+d$

After the same manner, $cd+36$ subtracted from $3aa+bc+24$ leaves $3aa+bc+24-cd-36$, that is, $3aa+bc-cd-12$.

More

More Examples of Subtraction in Compound Algebraic Integers.

Out of	$a+b$	$3c-8$
Subtract	$a-b$	$c+5$
Remainder,	$a+b-a+b$	$3c-8-c-5$
Remainder contracted,	$+2b$	$2c-13$

Out of	$5a-4b$	$29e$
Subtract	$3a-3b$	$-3e+7$
Remainder,	$5a-4b-3a+3b$	$29e+3e-7$
Remainder contracted,	$2a-b$	$32e-7$

Out of	$aa+2ba+bb$	$-2cd+6$
Subtract	$+4ba$	$+cd-2$
Remainder,	$aa+2ba+bb-4ba$	$-2cd+6-cd+2$
Remainder contracted,	$aa-2ba+bb$	$-3cd+8$

Out of	$5a^3+27$	$3aa+6$
Subtract	$-8+3a^3$	$-3dd$
Remainder,	$5a^3+27+8-3a^3$	$3aa+6+3dd$
Remainder contracted,	$2a^3+35$	

From	$a+b$	$aa-bb$
Subtract	$c-d$	$-cc+dd$
Remainder,	$a+b-c+d$	$aa-bb+cc-dd$

III. The reason of changing the Signs of the Quantity to be subtracted into their contraries, to wit $+$ into $-$, and $-$ into $+$ (according to the Rule before given) will be manifest from a serious Consideration of the definition of Subtraction, which requires that the Sum of the Quantity subtracted and the Remainder be equal to the quantity from which the Subtraction is made: for first, (according to the said Rule) the Remainder is always compos'd of both the quantities propos'd for Subtraction, with this Caution, that the Signs $+$ and $-$ in the quantity to be Subtracted be changed into the contrary Signs; Secondly, (according to Algebraical Addition) the quantity to be subtracted with its own signs being added to it self with contrary signs, will destroy or extinguish it self; therefore the Sum of the Remainder and the Quantity to be Subtracted will necessarily be equal to the Quantity from which the Subtraction was made: And therefore the certainty of the said Rule of Algebraical Subtraction, and the Reason of changing the Signs of the Quantity to be subtracted into their contraries, to wit, $+$ into $-$, and $-$ into $+$, is manifest: So if from $a+b$ there be subtracted $a-b$, the Remainder (according to the Rule of Algebraical Subtraction before given) will be $a+b-a+b$, to which if $a-b$ (the quantity subtracted) be added, it is evident that $a-b$ will destroy $-a+b$, and so the Sum will be $a+b$, to wit, the quantity from which $a-b$ was subtracted.

CHAP.

CHAP. IV.

Multiplication in Algebraic Integers.

I. A*Lgebraical Multiplication* does by two Quantities, whether they be expreſs'd by Letters wholly, or partly by Letters and partly by Numbers, find out a third Quantity, which is called the Product, the Fact, or the Rectangle.

The Quantities given to be multiplied one by the other are called Factors; or (as in vulgar Arithmetic) either of them may be called the Multiplicand, and the other the Multiplicator or Multiplier.

II. When two Simple (or ſingle) Quantities expreſs'd by Letters, whether like or unlike, be to be multiplied by one another, and have no Numbers prefix'd to them, join the Letters of both Quantities together, like Letters in a Word, it matters not in what order they be written; then the new Quantity repreſented by the Letters ſo ſet together is the Product ſought.

As for Example, If the Number or Line a be to be multiplied by it ſelf, to wit, by a, I write aa for the Product: So alſo to multiply a by b, I write ab or ba for the Product; in like manner if I would multiply abc by bc, I write $abcbc$, or $abbcc$, or $accbb$, &c. for the Product.

And if a, b, and c, be to be multiplied one into another; firſt a multiplied by b produces ab, then ab multiplied by c produces abc, or bac, or bca, to wit, the Product made by the continual Multiplication of the three Quantities a, b, and c.

Again, if aa be to be multiplied by ba, the Product will be $aaab$; which may alſo be written thus, a^3b; where the Learner muſt diligently note that the Figure 3 which ſtands next after but a little higher than a, muſt not be taken as a Number prefix'd to b, but as an Index to ſhew the number of Dimenſions in a^3, or aaa, (as before has been ſaid in *Sect.* XVI. and XVII. *Chap.* I.)

Likewiſe, if aaa be to be multiplied by aaa, or a^3 by a^3, the Product will be $aaaaaa$, or a^6, in which latter way of expreſſing the Product, the Index 6 ſtanding at the Head of a is the Sum of 3 and 3 the Indices of the Quantities a^3 and a^3 propos'd to be multiplied.

So the Product made by the Multiplication of $bbbb$ by bbb or b^4 by b^3 will be $bbbbbbb$, or b^7 (7 being the Sum of the Indices 4 and 3.).

Likewiſe if theſe three Quantities be to be multiplied continually, to wit, $aaaaa$, $bbbb$ and ccc, the Product may be expreſs'd thus, $aaaaabbbbccc$, or compendiouſly thus, $a^5b^4c^3$: and ſo of others.

More Examples of Multiplication in ſimple Algebraic Integers, according to the preceding Sect. II.

Multiplicand,	b	d	ac	ccc
Multiplicator,	c	d	d	cc
Product,	bc	dd	acd	$ccccc$

Multiplicand,	$aabc$	def	$aabbcc$
Multiplicator,	bca	abc	$aabbcc$
Product,	$aaabbcc$	$abcdef$	$a^4b^4c^4$

III. If two ſimple Quantities, whether like or unlike, having Numbers prefix'd before them, be to be multiplied one by the other; firſt multiply the Numbers prefix'd, one into the other, then to this Product annex the Letters of both Quantities, by ſetting them immediate-

immediately one after another, (as before in *Sect.* II.) so this new Quantity shall be the Product sought.

As, for Example, if it be desired to multiply $2a$ by $3b$; first I multiply 2 by 3, and the Product is 6; to which annexing ab, (to wit, the Letters found in both Quantities given to be multiplyed) there arises $6ab$ the Product sought; which shews that six times the Product of the Multiplication of any two Numbers, or Right-lines, a and b, is equal to the Product made by the Multiplication of the Double of a by the Triple of b.

Multiply	$2a$
by	$3b$
Product,	$6ab$

In like manner, if $2b$ be multiplyed be c the Product will be $2bc$, or $2cb$; for 2 which is prefix'd to b in the Multiplicand, being multiplied by 1, which is suppos'd to be prefix'd to the Multiplier c, makes 2, to which annexing bc, there is found $2bc$ for the Product sought.

Multiply	$2b$
by	c
Product,	$2bc$

More Examples of Multiplication in Simple Algebraic Integers, according to Sect. III.

Multiply	$4b$	$12ac$	$5ddfg$
by	$2a$	$3d$	dgb
Product,	$8ab$	$36acd$	$5d^3fggb$

Multiply	aaa	$3a^3$	$16aab$
by	$3bbb$	b^3	4
Product,	$3aaabbb$	$3a^3b^3$	$64aab$

IV. The Multiplication of Compound quantities depends upon the precedent Rules of multiplying simple quantities; for when a Compound quantity is to be multiplied by a simple (or single) quantity, every Member of that must be multiplied by this; also, when two compound quantities are to be mutually multiplied, every Member of the one must be multiplied into every Member of the other. It matters not whether you begin to multiply at the right Hand or the left, nor in what order the particular Products be set; (for quantities express'd by Letters retain their peculiar and unaltered values wheresoever they stand;) but due regard must be had to the Signs $+$ and $-$, one of which always belongs to every particular Product, and may be discovered by this Rule, *viz.* $+$ multiplied by $+$, or $-$ by $-$, makes $+$ in the Product; but $+$ multiplied by $-$, or $-$ by $+$, makes $-$ in the Product; lastly, all the particular Products added together (according to the Rules in the preceding *Chap.* 2.) make the total Product sought: All which will be made manifest by the following Examples.

First, if a Compound quantity, as $a+b$, be to be multiplied by a simple quantity, as c, I begin at the left Hand, and multiplying $+a$ by $+c$ the Product is $+ac$, (for $+$ multiplied by gives $+$;) likewise $+b$ multiplied by $+c$ produces $+bc$; which two Products added together make $ac+bc$, which is the Product of the Multiplication of $a+b$ by c.

Multiply	$a+b$
by	c
Product,	$ac+bc$

So if $a-b$ be to be multiplied by c, the Product will be $ac-bc$. For $+a$ multiplied by $+c$ produces $+ac$; and $-b$ multiplied by $+c$ produces $-bc$; (for according to the Rule, $-$ multiplied by $+$ gives $-$:) Therefore $+ac-bc$ or $ac-bc$ is the Product sought.

Multiply	$a-b$
by	c
Product,	$ac-bc$

After

After the same manner, if it be desired to multiply $a+b$ by $c+d$, the Product will be found $ac+bc+ad+bd$. For, first $a+b$ being multiplied by c, (as in the first Example) produces $+ac+bc$; likewise $a+b$ again multiplied by d, produces $+ad+b$; then adding those Products together, the Sum is $ac+bc+ad+bd$, which is the required Product of $a+b$ multiplied by $c+d$.

Multiply	$a+b$
by	$c+d$
	$+ac+bc$
	$+ad+bd$
Product,	$+ac+bc+ad+bd$

Again, if $a-b$ be multiplied by $c-d$ the Product will be $ac-bc-ad+bd$: For First, $a-b$ multiplied by c produces $ac-bc$, (as in the last Example but one;) then $a-b$ again multiplied by $-d$ produces $-ad+bd$; (for according to the Rule, $+a$ multiplied by $-d$ produces $-ad$, and $-b$ by $-d$ produces $+bd$.) Lastly, those particular Products added together make $ac-bc-ad+bd$, which is the Product of $a-b$ multiplied by $b-c$.

Multiply	$a-b$
by	$c-d$
Product,	$ac-bc-ad+bd$

Likewise, if $a+b$ be multiplied by $a-b$, the Product will be $aa+bb$: For first, $a+b$ multiplied by a produces $aa+ba$; then $a+b$ multiplied by $-b$ produces $-ba-bb$; lastly, the said Products $aa+ba$ and $-ba-bb$ added together make $aa-bb$; (for $+ba$ and $-ba$ by Addition do quite vanish;) Therefore $aa-bb$ is the Product of $a+b$ multiplied by $a-b$.

Multiply	$a+b$
by	$a-b$
	$aa+ba$
	$-ba-bb$
Product,	$aa \quad -bb$

Moreover, if $aa-ab+bb$ be multiplied by $a+b$, the Product will be only $aaa+bbb$; for the rest of the particular Products will vanish by Addition.

And if $a+b$ be multiplied by it self, to wit, by $a+b$, the Product will be $aa+2ab+bb$, which is the Square of $a+b$.

Likewise the Square of $a-b$ will be found $aa-2ab+bb$.

Nor will the Operation be otherwise when Numbers are prefixed to compound Quantities proposed to be multiplied, respect being had to the Third *Sect.* of this *Chap.* as, for Example, to multiply $3a-2e$ by $3a-2e$; First, $3a-2e$ multiplied by $3a$ produces $9aa-6ae$, and $3a-2e$ again multiplied by $-2e$ produces $-6ae+4ee$; which particular Products added together make $9aa-12ae+4ee$ which is the Square of $3a-2e$.

Multiply	$3a-2e$
by	$3a-2e$
	$+9aa-6ae$
	$-6ae+4ee$
Product,	$9aa-12ae+4ee$

When absolute Numbers are members of Quantities to be multiplied, the Rules of Multiplication in vulgar Arithmetic and those before given must be mixtly observed; as,

If it be desired to multiply	$3a+6$
By the absolute Number	5
The Product will be	$15a+30$

For five times $3a$ makes $15a$, and five times 6 makes 30.

Likewise, if $2aa-3$ be multiplied by $a-6$, the Product will be $2aaa-12aa-3a+18$, and the work will stand as here you see;

Multipicand,	$2aa-3$
Multiplicator,	$a-6$
	$+2aaa-3a$
	$-12aa+18$
Product,	$2aaa-12aa-3a+18$

For further illustration of the Multiplication of Algebraic Integers, the Learner may peruse the following Examples; in every one of which, as also in those afore-going, I begin to multiply at the left Hand, because in Algebraical Multiplication it being a thing

indifferent

indifferent to begin the work either at the right Hand or the left, it will be eaſier to write forward than backward. And as to the placing of the particular Products, there is no neceſſity of obſerving any Order therein; for whether they be written upon one, two, or more Lines, they retain the ſame values, and muſt by Algebraical Addition be collected into one Sum, to make the total Product: And therefore you may either write the particular Products all upon one Line when there is room, or elſe upon ſo many ſeveral Lines as there be particular Multipliers, ſetting like Products (when they happen) under one another to facilitate their Addition; or otherwiſe, as you ſhall find it moſt convenient.

More Examples of Multiplication in Compound Algebraic Integers, according to Sect. IV.

Multiplicand,	$a+e$	$2b-3d$	$5g-8$
Multiplicator,	d	f	6
Product,	$da+de$	$2bf-3fd$	$30g-48$

Multiplicand,	$5a+3c$	$2b+3$
Multiplicator,	$3a-2c$	$4b-6$
	$+15aa+9ca$ $-10ca-6cc$	$8bb+12b$ $-12b-18$
Product,	$15aa+9ca-10ca-6cc$	$8bb+12b-12b-18$
Product contracted,	$15aa-ca-6cc$	$8bb-18$

Multiplicand,	$3dd+4de+ee$
Multiplicator,	$3dd-ee$
	$+9dddd+12ddde+3ddee$ $-3ddee-4deee-eeee$
Product,	$9dddd+12ddde+3ddee-3ddee-4deee-eeee$
Product contracted,	$9d^4+12d^3e-4de^3-e^4$

Multiplicand,	$a+e$	$a+e$
Multiplicator,	$a+e$	$a-e$
	$aa+ae$ $+ae+ee$	$aa+ae$ $-ae-ee$
Product,	$aa+2ae+ee$	$aa-ee$

Multiplicand,	$4aaa+3aa-2a+1$
Multiplicator,	$aa-5a+6$
	$4aaaaa+3aaaa-2aaa+aa$ $-20aaaa-15aaa+10aa-5a$ $+24aaa+18aa-12a+6$
Product,	$4aaaaa-17aaaa+7aaa+29aa-17a+6$

Again,

Again,

$$\begin{array}{ll}
\text{Multiplicand,} & 2aa+3ba-bc \\
\text{Multiplicator,} & 3aa-3ba-cc \\
\hline
 & 6aaaa+9baaa-3bcaa \\
 & \quad -4baaa-6bbaa+2bbca \\
 & \qquad\qquad -2ccaa-3bcca+bccc \\
\hline
\text{Prod.} & 6aaaa+5baaa\left.\begin{array}{l}-3bc\\-6bb\\-2cc\end{array}\right\}aa\left.\begin{array}{l}+2bbc\\-3bcc\end{array}\right\}a+bccc
\end{array}$$

V. Sometimes when Compound quantities be to be multiplied one by the other, it will be very commodious to omit the Operation, and to ſet only the word *into*, or × (the Sign of Multiplication) between the Quantities to be multiplied, to ſignifie the Product of their Multiplication : But in ſuch Caſe, to avoid Miſtake, it will be convenient to draw a Line over each Compound quantity, to ſhew that every Member of the one is to be multiplied by every Member of the other.

As to multiply $4aaa+3aa-2a+1$ by $aa-5a+6$, I write

$$\overline{4aaa+3aa-2a+1} \text{ into } \overline{aa-5a+6}$$

Or, $$\overline{4aaa+3aa-2a+1} \times \overline{aa-5a+6}$$

But that + multiplied by —, or — by + makes — ; alſo, that — multiplied by — makes + in the Multiplication of compound Quantities, I ſhall hereafter make manifeſt in the laſt *Sect.* of *Chap.* XI.

CHAP. V.

Diviſion in Algebraic Integers.

I. ALgebraical *Diviſion* does by two Quantities, (whether they be expreſs'd wholly by Letters, or partly by Letters and partly by Numbers,) whereof one is called the Dividend, and the other the Diviſor, find out a Third called the Quotient ; to wit, ſuch a Quantity, that if it be multiplied by the Diviſor, the Product will be equal to the Dividend.

II. The Nature of Diviſion is to reſolve or undo that which is compoſed or done by Multiplication ; for the Dividend always repreſents the Fact or Product in Multiplication, the Diviſor one of the two Factors or Multipliers, and the Quotient the other. As, if 12 be to be divided by 2, the Dividend 12 repreſents the Fact or Product made by the Multiplication of two Numbers, one of which is the Diviſor 2, and the other is the Quotient ſought, to wit, 6.

III. Every Fraction is equal to the Quotient of the Numerator divided by the Denominator : So $\frac{3}{4}$ is the Quotient of 3 divided by 4 ; for, according to the Proof of Diviſion, if the Quotient $\frac{3}{4}$ be multiplied by the Diviſor 4, the Product will be equal to the Dividend 3. Upon this ground, Diviſion in Algebraic Integers, whether Simple or Compound is moſt commonly performed ; *viz.* by ſetting the Dividend as the Numerator of a Fraction, and the Diviſor as a Denominator ; for this Fraction is equal to the Quotient ſought.

As for Example, to divide the Quantity a by b, I write $\frac{a}{b}$, which ſignifies that that a is divided by b ; or $\frac{a}{b}$ is equal to the Quotient of the Quantity a divided by the Quantity b.

In like manner, if b be propos'd to be divided by ac, I write $\frac{b}{ac}$ to repreſent the Quotient; alſo, if ac be to be divided by b, I write $\frac{ac}{b}$ to ſignifie the Quotient.

Again, If $2ab$ be given to be divided by $3cd$, the Quotient will be $\frac{2ab}{3cd}$; and if a be to be divided by 5, I write for the Quotient $\frac{a}{5}$; alſo to divide 1 by a, I write $\frac{1}{a}$ to ſignifie the Quotient.

So alſo, if $a+b$ be given to be divided by c, the Quotient may be repreſented by $\frac{a+b}{c}$; and if $3a$ be to be divided by $2b-c$, the Quotient is $\frac{3a}{2b-c}$.

More Examples of Diviſion in Algebraic Integers, according to the foregoing Sect. III.

Dividend,	bb	$2de$	$3abc$	a^4b
Diviſor,	a	fg	$2dd$	$2d^3$
Quotient,	$\frac{bb}{a}$	$\frac{2de}{fg}$	$\frac{3abc}{2dd}$	$\frac{a^4b}{2d^3}$

Dividend,	$aa+bb$	$2ab-3bd$	aaa
Diviſor,	c	$d+e$	$a+b-c$
Quotient,	$\frac{aa+bb}{c}$	$\frac{2ab-3bd}{d+e}$	$\frac{aaa}{a+b-c}$

Dividend,	$4aa$	$2cc+5dd$
Diviſor,	3	3
Quotient,	$\frac{4aa}{3}$, or $\frac{4}{3}aa$	$\frac{2cc+5dd}{3}$, or, $\frac{2}{3}cc+\frac{5}{3}dd$.

IV. When the Dividend is equal to the Diviſor, the Quotient is 1; for every Quantity contains it ſelf once, and therefore being divided by it ſelf gives 1 in the Quotient: As to divide 4 by 4 the Quotient is 1; likewiſe, a divided by a gives 1 for the Quotient; alſo, if $a+b$ be divided by $a+b$ the Quotient is 1; and if $3a+2cd$ be divided by $3a+2cd$ the Quotient is 1. The like is to be underſtood of others.

V. When the Quotient is expreſſed Fraction-wiſe, (according to *Sect* III) if the ſame letter or letters be found equally repeated in every member of the Numerator and Denominator, caſt away thoſe letters, ſo the remaining Quantities ſhall ſignifie the Quotient.

As, for Example, If ab be to be divided by a, the Quotient expreſt Fraction-wiſe will be $\frac{ab}{a}$; But becauſe the letter a is found in the Numerator and Denominator, I caſt away a out of both, ſo b only is left, which is the Quotient of ab divided by a.

Likewiſe, If aa be divided by a the Quotient is $\frac{aa}{a}$, that is, a; (by caſting away a out of the Numerator and Denominator.)

Again, If aaa be to be divided by aa, the Quotient will be $\frac{aaa}{aa}$, that is, a; by caſting away aa out of the Numerator and Denominator. And if abc be to be divided by ab, the Quotient expreſt Fraction-wiſe will be $\frac{abc}{ab}$, that is, c, after ab is caſt out of the Numerator and Denominator.

After the ſame manner, if a^5 be propos'd to be divided by a^3, (that is, $aaaaa$ by aaa) the Quotient will be a^2, or aa, by expunging a^3 (or aaa) out of the Dividend and Diviſor.

This

This Contraction of Division is like to the reducing of a Fraction exprest by large numbers to more simple Terms, by dividing the Numerator and also the Denominator by a common Divisor.

Again, If $ab+ac$ be to be divided by $ad-af$, the Quotient exprest Fraction-wise according to the preceding *Sect.* III. will stand thus, $\frac{ab+ac}{ad-af}$, where because the letter a is found in every member of the Numerator and Denominator, it may be quite struck out, and then the new Quotient will be $\frac{b+c}{d-f}$, which Fraction is equal to the former, and exprest by more simple Terms.

Dividend,	$ab+ac$
Divisor,	$ad-af$
Quotient,	$\left\{\frac{ab+ac}{ad-af}\right.$
Quotient contracted,	$\left\{\frac{b+c}{d-f}\right.$

Likewise, If $ab+a$ be divided by a, the Quotient (according to *Sect.* III.) will be $\frac{ab+a}{a}$, that is, $b+1$; for by casting away a, there will remain $\frac{b+1}{1}$, that is, $b+1$; (for $\frac{b}{1}$ is but b; and $\frac{1}{1}$ is 1;) but that $b+1$ is the true Quotient it will appear by the proof of Division, for $b+1$ Multiplied by the Divisor a will produce the Dividend $ab+a$.

So also to divide $3bc-2b$ by $2bb+b$, I write $\frac{3c-2}{2b+1}$ for the Quotient; where observe, that altho' the letter b be cast out of every Member of the given Dividend and Divisor, yet the number prefixt to the letter cast out must stand still in the new Quotient.

But note diligently, That in this kind of Division of Compound Algebraic Integers, a letter cannot be cancell'd or cast away, unless it be found in every Member of the Dividend and Divisor; and therefore this Quotient $\frac{bc+cd}{c+f}$ cannot be contracted by casting away any letter.

More Examples of Contractions in Algebraic Division, according to the Preceding Sect. V.

Dividend,	aab	$ddef$	abc	a^7
Divisor,	aa	ef	b	a^3
Quotient,	$\frac{aab}{aa}$	$\frac{ddef}{ef}$	$\frac{abc}{b}$	$\frac{a^7}{a^3}$
Quotient contracted,	b	dd	ac	a^4

Dividend,	$ab+ac-a$	$ab-2a$
Divisor,	a	$3a$
Quotient,	$\frac{ab+ac-a}{a}$	$\frac{ab-2a}{3a}$
Quotient contracted,	$b+c-1$	$\frac{b-2}{3}$, or, $\frac{1}{3}b-\frac{2}{3}$.

Dividend,	$2abd+3bd$	$2ba^3+caa-3aa$
Divisor,	$3bb-b$	$baa-daa+aa$
Quotient,	$\frac{2ad+3d}{3b-1}$	$\frac{2ba+c-3}{b-d+1}$

VI. If

VI. If an Algebraic Integer, whether Simple or Compound, be to be divided by a simple Quantity, and there be such numbers prefix'd to the letters in the Dividend and Divisor as may all be severally divided by some number as a common Divisor without leaving a Remainder, set the Quotients arising by the Division of those numbers by their common Divisor, before the letters respectively, instead of the numbers that were first prefix'd: As, for Example, if $8a$ be to be divided by $6b$; First, the Quotient exprest Fraction-wise (according to *Section* III. of this *Chap.*) will be $\frac{8a}{6b}$, then dividing the prefixed Numbers 8 and 6 by their common Divisor 2, I set the Quotients 4 and 3 instead of 8 and 6 before a and b; so the Quotient sought is $\frac{4a}{3b}$.

In like manner, $6abc - 3dbc$ Divided by $9fbc$ gives the Quotient $\frac{2a-d}{3f}$;

Dividend,	$6abc - 3dbc$
Divisor,	$9fbc$
Quotient,	$\frac{6abc - 3dbc}{9fbc}$
Quotient contracted	$\frac{2a-d}{3f}$

For first, the Dividend and Divisor being set Fraction-wise will stand thus, $\frac{6abc-3dbc,}{9fbc}$; then, (according to *Sect.* V.) bc is to be cast out of the Numerator and Denominator; lastly, the prefixed numbers 6, 3, and 9 being divided by their common Divisor 3, give 2, 1, and 3, which being set before the remaining letters a, d and f respectively, give the contracted Quotient $\frac{2a-1d}{3f}$ or $\frac{2a-d}{3f}$.

More Examples of Contractions in Division, according to Sect. V. *and* VI.

Dividend,	$4cd$	$27ab$	$16gb$
Divisor,	$2c$	$9ad$	$8gb$
Quotient,	$\frac{4cd}{2c}$	$\frac{27ab}{9ad}$	$\frac{16gb}{8gb}$
Quotient contracted,	$2d$	$\frac{3b}{d}$	2

Dividend,	$18aaaa$	$30b^5c^4dd$
Divisor,	$6aa$	$5bbccd$
Quotient,	$\frac{18aaaa}{6aa}$	$\frac{30b^5c^4dd}{5bbccd}$
Quotient contracted	$3aa$	$6b^3ccd$

Dividend,	$28bbc + 16bbd$
Divisor,	$20bb$
Quotient,	$\frac{28bbc + 16bbd}{20bb}$
Quotient contracted,	$\frac{7c+4d}{5}$, or, $\frac{7}{5}c + \frac{4}{5}d$.

VII. If every Member of a Compound quantity be multiplied by one and the same simple quantity, it is evident from the Nature of Multiplication and Division, that if the Product of that Multiplication be divided by the said Compound Quantity, the Quotient will be the simple Quantity.

As, for Example, If $b+c$ be multiplied by a the Product will be $ba+ca$, and therefore $ba+ca$ divided by the Factor $b+c$ will give the other Factor a. And for

for the ſame Reaſon, $2bca+a$, that is $2bca+1a$, divided by $2bc+1$ will give the Quotient a.

Likewiſe, If $6a+5a-a$ (that is $10a$) be divided by $6+5-1$ (that is, 10,) the Quotient will be a.

Again, If $2ba+2ca+2da$ be divided by $b+c+d$, the Quotient will be $2a$; and if $2baa+caa-daa-aa$ be divided by $2b+c-d-1$, the Quotient will be aa.

More Examples of Contractions in Diviſion, according to the preceding Sect. VII.

Dividend,	$2da+3ca$	$23b+18b+1b$
Diviſor,	$2d+3c$	$23+18+1$
Quotient,	a	b

Dividend,	$2baa-3caa$	$2af-2bf+2cf-6f$
Diviſor,	$2b-3c$	$a-b+c-3$
Quotient	aa	$2f$

VIII. When the Dividend and Diviſor are Compound whole Quantities, the precedent Rules of Algebraical Diviſion will not always give the Quotient in the leaſt Terms; but the ſimpleſt Quotient may be found out by one of theſe two ways, *viz.*

1. When the Dividend and Diviſor are Algebraic Integers, and there is a poſſibility of expreſſing the Quotient by an Algebraical Integer, it may be found out by the general Method of Diviſion handled in the next following *Section*, which way is like that of dividing whole Numbers in Vulgar Arithmetic; but if the Learner find it difficult, he may wave it until he has proceeded as far as the 8. *Chapter* of the 2. *Book.*

2. The Quotient, whether it happen to be an Algebraic Integer, or a Fraction, may be found out in its leaſt Terms by the Method hereafter delivered in *Sect.* 7. *Chap.* 8. of the Second Book; where the manner of finding out all the *Aliquot Parts* or juſt Diviſors, every one of which will divide the Dividend and Diviſor propos'd without any Remainder is exhibited.

IX. In this *Section* a general Method of Diviſion in Algebraical Integers is handled. As to the order of the Work, it agrees with that form of Diviſion in whole Numbers which I have explained in Mr. *Wingate*'s Arithmetic, but the Work it ſelf depends upon the preceding Rules of Algebraical *Diviſion*, *Multiplication* and *Subtraction*, as alſo upon this Rule for diſcovering the due Sign belonging to every particular Quotient, *viz.* + divided by +, or — by —, gives + in the Quotient; but + divided by —, or — by +, gives — in the Quotient. Whether the Operation be begun at the Right Hand or the Left, it matters not; but becauſe 'tis eaſier to Write forwards than backwards, I ſhall (as in Vulgar Arithmetic) begin to Divide at the Left Hand, and proceed towards the Right.

Example 1. Let it be required to divide $ac+ad+bc+bd$ by $c+d$.

Having placed the Dividend and Diviſor in ſuch order as you ſee in the next Page, firſt I divide $+ac$ by $+c$, (according to *Sect.* 5. of this *Chap.*) and there ariſes $+a$, ($+a$, becauſe + divided by + gives +,) therefore I write $+a$ or a in the Quotient; then Multiplying the whole Diviſor $c+d$ by the ſaid Quotient a, I write the Product $ac+ad$ under the two firſt Members of the Dividend towards the Left Hand, to wit, under $ac+ad$; that done, drawing a Line under the ſaid Product $ac+ad$, I ſubtract the ſame from $ac+ad$, (the two firſt Members of the Dividend) and there remains o, which I ſet under the Line, as you may ſee in the Page following.

$$\begin{array}{llr} \text{Divisor.} & \text{Dividend} & \text{Quotient.} \\ c+d) & ac+ad+bc+bd & (a+b \\ & ac+ad & \\ \hline & 0\quad 0 \quad +bc+bd & \\ & \quad\quad +bc+bd & \\ \hline & \quad\quad 0 \quad 0 & \end{array}$$

Then there remains to be divided $+bc+bd$ which I bring down to the Remainder o, and renew the Work, *viz.* I divide $+bc$ by $+c$, and there ariſes $+b$ which I write in the Quotient next after a; then multiplying the whole Diviſor $c+d$ by the ſaid Quotient b, the Product is $bc+bd$, which being ſubſcribed, and ſubtracted from that which remained to be divided, there remains o. So the Diviſion is finiſhed, and the Quotient is found $a+b$; but that it is a true Quotient the Proof will make manifeſt; for $a+b$ multiplied by the Diviſor $c+d$ produces the Dividend $ac+ad+bc+bd$.

Example 2. In like manner, if $aa-bb$ be to be divided by $a+b$ the Quotient will be found $a-b$; For firſt, aa divided by a gives a in the Quotient, by which multiplying the whole Diviſor $a+b$ the Product is $aa+ab$, which ſubtracted from the Dividend $aa-bb$, there remains to be divided $-bb-ab$. Now I renew the Work, and divide $-bb$ by its correſpondent Diviſor $+b$, (not by a, becauſe the Quotient will be a Fraction, which is to be avoided when there is a poſſibility) and there ariſes $-b$ to be written next after a in the Quotient, I ſay $-b$, not $+b$; for according to the Rule before given, $-$ divided by $+$ gives $-$ in the Quotient; then multiplying the whole Diviſor $a+b$ by $-b$ (laſt ſet in the Quotient) the Product is $-ab-bb$, or $-bb-ab$, which ſubtracted from $-bb-ab$ that remained to be divided, there remains o; ſo the Diviſion is finiſh'd and the Quotient is found $a-b$, to wit, ſuch a Quantity that if it be multiplied by the Diviſor $a+b$, it will produce the Dividend $aa-bb$.

$$\begin{array}{l} a+b)\ aa-bb\ (a-b \\ \quad\quad aa+ab \\ \hline \quad\quad\quad -bb-ab \\ \quad\quad\quad -bb-ab \\ \hline \quad\quad\quad 0 \quad 0 \end{array}$$

Example 3. Again, If it be deſired to divide $aaa+bbb$ by $aa-ba+bb$, the Quotient will be found $a+b$, and the Work will ſtand thus:

$$\begin{array}{l} aa-ba+bb)\ aaa+bbb\ \ldots\ldots\ldots\ (a+b \\ \quad\quad\quad\quad aaa-baa+bba \\ \hline \quad\quad\quad\quad +bbb+baa-bba \\ \quad\quad\quad\quad +bbb+baa-bba \\ \hline \quad\quad\quad\quad\quad 0 \quad 0 \quad 0 \end{array}$$

In which Example, firſt (as before) I begin at the firſt Term of the Dividend towards the Left Hand, and dividing aaa by aa, (not by $-ba$ nor by $+bb$, becauſe each of theſe will give a Fraction in the Quotient) there ariſes a, which I ſet in the Quotient; then Multiplying the whole Diviſor $aa-ba+bb$ by the ſaid Quotient a, the Product is $aaa-baa+bba$, which I ſubtract from the Dividend $aaa+bbb$; ſo there remains to be yet divided $+bbb+baa-bba$.

Now I renew the Work, and divide $+bbb$ by its correſpondent Diviſor $+bb$, (not by $+aa$, nor by $-ba$, becauſe each of theſe gives a Fraction) and there ariſes $+b$, which I write next after a in the Quotient; then multiplying the whole Diviſor $aa-ba+bb$ by the ſaid Quotient $+b$, the Product is $bbb+baa-bba$, which I ſet under, and ſubtract from the Quantity that remained to be divided, ſo there remains o, and the Quotient ſought is $a+b$: But that it is a true Quotient the Proof will diſcover; for if the Diviſor $aa-ba+bb$ be multiplied by the Quotient $a+b$, it will produce the Dividend $aaa+bbb$.

Exam-

Example 4. In like manner, if $aaa-bbb$ be divided by $aa+ba+bb$, the Quotient will be $a-b$, and the work will stand thus;

$$\begin{array}{lll} \text{Divisor.} & \text{Dividend.} & \text{Quotient.} \\ aa+ba+bb) & aaa-bbb \;\ldots\ldots\ldots & (a-b \\ & aaa+baa+bba & \\ \hline & -bbb-baa-bba & \\ & -bbb-baa-bba & \\ \hline & 0 \quad 0 \quad 0 & \end{array}$$

Example 5. Again, If $9dddd+12ddde-4deee-eeee$ be to be divided by $3dd-ee$, the Quotient will be found $3dd+4de+ee$, as will be manifest by the subsequent Operation.

$$\begin{array}{ll} 3dd-ee) & 9dddd+12ddde-4deee-eeee\;(3dd+4de+ee \\ & 9dddd-\;3ddee \\ \hline & +12ddde+3ddee-4deee \\ & +12ddde \qquad\quad -4deee \\ \hline & +3ddee-\;eeee \\ & +3ddee-\;eeee \\ \hline & 0 \quad 0 \end{array}$$

In which Example, first I divide $9dddd$ by $3dd$, and it gives $3dd$, which I write in the Quotient; then multiplying the whole Divisor $3dd-ee$ by the said Quotient $3dd$, the Product is $9dddd-3ddee$, which I write under the two first Members of the Dividend, and subtract the same from the said two Members, so there remains $+12ddde$ $+3ddee$; to which I bring down $-4deee$ (the next Member of the Dividend) and it makes $+12ddde+3ddee-4deee$ which comes now to be divided; therefore I renew the work, and dividing $+12ddde$ by $+3dd$, it gives $+4de$, which I set in the Quotient next after $3dd$; then multiplying the whole Divisor, $3dd-ee$ by the said Quotient $+4de$, the Product is $+12ddde-4deee$, which I write under $+12ddde+3ddee$ $-4deee$ (the Quantity last set apart to be divided;) and having drawn a Line under the said Product I subtract it from the said particular Dividend, so there remains $+3ddee$ which I write underneath the Line; that done, to the said Remainder $+3ddee$ I bring down $-eeee$, (the last Member of the total Dividend) and it makes $+3ddee$ $-eeee$ which is yet to be divided: Therefore I renew the Work, and dividing $+3ddee$ by $+3dd$, it gives $+ee$ which I set in the Quotient next after $+4de$; (or I might here divide $+3ddee$ by $-ee$ in regard it will give an Algebraical Integer in the Quotient, as I shall shew in the next Example:) then multiplying the Divisor $3dd-ee$ by $+ee$, (last set in the Quotient,) and subtracting the Product $+3ddee-eeee$ from the Quantity that remained to be divided, there now remains o. So the Division is finished without any Quantity remaining, and the entire Quotient is $+3dd+4de+ee$.

Note. By this general Method of Division the Quotient may oftentimes be found out and express'd various ways, both as to the Order and Multitude of Members in the Quotient, but yet the entire Quotient in each Form will have one and the same value, as will appear by the following manner of Dividing the two Quantities propos'd in the last Example.

Let it therefore be again propos'd to divide $9dddd+12ddde-4deee-eeee$ by $3dd-ee$.

First, I work as before in the last Example to find out the two first Members in the Quotient, to wit, $3dd+4de$, and then there remains to be divided $+3ddee-eeee$ which you see stands at this Mark ∗ in the following Operation: Now because $+3ddee$ divided by $-ee$ gives an Algebraic Integer for the Quotient, to wit, $-3dd$, therefore I write $-3dd$ in the Quotient; then multiplying the whole Divisor $3dd-ee$ by $-3dd$ (last set in the Quotient) I subtract the Product $+3ddee-9dddd$ from $+3ddee$ $-eeee$ which remained to be divided; so there remains to be yet divided $-eeee+9dddd$.

$$
\begin{array}{l}
3dd-ee)\ 9dddd+12ddde-4deee-eeee\ (\ 3dd+4de \\
\qquad\quad 9dddd-\ 3ddee \qquad\qquad\qquad (-3dd+ee+3dd \\
\hline
\qquad\qquad\quad +12ddde+3ddee-4deee \\
\qquad\qquad\quad +12ddde \qquad\quad -4deee \\
\hline
\qquad\qquad\qquad * \ +3ddee-eeee \\
\qquad\qquad\qquad\quad +3ddee-9dddd \\
\hline
\qquad\qquad\qquad\qquad\quad -eeee+9dddd \\
\qquad\qquad\qquad\qquad\quad -eeee+3ddee \\
\hline
\qquad\qquad\qquad\qquad\qquad +9dddd-3ddee \\
\qquad\qquad\qquad\qquad\qquad +9dddd-3ddee \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad o\ (-\ o
\end{array}
$$

Then I divide $-eeee$ (which ſtands immediately under the third black Line) by its correſpondent Diviſor $-ee$, (for it cannot be divided by $3dd$ ſo as to give an Integer in the Quotient,) and there ariſes $+ee$, which I ſet in the Quotient; then multiplying the whole Diviſor $3dd-ee$ by the ſaid Quotient $+ee$ the Product is $-eeee+3ddee$, which ſubtracted from $-eeee+9dddd$ (to wit, the Quantity that remained to be divided) there remains to be yet divided $+9dddd-3ddee$, (which ſtands immediately under the laſt black Line but one;) therefore I divide $+9dddd$ by $+3dd$ and it gives $+3dd$ to be ſet in the Quotient; then multiplying the whole Diviſor $3dd-ee$ by the ſaid $+3dd$, it makes $+9dddd-3ddee$, which ſubtracted from $+9dddd-3ddee$ (the Quantity that remained to be divided) leaves o; ſo the Diviſion is finiſhed without any Quantity remaining, and the Quotient is found $3dd+4de-3dd+ee+3dd$, that is, $3dd+4de+ee$: So that the Quotient found out by the latter Operation, after it is contracted by Algebraical Addition, is the ſame found out by the former way of dividing the Quantities given in the fifth Example.

Example 6. Again, If $yyyyyy-8yyyy-124yy-64$ be divided by $yy-16$, the Quotient will be found $yyyy+8yy+4$, and the Work will ſtand thus:

$$
\begin{array}{l}
\text{Diviſor.} \qquad\qquad \text{Dividend.} \qquad\qquad \text{Quotient.} \\
yy-16)\ yyyyyy-\ 8yyyy-124yy-64\ (yyyy+8yy+4 \\
\qquad\qquad yyyyyy-16yyyy \\
\hline
\qquad\qquad\quad +\ 8yyyy-124yy \\
\qquad\qquad\quad +\ 8yyyy-128yy \\
\hline
\qquad\qquad\qquad\quad +\ 4yy-64 \\
\qquad\qquad\qquad\quad +\ 4yy-64 \\
\hline
\qquad\qquad\qquad\qquad o\quad o
\end{array}
$$

If the Powers of the Root y in the laſt Example be expreſſed according to *Carteſius* his way, the work will ſtand thus:

$$
\begin{array}{l}
yy-16)\ y^6-\ 8y^4-124yy-64\ (\ y^4+8yy+4 \\
\qquad\qquad y^6-16y^4 \\
\hline
\qquad\qquad\quad +\ 8y^4-124yy \\
\qquad\qquad\quad +\ 8y^4-128yy \\
\hline
\qquad\qquad\qquad\quad +\ 4yy-64 \\
\qquad\qquad\qquad\quad +\ 4yy-64 \\
\hline
\qquad\qquad\qquad\qquad o\quad o
\end{array}
$$

But

But *Cartesius* in dividing the Quantities propos'd in the last Example works backwards, *viz.* from the right Hand of the Dividend towards the left, as you here see in the following Operation.

$$yy-16\ )\ y^6-8y^4-124yy-64\ (4+8yy+y^4$$
$$+4yy-64$$
$$-8y^4-128yy$$
$$+8y^4-128yy$$
$$y^6-16y^4$$
$$y^6-16y^4$$
$$\circ\quad\circ$$

More Examples are here added for the fuller exercise and illustration of Division in compound Algebraic Integers, according to the general Method in Sect. IX. *of this* Chapter.

Divisor. Dividend. Quotient.

$$2c-3d)\ 6ca-9da-8bc+12db\ (3a-4b$$
$$6ca-9da$$
$$\circ\quad\circ\ -8bc+12db$$
$$-8bc+12db$$
$$\circ\quad\circ$$

Divisor. Dividend. Quotient.

$$a-b)\ aaa-3aab+3abb-bbb\ (aa-2ab+bb$$
$$aaa-aab$$
$$-2aab+3abb$$
$$-2aab+2abb$$
$$+abb-bbb$$
$$+abb-bbb$$
$$\circ\quad\circ$$

Divisor. Dividend. Quotient.

$$2aa+3bb)\ 4aaaa+12aabb+9bbbb\ (2aa+3bb$$
$$4aaaa+6aabb$$
$$+6aabb+9bbbb$$
$$+6aabb+9bbbb$$
$$\circ\quad\circ$$

$$a+b)\ aaa-abb\ (aa-bb-ab+bb,\ \text{that is, } aa-ab$$
$$aaa+aab$$
$$-abb-aab$$
$$-abb-bbb$$
$$-aab+bbb$$
$$-aab-abb$$
$$+bbb+abb$$
$$+bbb+abb$$
$$\circ\quad\circ$$

Again,

Again,

$$
\begin{array}{llll}
\text{Divifor.} & \text{Dividend.} & & \text{Quotient.} \\
ab-aa) & aab^3+a^4b-2a^5 & & (abb+a^3+aab+a^3 \\
 & aab^3-a^3bb & & \\
\hline
 & +a^4b+a^3bb-2a^5 & & \\
 & +a^4b \qquad -a^5 & & \\
\hline
 & +a^3bb-a^5 & & \\
 & +a^3bb-a^4b & & \\
\hline
 & -a^5+a^4b & & \\
 & -a^5+a^4b & & \\
\hline
 & 0 \quad 0 & &
\end{array}
$$

$$
\begin{array}{lll}
\text{Divifor.} & \text{Dividend.} & \text{Quotient.} \\
\frac{2}{3}ab-\frac{1}{2}aa) & \frac{4}{9}aab^3+\frac{13}{12}a^4b-a^5. & (\frac{2}{3}abb+\frac{13}{8}a^3+\frac{1}{2}aab+\frac{3}{8}a^3 \\
 & \frac{4}{9}aab^3-\frac{1}{3}a^3bb & (\ \textit{viz.}\ \frac{2}{3}abb+\frac{1}{2}aab+2a^3 \\
\hline
 & +\frac{13}{12}a^4b+\frac{1}{3}a^3bb-a^5 & \\
 & +\frac{13}{12}a^4b \qquad -\frac{13}{16}a^5 & \\
\hline
 & +\frac{1}{3}a^3bb-\frac{3}{16}a^5 & \\
 & +\frac{1}{3}a^3bb-\frac{1}{4}a^4b & \\
\hline
 & -\frac{3}{16}a^5+\frac{1}{4}a^4b & \\
 & -\frac{3}{16}a^5+\frac{1}{4}a^4b & \\
\hline
 & 0 \quad 0 &
\end{array}
$$

If Algebraical Divifion according to this general Method will not work off juft without a Remainder, then you may write the Dividend and Divifor fraction-wife, according to *Sect.* III. of this *Chap.* Or fometimes the Quotient may be exprefs'd partly by Integers, and partly by a Fraction; as if $bb+bd+cc$ be to be divided by $b+d$, the Quotient may be exprefs'd either thus $\frac{bb+bd+cc}{b+d}$; or elfe thus, $b+\frac{cc}{b+d}$, which latter Quotient is found out by the help of the faid general Method; for after you have thereby difcovered as many Integers as can arife in the Quotient, you may fet the Remainder of the Dividend as a Numerator over the Divifor as a Denominator, fo this Fraction together with the faid Integer or Integers fhall be equal to the Quotient fought; as in this following Example.

$$
\begin{array}{lll}
\text{Divifor.} & \text{Dividend.} & \text{Quotient.} \\
a-b) & 2aac+3aaa-2abc-3aab+2cc & (2ac+3aa+\frac{2cc}{a-b}. \\
 & 2aac \qquad -2abc & \\
\hline
 & +3aaa \qquad -3aab & \\
 & +3aaa \qquad -3aab & \\
\hline
 & 0 \qquad 0\ +2cc &
\end{array}
$$

CHAP.

CHAP. VI.

Containing the Arithmetic of Algebraical Fractions.

Of the rise of Algebraic Fractions, and the manner of expressing Integers and mixed Quantities fraction-wise.

I. THE Operations about Algebraic Fractions are wrought like those of vulgar Fractions, by the help of the Rules of Algebraic Integers before delivered, as will appear by the following Rules of this *Chapter*.

II. From the manner of dividing Quantities according to *Sect.* 3. of the preceding *Chap.* 5. Algebraic Fractions arise; as, if a be to be divided by b, the Quotient is represented by the Fraction $\frac{a}{b}$: Likewise $\frac{a+b}{c-d}$, which imports as much as the Quotient of $a+b$ divided by $c-d$; also $\frac{2aa+3cd}{bb}$, and such like, are called Algebraical Fractions.

III. If the Numerator be equal to the Denominator, that Fraction (or Quotient express'd fraction-wise) is equal to 1, (to wit, Unity;) as before hath been said in *Sect.* 4. *Chap.* 5.

So $\frac{aa}{aa}=1.$ And $\frac{abc+dd}{abc+dd}=1.$

IV. When an Algebraic Integer is to be express'd fraction-wise, make it a Numerator, and set 1 for the Denominator; as if these Quantities ab and $aa-bb$ be to be set in the Form of Fractions they will stand thus;

$\frac{ab}{1}$. And $\frac{aa-bb}{1}$.

V. If an Algebraic Integer, as a, be to be set in the Form of a Fraction that shall have for its Denominator some Algebraical Integer prescribed, as d, multiply a by the Denominator d, and write the Product ad as a Numerator over the Denominator d, thus, $\frac{ad}{d}$; which Fraction is equal to the Integer a first proposed, and hath for its Denominator the prescribed Quantity d.

Likewise the Quantity a reduced to the Form of a Fraction whose Denominator is prescribed $b+c$ will stand thus, $\frac{ab+ac}{b+c}$.

Moreover, if $a+\frac{aa}{d}$ be to be reduced to the Form of a Fraction that shall have d for a Denominator; let a be multiplied by the Denominator d, and to the Product ad add the Numerator aa; then set that Sum, to wit, $ad+aa$ over the Denominator d, so there will be $\frac{ad+aa}{d}$ for the Fraction desired. More Examples of this Rule are these following.

$\frac{bc}{c}=b.$	$\frac{aa+ab}{a+b}=a.$	$\frac{dda}{a}=dd.$

$\frac{bc+bb}{c}=b+\frac{bb}{c}$	$\frac{ab-ac+dd}{b-c}=a+\frac{dd}{b-c}.$

How

How to reduce Algebraic Fractions to others of the same value in more simple Terms.

VI. When the same Letter or Letters be found in the Numerator and Denominator, let them be cast out of both; and if the Numbers prefix'd can be abbreviated by some common Divisor set the Quotients in the places of those Numbers prefix'd, so shall the new Fraction be of the same value with that first proposed: So this Fraction $\frac{abc}{abd}$ will be reduced to $\frac{c}{d}$; and this $\frac{12ab+8ac}{16ad}$ will be reduced to $\frac{3b+2c}{4d}$. This Rule hath already been explained in *Sect.* 5. and 6. of *Chap.* 5. and may be further illustrated by these following Examples.

$\frac{dd}{ac}=\frac{d}{c}$.	$\frac{12add}{4abc}=\frac{3dd}{bc}$.
$a+\frac{bcd}{cd}=a+b$.	$\frac{36aa}{4ba+16da}=\frac{9a}{b+4d}$.

VII. The searching out of the greatest common Divisor, for reducing an Algebraic Fraction to the smallest Terms, after the manner used in vulgar Arithmetic, is for the most part a tedious and intricate work, especially when the Numerator and Denominator are compound Quantities consisting of many Members; and therefore instead of that way of finding out a common Measure (or Divisor,) I shall by a clear Method in *Chap.* 8. of the Second *Book*, shew how to find out all such Divisors as will divide the Numerator and Denominator precisely without leaving a Remainder. But in the mean time I shall recommend to the Learners exercise the following Examples of Fractions abbreviated by Division according to the general Method in *Sect.* 9. *Chap.* 5. of this Book; which Examples, together with the Rule above-delivered in the 6. *Sect.* will be great helps to reduce Algebraical Fractions to lower terms, when there is a possibility.

Examples of Fractions reduced to their smallest Terms.

$\frac{aa+ab}{a+b}=a$	$\frac{aa-ab}{a-b}=a.$
$\frac{aac-aad}{c-d}=aa$	$\frac{aa+2ba+bb}{a+b}=a+b.$
$\frac{a^4+2b^2a^2+b^4}{aa+bb}=aa+bb$	$\frac{aa-2ba+bb}{a-b}=a-b.$
$\frac{a^4-2b^2a^2+b^4}{aa-bb}=aa-bb$	$\frac{aa-bb}{a+b}=a-b.$
$\frac{aaaa-bbbb}{aa+bb}=aa-bb$	$\frac{aa-bb}{a-b}=a+b.$
$\frac{aaaa-bbbb}{aa-bb}=aa+bb$	$\frac{aaa+bbb}{aa-ba+bb}=a+b.$

$\dfrac{aaa+bbb}{a+b}=aa-ba+bb$	$\dfrac{aaa-bbb}{aa+ba+bb}=a-b.$
$\dfrac{aaa-bbb}{a-b}=aa+ba+bb$	$\dfrac{aaa-abb}{aa-ab}=a+b.$
$\dfrac{aaa-abb}{aa+ab}=a-b$	$\dfrac{aaaa-bbbb}{aaa-aab+abb-bbb}=a+b.$

More Examples of Fractions abbreviated.

$\dfrac{aa+ab}{ad+bd}=\dfrac{a}{d}.$ (By the common Divisor $a+b$)

$\dfrac{aa-ab}{ac-bc}=\dfrac{a}{c}.$ (By the common Divisor $a-b$)

$\dfrac{aac-aad}{cd-dd}=\dfrac{aa}{d}.$ (By the common Divisor $c-d$)

$\dfrac{aaa-abb}{aa+2ab+bb}=\dfrac{aa-ab}{a+b}.$ (By $a+b$.)

$\dfrac{aaa-bbb}{aa-bb}=\dfrac{aa+ba+bb}{a+b}.$ (By $a-b$.)

$\dfrac{a^4-b^4}{aa+ab}=\dfrac{aaa-aab+abb-bbb.}{a}$ (By $a+b$.)

How to find out the smallest Quantity that can be divided by two or more given Quantities severally without a Remainder.

VIII. Two or more Algebraic Quantities whether Simple or Compound being proposed, the smallest Quantity that can be divided by every one of those given, without a Remainder, may be found out by the following Operation, (which is grounded upon 36 *Prop.* 7. *Elem.* Euclid.) and the Use thereof will hereafter appear.

As for Example, if it be desired to find the smallest Quantity that can be divided by aac and cd; set them in the Form of a Fraction thus, $\dfrac{aac}{cd}$, and reduce the Fraction to its primitive or equivalent Fraction in the smallest Terms $\dfrac{aa}{d}$, which being set near the former, multiply cross-wise, *viz.* aac by d, or aa by cd, and there will arise one and the same Product, to wit $aacd$ the Quantity sought; which is the smallest Quantity that can be divided severally by aac and cd without leaving any Remainder.

$$\frac{aac}{cd}\times\frac{aa}{d}$$
$$aacd$$

In

In like manner to find the smallest Quantity that can be divided by $ab+ac$ and $ad-af$ severally, I set them Fraction-wise thus, $\frac{ab+ac}{ad-af}$, this reduced to its lowest Terms gives $\frac{b+c}{d-f}$; then I multiply cross-wise (as before) *viz.* $ab+ac$ by $d-f$ or $ad-af$ by $b+c$, and there arises $abd+acd-fab-fac$, which is the smallest Quantity that can be divided by $ab+ac$ and $ad-af$, so as to leave no Remainder.

$$\frac{ab+ac}{ad-af} \times \frac{b+c}{d-f}$$
$$abd+acd-fab-fac$$

IX. But if the given Quantities cannot be reduced to lower Terms, then multiply them one into another, and their Product is the Quantity desired. So to find the smallest Quantity that can be divided by $bb+cc$ and $dd+ff$ severally without leaving a Remainder; because $\frac{bb+cc}{dd+ff}$ cannot be reduced to more simple Terms, I multiply $bb+cc$ by $dd+ff$, and there is produced $bbdd+ccdd+bbff+ccff$ the Quantity sought.

$$\frac{bb+cc}{dd+ff} \times \frac{bb+cc}{dd+ff}$$
$$bbdd+ccdd+bbff+ccff$$

X. When three or more Quantities are given, the smallest Quantity that can be divided by them severally without leaving a Remainder may be found out in this manner; *viz.* To find out the least Quantity that can be divided by $aaa-abb$, $aa+2ab+bb$ and $aa-bb$; I first seek (after the Manner of the second Example in *Sect.* 8.) the smallest Quantity that can be divided by $aaa-abb$, and $aa+2ab+bb$, so I find $aaaa-aabb+aaab-abbb$; and because this Quantity may be also divided by $aa-bb$ (the third Quantity proposed) it is manifest that $aaaa-aabb+aaab-abbb$ is the Quantity sought.

$$\frac{aaa-abb}{aa+2ab+bb} \times \frac{aa-ab}{a+b}$$
$$aaaa-aabb+aaab-abbb$$

In like manner, if there be given these four Quantities, $aaaa-bbbb$; $aa+ab$; $aaaa+aabb$; and $a+b$; First, I find out (as before) the smallest Quantity $aaaaa-abbbb$ that can be divided by the first and second Quantities $aaaa-bbbb$ and $aa+ab$;

$$\frac{aaaa-bbbb}{aa+ab} \times \frac{aaa-aab+abb-bbb}{a}$$
$$aaaaa-abbbb$$

Then because the said $aaaaa-abbbb$ cannot be divided by the third Quantity $aaaa+aabb$, I seek the smallest Quantity that can be divided by $aaaaa-abbbb$ and $aaaa+aabb$, so I find (in like manner as before) $aaaaaa-aabbbb$, which, because it is divisible by the fourth Quantity proposed, to wit, by $a+b$ shall be the Quantity sought; *viz.* a^6-aab^4 is the smallest Quantity that can be divided by every one of these four Quantities, a^4-b^4; $aa+ab$; a^4+aabb; and $a+b$. And so of others.

$$\frac{aaaaa-abbbb}{aaaa+aabb} \times \frac{aa-bb}{a}$$
$$aaaaaa-aabbbb$$

How to reduce Algebraical Fractions which have different Denominators, into other Fractions of the same value that may have a common Denominator.

XI. When two Fractions having different Denominators are to be reduced into two other Fractions of the same Value that shall have a common Denominator; multiply the Numerator of the first Fraction by the Denominator of the second, and the Product shall be a new Numerator correspondent to the Numerator of that first Fraction; Also, multiply

Multiply the Numerator of the second Fraction by the Denominator of the first, and the Product is a new Numerator correspondent to the Numerator of the second Fraction; lastly, multiply the Denominators one by the other, and the Product shall be a common Denominator to both the new Numerators.

As, for Example, to reduce $\frac{ab}{c}$ and $\frac{bd}{a}$ (whose Denominators c and a are unlike) into two other Fractions that may be of the same value with those given, and have a common Denominator; First, I multiply cross-wise, *viz.* the Numerator ab by the Denominator a, and the Product is aab for a new Numerator instead of ab; likewise I multiply the Numerator bd by the Denominator c, and the Product is bdc, for a new Numerator instead of bd; lastly, the Denominators c and a multiplyed one by the other produce ac for a Denominator to each of those new Numerators aab and bdc: So the Fractions $\frac{aab}{ac}$ and $\frac{bdc}{ac}$ are found out which have a common Denominator ac, and are equal in value to the Fractions first given, *viz.* $\frac{aab}{ac}$ is equal to $\frac{ab}{c}$, and $\frac{bdc}{ac}$ is equal to $\frac{bd}{a}$, as was required.

$$\frac{ab}{c} \times \frac{bd}{a}$$

$$\frac{aab}{ac}, \quad \frac{bdc}{ac}$$

In like manner $\frac{aa}{7bc}$ and $\frac{2bb}{5d}$ (which have unlike Denominators) will be reduced into $\frac{5daa}{35bcd}$ and $\frac{14bbbc}{35bcd}$ which have a common Denominator.

Also, $\frac{12}{a}$ and $\frac{b}{5}$ will be reduced into these $\frac{60}{5a}$ and $\frac{ba}{5a}$.

Again, to reduce $\frac{aa+2bb}{c+d}$ and $\frac{3cc-dd}{ff}$ to a common Denominator, I multiply cross-wise (as before,) *viz.* $aa+2bb$ by ff, and $3cc-dd$ by $c+d$; so the Products are $aaff+2bbff$, and $3ccc-cdd+3ccd-ddd$ for New Numerators; then multiplying the Denominators $c+d$ and ff one into the other, the Product is $cff+dff$ for a common Denominator, and the Fractions sought are $\frac{aaff+2bbff}{cff+dff}$ and $\frac{3ccc-cdd+3ccd-ddd}{cff+dff}$.

XII. When three or more Fractions having unlike Denominators are to be reduced into as many other Fractions that may be of the same value, and have a common Denominator; multiply the Numerator of each Fraction into all the Denominators except its own, so the Products made by that continual Multiplication shall be new Numerators; multiply also all the Denominators one into another, and the Product shall be a Denominator to every one of the new Numerators.

As, for Example, To reduce these three Fractions $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{2ef}{g}$ into three others that may be of the same value and have a common Denominator; I multiply the Numerator a into the Denominators d and g, so the Product adg is a new Numerator instead of a; again, I multiply the Numerator c into the Denominators b and g, and the Product cbg is a Numerator instead of c; likewise, multiplying the Numerator $2ef$ into the Denominators b and d, the Product $2bdef$ is a Numerator instead of $2ef$; lastly, the Denominators b, d and g multiplied one into another produce bdg for a common Denominator to those three new Numerators, and the three Fractions sought are $\frac{adg}{bdg}$, $\frac{cbg}{bdg}$ and $\frac{2bdef}{bdg}$.

$$\frac{a}{b}, \quad \frac{c}{d}, \quad \frac{2ef}{g}$$

$$\frac{adg}{bdg}, \quad \frac{cbg}{bdg}, \quad \frac{2bdef}{bdg}$$

In

In like manner these three Fractions $\frac{aa+8}{bb}$, $\frac{9}{aa-8}$, and $\frac{dd}{7}$ will be reduced to to these three, to wit, $\frac{7aaaa-448}{7aabb-56bb}$, $\frac{63bb}{7aabb-56bb}$, and $\frac{aaddbb-8ddbb}{7aabb-56bb}$ which have for a common Denominator $7aabb-56bb$.

XIII. But if the Denominators of the given Fractions can be reduced to lower Terms, then those Fractions may oftentimes be reduced more compendiously than by the Rules in the two last preceding *Sections*, into others in the smallest Terms that have a common Denominator, in this manner; *viz.* Seek (by the Rules in *Sect.* 8. and 10. of this *Chap.*) the smallest quantity that can be divided by every one of the Denominators without a Remainder, which quantity reserve for a common Denominator; then for the Numerators divide the common Denominator by the Denominator of the first Fraction, and multiply the Quotient by the Numerator of the first Fraction, so shall the Product be a new Numerator instead of that first Numerator; work in like manner to find out the other Numerators, and set every one of them over the common Denominator before found out.

As, for Example, to reduce these Fractions $\frac{bbbd}{aac}$ and $\frac{aaa}{cd}$ to a common Denominator; I seek first of all the smallest quantity that can be divided by the Denominators aac and cd, and I find that quantity to be $aacd$, which shall be the common Denominator; then I divide the said $aacd$ by each of the given Denominators aac and cd, and multiply the Quotients d and aa by the given Numerators $bbbd$ and aaa, so the Products $bbbdd$ and $aaaaa$ shall be the new Numerators, which being severally set over the common Denominator $aacd$, there will arise $\frac{bbbdd}{aacd}$ and $\frac{aaaaa}{aacd}$ for the Fractions sought.

Likewise, to reduce $\frac{bbbb}{aac-aad}$ and $\frac{aaa+bbb}{cd-dd}$ to a common Denominator, having first found the common Denominator $aacd-aadd$, to wit, the least quantity that can be divided by the given Denominators $aac-aad$ and $cd-dd$, I divide the said common Denominator by the said given Denominators severally, and the Quotients d and aa I multiply by the Numerators $bbbb$ and $aaa+bbb$, and then setting the Products severally over the common Denominator, the Fractions sought will be $\frac{bbbbd}{aacd-aadd}$ and $\frac{aaaaa+aabbb}{aacd-aadd}$.

Again, to reduce these three Fractions, to wit, $\frac{a-b}{aaa-abb}$, $\frac{bb}{aa+2ab+bb}$ and $\frac{aa-ab}{aa-bb}$ to a common Denominator; First (as in the first Example in *Sect.* 10. of this *Chap.*) I seek the smallest quantity that can be just divided by every one of the three given Denominators, and I find $aaaa+aaab-aabb-abbb$, for a common Denominator; then dividing this quantity found by every one of the three given Denominators (according to the general Method in *Sect.* 9. *Chap.* 5.) the Quotients will be $a+b$, $aa-ab$ and $aa+ab$, that done, I multiply the first of those Quotients by the Numerator of the first Fraction; also the second Quotient by the second Numerator, and the third Quotient by the third Numerator; so the Products $aa-bb$, $aabb-abbb$ and $aaaa-aabb$ shall be new Numerators, which being severally set over the common Denominator first found, will give the Fractions sought, to wit, these:

$$\frac{aa-bb}{aaaa+aaab-aabb-abbb},$$

$$\frac{aabb-abbb}{aaaa+aaab-aabb-abbb},$$

$$\frac{aaaa-aabb}{aaaa+aaab-aabb-abbb}.$$

Nor

Nor will the Operation be otherwise to reduce these four Fractions to wit, $\frac{a^5}{a^4-b^4}$, $\frac{a^3-a^2b}{a^2+ab}$, $\frac{a^5-b^5}{a^4+a^2b^2}$ and $\frac{a^2+ab+b^2}{a+b}$, into these four following Fractions having a common Denominator.

1. $\frac{a^7}{a^6-a^2b^4}$,
2. $\frac{a^7-2a^6b+2a^5b^2-2a^4b^3+a^3b^4}{a^6-a^2b^4}$,
3. $\frac{a^7-a^5b^2-a^2b^5+b^7}{a^6-a^2b^4}$,
4. $\frac{a^7+a^5b^2-a^4b^3-a^2b^5}{a^6-a^2b^4}$.

For first by the help of the given Denominators, the smallest common Denominator a^6-aab^4 is found out by the Operation in the last Example of the preceding *Sect.* 10. (of this *Chap.*) then the said common Denominator being divided severally by the given Denominators a^4-b^4, $aa+ab$, a^4+aabb, and $a+b$; the Quotients are aa, $a^4-a^3b+aabb-ab^3$, $aa-bb$, and $a^5-a^4b+a^3bb-aab^3$; which multiplied respectively by the given Numerators a^5, a^3-aab, a^5-b^5, and $aa+ab+bb$, will produce those new Numerators which are before set over the common Denominator a^6-aab^4.

Addition of Algebraical Fractions.

XIV. If two or more Fractions to be added have one common Denominator, add the Numerators together, and set their Sum as a new Numerator over the common Denominator, so shall this new Fraction be the Sum of the Fractions given to be added.

As, for Example, to add $\frac{aa}{c}$ to $\frac{bb}{c}$, the Sum will be $\frac{aa+bb}{c}$.

So also, $\frac{2ab}{c+d}$ added to $\frac{3bb}{c+d}$ makes $\frac{2ab+3bb}{c+d}$.

Likewise the Sum of $\frac{5a-3b}{c+d}$ and $\frac{2b-3a}{c+d}$ will be found $\frac{2a-b}{c+d}$; (For the given Numerators $5a-3b$ and $2b-3a$ added together make $2a-b$.)

Again, the Sum of $\frac{a-b+24}{c+5}$, $\frac{a+b-24}{c+5}$ and $\frac{4a}{c+5}$ will be found $\frac{6a}{c+5}$.

And if these be added, to wit, $\frac{3ab}{b+c+d}$, $a+\frac{3ac}{b+c+d}$, and $\frac{3ad}{b+c+d}$; the Sum will be $a+\frac{3ab+3ac+3ad}{b+c+d}$; that is, $4a$. (For by Division, $\frac{3ab+3ac+3ad}{b+c+d}=3a$.)

XV. But if the Fractions propos'd to be added together have unlike Denominators, first reduce them to a common Denominator, and then add them as before; as to add $\frac{ab}{c}$ to $\frac{bd}{a}$, first I reduce them to $\frac{aab}{ac}$ and $\frac{bdc}{ac}$ which have the same Denominator ac; then setting the Sum of the Numerators aab and bdc over the common Denominator ac, there will be $\frac{aab+bdc}{ac}$ for the Sum required.

So also to add $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{2ef}{g}$, their Sum will be found $\frac{adg+cbg+2bdef}{bdg}$.

Likewise, to add these three Fractions $\frac{a-b}{aaa-abb}$, $\frac{bb}{aa+2ab+bb}$, and $\frac{aa-ab}{aa-bb}$; first I reduce them to three others of the same value under a common Denominator, (as in the third Example of the preceding 13. *Sect.*) and then setting the Sum of the three new Numerators over the common Denominator, I find the Sum of the given Fractions to be $\frac{aaaa+aa-abbb-bb}{aaaa+aaab-aabb-abbb}$.

XVI. When mixed quantities are to be added together, collect the Fractions into one Sum, and the Integers into another, then those two Sums added together give the Sum desired; as for Example:

To add these mixed quantities . . . $\frac{aa}{b}-a$ and $\frac{dd}{c}+d$.

The Sum of the Fractions, after they are reduced to a common Denominator, is . . . . $\frac{caa+bdd}{bc}$

To which Sum adding the Integers in the mix'd quantities proposed, the Sum desir'd will be $\frac{caa+bdd}{bc}-a+d$.

Or, when mixed quantities are to added together, you may reduce them to improper Fractions, (by *Sect.* 5. of this *Chap.*) and then add these together as in the preceding Examples, as,

To add those mixed quantities in the last Example, to wit, . . . $\frac{aa}{b}-a$ and $\frac{dd}{c}+d$;

I First reduce them to these Fractions . . . . $\frac{aa-ba}{b}$ and $\frac{dd+cd}{c}$;

Which reduced to a common Denominator produce these . . $\frac{caa-cba}{bc}$ and $\frac{bdd+bcd}{bc}$

Which two last Fractions added together give the Sum required, to wit, ... $\frac{caa-cba+bdd+bcd}{bc}$

Which is equal to the Sum before found, to wit, ... $\frac{caa+bdd}{bc}-a+d$.

Subtraction of Algebraical Fractions.

XVII. If the two Fractions given have the same Denominator, subtract the Numerator of the Fraction prescribed to be subtracted, from the other Numerator, and set the Remainder as a new Numerator over the common Denominator, so shall this new Fraction be the remainder sought.

As, for Example, If from $\frac{aa}{c}$ you desire to subtract $\frac{bb}{c}$, take bb from aa, and set the Remainder $aa-bb$ as a Numerator over the common Denominator c; so $\frac{aa-bb}{c}$ shall be the Remainder sought.

In like manner, If from $\frac{2ab}{b-c}$ you would subtract $\frac{2ac}{b-c}$, the Remainder will be $\frac{2ab-2ac}{b-c}$, that is, (by Division) $2a$.

Again, if from $\frac{8aa-7b+6}{a+b}$ it be desired to subtract $\frac{3aa+12b-18}{a+b}$, the

Remainder

Remainder will be found $\frac{5aa-19b+24}{a+b}$. (For $3aa+12b-18$ subtracted from $8aa-7b+6$, leaves $5aa-19b+24$.)

So also, from $d+\frac{bb}{b+d}$ subtracting $\frac{bd}{b+d}$, there remains $\frac{dd+bb}{b+d}$. For, (by *Sect.* 5. of this *Chap.*) $d+\frac{bb}{b+d}$ will be reduced to $\frac{db+dd+bb}{b+d}$; from which subtracting $\frac{bd}{b+d}$, the Remainder is $\frac{dd+bb}{b+d}$.

XVIII. But if the two Fractions given have different Denominators, first reduce them to a common Denominator, and then subtract as before; so as from $\frac{dd}{c}$ it be desired to subtract $\frac{aa}{b}$, I reduce them to $\frac{ddb}{cb}$ and $\frac{aac}{cb}$, which have the same Denominator cb; then from $\frac{ddb}{cb}$ subtracting $\frac{aac}{cb}$, there remains $\frac{ddb-aac}{cb}$, which is the Remainder sought.

After the same manner, If from $\frac{aa+d}{b-c}$ you would take away $\frac{aa}{b}$, there will remain $\frac{db+aac}{bb-bc}$.

Likewise from $\frac{aaa+bbb}{cd-dd}$ to take away $\frac{bbbb}{aac-aad}$, I first reduce these given Fractions to a common Denominator, (as in the second Example of *Sect.* 13. of this *Chap.*) and so I find $\frac{aaaaa+aabbb}{aacd-aadd}$ and $\frac{bbbbd}{aacd-aadd}$, which latter Fraction subtracted from the former there remains $\frac{aaaaa+aabbb-bbbbd}{aacd-aadd}$.

Again, If from a it be desired to subtract $\frac{aa-ab}{a+b}$, I reduce a into the form of a Fraction whose Denominator shall be $a+b$, and so instead of a, I find $\frac{aa+ab}{a+b}$, from which subtracting $\frac{aa-ab}{a+b}$, there remains $\frac{2ab}{a+b}$

Multiplication of Algebraical Fractions.

XIX. When two Algebraic Fractions are given to be multiplied one by the other, multiply their Numerators one into the other, and take the Product for a new Numerator; likewise multiplying the Denominators one into the other, this Product shall be a new Denominator, and the new Fraction is the Product sought.

As, for Example, to multiply $\frac{2a}{c}$ by $\frac{b}{3d}$, I multiply (as in vulgar Fractions) the Numerator $2a$ by the Numerator b, and the Product $2ab$ is a new Numerator; likewise I multiply the Denominators, $3d$ and c one into the other, and the Product $3dc$ shall be a new Denominator; so $\frac{2ab}{3dc}$ is the Product sought.

In like manner, $\frac{aa-bb}{c}$ multiplied by $\frac{2ab}{b+c}$ gives the Product $\frac{2aaab-2abbb}{bc+cc}$

XX. When either or both the given Terms are mixed Quantities, reduce the mixt Quantity to the form of a Fraction (by the Rule in *Sect.* 5. of this *Chap.*) and then multiply as before: So to multiply $c+\frac{bb}{d}$ by $a+\frac{ad}{c-d}$, I first Reduce

those

those mixt Quantities to these Fractions, $\frac{cd+bb}{d}$ and $\frac{ac}{c-d}$, then multiplying the Numerator $cd+bb$ by the Numerator ac, the Product is $accd+acbb$ for a new Numerator; also multiplying the Denominators d and $c-d$ one by the other, the Product is $dc-dd$ for a new Denominator, and the Product sought is $\frac{accd+acbb}{dc-dd}$.

XXI. When an Integer is to be Multiplied by a Fraction, express the Integer Fraction-wise by giving it unity, (to wit, 1) for a Denominator, (according to *Sect.* 4. of this *Chap.*) and then multiply as in the preceding Examples.

As, to multiply a by $\frac{b}{c}$, that is, $\frac{a}{1}$ by $\frac{b}{c}$, the Product will be $\frac{ab}{c}$. Likewise to Multiply $aa-bb$ by $\frac{aa+bb}{cd+fg}$, I reduce $aa-bb$ to $\frac{aa-bb}{1}$, then multiplying the Numerator $aa+bb$ by the Numerator $aa-bb$, the Product $aaaa-bbbb$ shall be a New Numerator; Likewise the Denominator $cd+fg$ multiplied by the Denominator 1 gives $cd+fg$ for a New Denominator, and the New Fraction $\frac{aaaa-bbbb}{cd+fg}$ is the Product sought.

XXII. But oftentimes there may be this useful Contraction in the Multiplication of Fractions, *viz.* When the Numerator of the one and the Denominator of the other may be severally divided by some common Divisor without a Remainder, take the Quotients instead of the said Numerator and Denominator, and then multiply as in the preceding Examples.

As, for Example, to multiply $\frac{aa+2ab+bb}{cd-dd}$ by $\frac{dd}{a+b}$:

Forasmuch as the Numerator of the first Fraction and the Denominator of the latter may be divided severally by $a+b$ without a Remainder, I set the Quotients $a+b$ and 1 in the places of $aa+2ab+bb$ and $a+b$, and by that exchange these Fractions will arise, to wit;

$$\frac{a+b}{cd-dd} \text{ and } \frac{dd}{1}.$$

In like manner, because $cd-dd$ the Denominator of the first of the two Fractions last above-written, and dd the Numerator of the latter Fraction, may be severally divided by d without a Remainder, I set the Quotients $c-d$ and d in the Places of $cd-dd$, and dd, and so these new Fractions arise, to wit;

$$\frac{a+b}{c-d} \text{ and } \frac{d}{1},$$

Then I multiply (as before) the Numerators $a+b$ and d, one by the other, and the Product $da+db$ is a New Numerator: Also multiplying the Denominator $c-d$ by the Denominator 1, the Product $c-d$ is a new Denominator, and the new Fraction $\frac{da+db}{c-d}$ is the Product sought; being equal to that which would be made by the mutual Multiplication of $\frac{aa+2ab+bb}{cd-dd}$ and $\frac{dd}{a+b}$ the Fractions first proposed to be Multiplied.

So as also, If it be desired to Multiply $a+\frac{bb}{a-b}$ by $a-2b+\frac{bb}{a}$, that is, $\frac{aa-ab+bb}{a-b}$ by $\frac{aa-2ab+bb}{a}$; Forasmuch as the Numerator $aa-2ab+bb$ of the latter Fraction, and the Denominator $a-b$ of the former, being severally divided by their common Divisor $a-b$ will give the Quotients $a-b$ and 1; therefore I set these in the places of $aa-2ab+bb$ and $a-b$, whence these Fractions will arise, to wit,

$$\frac{aa-ab+bb}{1} \text{ and } \frac{a-b}{a}:$$

Which

Which being multiplied one by the other will give $\frac{aaa-2aab+2abb--bbb}{a}$, or $aa-2ab+2bb-\frac{bbb}{a}$, the Product fought.

Again, this Fraction $\frac{aac-aad-bbc+bbd}{aa+2ab+bb}$ multiplied by $\frac{aaa-abb}{cd-dd}$, will produce $\frac{aaaa-aaab-aabb+abbb}{ad+bd}$; For the Numerator of the firft Fraction and the Denominator of the latter being feverally divided by their common Divifor $c-d$ the Quotients will be $aa-bb$ and d; Alfo, the Denominator of the firft Fraction and the Numerator of the fecond being feverally divided by their common Divifor $a+b$, the Quotients will be $a+b$ and $aa-ab$; then fetting the two former Quotients in the places of the two firft Dividends, and the two latter Quotients in the places of the two latter Dividends, thefe two Fractions will arife, to wit;

$$\frac{aa-bb}{a+b} \text{ and } \frac{aa-ab}{d}:$$

Laftly, multiplying the Numerators $aa-bb$ and $aa-ab$ one into the other; as alfo the Denominators $a+b$ and d, (as in former Examples,) you will find the Product fought, to wit;

$$\frac{aaaa-aaab-aabb+abbb}{ad+bd}.$$

XXIII. When a Fraction is to be multiplied by fome Integer that happens to be the fame with the Denominator of the Fraction, take the Numerator for the Product required. As, for Example, to multiply $\frac{aa+ab+bb}{a+d}$ by $a+d$; I write $aa+ab+bb$ for the Product of their multiplication.

Likewife, If $\frac{b}{c}$ be to be multiplied by the Denominator c; I write the Numerator b for the Product. The reafon of this Contraction is Evident; for if $\frac{b}{c}$ be multiplied by c, or $\frac{c}{1}$, in the ordinary way, the Product will ftand thus, $\frac{bc}{c}$, which, by cafting away the common Factor c out of the Numerator and Denominator, gives b for the Product; to wit, the Numerator of the given Fraction $\frac{b}{c}$.

Hence alfo, if an Algebraical Fraction be to be multiplied by fome letter or letters that are found among others in every Member of the Denominator, that multiplication needs no other work but the cafting away fuch letter or letters out of the Denominator: As to multiply $\frac{ab}{cd}$ by c, the Product is $\frac{ab}{d}$; where obferve, that becaufe the multiplier c is found in the given Denominator cd, I ftrike it quite out.

Likewife, to multiply $\frac{ab}{cd}$ by d, I write $\frac{ab}{c}$ for the Product: And to multiply $\frac{bbb-ccc}{3faa-3gaa}$ by $3aa$, I cancel $3aa$ in the Denominator, and write $\frac{bbb-ccc}{f-g}$ for the Product required.

Note. The taking of $\frac{2}{3}$ parts of the quantity a, imports the fame thing with the multiplying of a by $\frac{2}{3}$, and the Product may be expreft either thus, $\frac{2a}{3}$; or thus, $\frac{2}{3}a$.

Likewife $\frac{2}{5}$ of $b+c$, or the Product of $b+c$ multiplied by $\frac{2}{5}$, may be expreft either thus $\frac{2b+2c}{5}$, or thus, $\frac{2}{5}b+\frac{2}{5}c$. And fo of others.

Divifion

Diviſion in Algebraical Fractions.

XXIV. When the two given Fractions, to wit, the Dividend and Diviſor, have a common Denominator, caſt away the Denominator, and divide the Numerator of the Dividend by the Numerator of the Diviſor; ſo that which ariſes ſhall be the Quotient ſought. As, to divide $\frac{aab}{c}$ by $\frac{bb}{c}$; I caſt away the common Denominator c, and divide aab by bb, ſo the Quotient ſought is $\frac{aab}{bb}$, that is, $\frac{aa}{b}$.

In like manner, $\frac{aabb}{d}$ divided by $\frac{ab}{d}$ gives $\frac{aabb}{ab}$, that is, ab for the Quotient.

Again, If $\frac{aaa-abb}{c-d}$ be divided by $\frac{aa+2ab+bb}{c-d}$ there will ariſe $\frac{aaa-abb}{aa+2ab+bb}$, which abbreviated (by dividing the Numerator and Denominator ſeverally by their common Diviſor $a+b$) gives $\frac{aa-ab}{a+b}$ the Quotient ſought.

XXV. If the given Fractions have not a common Denominator, then (as in Diviſion of vulgar Fractions) multiply the Numerator of the Dividend by the Denominator of the Diviſor, and the Product ſhall be a new Numerator; alſo, multiply the Denominator of the Dividend by the Numerator of the Diviſor, and the Product ſhall be a new Denominator; ſo the new Fraction is the Quotient ſought.

As, for Example, to divide $\frac{ab}{c}$ by $\frac{dd}{a}$, I multiply ab by a, and the Product is aab for a new Numerator; alſo, multiplying c by dd, the Product is ddc for a new Denominator; ſo the Quotient ſought is $\frac{aab}{ddc}$.

$$\frac{dd}{a}\Big)\ \frac{ab}{c}\ \Big(\frac{aab}{ddc}$$

Likewiſe, If $\frac{aa-bb}{c+d}$ be divided by $\frac{c-d}{aa+bb}$, the Quotient will be $\frac{aaaa-bbbb}{cc-dd}$; For $aa-bb$ the Numerator of the Dividend being multiplied by $aa+bb$ the Denominator of the Diviſor, the Product $aaaa-bbbb$ is the new Numerator: and $c+d$ the Denominator of the Dividend being multiplied by $c-d$ the Numerator of the Diviſor produces $cc-dd$ for a new Denominator; whence the Quotient ſought is $\frac{aaaa-bbbb}{cc-dd}$.

XXVI. But oftentimes there may be this uſeful Contraction in the Diviſion of Fractions, *viz*, when either the two Numerators, or the two Denominators may be divided by ſome common Diviſor without a Remainder, ſet the Quotients ariſing out of ſuch Diviſion (or imagine them to be ſet) in the places of the ſaid Numerators or Denominators that were divided, and then divide as in the former Examples.

As, to divide $\frac{aa-ab}{cc}$ by $\frac{a-b}{cd}$; Foraſmuch as the Numerators $aa-ab$ and $a-b$ may be reduced to more ſimple Terms, to wit, a and 1, (for $aa-ab$ and $a-b$ being ſeverally divided by their common Meaſure $a-b$ give a and 1. And, becauſe the Denominators cc and cd may likewiſe be reduced to more ſimple Terms c and d, (by dividing the ſaid cc and cd by their common Diviſor c,) therefore in the places of the two given Numerators $aa-ab$ and $a-b$ I ſet the two former Quotients a and 1, and in the places of the two given Denominators cc and cd I ſet the two latter Quotients c and d; ſo there will be $\frac{a}{c}$ and $\frac{1}{d}$ for a new Dividend and Diviſor; then (as before) I multiply a by d, and the Product is ad or da for a new Numerator; Alſo, c multiplied by 1 gives c for a new Denominator, and the new Fraction $\frac{da}{c}$ is the

$$\frac{1}{d}\Big)\ \frac{a}{c}\ \Big(\frac{da}{c}$$

the Quotient sought; which is equal to that which would arise by dividing $\frac{aa-ab}{cc}$ by $\frac{a-b}{cd}$, to wit, the Fractions first proposed.

Again, If it be desired to divide $\frac{aaaa-bbbb}{aa-2ab+bb}$ by $\frac{aa+ab}{a-b}$; Forasmuch as the Numerators $aaaa-bbbb$ and $aa+ab$ may be reduced to $aaa-aab+abb-bbb$ and a by their common Divisor $a+b$; and the Denominators $aa-2ab+bb$ and $a-b$ may be reduced to $a-b$ and 1, by the common Divisor $a-b$; therefore instead of multiplying $aaaa-bbbb$ by $a-b$, I multiply the said $aaa-aab+abb-bbb$ by 1, and the Product is $aaa-aab+abb-bbb$ for a new Numerator; and instead of multiplying $aa-2ab+bb$ by $aa+ab$, I multiply $a-b$ by a; so the Product $aa-ab$ shall be a new Denominator, whence the Quotient sought is $\frac{aaa-aab+abb-bbb}{aa-ab}$.

In like manner, If $\frac{aaaa-625}{aa-10a+25}$ be divided by $\frac{aa+5a}{a-5}$, the Quotient will be $\frac{aaa-5aa+25a-125}{aa-5a}$; For $aaaa-625$ and $aa+5a$ may be reduced to $aaa-5aa+25a-125$, and a by the common Divisor $a+5$; Also, $aa-10a+25$ and $a-5$ may be reduced to $a-5$ and 1 by the common Divisor $a-5$ and 1; whence instead of the Fractions given we may divide

$$\frac{aaa-5aa+25a-125}{a-5} \text{ by } \frac{a}{1},$$

and the Quotient sought will be $\frac{aaa-5aa+25a-125}{aa-5a}$.

Again, to divide $aaa-2aab+abb$ by $\frac{aa-ab}{a+b}$, I set 1 for a Denominator under the Dividend $aaa-2abb+abb$, and it stands thus $\frac{aaa-2aab+abb}{1}$, then forasmuch as the Numerators $aaa-2aab+abb$ and $aa-ab$ may be reduced to $a-b$ and 1, (by the common Divisor $aa-ab$) therefore instead of the given Dividend and Divisor we may take $\frac{a-b}{1}$ and $\frac{1}{a+b}$, whence the Quotient sought will be found $aa-bb$.

So also, If $a+\frac{3abb}{a+4b}$ be to be divided by $a+b$, that is, $\frac{aaa+4aab+3abb}{a+4b}$ by $\frac{a+b}{1}$, the Quotient will be found $\frac{aa+3ab}{a+4b}$: And $\frac{xx+5x}{x-5}$ divided by $xx+5x$, gives the Quotient $\frac{1}{x-5}$: Lastly, $\frac{xx+5x}{x-5}$ divided by $x+5$ gives the Quotient $\frac{x}{x-5}$.

CHAP. VII.

The Rule of Three in Quantities represented by Letters.

I. AS in Vulgar Arithmetic ſo here in Algebraical, if three Quantities be given to find out a fourth in a direct Proportion, that is, when the Nature of the Queſtion is ſuch; that as the firſt Term is in proportion to the ſecond, ſo is the third to the fourth ſought; then (reſpect being had to the preceding Rules of Algebraical Multiplication and Diviſion) multiply the ſecond and third Terms one into another, and divide the Product by the firſt Term; ſo the Quotient ſhall be the fourth Proportional ſought.

As for Example, If the Quantity a give b, what ſhall c give, in a direct Proportion? Or, to the ſame Effect, find out a Quantity which ſhall have the ſame proportion to c, as b has to a; here I multiply b by c, and then dividing the Product bc by a, the Quotient, $\frac{bc}{a}$ is the fourth Proportional ſought; as will appear by the Proof of the Rule of Three direct: For if the fourth Term $\frac{bc}{a}$ be multiplied by the firſt Term a, the Product will be $\frac{abc}{a}$, which (by *Sect.* 5. *Chap.* 5.) is equal to bc, to wit, the Product of the ſecond Term multiplied by the third.

$$a \,.\, b :: c \,.\, \frac{bc}{a}$$

The Proof,

$$\frac{abc}{a} = bc.$$

In like manner, If $a+b$ give d, what ſhall $c+d$ give, in a Direct proportion? Anſwer, $\frac{dc+dd}{a+b}$

Again, If 4 gives 3, what ſhall $8aa$ give? Anſw. $\frac{24aa}{4}$ that is, $6aa$.

Moreover, If $aaa-aab+abb-bbb$ give $aa+bb$, what ſhall $aa-bb$ give? Anſw. $a+b$: For the ſecond and third Term being multiplied one by the other will produce $aaaa-bbbb$, which divided by the firſt Term $aaa-aab+abb-bbb$ (according to the general Method of Diviſion in *Sect.* 9. *Chap.* 5.) gives $a+b$ the fourth Proportional ſought.

II. When any one of the three given Quantities is an Algebraic Fraction, ſet the other two if they be Integers, in the form of Fractions, by placing 1 as a Denominator under each Integer.

Alſo, when any one of the three given Quantities is compos'd of an Integer and a Fraction, let it be reduced into the Form of a Fraction, (by *Sect.* 5. *Chap.* 6.) then if the Proportion be Direct, multiply and divide as before.

As for Example, If $a+\frac{bb}{c}$ give cd, what ſhall $\frac{ab}{f}$ give in a direct proportion? Anſw. $\frac{abccd}{acf+bbf}$: For firſt, $a+\frac{bb}{c}$ being reduced to the Form of a Fraction will ſtand thus $\frac{ac+bb}{c}$; alſo cd ſet Fraction-wiſe is $\frac{cd}{1}$; then multiplying the third Term $\frac{ab}{f}$ by the ſecond Term $\frac{cd}{1}$, the Product is $\frac{abcd}{f}$, which divided by the firſt Term $\frac{ac+bb}{c}$ gives $\frac{abccd}{acf+bbf}$ for the fourth Proportional ſought.

In like manner, If $\frac{ab}{c}$ give d, then $\frac{bb}{d}$ will give $\frac{cdbb}{abd}$, that is, $\frac{cb}{a}$, (for $\frac{cdbb}{abd}$ being abbreviated according to *Sect.* 5. *Chap.* 5. gives $\frac{cb}{a}$.

Alſo,

Alſo, If $\frac{a+c}{d}$ give $\frac{aa}{bb}$; then $\frac{bb}{a-c}$ will give $\frac{daa}{aa-cc}$.

III. If after the three given Quantities are ordered or ſet in the Rule according to the uſual manner in Vulgar Arithmetic; the Proportion flows backwards, *viz.* if the Nature of the Queſtion be ſuch, that as the third Term is in proportion to the ſecond, ſo is the firſt to the fourth Term ſought; then (as in the Inverſe or backward Rule of Three in Vulgar Arithmetic) multiply the firſt and ſecond Terms one by the other, and divide the Product by the third, ſo the Quotient ſhall be the fourth Proportional ſought. But I ſhall not need to give Examples of this Rule, nor to make application of Algebraical Arithmetic to the Double Rule of Three, Rules of Fellowſhip and Alligation; ſince he that underſtands the manner of working thoſe Rules in Vulgar Arithmetic, as alſo the Rules of Algebraical Arithmetic before delivered, cannot miſs of performing the like work Algebraically when there is occaſion.

CHAP. VIII.

An Introduction to the Extraction of ROOTS *out of Algebraical Quantities.*

I. IT is not my deſign in this Chapter to treat of the Extraction of *Roots* in general, (that Doctrine being hereafter handled in the third and fourth Chapter of the ſecond Book) but chiefly to ſhew how to extract the Roots or ſides of Simple Powers expreſs'd by Letters, as alſo of Squares formed from Rational Binomial Roots, in order to the Explication of divers Equations in the following Chapters: For I would not willingly affright the Learner with tedious and intricate Operations until he has had a conſiderable Taſte of the practice of Algebra in the ſolving of Arithmetical Queſtions.

II. As in Vulgar Arithmetic, the Extraction of the Square root of a given Number imports nothing elſe but the finding out ſuch a Number that being multiplied by it ſelf will produce the given Number; ſo the Extracting of the Square root of the Quantity aa implies only the finding out ſuch a Quantity, which if it be multiplied by it ſelf will produce aa; and ſince a multiplied by a produces aa, therefore a is the Root or ſide of the Square aa.

Likewiſe the ſquare Root of $4bb$ is $2b$; for $2b$ multiplied by $2b$ produces $4bb$: And for the ſame Reaſon, the ſquare Root of $\frac{1}{4}aa$ (or $\frac{aa}{4}$) is $\frac{1}{2}a$; (or $\frac{a}{2}$.) Alſo, the ſquare Root of $bbaa$ is ba; and the ſquare Root of $aaaa$ is aa.

Moreover, Foraſmuch as aa, or the Square of the Root a, being multiplied by the Root a produces aaa, or the Cube of a, therefore the Cubic Root of aaa being extracted there will come forth again the Root a. In like manner, the Cubic Root of $8aaa$ is $2a$; for $2a$ multiplied cubically, (that is, firſt by it ſelf and then again by the Product) produces $8aaa$.

III. The like is to be underſtood in the Extraction of the Root of a compound Power; For, as the Binomial Root $a+b$, which may repreſent the Sum of the two parts into which ſome Number or Right-line is divided, being ſquared or multiplied by it ſelf, produces the Square $aa+2ab+bb$; So the ſquare Root of $aa+2ab+bb$ being extracted, there will ariſe the Root $a+b$. Here the Learner may obſerve, That if a Number or Right-line be divided into any two parts, (a and b) the Square ($aa+2ab+bb$) which is made of $a+b$ the Sum of the Parts, is compoſed of (aa and bb) the Squares of the Parts, and of ($2ab$) the Double Product made by the Multiplication of the Parts (a and b) one into the other.

$$\begin{array}{ll} a+b. & \text{The Root.} \\ a+b & \\ \hline aa+ab & \\ \quad +ab+bb & \\ \hline aa+2ab+bb. & \text{The Square.} \end{array}$$

So the Square of 8, or of $5+3$, is equal to $25+9+30$, that is, 64.

Again, As the Binomial, or (as ſome call it) the Reſidual Root $a-b$, or $b-a$ being multiplied by it ſelf produces the Square $aa-2ab+bb$; So the ſquare Root of $aa-2ab+bb$ being extracted, there will come forth the Root $a-b$, or $b-a$; (for either of theſe Roots will produce the ſame Square.) Here alſo the Learner may obſerve, That if a Number or Right-line be divided into any two parts, (a and b) the Square ($aa-2ab+bb$) which is made by the Multiplication of ($a-b$, or $b-a$) the Difference of the Parts into it ſelf, is equal to ($aa+bb$) the Sum of the Squares of the Parts, leſs by ($2ab$) the double Product of the Multiplication of the Parts one into the other: So the Square of $5-3$, that is, of 2, is equal to $25+9-30$, that is, 4.

$$\begin{array}{ll} a-b. & \text{The Root.} \\ a-b & \\ \hline aa-ab & \\ \quad -ab+bb & \\ \hline aa-2ab+bb. & \text{The Square.} \end{array}$$

IV. From what has been ſaid in the laſt Section, this Theorem may be inferr'd, *viz.* If a compound Quantity conſiſts of three ſuch Members or ſimple Quantities, that two of them are Squares, each of them having the ſign + prefix'd to it, and the third is the double Product made by the mutual Multiplication of the Roots of thoſe ſimple Squares, the ſaid double Product alſo having the ſign + prefix'd to wit; that compound Quantity ſhall be a Square whoſe Root is the Sum of the two Roots of the ſaid two ſimple Squares: But if the ſaid double Product has the Sign — prefix'd to it, then the difference of the ſaid Roots ſhall be the Root of the ſaid compound Square.

Hence $aa+6a+9$ will be found a Square, whoſe Root is $a+3$; for it is evident that aa and 9 are Squares, whoſe Roots are a and 3; and $6a$ is the double Product of the Multiplication of thoſe Roots a and 3 one by the other.

Likewiſe, $9bb+6bc+cc$ is a Square, whoſe Root is $3b+c$; for $9bb$ and cc are Squares whoſe Roots are $3b$ and c, and $6bc$ is the double Product of the Multiplication of the Roots $3b$ and c one into the other. Alſo, $aaaa+baa+\frac{1}{4}bb$ will be found a Square, whoſe Root is $aa+\frac{1}{2}b$.

Moreover, (agreeable to the latter Caſe in the Theorem) This compound Quantity $aa-10a+25$ will be diſcovered to be a Square whoſe Root is $a-5$, or $5-a$. And $bbaa-2bca+cc$ is a quare whoſe Root is $ba-c$, or $c-ba$; For from either of theſe Roots the ſame Square $bbaa-2bca+cc$ will be produced by Algebraical Multiplication.

If the Learner be well vers'd in this Theorem, he may oftentimes diſcern at firſt ſight whether a compound Quantity that conſiſts of three Members or ſingle Quantities be a Square or not; and if a Square, what its Root is.

V. If a Quantity out of which a Root is to be extracted be ſuch, that the Root cannot any manner of way be exactly extracted; that Root is uſually deſign'd or repreſented by prefixing the radical ſign before the Quantity propoſed. So to extract the ſquare Root of the Quantity a, (whether it repreſents a plane Number or a Superficies) I write $\sqrt{a}$, or $\sqrt{(2)}a$, which ſignifies that the ſquare Root of a is extracted or to be extracted.

So alſo, $\sqrt{:\overline{aa+bb}}:$ or, $\sqrt{(2)}:\overline{aa+bb}$: denotes the ſquare Root of the Sum of the Squares aa and bb.

Likewiſe, to extract the Cubic Root of b, I write $\sqrt{(3)}b$; as alſo $\sqrt{(3)}\,aab$, to ſignifie the Cubic Root of aab; which kind of Roots are called Surd or Irrational Quantities. (As hereafter in *Chap.* 9. of the II. *Book* will be more fully declared.)

VI. When it is required to extract the Root of a Fraction, the Root of the Numerator and the Root of the Denominator ſhall give a new Fraction which is the Root ſought. As for Example, If the ſquare Root of $\frac{aa}{bb}$ be deſired; foraſmuch as the ſquare Root of aa is a, and the ſquare Root of bb is b; I write $\frac{a}{b}$ for the Root ſought.

In

In like manner, the ſquare Root of $\frac{aabb}{dd}$ is $\frac{ab}{d}$; (for the ſquare Root of $aabb$ is ab, and the Root of dd is d.)

Again, the ſquare Root of $\frac{aa+6a+9}{bb}$ is $\frac{a+3}{b}$; For (by the foregoing *Sect.* 4.) the ſquare Root of the Numerator $aa+6a+9$ is $a+3$; and the ſquare Root of the Denominator bb is b. Alſo, the ſquare Root of $\frac{9bb+6bc+cc}{\frac{1}{4}dd}$ is $\frac{3b+c}{\frac{1}{2}d}$; and the Cubic Root of $\frac{27ddd}{64}$ is $\frac{3d}{4}$, or $\frac{3}{4}d$.

VII. But if the Root ſought cannot be extracted out of the Numerator and Denominator as before, the Radical ſign is to be ſet before the given Fraction; as to extract the ſquare Root of $\frac{aa}{b}$ I write $\sqrt{\frac{aa}{b}}$; or becauſe the ſquare Root of the Numerator is a, the ſquare Root of $\frac{aa}{b}$ may be expreſs'd thus $\frac{a}{\sqrt{b}}$; likewiſe the ſquare Root of $\frac{aa+bb}{aabb}$ may be written either thus, $\sqrt{\frac{aa+bb}{aabb}}$; or thus, $\frac{\sqrt{:aa+bb:}}{ab}$

CHAP. IX.

Which teaches how by any two of the three Members of a Square formed from a Binomial Root, to find out the third Member.

I. FRom *Sect.* 3. of the precedent 8. *Chap.* it is evident that every Square formed from a Binomial Root, that is, a Root of two Names or Parts, conſiſts of three Members or diſtinct Quantities, to wit, two Affirmative Squares, and the double of the Product made by the mutual Multiplication of the two Roots of thoſe ſquares; which double Product is ſometimes Affirmative, and ſometimes Negative: So each of theſe compound Squares $9aa+12a+4$; and $9aa-12a+4$, whoſe Roots are $3a+2$, and $3a-2$, (or $2-3a$) conſiſts of two Squares, to wit, $9aa$ and 4, together with $12a$, the double Product of $3a$ multiplied by 2; which $3a$ and 2 are the Roots of the ſaid Squares $9aa$ and 4: Now if any two of the three Members of a Square formed from a Binomial root be given, we may find out the third Member by one of theſe two following Rules.

II. When two Affirmative Squares are given as two of the three Members or Parts of a compound Square formed from a Binomial root to find out the third or mean Member; extract the Square root out of each of thoſe given Squares, then the double of the Product made by the Multiplication of thoſe Roots one into the other ſhall be the mean or middle Member ſought, which if it be annexed to the two given Squares either by $+$ or $-$, will make a compleat compound Square having a Binomial Root.

As for Example, If the Squares $9aa$ and 4 be given, firſt I extract their Roots which are $3a$ and 2, then multiplying theſe Roots one by the other the Product is $6a$, which doubled makes $12a$, the middle Member ſought; this joined by $+$ to the Sum of the given Squares $9aa$ and 4 makes the compound Square $9aa+4+12a$, or $9aa+12a+4$, whoſe Root is $3a+2$: But if the ſaid double Product $12a$ be joined to the Sum of the Squares by $-$, there will ariſe the compound ſquare $9aa+4-12a$, or $9aa-12a+4$; whoſe Root is $3a-2$, or, $2-3a$.

In like manner, If $4aa$ and $9bb$ be propos'd as two of the three Members of a compound Square that has a Binomial Root, the third Member will be found $12ab$; and the Square ſought will be either $4aa+12ab+9bb$, whoſe Root is $2a+3b$; or elſe $4aa-12ab+9bb$, whoſe Root is $2a-3b$, or $3b-2a$.

III. When

III. When the double Product and either of the two Affirmative Squares aforesaid are given as two of the three Members of a compound Square having a Binomial Root, to find out the other Square or third Member; divide half the said double Product by the Root of the given Square, and the Square of the Quotient shall be the third Member sought, which added by + to the two given Members will compleat the compound Square.

As for Example, If $9aa+12a$ be proposed; the half of $12a$ is $6a$; this divided by $3a$ (the square Root of $9aa$) gives 2 whose Square is 4, which added by + to $9aa+12a$ makes $9aa+12a+4$, which is a compleat Compound Square, whose Root is $3a+2$.

In like manner, If $12a+4$ be given; the half of $12a$ is $6a$, which divided by 2, (the square Root of 4) gives $3a$, whose Square is $9aa$, which added by + to $12a+4$, makes the compound Square $12a+4+9aa$, that is, $9aa+12a+4$, whose Root is $3a+2$.

Again, If $aa-2ba$ be given; the half of $2ba$ is ba, which divided by a, (the square Root of aa) gives the Quotient b, whose Square is bb; which added to $aa-2ba$ makes the Square $aa-2ba+bb$, whose Root, because — is prefix'd to $2ba$, shall be $a-b$, or, $b-a$; but if + had been prefix'd to $2ba$, then the Root would have been $a+b$, or $b+a$.

Note: If the said Affirmative Square given be express'd by Letters, and has only 1 (to wit, Unity) prefix'd to it, then instead of the Rule above delivered in this *Sect.* 3. there may be this *Compendium*, *viz.* The Square of half that Quantity which in the double Product given is drawn into the Root of the given Square shall be the third Member sought to compleat the compound Square: As in the last Example, where $aa-2ba$ was given, because 1 is prefix'd (or must be imagined to be prefix'd) to aa; I take the half of $2b$ to wit, b, which multiplied by it self gives bb, which added by + to $aa-2ba$, will make (as before) the compleat Compound Square $aa-2ba+bb$. So also to make $aa+6da$ a compleat Square, I take the half of $6d$ which is $3d$, whose Square $9dd$ added by + to $aa+6da$ makes the compound Square $aa+6da+9dd$, whose Root is $a+3d$. This will be further illustrated in the next *Section*.

IV. If a compound Quantity consists of two such Quantities that one of them is an Affirmative Square express'd by Letters, before which 1 is prefix'd, (or suppos'd to be prefix'd) and the other is the Product made by the Multiplication of the Root of that Square by some Quantity, which is usually called the Coefficient; that compound Quantity may be made a compleat Square thus, *viz.* Add by the Sign + the Square of half the Coefficient to the compound Quantity given, so shall the Sum be a Square, whose Root, when + is prefix'd to the said Product, is the Sum of the Roots of the Square given and the Square added: But when — is prefix'd to the said Product, then the Root of the compound Square found shall be the difference of those two Roots.

As for Example, If the compound Quantity $aa+ca$ be proposed, I take the half of the Coefficient c, to wit, $\frac{1}{2}c$; then the Square of $\frac{1}{2}c$ is $\frac{1}{4}cc$, which added to $aa+ca$ makes $aa+ca+\frac{1}{4}cc$, which is a Square whose Root or Side is $a+\frac{1}{2}c$, to wit, the Sum of the Roots of the Squares aa and $\frac{1}{4}cc$; But if the said $\frac{1}{4}cc$ be added to $aa-ca$, then there will arise the Square $aa-ca+\frac{1}{4}cc$, whose Root is $a-\frac{1}{2}c$, or $\frac{1}{2}c-a$.

In like manner, To make $aa+5ba$ a compleat Square, and to discover its Root; I take the half of $5b$, to wit, $\frac{5}{2}b$, the Square whereof is $\frac{25}{4}bb$, which added to the given compound Quantity $aa+5ba$ makes $aa+5ba+\frac{25}{4}bb$, which is a Square whose Root is $a+\frac{5}{2}b$, as will easily appear by multiplying the said Root into it self.

So also, To make $aa-12a$ a perfect Square, I add 36 (the Square of half the Coefficient 12) to $aa-12a$, and it makes the compound Square $aa-12a+36$, whose Root is $a-6$, or $6-a$.

Again, To find what Quantity must be added to $aaaa+aa$, or $aaaa+1aa$, to make a compleat Square; I take $\frac{1}{2}$, to wit, half the Coefficient 1 which is prefix'd to aa, (the square Root of $aaaa$) and then the Square of the said $\frac{1}{2}$ is $\frac{1}{4}$; this added to $aaaa+1aa$ makes the Square $aaaa+1aa+\frac{1}{4}$, or, $aaaa+aa+\frac{1}{4}$, whose Root is $aa+\frac{1}{2}$, to wit, the Sum of the Roots of the Squares $aaaa$ and $\frac{1}{4}$.

After

After the same manner, to make this Compound Quantity a compleat Square, ... $aa+\frac{2b+3c}{d}a.$

I take the half of the Coefficient $\frac{2b+3c}{d}$, to wit, ... $\frac{2b+3c}{2d},$

Then the Square of that half Coefficient is . . . . $\frac{4bb+12bc+9cc}{4dd}$

Which Square added to the Compound Quantity proposed, makes . . . . $aa+\frac{2b+3c}{d}a+\frac{4bb+12bc+9cc}{4dd},$

Which last Compound Quantity is a Square, whose Root is . . . . $a+\frac{2b+3c}{2d}.$

Likewise, If it be desired to make this compound Quantity a compleat Square, to wit, $aaaaaa+baaa$, I add to it the Square of half the Coefficient b, to wit, $\frac{1}{4}bb$; so there will be $aaaaaa+baaa+\frac{1}{4}bb$ the Square desired, whose Root is $aaa+\frac{1}{2}b$.

CHAP. X.

A Collection of easie Questions to exercise the Rules hitherto delivered.

I. There are two Quantities, whereof the greater is a (or 3,) the lesser is e (or 2,) What is their Sum? What is their difference? What is the Product of their Multiplication? What is the Quotient of the greater divided by the lesser? What is the Quotient of the lesser divided by the greater? What is the Sum of their Squares? What is the difference of their Squares? What is the Sum of the Sum and difference of the two Quantities first proposed? What is the difference of their Sum and Difference? What is the Product made by the Multiplication of the Sum by the Difference? What is the Square of the Sum? What is the Square of the Difference? What is the Sum of the Squares of the Sum and Difference? What is the Difference between the Square of the Sum, and the Square of the Difference? What is the Square of the Product of the Multiplication of the said two Quantities?

	Answers by Letters,	*by Numbers.*
1. The Sum of the two Quantities proposed is	$a+e$	5
2. Their Difference, or the excess of the greater above the less, is	$a-e$	1
3. The Product of their Multiplication is	ae	6
4. The Quotient of the greater divided by the less is . .	$\frac{a}{e}$	$\frac{3}{2}$
5. The Quotient of the lesser divided by the greater is	$\frac{e}{a}$	$\frac{2}{3}$
6. The Sum of their Squares is	$aa+ee$	13
7. The Difference of their Squares is	$aa-ee$	5
8. The Sum of the Sum and Difference of the two Quantities first proposed is	$2a$	6
9. The difference of their Sum and Difference is	$2e$	4
10. The Product of the Multiplication of the Sum by the Difference is	$aa-ee$	5
11. The Square of the Sum is	$aa+2ae+ee$	25
12. The Square of the Difference is	$aa-2ae+ee$	1

13. The

13. The Sum of the Squares of the Sum and Difference is	$2aa+2ee$	26
14. The difference between the Square of the Sum and the Square of the Difference is	$4ae$	24
15. The Square of the Product of the Multiplication of the two Quantities is	$aaee$	36

In like manner, If the greater of two Quantities be c, (or 4,) and the lesser be $\frac{b-d}{c}$; (which we may suppose to represent $\frac{20-12}{4}$, that is, 2; by putting b for 20, and d for 12;) then

1. The Sum of those two Quantities will be	$c+\frac{b-d}{c}$	6
2. Their Difference is	$c-\frac{b-d}{c}$	2
3. The Product of their Multiplication is	$b-d$	8
4. The Quotient of the greater divided by the less is	$\frac{cc}{b-d}$	2
5. The Quotient of the lesser divided by the greater is	$\frac{b-d}{cc}$	$\frac{1}{2}$
6. The Sum of their Squares is	$cc+\frac{bb-2bd+dd}{cc}$	20
7. The Difference of their Squares is	$cc-\frac{bb-2bd+dd}{cc}$	12
8. The Sum of the Sum and Difference of the two Quantities is	$2c$	8
9. The Difference between the Sum and Difference is	$\frac{2b-2d}{c}$	4
10. The Product of the Sum multiplied by the Difference is	$cc-\frac{bb-2bd+dd}{cc}$	12

II. There are two Quantities whose Sum is b, (or 20,) and the greater of them is put a, (or 12;) What is the Lesser? What is their Difference? What is the Product of their Multiplication? What is the Sum of their Squares? What is the Difference of their Squares?

1. If from the Sum of two Quantities the greater be subtracted, the Remainder shall be the lesser; therefore the lesser Quantity sought is	$b-a$	8
2. If from the greater Quantity a, the lesser $b-a$ be subtracted, the Remainder or Difference will be	$2a-b$	4
3. The Product of the Multiplication of the two Quantities is	$ba-aa$	96
4. The Sum of their Squares is	$2aa+bb-2ba$	208
5. The Difference of their Squares is	$2ba-bb$	80

1. But if the Sum of two Quantities be represented by	b	20
2. And for the lesser of them there be put	e	8
3. The greater Quantity shall be	$b-e$	12
4. Their Difference shall be	$b-2e$	4
5. The Product of their Multiplication	$be-ee$	96
6. The Sum of their Squares	$2ee+bb-2be$	208
7. The Difference of their Squares	$bb-2be$	80

III. There

III. There are two Quantities whose Difference is d, (or 4,) and if for the Greater Quantity there be put a, (or 12;) What is the Lesser? What is their Sum? What is the Product of their Multiplication? What is the Sum of their Squares? What is the Difference of their Squares?

1. By subtracting the Difference from the Greater quantity, the Lesser will be	$a-d$	8
2. The Sum of the two Quantities is . . .	$2a-d$	20
3. The Product of their Multiplication is . .	$aa-da$	96
4. The Sum of their Squares is	$2aa+dd-2da$	208
5. The Difference of their Squares is	$2da-dd$	80

1. But if the Difference of two quantities be . . .	d	4
2. And for the Lesser quantity you put	e	8
3. The Greater shall be the sum of the Difference and the Lesser, to wit, ...	$d+e$	12
4. The Sum of the two Quantities is	$d+2e$	20
5. The Product of their Multiplication is . .	$de+ee$	96
6. The Sum of their Squares is	$dd+2de+2ee$	208
7. The Difference of their Squares is	$dd+2de$	80

IV. There are two Quantities, whereof the Greater has such Proportion to the Lesser as r (3) to s, (2,) now if for the greater quantity there be put a, (15,) What is the Lesser? What is their Sum? What is their Difference? What is the Product of their Multiplication? What is the Sum of their Squares? What is the Difference of their Squares?

1. First, say by the Rule of Three, If r give s, what will a give? *Answ.* $\frac{sa}{r}$, which is the Lesser quantity sought	$\frac{sa}{r}$	10
2. Then the Sum of the two quantities will be . .	$a+\frac{sa}{a}$	25
3. Their Difference is	$a-\frac{sa}{r}$	5
4. The Product of their Multiplication is . . .	$\frac{saa}{r}$	150
5. The Sum of their Squares is	$aa+\frac{ssaa}{rr}$	325
6. The Difference of their Squares is	$aa-\frac{ssaa}{rr}$	125

But if the Lesser of two Quantities be e (10,) and has such Proportion to the Greater as s (2,) to r (3;) Then

1. The Greater Quantity will by the Rule of Three be found	$\frac{re}{s}$	15
2. And the Sum of the two Quantities will be . .	$\frac{re}{s}+e$	25
3. Their Difference is	$\frac{re}{s}-e$	5
4. The Product of their Multiplication is . . .	$\frac{ree}{s}$	150
5. The Sum of their Squares is	$\frac{rree}{ss}+ee$	325
6. The Difference of their Squares is	$\frac{rree}{ss}-ee$	125

V. There are two Quantities, the Product of whose Multiplication is b (20;) and if for the Greater quantity there be put a (5,) What is the Lesser? What is their Sum? What is their Difference? What is the Sum of their Squares? What is the Difference of their Squares?

1. The Product b divided by the Greater quantity a gives the Lesser, to wit, ...	$\frac{b}{a}$	4
2. The Sum of the two Quantities is . . .	$a+\frac{b}{a}$	9
3. Their Difference is	$a-\frac{b}{a}$	1
4. Then the Sum of their Squares is	$aa+\frac{bb}{aa}$	41
5. The Difference of their Squares is	$aa-\frac{bb}{aa}$	9

But if the Product of the multiplication of two Quantities be b (20,) and for the Lesser there be put e (4.)

1. The Greater quantity will be	$\frac{b}{e}$	5
2. The Sum of the two quantities is	$\frac{b}{e}+e$	9
3. The Difference is	$\frac{b}{e}-e$	1
4. The Sum of their Squares is	$\frac{bb}{ee}+ee$	41
5. The difference of their Squares is	$\frac{bb}{ee}-ee$	9

VI. The extraction of Roots may be exercised by these following Questions, respect being had to *Sect.* 28. *Chap.* 1. as also *Chap.* 8.

1. What is the Square Root of $144aa$? *Answ.* $12a$.
2. What is the Square Root of $\frac{169}{16}aabb$? *Answ.* $\frac{13}{4}ab$.
3. What is the Square Root of $9aa-6ab+bb$? *Answ.* $3a-b$, or, $b-3a$.
4. What is the Square Root of $\frac{4aa+16ab+16bb}{9cc}$? *Answ.* $\frac{2a+4b}{3c}$.
5. What is the Cubic Root of $125aaabbb$? *Answ.* $5ab$.

6. If b be put for 65, and c for 8. what number is signified by $\sqrt{:}\overline{b+\frac{1}{4}cc{:}}-\frac{1}{2}c$? *Answ.* 5.

7. The same things being put as in the last Question, what number is signified by $\sqrt{:}\overline{b+\frac{1}{4}cc{:}}+\frac{1}{2}c$? *Answ.* 13.

8. If d be put for 8, and f for 48, what number is signified by $\sqrt{:}\overline{\sqrt{f+\frac{1}{4}dd}-\frac{1}{2}d{:}}$ *Answ.* 2.

9. But the same things being put as in the last Question, this quantity $\sqrt{:}\overline{\sqrt{f+\frac{1}{4}dd}+\frac{1}{2}d{:}}$ signifies $\sqrt{12}$, or, 3. 464, &c. that is $3\frac{464}{1000}$, &c.

10. If g be put for 4, and b for 837, what Number is signified by $\sqrt{(3)}{:}\overline{\sqrt{b+\frac{1}{4}gg}-\frac{1}{2}g{:}}$ *Answ.* 3.

11. But the same things being put as in the last Question, this Quantity $\sqrt{(3)}{:}\overline{\sqrt{b+\frac{1}{4}gg}+\frac{1}{2}g{:}}$ signifies $\sqrt{(3)}\,31$, or, 3. 141, &c.

VII. The Rules of the ninth *Chap.* may be exercised by these following Questions.

1. What Quantity is that which if it be added to $aa+25$, will make the Sum a Square? *Answ.* The Quantity to be added may be either $+10a$, or $-10a$; and

and the Square sought is either $aa+10a+25$, whose Root or side is $a+5$; or else the Square is $aa-10a+25$, whose Root is $a-5$, or $5-a$.

2. What Quantity is that which if it be added to $\frac{9}{16}aa+\frac{4}{9}bb$, will make the Sum a Square? *Answ.* The Quantity to be added may be either $+ab$, or $-ab$; and the Square is either $\frac{9}{16}aa+ab+\frac{4}{9}bb$, whose Root is $\frac{3}{4}a+\frac{2}{3}b$: Or else the Square is $\frac{9}{16}aa-ab+\frac{4}{9}bb$, whose Root is $\frac{3}{4}a-\frac{2}{3}b$; or $\frac{2}{3}b-\frac{3}{4}a$.

3. What Quantity is that which if it be added to $aa+3a$ will make the Sum a Square? *Answ.* The Quantity to be added is $\frac{9}{4}$; and the Square is $aa+3a+\frac{9}{4}$, whose Root is $a+\frac{3}{2}$.

4. What Quantity is that which together with $aaaa-2bbaa$ will make a perfect Square? *Answ.* The Quantity to be added is $bbbb$; and the Square is $aaaa-2bbaa+bbbb$, whose Root is $aa-bb$, or $bb-aa$.

5. What Quantity is that which if it be added to $aa+\frac{bb}{c}a$ will make the Sum a Square? *Answ.* The Quantity to be added is $\frac{bbbb}{4cc}$; and the Square is $aa+\frac{bb}{c}a+\frac{bbbb}{4cc}$, whose Root is $a+\frac{bb}{2c}$.

6. What Quantity is that which together with $aaaaaa-aaa$ will make a compleat Square? *Answ.* The Quantity to be added is $\frac{1}{4}$; and the Square sought is $aaaaaa-aaa+\frac{1}{4}$, whose Root is $aaa-\frac{1}{2}$, or $\frac{1}{2}-aaa$.

CHAP. XI.

Concerning an Equation, and the Reduction of Equations.

I. AN *Equation* in the Algebraical Art is a mutual Comparing of two Equal quantities or things of different Denominations: as, If the value of three Shillings be compared to thirty six pence of *English* Money, that comparison imports an Equation, which may be Symbolically exprest thus, $3s=36d$, that is, three Shillings are equal to thirty six pence. Likewise, forasmuch as nine Crowns are of equal value with the Sum of two Pounds and five Shillings of *English* Money; the comparing of these two Sums to one another is nothing else but an Equation which may be briefly exprest thus, $9c=2l+5s$. In each of which Equations the Moneys compared are of different kinds; for Equations between equal things of one and the same name, as $2s=2s$, or $5=5$, and such like, are fruitless.

After the same manner, this Equation $a=b+c$ may signifie that some Number or line represented by a is equal to two other Numbers or Lines b and c taken together as one; or, if the number or Line a be divided into two parts b and c, then also $a=b+c$; for the whole is equal to all its parts.

II. Every Equation consists of two Parts, which are usually separated one from another by this Character $=$; so in the first Equation in the preceding *Sect.* $3s$ is the first Part, and $36d$ the latter; also in the second Equation, $9c$ is the first Part, and $2l+5s$ is the latter; likewise in the last Equation of the same *Section*, a is the first Part, and $b+c$ the latter.

III. The single Quantities or things, whereof each part of an Equation is composed, are called the Terms of an Equation; as in this Equation, $a=b+c$, the Terms are a, b and c.

IV. How Equations are found out, the Resolution of Questions will hereafter shew; but when known quantities are intermingled with unknown in an Equation, the first Scope is to clear the Equation from all superfluous quantities, and to separate the known quantities from the unknown, that at length an Equation may remain in the feweſt and ſimpleſt Terms, ſo diſpoſed, that the unknown quantity or quantities may poſſeſs one part of the Equation, and the known the other, this work is called *Reduction*, and how 'tis perform'd the Examples in the following *Sections* will make manifeſt.

Reduction by Addition.

V. Reduction by Addition is grounded upon this Axiom, (or common Notion) *viz.* If equal quantities, or one and the ſame quantity, be added to equal quantities, the wholes or totals ſhall be equal. As, for Examples;

If the letter a repreſent ſome number unknown, and it be granted or found out that . . . . $a-3=12$

Then by adding $+3$ to each part of that Equation, this ariſes, to wit, ... $a-3+3=12+3$

That is, (becauſe -3 and $+3$ added together make o,) . . . . $a=15$

In like manner, to reduce this Equation. . . . . $3a-4=6-a$

I add $+4$ to each part, and there ariſes . . $3a-4+4=6-a+4$

Which Equation contracted makes . . . . $3a=10-a$

Then by adding $+a$ to each part of the laſt Equation, this ariſes, ... $3a+a=10-a+a$

That is, after each part is contracted, ... $4a=10$

Again, If this Equation be propos'd to be reduced . . . . $aa-b=d+b$

By adding $+b$ to each part, this Equation ariſes, ... $aa-b+b=d+b+b$

Which laſt Equation, after due contraction gives . . . . $aa=d+2b$

So alſo, If . . . . $a-b=o$

By adding $+b$ to each Part, there ariſes . . . . $a=b$

Likewiſe, If . . . . $b-a=o$

By adding a to each part there ariſes . . . . . . $b=a$

Moreover, If . . . . $aa-bb-cc=dd$

Then by adding $bb+cc$ to each part of this Equation comes forth, ... $aa=dd+bb+cc$

Laſtly, If . . . . $aa-bb=cc-da$

By adding $+bb$ to each part, this Equation ariſes, ... $aa=cc-da+bb$

And by adding $+da$ to each part of the laſt Equation, this ariſes, to wit, ... $aa+da=cc+bb$

From the premiſes it is evident, That if in any Equation any Quantity which has the ſign $-$ prefixed to it, be transfer'd to the other part of the Equation with the ſign $+$, that work effects the ſame thing as the adding of that Quantity to each part of the Equation, and is called *Tranſpoſition*.

Reduction by Subtraction.

V. If from equal Quantities you take away equal Quantities, or one and the ſame Quantity, the Quantities remaining will be equal; therefore,

If it be taken for granted that . . . . $a+3=12$

Then by ſubtracting $+3$ from each part, there ariſes . . . . $a=9$

In like manner, If . . . . $b+a=4b$

I Subtract $+b$ from each Part, and there ariſes . . . . $a=3b$

Again, If . . . . $bb+2aa=aa+cc$

Firſt, I ſubtract bb from each part, and there remains . . . . $2aa=aa+cc-bb$

Then aa Subtracted from each part of the laſt Equation, leaves this, to wit, . . . $aa=cc-bb$

So alſo, If . . . . $aa+b+c=2ca+df$

By ſubtracting $+b+c$ from each part, there ariſes . . . . $aa=2ca+df-b-c$

And by ſubtracting $2ca$ from each part of the laſt Equation, this ariſes, to wit, . . . $aa-2ca=df-b-c$

Hence it is evident, That if *in any* Equation any Quantity which has the ſign $+$ prefixed to it be transferr'd to the other part of the Equation with the ſign $-$, that work Effects the ſame thing as the ſubtracting of that Quantity from each part of the Equation, and is alſo called *Tranſpoſition.*

Reduction by Multiplication.

VII. If equal Quantities be multiplied by equal Quantities, or by one and the ſame Quantity, the Products ſhall be equal: Hence Equations expreſt by Algebraical Fractions are reduced to other Equations conſiſting altogether of Integers.

As, for Example, If . . . . $\frac{a}{5}=6$

Then by multiplying each part by 5, this Equation is produced . . . . $a=30$

Again, to reduce this Equation to another in Integers, *viz.* . . . . $a=\frac{dd}{a-b}$

I multiply each part by $a-b$ and there comes forth . . . . $aa-ab=dd$

Likewiſe, to reduce this Equation to another in Integers, ... $\frac{3aa}{c}=\frac{dd}{b}$

Firſt, I multiply each part by the Denominator b, and there will be produced . . $\frac{3aab}{c}=dd$

Then Multiplying each part of the laſt Equation by the Denominator c, I find this Equation . . . . $3aab=cdd$

Hence it is manifeſt, That an Equation whereof each part is a Fraction, may be reduced to another Equation in Integers, by multiplying croſs-wiſe, as in the Reduction of of Fractions to a common Denominator, and then omitting the common Denominator, a new Equation may be instituted between the new Numerators only.

When either part of an Equation is compos'd of Integers and Fractions, first reduce that part into a Fraction, (after the manner of the latter Example in *Sect.* 16. *Chap.* 6.) and then multiply as in the preceding Examples: as,

If this Equation be proposed, ... $\frac{aa}{b}+c+d=bc+\frac{dd}{a}$

First, I reduce that Equation to this, . . . $\frac{aa+bc+bd}{b}=\frac{bca+dd}{a}$

Which last Equation reduced by Multiplication as in the preceding Examples, gives $aaa+abc+abd=bbca+bdd$

But here is to be noted, that in reducing Equations which consist of Fractions into other Equations in Integers, the Operation may oftentimes be facilitated by the same compendium that has before been shewn in the Division of Fractions (in *Sect* 26. *Chap.* 6.) *viz.* When either the Numerators or Denominators can be reduced to more simple Terms by some common Divisor, set the Quotients in the Places of those Numerators or Denominators; and then reduce these new Fractions into an Equation in Integers, by multiplying cross-wise as before: As for Example,

To reduce this Equation to another in Integers, ... $\frac{aaa}{aa-bb}=\frac{ba-bb}{a+b}$

First, after the Denominators $aa-bb$ and $a+b$ are reduced to $a-b$ and 1, by the common Divisor $a+b$, this New Equation arises, ... $\frac{aaa}{a-b}=\frac{ba-bb}{1}$

Whence by multiplying cross-wise, (as in the preceding Examples) this Equation in Integers is produced, ... $aaa=baa-2bba+bbb$

Again, to reduce this Equation to another in Integers, ... $\frac{bba-cca}{a+b}=\frac{bbb-bcc}{a}$

First, the Numerators reduced to a and b by the common Divisor, $bb-cc$ will give . . . . $\frac{a}{a+b}=\frac{b}{a}$

Whence by Multiplying cross-wise, this Equation is produced . . . . . . $aa=ba+bb$

In like manner, to reduce this Equation, . . . $\frac{baa-caa}{cc-ca}=\frac{bb-bc}{c}$

First, I reduce the Numerators to aa and b, by the common Divisor $b-c$; also the Denominators to $c-a$ and 1, by the common Divisor c; which new Numerators and Denominators constitute this Equation, ... $\frac{aa}{c-a}=\frac{b}{1}$

Whence by multiplying cross-wise, this Equation is produced . . . . . . $aa=bc-ba$

So also to reduce this Equation . . . . $\frac{ba^3-ca^3}{aa-ba+bb}=bc-cc$

First, I set 1 for a Denominator under the Integer $bc-cc$, so the Equation proposed will stand thus, ... $\frac{ba^3-ca^3}{aa-ba+bb}=\frac{bc-cc}{1}$

Then

Then, after the Numerators $ba^3 - ca^3$ and $bc - cc$ are reduced to a^3 and c,) by the common Divisor $b - c$, this Equation arises . . . . $\frac{a^3}{aa - ba + bb} = \frac{c}{1}$

Which last Equation, by multiplying cross-wise, gives this in Integers, ... $aaa = caa - cba + cbb$

When one part of an Equation is a Surd quantity, (that is, such which has a Radical sign prefixt to it, as, $\surd$, or $\surd(3)$, *&c.)* and the other part is a rational Quantity; that Equation may be reduced to another which shall be free from any Surd quantity, by casting away the Radical sign, and multiplying the rational part of the given Equation either quadraticaly or cubicaly, *&c.* according to the import of the Radical sign; as,

If there be proposed . . . . $\surd a = 6$

Forasmuch as the Squares of equal Roots or Sides are also equal, therefore by squaring each part of that Equation, this is produced, to wit, ... $a = 36$

Likewise, If . . . . $\surd a = bc$

By multiplying each part into it self, this Equation is produced, ... $a = bbcc$

Again, If . . . . $\surd a = \surd 5$

By squaring each part, there comes forth . . . . $a = 5$

And, If . . . . $\surd a = \surd{:}\overline{bcc - b}{:}$

By squaring each part, which is done by casting away $\surd$, there will arise . . . . $a = bcc - b$

So also if this Equation be proposed, ... $\surd ca = b - d$

By multiplying each part into it self, this Equation is produced, ... $ca = bb - 2bd + dd$

And, If . . . . $\surd(3)a = 8.$

By multiplying each part into it self cubically, there arises . . . . $a = 512.$

Also, If . . . . $\surd(3)a = \surd(3){:}\overline{b+c}{:}$

By casting away $\surd(3)$ from each part it gives . . . . $a = b + c.$

Reduction by Division.

VIII. If equal Quantities be divided by equal Quantities, or by one and the same Quantity, there will come forth equal Quotients. Hence Equations are reduced to others of lower Degrees: As, for example;

If it be granted or found out that . . . . . . $aa = 5a$

Then by dividing each part by a, you will find . . $a = 5$

Again, If . . . . $aaa + baa = bba$

By dividing each part by a; this Equation arises . . . . $aa + ba = bb.$

Also, If . . . . $5a = 15$

By dividing each part by 5, there arises . . . . . . $a = 3$

Likewise, If . . . . $ba = bc$

By dividing each part by b, this Equation arises, ... $a = c$

Again, If . . . . $ba - ca = cc$

By dividing each part by $b - c$, there arises . . . $a = \frac{cc}{b - c}$

Also, If . . . . $baa + caa = bd + cd$

By dividing each part by $b + c$, there arises . . . . $aa = d$

More-

Moreover, If . . . . $3aa+4a=39$
By dividing each part by 3, there arises . . . $aa+\frac{4}{3}a=13$
Likewise, If . . . . $caa-ba=cdd$
By dividing each part by c, there arises . . . . $aa-\frac{b}{c}a=dd$

Reduction by Extraction of ROOTS.

IX. Forasmuch as the Sides or Roots of equal Squares and Cubes, &c. are also equal between themselves; therefore,

If there be proposed . . . . $aa=36$
By extracting the Square Root of each part, there arises . . . . $a=6$
In like manner, If . . . . $aa=bb+2bc+cc$
By extracting the Square Root of each part, there comes forth . . . . $a=b+c$
Again, If . . . . $aa=29$
By extracting the Square Root of each part, there will arise . . . . $a=\sqrt{29}$
Likewise, If . . . . $aa=bb-dd$
Then, by extracting the Square Root out of each part, there arises . . . . $a=\sqrt{:\overline{bb-dd}:}$
Again, If . . . . $aaa=27$
Then, the Cubic Root being extracted out of each part there comes forth . . . . $a=3$
Also, If . . . . $aaa=12$
By extracting the Cubic Root out of each part, this Equation will arise . . . . $a=\sqrt{(3)}12.$
Likewise, If . . . . $aaa=bbc+cdd$
Then, the Cubic Root extracted out of each part, gives . . . . $a=\sqrt{(3)}:\overline{bbc+cdd}:$

X. By the help of some of the foregoing Reductions, I shall here shew (after the manner of *Fran. van Scooten* in his *Principia Mathes. Universal.*) the certainty of the Rule before given concerning + and — in the Algebraical Multiplication of Compound Quantities: *viz.* That + multiplied by —, or — by + makes —; also, That — multiplied by — makes +.

First, let $a-b$ be to be multiplied by c, then the Product according to Algebraical Multiplication is $ac-bc$: now it must be proved that $-b$ multiplied by $+c$ makes $-bc$; to which end, let f be put equal to $a-b$, and then if it be proved that $ac-bc=fc$, it is evident that $ac-bc$ is the true Product sought; and consequently, $-b$ multiplied by $+c$ makes $-bc$: But that $ac-bc=fc$ may be proved thus,

Forasmuch as by supposition, ... $a-b=f$
Therefore by adding b to each part, it makes . . . $a=f+b$
And by multiplying each part of the last Equation by c, there will be produced . . . . $ac=fc+bc$
Wherefore, by subtracting bc from each part of the last Equation there remains . . . $ac-bc=fc$
Which was to be proved.

After the same manner it may be proved that — multiplied by — makes +: For, If $a-b$ be to be multiplied by $c-d$, and there be put (as before) $f=a-b$, it may be shewn that $ac-bc-ad+bd$ is equal to $\overline{a-b}\times\overline{c-d}$ the Product sought; and therefore $-b$ multiplied by $-d$ produces $+bd$. For,

By

By supposition	$f = a-b$
Therefore, by multiplying each part into $c-d$ . . .	$f\times\overline{c-d} = \overline{a-b}\times\overline{c-d}$
That is, ...	$fc-fd = \overline{a-b}\times\overline{c-d}$
But it has been proved in the former Example, that	$ac-bc = fc$
Therefore instead of fc in the third Equation of this latter Example, taking $ac-bc$ (equal to fc) there arises	$ac-bc-fd = \overline{a-b}\times\overline{c-d}$
Again, If each part of the first Equation be multiplied by d, this will be produced,	$fd = ad-bd$
Wherefore, If from $ac-bc$ in the fifth Equation there be subtracted $ad-bd$ instead of fd equal to $ad-bd$, there will remain according to the Rule of Algebraical Subtraction	$ac-bc-ad+bd = \overline{a-b}\times\overline{c-d}$

Which was to be proved.

CHAP. XII.

Which shews in what Order the Reductions in the foregoing Chap. 11. *are to be used to resolve Equations, or at least to prepare them for Resolution.*

I. BY the help of the precedent Reductions, either the value of the unknown Root or Quantity sought in an Equation will be found equal to some known Quantity or Quantities, and consequently the Quantity sought is then known also; or else a new Equation will be discovered, from whence the same Quantity sought may be made known by some other Rule or Rules hereafter delivered: But in the use of those Reductions, the work may oftentimes be facilitated by an orderly process, which is the Scope of the five following *Sections*; where I assume the Vowel a to stand for the unknown Root or Quantity sought, and Consonants for known Quantities.

II. If in any Equation the Quantity sought, or any Power or Degree of it be found in a Fraction, reduce that Equation to another that may be express'd altogether by Integers, (by *Sect.* 7. *Chap.* 11.) As for Example;

If this Equation be proposed, ...	$\dfrac{b-a}{c} = d+f-g$
By multiplying each part thereof by the Denominator c, this Equation arises in Integers, ...	$b-a = cd+cf-cg$
After the same manner, this Equation multiplied by 4, ...	$\dfrac{aa}{4}+6 = 15.$
Will be reduced into	$aa+24 = 60.$
Likewise this Equation	$\dfrac{aa+bb}{d}+b+c = a-c$
Will be reduced to	$aa+bb+db+dc = da-dc.$

III. When Quantities given or known be intermingled with those that are sought in an Equation, let Quantities be transfer'd from one part of the Equation to the other under a contrary Sign, (according to *Sect.* 5. and 6. of *Chap.* 11.) until at length the

unknown

unknown Quantity may make one part of an Equation, and all the known Quantities the other: As for Example;

If there be propoſed . . . . $2a-26=8$

By tranſpoſition of -26 to the other part of the Equation, under the contrary ſign $+$, there will ariſe . . . . $2a=8+26=34$.

In like manner, If . . . . $aa+24=60$

By tranſpoſition of $+24$, under the contrary ſign $-$ it gives . . . . $aa=60-24$

That is, ... $aa=36$

Again, If . . . . $6a-4=20-a$

Firſt, by tranſpoſition of -4, this Equation ariſes, ... $6a=20-a+4$

Then by tranſpoſition of $-a$, I find . . . . $6a+a=20+4$

Which laſt Equation being contracted by Addition, gives . . . . $7a=24$

Likewiſe, If . . . . $b-a=cd-cf$

After due Tranſpoſition, this Equation will ariſe, ... $b+cf-cd=a$

Or, ... $a=b+cf-cd$

IV. When ſome Power or Degree of the Quantity ſought happens to be multiplied into every Term or Member of an Equation, divide every Term by that Degree, ſo will that Degree or Power quite vaniſh, and conſequently the Equation will be depreſſed, that is, reduced to lower Degrees or more ſimple Terms: As for Example,

If there be propoſed . . . . $aa+3a=20a$

Foraſmuch as a is drawn into every Term of that Equation, I divide every Term by a, and there ariſes . . . . $a+3=20$

Whence by equal ſubtraction of 3 I find . . . . . . $a=17$

In like manner, If . . . . $aaa=3aa$

By caſting away aa, that is, by dividing each part by aa, there will ariſe . . . . . $a=3$

Again, If . . . . $aaaa+baaa=ddaa$

By expunging aa out of every Term, there ariſes . . . . $aa+ba=dd$

V. When ſome known Quantity is multiplied into the higheſt Power or Degree of the Quantity unknown or ſought in an Equation; divide each part of the Equation by that known Quantity, to the end the ſaid higheſt unknown Power may have no Co-efficient or Fellow-multiplier but 1, (or Unity; As for Example,

If there be propoſed . . . . $5a=60$

Becauſe the unknown Quantity a is multiplied by 5, I divide each part of the Equation by 5, and there ariſes . . . . . $a=12$

Again, If . . . . $ca=cc+dd$

Becauſe c is drawn into a the Root ſought, I divide every Term of the Equation by c, and there ariſes . . . . $a=c+\frac{dd}{c}$

Like-

Likewise, If	$2ba+3ca=2ddb+3cdd$
Because $2b+3c$ is drawn into the unknown Root a, I divide each part by $2b+3c$, and there arises . . .	$a=dd$

So also, If	$4aa=60$
By dividing each part by 4 which is drawn into aa, there arises . . .	$aa=15$

Again, If	$3aa-5a=24$
Because 3 is drawn into aa which is the highest unknown Power in the Equation, I divide every Term by 3, and there arises	$aa-\frac{5}{3}a=8$

Likewise, If	$2ccaa-4dda=5bbcc$
Because $2cc$ is drawn into aa which is the highest unknown Power in the Equation, I divide every Term by $2cc$, and there arises	$aa-\frac{2dd}{cc}a=\frac{5}{2}bb.$

Again, If	$2bbaa+3cdaa-dda=ccdd$
Because $2bb+3cd$ is drawn into aa the highest unknown Degree in the Equation, I divide each part by $2bb+3cd$, and there arises	$aa-\frac{dd}{2bb+3cd}a=\frac{ccdd}{2bb+3cd}$

Also, If	$3aaa+24aa-6a=1200$
Because 3 is drawn into aaa the highest unknown Power in the Equation, I divide each part by 3, and there arises	$aaa+8aa-2a=400$

VI. If there be a surd Quantity in an Equation, that is, if a Radical sign as $\sqrt{}$; or $\sqrt{(3)}$ be prefixed before some Quantity; first by Transposition (according to *Sect.* 5. or 6. of *Chap.* 11.) make the surd Quantity sole possessor of one part of an Equation, then cast away the Radical sign, and exalt the other part of the Equation to the same Degree or Power which is denoted by the Radical sign, by multiplying Quadratically or Cubically, &c. so at length an Equation will be found express'd altogether by rational Quantities: As for Example;

If this Equation be proposed	$\sqrt{a}=3$
By squaring each part, there will be produced . . .	$a=9$

In like manner, If	$\sqrt{ba}=3bc$
By multiplying each part into it self quadraticaly, there comes forth . . .	$ba=9bbcc$
Then dividing each part of the last Equation by b, there arises . . .	$a=9bcc$

Again, If	$b+\sqrt{ba}=c$
First by transposition of b there arises	$\sqrt{ba}=c-b$
Then by squaring each part of the last Equation, there will be produced . .	$ba=cc-2cb+bb.$
Whence, by dividing each part by b, there arises	$a=\frac{cc}{b}-2c+b$

Likewiſe, If	$-d+\sqrt{ba+da}=b$
Firſt by tranſpoſition of $-d$, this Equation ariſes, ...	$\sqrt{ba+da}=b+d$
Then by ſquaring each part, there will be produced	$ba+da=bb+2bd+dd$
Laſtly, by dividing each part of the laſt Equation by $b+d$, there ariſes . . .	$a=b+d$

Again, If	$\sqrt[3]{9a}=3$
By multiplying each part Cubically, there will be produced	$9a=27$
And, by dividing each part of the laſt Equation by 9 there ariſes	$a=3$

Likewiſe, If	$\sqrt[3]{ba-ca}+c=b$
Firſt, by tranſpoſition of $+c$ this Equation ariſes, ...	$\sqrt[3]{ba-ca}=b-c$

Then multiplying each part of the laſt Equation cubically, this Equation will be produced, to wit,

$$ba-ca=bbb-3bbc+3bcc-ccc:$$

Whence, by dividing each part by $b-c$, the value of a will be diſcovered, *viz.*

$$a=bb-2bc+cc.$$

VII. When after the uſing of all, or any of the foregoing Rules of this Chapter an Equation ariſes between a perfect Square, Cube or other higher Power of the Quantity ſought, and ſome known Quantity; then extract ſuch a Root out of each part of the ſaid Equation as the Index of the ſaid unknown Power denotes, ſo will the value of the unknown Root or Quantity ſought be made known: As, for Example;

If this Equation be propoſed, to wit, .	$\frac{6aa}{5}+8=128$
Firſt by ſubtracting 8 from each part, this Equation ariſes, ...	$\frac{6aa}{5}=120$
Then each part of the laſt Equation being multiplied by 5, gives	$6aa=600$
And by dividing each part of the laſt Equation by 6, this ariſes, ...	$aa=100$
Laſtly, the ſquare Root of each part of the laſt Equation being extracted, the value of a will be diſcovered, to wit, ...	$a=10$

Again, If	$\frac{3aaaa}{4}-8a=154a$
Then by tranſpoſition of $-8a$ there ariſes	$\frac{3aaaa}{4}=162a$
And by multiplying each part of the laſt Equation by 4, this will be produced,	$3aaaa=648a$
And by dividing each part of the laſt Equation by a this ariſes, to wit, ...	$3aaa=648$
Likewiſe each part of the laſt Equation divided by 3 gives	$aaa=216$
Laſtly, by extracting the Cubic Root out of each part of the laſt Equation, the value of a will be diſcovered, to wit,	$a=6$

Like-

Likewise, If	$aa+2ba+bb = cc$
The square Root extracted out of each part, gives	$a+b = c$
And then by transposition of b, the value of a is discovered, to wit, ...	$a = c-b$

CHAP. XIII.

Which shews how to convert Analogies into Equations, and Equations into Analogies.

I. IF four right-lines or numbers be Proportionals, the Product made by the Multiplication of the two Extreams is equal to the Product of the two means. And if three right-lines or numbers be Proportionals, the Product of the Extreams is equal to the Square of the mean, (by *Prop.* 16. and 17. of 6. *Elem.* and by 19. and 20. of 7. *Elem. Euclid.*) Hence Analogies may be converted into Equations, as in the following Examples; where for the greater evidence let a represent 2; b, 6; c, 12; and d, 3; Then

1. Let there be four Proportionals, suppose these, ...	$d \;.\; b :: d-a \;.\; a$ $3 \;.\; 6 :: 1 \;.\; 2$
Then by the Theorem above express'd, this Equation will follow, ...	$da = bd-ba$
Now to find the value of a in that Equation, first by transposition of $-ba$ this Equation arises, ...	$da+ba = bd$
Then each part divided by $d+b$ gives	$a = \dfrac{bd}{d+b}$

2. If there be three continual proportionals, suppose these, ...	$4a \;,\; c \;,\; 9a \;\div$ $8 \;,\; 12 \;,\; 18 \;\div$
That is, If	$4a \;.\; c :: c \;.\; 9a$
Then, by the latter part of the said Theorem, this Equation will follow, ...	$36aa = cc$
Now to find the value of a in that Equation, extract the square Root out of each part, and there arises	$6a = c$
Lastly, each part of the last Equation divided by 6 gives	$a = \dfrac{c}{6}$, or $\frac{1}{6}c$.

II. If the Product of the multiplication of two Quantities be found equal to the Product of two other Quantities, that Equation may be resolved into Proportionals; for as either of the Factors in either of the two equal Products is to a Factor of the same kind in the other Product, so is the remaining Factor in this latter Product to the other Factor in the former. Hence Equations may oftentimes be resolved into Proportionals; as,

If there be proposed	$3ba = cd$
From that Equation this Analogy may be infer'd, *viz.* As	$3b \;.\; c :: d \;.\; a$

Again, If	$bd = da+ba$
That Equation may be resolved into these Porportionals, *viz.* As	$d+b \;.\; b :: d \;.\; a$

Likewise, If	$6da = bb$
Then it shall be, As	$6d \;.\; b :: b \;.\; a$

III. When

III. When there happens to be an Equation between an Algebraical Fraction and an Integer, and the Numerator of the Fraction can be resolved into two such Quantities that being mutually multiplied will produce the said Numerator, then that Equation may be resolved into Proportionals in this manner, *viz.* Let the Denominator of the Fraction, and the Integer to which the Fraction is equal, be made the extream Terms of an Analogy; and let the two Quantities which being mutually multiplied will constitute the Numerator be made the mean Terms; but with this caution in Geometrical Questions, that the first and second Terms be of one and the same kind, that is, either both Lines, or both Planes, or both Solids. As for Example;

If this Equation be proposed, ... $\frac{cd}{3b} = a$

It may be resolved into these Proportionals, . $3b \,.\, c :: d \,.\, a$

But that they are Proportionals, I prove thus;

First, It is evident that these are Proportionals, (because the Product of the extreams is equal to the Product of the means) . . . . $3b \,.\, c :: d \,.\, \frac{cd}{3b}$

And by the Equation proposed, ... $a = \frac{cd}{3b}$

Therefore . . . . $3b \,.\, c :: d \,.\, a \,.\, (\frac{cd}{3b})$

Again, If . . . . $\frac{bb}{b+c} = a$

That Equation may be resolved into these Proportionals, ... $b+c \,.\, b :: b \,.\, a$

Likewise this Equation . . . . $\frac{cc-bb}{5b+2c} = a$

may be resolved into this Analogy, . . $5b+2c \,.\, c+b :: c-b \,.\, a$

And this Equation . . . . $\frac{bb+2bc+cc}{54d} = a$

may be converted into these Proportionals, $54d \,.\, b+c :: b+c \,.\, a$

Also, this Equation . . . . $\frac{bbc}{36d} = aa$

may be resolved into these Proportionals, . . . $36d \,.\, b :: bc \,.\, aa$

Or into these, ... $36d \,.\, c :: bb \,.\, aa$

But this Equation . . . . $\frac{b}{c} = a$

cannot be resolved into Proportionals any otherwise than thus, ... $c \,.\, \sqrt{b} :: \sqrt{b} \,.\, a$

Nor can this Equation . . . . $\frac{bb+cd}{g} = a$

be converted into Proportionals, unless thus, $g \,.\, \sqrt{:bb+cd:} :: \sqrt{:bb+cd:} \,.\, a$

CHAP.

CHAP. XIV.

Various Arithmetical Questions Algebraically resolved; whereby most of the Rules hitherto delivered are exercis'd, in the Invention and Resolution of pure or simple Equations.

I. Equations may be divided into two kinds, *viz.* { 1. Pure or Simple, 2. Affected or Compounded.

II. A pure or simple Equation is of two kinds, *viz.* First, when the Quantity sought is express'd by a simple Root only, as a; as in this Equation, $6a=12$: Secondly, when the Quantity sought is express'd by a simple Power only, as aa, or aaa, &c. as in this Equation, $3aaa=24$; likewise in this, $2aaaa=32$, and such like.

III. An affected or compounded Equation is that, wherein there are two or more different Degrees or Powers of the Quantity sought; as in this Equation, $aa+6a=27$, where aa and a express two different Degrees or Powers of the Quantity sought; the one signifying a Square, and the other its Root or Side: also in this Equation, $aaa+6aa-2a=28$, there are three unlike Powers or Degrees of the Quantity sought, to wit, aaa, aa, and a.

IV. The Invention and Resolution of pure or simple Equations is copiously illustrated by Arithmetical Questions in this Chapter, as also in the second and third Books of my *Algebraical Elements*; and the Resolution of Affected or Compound Equations in Numbers is handled in the 15, 16, and 17. Chapters of this Book, as also in the 10, and 11. Chapters of the Second Book. But how Algebraical Operations are applicable to the solving of Geometrical Problems, I shall shew in my fourth Book of *Algebraical Elements*.

V. When an Arithmetical Question is proposed, the number sought must first of all be assumed or supposed to be known; and you may represent it by the Letter a, or any other Vowel: You may likewise represent the given Numbers by Consonants, as, b, c, d, &c. *Renates des Cartes* puts for given Quantities the former Letters of the Alphabet, as, a, b, c, d, &c. but for Quantities sought the latter Letters, z, y, x, &c. Then with the Letters representing the Numbers given and sought, an orderly process must be made, by adding, subtracting, multiplying or dividing, *&c.* according to the Import of the Question, until at length an Equation be found out between the Number sought or some Power or Powers of it, and some Number or Numbers given: Lastly, when the Equation so found out is a pure or simple Equation, the Number sought may be discovered by some of the Reductions in the foregoing 12, and 13. Chapters; but when the Equation is Affected or Compounded, the Resolution thereof belongs either to the 15. Chapter of this first Book, or the 10, and 11. Chapters of the second Book.

VI. In the Resolution of every Question, I proceed from the Beginning to the End by steps numbred in the Margin, by 1, 2, 3, 4, 5, *&c.* And because *Numeral Algebra* is more easie for Learners than the *Literal*, (though not so useful for the Reasons before given in *Sect.* 8. *Chap.* 1.) I have in many Questions express'd the Operation belonging to every step in both kinds of *Algebra*, that the one may explain the other: So in the second step of the Resolution of the following first Question, the lesser Number sought is express'd by *Numeral Algebra* thus, $26-a$; but by *Literal Algebra* thus, $b-a$. Also, in the fourth step, the Equation by numeral Algebra is $2a-26=8$; but by literal Algebra it is $2a-b=c$.

VII. When an Equation is found out in any of the following Questions, I take it for granted that the Reader knows how to reduce it, if need be, according to the Rules in the foregoing 11, 12, and 13. Chapters, that I may avoid tedious repetitions of what has been already explain'd. These things premised, I proceed to the Questions themselves.

QUEST.

QUEST. 1.

There are two Numbers whose Sum is 26, (or b,) and their difference, (to wit, the excess of the greater above the lesser) is 8, (or c;) What are the Numbers?

RESOLUTION:	*Numeral,*	*Literal.*
1. For the greater Number put	a	a
2. Then subtracting that Number a from the given Sum, the Remainder will be the lesser Number, to wit, ...	$26-a$	$b-a$
3. And by subtracting the lesser number from the greater, the Remainder will be their difference, to wit, ...	$2a-26$	$2a-b$
4. Which difference found out in the last step must be equal to the given difference 8, (or c) whence this Equation arises, ...	$2a-26=8$	$2a-b=c$
5. From which Equation, after it is duly reduced according to *Sect.* 3. and 5. of *Chap.* 12. the greater number sought will be discovered, to wit, ...	$a=17$	$a=\frac{1}{2}b+\frac{1}{2}c$
6. And consequently from the fifth and second steps the lesser Number is also discovered, to wit, ...	9, that is,	$\frac{1}{2}b-\frac{1}{2}c$.

So the Numbers sought are found 17 and 9, whose Sum is 26, and their difference is 8, as was prescribed.

Moreover, If the two last steps of the literal Resolution be express'd by words, they will give this

THEOREM.

Half the difference of any two Numbers added to half their Sum, gives the greater Number: But half the difference of any two Numbers subtracted from half their Sum, leaves the lesser Number.

Therefore the Sum and difference of any two Numbers being given severally, the Numbers themselves are also given by the said Theorem; but it presupposes that the Number given for the Difference must be less than the Number given for the Sum.

Note here once for all, That the Numbers given in a Question cannot always be chosen at pleasure, but sometimes, they must be subject to one or more Determinations, which for the most part (though not always) are discoverable by the Theorem or Canon that results from the Resolution. But how Limits or Determinations are discovered, I shall have occasion to shew hereafter in my second, third, and fourth Books of *Algebraical Elements.*

QUEST. 2.

There are two Numbers whose Sum is 40, (or b,) and the greater Number has such proportion to the lesser as 3 to 2, or, as r to s;) What are the Numbers?

1. For the greater Number sought put . . .	a	a
2. Then to find the lesser Number, say by the Rule of Three, If 3 . 2 :: a . $\frac{2a}{3}$ Or, r . s :: a . $\frac{sa}{r}$ whence the lesser Number is	$\frac{2a}{3}$	$\frac{sa}{r}$

3. There-

3. Therefore the Sum of the two Numbers sought is	$\frac{5a}{3}$	$a+\frac{sa}{r}$
4. Which Sum found out in the last step must be equal to the given Sum 40, (or *b*,) whence this Equation	$\frac{5a}{3}=40$	$a+\frac{sa}{r}=b.$
5. Which Equation, after due Reduction according to *Sect.* 2. and 5. of *Chap.* 12. gives the greater Number	$a=24$	$a=\frac{rb}{r+s}$
6. And from the fifth, first, and second steps, the lesser Number is also discovered, to wit,	16, or	$\frac{sb}{r+s}$

So the Numbers sought are found 24 and 16, which will satisfie the Conditions in the Question; for their Sum is 40, and the greater has such proportion to the less as 3 to 2, as was prescribed.

Moreover, If the two last steps of the literal Resolution be resolved into Proportionals, according to *Sect.* 3. *Chap.* 13. there will arise this

THEOREM.

As the Sum of both the Terms which express the Reason (or Proportion) of two Numbers, is to the Sum of the same two Numbers; so is the greater Term to the greater Number; and so is the lesser Term to the lesser Number.

Therefore the Sum of two Numbers being given, as also their Reason, or Proportion; the Numbers shall also be given severally by the said Theorem.

QUEST. 3.

There are two Numbers whose difference is 8, (or *d*,) and the greater Number has such proportion to the lesser as 3 to 2, (or as *r* to *s*;) what are the Numbers?

1. For the greater Number put	a	a
2. Then to find the lesser Number say by the Rule of Three, If $3 . 2 :: a . \frac{2a}{3}$ Or if $r . s :: a . \frac{sa}{r}$ whence the lesser Number is	$\frac{2a}{3}$	$\frac{sa}{r}$
3. Therefore by subtracting the lesser Number from the greater, the Remainder shall be their difference, to wit, ...	$\frac{a}{3}$	$a-\frac{sa}{r}$
4. Which difference must be equal to the given difference 8 (or *d*,) hence this Equation arises	$\frac{a}{3}=8$	$a-\frac{sa}{r}=d.$
5. Which Equation, after due Reduction, discovers the greater Number sought, to wit,	$a=24$	$a=\frac{rd}{r-s}$
6. And from the fifth, first, and second steps the lesser number will be also made known, to wit,	. . . $=16$	$=\frac{sd}{r-s}$

So the Numbers sought are found 24 and 16, which will solve the Question; for their difference is 8, and they are in the proportion of 3 to 2, as was prescribed.

Moreover, If the two last steps of the literal Resolution be converted into Proportionals (according to *Sect.* 3. *Chap.* 13.) there will arise this

THEOREM.

As the difference of the two Terms which express the Reason or Proportion of two Numbers is to the difference of the same two Numbers, so is the greater Term to the greater Number; and so is the lesser Term to the lesser Number.

Therefore the Difference and Reason of two Numbers being severally given, the Numbers themselves shall be also given by the said Theorem.

QUEST.

QUEST. 4.

There are two Numbers whoſe Sum is 7, (or b,) and the difference of their Squares is 21, (or d;) what are the Numbers?

1. For the greater Number ſought put	a	a
2. Then ſubtracting the greater Number from the given Sum, the Remainder is the leſſer Number, to wit, ...	$7-a$	$b-a$
3. Therefore from the firſt ſtep the Square of the greater Number is	aa	aa
4. And from the ſecond ſtep the Square of the leſſer Number is	$aa-14a+49$	$aa-2ba+bb$
5. Therefore the difference of the Squares of the two numbers ſought ſhall be . . .	$14a-49$	$2ba-bb$
6. Which difference muſt be equal to the given difference 21 (or d,) whence this Equation ariſes	$14a-49=21$	$2ba-bb=d$
7. Which Equation, after due Reduction according to *Sect.* 3, and 5. of *Chap.* 12. diſcovers the greater number ſought, to wit,	$a=5$	$a=\frac{bb+d}{2b}$
8. And from the ſeventh and ſecond ſteps, the leſſer number will be alſo made known, to wit, ...	$=2$	$=\frac{bb-d}{2b}$

So the Numbers ſought are found 5 and 2, which will ſolve the Queſtion; for their Sum is 7, and the difference of their Squares is 21, (to wit, 25—4;) as was preſcribed.

Moreover, If the two laſt ſteps of the literal Reſolution be expreſs'd by words, they will give this

THEOREM.

If to the Square of the Sum of any two numbers the difference of their Squares be added, and the Sum of that addition be divided by the double Sum of the two Numbers, the Quotient will be the greater Number: But if from the Square of the Sum of two Numbers the difference of their Squares be ſubtracted, and the Remainder be divided by the double Sum of the two Numbers, the Quotient will give the leſſer Number.

Therefore the Sum of two numbers being given, as alſo the difference of their Squares, the numbers themſelves ſhall be given ſeverally; but it preſuppoſes the ſquare of the given Sum to exceed the given difference.

QUEST. 5.

There are two numbers whoſe difference is 3, (or c,) and the difference of their Squares is 21, (or d;) what are the Numbers?

1. For the leſſer number ſought put	a	a
2. To which adding the given difference 3, (or c,) the Sum will make the greater number, to wit, ...	$a+3$	$a+c$
3. Therefore the ſquare of the greater number is	$aa+6a+9$	$aa+2ca+cc$
4. And the ſquare of the leſſer number is	aa	aa
5. Therefore the difference of thoſe Squares is	$6a+9$	$2ca+cc$
6. Which difference muſt be equal to the given difference of the ſquares; whence this Equation ariſes, to wit, ...	$6a+9=21$	$2ca+cc=d$
7. Which Equation, after due Reduction (according to *Sect.* 3, and 5. of *Chap.* 12.) diſcovers the leſſer number, to wit, . . .	$a=2$	$a=\frac{d-cc}{2c}$
8. And from the ſeventh and ſecond Equations, the greater number will be found . . .	$=5$	$=\frac{d+cc}{2c}$

So

So the Numbers sought are 5 and 2, which will solve the Question; for their difference is 3, and the difference of their Squares is 21; as was prescribed.

Moreover, the two last steps of the literal Resolution afford this

THEOREM.

If to the difference of the Squares of any two Numbers the Square of their difference be added, and the Sum of that Addition be divided by the double of the difference of those two Numbers, the Quotient will give the greater Number: But if from the difference of the Squares of two Numbers the Square of their difference be subtracted, and the Remainder be divided by the double of the difference of those two Numbers, the Quotient shall be the lesser Number.

Therefore the difference of any two Numbers being given, as also the difference of their Squares, the Numbers themselves shall also be given severally by this Theorem; but it presupposes the given difference of the Squares of the two Numbers to exceed the Square of the given difference of the same two Numbers.

QUEST. 6.

A certain Person being asked what was the Age of every one of his four Sons, answered; the eldest was four Years (or b) elder than the second, the second was four Years elder than the third, the third was four Years elder than the fourth or youngest; and the double of the youngest Sons Age was equal to the Age of the eldest; what was the Age of each Son?

1. For the Age of the eldest Son put	a	a
2. Then from the Age of the eldest Son subtracting 4 (or b) there will remain the second Sons Age, to wit, ...	$a-4$	$a-b$
3. Likewise from the second Son's Age subtracting 4 (or b) the Remainder will be the third Son's Age, to wit, ...	$a-8$	$a-2b$
4. Again, from the third Son's Age subtracting 4 (or b) there will remain the fourth or youngest Son's Age, to wit, ...	$a-12$	$a-3b$
5. But according to the Question, the double of the Age in the fourth step must be equal to the Age in the first step, whence this Equation will arise, ...	$2a-24=a$	$2a-6b=a$
6. Which Equation duly reduced discovers the Age of the eldest Son, to wit, ...	$a=24$	$a=6b$

Wherefore the Ages of the four Sons were 24, 20, 16, and 12; for the first exceeds the second by 4, which is also the excess of the second above the third, the third above the fourth, and the double of the fourth is equal to the first, as was prescribed in the Question.

Moreover the last step of the literal Resolution shews, that if instead of 4, any other Number be given for the common difference of the four Sons Ages, then six times that common difference will give the eldest Sons Age, which shall be equal to the double of the Age of the youngest.

QUEST. 7.

A Merchant began to Trade with a certain Number of Pounds: By his first Voyage he doubled that Stock; by his second he lost 1200 Pounds (or b) by his third he doubled his remaining Stock; by his fourth he lost again 1200 Pounds, and then had no money left. The Question is, to find how many Pounds the Merchant began to Trade with?

1. For the number of Pounds which the Merchant began to trade with put	a	a
2. Then the double of that number gives the number of Pounds he had at the end of his first Voyage, to wit, ...	$2a$	$2a$
3. From which last number subtracting 1200 (or b,) the Remainder shews the number of Pounds that remained to the Merchant at the end of his second Voyage, to wit, . .	$2a-1200$	$2a-b$
4. Which remaining number being doubled gives the number of Pounds which the Merchant had at the end of his third Voyage, to wit, ...	$4a-2400$	$4a-2b$
5. From which last number subtracting again 1200 (or b) Pounds lost by the fourth Voyage, the Remainder must be equal to nothing; hence this Equation, ...	$4a-3600=0$	$4a-3b=0$
6. Which Equation, after due Reduction, gives	$a=900$	$a=\frac{3}{4}b.$

Whence it is found that the Merchant began to trade with 900 Pounds; which number will satisfie the Conditions in the Question.

Moreover the last step of the literal Resolution shews, that if instead of 1200 any other number were given, the Merchants stock at first would be three Quarters of that given number.

QUEST. 8.

A Gentleman hired a Servant for a Year, for 120 Shillings (or c,) together with a livery Cloak valued at a certain number of Shillings: But when $\frac{7}{12}$ (or d) parts of the Year were expired, the Master falling at variance with his Servant puts him away, and gives him the Cloak with 50 Shillings, (or f;) and so the Servant received full satisfaction for the time of his service. The Question is, to find how many Shillings the Cloak was valued at?

1. For the number of Shillings which the Cloak was valued at put	a	a
2. Then to find what part of the value of the Cloak was due to the Servant when $\frac{7}{12}$ (or d) parts of the Year were expired, say by the Rule of Three, If $1 : a :: \frac{7}{12} . (\frac{7a}{12}$ Or, if $1 . a :: d . (da$ whence the desired part of the value of the Cloak is found	$\frac{7a}{12}$	da
3. Find likewise what part of the 120 (or c) Shillings was due to the Servant when $\frac{7}{12}$ (or d) parts of the Year were expired, and say, If $1 . 120 :: \frac{7}{12} . (70$ Or, $1 . c :: d . (cd$ whence the part desired is found	70	cd

4. Now forasmuch as the Cloak together with the 50 Shillings the Servant received, ought to be equal to the part of the Cloak, together with the part of the 120 Shillings that was due to him at the time he left his service; therefore from the premises there arises this Equation:

$$a+50=\frac{7a}{12}+70; \quad \text{Or,} \quad a+f=da+cd.$$

5. Which

5. Which Equation after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. will give the desired value of the Cloak, to wit,

$$a = 48 = \frac{cd \backsim f}{1 \backsim d}:$$

Whence it is evident that the Cloak was valued at 48 Shillings; and the last Equation discovers this

CANON.

Multiply the Money which the Servant was to receive besides the Cloak for a Years Wages, by the time he served; then divide the difference between that Product and the Money he received when he left his service by the difference between 1 (or unity) and the same time he served; so the Quotient gives the value of the Cloak.

By which Canon the value of the Cloak will be found to be 48 *s.* as above.

The Proof.

$$48 + 50 = 98.$$

$$\tfrac{7}{12} \text{ of } 48, + \tfrac{7}{12} \text{ of } 120 = 98.$$

QUEST. 9.

A certain Man finding divers poor Persons at his Door, gave every one of them three pence (or *b*,) and had six pence (or *c*) left; but if he would have given them four pence (or *f*) a piece, he should have wanted two pence (or *g*.) How many poor Persons were there?

1. For the number of poor Persons put . . . . a
2. Then forasmuch as that number multiplied by 3 (or *b*) and the Product increased with 6 (or *c*) makes the whole number of pence that the giver had: And, because if the same number of poor Persons be multiplyed by 4 (or *f*,) the Product less by 2 (or *g*) must also make the same number of pence: hence this Equation;

$$3a + 6 = 4a - 2:$$

$$\text{Or,}\quad ba + c = fa - g.$$

3. Which Equation after due Reduction according to *Sect.* 3, and 5. of *Chap.* 12. discovers the number of poor Persons to be 8: *viz.*

$$8 = \frac{c+g}{f-b} = a.$$

QUEST. 10.

One being asked what a Clock it was, answer'd, That the time then past from Noon was equal to $\frac{33}{40}$ (or, *b*) parts of the time remaining until midnight: What was the present Hour? supposing the time between Noon and Midnight to be divided into 12 (or *c*) equal Hours.

1. For the Hour sought after noon put . . . :	a	a
2. Which subtracted from 12 (or *c*) leaves the time remaining until midnight, to wit,	$12-a$	$c-a$
3. Then $\frac{33}{40}$ (or *b*) parts of the said remaining time will be	$\frac{396}{40}-\frac{33}{40}a$	$bc-ba$
4. Therefore from the first and third steps (according to the Question) this Equation arises, to wit, ...	$a=\frac{396}{40}-\frac{33}{40}a$	$a=bc-ba$
5. Which Equation after due Reduction according to *Sect* 2, 3, and 5. of *Chap.* 12. gives the Hour sought, to wit, ...	$a=5\frac{31}{73}$	$a=\frac{bc}{b+1}$ . .

So the time sought was $5\frac{31}{73}$ Hours after noon, and consequently the remaining time until midnight was $\frac{480}{73}$ Hours, whereof $\frac{33}{40}$ is equal to the said $5\frac{31}{73}$; as was prescribed in the Question.

QUEST.

QUEST. 11.

A General of an Army having set his Soldiers in a Square Battel, there happened to be 500 (or *b*) Soldiers to spare; but to increase the Square so as that its side might consist of 1 (or *c*) Soldier more than the side of the former Square, there would be 29 (or *d*) Soldiers wanting. The Question is, to find how many Soldiers the General had in his Army.

1. For the Number of Soldiers that made the side of the first Square, put	a	a
2. Then that side multiplied by it self gives the Number of Soldiers in the first square Battel, to wit, ...	aa	aa
3. Therefore the number of Soldiers in the whole Army was	$aa+500$	$aa+b$
4. Then to the end the side of another Square may exceed the side of the former by 1 (or *c*,) let it be	$a+1$	$a+c$
5. Which latter side multiplied by it self gives the Number of Soldiers in the latter square Battel, to wit, ...	$aa+2a+1$	$aa+2ca+cc$

6. But the number of Soldiers in the last step exceeded the number of Soldiers in the Generals Army by 29 (or *d*;) therefore subtracting 29 (or *d*) from the number in the last step, the Remainder must be equal to the number in the third step: hence this Equation arises, to wit,

$$aa+2a+1-29 = aa+500,$$

$$\text{Or,}\quad aa+2ca+cc-d = aa+b.$$

7. Which Equation after due Reduction (according to *Sect.* 3, and 5. of *Chap.* 12.) makes known the side of the first Square, *viz.*

$$a = 264 = \frac{b+d}{2c} - \tfrac{1}{2}c.$$

8. Lastly, If the side or number found out in the last step be multiplied by it self, and the Product be increased with 500 (or *b*,) there will come forth the number of Soldiers that were in the Generals Army, to wit,

$$70196 = \frac{bb+2bd+dd}{4cc} + \tfrac{1}{4}cc + \tfrac{1}{2}b - \tfrac{1}{2}d.$$

Whence it is manifest that the General had 70196 Soldiers in his Army: Also, the side of the first square Battel consisted of 264 Soldiers; and the side of the latter 265; this multiplied by it self produces 70225, which exceeds the said 70196 by 29: Moreover, the said 70196 exceeds the Square of 264 by 500; as the Question requires.

QUEST. 12.

Two Persons, *A* and *B*, discourse of their Money in this manner, *viz.* *A* saith, if *B* would give him a Crown (or *c*,) then *A* should have as many Crowns as *B* had left; but *B* saith, if *A* would give him a Crown, then *B* should have twice as many Crowns as *A* had left. How many Crowns had each Person?

1. For the number of Crowns which *A* had, put	a
2. Then, according to the Question, if that number be increased with 1 Crown (or *c*,) the Sum will be the number of Crowns that remained to *B* after he had given 1 Crown to *A*, to wit, ...	$a+c$
3. And consequently, by adding 1 Crown (or *c*) to the said number of Crowns that remained to *B* after he had given 1 Crown to *A*, the Sum will be the number of Crowns which *B* had at first, to wit,	$a+2c$

4. Again,

4. Again, according to the Question, if 1 Crown (or c) be added to the said $a+2c$ in the last step, and subtracted from a in the first step, the Sum must be equal to the double of the Remainder; hence this Equation, ...	$a+3c=2a-2c$
5. Which Equation, after due Reduction, discovers the number of Crowns that *A* had at first, to wit, ...	$a=5c$
6. And from the fifth and third steps, the number of Crowns which *B* had at first will also be made known, to wit, ...	$a+2c=7c$

So it is found that *A* had 5 Crowns, and *B* 7 Crowns, as will be evident by

The Proof.

$$5+1=7-1=6$$
$$7+1=4+4=8$$

QUEST. 13.

A Vintner having two sorts of *French* Wines, to wit, one sort worth 10*d.* (or b) the Quart, and the other 6 *d* (or c) *per* Quart, would have a mixed Quantity of both sorts to consist of 100 Quarts (or m) that might be worth 7 *d.* (or f) *per* Quart. The Question is, to find what Quantity of each sort of Wine must be taken to make that mixture?

1. For the number of Quarts that must be taken of the better sort of Wine to make the mixture, put	a	a
2. Which number subtracted from 100 (or m) leaves the number of Quarts of the worser sort of wine in the mixture, to wit,	$100-a$	$m-a$
3. Then find the worth of the better sort of Wine in the mixture at 10 *d.* (or b) *per* Quart, and say by the Rule of Three, If 1 . 10 :: a . ($10a$, Or, if 1 . b :: a . (ba. So the Quantity of the better sort of Wine in the mixture is found worth	$10a$	ba
4. Find likewise the worth of the worser sort of Wine in the mixture at 6 *d.* (or c) *per* Quart, and say, If 1 . 6 :: $100-a$. ($600-6a$, Or, 1 . c :: $m-a$. ($cm-ca$. So the Quantity of the worser sort of Wine in the mixture is found worth . . .	$600-6a$	$cm-ca$
5. Therefore the Sum of the values of both the Quantities mentioned in the two last steps is	$4a+600$	$ba+cm-ca$

6. Which Sum must be equal to the Product made by the Multiplication of 100 (or m) the total mixed Quantity, by 7 (or f) the prescribed mean price; hence this Equation arises, to wit,

$$4a+600=700,$$

Or, ... $ba+cm-ca=fm.$

7. Which Equation, after due Reduction, discovers the value of a, to wit, the number of Quarts that must be taken of the better sort of Wine to make the mixture, *viz.*

$$a=25=\frac{fm-cm}{b-c}.$$

8. And from the seventh and second steps the number of Quarts that ought to be taken of the worser sort of Wine to make the mixture will also be made known, *viz.*

$$75=\frac{bm-fm}{b-c}.$$

9. From the two last steps it is evident, That 25 Quarts of the better sort of Wine, and 75 Quarts of the worser sort, must be taken to make the prescribed mixture; for those

Quantities

quantities at their respective prices will be worth in the whole 700 pence, which is also the just worth of 100 quarts at 7 pence *per* quart.

Moreover, If the latter parts of the two last Equations be resolved into Proportionals, (according to *Sect.* 3. *Chap.* 13.) and be express'd by words, they will give this following

THEOREM.

As the difference between the given prices of two sorts of Wines or other things whereof a mixture is desired, is to the total Quantity required to be in the mixture; So is the excess by which some mean price prescribed for the total Quantity mixed exceeds the lesser of the two given prices, to the Quantity to be taken of the better sort of Wine: And so is the excess of the greater of the two given prices above the mean price, to the Quantity that is to be taken of the worser sort of Wine.

This Theorem contains the substance of the Rule of Alligation-alternate in Vulgar Arithmetic. But how Questions of this nature, when three or more things are to be mixed, may be solved more generally than by that Rule, I shall hereafter shew in *Chap.* 13. of my second Book of *Algebraical Elements.*

QUEST. 14.

A Cistern in a certain Conduit is supplied with Water by two Pipes, of such capacities, that by both their Cocks *A* and *B* set open at once the Cistern will be filled in 12 (or *b*) Hours; but by the Cock *A* alone in 20 (or *c*) Hours: The Question is, to find in what time the Cistern will be filled by the Cock *B* alone?

1. Suppose the time sought to be	a	a
2. Then find what part of the Cistern will be filled by the Cock *B* alone in 12 (or *b*) Hours, and say by the Rule of Three, If a . 1 :: 12 . $(\frac{12}{a}$, Or, if a . 1 :: b . $(\frac{b}{a}$; whence the said part is found	$\frac{12}{a}$	$\frac{b}{a}$
3. Find likewise what part of the Cistern will be filled by the Cock *A* alone in 12 (or *b*) Hours, and say, If 20 . 1 :: 12 . $(\frac{3}{5}$, Or, if c . 1 :: b . $(\frac{b}{c}$; whence the said part is found	$\frac{3}{5}$	$\frac{b}{c}$
4. But those parts found out in the second and third steps must be equal to the whole Cistern, to wit, 1; hence this Equation arises,	$\frac{12}{a}+\frac{3}{5}=1.$	$\frac{b}{a}+\frac{b}{c}=1.$
5. Which Equation, after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. discovers the value of a, to wit, the time sought, *viz.*	$a=30$	$a=\frac{bc}{c-b}$

Whence it appears, that by the Cock *B* set open alone the Cistern would be filled in 30 Hours: And, if the last Equation of the literal Resolution be resolved into Proportionals according to *Sect.* 3. *Chap.* 13. there will arise this following

CANON.

As the difference of the two numbers or spaces of Time given in the Question is to either of them, so is the other to the Time sought, *viz.*

As 8 (20—12) . 12 :: 20 . 30,

Or, as . . . : $c-b$. b :: c . $\frac{bc}{c-b}$.

The

The Proof may be made by solving this Question, *viz.*

If a Cistern will be filled with Water by a Cock *A* in 20 hours, and by another Cock *B* in 20 hours; in what time will the Cistern be filled by both Cocks set open at once? *Answ.* 12 hours.

First find what part or parts of the Cistern will be filled by each Cock in one and the same time; then it shall be, As the Sum of those parts is to that common time, so is the whole Cistern (to wit, 1,) to the time wherein the whole Cistern will be filled by both Cocks set open at once; *viz.*

	ho.		*Cist.*		*ho.*		
First, If	30	.	1	::	20	.	($\frac{2}{3}$ Cistern.
						add	1 Cistern.
							Sum, $1\frac{2}{3}$ Cist.

So it is found that $1\frac{2}{3}$ Cistern will be filled in 20 hours by both Cocks *A* and *B* set open at once; then say again by the Rule of Three,

Cist.		*ho.*		*Cist.*		
$1\frac{2}{3}$	.	20	::	1	.	(12 hours.

If the Operation of this latter Question be formed Algebraically by Letters, it will afford this

CANON.

As the Sum of the two given numbers expressing spaces of time in the latter Question, is to either of them; So is the other to the time sought.

QUEST. 15.

A Shepherd in the time of War driving a Flock of Sheep, fell into the hands of three Companies of plundering Soldiers, who compell'd him to deliver the half of his flock with half a Sheep over and above to the first Company; also half of his remaining flock with half a Sheep to the second Company; likewise the half of the rest of the flock with half a Sheep to the third Company: All which Divisions the Shepherd exactly perform'd without killing a Sheep, and then there remained only 20 (or *b*) Sheep for himself. The question is, to find How many Sheep the Shepherd had in his flock at first?

1. Let the Number of Sheep which the Shepherd had in his Flock at first be represented by	a
2. Then the half of that number is $\frac{1}{2}a$, to which adding $\frac{1}{2}$, (that is, half a Sheep) the sum will be the Number of Sheep delivered to the first Company of Soldiers, to wit, ...	$\frac{1}{2}a+\frac{1}{2}$
3. And by subtracting the said $\frac{1}{2}a+\frac{1}{2}$ from a, the remainder will be the number of Sheep that were left to the Shepherd after he had satisfied the first Company of Soldiers, to wit, ...	$\frac{1}{2}a-\frac{1}{2}$
4. Then the half of that remaining Flock is $\frac{1}{4}a-\frac{1}{4}$, to which adding $\frac{1}{2}$, (that is, $\frac{1}{2}$ Sheep,) the sum will be the Number of Sheep delivered to the second Company of Soldiers, to wit, ...	$\frac{1}{4}a+\frac{1}{4}$
5. Which $\frac{1}{4}a+\frac{1}{4}$ being subtracted from $\frac{1}{2}a-\frac{1}{2}$ in the third step, the remainder will be the number of Sheep that were left to the Shepherd after he had satisfied the second Company of Soldiers, to wit, ...	$\frac{1}{4}a-\frac{3}{4}$
6. Then the half of the remaining flock in the last step is $\frac{1}{8}a-\frac{3}{8}$, to which adding $\frac{1}{2}$, (to wit, $\frac{1}{2}$ Sheep) the Sum will be the number of Sheep delivered to the third Company, to wit . . .	$\frac{1}{8}a+\frac{1}{8}$
7. Which $\frac{1}{8}a+\frac{1}{8}$ being subtracted from $\frac{1}{4}a-\frac{3}{4}$ in the fifth step, the remainder will be the number of Sheep that were left to the Shepherd after he had satisfied all the three Companies, to wit,	$\frac{1}{8}a-\frac{7}{8}$
8. But the remainder in the last step must be equal to 20 (or *b*) the number given in the Question; hence this Equation, . . .	$\frac{1}{8}a-\frac{7}{8}=b=20.$
9. Which Equation, after due Reduction, discovers the Number sought, to wit, ...	$a=8b+7=167$

So it appears that the Shepherd had 167 Sheep in his Flock at first.

The

The Proof.

1. The half of 167 is $83\frac{1}{2}$, to which adding $\frac{1}{2}$, the sum is 84, which was the number of Sheep delivered to the first Company of Soldiers; and then there remained 83 Sheep to the Shepherd.

2. Again, the half of 83 is $41\frac{1}{2}$, which increased with $\frac{1}{2}$ makes 42, the number of Sheep delivered to the second Company; and then there remained 41 Sheep to the Shepherd.

3. Lastly, the half of 31 is $40\frac{1}{2}$, which increased with $\frac{1}{2}$ makes 21, which was the number of Sheep delivered to the third Company; and so there remained 20 Sheep to the Shepherd, as the Question declares.

Moreover, the Equation in the last step of the Resolution shews, That if any whole number instead of 20 be prescribed in the Question, that number multiplied by 8, and the Product increased with 7 will give a number capable of the like Division as 167 that answered the Question: So if there had been but one Sheep left for the Shepherd, then his Flock at first was 15 Sheep; if 2 had been left, his Flock at first was 23; if 3 Sheep had been left, then he had 31 when he first met with the Soldiers; and so by a continual addition of 8, all the odd Numbers capable of that Division the Question requires may be orderly found out. But to have nothing left after such Division is made, the Number first to be divided is 7.

It is also Evident, that by continuing the Resolution an odd Number may be found out, that shall be capable of being divided according to the import of the Question, as many times as shall be desired.

QUEST. 16.

Two Merchants, *A* and *B*, were Co-partners in Traffic: the sum of their Stocks was 300 *l.* (or *b*;) the Stock of *A* continued in Company 9 (or *c*) Months, and the Stock of *B* 11 (or *d*) Months; they gained a certain sum of Money which they divided equally. The Question is, to find what each Merchants Stock was at first?

1. For the Stock of *A* when he entred Partnership, put	a	a
2. Then subtracting that stock from the Joynt stock 300 *l.* (or *b*) the Remainder will be the Stock of *B*, to wit, ...	$300 - a$	$b - a$
3. The first stock multiplied by the time it continued in Company produces . . .	$9a$	ca
4. And the other stock multiplied by its time produces.	$3300 - 11a$	$db - da$

5. Now forasmuch as the Merchants divided the gain equally, therefore the Products in the third and fourth steps must be equal to one another, (according to the nature of the Rule of Fellowship with Time.) Hence this Equation arises;

$$9a = 3300 - 11a,$$

Or, . . . $$ca = db - da$$

6. Which Equation, after due Reduction, according to *Sect* 3, and 5. of *Chap.* 12. will discover the Stock which *A* put in, *viz.*

$$a = 165 = \frac{db}{c+d}.$$

7. And from the 6, and 2. steps the stock which *B* put in will also be made known, to wit,

$$135 = \frac{cb}{c+d}.$$

So it is found that the stock of *A* was 165 *l.* and that of *B*, 135 *l.* For, $165 \times 9 = 135 \times 11$.

Moreover, If the latter parts of the two Equations in the sixth and seventh steps be resolved into Proportionals, according to *Sect.* 3. *Chap.* 13. there will arise this

CANON.

As the sum of both spaces of time given in the Question, is to the given sum of the two particular stocks sought; so is the greater time to the particular stock belonging to the lesser time: and so is the lesser time to the stock belonging to the greater time.

QUEST.

QUEST. 17.

A certain Man being asked how many Years old he was, anſwered, If $\frac{1}{20}$ (or b) part of the Number of Years he had lived, were multiplied by $\frac{5}{8}$ (or c) parts of the ſame number, the Product would give his Age. What was his Age?

1. For the Number of the Years ſought put . .	a	a
2. Then according to the Queſtion, multiplying $\frac{1}{20}a$ by $\frac{5}{8}a$ (or ba by ca) the Product will be	$\frac{1}{32}aa$	$bcaa$
3. Which Product muſt be equal to the number of Years ſought, *viz.*	$\frac{1}{32}aa = a$	$bcaa = a$
4. Then, by reducing that Equation according to *Sect.* 4, and 5. of *Chap.* 12. the number of years ſought will be diſcovered, *viz.* .	$a = 32$	$a = \frac{1}{bc}$

Whence it is manifeſt that the Reſpondent was 32 Years of Age; for if $1\frac{3}{5}$, that is, $\frac{1}{20}$ of 32, be multiplied by 20, that is, $\frac{5}{8}$ of 32; the Product will be 32, to wit, the Number of Years ſought. It is alſo evident by the laſt Equation in the literal Reſolution, that if 1 (to wit Unity) be divided by the Product made by the multiplication of the two numbers given in the Queſtion, the Quotient will be the number ſought.

QUEST. 18.

There are two Numbers, the greater of which has ſuch proportion to the leſſer as 3 to 2, (or as r to s;) and the ſum of the ſaid numbers has ſuch proportion to the ſum of their Squares, as 1 to 13, (or as b to c.) What are the Numbers?

1. For the greater Number ſought put . . .	a	a
2. Then, (according to *Queſt.* 2. in *Sect.* 4. *Chap.* 10.) the ſum of the two Numbers will be found	$\frac{5a}{3}$	$a+\frac{sa}{r}$
3. And (according to *Queſt.* 5. in the ſaid *Sect.* 4. *Chap.* 10) the ſum of the Squares of the two Numbers ſought will be . .	$\frac{13aa}{9}$	$aa+\frac{ssaa}{rr}$
4. Again, by the help of the latter Proportion given in the Queſtion, and of the ſum found in the ſecond ſtep, ſearch out the ſum of the Squares of the two numbers ſought; *viz.* ſay by the Rule of Three, If $1 . 13 :: \frac{5a}{3} . (\frac{65a}{3}$ Or, if $b . c :: a+\frac{sa}{r} . \frac{cra+csa}{br}$ whence the ſum of the ſaid Squares is found	$\frac{65a}{3}$	$\frac{cra+csa}{br}$

5. But the ſum of the Squares found out in the third ſtep muſt be equal to the ſum in the fourth; hence this Equation, *viz.*

$$\frac{13aa}{9} = \frac{65a}{3}$$

Or, . . . $$aa+\frac{ssaa}{rr} = \frac{cra+csa}{br}.$$

6. Which Equation, after due Reduction, will diſcover the greater of the two Numbers ſought, *viz.*

$$a = 15 = \frac{crr+crs}{brr+bss}.$$

7. Whence, by the help of the firſt proportion given in the Queſtion, the leſſer Number ſought will alſo be made known, *viz.*

$$10 = \frac{css+crs}{brr+bss}.$$

So the Numbers sought are 15 and 10; for they are in the given Reason of 3 to 2; and their Sum 25 is to 325 the Sum of their Squares, as 1 to 13; as was prescribed.

Moreover, the Letters in the latter parts of the two last Equations give a Canon to find out the Numbers required.

QUEST. 19.

There are two Numbers, the Greater of which has such proportion to the Lesser, as 3 to 2, (or as r to s;) and the Sum of the said Numbers has such proportion to the Product of their Multiplication, as 1 to 6, (or as b to c.) What are the numbers?

1. For the greater number sought put . .	a	a
2. Then (according to *Quest.* 2. in *Sect.* 4. *Chap.* 10.) the Sum of the two numbers will be	$\frac{5a}{3}$	$a+\frac{sa}{r}$
3. And (by *Quest.* 4. in *Sect.* 4. *Chap.* 10.) the Product of their Multiplication is . .	$\frac{2aa}{3}$	$\frac{saa}{r}$
4. Again, by the help of the latter proportion given in the Question, and of the Sum found in the second step, search out the Product of the multiplication of the two numbers sought; *viz.* say by the Rule of Three, If $1 \,.\, 6 :: \frac{5a}{3} \,.\, 10a$, Or, if $b \,.\, c :: a+\frac{sa}{r} \,.\, \frac{cra+csa}{br}$; whence the Product is found	$10a$	$\frac{cra+csa}{br}$

5. But the Products found out in the two last steps must be equal to one another; hence this Equation, *viz.*

$$\frac{2aa}{3} = 10a,$$

$$\text{Or,} \quad \frac{saa}{r} = \frac{cra+csa}{br}.$$

6. Which Equation, after due Reduction, discovers the greater of the two Numbers sought, *viz.*

$$a = 15 = \frac{cr+cs}{bs}.$$

7. Whence, by the help of the first Proportion given in the Question, the lesser number sought will also be made known, *viz.*

$$10 = \frac{cr+cs}{br}.$$

So the numbers sought are found 15 and 10; but that they will solve the Question the Proof will make manifest: For the greater is to the lesser as 3 to 2; and their Sum 25, is to 150 the Product of their Multiplication, as 1 to 6; as was prescribed.

Moreover, the two last Equations give a Canon to find out the Number sought.

QUEST. 20.

There are two Numbers, the greater of which has such Proportion to the lesser as 2 to 1 (or as r to s,) and the sum of the Squares of the said Numbers is 125 (or b;) What are the Numbers?

1. For the greater number sought put	a	a
2. (Then according to *Quest.* 1. in *Sect.* 4. *Chap.* 10.) the lesser Number will be found	$\frac{a}{2}$	$\frac{sa}{r}$
3. Therefore the Sum of their Squares shall be	$\frac{5aa}{4}$	$aa+\frac{ssaa}{rr}$

4. Which Sum muſt be equal to 125 (or b) the given ſum of the Squares; hence this Equation, ...	$\frac{5aa}{4} = 125$	$aa + \frac{ssaa}{rr} = b$
5. Which Equation, after due Reduction (according to *Sect.* 2, 5, and 7, of *Chap.* 12.) will diſcover the greater number ſought, *viz.*	$a = 10$	$a = \sqrt{\frac{rrb}{rr+ss}}$
6. But if a had been put for the leſſer number, it would by the like proceſs have been found	$= 5$	$= \sqrt{\frac{ssb}{rr+ss}}$

From the two laſt ſteps the numbers ſought are found 10 and 5, which will ſolve the Queſtion: For the greater is to the leſſer as 2 to 1, and the ſum of their Squares is 125; as was preſcribed.

Moreover, to find out the Numbers ſought, the two laſt ſteps of the literal Reſolution give this

CANON.

Multiply ſeverally the Squares of the Terms of the given Reaſon, by the given Sum of the Squares of the number ſought; then divide the Products ſeverally by the Sum of the Squares of the ſaid Terms; laſtly, extract the ſquare Root out of each Quotient, ſo ſhall theſe ſquare Roots be the Numbers ſought.

QUEST. 21.

There are two Numbers, the greater of which has ſuch proportion to the leſſer as 2 to 1, (or as r to s;) and the difference of the Squares is 75, (or d:) What are the Numbers?

1. For the greater Number ſought put . . .	a	a
2. Then (according to *Queſt.* 1. in *Sect.* 4. *Chap.* 10) the leſſer number will be . .	$\frac{a}{2}$	$\frac{sa}{r}$
3. Therefore the difference of their Squares is .	$\frac{3aa}{4}$	$aa - \frac{ssaa}{rr}$
4. Which Difference muſt be equal to the given Difference 75 (or d;) hence this Equation, *viz.*	$\frac{3aa}{4} = 75$	$aa - \frac{ssaa}{rr} = d$
5. Which Equation, after due Reduction, diſcovers the greater Number, *viz.* . .	$a = 10$	$a = \sqrt{\frac{rrd}{rr-ss}}$
6. But if a had been put for the leſſer Number it would have been found by the like proceſs	$= 5$	$= \sqrt{\frac{ssd}{rr-ss}}$

So the Numbers ſought are 10 and 5, which will ſolve the Queſtion: For the greater is to the leſſer as 2 to 1, and the difference of their Squares is 75; as was preſcribed.

Moreover, to find out the numbers ſought, the two laſt ſteps of the literal Reſolution give this

CANON.

Multiply ſeverally the Squares of the Terms of the given Reaſon by the given Difference of the Squares, then divide the Products ſeverally by the Difference of the Squares of the ſaid Terms, laſtly extract the ſquare Root of each Quotient, ſo ſhall theſe ſquare Roots be the Numbers ſought.

QUEST. 22.

There are two numbers, the ſum of whoſe Squares is 125 (or b) and the Difference of their Squares is 75 (or d;) what are the Numbers?

1. For the greater number put	a	a
2. Then its Square will be	aa	aa
3. Which ſubtracted from 125 (or b) the given Sum, leaves the Square of the leſſer Number, to wit, ...	$125 - aa$	$b - aa$

4. And

4. And from the second and third steps by subtracting the lesser Square from the greater, their Difference is	$2aa - 125$	$2aa - b$
5. Which Difference must be equal to the given Difference 75 (or *d*,) whence this Equation arises, ...	$2aa - 125 = 75$	$2aa - b = d$
6 From which Equation after due Reduction, according to *Sect.* 3, 5, and 7. of *Chap.* 12. the greater Number sought will be made known, *viz.*	$a = 10$	$a = \sqrt{\frac{b+d}{2}}$
7. But if *a* had been put for the lesser Number sought, it would by the like process have been found	$= 5$	$= \sqrt{\frac{b-d}{2}}$

So the Numbers sought are found 10 and 5, which will solve the Question; for the sum of their Squares is 125, and the difference of their Squares is 75, as was prescribed.

Moreover, to find out the Numbers sought, the two last steps of the literal Resolution give this

CANON.

The square Root of half the Sum of the given sum and difference of the Squares of the two Numbers sought, is equal to the greater Number, and the square Root of half the difference of the said given Sum and Difference gives the lesser Number.

QUEST. 23.

There are two Numbers, the sum of whose Squares is 340 (or *b*;) and the Product made by the multiplication of the two Numbers is equal to $\frac{6}{7}$ (or *c*) parts of the Square of the greater Number; what are the Numbers?

1. For the greater Number put	a	a
2. Then its square is	aa	aa
3. And $\frac{6}{7}$ (or *c*) parts of that Square is . . .	$\frac{6aa}{7}$	caa
4. Therefore also (according to the condition in the Question) the Product of the multiplication of the two numbers sought, shall be	$\frac{6aa}{7}$	caa
5. Which Product divided by the greater number *a* will give the lesser number, to wit,	$\frac{6a}{7}$	ca
6. Therefore from the last step the Square of the lesser number is	$\frac{36aa}{49}$	$ccaa$
7. And by adding together the Squares in the second and sixth steps, their sum will be .	$\frac{85aa}{49}$	$ccaa + aa$
8. Which sum must be equal to the given sum 340 (or *b*,) whence this Equation arises .	$\frac{85aa}{49} = 340$	$ccaa + aa = b$
9. From which Equation, after it is duly reduced according to *Sect.* 2, 5, and 7. of *Chap.* 12. the greater number sought will be made known, *viz.*	$a = 14$	$a = \sqrt{\frac{b}{cc+1}}$
10. And from the ninth and fifth steps the lesser number will also be discovered, . .	$= 12$	$= \sqrt{\frac{bcc}{cc+1}}$

So the two numbers sought are found 14 and 12, which will solve the Question; for the sum of their Squares 196 and 144 is 340; also, 14 multiplied by 12 makes 168, which is equal to $\frac{6}{7}$ of the greater Square 196.

QUEST. 24.

A Merchant bought a certain Number of Yards of linnen Cloth at 12 pence (or *b*) *per* Yard; and if the number of pence paid for all the Cloth be multiplied by the number of of Yards bought, the Product will be 30000, (or c.) The Question is, to find the number of Yards bought.

1. For the number of Yards bought put . .	a	a
2. Then the number of pence paid for the whole Cloth will be	$12a$	ba
3. Which Number multiplied by a (the number of Yards bought, produces	$12aa$	baa
4. Which Product must, according to the Question, be equal to 30000 (or c;) therefore	$12aa = 30000$	$baa = c$
5. From which Equation, after due Reduction, the number of Yards sought will be discovered, *viz.*	$a = 50$	$a = \sqrt{\frac{c}{b}}$

So it is found that the Merchant bought 50 Yards of Cloth, which at 12 *d. per* Yard makes 600 *d* this 600 multiplied by 50 (the Number of Yards bought,) produces 30000; as was prescribed in the Question.

QUEST. 25.

Two Merchants, *A* and *B*, were Co-partners in Traffic; *A* brought in a certain number of pounds, which continued in Company 4 (or c) Months, *B* brought in 100 (or b) pounds, which continued in Company such a time, that if it be multiplied by the Stock of *A* it makes 50 (or d.) At the end of their Partnership they had gained 60 Pounds, whereof *A* had 40 (or r) Pounds for his share, and *B* the rest, to wit, 20 (or s) Pounds. What was the Stock which *A* put in at first, and how many Months did the Stock of *B* continue in Company?

1. For the Stock of *A* put	a	a
2. Then multiplying that stock by the time it continued in Company, to wit, by 4 (or c,) it makes	$4a$	ca
3. Then divide 50 (or d) the Product given in the Question, by a the (stock of *A*,) and the Quotient will give the time that the stock of *B* continued in Company, to wit, . . .	$\frac{50}{a}$	$\frac{d}{a}$
4. The stock of *B*, to wit, 100 *l.* (or b) multiplied by its time $\frac{50}{a}$ (or $\frac{d}{a}$) produces . .	$\frac{5000}{a}$	$\frac{bd}{a}$

5. Then according to the Nature of the Rule of Fellowship with Time, this Analogy will arise, *viz.* As the Product made by the mutual multiplication of the Stock and Time of *A*, is to the Product of the Stock and Time of *B*; so is the gain of *A* to the gain of *B*: *viz.*

$$\text{As, } 4a \,.\, \frac{5000}{a} :: 40 \,.\, 20,$$

$$\text{Or, } ca \,.\, \frac{bd}{a} :: r \,.\, s.$$

6. Which Analogy (according to *Sect.* 1. *Chap.* 13.) may be converted into this Equation,

$$viz. \quad 80a = \frac{200000}{a},$$

$$\text{Or,} \quad sca = \frac{rbd}{a}.$$

7. From which Equation, (after due Reduction according to *Sect.* 2, 5, and 7. of *Chap.* 12.) the Stock of *A* will be discovered, *viz.*

$$a = 50 = \sqrt{\frac{rbd}{sc}}.$$

8. And

8. And from the seventh and third steps, the Time that the Stock of *B* continued in Company will also be made known, *viz.*

$$\frac{50}{50} = 1 = \sqrt{\frac{scd}{rb}}.$$

9. So it is found that the Stock which *A* put in at first was 50 *l.* and the time during which the Stock of *B* continued in Company was one Month; as will appear by

The Proof.

$$50 \times 4 = 200$$
$$100 \times 1 = 100$$

Then if . . . . . . 300 . 60 :: $\begin{cases} 200 & . \ 40 \\ 100 & . \ 20. \end{cases}$

QUEST. 26.

Certain Noble-men made a Progress for their Pleasure; every noble Man carried along with him the same Sum of Pounds; the Number of the Noble-men was equal to the number of Servants which attended upon each Noble-man; the number of Pounds that each Noble-man had was the double of the number of all their Servants; and the sum of all their Mony was 3456 Pounds: the Question is, to find out the Number of Noble-men; also, how many Pounds and Servants each Noble man had?

1. For the number of Noble-men put . . . . a
2. Then (according to the Question) the number of Servants that attended upon each Noble-Man was also . . . . a
3. Therefore the Number of all the Servants was . . . . . . aa
4. Which last Number doubled gives the number of Pounds that each Nobleman had, to wit, ... $2aa$
5. And if the said Number of Pounds be multiplied by the number of Noble-men, it produces the Sum of all their Money, to wit, . . $2aaa$
6. Which sum must be equal to the given sum 3456, therefore . . $2aaa = 3456$
7. Therefore by taking the half of that Equation, there arises . . $aaa = 1728$
8. Lastly, by extracting the Cubic Root of each part of the last Equation, the Number of Noble-men is discovered, to wit, . . . $a = 12$

So it is found that there were 12 Noble-Men; also every one of them had 12 Servants and 288 Pounds, as will appear by

The Proof.

$$12 \times 12 = 144.$$
$$144 \times 2 = 288$$
$$288 \times 12 = 3456.$$

QUEST. 27.

A Merchant bought as many Pounds of Pepper for one Crown as was half the number of Crowns he laid out, then in selling the Pepper he received for every 25 lb of Pepper as many Crowns as he paid for all the Pepper; and in conclusion he had 20 Crowns, The Question is, to find how many Crowns he laid out.

1. For the Number of Crowns which the Merchant laid out, let there be put . . . . a
2. Then the Number of Pounds of Pepper which he bought for one Crown was . . . . $\frac{a}{2}$
3. Whence the whole quantity of Pepper bought will be found $\frac{aa}{2}$, for, If $1 \;.\; \frac{a}{2} :: a \;.\; (\frac{aa}{2}$ . . . . $\frac{aa}{2}$

4. Then

4. Then find how many Crowns the Merchant received for the total quantity of Pepper sold; saying by the Rule of Three, If $25 . a :: \frac{aa}{2} . (\frac{aaa}{50};$ whence the number of Crowns for which all the Pepper was sold is found . . . . $: \frac{aaa}{50}$
5. Which number of Crowns found out in the last step, must be equal to 20 the number of Crowns given in the Question; hence this Equation, ... $\frac{aaa}{50} = 20$
6. From which Equation, after it is reduced according to *Sect.* 2, and 7. of *Chap.* 12. there will come forth the first cost of the Pepper, to wit, ... $: a = 10$

So the number of Crowns which the Merchant laid out was 10, as will appear by the Proof; for first, the half of 10, to wit, 5, will be the number of Pounds of Pepper which he bought for 1 Crown; then say,

If $1 . 5 :: 10 . 50$ || Pounds of Pepper bought,
If $25 . 10 :: 50 . 20$ || Crowns received for Pepper sold.

QUEST. 28.

There are two Numbers, the greater of which has such proportion to the lesser as 3 to 2, (or as r to s;) and the Sum of the Cubes of the two Numbers is 4375, (or b;) what are the numbers?

1. For the greater Number put	a	a
2. Then (according to *Quest.* 1. in *Sect.* 4. of *Chap.* 10) the lesser number will be found	$\frac{2a}{3}$	$\frac{sa}{r}$
3. Therefore from the first step, the Cube of the greater number is	aaa	aaa
4. And from the second step the Cube of the lesser Number is	$\frac{8aaa}{27}$	$\frac{sssaaa}{rrr}$
5. Therefore from the third and fourth steps, the Sum of the Cubes of both Numbers is	$\frac{35aaa}{27}$	$\frac{sssaaa}{rrr} + aaa$

6. Which Sum must be equal to the given Sum 4375, (or b;) whence this Equation arises, *viz.*

$$\frac{35aaa}{27} = 4375. \quad \text{Or, } \frac{sssaaa}{rrr} + aaa = b.$$

7. From which Equation, after due Reduction, (according to *Sect.* 2, 5, and 7. of *Chap.* 12.) the greater number sought will be made known, *viz.*

$$a = 15 = \sqrt{(3)}\frac{rrrb}{sss+rrr}.$$

8. And from the seventh and second steps, the lesser number will also be discovered, to wit,

$$10 = \sqrt{(3)}\frac{sssb}{sss+rrr}.$$

So the numbers sought are found 15 and 10, which will solve the Question; for they are in the given Reason of 3 to 2; and the Sum of the Cubes of the said 15 and 10, to wit, of 3375 and 1000 makes 4375; as was prescribed.

Moreover, to find the Numbers sought, the latter parts of the Equations in the seventh and eighth steps give this

CANON.

Multiply severally the Cubes of the Terms of the given Reason (or Proportion) by the given Sum of the Cubes of the Numbers sought; divide the Products severally by the Sum of the Cubes of the said Terms; lastly, extract the Cubic Root of each of the Quotients, so these Roots shall be the Numbers sought.

CHAP. XV.

Concerning the Resolution of such adfected or compounded Equations wherein their are two different Powers of the Quantity sought, and those Powers such, that the higher of them is a Square whose Side or Square Root is the lower Power.

I. THE Equations treated of in this Chapter fall under three Heads or Forms hereunder specified, which I shall first explain, and then shew how they may be Arithmetically resolved.

Equations of the first Form.

$aa + 6a = 55.$	$aa + ca = b.$
$aaaa + 8aa = 48.$	$aaaa + daa = f.$
$aaaaaa + 4aaa = 837.$	$aaaaaa + gaaa = h.$

Equations of the second Form.

$aa - 10a = 24.$	$aa - ba = k.$
$aaaa - 6aa = 27.$	$aaaa - paa = d.$
$aaaaaa - 2aaa = 48.$	$aaaaaa - maaa = g.$

Equations of the third Form.

$10a - aa = 24.$	$ca - aa = n.$
$5aa - aaaa = 4.$	$raa - aaaa = s.$
$9aaa - aaaaaa = 8.$	$daaa - aaaaaa = t.$

II. Every Equation which falls under any of the said three Forms, consists of three distinct Terms or Members, whereof two are unknown, and the third is known; of the two unknown Terms, one is a Square, (by which in this place I mean a square number) which is called the highest Term in the Equation; and the other unknown Term is the Product made by the Multiplication of the square Root of the said square number by some known number, which Product is called the middle Term; and the third or lowest Term is a number purely known: So in this Equation $aa+6a=55$, the highest Term is aa, which may represent an unknown square number whose Root is a; the middle term is $6a$, which is the Product of the Multiplication of the said unknown Root a by the known number 6; and the lowest Term (or known part of the said Equation) is the number 55, which for distinction sake is usually called the Absolute number given.

The like may be observed in this Equation $aa+ca=b$, where we may suppose b and c to represent two known numbers, and a some number unknown; then the highest Term is the Square aa; the middle Term is ca, to wit, the Product made by the Multiplication of a the Root of the said square aa by the known number c; and the lowest Term of the said Equation is the known Absolute number b.

Again, in this Equation $5aa-aaaa=4$, the highest Term is the square number $aaaa$; the middle term is $5aa$, to wit, the Product made by the Multiplication of aa the square Root of the said square Number $aaaa$ into the known number 5; and the lowest Term is the absolute number 4.

III. In every Equation which falls under any of the three before-mentioned Forms, there are two different Powers or Degrees of the number sought, and those such, that the Index or Exponent of the higher Power is the double of the Index of the lower: As in this Equation $aa+6a=55$, the Index or Number of Dimensions in aa is 2, which is the double of 1 the Index of a (in the middle Term $6a$:) so also in this Equation $5aa-aaaa=4$, the Index of the highest Term $aaaa$ is 4, which is the double of 2 the Index of aa in the middle Term. Likewise in this Equation $aaaaaa+4aaa=837$, the Index of the highest Term $aaaaaa$ is 6, which is the double of 3 the Index of aaa in the

the middle Term. But in this Equation $aaa+6a=39$ the Index of the highest Term aaa is not the double of the Index of a in the middle Term, (for the Index of the former is 3, and of the latter 1;) and therefore the Equation last proposed cannot be ranked under any of the three Forms aforesaid, and consequently it is not resolvable by the following Rules of this Chapter, but belongs to the 10 and 11 Chapters of my second Book.

IV. Known Numbers which are drawn into, or multiplied by some Degree or Power of the Number sought are by *Vieta* and others called Coefficients, *viz.* Fellow-factors, or Copartners in Multiplication with unknown Powers: So in this Equation $aa+6a=55$ the Number 6 is called the Co-efficient, to wit, the Fellow-multiplier with the unknown Number a to make the Product $6a$. Likewise in this Equation $aa+ca=b$, we may suppose the Letters b and c to represent known Numbers, and the Letter a some unknown Number whose Co-efficient is c.

But sometimes the Co-efficient will happen to be express'd by many Letters, as in this Equation $aa+\frac{sca}{r}$ (or $\frac{sc}{r}a$) $=\frac{15sscc}{4rr}$, where a only is supposed to be unknown, and the known Number $\frac{sc}{r}$ is the Co-efficient, which signifies but one Number, to wit, the Quotient that arises, when the Product of the Number s multiplied by the number c is divided by the number r, *viz.* if $s=2$; $c=4$; and $r=1$, then $\frac{sc}{r}$ or 8 is the Co-efficient, and consequently $\frac{sc}{r}a$ is the same with $8a$.

Likewise in this Equation $\frac{2r+s}{s}a$ (or $\frac{2ra+sa}{s}$) $-aa=\frac{2r}{s}$, the Co-efficient is $\frac{2r+s}{s}$, which is to be esteemed but as one number; to wit, the Quotient that arises by dividing the Sum of $2r$ and s by s; so that if we suppose $r=3$ and $s=2$, then the Equation last proposed may be express'd thus, $4a-aa=3$.

Note. When no known number appears to be drawn into the middle Term of the Equation, then 1 (or Unity) must in that case be always taken for the Co-efficient; so in this Equation $aa+a=30$, the middle Term a implies $1a$, to wit, the Product of a multiplied by 1, and therefore 1 is the Co-efficient.

Note also. When the highest unknown Power or Degree is multiplied by any number greater than 1, then every Term or Member of the Equation must be divided by that number, to the end the said highest unknown Power may be clear'd from any Co-efficient unless it be 1; as before has been shewn in *Sect.* 5 *Chap.* 12.

These things being premised by way of Explication, I proceed to the Resolution of Equations which fall under any of the three Forms before specified.

V. *The Arithmetical Resolution of Equations which fall under the first of the three Forms before specified in* Sect. I. *of this* Chapter.

QUEST. 1.

1 What is the number represented by a in this Equation? . . . $aa+6a=55$

2 Which Equation, if c be assumed to signifie 6, and b 55, may be express'd thus, ... $aa+ca=b$

RESOLUTION.

3. To resolve the said Equation imports the same thing as to solve this Question, *viz.* There is an unknown number (represented by a) which is such, that if to its Square you add the Product made by the Multiplication of that unknown number by 6, (or c,) the Sum will be 55, (or b;) what is that unknown number a? *Answ.* 5; found out thus,

4. Let the Square of half the Co-efficient 6 (or c) be added to each part of the Equation proposed, to the end its first part may be made a compleat Square, (according to *Sect.* 4. *Chap.* 9) whence this Equation arises,

$$aa+6a+9=64, \quad \text{or,} \quad aa+ca+\tfrac{1}{4}cc=b+\tfrac{1}{4}cc.$$

5. Then

5. Then by extracting the square Root of each part of the last Equation (according to *Sect.* 4 and 5. of *Chap.* 8.) this Equation arises;

$$a + 3 = 8,$$

Or, $$a + \frac{1}{2}c = \sqrt{:b+\frac{1}{4}cc:}$$

6. Wherefore by transposition (or equal subtraction) of 3, or $\frac{1}{2}c$, the number a sought will be made known, *viz.*

$$a = 5 = \sqrt{:b+\frac{1}{4}cc:} - \frac{1}{2}c.$$

I say the number a sought is 5, which will solve the Question proposed, as will appear by

The Proof.

If	$a = 5,$
Then consequently	$aa = 25,$
And	$6a = 30;$
Therefore	$aa+6a = 55.$

Which was the Equation proposed.

Note. Every Equation which falls under this first Form may be expounded by either of two Roots, whereof one is Affirmative or greater than nothing, and the other Negative or less than nothing. As in the Equation proposed, to wit, $aa+6a=55$; forasmuch as according to the Rules of Algebraical Multiplication, — multiplied by — produces +, and so in this Sense the square Root of 64 may be — 8 as well as + 8; therefore the square Root of the Equation $aa+6a+9=64$ in the fourth step may be this, to wit, ... $a+3 = -8.$

Whence, by transposition of +3, a Negative Root or value of a is discovered, to wit, ... $a = -11.$

I say the Root a in the Equation $aa+6a=55$ may be expounded by — 11. (besides + 5,) as will be manifest by

The Proof.

If	$a = -11$	Here the Rules of + and — in Algebraical Multiplication and Addition are to be respected.
Then	$aa = +121$	
And	$6a = -66,$	
Therefore, as before,	$aa+6a = +55.$	

Negative Roots are oftentimes of good use to find out Affirmative Roots, as hereafter will appear in *Chap.* 11. of the second Book.

QUEST. 2.

1. What is the number represented by a in this Equation? . . $aaaa+8aa = 48,$
2. Which Equation, if d be put for 8, and f for 48, may be express'd thus, ... $aaaa+daa = f.$

RESOLUTION.

3. To resolve the said Equation imports the same thing as to solve this Question, *viz.* There is an unknown number represented by a, which is such, that if to its Biquadrate or squared Square you add the Product made by the Multiplication of the Square of that unknown number a by 8, (or d,) the Sum will be 48, (or f;) what is the unknown number a? *Answ.* 2. found out in the same manner as before in *Quest.* 1. *viz.*
4. Let the Square of half the Co-efficient 8 (or d) be added to each part of the Equation proposed, to the end its former part may be made a compleat Square, according to *Sect.* 4. *Chap.* 9. whence this Equation arises;

$$aaaa+8aa+16 = 64,$$

Or, $$aaaa+daa+\frac{1}{4}dd = f+\frac{1}{4}dd.$$

5. Then by extracting the square Root of each part of the last Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation arises,

$$aa+4 = 8,$$

Or, $$aa+\frac{1}{2}d = \sqrt{:f+\frac{1}{4}dd:}$$

6. Whence by equal subtraction or transposition of 4 (or $\frac{1}{2}d$) there will arise

$$aa = 4$$

Or, $$aa = \sqrt{f+\frac{1}{4}dd:}-\frac{1}{2}d.$$

7. There-

7. Therefore by extracting the square Root of each part of the last Equation, the number a sought, will be made known, *viz.*

$$a = 2 = \sqrt{(2)}:\sqrt{f+\tfrac{1}{4}dd}-\tfrac{1}{2}d:$$

I say the number a sought is 2, which will solve the Question proposed, as will appear by

The Proof.

If	$a = 2$,
Then consequently	$aa = 4$,
And	$aaaa = 16$,
Also	$8aa = 32$,
Therefore	$aaaa+8aa = 48$.

Which was the Equation propos'd to be resolved.

QUEST. 3.

1. What is the number represented by a in this Equation? . . . . $aaaaaa+4aaa = 837$.
2. Which Equation, if g be put for 4, and h for 837, may be express'd thus . . . . . $aaaaaa+gaaa = h$.

RESOLUTION.

3. To resolve the said Equation imports the same thing as to solve this Question, *viz.* There is an unknown number represented by a, which is such, that if to its cubed Cube or sixth Power, you add the Product made by the Multiplication of the Cube of that unknown number by 4 (or g) the Sum will be 837, what is that unknown number a? *Answ.* 3. found out in the same manner as before, *viz.*
4. By adding the square of half the Co-efficient 4 (or g) to each part of the Equation proposed, this Equation arises;

$$aaaaaa+4aaa+4 = 841.$$
Or, $$aaaaaa+gaaa+\tfrac{1}{4}gg = h+\tfrac{1}{4}gg.$$

5. And by extracting the square Root of each part of the last Equation this arises;

$$aaa+2 = 29.$$
Or, $$aaa+\tfrac{1}{2}g = \sqrt{:h+\tfrac{1}{4}gg:}$$

6. Whence by transposition of 2 (or $\tfrac{1}{2}g$) this Equation arises;

$$aaa \quad 27.$$
Or, $$aaa = \sqrt{:h+\tfrac{1}{4}gg:}-\tfrac{1}{2}g.$$

7. Therefore by extracting the Cubic Root of each part of the last Equation the number a sought will be made known, *viz.*

$$a = 3 = \sqrt{(3)}:\sqrt{h+\tfrac{1}{4}gg}-\tfrac{1}{2}g:$$

I say the number a sought is 3, which will solve the Question proposed, as will appear by

The Proof.

If	$a = 3$,
Then consequently	$aaa = 27$,
And	$aaaaaa = 729$,
Also	$4aaa = 108$,
Therefore	$aaaaaa+4aaa = 837$.

Which was the Equation propos'd to be resolved.

VI. From the Resolution of the three last Questions the following Canon is deduced for the resolving of all Equations which fall under the first of the three Forms before specified in *Sect.* 1. of this *Chapter.*

CANON.

Add the square of half the Co-efficient, or (which is the same thing) a quarter of the square of the whole Co-efficient, to the given absolute number.

Extract the square Root of that Sum.

From

From the said Square Root subtract half the Co-efficient, and reserve the Remainder.

Lastly, when the unknown number which is multiplied by the Co-efficient in the middle Term of the Equation is express'd by a single Letter only, as a, then the Remainder before reserved is the number sought; but if the said unknown number in the middle Term be a Square, as aa, then the Square Root of the Remainder reserved is the number sought; if a Cube, as aaa, then the Cubic Root of the said Remainder shall be the number sought; if any higher Power, then the Root for the kind must be extracted out of the said Remainder, which Root shall be the number sought.

An Example of the Canon.

1. Let the preceding *Quest.* 1. be here repeated, *viz.* What is the number represented by a in this Equation?	$aa+6a=55$
2. Or, what is the value of a in this Equation,	$aa+ca=b$

RESOLUTION.

3. To the given absolute number	55	$b.$
4. Add the Square of half the Co-efficient 6, to wit, the Square of 3, which is	9	$\frac{1}{4}cc.$
5. The Sum is	64	$b+\frac{1}{4}cc$
6. The Square Root of that Sum is	8	$\sqrt{:b+\frac{1}{4}cc:}$
7. From that Square Root subtract half the Co-efficient 6, to wit,	3	$\frac{1}{2}c.$
8. The Remainder is the number a sought, to wit,	5	$\sqrt{:b+\frac{1}{4}cc:}-\frac{1}{2}c.$

Whence it is manifest that the Answer is the same as was before found to *Quest.* 1.

A second Example of the Canon.

1. Let the preceding *Quest.* 2. be here repeated, *viz.* What is the number represented by a in this Equation?	$aaaa+8aa=48$
2. Or what is the value of a in this Equation,	$aaaa+daa=f$

RESOLUTION.

3. To the given absolute number	48	$f.$
4. Add the Square of half the Co-efficient 8, to wit, the Square of 4, which is	16	$\frac{1}{4}dd$
5. The Sum is	64	$f+\frac{1}{4}dd.$
6. The square root of that Sum is	8	$\sqrt{:f+\frac{1}{4}dd:}$
7. From which square root subtract half the Co-efficient 8, to wit,	4	$\frac{1}{2}d.$
8. The Remainder is the value of aa, to wit	4	$\sqrt{f+\frac{1}{4}dd}-\frac{1}{2}d$
9. Lastly, the square Root of the said Remainder gives the number a,	2	$\sqrt{(2):\sqrt{f+\frac{1}{4}dd}-\frac{1}{2}d:}$

Whence it is evident that the Answer is the same as was before found to *Quest*, 2.

A third Example of the Canon.

1. Let the preceding *Quest.* 3. be here repeated, *viz.* What is the number represented by a in this Equation?	$aaaaaa+4aaa=837.$
2. Or what is the value of a in this Equation,	$aaaaaa+gaaa=h.$

RESOLUTION.

3. To the absolute Number	837	$h.$
4. Add the Square of half the Coefficient 4, to wit,	4	$\frac{1}{4}gg.$
5. The Sum is	841	$h+\frac{1}{4}gg.$
6. The square Root whereof is	29	$\sqrt{:h+\frac{1}{4}gg:}$
7. From that square Root subtract half the Co-efficient 4, to wit,	2	$\frac{1}{2}g.$

8. The

8. The Remainder is the value of aaa, to wit, .	27	$\sqrt{b+\frac{1}{4}gg}-\frac{1}{2}g$
9. Therefore the Cubic Root of that Remainder shall be the number a sought, . . .	3	$\sqrt{(3)}:\sqrt{b+\frac{1}{4}gg}-\frac{1}{2}g:$

Whereby it is manifest that the Answer is the same as was before found to *Quest.* 3.

Example 4.

If . . . . $aa+a=b$ (or 35,) what is $a=$?

Answ. . . . . . $a=\sqrt{:b+\frac{1}{4}:}-\frac{1}{2}=5\frac{4371}{10000}$, &c.

For the Co-efficient drawn into the middle Term a being 1, its half is $\frac{1}{2}$, the Square whereof is $\frac{1}{4}$, which added to the absolute number 35 makes $35\frac{1}{4}$, whose Square Root is $5\frac{9371}{10000}$, &c. from which subtracting $\frac{1}{2}$, (or $\frac{5}{10}$) to wit, half the Co-efficient 1, the Remainder $5\frac{4371}{10000}$, &c. is the number a sought, which here happens to be irrational, that is, inexpressible by any true number, but by continuing the extraction of the said Square Root of the said $35\frac{1}{4}$, you may approach infinitely near the exact number a.

Example 5.

If . . . . $aa+\frac{15}{2}a=\frac{143}{2}$, what is $a=$?

Answ. . . . . . . $a=\sqrt{:\frac{143}{2}+\frac{225}{16}:}-\frac{15}{4}=\frac{11}{2}$.

The Learner must remember to reduce a Fraction to its least Terms, before he goes about to extract any Root out of it.

Example 6:

If . . . . $\begin{cases} r=1, \\ s=2, \\ c=4, \end{cases}$

And if . . . . . $aa+\frac{sc}{r}a=\frac{15sscc}{4rr}$,

What is . . . . . . $a=$?

Answ. . . . . $a=\frac{3sc}{2r}=12$.

Example 7.

If . . $aaaa+\frac{11}{5}aa=\frac{87362}{475}$, what is $a=$?

Answ. . . . $a=\frac{11}{5}$.

VII. *The Arithmetical Resolution of Equations which fall under the second of the three Forms before expressed in* Sect. 1. *of this* Chapter.

QUEST. 1.

1. What is the number represented by a in this Equation?	. . . $aa-10a=24$.
2. Which Equation, by assuming b to represent 10, and k to signifie 24, may be express'd thus, ...	. . . $aa-ba=k$.

RESOLUTION.

3. Let the Square of half the Co-efficient 10 (or b) be added to each part of the Equation proposed, to the end its first part may be made a compleat Square, (according to *Sect.* 4. *Chap.* 9.) whence this Equation arises;

$$aa-10a+25=49,$$

$$\text{Or,}\quad aa-ba+\tfrac{1}{4}bb=k+\tfrac{1}{4}bb.$$

4. Then by extracting the Square Root of each part of the last Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation arises;

$$a-5=7,$$

$$\text{Or,}\quad a-\tfrac{1}{2}b=\sqrt{:k+\tfrac{1}{4}bb:}$$

5. Wherefore by equal addition of 5, or $\frac{1}{2}b$, the number a sought will be made known, *viz.* $a=12=\frac{1}{2}b+\sqrt{:k+\frac{1}{4}bb:}$

6. But forasmuch as the Square Root of $aa-10a+25$ in the third step may be $5-a$ as well as $a-5$, (for either of those Roots being multiplied by it self will produce the

the same Square $aa-10a+25$,) therefore let $5-a$ be set instead of $a-5$ in the fourth step; whence this Equation arises, *viz.*

$$5-a=7,$$
Or, $\frac{1}{2}b-a=\sqrt{:k+\frac{1}{4}bb:}$

7. Therefore by transposition, another value of a arises, to wit,

$$a=-2=\frac{1}{2}b-\sqrt{:k+\frac{1}{4}bb:}$$

Which latter value of a is less than nothing, and such it will always be, as may easily be proved from the last Equation. For $k+\frac{1}{4}bb$ is manifestly greater than $\frac{1}{4}bb$, and consequently the Square Root of the former will be greater than the Square Root of the latter, *viz.* $\sqrt{:k+\frac{1}{4}bb:}$ is greater than $\frac{1}{2}b$, therefore $\frac{1}{2}b-\sqrt{:k+\frac{1}{4}bb:}$ (that is a) will be less than nothing, for if a greater Quantity be subtracted from a less, the Remainder will be a negative Quantity, that is less than nothing, as before has been shewn in *Algebraical Subtraction.* From the premises it is evident that the Equation propounded, to wit, $aa-10a=24$ (and likewise every Equation which falls under the second form of Equations before-mentioned) is explicable by two Roots, whereof one is real or affirmative, whose value is before express'd in the fifth step; and the other negative or less than nothing, the value whereof is express'd in the seventh step.

I say the real or true number a sought in the Question proposed is 12, as will appear by

The Proof.

If . . . . $a=12$,
Then consequently . . . . . . $aa=144$,
And . . . . $10a=120$,
Therefore . . . . $aa-10a=24$.
Which was the Equation proposed.

Moreover, according to the Rules of *Algebraical* Multiplication and Subtraction, the negative value of a, to wit -2 before found, will constitute the Equation first proposed:

For if . . . . $a=-2$,
Then consequently . . . $aa=+4$,
And . . . . $10a=-20$,
Therefore . . . . $aa-10a=+24$;

QUEST. 2.

1. What is the number represented by a in this Equation? . . . . $aaaa-6aa=27$
2. Which Equation, if p be put for 6, and d for 27, may be express'd thus, ... $aaaa-paa=d$

RESOLUTION.

3. Let the Square of half the Co-efficient 6 (or p) be added to each part of the Equation proposed, to the end its first part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation arises;

$$aaaa-6aa+9=36,$$
Or, $aaaa-paa+\frac{1}{4}pp=d+\frac{1}{4}pp.$

4. Then by extracting the Square Root of each part of the last Equation (according to *Sect.* 4. and 5. of *Chap.* 8.) this Equation arises, *viz.*

$$aa-3=6,$$
$$aa-\frac{1}{2}p=\sqrt{:d+\frac{1}{4}pp:}$$

5. Whence, by equal Addition of 3 (or $\frac{1}{2}p$) there will arise

$$aa=9,$$
Or, $aa=\sqrt{:d+\frac{1}{4}pp:}+\frac{1}{2}p.$

6. Wherefore by extracting the Square Root of each part of the last Equation, the number a sought will be made known, *viz.*

$$a=3=\sqrt{(2):\sqrt{d+\frac{1}{4}pp}+\frac{1}{2}p:}$$

I say the number a sought is 3, which will solve the Question proposed, as will appear by

The

The Proof.

If . . . . $a = 3$,
Then consequently . . . . . . $aa = 9$,
And . . . . $aaaa = 81$,
Also . . . . $6aa = 54$,
Therefore . . . . . . $aaaa - 6aa = 27$.
Which was the Equation proposed to be resolved.

QUEST. 3.

1. What is the number represented by a in this Equation? . . . . $aaaaaa - 2aaa = 48$
2. Which Equation, if m be put for 2, and g for 48, may be exprest thus, ... $aaaaaa - maaa = g$

RESOLUTION.

3. Let the Square of half the Co-efficient 2 (or m) be added to each part of the Equation proposed, to the end its former part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation arises;

$$aaaaaa - 2aaa + 1 = 49,$$

Or, $$aaaaaa - maaa + \tfrac{1}{4}mm = g + \tfrac{1}{4}mm.$$

4. Then by extracting the Square Root of each part of the last Equation (according to *Sect.* 4, and 5 of *Chap.* 8.) this Equation arises;

$$aaa - 1 = 7,$$

Or, $$aaa - \tfrac{1}{2}m = \sqrt{:g + \tfrac{1}{4}mm:}$$

5. Whence by equal Addition of 1 (or $\frac{1}{2}m$) there arises

$$aaa = 8,$$

Or, $$aaa = \sqrt{:g + \tfrac{1}{4}mm:} + \tfrac{1}{2}m.$$

6. Wherefore by extracting the Cubic Root of each part of the last Equation, the number a sought will be made known, *viz.*

$$a = 2 = \sqrt{(3)}:\overline{\sqrt{g + \tfrac{1}{4}mm.} + \tfrac{1}{2}m}:$$

I say the number a sought is 2, which will solve the Question proposed; as will appear by

The Proof.

If . . . . $a = 2$,
Then consequently . . . . . $aaa = 8$,
And . . . . $aaaaaa = 64$,
Also . . . . $2aaa = 16$,
Therefore . . . . . $aaaaaa - 2aaa = 48$.
Which was the Equation proposed to be resolved.

VIII. From the Resolution of the three last Questions the following Canon is deduced, for the resolving of all Equations which fall under the second of the three Forms before specified, in *Sect.* 1. of this *Chap.*

CANON.

Add the Square of half the Co-efficient, or, (which is the same thing) a quarter of the Square of the whole Co-efficient, to the given Absolute Number.

Extract the Square Root of that Sum.

To the said Square Root add half the Co-efficient, and reserve this Sum.

Lastly, when the unknown number which is drawn into the Co-efficient in the middle term of the Equation is exprest by a single Letter only, as a, then the Sum before reserved is the Number sought; but if the said unknown number in the middle term be a Square, as aa, then the Square Root of the Sum reserved is the number sought; if a Cube, as aaa, then the Cubic Root of the said Sum shall be the number sought; if any higher Power, then the Root for the kind must be extracted out of the said Sum, which Root shall be the number sought.

An Example of the said Canon.

1. Let the preceding *Queſt.* 1. in *Sect.* 7. of this *Chap.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . . $aa - 10a = 24$
2. Or, what is the value of *a* in this Equation? . . $aa - ba = k$

RESOLUTION.

3. To the given abſolute number . . .	24	$k.$
4. Add the Square of half the Co-efficient 10, to wit, the Square of 5, which is . . .	25	$\frac{1}{4}bb.$
5. The Sum is	49	$k+\frac{1}{4}bb.$
6. The Square Root of that Sum is	7	$\sqrt{:k+\frac{1}{4}bb:}$
7. To which Square Root add half the Co-efficient 10, to wit, ...	5	$\frac{1}{2}b.$
8. The Sum is the number *a* ſought, to wit,	12	$\sqrt{:k+\frac{1}{4}bb:}+\frac{1}{2}b.$

Whence it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 1. in *Sect.* 7.

A Second Example of the Canon in Sect. 8.

1. Let the preceding *Queſt.* 2. in *Sect.* 7. of this *Chap.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . . $aaaa - 6aa = 27$
2. Or, What is the value of *a* in this Equation? . . $aaaa - paa = d.$

RESOLUTION.

3. To the given abſolute number . . .	27	$d.$
4. Add the Square of half the Co-efficient 6, to wit, the Square of 3, which is . .	9	$\frac{1}{4}pp.$
5. The Sum is	36	$d+\frac{1}{4}pp.$
6. The Square Root of that Sum is	6	$\sqrt{:d+\frac{1}{4}pp:}$
7. To which Square Root add half the Co-efficient 6, to wit, ...	3	$\frac{1}{2}p.$
8. The Sum is the value of *aa*, to wit, .	9	$\sqrt{:d+\frac{1}{4}pp:}+\frac{1}{2}p.$
9. Therefore the Square Root of the ſaid Sum ſhall be the number ſought, to wit, . .	3	$\sqrt{(2):\sqrt{d+\frac{1}{4}pp}+\frac{1}{2}p:}$

Whence it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 2. in *Sect.* 7.

A Third Example of the Canon in Sect. 8.

1. Let the Preceding *Queſt.* 3. in *Sect.* 7. of this *Chap.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . . $aaaaaa - 2aaa = 48,$
2. Or, What is the value of *a* in this Equation? . . $aaaaaa - maaa = g.$

RESOLUTION.

3. To the given abſolute number . . .	48	$g.$
4. Add the Square of half the Co-efficient 2, to wit, the Square of 1, which is . .	1	$\frac{1}{4}mm.$
5. The Sum is	49	$g+\frac{1}{4}mm.$
6. The Square Root of that Sum is . .	7	$\sqrt{:g+\frac{1}{4}mm:}$
7. To which Square Root add half the Co-efficient 2, to wit, ...	1	$\frac{1}{2}m.$
8. The Sum is the value of *aaa*, to wit, . .	8	$\sqrt{:g+\frac{1}{4}mm:}+\frac{1}{2}m.$
9. Therefore the Cubic Root of the ſaid Sum ſhall be the number *a* ſought, to wit, . .	2	$\sqrt{(3):\sqrt{g+\frac{1}{4}mm}+\frac{1}{2}m:}$

Whereby it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 3. in *Sect.* 7.

Example

Example 4.

If . . $aa - a = g$ (or 1122,) what is $a = ?$

Answ. . . . $a = \sqrt{:g + \frac{1}{4}:} + \frac{1}{2} = 34.$

Example 5.

If . . . . $aa - \frac{13}{5}a = 373\frac{17}{45}$, what is $a = ?$

Answ. . . . . . . $a = 20\frac{2}{3}$.

Example 6.

If . . . . $\begin{cases} r = 1, \\ s = 2, \\ c = 4, \end{cases}$

And if . . . . $aa - \frac{sc}{r}a = \frac{15sscc}{4rr}$,

What is . . . . $a = ?$

Answ. . . . . $a = \frac{5sc}{2r} = 20.$

IX. *The Arithmetical Resolution of Equations which fall under the last of the three Forms before exprest in* Sect. I. *of this* Chapter.

QUEST. 1.

1. What is the Number represented by a in this Equation? . . . $10a - aa = 24,$
2. Which Equation, if c be assumed to signifie 10, and n put for 24, may be exprest thus, ... $ca - aa = n.$

RESOLUTION.

3. Let the Equation proposed, by transposition of its Terms, be reduced to an Equation of the second of the three Forms before exprest in *Sect.* 1. *viz.* First by transposition of $- aa$, this Equation arises;

$$10a = 24 + aa,$$

Or, $$ca = n + aa.$$

4. Likewise by transposition of 24 (or n) this Equation arises;

$$10a - 24 = aa,$$

Or, $$ca - n = aa.$$

5. And from the last Equation by Transposition of $10a$ (or ca) there will arise

$$- 24 = aa - 10a,$$

Or, $$- n = aa - ca.$$

6. Which last Equation, by transposing each part of it to the contrary Coast, may be exprest thus;

$$aa - 10a = - 24,$$

Or, $$aa - ca = - n.$$

7. Now let the following process be made as before in the Resolution of Equations of the second Form (in *Sect.* 7.) *viz.* Let the Square of half the Co-efficient 10 (or c) be added to each part of the last Equation, to the end its former part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation arises;

$$aa - 10a + 25 = 25 - 24 = 1,$$

Or, $$aa - ca + \tfrac{1}{4}cc = \tfrac{1}{4}cc - n.$$

8. Then by extracting the Square Root of each part of the last Equation, (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation arises, *viz.*

$$a - 5 = 1,$$

Or, ... $$a - \tfrac{1}{2}c = \sqrt{:\tfrac{1}{4}cc - n:}$$

9. Whence by equal addition of 5 (or $\frac{1}{2}c$) one value of a will be made known, *viz.*

$$a = 6 = \tfrac{1}{2}c + \sqrt{:\tfrac{1}{4}cc - n:}$$

10. But forasmuch as the Square Root of $aa - 10a + 25$ in the seventh step may be $5 - a$ as well as $a - 5$, (for either of those Roots being multiplied into it self, will

 produce

produce $aa - 10a + 25$,) therefore let $5 - a$ be set instead of $a - 5$ in the eighth step, whence this Equation will arise, *viz.*

$$5 - a = 1,$$

Or, $$\tfrac{1}{2}c - a = \sqrt{\tfrac{1}{4}cc - n}:$$

11. Whence by due Transposition another value of a is discovered, to wit,

$$a = 4 = \tfrac{1}{2}c - \sqrt{\tfrac{1}{4}cc - n}:$$

12. I say the Number a sought may be either 6 or 4, for either of these numbers will constitute the Equation proposed, as will appear by

The Proof.

If	$a = 6$,
Then consequently	$aa = 36$,
And	$10a = 60$,
Therefore	$10a - aa = 24$.

Which was the Equation propos'd to be resolved.

Again,

If	$a = 4$,
Then consequently	$aa = 16$,
And	$10a = 40$,
Therefore	$10a - aa = 24$; as before.

13. But to the end that both the values of a before exprest in the ninth and eleventh Equations may be real or Affirmative Numbers, (that is, each greater than nothing) the given Numbers in the Equation proposed, and likewise in every Equation of the Third Form aforesaid must be subject to this following

DETERMINATION.

The Absolute number given must not exceed the Square of half the Co-efficient.

The Reason of this Determination is Evident by the said ninth aud eleventh Equations; for the latter part of each of them shews, that the given Absolute Number is to be subtracted from the Square of half the Co-efficient, and therefore it ought to be less, or equal to the said Square: Therefore when in any Equation of the third Form, the given Absolute number exceeds the Square of half the Co-efficient that Equation is impossible, and likewise the Question that produced it.

It is also evident by the said ninth and eleventh Equations, That when it happens that $n = \tfrac{1}{4}cc$, then $\tfrac{1}{4}cc - n = 0$, and consequently each value of a is equal to $\tfrac{1}{2}c$; *viz.* When the Absolute number happens to be equal to the Square of half the Co-efficient, then the two values of a will be equal to one another, each value in that case being equal to half the Co-efficient: But when it happens that the Absolute number is less than the Square of half the Co-efficient, then those two Roots or values of a will be unequal. But here is to be noted, that although in this latter case the Equation be always explicable by either of those two unequal Roots or Numbers, yet the Question that produced the Equation will sometimes be answered only by one of those Roots or Numbers, (as hereafter will appear in *Quest.* 10. *Chap.* 16. and by the latter way of resolving the 16. *Quest.* of the same *Chap.*)

QUEST. 2.

1. What is the Number represented by a in this Equation? . . . . $5aa - aaaa = 4$.
2. Which Equation, if r be put for 5, and s for 4, may be exprest thus . . . . $raa - aaaa = s$.

RESOLUTION.

3. Let the Equation propos'd, by Transposition of its Terms (after the same manner as in the third, fourth, fifth, and sixth steps of the preceding *Quest.* 1. *Sect.* 9.) be reduced to an Equation of the second of the three Forms before exprest in *Sect.* 1. so this Equation will arise, *viz.*

$$aaaa - 5aa = -4,$$

Or, $$aaaa - raa = -s.$$

4. Then

4. Then by adding (as in the former Examples) the Square of half the Co-efficient 5 (or r) to each part of the last Equation, there arises

$$aaaa - 5aa + \tfrac{25}{4} = \tfrac{25}{4} - 4 = \tfrac{9}{4},$$

Or, $$aaaa - raa + \tfrac{1}{4}rr = \tfrac{1}{4}rr - s.$$

5. And by extracting the Square Root of each part of the last Equation this arises;

$$aa - \tfrac{5}{2} = \tfrac{3}{2},$$

Or, $$aa - \tfrac{1}{2}r = \sqrt{:\tfrac{1}{4}rr - s:}$$

6. Whence by equal Addition of $\frac{5}{2}$ (or $\frac{1}{2}r$,) this Equation arises, *viz.*

$$aa = \tfrac{8}{2} \text{ or } 4,$$

Or, $$aa = \tfrac{1}{2}r + \sqrt{:\tfrac{1}{4}rr - s:}$$

7. Therefore by extracting the Square Root of each part of the last Equation, one value of a will be made known, *viz.*

$$a = 2 = \sqrt{(2):\tfrac{1}{2}r + \sqrt{\tfrac{1}{4}rr - s}:}$$

8. But forasmuch as the Square Root of $aaaa - 5aa + \frac{25}{4}$ in the fourth step may be $\frac{5}{2} - aa$, as well as $aa - \frac{5}{2}$, (for either of those Roots being multiplied by it self produce $aaaa - 5aa + \frac{25}{4}$;) therefore let $\frac{5}{2} - aa$ be set instead of $aa - \frac{5}{2}$ in the fifth step, whence this Equation will arise;

$$\tfrac{5}{2} - aa = \tfrac{3}{2},$$

Or, $$\tfrac{1}{2}r - aa = \sqrt{:\tfrac{1}{4}rr - s:}$$

9. Whence by due Transposition this Equation arises;

$$aa = \tfrac{2}{2} \text{ or } 1,$$

Or, $$aa = \tfrac{1}{2}r - \sqrt{:\tfrac{1}{4}rr - s:}$$

10. Wherefore by extracting the Square Root of each part of the last Equation, another value of a is discovered, to wit,

$$a = 1 = \sqrt{(2):\tfrac{1}{2}r - \sqrt{\tfrac{1}{4}rr - s}:}$$

I say the Number a sought may be either 2 or 1, for either of these numbers will constitute the Equation proposed, as will appear by

The Proof.

If	$a =$	2,
Then consequently	$aa =$	4,
And	$aaaa =$	16,
Also	$5aa =$	20,
Therefore	$5aa - aaaa =$	4.

Which was the Equation propos'd to be resolv'd.

Again,	If	$a =$	1,
	Then	$aa =$	1,
	And	$aaaa =$	1,
	Also	$5aa =$	5,
	Therefore,	$5aa - aaaa =$	4; as before.

QUEST. 3.

1. What is the number represented by a in this Equation? $\succ$ $9aaa - aaaaaa = 8.$
2. Which Equation, if d be put for 9, and t for 8 may be exprest thus . . . . $\Big\}$ $daaa - aaaaaa = t.$

RESOLUTION.

3. Let the Equation propos'd, by transposition of its Terms (after the same manner as in the third, fourth, fifth and sixth steps of the preceding *Quest.* 1. *Sect.* 9.) be reduced to an Equation of the second of the three forms before exprest in *Sect.* 1. so this Equation will arise, *viz.*

$$aaaaaa - 9aaa = -8,$$

Or, $$aaaaaa - daaa = -t.$$

4. Then by adding the Square of half the Co-efficient 9 (or d) to each part of the last Equation, there arises.

$$aaaaaa - 9aaa + \tfrac{81}{4} = \tfrac{81}{4} - 8 = \tfrac{49}{4},$$

Or, $$aaaaaa - daaa + \tfrac{1}{4}dd = \tfrac{1}{4}dd - t.$$

5. And

5. And by extracting the Square Root of each part of the last Equation this arises,

$$aaa - \tfrac{9}{2} = \tfrac{7}{2},$$

Or, $$aaa - \tfrac{1}{2}d = \sqrt{:\tfrac{1}{4}dd - t:}$$

6. Whence by equal addition of $\frac{9}{2}$ (or $\frac{1}{2}d$) this Equation arises;

$$aaa = \tfrac{16}{2} \text{ or } 8,$$

Or, $$aaa = \tfrac{1}{2}d + \sqrt{:\tfrac{1}{4}dd - t:}$$

7. Therefore by extracting the Cubic Root of each part of the Equation, one value of a will be made known, *viz.*

$$a = 2 = \sqrt{(3)}:\overline{\tfrac{1}{2}d + \sqrt{:\tfrac{1}{4}dd - t:}}$$

8. But forasmuch as the Square Root of $aaaaaa - 9aaa + \frac{81}{4}$ in the fourth step may be $\frac{9}{2} - aaa$ as well as $aaa - \frac{9}{2}$, (for either of these Roots being multiplied by it self, will produce the same Square $aaaaaa - 9aaa + \frac{81}{4}$,) therefore let $\frac{9}{2} - aaa$ be set instead of $aaa - \frac{9}{2}$ in the fifth step, whence this Equation will be made, *viz.*

$$\tfrac{9}{2} - aaa = \tfrac{7}{2},$$

Or, $$\tfrac{1}{2}d - aaa = \sqrt{:\tfrac{1}{4}dd - t:}$$

9. Whence by due transposition this Equation arises, *viz.*

$$aaa = \tfrac{2}{2} = 1,$$

Or, $$aaa = \tfrac{1}{2}d - \sqrt{:\tfrac{1}{4}dd - t:}$$

10. Wherefore by extracting the Cubic Root of each part of the last Equation, another value of a is made known, *viz.*

$$a = 1 = \sqrt{(3)}:\overline{\tfrac{1}{2}d - \sqrt{\tfrac{1}{4}dd - t:}}$$

I say the Number a sought is either 2 or 1, for either of these numbers will constitute the Equation proposed; as will appear by

The Proof.

If	$a =$	2,
Then consequently	$aaa =$	8,
And	$aaaaaa =$	64,
Also	$9aaa =$	72,
Therefore	$9aaa - aaaaaa =$	8.

Which was the Equation proposed to be resolved.

Again,

If	$a =$	1,
Then consequently	$aaa =$	1,
And	$aaaaaa =$	1,
Also	$9aaa =$	9,
Therefore	$9aaa - aaaaaa =$	8; as before.

X. From the Resolution of the three last Questions the following Canon is deduced for the resolving all Equations which fall under the last of the three Forms before specified in *Sect.* 1. of this *Chap.*

CANON.

From the Square of half the Co-efficient, or (which is the same thing) from a quarter of the Square of the whole Co-efficient, subtract the Absolute number given.

Extract the Square Root of that Remainder.

Add the said Square Root to half the Co-efficient, and also subtract it from half the Co-efficient, reserving the Sum and Remainder.

Lastly, when the unknown number which is multiplied by the Co-efficient in the middle term of the Equation is exprest by a single letter only, as a, then the Sum and Remainder before reserved are the two Numbers sought, each of which will constitute the Equation proposed; but if the said unknown number in the middle term be a Square, as aa, then the Square Root severally extracted out of the Sum and Remainder reserved shall be the two Numbers sought; if a Cube, as aaa, then the Cubic Root severally extracted out of the said Sum and Remainder shall be the two Numbers sought; if any higher Power, then the Root for the kind must be extracted severally out of the said Sum and Remainder, which Roots shall be the two Numbers sought.

An

An Example of the said Canon.

1. Let the preceding *Queſt.* 1. in *Sect.* 9 of this *Chap.* be here repeated, *viz.* What is the number repreſented by a in this Equation ? . .	$10a - aa = 24$,
2. Or, What is the value of a in this Equation ?	$ca - aa = n$.

RESOLUTION.

3. From the Square of half the Co-efficient 10, to wit, the Square of 5, which is . . .	25	$\frac{1}{4}cc$.
4. Subtract the given abſolute number . . .	24	n.
5. The remainder is	1	$\frac{1}{4}cc - n$.
6. The Square Root of that remainder is .	1	$\sqrt{:\frac{1}{4}cc - n:}$
7. To which Square Root add half the Co-efficient 10, to wit, ...	5	$\frac{1}{2}c$.
8. The Sum is the greater value of a ſought, to wit, ...	6	$\frac{1}{2}c + \sqrt{:\frac{1}{4}cc - n:}$
9. But ſubtracting the ſaid Square Root from half the Co-efficient, the remainder is the leſſer value of a, to wit	4	$\frac{1}{2}c - \sqrt{:\frac{1}{4}cc - n:}$

Either of which Numbers 6 and 4 found out in the two laſt ſteps will conſtitute the Equation propoſed, as before has been proved in the *Anſwer* to *Queſt.* 1. in *Sect.* 9. of this *Chap.*

A Second Example of the Canon in Sect. 10.

1. Let the preceding *Queſt.* 2. in *Sect.* 9. of this *Chap.* be here repeated, *viz.* What is the number repreſented by a in this Equation ?	$5aa - aaaa = 4$
2. Or, What is the value of a in this Equation ?	$raa - aaaa = s$.

RESOLUTION.

3. From the Square of half the Co-efficient 5, to wit, the Square of $\frac{5}{2}$, which is . .	$\frac{25}{4}$	$\frac{1}{4}rr$.
4. Subtract the given abſolute number .	4	s.
5. The remainder is	$\frac{9}{4}$	$\frac{1}{4}rr - s$.
6. The Square Root of that remainder is . .	$\frac{3}{2}$	$\sqrt{:\frac{1}{4}rr - s:}$
7. To which Square Root add half the Co-efficient 5, to wit, ...	$\frac{5}{2}$	$\frac{1}{2}r$.
8. The Sum is the greater value of aa, to wit,	4	$\frac{1}{2}r + \sqrt{:\frac{1}{4}rr - s:}$
9. But ſubtracting the ſaid Square Root from half the Co-efficient, the remainder is the leſſer value of aa, to wit, ...	1	$\frac{1}{2}r - \sqrt{:\frac{1}{4}rr - s:}$
10. Therefore the Square Root of the Sum in the 8*th* ſtep is the greater value of a, to wit,	2	$\sqrt{(2):\frac{1}{2}r + \sqrt{\frac{1}{4}rr - s}:}$
11. And the ſquare root of the remainder in the ninth ſtep is the leſſer value of a, to wit,	1	$\sqrt{(2):\frac{1}{2}r - \sqrt{\frac{1}{4}rr - s}:}$

Either of which numbers 2 and 1 found out in the two laſt ſteps will conſtitute the Equation propoſed, as before has been proved in the *Anſwer* to *Queſt.* 2. in *Sect.* 9 of this *Chap.*

A Third Example of the Canon in Sect. 10.

1. Let the preceding *Queſt.* 3. in *Sect.* 9. of this *Chap.* be here repeated, *viz.* What is the number repreſented by a in this Equation ?	$9aaa - aaaaaa = 8$
2. Or, What is the value of a in this Equation ? . . .	$daaa - aaaaaa = t$

RESOLUTION.

3. From the Square of half the Co-efficient 9, to wit, the Square of $\frac{9}{2}$, which is, . .	$\frac{81}{4}$	$\frac{1}{4}dd$.
4. Subtract the given abſolute number . .	8	t.

5. The

5. The remainder is	$\frac{49}{4}$	$\frac{1}{4}dd - t.$
6. The Square Root of that remainder is . .	$\frac{7}{2}$	$\sqrt{:\frac{1}{4}dd - t:}$
7. To which Square Root add half the Co-efficient 9, to wit, ...	$\frac{9}{2}$	$\frac{1}{2}d.$
8. The sum is the greater value of *aaa*, to wit,	8	$\frac{1}{2}d + \sqrt{:\frac{1}{4}dd - t:}$
9. But subtracting the said Square Root from half the Co-efficient, the remainder is the lesser value of *aaa*, to wit, ...	1	$\frac{1}{2}d - \sqrt{:\frac{1}{4}dd - t:}$
10. Therefore the Cubic Root of the sum in the eight step is the greater value of *a*, to wit,	2	$\sqrt{(3)\frac{1}{2}d + \sqrt{\frac{1}{4}dd - t}:}$
11. And the Cubic Root of the remainder in the ninth step is the lesser value of *a*, to wit,	1	$\sqrt{(3):\frac{1}{2}d - \sqrt{\frac{1}{4}dd - t}:}$

Either of which numbers 2 and 1 found out in the two last steps will constitute the Equation proposed, as before has been proved in the *Answer* to *Quest*. 3. in *Sect*. 9. of this *Chap*.

Example 4.

1. If b, d, f, g represent such known Numbers that bf is greater than dg; and,

2. If $\frac{bg + 2bf + df}{bg + dg + bf + df}a - aa = \frac{bf - dg}{bg + dg + bf + df}$;
What is a equal to?

Answ. a is equal to 1, and also to $\frac{bf - dg}{bg + dg + bf + df}$.

Which values of a are also found out by the Canon in the Tenth *Section* of this *Chap*. but I shall leave the Operation as an exercise for the industrious Learner, and in the next place shew the use of the Rules before delivered in this fifteenth *Chap*. in the Resolution of various Arithmetical Questions.

CHAP. XVI.

Various Arithmetical Questions, producing Equations that fall under some of the three Forms in Sect. 1. *of the foregoing* Chap. 15. *and are resolvable by their respective Canons in* Sect. 6, 8, *and* 10, *of the same* Chap.

QUEST. 1.

THere are two Numbers whose difference is 16 (or c,) and the Product of their Multiplication is 36 (or b;) what are the Numbers?

RESOLUTION.	*Numeral.*	*Literal.*
1. For the lesser of the two numbers sought put	a	a
2. Then by adding to the said lesser number the given difference 16 (or c,) the greater number sought will be	$a + 16$	$a + c$
3. Therefore from the two last steps the Product made by the mutual Multiplication of the two Numbers sought will be . . .	$aa + 16a$	$aa + ca$

4. Which Product must be equal to the given Product 36 (or b) whence this Equation arises, *viz*. $aa + 16a = 36$,
Or, $aa + ca = b$.

5. Which Equation being resolved by the Canon in *Sect*. 6. of *Chap*. 15. the value of a, or the lesser number sought by this Question will be discovered, *viz*.

$$a = 2 = \sqrt{:b + \frac{1}{4}cc:} - \frac{1}{2}c.$$

6. To

6. To which leſſer number adding the given difference 16 (or c) the greater number ſought will alſo be made known, *viz.*

$$2+16=18=\sqrt{:b+\tfrac{1}{4}cc:}+\tfrac{1}{2}c.$$

Otherwiſe thus,

1. For the greater of the two numbers ſought put	a	a
2. Then by ſubtracting from the ſaid greater number the given difference 16, (or c) the leſſer number ſought will be	$a-16$	$a-c$
3. Therefore from the two laſt ſteps, the Product made by the mutual Multiplication of the two numbers ſought will be . .	$aa-16a$	$aa-ca$

4. Which Product muſt be equal to the given Product 36, (or b;) whence this Equation ariſes, *viz.*

$$aa-16a=36, \qquad \text{or,} \qquad aa-ca=b.$$

5. Which Equation being reſolved by the Canon in *Sect.* 8. of *Chap.* 15. the value of a, to wit, the greater number ſought will be diſcovered, *viz.*

$$a=18=\sqrt{:b+\tfrac{1}{4}cc:}+\tfrac{1}{2}c.$$

6. And by ſubtracting from the ſaid greater number the given difference 16 (or c,) the leſſer number ſought will alſo be diſcovered, *viz.*

$$18-16=2=\sqrt{:b+\tfrac{1}{4}cc:}-\tfrac{1}{2}c.$$

From either of thoſe ways of Reſolution, the numbers ſought are found 18 and 2, which will ſolve the Queſtion propoſed; for their difference is 16, and the Product of their Multiplication is 36, as was preſcribed.

Moreover, the two laſt ſteps of each Reſolution by Literal Algebra give one and the ſame Canon to ſolve the Queſtion propoſed.

CANON.

To the given Product add the ſquare of half the given difference, and extract the ſquare Root of that ſum; then to the ſaid ſquare Root adding half the given difference, and from the ſaid ſquare Root ſubtracting the ſaid half difference, the ſum and Remainder ſhall be the two numbers ſought.

Therefore the difference and the Rectangle (or Product of the Multiplication) of any two numbers being ſeverally given, the numbers themſelves ſhall alſo be given by the ſaid Canon.

QUEST. 2.

There are three numbers in Geometrical proportion continued; the difference of the extremes, that is, of the firſt and third is 16 (or c.) and the mean is 6 (or m;) what are the extreme Proportionals?

RESOLUTION.

1. For the leſſer of the two extreme Proportionals ſought put	a	a
2. Then by adding to the ſaid leſſer extreme the given difference of the extremes, to wit, 16 (or c,) the greater extreme will be	$a+16$	$a+c$
3. Therefore the Rectangle contained under the extreme Proportionals,) to wit, the Product made by their mutual Multiplication) ſhall be	$aa+16a$	$aa+ca$

4. Which Rectangle (or Product) muſt (by *Sect.* 1. *Chap.* 13.) be equal to the ſquare of the given mean Proportional 6 (or m), hence this Equation;

$$aa+16a=36, \qquad \text{or,} \qquad aa+ca=mm.$$

5. Which Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, or the leſſer of the two extreme Proportionals ſought will be made known, *viz.*

$$a=2=\sqrt{:mm+\tfrac{1}{4}cc:}-\tfrac{1}{2}c.$$

6. To which lesser extreme Proportional adding 16 (or c) the given difference of the extremes, the greater of the two extreme Proportionals will also be discovered, *viz.*

$$2+16 = 18 = \sqrt{:mm+\tfrac{1}{4}cc:} + \tfrac{1}{2}c.$$

I say the two extreme Proportionals sought are 2 and 18, between which the given number 6 is a mean Proportional; for, as 2 is to 6, so is 6 to 18.

Moreover, the two last steps of the Resolution give the following Canon to find out the extreme Proportionals sought.

CANON.

To the Square of the given mean Proportional add the Square of half the given difference of the extremes, and extract the square Root of that Sum; then to the said square Root adding half the said difference, and from the said square Root subtracting the same half difference, the Sum and Remainder shall be the extreme Proportionals sought.

Therefore if of three numbers in continual proportion the mean be given, as also the difference of the extremes, the extremes shall be given severally by the said Canon.

QUEST. 9.

There are two numbers whose Sum is 20 (or c,) and the Product of their Multiplication is 36 (or n;) what are the numbers?

RESOLUTION.

1. For one of the numbers sought put . . .	a	a
2. Then by subtracting that number from the given Sum 20 (or c,) the Remainder will be the other number sought, to wit, . .	$20-a$	$c-a$
3. Therefore the Product of the Multiplication of those two numbers will be . . .	$20a-aa$	$ca-aa$

4. Which Product must be equal to the given Product 36 (or n,) whence this Equation arises, *viz.* $20a-aa = 36$,
Or, $ca-aa = n$.

5. Which Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers sought by this Question will be discovered, *viz.*

$$a = \begin{cases} 18 = \tfrac{1}{2}c + \sqrt{:\tfrac{1}{4}cc - n:} \\ 2 = \tfrac{1}{2}c - \sqrt{:\tfrac{1}{4}cc - n:} \end{cases}$$

I say the numbers sought are 18 and 2, for their Sum is 20, and the Product of their Multiplication is 36, as was prescribed.

Moreover, if the two values of a, which are express'd by Letters in the last step of the Resolution, be express'd by Words, they will give the following Canon to solve the Question proposed.

CANON.

From the Square of half the given Sum subtract the given Product, and extract the square Root of the Remainder; then to the said half Sum adding the said square Root, and from the said half Sum subtracting the same square Root, the Sum and Remainder shall be the two numbers sought.

Therefore the Sum and Rectangle (or Product of the Multiplication) of any two numbers being severally given, the numbers themselves shall also be given severally by the said Canon.

QUEST. 4.

There are three numbers in continual proportion; the sum of the extremes is 20, (or c,) and the mean proportional is 6, (or m;) what are the extremes?

RESOLUTION.

1. For one of the two extreme proportionals sought put	a	a

2. Then

2. Then by subtracting that extreme from 20 (or c) the given Sum, the Remainder will be the other extreme, to wit, ...	$20-a$	$c-a$
3. Therefore the Rectangle contained under the extreme proportionals, (to wit, the Product of their Multiplication) shall be	$20a-aa$	$ca-aa$

4. Which Rectangle (or Product) must (according to *Sect.* 1. *Chap.* 13.) be equal to the Square of the given mean Proportional 6 (or m,) whence this Equation arises, *viz.*

$$20a-aa=36,$$

Or, $$ca-aa=mm.$$

5. Which Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers sought by this Question will be discovered, *viz.*

$$a=\begin{cases} 18=\frac{1}{2}c+\sqrt{:\tfrac{1}{4}cc-mm:} \\ 2=\frac{1}{2}c-\sqrt{:\tfrac{1}{4}cc-mm:} \end{cases}$$

I say the two extreme Proportionals sought are 18 and 2, between which the given number 6 is a mean Proportional; for, as 18 is to 6, so is 6 to 2.

Moreover, if the two values of a which are express'd by Letters in the last step of the Resolution be express'd by words, they will give the following Canon to find out the extreme Proportionals sought.

CANON.

From the Square of half the given Sum of the extreme Proportionals subtract the Square of the given mean, and extract the square Root of the Remainder; then to the said half Sum adding the said Square Root, and from the said half Sum subtracting the same square Root, the Sum and Remainder shall be the two extreme Proportionals sought.

Therefore if of three Numbers in continual proportion the mean be given, as also the Sum of the extremes, the extremes themselves shall be given severally by the said Canon.

QUEST. 5.

There are two Numbers whose difference is 15, (or d,) and if the Product of the Multiplication of the said two Numbers be divided by 2, (or c,) the Quotient will give the Cube of the lesser Number; what are the Numbers?

RESOLUTION.

1. For the lesser Number sought put . . .	a	a
2. To which adding the given difference 15 (or d,) the Sum shall be the greater Number, to wit, ...	$a+15$	$a+d$
3. Therefore the Product of the Multiplication of the two Numbers is	$aa+15a$	$aa+da$
4. Which Product being divided by 2 (or c) the Quotient will be	$\frac{aa+15a}{2}$	$\frac{aa+da}{c}$
5. From the first step the Cube of the lesser Number is	aaa	aaa

6. Which Cube must (as the Question requires) be equal to the Quotient in the fourth step, whence this Equation;

$$aaa=\frac{aa+15a}{2},$$

Or, $$aaa=\frac{aa+da}{c}.$$

7. Which Equation being duly reduced (according to *Sect.* 2, 4, 3, 5 of *Chap.* 12.) there will arise $$aa-\tfrac{1}{2}a=\tfrac{15}{2},$$

Or, $$aa-\frac{1}{c}a=\frac{d}{c}.$$

8. Therefore the last Equation being resolved by the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the lesser number sought will be discovered, *viz.*

$$a=3=\sqrt{:\frac{d}{c}+\frac{1}{4cc}:}+\frac{1}{2c}.$$

9. To

9. To which lesser number adding the given difference 15 (or d) the Sum shall be the greater number sought, to wit,

$$3+15=18=\sqrt{:\frac{d}{c}+\frac{1}{4cc}:}+\frac{1}{2c}+d.$$

10. I say the two numbers sought are 3 and 18, which will satisfie the conditions in the Question, for their difference is 15, and if the Product of their Multiplication 54 be divided by 2, the Quotient is 27, which is the Cube of the lesser number 3, as was required.

11. But if the Equation in the eighth step be express'd by words, it will give the following Canon to find out the lesser number sought, to which adding the given difference, the greater number is also given.

CANON.

Divide the given difference by the given Divisor, also divide 1 (or Unity) by the quadruple of the Square of the given Divisor, add those two Quotients together, and extract the square Root of the Sum; then to this square Root add the Quotient that arises by dividing 1 by the double of the given Divisor; so shall the Sum be the lesser of the two numbers sought, which increased with their given difference will give the greater number.

QUEST. 6.

There are two numbers whose difference is 2 (or d,) and the Sum of their Squares is 130 (or c;) what are the numbers?

RESOLUTION.

1. For the lesser number sought put . . .	a	a
2. Then to that lesser number adding the given difference 2 (or d) the Sum shall be the greater number, to wit, . . .	$a+2$	$a+d$
3. Therefore from the first step the Square of the lesser number is	aa	aa
4. And from the second step the Square of the greater number is	$aa+4a+4$	$aa+2da+dd$
5. Therefore from the two last steps the Sum of the Squares of the two numbers sought is	$2aa+4a+4$	$2aa+2da+dd$

6. Which Sum must be equal to the given Sum of the Squares 130 (or c,) whence this Equation arises, *viz.*

$$2aa+4a+4=130,$$
Or, $$2aa+2da+dd=c.$$

7. Which Equation, after due Reduction according to the Rules of the twelfth *Chap.* will give this Equation, *viz.* $aa+2a=63$,
Or, $aa+da=\frac{1}{2}c-\frac{1}{4}dd$.

8. Therefore the Equation in the last step being resolved according to the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the lesser number sought by the Question will by made known, *viz.*

$$a=7=\sqrt{:\tfrac{1}{2}c-\tfrac{1}{4}dd:}-\tfrac{1}{2}d.$$

9. To which lesser number adding the given difference 2 (or d) the Sum shall be the greater number sought, to wit,

$$7+2=9=\sqrt{:\tfrac{1}{2}c-\tfrac{1}{4}dd:}+\tfrac{1}{2}d.$$

10. I say the two numbers sought are 9 and 7; for their difference is 2, and the Sum of their Squares is 130, as was prescribed by the Question.

11. Moreover, from the eighth and ninth step arises this

CANON.

From half the given Sum subtract the Square of half the given difference, and extract the square Root of the Remainder; then from this square Root subtract half the given difference, the Remainder shall be the lesser number sought, to which adding the given difference the Sum shall be the greater Number.

QUEST.

QUEST. 7.

There are two Numbers whose Sum is 14 (or b,) and the Sum of their Squares is 100 (or c,) what are the Numbers?

RESOLUTION.

1. For one of the Numbers sought put . . .	a	a
2. Which subtracted from the given Sum 14 (or b) leaves the other Number . . .	$14-a$	$b-a$
3. The Square of the first number is . . .	aa	aa
4. The Square of the other Number is . . .	$aa-28a+196$	$aa-2ba+bb$
5. The Sum of the said Squares is	$2aa-28a+196$	$2aa-2ba+bb$

6. Which Sum must be equal to 100 (or c) the given Sum of the Squares, whence this Equation arises, *viz.* $2aa-28a+196=100$,

Or, $2aa-2ba+bb=c$.

7. Which Equation, after due Reduction, according to the Rules of the twelfth *Chap.* will give this following Equation;

$$14a-aa=48,$$

Or, $ba-aa=\frac{1}{2}bb-\frac{1}{2}c$.

8. Which Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers sought by this Question, will be discovered, *viz.*

$$a=\begin{cases}8=\frac{1}{2}b+\sqrt{:\frac{1}{2}c-\frac{1}{4}bb}.\\ 6=\frac{1}{2}b-\sqrt{:\frac{1}{2}c-\frac{1}{4}bb}.\end{cases}$$

9. I say the Numbers sought are 8 and 6; for their Sum is 14, and the Sum of their Squares is 100, as was prescribed.

10. Moreover, if the two values of a which are express'd by Letters in the eighth step be express'd by words there will arise this

CANON.

From half the given Sum of the Squares subtract the Square of half the given Sum of the two numbers, and extract the square Root of the Remainder; then adding the said square Root to the said half Sum of the Numbers, the Sum of this Addition shall be the greater Number; but subtracting the said square Root from the said half Sum of the Numbers, the Remainder shall be the lesser Number.

QUEST. 8.

There are three Numbers in Geometrical proportion continued, and such, that if the difference between the sum of the extremes and the mean be multiplied by the sum of the extremes, the Product will be 1120 (or b;) but if the said difference be multiplied by the sum of all the three Proportionals, the Product will be 1456 (or c;) what are the Proportionals?

RESOLUTION.

1. For the difference of the Sum of the Extremes and Mean put	a	a
2. Then, according to the Question, the sum of the extremes is	$\frac{1120}{a}$	$\frac{b}{a}$
3. From which sum if the difference in the first step be subtracted, the Remainder will be the mean proportional, to wit, . . .	$\frac{1120}{a}-a$	$\frac{b}{a}-a$
4. Therefore from the two last steps the sum of all three proportionals is	$\frac{2240}{a}-a$	$\frac{2b}{a}-a$

5. But (according to the Question) if the sum of all the three proportionals be multiplied by the difference of the sum of the extremes and the mean, the Product must be equal to 1456 (or c;) therefore from the first and fourth steps this following Equation arises, *viz.*

$$2240-aa=1456,$$

Or, $2b-aa=c$.

6. Which

6. Which Equation being reduced according to the Rules of the twelfth *Chap.* the value of a will be discovered, *viz.*

$$a = 28 = \sqrt{:2b-c.}$$

7. Therefore from the sixth and second steps, the Sum of the extremes is also known, *viz.*

$$40 = \frac{b}{\sqrt{:2b-c:}} = \text{the Sum of the extremes.}$$

8. And from the sixth and third steps, the mean proportional is also given, *viz.*

$$12 = \frac{c-b}{\sqrt{:2b-c:}} = \text{the mean.}$$

9. Lastly, the Sum of the extremes of three continual proportionals being given 40, as also the mean 12, the extremes shall also be given severally by the Canon of the fourth *Question* of this *Chap.* to wit, 4 and 36; therefore the three continual proportionals sought are 4, 12 and 36, which will satisfie the conditions in the Question proposed, as will appear by

The Proof.

I. $4, 12, 36$ are $\div$; for, $4 \times 36 = 12 \times 12$.

II. $\overline{4+36-12}$ into $\overline{36+4} = 1120$.

III. $\overline{4+36-12}$ into $\overline{4+12+36} = 1456$.

QUEST. 9.

There are two Numbers whose Sum is 10 (or b,) and the Sum of their Cubes is 520 (or c;) what are the Numbers?

RESOLUTION.

1. For one of the Numbers sought put . . . . a | a
2. Then by subtracting that Number from the given Sum 10 (or b,) the other Number remains, to wit, ... $10-a$ | $b-a$
3. The Cube of the former is . . . . aaa | aaa
4. And from the second step the Cube of the latter Number is

$$1000 - 300a + 30aa - aaa,$$

Or, $bbb - 3bba + 3baa - aaa.$

4. Therefore the Sum of the two Cubes in the third and fourth steps is

$$1000 - 300a + 30aa,$$

Or, $bbb - 3bba + 3baa.$

5. Which Sum must be equal to 520 (or c) the given Sum of the Cubes, whence this Equation arises, *viz.* $1000 - 300a + 30aa = 520$,

Or, $bbb - 3bba + 3baa = c.$

6. Which Equation, after due Reduction according to the Rules of the twelfth *Chap.* will give this Equation; $16 = 10a - aa$,

Or, $$\frac{bbb-c}{3b} = ba - aa.$$

7. Therefore the last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers sought by this Question, will be discovered, *viz.*

$$a = \begin{cases} \frac{1}{2}b + \sqrt{:\frac{c}{3b} - \frac{bb}{12}:} = 8. \\ \frac{1}{2}b - \sqrt{:\frac{c}{3b} - \frac{bb}{12}:} = 2. \end{cases}$$

8. I say the two Numbers sought are 8 and 2; for their Sum is 10, and the Sum of their Cubes is 520, as was prescribed.
9. Moreover, if the two values of a which are express'd by Letters in the seventh step be express'd by words, they will give this

CANON.

From the Quotient that arises by dividing the given Sum of the two Cubes, by the triple of the given Sum of their sides, subtract $\frac{1}{12}$ of the Square of the last mentioned Sum,

Sum, and extract the ſquare Root of the Remainder; then adding the ſaid ſquare Root to half the ſaid Sum of the Sides of the two Cubes, and alſo ſubtracting the ſaid ſquare Root from the ſaid half Sum, the Sum and Remainder ſhall be the Sides or numbers ſought.

QUEST. 10.

There are two numbers whoſe Sum is 10 (or b,) and the proportion which their difference beareth to the Sum of their Squares is as 2 to 29, (or as r to s; what are the Numbers?

RESOLUTION.

1. For the greater number ſought put . . .	a	a
2. Which ſubtracted from the given Sum 10 (or b) leaves the leſſer number . . .	$10-a$	$b-a$
3. Therefore the difference of the two numbers is	$2a-10$	$2a-b$
4. And from the firſt ſtep the ſquare of the greater number is	aa	aa

5. And from the ſecond ſtep the Square of the leſſer number is

$$100 - 20a + aa,$$
$$\text{Or,}\quad bb - 2ba + aa.$$

6. And from the two laſt ſteps the Sum of the Squares of the two numbers ſought is

$$100 - 20a + 2aa,$$
$$\text{Or,}\quad bb - 2ba + 2aa.$$

7. Then according to the Queſtion, the difference in the third ſtep muſt be to the ſum of the ſquares in the ſixth ſtep as 2 to 29, (or as r to s;) *viz.*

$$2 \,.\, 29 :: 2a - 10 \,.\, 100 - 20a + 2aa,$$
$$\text{Or,}\quad r \,.\, s :: 2a - b \,.\, bb - 2ba + 2aa.$$

8. Which Analogy may be converted into this following Equation, (according to the Theorem in *Chap.* 1. *Sect.* 13.) *viz.*

$$200 - 40a + 4aa = 58a - 290,$$
$$\text{Or,}\quad rbb - 2rba + 2raa = 2sa - sb.$$

9. Which Equation, after due Reduction according to the Rules in the 12 *Chap.* will produce this Equation;

$$\tfrac{245}{2} = \tfrac{49}{2}a - aa,$$
$$\text{Or,}\quad \frac{rbb+sb}{2r} = \frac{s+rb}{r}a - aa.$$

10. Therefore by reſolving the Equation in the laſt ſtep according to *Sect.* 10. *Chap.* 15. the two values of a, or the two Roots of that Equation will be made known, *viz.*

$$a = \begin{cases} \tfrac{35}{2} = \dfrac{s}{2r} + \dfrac{b}{2} + \sqrt{:\dfrac{ss}{4rr} - \dfrac{bb}{4}:} \\ 7 = \dfrac{s}{2r} + \dfrac{b}{2} - \sqrt{:\dfrac{ss}{4rr} - \dfrac{bb}{4}:} \end{cases}$$

11. The leſſer of which two Roots or Numbers, to wit 7, is the greater number ſought by this Queſtion; and conſequently, the ſaid 7 being ſubtracted from the given ſum 10, the Remainder 3 is the leſſer number ſought.

I ſay 7 and 3 will ſolve the Queſtion, for their ſum is 10; and their difference 4 is to the ſum of their ſquares 58, as 2 to 29; which was preſcribed.

12. *Note.* Altho the value of a in the Equation in the ninth ſtep may be either $\tfrac{35}{2}$ or 7, (for that Equation may be expounded by $\tfrac{35}{2}$ as well as 7,) yet 7 only, to wit, the leſſer value of a, ſhall be the greater number ſought by this Queſtion.

For that the greater value of a, to wit, $\dfrac{s}{2r} + \dfrac{b}{2} + \sqrt{:\dfrac{ss}{4rr} - \dfrac{bb}{4}:}$ can never be equal to either of the two numbers ſought, I prove thus; Firſt, it is manifeſt by each of the values of a expreſs'd by Letters in the tenth ſtep, That if $\dfrac{s}{2r} = \dfrac{b}{2}$, then conſequently $\dfrac{ss}{4rr} = \dfrac{bb}{4}$, and the two values of a are equal one to the other, each

being

being equal to $\frac{s}{2r} + \frac{b}{2}$, that is, b; and therefore in this first case, neither of the two values of a can possibly be equal to either of the two numbers sought; for that which is equal to the sum of two numbers must needs be greater than either of them.

Secondly, If $\frac{s}{2r} \sqsubset \frac{b}{2}$, which is a necessary Determination to make the Question possible, then the greater value of a, that is, $\frac{s}{2r} + \frac{b}{2} + \sqrt{:\frac{ss}{4rr} - \frac{bb}{4}:}$ is manifestly greater than b the given sum of the two numbers sought, and therefore it cannot be equal to either of them. Wherefore the said greater value of a cannot in any case be equal to either of the two numbers sought. Which was to be proved.

But the said lesser value of a is the greater of the two numbers sought, and consequently they are given severally by this following

CANON.

13. From the Quotient that arises by dividing the Square of the latter term of the given Reason by the Quadruple of the Square of the first Term, subtract a quarter of the Square of the given Sum of the two numbers sought, and extract the square Root of the Remainder; then subtract that square Root from the Sum of the Quotient that arises by dividing the latter Term of the given Reason by the double of the first, and the half of the given sum of the two numbers, so the Remainder shall be the greater number sought; which subtracted from the said given sum leaves the lesser number.

14. From the premises this following Question may easily be solved, *viz.* The sum of two numbers being given, suppose $\frac{4}{5}$ (or b,) and their difference being equal to the sum of their Squares, to find the numbers.

First, suppose $r = s = 1$; (because the Terms of the Proportion in this Question are equal to one another,) then the two values of a before express'd in the tenth step will be converted into these, *viz.*

$$a = \frac{6}{5} = \frac{1+b}{2} + \sqrt{:\frac{1-bb}{4}},$$

$$a = \frac{3}{5} = \frac{1+b}{2} - \sqrt{:\frac{1-bb}{4}}.$$

The lesser of which values of a, to wit, $\frac{3}{5}$, is the greater of the two numbers sought, and therefore the said $\frac{3}{5}$ being subtracted from $\frac{4}{5}$ the given sum, leaves $\frac{1}{5}$ for the lesser number. I say $\frac{3}{5}$ and $\frac{1}{5}$ will solve the Question, for their difference $\frac{2}{5}$ is equal to the Sum of their Squares.

QUEST. 11.

There are two numbers, the Product of whose Multiplication is 48 (or p,) and the difference of their Squares is 28 (or d;) what are the numbers?

RESOLUTION.

1. For the greater number put	a	a
2. Then dividing 48 (or p) by a, the Quotient is the lesser number, to wit, . . .	$\frac{48}{a}$	$\frac{p}{a}$
3. From the first step the square of the greater number is	aa	aa
4. And from the second step the Square of the lesser number is	$\frac{2304}{aa}$	$\frac{pp}{aa}$
5. Therefore the difference of the said Squares is	$\frac{aaaa-2304}{aa}$	$\frac{aaaa-pp}{aa}$

6. Which difference must be equal to the given difference of the squares, whence this Equation arises, *viz.*

$$\frac{aaaa - 2304}{aa} = 28,$$

$$\text{Or,} \quad \frac{aaaa - pp}{aa} = d.$$

7. Which

7. Which Equation, after due Reduction according to the Rules of the twelfth *Chap.* will produce this; $aaaa - 28aa = 2304$,

Or, $aaaa - daa = pp$.

8. Therefore by resolving the last Equation according to the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the greater number sought will be discovered, *viz.*

$$a = 8 = \sqrt{(2)}:\overline{\sqrt{pp+\tfrac{1}{4}dd}+\tfrac{1}{2}d}:$$

Whence the greater number is found 8, by which if the given Product 48 be divided, the Quotient 6 is the Jesser Number sought.

I say, the Numbers 8 and 6 will solve the Question; for the Product of their Multiplication is 48, and the difference of their Squares 64 and 36 is 28, as was prescribed.

Moreover, the Equation in the eighth step gives a Canon to find the greater of the two numbers sought, by the help whereof and the given Product the lesser number shall be also given.

CANON.

9 To the Square of the given Product add the Square of half the given difference of the Squares, and extract the Square Root of that Sum; then to the said Square Root add the said half difference, and extract the Square Root of this Sum, so shall the last Square Root be the greater of the two Numbers sought; lastly, by the said greater number divide the given product of the multiplication of both numbers, and the Quotient shall be the lesser Number.

QUEST. 12.

There are two Numbers the Product of whose Multiplication is 48 (or p,) and the Sum of their Squares is 100 (or c;) what are the Numbers?

RESOLUTION.

1. For one of the numbers sought put	a	a
2. Then dividing 48 (or p) by a, the Quotient will give the other number, to wit, . .	$\frac{48}{a}$	$\frac{p}{a}$
3. From the first step, the Square of one of the Numbers is	aa	aa
4. And from the second step the Square of the other Number is	$\frac{2304}{aa}$	$\frac{pp}{aa}$
5. Therefore the Sum of the said Squares is . >	$\frac{aaaa+2304}{aa}$	$\frac{aaaa+pp}{aa}$

6. Which Sum must be equal to the given Sum of the Squares, whence this Equation arises, *viz.*

$$\frac{aaaa+2304}{aa} = 100,$$

Or, $$\frac{aaaa+pp}{aa} = c$$

7. From which Equation, after due Reduction by the Rules in *Chap.* 12, this will arise,

$$2304 = 100aa - aaaa,$$
$$pp = caa - aaaa.$$

8. Which last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the Numbers sought, will be discovered, *viz.*

$$a = \begin{cases} 8 = \sqrt{(2)}:\overline{\tfrac{1}{2}c+\sqrt{\tfrac{1}{4}cc-pp}}: \\ 6 = \sqrt{(2)}:\overline{\tfrac{1}{2}c-\sqrt{\tfrac{1}{4}cc-pp}}: \end{cases}$$

9. I say, 8 and 6 are the Numbers required; for the Product of their Multiplication is 48, and the Sum of their Squares 64 and 36 is 100, as was prescribed. From the last step also arises this

CANON.

From the Square of half the given Sum of the Squares of the two numbers sought subtract the Square of the given Product of their Multiplication, and extract the square Root of the Remainder, then to half the said Sum add the said Square Root, and from the

the said half Sum subtract the said Square Root; lastly, extract the Square Root of the Sum of that Addition, and also of the Remainder of the latter Subtraction, so shall these two Square Roots be the numbers sought by the Question propos'd.

QUEST. 13.

There are two Numbers whose Sum is 14 (or b,) and if the Sum of their Squares be multiplied by the Sum of their Cubes, the Product is 72800 (or c;) what are the Numbers?

RESOLUTION.

1. For one of the Numbers sought put . . .	$a+7$	$a+\frac{1}{2}b$
2. Then, that their Sum may be 14 (or b,) the other number must be	$-a+7$	$-a+\frac{1}{2}b$
3. The Square of the first Number is . . .	$aa+14a+49$	$aa+ba+\frac{1}{4}bb$
4. The Square of the latter Number is . . .	$aa-14a+49$	$aa-ba+\frac{1}{4}bb$
5. Therefore the Sum of their Squares is . . .	$2aa+98$	$2aa+\frac{1}{2}bb$

6. Again, the Cube of the first Number will be

$$aaa+21aa+147a+343,$$

Or, $$aaa+\tfrac{3}{2}baa+\tfrac{3}{4}bba+\tfrac{1}{8}bbb.$$

7. And the Cube of the latter Number will be

$$-aaa+21aa-147a+343,$$

Or, $$-aaa+\tfrac{3}{2}baa-\tfrac{3}{4}bba+\tfrac{1}{8}bbb.$$

8. Therefore the Sum of the Cubes in the two last steps is

$42aa+686$, Or, $3baa+\frac{1}{4}bbb$.

9. Which Sum of the Cubes in the last step being Multiplied by the Sum of the Squares in the fifth step, produces $84aaaa+5488aa+67228$,

Or, $6baaaa+2bbbaa+\frac{1}{8}bbbbb$.

10. Which Product in the last step must be equal to 72800 (or c) the Product given in the Question, whence this Equation arises, *viz.*

$$84aaaa+5488aa+67228=72800,$$

Or, $$6baaaa+2bbbaa+\tfrac{1}{8}bbbb=c.$$

11. And from that Equatien, after due Reduction according to the Rules of the twelfth *Chapter*, this will arise; $aaaa+\frac{196}{3}aa=\frac{199}{3}$,

Or, $$aaaa+\tfrac{1}{3}bbaa=\frac{c}{6b}-\tfrac{1}{48}bbbb.$$

12. Which Equation being resolved by the Canon in *Sect* 6. of *Chap.* 15. the value of a will be discovered, *viz.*

$$a=1=\sqrt{(2)}:\overline{\sqrt{\overline{\frac{c}{6b}+\tfrac{1}{144}bbbb.}}-\tfrac{1}{6}bb}:$$

13. Therefore from the twelfth, first and second steps the two numbers sought are made known:

$$7+1=8=\tfrac{1}{2}b+\sqrt{(2)}:\overline{\sqrt{\overline{\frac{c}{6b}+\tfrac{1}{144}bbbb.}}-\tfrac{1}{6}bb}:$$

$$7-1=6=\tfrac{1}{2}b-\sqrt{(2)}:\overline{\sqrt{\overline{\frac{c}{6b}+\tfrac{1}{144}bbbb.}}-\tfrac{1}{6}bb}:$$

I say the numbers sought are 8 and 6; for their Sum is 14, and if 100 the Sum of their Squares be multiplied by 728, the Sum of their Cubes, the Product will be 72800, as was prescribed.

Moreover, the thirteenth step gives a Canon to find out the Numbers sought.

CANON.

Divide the given Product by six times the given Sum; then to the Quotient add $\frac{1}{144}$ of the Biquadrate of the given Sum, and extract the Square Root of the Sum of that addition; then from the said Square Root subtract $\frac{1}{6}$ of the Square of the given Sum, and extract the Square Root of the Remainder; lastly, add this Square Root to half the given Sum and Subtract it from the said half Sum, so shall the Sum and Remainder be the two numbers sought.

QUEST.

QUEST. 14.

There are two numbers the Product of whose Multiplication is 20 (or b,) and the sum of their Cubes is 189 (or c;) what are the numbers:

RESOLUTION.

1. For one of the Numbers sought put . . .	a	a
2. Then, by dividing the given Product 20 (or b) by a, the other Number will be .	$\frac{20}{a}$	$\frac{b}{a}$
3. Therefore from the first step, the Cube of the first Number is	aaa	aaa
4. And from the second step the Cube of the other number is	$\frac{8000}{aaa}$	$\frac{bbb}{aaa}$
5. Therefore the Sum of the said Cubes is . .	$\frac{aaaaaa+8000}{aaa}$	$\frac{aaaaaa+bbb}{aaa}$

6. Which sum must be equal to 189 (or c) the Sum given in the Question, whence this Equation arises, *viz.*

$$\frac{aaaaaa + 8000}{aaa} = 189,$$

$$\text{Or,} \quad \frac{aaaaaa + bbb}{aaa} = c.$$

7. Which Equation being reduced according to *Sect.* 2, 3, and 5. of *Chap.* 12. there will arise $8000 = 189aaa - aaaaaa$,

Or, $bbb = caaa - aaaaaa.$

8. And by resolving the Equation in the last step by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers sought by this Question, will be made known, *viz.*

$$a = \begin{cases} 5 = \sqrt{(3)}:\tfrac{1}{2}c+\sqrt{\tfrac{1}{4}cc - bbb}: \\ 4 = \sqrt{(3)}:\tfrac{1}{2}c-\sqrt{\tfrac{1}{4}cc - bbb}: \end{cases}$$

9. I say, the numbers sought are 5 and 4; for the Product of their Multiplication is 20, and the Sum of their Cubes 125 and 64 is 189, as was prescribed.

Moreover, from the two values of a exprest by Letters in the eighth step, the following Canon arises to find out the number sought.

CANON.

From the Square of half the given Sum subtract the Cube of the given Product, and extract the Square Root of the Remainder; then add the said square Root to half the given Sum, and also subtract it from the said half Sum; lastly, extract the Cubic Root of the Sum of that Addition, and likewise extract the Cubic Root of the latter Remainder, so shall these Cubic Roots be the Numbers sought.

QUEST. 15.

There are two numbers the Product of whose Multiplication is 20 (or b,) and the difference of their Cubes is 61 (or d;) what are the numbers?

RESOLUTION.

1. For the greater of the two Numbers sought put	a	a
2. Then, by dividing the given Product 20 (or b) by a, the lesser number will be . .	$\frac{20}{a}$	$\frac{b}{a}$
3. Therefore from the first step the Cube of the greater number is	aaa	aaa
4. And from the second step the Cube of the lesser number is	$\frac{8000}{aaa}$	$\frac{bbb}{aaa}$
5. Therefore from the two last steps, the difference of the Cubes of the two Numbers sought is	$\frac{aaaaaa-8000}{aaa}$	$\frac{aaaaaa-bbb}{aaa}$

6. Which difference must be equal to 61 (or d) the difference given in the Question, whence this Equation arises, *viz.*

$$\frac{aaaaaa - 8000}{aaa} = 61, \qquad \text{Or,} \quad \frac{aaaaaa - bbb}{aaa} = d.$$

7. Which

7. Which Equation, after due Reduction, (according to *Sect.* 2, 3, and 5. of *Chap.* 12.) will give this that follows, *viz.* $aaaaaa - 61aaa = 8000$,
Or, $aaaaaa - daaa = bbb$.

8. Therefore by resolving the Equation in the last step by the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the greater Number sought will be made known, *viz.*

$$a = 5 = \sqrt{(3)}: \tfrac{1}{2}d + \sqrt{\tfrac{1}{4}dd + bbb}:$$

9. Whence the greater Number sought is found 5, by which if the given Product 20 be divided, the Quotient will give 4 for the lesser number required.

I say the Numbers 5 and 4 will solve the Question proposed; for the Product of their Multiplication is 20, and the difference of their Cubes 125 and 64 is 61, as was prescribed.

Moreover, the Equation in the eighth step gives a Canon to find out the greater of the two numbers sought, by the help whereof and the given Product the lesser number is also given.

CANON.

To the Square of half the given difference add the Cube of the given Product, and extract the Square Root of the Sum of that Addition, then add the said Square Root to half the given difference and extract the Cubic Root of this Sum, so shall the said Cubic Root be the greater of the two numbers sought; by which greater number if the given Product be divided the Quotient shall be the lesser number sought.

QUEST. 16.

A Merchant having bought certain Cloths, sells them at $17\frac{1}{4}\,l.$ (or b) the Cloth, and then found that by every 100 *l.* (or c) that he had laid out, he gained as many Pounds as he paid for one Cloth; what was the first cost of a Cloth?

RESOLUTION.

1. For the first cost of one Cloth put . . .	a	a
2. Which first cost being subtracted from the money for which the Merchant sold one Cloth, there will remain the gain of one Cloth, to wit, ...	$17\frac{1}{4} - a$	$b - a$

3. Then find what was gained in laying out 100 *l.* (or c,) *viz.* say by the Rule of Three,

$$\text{If } a \;.\; 17\tfrac{1}{4} - a \;::\; 100 \;.\; \frac{1725 - 100a}{a},$$

$$\text{Or, } a \;.\; b - a \;::\; c \;.\; \frac{cb - ca}{a}.$$

Whence the gain of 100 *l.* is found $\frac{1725 - 100a}{a}$, or $\frac{cb - ca}{a}$.

4. But according to the Question the gain of 100 *l.* (or c) must be equal to the first cost of one Cloth, therefore from the first and third steps this Equation arises, *viz.*

$$a = \frac{1725 - 100a}{a}, \quad \text{Or, } a = \frac{cb - ca}{a}.$$

5. Which Equation, after due Reduction (according to *Sect.* 2, and 3. of *Chap.* 12.) will give this that follows, *viz.* $aa + 100a = 1725$,
Or, $aa + ca = cb$.

6. Therefore by resolving the Equation in the last step by the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the first cost of a Cloth will be discovered, *viz.*

$$a = 15 = \sqrt{:cb + \tfrac{1}{4}cc:} - \tfrac{1}{2}c.$$

I say the first cost of a Cloth was 15 *l.* as will appear by the Proof: For if a Cloth be bought for 15 *l.* and sold for $17\frac{1}{4}\,l.$ the gain is $2\frac{1}{4}\,l.$ Then if 15 *l.* gain $2\frac{1}{4}\,l.$ it will follow that 100 *l.* will gain 15 *l.* which is equal to the first cost of a Cloth; as was prescribed.

Another

Another way of resolving the preceding Quest. 16.

1. Let the same things be given as before, then for the gain of one Cloth put . .	a	a
2. Which gain being Subtracted from the Money for which one Cloth was sold, will leave the first cost of a Cloth, to wit, . .	$17\frac{1}{4} - a$	$b - a$

3. Then find what was gained in laying out 100 *l.* (or *c*,) and say by the Rule of Three,

$$\text{If} \quad 17\tfrac{1}{4} - a \;.\; a \;::\; 100 \;.\; \frac{100a}{17\frac{1}{4} - a},$$

$$\text{Or,} \quad b - a \;.\; a \;::\; c \;.\; \frac{ca}{b - a}.$$

Whence the gain of 100 *l.* is found $\quad \frac{100a}{17\frac{1}{4} - a} \quad$ or $\quad \frac{ca}{b - a}$.

4. But, according to the Question, the gain of a 100 *l.* (or *c*) must be equal to the first cost of one Cloth; therefore from the second and third steps this Equation arises, *viz.*

$$\frac{100a}{17\frac{1}{4} - a} = 17\tfrac{1}{4} - a, \qquad \text{Or,} \quad \frac{ca}{b - a} = b - a.$$

5. Which Equation, after due Reduction according to *Sect.* 2, and 3. of *Chap.* 12. will give this that follows, *viz.*

$$\tfrac{269}{2}a - aa = \tfrac{4761}{16}, \qquad \text{Or,} \quad ca + 2ba - aa = bb.$$

6. Therefore by resolving the Equation in the last step by the Canon in *Sect* 10 *Chap.* 15. the two values of a, or the two Roots of that Equation will be made known, *viz.*

$$a = \begin{cases} \frac{529}{4} = \frac{1}{2}c + b + \sqrt{:\frac{1}{4}cc + cb:} \\ \frac{9}{4} = \frac{1}{2}c + b - \sqrt{:\frac{1}{4}cc + cb:} \end{cases}$$

The lesser of which two Roots or Numbers, to wit, $\frac{9}{4}$ or $2\frac{1}{4}$ *l.* is the gain of a Cloth, which subtracted from $17\frac{1}{4}$ *l.* leaves 15 *l.* for the first cost of a Cloth, as before.

Note. Although the value of a in the Equation in the fifth step may be either $\frac{529}{4}$ or $\frac{9}{4}$, (for that Equation may be expounded by $\frac{529}{4}$ as well as $\frac{9}{4}$,) yet $\frac{9}{4}$ only, to wit, the lesser value of a shall be the gain of a Cloth; for $\frac{529}{4}$ is greater than $17\frac{1}{4}$, and consequently the gain of one Cloth would exceed the Money for which one Cloth was sold. Which absurdity appears also by the greater value of a as 'tis exprest by Letters in the sixth step, for $\frac{1}{2}c + b + \sqrt{:\frac{1}{4}cc + cb:}$ is manifestly greater than b.

QUEST. 17.

Each of two Captains, whereof one had a lesser number of Soldiers in his Company by 40 (or b) than the other, distributed equally among the Soldiers of his own Company 1200 (or c) Crowns, whereby it happened that the Soldiers of the lesser Company had 5 (or d) Crowns a piece more than the Soldiers of the greater Company; the Question is to find the number of Soldiers in each Company, and how many Crowns each Soldier received.

RESOLUTION.

1. For the number of Soldiers in the lesser Company put	a	a
2. To which adding 40 (or b) the sum will give the number of Soldiers in the greater Company, to wit, ...	$a+40$	$a+b$
3. Then if 1200 (or c) Crowns be equally divided among the Soldiers of the lesser Company, the Quotient or share of every Soldier will be	$\frac{1200}{a}$	$\frac{c}{a}$
4. Likewise, if 1200 (or c) Crowns be equally divided among the Soldiers of the greater Company, the Quotient or share of every Soldier will be	$\frac{1200}{a+40}$	$\frac{c}{a+b}$
5. To which latter Quotient adding 5 (or d) Crowns, the sum is	$\frac{5a+1400}{a+40}$	$\frac{da+db+c}{a+b}$

6. But

6. But according to the Question the Sum in the last step must be equal to the Quotient in the third step, whence this Equation arises, *viz.*

$$\frac{5a+1400}{a+40} = \frac{1200}{a}, \quad \text{Or,} \quad \frac{da+db+c}{a+b} = \frac{c}{a}.$$

7. From which Equation after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. this will arise, *viz.* $aa+40a = 9600$,

$$\text{Or,} \quad aa + ba = \frac{bc}{d}.$$

8. Therefore the Equation in the last step being resolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the number of Soldiers in the lesser Company will be discovered, *viz.*

$$a = 80 = \sqrt{:\frac{bc}{d}+\frac{bb}{4}:} - \tfrac{1}{2}b.$$

From the eighth, first, and second steps it is evident that the lesser Company consisted of 80, and the greater 120 Soldiers; which numbers will satisfie the Conditions in the Question. For the difference of the two Companies is 40 Soldiers; also $\frac{1200}{80} = 15$, and $\frac{1200}{120} = 10$; whence it is manifest that the Soldiers of the lesser Company received 15 Crowns a piece, the Soldiers of the greater Company 10 Crowns a piece, and consequently the Soldiers of the lesser Company had 5 Crowns a piece more than the Soldiers of the greater Company, as was prescribed.

QUEST. 18.

Two Merchants sell Linnen-Cloth in this manner, *viz.* each sells 60 (or b) Ells, and the first Merchant selling 2 (or c) Ells less for one pound than the second, receives for his 60 Ells 5 (or d) pounds more than the second Merchant for his 60 Ells. The Question is to find how many Ells each Merchant sold for 1 Pound?

RESOLUTION.

1. For the number of Ells which the first Merchant sold for 1 *l.* put	a	a
2. To which number of Ells adding 2 (or c,) the Sum will be the number of Ells which the latter Merchant sold for 1 *l.* to wit, . .	$a+2$	$a+c$
3. Then find how much Money the first Merchant received for his 60 Ells, *viz.* say by the Rule of Three, If a . 1 :: 60 . $\frac{60}{a}$; Or, a . 1 :: b . $\frac{b}{a}$. whence the first Merchants total Mony is found	$\frac{60}{a}$	$\frac{b}{a}$
4. Find likewise how much Money the latter Merchant received for his 60 Ells, *viz.* say, If $a+2$. 1 :: 60 . $\frac{60}{a+2}$; Or, $a+c$. 1 :: b . $\frac{b}{a+c}$ whence the latter Merchants total Money is found	$\frac{60}{a+2}$	$\frac{b}{a+c}$
5. To which latter Sum of Money adding 5 (or d) pounds, the Sum will be . .	$\frac{5a+70}{a+2}$	$\frac{da+dc+b}{a+c}$

6. But according to the Question the Sum of Money it in the last step must be equal to the Sum in the third step, whence this Equation arises, *viz.*

$$\frac{5a+70}{a+2} = \frac{60}{a}, \quad \text{Or,} \quad \frac{da+dc+b}{a+c} = \frac{b}{a}:$$

7. Which

7. Which Equation, after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. will give this that follows, *viz.* $aa + 2a = 24$,

Or, $aa + ca = \frac{bc}{d}$.

8. Which Equation in the last step being resolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit the Number of Ells which the first Merchant sold will be made known, *viz.* $a = 4 = \sqrt{:}\,\frac{bc}{d} + \frac{cc}{4} : -\frac{1}{2}c$.

I say the first Merchant sold 4 Ells for 1 Pound, and the second 6 Ells for 1 Pound, as will appear by the Proof. For if 4 Ells give 1 Pound, then 60 Ells will give 15 Pounds Again, if 6 Ells give one Pound, then 60 Ells will give 10 Pounds. Whence it is manifest that the first Merchant sold his 60 Ells for 5 Pounds more than the second sold his 60 Ells, and sold 2 Ells less for 1 pound than the second Merchant sold for one Pound.

QUEST. 19.

Two Societies, whereof one exceeds the other by 4 (or b) men, divide two equal sums of Crowns; the Men of the lesser Society have 8 (or c) Crowns a piece more than those of the greater: And the number of Crowns which each Society receives exceeds the number of Men of both Societies by 172 (or d.) The Question is, to find the number of Men in each Society, and the number of Crowns which each Society had?

RESOLUTION.

1 For the number of Men of the lesser Society put	a	a
2. To which number adding 4 (or b,) the sum will be the Number of Men of the greater Society, to wit, ...	$a+4$	$a+b$
3. Then, according to the Question, if 172 (or d) be added to the Sum of the Men of both Societies, it will give the number of Crowns shared by each Society, to wit, .	$2a+176$	$2a+b+d$
4. Which number of Crowns being divided by (a) the number of Men of the lesser Society, the Quotient or share of every Man in that Society will be	$\frac{2a+176}{a}$	$\frac{2a+b+d}{a}$
5. Likewise if the same number of Crowns before exprest in the third step be divided by $a+4$, (or $a+b$, the number of Men of the greater Society,) the Quotient will give the share of every man in this Society to wit,	$\frac{2a+176}{a+4}$	$\frac{2a+b+d}{a+b}$
6. To which Quotient in the last step adding 8 (or c) the Sum will be	$\frac{10a+208}{a+4}$	$\frac{2+ab+d+ca+cb}{a+b}$

7. But, according to the Question, the sum in the last step must be equal to the Quotient in the fourth step, whence this Equation arises, *viz.*

$$\frac{10a+208}{a+4} = \frac{2a+176}{a}, \quad \text{Or,} \quad \frac{2a+b+d+ca+cb}{a+b} = \frac{2a+b+d}{a}$$

8. From which Equation, after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. this Eqution will arise, *viz.* $aa + 3a = 88$,

Or, $aa + \frac{cb-2b}{c}a = \frac{bb+bd}{c}$.

9. Therefore by resolving the last Equation according to the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the number of Men in the lesser Society will be discover'd, *viz.*

$$a = 8 = \sqrt{:}\,\frac{cbd+\frac{1}{4}ccbb+bb}{cc} : -\frac{b}{2} + \frac{b}{c}.$$

10. Lastly, from the ninth, first, second, and third steps, it is manifest that the number of men in the lesser Society was 8, that of the greater 12, and the number of Crowns divided by each Society 192; which numbers will satisfie the Conditions in the Question,

Question as will appear by the Proof: For $\frac{192}{8} = 24$, and $\frac{192}{12} = 16$; whence it is evident that the Men of the lesser Society had 8 Crowns a piece more than those of the greater; also 192, the number of Crowns which each Society divided, exceeded 20 the number of Men in both Societies by 172, and 12 the number of Men in the greater Society exceeded 8 the number of Men in the lesser by 4; as was prescribed.

QUEST. 20.

A Grasier having bought certain Oxen for 270 (or b) Pounds, finds, that if he had paid that sum for 5 (or c) Oxen fewer, every Ox would have cost him $\frac{3}{4}$ *l.* (or d) more than he paid for an Ox: What was the number of Oxen bought?

RESOLUTION.

1. For the number of Oxen bought put . .	a	a
2. Then find out the cost of an Ox, and say, If $a \,.\, 270 :: 1 \,.\, \frac{270}{a}$; Or, $a \,.\, b :: 1 \,.\, \frac{b}{a}$. whence the price of an Ox is : . . .	$\frac{270}{a}$	$\frac{b}{a}$
3. Subtract 5 (or c) from the number of Oxen bought, and then find what the rest would cost a piece, saying, If $a-5 \,.\, 270 :: 1 \,.\, \frac{270}{a-5}$; Or, $a-c \,.\, b :: 1 \,.\, \frac{b}{a-c}$. Whence the price of an Ox is found . .	$\frac{270}{a-5}$	$\frac{b}{a-c}$

4. Then according to the Question, the last mentioned price of an Ox must exceed that in the second step by $\frac{3}{4}$ *l.* (or d;) therefore if the former price be subtracted from the later, the remainder must be equal to $\frac{3}{4}$ or d;) whence this Equation arises, *viz.*

$$\frac{270}{a-5} - \frac{270}{a} = \frac{3}{4}; \quad \text{Or,} \quad \frac{b}{a-c} - \frac{b}{a} = d.$$

5. Which Equation, after due Reduction according to the Rules in *Chap.* 12. will give this that follows,

$$aa - 5a = 1800,$$

$$\text{Or,} \quad aa - ca = \frac{bc}{d}.$$

6. Therefore the Equation in the last step being resolved by the Canon in *Sect.* 12. *Chap.* 15. the value of a, to wit the Number of Oxen bought will be discovered, *viz.*

$$a = 45 = \sqrt{:\frac{bc}{d} + \frac{cc}{4}:} + \tfrac{1}{2}c.$$

I say the Number of Oxen bought was 45, and every Ox cost 6 Pounds, as will appear by the Proof: For first, $\frac{270}{45} = 6$; then from 45 Oxen subtracting 5, the remaining 40 Oxen valued at 270 *l.* will yield $6\frac{3}{4}$ *l.* a piece, which exceeds the former price 6 *l.* by $\frac{3}{4}$ *l.* as was prescribed.

QUEST. 21.

A Merchant buyes linnen Clothes of two sorts, *viz.* 90 (or b) Ells of one sort, together with 40 (or c) Ells of a worser sort for 42 (or d) Pounds; and he finds that in laying out 1 Pound upon each sort he has $\frac{1}{2}$ (or m) of an Ell more of the worser sort than the other: What was the price of an Ell of each sort.

RESOLUTION.

1. For the Number of Ells of the better sort of Cloth which the Merchant bought for 1 *l.* put	a	a
2. Then according to the Quest. the number of Ells of the worser sort bought for 1 *l.* will be	$a+\frac{1}{2}$	$a+m$

3. Find

3. Find the coſt of all the Ells of the worſer ſort, and ſay, If $a+\frac{1}{3}$. 1 :: 40 : $\frac{40}{a+\frac{1}{3}}$; Or, $a+m$. 1 :: c : $\frac{c}{a+m}$. whence the ſaid full Coſt is found . . .	$\frac{40}{a+\frac{1}{3}}$	$\frac{c}{a+m}$
4. Find likewiſe the coſt of all the Ells of the better ſort, and ſay, If a . 1 :: 90 : $\frac{90}{a}$; Or, a . 1 :: b : $\frac{b}{a}$. whence the ſaid full Coſt is	$\frac{90}{a}$	$\frac{b}{a}$
5. Then the two ſums of Money found out in the third and fourth ſteps being added together will give the full coſt of both ſorts of Cloth, to wit, ...	$\frac{130a+30}{aa+\frac{1}{3}a}$	$\frac{ca+ba+bm}{aa+ma}$

6. Which total Coſt expreſs'd in the laſt ſtep, muſt (according to the Queſtion) be equal to 42 (or d;) whence this Equation ariſes, *viz.*

$$42 = \frac{130a+30}{aa+\frac{1}{3}a}; \qquad \text{Or,} \qquad d = \frac{ca+ba+bm}{aa+ma}.$$

7. Which Equation, after due Reduction (according to the Rules in *Chap.* 12.) will give this that follows, *viz.* $aa - \frac{58}{21}a = \frac{5}{7}$,

$$\text{Or,} \qquad aa - \frac{c+b-dm}{d}a = \frac{mb}{d}.$$

In which laſt Equation, if inſtead of the known Co-efficient $\frac{c+b-dm}{d}$ we take f, that Equation may be expreſs'd thus;

$$aa - fa = \frac{mb}{d}.$$

8. Therefore by reſolving the laſt Equation according to the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the number of Ells of the better ſort of Cloth which were bought for 1 *l.* will be diſcovered, *viz.*

$$a = 3 = \sqrt{:\frac{mb}{d} + \frac{ff}{4}:} + \frac{1}{2}f.$$

Thus it is found that 3 Ells of the better ſort of Cloth did coſt 1 *l.* and conſequently 1 Ell coſt $\frac{1}{3}$ *l.* and 90 Ells 30 *l.* which ſubtracted from 42 *l.* (the full coſt of both ſorts,) leaves 12 *l.* for the full coſt of 40 Ells of the worſt ſort; and conſequently 1 Ell coſt $\frac{3}{10}$ *l*, and at this rate 1 *l.* will buy $3\frac{1}{3}$ Ells, which is more by $\frac{1}{3}$ of an Ell than was bought of the better ſort of Cloth for 1 *l.* Therefore all the Conditions in the Queſtion are ſatisfied.

QUEST. 22.

A Merchant having Spices, to wit, 80 ℔ weight (or b) of Mace, and 100 ℔ weight (or c) of Cloves, ſells both Quantities for 65 (or d) Pounds in Money; whereby it happened that he ſold a quantity of Mace for 10 *l.* (or m,) and the like quantity of Cloves with 60 ℔ weight (or n) more of Cloves for 20 *l.* (or r.) The Queſtion is, to find how many ℔ weight of Mace he ſold for 10 *l.*

RESOLUTION.

1. Let the number of ℔ weight of Mace that the Merchant ſold for 10 *l.* be repreſented by	a	a
2. To which number adding 60, the ſum will give the number of ℔ weight of Cloves that he ſold for 20 *l.* to wit, ...	$a+60$	$a+n$

3. Then

3. Then find how much Money 80 ℔ weight of Mace was sold for, and say, If $a \,.\, 10 :: 80 \,.\, \frac{800}{a}$; Or, $a \,.\, m :: b \,.\, \frac{mb}{a}$. whence the Money for which the said 80 ℔ of Mace was sold is	$\frac{800}{a}$	$\frac{mb}{a}$
4. Find likewise how much Money 100 ℔ weight of Cloves was sold for, and say, If $a+60 \,.\, 20 :: 100 \,.\, \frac{2000}{a+60}$; Or, $a+n \,.\, r :: c \,.\, \frac{rc}{a+n}$. whence the Money for which the said 100 ℔ of Cloves was sold is	$\frac{2000}{a+60}$	$\frac{rc}{a+n}$
5. The sum of both the said sums of Money found out in the third and fourth steps is	$\frac{2800a+48000}{aa+60a}$	$\frac{mba+mbn+rca}{aa+na}$

6. Which Sum in the last step must (according to the Question) be equal to 65 *l.* (or d,) hence this Equation arises, *viz.*

$$65 = \frac{2800a+48000}{aa+60a}; \quad \text{Or,} \quad d = \frac{mba+mbn+rca}{aa+na}.$$

7. Which Equation, after due Reduction (according to *Sect.* 2. 3, 5. *Chap.* 12.) will give this following Equation, *viz.* $aa+\frac{220}{13}a = \frac{9600}{13}$,

$$\text{Or,} \quad aa+\frac{dn-mb-rc}{d}a = \frac{mbn}{d}.$$

In which last Equation if we take f instead of the known Co-efficient $\frac{dn-mb-rc}{d}$, and g instead of the known number $\frac{mbn}{d}$, that Equation may be express'd thus,

$$aa + fa = g.$$

8. Therefore by resolving the last Equation according to the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the number of ℔ weight of Mace that was sold for 10 *l.* will be made known, *viz.*

$$a = 20 = \sqrt{:g+\tfrac{1}{4}ff:} - \tfrac{1}{2}f.$$

Thus it is found that 20 ℔ weight of Mace was sold for 10 *l.* and consequently 80 ℔ weight for 40 *l.*

Moreover, adding 60 to 20 (before found,) the Sum 80 is the number of ℔ weight of Cloves that was sold for 20 *l.* and consequently 100 ℔ of Cloves was sold for 25 *l.* which added to 40 *l.* (the price of 80 ℔ of Mace,) makes 65 *l.* the prescribed sum of Money for both Quantities of Spices sold.

QUEST. 23.

Two Merchants entred into Partnership; the first brought in a certain sum of Pounds which continued in Company 12 (or b) Months, and the second put in 30 *l.* (or c) for 17 (or d) Months; they gained together $18\frac{1}{2}$ *l.* (or m) whereof the first Merchant had 26 *l.* (or n) for his Principal and Gain. It is required to find how many Pounds the first Merchant brought into the common Stock?

RESOLUTION.

1. For the first Merchant's Stock put . . .	a	a
2. Which Stock being multiplied by the time it continued in Company, produces . . .	$12a$	ba
3. The second Merchant's Stock being multiplied by the time it remained in Company, produces	510	cd

4. Then

4. Then proceeding with those two Products according to the Rule of Fellowship with Time, find the gain of the first Merchant, and say,

$$\text{If} \quad 12a+510 \;.\; 18\tfrac{3}{4} \;::\; 12a \;.\; \frac{225a}{12a+510};$$

$$\text{Or,} \quad ba+cd \;.\; m \;::\; ba \;.\; \frac{mba}{ba+cd}.$$

Whence the gain of the first Merchant is found $\frac{225a}{12a+510}$ or $\frac{mba}{ba+cd}$.

5. Which gain added to the first Merchants Stock a, gives for the Sum of his Stock and gain,

$$\frac{12aa+735a}{12a+510}; \qquad \text{Or,} \quad \frac{baa+cda+mba}{ba+cd}.$$

6. Which Sum must be equal to the 26 *l.* (or n) given in the Question, whence this Equation arises, *viz.*

$$\frac{12aa+735a}{12a+510}=26; \qquad \text{Or,} \quad \frac{baa+cda+mba}{ba+cd}=n.$$

7. Then by reducing that Equation according to the Rules in *Chap.* 12. there will arise,

$$aa+35\tfrac{1}{4}a=1105,$$

$$\text{Or,} \quad aa+\frac{cd+mb-nb}{b}a=\frac{ncd}{b}.$$

8. Which last Equation being resolved by the Canon in *Sect.* 6. of the 15 *Chap.* the value of a, to wit, the first Merchant's Stock will be found 20 Pounds, *viz.* If instead of the known Co-efficient $\frac{cd+mb-nb}{b}$ we take f, and g instead of the given number $\frac{ncd}{b}$; Then by the said Canon,

$$a=20=\sqrt{:g+\tfrac{1}{4}ff:}-\tfrac{1}{2}f.$$

Whence the first Merchants Stock is found 20 *l.* The Proof may be made by the Rule of Fellowship with Time, in manner following.

$$20\times12=240$$
$$30\times17=510$$
$$750 \;.\; 18\tfrac{3}{4} \;::\; \begin{cases}240 \;:\; 6\\ 510 \;.\; 12\tfrac{3}{4}.\end{cases}$$

QUEST. 24.

Two Merchants entred into Partnership, the first put in a certain number of Pounds for 3 (or b) Months; the second put in 50 *l.* (or c) more than the first for 5 (or d) Months: They gained together 140 *l.* (or m,) whereof the first Merchant had such part, that if 60 *l.* (or n) be added to it, the Sum will be equal to the Stock wherewith he entred Partnership: What was the Stock and gain of each Merchant?

RESOLUTION.

1. For the Stock of the first Merchant put . .	a	a
2. To which adding 50 *l.* (or c,) the Sum will give the second Merchant's Stock, to wit,	$a+50$	$a+c$
3. Then multiplying the first Merchant's Stock by the time it remained in Company, the Product is	$3a$	ba
4. Likewise by multiplying the second Merchant's Stock by the time it continued in Company, the Product is	$5a+250$	$da+dc$

5. Then proceeding with those two Products according to the Rule of Fellowship with Time, find the first Merchant's Gain, and say,

$$\text{If} \quad 8a+250 \;.\; 140 \;::\; 3a \;.\; \frac{420a}{8a+250};$$

$$\text{Or,} \quad ba+da+dc \;.\; m \;::\; ba \;.\; \frac{mba}{ba+da+dc}.$$

Whence the gain of the first Merchant is found $\frac{420a}{8a+250}$; Or, $\frac{mba}{ba+da+dc}$.

6. To

6. To which gain add 60 (or n,) so the Sum will be

$$\frac{900a+15000}{8a+250}; \quad \text{Or,} \quad \frac{mba+nba+nda+ndc}{ba+da+dc}.$$

7. But, according to the Question, the Sum in the last step must be equal to (a) the first Merchant's Stock, whence this Equation arises;

$$\frac{900a+15000}{8a+250} = a = \frac{mba+nba+nda+ndc}{ba+da+dc}.$$

8. Which Equation, after due Reduction according to the Rules in *Chap.* 12. will produce this following Equation, *viz.* $aa - 81\frac{1}{4}a = 1875$,

$$\text{Or,} \quad aa - \frac{mb+nb+nd-dc}{b+d}a = \frac{ndc}{b+d}.$$

9. In which Equation the value of a, to wit, the first Merchant's Stock, will be discovered by the Canon in *Sect.* 8, *Chap.* 15. viz. $a = 100\,l.$ And consequently from the premises the second Merchant's Stock was 150 *l.* the gain of the first 40 *l.* and the gain of the second 100 *l.* All which will be evident by the following Proof wrought by the Rule of Fellowship with Time.

$$100 \times 3 = 300$$
$$150 \times 5 = 750$$
$$1050 \;.\; 140 \;::\; \left\{ \begin{array}{l} 300 \;.\; 40 \\ 750 \;.\; 100. \end{array} \right.$$

QUEST. 25.

A Citizen having bought a House for a certain sum of Pounds, sells it for 64 *l.* (or d,) and finds that his loss in 100 Pounds (or c) was equal to a fourth part (or m) of the Money that he paid for the House. What number of Pounds did the Citizen pay for the House?

RESOLUTION.

1. For the number of pounds which the Citizen paid for the House, put . . . . a | a
2. Then will the whole loss by sale of the House be $a-64$ | $a-d$
3. Find how much was lost by 100 *l.* (or c,) and say,

$$\text{If} \quad a \;.\; a-64 \;::\; 100 \;.\; \frac{100a-6400}{a};$$

$$\text{Or,} \quad a \;.\; a-d \;::\; c \;.\; \frac{ca-cd}{a}.$$

Whence the loss *per Cent.* is found $\frac{100a-6400}{a}$; Or, $\frac{ca-cd}{a}$.

4. But according to the Question the loss *per Cent.* was equal to $\frac{1}{4}$ part of the Money which the Citizen paid for the House, therefore from the first and third steps this Equation arises, *viz.*

$$\frac{100a-6400}{a} = \frac{a}{4}; \quad \text{Or,} \quad \frac{ca-cd}{a} = ma.$$

5. Which Equation, after due Reduction according to the Rules in *Chap.* 12. will give

$$400a - aa = 25600; \quad \text{Or,} \quad \frac{c}{m}a - aa = \frac{cd}{m}.$$

6. Therefore by resolving the said Equation according to the Canon in *Sect.* 10. *Chap.* 15. both the values of a will be discovered, either of which will solve the Question; which values or numbers are these following, *viz.*

$$a = \left\{ \begin{array}{l} 320 = \frac{c}{2m} + \sqrt{:}\frac{cc-4cdm}{4mm}: \\ 80 = \frac{c}{2m} - \sqrt{:}\frac{cc-4cdm}{4mm}: \end{array} \right.$$

I say either of the numbers 320 and 80 will satisfie the Conditions in the Question, as will be evident by the Proof: For if a House cost 320 *l.* and be sold for 64 *l.* the loss is 256 *l.* and 100 *l.* at that rate of loss will lose 80, which is $\frac{1}{4}$ part of the first Cost 320 *l.*

Again,

Again, if a House cost 80 *l.* and be sold for 64 *l.* the loss is 16 *l.* and 100 *l.* at this rate of loss will lose 20 *l.* which is likewise $\frac{1}{5}$ part of the first Cost 80 *l.*

QUEST. 26.

Two Merchants entred into Partnership; the Sum of their Stocks was 165 (or b) Pounds: the first Merchant's Stock continued in Company 12 (or c) Months, and the Stock of the second 8 (or d) Months: they gained a certain sum of Pounds, which together with their Stocks they divided between themselves in such manner, that the first Merchant received 67 (or f) Pounds for his Stock and Gain, and the second 126 (or g) Pounds for his Stock and Gain. It is desired to find out each Merchant's Stock and Gain.

RESOLUTION.

1. For the first Merchant's Stock put . . .	a	a
2. Then, by subtracting that Stock (a) from 165 (or b,) there remains the second Merchant's Stock; to wit, ...	$165-a$	$b-a$
3. And if you subtract (a) the first Merchant's Stock from 67 (or f) the sum of his Stock and Gain, there will remain his Gain only; to wit, ...	$67-a$	$f-a$
4. Likewise, if you subtract the second Merchant's Stock (in the second step) from 126 (or g) the Sum of his Stock and Gain, there will remain his Gain only; to wit,	$a-39$	$a+g-b$

5. Now according to the Nature of the Rule of Fellowship with Time, the Gain of the first Merchant $67-a$ must be in such proportion to $a-39$ the Gain of the second, as the Product of the first Merchant's Stock a multiplied by its time 12 Months, is to the Product of the second Merchant's Stock $165-a$ multiplied by its time 8 Months: Hence this Analogy, *viz.*

$$67-a \;.\; a-39 \;::\; 12a \;.\; 1320-8a,$$

That is, $$f-a \;.\; a+g-b \;::\; ca \;.\; db-da.$$

6. Which Analogy, by comparing the Product made by the Multiplication of the Means one into the other, to the Product of the Extremes, produces this Equation, *viz.*

$$12aa-468a = 8aa-1856a+88440,$$

That is, $$caa+cga-cba = daa-dba-dfa+dbf.$$

7. From which Equation after due Reduction this arises, *viz.*

$$aa+347a = 22110,$$

That is, $$aa + \frac{db+df+cg-cb}{c-d}a = \frac{dbf}{c-d}.$$

8. Wherefore by resolving the last Equation according to the Canon in *Sect.* 6. *Chap.* 15. the value of a, that is, the number of Pounds expressing the first Merchant's Stock will be found 55; which subtracted from 165 *l.* the sum of both their Stocks, leaves 110 *l.* for the second Merchant's Stock: then each of their Stocks being subtracted from their respective Stock and Gain, *viz.* 55 *l.* from 67 *l.* and 110 *l.* from 126 *l.* there remains 12 *l.* for the Gain of the first Merchant, and 16 *l.* for the Gain of the second; whence the total Gain was 28 *l.* Which numbers will solve the Question, as may easily be proved by the Rule of Fellowship with Time; thus,

$$55 \times 12 = 660$$
$$110 \times 8 = 880$$

$$1540 \;.\; 28 \;::\; \begin{cases} 660 \;.\; 12 \\ 880 \;.\; 16. \end{cases}$$

QUEST. 27.

A certain Foot-man A departs from *London* towards *Lincoln*, and at the same time another Foot-man B departs from *Lincoln* toward *London*, each keeping the same Road. When they met, A says to B, I find that I have travelled 20 (or c) miles more than you, and have gone as many miles in $6\frac{1}{4}$ (or d) days, as you have gone miles

Miles in all hitherto: 'Tis true saith *B*, I am not so good a Foot man as you, but I find that at the end of 15 (or f) days hence, I shall be at *London*, if I travel as many Miles in every one of those 15 days, as I have done in every day hitherto. The Question is, to find how many Miles those two Cities are distant one from another, and how many Miles each Foot-man had travelled when they met one another.

RESOLUTION.

1. For the desired distance between the two Cities put	a	a
2. Then forasmuch as the number of Miles each Foot-man had travelled when they met, being added together make the Sum (a,) and the difference between those two numbers was 20 (or c,) for *A* had travelled 20 Miles more than *B*: Therefore (by the Theorem at the end of *Quest.* 1. *Chap.* 14.) the number of Miles which *A* had travelled was	$\frac{1}{2}a+10$	$\frac{1}{2}a+\frac{1}{2}c$
3. And (by the same Theorem) the number of Miles which *B* had travelled was . .	$\frac{1}{2}a-10$	$\frac{1}{2}a-\frac{1}{2}c$
4. Then say, If in $6\frac{2}{3}$ days *A* had travelled $\frac{1}{2}a-10$ Miles, how many Miles did he travel in one day? so by the Rule of Three, you will find	$\frac{\frac{1}{2}a-10}{6\frac{2}{3}}$	$\frac{\frac{1}{2}a-\frac{1}{2}c}{d}$
5. Say again, If in 15 days *B* must travel $\frac{1}{2}a+10$ Miles, (that is, all the Miles which *A* had travelled,) how many Miles must *B* travel in one day? so you will find . .	$\frac{\frac{1}{2}a+10}{15}$	$\frac{\frac{1}{2}a+\frac{1}{2}c}{f}$
6. Say again, If $\frac{\frac{1}{2}a+10}{15}$ Miles were travelled by *B* in one day, in how many days did he travel $\frac{1}{2}a-10$ Miles? so you will find	$\frac{7\frac{1}{2}a-150}{\frac{1}{2}a+10}$	$\frac{\frac{1}{2}fa-\frac{1}{2}fc}{\frac{1}{2}a+\frac{1}{2}c}$
7. Say again, If $\frac{\frac{1}{2}a-10}{6\frac{2}{3}}$ Miles were travelled by *A* in one day, in how many days did he travel $\frac{1}{2}a+10$ Miles? so you will find	$\frac{3\frac{1}{3}a+66\frac{2}{3}}{\frac{1}{2}a-10}$	$\frac{\frac{1}{2}da+\frac{1}{2}dc}{\frac{1}{2}a-\frac{1}{2}c}$.

8. But the numbers of days found out in the two last steps must be equal to one another; for when *A* and *B* met, each had travelled the same number of days, because they began their Journey at one and the same time: Hence this Equation arises, *viz.*

$$\frac{3\frac{1}{3}a+66\frac{2}{3}}{\frac{1}{2}a-10}=\frac{7\frac{1}{2}a-150}{\frac{1}{2}a+10};$$

That is, $$\frac{\frac{1}{2}da+\frac{1}{2}dc}{\frac{1}{2}a-\frac{1}{2}c}=\frac{\frac{1}{2}fa-\frac{1}{2}fc}{\frac{1}{2}a+\frac{1}{2}c}.$$

9. In which Equation, if you double both the Numerators and Denominators, and then reduce the Equation resulting, to a common Denominator, and cast away the common Denominator, the new Numerators being compared to one another will give this following Equation, *viz.*

$$\frac{20}{3}aa+\frac{800}{3}a+\frac{8000}{3}=15aa-600a+6000;$$

That is, $$daa+2dca+dcc=faa-2fca+fcc.$$

10. Which last Equation duly reduced gives this that follows, *viz.*

$$104a-aa=400,$$

That is, $$\frac{2dc+2fc}{f-d}a-aa=cc.$$

11. Wherefore by resolving the Equation in the last step according to the Canon in *Sect.* 10. *Chap.* 15. the two values of a will be found these, *viz.*

$$a=100=\frac{dc+fc+\sqrt{4dfcc}}{f-d}$$

$$a=4=\frac{dc+fc-\sqrt{4dfcc}}{f-d}.$$

12. But

12. But altho by either of those values of a, to wit, 100 and 4, the Equation in the tenth step may be expounded, yet the greater value only is the desired number of Miles expressing the distance between the two Cities; for 'tis evident by the Question, that 20 is but part of the number of Miles between the two Cities, and therefore 4 the lesser value of a is much less than the said Distance: Wherefore 100 the greater value of a is the desired number of Miles between the two Cities. And consequently the second, third, fourth and fifth steps being resolved into numbers, will shew, that when the two Foot-men A and B met one another, A had travelled 60 Miles, and B 40 Miles: Also, A travelled 6 Miles, and B 4 Miles every day; as will easily appear by the Proof.

13. But the numbers in this Question must not be given at random, for the Denominator of the Fraction $\frac{2dc+2fc}{f-d}$ in the Equation in the tenth step shews that the number d must be less than the number f, otherwise the Question is impossible; as may easily be infer'd from the literal Equation in the ninth step: for if in that Equation d be supposed greater than f, then consequently dcc is greater than fcc, and after due transposition this Equation will arise, *viz.* $dcc-fcc=faa-daa-2dca-2fca$; where if d be greater than f, then the first part of the Equation will be a real Quantity, that is, greater than nothing, and the latter part less than nothing; but to affirm that a Quantity greater than nothing is equal to a Quantity less than nothing is absurd; the like absurdity will follow if we suppose $d=f$.

14. Having shew'd that d must necessarily be less than f, I shall prove that the lesser value of a, as it is express'd by Letters in the eleventh step can never be equal to the whole distance between the two Cities. For if we should suppose the lesser value to be equal to the said distance, it must necessarily be greater than c, which the Question shews to be but part of the said distance: But from that Supposition, it will follow by undeniable consequence, that d is greater than f, which is contrary to what has been before proved. Now to prove the said consequence;

15. Suppose the lesser value of a to exceed c, viz.	$\frac{dc+fc-\sqrt{4dfcc}}{f-a} \sqsubset c$
16. Then by multiplying each part by $f-d$, it follows that	$dc+fc-\sqrt{4dfcc} \sqsubset fc-dc$
17. And by adding $\sqrt{4dfcc}$ to each part, . . .	$dc+fc \sqsubset fc-dc+\sqrt{4dfcc}$
18. And by adding dc to each part, ...	$2dc+fc \sqsubset fc+\sqrt{4dfcc}$
19. And by subtracting fc from each part, . . .	$2dc \sqsubset \sqrt{4dfcc}$
20. And by squaring each part, ...	$4ddcc \sqsubset 4dfcc$
21. And by dividing each part by $4dcc$, . . .	$d \sqsubset f$

22. Thus from a Supposition that the lesser value of a in the eleventh step is greater than c, it follows by just consequence that d is greater than f, which is impossible, for it has before been proved that d must be less than f. And because the Series of Inferences deduced from the said Supposition ends in an impossibility, therefore that which was supposed cannot be true; *viz.* The lesser value of a is not greater than c, and consequently it cannot be equal to the distance between the two Cities. Which was to be proved.

23. Again, by supposing d to be less than f, as it ought to be, to the end the Question may be possible, we may prove the lesser value of a to be lesser than c, by returning backwards from the 21 step to the 15, in this manner, *viz.*

24. Suppose	$d \sqsupset f$
25. Then by multiplying each part by $4dcc$, . .	$4ddcc \sqsupset 4dfcc$
26. And by extracting the square Root out of each part, ...	$2dc \sqsupset \sqrt{4dfcc}$
27. And by adding fc to each part, . . .	$2dc+fc \sqsupset fc+\sqrt{4dfcc}$
28. And by subtracting dc from each part, . .	$dc+fc \sqsupset fc-dc+\sqrt{4dfcc}$
29. And by subtracting $\sqrt{4dfcc}$ from each part,	$dc+fc-\sqrt{4dfcc} \sqsupset fc-dc$
30. Wherefore by dividing each part by $f-d$, it is manifest that the lesser value of a is less than c, viz.	$\frac{dc+fc-\sqrt{4dfcc}}{f-d} \sqsupset c$

Which was to be proved. Wherefore the lesser value of a cannot possibly be equal to the distance between the two Cities, for the said distance must necessarily be greater than part of it self.

31. But

31. But it may be objected, That altho f be greater than d, yet how does it appear that $dc+fc$ is greater than $\sqrt{4dfcc}$, to the end that this may be subtracted from that, as the lesser value of a requires, to make it self a possible Root of the Equation in the tenth step? In answer to this Objection, I shall in the next place prove that $dc+fc$ is greater than $\sqrt{4dfcc}$.

32. Forasmuch as these Quantities are Proportionals, (for the Product of the Extremes is equal to the Product of the means,) $dd \,.\, df :: df \,.\, ff$

33. Therefore (*per* 25 Prop. 5. Elem. *Euclid.*) $dd+ff \sqsubset 2df$

34. And by multiplying all in the last step by cc, $ddcc+ffcc \sqsubset 2dfcc$

35. And by adding $2dfcc$ to each part, . . . $ddcc+ffcc+2dfcc \sqsubset 4dfcc$

36. Wherefore by extracting the square Root out of each part in the last step, . . . $dc+fc \sqsubset \sqrt{4dfcc}$.

Which was to be proved.

CHAP. XVII.

Concerning Arithmetical PROGRESSION.

I. ARithmetical Progression is, when many numbers (or other Quantities of one and the same kind) proceed by a common difference or excess; as in these, 2, 4, 6, 8, 10, 12, 14, &c. here 2 is the common difference betwixt 2 and 4, 4 and 6, 6 and 8, 8 and 10, &c. So 1, 2, 3, 4, 5, 6, &c. are in Arithmetical Progession, 1 being the common difference: Likewise 3, 7, 11, 15, 19, &c. or 19, 15, 11, 7, and 3, where 4 is the common difference.

II. Arithmetical Progression is either continued, as in the Examples above express'd, where every two terms that stand next to one another, have one common difference; or else discontinued or interrupted, as in these numbers, 3, 5 : 9, 11, where 5 exceeds 3 by 2, and so does 11 exceed 9; but 9 does not exceed 5 by 2, for the excess of 9 above 5 is 4. In like manner 18, 14 : 21, 17, are in Arithmetical Progression discontinued.

III. For the better Manifestation of the following Propositions concerning Arithmetical Progression, let there be a rank of numbers in a continued Arithmetical Progression, as, 3, 7, 11, 15, 19, 23, 27, &c. which numbers may be represented by a, b, c, d, e, f, g, &c. Also, let 105 the sum of all the Terms of the Progression be represented by Z; the common excess or difference 4 by X; and the number of Terms 7 by T: all which are here orderly express'd underneath.

$$\text{Quantities in Arithmetical Progression continued:}\left\{\begin{array}{l} 3 = a = a \\ 7 = b = a + X. \\ 11 = c = a + 2X. \\ 15 = d = a + 3X. \\ 19 = e = a + 4X. \\ 23 = f = a + 5X. \\ 27 = g = a + 6X. \end{array}\right.$$

$$\begin{array}{ll} \text{The Sum of all the Terms is . . .} & 105 = Z = Z \\ \text{The common difference is .} & 4 = X = X \\ \text{The number of Terms is . .} & 7 = T = T. \end{array}$$

IV. Whence it is manifest, that if a be put for the first and least Term of an Arithmetical Progression continued, and X for the common difference, then (according to the Definition in *Sect.* 1.) the second Term shall be $a+X$, the third $a+2X$, the fourth $a+3X$, the fifth $a+4X$, &c. Moreover, according to the Suppositions in *Sect.* 3. $a=a$. $b=a+X$. $c=a+2X$. $d=a+3X$. $e=a+4X$, &c.

V. Therefore it follows, that the last and greatest Term of every Arithmetical Progression continued is compos'd of the first (to wit, the least) term, and of the Product of the common difference multiplied by a number less by 1 (or Unity) than the number

number of Terms; as g, or $a+6X$ is compos'd of the first Term a and the Product of X multiplied by 6, which is less by 1 than 7 the number of Terms.

VI. Therefore the first and last Terms, as also the number of Terms being severally given, the common difference shall be also given; for if the first, (to wit, the smallest) Term be subtracted from the last, and the Remainder be divided by a number less by 1 (or Unity) than the number of Terms, the Quotient is the common difference, *viz.* $\frac{g-a}{T-1} = X$.

VII. It is also manifest from *Sect.* 3. That if the first (to wit, the least) Term be equal to the common difference, then the last Term is equal to the Product of the common difference (or first Term) multiplied by the number of Terms, *viz.* If $a = X$, then $g = X+6X = 7X$.

VIII. Therefore in an Arithmetical Progression continued whose first or least Term is equal to the common difference, if the last Term and the number of Terms be severally given, the first Term (or the common difference) shall also be given: For if the last Term be divided by the number of Terms, the Quotient is the first Term or common difference; as, if $a = X$, then $g = X+6X = 7X$; therefore $\frac{7X}{7} = X = a$.

IX. It is also manifest from *Sect.* 7. That when the common difference divides any Term just without any Remainder, then the common difference is the same with the least Term in that Progression, and the Quotient is the number of Terms; but if any number remain after the Division is finished, then that Remainder is the least Term, and the Quotient increased with 1 (or Unity) gives the number of Terms (*per* Sect. 4, & 5.) Therefore if any term greater than the least be given, as also the common difference, the least term, as also the number of terms in that Progression shall also be given; as if 27 be some term greater than the least, and 3 the common difference, by dividing 27 by 3, the Quotient 9 is the number of terms, and the least term is equal to the common difference 3; as in this Progression, 3,6,9,12,15,18,21,24,27.

But if 27 be given as before, and 4 be prescribed for the common difference, then 27 divided by 4 gives 6 in the Quotient, and there remains 3 for the least term, and 7 (to wit $6+1$) is the number of terms; as in this Progression, 3, 7, 11, 15, 19, 23, 27.

X. If three Numbers, suppose a,b,c, be in a continued Arithmetical Progression, *viz.* If the Excess of c above b be equal to the Excess of b above a, the Sum of the Extremes, that is, of the first and last terms shall be equal to the double of the mean or middle term; *viz.* $a+c = 2b$. For,

1. By Supposition, ... $c-b = b-a$,
2. Therefore by adding b to each part, it gives . . . . $c = 2b-a$,
3. And by adding a to each part of the last Equation . . . $a+c = 2b$.

Which was to be proved.

XI. If four Numbers; suppose a, b, c, d, be in Arithmetical Progression whether continued or interrupted, *viz.* If the excess of b above a be equal to the excess of d above c, the Sum of the Extremes shall be equal to the Sum of the Means, *viz.* $a+d = b+c$. For,

1. By Supposition, ... $d-c = b-a$,
2. Therefore by equal addition of a, ... $a+d-c = b$,
3. Therefore by equal addition of c, ... $a+d = b+c$.

Which was to be proved.

XII. If there be as many numbers as you please in a continued Arithmetical Progression, the Sum of the Extremes is equal to the Sum of any two Means equally distant from the Extremes, and also to the double of the Mean when the number of Terms is odd.

Let a,b,c,d,e,f, be in Arithmetical Progression continued, and increasing from a; I say the Sum of the Extremes a and f is equal to the Sum of any two terms equally distant from the extremes, that is, to the Sum of b and e, and to the Sum of c and d. For,

1. By Supposition, in regard of the continued Progression, . $f-e = b-a$,
2. Therefore by equal addition of e and a to each part, . . $a+f = b+e$,
3. Again, by supposition . . . . $c-b = e-d$,

4. Therefore by equal addition of d and b, to each part $c+d = b+e$,
5. Therefore from the second and fourth steps (*per* 1. *Axiom.* 1. *Elem. Euclid.*) $a+f = c+d = b+e$.

Which was to be proved.

And if more numbers were propos'd the Demonstration would not be otherwise; therefore the first part of the Theorem is manifest.

But if the number of Terms be odd as in this continued Progression, a, b, c, d, e, f, g, then the Sum of the Extremes a and g is equal to the double of the middle Term d, *viz.* $a+g = 2d$; which I prove thus:

1. By supposition, in regard of the continued Progression, ... $d-c = e-d$,
2. And consequently by equal addition of c and d, . . $2d = c+e$,
3. But by what has been proved concerning the first part of the Theorem in this twelfth *Sect.* . . . $a+g = c+e$,
4. Therefore from the two last steps, (*per Axiom.* 1. *Elem.* 1. *Euclid.*) $a+g = 2d$.

Which was to be demonstrated. Therefore the Theorem is every way manifest.

XIII. In every Arithmetical Progression continued, the Sum of the Extremes multiplied by the number of terms produces the double of the Sum of all the terms.

The number of terms is either even or odd: First, let there be an even number of terms, *viz.* suppose these six numbers a, b, c, d, e, f, to be in Arithmetical Progression continued;

I say, ... $6a+6f = \begin{cases} 2a+2b+2c+2d, \\ +2e+2f. \end{cases}$

DEMONSTRATION.

1. It is evident that . . . . $2a+2f = 2a+2f$,
2. And by *Sect.* 12. . . . . $2a+2f = 2b+2e$,
3. Likewise, by the same *Sect.* . . . . $2a+2f = 2c+2d$,
4. Therefore by adding the three last Equation together, $6a+6f = \begin{cases} 2a+2b+2c, \\ +2d+2e+2f. \end{cases}$

Which was to be demonstrated. And so of others when the number of terms is even.

Secondly, let there be an Arithmetical Progression consisting of an odd number of terms, suppose these five, a, b, c, d, e.

I say, ... $5a+5e = 2a+2b+2c+2d+2e$.

DEMONSTRATION.

1. It is manifest that . . . . . . $2a+2e = 2a+2e$,
2. And by *Sect.* 12. . . . . $2a+2e = 2b+2d$,
3. Likewise by *Sect.* 12. . . . . $a+e = 2c$,
4. Therefore by adding the three last Equations together, ... $5a+5e = 2a+2b+2c+2d+2e$.

And so of others when the number of terms is odd.

XIV. Therefore from the last *Sect.* the first and last terms, as also the number of terms in an Arithmetical Progression continued being given, the sum of all the terms shall be also given: For if the sum of the first and last terms be multiplied by the number of terms the Product is the double sum of all the terms, and consequently the half of that Product is the sum it self. For example, If a, b, c, d, e, f, g, be in Arithmetical Progression continued, and T be put for the number of terms, also Z for their sum (as before;) Then $Ta+Tg = 2Z$, and consequently $\frac{1}{2}Ta+\frac{1}{2}Tg = Z$.

XV. Mr. *William Oughtred* in *Prob.* 4. *Chap.* 19. of his incomparable *Clavis Mathemat.* has very elegantly handled 20 Propositions about Arithmetical Progression continued, which (for the more ample Illustration of the preceding Rules in this Book,) I shall explain in this *Section*, using his own Symbols, which are these, *viz.*

$$\left.\begin{array}{l} \alpha \\ \omega \\ T \\ X \\ Z \end{array}\right\} \text{Stands for} \left\{\begin{array}{l} \text{The least (or first) term.} \\ \text{The greatest (or last) term.} \\ \text{The number of Terms.} \\ \text{The common difference of the Terms.} \\ \text{The sum of all the terms.} \end{array}\right.$$

Any

Any three of these five things being given, the other two shall be also given, by the respective Canons of the following 20 Propositions, which Mr. *Oughtred* states thus:

Given,	Sought,	By Propos.
α, ω, T	Z and X	1 and 2
α, ω, X	T and Z	3 and 4
α, ω, Z	T and X	5 and 6
α, T, X	ω and Z	7 and 8
α, T, Z	ω and X	9 and 10
α, X, Z	ω and T	11 and 12
ω, T, X	α and Z	13 and 14
ω, T, Z	α and X	15 and 16
ω, X, Z	α and T	17 and 18
T, X, Z	α and ω	19 and 20

PROP. I.

1. . . . . $\left\{ \begin{array}{l} \alpha, \omega, \text{T are given severally;} \\ \quad \text{Z is sought.} \end{array} \right.$

RESOLUTION.

2. By *Sect.* 14. of this *Chap.* . . . . $T\omega + T\alpha = 2Z$.

Which Equation, if express'd by words, gives this

CANON.

Multiply the Sum of the first and last Terms by the number of Terms, the Product shall be the double of the Sum of all the Terms, and consequently the half of that Product is the required Sum of all the Terms.

Which Canon may be exemplified by the following (or any other) rank of numbers in Arithmetical Progression continued, *viz.*

3, 7, 11, 15, 19, 23, 27.

PROP. II.

1. . . . . $\left\{ \begin{array}{l} \alpha, \omega, \text{T are given severally;} \\ \quad \text{X is sought.} \end{array} \right.$

RESOLUTION.

2. By *Sect.* 6. of this seventeenth *Chap.* . . . . $\frac{\omega - \alpha}{T - 1} = X$.

Which Equation gives this following

CANON.

Divide the excess of the greatest (or last) Term above the least, by the number of Terms lessened by 1 (or Unity,) and the Quotient is the common difference required.

Which Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progression continued, *viz.*

3, 7, 11, 15, 19, 23, 27.

From the Equation in the second step of *Prop.* 1. and the Equation in the second step of *Prop.* 2. the Canons of all the following 18 Propositions are deduced.

PROP. III.

1. . . . . $\left\{ \begin{array}{l} \alpha, \omega, \text{X are given severally;} \\ \quad \text{T is sought.} \end{array} \right.$

RESOLUTION.

2. The Letters put for the things given and sought, without any other Letter, are contained in the Equation in the second step of *Prop.* 2. therefore the work here is only to set T alone in that Equation; which may be done thus, *viz.*

 3. By

3. By the Canon of *Prop.* 2. . . . . $\frac{\omega-\alpha}{T-1}=X,$

4. Therefore by multiplying each part of that Equation by $T-1$, this arises, *viz.* . . . . $\omega-\alpha=TX-X,$

5. And by addition of X to each part of the last Equation, this arises ; . . . . $\omega-\alpha+X=TX,$

6. Therefore each part of the last Equation being divided by X, the number T will be made known, *viz.* . . $\frac{\omega-\alpha}{X}+1=T.$

The last Equation gives this following

CANON.

From the last (to wit, the greatest) Term subtract the first, and divide the Remainder by the common difference ; then to the Quotient add 1 (or Unity) so shall the Sum be the required number of Terms.

This Canon may be exemplified by the following (or any other) Rank of Numbers in Arithmetical Progression continued :

3, 7, 11, 15, 19, 23, 27.

PROP. 4.

1. . . . . $\left\{ \begin{array}{l} \alpha,\ \omega,\ X \text{ are given severally;} \\ Z \text{ is required.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega+T\alpha=2Z,$

3. And by the Canon of *Prop.* 3. . . . . $\frac{\omega-\alpha}{X}+1=T,$

4. Now if instead of T in the first part of the Equation in the second step, you multiply into $\omega+\alpha$ that which in the last Equation is found equal to T, the former Equation will be converted into this, *viz.*

$$\frac{\omega\omega-\alpha\alpha}{X}+\omega+\alpha=2Z.$$

Which in words is this following

CANON.

From the Square of the greatest (or last) Term subtract the Square of the least (or first,) then dividing the Remainder by the common difference, and to the Quotient adding the Sum of the first and last Terms, the half of the Sum of this Addition shall be the required Sum of all the Terms.

The Canon may be exemplified by the following (or any other) Rank of Numbers in Arithmetical Progression continued :

3, 7, 11, 15, 19, 23, 27.

PROP. V.

1. . . . . $\left\{ \begin{array}{l} \alpha,\ \omega,\ Z, \text{ are given severally ;} \\ T \text{ is required.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega+T\alpha=2Z,$

3. Therefore by dividing each part of that Equation by $\omega+\alpha$, this arises, *viz.* . . . . $T=\frac{2Z}{\omega+\alpha},$

Which Equation gives this following

CANON.

Divide the double of the Sum of all the Terms by the Sum of the first and last Terms, the Quotient is the number of Terms sought ; as may be proved by this following (or any other) Rank of numbers in Arithmetical Progression :

3, 7, 11, 15, 19, 23, 27.

PROP.

PROP. VI.

1. . . . . $\left\{ \begin{array}{l} \alpha,\ \omega,\ Z \text{ are given severally;} \\ X \text{ is required.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 4. . . . . $\frac{\omega\omega-\alpha\alpha}{X}+\omega+\alpha=2Z$,
3. Which Equation multiplied by X produces, . . $\omega\omega-\alpha\alpha+\omega X+\alpha X=2ZX$,
4. And by subtracting $\omega X+\alpha X$ from each part of the last Equation, this arises, *viz.* . . . . . $\omega\omega-\alpha\alpha=2ZX-\omega X-\alpha X$,
5. Therefore by dividing each part of the last Equation by the Co-efficients that are drawn into X, you will find, ... $\frac{\omega\omega-\alpha\alpha}{2Z-\omega-\alpha}=X$.

Which last Equation gives this

CANON.

From the Square of the last Term subtract the Square of the first (to wit, the least) Term; divide the Remainder by the excess whereby the double Sum of all the Terms exceeds the Sum of the first and last Terms, so shall the Quotient be the common difference required.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progression:

3, 7, 11, 15, 19, 23, 27.

PROP. VII.

1. . . . . $\left\{ \begin{array}{l} \alpha,\ T,\ X \text{ are given severally;} \\ \omega \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 2. . . . . $\frac{\omega-\alpha}{T-1}=X$,
3. Therefore by multiplying each part of the said Equation by $T-1$, this will be produced, . . $\omega-\alpha=TX-X$,
4. And by adding α to each part of the last Equation this arises, *viz.* . . . . $\omega=TX+\alpha-X$.

Which last Equation gives this

CANON.

To the Product made by the Multiplication of the number of Terms into the common difference, add the first (to wit, the least) Term, and from the Sum subtract the said difference, so shall the Remainder be the last Term sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. VIII.

1. . . . . $\left\{ \begin{array}{l} \alpha,\ T,\ X \text{ are given severally;} \\ Z \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega+T\alpha=2Z$,
3. And by the Canon of *Prop.* 7. . . . . . . $TX+\alpha-X=\omega$.
4. Now to find an Equation that may consist only of the things given and sought in this *Prop.* 8. multiply each part of the Equation in the third step by T, and there will be produced

$$TTX+T\alpha-TX=T\omega.$$

5. Then

5. Then if instead of $T\omega$ in the second step, you take that which in the fourth is found equal to $T\omega$, the Equation in the second step will be reduced to this, to wit,

$$TTX+2T\alpha-TX=2Z,$$

That is, $\overline{TX+2\alpha-X}$ into $T=2Z.$

Which last Equation gives this

CANON.

6. To the Product of the Multiplication of the number of Terms by the common difference, add the double of the first (to wit, the least) Term, and from the Sum of that Addition subtract the common difference; then multiply the Remainder by the number of Terms; so shall the Product be the double Sum of all the Terms, and consequently the half of that Product is the required Sum of all the Terms.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. IX.

1. . . . . . α, T, Z are given severally; ω is sought.

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . . $T\omega+T\alpha=2Z,$
3. Therefore by equal subtraction of $T\alpha$, . . $T\omega=2Z-T\alpha,$
4. Therefore by dividing each part of the last Equation by T, this arises; . . . $\omega=\dfrac{2Z-T\alpha}{T}.$

Which last Equation gives this

CANON.

From the double of the Sum of all the Terms subtract the Product of the Multiplication of the number of Terms by the first (to wit, the least) Term, and divide the Remainder by the number of Terms; so shall the Quotient be the last Term sought.

This Canon may be exemplified by the following (or any other) Rank of Numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. X.

1. . . . . . α, T, Z are given severally; X is sought.

RESOLUTION.

2. By the Canon of *Prop.* 8. . . . . . $TTX+2T\alpha-TX=2Z,$
3. Therefore by equal subtraction of $2T\alpha$ from each part, this will arise; to wit, $TTX-TX=2Z-2T\alpha,$
4. And by dividing each part of the last Equation by $TT-T$, the common difference X will be made known, *viz.* . . $X=\dfrac{2Z-2T\alpha}{TT-T}.$

Which last Equation gives this

CANON.

From the double Sum of all the Terms subtract the double Product made by the Multiplication of the number of Terms by the least Term, and divide the Remainder by the excess of the Square of the Number of Terms above the number of Terms, so shall the Quotient be the common difference sought.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XI.

1. . . . . . α, X, Z are given severally; ω is sought.

RESOLUTION.

2. By the Canon of *Prop.* 4. . . . . . $\dfrac{\omega\omega-\alpha\alpha}{X}+\omega+\alpha=2Z,$

3. There-

3. Therefore by multiplying that Equation by X, this will be produced; to wit, ... $\omega\omega - \alpha\alpha + X\omega + X\alpha = 2ZX$,
4. And by transposition of $-\alpha\alpha$, this arises; . . $\omega\omega + X\omega + X\alpha = 2ZX + \alpha\alpha$,
5. And from the last Equation by transposition of $X\alpha$ this arises; . . . . $\omega\omega + X\omega = 2ZX + \alpha\alpha - X\alpha$,
6. Which last Equation falling under the first of the three Forms in *Sect.* 1. *Chap.* 15 of this Book, the value of ω shall be given by the Canon in *Sect.* 6. of the same *Chap. viz.*

$$\omega = \sqrt{:\tfrac{1}{4}XX + 2ZX + \alpha\alpha - X\alpha :} - \tfrac{1}{2}X.$$

Which Equation gives this

CANON.

From the sum of these three numbers, to wit, the Square of half the common difference; the double Product of the Multiplication of the sum of all the terms by the common difference; and the Square of the first (to wit, the least) term, subtract the Product of the first term multiplied by the common difference, and extract the square Root of the Remainder; then from the said square Root subtract half the common difference, so shall this last Remainder be the last and greatest term sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XII.

1. . . . . $\left\{ \begin{array}{l} \alpha,\ X,\ Z \text{ are given severally;} \\ T \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. The Canon of *Prop.* 8. gives this Equation, . . $XTT + 2\alpha T - XT = 2Z$,
3. Where in regard X is drawn into TT (which is the highest degree of the Quantity sought,) let every term of the Equation be divided by X, whence this Equation will arise; . . . $TT + \dfrac{2\alpha T - XT}{X} = \dfrac{2Z}{X}$.
4. Now it must be discovered from the things given whether 2α exceeds X, or is less, or equal to X. First then suppose $2\alpha \sqsubset X$, and then the last Equation may be express'd thus;

$$TT + \frac{2\alpha - X}{X}T = \frac{2Z}{X}.$$

5. Which Equation falling under the first of the three Forms in *Sect.* 1. *Chap.* 15. the value of T shall be given by the Canon in *Sect.* 6. of the same *Chap. viz.*

$$T = \sqrt{:\frac{\alpha\alpha - \alpha X + \tfrac{1}{4}XX + 2ZX}{XX}:} - \frac{2\alpha - X}{2X}.$$

6. Secondly, If $2\alpha \sqsupset X$, then the Equation in the third step shall be express'd thus;

$$TT - \frac{X - 2\alpha}{X}T = \frac{2Z}{X}.$$

7. Which Equation falling under the second of the three Forms in *Sect.* 1. *Chap.* 15. the value of T shall be given by the Canon in *Sect.* 8. of the same *Chap. viz.*

$$T = \sqrt{:\frac{\tfrac{1}{4}XX - \alpha X + \alpha\alpha + 2ZX}{XX}:} + \frac{X - 2\alpha}{2X}.$$

8. Lastly, If $2\alpha = X$, then the Equation in the third step will be express'd thus;

$$TT = \frac{2Z}{X}; \qquad \text{Whence,} \qquad T = \sqrt{\frac{2Z}{X}}.$$

The three Equations in the 5, 7, and 8 steps give a threefold Canon to solve this 12 *Prop. viz.*

Canon I. *When the double of the least term exceeds the common difference.*

9. To the Square of the excess of the least term above half the common difference add the double Product of the Multiplication of the Sum of all the Terms by the common difference, divide the Sum of that Addition by the square of the common difference and extract the square Root of the Quotient; then from the double of the least term subtract the common difference and divide the Remainder by the double of the common difference: lastly, subtracting this Quotient from the square Root before found, the Remainder shall be the number of terms sought.

This

This Canon may be exemplified by the following or the like Series of Numbers in Arithmetical Progression continued, where the double of the least Term exceeds the common difference of the Terms:

3, 5, 7, 9, 11, 13, 15, &c.

Canon II. *When the double of the least Term is less than the common difference of the Terms.*

10. To the Square of the excess of half the common difference above the least Term, add the double Product of the Multiplication of the Sum of all the Terms by the common difference; divide the Sum of that Addition by the Square of the common difference, and extract the square Root of the Quotient; then from the common difference subtract the double of the least Term, and divide the Remainder by the double of the common difference; lastly, adding this Quotient to the square Root before found, the Sum shall be the number of Terms sought.

This Canon may be exemplified by the following or the like Rank of numbers in Arithmetical Progression continued, where the double of the least Term is less than the common difference:

2, 7, 12, 17, 22, 27, 32, 37.

Canon. III. *When the double of the least Term is equal to the common difference of the Terms.*

11. Divide the double of the Sum of all the Terms by the common difference, so shall the square Root of the Quotient be the number of Terms sought.

This Canon may be exemplified by the following Rank of numbers in Arithmetical Progression continued, where the double of the least Term is equal to the common difference of the Terms:

3, 9, 15, 21, 27, 33, 39.

PROP. XIII.

1. . . . . $\left\{ \begin{array}{l} \omega,\ T,\ X \text{ are given severally;} \\ \alpha \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 7. . . . . $TX - X + \alpha = \omega$;
3. Therefore by transposition of TX—X, this Equation will arise, which makes known the value of α; $\}$ $\alpha = \omega + X - TX$.

Which Equation gives this

CANON.

To the last, (that is, the greatest) Term add the common difference, and from the Sum subtract the Product of the number of Terms multiplied by the common difference; so shall the Remainder be the first (or least) Term sought.

This Canon may be exemplified by the following or any other Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XIV.

1. . . . . $\left\{ \begin{array}{l} \omega,\ T,\ X \text{ are given severally;} \\ Z \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega + T\alpha = 2Z$,
3. And by the Canon of *Prop.* 13. . . . . $\omega + X - TX = \alpha$,
4. Which latter Equation if it be multiplied by T, will produce $T\omega + TX - TTX = T\alpha$;
5. Then if instead of $T\alpha$ in the Equation in the second step, you take that which in the fourth step is found equal to $T\alpha$, the Equation in the second step will be converted into this; $\}$ $2T\omega + TX - TTX = 2Z$,
6. That is . . . . $\overline{2\omega + X - TX} \text{ into } T = 2Z$.

Which Equation gives this

CANON.

To the double of the last (to wit, the greatest) Term, add the common difference; from the Sum subtract the Product of the number of Terms multiplied by the common difference;

difference: then multiply the Remainder by the number of Terms, the Product shall be the double of the Sum of all the Terms, and consequently the half of that Product is the required Sum of all the Terms.

This Canon may be exemplified by the following (or any other Rank) of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27, 31.

PROP. XV.

1. . . . . $\left\{\begin{array}{l}\omega,\ T,\ Z \text{ are given severally;}\\ \alpha \text{ is sought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 9. . . . . $\frac{2Z - T\alpha}{T} = \omega,$
3. Therefore multiplying each part of that Equation by T, this will arise; . . . . $2Z - T\alpha = T\omega,$
4. And by transposition of $-T\alpha$ in the last Equation this will arise; . . . . $2Z = T\omega + T\alpha,$
5. Likewise by transposition of $T\omega$, this Equation arises, $2Z - T\omega = T\alpha,$
6. Therefore each part of the last Equation being divided by T, the value of α will be made known, *viz.* $\frac{2Z}{T} - \omega = \alpha.$

Which Equation gives this

CANON.

Divide the double Sum of all the Terms by the number of Terms, and from the Quotient subtract the last (to wit, the greatest term; so shall the Remainder be the first and least term sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XVI.

1. . . . . $\left\{\begin{array}{l}\omega,\ T,\ Z \text{ are given severally;}\\ X \text{ is sought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 14. . . . . $\overline{2\omega + X - TX} \text{ into } T = 2Z,$
3. That is . . . . $2T\omega + TX - TTX = 2Z,$
4. Therefore by due transposition this Equation will arise, $2T\omega - 2Z = TTX - TX,$
5. Therefore by dividing all in the last Equation by TT—T, the value of X will be made known, *viz.* $\frac{2T\omega - 2Z}{TT - T} = X.$

Which Equation gives this

CANON.

From the double Product of the Multiplication of the number of Terms by the greatest Term, subtract the double of the Sum of all the Terms; divide the Remainder by the excess of the Square of the number of Terms above the number of Terms, so shall the Quotient be the common difference sought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progression continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XVII.

1. . . . . $\left\{\begin{array}{l}\omega,\ X,\ Z \text{ are given severally;}\\ \alpha \text{ is sought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 6. . . . . $\frac{\omega\omega - \alpha\alpha}{2Z - \omega - \alpha} = X,$
3. Therefore each part of that Equation being multiplied by $2Z - \omega - \alpha$, there will arise, ... $\omega\omega - \alpha\alpha = 2XZ - X\omega - X\alpha,$
4. Whence by equal addition of $X\omega + X\alpha$ you will find, $\omega\omega + X\omega + X\alpha - \alpha\alpha = 2ZX,$

Now

Now before known Quantities can be separated from unknown in the last Equation, we must discover from the things given in the Proposition, whether $\omega\omega + X\omega$ be equal, greater, or less than $2ZX$? First therefore,

5. Suppose . . . . $\omega\omega + X\omega = 2ZX$,
6. And then by setting $\omega\omega + X\omega$ in the place of $2ZX$ in the Equation in the fourth step, there will arise, ... $\omega\omega + X\omega + X\alpha - \alpha\alpha = \omega\omega + X\omega$,
7. Whence by subtracting $\omega\omega + X\omega$ from each part, and by transposition of $-\alpha\alpha$, this Equation arises; . . . . . $X\alpha = \alpha\alpha$,
8. Which last Equation being divided by α, gives . . . . $X = \alpha$.

From the premises arises this

CANON I.

9. When the sum of the Square of the last (to wit, the greatest) term and the Product of the multiplication of the said last term by the common difference of the terms is equal to the double of the Product made by the multiplication of the sum and common difference of the terms, then the said difference is equal to the first or least term sought.

This Canon may be exemplified by the following Series of numbers in Arithmetical Progression continued:

2, 4, 6, 8, 10, 12, 14.

10. Secondly, suppose . . . . $\omega\omega + X\omega \sqsubset 2ZX$.
11. Then from the Equation in the fourth step, after due Reduction, there will arise, $\alpha\alpha - X\alpha = \omega\omega + X\omega - 2ZX$,
12. In which last Equation all things are known but α, and the said Equation falls under the second of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the value of α, to wit, the first (or least) term sought shall be given by the Canon in *Sect.* 8. of the same *Chap.* *viz.*

$$\alpha = \tfrac{1}{2}X + \sqrt{:\overline{\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX}:}$$

From the tenth and twelfth steps arises

CANON II.

13. If the sum of the Square of the last (to wit, the greatest) term, and the Product of the multiplication of the said last term by the common difference of the terms, exceeds the double of the Product made by the multiplication of the sum and common difference of the terms; then to the sum first mentioned add the Square of half the common difference; from this sum subtract the double Product above mentioned, and extract the square Root of the Remainder: lastly, add the said square Root to half the common difference, so shall the Sum be the first (or least) term sought.

This Canon may be exemplified by the following Progression:

3, 5, 7, 9, 11, 13.

14. Thirdly, suppose : . . . . $\omega\omega + X\omega \sqsupset 2ZX$,
15. But in this third case, to the end a possible Equation may arise, this Determination is necessary, *viz.* . . . . $\omega\omega + X\omega + \tfrac{1}{4}XX$, not $\sqsupset 2ZX$,
16. Then from the Equation in the fourth step by transposition of $\omega\omega + X\omega$, this will arise; $X\alpha - \alpha\alpha = 2ZX - \omega\omega - X\omega$.
17. In which last Equation all things are known but α, and the Equation falls under the last of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the two values of α in that Equation shall be given by the Canon in *Sect.* 10. of the same *Chap.* *viz.*

$$\alpha = \tfrac{1}{2}X + \sqrt{:\overline{\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX}:}$$

Or, $$\alpha = \tfrac{1}{2}X - \sqrt{:\overline{\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX}:}$$

18. Whence it is manifest, that if in this third Case it happens that $\omega\omega + X\omega + \tfrac{1}{4}XX = 2ZX$, then $\quad = \tfrac{1}{2}X$; that is to say, the first (or least) term sought shall be equal to half the given difference of the terms. But if in the said third Case it happens that

that $\omega\omega + X\omega + \frac{1}{4}XX \sqsubset 2ZX$, then there will be two unequal Roots or values of α, to wit, those above express'd, by either of which the Equation in the sixteenth step may be expounded; yet (as may easily be apprehended) only one of those values of α can be such a first (or least) term as will agree with the things given in the Proposition: But which of those two values of α is the least term sought, you may discover by the Proof formed thus, *viz.* First, by the help of one of those unequal values of α found out as above, together with the given last (to wit, the greatest) term and the given common difference of the terms, you may find out (by the Canon of the third *Prop.*) the number of terms, (which must always be a whole number,) and then by the same value of α, together with the said last term and the number of Terms you may by the Canon of *Prop.* 1. find out the sum of all the terms; then if this sum be equal to the sum given in the *Propos.* propos'd, that value of α by which the Proof was made, is the least term sought. But if that Proof will not succeed, then the other value of α shall be the least term sought; as will be evident by the Proof made as before.

From the five last steps there will arise

CANON. III.

19. When the sum of the Square of the last (to wit, the greatest) term, and the Product of the Multiplication of the said last term by the common difference, is less than the double of the Product made by the multiplication of the sum and common difference of the terms; but the Aggregate of the sum first mentioned and the square of half the common difference is not less than the said double Product; then from the said Aggregate subtract the said double Product and extract the square Root of the Remainder, that done, add the said square Root to half the common difference of the terms, and also subtract the said square Root from the said half difference, so the Sum or else the Remainder, (*viz.* such of them, which by the Proof made according to the direction in the preceding eighteenth step will be found to agree with the things given in the Proposition,) shall be the first (or least) term sought.

This Canon may be exemplified by the two following Ranks of numbers in Arithmetical Progression continued:

I.	2, 5, 8, 11, 14, 17.
II.	2, 7, 12, 17, 22, 27.

PROP. XVIII.

1. . . . . $\left\{ \begin{array}{l} \omega,\ X,\ Z \text{ are given severally;} \\ \quad\ T \text{ is required.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 14 . . . . $2\omega T + XT - XTT = 2Z.$

3. Therefore dividing every member of the said Equation by X, (because it is drawn into TT the highest degree of the number sought,) this following Equation will arise, *viz.*

$$\frac{2\omega T + XT}{X} - TT = \frac{2Z}{X},$$

That is, $$\frac{2\omega + X}{X}T - TT = \frac{2Z}{X}.$$

4. In which all things are known but T, and the said Equation falls under the last of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the two values of T will be made known by the Canon in *Sect.* 10. of the same *Chap. viz.*

$$T = \frac{\omega + \frac{1}{2}X}{X} + \sqrt{:}\frac{\omega\omega + \omega X + \frac{1}{4}XX - 2ZX}{XX};$$

$$\text{Or, } T = \frac{\omega + \frac{1}{2}X}{X} - \sqrt{:}\frac{\omega\omega + \omega X + \frac{1}{4}XX - 2ZX}{XX};$$

5. But altho the Equation in the third step may be expounded by either of the two Roots or values of T above express'd in the fourth step, yet only one of them can be the number of terms sought; but which of the said numbers, or values of T will solve the Proposition you may discover thus: First, If one of the two numbers or values of T before found out be a Fraction or a mixt number, that value cannot be the number of terms sought; for the number of terms in an Arithmetical Progression is alwnys a whole number. Secondly, If both the values of T happen to be whole numbers, then the true number of terms sought may be discovered by this Proof; *viz.* First, by the help of one of those values of T in whole numbers, together with the given last (or greatest) term, and the given common difference, find out (by the Canon of *Prop.* 13.) the first (to wit, the least) term; and then by the same number T, together with the first and last terms, find out (by the Canon of *Prop.* 1.) the sum of all the terms; lastly, If the sum so found out be equal to the sum given in the Proposition propos'd, then that number or value of T by which the Proof was made shall be the true number of terms sought. But if the Proof will not succeed to find out a number equal to the sum first given, then the other value of T is the number of terms sought; which will be evident by the Proof made therewith in the same manner as before.

From the premisses there arises this

CANON.

6. From the Square of the sum of the last (to wit, the greatest) term, and half the common difference, subtract the double of the Product of the Multiplication of the sum of all the terms by the common difference; divide the Remainder by the square of the said difference, and extract the square Root of the Quotient. That done, add the said square Root to the Quotient which arises by dividing the sum of the last term and half the common difference by the difference it self, and also subtract the said square Root from the said Quotient; so the Sum, or else the Remainder (*viz.* such of them which according to the preceding fifth step will be found to agree with the things given in the *Propos.*) shall be the number of terms sought.

This Canon may be exemplified by the three following Progressions; in the first of which the greater of the two values of T (in the fourth step) is the number of terms sought; but in each of the two latter Progressions the lesser value of T is the number of terms sought.

I.	2, 7, 12, 17, 22, 27, 32.
II.	2, 5, 8, 11, 14, 17, 20.
III.	12, 20, 28, 36, 44, 52, 60.

PROP. XIX.

1\. . . . . $\left\{\begin{array}{l}\text{T, X, Z are given severally;}\\ \quad a \text{ is sought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 10. . . . . $\dfrac{2Z-2Ta}{TT-T} = X,$

3. Therefore multiplying each part of that Equation by TT—T, this will be produced, to wit, . . $\}$ $2Z-2Ta = TTX-TX$

4. In which last Equation all things are known but a, whose value after due Reduction of that Equation will be found out, *viz.* . . . . $\}$ $a = \dfrac{Z}{T} + \tfrac{1}{2}X - \tfrac{1}{2}TX,$

Which in words gives this

CANON.

5. Divide the given sum of all the terms by the given number of terms, to the Quotient add half the given difference of the terms, and from the sum of that addition subtract half the Product of the Multiplication of the said number of terms by the common difference; so shall the Remainder be the first (to wit, the least) term required.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progression continued:

2, 7, 12, 17, 22, 27, 32.

PROP.

PROP. XX.

1. . . . . $\left\{ \begin{array}{l} T,\ X,\ Z \text{ are given severally;} \\ \omega \text{ is sought.} \end{array} \right.$

RESOLUTION.

2. By the Canon of *Prop.* 16. . . . . $\frac{2T\omega - 2Z}{TT - T} = X,$

3. Therefore multiplying each part of that Equation by TT—T, this will be produced, to wit, . . $2T\omega - 2Z = TTX - TX,$

4. In which last Equation all things are known but ω, whose value, after due Reduction of that Equation, will be discovered, *viz.* . . . . $\omega = \frac{Z}{T} + \frac{1}{2}TX - \frac{1}{2}X.$

Which in words gives this

CANON.

5. Divide the given sum of all the terms by the given number of terms; to the Quotient add half the Product of the Multiplication of the number of terms by the common difference given, and from the sum of that Addition subtract half the said difference; the Remainder shall be the last (to wit, the greatest) term required.

This Canon may be exemplified by the following or any other Rank of numbers in Arithmetical Progression continued:

2, 5, 8, 11, 14, 17, 20.

Questions to exercise some of the Canons of the preceding Propositions.

Quest. 1. Suppose 40 Stones be so placed in a streight line, that the first is distant from a Basket one Yard, the second two, the third three, and the rest in the same excess; now if some Footman undertakes to go from the Basket to fetch into it every Stone one after another, how many Yards must he go to perform that work? *Answ.* 1640 Yards.

Forasmuch as the Footman must go 2 Yards (to wit, one forwards, and the same backwards,) to fetch the first Stone into the Basket; 4 Yards for the second; 6 for the third, *&c.* here is an Arithmetical Progression continued whose first (or least) term is 2, the common difference of the terms is also 2, and the number of Terms is 40; therefore the sum of all the terms, to wit, the number of Yards sought will be found 1640, by the Canon of the preceding eighth *Prop.*

Quest. 2. Two Footmen, *A* and *B*, depart at the same time from *London* towards *York*, and travel in this manner, *viz.* *A* travels 8 (or *c*) Miles every day; *B* travels 1 Mile the first day, 2 Miles the second day, 3 Miles the third day, and so forward; travelling every day one Mile more than in the day next preceding: The Question is, to find in how many days *B* will overtake *A*? *Answ.* At the end of 15 days, found out by this following

RESOLUTION.

1. For the number of days that *B* had travelled when he overtook *A*, put . . . a

2. Then to find how many Miles *B* had travelled when he overtook *A*, there is an Arithmetical Progression continued wherein the first and least term is 1, (to wit, 1 Mile which *B* travelled the first day,) also the common difference is 1, (for the Question saith that *B* travelled every day 1 Mile more than in the day next preceding,) and the number of terms is *a*, (which we assumed for the number of days that *B* had travelled when he overtook *A*;) therefore the sum of all the terms (or number of Miles that *B* had travelled) will by the Canon of the preceding *Prop.* 8. be found to be . . . . $\frac{1}{2}aa + \frac{1}{2}a$

3. And because *A* travelled 8 (or *c*) Miles daily, and had travelled the same number of days as *B* when *B* overtook *A*, therefore 8 (or *c*) multiplied by *a* produces the number of Miles that *A* had then travelled; to wit, ... ca

4. But

4. But when *B* overtook *A*, each had travelled the same number of Miles; therefore the numbers found out in the two last steps must be equal the one to the other, *viz.* . . . . $\frac{1}{2}aa + \frac{1}{2}a = ca$

5. Which Equation after due Reduction gives . . . . $a = 2c - 1$

Which in words is this

CANON.

From the double of the number of Miles that *A* travelled daily, subtract 1 (or Unity,) so shall the Remainder be the number of days sought.

Whence the number of days required will be found 15; for the double of 8 is 16, from which subtracting 1, the Remainder 15 is the number of days sought; *viz.* *B* will overtake *A* at the end of 15 days, as will be evident by

The Proof.

If 15 be the number of terms, and 1 the first (or least) term, as also the common difference of the terms of an Arithmetical Progression continued; the sum of all the terms will (*per* Canon of *Prop.* 8.) be found 120, being the number of Miles which *B* had travelled in 15 Days, (according to the Progression of 1 Mile the first Day, 2 Miles the second, 3 Miles the third, &c.) Also, *A* travelling 8 Miles every day, would in 15 days have travelled 120 Miles. Therefore the conditions in the Question are satisfied.

Quest. 3. A Merchant discharged a Debt of 1370 *l.* by several Payments made in this manner, *viz.* the first payment was $1\frac{1}{2}$ *l.* the second payment exceeded the first by $\frac{1}{6}$ *l.* the third exceeded the second by the same excess, and the rest of the payments in like manner. The Question is, to find how many payments the Merchant made in discharging the said Debt? *Answ.* 120, found out thus:

There is given in the Question $1\frac{1}{2}$, to wit, the first and least term of an Arithmetical Progression continued; also $\frac{1}{6}$ the difference of the terms, and 1370 the sum of all the terms, to find the number of terms, which (by Canon 1 of the foregoing *Prop.* 12. of this *Chap.*) will be found 120.

Quest. 4. If a Debt of 1370 *l.* was discharged by several Payments made in such manner, that the second payment exceeded the first by $\frac{1}{6}$ *l.* the third the second, the fourth the third, &c. in the same excess, *viz.* every following payment exceeded the next preceding by $\frac{1}{6}$ *l.* and that the last payment was $21\frac{1}{3}$ *l.* What was the first (to wit, the least) Payment, and how many several Payments did the Debitor make? *Answ.* The first and least Payment was $1\frac{1}{2}$ *l.* (found out by the Canon 2. of *Prop.* 17.) and the number of Payments was 120, found out by the Canon of *Prop.* 18.

Quest. 5. A Footman travelled 124 Miles in 8 Days at this rate, *viz.* The second Days journey exceeded the first by 3 Miles, the third the second by 3 Miles, and so forward in that excess; How many Miles was his first Days journey, and how many his last? *Answ.* 5, and 26 Miles; found out by the Canons of *Prop.* 19 and 20.

Quest. 6. A Draper bought 20 Cloths for 20 Crowns a piece, and sold the first Cloth for a certain number of Crowns; the second for two Crowns more than the first; the third for two Crowns more than the second; and so by increasing the price of every following Cloth by two Crowns more than the next preceding Cloth, he sold the last Cloth for 41 Crowns. It is desired to find the number of Crowns for which he sold the first Cloth, and what he gained or lost by all the Cloths.

This Question implies an Arithmetical Progression, whose number of Terms is 20; the common difference of the Terms is 2; and the last Term is 41: Therefore by the Canon of *Prop.* 13. of this *Chap.* the first and least term will be found 3; and then by the Canon of *Prop.* 1. (or by the Canon of *Prop.* 14.) the sum of all the terms will be found 440. Whence it is manifest that the Draper gained 40 Crown by the 20 Cloths; for he bought them for 400 Crowns, and sold them for 440.

Quest. 7. One distributed 456 Pence among a certain number of poor Persons in this manner, *viz.* To the first he gave 6 Pence, to the last 51 Pence; the number of Pence given to the second exceeded that given to the first, the third the second, and so forward to the last by an equal excess. The Question is, to find how many poor persons there were; and how many Pence every one between the first and last received?

To

To solve this Question, an Arithmetical Progression must be conceived, whose first Term is 6; the last Term is 51; and the sum of all the Terms 456: then by the Canon of *Prop.* 5. the number of Terms will be found 16; and by the Canon of *Prop.* 6. the common difference of the Terms will be found 3; wherefore there were 16 poor Persons: and if this Arithmetical Progression, to wit, 6, 9, 12, *&c.* be continued to the sixteenth Term inclusive, it will shew the number of Pence which every one of the poor Persons received; and all those 16 Terms or Numbers being added together, make the given sum 456.

Quest. 8. A Stationer sold 7 (or t) Reams of Paper, the particular prices whereof were certain numbers of Shillings in Arithmetical Progression; the price of the second Ream, that is, of that next above the cheapest, was 8 (or b) Shillings; and the price of the last or dearest Ream was 23 (or c) Shillings: what was the price of each Ream?

RESOLUTION.

1. For the price of the cheapest or first Ream put	a	a
2. Then because the price of the second Ream was 8, (or b,) therefore by subtracting a from 8, (or b,) there remains the common difference of the Terms of the Progression, *viz.*	$8-a$	$b-a$
3. Then by the help of the least term, the common difference of the terms, and the number of terms, seek (by the Canon of *Prop.* 7. of this *Chap.*) the last and greatest term, which will be found .	$48-5a$	$2a-ta+tb-b$

4. Which greatest Term last found out must be equal to 23 (or c,) hence this Equation arises, *viz.*

$$48-5a=23; \qquad \text{Or,} \qquad 2a-ta+tb-b=c.$$

5. From which Equation after due Reduction this arises, *viz.*

$$a=5=\frac{tb-b-c}{t-2}.$$

Which in words is this

CANON.

From the Product of the price of the second Ream of Paper (to wit, of that next above the cheapest, multiplied by the number of Reams, subtract the sum of the prices of the second and last Reams; then divide the Remainder by the excess of the number of Reams above 2: so shall the Quotient be the price of the first (or cheapest) Ream

Whence, by the help of the numbers given in the Question, these following numbers in Arithmetical Progression will be discovered, which solve the Question, *viz.* 5, 8, 11, 14, 17, 20, 23.

Quest. 9. One being asked what were the several ages of his five (or t) Children, answered, that the age of the eldest exceeded that of the second by 2 (or x) Years; and by the same excess the second exceeded the third, the third the fourth, the fourth the fifth or youngest Child's age; and if the age of the eldest Child were multiplied by the age of the youngest it would produce 128 (or c) Years. It's desired to find out the age of every one of the five Children.

The numbers sought by the Question are in Arithmetical Progression.

RESOLUTION.

1. For the age of the youngest Child (being the least Term of the Arithmetical Progression in the Question,) put	a	a
2. Then by the help of a, x and t, *viz.* the age of the youngest Child, the common difference of their ages, and the number of Children, seek (by the Canon of *Prop.* 7. of this *Chap.*) the age of the eldest, that is, the greatest Term of the Progression, so you will find	$a+8$	$a+tx-x$
3. Therefore the Product of the multiplication of the first and last Terms of the Progression is .	$aa+8a$	$aa+txa-xa$

4. Which

4. Which Product must be equal to 128 (or c,) the Product given in the Question; hence this Equation, *viz.* $aa + 8a = 128$; Or, $aa + txa - xa = c$.

5. Wherefore, by resolving the last Equation according to the Canon in *Sect. 6. Chap. 15.* the value of a, that is, the age of the youngest Child will be discovered, *viz.*

$$a = 8 = \frac{\sqrt{:ttxx - 2txx + xx + 4c:} - tx - x}{2}.$$

Which in words is this

CANON.

From the Product of the number of Children multiplied into the common difference of their ages subtract the said difference; then to the Square of the Remainder add four times the Product of the age of the eldest Child multiplied into the age of the youngest, and extract the square Root of the sum of that Addition: then from the said square Root subtract the Product of the common difference of their Ages multiplied into the excess of the number of Children above Unity; so the half of the Remainder shall be the age of the youngest Child.

Whence these five numbers are discovered, *viz.* 8, 10, 12, 14, 16; which shew the number of Years expressing the age of every one of the five Children: for the Product of the first and last numbers is 128, and the common difference is 2, as was required.

Quest. 10. If the sum of 6 (or t) numbers or terms in Arithmetical Progression be 48 (or z,) and the Product of the common difference multiplied into the least Term be equal to the number of Terms; what are the Numbers of that Progression?

RESOLUTION.

1. For the common difference of the Terms put . .	a	a
2. Then according to the condition in the Question, if the number of Terms be divided by the common difference, the Quotient is the least Term, to wit,	$\frac{6}{a}$	$\frac{t}{a}$
3. Now by the help of the common difference, the least Term, and the number of Terms, seek (by by eighth *Prop.* of this *Chap.*) the double sum of all the Terms, so you will find . : . ; . .	$30a + \frac{72}{a}$	$tta + \frac{2tt}{a} - ta$

4. Which double Sum must be equal to twice 48, the Sum given in the Question; hence this Equation arises, *viz.*

$$30a + \frac{72}{a} = 96;$$

That is, . . $tta + \frac{2tt}{a} - ta = 2z.$

5. Which Equation duly reduced gives

$$\tfrac{16}{5}a - aa = \tfrac{12}{5};$$

That is, . . $\frac{2z}{tt-t}a - aa = \frac{2t}{t-1}.$

6. Wherefore by resolving the last Equation according to the Canon in *Sect.* 10. *Chap.* 15 the two values of a will be found these, *viz.*

$$a = 2 = \frac{z + \sqrt{:zz + 2ttt - 2tttt:}}{tt - t}$$

$$a = \tfrac{6}{5} = \frac{z - \sqrt{:zz + 2ttt - 2tttt:}}{tt - t}$$

7. Each of which values of a, to wit, 2 and $\frac{6}{5}$ may be taken for the common difference sought. Then because 6 is prescribed in the Question for the Product of the least Term multiplied into the common difference, let 6 be divided by the said 2 and $\frac{6}{5}$ severally, and the Quotients 3 and 5 shall be the two least Terms of two Arithmetical Progressions, each of which will solve the Question: And therefore

The six numbers sought may be either these, 3, 5, 7, 9, 11, 13;
Or these, ... 5, $6\frac{1}{5}$, $7\frac{2}{5}$, $8\frac{3}{5}$, $9\frac{4}{5}$, 11.

In each of which Progressions, the number of Terms is 6; the sum of all the Terms is 48; and the common difference multiplied by the least Term produces the number of Terms. Which was prescribed in the Question.

The End of the First BOOK.

THE ELEMENTS OF THE Algebraical ART.

BOOK II.

CHAP. I.

Concerning the Genesis or Production of Powers from Roots Binomial, Trinomial, &c.

I. I Shall take it for granted, that the Reader of this Second Book of Algebraical Elements is well exercised in the First; and therefore without making any repetition of what has been there explained at large, I shall proceed to the handling of new matter in this Mysterious Art. First then, forasmuch as the Extraction of Roots is undoubtedly the hardest Lesson in Vulgar Arithmetic, and the Reason of the Rules delivered in most Treatises of Arithmetic for extracting of the Square and Cubic Roots is known but to few practical Arithmeticians, I shall explain what our learned Divine and famous Mathematician Mr. *William Oughtred*, hath succinctly delivered upon this Subject in the twelfth, thirteenth, and fourteenth Chapters of his Incomparable *Clavis Mathematicæ*; to which end in this and the following second Chapters I shall first shew the Genesis or Production of Powers from Roots Binomial, Trinomial, *&c.* and then in the third and fourth Chapters their Analysis, or the Extraction of the Root or Side out of any given Power, whether it be express'd by the Number or Letters.

II. If a Line or Number be divided into any two parts, suppose a the greater and e the lesser, these connected by the Sign + or — do constitute a Binomial Root, as $a+e$ or $a-e$, the latter of which some call a Residual Root, because it imports a Remainder, *viz.* the difference of the two Names or Parts of the Root. In like manner these Compound Quantities $a+b+c$, $a-b-c$; and the like, may be called Trinomial Roots, because each of them consists of three Names or Parts; and $a+b+c+d$ a Quadrinomial Root, that is, a Root consisting of four Parts: And so of others.

III. From a Root Binomial, Trinomial, *&c.* Algebraical Powers may be produced in like manner as from a simple Root, *viz.* by a continued Multiplication of the Root into it self. As for Example: The Binomial Root $a+e$ being multiplied by it self, that is, $a+e$ by $a+e$, produces $aa+2ae+ee$, the Square of $a+e$. Again, if the Square $aa+2ae+ee$ be multiplied by its Root $a+e$, the Product will be $aaa+3aae+3aee+eee$, which is the Cube of the Root $a+e$; and if the said Cube be multiplied by its Root $a+e$, it will produce the fourth Power: and so you may proceed to find a fifth, sixth, or what Power you please from the Binomial Root $a+e$. But for the greater evidence view the following Operation.

$$\begin{array}{ll}
\text{Binomial Root, } . & a+e \\
 & a+e \\ \hline
 & aa+ae \\
 & \quad +ae+ee \\ \hline
\text{Square, } . . . . & aa+2ae+ee. \\
 & a+e \\ \hline
 & aaa+2aae+aee \\
 & \quad + aae+2aee+eee \\ \hline
\text{Cube, } . . . . & aaa+3aae+3aee+eee \\
 & a+e \\ \hline
 & aaaa+3aaae+3aaee+aeee \\
 & \quad + aaae+3aaee+3aeee+eeee \\ \hline
\text{Biquadrate, } . . & aaaa+4aaae+6aaee+4aeee+eeee.
\end{array}$$

After the same manner, if the Residual Root $a-e$ be multiplied by it self, the Product will be $aa-2ae+ee$ the Square of $a-e$. Again, if the Square $aa-2ae+ee$ be multiplied by its Root $a-e$, the Product will be $aaa-3aae+3aee-eee$, which is the Cube of the Root $a-e$. And so you may proceed to find a fourth, fifth, or what Power you please from the Residual Root $a-e$; view the following Work.

$$\begin{array}{ll}
\text{Residual Root } . . & a-e \\
 & a-e \\ \hline
 & aa-ae \\
 & \quad -ae+ee \\ \hline
\text{Square, } . . . . & aa-2ae+ee \\
 & a-e \\ \hline
 & aaa-2aae+aee \\
 & \quad -aae+2aee-eee \\ \hline
\text{Cube, } . . . . & aaa-3aae+3aee-eee. \\
 & a-e \\ \hline
 & aaaa-3aaae+3aaee-aeee \\
 & \quad -aaae+3aaee-3aeee+eeee \\ \hline
\text{Biquadrate, } : . & aaaa-4aaae+6aaee-4aeee+eeee.
\end{array}$$

By those two Examples it is manifest, that the Powers from the Residual Root $a-e$ differ only in the Signs $+$ and $-$ from like Powers formed from the Binomial Root $a+e$; for in every Power of a Residual Root, the Signs prefix'd before the Parts or Members of the Power are alternately $+$ and $-$; *viz.* the greatest or first Member is Affirmative, the second Negative, the third Affirmative, the fourth Negative, and so forwards: as you may see in the Cube of $a-e$, where aaa the greatest extreme Member is Affirmative; the next Number in order being $-3aae$ is Negative; the third Member $+3aee$ is Affirmative; and the last (to wit, the least) Member $-eee$ is Negative. But in every Power produced from a Binomial Root, whose Parts are connected by $+$, as $a+e$, all the Members of the Power are Affirmative.

IV. If according to the Construction in the last preceding *Section* a Scale or Rank of Powers be formed from a Binomial Root, as from $a+e$, the Members of each Power to the tenth inclusive will be such as you see in the following Table, where the two last Powers are compendiously express'd according to *Cartesius* his way.

A

A Table of Powers produced from the Binominal Root $a+e$.

The Root.	a	e									
(2)	aa	$2ae$	ee								
(3)	aaa	$3aae$	$3aee$	eee							
(4)	$aaaa$	$4aaae$	$6aaee$	$4aeee$	$eeee$						
(5)	$aaaaa$	$5aaaae$	$10aaaee$	$10aaeee$	$5aeeee$	$eeeee$					
(6)	$aaaaaa$	$6aaaaae$	$15aaaaee$	$20aaaeee$	$15aaeeee$	$6aeeeee$	$eeeeee$				
(7)	$aaaaaaa$	$7aaaaaae$	$21aaaaaee$	$35aaaaeee$	$35aaaeeee$	$21aaeeeee$	$7aeeeeee$	$eeeeeee$			
(8)	$aaaaaaaa$	$8aaaaaaae$	$28aaaaaaee$	$56aaaaaeee$	$70aaaaeeee$	$56aaaeeeee$	$28aaeeeeee$	$8aeeeeeee$	$eeeeeeee$		
(9)	a^9	$9a^8e$	$36a^7ee$	$84a^6e^3$	$126a^5e^4$	$126a^4e^5$	$84a^3e^6$	$36aae^7$	$9ae^8$	e^9	
(10)	a^{10}	$10a^9e$	$45a^8ee$	$120a^7e^3$	$210a^6e^4$	$252a^5e^5$	$210a^4e^6$	$120a^3e^7$	$45aae^8$	$10ae^9$	e^{10}

V. By the foregoing Table it is evident, that the Square of $a+e$ consists of $aa+2ae+ee$; which shews, that if a Number be divided into any two parts, the Square of that Number shall be equal to the Squares of the parts, and to twice the Product made by the Multiplication of the parts one into the other; as if 12 be divided into 10 and 2, which may be signified by a and e, then

The Square of 10 is	100	aa
Product of 10 multiplied by 2 is 20, which doubled makes . .	40	$2ae$
The Square of 2 is	4	ee

Which three Numbers, to wit, 100, 40, and 4, added together make the Square of 12, *viz.* $144 = aa+2ae+ee$.

In like manner the said Table shews, that the Cube or third Power of the Binominal Root $a+e$ consists of the Cubes of the Names or Parts of the Root a and e, together with the triple of the solid Product made by the Multiplication of the Square of the greater part a into the lesser part e, and the triple of the solid Product made by the Multiplication of the greater part a into the Square of the lesser part e. This may be illustrated by Numbers thus: Suppose 12 to be divided into 10 and 2, which may (as before) be represented by a and e; then the Cube of 12 or of $a+e$, will be equal to the Sum of these four solid Numbers, *viz.*

 The

The Cube of 10 is	1000	*aaa*
The Square of 10 is 100, which multiplied by 2 produces 200, this tripled makes	600	3*aae*
Again, 10 multiplied by 4 the Square of 2 produces 40, the Triple whereof is	120	3*aee*
The Cube of 2 is	8	*eee*

Which four Numbers, *viz.* 1000, 600, 120, and 8, added together make the Cube of 12, (or 12×12×12) that is . 1728=*aaa*+3*aae*+3*aee*+*eee*

After the ſame manner the reſt of the Powers in the Table might be expreſs'd by Words. Whence 'tis evident, that this literal Method diſcovers many Properties in Powers, which in Numeral Calculations do lie in obſcurity.

VI. Moreover, by a bare Inſpection into the ſaid Table it may be perceived, that the Number prefix'd to every one of the mean Members of every Power produced from the Binomial Root *a*+*e*, is compoſed of the two Numbers prefix'd to the next ſuperiour and inferiour Members of the next preceding Power. As for example: If you conceive the Line upon which 3*aae* is ſet to be continued forth at length, it will paſs between *aa*, that is, 1*aa* and 2*ae*, in the foregoing ſecond Power (or Square.) Now I ſay that the number 3 prefix'd to *aae* is the ſum of 1 and 2 the Numbers prefix'd to *aa* and *ae*. Likewiſe the number 6 prefix'd to *aaee*, one of the Members of the fourth Power, is compoſed of 3 and 3, the Numbers prefix'd to *aae* and *aee* in the third Power. Again, the number 15 prefix'd to *aaaaee* is the ſum of 5 and 10, the Numbers prefix'd to *aaaae* and *aaaee* in the fifth Power. Hence a Table may be made to ſhew what Numbers are to be prefix'd to the mean Numbers of every Power.

A

2 For the Square.

3 . 3 For the Cube.

4 . 6 . 4 For the fourth Power.

5 . 10 . 10 . 5 For the fifth Power.

6 . 15 . 20 . 15 . 6 For the ſixth Power.

7 . 21 . 35 . 35 . 21 . 7 For the ſeventh Power.

8 . 28 . 56 . 70 . 56 . 28 . 8 For the eighth Power.

9 . 36 . 84 . 126 . 126 . 84 . 36 . 9 For the ninth Power.

10 . 45 . 120 . 210 . 252 . 210 . 120 . 45 . 10 For the tenth Power.

B C

In this Table the Numbers from *A* to *B*, and likewiſe from *A* to *C*, do proceed from 2 in an Arithmetical Progreſſion, having 1 (to wit, Unity) for a common difference; and every one of the mean Numbers ſtanding between the ſame Term of each Progreſſion, is compoſed of the two Numbers which ſtand next above each mean Number reſpectively: As 6, which ſtands between 4 and 4, is the Sum of 3 and 3, which ſtand above and on each ſide of 6: likewiſe 10, which is ſet between 5 and 5, is the Sum of 6 and 4 which ſtand above 10; and ſo of the reſt. So that this Table may be eaſily continued further at pleaſure.

VII. Any Power of a Binominal or Reſidual Root expreſs'd by Letters, may without a continued Multiplication of the Root into it ſelf be eaſily formed by the following Method, which is deduced from the Premiſes, *viz.* Suppoſe the fifth Power of the Bi-

Binominal Root $a+e$ be desired; First, I write all the simple Powers of a, descending orderly from the fifth Power downwards to the Root a; as $aaaaa$, $aaaa$, aaa, aa, and a, as here you see in the first Columel: then to all those Powers, except the uppermost $aaaaa$, I joyn such simple Powers of e, that the Sum of the Indices of both Powers may make 5, *viz.* To $aaaa$ I joyn e; to aaa, ee, to aa, eee; and to a, $eeee$; then I write $eeeee$ underneath; so that there are six distinct Members or Terms, every one of which consists of five Dimensions, as you see in the second Columel. That done, by the Table in the foregoing *Sect.* 6. I find that the Numbers 5, 10, 10, and 5 are to be prefix'd before the mean Members of the fifth Power; and accordingly I set 5 before $aaaae$, 10 before $aaaee$, likewise 10 before $aaeee$, and 5 before $aeeee$; lastly, by prefixing +, or supposing it to be prefix'd before every one of the said five Members, the fifth Power of the Binominal Root $a+e$ is compleated, as you see in the third Columel, and in every respect agrees with the fifth Power in the Table in the foregoing *Sect.* 4. But if the Signs + and — be alternately prefix'd before the Members of the said fifth Power, according to what has been said at the latter end of *Sect.* 3. it will be the fifth Power of the Residual Root $a-e$.

(1)	(2)	(3)
$aaaaa$	$aaaaa$	$aaaaa$
$aaaa$	$aaaae$	$5aaaae$
aaa	$aaaee$	$10aaaee$
aa	$aaeee$	$10aaeee$
a	$aeeee$	$5aeeee$
	$eeeee$	$eeeee$

VIII. Lastly, from a Root consisting of three, four, or any number of parts, the Square, Cube, or any higher Power of the Root may be produced by a continued Multiplication of the Root into it self: As the Trinomial Root $a+b+c$ being multiplied by it self, its Square will be found $aa+2ab+2ac+bb+2bc+cc$; and this Square multiplied again by its Root $a+b+c$ produces the Cube of the same Root, that is, $aaa+3aab+3aac+3abb+6abc+3acc+bbb+3bbc+3bcc+ccc$. After the same manner Powers may be produced from a Root consisting of four, or any Number of Parts. And if the Constitution of Powers express'd by Letters be seriously considered, it will be some help to discover whether an Algebraic Quantity consisting of more than three Members or Terms be a perfect Power or not, and also give some light to discover its Root.

CHAP. II.

Concerning the Composition of Powers in Numbers from a Binominal Root.

Sect. I. *Of the Composition of a Square from a Number given for the Side or Root.*

1. Suppose the Square of the Root 28 be desired: First, write down the Root 28 in such manner that there may be space enough to set one Figure between 2 and 8, and let a Line be drawn under them; as also two downright Lines, the one next after 2, and the other after 8, to the end the Numbers which are to be found out may be orderly placed for Addition: then let the Root 28 be conceived to be divided into these two parts 20 and 8, and let a be put for the greater part, and e for the lesser. Now forasmuch as the Square of $a+e$ is $aa+2ae+ee$, therefore the Square of 28, or of 20 + 8 may be composed thus, *viz.* The Square of 20 is 400, (or aa;) the double of 20 is 40, (or $2a$) which multiplied by 8 (or e) produces 320, (that is, $2ae$;) and the Square of 8 is 64 (or ee.) Lastly, the said three Numbers 400, 320, and 64, being set under one ano-

$a=20$
$e=8$

2	8		Root proposed.
4	00	aa	
3	20	$2ae$	
	64	ee	
7	84		Square required.

another in ſuch order, that Units may ſtand under Units, Tens under Tens, &c. and added together the Sum makes 784, the Square of the Root 28; as may eaſily be proved by multiplying 28 into it ſelf.

2. When the given Number or Root whoſe Square is deſired conſiſts of three or more places, as 47803; firſt, the Square of the two foremoſt Figures towards the left Hand, that is, of 47, muſt be found out in like manner as before in the firſt Example, ſo there will be produced 2209 for the Square of 47, as you ſee in the following Example 2. Secondly, write 47 in a void place, and annex a Cypher to it, ſo it makes 470, this Number muſt now be eſteemed *a*, and 8 the next following Character of the Root muſt be taken for *e*; and then according to theſe values of *a* and *e* the Numbers ſignified by *aa*, *2ae*, and *ee*, being added together make 228484 for the Square of 478, (as you ſee here underneath.) Where obſerve, that to find the Square of 470 (that is, of *a*) you need only annex two Cyphers to 2209, which was before found for the Square of 47. Thirdly, annex a Cypher to 478 in a void place, and it makes 4780 for a new Value of *a*, and the next following Character of the Root, to wit 0, is the new Value of *e*, then according to theſe Values of *a* and *e*, the Value of $aa+2ae+ee$ is 22848400, to wit *aa* only; for $e=0$, and conſequently $2ae+ee=0$: ſo the ſaid 22848400 is found for the Square of 4780. Laſtly, by annexing a Cypher to 4780 it makes 47800 for a new Value of *a*, and 3 the laſt Figure of the Root is the new Value of *e*; then according to theſe Values of *a* and *e* the Sum of the Numbers ſignified by *aa*, *2ae*, and *ee*, makes 2285126809, which is the Square of the ſaid given Root 47803, as may eaſily be proved by multiplying the ſaid Root by it ſelf. Compare the following Example with the precedent Directions.

Example 2. of Sect. I.

	4	7	8	0	3	Root propoſed.
$a=40$	16	00				*aa*
$e=7$	5	60				*2ae*
		49				*ee*
$a=470$	22	09	00			*aa*
$e=8$		75	20			*2ae*
			64			*ee*
$a=4780$	22	84	84	00		*aa*
$e=0$				00		*2ae*
				00		*ee*
$a=47800$	22	84	84	00	00	*aa*
$e=3$			28	68	00	*2ae*
					09	*ee*
	22	85	12	68	09	Square required.

Sect. II. *Of the Compoſition of a Cube from a Number given for the Side or Root.*

1. Let the Cube of the Root 28 be deſired: Firſt, I write the Root 28 in ſuch manner, that there may be ſpace enough to ſet two Figures between 2 and 8; then having drawn a Line under 28, and down-right Lines as before in the Square, I conceive the Root 28 to be divided into 20 and 8, that is, *a* and *e*. Now foraſmuch as the Cube of $a+e$ is compoſed of theſe four Members, *viz.* *aaa*, *3aae*, *3aee*, and *eee*, (as appears by the Table in *Sect.* 4. *Chap.* 1.) therefore the Cube of 20+8 (that is, of 28) may be compoſed thus, *viz.* Firſt, the Cube of 20 is 8000, (that is, *aaa*.) Secondly, the triple of the Square of 20 being mul-

	2	8	Root propoſed.
$a=20$	8	000	*aaa*
$e=8$	9	600	*3aae*
	3	840	*3aee*
		512	*eee*
	21	952	Cube deſired.

multiplied by 8 produces 9600, (that is, *3aae*;) thirdly, the triple of 20 being multiplied by the Square of 8 produces 3840, (that is, *3aee*;) fourthly, the Cube of 8 is 512, (that is, *eee*; lastly, the said four Numbers 8000,9600,3840,512, being set under one another in such order that Units may stand under Units, Tens under Tens, *&c.* and added together make 21952, the Cube of the given Root 28.

2. When the given Number or Root whose Cube is desired consists of three or more places, as 28503; First, the Cube of the two foremost Figures, that is, of 28, must be found out in like manner as before in *Example* 1. so there will be produced 21952. Secondly, write 28 in a void place, and annexing a Cypher to it, it makes 280, this Number must now be esteemed *a*, and 5 the next following Character of the Root must be taken for *e*; then according to these values of *a* and *e* the Numbers signified by *aaa*, *3aae*, *3aee*, and *eee*, being added together make 23149125 for the Cube of 285, (as you see in *Example* 2.) where observe that to find the Cube of 280, that is, of *a*, you need only annex three Cyphers to 21952, which was before found for the Cube of 28. Thirdly, annex a Cypher to 285 after it is set in a spare place, and it makes 2850 for a new value of *a*, and the next following Character of the Root, to wit, 0, is the new value of *e*: Then according to these values of *a* and *e*, the value of $aaa+3aae+3aee+eee$ is 23149125000, that is, *aaa* only; for $e=0$, and consequently $3aae+3aee+eee=0$, so the said 23149125000 is found for the Cube of 2850. Lastly, by annexing a Cypher to 2850 it makes 28500 for a new value of *a*, and 3 the last Figure of the Root is the new value of *e*; then according to these values of *a* and *e* the Sum of the Numbers signified by *aaa*, *3aae*, *3aee*, and *eee*, makes 23156436019527, which is the Cube of the given Root 28503, as may easily be proved by multiplying the said Root into it self Cubically. Compare the following Example with the precedent Directions.

Example 2. *of* Sect. II.

	2	8	5	0	3	Root proposed.
$a=20$	8	000				*aaa*
$e=8$	9	600				*3aae*
	3	840				*3aee*
		512				*eee*
$a=280$	21	952	000			*aaa*
$e=5$	1	176	000			*3aae*
		21	000			*3aee*
			125			*eee*
$a=2850$	23	149	125	000		*aaa*
$e=0$				000		*3aae*
				000		*3aee*
				000		*eee*
$a=28500$	23	149	125	000	000	*aaa*
$e=3$		7	310	250	000	*3aae*
				769	500	*3aee*
					27	*eee*
	23	156	436	019	527	Cube desired.

Sect.

Sect. III. *Of the Composition of a Biquadrate, or the fourth Power, from a Number given for the Root.*

1. Let the Root 28 be proposed, and its Biquadrate or fourth Power desired. First, I write the Root 28 in such manner that there may be space enough to set three Figures between 2 and 8; then having drawn a Line under 28, and downright Lines, as in former Examples, I conceive the Root 28 to be divided into 20 and 8, that is, *a* and *e*; now forasmuch as the Biquadrate, or fourth Power produced from the Binomial Root $a+e$ is $aaaa + 4aaae + 6aaee + 4aeee + eeee$, (as appears by the Table in *Sect.* 4. *Chap.* 1.) therefore the fourth Power of 20+8, (that is, of 28) may be composed thus, *viz.* First, the fourth Power of 20 is 160000, (that is *aaaa*;) secondly, four times the Cube of 20 being multiplied by 8 produces 256000, (that is, *4aaae*;) thirdly, six times the Square of 20 being multiplied by the Square of 8 produces 153600, (that is, *6aaee*,) fourthly, four times 20 multiplied by the Cube of 8 produces 40960, that is, *4aeee*; fifthly, the fourth Power of 8 is 4096, (that is, *eeee*;) lastly, the Sum of all the said five Numbers, to wit, 160000, 256000, 153600, 40960, and 4096 makes 614656, which is the fourth Power of 28 the Root proposed; as will easily appear by the Multiplication of 28 four times into it self.

	2	8	Root proposed.
$a=20$	16	0000	*aaaa*
$e=8$	25	6000	*4aaae*
	15	3600	*6aaee*
	4	0960	*4aeee*
		4096	*eeee*
	61	4656	Biquadrate desired.

2. When the given Number or Root whose fourth Power is desired consists of three Places, as 285; First, the fourth Power of the two foremost Figures 28 must be found out, in like manner as in Example 1. of this *Sect.* so there will be produced 614656 for the fourth Power of 28. Secondly, let 28 be set in a void place, and annex a Cypher to it, so it makes 280, which must now be esteemed *a*, and 5 the next following Character of the Root must be taken for *e*; and then according to these values of *a* and *e* the Numbers signified by *aaaa*, *4aaae*, *6aaee*, *4aeee*, and *eeee* being added together make 6597500625, which is the fourth Power of the given Root 285, and the work will stand as you see in the following Example 2. After the same manner the work is to be continued when the given Root consists of more than three places, as is manifest by the following Example 3.

Example 2. *of* Sect. III.

	2	8	5	Root proposed.
$a=20$	16	0000		*aaaa*
$e=8$	25	6000		*4aaae*
	15	3600		*6aaee*
	4	0960		*4aeee*
		4096		*eeee*
$a=280$	61	4656	0000	*aaaa*
$e=5$	4	3904	0000	*4aaae*
		1176	0000	*6aaee*
		14	0000	*4aeee*
			625	*eeee*
	65	9750	0625	Biquadrate required.

Example

Example 3. *of* Sect. III.

	2	8	0	5	Root proposed.
$a=20$	16	0000			*aaaa*
$e=8$	25	6000			4*aaae*
	15	3600			6*aaee*
	4	0960			4*aeee*
		4096			*eeee*
$a=280$	61	4656	0000		*aaaa*
$e=0$			0000		4*aaae*
			0000		6*aaee*
			0000		4*aeee*
			0000		*eeee*
$a=2800$	61	4656	0000	0000	*aaaa*
$e=5$		4390	4000	0000	4*aaae*
		11	7600	0000	6*aaee*
			140	0000	4*aeee*
				625	*eeee*
	61	9058	1740	0625	Biquadrate desired.

Sect. IV. *Of the Composition of the fifth Power from a Number given for its Root.*

1. Let the Root 28 be proposed, and its fifth Power desired: First, let the Root 28 be written in such manner, that there may be space enough to set 4 Figures between 2 and 8; then having drawn a Line under 28, and downright Lines, as in the Examples of the precedent Section, let 28 be conceived to be divided into 20 and 8, that is, *a* and *e*; now forasmuch as the fifth Power produced from the Binomial Root $a+e$ is *aaaaa*+5*aaaae*+10*aaaee*+10*aaeee*+5*aeeee*+*eeeee*, (as is manifest by the Table in *Sect.* 4. *Chap.* 1.) Therefore the fifth Power of 20+8 (that is, of 28) may be composed thus; First, the fifth Power of 20 is 3200000, (that is, *aaaaa*;) secondly, five times the fourth Power of 20 being multiplied by 8 produces 6400000, (that is, 5*aaaae*;) thirdly, ten times the Cube of 20 being multiplied by the Square of 8 produces 5120000, that is, 10*aaaee*;) fourthly, ten times the Square of 20 multiplied by the Cube of 8 produces 2048000, (that is, 10*aaeee*;) fifthly, five times 20 multiplied by the fourth Power of 8 produces 409600, (that is, 5*aeee*;) sixthly, the fifth Power of 8 is 32768, (that is, *eeeee*; lastly, the Sum of all those six Numbers, *viz.* 3200000, 6400000, 5120000, 2048000, 409600, and 32768 makes 17210368, which is the fifth Power of 28 the Root proposed, as will easily appear by multiplying 28 five times into it self.

	2	8	
$a=20$	32	00000	*aaaaa*
$e=8$	64	00000	5*aaaae*
	51	20000	10*aaaee*
	20	48000	10*aaeee*
	4	09600	5*aeeee*
		32768	*eeeee*
	172	10368	

2. When the given number or Root, whose fifth Power is desired, consists of three places, as 285; First, the fifth Power of the two foremost Figures 28 must be found out in like manner as in Example 1. of this *Sect.* so there will be produced 17210368 for the fifth Power of 28. Secondly, let 28 be set in a void place, and annex a Cypher to it, so it makes 280, which must now be esteemed *a*, and 5 the next following Character of the Root must be taken for *e*; then according to these values of *a* and *e* the Numbers signified by *aaaaa*, 5*aaaae*, 10*aaaee*, 10*aaeee*, 5*aeeee*, and *eeeee*, being added together make 1880287678125, which is the fifth Power of the given Root 285, and the work will stand as you see in the following Example 2. Nor will the Operation be more difficult (though more laborious) to find the fifth Power of a Number (or Root) consisting of four or more places.

Example 2. *of* Sect. IV.

	2	8	5	Root propoſed.
$a=20$	32	00000		*aaaaa*
$e=\ 8$	64	00000		5*aaaae*
	51	20000		10*aaaee*
	20	48000		10*aaeee*
	4	09600		5*aeeee*
		32768		*eeeee*
$a=280$	172	10368	00000	*aaaaa*
$e=\ 5$	15	36640	00000	5*aaaae*
		5488	00000	10*aaaee*
		980	00000	10*aaeee*
		8	75000	5*aeeee*
			3125	*eeeee*
	188	20876	78125	Fifth Power deſired.

By the Precedent Rules and Examples of this Chapter, the Ingenious Reader will eaſily apprehend how to compoſe the ſixth, ſeventh, or any higher Power, from a Root given in Number, and conſidered as a Binomial $a+e$, as before hath been directed. The main Buſineſs conſiſting in a right underſtanding of the Number ſignified by a and e, and in finding out the Numbers anſwering to the Members of the deſired Power of $a+e$, according to the Table in *Sect.* 4. of the precedent *Chap.* 1.

CHAP. III.

Concerning the Reſolution of Powers expreſt by Numbers, or the Extraction of all kinds of Roots out of Powers given in Numbers.

Sect. I. *Of the Extraction of the Square Root out of a Number given.*

1. LEt it be obſerved in general, that the Reſolution of every Power given in Numbers conſiſts in a Regular Subtraction of thoſe Numbers which are ſuppoſed to be added together in the Compoſition of each Power reſpectively, according to the Rules of the laſt preceding Chapter, wherein I preſuppoſe the Reader to be well exerciſed. And for the more ready Extraction of any Root, it will be convenient to have in a readineſs the reſpective Powers of the nine ſingle Figures; as if the Square Root be deſired, then the Squares of 1, 2, 3, 4, 5, 6, 7, 8, 9, will be uſeful, which Roots and Squares are expreſt in the following Tabulet.

ROOTS.	1	2	3	4	5	6	7	8	9
SQUARES.	1	4	9	16	25	36	49	64	81

2. When a whole Number is propoſed, and its Square Root deſired, the Number propoſed muſt be prepared for Extraction, by diſtributing it into parts or members after this manner, *viz.* Firſt, ſet a point over the firſt or Units place of the given Number, then paſſing over the ſecond place ſet another Point over the third; alſo paſſing over the fourth place ſet another Point over the fifth: and in that order if there be more places in the given Number, Points are to be ſet, ſo that between every two Points which ſtand next to one another, there will be one place without any Point over it. As for Example: If the Square Root of 119025 be deſired, I ſet Points as here you ſee, whereby the ſaid Number is diſtributed into 3 Members, to wit 11,90,25. In like manner if the Square Root of 785 be deſired,

$\dot{1}1\dot{9}0\dot{2}5$

the

the Points will ſtand as you ſee here, whereby the ſaid 784 is diſtributed into two Members 7 and 84. The Points ſet as aforeſaid ſhew the number of places that will be found in the Root; for if there be two Points, there will be two places in the Root; if three Points, then the Root will conſiſt of three places, &c. The Points alſo ſhew what Member of the Number given belongs to the finding out of every ſingle Character of the Root ſought, as is evident by the Rules in *Sect.* 1. of the precedent *Chap.* 2. Theſe things being premiſed as preparatory to the Extraction of the Square Root, I ſhall proceed to Examples.

$\dot{7}\ 8\ \dot{4}$

Example 1.

3. Let it be required to extract the Square Root of 784. By the Preceding Rule 2. it is evident that the deſired Root conſiſts of two places, *viz.* of ſome number of Tens under 100, and of ſome number of Units under 10; which two numbers (agreeable to the compoſition of a Square in *Sect.* 1. of the ptecedent *Chap.* 2.) may be repreſented by *a* and *e*, ſo that *a* and *e* ſignifie the Root ſought; and conſequently the Square of $a+e$, that is, $aa+2ae+ee$ is equal to the propoſed Number 784. Now to find out the number of Tens, (that is, *a*) in the Root; (after a crooked Line is drawn on the right hand of the given Number, that the Root, like the Quotient in Diviſion, may be ſet next after the ſaid crooked Line, as alſo a downright Line next after each of the Points, as here you ſee;) the firſt work in the Extraction is always to ſubtract the greateſt Square whole Number contained in the firſt Member towards the left hand from the ſaid Member, and to write the Root of the ſaid ſquare Number in the Quotient for the firſt ſingle Figure of the deſired Root: ſo 4 being the greateſt Square contained in the firſt Member 7, I ſubſcribe 4 under 7, and ſet 2 the Root of the ſaid 4 in the Quotient, then after a Line is drawn under 4, I ſubtract 4 from 7, or 400 from 784, and there remains the *Reſolvend* 384, that is, that part of the given Number 784, which is yet to be reſolved. Now obſerve, that the ſaid 2 in the Quotient, in reſpect of the next following unknown Character of the Root, is really 20, which is the Number ſignified by *a* in the Compoſition; and the Square of 20, to wit 400, is *aa*, which being the firſt Number found in the Compoſition, is the firſt Number to be ſubtracted in the Reſolution. Obſerve alſo, that the next ſingle Character of the Root, whither it happen to be a Figure or a Cypher, is called *e*, which is yet unknown.

$\dot{7}$	$\dot{8}4$	(2
4		
3	84	

4. Then I proceed to find the value of *e*, that is, the greateſt ſingle Character with this Condition, that the ſum of the Numbers ſignified by $2ae$ and ee may not exceed the *Reſolvend* 384; for from this Number that ſum muſt be ſubtracted. Now becauſe (for the reaſon aforeſaid) *a* is 20, therefore $2a$ is 40, which muſt be eſteemed a *Diviſor*, and ſet under the *Reſolvend*; then I divide the ſaid *Reſolvend* 384 by 40, and find the Quotient 9 for the Number *e*, provided it will anſwer the Condition before mentioned; and therefore I make Tryal (in a waſt Paper) to ſee whether 9 will ſatisfie the ſaid Condition or not in this manner, *viz.* If *e* be 9, and $2a$ 40, then conſequently $2ae$ is 360, and ee is 81; therefore $2ae+ee=441$, this ought to be ſubtracted from the *Reſolvend* 384; but 441 exceeds 384, and therefore cannot be ſubtracted from it, ſo as to leave a real Remainder; whence I conclude, that *e* muſt be leſs than 9: and therefore I make tryal with 8 in like manner as before with 9, *viz.* If $e=8$ and $2a=40$, then conſequently $2ae=320$, and $ee=64$; therefore $2ae+ee=384$, which may be ſubtracted from the *Reſolvend* 384; wherefore I conclude that *e*, that is the Figure which muſt follow 2 in the Quotient) is 8, which I ſet in the Quotient: then I ſubſcribe 320 and 64 (before found) under the *Reſolvend* 384, (in ſuch order that Units may ſtand under Units, and Tens under Tens) and adding the ſaid 320 and 64 together, the ſum is 384, which ſome Authors call the *Gnomen*, others, the *Ablatitium*) which ſubtracted from the *Reſolvend* 384 leaves 0; ſo the whole Extraction is finiſh'd,

	$\dot{7}$	$\dot{8}4$	(28
Subtract	4	00	aa
	3	84	2 *Reſolvend*
$a=20$		40	$2a$ *Diviſor*
$e=\ 8$	3	20	$2ae$
		64	ee
Subtract	3	84	*Ablatitium*
		00	

 and

and the Square Root of the given Number 784 is found 28, which is the true Root ſought, for 28 multiplied by 28 produces 784.

NOTE 1.

The firſt Operation in the Extraction of the Square Root is always to ſubtract the greateſt ſquare whole Number, (that is, aa) contained in the firſt Member (towards the left hand) of the given number, from the ſaid Member, and to ſet the Root of the ſaid Square in the Quotient, (as has been ſhewn in the third ſtep) which Root is the firſt Figure of the Root ſought. This Work is no more repeated in the whole Extraction, but the work in the fourth ſtep is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

After the firſt Figure of the Root ſought is known, and ſet in the Quotient, let it be written in a void place, and multiplied by 10, (by annexing to the ſaid firſt Figure a Cypher towards the right hand) then is the Product to be taken for the value of a, in order to the finding out of the firſt *Diviſor*. Alſo when the firſt and ſecond Characters of the Root are ſet in the Quotient, and there be yet another to come forth, then the Number conſiſting of thoſe two Characters with a Cypher annexed to them, is to be taken for a new value of a, in order to the finding out of the ſecond *Diviſor*. Likewiſe, when the firſt, ſecond, and third Characters of the Root are ſet in the Quotient, and there be yet another to come forth, then the number conſiſting of thoſe three Characters with a Cypher annexed to them, is to be taken for a new value of a; and ſo forwards, when there be more Characters in the Root. The reaſon of which Work is manifeſt from the Compoſition of Powers in the precedent *Chap.* 2.

But the Letter e repreſents every ſingle unknown Figure or Cypher next following that part of the Root which is already diſcovered and ſet in the Quotient. This Note concerning the Eſtimation of a and e is to be obſerved not only in the Extraction of the Square Root, but of any Root whatever.

NOTE 3.

After the Number ſignified by a is found out by Note 2. the *Diviſor*, which ſhews how to begin the Tryal in ſearching out the unknown ſingle Character repreſented by e, is conſequently known: for in the Reſolution of every Power produced from the Binomial Root $a+e$, the *Diviſor* conſiſts of ſuch Powers of a as are multiplied into the Powers of e; and becauſe the Square Root of $a+e$ is $aa+2ae+ee$, therefore in the Extraction of the Square Root the *Diviſor* is $2a$; ſo that when the Number a is known, the *Diviſor* $2a$ is conſequently known.

NOTE 4.

When the *Diviſor* is found out by Note 3. as alſo the *Ablatitium*, (that is, the Number to be ſubtracted) which in the Extraction of the Square Root is compos'd of $2ae$ and ee, the two numbers ſignified by $2ae$ and ee muſt each of them be ſet in ſuch order under the preſent *Reſolvend*, (that is, the number remaining to be reſolved) that Units may ſtand under Units, Tens under Tens, &c. to the end that the *Ablatitium* may be rightly compoſed and ſubtracted from the preſent *Reſolvend*.

NOTE 5.

When the *Diviſor* is not contained once in the particular or preſent *Reſolvend*, a Cypher (to wit, 0) muſt be ſet in the Quotient; and then the *Reſolvend* muſt be augmented with the next Member (towards the right hand) of the Power propoſed, for a new particular *Reſolvend*. Alſo a new *Diviſor* muſt be found out by Note 3, and the like is to be done as often as the *Diviſor* is not contained once in the particular *Reſolvend*. The Practice of theſe Notes will be ſhewn in the following Example.

Example

Example 2.

5. If the Square Root of 2285126809 be desired, it will be found 47803 by the precedent Rules, and the work will stand as here you see underneath.

	22	85	12	68	09	(47803. *Root.*
Subtract	16					*aa*
	6	85				*Resolvend.*
$a = 40$		80				2*a*. *Divisor.*
$e = 7$	5	60				2*ae*
		49				*ee*
Subtract	6	09				*Ablatitium.*
		76	12			*Resolvend.*
$a = 470$		9	40			2*a*. *Divisor.*
$e = 8$		75	20			2*ae*
			64			*ee*
Subtract		75	84			*Ablatitium.*
			28	68		*Resolvend.*
$a = 4780$			95	60		2*a*. *Divisor.*
$e = 0$			28	68	09	*Resolvend.*
$a = 47800$			9	56	00	2*a*. *Divisor.*
$e = 3$			28	68	00	2*ae*
					9	*ee*
Subtract			28	68	09	*Ablatitium.*
			00	00	00	

Explication of Example 2.

The first Figure of the Root is 4, (by the foregoing Note 1.) whose Square 16 subtracted from 22 the first Member towards the Left-hand of the number proposed leaves 6, to which the second Member 85 being annexed, there arises 685 for the next *Resolvend*: Or to cause the same Effect, suppose 0 to be annexed to 4 the first Figure of the Root, and it makes 40, (that is, *a*,) whose Square 1600 (or *aa*) subtracted from 2285 the two first Members of the Number first proposed, leaves (as before) the *Resolvend* 685.

Then, the first Figure of the Root being found 4, the value of *a* is 40, (by Note 2.) which doubled gives 80 for a Divisor to the *Resolvend* 685 by Note 3.) and then by dividing and making Tryal as is directed in the precedent fourth step, the number *e* will be found 7 for the second Figure of the Root, and consequently the numbers signified by 2*ae* and *ee* are 560 and 49; these being set orderly and added together (according to Note 4.) make the *Ablatitium* 609, which subtracted from the said *Resolvend* 685, there remains 76, to which annexing 12 the third Member of the Number first proposed, it makes 7612 for a new *Resolvend*.

Again, the two formost figures of the Root being found 47, the new value of *a* is 470, (by Note 2.) which doubled gives 940 for a Divisor to the said *Resolvend* 7612, (by Note 3.) then by dividing and making Tryal as is directed in the fourth step, the value of *e* is found 8 for the first Figure of the Root; whence the number signified by 2*ae* and *ee* are 7520 and 64; these being set orderly and added together (according to Note 4.) make the *Ablatitium* 7584, which subtracted from the *Resolvend* 7612 before-mentioned, leaves 28, to which annexing 68 the fourth Member of the Number first proposed, it makes 2868 for a new *Resolvend*.

Again

Again, the three foremost figures of the Root being 478, the value of a is 4780, (by Note 2.) which doubled gives 9560 for a divisor to the said *Resolvend* 2868, by Note 3.) then by dividing as aforesaid the value of e is found 0; therefore, (according to Note 5.) I set 0 in the Quotient, and because in this case the *Ablatitium* is also 0, the *Resolvend* 2868 from which the said *Ablatitium* ought to be subtracted remains the same without alteration; therefore by annexing 09 the last member of the number first proposed, to the said 2868 it makes 286809 for a new (and the last) Resolvend. Lastly, by proceeding as before, the last Figure of the Root will be found 3; so that the Square Root sought is 47803; for this multiplied by it self produces 2285126809, the number whose Square Root was desired.

The Premisses may suffice to shew a perfect Method of extracting the Square Root of a whole number having an exact Square Root, which I have explain'd at large, that the Reason and certainty of the Rules might be apparent. But this Method may be contracted into more practical and compendious Rules, as I have shewn in the 32 *Chap.* of Mr. *Wingate*'s common Arithmetic.

6. But when a whole Number has not a Square Root exactly expressible by any rational or true Number, then to approach infinitely near the exact Root, first, pairs of Cyphers, as 00, 0000, 000000, or 00000000, *&c.* are to be annexed to the Number given; then esteeming the number given with the Cyphers annexed to be one whole Number, let its Square Root be extracted according to the Precedent (or other practical) Rules; that done, look how many Points were set over the Number first given, for so many of the foremost places in the Quotient are to be taken for the Integers in the Root, and the rest following those Integers express the Fractional part of the Root in Decimal parts. As for Example: If the Square Root of 12 be desired, I annex six Cyphers to 12, thus 12.000000, and then the Square Root of 12.000000 being extracted, it will be found 3.464, that is, $3\frac{464}{1000}$. But because after the Extraction is finished, there happens to be a Remainder, I conclude that $3\frac{464}{1000}$ is less than the true Root, but $3\frac{465}{1000}$ is greater than it. So that by annexing three pairs of Cyphers you will not miss $\frac{1}{1000}$ part of an Unit of the true Root, and by annexing eight Cyphers you will not want $\frac{1}{10000}$ part: and in that order you may approach as near as you please, when you cannot obtain the exact Square Root of a whole Number given.

7. The Square Root of a Vulgar Fraction is found out thus, *viz.* First, if the Fraction be not in its least terms, let it be reduced to the least Terms; then extract the Square Root of the Numerator for a new Numerator, and the Square Root of the Denominator for a new Denominator, so shall this new Fraction be the Square Root of the Fraction proposed. As for Example: The Square Root of $\frac{9}{16}$ is $\frac{3}{4}$; likewise the Square Root of $\frac{1}{4}$ is $\frac{1}{2}$.

But when either the Numerator or Denominator of a vulgar Fraction has not a perfect Square Root, then to find the Square Root of that Fraction very near; first reduce the Fraction to a Decimal Fraction, whose Numerator may consist of an even number of places, *viz.* of two, four, or six places, *&c.* then extract the Square Root that Decimal as if it were a whole Number, and the Root that comes forth shall be a Decimal Fraction, expressing nearly the Square Root of the Fraction proposed. As for Example: If the Square Root of $\frac{13}{16}$ be desired, I first reduce it to this Decimal Fraction, .81250000; (for as 16. 13 :: 100000000. 81250000) then by extracting the Square Root of .81250000 as if it were a whole Number, I find .9013, that is, $\frac{9013}{10000}$, which is near the Square Root of $\frac{13}{16}$, for it wants not $\frac{1}{10000}$ part of an Unit of the exact Square Root of $\frac{13}{16}$.

8. Lastly, if the Square Root of a mixt number be desired, first reduce it to an improper Fraction, and then extract the Square Root of that improper Fraction as before; but if it has not an exact Square Root, then reduce the Fractional part of the mixt number first proposed to a Decimal Fraction of an even number of places, and after this Decimal is annexed to the Integers of this mixt number, extract the Square Root out of the whole, then so many Points as were set over the Integers, so many of the foremost places in the Quotient are to be taken for the Integers in the Root, and the rest express the Fractional part of the Root in Decimal Parts. As for Example: The Square Root of $34\frac{13}{64}$, that is, of $\frac{2189}{64}$, will be found $\frac{47}{8}$ or $5\frac{7}{8}$; and the Square Root of $7\frac{2}{3}$, that is, of 7.666666, *&c.* is 2.708, *&c.* that is: $2\frac{708}{1000}$, *&c.*

Sect.

Sect. II. *Of the Extraction of the Cubic Root out of a Number given.*

1. For the more ready extraction of the Cubic Root of a number given, the following Tabulet will be uſeful, which ſhews at firſt ſight the Cubic Root of any Cubical whole Number leſs than 1000.

ROOTS.	1	2	3	4	5	6	7	8	9
CUBES.	1	8	27	64	125	216	343	512	729

2. When a whole number is propoſed, and its Cubic Root deſired, the Number given muſt be prepared for Extraction, by diſtributing it into parts or members after this manner; *viz.* Firſt, a point is to be ſet over the Units place of the given number; then paſſing over the ſecond and third places towards the left hand, another point is to be ſet over the fourth place; alſo paſſing over the fifth and ſixth places another point is to be ſet over the ſeventh place: and in that order as many points are to be ſet as the number propos'd will admit, and conſequently between every two adjacent points there will be two Places without Points. So if the Cubic Root of 1331 be deſired, after Points are ſet as is above directed, the ſaid 1331 will be diſtributed into 2 Members, to wit, 1, and 331. In like manner if the Cubic Root of 21952 be required, the Points will ſtand as you ſee in the Example, and the ſaid 21952 will be diſtributed into two members 21 and 952; likewiſe this Number 941192 being pointed in the ſame order will be diſtributed into the two members 941 and 192; and this number 23156436019527 into theſe five members, 23, 156, 436, 019, 527. The points ſhew the number of places that will be found in the Root; for ſo many points as there be, ſo many places will the Root conſiſt of; they likewiſe ſhew what member of the Number propos'd belongs to the Extraction of every ſingle Character of the Root ſought.

$$\dot{1}33\dot{1}$$
$$2\dot{1}95\dot{2}$$
$$94\dot{1}19\dot{2}$$
$$2\dot{3}15\dot{6}43\dot{6}01\dot{9}52\dot{7}$$

3. The given number whoſe Cubic Root is deſired may be conceived to be produced from the Cubical Multiplication of the Binomial Root $a+e$, and then the ſaid number will be compos'd of theſe four members or ſolid numbers, *viz.* aaa, $3aae$, $3aee$, and eee, (as appears by the third Power in the Table in *Sect.* 4. *Chap.* 1.) Now becauſe the Reſolution of the Cubic number, *viz.* the Extraction of the Cubic Root, is deducible from the ſteps of the Compoſition of a Cubic number from its Root, (for ſuch numbers as are added in the Compoſition are to be ſubtracted in the Reſolution,) reſpect muſt be had to *Sect.* 2. *Chap.* 2. of this Book.

Example 1.

4. Let it be required to extract the Cubic Root of 21952. By the precedent ſecond Rule it is evident that the deſired Root conſiſts of two places, *viz.* of ſome number of Tens under 100, and of ſome number of Units under 10, which two Numbers, (agreeable to the Compoſition of a Cube in *Sect.* 2 of the precedent *Chap.* 2.) may be repreſented by a and e, ſo that $a+e$ ſignifies the Root ſought, and conſequently the Cube of $a+e$, that is, $aaa+3aae+3aee+eee$ is equal to the given number 21952. Now to find out the Number of Tens, (that is, a) in the Root, (after a crooked line is drawn on the right hand of the given number, that the Root, like the Quotient in Diviſion may be ſet next after the ſaid crooked Line, as alſo a downright Line next after each of the Points, as here you ſee.) The firſt Work in the Extraction is always to ſubtract the greateſt Cubic whole number contained in the firſt Member towards the Left hand, from the ſaid member, and to write the Root of the ſaid Cube number in the Quotient for the firſt ſingle Figure of the deſired Cubic Root: So 8 being the greateſt Cube contained in the firſt member 21, I ſubſcribe 8 under 21, and ſet 2 the Cubic Root of the ſaid 8 in the Quotient, then after a line is drawn under 8, I ſubtract 8 from 21, or, 8000 from 21952, and there remains the *Reſolvend* 13952, that is, that part of the propoſed number 21952 which is yet to be reſolved. Now obſerve, that the ſaid 2 in the Quotient,

$2\dot{1}$	$95\dot{2}$	(2
8		
13	952	

in

in reſpect of the next following unknown Character of the Root, is really 20, which is the number ſignified by a in the Compoſition, and the Cube of 20, to wit 8000, is aaa, which being the firſt Number found in the Compoſition, is firſt to be ſubtracted in the Reſolution. Obſerve alſo, that the next ſingle Character of the Root, whether it happen to be a Figure or a Cypher is called e, which is yet unknown.

5. Then I proceed to find the value of e, that is, the greateſt ſingle Character with this condition, that the Sum of the Numbers ſignified by $3aae$, $3aee$, and eee, may not exceed the remaining *Reſolvend* 13952, for from this Number that ſum muſt be ſubtracted. Now becauſe (for the Reaſon aforeſaid) a is 20, therefore $3aa=1200$, and $3a=60$; then ſubſcribing the ſaid 1200 and 60 under the *Reſolvend* 13952, (in ſuch order that Units may ſtand under Units, and Tens under Tens, &c.) and adding them together the Sum is 1260, which muſt be eſteemed a *Diviſor*, and ſet under the *Reſolvend*. Then by ſuppoſing I were to divide the ſaid *Reſolvend* 13952 by 1260, I find the Quotient exceeds 9, but e always repreſents a ſingle Figure or a Cypher, and therefore it cannot exceed 9; wherefore I make tryal with 9 (in a void place) to ſee whether it will anſwer the before mentioned Condition, to which e is ſubject, in this manner, *viz.* Foraſmuch as it was before found that $3aa=1200$ and $3a=60$, it will follow, if we ſuppoſe $e=9$, that $3aae=10800$, alſo $3aee=4860$, and $eee=729$; therefore $3aae+3aee+eee=16389$: this ought to be ſubtracted from the *Reſolvend* 13952 but 16389 exceeds 13952 and therefore cannot be really ſubtracted from it; whence I conclude that e muſt be leſs than 9; and therefore I make tryal with 8 in like manner as before with 9, *viz.* having before found that $3aa=1200$. and $3a=60$, it will follow if we ſuppoſe $e=8$, that $3aae=9600$, alſo $3aee=3840$, and $eee=512$; therefore $3aae+3aee+eee=13952$, which may be ſubtracted from the *Reſolvend* 13952; wherefore I conclude that e (that is, the Figure which muſt follow 2 in the Quotient) is 8, which I ſet in the Quotient: then I ſubſcribe the three Numbers before found, to wit, 9600, 3840, and 512, under the *Reſolvend* 13942, (in ſuch order that the Units may ſtand under Units, Tens under Tens, &c.) and adding together the ſaid three Numbers ſo ſubſcribed, their Sum makes 13952, (the *Ablatitium*) which ſubtracted from the *Reſolvend* 13952, leaves 0. So the Extraction is finiſh'd, and 28 is found to be the Cubic Root of the propoſed Number 21952; for 28 multiplied into itſelf cubically, *viz.* 28×28×28 produces 21952.

	21	952	(28
Subtract	8		aaa
	13	952	*Reſolvend*
$a=20$	1	200	$3aa$
		60	$3a$
	1	260	*Diviſor*
$e=8$	9	600	$3aae$
	3	840	$3aee$
		512	eee
	13	952	*Ablatitium.*
		000	

NOTE 1.

The firſt Operation in the Extraction of the Cubic Root is always to ſubtract the greateſt Cubic whole Number, (that is, aaa) contained in the firſt Member (towards the left hand) of the given Number; from the ſaid Member, and to ſet the Root of the ſaid Cube-number in the Quotient; which Root is the firſt Figure of the Root ſought, as hath been ſhewn in the fourth ſtep. This Work is no more repeated in the whole Extraction, but the Work in the fifth ſtep is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

The Number ſignified by a is to be found out by Note 2 in *Sect.* 1. of this *Chap.* and then the *Diviſor* for the finding of the unknown ſingle Character repreſented by e is conſequently known: For in the Reſolution of every Power produced from the Binomial Root $a+e$, the *Diviſor* conſiſts of ſuch Powers of a as are multiplied into the Powers of e; and becauſe the Cube of $a+e$ is $aaa+3aae+3aee+eee$, therefore in the Extaction of the Cubic Root the *Diviſor* is compoſed of $3aa$ and $3a$, ſo that when the Number a is known, the *Diviſor* $3aa+3a$ is conſequently known.

NOTE

NOTE 3.

When the *Divisor* is found out by the precedent Note 2. as also the *Ablatitium*, which in the Extraction of the Cubic Root is compos'd of $3aae$, $3aee$, and eee; the Numbers signified by the said $3aae$, $3aee$, and eee, must each of them be set in such order under the particular or present *Resolvend*, that Units may stand under Units, Tens under Tens, &c. to the end the *Ablatitium* may be rightly composed and subtracted from the *Resolvend*.

NOTE. 4.

When the *Divisor* is not contained once in the particular or present *Resolvend*, a Cypher (to wit, o) must be set in the Quotient; and then the *Resolvend* must be augmented with the next Member (towards the right Hand) of the Power proposed, for a new particular *Resolvend*. Also a new *Divisor* must be found out by Note 2. of this *Sect.* and the like is to be done as often as the *Divisor* is less than the *Resolvend*.

The Practice of these Notes will be shewn in the following Example.

Example 2.

6. If the Cubic Root of 23156436019527 be desired, it will be found 28503 by the precedent Rules, and the Work will stand as here you see underneath.

	23	156	436	019	527	(28503 *Root.*
Subtract	8					*aaa*
	15	156				*Resolvend.*
$a=20$	1	200				$3aa$
		60				$3a$
	1	260				*Divisor.*
$e=8$	9	600				$3aae$
	3	840				$3aee$
		512				eee
Subtract	13	952				*Ablatitium.*
	1	204	436			*Resolvend.*
$a=280$		235	200			$3aa$
			840			$3a$
		236	040			*Divisor.*
$e=5$	1	176	000			$3aae$
		21	000			$3aee$
			125			eee
Subtract	1	197	125			*Ablatitium.*
	0	007	311	019		*Resolvend.*
$a=2850$		24	367	500		$3aa$
$e=0$			8	550		$3a$
		24	376	050		*Divisor.*
		7	311	019	527	*Resolvend.*
$a=28500$		2	436	750	000	$3aa$
				85	500	$3a$
		2	436	835	500	*Divisor.*
$e=3$		7	310	250	000	$3aae$
				769	500	$3aee$
					27	eee
Subtract		7	091	019	527	*Ablatitium.*
		0	000	00	000	

Explication of Example 2.

The first Figure of the Root is 2 (by Note 1.) whose Cube 8 subtracted from 23, the first Member of the Number propos'd leaves 15, to which the second Member 156 being

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annexed, there ariſes 15156 for the next *Reſolvend.* Or to cauſe the ſame effect, ſuppoſe 0 to be annexed to 2, the firſt Figure of the Root, and it makes 20, (that is, *a*) whoſe Cube 8000 (or *aaa*) ſubtracted from 23156, the two foremoſt Members of the Number firſt propos'd, leaves (as before) the *Reſolvend* 15156.

Then the firſt Figure of the Root being found 2, the value of *a* is 20, and the *Diviſor* is 1260, (by Note 2.) and then by dividing and making tryal, as is directed in the foregoing fifth ſtep, the Number *e* will be found 8 for the ſecond Figure of the Root, and conſequently the Numbers ſignified by 3*aae*, 3*aee*, and *eee*, are 9600, 3840, and 512; theſe being ſet orderly and added together (according to Note 3.) make the *Ablatitium* 13952, which ſubtracted from the *Reſolvend* 15156 leaves 1204, to which annexing 436, the third Member of the Number firſt propoſed, it makes 1204436 for a new *Reſolvend.* The reſt of the Operation in Example 2. being but a Repetition of what has been directed for finding out the ſecond Figure of the Root, I ſhall leave it to the Learner's Practice.

The precedent Rules and Notes in this *Sect.* 2. for extracting the Cubic Root of a whole Number, having an exact Cubic Root, are expreſs'd at large, that the Reaſon of the Work might be apparent; but this Method may be contracted into more practical and compendious Rules, as I have ſhewn in the 33 *Ch.* of Mr. *Wingate*'s Common Arithmetic.

7. But when a whole Number has not a Cubic Root exactly expreſſible by any rational or true Number, then to approach infinitely near the exact Root, firſt, Ternaries of Cyphers, *viz.* three, or ſix, or nine, or twelve, &c. Cyphers are to be annexed to the whole Number given; then eſteeming the Number given with the Cyphers annexed to be one whole Number, let its Cubic Root be extracted by the precedent (or other practical) Rules. That done, look how many Points were ſet over the Number firſt given, for ſo many of the foremoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt following thoſe Integers expreſs the Fractional part of the Root in Decimal Parts. As for Example: If the Cubic Root of 8302348 be deſired, I annex ſix Cyphers to 8302348 thus, 8302348.000000, and then the Cubic Root of $\dot{8}30\dot{2}34\dot{8}00\dot{0}00\dot{0}$ being extracted, it will be found 202.48, that is, $202\frac{48}{100}$; but becauſe after the Extraction is finiſh'd there happens to be a Remainder, I conclude that $202\frac{48}{100}$ is leſs than the true Cubic Root ſought, but $202\frac{49}{100}$ is greater than it, ſo that by annexing ſix Cyphers you will not miſs $\frac{1}{100}$ part of an Unit of the true Root, and by annexing nine Cyphers you will not want $\frac{1}{1000}$ part; and in that order you may approach as near as you pleaſe when you cannot obtain the exact Cubic Root of a whole Number given.

8. The Cubic Root of a Vulgar Fraction is found out thus, *viz.* firſt, if the Fraction be not in its leaſt Terms, let it be reduced to the leaſt Terms; then extract the Cubic Root of the Numerator for a new Numerator, and the Cubic Root of the Denominator for a new Denominator, ſo ſhall this new Fraction be the Cubic Root of the Fraction propoſed. As for Example: The Cubic Root of $\frac{8}{27}$ is $\frac{2}{3}$, and the Cubic Root of $\frac{1}{8}$ is $\frac{1}{2}$.

9. But when either the Numerator or Denominator of a Vulgar Fraction has not a perfect Cubic Root, then to find the Cubic Root of that Fraction very near, firſt reduce the Fraction to a Decimal Fraction, whoſe Numerator may conſiſt of Ternaries of places, *viz.* either of three, ſix, nine, or twelve, &c. places, and then extract the Cubic Root of that Decimal as if it were a whole Number, and the Root that comes forth ſhall be a Decimal Fraction expreſſing nearly the Cubic Root of the Vulgar Fraction propoſed. As for Example: If the Cubic Root of $\frac{2}{3}$ be deſired, I firſt reduce it to this Decimal Fraction, .666666666666, and then by extracting the Cubic Root of the ſaid Decimal as if it were a whole Number, I find .8735, that is, $\frac{8735}{10000}$; which is near the Cubic Root of $\frac{2}{3}$, for it wants not $\frac{1}{10000}$ part of an Unit of the exact Cubic Root of $\frac{2}{3}$.

10. Laſtly, if the Cubic Root of a mixt Number, that is, of a whole Number with a Fraction in its leaſt Terms, be deſired; firſt reduce it to an improper Fraction, and then extract the Cubic Root of that improper Fraction in like manner as before in the eighth ſtep; but if it has not an exact Cubic Root, then reduce the Fractional part of the mixt Number firſt propoſed to a Decimal Fraction, whoſe Numerator may conſiſt of Ternaries of places, and after this Decimal is annexed to the Integers of the mixt Number, extract the Cubic Root out of the whole, then ſo many Points as were ſet over the Integers, ſo many of the foremoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt expreſs the Fractional part of the Root in Decimal parts. As for Example: The

Cubic

Cubic Root of $12\frac{19}{27}$, that is, of $\frac{343}{27}$, will be found $\frac{7}{3}$ or $2\frac{1}{3}$; and the Cubic Root of $2\frac{1}{3}$, that is, of 3.375000000, &c. will be found 1.334, &c. that is, $1\frac{334}{1000}$, &c.

Sect. III. *Of the Extraction of the Biquadratic Root out of a Number given.*

1. The briefest way to extract the Root of a Biquadratic Number, that is, of a Number produced by the Multiplication of some Number or Root four times into it self, is first to extract the Square Root of the Number proposed, and then to extract the Square Root of that Root. As for Example: If the Root of the Biquadratic Number, or fourth Power 256 be desired; first, the Square Root of 256 being extracted is 16, and then the Square Root of 16 is 4, which is the Root of the fourth Power 256; for 4×4×4×4 produces 256. But my purpose being to explain the general Method for the Extraction of all kinds of Roots, I shall upon that Foundation shew how to extract the Root of a Biquadratic Number.

2. For the more ready Extraction of the Biquadratic Root, the following Tabulet will be useful, which shews at first sight the Root of any Biquadratic whole Number under 10000.

Roots . . .	1	2	3	4	5	6	7	8	9
Fourth Powers	1	16	81	256	625	1296	2401	4096	6561

3. When a whole Number is proposed, and it is desired to extract the Biquadratic Root of that Number, set Points over the given Number in this manner, *viz.* first, set a Point over the Units place, then passing over the three next places towards the left Hand set another Point over the fifth place, and in that order as many Points are to be set as the given Number will admit, that there may be three places between every two adjacent Points. So if the Biquadratic Root of 614656 be desired, after Points are set as is above directed, the said 614656 will be distributed into two Members, to wit, 61 and 4656. In like manner this Number 6597500625 being pointed in the same order will be distributed into these three Members, 65, 9750, and 0625. The Points shew the number of places that will be found in the Root, as also what Member of the Number propos'd belongs to the Extraction of every single Character of the Root sought.

$6\dot{1}465\dot{6}$

$6\dot{5}975\dot{0}062\dot{5}$

4. The given Number, whose Biquadratic Root is desired may be conceived to be produced from the Multiplication of the Binomial Root $a+e$ four times into it self, and then the said Number will be composed of these five Members or Numbers, *viz.* $aaaa$, $4aaae$, $6aaee$, $4aeee$, $eeee$, (as is manifest by the fourth Power in the Table in *Sect.* 4. *Chap.* 1. of this Book.) Now because the Resolution of a Biquadratic Number, *viz.* the Extraction of the Biquadratic Root is deducible from the steps of the Composition of a Biquadratic Number from its Root, (for such Numbers as are added in the Composition are to be subtracted in the Resolution) respect must be had to *Sect.* 3. *Chap.* 2. of this Book.

Example.

5. Let it be required to extract the Biquadratic Root of 614656. After the Number given is prepared by Punctations as before is directed, I seek in the Tabulet in the precedent second step of this *Sect.* 3. for the greatest Biquadratic whole Number contained in 61, the first Member (towards the left Hand) of the Number proposed, and finding it to be 16, I subscribe 16 under 61, and write 2 the Root of the said fourth Power 16 in the Quotient, for the first Figure of the Root sought; then after a Line is drawn under 16 I subtract 16 from 61, or 160000 from 614656, and there remains to be resolved 454656.

			(2
$6\dot{1}$	$465\dot{6}$		
16			
—	—		
45	4656		

The *Divisor* for the finding out of *e*, that is, every Character which is to follow 2, the first Figure of the Root, is always in the Extraction of the Biquadratic Root composed of these Numbers, *viz.* 4*aaa*, 6*aa*, and 4*a*, for these are all the Powers of *a* that are drawn into the Powers of *e* in the fourth Power of $a+e$; (as is evident by the Table in *Sect.* 4. *Chap.* 1.) and because the first Figure of the Root is found 2, and consequently (by Note 2. in *Sect.* 1. of this *Chap.*) the Number signified by *a* is 20, therefore the Sum of the Numbers signified by 4*aaa*, 6*aa*, and 4*a*, is 34480, which is the *Divisor*; then supposing I were to divide the *Resolvend* 454656 by the *Divisor* 34480, I find the Quotient exceeds 9; but in regard *e* always represents either a single Figure or a Cypher, it cannot exceed 9: and therefore I make tryal (in a wast Paper) with 9, to see whether it will constitute an *Ablatitium* that does not exceed the *Resolvend* 454656, *viz.* I suppose $e=9$; then because *a* was before found 20, the *Ablatitium*, which in the Extraction of the Biquadratic Root is always compos'd of 4*aaae*, 6*aaee*, 4*aeee*, and *eeee*, will exceed the *Resolvend*, from which it ought to be subtracted. But if $e=8$, then the *Ablatitium* will be equal to the *Resolvend*, and consequently that being subtracted from this, there will remain 0, wherefore I set 8 in the Quotient, and conclude that the Biquadratic Root of the given Number 614656 is 28; for 28×28×28×28 produces 614656.

	61	4656	(28. *Root*
Subtract	16		*aaaa*
	45	4656	*Resolvend.*
$a=20$	3	2000	4*aaa*
		2400	6*aa*
		80	4*a*
	3	4480	*Divisor.*
$e=8$	25	6000	4*aaae*
	15	3600	6*aaee*
	4	0960	4*aeee*
		4096	*eeee*
Subtract	45	4656	*Ablatitium.*
	0	0000	

Sect. IV. *Of the Extraction of the Root of the fifth Power given in Number.*

1. For the more ready Extraction of the Root of any fifth Power given in Number, this Tabulet will be useful, which shews at first sight the fifth Powers of every single Figure, and consequently any fifth Power in Number under 100000 being given, its Root is hereby discovered.

Roots.	*5th Powers.*
1	1
2	32
3	243
4	1024
5	3125
6	7776
7	16807
8	32768
9	59049

2. When a whole Number is given for a fifth Power, and its Root desired, that is, such a Number which being multiplied five times into it self will produce the given Number, it must be prepared for Extraction by Punctations in this manner, *viz.* First, let a Point be set over the Units place of the given Number, then passing over the four next places towards the left Hand, set another Point over the sixth place; and in that order as many Points are to be set as the given Number will admit, that there may be four places between every two adjacent Points. So if the Root of the fifth Power 17210368 be desired, after Points are set as is above directed, the said 17210368 will be distributed into two Members, to wit, 172 and 10368. In like manner this Number 1880287678125 will be distributed into these three Members, 188, 02876, and 78125. The Points (as before hath been said) shew the number of Places that will be found in the Root, as also what Member of the Number given belongs to the Extraction of every single Character of the Root sought.

17210368

1880287678125

3. Every

3. Every Number confidered as a fifth Power may be conceived to be produced from the Multiplication of the Binomial Root $a+e$ five times into it felf, and then the faid Number will be compofed of thefe fix Members or Numbers, *viz.* $aaaaa$, $5aaaae$, $10aaaee$, $10aaeee$, $5aeeee$, and $eeeee$, (as is manifeft by the fifth Power in the Table in *Sect.* 4. *Chap.* 1. of this Book.) Now becaufe the Refolution of the fifth Power, *viz.* the Extraction of $\sqrt{(5)}$ out of a given Number, is deducible from the fteps of the Compofition of a fifth Power from its Root given in Number; (for fuch Numbers as are added in the Compofition are to be fubtracted in the Refolution) the Learner muft be exercis'd in *Sect.* 4. *Chap.* 2. of this Book.

Example.

Let it be required to extract $\sqrt{(5)}$ out of 17210368, *viz.* to find a Root or Number, which being multiplied five times into it felf will produce 17210368. After the given Number is prepared by Punctations as before is directed, I feek in the Tabulet in the firft ftep of this Section 4. for the greateft fifth Power contained in 172 the firft Member (towards the left Hand) of the given Number, and finding it to be 32, I fubfcribe 32 under 172, and write 2 the Root of the faid fifth Power 32 in the Quotient, for the firft Figure of the Root fought; then after having drawn a Line under 32, I fubtract 32 from 172, or 3200000 from 17210368, and there remains to be refolved 14010368.

172	10368	(2
32		
140	10368	

Then to difcover the *Divifor*, which fhews how to begin the tryal in the finding out of e, that is, every Character (whether it be a Figure or Cypher) which is to follow the firft Figure of the Root, I take fuch Powers of a as are multiplied into the Powers of e in the fifth Power produced from $a+e$, *viz.* $5aaaa$, $10aaa$, $10aa$, and $5a$; fo the Sum of thefe four Numbers make the *Divifor*. And becaufe the firft Figure of the Root is found 2, and confequently (by Note 2. in *Sect.* 1. of this *Chap.*) the Number fignified by a is 20, therefore the Sum of the Numbers fignified by $5aaa$, $10aaa$, $10aa$, and $5a$ is 884100, which is the *Divifor*; then fuppofing I were to divide the *Refolvend* 14010368 by the *Divifor* 884100, I find the Quotient exceeds 9; but in regard e always reprefents a fingle Figure or Cypher, it cannot exceed 9; therefore I make tryal (in a void place) with 9, to fee whether it will conftitute an *Ablatitium* that does not exceed the *Refolvend* 14010368, *viz.* I fuppofe $e=9$, then becaufe a was found 20, the *Ablatitium* $5aaaae+10aaaee+10aaeee+5aeeee$ exceeds the *Refolvend* from which it ought to be fubtracted. But if $e=8$, then the *Ablatitium* will be equal to the *Refolvend*, and confequently that being fubtracted from this, there will remain 0, wherefore I fet 8 in the Quotient; fo 28 is found to be the $\sqrt{(5)}$ of the given Number 17210368, for $28\times28\times28\times28\times28$ produces 17210368. Compare this following Work with the precedent Rules of *Sect.* 4.

	172	10368	(28. *Root.*
	32	00000	$aaaaa$
	140	10368	*Refolvend.*
$a=20$	8	00000	$5aaaa$
		80000	$10aaa$
		4000	$10aa$
		100	$5a$
	8	84100	*Divifor.*
$e=8$	64	00000	$5aaaae$
	51	20000	$10aaaee$
	20	48000	$10aaeee$
	4	09600	$5aeeee$
		32768	$eeeee$
	140	10368	*Ablatitium.*
	000	00000	

By the precedent Rules and Examples of this *Chap.* the Ingenious Reader will eafily perceive how to extend this general Method to the Extraction of the Roots of all kinds of of Powers in Numbers, *viz.* of the sixth, seventh, eighth, *&c.* Powers; as also to find out the Roots infinitely near of such Powers as have not Roots exactly expressible by any rational or true Number.

CHAP. IV.

Concerning the Extraction of Roots out of Powers expres's'd by Letters.

I. IN a Series or Scale of Powers produced from a Root, suppose from a, as in this Series, $a, aa, aaa, aaaa, aaaaa, a^6, a^7, a^8$, *&c.* those Powers only whose Indices are even Numbers are Squares; as $aa, aaaa, a^6, a^8$, *&c.* (whose Indices are 2, 4, 6, 8, *&c.*) are Squares. And those Powers only whose Indices are divisible by 3, are Cubes, as $aaa, aaaaaa, a^9$, *&c.* (whose Indices are 3, 6, 9, *&c.*) are Cubes. Therefore every Power whose Index is a Prime Number greater than 3, as $aaaaa, a^7, a^{11}$, *&c.* (whose Indices are 5, 7, 11, *&c.*) is neither a Square nor a Cube. But every Power whose Index is divisible by 6, as a^6, a^{12}, a^{18}, *&c.* is both a Square and a Cube, because the Index is divisible both by 2 and by 3.

II. If a Simple Quantity be express'd by the same Letter repeated an even number of times, the Square Root thereof is easily extracted; for the Root must be such that its Index may be the half of the Index of the Quantity proposed: As, $\sqrt{aa}$ (that is, the Square Root of aa) is a; for 1, the Index of the Root a is the half of 2, the Index of the Square aa. In like manner $\sqrt{aaaa}$ is aa, whose Index 2 is the half of 4, the Index of the Square $aaaa$. Again, $\sqrt{aaaaaa}$ is aaa, whose Index 3 is the half of 6, the Index of the Square a^6.

III. And with the like facility you may extract the Cubic Root of a Simple Quantity, which is express'd by one and the same Letter repeated such a Number of times as is divisible by 3; for the Cubic Root must be such that its Index may be $\frac{1}{3}$ of the Index of the Cube proposed: As $\sqrt{(3)}aaa$ (that is, the Cubic Root of the Quantity aaa) is a, whose Index 1 is $\frac{1}{3}$ of 3 the Index of aaa. In like manner $\sqrt{(3)}$ a^6 is aa, whose Index 2 is $\frac{1}{3}$ of 6 the Index of the Cube a^6.

IV. If the Index of a Simple Power express'd by the same Letter be some Prime Number greater than 3, as 5, 7, 11, *&c.* then neither $\sqrt{(2)}$ nor $\sqrt{(3)}$, nor any other Root, except that denoted by such Index or Prime Number can be exactly extracted out of the said Power: so no Root can be exactly extracted out of $aaaaa$ or a^5, but $\sqrt{(5)}$, which is a; nor any Root out of a^7 but $\sqrt{(7)}$, which is also a. But when the Root cannot be exactly extracted, the Sign of the Root is to be prefix'd to the Quantity; as to express the Square Root of $aaaaa$ or a^5, I write $\sqrt{aaaaa}$ or $\sqrt{a^5}$. Likewise I express the Cubic Root of a^5 thus, $\sqrt{(3)}\ a^5$; and $\sqrt{(4)}$ of a^7 thus, $\sqrt{(4)}a^7$; and so of others.

V. When some Power of an unknown Simple Root a is found equal to some known Number, and the Index of that unknown Power is not a Prime Number, then the value of the Root a in Number may oftentimes be discovered by two or more Extractions, more easily than by one single Extraction of a Root out of the said unknown Number. As for Example:

If there be proposed or found out	$aaaaaa = 729$,
Then to find out the value of a you need not extract the $\sqrt{(6)}$ of 729, by the general Method before delivered in *Chap.* 3. but first by that Method extract the Square Root of 729, and then by *Sect.* 2. of this *Chap.* the Square Root of $aaaaaa$, so those two Roots compared give this Equation, *viz.*	$aaa = 27$
Lastly, by extracting the Cubic Root of each part of the last Equation, the value of a the Root sought is discovered, *viz.* . .	$a = 3$

Or

Or thus,

First, by extracting the Cubic Root of each part of the Equation proposed, there arises . . . . $aa=9$

And then by extracting the Square Root of each part of the last Equation, the same value of the Root a is found out as before, to wit, . $a=3$

In like manner if . . . . $a^9=19683$

First by extracting the Cubic Root, it gives . . . . $a^3=27$

And again, by extracting the Cubic Root of that Root the Root a is made known, *viz.* . . . . $a=3$

VI. When two or more Squares, Cubes, or other Powers expreſs'd by different Letters, be multiplied one into another, then if the Root of each Power, *viz.* the Square Root if they be Squares, or the Cubic Root if they be Cubes, &c. be extracted, the Product made by the Multiplication of theſe Roots one into another, ſhall be a like Root of the Power or Product firſt given. As for Example: $\sqrt{aabb}$ is ab, which is the Product of the Square Roots of aa and bb. Likewiſe, $\sqrt{(3)aaabbb}$ is ab, which is the Product of the Cubic Roots of aaa and bbb.

Again, $\sqrt{aabbcc}$ is abc, which is the Product of the Square Roots of aa, bb, and cc. In like manner, $\sqrt{(3)27aaabbb}$ is $3ab$, which is the Product of the Cubic Roots of 27 aaa and bbb; and $\sqrt{16aabbcc}$ is $4abc$, which is the Product of the Square Roots of 16 aa, bb, and cc. The like is to be underſtood of others.

But if the Square Root of $5aabb$ be deſired, becauſe 5 is not a Square, the ſaid Root is to be expreſs'd either thus, $\sqrt{5aabb}$; or thus, $\sqrt{5}\times ab$; or thus, $ab\sqrt{5}$. In like manner, to denote the Square Root of $aaabbb$ I write $\sqrt{a^3b^3}$. And to ſignifie the Cubic Root of $aabb$ I write $\sqrt{(3)aabb}$; but the Cubic Root of $3aaabbb$ may be written either thus, $\sqrt{(3)3a^3b^3}$; or thus, $\sqrt{(3)3}\times ab$; or thus, $ab\sqrt{(3)3}$.

Concerning the Extraction of Roots out of Compound Quantities expreſs'd by Letters.

VII. Before the Learner enters upon the Extraction of Roots out of Compound Squares, Cubes, or other Powers expreſs'd by Letters, he ought to be well exercis'd in the eighth and ninth Chapters of my firſt Book of *Algebraical Elements*; as alſo in the foregoing firſt, ſecond, and third Chapters of this Book, and in the precedent Rules of this Chapter; all which well underſtood will render the following Rules and Examples of this Chapter very plain and eaſie.

VIII. *Rules for the Extraction of Square Roots out of Compound Quantities expreſs'd by Letters.*

Rule 1. Set the particular Members of the Compound Algebraic Quantity, whoſe Square Root is required, in ſuch order, that one of the Simple Squares may ſtand outermoſt towards the left Hand; and next after the ſame ſuch other Member or Members, wherein you find the ſame Letter or Letters as are in the ſaid Simple Square. Then the Square Root of the ſaid Simple Square is to be ſet in the Quotient for the firſt Number of the Compound Root ſought, and the Square it ſelf is the firſt Quantity to be ſubtracted from the Compound Quantity propoſed. This is the firſt Work, which is no more to be repeated in the whole Extraction.

Rule 2. Double the Root before ſet in the Quotient for the firſt *Diviſor*; likewiſe to find every following *Diviſor* double every Simple Quantity that ſtands in the Quotient, and take the Sum of the Products for the *Diviſor.*

Rule 3. When the *Diviſor* is found out, divide only the firſt Simple Quantity (towards the left Hand) in the *Reſolvend*, by the firſt Simple Quantity in the *Diviſor*, and ſet that which comes forth next after the Member or Members of the Root ſought that was before found out.

Rule 4. After the firſt Simple Square is ſubtracted (according to *Rule* 1.) then every following *Ablatitium*, that is, the Sum of the Quantities to be ſubtracted from the reſpective *Reſolvend*, muſt be compoſed of theſe two Products, *viz.* the Product made by the Multiplication of the whole *Diviſor* by that particular Quantity which was laſt ſet in the Quotient, and the Square of the ſame Simple Quantity.

The Practice of theſe Rules will be apparent in the following Examples.

Example

Example 1.

Let it be required to extract the Square Root of $aa+2ab+bb$.

First, I extract the Square Root of aa, and it is a, which I set in the Quotient; then multiplying a by it self, I set the Product aa under, and subtract it from the Qnantity first proposed, and there remains $2ab+bb$. This is the first work which answers to Rule 1. and is no more to be repeated.

The Square,	$aa+2ab+bb$	$(a+b$ The Root.
Subtract	aa	
Remainder,	$+2ab+bb$	
Divisor,	$+2a)$	
Subtract	$+2ab+bb$	
Remainder,	o o	

Secondly, the Divisor (according to Rule 2.) is $2a$, which I set under $2ab$.

Thirdly, I divide $+2ab$ by the Divisor $+2a$, and the Quotient is $+b$, which I set next after a, (the particular Root before found out) according to Rule 3.

Fourthly, I multiply the Divisor $+2a$ by $+b$, (that was last set in the Quotient) and the Product is $+2ab$, to which adding $+bb$, (the Square of $+b$) the Sum is $+2ab+bb$, which (according to Rule 4.) I set under and subtract from the Resolvend $+2ab+bb$, and there remains o: So the Extraction being finish'd, the Root sought is found $a+b$; for if it be multiplied by it self it produces $aa+2ab+bb$, the Quantity first proposed.

Note. By what I have said in the eighth and ninth Chapters of my First Book of *Algebraical Elements*, 'tis easie to discover at first sight whether a Compound Algebraic Quantity consisting of three Terms be a perfect Square or not, and if a Square what its Root is. Neverthelefs in this first Example I have express'd the Work at large according to the four Rules before given, that the like Opertion may the more easily be perceived in the following Examples.

Example 2.

If the Square Root of $aa-2ab+2ac-2bc+bb+cc$ be desired, it will be found $a-b+c$ by the precedent Rules, and the Work stands as here you see underneath.

The Square,	$aa-2ab+2ac-2bc+bb+cc$	$(a-b+c$ The Root.
Subtract	aa	
Remainder,	$-2ab+2ac-2bc+bb+cc$	
Divisor,	$+2a)$	
Subtract	$-2ab+bb$	
Remainder,	$+2ac-2bc+cc$	
Divisor,	$+2a-2b)$	
Subtract	$+2ac-2bc+cc$	
Remainder,	o o o	

Example 3.

In like manner the Square Root of $64aabb+32abc-144ab+4cc-36c+81$ will be found $8ab+2c-9$, as is manifest by the following Operation.

The Square,	$64aabb+32abc-144ab+4cc-36c+81$	$(8ab+2c-9$
Subtract	$64aabb$	
Remainder,	$+32abc-144ab+4cc-36c+81$	
Divisor,	$+16ab)$	
Subtract	$+32abc \quad +4cc$	
Remainder,	$-144ab \quad -36c+81$	
Divisor,	$+16ab \quad +4c)$	
Subtract	$-144ab \quad -36c+81$	
Remainder,	o o o	

Exam-

Example 4.

Again, the Square Root of $dddd+2dddb+3ddbb+2dbbb+bbbb$ will be found $dd+db+bb$; and the Extraction ſtands thus:

The Square,	$d^4+2d^3b+3d^2b^2+2db^3+b^4$	$(dd+db+bb.$
Subtract	d^4	
Remainder,	$+2d^3b+3d^2b^2+2db^3+b^4$	
Diviſor,	$+2d^2)$	
Subtract	$+2d^3b+d^2b^2$	
Remainder,	$+2d^2b^2+2db^3+b^4$	
Diviſor,	$+2d^2\ \ +2db)$	
Subtract	$+2d^2b^2+2db^3+b^4$	
Remainder,	o o o	

IX. *Rules for the Extraction of Cubic Roots out of Compound Quantities expreſs'd by Letters.*

Rule 1. Set the particular Members or Parts of the Compound Algebraic Quantity whoſe Cubic Root is required, in ſuch order, that one of the Simple Cubes may ſtand outermoſt towards the left Hand, and next after the ſame ſuch other Members wherein you find the ſame Letter or Letters as are in the ſaid Simple Cube; then the Cubic Root of the ſaid Simple Cube is to be ſet in the Quotient for the firſt Member of the Root ſought, and the Simple Cube it ſelf is the firſt Quantity to be ſubtracted from the Compound Quantity propoſed. This is the firſt Work, and no more to be repeated in the whole Extraction.

Rule 2. The firſt Diviſor muſt be compoſed of the Triple of the Square of the Root before ſet in the Quotient, (which Triple Square I call the firſt part of the Diviſor) and the Triple of the ſame Root, (which Triple Root I call the latter part of the Diviſor.) Likewiſe every following Diviſor muſt be compoſed of the Triple of the Square of the Sum of all the ſingle Quantities or Parts of the Root already found out and ſet in the Quotient, and of the Triple of the ſame Sum.

Rule 3. When the Diviſor is found out, divide only the firſt Simple Quantity (towards the left Hand) in the *Reſolvend*, by the firſt Simple Quantity in the Diviſor, and ſet that which comes forth in the Quotient next after the Member or Members of the Root ſought before found out.

Rule 4. After the firſt Simple Cube is ſubtracted (according to Rule 1.) then every following *Ablatitium*, that is, the Sum of the Quantities to be ſubtracted from the *Reſolvend*, muſt be compoſed of theſe three Products, *viz.* Firſt, the Product made by the Multiplication of the firſt Part of the Diviſor, (to wit, the Triple Square mentioned in Rule 2.) by the ſimple Quantity laſt ſet in the Quotient. Secondly, the Product made by the Multiplication of the latter part of the Diviſor, (to wit, the Triple Root or Sum mentioned in Rule 2.) by the Square of the ſame ſimple Quantity. And thirdly, the Cube of the ſaid ſimple Quantity laſt ſet in the Quotient.

The Practice of theſe Rules will appear in the following Examples.

Example 1.

Let it be required to extract the Cubic Root out of $aaa+3aae+3aee+eee$.

Firſt, beginning at the left Hand I extract the Cubic Root of aaa, and it is a, which I ſet in the Quotient, then multiplying the ſaid Root a Cubically it makes aaa, which I ſubtract from the Compound Quantity firſt propoſed for Extraction, and there remains to be reſolved $+3aae+3aee+eee$. This is the firſt Work, which anſwers to Rule 1. and is no more to be repeated in the whole Extraction.

The Cube,	$aaa+3aae+3aee+ee$	$(a+e.$ The Root.
Subtract	aaa	
Remainder,	$+3aae+3aee+eee$	
Diviſor,	$+3aa\ +3a)$	
Subtract	$+3aae+3aee+eee$	
Remainder,	o o o	

X Se-

Secondly, I seek a Divisor thus, *viz.* to $+3aa$, which is the triple of aa the Square of the Root a, I add $+3a$, the triple of the said Root a, and the Sum $3aa+3a$ is the Divisor, which I set underneath the remaining *Resolvend*, according to Rule 2.

Thirdly, according to Rule 3. I divide $+3aae$ by $+3aa$, and it gives $+e$, which I set in the Quotient next after a.

Fourthly, to find out the *Ablatitium* (or Quantity next to be subtracted) I make a threefold Multiplication, *viz.* First, I multiply $+3aa$ (the first part of the Divisor) by $+e$ the Root last set in the Quotient, and the Product is $+3aae$. Secondly, I multiply $+3a$, the latter part of the Divisor by $+ee$, the Square of the said Root e, and the Product is $+3aee$. Thirdly, I multiply the said Root e Cubically, and the Product is eee. Lastly, I subtract the Sum of the said three Products from the *Resolvend*, and there remains 0. So the Extraction is finish'd, and $a+e$ is the true Cubic Root sought; for if it be multiplied cubically, it will produce $aaa+3aae+3aee+eee$ first proposed.

Example 2.

In like manner the Cubic Root extracted out of $125aaa+225aae+135aee+27eee$ is $5a+3e$, and the Work stand thus:

The Cube.	$125aaa+225aae+135aee+27eee$	$(5a+3e.$ Root.
Subtract	$125aaa$	
Remainder,	$+225aae+135aee+27eee$	
Divisor,	$+75aa+15a)$	
Subtract	$+225aae+135aee+27eee$	
Remainder,	0 0 0	

Example 3.

So the Cubic Root of $27a^6-54a^5+171a^4-188a^3+285aa-150a+125$ will be found $3aa-2a+5$, and the Operation stands thus:

Cube,	$27a^6-54a^5+171a^4-188a^3+285aa-150a+125$	$(3aa-2a+5.$ Root.
Subtract	$27a^6$	
Remainder,	$-54a^5+171a^4-188a^3+285aa-150a+125$	
Divisor,	$+27a^4+9a^2)$	
Subtract	$-54a^5+36a^4-8a^3$	
Remainder,	$+135a^4-180a^3+285aa-150a+125$	
Divisor,	$\left\{\begin{array}{l} +27a^4-36a^3+12aa \\ \qquad\qquad +9aa-6a \end{array}\right.$	
Add these,	$\left\{\begin{array}{l} +135a^4-180a^3+60aa \\ \qquad\qquad +225aa-150a \\ \qquad\qquad\qquad +125 \end{array}\right.$	
Subtract	$+135a^4-180a^3+285aa-150a+125$	
Remainder,	0 0 0 0 0	

If there be occasion to extract the Root of the fourth, fifth, or other higher Compound Power, the Divisors and Ablatitious Quantities may be drawn out of the Table in *Sect.* 4. *Chap.* 1. of this Book.

X. *Concerning the Extraction of Roots out of Algebraical Fractions.*

1. Forasmuch as in the Extraction of Roots out of Fractions, the Root of the Numerator and Denominator being severally extracted gives the Root sought; therefore if the Square Root of $\frac{aabb}{cc}$ be to be extracted, I write $\frac{ab}{c}$ for the Root sought; for the Square Root of the Numerator $aabb$ is ab, and the Square Root of the Denominator cc is c.

In like manner if the Square Root of $\frac{aaaa-2aabb+bbbb}{aa+4ab+4bb}$ be deſired, by extracting the Square Root out of the Numerator and Denominator, there ariſes $\frac{aa-bb}{a+2b}$ for the Root ſought.

And for the ſame Reaſon the Cubic Root of this Fraction . . . . $\frac{27a^6-54a^5+171a^4-188a^3+285aa-150a+125}{aaa-9aa+27a-27}$ will be $\frac{3aa-2a+5}{a-3}$, which is found by extracting the Cubic Root out of the Numerator and Denominator of the Fraction propoſed.

2. But if the Root ſought cannot be extracted out of the Numerator and Demoninator, then the Radical Sign $\surd$ with the Index of the Power, if it exceed a Square, is to be prefix'd to the Fraction; as to denote the Square Root of $\frac{ccxx}{4bb}-ac$, that is, of $\frac{ccxx-4abbc}{4bb}$, I write $\surd\frac{ccxx-4abbc}{4bb}$, or (becauſe the Square Root of the Denominator is $2b$) the Square Root of the Quantity propoſed may be expreſs'd thus $\frac{\sqrt{ccxx-4abbc}}{2b}$; likewiſe the Cubic Root of $\frac{a^3b^3}{aa+bb}$ may be deſigned either thus, $\surd(3)\frac{a^3b^3}{aa+bb}$, or (becauſe the Numerator is a Cube) thus, $\frac{ab}{\surd(3)aa+bb}$. The like is to be underſtood in expreſſing the irrational Roots of higher Powers.

CHAP. V.

Concerning Geometrical Proportion.

I. THE *Difference* of two Numbers is found out by Subtraction; but the *Ratio*, *Reaſon*, or *Habitude* of one Number to another is diſcovered by dividing the Antecedent (or firſt Number) by the Conſequent (or ſecond Number;) for the Quotient denominates the *Ratio*, Reaſon, or (as ſome call it) the Proportion which the Antecedent has to the Conſequent. As if 6 be compared to 2, then $\frac{6}{2}$, that is $\frac{3}{1}$, or 3, ſhews that 6 has triple Reaſon to 2, *viz.* 6 contains 2 thrice, or 6 is in proportion to 2 as 3 to 1; but if 2 be compared to 6, then $\frac{2}{6}$ or $\frac{1}{3}$ ſhews, that 2 has ſubtriple Reaſon to 6, *viz.* 2 is $\frac{1}{3}$ part of 6, or 2 is in proportion to 6 as 1 to 3. In like manner if the Quantity a be compared to the Quantity b, then $\frac{a}{b}$ expreſſes the *Ratio* or Reaſon of a to b, and $\frac{b}{a}$ ſhews the Reaſon of b to a.

Note, that the Reaſon of two Numbers or Quantities ought to be expreſs'd by the ſmalleſt Terms or Quantities that can poſſibly be found to expreſs that Reaſon. So the Denominator of the Reaſon of 16 to 12 is $\frac{4}{3}$, where 16 and 12 are firſt reduced to the ſmalleſt Terms 4 and 3, (by dividing the 16 and 12 ſeverally by their greateſt common Diviſor 4) and then dividing the Antecedent 4 by the Conſequent 3, the Quotient $\frac{4}{3}$ expreſſes the Reaſon or Proportion of 16 to 12, *viz.* 16 is to 12 as 4 to 3. In like manner the Reaſon of bb to ba, or of bbb to bba is $\frac{b}{a}$.

II. Quantities which proceed by equal Differences are ſaid to be in a continued Arithmetical Progreſſion, (as has been ſhewn in *Chap.* 17. *Book* 1. of my *Algebraical Elements*;) but Quantities which proceed by equal Reaſons (or Proportions) are ſaid to be in a continued Geometrical Progreſſion or Proportion. So theſe Numbers 2, 6, 18, 54, 16, are continually proportional, becauſe the Reaſon (or Proportion) of the firſt the ſecond is equal to the Reaſon of the ſecond to the third, alſo of the third to the fourth, and ſo forward; *viz.* $\frac{2}{6}$ (or $\frac{1}{3}$) $=\frac{6}{18}=\frac{18}{54}=\frac{54}{162}$; or backward, $\frac{162}{54}=\frac{54}{18}=\frac{18}{6}=\frac{6}{2}$ (or 3.) In like manner if theſe Quantities a, b, c, d, e, be ſuch, that $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}$; or backwards, if $\frac{e}{d}=\frac{d}{c}=\frac{c}{b}=\frac{b}{a}$, then thoſe Quantities are continually proportional; *viz.* as the firſt is in proportion to the ſecond, ſo is the ſecond to the third, the third to the fourth, &c.

But if there be four ſuch Quantities, that the Reaſon (or Proportion) of the firſt to the ſecond, is equal to the Reaſon of the third to the fourth; but the Reaſon of the ſecond to the third, is not equal to the Reaſon of the firſt to the ſecond, then thoſe Quantities are ſaid to be in Geometrical Proportion diſcontinued or interrupted; ſuch are theſe four Numbers $2 . 6 :: 12 . 36$; for $\frac{2}{6}$ (or $\frac{1}{3}$) $=\frac{12}{36}$, but $\frac{6}{12}$ (or $\frac{1}{2}$) is not equal to $\frac{2}{6}$ or $\frac{1}{3}$. In like manner, if a, b, c, d, be ſuch Quantities that $\frac{a}{b}=\frac{c}{d}$, but $\frac{b}{c}$ is not equal to $\frac{a}{b}$, (or $\frac{c}{d}$;) then are a, b, c, d, diſcontinual Proportionals.

III. If three Quantities be Proportionals, the Product made by the mutual Multiplication of the Extremes is equal to the Square of the Mean; as,

If there be propoſed . . . . $\left\{\begin{array}{l} 18, 6, 2, \div \\ a, b, c, \div \end{array}\right.$

Then this Equation enſues . . . . $ac=bb=36$

For ſince by ſuppoſition . . . . $a . b :: b . c$

It follows (by *Sect.* 1. and 2.) that . . . . $\left\{\frac{a}{b}=\frac{b}{c}=3\right.$

Whence by multiplying each part by c, ... $\left\{\frac{ac}{b}=b=6\right.$

And by multiplying each part of the laſt Equation by b, it produces . . . . $\left\{ac=bb=36\right.$

Which was to be proved.

IV. If four Quantities be Proportionals, whether they be continual or diſcontinual, the Product made by the mutual Multiplication of the Extremes is equal to the Product of the Means; and conſequently if the Product of the Means be divided by either of the Extremes, the Quotient is the other Extreme. As for Example:

Let four diſcontinual Proportionals be propoſed, . . $\left\{\begin{array}{l} d . c :: b . a \\ 12 . 4 :: 15 . 5 \end{array}\right.$

Then by the foregoing *Sect.* 2. . . . . $\left\{\frac{d}{c}=\frac{b}{a}=3\right.$

And by multiplying each part of that Equation by a this is produced, *viz.* . . . . $\left\{\frac{da}{c}=b=15\right.$

And by multiplying each part of the laſt Equation by c, the firſt part of the Propoſition is manifeſt, *viz.* . . $\left\{da=cb=60\right.$

And by dividing each part by d there ariſes . . . . $\left\{a=\frac{cb}{d}=5\right.$

Which laſt Equation being compared with the four Proportionals firſt propoſed does ſhew, that if three Quantities d, c, b, be given, to find ſuch a fourth as ſhall have the ſame Proportion to b as c has to d, then the Product of the ſecond and third Terms, to wit cb, being divided by the firſt Term d will give the fourth Proportional ſought, which is the very Operation in the Rule of Three Direct.

V. If three Quantities a, b, c be Proportionals, and the firſt and ſecond, to wit a and b be given ſeverally, the third is alſo given; for by *Sect.* 3. of this *Chap.* $ac=bb$, whence by dividing each part by a there ariſe $c=\frac{bb}{a}$ which ſhews, that if the Square of the Mean or ſecond Term be divided by the firſt, the Quotient is the third Proportional; hence a, b, and $\frac{bb}{a}$ are continual Proportionals. In like manner if three Quantities in continual Proportion be given ſeverally, and a fourth Proportional be deſired, the

the Square of the third Term divided by the ſecond gives the fourth: as if there be given theſe three, $a, b, \frac{bb}{a} \div$; then by dividing the Square of $\frac{bb}{a}$, to wit, $\frac{bbbb}{aa}$ by b, the Quotient $\frac{bbb}{aa}$ ſhall be the fourth continual Proportional: Hence $a, b, \frac{bb}{a}, \frac{bbb}{aa}$ are continual Proportionals. Likewiſe if the Square of the fourth continual Proportional be divided by the third, the Quotient will be the fifth; ſo to thoſe four continual Proportionals this fifth will be found, to wit, $\frac{bbbb}{aaa}$; and ſo forwards infinitely. Therefore,

VI. If Numbers, how many ſoever, be continually Proportionals, and the leaſt Term be eſteemed the firſt, that next greater than the leaſt the ſecond, and ſo forwards; then the ſecond Term is produced by the Multiplication of the firſt into the Reaſon of the ſecond Term to the firſt, the third Term is produced by the Multiplication of the firſt into the Square of the ſame Reaſon, the fourth Term is produced by the Multiplication of the firſt into the Cube of the ſame Reeſon; and in like manner every following Term is produced by the Multiplication of the firſt into ſuch a Power of the Reaſon of the ſecond Term to the firſt, as has fewer dimenſions by one than the Number of Terms has Units: as in theſe following ſix continual Proportionals, to wit,

$$a,\ b,\ \frac{bb}{a},\ \frac{bbb}{aa},\ \frac{bbbb}{aaa},\ \frac{bbbbb}{aaaa}\ \div$$
$$2,\ 6,\ 18,\ 54,\ 162,\ 486\ \div$$

Suppoſing a to be the firſt and leaſt Term, the ſecond Term b is equal to the Product of the firſt Term a into $\frac{b}{a}$, to wit, the Reaſon of the ſecond Term to the firſt; alſo the third Term $\frac{bb}{a}$ is produced by the Multiplication of the firſt Term a into the Square of the ſame Reaſon, that is, into $\frac{bb}{aa}$; and the fourth Term $\frac{bbb}{aa}$ is produced by the Multiplication of the firſt Term a into the Cube of the ſame Reaſon, that is, into $\frac{bbb}{aaa}$; and the fifth Term $\frac{bbbb}{aaa}$ is produced by the Multiplication of the firſt Term a into the fourth Power of the ſame Reaſon, that is, into $\frac{bbbb}{aaaa}$: and ſo forwards.

But if the greateſt Term be eſteemed the firſt, that next leſs than the greateſt the ſecond, and ſo downwards; then the ſecond Term is equal to the Quotient that ariſes by dividing the firſt (or greateſt) Term by the Reaſon of the firſt to the ſecond; the third is equal to the Quotient that ariſes by dividing the firſt Term by the Square of the ſame Reaſon; the fourth Term is equal to the Quotient that ariſes by dividing the firſt Term by the Cube of the ſame Reaſon; and in like manner every Term beneath the greateſt is equal to the Quotient that ariſes by dividing the firſt (or greateſt) Term by ſuch a Power of the Reaſon of the greateſt to the greateſt but one (or ſecond) Term, as has fewer Dimenſions by one than the number of Terms: as in theſe following ſix continual Proportionals, to wit,

$$\frac{bbbbb}{aaaa},\ \frac{bbbb}{aaa},\ \frac{bbb}{aa},\ \frac{bb}{a},\ b,\ a,\ \div$$
$$486\ \ 162,\ 54,\ 18,\ 6,\ 2\ \div$$

If we ſuppoſe $\frac{bbbbb}{aaaa}$ to be the firſt and greateſt Term, then the ſecond Term $\frac{bbbb}{aaa}$ is equal to the Quotient of the firſt Term $\frac{bbbbb}{aaaa}$ divided by $\frac{b}{a}$, to wit, by the Reaſon of the firſt Term to the ſecond; alſo the third Term $\frac{bbb}{aa}$ is equal to the Quotient of the firſt Term $\frac{bbbbb}{aaaa}$ divided by $\frac{bb}{aa}$, that is, by the Square of the Reaſon $\frac{b}{a}$; and the

fourth

fourth Term $\frac{bb}{a}$ is equal to the Quotient of the firſt Term $\frac{bbbbb}{aaaa}$ divided by $\frac{bbb}{aaa}$ the Cube of the ſame Reaſon. And ſo of the reſt.

VII. From the laſt preceding Section it follows, that if in a Series or Rank of Numbers which are in continual proportion, the firſt Term, the ſecond Term, and the Number of Terms be given ſeverally, the laſt Term ſhall be alſo given by this Rule, *viz.* firſt, (according to the *Note* in *Sect.* 1. of this *Chap.*) find out the ſmalleſt Numbers that may ſhew the Reaſon of the greater of the two given Terms to the leſs; then eſteeming the ſaid Reaſon as a Root, find ſuch a Power thereof whoſe Index may be equal to the given multitude of Terms leſs by Unity, which Power multiplied by the firſt Term, when the firſt Term is leſs than the ſecond, gives the laſt, to wit, the greateſt Term. But when the firſt Term is greater than the ſecond, then the firſt Term divided by the ſaid Power gives the laſt Term. As if there be given a and b, the firſt and ſecond of ſix Numbers in continual proportion, and that b is greater than a; then the Reaſon of b to a is $\frac{b}{a}$, (by *Sect.* 1. of this *Chap.*) and the fifth Power of $\frac{b}{a}$ is $\frac{bbbbb}{aaaaa}$, this multiplied by the firſt Term a produces $\frac{bbbbb}{aaaa}$, which is the ſixth Proportional ſought, (as is evident by *Sect.* 6.) but if the firſt Term a be greater than the ſecond Term b, then the Reaſon of a to b is $\frac{a}{b}$, whoſe fifth Power is $\frac{aaaaa}{bbbbb}$, by which if you divide the firſt Term a, the Quotient is the ſixth Term $\frac{bbbbb}{aaaa}$.

This Rule may be exemplified by the four following Ranks of Numbers in continual Proportion.

$$2 \;,\; 6 \;,\; 18 \;,\; 54 \;,\; 162 \;,\; 468 \;∺$$
$$3072 \;,\; 768 \;,\; 192 \;,\; 48 \;,\; 12 \;,\; 3 \;∺$$
$$2 \;,\; 3 \;,\; \tfrac{9}{2} \;,\; \tfrac{27}{4} \;,\; \tfrac{81}{8} \;,\; \tfrac{243}{16} \;∺$$
$$\tfrac{1024}{81} \;,\; \tfrac{256}{27} \;,\; \tfrac{64}{9} \;,\; \tfrac{16}{3} \;,\; 4 \;,\; 3 \;∺$$

VIII. If there be given two Integers expreſſing a Reaſon in the leaſt Terms, and it be deſired to find out a given multitude of continual Proportionals in the ſame Reaſon, and that all the Terms may be Integers; Firſt, to thoſe two Integers, or firſt and ſecond Proportionals given, find out (by *Sect.* 5. or 6. of this *Chap.*) ſo many Proportionals as with thoſe given may make the deſired multitude: then multiply every Term by the Denominator of the laſt Term, ſo ſhall the Products be continual Proportionals in Integers in the ſame Reaſon as the two Terms firſt given. As for Example: If a and b be given, and it be deſired to find three Proportionals in Integers in the Reaſon of a to b, firſt, to a and b I find a third Proportional, which (by *Sect.* 5.) is $\frac{bb}{a}$ then a, b, $\frac{bb}{a}$ being multiplied ſeverally by the Denominator a, the Products aa, ab, bb, are Proportionals expreſs'd by Integers, and in the Reaſon of a to b, as was deſired.

Hence if $a=2$, and $b=3$; then aa, ab, and bb will give 4, 6, and 9, which are continual Proportionals in Integers in the given Reaſon of 2 to 3.

So if four continual Proportionals in the Reaſons of a to b, be deſired; firſt, (by *Sect.* 5. or 6.) theſe will be found continual Proportionals, to wit, a, b, $\frac{bb}{a}$, $\frac{bbb}{aa}$ which multiplied ſeverally by aa, (the Denominator of the laſt Term) will produce aaa, aab, abb, bbb, which are four continual Proportionals in Integers in the given Reaſon of a to b. Hence if $a=2$, and $b=3$, then aaa, aab, abb, and bbb, will give 8, 12, 18, and 27, which are continual Proportionals in Integers in the given Reaſon of 2 to 3.

In like manner theſe five Quantities $aaaa$, $aaab$, $aabb$, $abbb$, and $bbbb$, will be found continual Preportionals in the Reaſon of a to b; ſo that if $a=2$, and $b=3$, then thoſe five Proportionals will give theſe five, to wit, 16, 24, 36, 54, and 81 ∺ in the Reaſon of 2 to 3. After the ſame manner you may proceed infinitely.

IX. If

IX. If there be Quantities in continual Proportion, how many soever, the Product made by the Multiplication of the Extremes is equal to the Product of any two Means equally distant from the Extremes; and also to the Square of the Mean Term, when the number of Terms is odd. As for Example: If a,b,c,d,e,f, be continual Proportionals, I say, the Product of the Extremes a and f, to wit af, is equal to the Product of any two Terms equally distant from the Extremes, *viz.* to the Product cd, and to the Product be: For,

1. By supposition, (and by *Sect.* 1. and 2.) . . . . $\frac{a}{b}=\frac{e}{f}$
2. Therefore by multiplying each part by f, it produces . $\frac{af}{b}=e$
3. And by multiplying each part of the last Equation by b, it gives $af=be$
4. Again, by supposition . . . . $\frac{b}{c}=\frac{d}{e}$
5. Therefore (by multiplying in like manner as before) . . $cd=be$
6. Therefore from the third and fifth Equation (*per* 1. *Axiom.* 1. *Elem Euclid.* . . . . $af=cd=be$

Which was to be proved. And if more continual Proportionals even in multitude were proposed, the Demonstration would not be otherwise.

But if the multitude of Terms be odd, as in these seven Quantities which we may suppose to be continually proportional, $a,b,c,d,e,f,g \div$; then the Product made by the Multiplication of the two Extremes a and g is equal to the Square of the middle Term d, *viz.* $ag=dd$. For,

1. By supposition (and by *Sect.* 1. and 2.) . . . . $\frac{c}{d}=\frac{d}{e}$
2. Therefore by multiplying each part of that Equation by d, it makes . . . . $c=\frac{dd}{e}$
3. And by multiplying each part of the last Equation by e, it produces . . . . $ce=dd$
4. And by what has been already proved in the first part of this Proposition, ... $ce=ag$
5. Therefore from the two last Equations (*per* 1.*Ax.*1 *Elem.Eucl.*) $ag=dd$

Which was to be proved. Therefore the Proposition is every way manifest; but for further Illustration:

Let there be proposed these six continual Proportionals in Numbers, to wit, . . .	$2, 6, 18, 54, 162, 486 \div$
Then according to the first part of the Proposition, ...	$2\times486=6\times162=18\times54=972$
Again, let there be proposed these seven continual Proportionals, to wit, ...	$2, 6, 18, 54, 162, 486, 1458$
Then according to the latter part of the Proposition,	$2\times1458=54\times54=2916.$

X. If four Quantities be Proportionals, $a.b::c.d$, they shall be also Alternly, and Inversly, and Composedly, and Dividedly, and Conversly, Proportionals, *viz.*

If	$a.b::c.d$ $6.4::12.8$	
Then Alternly,	$a.c::b.d$ $6.12::4.8$	*Per* 16. *Prop.* 5. *Elem. Eucl.*
And Inversly,	$c.a::d.b$ $12.6::8.4$	*Per Cor. of Prop.* 4. *Elem.* 5.
And Composedly,	$a+b.b::c+d.d$ $10.4::20.8$	*Per* 18. *Prop.* 5. *Elem.*
And Dividedly,	$a-b.b::c-d.d$ $2.4::4.8$	*Per* 17. *Prop.* 5. *Elm.*
And Conversly,	$a.a+b::c.c+d$ $6.10::12.20$	*Per Cor. of Prop.* 19. *Elem.* 5.

But

But that the Learner may the better perceive the meaning and uſe of theſe ways of arguing about Proportionals, I ſhall apply ſome of them to the Reſolution of this following

QUEST.

The Difference (b) between the greater extreme and mean of three Quantities continually proportional being given, as alſo the Difference (c) between the mean and the leſſer Extreme, to find the Proportionals; but the firſt Difference muſt be greater than the latter.

RESOLUTION.

1. For the mean Proportional ſought put	a
2. To which adding the given Difference (b) the Sum is the greater Extreme, to wit, ...	$a+b$
3. But if from the Mean (a) the given Difference (c) be ſubtracted, the Remainder is the leſſer Extreme, to wit,	$a-c$
4. Then (according to the Queſtion) theſe three Quantities $a+b$, a, and $a-c$ muſt be in continual proportion, *viz.*	$a+b \,.\, a :: a \,.\, a-c$
5. Therefore by Diviſion of Reaſon, ...	$b \,.\, a :: c \,.\, a-c$
6. And alternately (or by Permutation)	$b \,.\, c :: a \,.\, a-c$
7. And by Diviſion of Reaſon, ...	$b-c \,.\, c :: c \,.\, a-c$
8. Wherefore by Converſion of Reaſon, ...	$b-c \,.\, b :: c \,.\, a$

Which laſt Analogy if it be expreſs'd by Words gives this

CANON.

As the Difference between the two given Differences is to either of them, ſo is the other to the mean Proportional ſought.

Therefore if $36=b$; and $12=c$; the Canon will diſcover 18 for the mean Proportional ſought, (to wit, a in the Reſolution) which increaſed with 36, and leſſened by 12, gives 54 and 6 for the Extremes. Therefore the three Proportionals ſought are manifeſtly 54, 18, and 6.

Note. If the Analogy in the fourth ſtep of the Reſolution be converted into an Equation, by comparing the Product made by the mutual Multiplication of the Extremes to the Product of the Means, that Equation after due Reduction will give the ſame Canon as above; ſo that the Argumentation in the four laſt ſteps of the Reſolution is not of neceſſity, but only to ſhew how without the help of any Equation the Number ſought may ſometimes be made the fourth Term of an Analogy, whoſe three firſt Terms are known, whence by the Rule of Three the Number ſought is alſo known. Which ways of inferring one Analogy out of another are more proper when the Nature of a Queſtion will admit the ſame, than the common way of proceeding by Equations, eſpecially in the Reſolution of Geometrical Problems, where every ſtep ought to be expreſs'd in the moſt ſimple Terms, to the end the Compoſition of the Problem may the more eaſily be formed by the ſteps of the Reſolution; but in a retrograde or backward Order, as I ſhall hereafter ſhew in the fourth Book of my *Algebraical Elements.*

XI. If Proportionals be multiplied or divided by Proportionals, the Products alſo or Quotients ſhall be Proportionals; as,

If theſe four Proportional Numbers, to wit, ...	$a \,.\, b :: ca \,.\, cb.$ $2 \,.\, 4 :: 3\times2 \,.\, 3\times4$
be multiplied by theſe four Proportional Numbers, ...	$d \,.\, f :: gd \,.\, gf.$ $5 \,.\, 6 :: 7\times5 \,.\, 7\times6$
there will be produced theſe four Proportional Numbers, to wit, . . .	$ad \,.\, bf :: cgad \,.\, cgbf$ $2\times5 \,.\, 4\times6 :: 3\times7\times2\times5 \,.\, 3\times7\times4\times6$

whereby the firſt part of the Propoſition is manifeſt.

And if theſe four Proportional Numbers, to wit, ...	$ad \,.\, bf :: cgad \,.\, cgbf.$
be divided by theſe four Proportionals, to wit, ...	$d \,.\, f :: gd \,.\, gf.$
the Quotients will be theſe four Proportionals, to wit, ...	$a \,.\, b :: ca \,.\, cb.$

whereby the latter part of the Propoſition is manifeſt.

Hence

Hence it may easily be proved, that the Squares, Cubes, fourth Powers, fifth Powers, &c. of proportional Numbers shall be also Proportionals; as,

If . . . . $a \,.\, b :: ca \,.\, cb$

Then their Squares also shall be proportionals, *viz.* $aa \,.\, bb :: ccaa \,.\, ccbb$

And the Cubes of the first four Proportionals shall also be Proportionals, *viz.* . . . . . . $aaa \,.\, bbb :: cccaaa \,.\, cccbbb$

And so of higher Powers.

XII. In every Series or Rank of Quantities continually proportional, all the mean Terms between the first and the last are both Antecedents and Consequents of Reasons; as

If . . . . $a, b, c, d, e, f \div$

That is, ... $a \,.\, b :: b \,.\, c :: c \,.\, d :: d \,.\, e :: e \,.\, f$

It is evident that every Term except the last (f) is a Antecedent of a Reason, and every Term except the first (a) is a Consequent; wherefore if (s) be put for the sum of all the Terms in the Series, then $s-f$ shall be the sum of all the Antecedents, and $s-a$ the sum of all the Consequents. Therefore,

From the premisses (*per* 12 *Prop.* 5. *Elem. Eucl.*) this Analogy arises, *viz.* . . . . $a \,.\, b :: s-f \,.\, s-a$

Whence by comparing the Product of the Extremes to the Product of the Means . . . . $as-aa=bs-bf$

Therefore by due Transposition in that Equation when b is greater than a, ... $bf-aa=bs-as$

And by dividing each part of the last Equation by $b-a$, there arises . . . . $\frac{bf-aa}{b-a}=s$

But if a exceed b, then there will arise . : . $\frac{aa-bf}{a-b}=s$

Which two last Equations give a Cannon to find the sum of all the Terms of a Geometrical Progression, the first, second, and last Term being severally given.

CANON.

Divide the difference between the square of the first Term, and the Product made by the Multiplication of the second Term into the last, by the difference of the first and second Terms, so the Quotient shall be the sum of all the Terms of the Geometrical Progression proposed.

Examples in Numbers.

Let the Values of these . . . . . . $a, b, c, d, e, f \div$
be express'd by these Numbers, . . $32, 48, 72, 108, 162, 243 \div$

Then by the Canon . . . . . . $\frac{bf-aa}{b-a}=665$ the sum of all.

But if the Values of the same Proportionals . . . . $a, b, c, d, e, f \div$
be expounded by these Numbers . . $243, 162, 108, 72, 48, 32 \div$

Then by the Canon . . . . $\frac{aa-bf}{a-b}=665$ the sum of all.

XIII. If what has been said in the eight *Sect.* of this *Chap.* be compared with the Table in *Sect.* 4. *Chap.* 1. of this Book, it will be manifest, that if we cast away the Numbers of Multitude which are prefix'd to all the mean Terms or Members belonging to any Compound Power produced from a Binomial Root, suppose from $a+e$, then all the Members or simple Quantities whereof the said Compound Power is composed, are in continual Proportion. As for Example: The Members whereof the square of $a+e$ is composed are aa, $2ae$, and ee; now if 2 which is prefix'd to ae be cast away, then aa, ae, and ee are Continual Proportionals, (as is evident by the preceeding eight *Sect.* of this *Chap.*)

Again, it appears by the said Table, that the Members whereof the Cube of $a+e$ is composed are aaa, $3aae$, $3aee$, and eee; here if 3 and 3 which are prefix'd to the mean Terms be cast away, then these four Quantities aaa, aae, aee, and eee will be in Continual Proportion.

Likewise, forasmuch as the fourth Power of $a+e$ is composed of these Members, *aaaa*, 4*aaae*, 6*aaee*, 4*aeee*, and *eeee*; by casting away the Numbers of Multitude 4, 6, and 4, these five Quantities *aaaa*, *aaae*, *aaee*, *aeee*, and *eeee*, shall be continual Proportionals. And so of higher Powers infinitely.

XIV. Forasmuch as by the last preceding *Sect.* these Quantities are in continual proportion to wit, } $\because aa$, ae, ee $\div$

Therefore their square Roots also shall be in continual proportion, (*per* 22 *Prop.* 6. *Elem. Eucl.*) to wit, } $\because a$, $\sqrt{ae}$, e $\div$

Hence if a mean Proportional between any two given Numbers a and e be desired, it shall be $\sqrt{ae}$; as if $a=12$ and $e=3$, then $ae=36$, and $\sqrt{ae}$ or $\sqrt{36}$, that is, 6, is a mean Proportional between 12 and 3; for as 12 is to 6, so is 6 to 3.

Again, forasmuch as these Quantities are in continual Proportion, to wit, ... } *aaa*, *aae*, *aee*, *eeee* $\div$

Therefore their Cubic Roots also shall be continual Proportionals, (*per* 37 *Prop.* 11 *Elem. Eucl.*) to wit, } a, $\sqrt{(3)}aae$, $\sqrt{(3)}aee$, e $\div$

Hence if two mean Proportionals between any two given Numbers (a the greater and e the lesser) be desired, then $\sqrt{(3)}\ aae$ shall be the greater Mean, and $\sqrt{(3)}aee$ the lesser; as if $a=54$ and $e=2$, then $aae=5832$, and $\sqrt{(3)}aae=\sqrt{(3)}5832$; therefore $\sqrt{(3)}\ 5832$, that is, 18 is the greater Mean sought; also $aee=216$, and therefore $\sqrt{(3)}216$, that is, 6 is the lesser Mean: so that 18 and 6 are the two desired Mean Proportionals between 54 and 2; for 54, 18, 6, and 2, are in continual proportion. But when one Mean next to either of the Extremes is found out, the other Mean may be found out by *Sect.* 5. of this *Chap.* without extracting any Root.

After the same manner by the help of the said Table in *Sect.* 4. *Chap.* 1. of this Book, continued to higher Powers if need be, you may find out as many mean Proportional Numbers as shall be desired between any two given Numbers. As, if you would find five mean proportional Numbers between 1458 (or a) and 2 (or e;) look into the said Table for the sixth Power, (to wit, a Power whose Index exceeds by Unity the number of Means sought) and you will find *aaaaaa*, 6*aaaaae*, 15*aaaaee*, 20*aaaeee*, 15*aaeeee*, 6*aeeeee*, and *eeeeee*; then casting away 6, 15, 20, 15, and 6, which are prefix'd to the mean terms, and extract $\sqrt{(6)}$ out of every one of those six Terms after the said Numbers prefix'd are cast away, there will arise a, $\sqrt{(6)}aaaaae$, $\sqrt{(6)}aaaaee$, $\sqrt{(6)}aaaeee$, $\sqrt{(6)}aaeeee$, $\sqrt{(6)}aeeee$, and $e\div$; now to find the five mean proportional Numbers answering to those five proportional Roots express'd by Letters which fall between a and e, it will be convenient to find the smallest Mean first, *viz.* forasmuch as a was put for 1458, and e for 2; therefore $aeeeee=46656$, and $\sqrt{(6)}aeeeee=\sqrt{(6)}46656$, that is, 6 shall be the least Mean sought: then 2 being the first Proportional, or lesser Extreme, and 6 the second, the third will (by *Sect.* 5. of this *Chap.*) be found 18, the fourth 54, the fifth 162, the sixth 486, and the seventh, to wit, the greater Extreme, was first given 1458: so that between 2 and 1458 five mean Proportionals are found out, as was desired; and the seven continual Proportionals are these, to wit, 2, 6, 18, 54, 162, 486, and 1458.

Many other admirable Properties adherent to Numbers in Geometrical Proportion continued, are deducible from the said Table of Powers in *Sect.* 4. *Chap.* 1. of this Book, as will partly appear by the Theorems in the following sixth Chapter, which I find dispersed in several Algebraical Treatises.

CHAP. VI.

Various Theorems about Quantities in Continual Proportion.

Theorem 1.

IF three Numbers be Proportionals, the Solid Number made by the Continual Multiplication of all the three is equal to the Cube of the Mean.

Let three Proportionals be exposed in Integers according to *Sect.* 8. or 13. of the preceding *Chap.* 5. $\left.\begin{matrix} aa, & ae, & ee \div \\ 9, & 6, & 4 \div \end{matrix}\right.$

Thence it is evident, that *aaaeee*, the Product made by the Multiplication of all the three Proportionals one into another, is equal to the Cube of the Mean *ae*, as is affirmed by the Theorem.

Theorem 2.

If three Numbers be Proportionals, the Product made by the Multiplication of the Square of the first by the third, is equal to the Product of the Square of the second by the first.

As in these three, ... $\left.\begin{matrix} aa, & ae, & ee \div \\ 9, & 6, & 4 \div \end{matrix}\right.$

It is evident that $aaaa \times ee = aaee + aa = aaaaee$.

Theorem 3.

If three Numbers be Proportionals, the Square of the Sum of the Extremes is equal to both the Squares of the Extremes, together with twice the Square of the Mean.

As in these three, ... $\left.\begin{matrix} aa, & ae, & ee \div \\ 9, & 6, & 4 \div \end{matrix}\right.$

The Square of $aa+ee$ is $aaaa+2aaee+eeee$, which is manifestly equal to the Squares of *aa* and *ee*, together with twice the Square of *ae*.

Theorem 4.

If three Numbers be Proportionals, the Product of the lesser Extreme multiplied by the difference of the Extremes, is equal to the difference of the Squares of the mean and lesser Extreme.

As in these three, ... $\left.\begin{matrix} aa, & ae, & ee \div \\ 9, & 6, & 4 \div \end{matrix}\right.$

It is evident that $ee \times aa - ee = aaee - eeee$

Theorem 5.

If three Numbers be Proportionals, the Product of the greater Extreme multiplied by the difference of the Extremes, is equal to the difference of the Squares of the greater Extreme and the Mean.

As in these three . . . . $\left.\begin{matrix} aa, & ae, & ee \div \\ 9, & 6, & 4 \div \end{matrix}\right.$

It is evident that $aa \times aa - ee = aaaa - aaee$.

Theorem 6.

If three Numbers be Proportionals, the difference of the Squares of the Extremes is equal to the Square of the difference of the Extremes, together with twice the difference of the Squares of the mean and lesser Extreme.

As in these three, ... $\left.\begin{matrix} aa, & ae, & ee \div \\ 9, & 6, & 4 \div \end{matrix}\right.$

1. The difference of the Squares of the Extremes is $aaaa - eeee$
2. The square of $aa - ee$ (the difference of the Extremes) is . . . . $aaaa - 2aaee + eeee$
3. The double of the difference of the Squares of the mean and lesser Extreme is . . . . . $+2aaee - 2eeee$

Now the Sum of the two later of those three Quantities is manifestly equal to the first, as the Theorem affirms.

Theorem 7.

If three Numbers be Proportionals, the difference of the Squares of the greater Extreme and the Mean is equal to the Square of the difference of the Extremes, and to the difference of the Squares of the Mean and the leſſer Extreme.

As in theſe three, ... $\left\{ \begin{array}{l} aa\ ,\ ae\ ,\ ee\ \div \\ 9\ ,\ 6\ ,\ 4 \end{array} \right.$

1. The difference of the Squares of the greater Extreme and the Mean is . . . . $aaaa-aaee$
2. The Square of $aa-ee$ (the difference of the Extremes) is . . . . $aaaa-2aaee+eeee$
3. The difference of the Squares of the Mean and leſſer Extreme is . . . . $\cdot\ +aaee-eeee$

Now the Sum of the two latter of thoſe three Quantities is manifeſtly equal to the firſt, as the Theorem affirms.

Theorem 8.

If three Numbers be Proportionals, then as the firſt is to the third, ſo is the Square of the firſt to the Square of the ſecond; and ſo is the Square of the ſecond to the Square of the third.

As in theſe three, ... $\left\{ \begin{array}{l} aa\ ,\ ae\ ,\ ee\ \div \\ 9\ ,\ 6\ ,\ 4\ \div \end{array} \right.$

1. It is evident that . . . . $aa\ .\ ee :: aa\ .\ ee$
2. Therefore by drawing aa as a common Factor into the two latter Terms of that Analogy, this ariſes, $aa\ .\ ee :: aaaa\ .\ aaee$
3. And by drawing ee as a common Factor into the two latter Terms of the firſt Analogy, this ariſes, ... $aa\ .\ ee :: aaee\ .\ eeee$

By which two laſt Analogies the truth of the Theorem is manifeſt.

Theorem 9.

If three Numbers be Proportionals, then as the firſt is to the ſecond, (or as the ſecond is to the third) ſo is the difference of the firſt and ſecond, to the difference of the ſecond and third.

As in theſe three, ... $\left\{ \begin{array}{l} aa\ ,\ ae\ ,\ ee\ \div \\ 9\ ,\ 6\ ,\ 4 \end{array} \right.$

1. It is evident (as before hath been ſhewn in *Theorem* 4.) that, ... $ee \times \overline{aa-ee} = aaee-eeee$
2. And by Multiplication it will appear that, . . . $\overline{ae+ee} \times \overline{ae-ee} = aaee-eeee$
3. Therefore from the two laſt Equations (*per* 1 *Ax.* 1 *Elem. Eucl.*) . . . . $ee \times \overline{aa-ee} = \overline{ae+ee} \times \overline{ae-ee}$
4. Therefore by reſolving the laſt Equation into Proportionals, ... $aa-ee\ .\ ae-ee :: ae+ee\ .\ ee$
5. Therefore by diviſion of Reaſon, ... $aa-ae\ .\ ae-ee :: ae\ .\ ee$

Which was to be Demonſtrated.

Theorem 10.

If four Numbers be continually proportional, the Sum of the Means is a mean Proportional between the ſum of the firſt and ſecond, and the ſum of the third and fourth.

Let four continual Proportionals be expos'd in Integers, to wit, ... $\left\{ \begin{array}{l} aaa\ ,\ aae\ ,\ aee\ ,\ eee\ \div \\ 8\ ,\ 4\ ,\ 2\ ,\ 1\ \div \end{array} \right.$

Then according to the import of the Theorem, it muſt be proved that theſe three Quantities are Proportionals, *viz.*

$$aaa+aae\ .\ aae+aee\ .\ aee+eee\ \div$$

But that they are Proportionals it will be evident by Multiplication, for the Product of the Extremes is equal to the Square of the Mean: therefore the Truth of the Theorem is manifeſt.

Theorem

Theorem 11.

If four Numbers be continual Proportionals, the Sum of all is to the Sum of the Means, as the Sum of the firſt and third to the ſecond.

As in theſe four, ... $\left\{\begin{matrix} aaa, & aae, & aee, & eee \div \\ 8, & 4, & 2, & 1 \div \end{matrix}\right.$

1. The Sum of all four is . . . . $aaa + aae + aee + eee$
2. The Sum of the Mean is . . . . $+ aae + aee$
3. The Sum of the firſt and third is . . . . $aaa \quad + aee$
4. And the ſecond is . . . . $+ aae$

I ſay, thoſe four Quantities are Proportionals in ſuch order as they are above written; for it will appear by Multiplication, that the Product of the Extremes is equal to the Product of the Means: therefore the Theorem is manifeſt.

Theorem 12.

If four Numbers be in continual Proportion, the Sum of all is to the Sum of the Means, as the Sum of the Squares of the Means is to the Product of the Means or Extremes.

As in theſe four, ... $\left\{\begin{matrix} aaa, & aae, & aee, & eee \div \\ 8, & 4, & 2, & 1 \div \end{matrix}\right.$

1. The Sum of all is . . . . $a^3 + a^2e + ae^2 + e^3$
2. The Sum of the Mean is . . . . $+ a^2e + ae^2$
3. The Sum of the Squares of the Means is . . . . . . $+ a^4e^2 + a^2e^4$
4. The Product of the Means or Extremes is . . . . . . $+ a^3e^3$

I ſay, thoſe four Quantities are Proportionals, in ſuch order as they are above written; for it will appear by Multiplication, that the Product of the Extremes is equal to the Product of the Means: therefore the Theorem is manifeſt.

Theorem 13.

If four Numbers be continual Proportionals, the Sum of the Squares of the Mean is a mean Proportional between the Sum of the Squares of the firſt and ſecond, and the Sum of the Squares of the third and fourth.

As in theſe four, : . . . . $\left\{\begin{matrix} aaa, & aae, & aee, & eee \div \\ 8, & 4, & 2, & 1 \div \end{matrix}\right.$

1. The ſum of the Squares of the firſt and ſecond is . . . . $\}$ $a^6 + a^4e^2$
2. The Sum of the Squares of the Means is . . . . . . $a^4e^2 + a^2e^4$
3. The Sum of the Squares of the third and fourth is . . . . $\}$ $a^2e^4 + e^6$

I ſay, thoſe three Quantities are Proportionals in ſuch order as they are above written; for it will appear by Multiplication that the Square of the Mean (or ſecond Quantity) is equal to the Product of the Extremes: therefore the Theorem is manifeſt.

Theorem 14.

If four Numbers be continual Proportionals the Square of the Sum of the Means is equal to the Square of their difference, together with four times the Product of the Extremes or Means.

As in theſe four, ... $\left\{\begin{matrix} aaa, & aae, & aee, & eee \div \\ 8, & 4, & 2, & 1 \div \end{matrix}\right.$

1. The Square of $a^2e + ae^2$ (the ſum of the Means) is . . . . $\}$ $a^4e^2 + 2a^3e^3 + a^2e^4$
2. The Square of $a^2e - ae^2$ (the difference of the Means is . . . . $\}$ $a^4e^2 - 2a^3e^3 + a^2e^4$
3. The Quadruple of the Product of the Extremes or Means is . . . . $\}$. $+ 4a^3e^3$

Now it is Evident that the firſt of thoſe three Quantities is equal to the Sum of the ſecond and third: therefore the Theorem is manifeſt.

Theorem

Theorem 15.

If four Numbers be continual Proportionals, the Sum of their Squares shall be to the Sum of the Products of the first into the second, and the third into the fourth; as the sum of all the four Proportionals to the sum of the Means.

As in these four, ... $\left\{ \begin{matrix} aaa, & aae, & aee, & eee \div \\ 8, & 4, & 2, & 1 \div \end{matrix} \right.$

1. The sum of the Squares of the four Proportionals is . . . . $\}\ a^6+a^4e^2+a^2e^4+e^6$
2. The sum of the Products of the first into the second, and the third into the fourth is . . . . . $\}\ a^5e+ae^5$
3. The sum of all the four Proportionals is . . . . . $a^3+a^2e+ae^2+e^3$
4. The sum of the Means is . . . . a^2e+ae^2

I say, those four Quantities are Proportionals in such order as they are above seated, for it will appear by Multiplication, that the Product of the Extremes is equal to the Product of the Means. Therefore the Theorem is manifest.

Theorem 16.

If from the square of the sum of four Numbers in continual proportion the sum of their squares be subtracted, and from half the Remainder there be also subtracted the square of the sum of the two Means, this latter Remainder shall be the sum of the Products of the first Proportional into the second, and of the third into the fourth, and shall be to the sum of the squares of those four Proportionals, as the sum of the two Means is to the sum of all the Proportionals.

As in these four, ... $\left\{ \begin{matrix} aaa, & aae, & aee, & eee \div \\ 8, & 4, & 2, & 1 \div \end{matrix} \right.$

1. The square of the sum of the four Proportionals will by Multiplication be found
$$a^6+2a^5e+3a^4e^2+4a^3e^3+3a^2e^4+2ae^5+e^6.$$
2. The Sum of the squares of the four Proportionals is
$$a^6 \quad +a^4e^2 \quad +a^2e^4 \quad +e^6.$$
3. Which Sum of the squares being subtracted from the said square of the sum, the half of the Remainder will be
$$+a^5e+a^4e^2+2a^3e^3+a^2e^4+ae^5$$
4. The square of the sum of the two Means, to wit, of a^2e+ae^2 is
$$+a^4e^2+2a^3e^3+a^2e^4.$$
5. Which last mentioned square being subtracted from the half Remainder in the third step, there will remain the sum of the Products of the first Proportional into the second, and of the third into the fourth, to wit,
$$+a^5e+ae^5$$
6. Now according to the import and meaning of the Theorem it remains to prove, that the Remainder in the last step is to the sum of the squares in the second step, as the sum of the two mean Proportionals is to the sum of all four, *viz.* that

These four Quantities are Proportionals, $\left\{ \begin{array}{l} +a^5e+ae^5 \\ +a^6+a^4e^2+a^2e^4+e^6 \ :: \\ +a^2e+ae^2. \\ +a^3+a^2e+ae^2+e^3. \end{array} \right.$

7. But that they are Proportionals will be evident by Multiplication; for the Product of the Extremes is equal to the Product of the Means, each Product being
$$a^8e+a^7e^2+a^6e^3+a^5e^4+a^4e^5+a^3e^6+a^2e^7+ae^8.$$

Therefore the Theorem is manifest.

Theorem 17.

If four Numbers be Continual Proportionals, the sum of all their Squares shall be to the sum of the squares of the Means, as the sum of the Products of the first into the second, and the third into the fourth, to the Product of the Means or Extremes.

This is inferr'd from *Theorem* 12. and 15. by exchange of equal Reasons.

Theorem 18.

If four Numbers be Continual Proportionals, the sum of the squares of the Extremes shall be to the sum of the squares of the Means; as the Excess whereby the sum of the

the Products of the first into the second, and third into the fourth, exceeds the Product of the Means, is to the Product of the Means or Extremes.

This is inferr'd from *Theorem* 17. by Division of Reason.

Theorem 19.

If four Numbers be Continual Proportionals, the sum of the first and third shall be to the second; as the sum of the Squares of the Means is to the Product of the Means or Extremes.

This is deduced from *Theorem* 11. and 12. by exchange of equal Reasons.

Theorem 20.

If four Numbers be continual Proportionals, the sum of all their Squares shall be to the sum of the Products of the first into the second, and the third into the fourth; as the sum of the first and third is to the second.

This is deduced from *Theorem* 17. and 19. by exchange of equal Reasons.

Theorem 21.

If four Numbers be continual Proportionals, the sum of the Cubes of the Means is equal to the Product made by the Multiplication of the sum of the Extremes into the Product of the Means or Extremes.

As in these four, ... $\left\{ \begin{matrix} aaa, & aae, & aee, & eee & ∺ \\ 8, & 4, & 2, & 1 & ∺ \end{matrix} \right.$

1. The Sum of the Cubes of the Means is . $a^6e^3+a^3e^6$
2. The sum of the Extremes is . . . . . . a^3+e^3
3. The Product of the Means or Extremes is . a^3e^3

Now it is evident, that the first of those three Quantities is equal to the Product of the second Quantity multiplied by the third, as affirmed by the Theorem.

Theorem 22.

If four Numbers be continual Proportionals, the Cube of the sum of the Extremes is equal to the Cubes of the Extremes, together with the triple sum of the Cubes of the Means.

As in these four, ... $\left\{ \begin{matrix} aaa, & aae, & aee, & eee & ∺ \\ 8, & 4, & 2, & 1 & ∺ \end{matrix} \right.$

1. The Cube of a^3+e^3 (the sum of the Extremes) is . . . . $a^9+3a^6e^3+3a^3e^6+e^9$
2. The Cubes of the Extremes is . . . . . a^9+e^9
3. The triple sum of the Cubes of the Means is . $3a^6e^3+3a^3e^6$

Now it is manifest, that the first of those three Quantities is equal to the sum of the other two, as the Theorem affirms.

Theorem 23.

If four Numbers be continual Proportionals, the difference of the Cubes of the Extremes is equal to the triple of the difference of the Cubes of the Means, together with the Cube of the difference of the Extremes.

As in these four, ... $\left\{ \begin{matrix} aaa, & aae, & aee, & eee & ∺ \\ 8, & 4, & 2, & 1 & ∺ \end{matrix} \right.$

1. The difference of the Cubes of the Extremes is a^9-e^9
2. The Triple of the difference of the Cubes of the Means is . . . . $3a^6e^3-3a^3e^6$
3. The Cube of a^3-e^3 (the difference of the Extremes) is . . . . $a^9-3a^6e^3+3a^3e^6-e^9$

Now it is manifest, that the first of those three Quantities is equal to the sum of the other two; which was to be prov'd.

Theorem

Theorem 24.

If four Numbers be Continual Proportionals, the Cube of the Sum of the firſt and ſecond is equal to the Product made by the Multiplication of the ſquare of the firſt by the Aggregate of the ſum of the Extremes and the triple ſum of the Means.

As in theſe four, ... $\left\{\begin{matrix} aaa, & aae, & aee, & eee & ∺ \\ 8, & 4, & 2, & 1 & ∺ \end{matrix}\right.$

1. The Cube of the ſum of the firſt and ſecond to wit, of a^3+aae is . . . . . $a^9+3a^8e+3a^7e^2+a^6e^3$
2. The Square of the firſt is . . . . a^6
3. The Aggregate of the Extremes and the triple of the ſum of the Means is . . . $a^3+e^3+3a^2e+3ae^2$

Now it is evident that the firſt of thoſe three Quantities is equal to the Product made by the Multiplication of the third by the ſecond; which was to be proved.

Theorem 25.

If four Numbers be continual Proportionals, the Cube of the ſum of the Means is equal to the Product made by the Multiplication of the Product of the Extremes or Means into the Aggregate of the Extremes and the triple ſum of the Means.

As in theſe four, ... $\left\{\begin{matrix} aaa, & aae, & aee, & eee & ∺ \\ 8, & 4, & 2, & 1 & ∺ \end{matrix}\right.$

1. The Cube of the ſum of the Means, to wit, of a^2e+ae^2 is . . . . $a^6e^3+3a^5e^4+3a^4e^5+a^3e^6$
2. The Product of the Extremes or Means is . a^3e^3
3. The Aggregate of the Extremes and the triple ſum of the Means is . . . . . $a^3+e^3+3a^2e+3ae^2$

Now it is evident that the firſt of thoſe three Quantities is equal to the Product of the two latter; which was to be proved.

Theorem 26.

If four Numbers be continual Proportionals, the Product made by the Multiplication of the ſum of the Extremes by the Sum of the Squares of the Extremes, is equal to the Cubes of the four Proportionals.

As in theſe four, ... $\left\{\begin{matrix} aaa, & aae, & aee, & eee & ∺ \\ 8, & 4, & 2, & 1 & ∺ \end{matrix}\right.$

1. The ſum of the Extremes is . . . . . a^3+e^3
2. The ſum of the ſquares of the Extremes is . a^6+e^6
3. The Product of theſe two ſums is . . . $a^9+a^6e^3+a^3e^6+e^9$
4. The ſum of the Cubes of the four Propornals is . . . . $a^9+a^6e^3+a^3e^6+e^9$

But the Product in the third ſtep is manifeſtly equal to the ſum in the fourth; as the Theorem affirms.

Theorem. 27.

If five Number be continual Proportionals, the Product of the Mean (or third Proportional) into the ſum of the Extremes, is equal to the Squares of the ſecond and fourth.

As in theſe five, ... $\left\{\begin{matrix} aaaa, & aaae, & aaee, & aeee, & eeee \\ 16, & 8, & 4, & 2, & 1 \end{matrix}\right.$

1. The Product of the Mean into the Sum of the Extremes is . . . . $a^6e^2+a^2e^6$
2. And the ſum of the Squares of the ſecond and fourth is alſo . . . . $a^6e^2+a^2e^6$

Therefore the Theorem is manifeſt.

Theorem

Theorem 28.

If five Numbers be continual Proportionals, the sum of the first, third, and fifth, shall be to the third; as the sum of the Squares of the second, third, and fourth, is to the square of the third.

As in these five, ... $\left\{\begin{matrix} aaaa, & aaae, & aaee, & aeee, & eeee \\ 16, & 8, & 4, & 2, & 1 \end{matrix}\right.$

1. The sum of the first, third, and fifth is . . . . . $a^4+a^2e^2+e^4$
2. The third is . . . . a^2e^2 ::
3. The sum of the Squares of the second, third, and fourth is . . . . $a^6e^2+a^4e^4+a^2e^6$
4. The square of the third is . . . . a^4e^4.

I say, those four Quantities are Proportionals, in such order as they are above seated; for it will appear by Multiplication, that the Product of the Extremes is equal to the Product of the Means; each Product being $a^8e^4+a^6e^6+a^4e^8$: Therefore the Theorem is manifest.

Theorem 29.

If five Numbers be continual Proportionals, the sum of the Extremes more by the double of the Mean, the sum of the second and fourth, and the Mean, are also continual Proportionals.

As in these five, ... $\left\{\begin{matrix} aaaa, & aaae, & aaee, & aeee, & eeee \\ 16, & 8, & 4, & 2, & 1 \end{matrix}\right.$

1. The sum of the Extremes more by the double of the Mean is . . . . $a^4+e^4+2a^2e^2$
2. The sum of the second and fourth is . . . . . a^3e+ae^3
3. The Mean is . . . . a^2e^2

I say, those three Quantities are Proportionals; for it will be evident by Multiplication that the Product of the first and third is equal to the square of the second: therefore the Theorem is manifest.

Theorem 30.

If five Numbers be continual Proportionals, the Sum of the Extremes is to the Mean; as the difference of the Squares of the Extremes, to the difference of the Squares of the second and fourth.

As in these five, ... $\left\{\begin{matrix} aaaa, & aaae, & aaee, & aeee, & eeee \\ 16, & 8, & 4, & 2, & 1 \end{matrix}\right.$

1. The sum of the Extremes is . . . . a^4+e^4.
2. The Means is . . . . a^2e^2 : :
3. The difference of the Squares of the Extremes is . . a^8-e^8.
4. The difference of the Squares of the second and fourth is . . . . $a^6e^2a^2-a^2e^6$.

I say, those four Quantities are Proportionals in such order as they are above placed; for it will be evident by Multiplication, that the Product of the Extremes is equal to the Product of the Means, each Product being $a^{10}e^2-a^2e^{10}$: Therefore the Theorem is manifest.

Theorem 31.

If five Numbers be continual Proportionals, the sum of the Squares of the second and fourth shall be to the square of the Mean, as the difference of the Squares of the Extremes to the difference of the Squares of the second and fourth.

As in these five, ... $\left\{\begin{matrix} aaaa, & aaae, & aaee, & aeee, & eeee \\ 16, & 8, & 4, & 2, & 1 \end{matrix}\right.$

1. The sum of the Squares of the second and fourth is . $a^6e^2+a^2e^6$.
2. The Square of the Mean is . . . . a^4e^4 : :
3. The difference of the Squares of the Extremes is . a^8-e^8.
4. The difference of the Squares of the second and fourth is . . . . $a^6e^2-ae^6$.

I say, those four Quantities are Proportionals in such order as they are above seated; for it will be evident by Multiplication, that the Product of the Extremes is equal to the Product of the Means; therefore the Theorem is manifest.

Theorem 32.

If five Numbers be continual Proportionals, the Sum of the Exrremes shall be to the Mean, as the Sum of the Squares of the second and fourth is to the Square of the Mean. This is evident from the two last preceding Theorems by exchange of equal Reasons.

Theorem 33.

If five Numbers be continual Proportionals, the Sum of the Squares of the second and fourth shall be equal to the Product made by the Multiplication of the third into the Sum of the first and fifth.

As in these five, ... $\left.\right\}$ $aaaa$, $aaae$, $aaee$, $aeee$, $eeee$
16, 8, 4, 2, 1

1. The Sum of the Squares of the second and foutth is . . . . $\left.\right\}$ $a^6e^2+a^2e^6$
2. The Mean or third is . . . . a^2e^2
3. The Sum of the first and fifth is . . . . a^4+e^4

But the Product of the second and third of those three Quantities above written is equal to the first; therefore the Theorem is manifest.

CHAP. VII.

Questions about Quantities in Continual Proportion resolved by Literal Algebra.

QUEST. I.

THE Sum (b) of three Proportional Quantities being given, as also (c) the Sum of their Squares, to find the Proportionals.

RESOLUTION.

1. For the Mean Proportional sought put . . . . . . a
2. Then subtracting the said Mean from (b) the given Sum of all the three Proportionals, there will remain the Sum of the Extremes, to wit, ... $\left.\right\}$ $b-a$
3. Therefore the Square of the Sum of the Extremes is . $bb-2ba+aa$
4. From which Square if there be subtracted the double of the Square of the Mean, to wit, ... $\left.\right\}$ $2aa$
5. There will remain (as is manifest by *Th.* 3. of the preceding *Chap.* 6.) the Sum of the Squares of the Extremes, to wit, $\left.\right\}$ $bb-2ba-aa$
6. To which Sum of the Squares of the Extremes if you add (aa) the Square of the Mean, the Aggregate shall be the sum of the Squares of the three Proportionals sought, to wit, $\left.\right\}$ $bb-2ba$
7. Which sum in the last step must be equal to (c) the given sum of the Squares; hence this Equation, *viz.* . . $\left.\right\}$ $bb-2ba=c$
8. Which Equation after due Reduction gives . . . . $\left.\right\}$ $\frac{bb-c}{2b}=a$

And the last Equation in words is this

CANON.

From the Square of the given sum of the three Proportionals sought subtract the given sum of their Squares; then divide the Remainder by the double of the sum of the three Proportionals, and the Quotient is the mean Proportional.

Therefore if 14 be given for the sum of the three Numbers in continual proportion, and 84 for the sum of their Squares, the mean Proportional will be found 4 by the said Canon. Then the Mean being given 4, as also 10 the sum of the Extremes, the Ex-

Extremes will be found 2 and 8, (by the Canon of *Quest.* 4. *Chap.* 16. of my First Book of *Algebraical Elements*,) and therefore the three Proportionals sought are 2, 4, and 8.

QUEST. 2.

The Sum (b) of three proportional Quantities being given, as also (c) the Sum of the Squares of the Extremes, to find the Proportionals.

RESOLUTION.

1. For the mean Proportional sought put . . . . a
2. Then subtracting the said Mean from (b) the given Sum of all the three Proportionals, there will remain the Sum of the Extremes, to wit, ... $b-a$
3. Therefore the Square of the Sum of the Extremes is . $bb-2ba+aa$
4. From which square if you subtract the double of the Square of the Mean, to wit, ... $2aa$
5. There will remain (as is manifest by the third Theorem of the preceding sixth *Chap.*) the Sum of the Squares of the Extremes, to wit, ... $bb-2ba-aa$
6. Which Sum of the Squares of the Extremes must be equal to the given Sum (c,) hence this Equation, *viz.* $bb-2ba-aa=c$
7. From which Equation after due Reduction this will arise $bb-c=aa+2ba$
8. Therefore by resolving the last Equation, (according to the Canon in *Sect.* 6. *Chap.* 1. of my First Book of *Algebraical Elements*;) the value of (a) the mean Proportional will be made known, *viz.* . . . . $\sqrt{:2bb-c:}-b=a$

Which last Equation in words is this

CANON.

From the double of the Square of the given Sum of all the three Proportionals sought subtract the given Sum of the Squares of the Extremes; then from the square Root of the Remainder subtract the Sum of the three Proportionals, so shall this last Remainder be the mean Proportional sought.

Therefore if 14 be given for the Sum of three Continual Proportionals, and 68 for the Sum of the Squares of the Extremes, the mean Proportional will be found 4 by the said Canon. Then the Mean being given 4, as also 10 the Sum of the Extremes, the Extremes will be found 2 and 8, (by the Canon of *Quest.* 4. *Chap.* 15. of my First Book of *Algebraical Elements*;) and therefore the three Proportionals sought are 2, 8, and 4.

QUEST. 3.

The difference (b) of the Extremes of three proportional Quantities being given, as also (c) the Sum of the Squares of the three Proportionals; to find the Proportionals.

RESOLUTION.

1. For the Sum of the Extremes, (to wit, of the first and third Proportionals sought) put . . . . a
2. Then forasmuch as the difference of the Extremes is given (b,) and their Sum is assumed to be (a,) therefore (by the Theorem in *Quest.* 1. *Chap.* 14. of my First Book of *Algebraical Elements*) the greater Extreme shall be . . $\frac{1}{2}a+\frac{1}{2}b$
3. And by the same Theorem the lesser Extreme is . . . $\frac{1}{2}a-\frac{1}{2}b$
4. Then the Product made by the Multiplication of the Extremes in the second and third steps will give the Square of the Mean, to wit, ... $\frac{1}{4}aa-\frac{1}{4}bb$
5. And from the second step the Square of the greater Extreme is . . . . $\frac{1}{4}aa+\frac{1}{2}ab+\frac{1}{4}bb$
6. And from the third step the Square of the lesser Extreme is $\frac{1}{4}aa-\frac{1}{2}ab+\frac{1}{4}bb$
7. Therefore from the fourth, fifth, and sixth steps the Sum of the Squares of all the three Proportionals is . $\frac{1}{4}aa+\frac{1}{4}bb$

8. Which ſum in the laſt ſtep muſt be equal to (c) the ſum of the Squares given in the Queſtion, hence this Equation ariſes, to wit, ... $\frac{3}{4}aa+\frac{1}{4}bb=c$
9. Which Equation after due Reduction will give . . . . $aa=\frac{4c-bb}{3}$
10. Therefore by extracting the ſquare Root out of each part of the laſt Equation the ſum of the extreme Proportionals is diſcovered, to wit, ... $a=\sqrt{\frac{4c-bb}{3}}$

Which laſt Equation gives this

CANON.

From four times the given ſum of the ſquares of the three Proportionals ſought, ſubtract the ſquare of the given difference of the Extremes; then the ſquare Root of one third part of that Remainder ſhall be the ſum of the extreme Proportionals.

Then half the ſum of the Extremes increaſed with half their difference gives the greater Extreme, and half the ſaid ſum leſſened by half the ſaid difference leaves the leſſer Extreme.

Laſtly, the ſquare Root of the Product made by the mutual Multiplication of the Extreme is the mean Proportional.

Therefore if 16 be given for the difference of the Extremes of three Proportionals, and 364 for the ſum of the ſquares of all the three Proportionals, the Proportionals are alſo given ſeverally, to wit. 2, 6, 18 ∺

QUEST. 4.

One Extreme (b) of three Proportional Quantities being given, as alſo (c) the ſum of the ſquares of the other Extreme and the Mean, to find out that other Extreme and Mean

RESOLUTION.

1. For the extreme Proportional ſought put . . . . . a
2. Which multiplied by the given Extreme (b) produces the ſquare of the Mean, to wit, ... ba
3. But from the firſt ſtep the ſquare of the extreme Proportional ſought is . . . . aa
4. Therefore from the ſecond and third ſteps the ſum of the ſquares of the two Proportionals ſought is . . . $aa+ba$
5. Which ſum in the laſt ſtep muſt be equal to (c) the ſum given in the Queſtion; hence this Equation ariſes, *viz.* $aa+ba=c$
6. Which Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. of my Firſt Book of *Algebraic Elements*, will diſcover the extreme Proportional ſought, to wit, . . $a=\sqrt{:c+\frac{1}{4}bb:}-\frac{1}{2}b$

The laſt Equation in words is this

CANON.

To the given ſum add the ſquare of half the extreme Proportional given, and out of this ſum extract the ſquare Root; then this ſquare Root leſſened by half the given Extreme will give the other Extreme.

Therefore if 18 be given for one of the Extremes of three Proportionals, and 40 for the ſum of the ſquares of the other two Proportionals, the Canon will diſcover 2 for the Extreme ſought. Laſtly, the ſquare Root of the Product of the Extremes, to wit, 6 is the Mean ſought; therefore the three Proportionals are 18, 6, and 2.

QUEST. 5.

The difference (b) between the Extremes of three proportional Quantities being given, as alſo the Proportion which the difference of the ſquares of the Extremes has to the ſum of the ſquares of all the three Proportionals, ſuppoſe the difference be to the ſum as (r) to (s;) to find the Proportionals. But (r) muſt be leſs than (s.)

RESOLUTION.

1. For the ſum of the Extremes put . . . . a
2. Then foraſmuch as their difference is given . . . . . b
3. Therefore the difference of the ſquares of the Extremes ſhall be ba; (for the Product of the Multiplication of the ſum of any two Numbers into their difference is equal to the difference of their ſquares.) ba

4. Then

4. Then from the first and second steps (by the *Theorem* of *Quest.* 1. *Chap.* 14. of my First Book of *Algebraical Elements*) the greater Extreme shall be . . . . $\frac{1}{2}a + \frac{1}{2}b$
5. And (by the same *Theorem*) the lesser Extreme shall be $\frac{1}{2}a - \frac{1}{2}b$
6. Therefore from the fourth step the square of the greater Extreme is . . . . $\frac{1}{4}aa + \frac{1}{4}bb + \frac{1}{2}ba$
7. And from the fifth step the square of the lesser Extreme is $\frac{1}{4}aa + \frac{1}{4}bb - \frac{1}{2}ba$
8. And because the Product made by the mutual Multiplication of the Extremes is equal to the Square of the Mean, therefore the Extremes in the fourth and fifth steps being multiplied one by the other, will give the Square of the Mean, to wit, ... $\frac{1}{4}aa - \frac{1}{4}bb$
9. Therefore by adding together the Squares in the three last steps, the Sum of the squares of the three Proportionals shall be . . . . $\frac{3}{4}aa + \frac{1}{4}bb$
10. Then according to the Question as r is to s, so must the difference in the third step be to the sum in the ninth step; hence this Analogy arises, *viz.*

$$r \; . \; s \; :: \; ba \; . \; \tfrac{3}{4}aa - \tfrac{1}{4}bb$$

11. Whence by comparing the Product made by the mutual Multiplication of the Extremes to the Product of the Means this Equation comes forth, *viz.*

$$sba = \tfrac{3}{4}raa + \tfrac{1}{4}rbb.$$

12. From which Equation after due Reduction there will arise $\frac{4sb}{3r}a - aa = \frac{bb}{3}$
13. Therefore (*per Canon in Sect.* 10. *Chap.* 15. *Book* 1.) the two Roots or Values of a in the last Equation are these, to wit,

$$a = \frac{2sb + \sqrt{:4ssbb - 3rrbb:}}{3r} \text{ the greater; } a = \frac{2sb - \sqrt{:4ssbb - 3rrbb:}}{3r} \text{ the lesser.}$$

14. But the greater of those two Values of (a) is the desired sum of the extreme Proportionals sought; for if we should suppose the lesser Value to be the sum of the Extremes, it ought to exceed (b) the difference of the Extremes: but from that supposition it will follow that (r) is greater than $(s,)$ and consequently that the difference of the squares of the Extremes is greater than the sum of the squares of all the three Proportionals, which is impossible. Now to prove the said Consequence,
15. Suppose . . . . $\frac{2sb - \sqrt{:4ssbb - 3rrbb:}}{3r} \sqsubset b.$
16. Then by multiplying each part by $3r$, it follows that . . . . $2sb - \sqrt{:4ssbb - 3rrbb:} \sqsubset 3rb.$
17. And by adding $\sqrt{:4ssbb - 3rrbb:}$ to each part in the sixteenth step; . . $2sb \sqsubset 3rb + \sqrt{:4ssbb - 3rrbb:}$
18. And by subtracting $3rb$ from each part in the seventeenth step, ... $2sb - 3rb \sqsubset \sqrt{:4ssbb - 3rrbb:}$
19. And by squaring each part in the eighteenth step, ... $4ssbb - 12srbb + 9rrbb \sqsubset 4ssbb - 3rrbb.$
20. And by adding $3rrbb$ to each part in the nineteenth step, ... $4ssbb - 12srbb + 12rrbb \sqsubset 4ssbb.$
21. And by adding $12srbb$ to each part in the twentieth step, ... $4ssbb + 12rrbb \sqsubset 4ssbb + 12srbb.$
22. And by subtracting $4ssbb$ from each part in the twenty first step, ... $12rrbb \sqsubset 12srbb.$
23. Wherefore by dividing each part in the twenty second step by $12rbb$, . . $r \sqsubset s.$
24. Thus from a supposition that the lesser Value of (a) in the thirteenth step is greater than (b) the given difference of the Extremes, it follows by just consequence that (r) is greater than $(s,)$ which is impossible; for in regard the difference of the squares of the Extremes is less than the sum of the Squares of all 3 Proportionals, and that according to the Question the said difference is to the said sum as (r) to $(s,)$ therefore (r) is less than $(s.)$ And because the series of Inferences drawn from the said supposition ends in an Impossibility, therefore that which was supposed cannot be true, *viz.* The lesser

Value

value of (a) is not greater than (b) the given difference of the Extremes, and consequently it cannot be equal to the Sum of the Extremes; which was to be proved.

But by the like Argumentation it may be proved, that the greater value of (a) in the thirteenth step exceeds (b) the given difference of the Extremes; and if it be express'd by words, it will give the following Canon to find out the Sum of the extreme Proportionals sought; whence by the help of the given difference of the Extremes, the Extremes are severally given.

CANON.

From four times the square of the latter or greater Term (s) of the given Reason subtract thrice the square of the first Term $(r,)$ and multiply the Remainder by the square of the given difference of the extreme Proportionals sought; then add the square Root of that Product to the double of the Product made by the Multiplication of the latter Term (s) into the difference of the Extremes, and divide the Sum of that Addition by the triple of the first Term $(r;)$ so shall the Quotient be the Sum of the extreme Proportionals. Lastly, half the Sum of the Extremes increased with half their difference gives the greater Extreme, but the said half Sum lessened by the said half difference leaves the lesser Extreme.

As for Example: If 6 be given for the difference of the Extremes of three Continual Proportionals, and the difference of the squares of the Extremes has such proportion to the Sum of the Squares of all the three Proportionals as 5 to 7, then by the Canon the three Proportionals will be found 2, 4, and 8.

Again, if $2\frac{1}{4}$ be given for the difference of the Extremes, and the difference of the Squares of the Extremes be to the Sum of the Squares of all the three Proportionals, as 123 to 427, the Proportionals will be found 4, 5, and $6\frac{1}{4}$.

QUEST. 6.

The Sum (b) of the Extremes, and the Sum (c) of the Means of four Quantities in Continual Proportion being given, to find out the Proportionals; but (b) must exceed $(c.)$

RESOLUTION.

1. For one of the Means put . . . . a
2. Then by subtracting the Mean from (c) the given Sum of the Means, the Remainder is the other Mean, to wit, $c-a$
3. And by dividing the Square of the latter Mean by the former, the Quotient gives one of the Extremes, to wit, $\frac{cc-2ca+aa}{a}$
4. In like manner the square of the first Mean (a) being divided by the other Means $(c-a)$ gives the other Extreme, to wit, $\frac{aa}{c-a}$
5. Therefore from the third and fourth steps the Sum of the two Extremes is . . . . $\frac{ccc-3cca-3caa}{ca-aa}$
6. Which Sum must be equal to (b) the given Sum of the Extremes; hence this Equation arises, to wit, ... $\frac{ccc-3cca-3caa}{ca-aa}=b$
7. From which Equation after due Reduction this arises, to wit, ... $\frac{ccc}{3c+b}=ca-aa$
8. Wherefore by resolving the last Equation by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two values of $(a,)$ to wit, the mean Proportionals sought will be made known, *viz.*

$$a=\tfrac{1}{2}c+\sqrt{:\frac{cc}{4}-\frac{ccc}{3c+b}}:\text{ the greater Mean:}$$

$$a=\tfrac{1}{2}c-\sqrt{:\frac{cc}{4}-\frac{ccc}{3c+b}}:\text{ the lesser Mean.}$$

Which values of (a) give this

CANON.

Divide the Cube of the Sum of the Means by the Aggregate of the triple Sum of the Means and the Sum of the Extremes; subtract the Quotient from the square of half the sum of the Means, and extract the square Root of the Remainder; then the said square Root being added to and subtracted from half the sum of the Means, the Sum and Remainder shall be the Means sought.

Then

Then the Square of the lesser Mean being divided by the greater will give the lesser Extreme, and the Square of the greater Mean divided by the lesser gives the greater Extreme.

Therefore if 18 be given for the sum of the Extremes, and 12 for the sum of the Means of four continual Proportionals, the Proportionals are given severally by the said Canon, to wit, 2, 4, 8, and 16.

QUEST. 7.

The difference (b) of the Extremes, and the difference (c) of the Means of four Quantities continually proportional being given, to find out the four Proportionals.

RESOLUTION.

1. For the lesser mean Proportional put . . . . a
2. Which added to (c) the given difference of the Means gives the greater Mean, to wit, ... $c+a$
3. Then the Square of the said greater Mean being divided by the lesser, gives for the greater Extreme . . . . . $\frac{cc+2ca+aa}{a}$
4. Likewise by dividing (aa) the Square of the lesser Mean by the greater, there arises for the lesser Extreme . . $\frac{aa}{c+a}$
5. Therefore the difference of the two Extremes in the third and fourth steps is . . . . $\frac{ccc+3cca+3caa}{ca+aa}$
6. Which difference must be equal to (b) the given difference of the Extremes; hence this Equation arises, *viz.* $\frac{ccc+3cca+3caa}{ca+aa}=b$
7. From which Equation after due Reduction this arises, to wit, ... $\frac{ccc}{b-3c}=ca+aa$
8. Wherefore by resolving the last Equation by the Canon in *Sect.* 6. *Ch.* 15. *Book* 1. the value of (a,) to wit, the lesser mean Proportional sought will be made known, *viz.*

$$a=\sqrt{:\frac{cc}{4}-\frac{ccc}{b-3c}:}-\tfrac{1}{2}c.$$

Which Equation in words is this

CANON.

Divide the Cube of the given difference of the Means by the excess of the given difference of the Extremes above the triple of the difference of the Means; add the Quotient to the Square of half the difference of the Means; then from the square Root of that sum subtract half the difference of the Means, so shall this Remainder be the lesser Mean. Then to the lesser Mean add the difference of the Means, and the sum is the greater. Lastly, the Square of the greater Mean divided by the lesser gives the greater Extreme, and the Square of the lesser Mean divided by the greater gives the lesser Extreme.

Therefore if 52 be given for the difference of the Extremes of 4 continual Proportionals, and 12 for the difference of the Means, the Proportionals will be found 2, 6, 18, 54.

QUEST. 8.

The sum (b) of four Quantities in continual proportion being given, as also (c) the sum of their squares, to find the Proportionals.

RESOLUTION.

1. For the sum of the Means put . . . . a
2. Which subtracted from (b) the given sum of all the four Proportionals. leaves the sum of the Extremes, to wit, $b-a$
3. The square of (b) the given sum of all the four Proportionals is . . . . bb
4. Now (according to *Theor.* 16. of the preceding *Chap.* 6.) from the said square (bb) I subtract (c) the given sum of the squares of the four Proportionals, and from the half of the Remainder I also subtract (aa) the square of the sum of the Means, so this Quantity remains, to wit, . $\frac{1}{2}bb-\frac{1}{2}c-aa$
5. Which Remainder, to wit, $\frac{1}{2}bb-\frac{1}{2}c-aa$ (by the said *Theor.* 16.) shall be to the given sum of the squares of the four Proportionals, as the sum of the Means is to the sum of all the four Proportionals, hence this Analogy arises, *viz.*

$$\tfrac{1}{2}bb-\tfrac{1}{2}c-aa \,.\, c :: a \,.\, b$$

6. Which

6. Which Analogy, by comparing the Product made by the mutual Multiplication of the Extremes to the Product of the Means, will be converted into this Equation, *viz.* $\frac{1}{2}bbb - \frac{1}{2}bc - baa = ca$

7. Whence after due Reduction this Equation arises, to wit,

$$\tfrac{1}{2}bb - \tfrac{1}{2}c = aa + \frac{c}{b}a$$

Which Equation being resolved (*per Canon in Sect. 6. Chap.* 15. *Book* 1.) gives this following.

CANON.

From the square of the given sum of the four Proportionals subtract the given sum of their Squares, and to the half of the Remainder add the square of half the Quotient that arises by dividing the sum of the Squares of the four Proportionals by the sum of the four Proportionals. Then extract the square Root of the sum of that Addition, and from the said square Root subtract half the Quotient aforesaid, so shall the Remainder be the sum of the two desired mean Proportionals.

Then the sum of the Means of four continual Proportionals being given, as also the sum of the Extremes, the Proportionals shall be given severally by the Canon of the preceding *Quest.* 6. of this *Chap.*

So if 30 be given for the sum of four Proportionals, and 340 for the sum of their Squares; first, by the Canon above express'd the sum of the Means will be found 12, which subtracted from 30 the given sum of the four Proportionals, leaves 18 for the sum of the Extremes; then the sum of the Means being given 12, and the sum of the Extremes 18, the four Proportionals (by the Canon of the preceding sixth Question) will be found 2, 4, 8, 16.

QUEST. 9.

The sum (b) of four Quantities in continual proportion being given, as also (c) the sum of the squares of the Means; to find the Proportionals.

RESOLUTION.

1. For the sum of the Means put . . . . a

Then because (by *Theorem* 12. of the preceding *Chap.* 6.) the sum of the four Quantities continually proportional is to the sum of the Means, as the sum of the Squares of the Means is to the Product made by the mutual Multiplica- of the Means or Extremes, say by the Rule of Three,

If . . . . b . a :: c . $\frac{ca}{b}$

Whence the Product of the Means or Extremes is found $\Big\}\ \frac{ca}{b}$

3. And because if from the square of the sum of the Means there be subtracted the sum of the squares of the Means, there will remain the double Product of the Means or Extremes; therefore if from (aa) you subtract (c) the half of the Remainder shall be the Product of the Means or Extremes, to wit, $\Big\}\ \frac{1}{2}aa - \frac{1}{2}c$

4. Which Product, to wit, $\frac{1}{2}aa - \frac{1}{2}c$ must be equal to $\frac{ca}{b}$ the Product in the second step; hence this Equation arises, to wit, ... $\Big\}\ \frac{1}{2}aa - \frac{1}{2}c = \frac{ca}{b}$

5. From which Equation after due Reduction there arises $\Big\}\ aa - \frac{2c}{b}a = c$

Which last Equation being resolved (by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1.) gives this following

CANON.

To the given sum of the Squares of the Means add the Square of the Quotient that arises by dividing the said sum by the given sum of the four Proportionals, and out of the sum made by that Addition extract the square Root; then this square Root added to the aforesaid Quotient gives the Sum of the Mean Proportionals sought.

Then the Sum of the Means being given, as also the Sum of the Extremes, (for the Sum of the Means found out being subtracted from the given sum of all the four Proportionals leaves the Sum of the Extremes) the four Proportionals will be discovered by the Canon of the sixth Question of this Chapter.

There-

Therefore if 30 be given for the ſum of four Continual Proportionals, and 80 for the ſum of the Squares of the Means, the four Proportionals are alſo ſeverally given; to wit, 2,4,8,16, by the Canon above expreſs'd.

QUEST. 10.

The ſum (b) of four Quantities continually proportional being given, as alſo (c) the ſum of the ſquares of the Extremes, to find out the Proportionals.

RESOLUTION.

1. For the ſum of the Means put . . . . a
2. Which ſubtracted from (b) the given ſum of the four Proportionals leaves the ſum of the Extremes, to wit, . . $b-a$
3. Therefore the ſquare of the ſum of the Extremes is . $bb-2ba+aa$
4. From which Square if (c) the given ſum of the ſquares of the Extremes be ſubtracted, there will remain the double Product made by the mutual Multiplication of the Extremes or Means; therefore the Product of the Means is $\frac{bb-2ba+aa-c}{2}$
5. And becauſe if from aa the ſquare of the ſum of the Means there be ſubtracted $bb-2ba+aa-c$, the double Product of the Means, there will remain the ſum of the ſquares of the Means; therefore the ſum of the ſquares of the Means is $2ba-bb+c$
6. And becauſe by *Theor.* 12. in the preceding *Chap.* 6. the ſum of the ſquares of the Means is to the Product of the Means, as the ſum of all the 4 Proportionals is to the ſum of the Means; therefore from the premiſes this following Analogy ariſes, *viz.*

$$2ba-bb+c \; . \; \frac{bb-2ba+aa-c}{2} :: b \; . \; a \; .$$

7. From which Analogy by comparing the Product of the Extremes to the Product of the Means, this Equation ariſes, *viz.*

$$2baa-bba+ca=\frac{bbb-2bba+baa-bc}{2}$$

8. Which Equation after due Reduction gives this following Equation, *viz.*

$$aa+\frac{2c}{3b}a=\frac{bb-c}{3}$$

Whence (*per Canon in Sect. 6. Chap. 15. Book 1.*) there ariſes this following

CANON.

Divide the given ſum of the ſquares of the Extremes by the triple of the given ſum of all the four Proportionals, and to the ſquare of the Quotient add one third part of the exceſs of the ſquare of the ſum of the four Proportionals above the Sum of the ſquares of the Extremes; then from the ſquare Root of the Sum made by that Addition ſubtract the Quotient firſt found out; ſo ſhall the Remainder be the deſired ſum of the mean Proportionals.

Then the ſum of the Means being given, as alſo the ſum of the Extremes, (for the ſum of the Means being ſubtracted from the given ſum of the four Proportionals leaves the ſum of the Extremes) the four Proportionals will be diſcovered by the Canon of the ſixth Queſtion of this Chapter.

Therefore if 80 be given for the ſum of four continual Proportionals, and 2920 for the ſum of the Squares of the Extremes, the 4 Proportionals will be found 2,6,18,54.

QUEST. 11.

The ſum (b) of the ſquares of the Extremes of 4 Quantities in continual proportion being given, as alſo (c) the ſum of the ſquares of the Means, to find out the Proportionals.

RESOLUTION.

1. Add the two given ſums into one that you may have the ſum of the ſquares of the four Proportionals ſought, for which laſt mentioned Sum put . . . . d
2. Then for the ſum of the Squares of the firſt and ſecond Proportionals put . . . . a
3. Therefore the ſum of the Squares of the third and fourth Proportionals is . . . . $d-a$

4. Then because (by *Theorem* 13. of the preceding *Chap.* 6.) the sum of the Squares of the two Means is a mean Proportional between the Sum of the Squares of the first and second, and the sum of the Squares of the third and fourth, this Analogy is manifest, *viz.* . . . . $a \,.\, c :: c \,.\, d-a$
5. Therefore by comparing the Product made by the Multiplication of the Extremes of that Analogy to the Product of the Means this Equation arises, *viz.* . . . . $da-aa=cc$
6. Which Equation being resolved by the Canon in *Sect.* 10. *Chap.* 16. *Book* 1. gives this following

CANON.

Add the given Sum of the Squares of the Extremes to the given Sum of the Squares of the Means, and reserve half of the sum. From the square of this half sum subtract the Square of the sum of the Squares of the Means, and extract the square Root of the Remainder; add this square Root to the half sum before reserved, and also subtract it from the same half sum, so the sum shall be the sum of the Squares of the first and second Proportionals, and the Remainder shall be the sum of the Squares of the third and fourth.

Then (according to *Theor.* 3. of the preceding *Chap.* 6.) add severally the sum of the Squares of the first and second Proportionals, and the sum of the Squares of the third and fourth to the sum of the Squares of the Means, and out of each sum extract the square Root; so shall one of these Roots be the sum of the first and third Proportionals, and the other shall be the sum of the second and fourth. Which two last mentioned sums being added together give the sum of the four Proportionals sought.

Lastly, the sum of four Proportionals being given, as also the sum of the Squares of the Means, the Proportionals shall be given severally by the ninth Question of this *Chap.*

Therefore if 260 be given for the sum of the Squares of the Extremes of four continual Proportionals, and 80 for the sum of the Squares of the Means, the Proportionals will be found 16,8,4,2.

QUEST. 12.

The sum (b) of the Extremes of four Quantities in continual Proportion being given, as also (c) the sum of the Cubes of the Means; to find out the Proportionals.

RESOLUTION.

1. For one of the Extreme Proportionals, put . . . . . a
2. Then the other Extreme (by subtracting (a) from (b) the given sum of the Extremes) shall be . . . . $b-a$
3. Therefore the Product made by the mutual Multiplication of the Extremes is . . . . $ba-aa$
4. And because (*per Theorem* 21. of the preceding *Chap.* 6.) the Product made by the Multiplication of the Means or Extremes into the sum of the Extremes, is equal to the sum of the Cubes of the Means; therefore if you multiply $ba-aa$ by b, this Product shall be equal to (c) the given sum of the Cubes of the Means; hence arises this Equation, *viz.* . . $bba-baa=c$
5. And by dividing every Term of that Equation by (b,) there arises . . . . $ba-aa=\frac{c}{b}$

Which last Equation being resolved (by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) gives this following

CANON.

6. From the Square of half the given sum of the Extremes subtract the Quotient that arises by dividing the given sum of the Cubes of the Means by the sum of the Extremes, and extract the square Root of the Remainder, then half the sum of the Extremes being increased, and also lessened by the said square Root, gives the Extremes severally. Then you may find out the Means by a new Work thus;
7. Let the greater Extreme found out as above be . . . . f
8. And the lesser Extreme . . . . g
9. Then for the greater Mean put . . . . a
10. Therefore by dividing (aa) the square of the greater Mean by the greater Extreme (f,) the Quotient shall be the lesser Mean, to wit, ... $\frac{aa}{f}$

11. But

11. But the ſquare of the leſſer Mean is equal to the Product of the leſſer Extreme multiplied by the greater Mean; therefore from the three laſt preceding ſteps this Equation ariſes, *viz.* $\frac{aaaa}{ff}=ga$
12. Which Equation after due Reduction gives . . . . $aaa=ffg$
13. Therefore by extracting the Cubic Root out of each part of the laſt Equation the greater Mean is made known, *viz.* $a=\sqrt{(3)}ffg$

Which laſt Equation, together with that in the tenth ſtep, will give this

CANON.

14. Multiply the ſquare of the greater Extreme by the leſſer, then the Cubic Root of the Product ſhall be the greater Mean. Laſtly, the Square of the greater Mean divided by the greater Extreme gives the leſſer Mean.

Therefore if 18 be given for the ſum of the Extremes of four Numbers in continual proportion, and 576 for the ſum of the Cubes of the Means, then by the firſt Canon of this Queſtion the Extremes will be found 16 and 2. And laſtly, by the latter Canon the Means will be found 8 and 4. Wherefore the four continual Proportionals ſought are 16, 8, 4, 2.

QUEST. 13.

The ſum (b) of the Cubes of the Extremes of four Quantities in continual proportion being given, as alſo (c) the ſum of the Cubes of the Means, to find the four Proportionals.

RESOLUTION.

1. For the ſum of the Extremes put . . . . a
2. Therefore the Cube of that ſum is . . . . aaa
3. Then becauſe by *Theor.* 22. of the preceding *Chap.* 6. if four Quantities be continually proportional, the ſum of the Cubes of the Extremes more by the triple of the Cubes of the Means is equal to the Cube of the ſum of the Extremes; therefore if to (b) you add $3c$, it gives the Cube of the ſum of the Extremes, which Cube muſt be equal to aaa; hence this Equation $b+3c=aaa$
4. Therefore by extracting the Cubic Root out of each part of that Equation, the ſum of the Extremes is made known, *viz.* $\sqrt{(3)}:\overline{b+3c}:=a$

Which laſt Equation in words is this following

CANON.

Add the triple of the given ſum of the Cubes of the Means to the given ſum of the Cubes of the Extremes, and out of the ſum made by that Addition extract the Cubic Root, which ſhall be the ſum of the Extremes ſought.

Then the ſum of the Extremes being given, as alſo the ſum of the Cubes of the Means, the four Proportionals ſhall be given ſeverally by the Canon of the preceding twelfth Queſtion. As for Example, if 157472 be given for the ſum of the Cubes of the Extremes of four Numbers in continual proportion, and 6048 for the ſum of the Cubes of the Mean; firſt, by the Canon of this Queſtion the ſum of the Extremes will be found 56, and then by the Canon of the preceding twelfth Queſtion, the four Proportionals will be found 2, 6, 18, 54.

QUEST. 14.

The ſum of the Extremes (b) of five Quantities in continual Proportion being given, as alſo (c) the ſum of the three Means; to find the five Proportionals.

RESOLUTION.

1. For the third Proportional, that is, the middle Term of all the five, put . . . . a
2. Then ſubtract that middle Term (a) from (c) the given ſum of the three Means, and there will remain the ſum of the ſecond and fourth, *viz.* . . . . $c-a$
3. And becauſe by *Theorem* 29. of the preceding *Chap.* 6. the ſum of the Extremes of five continual Proportionals, together with the double of the Mean, the ſum of the ſecond and fourth, and the Mean, are alſo in continual proportion; therefore this Analogy is manifeſt, *viz.* . . . . $b+2a.c-a::c-a.a$

4. From which Analogy, by comparing the Product made by the Multiplication of the Extremes to the Product of the Means, this Equation is produced, *viz.* . . . . . $ba+2aa=cc-2ca+aa$

5. Which Equation after due Reduction gives . . . . . $aa+ba+2ca=cc$

Lastly, by resolving the last Equation according to the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. there will arise this following

CANON.

Add the sum of the Extremes to the double of the sum of the three Means, and take the half of the sum made by such Addition; then to the Square of the said half sum add the square of the sum of the three Means, and out of this sum extract the square Root; from which Root subtract the half sum first taken, and the Remainder shall be the middle (or third) Proportional of the five sought.

Then by subtracting the said third Proportional from the sum of the three Means, the Remainder is the sum of the second and fourth; by which sum and the third Proportional, the second and fourth shall be given severally, (by the Canon of *Quest.* 4. *Chap.* 16. *Book* 1.) Then the square of the second Proportional being divided by the third gives the first, and the Square of the fourth being divided by the third gives the fifth.

Therefore if 34 be given for the sum of the first and fifth of five continual Proportionals, and 28 for the sum of the three Means, the five Proportionals shall be given severally, *viz.* 2, 4, 8, 16, 32 ∻.

QUEST. 15.

The Sum (b) of the first, third, and fifth of five Quantities in continual proportion being given, as also (c) the sum of the second and fourth, to find the five Proportionals.

RESOLUTION.

1. For the third Proportional, that is, the middle Term of the 5, put a
2. Then subtract that middle Term (a) from the given sum (b,) and the Remainder is the sum of the first and fifth, *viz.* . . . $b-a$
3. And because (by *Theorem* 27. of the preceding *Chap.* 6.) the Product made by the Multiplication of the third or middle Term of five continual Proportionals into the sum of the first and fifth, is equal to the Squares of the second and fourth, therefore (from the first and second steps) the sum of the Squares of the second and fourth Proportionals is . . . . $ba-aa$
4. The square of the third Proportional (a) is equal to the Product of the second multiplied into the fourth, therefore the double of that Product is . . . . $2aa$
5. Therefore from the two last steps the Aggregate of the Squares and the double Product of the second and fourth Proportional is $aa+ba$
6. But the Aggregate of the Squares and the double Product of the second and fourth Proportional is equal to the Square of their sum, therefore the Aggregate in the fifth step must be equal to the Square of the given sum (c) *viz.* . . . . $aa+ba=cc$

Which Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. will give this following

CANON.

Add the Square of half the given sum of the first, third, and fifth Proportionals to the Square of the given sum of the second and fourth, then from the square Root of the sum made by that Addition subtract the said half sum, and the Remainder shall be the third Proportional.

Then by subtracting the said third Proportional from the given sum of the first, third, and fifth, the Remainder is the sum of the first and fifth; by which sum and the third (or mean) Proportional, the first and fifth (to wit, the Extremes) shall be given severally by the Canon of *Quest.* 4. *Chap.* 16. *Book* 1. Then the third Proportional being multiplied into the first and fifth severally, and the square Root being extracted out of each Product, these Roots shall be the second and fourth Proportionals.

Therefore if 42 be given for the sum of the first, third, and fifth of five Numbers in continual proportion, and 20 for the sum of the second and fourth, the five Proportionals will be found these, to wit, 2,4,8,16,32.

QUEST. 16.

The third Proportional (b) of five Quantities in continual proportion being given, as alſo (c) the ſum of the other four, to find out the five Proportionals.

RESOLUTION.

1. For the ſum of the ſecond and fourth Proportional put . . . a
2. Then ſubtract that Sum (a) from (c) the given ſum of the firſt, ſecond, fourth, and fifth Proportionals, and there will remain the ſum of the firſt and fifth, to wit, ... $c-a$
3. The ſquare of the third (that is, of the mean) Proportional (b) is equal to the Product of the ſecond multiplied into the fourth, therefore the double of that Product is, ... $2bb$
4. Which double Product ($2bb$) ſubtracted from (aa) the Square of the ſum of the ſecond and fourth Proportionals, leaves for the ſum of the Squares of the ſecond and fourth, ... $aa-2bb$
5. And becauſe (by *Theor.* 33. of the preceding *Chap.* 6.) the ſum of the Squares of the ſecond and fourth of 5 continual Proportionals is equal to the Product of the third (or mean) multiplied by the ſum of the firſt and fifth, therefore if ($aa-2bb$) the ſum of the Squares of the ſecond and fourth be divided by the mean (b) the Quotient ſhall be the ſum of the firſt and fifth, *viz.* $\frac{aa-2bb}{b}$
6. Which Sum found out in the laſt ſtep muſt be equal to the ſum of the firſt and fifth Proportionals found out in the ſecond ſtep; hence this Equation ariſes, *viz.* . . . . $\frac{aa-2bb}{b}=c-a$
7. Which Equation after due Reduction gives . . . . $aa+ba=2bb+bc$

Wherefore by reſolving the laſt Equation (according to the Canon in *Sect.* 6. *Chap.* 15. *Book.* 1.) there will come forth this following

CANON.

To the ſquare of the half of the given third (or mean) Proportional add the double of the ſquares of the ſaid Mean, as alſo the Product of the ſaid Mean multiplied into the given ſum of the other four Proportionals, and out of the ſum of that Addition extract the ſquare Root; this Root leſſened by half the given Mean, gives the ſum of the ſecond and fourth Proportionals.

Then from the given ſum of the firſt, ſecond, fourth, and fifth Proportionals ſubtract the ſum of the ſecond and fourth (found out as above) and the Remainder is the ſum of the firſt and fifth; by which ſum and the third (or mean) Proportional, the ſaid firſt and fifth ſhall be given ſeverally by the Canon of *Queſt* 4. *Chap.* 16. *Book* 1.

Laſtly, the ſquare Roots of the Product of the firſt multiplied into the third, and of the Product of the third into the fifth, ſhall be the ſecond and fourth Proportionals.

Therefore if 8 be given for the third of five Numbers in continual proportion, and 54 for the ſum of the other four, the five Proportionals will be found theſe, to wit, 2,4,8,16,32.

QUEST. 17.

The ſum (b) of the Extremes of five Quantities in continual Proportion being given, as alſo (c) the ſum of the Squares of three Means; to find the five Proportionals.

RESOLUTION.

1. For the Mean (or third Proportional put . . . . a
2. Then (by *Theor* 33. of the preceding *Chap.* 6.) the Mean (a) multiplied by (b) the given ſum of the Extremes, produces the ſum of the Squares of the ſecond and fourth Proportionals, *viz.* ba
3. Therefore if to (aa) the ſquare of the Mean you add (ba) the ſum of the Squares of the ſecond and fourth, there will come forth the ſum of the Squares of the ſecond, third, and fourth Proportionals, *viz.* . . . . $aa+ba$
4. Which ſum found out in the laſt ſtep muſt be equal to the given ſum (c;) hence this Equation ariſes, *viz.* . . . . . . $aa+ba=c$

Where-

Wherefore by resolving that Equation (according to the Canon in *Sect.* 6. *Chap.* 15. *Book* 1.) there will arise this following

CANON.

Add the square of half the given sum of the Extremes to the given sum of the Squares of the three Means, and out of the sum of that Addition extract the square Root; this Root lessened by half the sum of the Extremes will give the Mean (or third) Proportional.

Then the mean (or third) Proportional being given, and the sum of the Extremes, (*viz.* of the first and fifth) the said Extremes shall be given severally by the Canon of *Quest.* 4. *Chap.* 16. *Book* 1.

Lastly, the square Roots of the Products of the first into the third, and of the third into the fifth shall be the second and fourth Proportionals.

Therefore if 34 be given for the sum of the Extremes of five Numbers in continual proportion, and 336 for the sum of the Squares of the three Means, the five Proportionals shall be also given, to wit, 2, 4, 8, 16, 32.

QUEST. 18.

The Sum (b) of the Extremes of five Quantities in continual Proportion being given, as also (c) the Sum of the Squares of the second and fourth, to find the 5 Proportionals.

RESOLUTION.

1. For the mean Proportional put . . . . a
2. Then (by *Theorem* 33. of the preceding *Chap.* 6.) the Mean (a) multiplied by (b) the sum of the Extremes, produces the sum of the Squares of the second and fourth, *viz.* . . ba
3. Which sum must be equal to the given sum (c,) therefore $ba=c$
4. Wherefore by dividing each part of that Equation by (b,) the mean Proportional will be made known, *viz.* . . . . $a=\frac{c}{b}$

Which last Equation in words is this following

CANON.

Divide the given sum of the Squares of the second and fourth Proportionals by the given sum of the first and fifth, so shall the Quotient be the mean or third Proportional.

Then the mean (or third) Proportional being given, as also the sum of the first and fifth, these shall be given severally by the Canon of *Quest.* 4. *Chap.* 16. *Book* 1.

Lastly, the square Roots of the Products of the first into the third, and of the third into the fifth, shall be the second and fourth Proportionals.

Therefore if 34 be given for the sum of the Extremes of five Numbers in continual proportion, and 272 for the sum of the Squares of the second and fourth, the Proportionals will be discovered severally, *viz.* 2, 4, 8, 16, 32.

QUEST. 19.

A Vintner having a Vessel full of Wine containing 16 (or b) Gallons, draws out 4 (or c) Gallons, and then pours into the Vessel as much Water as he drew out Wine; then out of that mixt Quantity of Wine and Water he draws out the same number of Gallons as before, and pours in the same quantity of Water. Again, he makes a third draught of the same quantity as at first. The question is, to find how much pure Wine remained in the Vessel after the third draught.

RESOLUTION.

1. The Number of Gallons of Wine in the Vessel at first was b
2. Out of which Quantity (c) Gallons being drawn, there remained of pure Wine in the Vessel . . . . $b-c$
3. To which remaining quantity of pure Wine (c) Gallons of Water being added, the Vessel is again full, and contains (b) Gallons of Wine and Water together; out of which drawing again (c) Gallons, we must seek how much pure Wine was in this second draught, saying by the Rule of Three,

$$\begin{array}{lcccccc} & \textit{mixt} & & \textit{Wine} & & \textit{mixt} & \textit{Wine} \\ \text{If} & b & . & b-c & :: & c & . \left(\frac{bc-cc}{b}\right. \end{array}$$

Whence it is found, that the quantity of pure Wine in the second draught was . . . . $\frac{bb-cc}{b}$

4. Which

4. Which Quantity $\frac{bc-cc}{b}$ being subtracted from $b-c$, the Quantity of pure Wine in the Vessel before the second draught was made, there remains for the Quantity of pure Wine in the Vessel after the second draught . . . . . . $\Big\}\ \frac{bb-2bc+cc}{b}$

5. To which remaining Quantity of pure Wine add (c) Gallons of Water, so the Vessel is again full, and contains (b) Gallons of Wine and Water together; out of which drawing again (c) Gallons, we must seek how much pure Wine was in this third draught, saying,

$$\begin{array}{ccccccc} \textit{mixt} & & \textit{Wine} & & \textit{mixt} & & \\ \text{As } b & . & \frac{bb-2bc+cc}{b} & :: & c & . & \text{to a fourth Pro-} \end{array}$$

portional or Quantity of pure Wine in the third draught, which will be found . . : . . . . $\Big\}\ \frac{bbc-2bcc+ccc}{bb}$

6. Then by subtracting the said fourth Proportional or Quantity of pure Wine in the third draught, from $\frac{bb-2bc+cc}{b}$ the Quantity of pure Wine in the Vessel when the third draught was made, there remains for the desired Quantity of pure Wine in the Vessel after the third draught . . . . . . $\Big\}\ \frac{bbb-3bbc+3bcc-ccc}{bb}$

Which Quantity last found out is the Answer of the Question; and if it be resolved into Numbers it gives $6\frac{3}{4}$ for the number of Gallons of pure Wine that remained in the Vessel after the third draught. Moreover, if the first, second, fourth, and sixth steps of the Resolution be well examined and compared with *Sect.* 2, 5, and 6 *Chap.* 5. of this second Book, it will be manifest that the Quantity of pure Wine in the Vessel at first, and the several Quantities of Wine remaining in the Vessel after each draught are in continual Proportion:

$$\textit{Viz.} \left\{ \begin{array}{ccccccccc} b & . & b-c & . & \frac{bb-2bc+cc}{b} & . & \frac{bbb-3bbc+3bcc-ccc}{bb} & \div \\ 16 & . & 12 & . & 9 & . & 6\frac{3}{4} & \div \end{array} \right.$$

Of which continual Proportionals the first is the given Quantity of Wine in the Vessel at first; the second is the Excess of the same Quantity above the given Quantity drawn out at each draught; and then the fourth continual Proportional is the Quantity of pure Wine remaining in the Vessel when three draughts have been made, according to the import of the Question; but the fifth continual Proportional when four draughts, the sixth when five draughts, the seventh when six draughts, shall be the remaining Quantity of pure Wine sought by the Question. Lastly, the first and the second Terms of a rank of Numbers in continual proportion being given, any of the following Terms shall be given by the Rule in *Sect.* 5. and 6 *Chap.* 5. of this second Book.

QUEST. 20.

A Vintner having a Vessel full of Wine containing 16 (or b) Gallons, draws out a certain quantity, and then pours into the Vessel as much Water as he drew out Wine. Again, out of that mixt quantity of Wine and Water he draws out the same quantity as before, and pours in the same quantity of Water. Then he makes a third draught of the same quantity as at first, and after this third draught there remained $6\frac{3}{4}$ (or d) Gallons of pure Wine. The Question is, to find what quantity of pure Wine was drawn out at the first draught, or what quantity of Wine and Water together at the second or third draught, (for the three draughts were equal quantities.)

RESOLUTION.

1. For the Number of Gallons of Wine in the Vessel at first was b
2. For the Number of Gallons of Wine drawn out at the first draught put . . . . $\Big\}\ a$
3. Then the quantity of Wine remaining in the Vessel after the first draught was . . . . $\Big\}\ b-a$
4. By prosecuting the search as in the preceding nineteenth Question, saving that (a) is to

be

be used here instead of (c) there, you will find this Quantity, *viz.* $\frac{bbb - 3bba + 2baa - aaa}{bb}$ to be the Number of Gallons of pure Wine remaining in the Veſſel after the third draught, and therefore it muſt be equal to the given Quantity $6\frac{3}{4}$ (or d;) hence ariſes this Equation, *viz.*

$$\frac{bbb - 2bba + 3baa - aaa}{bb} = d.$$

5. Therefore by multiplying each part of that Equation by the Denominator bb, there will come forth this Equation in Integers, *viz.*

$$bbb - 3bba + 3baa - aaa = bbd.$$

6. And by extracting the Cubic Root out of each part of the laſt Equation, there ariſes $b - a = \sqrt{(3)}bbd.$

7. Wherefore from the laſt Equation after due Tranſpoſition the Value of (a) will be made known, *viz.* $a = b - \sqrt{(c)}bbd = 4.$

Whence it is manifeſt, that four Gallons were drawn out at every one of the three draughts. But if the Reſolution had been wrought out at large, as in the preceding nineteenth Queſtion, then it would appear, that if between (b) and (d,) *viz.* the quantity of Wine firſt given, and the quantity of Wine remaining after the laſt draught, there be found the greater of two mean Proportionals when three draughts are propoſed, or the greateſt of three Means when four draughts, and ſo forwards; then the Mean ſo found out being ſubtracted from the greater Extreme (b) leaves the Quantity drawn out at each draught. The manner of finding out mean proportional Numbers between any two Numbers given for Extremes, has already been ſhewn in *Sect.* 14. *Chap.* 5. of this ſecond Book.

If the Reader deſires more variety of Queſtions about Quantities in continual Proportion, he may conſult the *Algebra* of *Jac. de Billy*, intituled *Nova Geometriæ Clavis*, and the firſt Part of our Learned Dr. *Wallis* his Mathematical Works.

CHAP. VIII.

The manner of finding out all the Aliquot Parts *both of Numbers and Algebraical Quantities, as alſo the ſmalleſt Numbers that ſhall have given Multitudes of* Aliquot Parts.

I. IN the Reſolution of knotty Queſtions about Quantity, there is oftentimes great uſe of finding out all the *Aliquot Parts*, or juſt Diviſors, as well of Numbers, as of Quantities repreſented by Letters; and therefore in this Chapter I ſhall ſhew how that Work may be done; as alſo how to find out the leaſt Number that ſhall have a given Multitude of *Aliquot Parts*, according to the Method of *Fran. van Schooten*, in *Sect.* 2, 3, and 4. of his *Miſcellanies*, and in his *Principia Matheſ. Univerſal.*

II. A *Prime* or Incompoſit Number is that which can only be meaſured or divided by it ſelf or by Unity, and leave no Remainder; as 2,3,5,7,11,13, &c. are *Prime* Numbers.

III. A *Compoſit Number* is that which may be divided by ſome Number leſs than the Compoſit it ſelf, but greater than Unity; as 4,6,8,9,10, &c. are Compoſits.

IV. *Juſt Diviſors* are ſuch Numbers or Quantities as will divide a given Number or Quantity, and leave no Remainder; every one of which Diviſors, except that which is equal to the given Quantity is called an *Aliquot Part*, becauſe if it be taken *Aliquoties*, that is, certain times, it will preciſely conſtitute the given Quantity: As if 6 be a Number propoſed, its juſt Diviſors are 1,2,3, and 6; but the Aliquot Parts of 6 are only 1,2, and 3; for 6 cannot be a part of 6, but it may be a Diviſor to it ſelf, that is, 6 may be divided by 6, and the Quotient is Unity. Hence it is manifeſt, that the juſt Diviſors of a Number are more in multitude by one than the Number of its Aliquot Parts.

V. The Aliquot Parts of a whole Number may be found out in this manner, *viz.* Firſt, if the Number propoſed be even, divide it by 2, and reſerve the Diviſor. Again, if the Quotient be even divide it by 2, and reſerve the Diviſor; and continue the Diviſion of of every following Quotient by 2, until the Quotient be an odd number. But if either the number firſt propoſed, or the Quotient reſulting from ſuch Diviſion by 2 be odd, divide it by 3, if it will give an Integer Quotient, and continue the Diviſion by 3 in like manner as before by 2, ſo long as the Quotient is an Integer without any Fraction; likewiſe when the Diviſion by 3 ceaſeth, divide by 5,7,11,13,17,19,*&c.* that is, by every prime Number,until you find a Quotient leſs than the Diviſor; and if no ſuch Diviſor will give an Integer Quotient before the Quotient is leſs than the Diviſor,you may conclude the number firſt propoſed to be Incompoſit, (*viz.* ſuch as has no Diviſor but it ſelf or Unity) and that laſt Diviſor to be greater than the ſquare Root of the propoſed Number. Then by the help of the prime Diviſors to the given Number, all the reſt may be found out by the Operation directed in the following Examples.

Example 1.

Suppoſe it be deſired to find out all the Aliquot Parts and Diviſors of 360; firſt, I divide 360 by 2, and the Quotient is 180; this divided by 2 gives 90, which divided by 2 gives 45; this being an odd Number the Diviſion by 2 ceaſes. Then I divide the ſaid 45 by 3, and the Quotient is 15; this divided by 3 gives the Quotient 5, and ſo the Diviſion by 3 ceaſes; then I divide 5 by it ſelf, and the Quotient is Unity. Now by the help of thoſe Diviſors or prime Numbers, which (as may eaſily be proved) are ſuch, that if they be continually multiplied will produce the given number 360, all the reſt of the juſt Diviſors of the ſaid 360 may be found out thus.

360	180	90	45	15	5	1
2	2	2	3	3	5	

Firſt, I ſet every one of the ſaid prime Diviſors 2,2,2,3,3, and 5, at the head of a Columel, as you ſee in this Table; then I multiply the firſt Diviſor 2 by the ſecond Diviſor 2, and ſet the Product 4 under 2 in the ſecond Columel. Again, I multiply the ſaid 4 by 2, (which ſtands at the head of the third Columel) and ſet the Product 8 under 2 in the third Columel. Then I multiply every one of the Numbers in the firſt, ſecond,and third Columels,by 3, which ſtands at the head of the fourth Columel, and write the Products under 3 in the ſaid fourth Columel; except ſuch Products which happen to be the ſame with any of thoſe before written, (for one and the ſame Product muſt not be written twice;) ſo multiplying 2, 4, and 8, by 3, I ſet the Products 6, 12, and 24 under 3 in the fourth Columel. Again, I multiply every one of the Numbers in the firſt, ſecond, third, and fourth Columels by 3, (which ſtands at the top of the fifth Columel) and ſet the Products under the ſaid 3; except (as before) ſuch Products which happen to be the ſame with any of thoſe before written in any of the precedent Columels: ſo the Products written under 3 in the fifth Columel are 9, 18, 36, and 72. Laſtly, I multiply every one of the Numbers in the firſt, ſecond, third, fourth, and fifth Columels by 5, (which ſtands at the head of the laſt Columel) and write the ſeveral Products (except as before excepted) under the ſaid 5. So at length all the juſt Diviſors to the given Number 360 are found theſe, to wit, 1,2,3,4,5,6,7,8,9,10,12,15, 18,20,24,30,36,40,45,60,72,90,120, 180, and 360; every one of which Diviſors (except the greateſt, which is always equal to the Number firſt propoſed) is an Aliquot part of 360, which (as you ſee) hath 23 Aliquot parts and 24 Diviſors.

2	2	2	3	3	5
	4	8	6	9	10
			12	18	20
			24	36	40
				72	15
					30
					60
					120
					45
					90
					180
					360

Example 2.

Again, if it be required to find out all the Aliquot Parts and Divisors of 2310, the Operation will be like that in Example 1. For, first the prime Divisors will be found these, to wit, 2, 3, 5, 7, 11; then after the said prime Divisors are set at the heads of so many Columels, as you see in the Table in the Margin, the rest of the Divisors will be found out by Multiplication according to the foregoing directions; which in sum amounts to this, *viz.* each prime Divisor standing at the head of every Columel following the first, is to be multiplied by every one of the Numbers in the foregoing Columels, (except such which make the same Products as were before produced) and the Products are to be set under each prime Divisor respectively by which they were produced. So all the Divisors to the given Number 2310 are discovered to be these, to wit, 1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 21, 22, &c. as you see in this Table; every one of which Divisors, except the greatest, to wit, 2310, (which is the same with the Number proposed) is an Aliquot part of the said 2310, which has 31 Aliquot parts, but 32 Divisors.

2310	1155	385	77	11	1
2	3	5	7	11	

2	3	5	7	11
	6	10	14	22
		15	21	33
		30	42	66
			35	55
			70	110
			105	165
			210	330
				77
				154
				231
				462
				385
				770
				1155
				2310

Upon the same foundation the Divisors of Quantities exprest by Letters may be found out, as will appear by the following Examples. But this work requires that the Analyst be well exercis'd in the Rules of Algebraical Multiplication, Division, and the Extraction of Roots; for the finding out of the Primitive or Incomposit Divisors, when the given Quantity is compos'd of many large Members connexed by different Signs, is oftentimes both difficult and laborious.

Example 3.

Let it be required to find out all the Divisors and Aliquot parts of this Quantity *aaabbc*. First, I divide the said *aaabbc* by *a*, and the Quotient is *aabbc*, which divided by *a* gives *abbc*, this divided by *a* gives *bbc*; and so the Division by *a* ceases. Then I divide *bbc* by *b*, and the Quotient is *bc*; this divided by *b* gives *c*, which being a Primitive or Incomposit Quantity I divide by it self, and the Quotient is 1. So all the primitive Divisors of the proposed Quantity *aaabbc* are found *a*, *a*, *a*, *b*, *b*, and *c*; which are manifestly such as being multiplied continually will produce the given Quantity *aaabbc*.

aaabbc	*aabbc*	*abbc*	*bbc*	*bc*	*c*	1
a	*a*	*a*	*b*	*b*	*c*	

Now out of those Divisors, after they are set at the heads of so many Columels as you see in this Table, I search out the rest of the Divisors by Algebraical Multiplication, in like manner as in Example 1. So all the different Divisors to the given Quantity *aaabbc* are found these, to wit, 1, *a*, *aa*, *aaa*, *b*, *ab*, *aab*, *aaab*, *bb*, *abb*, *aabb*, *aaabb*, *c*, *ac*, *aac*, *aaac*, *bc*, *abc*, *aabc*, *aaabc*, *bbc*, *abbc*, *aabbc*, *aaabbc*; every one of which Divisors, except the last and greatest is an Aliquot part of the given Quantity *aaabbc*, which has 23 parts, and 24 Divisors.

a	*a*	*a*	*b*	*b*	*c*
	aa	*aaa*	*ab*	*bb*	*ac*
			aab	*abb*	*aac*
			aaab	*aabb*	*aaac*
				aaabb	*bc*
					abc
					aabc
					aaabc
					bbc
					abbc
					aabbc
					aaabbc

Note, That this third Example differs not from Example 1. saving that Algebraical Division and Multiplication is used here instead of vulgar Division and Multiplication in Numbers there.

Example

Example 4.

After the ſame manner 31 Aliquot parts and 32 Diviſors will be found to this Quantity $abcde$, viz. 1. a, b, ab, c, ac, bc, abc, d, ad, bd, &c. as you ſee them expreſt in the following Table.

	$abcde$	$bcde$	cde	de	e	1
Primitive Diviſors,	a	b	c	d	e	

a	b	c	d	e
	ab	ac	ad	ae
		bc	bd	be
		abc	abd	abe
			cd	ce
			acd	ace
			bcd	bce
			$abcd$	$abce$
				de
				ade
				bde
				$abde$
				cde
				$acde$
				$bcde$
				$abcde$

Compare this Example with the precedent Example 2.

Example 5.

Again, to find all the Diviſors of this compound Quantity $aaabc-abbbc$, firſt, I ſearch out all its prime Diviſors thus, *viz.* I divide the ſaid Compound Quantity by a, and the Quotient is $aabc-bbbc$; this divided by b gives $aac-bbc$, which divided by c gives the Quotient $aa-bb$; this divided by $a-b$ gives the Quotient $a+b$, which being a primitive Quantity I divide it by it ſelf, and the Quotient is 1. So the prime Diviſors are found a, b, c, $a-b$, and $a+b$, which are to be reſerved.

$aaabc-abbbc$	$aabc-bbbc$	$aac-bbc$	$aa-bb$	$a+b$	1
a	b	c	$a-b$	$a+b$	

Then (as in the foregoing Examples) I ſet the ſaid primitive Diviſors at the heads of ſo many Columels, and from thoſe Diviſors (according to the directions in Example 1.) I find out all the reſt by Multiplication; ſo at length it appears that $aaabc-abbbc$ the compound Quantity propoſed has 31 Aliquot parts and 32 Diviſors, wit, 1, a, b, ab, c, ac, bc, abc, $a-b$, $aa-ab$, $ab-bb$, &c. as you ſee them expreſt in the following Table.

a	b	c	$a-b$	$a+b$
	ab	ac	$aa-ab$	$aa+ab$
		bc	$ab-bb$	$ab+bb$
		abc	$aab-abb$	$aab+abb$
			$ac-bc$	$ac+bc$
			$aac-abc$	$aac+abc$
			$abc-bbc$	$abc+bbc$
			$aabc-abbc$	$aabc+abbc$
				$aa-bb$
				$aaa-abb$
				$aab-bbb$
				$aaab-abbb$
				$aac-bbc$
				$aaac-abbc$
				$aabc-bbbc$
				$aaabc-abbbc$

Example 6.

Again, to find out all the Divisors of this Quantity $aaabbc—2aabbbc+abbbbc$; first, (as before) I search out all the primitive Divisors, *viz.* I divide the Quantity propos'd by a, and the Quotient is $aabbc—2abbbc+bbbbc$, which divided by b gives the Quotient $aabc—2abbc+bbbc$; this divided again by b gives $aac—2abc+bbc$, which divided by c gives $aa—2ab+bb$. This last Quotient being a Square whose side is either $a—b$ or $b—a$, according as a is greater or less than b, I shall suppose a to be greater than b; and then dividing the said Square $aa—2ab+bb$ by its side $a—b$ the Quotient is also $a—b$. And lastly, by dividing $a—b$ by it self (because 'tis a Primitive Quantity) the Quotient is 1. Thus the primitive Divisors of the Quantity proposed are found a, b, b, c, $a—b$ and $a—b$. Then every one of them being set at the head of a Columel, and Multiplication made according to the Operation in the precedent Examples, the rest of the desired Divisors to the Quantity $aaabbc—2aabbbc+abbbbc$ will be found out; and at length all the Divisors to the said Quantity are discovered to be these, *viz.* 1. a, b, ab, bb, abb, c, ac, bc, abc, bbc, $abbc$, $a—b$, $aa—ab$, $ab—bb$, &c. as you see them exprest in the following Table.

a	b	b	c	$a—b$	$a—b$
	ab	bb	ac	$aa—ab$	$aa—2ab+bb$
		abb	bc	$ab—bb$	$aaa—2aab+abb$
			abc	$aab—abb$	$aab—2abb+bbb$
			bbc	$abb—bbb$	$aaab—2aabb+abbb$
			$abbc$	$aabb—abbb$	$aabb—2abbb+bbbb$
				$ac—bc$	$aaabb—2aabbb+abbbb$
				$aac—abc$	$aac—2abc+bbc$
				$abc—bbc$	$aaac—2aabc+abbc$
				$aabc—abbc$	$aabc—2abbc+bbbc$
				$abbc—bbbc$	$aaabc—2aabbc+abbbc$
				$aabbc—abbbc$	$aabbc—2abbbc+bbbbc$
					$aabbbc—2aabbbc+abbbbc$

Example 7.

In like manner, if it be desired to find out all the Divisors of this Quantity $aaaaaa+2aaaacc+aacccc$, that is, $a^6+2a^4cc+aac^4$; I divide it first by a, and the Quotient is $a^5+2a^3cc+ac^4$, this divided again by a gives $a^4+2aacc+c^4$. Now 'tis evident that this last Quotient cannot be divided by a or by c, or the like quantity, but because (by *Sect.* 4. *Chap.* 8. *Book* 1.) the said $a^4+2aacc+c^4$ is a Square, whose Root is $aa+cc$, I divide the Square by its Root $aa+cc$, and the Quotient is also the same Root said $aa+cc$, which being a primitive Quantity I divide it by it self, and the Quotient is 1. So the Divisors to be reserved are a, a, $aa+cc$ and $aa+cc$.

$a^6+2a^4cc+aac^4$	$a^5+2a^3cc+ac^4$	$a^4+2aacc+c^4$	$aa+cc$	1
a	a	$aa+cc$	$aa—cc$	

Then after those Divisors are set at the heads of so many Columels, (as you see in the following Table) I proceed to find out the rest of the Divisors by Multiplication according to the directions in Example 1. *viz.* I multiply each primitive Divisor standing at the head of every Columel following the first by every one of the Quantities in the preceding Columels, and set the Products under the respective primitive Divisor, with this caution, that one and the same Product be not written down twice. So at length I find all the different Divisors to be these, *viz.* 1, a, aa, $aa+cc$, a^3+acc, $a+aacc$, $a^4+2aacc+c^4$, $a^5+2a^3cc+ac^4$, and $a^6+2a^4cc+aac^4$; all which Divisors except the last are Aliquot parts of the proposed Quantity $a^6+2a^4cc+aac^4$.

a		$aa+cc$	$aa+cc$
	aa	a^3+acc	$a^4+2aacc+c^4$
		a^4+aacc	$a^5+2a^3cc+ac^4$
			$a^6+2a^4cc+aac^4$

VI. By

VI. By this skill of finding out all the Divisors of Quantities we may reduce two or more given quantities, when they are not prime between themselves, to others in the same Reason (or Proportion) with those given, and in the smallest Terms. As to reduce those three Quantities $aaa-abb$, $aab-bbb$, and $aaa+aab-abb-bbb$, to the smallest quantities in the same proportion with those proposed; first, I seek (by the Method before delivered) all the different Divisors to every one of those three given quantities, so I find the Divisors of the first quantity $aaa-abb$ to be these, 1, a, $a+b$, $a-b$, $aa+ab$, $aa-ab$, $aa-bb$, $aaa-abb$; and the Divisors of the second quantity $aab-bbb$ to be these, *viz.* 1, b, $a-b$, $ab-bb$, $a+b$, $ab+bb$, $aa+bb$, and $aab-bbb$; also the Divisors of the third quantity, $aaa+aab-abb-bbb$, to be these, to wit, 1, $a-b$, $a+b$, $aa-bb$, $aa+2ab+bb$, and $aaa+aab-abb-bbb$. Now because among those three companies of Divisors these three $a-b$, $a+b$, and $aa-bb$ are found in each company, we may by the help of any one of those three Divisors reduce the given quantities to others more simple, and in the same proportion with those given. But to find out the smallest Terms I divide the proposed quantities $aaa-abb$, $aab-bbb$, and $aaa+aab-abb-bbb$, severally by $aa-bb$, to wit, such of the said three Divisors which has most Dimensions, and there arise a, b, and $a+b$; which three quantities are the smallest Terms that can be found in the same proportion with the three quantities first proposed.

Note, The Quantities propos'd to be reduced are said to be Prime the one to the other, when they have no common Divisor besides 1, (to wit, Unity) in which case the quantities proposed are already in their smallest Terms.

VII. The finding out of Divisors may be very fitly be applied to the reducing of Fractions to their smallest Terms; as to abbreviate this Fraction.

$$\frac{aaa+aab-abb-bbb}{aaa-abb}$$

First, the Divisors of the Numerator (by the precedent Method) are found 1, $a-b$, $a+b$, $aa-bb$, $aa+2ab+bb$; and $aaa+aab-abb-bbb$. Likewise the Divisors of the Denominator are 1, $a+b$, $a-b$, $aa+ab$, $aa-ab$, $aa-bb$, and $aaa-abb$. Then because among those Divisors these three, to wit, $a+b$, $a-b$, and $aa-bb$, are common both to the Numerator and Denominator, I divide the Numerator and Denominator severally by $aa-bb$, to wit, that common Divisor which has most Dimensions; so there arises $a+b$ for a new Numerator, and a for a new Denominator, which gives this Fraction $\frac{a+b}{a}$ (or $1+\frac{b}{a}$) equal to that proposed, and in the smallest Terms, as was desired.

In like manner to abbreviate $\frac{aaa-abb}{aa+2ab+bb}$, because the greatest Divisor common to the Numerator and Denominator is $a+b$, I divide the Numerator and Denominator severally by $a+b$, and there arises $\frac{aa-ab}{a+b}$; which is equal to the Fraction proposed, and in the smallest Terms.

VIII. *Observations upon the Examples in the foregoing* Sect. V.

First, When two, three or four of the foremost Letters (towards the left hand) of a simple quantity are equal to one another, (*viz.* exprest by one and the same letter) then mark well how many equal letters stand foremost together, for so many Aliquot parts they will give. As in Example 3. in *Sect.* 5. where the quantity proposed is $aaabbc$, the three first letters a, a, a, (that is, aaa) give three Aliquot parts, to wit, 1, a, aa; but four Divisors, 1, a, aa, aaa. In like manner, if four equal letters stand foremost together, as a, a, a, a, or $aaaa$, they will afford these four parts, 1, a, aa, aaa; but five Divisors, to wit, 1, a, aa, aaa, $aaaa$. The like property ensues, when five or more equal letters stand foremost together.

Hence it is evident, that every Power has so many Aliquot Parts as there be Dimensions in the Power; as the Square aa, whose Index (or number of Dimensions) is 2, has two parts, to wit, 1 and a; likewise the Cube aaa, or a^3, has three parts; the fourth Power $aaaa$, or a^4, has four parts; and so forwards.

Secondly,

Secondly, It is evident from all the precedent Examples in *Sect.* 5. that when among the primitive Divisors (which are set at the tops of the Columels) a following Divisor differs from the next precedent primitive Divisor, then the multitude of Divisors in the Columel of the said following Divisor is more by 1 than the multitude of all the different Divisors in the precedent Columels. As in Example 3. in *Sect.* 5. where the quantity proposed is *aaabbc*, the letter (or primitive Divisor) *b*, which follows and is different from the next foregoing primitive Divisor *a*, gives four Divisors, to wit, *b*, *ab*, *aab*, and *aaab*; which are more in multitude by 1 than all the foregoing different Divisors *a*, *aa*, & *aaa*.

Again, in Example 4. *Sect.* 5. where the quantity proposed is *abcde*, the Divisors *b* and *ab* in the second Columel are more in number by 1 than *a* in the first. Likewise the Divisor *c*, *ac*, *bc*, and *abc*, in the third Columel, are more in multitude by 1 than *a*, *b*, and *ab*, to wit, all the Divisors in the first and second Columels. Also *d*, *ad*, *bd*, *abd*, *cd*, *acd*, *bcd*, and *abcd* in the fourth Columel, are more in multitude by 1 than all the Divisors in the first, second, and third Columels, and so forward. The reason is manifest, for every primitive Divisor which stands at the top of a following Columel, is multiplied into all the different Divisors severally in all the foregoing Columels; & therefore if that multiplying primitive Divisor be added to the number of those Products, the total multitude must necessarily be more by 1 than the multitude of different Divisors in all the foregoing Columels.

Thirdly, It is also evident, that when the said primitive Divisors are all different, than the numbers which express the multitude of Divisors in every Columel are in continual proportion increasing from Unity in a duple Reason. As in the fourth example in *Sect.* 5. where the primitive Divisors *a*, *b*, *c*, *d*, *e*, are all different, there is one Divisor in the first Columel, two in the second, four in the third, eight in the fourth, and sixteen in the fifth, which numbers of multitude, to wit, 1, 2, 4, 8, and 16, are manifestly in duple proportion. Therefore when all the primitive Divisors of a quantity proposed are different or unlike, then if so many of the foremost Terms of the said continual Proportionals 1, 2, 4, 8, 16, *&c.* be added together, as there be primitive Divisors, (to wit, those Incomposit quantities, which being continually multiplied will produce the quantity proposed) the sum shall be the number of Aliquot parts contained in that quantity, and the number of Divisors shall be more by 1 than that sum.

As for Example, if the number of Aliquot parts in the quantity *ab* be desired, I add 1 and 2 together, (to wit, the two first Terms of the said Geometrical Progression 1, 2, 4, 8, 16, *&c.*) and the sum 3 shews that *ab* contains three Aliquot parts, and four (that is, $3+1$) Divisors. Likewise if there be proposed the quantity *abc*, (which consists of three different letters) the sum of 1, 2, 4, (to wit, of the three first Terms of the said Geometrical Progression) is 7; which shews, that *abc* contains seven parts, but eight (or $7+1$) Divisors. Again, if *abcd* (which consists of four different letters) be proposed, the sum of 1, 2, 4, 8, (the four foremost Terms of the said Progression) is 15; which shews that the quantity *abcd* contains fifteen Aliquot parts, and fixteen (or $15+1$) Divisors, and so forward. But because the said Proportionals proceed in a duple reason from Unity, the sum of any number of Terms may be found out by this brief Rule, *viz.* the third Term (or Proportional) lessened by Unity (the first Term) gives the sum of the first and second Terms. Likewise the fourth Term lessened by 1 gives the sum of the first, second, and third Terms; and the fifth Term lessened by 1 gives the sum of the first, second, third, and fourth Terms, and so forward infinitely. All which may be further illustrated by the ten quantities, and their respective multitudes of Aliquot parts, exprest in the following Table.

Quantities given.	Multitude of Parts	Sums of Terms in continual Proportion, proceeding from 1 in duple Reason.
a	has 1 =	1
ab	. . . 3 =	$1+2$
abc	. . . 7 =	$1+2+4$
abcd	. . 15 =	$1+2+4+8$
abcde	. . 31 =	$1+2+4+8+16$
abcdef	. . 63 =	$1+2+4+8+16+32$
abcdefg	. . 127 =	$1+2+4+8+16+32+64$
abcdefgh	. . 255 =	$1+2+4+8+16+32+64+128$
abcdefghi	. . 511 =	$1+2+4+8+16+32+64+128+256$
abcdefghik	. 1023 =	$1+2+4+8+16+32+64+128+256+512$

Fourthly,

Fourthly, When two, three, or more equal Letters in a simple quantity stand together, and follow some different foregoing letter or letters, then as many Aliquot parts as the first of those following equal letters produces, (according to *Observat.* 2.) so many parts every one of the rest of the said following letters will produce. As in Example 3. in *Sect.* 5. where this quantity *aaabbc* is proposed, the three first letters *a*, *a*, *a* (or *aaa*) gives three parts (by *Observat.* 1.) And the first following letter *b*, in regard it differs from the next preceding letter *a*, gives four parts (by *Observat.* 2.) Now I say, the second *b* shall also give four parts, and if there had been a third *b*, or a fourth *b*, &c. every one of them would give four parts, to wit, as many as the first *b* produced.

In like manner, if this quantity *abbbbb* or ab^5 be proposed, the first letter *a* gives one part; then (by *Observat.* 2.) the next following letter *b* (in regard it differs from *a*) gives two parts. Now I say, every *b* following the first *b* will also give two parts, and so *bbbbb* will give ten, (to wit, five times two) parts, which added to one part noted for *a* makes 11 parts. Whence I conclude, that the quantity *abbbbb* contains 11 Aliquot parts and 12 Divisors. All which may be produced particularly by the Rule in the foregoing *Sect.* 5.

Again, if this quantity *abcddd* be proposed, first, (by *Observat.* 3.) *abc* will give seven parts, and (by *Observat.* 2.) the next following letter *d* gives eight parts; therefore (by this fourth *Observat.*) every *d* following the first *d* gives also eight parts, and consequently *ddd* gives 24 parts, which added to the seven parts before noted for *abc*, makes 31 parts. So that the Quantity *abcddd* has 31 Aliquot parts, and 32 Divisors; and the same number of Parts and Divisors will be found in the Number produced by the continual Multiplication of these five prime Numbers 2, 3, 5, 7, 7, 7.

Fifthly, From what has been said in the precedent Observations 'tis easie to discover how many Aliquot parts are contained in any simple Quantity design'd by letters, without producing the particular parts. As if *aaabbc* be proposed, first, three parts are to be noted for *aaa* (according to *Observat.* 1.) and eight parts more for *bb* (by *Observat.* 4.) which eight parts added to the three parts before noted make eleven parts; then for *c* twelve parts are to be noted, (to wit, 11 + 1, according to *Observat.* 2.) which added to the said 11 parts makes 23 parts. Whence I conclude, that the quantity *aaabbc* has 23 Aliquot parts and 24 Divisors, which are particularly exprest in Example 3. *Sect.* 5.

In like manner we may discover, that this quantity *aaaaabbbbcccdd*, or $a^5b^4c^3d^2$ has 359 Aliquot parts, and 360 Divisors. For first, I note 5 parts for a^5 (according to *Observat.* 1.) then (by *Observat.* 4.) *bbbb* or b^4 gives 24 parts, which added to the 5 parts before noted makes 29 parts. And because one single *c* gives 30 parts, to wit, 29 + 1 (by *Observat.* 2.) *ccc* or c^3 will give 90, to wit, three times 30 parts (by *Observat.* 4.) which added to 29 parts before noted, makes 119 parts. Lastly, because the letter *d* is written twice, and one single *d* gives 120, to wit, 119 + 1 parts, (by *Observat.* 2.) *dd* will give 240 parts (by *Observat.* 4.) which added to 119 parts before noted, makes 359 parts, which is the multitude of Aliquot parts the proposed quantity has, but its number of Divisors is 360.

And with the like facility we may discover the multitude of Parts and Divisors of a given number, after its primitive Divisors are found out. As for Example, to find how many Parts and Divisors 15876000 has, I search out by Division (in like manner as in the Examples in *Sect.* 5.) all the primitive Divisors, which being continually multiplied will produce the said given Number, and find them to be these, to wit, 2, 2, 2, 2, 2, 3, 3, 3, 3, 5, 5, 5, 7, 7, which may be noted by $a^5b^4c^3dd$; but this quantity (as before has been shewn) has 359 Aliquot parts and 360 Divisors, and therefore the said 15876000 has the same Number of Parts and Divisors, which may be particularly found out by the Method in the precedent Examples in *Sect.* 5.

Sixthly, If a quantity be composed of different Letters or Powers, and Unity be added severally to the Indices of those Powers, that is, to the numbers expressing how oft each Letter is found in that quantity, then the Numbers resulting by those Additions being multiplied one into the other continually, will produce a Number greater by Unity than the number of Aliquot parts that quantity has. As for Example, if *aaaabbb* or a^4b^3 be proposed, I add 1 to 4 and 3 severally, (because the Indices of *aaaa* and *bbb* are 4 and 3) and it makes 5 and 4; these multiplied one into the other make 20, which is greater by 1 than 19, the number of Aliquot parts that the proposed quantity a^4b^3 has. The reason of this Property is not difficult to be conceived; for since (by *Observat.* 1.)

aaaa

aaaa hath four parts, that is, five parts wanting one part; and *bbb* following *aaaa* has thrice five parts (by *Observat.* 4.) therefore the whole Quantity *aaaabbb* (or a^4b^3) has 4×5 parts wanting one part, *viz.* 19 parts; which numbers 4 and 5 exceed 3 and 4 the Indices of *bbb* and *aaaa* ſeverally by Unity.

Again, if *aaaabbbcc* be propoſed, the Indices of *aaaa*, *bbb*, and *cc*, are 4, 3, and 2, which increaſed ſeverally by 1 make 5, 4, and 3; theſe multiplied continually produce 60, which is greater by Unity than 59, the number of Aliquot parts which the propoſed Quantity *aaaabbbcc* has. For ſince (for the Reaſon in the laſt preceding Example) *aaaabbb* has 4×5 parts wanting one part, and *cc* following *aaaabbb* has (by *Obſervat.* 4.) $2 \times 4 \times 5$ parts, the propoſed Quantity *aaaabbbcc* has conſequently $3 \times 4 \times 5$ parts wanting one part, that is, 59 parts; which Numbers 3, 4, and 5 do ſeverally exceed the Indices of *cc*, *bbb*, and *aaaa*, by Unity.

Seventhly, From the preceding *Obſervat.* 6. it follows, that if a Compoſit Number be reſolved into any two or more of ſuch of its Factors, the leaſt of which exceeds Unity; and if from every one of thoſe Factors Unity be ſubtracted, the Remainders ſhall be Indices of ſo many ſeveral Powers expreſſible by different Letters, that being joyned together (that is, multiplied one into the other) will give a Quantity having a number of Aliquot parts leſs by Unity than the Compoſit Number propoſed. As for example, if 20 be propoſed; foraſmuch as 5 and 4 multiplied one by the other produce 20, I ſubtract 1 from 5 and 4 ſeverally; ſo the Remainders 4 and 3 do ſhew, that if the fourth Power of ſome Quantity *a*, as *aaaa*, be multiplied into the third Power of ſome other Quantity *b*, as into *bbb*, the Quantity produced, to wit, *aaaabbb* has 19 Aliquot parts, which 19 is leſs by Unity than 20 the Number propoſed. Again, becauſe the Product of 10 into 2 does alſo make 20, I ſubtract 1 from 10 and 2 ſeverally, ſo the Remainders 9 and 1 do ſhew, that if the ninth Power of ſome Quantity *a*, as a^9, be multiplied by ſome other different Quantity *b*, the Quantity produced, to wit, a^9b, has alſo 19 Aliquot parts. Hence it is manifeſt, that often times many Quantities may be found out, every one of which ſhall have a given multitude of Aliquot parts, as will appear in the next following Section.

IX. *The manner of finding out all ſuch Quantities as ſhall have a given Multitude of Aliquot Parts.*

If the multitude of Aliquot parts deſired be any of the Numbers of the ſecond Columel of the Table in *Obſervat.* 3. *Sect.* 8. the Quantity there ſtanding on the left hand of that number, and on the ſame Line with it, has the number of parts deſired. As if it be deſired to find a Quantity that has 63 Aliquot parts, that Table ſhews that *abcdef* has 63 parts; and therefore if ſix prime Numbers, ſuppoſe 2,3,5,7,11,13, be taken for the values of thoſe ſix Letters *a*, *b*, *c*, *d*, *e*, *f*, the Product made by the continual Multiplication of the ſaid prime Numbers, to wit, 30030, ſhall have 63 Aliquot parts, and 64 Diviſors.

But without reſpect to that Table, by the help of the Obſervations in the foregoing *Sect.* 8. many Quantities for the moſt part, and always one Quantity may eaſily be found out, that ſhall have a given Multitude of Aliquot parts, as will be made manifeſt by the following Examples.

Example 1.

Let it be required to find out all ſuch ſimple Quantities expreſſible by Letters, that may every one of them have 15 Aliquot parts and 16 Diviſors.

1. To the ſaid 15 I add 1 and it make 16, this I divide by 2 and the Quotient is 8, which divided by it ſelf gives 1; then from each of the Diviſors 2 and 8 (the Product of whoſe Multiplication makes the firſt Dividend 16) I ſubtract 1; ſo the Remainders 1 and 7 do ſhew, that if ſome letter, as *a*, be written once, and next after it another different letter *b* ſeven times, the Quantity ſo Compoſed, to wit, *abbbbbbb* (or ab^7) ſhall have 15 Aliquot parts, and 16 Diviſors, as was deſired.

16	8	1
2	8	

2. Again, I divide the ſaid 16 (to wit, 15+1) by 2, and the Quotient is 8; this divided again by 2 gives 4, which divided again by 2 gives 2, which divided by it ſelf gives 1; then from every one of the Diviſors 2, 2, 2, 2, I ſubtract 1; ſo the Remainders 1, 1, 1, 1 do ſhew, that if four different ſingle Letters be ſet together, as *abcd*, this Quantity ſhall have 15 Parts and 16 Diviſors, as before.

16	8	4	2	1
2	2	2	2	

3. Again,

3. Again, I divide 16 by 2, and the Quotient is 8; this divided by 2 gives 4, which divided by itſelf gives 1; then from every one of the Diviſors 2,2,4, I ſubtract 1, and the Remainders 1, 1, and 3 do ſhew, that if two different Letters *a* and *b* be joined together, and next after them a third different from each of them (as *c*) be written thrice, the Quantity ſo compoſed, to wit, *abccc*, ſhall have 15 Aliquot Parts, and 16 Diviſors, as before.

16	8	4	1
2	2	4	

4. Again, I divide 16 by 4, and the Quotient is 4; this divided by it ſelf gives 1: then from each of the Diviſors 4 and 4 I ſubtract 1, and the Remainders 3 and 3 do ſhew, that if ſome Letter *a* be written thrice, as *aaa*, and next after the ſame another Letter different from *a* (as *b*) be likewiſe written thrice, the Quantity ſo compoſed, to wit, *aaabbb*, or a^3b^3, ſhall have 15 Aliquot Parts and 16 Diviſors, as before.

16	4	1
4	4	

5. Laſtly, I divide 16 by it ſelf and the Quotient is 1; then from 16 I ſubtract 1, and the Remainder 15 ſhews, that if ſome Letter *a* be written 15 times, as *aaaaaaaaaaaaaaa*, or a^{15}, this Quantity ſhall have 15 Parts and 16 Diviſors, as before.

16	1
16	

Hence becauſe 16 cannot be divided by any other ways than thoſe five before expreſs'd, we may conclude that the five Quantities found out, and thoſe only, to wit, ab^7, $abcd$, abc^3, a^3b^3, and a^{15}, have each of them 15 Aliquot Parts and 16 Diviſors. All which Operations do clearly reſult from *Obſervat.* 6. and 7. in the precedent *Sect.* 8.

Example 2.

Let it be required to find out all ſuch Quantities expreſſible by Letters, which may every one of them have 23 Aliquot Parts and 24 Diviſors.

Firſt, as before I add 1 to 23, and it makes 24; this may be divided by its Factors in a ſevenfold manner before the Quotient be Unity, as here you ſee.

24	8	4	2	1
3	2	2	2	

24	6	2	1
4	3	2	

24	4	2	1
6	2	2	

24	4	1
6	4	

24	3	1
8	3	

24	2	1
12	2	

24	1
24	

Whence I conclude that ſeven different Quantities may be produced, every one of which ſhall have 23 Aliquot Parts and 24 Diviſors; now to find out the ſaid Quantities I ſubtract 1 (to wit, Unity) from every one of the Diviſors of the foregoing ſevenfold Diviſion, ſo the Diviſors 3,2,2,2, of the firſt Diviſion being ſeverally leſſened by Unity give 2,1,1,1; whence according to the precedent directions in Example 1 of this *Sect.* 9. this Quantity may be compoſed, to wit, *aabcd*; and by proceeding in like manner with the reſt of the Diviſors ſeven different Quantities, every one of which has 23 Aliquot Parts and 24 Diviſors, are diſcovered, and may be expreſs'd either

Thus,		Or thus,
	aabcd	a^2bcd
	aaabbc	a^3b^2c
	aaaaabc	a^5bc
	aaaaabbb	a^5b^3
	aaaaaaabb	a^7b^2
	aaaaaaaaaaab	$a^{11}b$
	aaaaaaaaaaaaaaaaaaaaaaa	a^{23}

Example 3.

Let it be required to find out a Quantity which has 42 Aliquot Parts.

Firſt, as before I add 1 to 42 and it makes 43, which being a prime Number (that is, ſuch as cannot be divided by any Number but by itſelf or Unity) does ſhew, that there is only one Quantity can be found that has 42 Aliquot Parts, *viz.* ſome Letter (as *a*) being written 42 times one after another, or a ſingle *a* with its Index 42, as a^{42}, does expreſs a Quantity (to wit, the forty ſecond Power of *a*) which has 42 Aliquot Parts, and 43 Diviſors. The like is to be underſtood of other Quantities, when the multitude of Aliquot Parts deſired being increaſed with Unity makes a prime Number.

 For

For further Illustration of the Premises, the Learner may view the following Table, which shews all the various Quantities express'd by Letters, that have a given multitude of Aliquot Parts not exceeding 50; and upon the grounds before explained the Table may be continued as far as you please.

Quantities.		Aliquot Parts.
a	has	1
aa	has	2
ab, a^3	have each	3
a^4	&c.	4
aab, a^5		5
a^6		6
a^3b, abc, a^7		7
$aabb, a^8$		8
a^4b, a^9		9
a^{10}		10
$a^2bc, a^3b^2, a^5b, a^{11}$		11
a^{12}		12
a^6b, a^{13}		13
a^4bb, a^{14}		14
$a^3bc, abcd, a^3b^3, a^7b, a^{15}$		15
a^{16}		16
$a^2b^2c, a^5b^2, a^8b, a^{17}$		17
a^{18}		18
$a^4bc, a^4b^3. a^9b, a^{19}$		19
a^6b^2, a^{20}		20
$a^{10}b, a^{21}$		21
a^{22}		22
$a^3b^2c, a^2bcd, a^5bc, a^5b^3, a^7b^2, a^{11}b, a^{23}$		23
a^4b^4, a^{24}		24
$a^{12}b, a^{25}$		25
$a^{10}b^2c^2, a^8b^2, a^{26}$		26
$a^6bc, a^6b^3, a^{13}b, a^{27}$		27
a^{28}		28
$a^4b^2c, a^5b^4, a^9b^2, a^{14}b, a^{29}$		29
a^{30}		30
$a^3bcd, a^3b^3c, a^7bc, abcde, a^7b^3, a^{15}b, a^{31}$		31
$a^{10}b^2a^{32}$		32
$a^{16}b, a^{33}$		33
a^5b^4, a^{34}		34
$a^2b^2cd, a^5b^2c, a^3b^2c^2, a^8bc, a^8b^3, a^5b^5, a^{11}b^2, a^{17}b, a^{35}$		35
a^{36}		36
$a^{18}b, a^{37}$		37
$a^{12}b^2, a^{38}$		38
$a^4bcd, a^4b^3c, a^9bc, a^7b^4, a^9b^3, a^{19}b, a^{39}$		39
a^{40}		40
$a^6b^2c, a^6b^5, a^{13}b^2, a^{20}b, b^{41}$		41
a^{42}		42
$a^{10}bc, a^{10}b^3, a^{21}b, a^{43}$		43
$a^4b^2c^2, a^8b^4, a^{14}b^2, a^{44}$		44
$a^{22}b, a^{45}$		45
a^{46}		46
$a^3b^2cd, a^5bcd, a^5b^3c, a^2bcde, a^3b^3c^2, a^7b^2c, a^{11}bc, a^7b^5, a^{11}b^3, a^{15}b^2, a^{23}b, a^{47}$		47
a^6b^6, a^{48}		48
$a^4b^4c, a^9b^4, a^{24}b, a^{49}$		49
$a^{16}b^2, a^{50}$		50

X. How

X. *How to find out the ſmalleſt Number that ſhall have a given multitude of Aliquot Parts.*

Firſt, by the foregoing *Sect.* 9. ſearch out all the Quantities expreſſible by Letters, every one of which may have the Number of Aliquot Parts deſired; then to the different Letters by which every one of thoſe Quantities is expreſs'd, aſſign the ſmalleſt prime Numbers, and find out by continual Multiplication the Products of thoſe prime Numbers correſpondent to the ſaid Quantities. Again, let the values of thoſe Letters be expreſs'd by the ſame prime Numbers varied as many ways as is poſſible, and find out their reſpective Products, as before. Laſtly, all thoſe Products being compared to one another, the leaſt of them ſhall be the ſmalleſt Number that has the preſcribed multitude of Aliquot Parts.

Example 1.

Let it be required to find the ſmalleſt Number that has 15 Aliquot Parts.

Firſt, all the different Quantities that can be found to have ſeverally 15 Aliquot Parts (as appears by the precedent *Sect.* 9) are theſe, to wit, $abcd$, a^3bc, a^3b^3, a^7b, a^{15}; then by aſſigning to a,b,c,d the ſmalleſt prime Numbers 2,3,5,7, for $abcd$ there will be found 210, (by multiplying 2,3,5,7 one into the other continually;) for a^3bc 120, for a^3b^3 216, for a^7b 384, and for a^{15} 32768; the leaſt of which Products is 120. But before we can determine whether 120 be the leaſt Number or not that has 15 Aliquot Parts, enquiry muſt be made by exchanging the values of thoſe Letters with the ſaid prime Numbers all manner of ways, *viz.* we may ſuppoſe $a=3$, $b=2$, $c=5$, and $d=7$; or $a=5$, $b=2$, $c=3$, and $d=7$: or again, $a=7$, $b=2$, $c=3$, $d=5$. And many otherways the values of a,b,c,d may be expreſs'd by the ſaid prime Numbers 2,3,5,7; and conſequently from thoſe Variations the Quantities $abcd$, a^3bc, b^3b^3, a^7b, a^{15} will be expounded by various Numbers, which muſt be compared together, and then the leaſt among them all is the Number ſought. So after all Variations are made, it will appear that a^3bc is that Quantity by which 120, the ſmalleſt Number having 15 Aliquot Parts and 16 Diviſors, will be found out.

Example 2.

Again, if the leaſt Number that has 23 Aliquot Parts, or 24 Diviſors, be deſired.

Firſt, by *Sect.* 9. all the Quantities which have ſeverally 23 Parts will be found theſe, to wit, a^2bcd, a^3bbc, a^5bc, a^5b^3, a^7b^2, $a^{11}b$, and a^{23}. Then by aſſuming for the values of a,b,c,d the leaſt prime Numbers 2,3,5,7: for a^2bcd there will be found 420, for a^3b^2c 360, for a^5bc 480, for a^5b^3 864, for a^7b^2 1152, for $a^{11}b$ 6144, and for a^{23} 8388608. And after all other poſſible Variations made with the ſaid Letters and prime Numbers, by taking ſometimes one, ſometimes another of the ſaid Numbers for the value of a, b, &c. it will at length appear that a^3b^2c finds out 360, the leaſt Number that has the deſired multitude of 23 Aliquot Parts and 24 Diviſors.

If there be not occaſion to find the leaſt, but any Number that has a given multitude of Aliquot Parts, ſuppoſe 15, then you may indifferently uſe any one of theſe five Quantities $abcd$, a^3bc, a^3b^3, a^7b, a^{15}, by aſſigning to a,b,c,d prime Numbers at pleaſure, and taking ſometimes one, ſometimes another of thoſe Numbers, or always new prime Numbers for the values of a,b,c,d; whence innumerable Numbers may be found out, every one of which ſhall have Aliquot Parts. As if we ſuppoſe $a=2$, $b=3$, and $c=5$, there will be found for a^3bc 120; but by putting $a=3$, $b=2$, and $c=5$, there will be found for a^3bc 270. Or alſo by aſſuming $a=7$, $b=11$, and $c=13$, there will be produced for a^3bc 49049. Or if we put $a=17$, $b=19$, and $c=23$, then $a^3bc=2146981$. And in like manner you may uſe every one of the other four Quantities $abcd$, a^3b^3, a^7b, and a^{15}. The like alſo is to be underſtood of every one of theſe a^2bcd, a^3b^2c, a^5bc, a^5b^3, a^7b^2, $a^{11}b$, and a^{23}, for the finding out innumerable Numbers, which have ſeverally 23 Aliquot Parts and 24 Diviſors.

Laſtly, to find the leaſt Number that has 42 Parts and 43 Diviſors; foraſmuch as a Quantity having this multitude of Parts and Diviſors can be deſigned only in one manner, *viz.* by writing a^{42}; let the leaſt prime Number 2 be taken for the value of a, and then ſeek the forty ſecond Power of the Root 2, by writing down 2 forty two times ſeparately, and multiplying thoſe Numbers one into another, according to the Rule of continual Multiplication, ſo the laſt Product will be 4398046511104, which is the leaſt Number that has the deſired multitude of 42 Aliquot Parts. And ſo of others.

For further illuſtration the Learner may view the following Table, which ſhews the leaſt Number that has any given multitude of Aliquot Parts under 51. *Note*, That the number of Diviſors to any number is always more by one than its number of Aliquot Parts; for albeit a number cannot properly be called a Part of itſelf, yet 'tis contained in it ſelf once, and therefore may be ſaid to be a Diviſor to itſelf.

Each number in the firſt of theſe Columels is the ſmalleſt that can be found to have ſuch a multitude of Aliquot Parts as is expreſs'd in the latter Columel.

2	has	1	Aliquot Part.
4	has	2	Aliquot Parts.
6		3	
16		4	
12		5	
64		6	
24		7	
36		8	
48		9	
1024		10	
60		11	
4096		12	
192		13	
144		14	
120		15	
65536		16	
180		17	
262144		18	
240		19	
576		20	
3072		21	
4194304		22	
360		23	
1296		24	
12288		25	
900		26	
960		27	
268435456		28	
720		29	
1073741824		30	
840		31	
9216		32	
196608		33	
5184		34	
1260		35	
68719476736		36	
786432		37	
36864		38	
1680		39	
1099511627776		40	
2880		41	
4398046511104		42	
15360		43	
3600		44	
12582912		45	
70368744177664		46	
2520		47	
46656		48	
6480		49	
589824		50	

CHAP.

CHAP. IX.

The Arithmetic both of Surd Numbers and Surd Quantities expreſs'd by Letters. The Conſtitution and Invention of ſix Binomials in numbers, agreeable to thoſe expounded in Prop. 49, 50, 51, 52, 53, 54. Elem. 10. Euclid. *with Rules to extract the Square Root out of every one of them; as alſo what Root you pleaſe out of any Binomial in Numbers, having ſuch a Binomial Root as is deſired.*

Sect. I. *Definitions concerning Surd Roots, and their Fundamental Operations.*

EVery Abſolute (or Ordinary) Number, whether it be a whole Number or a Fraction, or a whole Number with a Fraction annex'd to it, is called *Rational:* As 1, 2, 3, 4, &c. alſo $\frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{5}{8}, \frac{1}{5}, \frac{3}{7}$, &c. and $2\frac{1}{2}$ (or $\frac{5}{2}$) $5\frac{1}{3}$ (or $\frac{16}{3}$) $20\frac{1}{12}$, &c. are called Rational Numbers; ſo alſo $a, ab, \frac{bc}{a}, a+\frac{bc}{a}$, &c. repreſent Rational Quantities.

But when the Square Root, Cubic Root, or any other Root, cannot be perfectly extracted out of a Rational Number, that Root is called *Irrational* or *Surd*; and becauſe it cannot be exactly expreſs'd by any Rational Number, it is uſual to ſet ſome Character (which is called the Radical Sign) before the Rational Number out of which the Root ought to be extracted, to deſign or ſignifie the ſame Root: As $\sqrt{}$ or $\sqrt{(2)}$ prefix'd before any Rational Number, ſignifies the Square Root of that Number; $\sqrt{(3)}$ the Cubic Root, $\sqrt{(4)}$ the Biquadratic Root, $\sqrt{(5)}$ the Root of the fifth Power, &c.

Hence $\sqrt{(12)}$ or $\sqrt{(2)}12$ denotes or repreſents the Square Root of 12, which Root is called Irrational or Surd, becauſe it cannot be perfectly expreſs'd by any Rational Number, for 3 multiplied by itſelf produces 9, which is leſs than 12; and 4 multiplied by itſelf produces 16, which is greater than 12: and altho there be innumerable mixt Numbers conſiſting of 3, and ſome Fractions which fall between 3 and 4, yet none of them multiplied into itſelf quadraticaly can produce the whole Number 12.

In like manner $\sqrt{(3)}5$, which repreſents the Cubic Root of 5, is called an Irrational or Surd Number, becauſe no Number can be found, which being multiplied into itſelf cubically will produce 5 exactly: ſo alſo $\sqrt{a}, \sqrt{bc}, \sqrt{(3)}bb$, &c. repreſent Surd Quantities.

There are two ſorts of Irrational or Surd Numbers, Simple and compound: a Simple Surd Number is expreſs'd by one ſingle Term; ſuch are $\sqrt{5}, \sqrt{10}, \sqrt{(3)}16, \sqrt{(4)}8$, &c. but a Compound Surd Number conſiſts of many ſimple or ſingle Terms, and is formed by the Addition or Subtraction of Simple Terms, ſuch are $\sqrt{5}+\sqrt{2}, \sqrt{5}-\sqrt{2}, \sqrt{8}+\sqrt{6}-\sqrt{2}, \sqrt{(3)}:\overline{7+\sqrt{2}}$: which laſt is called an Univerſal Root, and ſignifies the Cubic Root of the Sum of 7, and the ſquare Root of 2. (See *Sect.* 28. *Chap.* 1. *Book* 1. concerning the deſigning of Surd Numbers.

The Arithmetic of Surd Numbers, and Surd Quantities deſign'd by Letters, depends chiefly upon theſe ſix primary or fundamental Operations in Simple Surds, *viz.*

1. The Reduction of Rational Numbers and Rational Quantities expreſs'd by Letters, to the form of Surd Roots, which ſhall have a given Radical Sign.
2. The Reduction of Simple Surd Roots having different Radical Signs, to other Surds which ſhall have one common Radical Sign, and be equal in value to the given Surds.
3. Multiplication in Simple Surds.
4. Diviſion in Simple Surds.
5. The Reduction of a given Surd Number or Quantity to another more ſimple, when it may be done.
6. How to diſcover whether two Simple Surd Numbers or Quantities be *Commenſurable* or not, *viz.* whether their *Reaſon* or *Proportion* can be expreſs'd by Rational Numbers or Quantities, or not. Theſe ſix Operations I ſhall handle in order.

Sect.

Sect. II. *How to reduce Rational Numbers and Quantities designed by Letters to the form of Surd Roots, which shall have the same Radical Sign with any Surd Root prescribed.*

Multiply the given Rational Number or Quantity into itself, so often as is requisite to produce a Power of the same degree with that Power which is denoted by the Radical Sign of the prescribed Surd, and then set the said Radical Sign before the Power produced by the said Multiplication.

As to reduce 6 to the form of a Surd Root which shall have the same Radical Sign with $\surd 12$ (or $\surd(2)12$,) I multiply 6 into it self quadraticaly, and it makes 36; then $\surd 36$ (that is, 6) and $\surd 12$ have the same Radical Sign, to wit, $\surd$ or $\surd(2)$.

Again, to reduce 5 to the same Radical Sign with $\surd(3)12$, I multiply 5 into itself cubically, (*viz.* 5 into 5, and the Product into 5) and it produces 125; then $\surd(3)125$ (that is, 5) and $\surd(3)12$ have the same Radical Sign, to wit $\surd(3)$.

Likewise to reduce 3 to the same Radical Sign with $\surd(4)12$, I seek the fourth Power of 3, which (by multiplying the Square of 3 into itself) will be found 81; then $\surd(4)81$ and $\surd(4)12$ are of the same kind. And so of others.

By the help of this Rule, when the Radical Sign of a Simple Surd Fraction has reference only to one of its Terms, we may reduce the Fraction to another, whose Radical Sign shall refer both to the Numerator and Denominator: As if $\frac{\surd 2}{5}$ be proposed, which signifies that $\surd 2$ is divided or to be divided by 5, we may take $\surd 25$ instead of 5, and then that Fraction will be reduced to this $\surd\frac{2}{25}$, whose Radical Sign refers as well to the Denominator as the Numerator, *viz.* $\surd\frac{2}{25}$ signifies that $\surd 2$ is divided by $\surd 25$.

Likewise $\frac{5}{\surd(3)4}$ may be reduced to $\surd(3)\frac{125}{4}$, by setting 125 the Cube of 5 for a Numerator instead of 5, and the Radical Sign $\surd(3)$ against the middle of the Fraction; so that $\surd(3)\frac{125}{4}$ (which signifies that $\surd(3)125$ is divided by $\surd(3)4$) imports as much as $\frac{5}{\surd(3)4}$ that is, 5 divided by $\surd(3)4$.

Nor will the Operation be otherwise in reducing Rational Quantities designed by Letters to the form of Surd Quantities; (respect being had to the Rules of Algebraical Multiplication before delivered.) As to reduce the Quantity a, so as it may have the same Radical Sign with $\surd b$, I multiply a into itself quadraticaly, and it makes aa; then $\surd aa$ (that is, a) and $\surd b$ have the same Radical Sign.

Again, to reduce $a+b$ to the same Radical Sign with $\surd bc$, I square $a+b$, and it makes $aa+2ab+bb$; then $\surd{:}aa+2ab+bb{:}$ (that is, $a+b$) and $\surd bc$ have the same Radical Sign.

Likewise to reduce b to the same Radical Sign with $\surd(3)ab$, I multiply b into itself cubically, and it makes bbb; then $\surd(3)bbb$ (that is, b) and $\surd(3)ab$ have the same Radical Sign, to wit, $\surd(3)$.

Hence also $\frac{a}{\surd b}$ may be reduced to $\surd\frac{aa}{b}$, and $\frac{\surd(3)ab}{3c}$ to $\surd(3)\frac{ab}{27ccc}$.

Sect. III. *How to reduce two simple Surd Numbers or Quantities having different Radical Signs, to two others that may have a common Radical Sign.*

This Reduction is like that of reducing Vulgar Fractions to a common Denominator; but how 'tis wrought I shall shew by Examples, first in Surd Numbers, and then in Surd Quantities express'd by Letters.

Example 1.

Let it be required to reduce $\surd(4)10$ and $\surd(6)7$ into two other Roots that may have a common Radical Sign, and be equal in value to those given.

First, divide the given Indices (4) and (6) by their greatest common Divisor (2), and set the Quotients (2) and (3) under their respective Dividends as here you see; then multiply cross-wise *viz.* the first Dividend or Index (4) by the second Quotient (3), (or the second Dividend (6) by the

$$\begin{array}{lccc} (2)\,) & \surd(4)10 & \times & \surd(6)7 \\ & (2) & & (3) \\ & \surd(12)1000 & & \surd(12)49 \end{array}$$

the first Quotient (2), and the Product is (12), before which setting √ it gives √(12), which is to be reserved for the common radical Sign sought. Then multiply the Powers of the given Roots according to the altern Quotients, *viz.* multiply the first Power 10 cubically, because the second Quotient is (3); and the latter Power 7 quadratically, because the first Quotient is (2): so the Products will be 1000 and 49, before each of which prefixing √(12) the common Radical Sign before found, there arise √(12)1000 and √(12)49, the two Surd Roots sought, which are equal in value to the given Surds respectively, *viz.* √(12)1000 is equal to √(4)10, and √(12)49 is equal to √(6)7; and the Surds found out have a common Radical Sign, as was required.

Example 2.

In like manner √(2)5 and √(3)6 will be reduced to √(6)125 and √(6)36; and the Work will stand as here you see underneath.

$$\begin{array}{cccc} (1)\,) & \surd(2)5 & \times & \surd(3)6 \\ & (2) & & (3) \\ & \surd(6)125 & & \surd(6)36 \end{array}$$

Example 3.

Again, if $\frac{\surd 7}{3}$ and $\frac{5}{\surd(3)4}$ be proposed to be reduced to a common Radical Sign, first by the Rule in the preceding *Sect.* 2. I reduce them to $\surd\frac{7}{9}$ (or $\surd(2)\frac{7}{9}$) and $\surd(3)\frac{125}{4}$, which according to the Rule in the first Example of this Section will be reduced to these, to wit, $\surd(6)\frac{343}{729}$ and $\surd(6)\frac{15625}{16}$, and the Work will stand as here you see.

$$\begin{array}{cccc} (1)\,) & \surd(2)\frac{7}{9} & \times & \surd(3)\frac{125}{4} \\ & (2) & & (3) \\ & \surd(6)\frac{343}{729} & & \surd(6)\frac{15625}{16} \end{array}$$

The like Work is to be done in reducing two Surd Quantities express'd by Letters, which have different Radical Signs, to two others which shall have a common Radical Sign, as will appear in the following Examples.

Example 4.

Suppose it be desired to reduce √(2)*a* and √(6)*aa* to a common Radical Sign.

First, I divide the given Indices (2) and (6) severally by their greatest common Divisor (2) and set the Quotient (1) and (3) under their respective Dividends, as here you see; then I multiply cross-wise, *viz.* the first Dividend (2) by the second Quotient (3), or the latter Dividend (6) by the first Quotient (1),

$$\begin{array}{cccc} (2)\,) & \surd(2)a & \times & \surd(6)aa \\ & (1) & & (3) \\ & \surd(6)aaa & & \surd(6)aa \end{array}$$

and the Product is (6); before which setting √ it gives √(6) for the common Radical Sign sought. Then I multiply the Powers of the given Roots according to the alternate Quotients, *viz.* the first Power *a* cubically, because the latter Quotient is (3), but the second Power *aa*, because the first Quotient (1) is a lateral Index, is not to be multiplied into itself at all. So the Products are *aaa* and *aa*, before each of which prefixing √(6), (the common Radical Sign before found) there arise √(6)*aaa* and √(6)*aa* the two Surd Roots sought; which are equal in value to the given Surds respectively, *viz.* √(6)*aaa* is equal to √(2)*a*, and √(6)*aa* is equal to √(6)*aa*; and the Surd Roots found out have a common Radical Sign, to wit, √(6). Therefore that is done which was required.

Example 5.

After the same manner √(4)3*b* and √(10)5*ac* will be reduced to √(20)243*bbbbb* and √(20)25*aacc*, and the Work will stand as here you see.

$$\begin{array}{cccc} (2)\,) & \surd(4)3b & \times & \surd(10)5ac \\ & (2) & & (5) \\ & \surd(20)243bbbbb & & \surd(20)25aacc \end{array}$$

Sect.

Sect. IV. *Multiplication in simple Surd Quantities.*

Before Addition and Subtraction can be performed in Surd Quantities, the manner of their Multiplication and Division must first be learnt; I shall therefore begin with Multiplication, which requires that the Surd Roots proposed to be multiplied be of the same kind; and therefore if they be of different kinds, they must first of all be reduced to the same radical Sign, (by the Rule in the foregoing *Sect.* 3.) Then,

1. Multiply the Numbers or Quantities standing next after their common radical Sign one into another, without any regard had to the said Sign; and to the Product of that Multiplication prefix the common radical Sign: so this new Root shall be the Product sought.

As for Example, to multiply $\sqrt{5}$ by $\sqrt{3}$, I multiply 5 by 3 and it makes 15; to which I prefix $\sqrt{}$, (the radical Sign of each of the Surds given to be multiplied) and then arises $\sqrt{15}$ for the Product sought.

Likewise if $\sqrt{6}$ be multiplied by $\sqrt{5}$ it produces $\sqrt{30}$.

Also $\sqrt{\frac{3}{4}}$ multiplied by $\sqrt{\frac{1}{3}}$ makes $\sqrt{\frac{3}{12}}$.

And $\sqrt{2\frac{1}{2}}$ (or $\sqrt{\frac{5}{2}}$) into $\sqrt{2\frac{1}{3}}$ (or $\sqrt{\frac{7}{3}}$) gives $\sqrt{\frac{35}{6}}$.

Again, $\sqrt{(3)}4$ multiplied by $\sqrt{(3)}5$ produces $\sqrt{(3)}20$.

Likewise $\sqrt{(4)}\frac{5}{2}$ into $\sqrt{(4)}2$ produces $\sqrt{(4)}5$.

And if $\sqrt{(2)}5$ be to be multiplied into $\sqrt{(3)}6$, the Product will be $\sqrt{(6)}4500$; for, first, the given Roots being of different kinds are reduced to these, to wit, $\sqrt{(6)}125$ and $\sqrt{(6)}36$, which multiplied one into another make $\sqrt{(6)}4500$.

After the same manner Multiplication in simple Surd Quantities express'd by Letters is performed: as if $\sqrt{a}$ be to be multiplied by $\sqrt{b}$, the Product will $\sqrt{ab}$. For (according to the Rule of Algebraical Multiplication) the quantity a multiplied by the quantity b produces ab, to which I prefix the given radical Sign $\sqrt{}$, and it gives $\sqrt{ab}$ the Product sought.

Likewise $\sqrt{ab}$ into $\sqrt{cd}$ produces $\sqrt{abcd}$.

And $\sqrt{\frac{2ab}{3c}}$ multiplied by $\sqrt{\frac{9ad}{2b}}$ makes $\sqrt{\frac{3aad}{c}}$.

Again, to multiply $\sqrt{(2)}d$ by $\sqrt{(3)}ab$, first, (by the Rule in the foregoing *Sect.* 3.) I reduce them to $\sqrt{(6)}ddd$ and $\sqrt{(6)}aabb$, which multiplied one into another give $\sqrt{(6)}dddaabb$ for the Product required.

2. When any Surd Root is to be multiplied into it self according to the Index of its own Power, *viz.* if a Surd square Root be to be squared, or a Surd cubic Root to be cubed, cast away the radical Sign, and take the number or quantity remaining for the Product sought, which in this case is always rational: as to multiply $\sqrt{5}$ into it self I cast away the radical Sign $\sqrt{}$, and take 5 for the Product or Square of $\sqrt{5}$, (or $\sqrt{5}$ into $\sqrt{5}$ makes $\sqrt{25}$, that is, 5.) Likewise the Square of $\sqrt{8}$ is 8, and the Square of $\sqrt{4}$ is 4

In like manner to multiply $\sqrt{(3)}5$ into it self cubically, I take 5 for the Product, to wit, the Cube of $\sqrt{(3)}5$: for $\sqrt{(3)}5$ into $\sqrt{(3)}5$ makes $\sqrt{(3)}25$, and this again into $\sqrt{(3)}5$ produces $\sqrt{(3)}125$, that is, 5.

Again, $\sqrt{(4)}12$ multiplied into itself biquadraticaly produces 12; for $\sqrt{(4)}12$ into $\sqrt{(4)}12$ makes $\sqrt{(4)}144$, (which is the Square of $\sqrt{(4)}12$;) then $\sqrt{(4)}144$ again into $\sqrt{(4)}12$ makes $\sqrt{(4)}1728$, (which is the Cube of $\sqrt{(4)}12$.) Lastly, $\sqrt{(4)}1728$ again into $\sqrt{(4)}12$ produces $\sqrt{(4)}20736$, that is 12, which is the fourth Power of $\sqrt{(4)}12$ the Root proposed.

The like is to be done in Surd Quantities express'd by Letters; as if $\sqrt{ab}$ be to be multiplied into itself, or squared, I cast away the radical Sign, and write ab for the Product or Square of $\sqrt{ab}$. Likewise if $\sqrt{(3)}bcd$ be to be multiplied into itself cubically, the Product or Cube thereof will be bcd.

3. When a Surd Quantity is given to be multiplied by a Rational Quantity, reduce the Rational into the form of a Surd of the same kind with the given Surd, (by the foregoing Rule in *Sect.* 2.) and then multiply according to the first Rule of this fourth Section; as to multiply $\sqrt{8}$ by 2, I first reduce 2 to $\sqrt{4}$, then $\sqrt{8}$ into $\sqrt{4}$ gives $\sqrt{32}$, the Product desired. Likewise $\sqrt{7}$ multiplied by 5, that is, by $\sqrt{25}$, gives the Product $\sqrt{175}$.

Again, if $\sqrt{(3)}6$ be to be multiplied by 2, I reduce 2 to $\sqrt{(3)}8$, (by multiplying 2 into it self cubically;) then $\sqrt{(3)}6$ multiplied by $\sqrt{(3)}8$, gives $\sqrt{(3)}48$ for the Product desired.

Like-

Likewise $\surd(4)8$ multiplied by 5, that is, by $\surd(4)625$, gives $\surd(4)5000$ for the Product sought.

After the same manner to multiply the Surd quantity $\surd a$ by the Rational quantity b, I first reduce b to $\surd bb$, then $\surd a$ into $\surd bb$, makes $\surd abb$ the Product sought. Likewise $\surd(3)a$ into b makes $\surd(3)abbb$, (b being first reduced to $\surd(3)bbb$.)

Again $\surd 3$ into $4a$ gives the Product $\surd 48aa$.

4. But when a Surd quantity is given to be multiplied by a Rational quantity, it will oftentimes be very convenient to omit their Multiplication, and only to connect them so as that the Rational quantity may stand on the left hand of the given Surd, to signifie the Product of their Multiplication; as to Multiply $\surd 8$ by 2, I write $2\surd 8$ for the Product, which signifies twice the square Root of 8. Likewise $20\surd 3$ represents the Product of the Multiplication of $\surd 3$ by 20, *viz.* it imports $\surd 3$ to be taken 20 times, which amounts to as much as $\surd 1200$, found out by the preceding third Rule of this Section.

Again, $\frac{8}{5}\surd 7$ signifies the Product of $\surd 7$ multiplied by $\frac{8}{5}$, (or $\frac{8}{5}$ by $\surd 7$;) and $\frac{2}{3}\surd\frac{7}{5}$ denotes the Product of $\frac{2}{3}$ multiplied into $\surd\frac{7}{5}$, (or $\surd\frac{7}{5}$ into $\frac{2}{3}$;) also 4 into $20\surd 3$ makes $80\surd 3$, that is, $20\surd 3$ taken four times. Likewise $2\surd(3)6$, signifies twice the Cubic Root of 6, and is of equal value with $\surd(3)48$. Likewise $\frac{5}{3}\surd(3)80$ denotes the Product of the Cubic Root of 80 multiplied by $\frac{5}{3}$, or $\frac{5}{3}$ of $\surd(3)80$, which is equivalent to $\surd(3)\frac{10000}{27}$; and $3\surd(3)5$ multiplied by 6 makes $18\surd(3)5$, that is, $\surd(3)29160$.

The like may be done in Surd quantities exprest by Letters; as if $\surd a$ be to be multiplied by b, I write $b\surd a$ to signifie the Product; also 5 into $b\surd a$ makes $5b\surd a$; and c into $b\surd a$ gives the Product $cb\surd a$; likewise $4a$ into $\surd 3$ makes $4a\surd 3$.

Again, if $\surd ab$ be to be multiplied by $b-d$, the Product may be exprest thus, $\overline{b-d}\times\surd ab$, or thus, $\overline{b-d}\surd ab$.

Also if $\surd(3)\frac{2ab}{c}$ be to be multiplied by d, the Product may be exprest thus, $d\surd(3)\frac{2ab}{c}$ and $\surd(3)a$ into b, makes $b\surd(3)a$, which is equivalent to $\surd(3)abbb$.

5. When two Rational quantities, whether they be equal or unequal, are multiplied severally into one common Surd Square Root, according to the method in the preceding fourth Rule, and it is desired to multiply those Products one into the other, (which Products are called Commensurable Quantities, for the reason hereafter given in *Sect.* 7.) multiply the Rational by the Rational, and that which is produced multiply by the said common Surd, omitting its Radical Sign; so the last Product is that which is sought, and will be intirely Rational.

As for example, to multiply $3\surd 5$ by $2\surd 5$ I multiply 3 by 2, and the Product 6 by 5, so it makes 30, which is the Product of $3\surd 5$ multiplied by $2\surd 5$, (or of $\surd 45$ into $\surd 20$.)

Likewise $2\surd 3$ multiplied by $2\surd 3$, (*viz.* the square of $2\surd 3$) makes 12; and $20\surd 3$ into $8\surd 3$ makes 480, (by multiplying 20, 8, and 3, one into another continually;) again, $\frac{8}{3}\surd 12$ into $5\surd 12$ produces 160.

After the same manner to multiply $a\surd c$ by $b\surd c$, I multiply a by b, and the Product ab by c; so there arises abc for the Product sought. The Reason of this Rule is evident, for $\surd aac$, (that is, $a\surd c$) multiplied into $\surd bbc$, (that is, $b\surd c$) makes $\surd aabbcc$, (that is, abc,) as before.

In like manner $5\surd b$ into $5\surd b$ produces $25b$, to wit, the Square of $5\surd b$; and $2a\surd b$ into $5a\surd b$ gives the Product $10aab$. Also $5a\surd 12d$ multiplied by $\frac{8}{3}a\surd 12d$ produces $160aad$.

But here is to be noted, that this fifth Rule of Multiplication takes place only when the common Surd Root into which Rational Numbers are multiplied is a Surd square Root; so that if $4\surd(3)5$ be to be multiplied by $2\surd(3)5$, the said fifth Rule will be ineffective, and the Product is to be found out by the following sixth Rule.

6. When two Rational Quantities, whether they be equal or unequal, are multiplied into two unequal Surd Roots of the same kind, or into one common Surd above the quadratic kind, according to the Method in the foregoing fourth Rule of this *Sect.* and it is desir'd to multiply those Products one into another, multiply the Rational by the Rational and the Surd by the Surd, and joyn these Products together, so as the Rational Product may stand on the left hand; then those 2 Products so connected shall be the Product sought.

As for Example, to multiply $5\surd 8$ by $2\surd 3$ I multiply 5 by 2, and the Product is 10; also $\surd 8$ into $\surd 3$ makes $\surd 24$; then those 2 Products connected make $10\surd 24$, (that is,

$\surd 2400$)

√2400) the Product sought. In like manner 2√8 into 2√3 makes 4√24, that is, √384.

Again, 20√5 multiplied by 18√3 produces 360√15; and 8√27 into 2√3 makes 16√81, that is, 144; also 5√(3)4 into 3√(3)5 produces 15√(3)20, that is, √(3)3375; likewise 4√(3)5 into 2√(3)5 makes 8√(3)25; and 3√(4)5 into 2√(4)6 makes 6√(4)30.

After the same manner to multiply *a*√*bc* into *g*√*ad*, first I multiply *a* by *g*, and it makes *ag*; then √*bc* into √*ad* produces √*bcad*. Lastly, *ag* into √*bcad* gives *ag*√*bcad*, the Product sought.

Likewise 2√*ab* multiplied by 3*c*√*bc* produces 6*c*√*abbc*; and 2√*a* into 2√*b* makes 4√*ab*.

Also $\frac{2bc}{a}$√*ddd* multiplied by $\frac{aa}{2c}$√*ac*, gives the Product *ab*√*acddd*; and *b*√(3)*dd* into *c*√(3)*f* makes *bc*√(3)*ddf*; again, *a*√(3)*c* into *b*√(3)*c* makes *ab*√(3)*cc*.

7. When a simple Surd Quantity whose Radical Sign has for its Index some even Number greater than 2 is to be squared, prefix a Radical Sign whose Index is half the given Index, before the Power of the given Surd; so shall this new Surd be the square of that given. As if √(4)5 be to be squared or multiplied into it self, take √(2)5 or √5, for the Square or Product sought. Likewise the Square of √(6)10 is √(3)10, and √(8)10 into √(8)10 makes √(4)10.

After the same manner to multiply √(4)*bc* into itself quadraticaly, I write √(2)*bc* or √*bc* for the Product or Square of √(4)*bc*. Likewise the Square of √(8)10*bc* is √(4)10*bc*, and √(10)*a* into √(10)*a* makes √(5)*a*. Moreover, 2*ab*√(4)*d* into 3√(4)*d* makes 6*ab*√*d*; for 2*ab* into 3 makes 6*ab* and √(4)*d* being squared makes √(2) or √*d*.

But when a simple Surd Quantity, whose Radical sign has for its Index some Ternary Number greater than 3, as 6, 9, &c. is to be multiplied into itself cubically, prefix a Radical Sign with an Index that may be a third part of the given Index before the Power of the given Surd Root, so shall this new Surd be the Cube of that given; as if √(6)64 be to be multiplied into itself cubically, then √(2)64 or √64 shall be the Cube sought. Likewise the Cube of √(9)512 is √(3)512.

More Examples to excercise the precedent Rules of Multiplication in Simple Surd Numbers.

Multiply	√5	√(3)4	√(4)8
by	√8	√(3)7	√(4)2
Product	√40	√(3)28	√(4)16 that is, 2.

Multiply	√32	Multiply these three continually, √(3)50
by	√32	√(3)50
		√(3)50
Product	32	50

Multiply	√27	12
by	6	√(3)5
Product	6√27 or √972	12√(3)5 or √(3)8640

Multiply	18√5	24√6¼	6√7
by	4√5	5√6¼	5√3
Product	360	765	30√21

Multiply	√8 } that is,	√(6)512	4√5
by	√(3)4 }	√(6)16	4√5
Product	. . . .	√(6)8192	80

Multiply	5√8	12√(3)4	√(4)12
by	4	2½	√(4)12
Product	20√8	30√(3)4	√12

More

More Examples to Exercise the precedent Rules of Multiplication in Simple Surd Quantities exprest by Letters.

Multiply	$\sqrt{12a}$	$\sqrt{\frac{8}{1}ab}$
by	$\sqrt{3a}$	$\sqrt{\frac{1}{2}ac}$
Product	$\sqrt{36aa}$ or $6a$	$\sqrt{4aabc}$ or $2a\sqrt{bc}$

Multiply	$\sqrt{a}$	that is,	$\sqrt{(6)aaa}$
by	$\sqrt{(3)aa}$		$\sqrt{(6)aaaa}$
Product			$\sqrt{(6)a^7}$

Multiply	$\sqrt{27aa}$	Multiply these three continually,	$\sqrt{(3)aa}$
by	$\sqrt{27aa}$		$\sqrt{(3)aa}$
			$\sqrt{(3)aa}$
Product	$27aa$		aa

Multiply	$\sqrt{3bc}$	$5b$
by	2	$\sqrt{(3)2a}$
Product	$2\sqrt{3bc}$ or $\sqrt{12bc}$	$5b\sqrt{(3)2a}$ or $\sqrt{(3)250abbb}$

Multiply	$3a\sqrt{5}$	$7\sqrt{bc}$	$\frac{8}{1}a\sqrt{bc}$
by	$2b\sqrt{5}$	$4\sqrt{bc}$	$\frac{1}{4}b\sqrt{bc}$
Product	$30ab$	$28bc$	$2abbc$

Multiply	$5\sqrt{ab}$	$3a\sqrt{5}$	$\frac{2bc}{a}\sqrt{d}.$
by	$3\sqrt{ac}$	$2b\sqrt{6}$	$\frac{aa}{2c}\sqrt{d}$
Product	$15\sqrt{aabc}$	$6ab\sqrt{30}$	abd

The certainty of the first Rule of this fourth Section, (upon which all the rest depend) for the Multiplication of two simple Surd Numbers of the same kind, may be demonstrated in manner following: First, let there be two square Roots given to be multiplied, suppose $\sqrt{5}$ and $\sqrt{3}$, then (by the said Rule) the Product of their Multiplication is $\sqrt{15}$; now we must prove that $\sqrt{15}$ is the true Product of $\sqrt{5}$ multiplied by $\sqrt{3}$.

Demonstration.

By the Definition of Multiplication these are Proportionals, *viz.* . . .	1	.	$\sqrt{5}$	::	$\sqrt{3}$	.	Product.
Therefore their Squares shall be also Proportionals, (*per* 22 *Prop.* 6 *Elem. Euclid.*) *viz.*	1	.	5	::	3	.	Square of the Product.
But these are Proportionals, (*per* 19 *Prop.* 7 *Elem. Euclid.*)	1	.	5	::	3	.	15

Therefore from the two last Analogies 15 is equal to the Square of the Product, and consequently $\sqrt{15}$ is the Product of $\sqrt{5}$ into $\sqrt{3}$; which was to be proved.

Likewise in Cubic Roots, if $\sqrt{(3)5}$ be to be multiplied by $\sqrt{(3)4}$, the Product (by the same Rule) is $\sqrt{(3)20}$. For,

By the Definition of Multiplication these are Proportionals, *viz.* .	1	.	$\sqrt{(3)5}$	::	$\sqrt{(3)4}$	.	Product.
Therefore their Cubes are also Proportionals, (*per Prop.* 37. *Elem.* 11. *Euclid.*) *viz.*	1	.	5	::	4	.	Cube of the Product.
But as	1	.	5	::	4	.	20

There-

Therefore 20 is equal to the Cube of the Product, and consequently the Cubic Root of 20, to wit, $\surd(3)20$, is the Product of $\surd(3)5$ multiplied by $\surd(3)4$; which was to be proved.

Moreover, (because (by *Sect.* 11. *Chap.* 5.) if four Numbers be Proportionals, their fourth Powers, fifth Powers, *&c.* are also Proportionals, this Demonstration may be extended to prove the certainty of the said Rule for multiplying any two simple Surd Numbers of the same kind.

Sect. V. *Division in simple Surd Quantities.*

As before in Multiplication, so here in Division, if the given Surd Roots, to wit, the Dividend and Divisor be not of the same kind, they must be reduced to a common Radical Sign by the preceding *Sect.* 3. Then,

1. Divide the Number or Quantity following the Radical Sign of the Dividend, by the Number or Quantity following the same Radical Sign of the Divisor, without any regard to the Sign, and to the Quotient prefix the said common Radical Sign; so this new Root shall be the Quotient sought.

As for Example, to divide $\surd 15$ by $\surd 3$, I divide 15 by 3, and there arises 5, before which I prefix $\surd$, (the Radical Sign common to the given Surds) so $\surd 5$ is the Quotient sought.

Likewise if $\surd 30$ by divided by $\surd 5$, the Quotient $\surd 6$.

Also $\surd\frac{3}{8}$ divided by $\surd\frac{3}{4}$ gives the Quotient $\surd\frac{1}{2}$.

And $\surd 5\frac{5}{8}$, or $\surd\frac{45}{8}$, divided by $2\frac{1}{4}$, or $\frac{9}{4}$, gives the Quotient $2\frac{1}{2}$.

Again, $\surd(3)20$ divided by $\surd(3)5$, gives the Quotient $\surd(3)4$; for 20 divided by 5 gives 4, before which setting $\surd(3)$ the Radical Sign belonging to each of the given Surds, there arises $\surd(3)4$ for the Quotient sought.

Likewise $\surd(4)5$ divided by $\surd(4)\frac{5}{2}$, gives the Quotient $\surd(4)2$.

Moreover, if $\surd(6)4500$ be given to be divided by $\surd(2)5$, the Quotient will be $\surd(3)6$; for first, the given Roots being of different kinds are reduced to these, to wit, $\surd(6)4500$ and $\surd(6)125$; then by dividing $\surd(6)4500$ by $\surd(6)125$ there arises $\surd(6)36$, whose square Root being extracted, (because 36 is a square Number, and the Index (6) an even Number) it gives $\surd(3)6$ for the Quotient sought.

After the same manner Division is performed in simple Surd Quantities exprest by Letters. As to divide $\surd ab$ by $\surd a$, I divide ab by a and there arises b, then setting $\surd$ before b it gives $\surd b$ for the Quotient sought, to wit, the Quotient that arises by dividing $\surd ab$ $\surd a$.

Also $\surd b$ divided by $\surd a$ gives the Quotient $\surd\frac{b}{a}$

Likewise $\surd abcd$ divided by $\surd ab$ gives the Quotient $\surd cd$.

Also $\surd\frac{3aad}{c}$ divided by $\surd\frac{2ab}{3c}$ gives the Quotient $\surd\frac{9ad}{2b}$

Again, to divide $\surd(6)dddaabb$ by $\surd(3)ab$, I first reduce them to $\surd(6)dddaabb$, and $\surd(6)aabb$, then I divide $\surd(6)dddaabb$ by $\surd(6)aabb$, and there arises $\surd(6)ddd$, that is, $\surd(2)d$ for the Quotient sought.

2. When a Rational Number or Quantity is to be divided by its square Root, that Root is the Quotient; as if 5 be divided by its square Root, to wit, by $\surd 5$, the Quotient will be $\surd 5$. Also 8 divided by $\surd 8$ gives $\surd 8$ for the Quotient.

In like manner if the Quantity bc be divided by its square Root, to wit, by $\surd bc$, the Quotient will be $\surd bc$. And $5a$ divided by $\surd 5a$ gives the Quotient $\surd 5a$.

3. When a Surd number or quantity is to be divided by a Rational number or quantity, or a rational number or quantity by a Surd, reduce the rational into the form of a Surd, (by *Sect.* 2. of this *Chap.*) and then divide according to the first rule of this *Sect.* 5.

As to divide $\surd 32$ by 2, I first reduce 2 to $\surd 4$; then by dividing $\surd 32$ by $\surd 4$ there arises $\surd 8$ for the Quotient.

Likewise $\surd 175$ divided by 5, that is $\surd 25$, gives the Quotient $\surd 7$.

Also 12, that is $\surd 144$, divided by $\surd 3$, gives the Quotient $\surd 48$.

Again, if $\surd(3)48$ be to be divided by 2, I first reduce 2 to $\surd(3)8$, then by dividing $\surd(3)48$ by $\surd(3)8$ there arises $\surd(3)6$ for the Quotient sought. Also $\surd(4)5000$ divided by 5, (that is, by $\surd(4)625$) gives the Quotient $\surd(4)8$. After

After the same manner to divide the quantity $\surd abb$ by b, I first reduce b to $\surd bb$; and then by dividing $\surd abb$ by $\surd bb$, there arises $\surd a$ the Quotient sought. Again, $\surd 48aa$ divided by $4a$, that is by $\surd 16aa$, gives the Quotient $\surd 3$. Also $\surd(3)abbb$ divided by b, that is by $\surd(3)bbb$, gives the Quotient $\surd(3)a$.

Likewise to divide the Rational Quantity $\frac{bc}{a}$ by $\surd(3)bbcc$, I first reduce $\frac{bc}{a}$ to $\surd(3)\frac{bbbccc}{aaa}$, then I divide $\surd(3)\frac{bbbccc}{aaa}$ by $\surd(3)bbcc$, and there arises $\surd(3)\ \frac{bc}{aaa}$ or $\frac{\surd(3)bc}{a}$ the Quotient sought.

4 When the Product of a Rational Number or Quantity multiplied into a Surd Number or Quantity is to be divided by the same Surd, the Quotient will be the said multiplying Rational Number or Quautity. As $5\surd 3$ divided by $\surd 3$ gives the Quotient 5; also $20\surd(3)4$ gives the Quotient 20.

In like manner $5a\surd b$ divided by $\surd b$ gives the Quotient $5a$; and $4b\surd(3)12$ divided by $\surd(3)12$ gives the Quotient $4b$.

5. When the Dividend and Divisor are the Products of two Rational Numbers or Quantities multiplied severally into one common Surd, according to the fourth Rule of Multiplication in *Sect.* 4. (which Products are called Commensurable Surd Roots, as hereafter will appear in *Sect.* 7. of this *Chap.*) divide the Rational part of the Dividend by the Rational part of the Divisor, and that which arises shall be the Quotient sought. As for Example, to divide $6\surd 3$ by $2\surd 3$, I divide 6 by 2, and there arises 3 the Quotient sought; (for $2\surd 3$ multiplied by 3 produces $6\surd 3$.)

Again, $5\surd 6$ divided by $2\surd 6$ gives the Quotient $\frac{5}{2}$ or $2\frac{1}{2}$.

Also $2\surd 6$ divided by $5\surd 6$ gives the Quotient $\frac{2}{5}$, and $2\surd 5$ divided by $2\surd 5$ gives the Quotient 1.

So also $8\surd(3)7$ divided by $4\surd(3)7$ gives the Quotient 2; and $3\surd(4)5$ divided by $4\surd(4)5$ gives $\frac{3}{4}$ for the Quotient.

In like manner to divide $4a\surd 7$ by $2a\surd 7$, I divide $4a$ by $2a$, and there arises 2 the Quotient sought; (for $2a\surd 7$ into 2 produces $4a\surd 7$: also $3\surd b$ divided by $5\surd 6$ gives the Quotient $\frac{3}{5}$, and $2\surd b$ divided by $2\surd 5$ gives the Quotient 1.

Again, $5a\surd 3b$ divided by $3a\surd 3b$ gives the Quotient $\frac{5}{3}$.

And $7ab\surd(3)dd$ divided by $3b\surd(3)dd$ gives the Quotient $\frac{7}{3}a$.

6. When the Dividend and Divisor are the Products of two Rational Numbers or Quantities multiplied into two unequal Surd Numbers or Quantities, according to the fourth Rule of Multiplication in the preceding *Sect.* 4. (which Products are called Incommensurable Surd Roots, as hereafter will appear;) divide the Rational part of the Dividend by the Rational part of the Divisor, and the Surd part by the Surd part, then connect the Quotients so as the Rational Quotient may stand on the left hand, and this new Quantity shall be the Quotient sought.

As for Example, if $4\surd 15$ be to be divided by $2\surd 5$, first I divide 4 by 2, and there arises 2; also I divide $\surd 15$ by $\surd 5$, and there arises $\surd 3$: then those two Quotients joyned together make $2\surd 3$ (or $\surd 12$) the Quotient sought.

In like manner $4\surd 12$ divided by $3\surd 2$ gives the Quotient $\frac{4}{3}\surd 6$; for 4 divided by 3 (to wit, the Rational by the Rational) gives $\frac{4}{3}$; and $\surd 12$ divided by $\surd 2$, (to wit, the Surd by the Surd) gives $\surd 6$: then by joyning together those two Quotients there arises $\frac{4}{3}\surd 6$, or $1\frac{1}{3}\surd 6$, (or $\surd\frac{32}{3}$) for the Quotient sought.

Again, $2\surd 7$ divided by $3\surd 5$ gives the Quotient $\frac{2}{3}\surd\frac{7}{5}$; and $2\surd 3$ divided by $2\surd 5$ gives the Quotient $1\surd\frac{3}{5}$ or $\surd\frac{3}{5}$.

Likewise to divide $4\surd(3)64$ by $2\surd(3)8$, I divide 4 by 2, and it gives 2: also $\surd(3)64$ divided by $\surd(3)8$ gives $\surd(3)8$, then those two Quotients joyned together make $2\surd(3)8$, that is 4, the Quotient sought. Moreover, $5\surd(3)20$ divided by $3\surd(3)4$ gives the Quotient $\frac{5}{3}\surd 35$.

After the same manner $4a\surd fb$ divided by $2a\surd f$ gives the Quotient $2\surd b$; for $4a$ divided by $2a$ gives 2, and $\surd fb$ divided by $\surd f$ gives $\surd b$; then connecting those two Quotients there arises $2\surd b$ for the Quotient sought.

So also $6ab\surd cd$ divided by $6a\surd df$ gives the Quotient $b\surd\frac{c}{f}$.

And

And $a\sqrt{(3)cc}$ divided by $b\sqrt{(3)dd}$, gives the Quotient $\frac{a}{b}\sqrt{(3)}\frac{cc}{dd}$.

The Demonstration of the aforesaid first Rule of Division (which is the rise of all the rest) may be formed like that of Multiplication in the preceding *Sect.* 4 if there be laid as a ground-work this Analogy, *viz.* As the Divisor is to 1 (or Unity) so is the Dividend to the Quotient. But waving the Demonstration, I shall give more Examples of Division in simple Surds, both in Numbers and Quantities exprest by Letters.

More Examples to exercise Division in simple Surd Numbers.

Dividend	$\sqrt{117}$	$\sqrt{(3)}16\frac{1}{3}$ or $\sqrt{(3)}\frac{49}{3}$	$\sqrt{(4)}256$
Divisor	$\sqrt{6\frac{1}{2}}$	$\sqrt{(2)}\ 2\frac{1}{3}$ or $\sqrt{(3)}\ \frac{7}{3}$	$\sqrt{(4)}16$
Quotient	$\sqrt{18}$	$\sqrt{(3)}\ 4\frac{2}{3}$ or $\sqrt{(3)}\frac{14}{3}$	2

Dividend	$\sqrt{(12)}6125$ } that is, { $\sqrt{(12)}6125$		$\sqrt{(6)}8192$
Divisor	$\sqrt{(4)}5$ } { $\sqrt{(12)}125$		$\sqrt{(2)}8$
Quotient	. . . . . . $\sqrt{(12)}49$ or $\sqrt{(6)}7$		$\sqrt{(3)}4$

Dividend	12	$5\sqrt{8}$	$16\sqrt{(3)}25$
Divisor	$\sqrt{12}$	$\sqrt{8}$	$\sqrt{(3)}25$
Quotient	$\sqrt{12}$	5	16

Dividend	$\sqrt{245}$	$\sqrt{(3)}686$	$\sqrt{(5)}23328$
Divisor	$3\frac{1}{2}$	$3\frac{1}{2}$	6
Quotient	$\sqrt{20}$	$\sqrt{(3)}16$	$\sqrt{(5)}3$

Dividend	$20\sqrt{14}$	$\frac{2}{3}\sqrt{20}$	$5\sqrt{(3)}3$
Divisor	$2\sqrt{14}$	$\frac{2}{15}\sqrt{20}$	$2\sqrt{(2)}2$
Quotient	10	5	$\frac{5}{2}$ or $2\frac{1}{2}$

Dividend	$15\sqrt{18}$	$3\sqrt{8}$	$6\sqrt{(3)}24$
Divisor	$3\sqrt{6}$	$3\sqrt{3}$	$9\sqrt{(3)}4$
Quotient	$5\sqrt{3}$	$\sqrt{\frac{8}{3}}$	$\frac{2}{3}\sqrt{(3)}6$

More Examples to Exercise Division in simple Surd Quantities exprest by Letters.

Dividend	$\sqrt{15bc}$	$\sqrt{(3)}4bbddd$	$\sqrt{(4)}32aa$
Divisor	$\sqrt{3a}$	$\sqrt{(3)}4bb$	$\sqrt{(4)}2aa$
Quotient	$5\frac{bc}{a}$	$\sqrt{(3)}ddd$ or d	$\sqrt{(4)}16$ or 2

Dividend	$\sqrt{(6)}675aaaaabbbbb$ } that is, { $\sqrt{(6)}675a^5b^5$
Divisor	$\sqrt{(2)}3ab$ } { $\sqrt{(6)}27a^3b^3$
Quotient	. . . . $\sqrt{(6)}25aabb$ or $\sqrt{(3)}5ab$

Dividend	$\sqrt{80aaabbb}$	$9bcdd$ (or $\sqrt{81bbccd^4}$)
Divisor	$4ab$ (or $\sqrt{16aabb}$)	$\sqrt{27bcd}$
Quotient	$\sqrt{5ab}$	$\sqrt{3bcdd}$

Dividend	bc	$b\sqrt{df}$	$2d\sqrt{(3)}bb$
Divisor	$\sqrt{bc}$	$\sqrt{df}$	$\sqrt{(3)}bb$
Quotient	$\sqrt{bc}$	b	$2d$

Divi-

Dividend	$12\sqrt{dc}$	$\frac{2bc}{a}\sqrt{d}$	$ab\sqrt{(3)f}$
Divisor	$3\sqrt{dc}$	$\frac{2c}{b}\sqrt{d}$	$b\sqrt{(3)f}$
Quotient	4	$\frac{bb}{a}$	a

Dividend	$2bc\sqrt{d}$	$b\sqrt{af}$	$6aa\sqrt{(3)bbbd}$
Divisor	$c\sqrt{a}$	$c\sqrt{f}$	$2a\sqrt{(3)d}$
Quotient	$db\sqrt{\frac{d}{a}}$	$\frac{b}{c}\sqrt{a}$	$3ab$

Note, By the help of Divisors Surd Quantities may oftentimes be reduced into others more simple, which being a very useful Work I shall explain it in the next Section.

Sect. VI. *How to reduce a Surd Quantity to another more simple, when it may be done.*

When the Power of a Surd Quantity, the Radical Sign being omitted, can be divided just without any Remainder, by a Power which has a Rational Root of the same kind with that which is denoted by the said Radical Sign; then divide the Surd Quantity proposed by that Rational Root, and prefix this Root before the Quotient; so you have a new Surd Quantity equal to that proposed, and in more simple Terms.

As if $\sqrt{63}$ be proposed, because 63 may be divided by the square Number 9 without any Remainder, I divide $\sqrt{63}$ by $\sqrt{9}$, (that is, by 3) and it gives the Quotient $\sqrt{7}$, before which I set the Rational Divisor 3, and it makes $3\sqrt{7}$, (that is, 3 into the square Root of 7, or thrice the square Root of 7) which is equal to $\sqrt{63}$ first proposed; (for the Quotient $\sqrt{7}$ multiplied by the Divisor 3 makes the Dividend $\sqrt{63}$; so that instead of $\sqrt{63}$ I write $3\sqrt{7}$.

Likewise instead of $\sqrt{50}$ we may write $5\sqrt{2}$, (which signifies five times the square Root of 2;) for in regard 50 divided by the Square 25 gives 2, I divide $\sqrt{50}$ by $\sqrt{25}$, that is, by 5, and the Quotient is $\sqrt{2}$: and because every Quotient multiplied by the Divisor, produces the Dividend, therefore $5\sqrt{2}$ shall be equal to the Dividend $\sqrt{50}$.

After the same manner instead of $\frac{\sqrt{75}}{2}$, or $\frac{75}{4}$, we may write $\frac{5}{2}\sqrt{3}$; for $\frac{75}{4}$ divided by the square Number $\frac{25}{4}$ gives the Quotient 3; and consequently $\sqrt{\frac{75}{4}}$ divided by $\sqrt{\frac{25}{4}}$, that is, by $\frac{5}{2}$, gives the Quotient $\sqrt{3}$: Therefore $\frac{5}{2}\sqrt{3}$ shall be equal to $\frac{\sqrt{75}}{2}$ or $\sqrt{\frac{75}{4}}$.

Again, instead of $\sqrt{(3)40}$ we may write $2\sqrt{(3)5}$, (which signifies twice the Cubic Root of 5;) for 40 divided by the Cube 8 gives the Quotient 5; and consequently $\sqrt{(3)40}$ divided by $\sqrt{(3)8}$, that is, by 2, gives $\sqrt{(3)5}$; Therefore $2\sqrt{(3)5}$ shall be equal to $\sqrt{(3)40}$.

Likewise for $\sqrt{(3)\frac{54}{8}}$, (or $\frac{\sqrt{(3)54)}}{2}$ we may write $\frac{3}{2}\sqrt{(3)2}$; for $\frac{54}{8}$ divided by the Cube $\frac{27}{8}$ gives 2; and consequently $\sqrt{(3)\frac{54}{8}}$ divided by $\sqrt{(3)\frac{27}{8}}$, that is, by $\frac{3}{2}$, will give $\sqrt{(3)2}$: Wherefore $\frac{3}{2}\sqrt{(3)2}$ shall be equal to $\sqrt{(3)\frac{54}{8}}$.

The like Operation is to be done in reducing Surd Quantities exprest by Letters to others more simple: as if $\sqrt{75aa}$ be proposed, forasmuch as $75aa$ divided by the Square $25aa$ gives the Quotient 3, and consequently $\sqrt{75aa}$ divided by $\sqrt{25aa}$, that is, by $5a$, will give $\sqrt{3}$; therefore the Divisor $5a$ multiplied into the Quotient $\sqrt{3}$, produces $5a\sqrt{3}$, equal to the Dividend $\sqrt{75aa}$, and therefore instead of $\sqrt{75aa}$, we may write $5a\sqrt{3}$.

After the same manner $\sqrt{10aabb}$ may be reduced to $ab\sqrt{10}$, also $\sqrt{5aa}$ to $a\sqrt{5}$, and $\sqrt{(3)4ddd}$ to $d\sqrt{(3)4}$.

Again, forasmuch as $aaab+aabb$ may be divided by the Square aa, and there arises $ab+bb$, and consequently $\sqrt{:\overline{aaab+aabb}:}$ divided by $\sqrt{aa}$, that is, by a, gives the Quotient $\sqrt{:\overline{ab+bb}:}$ therefore a into $\sqrt{:\overline{ab+bb}:}$ shall be equal to $\sqrt{:\overline{aaab+aabb}:}$ So that instead of $\sqrt{:\overline{aaab+aabb}:}$ we may write a into $\sqrt{:\overline{ab+bb}:}$ or $a\sqrt{:\overline{ab+bb}:}$

Like-

Likewise for $\sqrt{:aabbc+2afbbc+ffbbc}$: we may write $\overline{a+f}$ into $\sqrt{bbc}$, or $\overline{a+f}\sqrt{bbc}$; for $aabbc+2afbbc+ffbbc$ divided by the Square $aa+2af+ff$ gives bbc, and consequently $\sqrt{:aabbc+2afbbc+ffbbc}$: divided by $\sqrt{:aa+2af+ff}$: that is, by $a+f$, gives the Quotient $\sqrt{bbc}$. Therefore $\overline{a+f}\sqrt{bbc}$ imports as much as $\sqrt{:aabbc+2afbbc+gfbbc}$:

After the same manner instead of $\sqrt{(3)}\frac{27aaaabbb}{8b-8a}$ we may write $\frac{3ab}{2}$ into $\sqrt{(3)}\frac{a}{b-a}$ or $\frac{3ab}{2}\sqrt{(3)}\frac{a}{b-a}$; for since the Power of the Surd proposed is produced by the Multiplication of $\frac{a}{b-a}$ into the Cube $\frac{27aaabbb}{8}$, whose Cubic Root is $\frac{3ab}{2}$, and consequently $\sqrt{(3)}\frac{27aaaabbb}{8b-8a}$ divided by $\sqrt{(3)}\frac{27aaabbb}{8}$, that is, by $\frac{3ab}{2}$, gives the Quotient $\sqrt{(3)}\frac{a}{b-a}$. Therefore $\frac{3ab}{2}\sqrt{(3)}\frac{a}{b-a}$ shall be equal to $\sqrt{(3)}\frac{27aaaabbb}{8b-8a}$.

So also for $\sqrt{:\frac{aaoomm+4aammmp}{ppzz}}$: we may write $\frac{am}{pz}\sqrt{:oo+4mp}$: for if the Power of the Surd proposed be divided by the Square $\frac{aamm}{ppzz}$ the Quotient will be $oo+4mp$; and consequently if the Surd proposed be divided by $\sqrt{:\frac{aamm}{ppzz}}$: that is by $\frac{am}{pz}$, the Quotient will be $\sqrt{:oo+4mp}$: therefore the Divisor $\frac{am}{pz}$ multiplied into the Quotient $\sqrt{:oo+4mp}$: viz. $\frac{am}{pz}\sqrt{:oo+4mp}$: denotes as much as $\sqrt{:\frac{aaoomm+4aammmp}{ppzz}}$ the Surd proposed.

Likewise for $\sqrt{:\frac{oozz+4mpzz}{aa}}$: we may write $\frac{z}{a}\sqrt{:oo+4mp}$:

But when a Square or Cube, &c. by which the Division necessary to such Contraction is to be performed, cannot be readily discerned, first, (by the Rules of the preceding eighth Chapter) search out all the Divisors of the Power of the Surd Quantity proposed, and then see whether any of them be a Square or Cube, &c. to wit, such a Power as the Radical Sign denotes, which if you find you may use in the aforesaid manner to free the Surd Quantity in part from the Radical Sign.

As if $\sqrt{288}$ be proposed, because among the Divisors of 288 there are found the Square Numbers 4, 9, 16, 36, and 144, which dividing 288 will give the Quotients 72, 32, 18, 8 and 2; instead of $\sqrt{288}$ we may write $2\sqrt{72}$, or $3\sqrt{32}$, or $4\sqrt{18}$, or $6\sqrt{8}$, or lastly $12\sqrt{2}$.

In like manner if $\sqrt{:aaab+aabb}$: be proposed, because among the Divisors of the Quantity $aaab+aabb$, there is found the Square aa, the said $\sqrt{:aaab+aabb}$: may be reduced to $a\sqrt{:aa+bb}$: as before.

Again, for as much as $a^3b-aabb+2aabc+abcc-ab^3+bbcc-2b^3c+b^4$ is produced by the Multiplication of $ab+bb$ into the Square $aa+2ac+cc-2ab-2bc+bb$, whose Root is $a+c-b$; we may instead of $\sqrt{:a^3b-aabb+2aabc+abcc-ab^3+bbcc-2b^3c+b^4}$: write $\overline{a+c-b}$ into $\sqrt{:ab+bb}$: or $\overline{a+c-b}\sqrt{:ab+bb}$:

Likewise, because among the Divisors of $1200aabb$ there are found the Squares $4aabb$, $16aabb$, $25aabb$, $100aabb$, and $400aabb$; which dividing the said $1200aabb$ will give the Quotients 300, 75, 48, 12, and 3, we may for $\sqrt{1200aabb}$ write $2ab\sqrt{300}$, or $4ab\sqrt{75}$, or $5ab\sqrt{48}$, or $10ab\sqrt{12}$, or lastly $20ab\sqrt{3}$.

Sect. VII. *Two Surd Roots being given, to find whether they be Commensurable or Incommensurable.*

Commensurable Surd Roots are such whose Reason or Proportion to one another may be exprest by Rational Numbers or Quantities; and those Surd Roots whose Proportion cannot be exprest by Rational Numbers or Quantities, are called Incommensurable.

The

The Rule to try whether two Surd Roots of the ſame kind, (that is, ſuch as have a common Radical Sign) be Commenſurable or not, is this that follows, *viz.*

Divide the given Roots ſeverally by their greateſt common Diviſor, then if the Quotients be Rational Numbers or Quantities, the Roots propoſed are Commenſurable; but if the Quotient be Irrational or Surd, the given Roots are Incommenſurable.

As for Example, to try whether $\sqrt{12}$ and $\sqrt{3}$ be Commenſurable or not, I divide them ſeverally by their greateſt common Diviſor $\sqrt{3}$, and find the Quotient $\sqrt{4}$ and $\sqrt{1}$, that is, 2 and 1 to be Rational Numbers; whence I conclude that $\sqrt{12}$, that is $2\sqrt{3}$, has ſuch proportion to $\sqrt{3}$, that is $1\sqrt{3}$, as 2 to 1, *viz.* as a Rational Number to a Rational Number; and conſequently $\sqrt{12}$ and $\sqrt{3}$ (according to the Definition above given) are Commenſurable. But that $\sqrt{12}$ is to $\sqrt{3}$ as 2 to 1, may be demonſtrated thus, *viz.* It is evident (by reaſon of the common Factor $\sqrt{3}$) that $2\sqrt{3} \,.\, 1\sqrt{3} :: 2 \,.\, 1$, and (by Diviſion as above) $\sqrt{12}=2\sqrt{3}$, and $\sqrt{3}=1\sqrt{3}$; therefore $\sqrt{12} \,.\, \sqrt{3} :: 2 \,.\, 1$. Otherwiſe thus:

Foraſmuch as 12 and 3 divided ſeverally by their common Diviſor 3 give the Quotients 4 and 1; therefore as . . .	$12 \,.\, 3 :: 4 \,.\, 1$
Wherefore the ſquare Roots of thoſe Proportionals ſhall be Proportionals alſo, (*per 22 Prop. 6. Elem. Euclid.*) *viz.*	$\sqrt{12} \,.\, \sqrt{3} :: 2 \,.\, 1$

Which was to be demonſtrated.

After the ſame manner $\sqrt{18}$ and $\sqrt{8}$ will be found Commenſurable; for the former is to the latter as 3 to 2, to wit, as a Rational Number to a Rational Number: for if $\sqrt{18}$ and $\sqrt{8}$ be ſeverally divided by their greateſt common Diviſor $\sqrt{2}$, the Quotients will be $\sqrt{9}$ and $\sqrt{4}$, that is 3 and 2. Therefore $\sqrt{18}$ is to $\sqrt{8}$ as 3 to 2, and inſtead of $\sqrt{18}$ and $\sqrt{8}$ we may write $3\sqrt{2}$ and $2\sqrt{2}$, to wit, the Products of the Rational Quantities 3 and 2, multiplied into the common Diviſor $\sqrt{3}$.

Again, $\sqrt{48}$ and $\sqrt{75}$ (that is, $4\sqrt{3}$ and $5\sqrt{3}$) are Commenſurable; for the former is to the latter as 4 to 5, to wit, as a Rational Number to a Rational Number: for $\sqrt{48}$ and $\sqrt{75}$ being ſeverally divided by their greateſt common Diviſor $\sqrt{3}$, give the Quotients $\sqrt{16}$ and $\sqrt{25}$, to wit, 4 and 5. Therefore $\sqrt{48} \,.\, \sqrt{75} :: 4 \,.\, 5 :: 4\sqrt{3} \,.\, 5\sqrt{3}$.

Moreover, $\sqrt{(3)}320$ and $\sqrt{(3)}135$ (that is, $4\sqrt{(3)}5$ and $3\sqrt{(3)}5$) having ſuch proportion one to the other as 4 to 3 are Commenſurable; for $\sqrt{(3)}320$ and $\sqrt{(3)}135$ being ſeverally divided by their greateſt common Diviſor $\sqrt{(3)}5$, will give the Quotient $\sqrt{(3)}64$ and $\sqrt{(3)}27$, to wit, 4 and 3. Therefore $\sqrt{(3)}320 \,.\, \sqrt{(3)}135 :: 4 \,.\, 3 :: 4\sqrt{(3)}5 \,.\, 3\sqrt{(3)}5$.

So alſo $\sqrt{(4)}3888$ and $\sqrt{(4)}243$ (that is, $2\sqrt{(4)}243$ and $1\sqrt{(4)}243$) are Commenſurable, the former having ſuch proportion to the latter as 2 to 1; for if they be ſeverally divided by their greateſt Common Diviſor $\sqrt{(4)}243$, the Quotients will be $\sqrt{(4)}16$ and $\sqrt{(4)}1$, to wit, 2 and 1. Therefore $\sqrt{(4)}3888 \,.\, \sqrt{(4)}243 :: 2 \,.\, 1 :: 2\sqrt{(4)}243 \,.\, 1\sqrt{(4)}243$.

If two Surd Fractions, or mix'd Numbers ſtanding Fraction-wiſe, be propoſed, and have not a common Denominator, reduce them to their ſmalleſt common Denominator, and then try (in like manner as before) whether the new Surd Numerators be Commenſurable or not; for if theſe be Commenſurable, the Surd Fractions firſt propoſed ſhall be alſo Commenſurable. As if $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{24}{25}}$ be propoſed, I reduce them to $\sqrt{\frac{50}{75}}$ and $\sqrt{\frac{72}{75}}$; then I divide the new Numerators only, to wit, $\sqrt{50}$ and $\sqrt{72}$, by their greateſt Common Diviſor $\sqrt{2}$, and the Quotients $\sqrt{25}$ and $\sqrt{36}$, that is, 5 and 6 are Rational Numbers. Therefore $\sqrt{\frac{2}{3}}$ and $\sqrt{\frac{24}{25}}$ firſt propoſed are Commenſurable, and the former has ſuch proportion to the latter as 5 to 6. For,

$$\text{As} \quad \tfrac{50}{75} \,.\, \tfrac{72}{75} :: 50 \,.\, 72 :: 25 \,.\, 36$$
$$\text{Therefore} \quad \sqrt{\tfrac{50}{75}} \,.\, \sqrt{\tfrac{72}{75}} :: \sqrt{50} \,.\, \sqrt{72} :: 5 \,.\, 6$$
$$\text{And becauſe} \quad \sqrt{\tfrac{2}{3}} = \sqrt{\tfrac{50}{75}} \quad \text{and} \quad \sqrt{\tfrac{24}{25}} = \sqrt{\tfrac{72}{75}}$$
$$\text{Therefore} \quad \sqrt{\tfrac{2}{3}} \,.\, \sqrt{\tfrac{24}{25}} :: 5 \,.\, 6$$

But if either the Numerators or Denominators of two Surd Fractions or mix'd Numbers ſtanding Fraction-wiſe, (the Radical Sign being neglected) be Squares or Cubes, &c, *viz.* Powers of that kind which is denoted by the Radical Sign, then you need not reduce the Surd Fractions to a common Denominator, but try whether their Numerators or Denominators be Commenſurable or not; for if theſe be Commenſurable, the

Surd Fractions proposed shall be also Commensurable. As if $\sqrt{\frac{50}{16}}$ and $\sqrt{\frac{72}{25}}$ be proposed, because the Denominators (the Radical Sign being neglected) are Squares, (to wit, Powers of that kind which the Radical Sign denotes) and the Numerators $\sqrt{50}$ and $\sqrt{72}$ are Commensurable; (for if these be divided by their common Divisor $\sqrt{2}$, the Quotients are rational, to wit 5 and 6.) Therefore the Surd Fractions proposed are also Commensurable, and have such proportion as $\frac{5}{4}$ to $\frac{6}{5}$, (whose Denominators 4 and 5, to wit, $\sqrt{16}$ and $\sqrt{25}$, are the given Denominators) or as 25 to 24; and (according to the preceding *Sect.* 6.) the Surd Fractions proposed may be express'd thus, $\frac{5}{4}\sqrt{2}$ and $\frac{6}{5}\sqrt{2}$.

When two Surd Roots proposed be of different kinds, they must first of all be reduced to a common Radical Sign, (by the preceding *Sect.* 3. of this *Chap.*) before the Rules aforesaid be used, to try whether they be Commensurable or not. As if $\sqrt{(6)}64$ and $\sqrt{(3)}27$ be given, they may be reduced to $\sqrt{(6)}64$ and $\sqrt{(6)}729$, which divided by their greatest common Divisor $\sqrt{(6)}1$, the Quotient will be the same with the Dividends. Now if $\sqrt{(6)}64$ and $\sqrt{(6)}729$ be Rational, then the Surds first given are Commensurable; but $\sqrt{(6)}64$ is 2, and $\sqrt{(6)}729$ is 3. Therefore the Surd Roots proposed are Commensurable, and have such proportion as 2 to 3.

But if the Quotients arising by the Division of two Surd Roots by their greatest common Divisor as aforesaid, happen to be Irrational or Surd, then the Roots proposed are Incommensurable; such are $\sqrt{48}$ and $\sqrt{8}$, for if they be divided severally by their greatest common Divisor $\sqrt{8}$, the Quotients are $\sqrt{6}$ and 1: but $\sqrt{6}$ is Irrational, therefore the proportion which $\sqrt{48}$ has to $\sqrt{8}$ is not as a Rational Number to a Rational Number, and consequently $\sqrt{48}$ and $\sqrt{8}$ are Incommensurable, and so are all other Surd Roots whose proportion cannot be express'd by Rational Numbers.

I shall now shew how by the help of the preceding Rules we may discover whether two Surd Quantities express'd by Letters be Commensurable or not. As if $\sqrt{27aa}$ and $\sqrt{12aa}$ be proposed, they will be found Commensurable; for if they be severally divided by their greatest common Divisor $\sqrt{3aa}$, the Quotients $\sqrt{9}$ and $\sqrt{4}$, that is 3 and 2, are Rational Numbers, and shew that $\sqrt{27aa}$ is to $\sqrt{12aa}$ as 3 to 2, to wit, as a Rational Number to a Rational Number; wherefore $\sqrt{27aa}$ and $\sqrt{12aa}$ are Commensurable, and may be express'd thus, $3\sqrt{3aa}$ and $2\sqrt{3aa}$.

Note, If two Surd Quantities be divided by some common Divisor, though it be not the greatest, yet if there come forth Rational Quotients, we may thence conclude those Surd Quantities to be Commensurable, and oftentimes express them various ways. As if $\sqrt{27aa}$ and $\sqrt{12aa}$ be again proposed, by dividing them severally by their common Divisor $\sqrt{3}$, there will come forth the Quotients $\sqrt{9aa}$ and $\sqrt{4aa}$, that is, $3a$ and $2a$; whence it is evident, that $\sqrt{27aa}$ is to $\sqrt{12aa}$ as $3a$ to $2a$, to wit, as a Rational Quantity to a Rational Quantity, and consequently $\sqrt{27aa}$ and $\sqrt{12aa}$ are Commensurable. Moreover, according to this latter Division we may write $3a\sqrt{3}$ for $\sqrt{27aa}$, and $2a\sqrt{3}$ for $\sqrt{12aa}$.

Again, $\sqrt{:\overline{aaaa+aabb}:}$ and $\sqrt{:\overline{aabb+bbbb}:}$ are Commensurable; for each of them being divided by $\sqrt{:\overline{aa+bb}:}$ there arise $\sqrt{aa}$ and $\sqrt{bb}$, that is a and b, which are Rational Quantities, each of which being multiplied into the common Divisor $\sqrt{:\overline{aa+bb}:}$ will give, instead of the Surds proposed, $a\sqrt{\overline{aa+bb}}$ and $b\sqrt{\overline{aa+bb}}$, which have the same proportion to one another as there is between a and b.

Likewise $\sqrt{\frac{oozz+4mpzz}{aa}}$ and $\sqrt{\frac{aaoomm+4aammmp}{ppzz}}$ are Commensurable, for each of them being divided by their common Divisor $\sqrt{:\overline{oo+4mp}:}$ there will arise $\sqrt{\frac{zz}{aa}}$ and $\sqrt{\frac{aamm}{ppzz}}$ that is, $\frac{z}{a}$ and $\frac{am}{pz}$, (to wit, Rational Quantities) each of which multiplied into the common Divisor $\sqrt{:\overline{oo+4mp}:}$ will produce $\frac{z}{a}\sqrt{:\overline{oo+4mp}:}$ and $\frac{am}{pz}\sqrt{:\overline{oo+4mp}:}$ which are equal to, but more simply express'd than the Surd Quantites proposed, and have that proportion one to another as is between $\frac{z}{a}$ and $\frac{am}{pz}$.

So also $\sqrt{:\overline{aaaa+6aaa+21aa+72a+108}:}$ and $\sqrt{:\overline{aaaa-10aaa+37aa-120a+300}:}$ are Commensurable, for if they be severally divided by their common Divisor $\sqrt{:\overline{aa+12}:}$ there will arise $\sqrt{:\overline{aa+6a+9}:}$ and $\sqrt{:\overline{aa+10a+25}:}$ that is, $a+3$ and $a \backsim 5$, each of which

which multiplied into the common Divisor $\sqrt{aa+12}$ will produce $\overline{a+3\sqrt{:aa+12:}}$ and $\overline{a \backsim 5\sqrt{:aa+12:}}$ which have the same proportion between themselves, as that of $a+3$ to $a \backsim 5$, and are of the same value with the Surd Quantities first proposed.

Again, $\sqrt[3]{81abbb}$ and $\sqrt[3]{24abbb}$ are Commensurable; for if each of them be divided by their common Divisor $\sqrt[3]{3a}$, there will arise $\sqrt[3]{27bbb}$ and $\sqrt[3]{8bbb}$, that is, $3b$ and $2b$; therefore the Surds proposed may be reduced to $3b\sqrt[3]{3a}$ and $2b\sqrt[3]{3a}$, the former of which is to be the latter as $3b$ to $2b$: and so of others.

Sect. VIII. *Addition and Subtraction in simple Surd Quantities.*

When two or more equal Surd Roots are to be added together, multiply one of them by the Number which expresses the Multitude of the Roots proposed, and the Product shall be their sum: as the sum of $\sqrt{6}$ and $\sqrt{6}$ is $\sqrt{24}$; for $\sqrt{6}$ multiplied by 2, that is by $\sqrt{4}$, produces $\sqrt{24}$. Also $\sqrt[3]{6}$, $\sqrt[3]{6}$, and $\sqrt[3]{6}$, added into one make $\sqrt[3]{162}$; for $\sqrt[3]{6}$ multiplied by 3, that is, by $\sqrt[3]{27}$, makes $\sqrt[3]{162}$.

But when two unequal Surd Roots of the same kind, that is, such as have the same Radical Sign prefix'd before each of them, be to be added together; also when the lesser is to be subtracted from the greater, observe this Rule, *viz.* First, (by the preceding *Sect.* 7. of this *Chap.*) you must try whether they be Commensurable or not; then if they be Commensurable, that is, if after they have been severally divided by their greatest common Divisor, the Quotients be Rational Quantities, multiply the sum of those Rational Quantities by the said common Divisor, and the Product shall be the sum of the Surd Roots proposed; but if the Difference of those Rational Quotients be multiplied by the said common Divisor, the Product shall be the Difference of the Roots proposed.

As for Example, if the Sum and Difference of $\sqrt{50}$, and $\sqrt{8}$ be desired, first, I divide each of them by their greatest common Divisor $\sqrt{2}$, and the Quotients are $\sqrt{25}$ and $\sqrt{4}$, that is 5 and 2, (which are Rational Numbers expressing the proportion of the given Roots one to the other;) whose sum 7 multiplied by the common Divisor $\sqrt{2}$ produces $7\sqrt{2}$, or if you please $\sqrt{98}$, (for 7, to wit, $\sqrt{49}$ into $\sqrt{2}$, makes $\sqrt{98}$;) which is the desired sum of the given Roots $\sqrt{50}$ and $\sqrt{8}$. And if $5-2$, that is 3, (the Difference of the Rational Quotients before found) be multiplied by the said common Divisor $\sqrt{2}$, the Product will be $3\sqrt{2}$, that is $\sqrt{18}$; which is the desired Difference of $\sqrt{50}$ and $\sqrt{8}$, the Roots first proposed.

Likewise the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$ will be found $8\sqrt[3]{4}$, that is, $\sqrt[3]{2048}$; and their Difference $2\sqrt[3]{4}$, that is $\sqrt[3]{32}$, as will appear by the following Work, *viz.* first, I divide each of the given Roots $\sqrt[3]{500}$ and $\sqrt[3]{108}$ by their greatest common Divisor $\sqrt[3]{4}$, and the Quotients are $\sqrt[3]{125}$ and $\sqrt[3]{27}$, that is 5 and 3; then by multiplying 8 (to wit $5+3$, the sum of the Rational Quotients) by the common Divisor $\sqrt[3]{4}$, the Product $8\sqrt[3]{4}$, that is, $\sqrt[3]{2048}$; (for 8, to wit, $\sqrt[3]{512}$ into $\sqrt[3]{4}$ makes $\sqrt[3]{2048}$) which is the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$, the Roots proposed.

And by multiplying 2, (that is, $5-3$ the Difference of the Rational Quotients) by the said common Divisor $\sqrt[3]{4}$, the Product is $2\sqrt[3]{4}$, that is, $\sqrt[3]{32}$; (for 2, to wit, $\sqrt[3]{8}$ into $\sqrt[3]{4}$ makes $\sqrt[3]{32}$) which is the Difference of $\sqrt[3]{500}$ and $\sqrt[3]{108}$, the Roots proposed.

Here follow Contractions of the Work in the two last preceding Examples, with others of like nature, to illustrate the Rule before given for the Addition and Subtraction of such simple Surd Roots as are Commensurable.

Example 1.

What is the Sum and Difference of . . . . $\sqrt{50}$ and $\sqrt{8}$?

The Operation.

$\sqrt{2})\ \sqrt{50}\ (\sqrt{25}$, that is, 5. Therefore $5\sqrt{2}=\sqrt{50}$.

$\sqrt{2})\ \sqrt{8}\ (\sqrt{4}$, that is, 2. Therefore $2\sqrt{2}=\sqrt{8}$.

The Sum, $7\sqrt{2}=\sqrt{50}+\sqrt{8}$.

Or, $\sqrt{98}=\sqrt{50}+\sqrt{8}$.

The Difference, $3\sqrt{2}=\sqrt{50}-\sqrt{8}$.

Or, $\sqrt{18}=\sqrt{50}-\sqrt{8}$.

Example 2.

What is the Sum and Difference of . . . . . . $\surd(3)500$ and $\surd(3)108$?

The Operation.

I. $\surd(3)4)\ \surd(3)500\ (\surd(3)125$, that is, 5.
II. $\surd(3)4)\ \surd(3)108\ (\surd(3)\ 27$, that is, 3.

From Division I.	$5\surd(3)4=\surd(3)500.$
From Division II.	$3\surd(3)4=\surd(3)108.$
The Sum,	$8\surd(3)4=\surd(3)500+\surd(3)108.$
Or,	$\surd(3)2048=\surd(3)500+\surd(3)108.$
The Difference,	$2\surd(3)4=\surd(3)500-\surd(3)108.$
Or,	$\surd(3)32=\surd(3)500-\surd(3)108.$

Example 3.

What is the Sum and Difference of . . . . . . $\surd 147$ and $\surd 12$?

The Operation.

$\surd 3)\ \surd 147\ (\surd 49$, that is, 7. Therefore $7\surd 3=\surd 147.$
$\surd 3)\ \surd\ 12\ (\surd 4$, that is, 2. Therefore $2\surd 3=\surd\ 12.$

The Sum,	$9\surd 3=\surd 147+\surd 12.$
Or,	$\surd 243=\surd 147+\surd 12.$
The Difference,	$5\surd 3=\surd 147-\surd 12.$
Or,	$\surd 75=\surd 147-\surd 12.$

Example 4.

What is the Sum and Difference of . . . . . . $\surd(3)1715$ and $\surd(3)40$?

The Operation.

I. $\surd(3)5)\ \surd(3)1715\ (\surd(3)343$, that is, 7.
II. $\surd(3)5)\ \surd(3)\ 40\ (\surd(3)\ 8$, that is, 2.

From Division I.	$7\surd(3)5=\surd(3)1715.$
From Division II.	$2\surd(3)5=\surd(3)\ 40.$
The Sum,	$9\surd(3)5=\surd(3)1715+\surd(3)40.$
Or,	$\surd(3)3645=\surd(3)1715+\surd(3)40.$
The Difference,	$5\surd(3)5=\surd(3)1715-\surd(3)40.$
Or,	$\surd(3)625=\surd(3)1715-\surd(3)40.$

Note, When two Commensurable Surd Roots proposed to be added or subtracted are Fractions, or mix'd Numbers reduced into the form of Fractions, if they have not a common Denominator, reduce them into others which may have a common Denominator in the least Terms, then to find out the Rational Quotients divide only the two new Numerators severally by their greatest Common Divisor, and continue the Process as before. The Practice of this Note will be evident in the two following Examples.

Example 5.

What is the Sum and Difference of . . . . . $\left\{\begin{array}{l}\surd\frac{24}{25} \text{ and } \surd\frac{2}{3}.\\ \text{Or, } \surd\frac{72}{75} \text{ and } \surd\frac{50}{75}.\end{array}\right.$

The Operation.

$\surd\frac{2}{75})\ \surd\frac{72}{75}\ (\surd 36$, that is, 6. Therefore $6\surd\frac{2}{75}=\surd\frac{72}{75}.$
$\surd\frac{2}{75})\ \surd\frac{50}{75}\ (\surd 25$, that is, 5. Therefore $5\surd\frac{2}{75}=\surd\frac{50}{75}.$

The Sum,	$11\surd\frac{2}{75}=\surd\frac{72}{75}+\surd\frac{50}{75}.$
Or,	$\surd\frac{242}{75}=\surd\frac{72}{75}+\surd\frac{50}{75}.$
The Difference	$\surd\frac{2}{75}=\surd\frac{72}{75}-\surd\frac{50}{75}.$

Example 6.

What is the Sum and Difference of . . . . $\left\{ \begin{array}{l} \sqrt{12} \text{ and } \sqrt{\frac{27}{4}}\,? \\ \text{Or, } \sqrt{\frac{48}{4}} \text{ and } \sqrt{\frac{27}{4}}\,? \end{array} \right.$

The Operation.

$\sqrt{\frac{3}{4}})\ \sqrt{\frac{48}{4}}\ (\sqrt{16}$, that is, 4. Therefore $4\sqrt{\frac{3}{4}}=\sqrt{\frac{48}{4}}$.

$\sqrt{\frac{3}{4}})\ \sqrt{\frac{27}{4}}\ (\sqrt{9}$, that is, 3. Therefore $3\sqrt{\frac{3}{4}}=\sqrt{\frac{27}{4}}$.

The Sum, $7\sqrt{\frac{3}{4}}=\sqrt{\frac{48}{4}}+\sqrt{\frac{27}{4}}$.

Or, $\sqrt{\frac{147}{4}}=\sqrt{\frac{48}{4}}+\sqrt{\frac{27}{4}}$.

The Difference, $\sqrt{\frac{3}{4}}=\sqrt{\frac{48}{4}}-\sqrt{\frac{27}{4}}$.

When two ſimple Surd Roots given to be added or ſubtracted be Incommenſurable, neither their Sum nor their Difference can be expreſs'd by any ſimple Root, but they are to be added by $+$, and to be ſubtracted by $-$. As to add $\sqrt{5}$ and $\sqrt{3}$, I write $\sqrt{5}+\sqrt{3}$ for the Sum; but to ſubtract $\sqrt{3}$ from $\sqrt{5}$, I write $\sqrt{5}-\sqrt{3}$ for the Remainder. So alſo the Sum of $\sqrt{(3)}40$ and $\sqrt{(3)}12$ is $\sqrt{(3)}40+\sqrt{(3)}12$, and their Difference is $\sqrt{(3)}40-\sqrt{(3)}12$.

But Incommenſurable ſquare Roots may be added or ſubtracted by this following Rule, (which is deduced from *Prop.* 4. & 7. *lib.* 2. *Euclid.*)

To the Sum of the Squares of the given Surd ſquare Roots, add the double Product of the Multiplication of thoſe Roots one into another; ſo ſhall the ſquare Root of the Sum be the Sum of the Roots propoſed to be added. But if the ſaid double Product be ſubtracted from the ſaid ſum of the Squares, the ſquare Root of the Remainder ſhall be the Difference of the given Surd ſquare Roots. As if the Sum and Difference of $\sqrt{6}$ and $\sqrt{3}$ be deſired, their Sums ſhall be $\sqrt{:9+\sqrt{72}:}$ and their Difference $\sqrt{:9-\sqrt{72}:}$ for the Sum of the Squares of the given ſquare Roots $\sqrt{6}$ and $\sqrt{3}$ is 9, and the double Product of their Multiplication is $\sqrt{72}$, which I add to and ſubtract from 9; ſo the ſquare Root of the ſum, to wit, $\sqrt{:9+\sqrt{72}:}$ is the Sum deſired; and the ſquare Root of the Remainder, to wit, $\sqrt{:9-\sqrt{72}:}$ is the Difference.

After the ſame manner the Addition and Subtraction of ſimple Surd Quantities expreſs'd by Letters may be performed; as to add $\sqrt{75aa}$ and $\sqrt{27aa}$, firſt, (by the preceding *Sect.* 7.) I find them to be Commenſurable; for if $\sqrt{75aa}$ and $\sqrt{27aa}$ be ſeverally divided by their greateſt common Diviſor $\sqrt{3aa}$, the Quotients are $\sqrt{25}$ and $\sqrt{9}$, that is, 5 and 3, whoſe ſum 8 multiplied into the common Diviſor $\sqrt{3aa}$ makes $8\sqrt{3aa}$, (that is, $\sqrt{192aa}$) for the ſum of $\sqrt{75aa}$ and $\sqrt{27aa}$. But if the Difference of the ſame Rational Quotients 5 and 3, to wit 2, be multiplied into the ſaid common Diviſor $\sqrt{3aa}$, it makes $2\sqrt{3aa}$, (that is, $\sqrt{12aa}$) for the Difference of $\sqrt{75aa}$ and $\sqrt{27aa}$, the Roots firſt propoſed.

Or we may write $8a\sqrt{3}$ (inſtead of $8\sqrt{3aa}$) for the Sum, and $2a\sqrt{3}$ inſtead of $2\sqrt{3aa}$) for the Difference of $\sqrt{75aa}$ and $\sqrt{27aa}$ before propoſed; for theſe divided ſeverally by their common Diviſor $\sqrt{3}$, give Rational Quotients, to wit $\sqrt{25aa}$ and $\sqrt{9aa}$, that is, $5a$ and $3a$; whoſe Sum $8a$ multiplied into the common Diviſor $\sqrt{3}$, gives $8a\sqrt{3}$ for the Sum of $\sqrt{75aa}$ and $\sqrt{27aa}$; but if the Difference of the ſaid Rational Quotients $5a$ and $3a$, to wit, $2a$, be multiplied into the ſaid common Diviſor $\sqrt{3}$, the Product $2a\sqrt{3}$ is the Difference of the ſaid $\sqrt{75aa}$ and $\sqrt{27aa}$.

Again, to add $\sqrt{(3)}256aaa$ and $\sqrt{(3)}32aaa$, firſt, (by *Sect.* 7.) I find them to be Commenſurable, for if each of them be divided by their common Diviſor $\sqrt{(3)}4$, the Quotients are Rational, to wit, $\sqrt{(3)}64aaa$ and $\sqrt{(3)}8aaa$, that is, $4a$ and $2a$; theſe added together make $6a$, which multiplied into the common Diviſor $\sqrt{(3)}4$, makes $6a\sqrt{(3)}4$ (that is, $\sqrt{(3)}864aaa$) for the deſired Sum of $\sqrt{(3)}256aaa$ and $\sqrt{(3)}32aaa$; but if $2a$, the Difference of the ſame Rational Quotients $4a$ and $2a$, be multiplied into the ſaid common Diviſor $\sqrt{(3)}4$, the Product $2a\sqrt{(3)}4$, (that is, $\sqrt{(3)}32aaa$) ſhall be the Difference of $\sqrt{(3)}256aaa$ and $\sqrt{(3)}32aaa$ firſt propoſed.

More

More Examples of the Addition and Subtraction of Commensurable simple Surd Quantities express'd by Letters.

Example 1.

What is the Sum and Difference of . . . . $\sqrt{28aa}$ and $\sqrt{7aa}$?

The Operation.

I. $\sqrt{7}$) $\sqrt{28aa}$ ($\sqrt{4aa}$, that is, $2a$.
II. $\sqrt{7}$) $\sqrt{7aa}$ ($\sqrt{aa}$, that is, a.

From Division I.	$2a\sqrt{7}=\sqrt{28aa}$.
From Division II.	$a\sqrt{7}=\sqrt{7aa}$.
The Sum,	$3a\sqrt{7}=\sqrt{28aa}+\sqrt{7aa}$.
The Difference,	$a\sqrt{7}=\sqrt{28aa}-\sqrt{7aa}$.

Example 2.

What is the Sum and Difference of . . . . $\sqrt{45aabc}$ and $\sqrt{20aabc}$?

The Operation.

I. $\sqrt{5bc}$) $\sqrt{45aabc}$ ($\sqrt{9aa}$, that is, $3a$.
II. $\sqrt{5bc}$) $\sqrt{20aabc}$ ($\sqrt{4aa}$, that is, $2a$.

From Division I.	$3a\sqrt{5bc}=\sqrt{45aabc}$.
From Division II.	$2a\sqrt{5bc}=\sqrt{20aabc}$.
The Sum,	$5a\sqrt{5bc}=\sqrt{45aabc}+\sqrt{20aabc}$.
The Difference,	$a\sqrt{5bc}=\sqrt{45aabc}-\sqrt{20aabc}$.

Example 3.

What is the Sum and Difference of . . . . . $\sqrt{(3)}81abbb$ and $\sqrt{(3)}24abbb$?

The Operation.

I. $\sqrt{(3)}3a$) $\sqrt{(3)}81abbb$ ($\sqrt{(3)}27bbb$, that is, $3b$.
II. $\sqrt{(3)}2a$) $\sqrt{(3)}24abbb$ ($\sqrt{(3)}8bbb$, that is, $2b$.

From Division I.	$3b\sqrt{(3)}3a=\sqrt{(3)}81abbb$.
From Division II.	$2b\sqrt{(3)}3a=\sqrt{(3)}24abbb$.
The Sum,	$5b\sqrt{(3)}3a=\sqrt{(3)}81abbb+\sqrt{(3)}24abbb$.
The Difference,	$b\sqrt{(3)}3a=\sqrt{(3)}81abbb-\sqrt{(3)}24abbb$.

Example 4.

What is the Sum and Difference of . . . . $\left\{\begin{array}{l}\sqrt{\frac{15}{4}}aad \text{ and } \sqrt{\frac{125}{48}}aad,\\ \text{Or, } \sqrt{\frac{180}{48}}aad \text{ and } \sqrt{\frac{125}{48}}aad?\end{array}\right.$

The Operation.

I. $\sqrt{\frac{5}{48}d}$) $\sqrt{\frac{180}{48}aad}$ ($\sqrt{36aa}$, that is, $6a$.
II. $\sqrt{\frac{5}{48}d}$) $\sqrt{\frac{125}{48}aad}$ ($\sqrt{25aa}$, that is, $5a$.

From Division I.	$6a\sqrt{\frac{5}{48}d}=\sqrt{\frac{180}{48}aad}$.
From Division II.	$5a\sqrt{\frac{5}{48}d}=\sqrt{\frac{125}{48}aad}$.
The Sum,	$11a\sqrt{\frac{5}{48}d}=\sqrt{\frac{180}{48}aad}+\sqrt{\frac{125}{48}aad}$.
The Difference,	$a\sqrt{\frac{5}{48}d}=\sqrt{\frac{180}{48}aad}-\sqrt{\frac{125}{48}aad}$.

If two Surd Quantities express'd by Letters be Incommensurable, their sum is given by $+$, and their Difference by $-$; as to add $\sqrt{5a}$ and $\sqrt{3a}$, I write $\sqrt{5a}+\sqrt{3a}$ for the Sum; and to subtract $\sqrt{3a}$ from $\sqrt{5a}$, I write $\sqrt{5a}-\sqrt{3a}$ for the Remainder or Difference.

Sect. IX. *Addition and Subtraction in Compound Surd Roots.*

The Arithmetic of Compound Surds depends upon the Rules of the Simple, and the Rules of $+$ and $-$ in Algebraical Addition, Subtraction, Multiplication, and Division; but how those Rules are applied to the Arithmetic of Compound Surds, I shall shew in this and the following tenth and eleventh Sections, by Examples both in Surd Numbers and Surd Quantities express'd by Letters.

Examples

Examples of Addition and Subtraction in Commensurable simple Surd Numbers connected to Rational Numbers by + or —, as also in Compound Surd Numbers composed of Commensurable simple Surds.

To and from	$6+\sqrt{18}(3\sqrt{2})$	$\sqrt{192}(8\sqrt{3})+3$
Add and Subtr.	$4+\sqrt{8}(2\sqrt{2})$	$\sqrt{75}(5\sqrt{3})-3$
The Sum	$10+\sqrt{50}(5\sqrt{2})$	$\sqrt{507}(13\sqrt{3})+0$
Difference	$2-\sqrt{2}$	$\sqrt{27}(3\sqrt{3})+6$

To and from	$+\sqrt{242}(11\sqrt{2})-12$	$15-2\sqrt{2}(\sqrt{8})$
Add and Subtr.	$-\sqrt{50}(-5\sqrt{2})+8$	$7+\sqrt{2}$
Sum	$+\sqrt{72}(6\sqrt{2})-4$	$22-\sqrt{2}$
Difference	$+\sqrt{512}(16\sqrt{2})-20$	$8-3\sqrt{2}(\sqrt{18})$

To and from	$\sqrt{242}+\sqrt{192}$	that is,	$11\sqrt{2}+8\sqrt{3}$
Add and Subtr.	$\sqrt{50}+\sqrt{75}$		$5\sqrt{2}+5\sqrt{3}$
Sum	$\sqrt{512}+\sqrt{507}$	that is,	$16\sqrt{2}+13\sqrt{3}$
Difference	$\sqrt{72}+\sqrt{27}$		$6\sqrt{2}+3\sqrt{3}$

To and from	$\sqrt{320}-\sqrt{108}$	that is,	$8\sqrt{5}-6\sqrt{3}$
Add and Subtr.	$\sqrt{80}-\sqrt{27}$		$4\sqrt{5}-3\sqrt{3}$
Sum	$\sqrt{720}-\sqrt{243}$	that is,	$12\sqrt{5}-9\sqrt{3}$
Difference	$\sqrt{80}-\sqrt{27}$		$4\sqrt{5}-3\sqrt{3}$

To and from	$\sqrt{320}+\sqrt{108}$	that is,	$8\sqrt{5}+6\sqrt{3}$
Add and Subtr.	$\sqrt{80}-\sqrt{27}$		$4\sqrt{5}-3\sqrt{3}$
Sum	$\sqrt{720}+\sqrt{27}$	that is,	$12\sqrt{5}+3\sqrt{3}$
Difference	$\sqrt{80}+\sqrt{243}$		$4\sqrt{5}+9\sqrt{3}$

To and from	$\sqrt[3]{2058}+\sqrt[3]{54}$	that is,	$7\sqrt[3]{6}+3\sqrt[3]{2}$
Add and Subtr.	$\sqrt[3]{162}+\sqrt[3]{16}$		$3\sqrt[3]{6}+2\sqrt[3]{2}$
Sum	$\sqrt[3]{6000}+\sqrt[3]{250}$	that is,	$10\sqrt[3]{6}+5\sqrt[3]{2}$
Difference	$\sqrt[3]{384}-\sqrt[3]{2}$		$4\sqrt[3]{6}-\sqrt[3]{2}$

To and from	$\sqrt[4]{1875}+\sqrt[3]{250}$	that is,	$5\sqrt[4]{3}+5\sqrt[3]{2}$
Add and Subtr.	$\sqrt[4]{48}-\sqrt[3]{16}$		$2\sqrt[4]{3}-2\sqrt[3]{2}$
Sum	$\sqrt[4]{7203}+\sqrt[3]{54}$	that is,	$7\sqrt[4]{3}+3\sqrt[3]{2}$
Difference	$\sqrt[4]{243}+\sqrt[3]{686}$		$3\sqrt[4]{3}+7\sqrt[3]{2}$

EXPLICATION.

In the first Example the Rational Numbers 6 and 4 added together make 10, and their difference is 2; then forasmuch as $\sqrt{18}$ and $\sqrt{8}$ (that is, $3\sqrt{2}$ and $2\sqrt{2}$) are Commensurable, (for the former is to the latter as 3 to 2) their Sum is $\sqrt{50}$ (that is, $5\sqrt{2}$) and their Difference $\sqrt{2}$ (by *Sect.* 8.) Wherefore $10+\sqrt{50}(5\sqrt{2})$ is the Sum, and $2-\sqrt{2}$ the Difference of the two Binomials $6+\sqrt{18}$ and $4+\sqrt{8}$, proposed in the first Example.

Likewise in the second Example the two Commensurable Surd Roots $\sqrt{192}$ and $\sqrt{75}$, (that is, $8\sqrt{3}$ and $5\sqrt{3}$) added into one simple Surd make $\sqrt{507}$, (that is, $13\sqrt{3}$) but their Difference is $\sqrt{27}$, (that is, $3\sqrt{3}$;) also $+3$ and -3 added together make 0, but -3 subtracted from $+3$ makes $+6$. Wherefore $\sqrt{507}$ (that is, $13\sqrt{3}$) is the Sum, and $\sqrt{27}$ (that is, $3\sqrt{3}$) $+6$ is the Difference of the Binomial $\sqrt{192}+3$, and the Residual $\sqrt{75}-3$ proposed in the second Example.

Again, in the third Example, where $-\sqrt{50}+8$ is proposed to be added to $\sqrt{242}$ -12, and also to be subtracted from the same; first, $-\sqrt{50}$ added to $+\sqrt{242}$ (that is, $5\sqrt{2}$ to $+11\sqrt{2}$) makes $+\sqrt{72}$ (that is, $6\sqrt{2}$;) but $-\sqrt{50}$ subtracted from

from $+\sqrt{242}$ (that is, $-5\sqrt{2}$ from $+11\sqrt{2}$) leaves the Remainder or Difference $+\sqrt{512}$, (that is, $16\sqrt{2}$; alſo $+8$ added to -12 makes -4, but $+8$ ſubtracted from -12 leaves the Remainder or Difference -20. Wherefore $\sqrt{72}$ (that is, $6\sqrt{2}$)-4 is the Sum, and $\sqrt{512}$ (that is, $16\sqrt{2}$)-20 is the Difference of the two Reſiduals propoſed in the third Example. The Operation in the reſt of the preceding Examples is after the ſame manner.

Examples of Addition and Subtraction in Compound and Surd Numbers, partly Commenſurable and partly Incommenſurable.

To and from	$\sqrt{27}(3\sqrt{3})+\sqrt{8}$	$\sqrt{10}+\sqrt{8}(2\sqrt{2})$
Add and Subtr.	$\sqrt{12}(2\sqrt{3})+\sqrt{5}$	$\sqrt{3}-\sqrt{2}$
The Sum,	$\sqrt{75}(5\sqrt{3})+\sqrt{8}+\sqrt{5}$	$\sqrt{10}+\sqrt{3}+\sqrt{2}$
Or,	$\sqrt{75}(5\sqrt{3})+\sqrt{:13+\sqrt{160}:}$	$\sqrt{:13+\sqrt{120}:}+\sqrt{2}$
The Difference,	$\sqrt{3}+\sqrt{8}-\sqrt{5}$	$\sqrt{10}-\sqrt{3}+\sqrt{18}(3\sqrt{2})$
Or,	$\sqrt{3}+\sqrt{:13-\sqrt{160}:}$	$\sqrt{:13-\sqrt{120}:}+\sqrt{18}(3\sqrt{2})$
To and from	$\sqrt[3]{56}+\sqrt[3]{16}$	$\sqrt[4]{405}-\sqrt[3]{2}$
Add and Subtr.	$\sqrt[3]{7}-\sqrt[3]{12}$	$\sqrt[4]{80}+\sqrt[2]{5}$
Sum	$3\sqrt[3]{7}+\sqrt[3]{16}-\sqrt[3]{12}$	$5\sqrt[4]{5}+\sqrt[3]{5}-\sqrt[3]{2}$
Difference	$\sqrt[3]{7}+\sqrt[3]{16}+\sqrt[3]{12}$	$\sqrt[4]{5}-\sqrt[3]{5}-\sqrt[3]{2}$

EXPLICATION.

In the firſt of the four laſt preceding Examples the Sum of the two Commenſurable Surd Roots $\sqrt{27}$ and $\sqrt{12}$ (that is, $3\sqrt{3}$ and $2\sqrt{3}$) is $\sqrt{75}$, (that is, $5\sqrt{3}$;) but their Difference is $\sqrt{3}$: and the Sum of the two Incommenſurable Roots $\sqrt{8}$ and $\sqrt{5}$ is $\sqrt{8}+\sqrt{5}$, or $\sqrt{:13+\sqrt{160}:}$ but their Difference is $\sqrt{8}-\sqrt{5}$, or $\sqrt{:13-\sqrt{160}:}$ (according to the Rule before given in *Sect.* 8. for adding and ſubtracting two Incommenſurable ſquare Roots. Therefore $5\sqrt{3}+\sqrt{8}+\sqrt{5}$, or $5\sqrt{3}+\sqrt{:13-\sqrt{160}:}$ is the Sum, and $\sqrt{3}+\sqrt{8}-\sqrt{5}$, or $\sqrt{3}+\sqrt{:13-\sqrt{160}:}$ is the Difference of the two Binomials $\sqrt{27}+\sqrt{8}$ and $\sqrt{12}+\sqrt{5}$, propoſed in the ſaid firſt Example.

Again, in the third of the ſaid four Examples, where $\sqrt[3]{56}+\sqrt[3]{16}$ and $\sqrt[3]{7}-\sqrt[3]{12}$ are propoſed to be added and ſubtracted; the Sum of the two Commenſurable Surd Cubic Roots $\sqrt[3]{56}$ and $\sqrt[3]{7}$ is $3\sqrt[3]{7}$, and their Difference is $\sqrt[3]{7}$; alſo the Sum of the two Incommenſurable Cubic Roots $\sqrt[3]{16}$ and $-\sqrt[3]{12}$ is $\sqrt[3]{16}-\sqrt[3]{12}$; but $-\sqrt[3]{12}$ ſubtracted from $\sqrt[3]{16}$ leaves $\sqrt[3]{15}+\sqrt[3]{12}$. Wherefore $3\sqrt[3]{7}+\sqrt[3]{16}-\sqrt[3]{12}$ is the Sum, and $\sqrt[3]{7}+\sqrt[3]{16}+\sqrt[3]{12}$ is the Difference of the ſaid Binomial and Reſidual propoſed in the third Example.

Examples of Addition and Subtraction in Compound Surd Quantities expreſs'd by Letters.

Example 1.

To and From	$\sqrt{75aa}+\sqrt{8bb}$	viz.	$5a\sqrt{3}+2b\sqrt{2}$
Add and Subtr.	$\sqrt{12aa}+\sqrt{2bb}$		$2a\sqrt{3}+b\sqrt{2}$
The Sum is			$7a\sqrt{3}+3b\sqrt{2}$
The Difference is			$3a\sqrt{3}+b\sqrt{2}$

EXPLICATION.

Firſt, (by *Sect.* 7.) I find that $\sqrt{75aa}$ and $\sqrt{12aa}$ are Commenſurable, and may be reduced to $5a\sqrt{3}$ and $2a\sqrt{3}$; likewiſe $\sqrt{8bb}$ and $\sqrt{2bb}$ are Commenſurable, and may be reduced to $2b\sqrt{2}$ and $b\sqrt{2}$: then the ſum of $5a\sqrt{3}$ and $2a\sqrt{3}$ is $7a\sqrt{3}$; alſo the Sum of $2b\sqrt{2}$ and $b\sqrt{2}$ is $3b\sqrt{2}$: therefore the Sum of the two Binomials propoſed in the Example is $7a\sqrt{3}+3b\sqrt{2}$. But by ſubtracting $2a\sqrt{3}$ from $5a\sqrt{3}$, the Remainder is $3a\sqrt{3}$; and by ſubtracting $b\sqrt{2}$ from $2b\sqrt{2}$ the Remainder is $b\sqrt{2}$. Therefore the Difference of the two Binomials propoſed is $3a\sqrt{3}+b\sqrt{2}$.

Example

Example 2.

What is the Sum and Difference of this Binomial and Residual, ...	$\left\{ \begin{array}{l} \sqrt{(3)}1715a^3b^3+\sqrt{(3)}bcd, \\ \sqrt{(3)}\ 40a^3b^3-\sqrt{(3)}bcd? \end{array} \right.$
Those reduced give these, to wit, ...	$\left\{ \begin{array}{l} 7ab\sqrt{(3)}5+\sqrt{(3)}bcd, \\ 2ab\sqrt{(3)}5-\sqrt{(3)}bcd. \end{array} \right.$
The Sum, ...	$9ab\sqrt{(3)}5$
The Difference, ...	$5ab\sqrt{(3)}5+2\sqrt{(3)}bcd.$

Examples of Addition and Subtraction in Compound Surd Numbers altogether Incommensurable.

To and from	$\sqrt{10}+\sqrt{7}$
Add and Subtr.	$\sqrt{3}+\sqrt{2}$
Sum,	$\sqrt{10}+\sqrt{7}+\sqrt{3}+\sqrt{2}$
Or,	$\sqrt{:17+\sqrt{280}:}+\sqrt{:5+\sqrt{24}:}$
Difference,	$\sqrt{10}+\sqrt{7}-\sqrt{3}-\sqrt{2}$
Or,	$\sqrt{:17+\sqrt{280}:}-\sqrt{:5+\sqrt{24}:}$

To and from	$\sqrt{(3)}10+\sqrt{(3)}7$
Add and Subtr.	$\sqrt{(3)}3-\sqrt{(3)}2$
Sum,	$\sqrt{(3)}10+\sqrt{(3)}7+\sqrt{(3)}3-\sqrt{(3)}2$
Difference,	$\sqrt{(3)}10+\sqrt{(3)}7-\sqrt{(3)}3+\sqrt{(3)}2.$

Sect. X. *Of Multiplication in Compound Surds.*

Example 1.

Multiplicand, $\sqrt{180}+\sqrt{48}$
Multiplicator, $\sqrt{125}+\sqrt{12}$ that is, $\left\{ \begin{array}{l} 6\sqrt{5}+4\sqrt{3} \\ 5\sqrt{5}+2\sqrt{3} \end{array} \right.$

$$\begin{array}{r} 150+20\sqrt{15}\qquad\qquad \\ +12\sqrt{15}+24 \\ \hline \end{array}$$

Product, $150+32\sqrt{15}+24$
That is, $174+32\sqrt{15}.$

Example 2.

Multiplicand, $6-\sqrt{20}$
Multiplicator, $8-\sqrt{45}$ that is, $\left\{ \begin{array}{l} 6-2\sqrt{5} \\ 8-3\sqrt{5} \end{array} \right.$

$$\begin{array}{r} 48-16\sqrt{5}\qquad\qquad \\ -18\sqrt{5}+30 \\ \hline \end{array}$$

Product, $78-34\sqrt{5}$

Example 3.

Multiplicand $\sqrt{18}-3$
Multiplicator, $\sqrt{8}+2$ that is, $\left\{ \begin{array}{l} 3\sqrt{2}-3 \\ 2\sqrt{2}+2 \end{array} \right.$

$$\begin{array}{r} 12-6\sqrt{2}\qquad\qquad \\ +6\sqrt{2}-6 \\ \hline \end{array}$$

Product, $12\qquad -6$
That is, $6.$

Example 4.

Multiplicand, $4\sqrt{5}+3\sqrt{5}$
Multiplicator, $4\sqrt{5}+3\sqrt{5}$ that is, $\left\{ \begin{array}{l} 7\sqrt{5} \\ 7\sqrt{5} \end{array} \right.$

Product, 245

EXPLICATION.

In the first Example, the two Compound Surd Numbers propos'd to be multiplied are $\surd 180+\surd 48$ and $\surd 125+\surd 12$, which are reduced to $6\surd 5+4\surd 3$ and $5\surd 5+2\surd 3$; (by *Sect.* 6. of this *Chap.*) then $6\surd 5$ multiplied by $5\surd 5$, (according to Rule 5. in *Sect.* 4. of this *Chap.*) produces 150; also $4\surd 3$ multiplied by $5\surd 5$ (according to Rule 6. in *Sect.* 4.) produces $20\surd 15$; again, $6\surd 5$ into $2\surd 3$ makes $12\surd 15$, and $4\surd 3$ into $2\surd 3$ produces 24; lastly, those Products added together make $174+32\surd 15$, the Product sought. The rest of the Examples are wrought in like manner.

When the Multiplicand has not the same Radical Sign with the Multiplier, they must first be reduced to the same Radical Sign, (by *Sect.* 3. of this *Chap.*) and then the Multiplication is to be made by some of the Rules in *Sect.* 4 as will be manifest in the following Example.

Multiplicand,	$\surd(5)6+\surd(3)7+5$
Multiplicator,	$\surd 3$
Product,	$\surd(10)8748+\surd(6)1323+5\surd 3.$

EXPLICATION.

1. $\surd(5)6$ and $\surd 3$ are reduced to these having a common Radical Sign, to wit, $\surd(10)36$ and $\surd(10)243$, which multiplied one into the other produce $\surd(10)8748$.

2. $\surd(3)7$ and $\surd 3$ are reduced to $\surd(6)49$ and $\surd(6)27$, which multiplied one by the other produce $\surd(6)1323$.

3. The Rational Number 5 multiplied into $\surd 3$ makes $5\surd 3$ or $\surd 75$.

Lastly, those three simple Products added together give the Product sought, to wit, $\surd(10)8748+\surd(6)1323+5\surd 3(\surd 75.)$

Three Compendious Rules, very useful in the Multiplication of Binomials and Residuals.

1. Because $a+e$ multiplied by $a+e$ produces $aa+2ae+ee$, it is evident that the sum of the Squares of the Parts (or Names) or any Binomial, together with twice the Product of the Parts multiplied one into the other is equal to the Square of the Sum of the Parts. Therefore to multiply any Binomial by itself (or to square it) take the Squares of the Parts, and twice the Product of the Parts for the Square sought.

2. Because $a-e$ multiplied by $a-e$ produces $aa-2ae+ee$, it is manifest that the sum of the squares of the Parts of any Residual, less by the double Product of the Parts, is equal to the square of the difference of the Parts. Therefore to square any Residual from the Sum of the Squares of the Parts subtract twice the Product of the Parts, and take the remainder for the Square sought.

3. Because $a+e$ multiplied by $a-e$ produces $aa-ee$, it is evident that the difference of the Square of the Parts of any Binomial, is equal to the Product made by the Multiplication of the Sum of the Parts into their difference. Therefore if a Binomial be to be multiplied by its correspondent Residual, that is, by the difference of the Parts of the Binomial, take the difference of the Squares of the Parts for the Product sought. These three Rules will be exercised by the six Examples next following, and by divers other Examples in this and the following Sections of this Chapter.

Multiplicand,	$3+\surd 5$	$3-\surd 5$
Multiplicator,	$3+\surd 5$	$3-\surd 5$
Product	$9+6\surd 5+5$	$9-6\surd 5+5$
That is,	$14+6\surd 5$	$14-6\surd 5$

Multiplicand,	$3+\surd 5$	$\surd(3)27+\surd(3)8$
Multiplicator,	$3+\surd 5$	$\surd(3)27-\surd(3)8$
Product,	$9-5$	$\surd(3)729-\surd(3)64$
That is,	4	5

Multiplicand,	$\surd(6)7-\surd(6)5$	$\surd(10)7+\surd(10)3$
Multiplicator,	$\surd(6)7+\surd(6)5$	$\surd(10)7-\surd(10)3$
Product,	$\surd(3)7-\surd(3)5$	$\surd(5)7-\surd(5)3$

EX-

EXPLICATION.

In the first of the six last Examples the Binomial $3+\sqrt{5}$ multiplied into it self, or squared, produces $14+6\sqrt{5}$; for the Squares of the Parts 3 and $\sqrt{5}$ are 9 and 5, and twice the Product of 3 into $\sqrt{5}$ makes $6\sqrt{5}$, to wit $\sqrt{180}$; therefore (by the first of the three preceding Rules) $9+5+6\sqrt{5}$, that is, $14+6\sqrt{5}$ is the Square of the given Binomial $3+\sqrt{5}$.

In the second Example the Residual $3-\sqrt{5}$ squared or multiplied by itself produces $14-6\sqrt{5}$, (by the second of the said three Rules.)

In the third Example the Binomial $3+\sqrt{5}$ multiplied by its correspondent Residual $3-\sqrt{5}$ produces 4, which (by the last of the said three Rules) is equal to the difference of the Squares of the Parts 3 and $\sqrt{5}$.

Likewise in the fourth Example the Binomial $\sqrt{(3)}27+\sqrt{(3)}8$ multiplied by its correspondent Residual $\sqrt{(3)}27-\sqrt{(3)}8$ produces $\sqrt{(3)}729-\sqrt{(3)}64$, to wit, the difference of the Squares of the Parts of the given Binomial or Residual.

And in the fifth Example the Residual $\sqrt{(6)}7-\sqrt{(6)}5$, multiplied by its correspondent Binomial $\sqrt{(6)}7+\sqrt{(6)}5$, produces $\sqrt{(3)}7-\sqrt{(3)}5$; which is equal to the difference of the Squares of the parts of the given Residual or Binomial. For (by the seventh Rule in *Sect.* 4. of this *Chap.*) the Square of $\sqrt{(6)}7$ is $\sqrt{(3)}7$, and the Square of $\sqrt{(6)}5$ is $\sqrt{(3)}5$.

Examples of Multiplication in Compound Surd Quantities exprest by Letters.

$$\begin{array}{lll} \text{Multiplicand} & \sqrt{abb}+\sqrt{cff} & \text{that is, } b\sqrt{a}+f\sqrt{c} \\ \text{Multiplicator,} & \sqrt{add}+\sqrt{caa} & \phantom{\text{that is, }} d\sqrt{a}+a\sqrt{c} \\ & & \phantom{\text{that is, }} bda+fd\sqrt{ca} \\ & & \phantom{\text{that is, }} \qquad +ba\sqrt{ca}+fac \\ & \text{Product,} & \phantom{\text{that is, }} bda+\overline{fd+ba}\times\sqrt{ca}+fac. \end{array}$$

$$\begin{array}{ll|l} \text{Multiplicand,} & 2a+3a\sqrt{d} & \sqrt{bc}+a \\ \text{Multiplicator,} & 3c-2c\sqrt{d} & \sqrt{bc}-a \\ & 6ac+9ac\sqrt{d} & bc+a\sqrt{bc} \\ & \quad -4ac\sqrt{d}-6acd & \quad -a\sqrt{bc}-aa \\ \text{Product,} & 6ac+5ac\sqrt{d}-6acd & bc \quad -aa \end{array}$$

$$\begin{array}{ll|l} \text{Multiplicand,} & a+\sqrt{b} & \sqrt{ab}+\sqrt{c} \\ \text{Multiplicator,} & a+\sqrt{b} & \sqrt{ac}+\sqrt{d} \\ \text{Product,} & aa+2a\sqrt{b}+b & a\sqrt{bc}+c\sqrt{a}+\sqrt{abd}+\sqrt{cd} \end{array}$$

$$\begin{array}{lll} \text{Multiplicand,} & 3bb\sqrt{d}+d\sqrt{d} & \text{that is, } \overline{3bb+d}\times\sqrt{d} \\ \text{Multiplicator,} & 3bb\sqrt{d}+d\sqrt{d} & \phantom{\text{that is, }} \overline{3bb+d}\times\sqrt{d} \\ \text{Product,} & 9bbbbd+6bbdd+ddd & \text{or } \overline{9bbbb+6bbd+dd}\times d. \end{array}$$

The Operation in these six last Examples will be familiar to him that understands the Rules and Examples before delivered concerning the Multiplication of Surd Numbers and Quantities exprest by Letters.

Sect. XI. *Division in Compound Surds.*

Examples of Division where the Dividend is a Compound Quantity, and the Divisor a Simple Quantity.

$$\begin{array}{ll|l} \text{Dividend,} & \sqrt{21}+\sqrt{15} & \sqrt{(3)}14-\sqrt{(3)}28 \\ \text{Divisor,} & \sqrt{3} & \sqrt{(3)}7 \\ \text{Quotient,} & \sqrt{7}+\sqrt{5} & \sqrt{(3)}2-\sqrt{(3)}4 \end{array}$$

Dividend,	$12\sqrt{6}+6\sqrt{18}-2\sqrt{12}$	$\sqrt{20}-\sqrt{(3)}10$
Divisor	$3\sqrt{6}$	3
Quotient,	$4+2\sqrt{3}-\frac{2}{3}\sqrt{2}$	$\sqrt{\frac{20}{9}}-\sqrt{(3)}\frac{10}{27}$
Dividend,	$\sqrt{(4)}8+\sqrt{(5)}3$	$\sqrt{(4)}23328-\sqrt{(4)}10368$
Divisor,	$\sqrt{2}$	6
Quotient,	$\sqrt{(4)}2+\sqrt{(10)}\frac{9}{32}$	$\sqrt{(4)}18-\sqrt{(4)}8$

EXPLICATION.

The first Example is wrought according to Rule 1. in *Sect.* 5. of this *Chap.* For first, $\sqrt{21}$ divided by $\sqrt{3}$ gives the Quotient $\sqrt{7}$, then $\sqrt{15}$ divided by $\sqrt{3}$ gives the Quotient $\sqrt{5}$. Therefore $\sqrt{21}+\sqrt{15}$ divided by $\sqrt{3}$ gives $\sqrt{7}+\sqrt{5}$, the Quotient sought in the first Example.

The second Example is wrought like the first; for $\sqrt{(3)}14$ divided by $\sqrt{(3)}7$ gives $\sqrt{(3)}2$, and $-\sqrt{(3)}28$ divided by $\sqrt{(3)}7$ gives $-\sqrt{(3)}4$. Therefore $\sqrt{(3)}14-\sqrt{(3)}28$ divided by $\sqrt{(3)}7$, gives $\sqrt{(3)}2-\sqrt{(3)}4$, the Quotient sought in the second Example.

The third Example is wrought according to the fifth and sixth Rules of *Sect.* 5. of this *Chap.* For first, $12\sqrt{6}$ divided by $3\sqrt{6}$ give the Quotient 4, (by the said fifth Rule;) then $6\sqrt{18}$ divided by $3\sqrt{6}$ gives $2\sqrt{3}$, (by the said sixth Rule;) likewise $-2\sqrt{12}$ divided by $3\sqrt{6}$ gives $-\frac{2}{3}\sqrt{2}$; (for 2 divided by 3 gives $\frac{2}{3}$, and $\sqrt{12}$ divided by $\sqrt{6}$ gives $\sqrt{2}$.) Therefore $12\sqrt{6}+6\sqrt{18}-2\sqrt{12}$ divided by $3\sqrt{6}$ gives $4+2\sqrt{3}-\frac{2}{3}\sqrt{2}$, the Quotient sought in the third Example.

In the fourth Example $\sqrt{20}$ divided by $\sqrt{3}$, (that is, by $\sqrt{9}$) gives $\sqrt{\frac{20}{9}}$, or $\sqrt{2\frac{2}{9}}$; and $-\sqrt{(3)}10$ divided by 3, (that is, by $\sqrt{(3)}27$) gives $-\sqrt{(3)}\frac{10}{27}$.

In the fifth Example $\sqrt{(4)}8$ and $\sqrt{2}$ are first reduced to $\sqrt{(4)}8$ and $\sqrt{(4)}4$; then $\sqrt{(4)}8$ divided by $\sqrt{(4)}4$ gives $\sqrt{(4)}2$; likewise $\sqrt{(5)}3$ and $\sqrt{2}$ are reduced to $\sqrt{(10)}9$ and $\sqrt{(10)}32$; then $\sqrt{(10)}9$ divided by $\sqrt{(10)}32$ gives the Quotient $\sqrt{(10)}\frac{9}{32}$. Therefore $\sqrt{(4)}8+\sqrt{(5)}3$ divided by $\sqrt{2}$, gives $\sqrt{(4)}2+\sqrt{(10)}\frac{9}{32}$, the Quotient sought in the fifth Example. The sixth Example is wrought in like manner, and the Proof in these or the like Examples of Division may be made by Multiplication.

Propositions concerning Division in Surd Quantities, when the Divisor is a Binomial or Trinomial, &c.

When the Divisor is a Binomial or Residual consisting of two Square Roots or Biquadratic Roots, or of one Square Root or Biquadratic Root, and of a Rational Number; as also when the Divisor is a Trinomial or Quadrinomial, and none of its Radical Signs exceeds that of the Square Root, the work of Division in those cases is grounded upon these five following Propositions, *viz.*

1. If a Binomial consisting of two simple square Roots connected by $+$, be multiplied by its correspondent Residual, that is, by the difference of those Roots; or if a Residual consisting of two simple square Roots connected by $-$, be multiplied by its correspondent Binomial, that is, by the Sum of the same Roots, the Product will be entirely Rational. So the Binomial $\sqrt{5}+\sqrt{3}$ multiplied by $\sqrt{5}-\sqrt{3}$, (or the Residual $\sqrt{5}-\sqrt{3}$ by $\sqrt{5}+\sqrt{3}$) gives the Rational Product 2, (by the last of the three Rules before delivered in *Sect.* 10. of this *Chap.*)

Likewise $\sqrt{a}+\sqrt{b}$ multiplied by $\sqrt{a}-\sqrt{b}$ gives the Rational Product $a-b$.

2. If a Binomial consisting of two Biquadratic simple Roots connected by $+$, be multiplied by its correspondent Residual, to wit, by the difference of those Roots the Product will be also a Residual consisting of two square Roots connected by $-$, and if this Residual be multiplied by the sum of its Names (or Parts,) it will give a Product entirely Rational.

As for Example, the Binomial $\sqrt{(4)}5+\sqrt{(4)}3$ multiplied by $\sqrt{(4)}5-\sqrt{(4)}3$ makes $\sqrt{5}-\sqrt{3}$, which multiplied by $\sqrt{5}+\sqrt{3}$ gives the Rational Product 2.

Likewise $\sqrt{(4)}81-2$ or $\sqrt{(4)}81-\sqrt{(4)}16$ multiplied by $\sqrt{(4)}81+\sqrt{(4)}16$ makes $\sqrt{81}-\sqrt{16}$, which multiplied by $\sqrt{81}+\sqrt{16}$ gives the Rational Product 65.

3. If

3. If a Trinomial consisting of three simple square Roots connected by +, or by + and —, be multiplied by the same Trinomial, after any one Sign + is changed into —, or any one Sign — into +, the Product will consist of two Names (or Parts;) and then if this Product be multiplied by its correspondent Binomial or Residual, (according to the preceding *Prop.* 1.) the last Product will be entirely Rational.

As for Example, the Trinomial $\sqrt{5}+\sqrt{3}+\sqrt{2}$ multiplied by $\sqrt{5}+\sqrt{3}-\sqrt{2}$ gives $2\sqrt{15}+6$, and this multiplied by $2\sqrt{15}-6$ gives the Rational Product 24.

Likewise $\sqrt{30}-\sqrt{5}-\sqrt{3}$ multiplied by $\sqrt{30}+\sqrt{5}-\sqrt{3}$ produces $28-2\sqrt{90}$, and this multiplied by $28+2\sqrt{90}$ gives the Rational Product 424.

After the same manner $\sqrt{a}+\sqrt{b}-\sqrt{c}$ multiplied by $\sqrt{a}+\sqrt{b}+\sqrt{c}$ gives the Product $2\sqrt{ab}+a+b-c$, whose Rational Part $a+b-c$ we may suppose to be equal to some single Quantity d, and then the said Product will be a Binomial $2\sqrt{ab}+d$, which multiplied by its correspondent Residual $2\sqrt{ab} \backsim d$ gives a Product entirely Rational, to wit, $4ab \backsim dd$. And so of other Trinomials that are qualified as before is supposed.

4. If a Quadrinomial consisting of four simple square Roots connected by +, or by + and —, be multiplied by the same Quadrinomial after two Signs + are changed into —, or two Signs — into +, the Product will consist of three Names (or Parts; (then if this Product be multiplied by its correspondent Trinomial(according to *Prop.* 3.) there will come forth a Binomial or Residual. And lastly, this Binomial or Residual multiplied by its correspondent Residual or Binomial will give a Rational Product.

As for Example, the Quadrinomial $\sqrt{6}+\sqrt{5}+\sqrt{3}+\sqrt{2}$ multiplied by $\sqrt{6}+\sqrt{5}-\sqrt{3}-\sqrt{2}$, produces the Trinomial $6+2\sqrt{30}-2\sqrt{6}$; which multiplied by its correspondent Trinomial $6+2\sqrt{30}+2\sqrt{6}$, (according to the precedent *Prop.* 3.) gives the Binomial $132+24\sqrt{30}$; and this multiplied by its correspondent Residual $132-24\sqrt{30}$, gives the Rational Product 144.

After the same manner the Quadrinomial $\sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{d}$ multiplied by $\sqrt{a}-\sqrt{b}-\sqrt{c}-\sqrt{d}$ gives the Product $a+d-b-c-2\sqrt{ad}-2\sqrt{bc}$, whose Rational Part $a+d-b-c$ we may suppose to be equal to some single Quantity f, and then the said Product will be a Trimonial, to wit, $f-2\sqrt{ad}-2\sqrt{bc}$; this multiplied by it self after one of its Signs — is changed into + (according to *Prop.* 3.) will produce a Residual of two Names (or Parts,) and this Residual multiplied by its correspondent Binomial will give a Rational Product.

5 If two Numbers be given for a Dividend and Divisor, and each be multiplied by some Number, the first Product divided by the later will give the same Quotient that arises by dividing the given Dividend by the given Divisor As if 6 be to be divided by 2, if you multiply each by 4, and divide the first Product 24 by the later 8, the Quotient 3 is the same that arises by dividing 6 by 2. For (by 17 *Prop* 7. *Elem. Euclid.*) if a Number a multiplying two numbers b, c, produce two other Numbers ab and ac, the Numbers produced shall be in the same proportion that the numbers multiplied are, *viz.* as $b \,.\, c :: ab \,.\, ac$, and therefore $\frac{ab}{ac}=\frac{b}{c}$; also $\frac{ac}{ab}=\frac{c}{b}$. From the foregoing five Propositions the following Rule is deduced, *viz.*

6. *A Rule for Division in Surd Quantities when the Divisor is a Binomial, Trinomial or Quadrinomial of such kind as before is declared.*

Reduce the given Divisor to a new Divisor that may be a simple Rational Quantity; reduce also the given Dividend to a new Dividend, by multiplying the former by the same Quantity or Quantities that were Multiplicators in reducing the given Divisor to a Rational Quantity; then divide the new Dividend by the new Divisor, (according to the Method in the Examples at the beginning of this *Sect.* 11.) so the Quotient shall be the same with that which would arise by dividing the given Dividend by the given Divisor.

As for Example, to divide $\sqrt{8}+\sqrt{6}$ by $\sqrt{4}+\sqrt{2}$, I first multiply the Divisor $\sqrt{4}+\sqrt{2}$ by its correspondent Residual $\sqrt{4}-\sqrt{2}$, and it produces 2 for a new Divisor; also I multiply the Dividend $\sqrt{8}+\sqrt{6}$ by the said $\sqrt{4}-\sqrt{2}$, and it gives the Product $\sqrt{32}+\sqrt{24}-\sqrt{16}-\sqrt{12}$ for a new Dividend, this divided by 2 (the Divisor before found) gives $\sqrt{8}+\sqrt{6}-2+\sqrt{3}$ the Quotient sought, being equal to that which would arise by dividing $\sqrt{8}+\sqrt{6}$ by $\sqrt{4}+\sqrt{2}$, as will be evident by the Proof; for

for if the said Quotient $\sqrt{8}+\sqrt{6}-2-\sqrt{3}$ be multiplied by the given Divisor $\sqrt{4}+\sqrt{2}$, it produce the given Dividend $\sqrt{8}+\sqrt{6}$.

Likewise to divide $ab+b\sqrt{bc}$, by $a+\sqrt{bc}$, I multiply each by $a-\sqrt{bc}$ (the Residual Correspondent to the Divisor) and it produces $aa-bc$ for a new Divisor, and $aab-bbc$ for a new Dividend, this divided by that gives b for the Quotient sought; for b multiplied into the given Divisor $a+\sqrt{bc}$ makes the given Dividend $ab+b\sqrt{bc}$. Another way of finding out the Quotient in this last Example, is shewn in the first of the six Examples at the latter end of this *Sect.* 11.

Again, to divide 10 by $\surd(4)5+\surd(4)3$, I multiply each by $\surd(4)5-\surd(4)3$, and there comes forth a new Dividend $\surd(4)50000-\surd(4)30000$, and a new Divisor $\sqrt{5}-\sqrt{3}$; but this Divisor not being a Rational Number, I multiply again both the said new Dividend and Divisor by $\sqrt{5}+\sqrt{3}$, and it produces another new Dividend $\surd(4)1250000-\surd(4)750000+\surd(4)450000-\surd(4)270000$, and another new Divisor 2; by this I divide the last Dividend, and there arises $\surd(4)78125-\surd(4)46875+\surd(4)28125-\surd(4)16875$ the Quotient sought; for if it be multiplied by the proposed Divisor $\surd(4)5+\surd(4)3$ it will produce the given Dividend 10.

Again, to divide $\sqrt{8}$ by $\sqrt{3}+\sqrt{2}+1$, I first multiply the Divisor by $\sqrt{3}+\sqrt{2}-1$, and it makes $\sqrt{24}+4$, this multiplied by its correspondent Residual $\sqrt{24}-4$ gives the Product 8 for a new Divisor. Now because the given Divisor was first multiplied by $\sqrt{3}+\sqrt{2}-1$, and the Product by $\sqrt{24}-4$, the given Dividend must likewise be multiplied first by $\sqrt{3}+\sqrt{2}-1$, and the Product $\sqrt{24}+4-\sqrt{8}$ by $\sqrt{24}-4$, and there will be produced $8+\sqrt{128}-\sqrt{192}$ for a new Dividend; so instead of the given Dividend and Divisor we have other Numbers in the same proportion, *viz.* $8+\sqrt{128}-\sqrt{192}$ and 8. Therefore (by *Prop.* 5.) the former divided by the latter will give the Quotient sought, to wit, $1+\sqrt{2}-\sqrt{3}$; but that this is the true Quotient will appear by Multiplication, for if $1+\sqrt{2}-\sqrt{3}$ be multiplied by the proposed Divisor $\sqrt{3}+\sqrt{2}+1$, it will produce the given Dividend $\sqrt{8}$.

Note, Although the new Divisor and Dividend found out as aforesaid, may sometimes happen to be Negative Quantities, (that is, such whose values are less than nothing) yet Division being made by them with respect to the Rules of + and −, they will give the true Quotient sought. As for Example, suppose 30 be to be divided by $2+\sqrt{9}$, (that is 30 by 5;) first the Divisor $2+\sqrt{9}$ being multiplied by $2-\sqrt{9}$ gives $4-9$, that is, -5 for a new Divisor, and the Dividend 30 multiplied by the said $2-\sqrt{9}$ gives $60-\sqrt{8100}$ for a new Dividend, which divided by -5 gives $+6$, which is the same with the Quotient that arises by dividing 30 by $2+\sqrt{9}$, that is, by 5.

Again, let $4+\sqrt{25}$ be divided by $1+\sqrt{9}$, (that is, 9 by 4, where the Quotient is manifestly $2\frac{1}{4}$;) first, the Divisor $1+\sqrt{9}$ multiplied by $1-\sqrt{9}$ produces $1-9$, that is, -8 for a new Divisor; and the Dividend $4+2\sqrt{5}$ multiplied by the said $1-\sqrt{9}$ makes $4+\sqrt{25}-4\sqrt{9}-\sqrt{225}$ for a new Dividend, which divided by -8, (according to the Examples at the beginning of this *Sect.* 11.) gives $-\frac{1}{2}-\sqrt{\frac{25}{64}}+\frac{1}{2}\sqrt{9}+\sqrt{\frac{225}{64}}$ the Quotient sought, which after due contraction makes $2\frac{1}{4}$. For $\frac{1}{2}\sqrt{9}$, that is, $\sqrt{\frac{144}{64}}$ is equal to $\frac{12}{8}$, and $\sqrt{\frac{225}{64}}$ is $\frac{15}{8}$, which added to the said $\frac{12}{8}$ makes $\frac{27}{8}$; also $-\sqrt{\frac{25}{64}}$ is $-\frac{5}{8}$, which added to $-\frac{1}{2}$, (or $-\frac{4}{8}$) makes $-\frac{9}{8}$, this added to $\frac{27}{8}$ gives $\frac{18}{8}$ (or $2\frac{1}{4}$) the Quotient before found.

7. When the Divisor is a Binomial or a Residual, consisting of two simple Cubic or Biquadratic, &c. Roots, it may be reduced to a Rational Divisor by this following Proposition, *viz.*

If in the Proportion of the Names (or Parts) of a Binomial or Residual, there be found so many continual Proportionals in multitude as there be Units in the Index of the Radical Sign, and that the Radical Signs of the Parts of the Binomial or Residual, and also of the Proportionals be the same, but connected in the Binomial by +, and in the Proportionals by + and − alternately; or contrarily, in the Proportionals by +, and in the Residual by + and −; the Product made by the Multiplication of the Proportionals by the Binomial or Residual shall be Rational.

As for example, if there be proposed the Binomial $\surd(3)7+\surd(3)5$; find three continual Proportionals, that the first may be to the second, and the second to the third, as $\surd(3)7$ to $\surd(3)5$, which may be done by the help of *Sect.* 8. *Chap.* 5. of this Book; where it has been shewn, that aa, ae, and ee, are continual Proportionals in the Reason of a to e. Therefore if we suppose $\surd(3)7$ to be a, and $\surd(3)5$ to be e, then the Square

of

of $\surd(3)7$, to wit, $\surd(3)49$, ſhall be the firſt Proportional (aa); the Product of $\surd(3)7$ into $\surd(3)5$, to wit, $\surd(3)35$, ſhall be the ſecond Proportional (ae); and the Square of $\surd(3)5$, to wit, $\surd(3)25$, ſhall be the third Proportional (ee): ſo that theſe three Cubic Roots, to wit, $\surd(3)49$, $\surd(3)35$, and $\surd(3)25$, are continual Proportionals in the Reaſon of $\surd(3)7$ and $\surd(3)5$. Now I ſay, (according to the Propoſition) if $\surd(3)49 - \surd(3)35 + \surd(3)25$ be multiplied by $\surd(3)7 + \surd(3)5$, the Product ſhall be Rational; alſo if $\surd(3)49 + \surd(3)35 + \surd(3)25$ be multiplied by $\surd(3)7 - \surd(3)5$, the Product ſhall be Rational, as will appear by the following Operation.

Multiplicand,	$\surd(3)49 - \surd(3)35 + \surd(3)25$
Multiplicator,	$\surd(3)7 + \surd(3)5$
	$7 - \surd(3)245 + \surd(3)175$
	$+ \surd(3)245 - \surd(3)175 + 5$
The Product,	12 is Rational.

Multiplicand,	$\surd(3)49 + \surd(3)35 + \surd(3)25$
Multiplicator,	$\surd(3)7 - \surd(3)5$
	$7 + \surd(3)245 + \surd(3)175$
	$- \surd(3)245 - \surd(3)175 - 5$
The Product,	2 is Rational.

But for the greater Evidence of the certainty of this Propoſition in a Binomial and Reſidual conſiſting of any two ſimple Cubic Roots whatever, let there be propoſed this Binomial $\surd(3)b + \surd(3)d$, and ſuppoſe b greater than d; then three continual Proportionals in the Proportion of $\surd(3)b$ to $\surd(3)d$ will be found $\surd(3)bb$, $\surd(3)bd$, and $\surd(3)dd$; then multiply as before, *viz.*

Multiplicand,	$\surd(3)bb - \surd(3)bd + \surd(3)dd$
Multiplicator,	$\surd(3)b + \surd(3)d$
	$b - \surd(3)bbd + \surd(3)bdd$
	$+ \surd(3)bbd - \surd(3)bdd + d$
The Product,	$b + d$ is Rational.

Again,

Multiplicand,	$\surd(3)bb + \surd(3)bd + \surd(3)dd$
Multiplicator,	$\surd(3)b - \surd(3)d$
	$b + \surd(3)bbd + \surd(3)bdd$
	$- \surd(3)bbd - \surd(3)bdd - d$
Product,	$b - d$ is Rational.

Whence you may obſerve, that the firſt Rational Product is the ſum of the Names (or Parts,) omitting the Radical Signs, of the Cubic Binomial propoſed; and the latter Rational Product is the difference of the Parts, omitting the Radical Signs, of the Cubic Reſidual propoſed: ſo that the Rational Product made by the Multiplication of the ſaid Proportionals and Binomial or Reſidual may be diſcovered without any Multiplication.

8. Now that the uſe of the laſt preceding Propoſition may appear, let it be required to divide 10 by $\surd(3)7 - \surd(3)5$; firſt, becauſe the Index of the Radical Sign is 3, I ſeek three continual Proportionals in the Proportion of $\surd(3)7$ to $\surd(3)5$; which Proportionals as (before has been ſhewn (are $\surd(3)49$, $\surd(3)35$, and $\surd(3)25$; theſe I connect by $+$, becauſe the Parts of the given Diviſor are connected by $-$, and there ariſes $\surd(3)49 + \surd(3)35 + \surd(3)25$: then by this common Multiplicator I multiply as well the Dividend 10, as the Diviſor $\surd(3)7 - \surd(3)5$, and it produces $\surd(3)49000 + \surd(3)35000 + \surd(3)25000$ for a new Dividend, and 2 for a new Diviſor. Laſtly, by dividing the ſaid new Dividend by the new Diviſor, there ariſes $\surd(3)6125 + \surd(3)4375 - \surd(3)3125$ the Quotient ſought: for if it be multiplied by the great Diviſor $\surd(3)7 - \surd(3)5$, it will produce the given Dividend 10.

In like manner, to divide 10 by this Binomial $\sqrt{(3)}5+\sqrt{(3)}3$, firſt, I ſeek three continual Proportionals in the Reaſon of $\sqrt{(3)}5$ to $\sqrt{(3)}3$, which Proportionals will be found $\sqrt{(3)}25$, $\sqrt{(3)}15$, and $\sqrt{(3)}9$; theſe I connect by + and — alternately, becauſe the Parts of the given Diviſor are connected by +, *viz.* to the firſt Proportional I prefix +, to the ſecond —, and to the third +; ſo they make $\sqrt{(3)}25 - \sqrt{(3)}15+\sqrt{(3)}9$. By this as a common Multiplicator I multiply as well the Dividend 10 as the Diviſor $\sqrt{(3)}5+\sqrt{(3)}3$, and there ariſes a new Dividend $\sqrt{(3)}25000-\sqrt{(3)}15000+\sqrt{(3)}9000$, and a new Diviſor 8, by which I divide the ſaid new Dividend, and there comes forth $\sqrt{(3)}\frac{3125}{64}-\sqrt{(3)}\frac{1875}{64}+\sqrt{(3)}\frac{1125}{64}$, the Quotient ſought.

The ſame Method is to be obſerved when the Diviſor is a Binomial or a Reſidual conſiſting of two ſimple Biquadratic Roots.

As for Example, to divide 10 by $\sqrt{(4)}5+\sqrt{(4)}3$, (which has already been done after another manner in the third Example of the Rule in the ſixth ſtep of this Section;) firſt, becauſe the Index of the Radical Sign is 4, I ſearch out four continual Proportionals in the Reaſon of $\sqrt{(4)}5$ to $\sqrt{(4)}3$ in this manner, *viz.* Foraſmuch as (by *Sect.* 8. *Chap* 5. of this Book) theſe are continual Proportionals, to wit, *aaa*, *aae*, *aee*, and *eee*; I ſuppoſe $\sqrt{(4)}5$ to be a, and $\sqrt{(4)}3$ to be e, then I multiply $\sqrt{(4)}5$ into it ſelf cubically, and it gives the firſt Proportional $\sqrt{(4)}125$, to wit, *aaa*;) alſo I multiply the Square of $\sqrt{(4)}5$ into $\sqrt{(4)}3$, and it gives the ſecond Proportional $\sqrt{(4)}75$, (to wit, *aae*;) again, I multiply $\sqrt{(4)}5$ into the Square of $\sqrt{(4)}3$, and it gives the third Proportional $\sqrt{(4)}45$, (to wit, *aee*;) laſtly, I multiply $\sqrt{(4)}3$ into itſelf cubically, and it gives the fourth Proportional $\sqrt{(4)}27$, (to wit, *eee*:) Then becauſe the two Parts of the given Diviſor are connected by +, I connect thoſe four Proportionals by + and — alternately, ſo there ariſes this Compound Number $\sqrt{(4)}125-\sqrt{(4)}75+\sqrt{(4)}45-\sqrt{(4)}27$, by which as a common Multiplicator I multiply as well the given Dividend 10, as the given Diviſor $\sqrt{(4)}5+\sqrt{(4)}3$, and there ariſes a new Dividend $\sqrt{(4)}1250000-\sqrt{(4)}750000+\sqrt{(4)}450000-\sqrt{(4)}270000$, and a New Diviſor 2; which are the ſame in every reſpect with thoſe found in the place before cited.

After the ſame manner, when the Diviſor is a Binomial or a Reſidual having 5 or 6, *&c.* for the Index of the common Radical Sign of the Roots, it may be reduced to a new Diviſor that ſhall be Rational. But it muſt be remembred, that when the Roots are of different kinds they muſt firſt be reduced to a common Radical Sign.

But when the Diviſor cannot be reduced to a ſimple Rational Number by any of the foregoing Rules, (which are all that I have met with in Algebraical Authors) the Dividend may be ſet as a Numerator over the Diviſor as a Denominator, and the Fraction ſo conſtituted ſhall be equal to the Quotient. As for Example, if $\sqrt{48}+\sqrt{(3)}3$ be to be divided by $\sqrt{15}+\sqrt{(3)}6-\sqrt{3}$, the Quotient may be repreſented by this Fraction, to wit,

$$\frac{\sqrt{48}+\sqrt{(3)}3}{\sqrt{15}+\sqrt{(3)}6-\sqrt{3}}.$$

Examples of Diviſion in Compound Surd Quantities expreſt by Letters.

Diviſion in Compound Surd Quantities expreſt by Letters depends upon the Rules of ſimple Surds before delivered; as alſo upon the general Method of Diviſion in *Sect.* 9. *Chap.* 5. *Book* 1. as will appear by the following Examples, ſome of which I ſhall afterwards explain.

$$\begin{array}{lll} \text{Diviſor.} & \text{Dividend.} & \\ a+\sqrt{bc}) & ab+b\sqrt{bc}\ (b & \text{Quotient.} \\ & ab+b\sqrt{bc} & \\ \hline & 0 \quad 0 & \end{array}$$

$$\begin{array}{lll} a+\sqrt{bc}) & aa-bc & (a-\sqrt{bc}. \\ & aa+a\sqrt{bc} & \\ \hline & -bc-a\sqrt{bc} & \\ & -bc-a\sqrt{bc} & \\ \hline & 0 \quad 0 & \end{array}$$

$\sqrt{ab}-$

$$\begin{array}{l}
\sqrt{ab}-\sqrt{cd}\ )\ ab-cd\ (\ \sqrt{ab}+\sqrt{cd} \\
\qquad\qquad\quad \underline{ab-\sqrt{abcd}\qquad\qquad} \\
\qquad\qquad\qquad -cd+\sqrt{abcd} \\
\qquad\qquad\qquad \underline{-cd+\sqrt{abcd}} \\
\qquad\qquad\qquad\quad 0 \qquad\ 0
\end{array}$$

$$\begin{array}{l}
a+\sqrt{bc}\ )\ aaa+bc\sqrt{bc}\ (\ aa+bc-a\sqrt{bc} \\
\qquad\qquad\ \underline{aaa+aa\sqrt{bc}\qquad\qquad\qquad\quad} \\
\qquad\qquad\quad +bc\sqrt{bc}-aa\sqrt{bc} \\
\qquad\qquad\quad \underline{+bc\sqrt{bc}+abc\qquad\qquad\qquad} \\
\qquad\qquad\qquad\qquad\quad -aa\sqrt{bc}-abc \\
\qquad\qquad\qquad\qquad\quad \underline{-aa\sqrt{bc}-abc} \\
\qquad\qquad\qquad\qquad\qquad\ 0 \qquad\quad 0
\end{array}$$

$$\begin{array}{l}
aa+a\sqrt{bc}\ )\ aaab-abbc\ (\ ab-b\sqrt{bc} \\
\qquad\qquad\quad \underline{aaab+aab\sqrt{bc}\qquad} \\
\qquad\qquad\qquad -abbc-aab\sqrt{bc} \\
\qquad\qquad\qquad \underline{-abbc-aab\sqrt{bc}} \\
\qquad\qquad\qquad\quad 0 \qquad\quad 0
\end{array}$$

$$\begin{array}{l}
a-\sqrt{bc}\ )\ aab-bbc-ab\sqrt{bc}+\frac{bbc}{a}\sqrt{bc}\ (\ ab-\frac{bbc}{a} \\
\qquad\qquad\ \underline{aab\qquad\quad -ab\sqrt{bc}\qquad\qquad\qquad} \\
\qquad\qquad\qquad -bbc\qquad\qquad +\frac{bbc}{a}\sqrt{bc} \\
\qquad\qquad\qquad \underline{-bbc\qquad\qquad +\frac{bbc}{a}\sqrt{bc}} \\
\qquad\qquad\qquad\quad 0 \qquad\qquad\qquad 0
\end{array}$$

EXPLICATION.

In the firſt Example, firſt, ab divided by a gives the Quotient b, by which I multiply the whole Diviſor $a+\sqrt{bc}$, and it makes $ab+b\sqrt{bc}$, this ſubtracted from the given Dividend $ab+b\sqrt{bc}$, there remains 0; ſo the Quotient ſought is b.

In the third Example, firſt, ab divided by $\sqrt{ab}$ gives the Quotient $\sqrt{ab}$, by which I multiply the whole Diviſor $\sqrt{ab}-\sqrt{cd}$, and the Product is $ab-\sqrt{abcd}$, this ſubtracted from the given Dividend $ab-cd$, there remains to be yet divided $-cd+\sqrt{abcd}$; then I divide $-cd$ by $-\sqrt{cd}$, and it gives the Quotient $+\sqrt{cd}$, by which I multiply the whole Diviſor $\sqrt{ab}-\sqrt{cd}$, and it makes $-cd+\sqrt{abcd}$, this ſubtracted from the remaining Dividend $-cd+\sqrt{abcd}$ leaves 0; ſo the Diviſion is finiſh'd, and the Quotient ſought is $\sqrt{ab}+\sqrt{cd}$.

In the ſixth and laſt Example, firſt, aab divided by a gives the Quotient ab, this multiplying the whole Diviſor $a-\sqrt{bc}$ produces $aab-ab\sqrt{bc}$, which ſubtracted from the given Dividend leaves to be yet divided $-bbc+\frac{bbc}{a}\sqrt{bc}$; then I divide $+\frac{bbc}{a}\sqrt{bc}$ by $-\sqrt{bc}$, and it gives the Quotient $-\frac{bbc}{a}$, by which I multiply the whole Diviſor $a-\sqrt{bc}$, and it produces $-bbc+\frac{bbc}{a}\sqrt{bc}$, which ſubtracted from the remaining Dividend $-bbc+\frac{bbc}{a}\sqrt{bc}$ leaves nothing; ſo the Quotient ſought is $ab-\frac{bbc}{a}$.

The Arithmetic of Univerſal Surd Roots, both in Numbers and Quantities expreſs'd by Letters.

Sect. XII. *Multiplication in Univerſal Surds.*

Univerſal Roots are the Roots of Compound Numbers or Quantities. How to expreſs Univerſal Roots, and to find out their values, has already been ſhewn in *Sect.* 28. *Chap.* 1. *Book* 1. I ſhall therefore proceed to their Multiplication.

1. If the ſquare Root of any compound Number to be ſquared, or multiplied into itſelf, caſt away the univerſal Radical Sign $\surd$ or $\surd(2)$, as alſo the Line that is drawn over the compound Number, and the compound Number itſelf ſhall be the Square of the univerſal Root propoſed. Alſo the Cube of the Cubic Root of any compound Number is the compound Number itſelf, the Line drawn over it, and the univerſal Radical Sign $\surd(3)$ being caſt away; and ſo of others.

As for Example, the ſquare of this univerſal ſquare Root $\surd:\overline{12+\surd 3}:$ is $12+\surd 3$; likewiſe the ſquare of $\surd:\overline{12-\surd 3}:$ is $12-\surd 3$; alſo the ſquare of $\surd:\overline{15+\surd 3+\surd 2}:$ is $15+\surd 3+\surd 2$; and the ſquare of $\surd:\overline{15-\surd 3-\surd 2}:$ is $15-\surd 3-\surd 2$.

After the ſame manner the Cube of this univerſal Cubic Root $\surd(3):\overline{\surd 25+\surd 9}:$ is $\surd 25+\surd 9$, that is 8.

Likewiſe the Square of $\surd:\overline{aa+bb}:$ is $aa+bb$, and the Cube of $\surd(3):\overline{bbb+ccc}:$ is $bbb+ccc$; alſo the ſquare of $\surd:\overline{\frac{1}{2}c+\frac{1}{4}cc-n}:$ is $\frac{1}{2}c+\surd\overline{\frac{1}{4}cc-n}:$ and ſo of others.

2. When an univerſal Root is to be multiplied by a Rational Quantity, or by a ſimple or compound Surd, or by any univerſal Root; multiply the ſquare of the Multiplicand by the ſquare of the Multiplier, when the univerſal Radical Sign is Quadratic; or the Cube of the one by the Cube of the other, when the univerſal Radical Sign is Cubic, *&c.* then before that Cubic prefix the given univerſal Radical Sign; ſo ſhall this new univerſal Root be the Product ſought.

As for Example, if it be deſired to double or multiply by 2 this univerſal ſquare Root $\surd:\overline{10+\surd 40}:$ I take the ſquare of 2 which is 4, and the ſquare of $\surd:\overline{10+\surd 40}:$ which (by the foregoing firſt Rule of this *Sect.*) is $10+\surd 40$; then I multiply $10+\surd 40$ by 4, and it makes $40+4\surd 40$, or $40+\surd 640$, whoſe univerſal ſquare Root, to wit, $\surd:\overline{40+4\surd 40}:$ or $\surd:\overline{40+\surd 640}:$ is the Product of $\surd:\overline{10+\surd 40}:$ multiplied by 2, or the ſaid Product may be expreſs'd thus $2\surd:\overline{10+\surd 40}:$

Likewiſe if $\surd(3):\overline{\surd(3)64+\surd(3)27}:$ be to be doubled or multiplied by 2, I firſt multiply each of thoſe Numbers cubically, becauſe the Radical Sign of the given univerſal Root is $\surd(3)$, and their Cubes will be $\surd(3)64+\surd(3)27$ and 8; which multiplied one into the other make $8\surd(3)64+8\surd(3)27$, to which Product I prefix the univerſal Radical Sign $\surd(3)$ and it gives $\surd(3):\overline{8\surd(3)64+8\surd(3)27}:$ that is, $\surd(3):\overline{32+24}:$ or $\surd(3)56$, which is the Product ſought, to wit, the double of $\surd(3):\overline{\surd(3)64+\surd(3)27}:$

After the ſame manner if $\surd(3):\overline{\surd(3)64+\surd(2)36+3}:$ be to be multiplied by 5, or $\surd(3)125$, the Product will be $\surd(3):\overline{125\surd(3)64+125\surd(2)36+375}:$ that is, $\surd(3)1625$.

Again, to multiply $\surd:\overline{\surd 10+\surd 3}:$ by $\surd 5$, their Squares are $\surd 10+\surd 3$ and 5, which multiplied one into another make $5\surd 10+5\surd 3$, (that is, $\surd 250+\surd 75$) whoſe univerſal ſquare Root, to wit, $\surd:\overline{5\surd 10+5\surd 3}:$ (or $\surd:\overline{\surd 250+\surd 75}:$) is the Product of $\surd:\overline{\surd 10+\surd 3}:$ multiplied by $\surd 5$.

Likewiſe, to multiply $\surd:\overline{13+\surd 9}:$ by $\surd:\overline{5+\surd 16}:$ (that is, 4 by 3, where the Product is manifeſtly 12;) the Squares of the univerſal Roots propoſed are $13+\surd 9$ and $5+\surd 16$, which multiplied one into another make $65+5\surd 9+13\surd 16+\surd 144$; whoſe univerſal ſquare Root, to wit, $\surd:\overline{65+5\surd 9+13\surd 16+\surd 144}:$ that is, $\surd 144$, or 12, is the Product ſought.

Again, to multiply $\surd:\overline{\frac{7}{2}+\surd\frac{29}{4}}:$ into $\surd:\overline{\frac{7}{2}-\surd\frac{29}{4}}:$ I multiply their Squares $\frac{7}{2}+\surd\frac{29}{4}$ and $\frac{7}{2}-\surd\frac{29}{4}$ one into another, according to the laſt of the three compendious Rules in *Sect.* 10. of this *Chap.* and there comes forth $\frac{49}{4}-\frac{29}{4}$, that is 5, (to wit, the difference

difference between the Squares of $\frac{7}{2}$ and $\sqrt{\frac{29}{4}}$) Lastly, the Square Root of the said 5 is $\sqrt{5}$ for the Product sought.

So also to multiply $\sqrt{:5+\sqrt{2}:}$ by $\sqrt{5}+\sqrt{2}$, their Squares $5+\sqrt{2}$ and $7+2\sqrt{10}$ multiplied one into another give $35+10\sqrt{10}+7\sqrt{2}+2\sqrt{20}$, whose universal square Root to wit, $\sqrt{:35+10\sqrt{10}+7\sqrt{2}+2\sqrt{20}:}$ is the Product sought.

Moreover, to multiply $\sqrt{:\sqrt{144}+4:}-\sqrt{:\sqrt{4}+2:}$ by $\sqrt{:\sqrt{100}-1:}$ (that is, 2 by 3, which will produce 6) I first multiply the Square of $\sqrt{:\sqrt{144}+4:}$ by the Square of $\sqrt{:\sqrt{100}-1:}$ *viz.* $\sqrt{144}+4$ by $\sqrt{100}-1$, and it makes $\sqrt{14400}+4\sqrt{100}-\sqrt{144}-4$, before which I prefix the universal Radical Sign $\sqrt{}$, and it gives $\sqrt{:\sqrt{14400}+4\sqrt{100}-\sqrt{144}-4:}$ which is one of the Members of the Product sought; then I multiply in like manner $-\sqrt{:\sqrt{4}+2:}$ by $\sqrt{:\sqrt{100}-1:}$ and it makes $-\sqrt{:\sqrt{400}+2\sqrt{100}-\sqrt{4}-2:}$ for the latter Member of the Product sought. Lastly, both those Members being joined together give $\sqrt{:\sqrt{14400}+4\sqrt{100}-\sqrt{144}-4:}$ $-\sqrt{:\sqrt{400}+2\sqrt{100}-\sqrt{4}-2:}$ that is, $\sqrt{144}-\sqrt{36}$, that is, $12-6$ or 6, for the Product required.

3. Sometimes the fourth, fifth, and sixth Rules in *Sect.* 4. of this *Chap.* will be useful in the Multiplication of universal Surds. As if it be desired to multiply $3\sqrt{:2+\sqrt{5}:}$ by $4\sqrt{:2+\sqrt{5}:}$ (which are commensurable Roots, for they are in proportion one to the other as 3 to 4) I multiply 3 by 4, and the Product 12 into $2+\sqrt{5}$; so there is produced $24+12\sqrt{5}$ (that is, $24+\sqrt{720}$) for the Product sought.

Likewise, $5\sqrt{:6+\sqrt{9}:}$ multiplied by $2\sqrt{:6+\sqrt{9}:}$ (that is, 15 by 6) produces $60+10\sqrt{9}$, (that is, 90.)

Moreover, if $5\sqrt{:6+\sqrt{9}:}$ be to be multiplied by $3\sqrt{:19-\sqrt{9}:}$ (that is, 15 by 12) I first multiply 5 by 3 and it makes 15, then I multiply $\sqrt{:6+\sqrt{9}:}$ by $\sqrt{:19-\sqrt{9}:}$ and it produces $\sqrt{:105+13\sqrt{9}:}$ which latter Product multiplied into the former Product 15 makes $15\sqrt{:105+13\sqrt{9}:}$ (that is, 180) the Product sought.

4. Sometimes also the three Rules before delivered in *Sect.* 10. of this *Chap.* concerning the multiplying of Binomials and Residuals will be useful in the Multiplication of universal Surd Roots. As if this Binomial Root $\sqrt{:12+\sqrt{6}:}+\sqrt{:12-\sqrt{6}:}$ be to be squared or multiplied into itself, the Squares of the Parts are $12+\sqrt{6}$ and $12-\sqrt{6}$, whose Sum is 24; then the Product made by the Multiplication of the Parts one into the other, *viz.* $\sqrt{:12+\sqrt{6}:}$ into $\sqrt{:12-\sqrt{6}:}$ is $\sqrt{138}$, (for the difference of the Squares of 12 and $\sqrt{6}$ is 138, whose square Root is $\sqrt{138}$;) and the double of the said Product is $2\sqrt{138}$, which added to 24 (the Sum of the Squares of the Parts) makes $24+2\sqrt{138}$, which is the Square of $\sqrt{:12+\sqrt{6}:}+\sqrt{:12-\sqrt{6}:}$ Moreover, the square Root of the said $24+2\sqrt{138}$, to wit, $\sqrt{:24+2\sqrt{138}:}$ is the Sum of the two Parts $\sqrt{:12+\sqrt{6}:}$ and $\sqrt{:12-\sqrt{6}:}$ For when the Sum of two Numbers is multiplied into itself, the square Root of the Product is equal to the same Sum.

Likewise if $\sqrt{:10+\sqrt{36}:}-\sqrt{:10-\sqrt{36}:}$ that is 2, be to be squared or multiplied into itself, the Product will be found $20-2\sqrt{64}$, that is 4, and the square Root of this 4, to wit 2, is the difference of the two Roots or Parts $\sqrt{:10+\sqrt{36}:}$ and $\sqrt{:10-\sqrt{36}:}$ For when the difference of two Numbers is multiplied into itself, the square Root of the Product is equal to the said difference.

Again, if $6+\sqrt{:20-\sqrt{16}:}$ be to be multiplied into $6-\sqrt{:20-\sqrt{16}:}$ the Product will be found 20; for (according to Rule 3. in *Sect.* 10. of this *Chap.*) if $20-\sqrt{16}$, which is the Square of $\sqrt{:20-\sqrt{16}:}$ be subtracted from 36 the Square of 6, there will remain $16+\sqrt{16}$, that is, 20 the Product sought.

Likewise if $\sqrt{20}+\sqrt{:20-\sqrt{5}:}$ be to be multiplied into $\sqrt{20}-\sqrt{:20-\sqrt{5}:}$ the Product will be $\sqrt{5}$.

So also if $\sqrt{:5+\sqrt{:20-\sqrt{16}:}:}$ be to be multiplied by $\sqrt{:5-\sqrt{:20-\sqrt{16}:}:}$ (that is 3 by 1;) first, the Squares of the universal Roots proposed are $5+\sqrt{:20-\sqrt{16}:}$ and $5-\sqrt{:20-\sqrt{16}:}$ these multiplied one by the other, by taking the difference of the Squares of 5 and $\sqrt{:20-\sqrt{16}:}$ give the Product $5+\sqrt{16}$, whose universal

 square

ſquare Root, to wit, $\sqrt{:5+\sqrt{16}:}$ that is 3, is the Product of the two univerſal ſquare Roots propoſed to be multiplied.

5. The four preceding Rules of this Section are alſo to be obſerved in the Multiplication of univerſal Surd Roots expreſs'd by Letters. As if it be deſired to multiply $\sqrt{:aa+bb:}$ by a, I multiply their Squares $aa+bb$ and aa one into the other, and there comes forth $aaaa+aabb$, whoſe univerſal ſquare Root $\sqrt{:aaaa+aabb:}$ is the Product ſought; which may more compendiouſly be expreſs'd thus, $a\sqrt{:aa+bb:}$

Likewiſe to multiply $\sqrt{:oo+4mp:}$ into $\frac{z}{a}$, I write $\sqrt{\frac{oozz+4mpzz}{aa}}$, or $\frac{z}{a}\sqrt{:oo+4mp:}$ for the Product.

Again, if $\sqrt{:aa+12:}$ be to be multiplied by $a+3$, the Product may be ſignified by $\overline{a+3}$ into $\sqrt{:aa+12:}$ Or, after the Squares of the Quantities propoſed are multiplied one into the other, and the univerſal Radical Sign prefix'd, the Product may be expreſs'd thus, $\sqrt{:aaaa+6aaa+21aa+72a+108:}$

So alſo $\sqrt{bc}$ multiplied into $\sqrt{:aa+bb:}$ produces $\sqrt{:aabc+bbbc:}$ and $\sqrt{:\sqrt{bc}+\sqrt{a}:}$ multiplied by $\sqrt{:\sqrt{ba}-\sqrt{bc}:}$ produces $\sqrt{:b\sqrt{ca}+a\sqrt{b}-bc-\sqrt{abc}:}$ that is, $\sqrt{:\sqrt{bbca}+\sqrt{aab}-bc-\sqrt{abc}:}$

Again, after the manner of the preceding third Rule of this Section $a\sqrt{:bb-cc:}$ multiplied by $d\sqrt{:bb-cc:}$ produces $adbb-adcc$.

And $a\sqrt{:b+c:}$ into $d\sqrt{:b-c:}$ produces $ad\sqrt{:bb-cc:}$

Moreover, if this Binomial Root $\sqrt{:\sqrt{a}+\sqrt{bc}:}+\sqrt{:\sqrt{a}-\sqrt{bc}:}$ be to be ſquared or multiplied into itſelf, firſt, the Squares of the Names or Parts of the Binomial are $\sqrt{a}+\sqrt{bc}$ and $\sqrt{a}-\sqrt{bc}$, which added together make $2\sqrt{a}$; then the double Product of the Parts is $2\sqrt{:a+\sqrt{bc}:}$ (for the difference of the Squares of $\sqrt{a}$ and $\sqrt{bc}$ is $a-bc$, whoſe univerſal ſquare Root doubled is $2\sqrt{:a-bc:}$) which double Product added to $2\sqrt{a}$, (to wit, the ſum of the Squares of the Parts firſt found) makes $2\sqrt{a}+2\sqrt{:a-bc:}$ which is the Square or Product deſired; and if the ſquare Root of this Product be extracted, it gives $\sqrt{:2\sqrt{a}+2\sqrt{:a-bc:}:}$ which is equal to the Sum of the Parts of the Binomial Roots firſt propoſed to be ſquared.

Sect. XIII. *Diviſion in Univerſal Surds.*

Divide the Square of the Dividend by the Square of the Diviſor, when the univerſal Radical Sign is Quadratic, or the Cube of the one by the Cube of the other, when the univerſal Radical Sign is Cubic, *&c.* then prefix the given univerſal Sign to the Quotient, ſo ſhall this new Root be the Quotient ſought.

As for Example, if it be deſired to divide $\sqrt{:40+4\sqrt{40}:}$ by 2, I divide $40+4\sqrt{40}$, which is the Square of the Dividend, by 4 the Square of the Diviſor, (according to *Sect.* 11. of this *Chap.*) and there ariſes $10+\sqrt{40}$, whoſe ſquare Root univerſal, to wit, $\sqrt{:10+\sqrt{40}:}$ is the Quotient ſought.

Again, if it be deſired to divide $\sqrt{:40+4\sqrt{40}:}$ by $\sqrt{:10+\sqrt{40}:}$ firſt, I take their Squares, to wit, $40+4\sqrt{40}$ and $10+\sqrt{40}$ as a Dividend and Diviſor, then becauſe the Diviſor is a Compound Number, a new Dividend and Diviſor muſt be found out, ſuch that the new Diviſor may be a Rational Number; ſo (according to the Rule in the ſixth branch of *Sect.* 11. of this *Chap.*) there will be produced 240 and 60 for a new Dividend and Diviſor, which give the Quotient 4, whoſe ſquare Root is 2 the Quotient ſought, to wit, the Quotient of $\sqrt{:40+4\sqrt{40}:}$ divided by $\sqrt{:10+\sqrt{40}:}$

Likewiſe, to divide 20 by $\sqrt{:10-\sqrt{5}:}$ firſt, I reduce their Squares 400 and $10-\sqrt{5}$ to a new Dividend and Diviſor, to wit, $4000+400\sqrt{5}$ and 95; then I divide $4000+400\sqrt{5}$ by 95, and there ariſes $42\frac{2}{19}+\frac{80}{19}\sqrt{5}$, whoſe univerſal ſquare Root, to wit, $\sqrt{:42\frac{2}{19}+\frac{80}{19}\sqrt{5}:}$ the Quotient ſought.

Another Example (in Rational Numbers expreſs'd Surd-wiſe) may be this, *viz.* ſuppoſe it be deſired to divide $\sqrt{:4+\sqrt{25}:}$ by $\sqrt{:1+\sqrt{9}:}$ (that is, by 3 and 2, which gives the Quotient $1\frac{1}{2}$;) firſt, I reduce $4+\sqrt{25}$ and $1+\sqrt{9}$, the Squares of the given Dividend and

and Divisor, to a new Dividend and Divisor, to wit, $4+\sqrt{25}-4\sqrt{9}-\sqrt{225}$ and -8, these give the Quotient $\frac{9}{4}$, (as has been proved in the latter Example of the Note in the preceding *Sect.* 11.) the square Root whereof, to wit $\frac{3}{2}$, is the Quotient sought; for if the given Divisor $\sqrt{:1+\sqrt{9}:}$ be multiplied by the Quotient $\frac{3}{2}$, it will produce 3, which is equal to the given Dividend $\sqrt{:4+\sqrt{25}:}$

Again, to divide $\sqrt{(3)}\overline{:8\sqrt{(3)}64+8\sqrt{(2)}27:}$ by 2, I divide the Cube of the one by the Cube of the other, *viz.* $8\sqrt{(3)}64+8\sqrt{(2)}27$ by 8, and there arises $\sqrt{(3)}64+\sqrt{(2)}27$, whose universal Cubic Root, to wit, $\sqrt{(3)}\overline{:\sqrt{64}+\sqrt{(2)}27:}$ is the Quotient sought, to wit, the half of the Dividend proposed.

2. If the given universal Roots, to wit, the Dividend and Divisor be commensurable, then (according to the fifth Rule of *Sect.* 5. of this *Chap.*) divide the Rational part of the Dividend by the Rational part of the Divisor, and what arises is the Quotient sought. As to divide $21\sqrt{:6+\sqrt{9}:}$ by $3\sqrt{:6+\sqrt{9}:}$ I divide 21 by 3, and there arises 7 for the Quotient sought.

Likewise $183\sqrt{:\sqrt{3}-\sqrt{2}:}$ divided by $\frac{61}{8}\sqrt{:\sqrt{3}-2:}$ gives the Quotient 24.

3. Division in universal Surds express'd by Letters depends upon the Rules before given: as to divide $\sqrt{:aaaa+aabb:}$ by a, I divide the Square of the Dividend by the Square of the Divisor, *viz.* $aaaa+aabb$ by aa, and there arises $aa+bb$, whose square Root universal, to wit, $\sqrt{:aa+bb:}$ is the Quotient sought.

Again, if it be desired to divide $\sqrt{:\sqrt{bbca}+\sqrt{aab}-bc-\sqrt{abc}:}$ by $\sqrt{:bc+\sqrt{a}:}$ I divide the square of the Dividend by the square of the Divisor, *viz.* $\sqrt{bbca}+\sqrt{aab}-bc-\sqrt{abc}$ by $\sqrt{bc}+\sqrt{a}$, (according to the Method in the Examples at the latter end of *Sect* 11. of this *Chap.*) and there arises $\sqrt{ba}-\sqrt{bc}$, whose universal square Root, to wit, $\sqrt{:\sqrt{ba}-\sqrt{bc}:}$ is the Quotient sought.

Moreover, to divide $d\sqrt{:bb+cc:}$ by $3a\sqrt{:bb+cc:}$ because they are commensurable, I divide only the Rational part by the Rational, and there arises $\frac{d}{3a}$ for the Quotient.

4. Lastly, when the work of Division in universal Surds according to the foregoing Rules and Examples in this Section, happens to be intricate, or will not work off just without a Remainder, you may set the Power of the Dividend (the universal Radical Sign being omitted) as a Numerator, over the Power of the Divisor as a Denominator, and prefix the universal Radical Sign before the Line that separates the Numerator from the Denominator; then shall the universal Root so denoted signifie the Quotient sought.

As if it be desired to divide $\sqrt{:\sqrt{5}+\sqrt{8}-3:}$ by $\sqrt{:\sqrt{7}-\sqrt{2}+1:}$ the Quotient may be represented by this Fraction $\sqrt{:\frac{\sqrt{5}+\sqrt{8}-3}{\sqrt{7}-\sqrt{2}+1}:}$

Likewise if $\sqrt{:\sqrt{abb+bcd}:}$ be to be divided by $\sqrt{:\sqrt{ac-dd}:}$ you may write $\sqrt{:\frac{\sqrt{abb+bcd}}{\sqrt{ac-dd}}:}$ to signifie the Quotient.

Sect. XIV. *Addition and Subtraction in Universal Surds.*

1. When two universal Surds proposed to be added or subtracted are commensurable, they may be added or subtracted like simple Surds, (according to the Rule in *Sect.* 8. of this *Chap.*) As for Example, if the Sum and Difference of $\sqrt{:8+4\sqrt{3}:}$ and $\sqrt{:2+\sqrt{3}:}$ be desired; because each of them divided by their common Divisor $\sqrt{:2+\sqrt{3}:}$ gives $\sqrt{4}$ and $\sqrt{1}$, that is, 2 and 1, which are Rational Numbers expressing the proportion of the Surds proposed. Therefore the Sum of 2 and 1, to wit, 3 multiplied into the said common Divisor gives $3\sqrt{:2+\sqrt{3}:}$ for the Sum required, (which may also be express'd thus, $\sqrt{:18+\sqrt{243}:}$) and the difference of the said 2 and 1, to wit, 1 multiplied into the said common Divisor $\sqrt{:2+\sqrt{3}:}$ makes $\sqrt{:2+\sqrt{3}:}$ for the difference of the two Roots first proposed.

Another Example in Rational Numbers express'd Surd-wise, *viz.* let it be required to find out the Sum and Difference of $\sqrt{:99+9\sqrt{25}:}$ and $\sqrt{:44+4\sqrt{25}:}$ (that is, 12 and 8; first, those universal Roots being severally divided by the common Divisor

$\sqrt{:11}$

$\sqrt{:11+\sqrt{25}:}$ give the Quotients $\sqrt{9}$ and $\sqrt{4}$, to wit 3 and 2, which are Rational Numbers expreſſing the Proportion which the given Roots have one to another. Therefore $3+2$, to wit 5, multiplied into the common Diviſor $\sqrt{:21+\sqrt{25}:}$ gives $5\sqrt{:11+\sqrt{25}:}$ that is, $\sqrt{:275+\sqrt{15625}:}$ (to wit 20) which is the Sum of the Roots propoſed; and $3-2$, that is 1, multiplied into the ſaid $\sqrt{:11+\sqrt{25}:}$ gives $\sqrt{:11+\sqrt{25}:}$ that is 4, for the Difference of the given Roots.

Here follow Contractions of the work of Addition and Subtraction in the two laſt Examples, with others of like nature in Surd Quantities expreſs'd by Letters.

Example 1.

What is the Sum and Difference of . . . . . . $\sqrt{:8+4\sqrt{3}:}$ and $\sqrt{:2+\sqrt{3}:}$

The Operation.

I. $\sqrt{:2+\sqrt{3}:}$) $\sqrt{:8+4\sqrt{3}:}$ ($\sqrt{4}$, that is, 2.
II. $\sqrt{:2+\sqrt{3}:}$) $\sqrt{:2+\sqrt{3}:}$ ($\sqrt{1}$, that is, 1.
Therefore from I. $2\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:}$
And from II. $1\sqrt{:2+\sqrt{3}:} = \sqrt{:2+\sqrt{3}:}$
The Sum, $3\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:} + \sqrt{:2+\sqrt{3}:}$
The Difference $1\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:} - \sqrt{:2+\sqrt{3}:}$

Example 2.

What is the Sum and Difference of $\sqrt{:99+9\sqrt{25}:}$ and $\sqrt{:44+4\sqrt{25}:}$

The Operation.

I. $\sqrt{:11+\sqrt{25}:}$) $\sqrt{:99+9\sqrt{25}:}$ ($\sqrt{9}$, that is, 3.
II. $\sqrt{:11+\sqrt{25}:}$) $\sqrt{:44+4\sqrt{25}:}$ ($\sqrt{4}$, that is, 2.
Therefore from I. $3\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:}$
And from II. $2\sqrt{:11+\sqrt{25}:} = \sqrt{:44+4\sqrt{25}:}$
The Sum, $5\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:} + \sqrt{:44+4\sqrt{25}:}$
The Difference, $1\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:} - \sqrt{:44+4\sqrt{25}:}$

Example 3.

What is the Sum and Difference of $\sqrt{:aaaa+aabb:}$ and $\sqrt{:aabb+bbbb:}$
Thoſe reduced (by *Sect.* 6. of this *Chap*) give $a\sqrt{:aa+bb:}$ and $b\sqrt{:aa+bb:}$
Therefore their Sum is . . . . $\overline{a+b}$ into $\sqrt{:aa+bb:}$
And their Difference is . . . . $\overline{a \backsim b}$ into $\sqrt{:aa+bb:}$

Example 4.

What is the Sum and Difference of $\Big\}$ $\sqrt{\frac{oozz+4mpzz}{aa}}$ and $\sqrt{\frac{aaoomm+4aammmp}{ppzz}}$

By dividing each of them by their common Diviſor $\sqrt{:oo+4mp:}$ there will ariſe Rational Quotients, to wit, ... $\Big\}$ $\frac{z}{a}$ and $\frac{am}{pz}$

Therefore the Surds propoſed are Commenſurable and inſtead of them we may write . . . $\Big\}$ $\frac{z}{a}\sqrt{:oo+4mp:}$ and $\frac{am}{pz}\sqrt{:oo+4mp:}$

Therefore their Sum ſhall be . $\Big\}$ $\overline{\frac{z}{a}+\frac{am}{pz}}$ into $\sqrt{:oo+4mp:}$

That is, ... $\Big\}$ $\frac{pzz+aam}{apz}$ into $\sqrt{:oo+4mp:}$

And

And the Difference of the given Surds shall be $\left\{ \frac{pzz \backsim aam}{apz} \right.$ into $\sqrt{:oo+4mp:}$

Example 5.

What is the Sum and Difference of these two universal Roots?

$$\sqrt{:aaaa + 6aaa + 21aa + 72a + 108:} \quad \text{and,}$$
$$\sqrt{:aaaa + 10aaa + 37aa - 120a + 300:}$$

The Operation.

The given Roots are Commensurable, (as has been shewn in the last Example but one in *Sect.* 7. of this *Chap*) and may be express'd thus;

$$\overline{a + 2}\sqrt{:aa + 12:} \text{ and } \overline{a \backsim 5}\sqrt{:aa + 12:}$$

Therefore their Sum, supposing a to be greater than 5, shall be

$$\overline{2a - 2} \text{ into } \sqrt{:aa + 12:}$$

And their Difference shall be $\quad 8\sqrt{:aa + 12:}$

But if we suppose a to be less than 5, then the Sum of the given Surds will be $8\sqrt{:aa+12:}$ and their Difference $\overline{2a \backsim 2}\sqrt{:aa+12:}$ that is, $\overline{2a \backsim 2}$ into $\sqrt{:aa+12:}$

2. When the Root of a Residual is to be added unto, or subtracted from, the Root of its correspondent Binomial, those Roots may be connected together by + or —; and then the whole being multiplied into itself, the universal Root of the Product shall be the Sum or Difference of the Roots given to be added or subtracted, as before has been shewn in Rule 4. *Sect.* 12. of this *Chap.*

As if these two Roots be proposed to be added, to wit, $\sqrt{:12+\sqrt{6}:}$ and $\sqrt{:12-\sqrt{6}:}$ we may multiply this composed Number $\sqrt{:12+\sqrt{6}:} + \sqrt{:12-\sqrt{6}:}$ into itself, and there will be produced $24+2\sqrt{138}$, whose universal square Root, to wit, $\sqrt{:24+2\sqrt{138}:}$ shall be the Sum of the two Roots proposed to be added.

Likewise if $\sqrt{:12+\sqrt{6}:} - \sqrt{:12-\sqrt{6}:}$ be multiplied into itself, the Product will be $24-2\sqrt{138}$, whose universal square Root, to wit, $\sqrt{:24-2\sqrt{138}:}$ is the Difference of the two Roots proposed.

After the same manner the Sum of these two Roots, $\sqrt{:10+\sqrt{36}:}$ and $\sqrt{:10-\sqrt{36}:}$ will be found $\sqrt{:20+2\sqrt{64}:}$ (that is, $\sqrt{36}$, to wit 6;) but their Difference $\sqrt{:20-2\sqrt{64}:}$ (that is $\sqrt{4}$, to wit 2.)

Likewise the Sum of these Binomial Roots $\sqrt{:\sqrt{a}+\sqrt{bc}:}$ and $\sqrt{:\sqrt{a}-\sqrt{bc}:}$ will be found $\sqrt{:2\sqrt{a}+2\sqrt{:a-bc}:}$ and their Difference $\sqrt{:2\sqrt{a}-2\sqrt{:a-bc:}}$

3. But if the universal Roots proposed be not Commensurable, nor such Binomials and Residuals as are mentioned in the last preceding Rule, then they are to be added by +, and subtracted by —.

As if $\sqrt{:5+\sqrt{2}:}$ and $\sqrt{:5-\sqrt{3}:}$ be to be added, I write $\sqrt{:5+\sqrt{2}:} + \sqrt{:-\sqrt{3}:}$ for the Sum, and to subtract $\sqrt{:5-\sqrt{3}:}$ from $\sqrt{:5+\sqrt{2}:}$ I write $\sqrt{:5+\sqrt{2}:} - \sqrt{:5-\sqrt{3}:}$ for the Remainder.

Likewise the Sum of $\sqrt{:aa+bb:}$ and $\sqrt{:aa-cc:}$ is $\sqrt{:aa+bb:} + \sqrt{:aa-cc:}$ and their Difference is $\sqrt{:aa+bb:} - \sqrt{:aa-cc:}$

Sect. XV. *Concering the Constitution and Invention of six Binomials in Numbers, agreeable to those expounded in* Prop. 49, 50, 51, 52, 53, 54. Elem. 10. Eucl.

By way of preparation to the Construction of the six Binomials in Numbers I shall premise this

QUESTION.

To find two square Numbers whose Difference may be equal to a given Rational Number?

CANON.

Take any two Numbers, which multiplied one by the other will produce the given Num-

Number; then half the Sum of those two Numbers and half their difference shall be the Sides or Roots of the two Squares sought.

As if 5 be given for the difference of two Squares sought, I take 5 and 1; for the Product of their Multiplication is 5; then the half of their Sum is 3, and the half of their difference is 2; lastly, the Squares of the said 3 and 2 are 9 and 4, the Squares sought; for their difference is 5, as was prescribed.

Again, the same Number 5 being given for the difference of two Squares, take a Number at pleasure, as 2, by this divide the given Number 5, the Quotient is $\frac{5}{2}$, therefore the Product of the Multiplication of the Divisor 2 by the Quotient $\frac{5}{2}$ is 5; then according to the Canon, half the sum and half the difference of the said 2 and $\frac{5}{2}$, to wit, $\frac{9}{4}$ and $\frac{1}{4}$, shall be the Sides of the Squares sought; and consequently the squares themselves are $\frac{81}{16}$ and $\frac{1}{16}$, whose difference is 5, as was desired.

After the same manner innumerable pairs of squares may be found out in Rational Numbers, and the difference of each pair shall be equal to one and the same given Number.

The Reason of the Canon may be made manifest by this

Theorem.

The Product made by the Multiplication of any two unequal Numbers is equal to the difference of two squares, to wit, of the square of half the sum, and the square of half the difference of the same two unequal Numbers.

As if c be the greater, and b the lesser of two Numbers, then

The Square of $\frac{1}{2}c+\frac{1}{2}b$ is . . . . . . $\frac{1}{4}cc+\frac{1}{2}cb+\frac{1}{4}bb$,
The Square of $\frac{1}{2}c-\frac{1}{2}b$ is . . . . . . $\frac{1}{4}cc-\frac{1}{2}cb+\frac{1}{4}bb$,
The difference of those two Squares is . . . cb

Which difference is manifestly the Product of the Multiplication of the two proposed Numbers c and b; wherefore the Theorem, and consequently the Canon first given, is manifest.

The Definition of Binomial I.

When the greater Name (or Part) of a Binomial is a Rational Number, and the lesser part is a Surd square Root of some Rational Number, the square Root of the difference of the Squares of the parts is a Rational Number, the sum of the two parts is called a first Binomial.

Explication.

Let this Binomial be proposed, ...	$3+\sqrt{5}$
The Squares of the Names or Parts are	9 5
The difference of those Squares is	4
The square Root of that difference is	2

Because the greater part 3 is a Rational Number, and the lesser part $\sqrt{5}$ is a Surd square Root of a Rational Number 5, and the difference of the Squares of the Parts, *viz.* 4, is a Square whose Root 2 is a Rational Number; the Binomial proposed, to wit, $3+\sqrt{5}$, is called a first Binomial.

How to find out two such Numbers as may constitute a first Binomial.

1. By the Canon of the preceding Question at the beginning of this 15 *Sect.* find out two square Numbers, whose difference may be some Rational Number not a Square, such are these Squares, .	9 4
2. Their difference is	5
3. Take some Rational Number at pleasure for the greater part of the Binomial sought, as	6
4. Then say, By the Rule of Three if 9 the greater of the two squares found out in the first step, give 5 the difference in the second, what shall 36 the square of the Number taken in the third give? whence the fourth Proportional will be found 20, the square Root whereof is the lesser part, to wit, ...	$\sqrt{20}$
5. I say, The sum of the two Numbers found out in the third and fourth steps, is a first Binomial, to wit	$6+\sqrt{20}$

The

The Definition of Binomial II.

When the lesser part of a Binomial is a Rational Number, and the greater part is a Surd square Root of a Rational Number, and the square Root of the Difference of the Squares of the Parts is Commensurable to the greater Part, the Sum of the two Parts is called a second Binomial.

Explication.

Let this Binomial be proposed . . . . $\sqrt{18}+4$

The Squares of the Parts are . . . . $\left\{\begin{array}{l}18\\16\end{array}\right.$

The Difference of those Squares is 2

The square Root of the Difference is . . . . $\sqrt{2}$

Because the lesser Part 4 is a Rational Number, and the greater Part $\sqrt{18}$ is the Surd square Root of a Rational Number 18, and the square Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{2}$, is Commensurable to the greater Part $\sqrt{18}$; (for according to the Definition in *Sect* 7. of this *Chap.* $\sqrt{2} \,.\, \sqrt{18} :: 1 \,.\, 3$, that is, as a Rational Number to a Rational Number) the proposed Number $\sqrt{18}+4$ is a second Binomial.

How to find out two such Numbers as may constitute a second Binomial.

1. By the foregoing Canon find out two square Numbers, whose Difference may be some Rational Number not a Square; such are these Squares . . . . $\left\{\begin{array}{l}9\\4\end{array}\right.$
2. Their Difference is 5
3. Take some Rational Number at pleasure for the lesser Part of the Binomial sought, as 10
4. Then say, If 5 the Difference in the third step gives 9 the greater of the two Squares in the first; what shall 100 the Square of the Number taken in the third give? Whence you will find 180, whose square Root shall be the greater part, *viz.* . . . $\sqrt{180}$
5. I say, The Sum of the two Numbers found out in the third and fourth steps is a second Binomial, *viz.* . . . . $\sqrt{180}+10$

The Definition of Binomial III.

When each of the two Parts of a Binomial is a Surd Square of a Rational Number, and the square Root of the Difference of the Squares of the Parts is Commensurable to the greater Part, the Sum of the two Parts is called a third Binomial.

Explication.

Let this Binomial be proposed . . . . $50+\sqrt{32}$

The Squares of the Parts are . . . . $\left\{\begin{array}{l}50\\32\end{array}\right.$

The Difference of those Squares is 18

The square Root of that Difference is . . . . $\sqrt{18}$

Because the two Parts $\sqrt{50}$ and $\sqrt{32}$ are Surd square Roots of two Rational Numbers 50 and 32, and the square Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{18}$, is Commensurable to the greater Part $\sqrt{50}$; (for $\sqrt{18} \,.\, \sqrt{50} :: 3 \,.\, 5$, that is, as a Rational Number to a Rational Number) the proposed Number $\sqrt{50}+\sqrt{32}$ is a third Binomial.

How to find out two such Numbers as may constitute a third Binomial.

1. Find out two square Numbers whose Difference may be some Rational Number not a Square; such are these Squares . . . $\left\{\begin{array}{l}9\\4\end{array}\right.$
2. Their Difference is 5
3. Take some Rational Number not a Square, which may exceed the said Differences 5 by an Unit or two, *viz.* by 1, when the said Difference increased with 1 makes not a Square; but by 2, when the Difference increased with 1 makes a Square: So in this Example I take 6, because 5+1 makes not a Square . . . 6
4. Again, take some Rational Number at pleasure, as 12

5. The

5. The square thereof is 144
6. Then say, If 6 the Number taken in the third step gives 9 the greater of the two squares in the first, what shall 144 the square Number in the fifth give? whence the fourth Proportional is 216, whose square Root, to wit $\sqrt{216}$, shall be the greater part. } $\sqrt{216}$
7. Say again, If the said Square 9 gives 5 the Difference in the second step, what shall 216 the fourth Proportional found out in the sixth give? Whence you will find 120, whose square Root, to wit $\sqrt{120}$, shall be the lesser part . . . . } $\sqrt{120}$
8. I say, the sum of the two Numbers found out in the sixth and seventh steps is a third Binomial, to wit, ... } $\sqrt{216}+\sqrt{120}$

The Definition of Binomial IV.

When the greater part of a Binomial is a Rational Number, and the lesser part is a Surd square Root of a Rational Number, and the square Root of the Difference of the squares of the parts is Incommensurable to the greater part, the Sum of the two parts is called a fourth Binomial.

Explication.

Let this Binomial be proposed	$5+\sqrt{12}$
The Squares of the Parts are	$\begin{cases}25\\12\end{cases}$
The Difference of those Squares is	13
The square Root of that Difference is	$\sqrt{13}$

Because the greater part 5 is a Rational Number, and the lesser part $\sqrt{12}$ is a Surd square Root of a Rational Number 12, and the square Root of the Difference of the squares of the Parts, *viz.* $\sqrt{13}$, is Incommensurable to the greater part 5; (for $\sqrt{13}$ has not such proportion to 5 as a Rational Number to a Rational Number) the Number $5+\sqrt{12}$ above proposed is a fourth Binomial.

How to find out two such Numbers as may constitute a fourth Binomial.

1. Take any square Number, as 9
2. Divide that square Number 9 into two Numbers not squares, as into . . . . } 6 and 3
3. Take a Rational Number at pleasure for the greater part of the Binomial sought, as . . . . } 6
4. Then say, If 9 the square Number in the first step give 6 the greater of the two Numbers in the second, what shall 36 the square of the Number taken in the third give? So the fourth Proportional will be found 24, whose square Root, to wit $\sqrt{24}$, shall be the lesser part. . . . . } $\sqrt{24}$
5. I say, The Sum of the two Numbers found out in the third and fourth steps is a fourth Binomial, *viz.* . . . . . . } $6+\sqrt{24}$

The Definition of Binomial V.

When the lesser part of a Binomial is a Rational Number, and the greater part is a Surd square Root of some Rational Number, and the square Root of the Difference of the squares of the Parts is Incommensurable to the greater part, the Sum of the two Parts is called a fifth Binomial.

Explication.

Let this Binomial be proposed	$\sqrt{6}+2$
The squares of the Parts are	$\begin{cases}6\\4\end{cases}$
The Difference of those squares is	2
The square Root of the Difference is	$\sqrt{2}$

Because the lesser part 2 is a Rational Number, and the greater part $\sqrt{6}$ is a Surd square Root of a Rational Number 6, and the square Root of the Difference of the squares of the parts, *viz.* $\sqrt{2}$, is Incommensurable to the greater part $\sqrt{6}$; (for $\sqrt{2} . \sqrt{6} :: 1 . \sqrt{3}$, not as a Rational Number to a Rational Number) the proposed Number $\sqrt{6}+2$ is a fifth Binomial.

How

How to find out two ſuch Numbers as may conſtitute a fifth Binomiol.

1. Take any ſquare Number, as	9
2. Divide that ſquare Number 9 into two Numbers not ſquares, as into .	6 and 3
3. Take a Rational Number at pleaſure for the leſſer part of the Binomial ſought, as	2
4. Then ſay, If 6 the greater of the two Numbers in the ſecond ſtep gives 9 the ſquare Number in the firſt; what ſhall 4 the ſquare of the Rational Number taken in the third give? Whence you will find the fourth Proportional 6, whoſe ſquare Root, to wit $\surd 6$, ſhall be the greater part ſought, ...	$\surd 6$
5. I ſay, The ſum of the two Numbers found out in the third and fourth ſteps is a fifth Binomial, *viz.*	$2+\surd 6$

The Definition of Binomial VI.

When each of the two parts of a Binomial is a Surd ſquare Root of ſome Rational Number, and the ſquare Root of the Difference of the Squares of the Parts is Incommenſurable to the greater part, the Sum of the two parts is called a fixth Binomial.

Explication.

Let this Binomial be propoſed	$\surd 5+\surd 3$
The Squares of the Parts are	5 3
The difference of the Squares of the Parts is . . .	2
The ſquare Root of that difference is	$\surd 2$

Becauſe the two Parts $\surd 5$ and $\surd 3$ are Surd ſquare Roots of two Rational Numbers 5 and 3, and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\surd 2$, is Incommenſurable to the greater part $\surd 5$; (for $\surd 2$ has not ſuch a proportion to $\surd 5$ as a Rational Number to a Rational Number) the Number $\surd 5+\surd 3$ above propoſed is a fixth Binomial.

How to find out two ſuch Numbers as may conſtitute a ſixth Binomial.

1. Take two ſuch prime Numbers that their Sum may not be a Square as	7 and 5
2. Their Sum is	12
3. Take alſo any ſquare Number, as	9
4. Take again ſome Rational Number at pleaſure, as	6
5. The ſquare thereof is	36
6. Then ſay, If 9 the ſquare Number taken in the third ſtep, gives 12 the ſum of the two prime Numbers in the firſt, what ſhall 36 the ſquare in the fifth ſtep give? Whence you will find 48, whoſe ſquare Root, to wit, $\surd 48$, ſhall be the greater part, ...	$\surd 48$
7. Say again, If 12 the ſum of the two prime Numbers in the firſt ſtep, gives 7 the greater of thoſe prime Numbers, what ſhall 48 the fourth Proportional found out in the ſixth ſtep give? Whence you will find 28, whoſe ſquare Root, *viz.* $\surd 28$, ſhall be the leſſer part,	$\surd 28$
I ſay, the ſum of the two Numbers found out in the ſixth and ſeventh ſteps is a ſixth Binomial, *viz.*	$\surd 48+\surd 28$

If of every one of thoſe ſix Binomials the leſſer part be ſubtracted from the greater, by interpoſing the Sign —, the ſix Remainders anſwer to the ſix Lines which *Euclid* in *Prop.* 86, 87, 88, 89, 90, 91. of his *Elem.* 10. calls *Apotomes* or *Reſidual Lines*; as,

Out of Binomial		By changing + into — is made Reſidual	
	I. $3+\surd 5$		I. $3-\surd 5$
	II. $\surd 18+4$		II. $\surd 18-4$
	III. $\surd 50+\surd 32$		III. $\surd 50-\surd 32$
	IV. $5+\surd 12$		IV. $5-\surd 12$
	V. $\surd 6+2$		V. $\surd 6-2$
	VI. $\surd 5+\surd 3$		VI. $\surd 5-\surd 3$

The precedent Conſtructions of the ſaid ſix Binomials are demonſtrated in *Prop.* 49, 50, 51, 52, 53, 54. of 10 *Elem. Euclid.*

Now if any Binomial or Residual be given, we may easily find out another of the same kind in this manner, *viz* For the first and fourth Binomials, if it be made as the greater Name or Part to the lesser, so any Rational Number assumed for the greater Part of a new first or fourth Binomial, to a fourth Proportional Number, this Number shall be the lesser Part of the new first or fourth Binomial. But for the second and fifth, if it be made as the lesser part to the greater, so any Rational Number taken for the lesser part of a new second or fifth Binomial to a fourth Proportional, the Number so produced shall be the greater part of the new second or fifth Binomial. And lastly, for the third and sixth Binomials, if it be made as the greater Part to the lesser, (each of which is a Surd square Root) so any Surd square Root assumed for the greater Part of a new third or sixth Binomial, to a fourth Proportional, there will come forth the lesser part of a new third or sixth Binomial. (The reason of this Operation is manifest *per Prop.* 15. *Elem.* 10. *Eucl.*) And after a new Binomial is found out, its correspondent Residual is also made by changing the Sign $+$ into $-$, as before has been said.

As for Example, if a first Binomial $3+\sqrt{5}$ be proposed, to find another like to it; I take a Rational Number at pleasure, as 8, for the greater Part of the Binomial sought, then by the Rule of Three as 3 is to $\sqrt{5}$, so 8 to a fourth Proportional, to wit $\sqrt{\frac{320}{9}}$, for the lesser Part sought, therefore $8+\sqrt{\frac{320}{9}}$ shall be a new first Binomial, and $8-\frac{320}{9}$ a new first Residual; and so of the rest.

Sect. XVI. *Concerning the Extraction of the square Root out of Binomials and Residuals constituted in such manner as has been shewn in the preceding* Sect. 15.

Every one of the Binomials and Residuals, whose Construction has been shewn in the preceding *Sect.* 15. has a square Root, that is, such a Binomial or Residual that if it be multiplied into itself will produce the given Binomial or Residual; as may be evidently collected out of *Prop.* 55, 56, 57, 58, 59, and 60; also out of *Prop.* 92, 93, 94, 95, 96, and 97. of the tenth Book of *Euclid*'s Elements.

As for Example, a Binomial of the first kind, suppose $7+\sqrt{48}$, has a square Root, to wit $2+\sqrt{3}$, for this being squared (or multiplied into itself, produces that Binomial $7+48$, whose greater Part 7 is composed of 4 and 3, the Squares of the Parts of the Root $2+\sqrt{3}$; and the lesser part $\sqrt{48}$ is the double of the Product made by the Multiplication of 2 into $\sqrt{3}$, the Parts of the Root $2+\sqrt{3}$: all which is evident by the Multiplication of $2+\sqrt{3}$ into itself. The like effect will be found in every one of the rest of the Binomials constituted in the preceding *Sect.* 15. Therefore if a Binomial be proposed, and its square Root desired, there is given the Sum of the Squares of the Parts of the Root, (which Sum is the greater Part of the Binomial proposed) and the double of the Product of the Parts of the Root (which double Product is the lesser Part of the Binomial proposed) to find out the two Parts of the Root severally. And therefore in order to the Extraction of the square Root of a Binomial, it will be requisite to search out a Canon for the solving of this following

QUESTION.

The Sum (b) of the Squares of two Numbers being given, as also (c) the double Product of the Multiplication of the same two Numbers, to find the Numbers severally.

RESOLUTION.

1. For one of the two Numbers sought put	a
2. Then forasmuch as the double of the Product of their Multiplication is given c, therefore the Product itself is	$\frac{c}{2}$
3. Which Product divided by the first Number a gives the other Number	$\frac{c}{2a}$
4. Therefore the Square of the first Number is	aa
5. And the Square of the other Number is	$\frac{cc}{4aa}$
6. Therefore the Sum of the squares of the two Numbers is	$aa+\frac{cc}{4aa}$

7. Which

7. Which ſum muſt be equal to b the given ſum of the ſquares; hence this Equation, ... $aa + \frac{cc}{4aa} = b$
8. From this Equation after due Reduction, there will ariſe . $baa - aaaa = \frac{1}{4}cc$
9. And from the laſt Equation (*per* Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) there will ariſe this following Canon, to find out the two Numbers ſought, *viz.*

CANON 1.

$$\begin{cases} \sqrt{:\tfrac{1}{2}b + \sqrt{.\tfrac{1}{4}bb - \tfrac{1}{4}cc.}:} = \text{the greater Number.} \\ \sqrt{:\tfrac{1}{2}b - \sqrt{.\tfrac{1}{4}bb - \tfrac{1}{4}cc.}:} = \text{the leſſer Number.} \end{cases}$$

That is in words,

From a quarter of the ſquare of the given ſum of the ſquares, ſubtract a quarter of the ſquare of the double Product given, then add and ſubtract the ſquare Root of that Remainder to and from half the given ſum of the Squares, ſo ſhall the ſquare Roots of the Sum and Remainder of that Addition and Subtraction be the two Numbers ſought.

10. Moreover, becauſe $\frac{b + \sqrt{bb - cc}}{2} = \tfrac{1}{2}b + \sqrt{:\tfrac{1}{4}bb - \tfrac{1}{4}cc:}$

11. Therefore . . . . $\sqrt{:\frac{b + \sqrt{bb - cc}}{2}:} = \sqrt{:\tfrac{1}{2}b + \sqrt{.\tfrac{1}{4}bb - \tfrac{1}{4}cc}:}$

12. Likewiſe, becauſe $\frac{b - \sqrt{bb - cc}}{2} = \tfrac{1}{2}b - \sqrt{:\tfrac{1}{4}bb - \tfrac{1}{4}cc:}$

13. Therefore . . . . $\sqrt{:\frac{b - \sqrt{bb - cc}}{2}:} = \sqrt{\tfrac{1}{2}:b - \sqrt{:\tfrac{1}{4}bb - \tfrac{1}{4}cc:}}$

14. Therefore from the eleventh and thirteenth ſteps another Canon ariſes to ſolve the Queſtion, *viz.*

CANON 2.

$$\begin{cases} \sqrt{:\frac{b + \sqrt{bb - cc}}{2}:} = \text{the greater Number.} \\ \sqrt{:\frac{b - \sqrt{bb - cc}}{2}:} = \text{the leſſer Number.} \end{cases}$$

That is in words,

From the Square of the given Sum of the Squares ſubtract the Square of the double Product given, then add and ſubtract the ſquare Root of the Remainder to and from the given Sum of the Squares; ſo ſhall the ſquare Root of half the Sum and Remainder of that Addition and Subtraction be the two Numbers ſought.

By the help of either of thoſe Canons we may extract the ſquare Root of a Binomial or Reſidual, but I ſhall uſe the latter only, whence ariſes

A general Rule for the Extraction of the Square Root out of Binomials and Reſiduals.

From the Square of a greater part of a given Binomial or Reſidual ſubtract the Square of the leſſer, then add the ſquare Root of the Remainder to the greater part, and ſubtract it alſo from the ſame; laſtly, connect the ſquare Roots of the half of that Sum and Remainder by the Sign + if a Binomial be propoſed, but by — if a Reſidual: ſo you have the deſired ſquare Root of the given Binomial or Reſidual.

The Practice of this Rule will be ſhewn at large in the following Examples.

Example 1.

Let it be required to extract the ſquare Root out of this firſt Binomial $27 + \sqrt{704}$.

The Operation.

1. From the Square of the greater part 27, *viz.* from .	729
2 Subtract the Square of the leſſer part $\sqrt{704}$, to wit, .	704
3. The Remainder is	25
4. The ſquare Root of that Remainder is	5

5. To

5. To which square Root add the greater part 27
6. The Sum is 32
7. The half of that sum is 16
8. The square Root of the said half Sum is the greater part of the Root sought, to wit, ... 4
9. Then from the greater part of the given Binomial, *viz.* from . . 27
10. Subtract the square Root before found in the fourth step, to wit, . 5
11. The Remainder is 22
12. The half of which Remainder is 11
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, ... $\sqrt{11}$
14. I say, the two Names or Parts in the eighth and thirteenth steps being connected by + shall be the square Root sought, to wit. $4+\sqrt{11}$

But if — instead of + be prefix'd to the lesser part of the said Root, it will give $4-\sqrt{11}$, which is the square Root of the first Residual or Apotome $27-\sqrt{704}$.

The former of those two Roots answers to the Irrational Line called (in *Prop.* 37. & 55. *lib.* 10. *Elem. Eucl.*) a *Binomial Line*, and the latter answers to the Irrational Line called (in *Prop.* 74. & 92.) an *Apotome* or *Residual Line.*

The Proof of the Root above extracted out of the first Binomial is made by multiplying the Root into itself thus:

The Sum of the Squares of the Parts of $4+\sqrt{11}$, the Root found out is	$16+11$, that is, 27
The Product of the same Parts multiplied one into the other is	$4\sqrt{11}$, that is, $\sqrt{176}$
The double of the said Product is	$8\sqrt{11}$, that is, $\sqrt{176}$.
The Sum of the said Sum of the Squares of the Parts and the double Product is	. . $27+\sqrt{704}$

Whence it is manifest that $27+\sqrt{704}$ is the Square of $4+\sqrt{11}$, therefore this is the true square Root of that first Binomial; which was to be proved. Moreover, if the said double Product be subtracted from the said Sum of the Squares of the Parts, the Remainder $27-\sqrt{704}$ is the Square of $4-\sqrt{11}$; therefore this is the square Root of that first Residual.

Example 2.

Let it be required to extract the square Root of this second Binomial . $\sqrt{\frac{147}{4}}+6$.

The Operation.

1. From the Square of the greater part $\sqrt{\frac{147}{4}}$, *viz.* from . . $\frac{147}{4}$
2. Subtract the Square of the lesser part 6, to wit 36
3. The Remainder is . . . . $\frac{3}{4}$
4. The square Root of that Remainder is . . . . $\sqrt{\frac{3}{4}}$
5. To which square Root add the greater part, (by the Rule in *Sect.* 8. of this *Chap.*) . . . . $\sqrt{\frac{147}{4}}$
6. The Sum is . . . . $\sqrt{48}$
7. The half of which Sum is . . . . $\sqrt{12}$
8. The square Root of that half Sum is the greater part of the Root sought, to wit, ... $\sqrt{(4)12}$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\sqrt{\frac{147}{4}}$
10. Subtract the square Root before found in the fourth step, (by the said Rule in *Sect.* 8.) *viz.* . . . . . . $\sqrt{\frac{3}{4}}$
11. The Remainder is . . . . $\sqrt{27}$
12. The half of which Remainder is . . . . $\sqrt{\frac{27}{4}}$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, ... $\sqrt{(4)\frac{27}{4}}$
14. I say, the two parts in the eighth and thirteenth steps being connected by the Sign + shall be the Root sought, to wit, ... $\sqrt{(4)12}+\sqrt{(4)\frac{27}{4}}$

And if — instead of + be prefix'd to the lesser part of the said Root, it will give $\sqrt{(4)12}-\sqrt{(4)\frac{27}{4}}$, which is the square Root of the second Residual $\frac{147}{4}-6$.

The

The former of those two Roots answers to the Irrational Line called (in *Prop.* 38. & 56. *lib.* 10. *Elem. Eucl.*) a *first Bimedial*, and the latter answers to the Irrational Line called (in *Prop.* 75. & 93.) a *first Medial Residual.*

The Proof of the Root above extracted out of the second Binomial.

The Squares of the parts of $\surd(4)12+\surd(4)\frac{27}{4}$ the Root found out are . . . . $\surd 12$ and $\surd\frac{27}{4}$
Which Squares added together (as in Example 6. *Sect.* 8. of this *Chap.* is manifest) makes the Sum . . . . . $7\surd\frac{3}{4}$, that is, $\surd\frac{147}{4}$
The Product of the parts, *viz.* $\surd(4)12$ into $\surd(4)\frac{27}{4}$ is $\surd(4)81$, that is, 3.
The double of the said Product is 6
Therefore the Sum of the Sum of the Squares of the parts and the said double Product is . . . . $\surd\frac{147}{4}+6$
Whence it is manifest that $\surd\frac{147}{4}+6$ is the Square of $\surd(4)12+\surd(4)\frac{27}{4}$, therefore this is the true square Root of that second Binomial, which was to be proved. Moreover, if the said double Product be subtracted from the said Sum of the Squares of the Parts, the Remainder $\surd\frac{147}{4}-6$ is the square of $\surd(4)12-\surd(4)\frac{27}{4}$; therefore this is the square Root of that second Residual.

Example 3.

Let it be required to extract the square Root of this third Binomial $\surd\frac{245}{3}+\surd 80$.

The Operation.

1. From the Square of the greater part $\surd\frac{245}{3}$, *viz.* from . . . $\frac{245}{3}$
2. Subtract the square of the lesser part, to wit, ... 80
3. The Remainder is . . . . $\frac{5}{3}$
4. The square Root of that Remainder is . . . . $\surd\frac{5}{3}$
5. To which square Root add the greater part . . . . $\surd\frac{245}{3}$
6. The Sum is . . . . $\surd\frac{320}{3}$
7. The half of which Sum is . . . . $\surd\frac{80}{3}$
8. The square Root of that half Sum is the greater part of the Root sought, to wit, ... $\surd(4)\frac{80}{3}$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\surd\frac{245}{3}$
10. Subtract the square Root before found in the fourth step, to wit, ... $\surd\frac{5}{3}$
11. The Remainder is . . . . $\surd 60$
12. The half of which Remainder is . . . . $\surd 15$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, ... $\surd(4)15$
14. I say, the two parts in the eighth and thirteenth steps being connected by + shall be the square Root sought, to wit, ... $\surd(4)\frac{80}{3}+\surd(4)15$

And if — instead of + be prefix'd to the lesser part of the said Root, it gives $\surd(4)\frac{80}{3}-\surd(4)15$, which is the square Root of the third Residual $\surd\frac{245}{3}-\surd 80$.
The former of those two Roots answers to the Irrational Line called (in *Prop.* 39. & 57. *lib.* 10. *Elem. Eucl.*) a *second Bimedial*, and the latter answers to the Irrational Line called (in *Prop.* 76. & 94.) a *second Medial Residual.*

The Proof of the Root above extracted out of the third Binomial.

The Squares of the Parts of $\surd(4)\frac{80}{3}+\surd(4)15$, the Roots found out, are . . . . $\surd\frac{80}{3}$ and $\surd 15$
Which Squares added together make . . . . $7\surd\frac{5}{3}$, that is, $\surd\frac{245}{3}$
The Product of the parts, *viz.* $\surd(4)\frac{80}{3}$ into $\surd(4)15$ is $\surd(4)400$, that is, $\surd 20$
The double of the said Product is . . . . $\surd 80$
Therefore the Sum of the Sum of the Squares of the parts and the said double Product is . . . . $\surd\frac{245}{3}+\surd 80$

Whence it is manifest, that $\surd\frac{245}{3}+\surd 80$ is the Square of $\surd(4)\frac{80}{3}+\surd(4)15$; therefore this is the square Root of that third Binomial: which was to be proved More-

Moreover, if the ſaid double Product be ſubtracted from the ſaid Sum of the Squares of the Parts, the Remainder $\sqrt{\frac{245}{3}}-80$ is the Square of $\sqrt{(4)\frac{80}{3}}-\sqrt{(4)15}$; therefore this is the ſquare Root of that third Reſidual.

Example 4.

Let it be required to extract the ſquare Root of this fourth Binomial . $7+\sqrt{20}$.

The Operation.

1. From the Square of the greater part 7, *viz.* from	49
2. Subtract the Square of the leſſer part $\sqrt{20}$, to wit, ...	20
3. The Remainder is	29
4. The ſquare Root of that Remainder is	$\sqrt{29}$
5. To which ſquare Root add the greater part	7
6. The Sum is	$7+\sqrt{29}$
7. The half of which is	$\frac{7}{2}+\sqrt{\frac{29}{4}}$
8. The ſquare Root of that half Sum is the greater part of the Root ſought, to wit, ...	$\sqrt{:\frac{7}{2}-\sqrt{\frac{29}{4}}:}$
9. Again, from the greater part of the given Binomial, *viz.* from	7
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ...	$\sqrt{29}$
11. The Remainder is	$7-\sqrt{29}$
12. The half of which Remainder is	$\frac{7}{2}-\sqrt{\frac{29}{4}}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ...	$\sqrt{:\frac{7}{2}-\sqrt{\frac{29}{4}}:}$
14. I ſay, the two parts in the eighth and thirteenth ſteps (the former of which is a Binomial, and the latter a Reſidual) being connected by + ſhall be the ſquare Root ſought, to wit, ...	$\sqrt{:\frac{7}{2}+\sqrt{\frac{29}{4}}:}+\sqrt{:\frac{7}{2}-\sqrt{\frac{29}{4}}:}$

Which Root anſwers to the Irrational Line called (in *Prop.* 40. & 58. *lib.* 10. *Elem. Eucl.*) a *Major Line.*

And if the leſſer Name of the ſaid Root be ſubtracted from the greater, by interpoſing the Sign —, it gives $\sqrt{:\frac{7}{2}+\sqrt{\frac{29}{4}}:}-\sqrt{:\frac{7}{2}-\sqrt{\frac{29}{4}}:}$ which is the Root of the fourth Reſidual $7-\sqrt{20}$, and anſwers to the Irrational Line called (in *Prop.* 77. & 95. *lib.* 10. *Elem. Eucl.*) a *Minor Line.*

The Proof of the Root above extracted out of the fourth Binomial.

The Squares of the Parts of the Root found out are .	$\frac{7}{2}+\sqrt{\frac{29}{4}}$ and $\frac{7}{2}-\sqrt{\frac{29}{4}}$
Therefore the Sum of the Squares of the Parts is . . .	$\frac{7}{2}+\frac{7}{2}$, that is, 7.
The Product of the Parts will be found (by Rule 2. *Sect.* 12. of this *Chap.*)	$\sqrt{:\frac{49}{4}-\frac{29}{4}:}$ that is, $\sqrt{5}$
The double of the ſaid Product is	$\sqrt{20}$
Therefore the Sum of the ſaid Sum of the Squares of the Parts and the ſaid double Product is	$7+\sqrt{20}$

Whence it is manifeſt that $7+\sqrt{20}$ is the Square of $\sqrt{:\frac{7}{2}+\sqrt{\frac{29}{4}}:}+\sqrt{:\frac{7}{2}-\sqrt{\frac{29}{4}}:}$ therefore this is the ſquare Root of that fourth Binomial; which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid Sum of the Squares of the Parts, the Remainder $7-\sqrt{20}$ is the Square of $\sqrt{:\frac{7}{2}+\sqrt{\frac{29}{4}}:}-\sqrt{:\frac{7}{2}-\sqrt{\frac{29}{4}}:}$ therefore this is the ſquare Root of that fourth Reſidual $7-\sqrt{20}$.

Example 5.

Let it be required to extract the ſquare Root out of this fifth Binomial $\sqrt{20}+4$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, *viz.* from . .	20
2. Subtract the Square of the leſſer part 4, to wit, . . .	16
3. The Remainder is	4
4. The ſquare Root of that Remainder is	2
5. To which ſquare Root add the greater part	$\sqrt{20}$

6. The

6. The Sum is . . . . $\sqrt{20}+2$
7. The half of that Sum is . . . . $\sqrt{5}+1$
8. The ſquare Root of the ſaid half Sum is the greater part of the Root ſought, to wit, ... $\sqrt{:\sqrt{5}+1:}$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\sqrt{20}$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ... 2
11. The Remainder is . . . . $\sqrt{20}-2$
12. The half of which Remainder is . . . . $\sqrt{5}-1$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\sqrt{:\sqrt{5}-1:}$
14. I ſay, The two Parts in the eighth and thirteenth ſteps, (the former of which Parts is a Binomial, and the latter a Reſidual) being connected by $+$ ſhall be the ſquare Root ſought, to wit, ... $\sqrt{:\sqrt{5}+1:}+\sqrt{:\sqrt{5}-1:}$

Which Root anſwers to the Irrational Line called (in *Prop.* 41. & 59. *lib.* 10. *Elem. Eucl.*) a Line containing in Power a *Rational* and a *Medial Rectangle*. And if the leſſer Name of the ſaid Root be ſubtracted from the greater, by the interpoſition of the Sign $-$, it gives $\sqrt{:\sqrt{5}+1:}-\sqrt{:\sqrt{5}-1:}$ which is the ſquare Root of the fifth Reſidual $\sqrt{20}-4$, and anſwers to the Irrational Line which (in *Prop.* 78. & 96. *lib.* 10.) is called a Line making with a *Rational Space* the whole *Space Medial*.

The Proof of the Root above Extracted out of the fifth Binomial.

The ſquares of the Parts of $\sqrt{:\sqrt{5}+1:}+\sqrt{:\sqrt{5}-1:}$ (the Root found out) are . . . . $\sqrt{5}+1$ and $\sqrt{5}-1$

Therefore the Sum of the ſaid Squares of the Parts is $\sqrt{5}+\sqrt{5}$, that is, $\sqrt{20}$

The Product of the Parts multiplied one into the other (according to Rule 2. *Sect.* 12. of this *Chap.*) is $\sqrt{:5-1:}$ that is, 2.

The double of the ſaid Product is . . . . 4.

Therefore the Sum of the ſaid Sum of the Squares of the Parts and double Product is . . . . $\sqrt{20}+4$

Whence it is manifeſt that $\sqrt{20}+4$ is the Square of $\sqrt{:\sqrt{5}+1:}+\sqrt{:\sqrt{5}-1:}$ therefore this is the ſquare Root of that fifth Binomial; which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid Sum of the Squares of the Parts, the Remainder $\sqrt{20}-4$ is the ſquare of $\sqrt{:\sqrt{5}+1:}-\sqrt{:\sqrt{5}-1:}$ therefore this is the ſquare Root of the ſaid fifth Reſidual $\sqrt{20}-4$.

Example 6.

Let it be required to extract the ſquare Root of this ſixth Binomial $\sqrt{20}+\sqrt{8}$.

The Operation.

1. From the ſquare of the greater Part $\sqrt{20}$, *viz.* from 20
2. Subtract the ſquare of the leſſer part $\sqrt{8}$, to wit, ... 8
3. The Remainder is 12
4. The ſquare Root of that Remainder is . . . . . . $\sqrt{12}$
5. To which ſquare Root add the greater Part . . . . $\sqrt{20}$
6. The Sum is . . . . $\sqrt{20}+\sqrt{12}$
7. The half of which Sum is . . . . $\sqrt{5}+\sqrt{3}$
8. The ſquare Root of the ſaid half Sum is the greater part of the Root ſought, to wit, . . . $\sqrt{:\sqrt{5}+\sqrt{3}:}$.
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\sqrt{20}$
10. Subtract the ſquare Root before found in the fourth ſtep, *viz.* . . . . $\sqrt{12}$
11. The Remainder is . . . . $\sqrt{20}-\sqrt{12}$
12. The half of which Remainder . . . . $\sqrt{5}-\sqrt{3}$

13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, $\sqrt{:\sqrt{5}-\sqrt{3}:}$

14. I say, the two Parts in the eighth and thirteenth steps (the former of which Parts is a Binomial, and the latter a Residual) being connected by + shall be the square Root sought, to wit, $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$

Which Root answers to the Irrational Line which (in *Prop.* 42. & 60. *lib.* 10. *Elem. Eucl.*) is called a Line containing in Power two *Medial Rectangles.* And if the lesser part of the said Root be subtracted from the greater, by the interposing of the Sign −, it gives $\sqrt{:\sqrt{5}+\sqrt{3}:}-\sqrt{:\sqrt{5}-\sqrt{3}:}$ which is the Root of the sixth Residual $\sqrt{20}-\sqrt{8}$, and answers to the Irrational Line which (in *Prop.* 79. & 97. *lib.* 10. *Eucl.*) is called a Line making with a *Medial Rectangle* a whole *Space Medial.*

The Proof of the Root above extracted out of the sixth Binomial.

The Squares of the Parts of $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$ the Root sought are . . . $\sqrt{5}+\sqrt{3}$ and $\sqrt{5}-\sqrt{3}$

Therefore the Sum of the said Squares of the Parts is $\sqrt{5}+\sqrt{5}$, that is, $\sqrt{20}$

The Product of the Parts multiplied one into the other is $\sqrt{:5-3:}$ that is, $\sqrt{2}$.

The double of the said Product is . . . $\sqrt{8}$.

Therefore the Sum of the said Sum of the Squares of the Parts and double Product is . . . $\sqrt{20}+\sqrt{8}$.

Whence it is manifest that $\sqrt{20}+\sqrt{8}$ is the Square of $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$ therefore this is that square Root of the sixth Binomial; which was to be proved. Moreover, if the said double Product be subtracted from the said sum of the Squares of the Parts, the Remainder $\sqrt{20}-\sqrt{8}$ is the Square of $\sqrt{:\sqrt{5}+\sqrt{3}:}-\sqrt{:\sqrt{5}-\sqrt{3}:}$ therefore this is the square Root of that sixth Residual.

Note. In every Binomial and Residual constituted according to the preceding *Sect.* 15. the square Root of the Difference of the Squares of the Names or Parts is equal to the Difference of the Squares of the Parts of the Root of the Binomial or Residual.

As in the first Binomial $27+\sqrt{704}$, whose square Root has before been found $4+\sqrt{11}$, the Square of 27, to wit 729, exceeds 704, the Square of $\sqrt{704}$ by 25, whose square Root 5 is equal to the Difference of the Squares of the Parts of the Root of the Binomial proposed, to wit, the Difference between 16 and 11.

This Property may be demonstrated thus; let $b+\sqrt{d}$ represent a Binomial Root, whose greater Part is b; then the Square of that Root is $bb+2b\sqrt{d}+d$, this divided into its Names or Parts makes the Binomial $bb+d$ more $2b\sqrt{d}$; then the Squares of the Parts of this Binomial are $bbbb+2bbd+dd$ and $4bbd$, and the Difference of those Squares is $bbbb-2bbd+dd$, whose square Root $bb-d$ is manifestly the Difference of the Squares of the Parts of the Root $b+\sqrt{d}$ first proposed; which was to be shewn. The like Property may be demonstrated in a Residual.

How to extract the Square Root out of a Binomial design'd by Letters, if it has a Binomial Root.

By the same general Rule which has before been exercis'd in extracting the square Root out of Binomials express'd by Numbers, we may extract the square Root out of a Binomial design'd by Letters, when it has a Binomial Root, as will be evident by the following Examples; where for the more apparent distinction of the Parts of the given Binomial, instead of + I set the Word [*more*] between the Parts, and instead of − I set the Word [*less*] between the Parts of a given Residual.

Example 1.

Let it be required to extract the square Root out of this Binomial . . . $bb+d$ more $2b\sqrt{d}$

The Operation.

1. From the Square of the greater part, (which suppose to be $bb+d$) *viz.* from . . . $bbbb+2dbb+dd$
2. Subtract the Square of the lesser part $2b\sqrt{d}$, to wit, . . . $+4dbb$

3. The

3. The Remainder is . . . . $bbbb-2dbb+dd$
4. The square Root of that Remainder is . . . . $bb-d$
5. To which square Root add the greater part, to wit, . . . $bb+d$
6. The Sum is . . . . $2bb$
7. The half of which Sum is . . . . bb
8. The square Root of the said half Sum is the greater part of the Root sought, to wit, ... b
9. Then from the greater part of the given Binomial, *viz.* from $bb+d$
10. Subtract the square Root before found in the fourth step, to wit, $bb-d$
11. The Remainder is . . . . $+2d$
12. The half of which Remainder is . . . . $+d$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, ... $\surd d$
14. I say, The two Parts in the eighth and thirteenth steps being connected by the Sign $+$, shall be the square Root sought, to wit, $b+\surd d$

Which Root being squared, or multiplied into it self, will evidently produce the given Binomial $bb+d$ more $2b\surd d$.

Example 2.

Let it be required to extract the square Root out of this Binomial, ... $mm+\frac{pxx}{m}$ more $x\surd 4mp$

The Operation.

1. From the Square of the greater part $mm+\frac{pxx}{m}$ *viz.* from . . . . $mmmm+2mpxx+\frac{ppxxxx}{mm}$
2. Subtract the square of the lesser part $x\surd 4mp$, to wit, . $+4mpxx$
3. The Remainder is . . . . $mmmm-2mpxx+\frac{ppxxxx}{mm}$
4. The Square of that Remainder is . . . . . . $mm-\frac{pxx}{m}$
5. To which square Root add the greater part, to wit, $mm+\frac{pxx}{m}$
6. The Sum is . . . . $2mm$
7. The half of which Sum is . . . . mm
8. The square Root of the said half Sum is the greater part of the Root sought, to wit, ... m
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $mm+\frac{pxx}{m}$
10. Subtract the square Root before found in the fourth step, to wit, ... $mm-\frac{pxx}{m}$
11. The Remainder is . . . . $+\frac{2pxx}{m}$
12. The half of which Remainder is . . . . $+\frac{pxx}{m}$
13. The square Root of the said half Remainder is the lesser part of the Root sought, to wit, . . . $\surd\frac{pxx}{m}$ or $x\surd\frac{p}{m}$
14. I say, the two Parts in the eighth and thirteenth steps being connected by $+$ shall be the square Root sought, to wit, ... $m+x\surd\frac{p}{m}$

Which Binomial Root being squared, or multiplied into it self, will produce the given Binomial.

Example 3.

Let it be required to extract the square Root out of this Binomial, ... $\overline{a+b}\surd ab$ more $2ab$

The Operation.

1. From the ſquare of the greater part, *viz.* from . . $aaab+2aabb+abbb$
2. Subtract the Square of the leſſer part, to wit, ... $+4aabb$
3. The Remainder is . . . . $aaab-2aabb+abbb$
4. The ſquare Root of that Remainder is . . . . $\overline{a-b}\sqrt{ab}$
5. To which ſquare Root add the greater part, to wit, . . $\overline{a+b}\sqrt{ab}$
6. The Sum is . . . . $2a\sqrt{ab}$
7. The half of which Sum is . . . . $a\sqrt{ab}$
8. The ſquare Root of the ſaid half Sum is the greater part of the Root ſought, to wit, ... $\sqrt{:a\sqrt{ab}:}$ or $\sqrt{(4)}aaab$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\overline{a+b}\sqrt{ab}$
10. Subtract the ſquare Root before found in the fourth ſtep, *viz.* . . . . $\overline{a-b}\sqrt{ab}$
11. The Remainder is . . . . $2b\sqrt{ab}$
12. The half of which Remainder is . . . . $b\sqrt{ab}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit . . . . $\sqrt{:b\sqrt{ab}:}$ or $\sqrt{(4)}abbb$
14. I ſay, the two Parts in the eighth and thirteenth ſteps, being connected by $+$, ſhall be the ſquare Root ſought, to wit, ... $\sqrt{:a\sqrt{ab}:}+\sqrt{:b\sqrt{ab}:}$
15. Which Binomial Root may be alſo expreſs'd thus . $\sqrt{(4)}aaab+\sqrt{(4)}abbb$

The Proof may be made by multiplying the Root found out into it ſelf.

Example 4.

Again, if the ſquare Root of this Reſidual be deſired . $\overline{a+d}\sqrt{bc}$ leſs $2\sqrt{abcd}$

The Root being extracted by the precedent Method will be found . . . . $\sqrt{:a\sqrt{bc}:}-\sqrt{:d\sqrt{bc}:}$

Which Root may be alſo expreſs'd thus . . . . . . $\sqrt{(4)}aabc-\sqrt{(4)}ddbc$

But if it happen that when the Square of the leſſer part of the given Binomial or Reſidual is ſubtracted from the ſquare of the greater part, the ſquare Root of the Remainder and the greatet part are not commenſurable, (according to the Definition before given in *Sect.* 7. of this *Chap.*) there is no more to be done in ſuch a caſe, but to prefix before the given Binomial or Reſidual the Sign $\sqrt{}$, with a Line drawn over both its Parts, to denote the univerſal ſquare Root of the given Binomial or Reſidual. As to extract the ſquare Root out of this Reſidual $\sqrt{:\frac{1}{4}aa+bb:}-\frac{1}{2}a$, I write $\sqrt{:\sqrt{\frac{1}{4}aa+bb}-\frac{1}{2}a:}$ which kind of Roots are commonly called Univerſal.

Sect. 17. *Queſtions to exerciſe the foregoing Rules of this Chapter.*

QUESTION. 1.

To divide 100 into two ſuch parts, that if each part be divided by the other part, the Sum of the Quotient may make 3.

RESOLUTION.

1. For one of the parts ſought put . . . . a
2. Then conſequently the other part is . . . . . . $100-a$
3. Therefore according to the import of the Queſtion this Equation ariſes, *viz.* . . . . $\dfrac{a}{100-a}+\dfrac{100-a}{a}=3$
4. Which Equation duly reduced gives . . . . $1000a-aa=2000$
5. Wherefore by reſolving the ſaid Equation by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two values of a, which are the deſired parts of 100, will be found theſe, to wit, ... $a=\begin{cases}50+10\sqrt{5}\\50-10\sqrt{5}\end{cases}$

6. The

6. The Sum of the said Parts or Numbers found out is manifestly 100, so it remains only to prove that,

$$\frac{50+10\sqrt{5}}{50-10\sqrt{5}}+\frac{50-10\sqrt{5}}{50+10\sqrt{5}}=3$$

The Proof.

7. To add those two Surd Fractions in the sixth step into one Sum, reduce them to a common Denominator, *viz.* multiply $50+10\sqrt{5}$ by $50+10\sqrt{5}$, and the Product (by the first of the three compendious Rules in *Sect.* 10. of this *Ch.*) will be found — $3000+1000\sqrt{5}$

8. Likewise multiply $50-10\sqrt{5}$ by $50-10\sqrt{5}$ and the Product (by the second of the said three Rules) will be . . — $3000-1000\sqrt{5}$

9. Then take the Sum of those two Products for the Numerator of a Fraction or a Dividend, to wit, ... — 6000

10. Also multiply the two Denominators of the Surd Fractions in the sixth step one by the other, (according to the last of the three Rules above cited) and take the Product for a Denominator or Divisor, *viz.* . . . . — 2000

11. Lastly, the Numerator in the ninth step being set over the Denominator in the tenth gives the Sum of the two Surd Fractions or Quotients in the sixth step, *viz.* . . . — $\frac{6000}{2000}=3$

Which Sum is manifestly 3, as was to be proved.

Another Proof.

The Quotient that arises by dividing $50+10\sqrt{5}$ by $50-10\sqrt{5}$ (according to the Rule of Division in the sixth branch of *Sect.* 11. of this *Chap.*) is . . . . — $\frac{3}{2}+\sqrt{\frac{5}{4}}$

Likewise the Quotient that arises by dividing $50-10\sqrt{5}$ by $50+10\sqrt{5}$ is . . . . — $\frac{3}{2}-\sqrt{\frac{5}{4}}$

The Sum of those two Quotients is manifestly 3, (as before.)

QUESTION. 2.

To divide a given Number (suppose 6) into three such unequals Numbers in continual proportion, that the Sum of the Squares of the Extremes may be to the Square of the Mean in a given proportion; but the first Term of this proportion must exceed the double of the latter Term. Let it therefore be desired that the Sum of the Squares of the Extremes may be to the square of the Mean as 3 to 1.

RESOLUTION.

1. For the mean Proportional put . . . . — a
2. Then because the sum of all the three Proportionals must make 6, and the Mean is a, the sum of the Extremes shall be — $6-a$
3. Therefore the square of the sum of the Extremes is . . — $36-12a+aa$
4. But (by *Theor.* 3. *Chap.* 6. of this Book) the square of the Sum of the Extremes of three Numbers continually proportional is equal to the squares of the Extremes, together with the double square of the Mean; therefore from the square in the third step I subtract $2aa$ (the double square of the Mean) and there remains the sum of the squares of the Extremes, to wit, ... — $36-12a-aa$
5. But (according to the Question) the sum of the squares of the Extremes must be equal to the triple square of the Mean; therefore from the fourth and first step this Equation arises, *viz.* . . . . — $36-12a-aa=3aa$
6. From which Equation after due Reduction this arises, *viz.* — $aa+3a=9$
7. Therefore by resolving the last Equation (according to the Canon in *Sect.* 6. *Chap.* 15.) the value of a, that is, the mean Proportional sought will be discovered, *viz.* . — $\sqrt{\frac{45}{4}}-\frac{3}{2}=$ the Mean

8. And

8. And from the seventh and second steps the Sum of the Extremes will be also made known, *viz.* $\frac{15}{2} - \sqrt{\frac{45}{4}} =$ Sum of the Extremes.

9. Then (as is manifest by *Quest.* 4. *Chap.* 16. *Book* 1.) the Sum of the Extremes of three Numbers continually proportional being given, as also the Mean, the Extremes shall be given severally by this following

CANON.

From the Square of half the Sum of the Extremes subtract the Square of the Mean, and extract the square Root of the Remainder; then this square Root being added to, and subtracted from the said half Sum, will give the Extremes severally. Therefore,

10. From the square of the half of $\frac{15}{2} - \sqrt{\frac{45}{4}}$, that is, from . . $\frac{135}{8} - \frac{15}{4}\sqrt{\frac{45}{4}}$
11. Subtract the square of $\sqrt{\frac{45}{4}} - \frac{3}{2}$, *viz.* . . . . $\frac{108}{8} - \frac{12}{4}\sqrt{\frac{45}{4}}$
12. The Remainder is . . . . $\frac{27}{8} - \frac{3}{4}\sqrt{\frac{45}{4}}$
13. The square Root of that Remainder being extracted (by the general Rule before delivered in *Sect.* 16. of this *Chap.* for extracting the square Root out of Binomials) will be found . $\sqrt{\frac{45}{16}} - \frac{3}{4}$
14. Which square Root added to the half of $\frac{15}{2} - \sqrt{\frac{45}{4}}$, gives the greater Extreme sought, to wit, ... 3
15. But the said square Root subtracted from the half of $\frac{15}{2} - \sqrt{\frac{45}{4}}$, leaves the lesser Extreme, to wit . . . . $\frac{9}{2} - \sqrt{\frac{45}{4}}$
16. Wherefore (in the seventh, fourteenth and fifteenth steps) three Numbers continually proportional are found out, *viz.* 3, $\sqrt{\frac{45}{4}} - \frac{3}{2}$, and $\frac{9}{2} - \sqrt{\frac{45}{4}}$, whose Sum is 6; and the Sum of the Squares of the Extremes is equal to the triple of the Square of the Mean, as will appear by

The Proof.

First, The Product made by the Multiplication of the first and third Numbers one into the other, that is, of 3 into $\frac{9}{2} - \sqrt{\frac{45}{4}}$, is $\frac{27}{2} - 3\sqrt{\frac{45}{4}}$, which is also the square of the second Number $\sqrt{\frac{45}{4}} - \frac{3}{2}$, (as will easily appear by Multiplication;) therefore the said three Numbers are Proportionals.

Secondly, The Sum of the said three proportional Numbers is 6; for the Mean $\sqrt{\frac{45}{4}} - \frac{3}{2}$ added to $\frac{9}{2} - \sqrt{\frac{45}{4}}$ the lesser Extreme, makes 3, to which adding the greater Extreme 3, the Sum is 6.

Thirdly, The Sum of the Squares of the Extremes 3 and $\frac{9}{2} - \sqrt{\frac{45}{4}}$ is equal to the triple of the Square of the Mean $\sqrt{\frac{45}{4}} - \frac{3}{2}$; for the said Sum, as also the said triple Square will by Multiplication be found $\frac{81}{2} - 9\sqrt{\frac{45}{4}}$. Therefore all the Conditions in the Question are satisfied.

But that the necessity of Determination annexed to the Question may be made manifest, it remains to prove, that if three unequal Numbers be in continual proportion, the Sum of the Squares of the Extremes is greater than the double of the Square of the Mean. Therefore,

Let three unequal Numbers in continual proportion be exposed, suppose these, ... a, $\sqrt{ae}$, e ∺

Then their Squares shall be also Proportionals, (*per* 23 *Prop.* 6 *Elem. Eucl.*) *viz.* . . . . $aa \,.\, ae :: ae \,.\, ee$

Therefore (by 25 *Prop.* 5. *Elem. Eucl.*) . . . . $aa + ee \sqsupset 2ae$

But $aa + ee$ is the Sum of the Squares of the Extremes of the three Proportionals exposed, and $2ae$ is equal to the double Square of the Mean proportional; wherefore the Theorem is proved, and consequently the Determination is manifestly necessary to be annexed to the Question proposed, that there may be a possibility of finding out what is thereby desired. The Determination may also be easily inferr'd from the Canon in the foregoing ninth step..

QUESTION 3.

What is the Product made by the continual Multiplication of these four Numbers one into another, which differ by an equal Excess, to wit, Unity? . . . . .

$$\left\{\begin{array}{l} \sqrt{:\frac{5}{4} + \sqrt{101}:} - \frac{3}{2} \\ \sqrt{:\frac{5}{4} + \sqrt{101}:} - \frac{1}{2} \\ \sqrt{:\frac{5}{4} + \sqrt{101}:} + \frac{1}{2} \\ \sqrt{:\frac{5}{4} + \sqrt{101}:} + \frac{3}{2} \end{array}\right.$$

Answ.

Answ. The desired Product is exactly 100

For, (by the last of the three compendious Rules before delivered in *Sect.* 10. of this *Chap.* for the Multiplication of Binomials and Residuals) the Product of the first and fourth Number is . . $\sqrt{101}-1$

Likewise the Product of the second and third Number is . . $\sqrt{101}+1$

Lastly, the two last preceding Products being multiplied one into another (by the same Rule) make 100

QUESTION 4.

1. If a, b, c, be such Quantities, that . . . . $aa+ca=b$
What is the value of a?

2. *Answ.* By the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. . . . $a=\sqrt{:b+\frac{1}{4}cc:}-\frac{1}{2}c$
By which value of a the Equation propos'd may be expounded (as is usual) by the following

Demonstration.

3. If . . . . $a=\sqrt{:b+\frac{1}{4}cc:}-\frac{1}{2}c$
4. Then consequently by adding $\frac{1}{2}c$ to each part . . . $a+\frac{1}{2}c=\sqrt{:b+\frac{1}{4}cc:}$
5. And by multiplying each part of the last Equation into it self . . . . $aa+ca+\frac{1}{4}cc=b+\frac{1}{4}cc$
6. Wherefore by subtracting $\frac{1}{4}cc$ from each part, there remains . . . . $aa+ca=b$
Which was to be proved.

Note. This Demonstration is formed in the way of Composition by the steps of the Resolution of the same Question in *Sect.* 5. *Chap.* 15. *Book* 1. but in a retrograde or backward order; for the first step in the Composition (or Demonstration) is the last in the Resolution, the second step in the Composition is the last but one in the Resolution; and so by returning backwards by the steps of the Resolution, the Demonstration ends in the Equation propos'd to be resolved. But this is largely handled in my fourth Book of Algebraical Elements.

QUESTION. 5.

1. If a, b, k, be such Quantities that . . . . $aa-ba=k$
What is the value of a?

2. *Answ.* By the Canon in *Sect.* 8. *Chap.* 15. *Book* 1. . . $a=\frac{1}{2}b+\sqrt{:k+\frac{1}{4}bb:}$
By which value of a the Equation propos'd may be expounded, as appears by the following

Demonstration.

3. If . . . . $a=\frac{1}{2}b+\sqrt{:k+\frac{1}{4}bb:}$
4. Then by subtracting $\frac{1}{2}b$ from each part . . . . . . $a-\frac{1}{2}b=\sqrt{:k+\frac{1}{4}bb:}$
5. And by multiplying each part of the last Equation into it self, ... $aa-ba+\frac{1}{4}bb=k+\frac{1}{4}bb$
6. Wherefore by subtracting $\frac{1}{4}bb$ from each part, . $aa-ba=k$
Which was to be proved.

QUESTION 6.

1. If c and n be put for such known Quantities, that, ... n not $=\frac{1}{4}cc$
2. And if a be put for a Quantity unknown, and . . $ca-aa=n$
What is the value of a?
3. *Answ.* By the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. these two values of a will be found out, *viz.* . . . $a=\begin{cases}\frac{1}{2}c+\sqrt{:\frac{1}{4}cc-n:}\\ \frac{1}{2}c-\sqrt{:\frac{1}{4}cc-n:}\end{cases}$

By each of which values of a the Equation proposed in the second step may be expounded, *viz.* if either $\frac{1}{2}c+\sqrt{:\frac{1}{4}cc-n:}$ or $\frac{1}{2}c-\sqrt{:\frac{1}{4}cc-n:}$ be put equal to a, then $ca-aa=n$.

De-

DEMONSTRATION.

4. First, if . . . . $a=\frac{1}{2}c+\sqrt{:\frac{1}{4}cc-n:}$
5. Then by subtracting $\frac{1}{2}c$ from each part . . . . . $a-\frac{1}{2}c=\sqrt{:\frac{1}{4}cc-n:}$
6. And by multiplying each part of the last Equation into it self . . . . $aa-ca+\frac{1}{4}cc=\frac{1}{4}cc-n$
7. And by adding ca to each part . . . . $aa+\frac{1}{4}cc=\frac{1}{4}cc+ca-n$
8. And by subtracting $\frac{1}{4}cc$ from each part . . . . . . $aa=ca-n$
9. And by adding n to each part . . . . $aa+n=ca$
10. Wherefore by subtracting aa from each part . . . . $n=ca-aa$
11. That is, ... $ca-aa=n$

Which was to be proved.

Again, if . . . . $a=\frac{1}{2}c-\sqrt{:\frac{1}{4}cc-n:}$

12. Then by adding $\sqrt{:\frac{1}{4}cc-n:}$ to each part . $a+\sqrt{:\frac{1}{4}cc-n:}=\frac{1}{2}c$
13. And by subtracting a from each part . . . . $\sqrt{:\frac{1}{4}cc-n:}=\frac{1}{2}c-a$
14. And by multiplying each part of the last Equation into it self . . . . $\frac{1}{4}cc-n=\frac{1}{4}cc-ca+aa$
15. And by adding ca to each part . . . . . . $ca+\frac{1}{4}cc-n=\frac{1}{4}cc+aa$
16. And subtracting $\frac{1}{4}cc$ from each part . . . . . $ca-n=aa$
17. And by adding n to each part . . . . $ca=aa+n$
18. Wherefore by subtracting aa from each part . . . $ca-aa=n$

Which was to be proved.

QUESTION 7.

1. If b and c be put for such known Quantities, that c is greater than b, but less than $2b$; and if a be put for a Quantity unknown;
2. And if . . . . . $\sqrt{\frac{aa+3bb}{4}}+\sqrt{\frac{aa-3bb}{4}}=\sqrt{\frac{baa}{c}}$;

What is the value of a?

RESOLUTION.

3. Because the Squares of equal Quantities are also equal, by multiplying each part of the Equation in the second step into it self, this is produced, *viz.*

$$\frac{aa}{2}+\sqrt{\frac{a^4-9b^4}{4}}=\frac{baa}{c}$$

4. Then to the end the Surd Quantity in the Equation in the third step may solely make one part of an Equation, let $\frac{aa}{2}$ be subtracted from each part of that Equation, and this will remain, *viz.*

$$\sqrt{\frac{a^4-9b^4}{4}}=\frac{baa}{c}-\frac{aa}{2}=\frac{2baa-caa}{2c}$$

5. And to the end the Radical Sign in the first part of the last Equation may vanish, let each part be multiplied by it self, so an Equation in Rational Quantities will be produced, *viz.* $\frac{a^4-9b^4}{4}=\frac{4bba^4-4bca^4+cca^4}{4cc}$
6. And by reducing the last Equation to a common Denominator $4cc$, and then by multiplying each part by the same $4cc$, this Equation in Integers will be produced, *viz.*

$$cca^4-9b^4cc=4bba^4-4bca^4+cca^4$$

7. And from the Equation in the last preceding step, after due Reduction is made, to make those Quantities wherein a^4 is found to possess one part, this following Equation arises, *viz.* $4bca^4-4bba^4=9b^4cc$
8. Then by dividing each part of the last Equation by $4bc-4bb$, to the end that a^4 may stand alone, this Equation arises, *viz.*

$$a^4=\frac{9b^4cc}{4bc-4bb}=\frac{9b^3cc}{4c-4b}$$

9. But . . . . $\frac{9bbcc}{4}$ into $\frac{b}{c-b}=\frac{9b^3cc}{4c-4b}$

10. There-

10. Therefore from the two last preceding Equations, by exchanging equal Quantities, this Equation arises, *viz.*

$$a^4 = \frac{9bbcc}{4} \text{ into } \frac{b}{c-b}$$

11. And by extracting the square Root out of each part of the Equation in the tenth step, this arises;

$$aa = \frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}$$

12. Wherefore by extracting the square Root out of each part of the Equation in the eleventh step, the desired value of a is discovered, *viz.*

$$a = \sqrt{:\frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}:}$$

An Example of Quest. 7. *in Numbers.*

13. If . . . . $b = 16$
14. And . . . . $c = 25$
15. And . . . . $a =$ a Number unknown
16. And if . . . . $\sqrt{\frac{aa+3bb}{4}} + \frac{\sqrt{aa-3bb}}{4} = \sqrt{\frac{baa}{c}}$

What is the Number a?

17. *Answ.* From the thirteenth, fourteenth, and twelfth steps, $a = \sqrt{800}$, or $20\sqrt{2}$. By which value of a the Equation propos'd may be expounded, as will appear by

The Proof.

18. If $b = 16$, $c = 25$, and $a = \sqrt{800}$; then it will follow that

$$\frac{\sqrt{aa+3bb}}{4} + \sqrt{\frac{aa-3bb}{4}} = \sqrt{\frac{baa}{c}} \quad (= 8\sqrt{8}, \text{ or } \sqrt{512})$$

Note, The Numbers to express the values of b and c must not be taken at pleasure, but such that the Number c may exceed the Number b, and be less than $2b$, as is prescribed in the Question; the former part of which Determination is discovered by the Denominator $c-b$ of the Surd Fraction in the twelfth step, and the latter part of the Determination is manifest by the latter part of the Equation in the fourth step, where caa is to be subtracted from $2baa$, which cannot be done so as to leave a Remainder greater than nothing, unless c be less than $2b$.

Sect. XVIII. *An Explanation of* Fran. van Schooten's *General Rule to extract what Root you please out of any Binomial in Numbers, having such a Binomial Root as is desired.*

Preparation.

First, if the given Binomial has Fractions in it, must be freed from them by multiplying the Binomial by their Denominator. As for Example, to extract $\sqrt{(3,)}$ that is, the Cubic Root out of $\sqrt{242}+12\frac{1}{2}$, I multiply the Binomial by 2, and it makes $\sqrt{968}+25$; for $\sqrt{242}$ multiplied by $\sqrt{4}$, (that is, by 2) produces $\sqrt{968}$; and $12\frac{1}{2}$ into 2 makes 25. Likewise, if there be proposed $\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}$, I first multiply it by $\sqrt{5}$, and it makes $\sqrt{242}+\frac{25}{2}$, then this Binomial multiplied by 2 produces (as before) $\sqrt{968}+25$; and so of others.

Secondly, if neither of the two Parts of the given Binomial be Rational, it must be reduced by Multiplication or Division to another Binomial that shall have one of its Parts Rational; which Reduction may always be done by the Multiplication of either Part, but oftentimes more briefly by the Multiplication or Division of the lesser Number. As for Example, $\sqrt{242}+\sqrt{243}$ may be multiplied by $\sqrt{242}$, and it makes $242+\sqrt{58806}$; but more compendiously by $\sqrt{2}$, and there comes forth $22+\sqrt{486}$. After the same manner: $\sqrt{(3)}3993+\sqrt{(6)}17578125$ may be first multiplied by $\sqrt{(3)}3993$, and the Product again by $\sqrt{(3)}3993$, so there will be produced another Binomial, whose Rational Part is the absolute Number 3993; but more briefly by $\sqrt{(3)}9$, and there will be produced another Binomial whose Rational Part is 33; and yet more compendiously if the Binomial propos'd be divided by $\sqrt{(3)}3$, there will arise $11+\sqrt{125}$.

But here is to be noted, that when one part of a Binomial is Rational, whither it be of a Binomial first given, or of another deduced (as above) from that given, then also the Square of the other part ought to be rational, otherwise no Root can be extracted out of the Binomial, or the other deduced from it.

Thirdly, to extract $\sqrt{(6)}$ out of a given Binomial qualified as above is supposed, we must first extract the square Root, and then out of this the Cubic Root; and to extract $\sqrt{(9)}$ we must first extract $\sqrt{(3)}$, and then out of the Cubic Root found out we must again extract $\sqrt{(3)}$; and so of any other Root whose Index is a Composit Number. But as to the Extraction of the square Root out of a Binomial, a Rule has been already given and exemplified in the preceding *Sect.* 16, so that here there is need only that I shew how to extract $\sqrt{(3)}$, $\sqrt{(5)}$, $\sqrt{(7)}$, $\sqrt{(11)}$, and such like whose Indices are Prime Numbers.

Fourthly, to extract $\sqrt{(3)}$, $\sqrt{(5)}$, $\sqrt{(7)}$, or the like Root, whose Index is a Prime Number, we must first of all try whether out of the given Binomial there can be extracted a Binomial Root which has one part Rational, but that may be discovered by subtracting the Square of the lesser part of the given Binomial from the square of the greater, and extracting the Root out of the Remainder, to wit, the Cubic Root of $\sqrt{(3)}$ be to be extracted out of the given Binomial, or the Root of the fifth Power, if $\sqrt{(5)}$ be to be extracted: and so of others. For if the Root of the said Remainder be not a Rational Number, then the Binomial Root sought will certainly want a Rational part, *viz.* each of its parts will be Surd; in which case, in order to extract the Root, the given Binomial must be multiplied by the Difference of the Squares of the Parts, if the Question be concerning the Extraction of the Cubic Root; or by the Square of the said Difference, if $\sqrt{(5)}$ be sought; or by the Cube of the same Difference, if $\sqrt{(7)}$ be required; or by the fifth Power of the said Difference, if $\sqrt{(11)}$ be sought; and so of the rest. By which Multiplication another Binomial will always be produced, wherein the Root of the Difference of the Squares of the Parts will be the same with the Difference of the Squares of the Parts of the former Binomial.

As to extract the Cubic Root out of $25+\sqrt{968}$, I first subtract 625 the Square of 25, from 968 the Square of $\sqrt{968}$, and there remains 343, whose Cubic Root 7 is a Rational Number; which argues that the Root of the given Binomial, if there can be a Root extracted out of it, is a Binomial which has one of its Parts Rational.

Likewise, to extract the Cubic Root out of $22+\sqrt{486}$, we must subtract 484, the Square of 22, from 486, and extract the Cubic Root out of the Remainder 2; but because that cannot be done exactly, it shews that the Cubic Root of $22+\sqrt{486}$ wants a Rational Part; and therefore $22+\sqrt{486}$ must be multiplied by the said Remainder 2, that there may be a Binomial $44+\sqrt{1944}$, wherein the Cubic Root of the Difference of the Squares of the Parts in 2.

So to extract $\sqrt{(5)}$ out of $11+\sqrt{125}$, because 121 the Square of 11 subtracted from 125 leaves 4, which considered as a fifth Power has not an exact Rational Root, we must multiply $11+\sqrt{125}$ by 16 the Square of 4, that there may come forth $176+\sqrt{32000}$, where $\sqrt{(5)}$ of the Difference of the Squares of the Parts is 4.

Again, to extract $\sqrt{(7)}$ out of $338+\sqrt{114242}$, wherein the Difference of the Squares of the parts is 2; because this 2 is not the seventh Power of any Rational Number, the given Binomial may be multiplied by 8, that is, by the Cube of 2, and it make $2704+\sqrt{7311488}$, wherein the $\sqrt{(7)}$ of the Difference of the Squares of the Parts in 2.

The RULE.

When a Binomial given, or another deduced from it, (if need be) by the Precedent Preparation is such, that one of its parts and the Square of the other part, as also the Root of the Difference of the Squares of the Parts, (to wit, the Cubic Root when $\sqrt{(3)}$, or $\sqrt{(5)}$ when $\sqrt{(5)}$ is sought) are Rational whole Numbers; then out of a Binomial so qualified $\sqrt{(3)}$, or $\sqrt{(5)}$, or $\sqrt{(7)}$, *&c.* may be extracted, if it has such a Root, in manner following, *viz.*

First, extract the Root of the Difference of the Squares of the parts of the Binomial qualified as aforesaid, *viz.* the Cubic Root when $\sqrt{(3)}$ is sought, but $\sqrt{(5)}$ when $\sqrt{(5)}$, or $\sqrt{(7)}$ when $\sqrt{(7)}$, *&c.* which Root so extracted is to be reserved for a Dividend.

Se-

Secondly, find out a Rational Number a little greater than the Root sought with this caution, that the Rational Number found out may not exceed the said Root above $\frac{1}{2}$, which may easily be done by Vulgar Arithmetick, and take the said Rational Mumber for a Divisor.

Thirdly, divided the said Dividend by the said Divisor, and if the Rational part of the given Binominal be greater than the other part, add the Quotient to the said Rational Divisor, and the half of the greatest whole Number contained in the Sum shall be the Rational part of the Root sought; then from the square of that Rational part subtract the Root of the Difference of the squares of the parts, (to wit, the Dividend first found out as above) so the Remainder shall be the Square of the other part, when such a Root as was required can be extracted out of the given Binomial; which you may easily try by multiplying this Root found out into itself, according to the degree of the Power represented by the given Binomial: for the Root found out being multiplied into itself cubically, if $\sqrt{(3)}$ was sought, or five times into itself if $\sqrt{(5)}$ was sought, ought to produce the given Binomial.

But if the Rational part of the given Binomial be less than the other part, then after you have found out the Quotient as above, subtract it from the Rational Divisor, and the half of the greatest whole Number contained in the Remainder shall be the Rational Part of the Root sought; to the square of which part if there be added the Dividend first found out as above, the Sum will be the Square of the other part, when the Binomial proposed has a Root; but by multiplying the Root found out into itself as before, you may easily try whether it be a true Root or not.

Example 1. *To extract the Cubic Root out of* $20+\sqrt{392}$.

First, the Difference of the Squares of the Parts of the given Binomial, *viz.* the Excess of 400, the Square of 20, above 392, the Square of $\sqrt{392}$ is 8, whose Cubic Root I reserve for a Dividend.

Secondly, I seek a Rational Number that may be greater than the Cubic Root of $20+\sqrt{392}$, the given Binomial, yet so that the Excess may not be greater than $\frac{1}{2}$; to which end I extract the Square Root of 392, and find it to be greater than 19, but less than 20; then to 20 the Rational part of the given Binomial I add 19 and 20 severally and it makes 39 and 40, which are the nearest Rational whole Numbers that can express the true value of the given Binomial; whence the Cubic Root thereof will be found greater than 3, but less than $3\frac{1}{2}$: this $3\frac{1}{2}$ which (according to the Caution before given) exceeds the true Cubic Root of the given Binomial by an Excess not greater than $\frac{1}{2}$, I reserve for a Divisor.

Thirdly, I divide 2 (the Dividend before reserved) by the said Divisor $3\frac{1}{2}$, and the Quotient is $\frac{4}{7}$. Now because 20 the Rational part of the given Binomial is greater than the other part $\sqrt{392}$, I add the said Quotient $\frac{4}{7}$ to the said Divisor $3\frac{1}{2}$, and it makes the Sum $4\frac{1}{14}$, wherein the greatest whole Number is 4, whose half is 2 the Rational part of the Root sought: by the help of which Rational part the other part is easily discovered, for if from 4 the Square of the said 2 you subtract 2, the Cubic Root of the Difference of the Squares of the parts of the given Binomial, there will remain 2 the Square of the other part. So that $2+\sqrt{2}$ is the Cubic Root of $20+\sqrt{392}$ the Binomial proposed, as will appear by the Proof; for $2+\sqrt{2}$ being multiplied into it self cubically produces $20+\sqrt{392}$, and for the same reason $2-\sqrt{2}$ is the Cubic Root of $20-\sqrt{392}$.

Example 2. *To extract the Cubic Root out of* $44+\sqrt{1944}$.

First, the Cubic Root of the Difference of the Squares of the Parts is 2 for a Dividend. Secondly, the square Root of 1944 is greater than 44, but less than 45; these added severally to 44 the rational part of the given Binomial, make 88 and 89, whose Cubic Roots being extracted, do shew that the Cubic Root of the given Binomial is greater than 4, but less than $4\frac{1}{2}$; this Rational Number $4\frac{1}{2}$, which according to the Caution before given exceeds the true Root sought by an excess not greater than $\frac{1}{2}$, I take for a Divisor. Thirdly, I divide the said Dividend 2 by the said Divisor $4\frac{1}{2}$, and the Quotient is $\frac{4}{9}$, which I subtract from the said $4\frac{1}{2}$, (I subtract, because 44 the Rational part of the given Binomial is less than the other Part $\sqrt{1944}$) and there remains $4\frac{1}{18}$; then the half of 4, the greatest whole number contained in $4\frac{1}{18}$, is 2, which is the Rational Part of the Root sought. Lastly, to 4 the Square of the said 2 I add 2, the Cubic Root of the Difference of the Squares of the Parts, and it makes 6 the Square of the other part. So that $2+\sqrt{6}$ is the Cubic Root sought, as will appear by the Proof;

for if it be multiplied into itſelf cubically, it produces $44+\sqrt{1944}$ the Binomial propoſed; and for the ſame Reaſon $\sqrt{6}-2$ is the Cubic Root of $\sqrt{1944}-44$.

Example 3. *To Extract* $\sqrt{(5)}$ *out of* $176+\sqrt{32000}$.

Firſt, the Difference of the Squares of the Parts will be found 1024, whoſe $\sqrt{(5)}$ is 4 for a Dividend. Secondly, the Sum of the Parts will be found greater than 354, but leſs than 355; and conſequently $\sqrt{(5)}$ of the ſum of the Parts is greater than 3, but leſs than $3\frac{1}{4}$. Thirdly, by the ſaid $3\frac{1}{4}$ I divide the ſaid 4, and the Quotient is $1\frac{3}{13}$, which I ſubtract from the ſaid Diviſor $3\frac{1}{4}$, (becauſe the rational Part of the given Binomial is leſs than the other Part) and there remains $2\frac{1}{52}$; then the half of 2 (the greateſt whole Number contained in $2\frac{1}{52}$) is 1, the Rational Part of the Root ſought. Laſtly, the Square of the ſaid 1, to wit 1, added to 4 (the $\sqrt{(5)}$ of the Difference of the ſquares of the Parts of the given Binomials) makes 5 the ſquare of the other Part. So that $1+\sqrt{5}$ is the $\sqrt{(5)}$ of the given Binomial $176+\sqrt{32000}$, at leaſt if any $\sqrt{(5)}$ can be extracted out of the ſame; but $1+\sqrt{5}$ multiplied into itſelf five times makes $176+\sqrt{32000}$; therefore $1+\sqrt{5}$ is manifeſtly the deſired $\sqrt{(5)}$ of $176+\sqrt{32000}$.

Example 4. *To Extract* $\sqrt{(7)}$ *out of* $2704+\sqrt{7311488}$.

Firſt, the $\sqrt{(7)}$ of the Difference of the ſquares of the Parts in 2 for a Dividend. Secondly, the value of the given Binomial will be found greater than 5407, but leſs than 5408; whence the $\sqrt{(7)}$ thereof will be diſcovered to be greater than 3, but leſs than $3\frac{1}{2}$. Thirdly, by the ſaid $3\frac{1}{2}$ I divide the Dividend before found 2, and the Quotient is $\frac{4}{7}$, which I add to the Diviſor $3\frac{1}{2}$, (becauſe the Rational Part 2704 is greater than the other Part) and it makes the Sum $4\frac{1}{14}$; and therefore 2 the half of the greateſt whole Number contained in $4\frac{1}{14}$, is the Rational part of the Root ſought. Laſtly, from 4 the ſquare of the ſaid 2 I ſubtract 2, to wit $\sqrt{(7)}$, of the Difference of the Squares of the Parts of the given Binomial, and there remains 2 the ſquare of the other Part. So that $2+\sqrt{2}$ is the deſired $\sqrt{(7)}$ of the given Binomial $2704+\sqrt{7311488}$; for this is the ſeventh Power of $2+\sqrt{2}$, as will appear by Multiplication.

But here is to be noted, that when the given Binomial has been multiplied or divided by ſome Number, and thereby reduced to another Binomial, and the Root of this latter is found out, we muſt divide or multiply the Root found out by the Root of the Number by which the Binomial was multiplied or divided; ſo there will come forth the Root of the given Binomial.

As for Example, becauſe to extract the Cubic Root out of $\sqrt{242}+12\frac{1}{2}$, we firſt multiplied this Binomial by 2, and found $25+\sqrt{968}$, whoſe Cubic Root by the Rule before given will be found $1+\sqrt{8}$; this muſt be divided by $\sqrt{(3)}2$, and the Quotient $\sqrt{(3)}\frac{1}{2}+\sqrt{(6)}128$ ſhall be the Cubic Root of $\sqrt{242}+12\frac{1}{2}$ the Binomial propoſed.

But that the reaſon of the ſaid Diviſion by $\sqrt{(3)}2$ may the more clearly appear, let there be put $d=1+\sqrt{8}$, then it follows that $ddd=25+\sqrt{968}$, and $\frac{ddd}{2}=\sqrt{242}+12\frac{1}{2}$ (the Binomial propoſed.) Therefore by extracting the Cubic Root out of each part of the laſt Equation there ariſes $\sqrt{(3)}\frac{ddd}{2}$, that is, $\frac{d}{\sqrt{(3)}2}$, $=\sqrt{(3)}:\overline{\sqrt{242}+12\frac{1}{2}}$: But by ſuppoſition $d=1+\sqrt{8}$; therefore $1+\sqrt{8}$ divided by $\sqrt{(3)}2$, that is to ſay, $\sqrt{(3)}\frac{1}{2}+\sqrt{(6)}128$ ſhall be the Cubic Root of $\sqrt{242}+12\frac{1}{2}$; which was to be ſhewn.

Example 2. *To extract* $\sqrt{(3)}$ *out of* $\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}$.

Firſt, to prepare it for Extraction we multiplied by $\sqrt{5}$, and found $\sqrt{242}+12\frac{1}{2}$, whoſe $\sqrt{(3)}$ (as appears in the laſt preceding Example) is $\sqrt{(3)}\frac{1}{2}+\sqrt{(6)}128$, which by dividing by $\sqrt{(6)}5$ gives the Quotient $\sqrt{(6)}\frac{1}{20}+\sqrt{(6)}\frac{128}{5}$ for the deſired Cubic Root of $\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}$. The reaſon of which Diviſion by $\sqrt{(6)}5$ may be thus manifeſted, let there be put $d=\sqrt{(3)}\frac{1}{2}+\sqrt{(6)}128$; then it follows that $ddd=\sqrt{242}+12\frac{1}{2}=\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}$ into $\sqrt{5}$, whence $\frac{ddd}{\sqrt{5}}=\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}$; therefore the Cubic Root of each part of the laſt Equation being extracted there ariſes $\sqrt{(3)}\frac{ddd}{\sqrt{5}}$, that is, $\frac{d}{\sqrt{(6)}5}$ (for $\sqrt{(3)}$ of $\sqrt{5}$ is $\sqrt{(6)}5$) $=\sqrt{(3)}:\overline{\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}}$: But by ſuppoſition $d=\sqrt{(3)}\frac{1}{2}+\sqrt{(6)}128$;

$\sqrt{(6)}128$; therefore $\sqrt{(3)}\frac{1}{2}+\sqrt{(6)}128$ divided by $\sqrt{(6)}5$ gives the true Cubic Root of $\sqrt{\frac{243}{25}}+\sqrt{\frac{125}{4}}$; which was to be ſhewn.

Example 3. *To extract $\sqrt{(3)}$ out of $\sqrt{242}+\sqrt{243}$.*

Firſt, (according to the ſecond Rule of the precedent Preparation) I multiply it by $\sqrt{2}$, and there comes forth $22+\sqrt{486}$; this multiplied by 2 (according to the fourth preparatory Rule) makes $44+\sqrt{1944}$, whoſe Cubic Root (as before has been ſhewn) is $2+\sqrt{6}$, which muſt be divided by $\sqrt{2}$, and there will come forth $\sqrt{2}+\sqrt{3}$ for the Cubic Root ſought of $\sqrt{242}+\sqrt{243}$. But to manifeſt the Reaſon of dividing $2+\sqrt{6}$ by $\sqrt{2}$, let there be put $d=2+\sqrt{6}$, then it follows that $ddd=44+\sqrt{1944}=\overline{22+\sqrt{486}}$ into 2, whence $\frac{ddd}{2}=22+\sqrt{486}$, and this Equation divided by $\sqrt{2}$ (becauſe in the Preparation we multiplied by $\sqrt{2}$) gives $\frac{ddd}{\sqrt{8}}=\sqrt{242}+\sqrt{243}$; therefore $\sqrt{(3)}$ being extracted out of each Part of the laſt Equation, there ariſes $\sqrt{(3)}\frac{ddd}{\sqrt{8}}$, that is, $\frac{d}{\sqrt{(6)}8}$, or $\frac{d}{\sqrt{2}}$, $=\sqrt{(3)}:\overline{\sqrt{242}+243}$: But by ſuppoſition $d=2+\sqrt{6}$; therefore $2+\sqrt{6}$ divided by $\sqrt{2}$, *viz.* the Quotient $\sqrt{2}+\sqrt{3}$ ſhall be the Cubic Root of $\sqrt{242}+\sqrt{243}$; which was to be ſhewn.

Example 4. *To extract $\sqrt{(5)}$ out of $\sqrt{(3)}3993+\sqrt{(6)}17578125$.*

Firſt, (according to the ſecond Preparatory Rule) I divide the given Binomial by $\sqrt{(3)}3$, and then (according to the fourth Preparatory Rule) I multiply the Quotient $\sqrt{(3)}1331+\sqrt{6}1953125$ by 16, and there comes forth $176+\sqrt{32000}$, whoſe $\sqrt{(5)}$ (as has before been ſhewn) is $1+\sqrt{5}$. Now this Root $1+\sqrt{5}$ divided by $\sqrt{(5)}16$, and the Quotient multiplied by $\sqrt{(15)}3$ will diſcover the true $\sqrt{(5)}$ of $\sqrt{(3)}3993+\sqrt{(3)}17578125$; the reaſon of which Diviſion and Multiplication may be made manifeſt thus; let there be put $d=1+\sqrt{5}$, then it follows that $ddddd=176+\sqrt{32000}$; and by dividing each part of the laſt Equation by 16, (becauſe in the preparatory work we multiplied by 16) there ariſes $\frac{ddddd}{16}=\sqrt{(3)}1331+\sqrt{(6)}1953125$; and by multiplying each part of this Equation by $\sqrt{(3)}3$, there will be produced $\frac{ddddd\times\sqrt{(3)}3}{16}=\sqrt{(3)}3993+\sqrt{(6)}17578125$. Therefore $\sqrt{(5)}$ being extracted out of each part of the laſt Equation there will ariſe $\sqrt{(5)}\frac{ddddd\times\sqrt{(3)}3}{16}$, that is, $\frac{d\sqrt{(15)}3}{\sqrt{(5)}16}$ equal to $\sqrt{(5)}$ of $\sqrt{(5)}1331+\sqrt{(6)}17578125$. But by ſuppoſition $d=1+\sqrt{5}$, therefore $1+\sqrt{5}$ multiplied into $\sqrt{(15)}3$, and the Product divided by $\sqrt{(5)}16$; or $1+\sqrt{5}$ divided by $\sqrt{(5)}16$, and the Quotient multiplied $\sqrt{(15)}3$ produces the true $\sqrt{(5)}$ of $\sqrt{(3)}3993+\sqrt{(6)}17578125$; which was to be ſhewn.

The Demonſtration follows.

The certainty of the preceding Rule will be made manifeſt by the three following Propoſitions.

PROP. I.

If a Binomial, whereof one part and the Square of the other are rational Numbers, be multiplied into itſelf cubically, there will be produced another Binomial, the Square of whoſe leſſer Part being ſubtracted from the Square of the greater Part, leaves a Cubic Number, to wit, the Cube of the Difference of the Squares of the Parts of the Root or firſt Binomial.

To make this manifeſt, let there be propoſed the Binomial $b+\sqrt{d}$, this multiplied into itſelf cubically produces $bbb+3bb\sqrt{d}+3bd+d\sqrt{d}$, to wit, the Cube of $b+\sqrt{d}$. Here you are to note well, that although in that Cube there be four Parts or Members, yet they are to be eſteemed but as two, one of which, to wit, $bbb+3bd$, may deſign a Rational Number, and the other $3bb\sqrt{d}+d\sqrt{d}$ (or $3bb+d\times\sqrt{d}$) an Irrational or Surd Number, whoſe Square is Rational; whence it is manifeſt, firſt, that the Cube of a Binomial is alſo a Binomial, *viz.* $b+\sqrt{d}$ multiplied into itſelf cubically produces this

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Binomial $bbb+3bd$ more $3bb\sqrt{d}+d\sqrt{d}$ (or $\overline{3bb+d}\times\sqrt{d}$.) Secondly, the Rational part $bbb+3bd$ is manifestly composed of the Cube of the Rational part of the Root, and of the triple Product made by the Multiplication of the same Root into the Square of its other part. And lastly, the Difference of the Squares of the said Parts $bbb+3bd$ and $3bb\sqrt{d}+d\sqrt{d}$ is equal to the Cube of $bb-d$, or of $d-bb$, *viz.* to the Cube of the Difference of the Squares of the Parts of the Root $b+\sqrt{d}$. For the Squares of $bbb+3bd$ and $3bb\sqrt{d}+d\sqrt{d}$ are $bbbbbb+6bbbbd+9bbdd$ and $9bbbbd+6bbdd+ddd$; and if these Squares be subtracted one from the other, the Remainder is either $bbbbbb-3bbbbd+3bbdd-ddd$, which is the Cube of $bb-d$; or else the Remainder is $ddd-3bbdd+3bbbbd-bbbbbb$, which is the Cube of $d-bb$.

To illustrate this Proposition by Numbers, let there be put $b=2$ and $\sqrt{d}=6$; hence the Binomial $2+\sqrt{6}$ multiplied into it self cubically produces the Binomial $44+\sqrt{1944}$, wherein the Difference of the Squares of the Parts (*viz.* the Remainder when 1936 the Square of 44 is subtracted from 1944 the Square of $\sqrt{1944}$) is 8, to wit, the Cube of the Difference of the Squares of the Parts of the Binomial Root $2+\sqrt{6}$.

Likewise this Binomial $2+\sqrt{2}$ multiplied into itself cubically produces the Binomial $20+\sqrt{392}$, wherein the Differences of the Squares of the Parts, to wit 8, is the Cube of the Difference of the Squares of the Parts of the Root $2+\sqrt{2}$.

The same Properties adhere also to a Residual Root, *viz.* the Cube of the Residual Root $b \backsim \sqrt{d}$ is also a Residual, to wit, $\overline{bbb+3bd} \backsim \overline{3bb\sqrt{d}+d\sqrt{d}}$, (or $\overline{3bb+d}\times\sqrt{d}$;) and the Difference of the Squares of the Parts of the later Residual is equal to the Cube of the Difference of the Squares of the Parts of the Roots or first Residual.

PROP. 2.

If a Binomial, whereof one Part and the Square of the other are the Rational Numbers, be multiplied by the Difference of the Squares of the Parts, the Product will be another, Binomial, wherein the difference of the Squares of the Parts is a Cubic Number, to wit, the Cube of the Difference of the Squares of the Parts of the Root multiplied

To make this manifest, let there be proposed the Binomial $b+\sqrt{d}$, and suppose b greater than $\sqrt{d}$, then $b+\sqrt{d}$ multiplied by $bb-d$, the Difference of the Squares of the Parts, will produce this Binomial, to wit, $\overline{bbb-bd}$ more $\overline{bb\sqrt{d}-d\sqrt{d}}$, the Squares of whose Parts are $bbbbbb-2bbbbd+bbd$ and $bbbbd-2bbdd+ddd$; then this later Square subtracted from the former leaves $bbbbbb-3bbbbd+3bbdd-ddd$, which is the Cube of $bb-d$, the Difference of the Squares of the Parts of the first Binomial $b+\sqrt{d}$. The same Property would appear if we supposed b less than $\sqrt{d}$.

To illustrate this Proposition by Numbers, suppose $b=22$, and $\sqrt{d}=486$; whence the Binomial $22+\sqrt{486}$ multiplied by 2, the difference of the Squares of the Parts, produces the Binomial $44+\sqrt{1944}$, wherein the difference of the Squares of the Parts is 8, which is the Cube of 2, the Difference of the Squares of the Parts of the former Binomial $22+\sqrt{486}$.

PROP. 3.

If the Difference of the Squares of any two Numbers be divided by a Number which doth not exceed the Sum of those two Numbers above $\frac{1}{2}$; then the Quotient added to the said Divisor will give a Number greater than the double of the greater of the said two Numbers, but the Excess will be less than Unity. And if the said Quotient be subtracted from the said Divisor, the Remainder shall be greater than the double of the lesser of the two Numbers, but this Excess also shall be less than Unity.

To manifest this, let a represent the greater of two Numbers, and e the lesser; also let b represent some Fraction not greater than $\frac{1}{2}$; then I say, first, $a+e+b+\frac{aa-ee}{a+e+b}$ is greater than $2a$, but the Excess is less than 1, which I prove thus:

It is evident that $aa+ee+bb+2ae+2be+2ba+aa-ee$ is greater than $2aa+2ae+2ba$; therefore by dividing each of those two Compound Quantities by $a+e+b$, it follows, that the first Quotient $a+e+b+\frac{a+e+b}{aa-ee}$ shall be greater than the later Quotient $2a$; and if this Quantity be subtracted from that, the Remainder $\frac{2bc+bb}{a+e+b}$ will be less than 1. For by supposition b is not greater than $\frac{1}{2}$; therefore $2be$ is less than

$a+e$, and bb less than b; and consequently the Numerator $2be+bb$ is less than the Denominator $a+e+b$: wherefore $\frac{a+e+b}{2be+bb}$ is less than 1.

After the same manner it may be proved that $a+e+b-\frac{aa-ee}{a+e+b}$ is greater than $2e$, but this Excess also shall be less than 1; which was to be shewn.

Now to apply the preceding three Propositions to the Demonstration of the Rule before given, let it be required to extract the Cubic Root out of the Binomial $100+\sqrt{7803}$, whose Rational part 100 is greater than the other part $\sqrt{7803}$; Here we may suppose $bbb+3bd$ to be 100, and $3bb\sqrt{d}+d\sqrt{d}$ (or $\overline{3bb+d}\times\sqrt{d}$) to be $\sqrt{7803}$; so that $\overline{bbb+3bd}$ more $\overline{3bb+d}\times\sqrt{d}$ may design the given Binomial $100+\sqrt{7803}$; and its Cubic Root $b+\sqrt{d}$ the Root sought, whose greater part may be b, and the lesser $\sqrt{d}$. Then according to the Rule:

To extract $\sqrt{(3)}$ *out of* $100+\sqrt{7803}$.

First, from the Square of 100, that is, from . . .	10000
Subtract the Square of $\sqrt{7803}$, that is	7803
The Remainder is	2197
The Cubic Root of that Remainder is	13 $(=bb-d)$

Which Root 13 is (by *Prop.* 1.) equal to the Difference of the Squares of the Parts of the Binomial Root sought.

Secondly, find out a Rational Number greater than the Sum of the Parts of the Cubic Root sought, with this caution, that the Excess may not be above $\frac{1}{2}$, *viz.*

To the greater part of the given Binomial, that is, to .	100
Add the nearest value in whole Numbers of the other part $\sqrt{7803}$, that is, ...	88 or 89
So the Sum shews that the value in whole Numbers of the given Binomial falls between	188 and 189

Whence the Cubic Root of the given Binomial is greater than $5\frac{1}{2}$, but less than 6; so that the Excess of 6 above the true Root sought in less than $\frac{1}{2}$.

Thirdly, having found out (as above) 13, the true Difference of the Squares of the Parts of the Cubic Root sought; and 6 a Rational Number, which exceeds not the true Sum of the same Parts above $\frac{1}{2}$, we may by the help of *Prop.* 3. and 1 find out the Parts severally in this manner, *viz.*

Divide the said	13
By the said	*6*
And the Quotient is	$2\frac{1}{6}$
Which added to the said Divisor *6*, makes the Sum . . .	$8\frac{1}{6}$

Which Sum $8\frac{1}{6}$ does by (*Prop.* 3) exceed the double of the greater (to wit, the Rational) Part of the Cubic Root sought, but the Excess is less than 1: therefore $7\frac{1}{6}$ is less than the said double, but $8\frac{1}{6}$ is greater than the same; and consequently because the said greater Part is supposed to be a Rational whole Number, the double thereof must necessarily be 8, to wit, the greatest whole Number between $7\frac{1}{6}$ and $8\frac{1}{6}$, and therefore the said Part it self is 4, which being found out, it is easie to find the other Part; for (by *Prop.* 1.) if from 16 the Square of the said greater Part 4, there be subtracted 13 the Cubic Root of the Difference of the Squares of the Parts of the given Binomial, there will remain 3 the Square of the other part; so that the Cube Root found out is $4+\sqrt{3}$, which will appear by the Proof to be the true Cubic Root sought; for $4+\sqrt{3}$ being multiplied into it self cubically produces the given Binomial $100+\sqrt{7803}$. And for the same reason $4-\sqrt{3}$ is the Cubic Root of $100-\sqrt{7803}$.

Or more briefly the Proof may be made thus:

To the Cube of 4 the Rational Part of the Root found out, *viz.* to	64, that is, bbb
Add the Product of thrice that Part multiplied into the Square of the Surd Part found out, *viz.* the Product .	36, that is, $3bd$
And it makes the Sum	100, that is, $bbb+3bd$

Which

Which Sum is the same with the Rational part of the given Binomial, and therefore it proves that $4+\sqrt{3}$ is the Cubic Root sought.

In like manner to extract $\sqrt{(3)}$ out of $44+\sqrt{1944}$, where the Rational Part 44 is less than the other Part $\sqrt{1944}$, we may suppose (as before) $bbb+3bd$ to be 44, and $\overline{3bb+d}\times\sqrt{d}$ (that is, $3bb\sqrt{d}+d\sqrt{d}$) to be $\sqrt{1944}$; so that $bbb+3bd$ more $\overline{3bb+d}\times\sqrt{d}$ may design the given Binomial $44+\sqrt{1944}$, and its Cubic Root $b+\sqrt{d}$ the Root sought, whose lesser part may be b, and the greater $\sqrt{d}$; then according to the Rule,

To extract $\sqrt{(3)}$ *out of* $44+\sqrt{1944}$.

First, from the Square of $\sqrt{1944}$, *viz.* from	1944
Subtract the Square of 44, ...	1936
The Remainder is	8
The Cubic Root of that Remainder is	2 $(=d-bb)$

Which Root 2 is (by *Prop.* 1.) equal to the Difference of the Squares of the Parts of the Binomial Root sought.

Secondly, find out a Rational Number greater than the Sum of the Parts of the Cubic Root sought, with this caution, that the Excess may not be above $\frac{1}{2}$, which may be done thus, *viz.*

To the lesser part of the given Binomial, *viz.* to . .	44
Add the nearest value in whole Numbers of the other part $\sqrt{1944}$, that is, ...	44 or 45
So the Sum shews that the value in whole Numbers of the given Binomials falls between	88 and 89

Whence the Cubic Root of the given Binomial is greater than 4, but less than $4\frac{1}{2}$; so that the Excess of $4\frac{1}{2}$ above the true Root sought is less than $\frac{1}{2}$.

Thirdly, having found out 2, the true Difference of the Squares of the Parts of the Cubic Root sought; and $4\frac{1}{2}$ a Rational Number, which does not exceed the true Sum of the same Parts above $\frac{1}{2}$, we may by the help of *Prop.* 3. and 1. find out the Parts severally in this manner, *viz.*

Divide the said	2
By the said	$4\frac{1}{2}$
And it gives the Quotient	$\frac{4}{9}$
Which subtracted from the said Divisor $4\frac{1}{2}$, there remains .	$4\frac{1}{18}$

Which Remainder $4\frac{1}{18}$ does (by *Prop.* 3.) exceed the double of the lesser Part (which in this Example is the Rational Part of the Cubic Root sought, but the Excess is less than 1: therefore $3\frac{1}{18}$ is less than the said double, but $4\frac{1}{18}$ is greater than the same, and consequently because the said lesser Part is a Rational whole Number, the double thereof must necessarily be 4, to wit, the greatest whole Number between $3\frac{1}{18}$ and $4\frac{1}{18}$, and therefore the said Part it self is 2, which being found, it is easie to find the other Part; for if to 4 the Square of the said lesser Part 2, there be added 2 the Cubic Root of the Difference of the Squares of the Parts of the given Binomial, the Sum 6 shall be the Square of the other Part; so that the Cube Root found out is $2+\sqrt{6}$, which will appear to be the true Cubic Root sought; for $2+\sqrt{6}$ multiplied into it self cubically produces the given Binomial $44+\sqrt{1944}$. And for the same Reason $\sqrt{6}-2$ is the Cubic Root of $\sqrt{1944}-44$.

Or more briefly the Proof may be made thus:

To the Cube of 2 the Rational Part of the Root found out, *viz.* to	8, that is, bbb
Add the Product of thrice that Part multiplied into the Square of the Surd Part found out, *viz.* the Product . .	36, that is, $3bd$
And the Sum is	44, that is, $bbb+3bd$.

Which Sum is the same with the Rational Part of the given Binomial, and therefore it proves that $2+\sqrt{6}$ is the Cubic Root sought.

Lastly, what has here been shewn concerning the Demonstration of the Extraction of the Cubic Root, may easily be applied to the Extraction of the other Roots before mentioned, so that there is no need of further Discourse in this Matter.

CHAP.

CHAP. X.

An Explication of Simon Stevin's *General Rule, to extract one Root out of any possible Equation in Numbers, either exactly or very nearly true.*

I. EQuations falling under any of the Forms in the fourteenth and fifteenth Chapters of the First Book of these Elements, are capable (as has there been shewn) of perfect Resolutions in Numbers, *viz.* the value of the Root or Roots sought in any of those Equations may be found out and express'd exactly, either by some rational or irrational Number or Numbers; but the perfect Resolution of all manner of Compound Equations in Numbers I have not found in any Author. And since an Exposition of the General Method of *Vieta*, the Rules of *Huddenius* and others to that purpose, would make a large Treatise, and after all leave the curious Analyst dissatisfied, I shall not clog these Elements with a tedious Discourse upon those difficult Rules, which at the best are exceeding tedious in Operation, and some of them uncertain too; but rather pursue my first design, which was to explain Fundamentals, and such Rules as are certain and most important in this profound Art. However, I shall lead the industrious Learner to a few steps further, in order to his understanding the Resolution of all manner of Compound Equations in Numbers, and in this Chapter explain *Simon Stevin's* General Rule, which with the help of the Rules in the following eleventh Chapter will discover all the Roots of any possible Equation in Numbers, either exactly if they be Rational, or very nearly true if Irrational.

QUESTION. 1.

If . . . . $aaa+26a=40188$, what is the Number a?

RESOLUTION.

This Equation not falling under any of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. cannot be resolved by any of the Canons in that Chapter, and therefore according to *Simon Stevin's* General Method I search out the Number a by tryals thus, *viz.*

1. I suppose . . . . $a=1$
   Thence it follows that . . . . $aaa=1$
   And . . . . $26a=26$
   Therefore . . . . $aaa+26a=27$

Which 27 ought to have been 40188, but it's too little; whereby I find that by supposing a to be 1 I did not hit upon the true Number a, and therefore I make another tryal in like manner as before, *viz.*

2. I suppose . . . . $a=10$
   Thence it follows that . . . . $aaa=1000$
   And . . . . $26a=260$
   Therefore . . . . $aaa+26a=1260$

Which 1260 being yet too little, I make a third tryal, *viz.*

3. I suppose . . . . $a=100$
   Thence it follows that . . . . $aaa+26a=1002600$

Which 1002600 exceeds the just Result or absolute Number 40188 in the latter part of the Equation first propos'd, and therefore the true Number a is less than 100; but the second tryal shews it to be greater than 10, and therefore the whole Number which expresses the exact, or at least part of the value of a, must necessarily consist of two Characters, and consequently the first (towards the left Hand) must be one of these nine, 1, 2, 3, 4, 5, 6, 7, 8, 9; but because by the second Inquiry 10 was found too little, I now make tryal with 2 for the first Figure of the Root a, *viz.*

4. I suppose . . . . $a=20$
   Thence . . . . $aaa+26a=8520$

Which Result 8520 being yet less than the just Result 40188, I make tryal again, *viz.*

5. I suppose . . . . $a=30$
   Thence . . . . $aaa+26a=27780$

Which

Which is yet too little; therefore,

6. I suppose . . . . $a = 40$

Thence . . . . $aaa + 26a = 65040$

Which 65040 being greater than 40188, it shews me that the true Root or value of a is less than 40; but by the fifth Tryal it's greater than 30, and consequently the last Figure of the Root is 3.

Now the second Character of the Root must necessarily be one of these, *viz.* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; and because it has been discovered, that the true value of the Root a is greater than 30, the second Character cannot be 0, I therefore make tryal with 1, and suppose $a=31$; which proving too little, I make tryal with 32, 33, 34, &c. severally in like manner as before, and at length I find 34 to be the true Number a sought, by which the Equation propos'd may be expounded; for if $a=34$, then consequently $aaa+26a=40188$.

II. But if after tryals made (as before) the value of a the Root sought happens to fall between two whole Numbers that differ by Unity; then tryals are to be made with the lesser whole Number increased with $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c. until you have found the value of a in some mixt Number consisting of a whole Number and some certain tenth parts of an Unit. But if the said value of a happens not to be express'd exactly by the said lesser whole Number increased with certain tenth parts, then you are to make tryals with the said lesser whole Number increased with a Decimal Fraction, having for its Numerator a Number greater than 10, but less than 100; and for its Denominator 100, as with $\frac{11}{100}$, $\frac{12}{100}$, &c. and by proceeding in that manner you may find the exact value of the Root a, when its fractional part is exactly equal to some Decimal Fraction: or else approach infinitely near to the said exact value when 'tis Irrational or Surd, as in this following

QUESTION. 2.

If . . . . . . $aaaa+50a=184638.6801$; (or $184638\frac{6801}{10000}$;) what is the Number a?

RESOLUTION.

First, I suppose $a=1$, but this proving too little I put $a=10$, this also proving too little, I assume $a=100$, which after tryal I find to be greater than the true Number a, and consequently the Number a falls between 10 and 100; then making tryal with 20 I find it too little, but making tryal with 30 I find this too great, and therefore the true Root a falls between 20 and 30. Again, making tryal with 21 I find it too great, but 20 was before found too little; therefore the true Root a is between 20 and 21; then I make tryal with 20.1, (that is, $20\frac{1}{10}$) 20.2, 20.3, &c. and at length find 20.7 to be the true Number a sought; for if $a=20.7$ (that is, $20\frac{7}{10}$) it will make $aaaa+50a=184638.6801$ the Equation proposed.

But if 20.7 had proved too little, and 20.8 too great, then tryals must have been made with 20.71, (that is, $20\frac{71}{100}$) 20.72, 20.73, &c. In like manner, if 20.7 had been too little, but 20.71 (that is, $20\frac{71}{100}$) too great, then tryals must have been made with 20.701, (that is, $20\frac{701}{1000}$) 20.702, 20.703, &c. This will be partly exercis'd in resolving the Equation in this following

QUESTION. 3.

If . . . . $aaa+20aa=1954$, what is the Number a?

Answ. . . . . . $a=8.308$, &c. found out by tryals as before.

III. When the value of (a) the required Root of an Equation happens to be less than Unity, then trial is to be made with $\frac{1}{10}$; but if this prove too great, then with $\frac{1}{100}$, &c. Now suppose .1 (that is $\frac{1}{10}$) be too great, .01 (that is, $\frac{1}{100}$) too little, then tryal must be made with .02 | .03 | .04 |, &c. until you have found out the greatest Figure that must stand in the second place of the Decimal Fraction expressing the Root sought; supposing then such Figure to be found 8, *viz.* that .08 (or $\frac{8}{100}$) is less, but .09 (or $\frac{9}{100}$) is greater than the Root, tryal must be made with .081, (that is, $\frac{81}{1000}$) .082 | .083 | &c. as in this following

QUESTION 4.

If . . . . $aaa+3240a=269$, what is the Number a?

Answ. . . . $a=.083$, &c. that is, $\frac{83}{1000}$, &c.

IV. The

IV. The preceding Examples may ſuffice to ſhew the uſe of this general Method, when all the Terms of the unknown part of an Equation are Affirmative, (*viz.* when $+$ is prefix'd to each Term) in which caſe there is but one Affirmative Root; in the ſearch whereof by Tryals (as before) if the Numbers aſſumed ſeverally for the value of the Root ſought do aſcend greater and greater, then the abſolute Numbers reſulting from thoſe aſſumed Values will likewiſe aſcend; and contrarily, if the aſſumed Roots do deſcend from a greater to a leſs, the Reſults will likewiſe grow leſs and leſs: whence by comparing an abſolute Number reſulting from an aſſumed Root with the juſt abſolute Number of the Equation propos'd, you may certainly know (if the ſaid Reſult and juſt Abſolute be not equal to one another) whether you are to take a Number greater or leſs than that laſt before aſſumed.

But when the unknown part of an Equation conſiſts of affirmative and negative Terms mingled one with another, then the ſearch by Tryals will be more intricate and doubtful than before; for ſometimes it will be hard to diſcern whether a following aſſumed Root muſt be taken greater or leſs than that which was taken next before. Moreover, a compound Equation of this latter kind may happen to be ſuch, that it may be expounded by as many ſeveral affirmative Roots, as there be Unities in the Index of the higheſt unknown Power, *viz.* a Cubical Equation may be ſo conſtituted, that it ſhall have three different affirmative Roots, a Biquadratic Equation four ſeveral Roots; and ſo of higher Equations, as will be ſhewn in the following *Chap.* 11. But in what manner ſoever any poſſible Equation is conſtituted in Rational Numbers, this general Method will always find out one affirmative Root, either exactly true, or at leaſt very near the truth, as will further appear by the following Queſtions.

QUESTION 5.

If . . . . $aaa - 22aa + 157a = 360$, what is the Number a?

RESOLUTION.

1. I ſuppoſe . . . . $a = 1$
   Thence it follows that . . . . . $aaa - 22aa + 157a = 136$

Which 136 is leſs than the juſt abſolute Number 360, and therefore I make another Tryal, *viz.*

2. I ſuppoſe . . . . $a = 10$
   Thence it follows that . . . . . $aaa - 22aa + 157a = 370$

Which 370 exceeds the juſt abſolute Number 360, and therefore I conclude there is one affirmative value of a, (either Rational or Irrational) between 1 and 10; which value, after Tryals made with 2,3,4,5, I find to be 5; this will conſtitute the Equation propoſed, for if $a=5$, then $aaa - 22aa + 157a$ will exactly make 360.

But there are two other Roots or Values of a, to wit 8 and 9, each of which will likewiſe conſtitute the Equation firſt propoſed, but how they are found out will be ſhewn in *Sect.* 9. of the following *Chap.* 11.

QUESTION 6.

If . . . . $3200a - aaa = 46577$ (juſt,) what is the Number a?

RESOLUTION.

1. I ſuppoſe . . . . $a = 1$
   Thence . . . . $3200a - aaa = 3199$ (leſs than juſt)
2. I ſuppoſe . . . . $a = 10$
   Thence . . . . $3200a - aaa = 31000$ (leſs than juſt)
3. I ſuppoſe . . . . $a = 100$
   Thence . . . . $3200a - aaa = -680000$ (leſs than juſt)

Now becauſe the ſecond Reſult (or abſolute Number) $+31000$ is Affirmative, and the laſt Reſult 680000 is Negative, I make tryals with Numbers between 10 and 100 for the value of a; for if the Equation propoſed be poſſible, before the affirmative Reſults fall off to negatives, there will be a Root or Value of a producing an Affirmative Reſult either exactly equal, or very near to the juſt Reſult 46577; therefore,

4. I ſuppoſe . . . . $a = 20$
   Thence . . . . $3200a - aaa = 56000$ (greater than juſt)

Now becauſe by taking 20 for the value of a, the Reſult 56000 exceeds the juſt Reſult 46577; but by taking 10 for a, the Reſult 31000 happened to be leſs than the ſaid 46577, it ſhews there is one affirmative Root or value of a between 10 and 20, which Root, after tryals made with intermediate Numbers (as in former Examples) will be found 15, 7, *&c.* Moreover, becauſe by ſuppoſing $a=20$ the Reſult 56000 happened to exceed the juſt Reſult 46577, but by putting $a=100$ the Reſult -680000 proved to be leſs than the ſame 46577, it ſhews there is an Affirmative value of a between 20 and 100, which value after tryals made will be found 47; ſo that there are two affirmative Roots or values of a found out, to wit, 15, 7. *&c.* (or $15\frac{7}{10}$, *&c.*) and 47; the former of which will nearly, and the latter exactly conſtitute the Equation propoſed.

V. *Florimond de Beaune* in the latter of two ſmall Treatiſes printed in 1659, concerning the Nature, Conſtitution, and Limits of Equations, ſhews how to find out Limits within which the Roots of all compound Equations not aſcending above the Biquadratic kind are confined; which Limits when they may be diſcovered without much trouble, and are not very wide aſunder, will help to leſſen the tryals in the general Method before delivered. As in the laſt Example, where

The Equation propoſed was	$3200a-aaa=46577$
Firſt, becauſe aaa muſt be ſubtracted from $3200a$, and leave a Remainder equal to 46577, it preſuppoſes	. . . $aaa \sqsupset 3200a$
Therefore by dividing each part by a	. . . $aa \sqsupset 3200$
And by extracting the ſquare Root out of each part, it follows that	. . . . . $a \sqsupset 56.5$, *&c.*
Again, from the Equation propos'd by Tranſpoſition 'tis evident that	$3200a-46577=aaa$
Whence 'tis alſo manifeſt that	$3200a \sqsubset 46577$
And conſequently by dividing each part by 3200, ...	$a \sqsubset 14.5$, *&c.*

Thus it is found that the value of a the Root ſought is greater than 14.5, *&c.* but leſs than 56.5, *&c.* and therefore tryals according to the general Method aforeſaid need not be made with any Numbers that are not within thoſe Limits.

From the Premiſes 'tis evident that this general Method finds not a perfect Root of an Equation, unleſs ſuch Root be a whole Number, or elſe a Fraction exactly equal to ſome Decimal Fraction; or laſtly, a mixt Number compos'd of a whole Number and a perfect Decimal Fraction.

Note. When the Coefficients or known Numbers multiplied into any of the unknown Powers under the higheſt, (which muſt have no Coefficient but Unity) are Vulgar (not Decimal) Fractions, or mixt Numbers whoſe fractional parts are Vulgar Fractions; likewiſe, when the abſolute Number that ſolely poſſeſſes the latter part of the Equation propos'd is a Vulgar Fraction, or mixt Number whoſe Fractional part is a Vulgar Fraction; all thoſe Vulgar Fractions muſt be reduced to Decimal Fractions, or elſe the Equation muſt be reduced to another Equation in Integers (by *Sect.* 7. in the following *Chap.* 11.) before you enter upon the Reſolution by tryals as aforeſaid.

CHAP. XI.

Extractions out of the Algebraical Treatiſes of Vieta *and* Renates des Cartes, *concerning the Conſtitution and Reſolution of Compound Equations in Numbers, eſpecially thoſe which have many Roots.*

I. THE Scope of this Chapter is, firſt, to ſhew how to form an Equation that ſhall have as many different Roots or values of the Quantity ſought as ſhall be deſired; then how to free an Equation from Fractions, and to caſt away the ſecond Term; and laſtly, how to find out the Roots of all manner of Compound Equations in Numbers, either exactly if they be Rational, or very near the truth if Irrational.

But

But that the Learner may the more easily perceive my meaning, I shall premise a few Definitions in three Sections next following.

II. When the known absolute Number in an Equation solely possesses one part thereof, let it be transfer'd to the other part by the Sign —, and then there will be an Equation which has o or nothing for one part, and the other part is by *Cartesius* called the Sum of the Equation proposed. As for Example, if this Equation be proposed, *viz.* $aaa - 9aa + 26a = 24$, by transposition of 24 it makes $aaa - 9aa + 26a - 24 = 0$, whose first part is called the Sum of the Equation proposed.

III. In the Equations handled in this Chapter I put a, e, or y, to signifie an unknown Quantity, and by the first Term of an Equation is meant the highest unknown Power, to wit, that which has most Dimensions or Degrees of a; by the second Term that which has fewer Dimensions by one than the first, and so downwards. As in this Equation, $aaa - 9aa + 26a - 24 = 0$, the first Term is aaa, whose Index is 3; the second Term is $-9aa$, where the Index of aa is 2; the third Term is $+26a$, where the Index of a is 1; and the last Term is -24, the known absolute Number, whose Index is o.

IV. The Roots of an Equation are of three kinds, *viz.* either Affirmative, or Negative, or Impossible. An Affirmative Root is a Quantity greater than nothing, as $+5$ or $+20$. A negative Root (which *Cartesius* calls a false Root) expresses a Quantity whose Denomination is opposite to an affirmative, as -5 or -20; the former of which wants 5, and the latter 20, of being equal to nothing. Lastly, impossible Roots are such whose values cannot be conceived or comprehended either Arithmetically or Geometrically; as in this Equation. $a = 2 - \sqrt{-1}$, where $\sqrt{-1}$, that is, the square Root of -1, is no manner of way intelligible, for no Number can be imagined, which being multiplied by itself according to any Rule of Multiplication will produce -1.

V. These things premised, I shall proceed to the forming of Equations which shall have many Roots.

PROP. I.

To form an Equation which shall have two Affirmative Roots.

1. Suppose . . . . $\begin{cases} a = 2, \text{ that is, } a - 2 = 0 \\ a = 3, \text{ that is, } a - 3 = 0 \end{cases}$

2. Then by multiplying the said $a - 2 = 0$ by $a - 3 = 0$, this Equation is produced, *viz.* $\}$ . . $aa - 5a + 6 = 0$

3. That is, by transposition, ... $5a - aa = 6$

Which last Equation falls under the last of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. and may be expounded by either of the two Roots or values of a, which by the Canon in *Sect.* 10. of the same *Chap.* will be found 2, and 3, to wit, those from which the said Equation was produced by Multiplication, as above.

Again, if this Equation $aa + 6a - 55 = 0$, (that is, $aa + 6a = 55$) which has one affirmative Root, to wit 5, be multiplied by $a - 6 = 0$, there will be produced $aaa - 91a + 330 = 0$, (that is, $91a - aaa = 330$) which has two affirmative Roots or values of a, to wit 5 and 6, which may be found out by the Rule hereafter delivered in *Sect.* 9. of this *Chap.*

PROP. II.

To form an Equation which shall have one Affirmative and one Negative Root.

1. Suppose . . . . $\begin{cases} a = 3, \text{ that is, } a - 3 = 0 \\ a = -2, \text{ that is, } a + 2 = 0 \end{cases}$

2. Then by multiplying the said $a - 3 = 0$ by $a + 2 = 0$, this Equation is produced, *viz.* $\}$ . . $aa - a - 6 = 0$

3. That is, ... $aa - a = 6$

Which last Equation falls under the second of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. and may be expounded by either of these two Roots or values of a, whereof one is Affirmative and the other Negative; which after the manner of resolving *Quest.* 1. in *Sect.* 7. of the same *Chap.* will be found $+3$ and -2, to wit, those from which the said Equation was produced by Multiplication, as before.

PROP.

PROP. III.

To form an Equation which shall have three Affirmative Roots.

1. Suppose : . . . . $\left\{\begin{array}{l} a = 2, \text{ that is, } a-2 = 0 \\ a = 3, \text{ that is, } a-3 = 0 \\ a = 4, \text{ that is, } a-4 = 0 \end{array}\right.$

2. Then by multiplying the three last Equations (in each of which the latter part is o) one into another, this Equation will be produced, $aaa-9aa+26a-24 = 0$

3. That is, by transposition of -24, . . . $aaa-9aa+26a = 24$

Which Equation may be expounded by every one of these three affirmative Roots or values of a, to wit, 2, 3, and 4; which may be found out by the Rule in the following *Sect.* 9. of this *Chap.*

The same Equation may likewise be formed altogether by Letters thus, *viz.* let the said known Roots 2, 3, and 4, be represented by b, c, d; and then,

4. Suppose . . . . $\left\{\begin{array}{l} a = b, \text{ that is, } a-b = 0 \\ a = c, \text{ that is, } a-c = 0 \\ a = d, \text{ that is, } a-d = 0 \end{array}\right.$

5. Then by multiplying those three last Equations, in each of which the latter part is nothing, one into another, this Equation will be produced, *viz.*

$$aaa \begin{array}{l} -b \\ -c \\ -d \end{array}\Big\} aa \begin{array}{l} +bc \\ +bd \\ +cd \end{array}\Big\} a - bcd = 0$$

That is, ... $aaa - 9aa + 26a - 24 = 0$

PROP. IV.

To form an Equation which shall have three Affirmative Roots, and one Negative Root.

1. Suppose . . . . $\left\{\begin{array}{l} a = 2, \text{ that is, } a-2 = 0 \\ a = 3, \text{ that is, } a-3 = 0 \\ a = 4, \text{ that is, } a-4 = 0 \\ a = -5, \text{ that is, } a+5 = 0 \end{array}\right.$

2. Then by multiplying the four last Equations, in each of which the latter part is o, one into another, this following Equation will be produced, *viz.*

$$aaaa-4aaa-19aa+106a-120 = 0$$

That is, ... $aaaa-4aaa-19aa+106a = 120$

Which last Equation may be expounded by every one of these three Affirmative Roots or values of a, *viz.* 2, 3, and 4, and by one Negative Root -5; every one of which may be found out by the Rule in the following *Sect.* 9. of this *Chap.*

The same Equation may likewise be formed altogether by Letters thus, *viz* let the said known Roots 2,3,4, and -5, be represented by b, c, d, and $-f$; then,

3. Suppose . . . . $\left\{\begin{array}{l} a = b, \text{ that is, } a-b = 0 \\ a = c, \text{ that is, } a-c = 0 \\ a = d, \text{ that is, } a-d = 0 \\ a = -f, \text{ that is, } a+f = 0 \end{array}\right.$

4. Then by multiplying the four last Equations, in each of which the latter part is o, one into another, this following Equation will be produced, *viz.*

$$aaaa \begin{array}{l} -b \\ -c \\ -d \\ +f \end{array}\Bigg\} aaa \begin{array}{l} +bc \\ +bd \\ +cd \\ -bf \\ -cf \\ -df \end{array}\Bigg\} aa \begin{array}{l} -bcd \\ +bcf \\ +bdf \\ +cdf \end{array}\Bigg\} a - bcdf = 0$$

That is, ...

$$aaaa - 4aaa - 19aa + 106a - 120 = 0$$

After the same manner you may form an Equation, which shall have as many Roots as you please, either all Affirmative, or some of them Affirmative and some Negative.

VI. *Ob-*

VI. *Observations upon the preceding four Propositions.*

1. By what has been said 'tis evident, that sometimes an Equation may have as many Roots as there be Unities in the Index of the highest unknown Term; I say, sometimes, not always: for altho this Equation $aaa-6aa+13a-10=0$, as to its number of Terms and Signs, be like to the Equation formed in the preceding *Prop.* 3, so that one may think it has three Roots, yet it has only one affirmative Root, to wit 2, and no other Root either affirmative or negative can constitute the said Equation, for 'tis produced by the Multiplication of this impossible Equation $aa-4a+5=0$ by $a-2=0$; but that $aa-4a+5=0$, that is, $4a-aa=5$ is an impossible Equation, the Determination in *Sect.* 9. *Quest.* 1. *Chap.* 15. *Book* 1. makes manifest.

In like manner, altho this Equation $aaaa-60aaa+1650aa-22500a+115344=0$, as to its Number of Terms and Signs be like to an Equation that has four affirmative Roots, yet that Equation can be expounded only by two affirmative Roots, to wit, 12 and 18, and by no other Root either affirmative or negative; for 'tis made by the Multiplication of $aa-30a+216=0$, which has two affirmative Roots, 12 and 18, into this impossible Equation $aa-30a+534=0$.

2. Forasmuch as Division resolves or undoes that which is compos'd or done by Multiplication, if the Sum of an Equation which is produced by the Multiplication of two or more Equations one into another, (according to the Method in the preceding four Propositions) be divided by a Binomial compos'd of the unknown Quantity (a) less by the value of any one of the affirmative Roots, or more by the value of one of the negative Roots, the Quotient shall be an Equation in which the first Term has fewer Dimensions by one than the first Term of the Equation so divided. And if the Quotient be divided in like manner, there will come forth an Equation whose first Term has fewer Dimensions by one than the former Quotient. As for Example, let there be proposed the Equation in the preceding *Prop.* 4. to wit, $aaaa-4aaa-19aa+106a-120=0$, which was made by the continual Multiplication of $a-2=0$, $a-3=0$, $a+5=0$; I say, If the Equation proposed be divided by any one of the Binomials $a-2$, $a-3$, $a-4$, $a+5$, the Quotient will be an Equation wherein the first Term has only three Dimensions, which are fewer by one than those in $aaaa$ the first Term of the Equation proposed. So if the said $aaaa-4aaa-19aa+106a-120=0$ be divided by $a-2=0$, there will arise $aaa-2aa-23a+60=0$, as you see by the subsequent Division.

$$
\begin{array}{lllll}
a-2\ ) & aaaa-4aaa-19aa+106a-120 & (& aaa-2aa-23a+60 \\
 & aaaa-2aaa & & \\
 & \overline{\quad -2aaa-19aa} & & \\
 & \quad -2aaa+\ 4aa & & \\
 & \overline{\qquad -23aa+106a} & & \\
 & \qquad -23aa+\ 46a & & \\
 & \overline{\qquad\quad +\ 60a-120} & & \\
 & \qquad\quad +\ 60a-120 & & \\
 & \overline{\qquad\qquad 0 \quad 0} & &
\end{array}
$$

Likewise if the Quotient, to wit, the Equation $aaa-2aa-23a+60=0$, where the first Term aaa has three Dimensions, be divided by $a-3=0$, there will arise $aa+a-20$, whose first Term aa has but two Dimensions. And lastly, if the said latter Quotient $aa+a-20$ be divided by $a-4=0$, there will come forth a simple Equation, to wit, $a+5=0$, that is the negative Root $a=-5$.

The like Division may be practised with the literal Equations at the latter end of *Prop.* 3. and 4. in the preceding *Sect.* 5.

3. If a compleat Equation, that is, such in which all the Terms are extant, be produced by the Multiplication of possible Equations one into another, you may discover how many affirmative, and how many negative Roots that Equation has, by this Rule, *viz.* As often as — follows next after +, or + next after —, so often there is an affirmative Root; and as often as two Signs — or two Signs + stand next to one another, so often there is a negative Root. As for Example, in this Equation, (before formed in *Prop.* 4.)

to

to wit, $aaaa - 4aaa - 19aa + 106a - 120 = 0$, becauſe next after the firſt Term $+aaaa$ there follows $-4aaa$, it ſhews there is one Affirmative Root; and becauſe next after $-4aaa$ there comes $-19aa$, it ſhews that the Equation has one negative Root. Again, becauſe next after $-19aa$ there follows $+106a$, it hints there is another Affirmative Root; and becauſe next after $+106a$ there follows -120, it ſhews there is a third Affirmative Root: ſo that the ſaid Rule diſcovers the Equation propos'd to have three Affirmative Roots, and one negative Root.

4. It is alſo manifeſt from the manner of forming Equations according to the Propoſitions in the preceding *Sect.* 5. that in every Equation which has as many Affirmative Roots as there be Dimenſions in the firſt Term, the Co-efficient or known Quantity in the ſecond Term is equal to the ſum of all the Affirmative Roots; and the known Quantity in the third Term is equal to the ſum of the Products of every two of the ſaid Roots multiplied one by the other, and the known Quantity in the fourth Term is equal to the ſum of the Products of every three of the ſaid Roots; and ſo forward when there be more Terms: but the laſt Term, to wit, the abſolute Quantity given is equal to the Product of all the Roots multiplied one into another. As in the following Equation (before formed in *Prop.* 3.) *viz.*

$$aaa \left.\begin{matrix} -b \\ -c \\ -d \end{matrix}\right\} aa \quad \left.\begin{matrix} +bc \\ +bd \\ +cd \end{matrix}\right\} a - bcd = 0.$$

That is, $\quad aaa - 9aa + 26a - 24 = 0.$

Firſt, the Sum of 2, 3, and 4, (that is, of b, c, d) the three Roots of that Equation is 9, which is the known Number of the ſecond Term $-9aa$. Secondly, the Sum of the Products of every two of the ſaid Roots multiplied one by the other is 26, that is, $+bc+bd+cd$, which is the known Coefficient of the third Term $+26a$, or $\overline{+bc+bd+cd}$ into a. And laſtly, the Product of all the three Roots multiplied one into another is 24, or bcd, to which prefixing $-$ it makes -24, or $-bcd$, the laſt Term of the Equation propos'd.

The like Properties enſue when the Sum of the Numbers of Multitude of Affirmative and negative Roots is equal to the number of Dimenſions in the firſt Term of an Equation; ſaving that here in ſumming up all the Roots which compoſe thoſe known Quantities in the ſecond Term, and likewiſe the Products which compoſe the known Quantities in the following Terms, reſpect muſt be had to the Rules of Addition of $+$ and $-$ in ſuch manner as the Equation propoſed if it be found altogether by Letters will direct; as you may eaſily perceive by the Equation formed in *Prop.* 4. of the preceding *Sect.* 5.

VII. *How to free an Equation from Fractions, when 'tis incumbered therewith in the ſecond, third, or any of the following Terms. Which work is by* Vieta *called* Iſomœria.

The Rules in *Chap.* 12. *Book* 1. ſhew how to reduce an Equation ſo, as that the firſt Term may have no Coefficient but Unity; but if after any Equation is ſo reduced there happens to be any Fraction in the ſecond, third, or any of the following Terms, ſuch Equation may be reduced to another whoſe Terms ſhall be all Integers, by the Method in the five Examples next following.

Example 1.

1. Let this Equation be propos'd to be reduced to another in Integers, *viz.* . . . . $\quad aaa + \frac{3}{2}a = 225$

Operation.

2. Suppoſe $e = 2a$, ($2a$, becauſe 2 is the Denominator of the Fraction $\frac{3}{2}$), ... $\quad e = 2a$
3. Then divide each part of the laſt Equation by 2, (the Denominator aforeſaid) and there ariſes . . . . . . $\quad \frac{e}{2} = a$
4. And by multiplying each part of the Equation in the third ſtep cubically, there comes forth . . . . $\quad \frac{eee}{8} = aaa$
5. Again, by multiplying each part of the Equation in the third ſtep by $\frac{3}{2}$, (the Fraction in the ſecond Term of the Equation firſt propoſed) it makes . . . . $\quad \frac{3e}{4} = \frac{3}{2}a$

6. Then

6. Then add the two laſt Equations into one, and the Sum is . . . . $\frac{eee}{8}+\frac{3e}{4} = aaa+\frac{3}{2}a$
7. But by ſuppoſition in the firſt ſtep . . . . $225 = aaa+\frac{3}{2}a$
8. Therefore from the two laſt Equations (by 1 *Axiom.* 1 *Elem. Euclid.*) . . . . $\frac{eee}{8}+\frac{3e}{4} = 225$
9. Which laſt Equation being reduced to Integers (by *Sect.* 2. *Chap.* 12. *Book* 1.) gives . . . . $eee+6e = 1800$

Therefore an Equation is found out, which is altogether expreſs'd by Integers; and when the value of e in the laſt Equation is diſcovered, the value of a in the Equation propos'd is conſequently known; for by the third ſtep $a=\frac{1}{2}e$, therefore if e be 12, then a ſhall be 6.

Example 2.

Again, if this Equation be propos'd, . . . : . $aaa+\frac{3}{2}a = \frac{265}{2}$
It may be reduced in like manner as before in *Example* 1. to this, *viz.* . . . . $eee+6e = 1060$
And if e be 10, then a ſhall be 5.

Example 3.

So likewiſe this Equation . . . . $aaa+\frac{3}{2}aa = \frac{325}{2}$
May be reduced to this . . . . $eee+3ee = 1300$
And if e be 10, then a is 5.

Example 4.

1. Again, let there be propoſed . . . . , $aaa+\frac{11}{12}a = \frac{19}{4}$

Operation.

2. Suppoſe $e=12a$, (12a becauſe 12 is the Denominator of the Fraction $\frac{11}{12}$ in the ſecond Term) . . . . . $e = 12a$
3. Then divide each part of the laſt Equation by 12, (the Denominator aforeſaid) and there ariſes . . . $\frac{e}{12} = a$
4. And by multiplying cubically the laſt Equation, it produces . . . . $\frac{eee}{1728} = aaa$
5. And by multiplying the Equation in the third ſtep by $\frac{11}{12}$, it makes . . . . $\frac{11e}{144} = \frac{11}{12}a$
6. And by adding the two laſt Equations into one, the Sum makes . . . . $\frac{eee}{1728} + \frac{11e}{144} = aaa+\frac{11}{12}a$
7. But by the Equation propoſed . . . . $\frac{19}{4} = aaa+\frac{11}{12}a$
8. Therefore from the two laſt Equations (by 1. *Axiom.* 1. *Elem. Euclid.*) . . . . $\frac{eee}{1728} + \frac{11e}{144} = \frac{19}{4}$

Which Equation reduced to Integers gives . . . . . $eee+132e = 8208$

Thus an Equation is found out in Integers; and when the value of e is diſcovered, the value of a in the Equation propos'd is conſequently known; for by ſuppoſition in the ſecond ſtep e is to a as 12 to 1; therefore if e be 18, then a ſhall be $1\frac{1}{2}$.

Example 5.

1. Again, let there be propoſed . . $aaaa-10aaa+45\frac{5}{6}aa-104\frac{5}{6}a+89 = 0.$

Operation.

2. Suppoſe $e=6a$, (6a becauſe 6 is the Denominator of the Fraction $\frac{5}{6}$) . . . . $e = 6a$
3. Then by dividing each part of the laſt Equation by 6, there ariſes . . . . $\frac{e}{6} = a$
4. And by ſquaring the laſt Equation it makes . . . $\frac{ee}{36} = aa$
5. Likewiſe by ſquaring each part of the laſt Equation, there will be produced . . . . $\frac{eeee}{1296} = aaaa$
6. And by multiplying the Equation in the fourth ſtep by that in the third, the Product is . . . . $\frac{eee}{216} = aaa$

Mm

7. And

7. And by multiplying the last Equation by 10, it gives this, *viz.* . . . . $\frac{10eee}{216} = 10aaa$
8. And by multiplying the Equation in the fourth step by $45\frac{5}{6}$ it produces . . . . $\frac{275ee}{216} = 45\frac{5}{6}aa$
9. And by multiplying the Equation in the third step by $104\frac{1}{6}$, the Product will be . . . . $\frac{625e}{36} = 104\frac{1}{6}a$
10. Then by connecting the Quantities which stand in the first Parts of the Equations in the fifth, seventh, eighth, and ninth steps, together with 89, by the same Signs which respectively belong to each Term of the Equation proposed, the Sum shall be equal to the Sum of the same Equation, and consequently equal to nothing; hence this Equation arises, *viz.*

$$\frac{eeee}{1296} - \frac{10eee}{216} + \frac{275ee}{216} - \frac{625e}{36} + 89 = 0.$$

11. Which Equation being reduced to Integers (by *Sect.* 7. *Chap.* 11. *Book* 1.) gives

$$eeee - 60eee + 1650ee - 22500e + 115344 = 0.$$

Thus an Equation is found out whose Terms are all Integers; and the value of the Root e in this Equation is to the value of the Root a in the Equation proposed as 6 to 1; (for by supposition in the second step $e=6a$:) and therefore if e be 12, then a shall be 2; or if e be 18, then a shall be 3.

VIII. *How to take away the second Term of a Compound Equation.*

The Rule is this; Divide the Coefficient (that is, the known Quantity) multiplied into the second Term of an Equation proposed, by the Index (or Number of Dimensions) of the Power which is the first Term. Then if the Signs of the first and second Terms be unlike, (*viz.* if one be + and the other —) subtract the Quotient from the Affirmative Root sought; but if the Signs be like, (that is, both + or both —) add the said Quotient to the Affirmative Root; then Equate the said Sum or Remainder to some Letter to represent an unknown quantity, and proceed according to the Method in the following Examples; so at length a new Equation will arise, wherein the second Term is wanting.

Example 1.

1. Let there be proposed this Equation . . . . $aa-6a = 72$
2. That is, ... $aa-6a-72 = 0$
3. Here the number of Dimensions in the first Term aa is 2, and the known Number multiplied into a making the second Term $6a$ is 6; this divided by the said 2 gives 3, which subtracted from the Root a (because the Signs of the first and second Terms are unlike) leaves $a-3$, which is equal to some unknown number, let it be e; then,
4. By supposition . . . . $a-3 = e$
5. And consequently by adding 3 to each part of that Equation there arises . . . . $a = e+3$
6. And by squaring each part of the last Equation there comes forth . . . . $aa = ee+6e+9$
7. And by multiplying each part of the Equation in the fifth step by the Coefficient 6 in the proposed Equation, it makes . . . . $6a = 6e+18$
8. Then by subtracting the last Equation from that in the sixth step, there remains . . . . $aa-6a = ee-9$
9. And lastly, by subtracting 72 (the last Term of the Equation propos'd) from the Equation in the eighth step, there remains . . . . $aa-6a-72 = ee-81=0$

Thus you see an Equation is found out, to wit, $ee-81=0$, which is equal to the Equation proposed, and it wants the second Term; (for there is not any number of e in the Equation found out.) Now if the value of e be made known, then the value of a is consequently known; but the Equation found out, to wit, $ee-81=0$, that is, $ee=81$ gives $e=9$, and by the fifth step $a=e+3$, therefore $a=12$.

Example 2.

1. Again, let there be propoſed this Equation, *viz.* . . . $aa+6a = 216$
2. That is, ... $aa+6a-216 = 0$
3. Here (as before) I divide 6, the Coefficient in the ſecond Term $6a$, by 2, which denotes the Number of Dimenſions in the firſt Term aa, and the Quotient 3 I add to the Root a, (becauſe the firſt and ſecond Terms of the Equation have the ſame Sign +) and the Sum $a+3$ is equal to ſome unknown Number, let it be e; then,
4. By ſuppoſition . . . . $a+3 = e$
5. Therefore by ſubtracting 3 from each part of that Equation, there ariſes . . . . $a = e-3$
6. And by ſquaring the laſt Equation there comes forth . . . . $aa = ee-6e+9$
7. And by multiplying the Equation in the fifth ſtep by 6, it produces . . . . $6a = 6e-18$
8. Then by adding the two laſt Equations into one, the ſum is . . . . $aa+6a = ee-9$
9. And by ſubtracting 216 (the laſt Term of the Equation propos'd) from each part of the Equation in the eighth ſtep, there remains . . . . $aa+6a-216 = ee-225=0$

Thus an Equation is found out, to wit, $ee-225=0$, which wants a ſecond Term, (for there is no Number of e in that Equation;) and when the value of e is made known, the value of a in the Equation propos'd is known alſo; but the Equation $ee-225=0$, that is, $ee=225$ gives $e=15$, and by the fifth ſtep $a=e-3$; therefore $a=12$, that is, $15-3$.

Example 3.

1. Again, let this Equation be propos'd . . . $aaa-18aa-7a+696=0$
2. According to the Rule before given, I divide 18 the known Number of the ſecond Term $-18aa$ by 3, which denotes the number of Dimenſions in the firſt Term aaa, and the Quotient is 6, this I ſubtract from the Root a, (becauſe the Signs of the firſt and ſecond Terms are unlike) and the Remainder is $a-6$, which is equal to ſome unknown Number, ſuppoſe it be e; then,
3. By ſuppoſition . . . . $a-6=e$
4. Therefore by adding 6 to each part of that Equation there ariſes . . . . $a=e+6$
5. And by ſquaring the laſt Equation it makes . . . . $aa=ee+12e+36$
6. And by multiplying the two laſt Equations one by the other, the Product is . . . . $aaa=eee+18ee+108e+216$
7. And by multiplying the Equation in the fifth ſtep by 18, (the Coefficient in the ſecond Term of the Equation propos'd) it makes . . . . $18aa=18ee+216e+648$
8. Likewiſe the Equation in the fourth ſtep being multiplied by 7, (the Coefficient in the third Term of the Equation propos'd) produces . . . . . $7a=7e+42$
9. Then to the Equation in the ſixth ſtep adding 696, (to wit, the laſt Term of the Equation propos'd) the Sum is
$$aaa+696 = eee+18ee+108e+912$$
10. Likewiſe, by adding the eighth Equation to the ſeventh, it makes
$$18aa+7a = 18ee+223e+690$$
11. Laſtly, by ſubtracting the Equation in the tenth ſtep from that in the ninth, this following Equation remains, *viz.*
$$aaa-18aa-7a+696 = eee-115e+222 = 0.$$

Thus an Equation is found out, to wit, $eee-115e+222=0$, which wants the ſecond Term, (to wit, the Power ee;) and when the value of the Root e is made known, the value of the Root a ſhall be known alſo, for by the fourth ſtep $a=e+6$; therefore if e be 2, then a ſhall be 8; and if e be equal to $\sqrt{112}-1$, then a ſhall be equal to $\sqrt{112}+5$.

Example 4.

1. Again, let there be proposed . . . . $aaaa+6aaa+11aa+6a-100=0$
2. According to the Rule before given, I divide 6 the Coefficient in the second Term $+6aaa$ by 4, which denotes the number of Dimensions in the first Term $aaaa$, and the Quotient is $\frac{3}{2}$, which I add to the Root a, (because the Signs of the first and second Terms are like) and the sum is $a+\frac{3}{2}$, which is equal to some unknown Number, let it be e, then,
3. By supposition . . . . $a+\frac{3}{2}=e$
4. Therefore . . . . $a=e-\frac{3}{2}$
5. The Square of the last Equation is . . . $aa=ee-3e+\frac{9}{4}$
6. And the two last Equations multiplied one by the other make . . . . . $aaa=eee-\frac{9}{2}ee+\frac{27}{4}e-\frac{27}{8}$
7. And the Equation in the sixth step being multiplied by that in the fourth step, will produce . . . . $aaaa=eeee-6eee+\frac{27}{2}ee-\frac{27}{2}e+\frac{81}{16}$
8. And the Equation in the sixth step multiplied by 6 produces . . . . . . $6aaa=6eee-\frac{54}{2}ee+\frac{162}{4}e-\frac{162}{8}$
9. And the Equation in the fifth step multiplied by 11 produces . . . . . $11aa=11ee-33e+\frac{99}{4}$
10. And the Equation in the fourth step multiplied by 6 gives . . . . . . $6a=6e-9$
11. Now 'tis manifest, that if from the Sum of the first Parts of the four last Equations there be subtracted 100, the Remainder will be equal to the Sum of the Equation first propos'd equal to 0; therefore also if 100 be subtracted from the Sum of the latter parts of the said four Equations the Remainder shall be equal to 0, *viz.*
$$eeee-\tfrac{5}{2}ee-99\tfrac{7}{16}=0.$$
12. In which last Equation the second Term, to wit, the Power eee, is wanting, as was desired. And when the value of e is made known, the value of the Root a in the Equation proposed shall be known also; for by the fourth step $a=e-\frac{3}{2}$, but (by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1.) the value of e in the Equation in the eleventh step will be found $\sqrt{:1\frac{1}{4}+\sqrt{101}:}$ and therefore $a=\sqrt{:1\frac{1}{4}+\sqrt{101}:}-\frac{3}{2}$.

IX. *The use of the preceding Rules of this Chapter, in the Resolution of all manner of Compound Equations in Numbers.*

After an adfected or compound Equation different from any of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. is prepared for Resolution by the Rules of *Chap.* 12. *Book* 1. and reduced (if need be) to Integers, and the sum of all the Terms made equal to 0 (or nothing) according to *Sect.* 7. and 2. of this *Chap.* search out (by the Rules of *Chap.* 8. of this *Book*) all the just Divisors to the last Term (that is, the known absolute Number of the Equation so reduced.) Then try whether any one of those Divisors connected to the unknown Root a by — or +, will divide the total Sum of the said reduced Equation without leaving a Remainder; for when such Division succeeds, either the known part of the said Binomial Divisor is the desired value of the Root a, or at least the Quotient gives an Equation, whose first Term has fewer Dimensions by one than the Equation divided; and then the Root of this new Equation, if its first Term be a Square, may be found out by some of the Canons in *Sect.* 6,8,10. of *Chap.* 15. *Book* 1. But if the first Term contains three or more Dimensions, let this Equation be examined by Division, (as before,) and if none of those Divisions work off just without a Fraction, then by taking away the second Term, (by the Rule in *Sect.* 8. of this *Chap*) another Equation more simple, and such as may be resolved by some of the Canons in *Sect.* 6, 8, 10. *Chap.* 15. *Book* 1. will sometimes arise. But if none of those ways prove effectual, you may by the general Method in the foregoing *Chap.* 10. find out one affirmative Root very near a true Root, and then joyning this Root found out to the unknown Root a by the Sign —, you may by this Binomial divide the Equation, and proceed to find out the rest of the Roots very near the truth. All which will be made manifest by the following Questions.

QUESTION. 1.

If . . . $aaa - 9aa + 26a = 24$ }
That is, if . . . $aaa - 9aa + 26a - 24 = 0$ } What is the Number a?

RESOLUTION.

First, (by the Method in *Sect.* 5. *Chap.* 8. of this *Book*) I search out all the Numbers that will severally divide the last Term 24 without a Remainder, and find them to be these, *viz.* 1,2,3,4,6,8,12,24. Then by examining in order whether the total sum of the Equation propos'd may be divided by $a-1$ or $a+1$, by $a-2$ or $a+2$, &c. I find it may be exactly divided by $a-2$ without a Remainder, and the Quotient is $aa-7a+12$, as you see by this following Division.

$$a-2\ )\ aa a-9aa+26a-24\ (\ aa-7a+12$$
$$aaa-2aa$$
$$\overline{\quad -7aa+26a \quad}$$
$$-7aa+14a$$
$$\overline{\quad +12a-24 \quad}$$
$$+12a-24$$
$$\overline{\quad 0 \quad 0 \quad}$$

Therefore 2 the known Number in the Divisor $a-2$ is one real or affirmative Root of the Equation proposed; for as well the Divisor as the Dividend was supposed equal to nothing, *viz.* $a-2=0$, whence $a=2$; the Quotient also is consequently equal to 0, *viz.* $aa-7a+12=0$, that is, $7a-aa=12$; hence (by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) two other affirmative values of the Root a will be discovered, to wit, 4 and 3. So that three real values of a, to wit, 2, 3, and 4, are found out, by every one of which the Equation propos'd may be expounded, as the Proof will easily shew.

QUESTION 2.

If . . . $aaa - 22aa + 157a = 360$ }
That is, if . . . $aaa - 22aa + 157a - 360 = 0$ } What is $a=$?

RESOLUTION.

First, the Divisor of the last Term 360 will be found these, 1,2,3,4,5,6,8,9,10, 12,15,18,20,24,30,36,40,45,60,72,90,120,180, and 360; then by examining in order whether the sum of the Equation propos'd may be divided by $a-1$ or $a+1$, by $a-2$ or $a+2$, by $a-3$ or $a+3$, &c. I find that $a-5$ will precisely divide the said Sum without a Fraction, and therefore 5 is one affirmative Root or Value of a; then the Quotient $aa-17a+72a=0$, that is, $17a-aa=72$ affords two other affirmative values of a, to wit, 8 and 9. Thus you see three real values of a, to wit, 5, 8, and 9, are found out; by every one of which the Equation proposed, to wit, $aaa-22aa+157a=360$ may be expounded, as will appear by the Proof.

QUESTION. 3.

If . . . $91a - aaa = 330$ }
That is, if . . . $aaa - 91a + 330 = 0$ } What is $a=$?

RESOLUTION.

First, the Divisors of the last Term 330 will be found 1,2,3,5,6,10,11,15,22,30, 55,66,110,165, and 330; then by examining in order whether the sum of the Equation propos'd, to wit, $aaa-91a+330$ may be divided by $a-1$ or $a+1$, by $a-2$ or $a+2$, &c. I find it may be divided by $a-5$ and leave no Remainder; therefore $a-5=0$ gives $a=5$, which is one affirmative Root of the Equation propos'd, and the Quotient $aa+5a-66=0$, that is, $aa+5a=66$ affords another affirmative value of a, to wit 6. So that two real values of a are found out, by each of which the Equation propos'd may be expounded; for if $a=5$, or $a=6$, from either supposition it follows that $91a-aaa=330$.

QUESTION 4.

To find two Numbers whose Sum shall be 5, and that if the Sum of their Squares be multiplied by the Sum of their Cubes, the Product may be 455.

RESOLUTION.

This Question may be solved by the Canon of *Quest.* 13. *Chap.* 16. *Book* 1. but that Canon being raised from Positions that lie out of the common Road, I shall here solve the Question in the ordinary way, and so it will exercise the preceding Rules of this Chapter. First then,

1. For one of the Numbers sought put . . . . . a
2. Therefore the other Number is . . . . . . $5-a$
3. The square of the first Number is . . . . aa
4. The Square of the second is . . . . $aa-10a+25$
5. The Sum of those Squares is . . . . $2aa-10a+25$
6. The Cube of the first Number is . . . . aaa
7. The Cube of the second is . . . . $-aaa+15aa-75a+125$
8. Therefore the Sum of those Cubes is . . . . $+15aa-75a+125$
9. Which Sum being multiplied by the Sum of the Squares in the fifth step gives this Product, *viz.* $30aaaa-300aaa+1375aa-3125a+3125$.
10. But according to the Question, the Product in the last step must be equal to the given Product 455, hence this Equation arises,
$$30aaaa-300aaa+1375aa-3125a+3125 = 455.$$
11. And by subtracting 455 from each part of the last Equation this arises,
$$30aaaa-300aaa+1375aa-3125a+2670 = 0.$$
12. And by dividing every Term in the last Equation by 30 this arises.
$$aaaa-10aaa+45\tfrac{5}{6}aa-104\tfrac{1}{6}a+89 = 0$$
13. Then by supposing $e=6a$, and proceeding according to the *Example* 5. in *Sect.* 7. of this *Chap.* to free the Equation in the preceding twelfth step from Fractions, this will be produced, *viz.*
$$eeee-60eee+1650ee-22500e+115344 = 0.$$
14. Now the Divisors of the last Term 115344 will be found 1,2,3,4,6,8,9,12,18, 24,27, &c. and after tryals made by Division, (like as in the three last preceding Questions) I find that $e-12=0$ will precisely divide the sum of the Equation in the thirteenth step, and therefore 12 is one true value of e. Again, the Quotient of that Division being $eee-48ee+1074e-9612$, I seek the Divisors of the last Term 9612, and find them to be 1,2,3,4,6,9,12,18,27,36, &c. Then after tryals made (as before) I find that $e-18$ will exactly divide the said $eee-48ee+1074e-9612$, and therefore 18 is one other affirmative value of e; and because the Quotient of the last mentioned Division, to wit, $ee-30e+534=0$, that is, $30e-ee=534$, is an impossible Equation, (as is evident by the Determination in *Sect.* 9. *Quest.* 1. *Chap* 15. *Book* 1.) I conclude that the Equation in the thirteenth step has no other Root or Value of e besides 12 and 18 before found. But because by supposition in the thirteenth step $e=6a$, $\frac{1}{6}$ of 12 and likewise of 18, that is, 2 and 3, shall be the true values of a to solve the Question, for their sum is 5; and if 13 the sum of their Squares be multiplied by 35 the sum of their Cubes, the Product is 455, as was desired.

Sometimes the taking away of the second Term of an Equation (by the Rule in *Sect.* 8. of this *Chap.*) will be an Expedient to find out an Equation resolvable by some of the Canons in *Sect.* 6, 8, and 10. *Chap.* 15. *Book* 1. when tryals by Division (as before) will be in vain, as will appear by the following fifth Question, which I find resolved two manner of ways in *Pag.* 319. of *Cartesius* his Geometry, set forth with Comments by *Fran. van. Schooten*, and Printed at *Amsterdam* 1659.

QUESTION. 5.

To find four Numbers in Arithmetical Progression continued, such that their common Difference may be Unity, and the Product made by their continual Multiplication 100.

RESOLUTION.

1. For the first Number put . . . . a
2. Then the second shall be . . . . $a+1$
3. The third . . . . $a+2$
4. And the fourth . . . . $a+3$
5. Therefore the Product of their continual Multiplication is . . . . $aaaa+6aaa+11aa+6a$

6. Which

6. Which Product must be equal to 100, therefore . . . . $\Big\}\ aaaa+6aaa+11aa+6a = 100$

7. That is, ... $aaaa+6aaa+11aa+6a-100=0$

8. Of which Equation the last Term 100 may be divided by 1,2,4,5,10,20,25,50, and 100; but Division being tried by $a-$ or $+1$, by $a-$ or $+2$, by $a-$ or $+4$, &c. it proves ineffectual. Then by taking away the second Term, (as in *Example* 4. *Sect.* 8. of this *Chap.*) this Equation arises, *viz.* $eeee-2\frac{1}{2}ee-99\frac{7}{16}=0$, in which the Root e (by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1.) will be found equal to $\sqrt{:1\frac{1}{4}+\sqrt{101}:}$ but in taking away the second Term a was put equal to $e-\frac{3}{2}$, and therefore $a=\sqrt{:1\frac{1}{4}+\sqrt{101}:}-\frac{3}{2}$; and consequently from the first, second, third, and fourth steps,

$$\text{The four Numbers sought are these,}\left\{\begin{array}{l}\sqrt{:1\frac{1}{4}+\sqrt{101}:}-\frac{3}{2}\\ \sqrt{:1\frac{1}{4}+\sqrt{101}:}-\frac{1}{2}\\ \sqrt{:1\frac{1}{4}+\sqrt{101}:}+\frac{1}{2}\\ \sqrt{:1\frac{1}{4}+\sqrt{101}:}+\frac{3}{2}\end{array}\right.$$

Which Numbers exceed one another by Unity, and the Product of their Multiplication is 100, as before has been proved in *Quest* 3. *Sect.* 17. *Chap.* 9. of this Book.

Another way of Resolving Quest. 5.

For the first number put $a-1\frac{1}{2}$, for the second $a-\frac{1}{2}$, for the third $a+\frac{1}{2}$, and for the fourth $a+1\frac{1}{2}$; then by multiplying these four Numbers one into another, and comparing the Product to 100, this Equation arises, *viz.* $aaaa-2\frac{1}{2}aa=99\frac{7}{16}$; whence the four Numbers sought will be found the same as before.

QUESTION 6.

1. . . . If . . . . $8a^3+63aa-a^4-341a = 1304$

2. That is, if . . . . $a^4-8a^3-63aa+341a+1304 = 0$;

What is the Number a?

RESOLUTION.

3. The Divisors of the last Term 1304 are 1,2,4,8,163,326, and 1304; then after tryals made by Division, (as in the preceding Questions) I find $a-8=0$ will exactly divide the sum of the Equation proposed without any Remainder, and therefore 8 is one affirmative value of the Root a. Again, because the Divisors of 163 the last Term of this Equation $aaa-63a-163=0$, (which was the Quotient of the said Division) are only Unity and 163, I try to divide the Equation last mentioned by $a-1$ and $a+1$, likewise by $a-163$ and $a+163$; but none of these Divisions working off just without a Fraction, and there being no second Term to be taken away, I search out one affirmative value of a out of the said Equation $aaa-63a-163=0$, (that is, $aaa-63a=163$) by the general Method in the foregoing *Chap.* 10. and thereby discover $a=9.0055$, &c. then I divide the said Cubic Equation $aaa-63a-163=0$, by $a-9.0055=0$, and the Quotient (the Remainder after the Division is ended being neglected) is $aa+9.0055a+18.09903025=0$; but this Equation cannot possibly have any affirmative Root, and therefore I conclude that the Equation first propos'd to be resolved has only two affirmative Roots or Values of a, to wit, 8 and 9.0055, &c. found out as above.

By the like Operation it will appear, that this Equation $a^4-17a^3-212aa+4979a-21131=0$ may be expounded by every one of these three Roots or Values of a, to wit 11, 7.1125, &c. and 15.8874, &c. but by no other affirmative Root.

When the Index of the Power of the unknown Quantity in every Term of an Equation is an even number, the Resolution of such Equation will admit of a Contraction, which will be made manifest by this following

QUESTION. 7.

1. If . . . . . $a^6-29a^4+244a^2-576 = 0$; What is $a=$?

RESOLUTION.

2. Here because the Indices of the unknown Powers are even Numbers, to wit, 6, 4, and 2, put . . . . $\Big\}\ e=a^2$

. 3. Then

3. Then for . . . . $\left\{\begin{array}{l}+a^6\\-29a^4\\+244a^2\end{array}\right\}$ write $\left\{\begin{array}{l}+e^3\\-29e^2\\+244e\end{array}\right.$

4. To which Powers of e joyn -576, the last Term of the given Equation, and it makes $e^3-29e^2+244e-576=0$.

5. Which last Equation being resolved by Division, (in like manner as in the preceding Examples of this Section) there will be found three affirmative values of the Root e, *viz.* 4, 9, and 16; then because e was put equal to aa, the square Root of 4, 9, and 16, that is, 2, 3, and 4, shall be three Roots or Values of a in the Equation first proposed, to wit, $a^6-29a^4+244a^2-576=0$, as may easily be proved.

I might here shew how to reduce a Biquadratic Equation, not falling under any of the three Forms in *Sect.* 1. *Chap.* 15. *Book.* 1. to a Cubic Equation, and sometimes into two Quadratic Equations, but I shall spare that labour for these Reasons: First, that Reduction being subject to many cases, is very tedious and troublesom. Secondly, such a Biquadratic Equation is seldom capable of being reduced into two Quadratic Equations; and when 'tis reduced to a Cubic Equation, this may happen to be such as its Root or Roots in Numbers cannot be perfectly found out by any Rules hitherto publish'd by any Author. Thirdly, by the Method in this ninth Section all the Roots of any Cubic, Biquadratic, or other Equation of higher degrees, may be found out in Numbers, either exactly if they be Rational, or as near the truth if they be Irrational, as shall be needful for any practical use. And lastly, my undertaking (as I have before hinted) is not to handle all, but the most useful Rules only in this profound Art.

Note. The Resolutions of the preceding Questions of this ninth Section do clearly shew, that there is no small labour in making tryals with the Divisors of the last Term of an Equation to find its Root or Roots; and therefore to lessen that work, first, it will be convenient to make some tryals by the general Method in the foregoing *Chap.* 10. to find out Limits within which the Root or Roots of an Equation do fall, or to argue the same from some things given in a Question producing the said Equation, and then to make tryals only with such Divisors of the last Term as fall within those Limits; but when all Contractions are used, the work is sufficiently laborious, so that one chief Scope of an Analyst in resolving a knotty Question must be to frame his Positions with such artifice, that the Resolution may end in as simple an Equation as is possible. And altho one way of Resolution may produce an Equation composed of high Powers, yet oftentimes by another way you may come to a more simple Equation, as may partly appear by the foregoing fourth and fifth Questions of this Section; but the skill of finding out the most simple and facil ways of Resolution, is not attainable (as I conceive) by any certain or constant Method, but rather by much use and exercise in the solving of Questions.

Sect. X. *Concerning the Resolution of certain Cubic Equations in Numbers by two Rules, the Invention whereof* Cardanus *attributes to* Scipio Ferreus.

1. All Cubic Equations, after the second Term is taken away, when there happens to be any, (by the Rule in *Sect.* 8. of this *Chap.*) are reducible to these three following Forms, in which a represents the Root or Quantity sought, but p and q known Quantities.

$aaa = -6a+20$	$aaa = -pa+q$
$aaa = +6a+40$	$aaa = +pa+q$
$aaa = +91a-330$	$aaa = +pa-q$

2. Now let it be required to resolve the first of those Equations, *viz.*
If $aaa = -6a+20$, or $aaa = -pa+q$;
What is the value of a?

Preparation.

3. Suppose . . . . $a = e - y$
4. Suppose also . . . . $20 = eee - yyy$
5. And . . . . $6 = 3ey$
6. Then by multiplying each part of the Equation in the third step into it self Cubically, this is produced, *viz.* $aaa = eee - 3eey + 3eyy - yyy$

7. And

7. And by multiplying the Equations in the third and fifth steps one into the other, it makes . . . . . } $6a = 3eey - 3eyy$

8. And by subtracting the Equation in the seventh step from that in the fourth, there remains . . . . } $20 - 6a = eee - 3eey + 3eyy - yyy$

9. Therefore by the sixth and eighth steps 'tis manifest that . . . . } $aaa = eee - 3eey + 3eyy - yyy = 20 - 6a$

10. From the premisses it's evident, that if in the Equation propos'd to be resolv'd to wit, $aaa = -6a + 20$, or $aaa = -pa + q$, we suppose the Root a sought to be equal to the difference of two unknown numbers e and y; also the absolute number 20 (or q) to be equal to the difference of the Cubes of the same two numbers, and the Co-efficient 6 (or p) to be equal to the triple Product of their Multiplication: then as well aaa as $20 - 6a$ (that is, $q - pa$) shall be equal to the Cube of the difference of those two numbers, *viz.* to the Cube of $e - y$; and therefore when two such numbers are found out, their difference shall be the Root or number a sought. But to find out the said two numbers (e and y) there is given the Product of their Multiplication, to wit 2, (or $\frac{1}{3}p$) that is, one third part of the Co-efficient, as also 20 (or q) the difference of the Cubes of the same two numbers. And therefore the numbers themselves shall be given severally by the Canon of *Quest.* 15. *Chap.* 16. *Book* 1. and consequently the Root a sought shall be given also, as will be made manifest by this following

Operation.

11. To the Square of half the given Absolute number 20 (or q) *viz.* to . .	100	$\frac{1}{4}qq$
12. Add the Cube of 2 (or $\frac{1}{3}p$) *viz.* the Cube of $\frac{1}{3}$ of the Co-efficient 6 (or p) which Cube is .	8	$\frac{1}{27}ppp$
13. The Sum is	108	$\frac{1}{4}qq + \frac{1}{27}ppp$
14. The Square Root of that Sum is . . .	$\sqrt{108}$	$\sqrt{:\frac{1}{4}qq + \frac{1}{27}ppp:}$
15. To that Square Root add half the Absolute number 20 (or q) and the Sum is	$10 + \sqrt{108}$	$\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}$
16. The Cube Root of that Sum is the greater number e sought, *viz.* . .	$\sqrt[3]{:10 + \sqrt{108}:}$	$\sqrt[3]{:\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}:}$
17. Again, from the square Root in the fourteenth step subtract half the absolute number 20 (or q) and the Remainder is .	$-10 + \sqrt{108}$	$-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}$
18. Then the Cubic Root of that Remainder shall be the lesser number y sought, *viz.* . . .	$\sqrt[3]{:-10 + \sqrt{108}:}$	$\sqrt[3]{:-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}:}$

19. And then the difference of the two Cubic Roots found out in the sixteenth and eighteenth steps shall be the value of the Root a in the Equation proposed, *viz.*

$a = \sqrt[3]{:1 + \sqrt{108}:} - \sqrt[3]{:-10 + \sqrt{108}:}$ that is,

$a = \sqrt[3]{:\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}:} - \sqrt[3]{:-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}:}$

20. It remains to make tryal whether the Binomial $10 + \sqrt{108}$ has a perfect Cubic Root or not; so by the Rule in *Sect.* 18. *Chap.* 9. of this second Book, it will appear that $1 + \sqrt{3}$ is the Cubic Root of $10 + \sqrt{108}$, and $\sqrt{3} - 1$ is the Cubic Root of $\sqrt{108} - 10$; and consequently the value of the Root a before found out in the nineteenth step is expressible by a rational number; for if $\sqrt{3} - 1$ be subtracted

 from

from $1+\sqrt{3}$, the Remainder 2 is the desired value of a in the Equation proposed; for if $a=2$, then $aaa=20-6a$, or $aaa+6a=20$.

21. In like manner by the Canon in the foregoing nineteeth step the value of a in this Equation $aaa+27a=64$, will be found this that follows, *viz.*

$$a = \sqrt{(3)}:\overline{32+\sqrt{1753}}:-\sqrt{(3)}:\overline{-32+\sqrt{1753}}:$$

But this value of a cannot be exprest by any rational number, because the Binomial $32+\sqrt{1753}$ has not a perfect Cubic Root, and therefore the said value must either rest in that Surd Form, or else be exprest by some rational number near the true value, which will be found 2. 05, &c. that is, $2\frac{5}{100}$, &c.

22. In the next place let it be required to resolve a Cubic Equation of the second of the three Forms before mentioned, *viz.*

If . . . . $aaa = 6a+40$; or, $aaa = pa+q$;

What is the value of a?

Preparation.

23. Suppose . . . . $a = e+y$

24. Suppose also, ... $40 = eee+yyy$

25. And . . . . $6 = 3ey$

26. Then by multiplying each part of the Equation in the twenty third step into itself cubically, this is produced, $aaa = eee+3eey+3eyy+yyy$

27. And the Equations in the twenty third and twenty fifth steps being mutually multiplied one by the other will produce . . . . $6a = 3eey+3eyy$

28. And the Sum of the Equation in the twenty fourth and twenty seventh steps makes . . . . $6a+40 = eee+3eey+3eyy+yyy$

29. Therefore by the twenty sixth and twenty eight steps 'tis evident that . $aaa = eee+3eey+3eyy+yyy=6a+40$

30. By the eight last preceding steps 'tis manifest, That if in the Equation propos'd to be resolved, to wit, $aaa=6a+40$, or $aaa=pa+q$, we suppose the Root a sought to be equal to the sum of two unknown numbers, e and y, also the absolute number 40 (or q) to be equal to the sum of the Cubes of the same two numbers, and the Co-efficient 6 (or p) to be equal to the triple Product of their Multiplication, then as well aaa as $6a+40$ (that is, $pa+q$) shall be equal to the Cube of $e+y$; and therefore when two such numbers are found out, their sum shall be the Root or number a sought. But to find out the said two numbers (e and y) there is given the Product of their Multiplication, to wit 2 (or $\frac{1}{3}p$) that is, $\frac{1}{3}$ part of the Co-efficient, as also 40 (or q) the sum of the Cubes of the same two numbers, and therefore the numbers shall be given severally by the Canon of *Qu.* 14. *Ch.* 16. *Book* 1, and consequently the Root a sought shall be given also. All which will be made manifest by this following

Operation.

31. From the Square of half the given absolute number 40 (or q) *viz.* from	400	$\frac{1}{4}qq$
32. Subtract the Cube of 2 (or $\frac{1}{3}p$) *viz.* the Cube of $\frac{1}{3}$ of the Co-efficient, which Cube is	8	$\frac{1}{27}ppp$
33. The Remainder is	392	$\frac{1}{4}qq-\frac{1}{27}ppp$
34. The square Root of that Remaind. is	$\sqrt{392}$	$\sqrt{}:\overline{\frac{1}{4}qq-\frac{1}{27}ppp}:$
35. Which square root added to half the absolute number 40 (or q) makes the sum	$20+\sqrt{392}$	$\frac{1}{2}q+\sqrt{}:\overline{\frac{1}{4}qq-\frac{1}{27}ppp}:$
36. The Cubic Root of the sum in the last step is the value of e .	$\sqrt{(3)}:\overline{20+\sqrt{392}}:$	$\sqrt{(3)}:\overline{\frac{1}{2}q+\sqrt{\frac{1}{4}qq-\frac{1}{27}ppp}}:$
37. The square root in the thirty fourth step being subtracted from half the absolute number 40 (or q) leaves	$20-\sqrt{392}$	$\frac{1}{2}q-\sqrt{\frac{1}{4}qq-\frac{1}{27}ppp}$
38. The Cubic Root of that Remainder is the value of y	$\sqrt{(3)}:\overline{20-\sqrt{392}}:$	$\sqrt{(3)}:\overline{\frac{1}{2}q-\sqrt{\frac{1}{4}qq-\frac{1}{27}ppp}}:$

36 Then

39. Then the ſum of the two Cubic Roots found out in the thirty ſixth and thirty eighth ſteps ſhall be the value of the Root a in the Equation propos'd to be reſolved

$$a = \sqrt{(3)}\overline{:2+\sqrt{392}:}+\sqrt{(3)}\overline{:20+\sqrt{302}:}\ \text{that is,}$$

$$a = \sqrt{(3)}\overline{:\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}ppp}:}+\sqrt{(3)}\overline{:-\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}ppp}:}$$

40. It remains to make tryal whether the Binomial $20+\sqrt{392}$ has a perfect Cubic Root or not; ſo by the Rule in *Sect.* 18. *Chap.* 9. of this ſecond Book, you will find $2+\sqrt{2}$ to be the Cubic Root of $20+\sqrt{392}$, and $2-\sqrt{2}$ the Cubic Root of $20-\sqrt{392}$, and conſequently the value of the Root a before found out in the thirty ninth ſtep is expreſſible by a rational Number; for if $2-\sqrt{2}$ be added to $2+\sqrt{2}$, the ſum 4 is the deſired value of a in the Equation propoſed to be reſolved: for if $a=4$, then $aaa=6a+40$.

41. Another Example of reſolving a Cubic Equation of the ſecond Form may be this, *viz.* Let it be required to find the value of a in this Equation $aaa=12a+18$, that is, $aaa=pa+q$, then the Cannon expreſt by the Literal Equation in the thirty ninth ſtep will give

$$a=\sqrt{(3)}\overline{:9+\sqrt{17}:}+\sqrt{(3)}\overline{:9-\sqrt{17}:}$$

But this value of a is inexpreſſible by any rational number, becauſe the Binomial $9+\sqrt{17}$ has not a perfect Cubic Root, and therefore the ſaid value muſt either reſt in that Surd Form, or elſe be expreſs'd by ſome rational number near the true value, which will be found 4. 05, &c. that is, $4\frac{1}{20}$, &c.

The premiſſes do clearly ſhew the riſe of two Rules delivered by *Cardanus* in his Algebraical Treatiſe entituled *Ars magna*, which Rules are mentioned in divers Authors, and the Subſtance of them is contained in the two literal Equations in the foregoing nineteenth and thirty ninth ſteps; the former of which Equations is a Canon to find out the Root of any Cubic Equation in Numbers, which falls under the firſt of the three Forms before mentioned, and to expreſs the ſame perfectly either by ſome rational or irrational Number; and the later of thoſe literal Equations finds out the like exact Root of any Cubic Equation of the ſecond Form, except in one caſe only, *viz.* when the Square of half the abſolute Number (q) which is the laſt term of the Equation, is leſs than the Cube of one third part of the known Co-efficient (p). But no Author that I have met with, gives a certain Rule, either to find out the Root in that caſe if it be an irrational number; or the two affirmative Roots of a Cubic Equation of the third Form, if each of theſe alſo be irrational. *Huddenius* indeed ſaith in *pag.* 503. of *Carteſius* his Geometry before mentioned, he had a Rule (which he intended to publiſh) by which all irrational Roots, as well of numeral as of literal Equations, may be found out, but that much deſired Rule is not yet come to light. But when a Cubic Equation of what kind ſoever has one Root expreſſible by a rational Number, both that and the reſt of the Roots, when the Equation is capable of more than one, may be exactly found out by the help of the Diviſors of the laſt term, according to *Sect.* 9 of this *Chap.*

CHAP. XII.

Of the Method of reſolving Queſtions wherein many Quantities are ſought, by aſſuming different Letters to repreſent the ſaid Quantities ſeverally.

I. Hitherto in the Algebraical Reſolution of a Queſtion, wherein two or more Quantities have been ſought, I have aſſumed only one letter, as a or e, to repreſent ſome one of the unknown Quantities, and formed the Poſitions for the reſt by the help of that letter and the Quantities given in the Queſtion. But many Queſtions may be more eaſily reſolved by aſſuming a peculiar letter to repreſent every one of the Quantities ſought; as a for one unknown Quantity, e for a ſecond, y for a third, &c. By this Method alſo thoſe intricate and obſcure ways of reſolving Queſtions by ſecond Roots, or (as *Simon Stevin* calls them) poſtpoſed Quantities, will be avoided.

In handling the following Method I ſhall give three principal Rules, and explain them by Examples; but to preſcribe Rules for all Caſes, is (as I conceive) an impoſſible Work.

RULE I.

When many Quantities are ſought by a Queſtion, firſt let them be ſeverally repreſented by different Letters; then after you have well conſidered the Condition in the Queſtion, abſtract it from words, and expreſs the Tenor thereof by Equations; that done by the help of tranſpoſition find what the firſt, that is, any ſingle Letter repreſenting a number or quantity ſought in the firſt Equation is equal to; then whereſoever that firſt Letter is found in the other Equations, take inſtead of it thoſe Quantities to which the ſaid firſt Letter was found equal: ſo ſuch firſt Letter will quite vaniſh out thoſe other Equations. Again, by Tranſpoſition ſet a ſecond Letter alone in one of thoſe Equations out of which the firſt Letter was expell'd, and proceed as before; ſo at length one of the numbers ſought will be made known, by the help whereof the reſt will eaſily be diſcovered. This work will be better underſtood by Examples than many Words, and therefore I ſhall proceed to Queſtions.

QUESTION I.

A Factor exchanged 6 *French* Crowns and 2 Dollars for 45 Shillings of *Engliſh* Money; alſo at another time he exchanged 9 *French* Crowns and 5 Dollars (each of theſe being of the ſame value with the former) for 76 Shillings: I demand the value of a *French* Crown, and alſo of a Dollar, in *Engliſh* Money?

Let a repreſent the deſired value of a Crown, and e the value of a Dollar, then the Queſtion being abſtracted from Words may be ſtated thus.

1. If . . . . $6a + 2e = 45$
2. And . . . . $9a + 5e = 76$

What are the Numbers a and e? ||

RESOLUTION.

3. By Tranſpoſition of $2e$ in the firſt Equation this ariſes . $6a = 45 - 2e$
4. And by dividing each part of the third Equation by 6, it gives . . . . $a = \frac{45 - 2e}{6}$
5. The fourth Equation multiplied by 9 produces . . $9a = \frac{405 - 18e}{6}$
6. Then if inſtead of $9a$ in the ſecond Equation you take the later Part of the fifth, this will ariſe . . . . $\frac{405 - 18e}{6} + 5e = 76$
7. The ſixth Equation, after due Reduction, diſcovers the value of a Dollar, *viz.* . . . . $e = 4\frac{1}{4}$
8. The ſeventh Equation multiplied by 2 gives . . . $2e = 8\frac{1}{2}$
9. And by ſetting the later part of the eighth Equation in the place of $2e$ in the firſt, this Equation ariſes . $6a + 8\frac{1}{2} = 45$
10. From which laſt Equation, after due Reduction, the value of a or one *French* Crown is diſcovered, *viz.* . $a = 6\frac{1}{12}$

Thus by the ſeventh and tenth Equations it is found that a Dollar was valued at 4 *s.* 3 *d.* and a *French* Crown at 6 *s.* 1 *d.* which numbers will ſatisfie the Conditions in the Queſtion, as may eaſily be proved.

QUESTION 2.

Three Men had every one of them a certain number of Pounds in his Purſe; the ſum of the firſt and ſecond mans Money was 5 (or b) Pounds, the Sum of the ſecond and third mans Money was 12 (or c) Pounds, and the Sum of the third and firſt mans Money was 11 (or d) Pounds: How many Pounds had every one in his Purſe?

Let the three numbers of Pounds ſought be repreſented by a, e, and y; then reſpect being had to the numbers given, the Queſtion may be ſtated thus, *viz.*

1. If . . . . $a + e = b\ (= 5)$
2. And . . . . $e + y = c\ (= 12)$
3. And . . . . $y + a = d\ (= 11)$

What are the Numbers a, e, and y? ||

RESOLUTION.

4. By Transposition of a in the first Equation there will arise $e = b - a$
5. Then by taking the later part of the fourth Equation instead of e in the second, this Equation arises . . $b - a + y = c$
6. And by Transposition of $b - a$ in the 5th Equation it gives $y = c - b + a$
7. And by taking the later part of the sixth Equation instead of y in the third, this arises . . . . $c - b + a + a = d$
8. From which seventh Equation, after due Reduction, the Number a will be made known, *viz.* . . . . . $a = \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c$
9. Again, if instead of a in the first Equation we take the later part of the eighth, this arises . . . . $\frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c + e = d$
10. Then from the ninth, after due Reduction, the Number e will be made known, *viz.* . . . . $e = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d$
11. Again, if instead of a in the third Equation we take the later part of the eighth, this arises . . . . . $y + \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c = d$
12. Lastly, from the eleventh Equation, after due Reduction, the Number y will be made known, *viz.* . . . $y = \frac{1}{2}d + \frac{1}{2}c - \frac{1}{2}b$

The eighth, tenth, and twelfth Equations give this

CANON.

From the sum of every two of the three Numbers given subtract the remaining number, then the halves of the three remainders shall be the numbers sought. Whence the numbers sought, to wit, a, e, and y, will be found 2, 3, and 9; for $2+3=5$, also $3+9=12$, and $9+2=11$, as was required.

The foregoing *Resolution* of this *Quest.* 2. is formed according to *Rule* 1. but the same Canon may be more expeditiously discovered by this following *Resolution, viz.*

The Sum of the first, second, and third Equations which state the Question is . . . . $2a + 2e + 2y = b + c + d$

The half of that Sum is . . . . $a + e + y = \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d$

Then from that half sum subtract the first Equation, and the Remainder will be . . . . $y = \frac{1}{2}c + \frac{1}{2}d - \frac{1}{2}b$

Again, from the said half sum subtract the second Equation, and the Remainder is . . . $a = \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c$

Lastly, from the said half sum subtract the third Equation, and the Remainder gives . . . . $e = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d$.

Which three last Equations do manifestly give the same values of a, e, and y, as were found out by the former Resolution.

QUESTION. 3.

Three men discoursing of their Moneys in this manner; the first says to the other two, if 100 *l.* were added to his Money, the sum would be equal to both their Moneys; the second says to the other two, if 100 *l.* were added to his Money, the sum would be equal to the double of both their Moneys; the third says to the other two, if 100 *l.* were added to his Money, the sum would be equal to the triple of both their Moneys: The Question is, to find how many Pounds each Man had.

Let the three numbers of Pounds sought be represented by a, e, and y; then the Question may be stated thus, *viz.*

1. If . . . . $a + 100 = e + y$
2. And . . . . $e + 100 = 2a + 2y$
3. And . . . . $y + 100 = 3a + 3e$

What are the Numbers a, e, and y? ||

RESOLUTION.

4. From the first Equation by Transposition of y, this arises, ... $a + 100 - y$? || e
5. Then if instead of e in the second Equation there be taken that which is equal to e, to wit, the first part of the fourth, this will arise, $a + 100 - y + 100 = 2a + 2y$
6. That is, after due Reduction, ... $200 = a + 3y$

7. Again,

7. Again, if instead of $3e$ in the third Equation there be taken the triple of the first part of fourth Equation, this will arise, to wit. . $y+100 = 3a+3a+300-3y$
8. Which last Equation after due Reduction gives $y = \frac{3}{2}a+50$
9. Then if instead of $3y$ in the sixth Equation, there be set the triple of the latter part of the eighth, this will come forth, *viz.* . . . $200 = a+\frac{9}{2}a+150$
10. From the ninth Equation, after due Reduction the number a will be discovered, *viz* . $a = 9\frac{1}{11}$
11. Again, if instead of a in the sixth Equation, there be taken $9\frac{1}{11}$, to wit, the value of a found out in the tenth, it will give . . . $200 = 9\frac{1}{11}+3y$
12. The eleventh Equation duly reduced discovers the Number y, *viz.* . . . . $y = 63\frac{7}{11}$
13. From the fourth, tenth, and twelfth Equations by exchange of equal Quantities this Equation arises, *viz.* . . . . $9\frac{1}{11}+100-63\frac{7}{11} = e$
14. The thirteenth reduced gives . . . . . $e = 45\frac{5}{11}$

From the 10th. 14th and 12th. Equations the three numbers sought a, e and y are discovered, *viz.* the first man had $9\frac{1}{11}$ *l.* the second $45\frac{5}{11}$ *l.* and the third $63\frac{7}{11}$ *l.* which numbers will satisfie the Question, as may easily be proved.

If 121 be given instead of 100 in this third Question, then the three numbers sought will be whole Numbers, to wit, 11, 55, 77.

RULE II.

When the same Quantity, suppose a, is found in two several Equations, and equal numbers are prefixed to those Quantities, then if their signs be both + or both —, subtract the lesser Equation from the greater; but if one of the signs be +, and the other —, add those two Equations together; so the said Quantity a will quite vanish, as will appear by the Resolution of the following Question.

QUESTION 4.

The sum of two Numbers being given 12 (or b) and their difference 8 (or c) to find the Numbers.

Let a be put for the greater Number, and e for the lesser, and the Question may be stated thus:

1. If . . . . $a+e = b\ (= 12)$
2. And . . . . $a-e = c\ (= 8)$

What are the Numbers a, and e?

RESOLUTION.

3. For as much as a or $+1a$ is found in each of the Equations proposed, therefore (according to *Rule* 2.) I subtract the lesser Equation from the greater; whence the letter a quite vanishes, and there remains $2e = b-c\ (= 4)$
4. Then by dividing each part of the third Equation by 2, the number e is made known, *viz.* . . $e = \frac{1}{2}b-\frac{1}{2}c\ (= 2)$
5. And by taking the latter part of the fourth Equation instead of e in the first, there remains . . $a+\frac{1}{2}b-\frac{1}{2}c=b\ (= 12)$
6. Lastly, the fifth Equation duly reduced discovers the number a, *viz.* . . . . $a = \frac{1}{2}b+\frac{1}{2}c\ (= 10)$

The 6th. and 4th. Equations discover a Canon to find out the numbers sought, which in this Example are 10 and 2, and the Canon is the same with that before found in *Quest.* 1. *Chap.* 14. *Book* 1.

Otherwise thus.

5. For as much as $a+e$ is found in the first Equation, and $-e$ in the second, therefore by adding those two Equations together, (according to *Rule* 2.) the letter e vanishes, and the sum is . . . . . $2a = b+c\ (= 20)$

8. There-

8. Therefore by dividing each part of the seventh Equation by 2, there arises the same value of a, which was before found in the sixth Equation, *viz.* $\Big\}\ a = \frac{1}{2}b + \frac{1}{2}c\ (= 10)$

9. And by setting the latter part of the eighth Equation in the place of a in the first, this arises, . . $\Big\}\ \frac{1}{2}b + \frac{1}{2}c + e = b\ (= 12)$

10. Which last Equation reduced discovers the same value of e, which was before found in the fourth Equation, *viz.* . . . . $\Big\}\ e = \frac{1}{2}b - \frac{1}{2}c\ (= 2)$

RULE III.

When the same quantity, suppose a, is found in two several Equations, but the numbers prefix'd to those equal quantities are unequal, those two Equations may be reduced into two others which shall have equal numbers prefix'd to the said Quantity a, by this Rule, *viz.* Multiply all the quantities in the first Equation by the number which is prefix'd to the said quantity a in the second; multiply likewise all the quantities in the second Equation by the number which is prefix'd before the same quantity a in the first; so by such alternate Multiplication two new Equations will be produced, wherein the numbers prefix'd to the said quantity a will be equal to one another: and then by adding or subtracting, according to the import of *Rule* 2. of this *Chap.* that quantity a will quite vanish. That done, renew the like work to expel the same quantity out of the rest of the Equations; and proceed in like manner with a second quantity, until at length the value of some one quantity be made known. This I shall make plain by the Resolution of five *Questions* next following.

QUESTION. 5.

To find two Numbers that if the Quadruple of the greater be increased with the triple of the less it may make 36; but if the triple of the greater be lessened by the double of the less, the remainder may be 10.

Put a for the greater number, and e for the lesser, then the Question may be stated thus, *viz.*

1. If . . . . $4a + 3e = 36$
2. And . . . . $3a - 2e = 10$

What are the Numbers a and e? ||

RESOLUTION.

3. The first Equation multiplied by 3, which is prefix'd to a in the second, produces . . . . $\Big\}\ 12a + 9e = 108$

4. The second Equation multiplied by 4, which is prefix'd to a in the first, makes . . . . $\Big\}\ 12a - 8e = 40$

5. Now for as much as the quantity $12a$ is found both in the fourth and third Equations, and is affirmative in each, therefore according to *Rule* 2. I subtract the lesser Equation from the greater, so the quantity $12a$ vanishes, and this Equation remains $\Big\}\ 9e + 8e = 68$

6. The fifth Equation after due Reduction discovers the number e, *viz.* . . . . $\Big\}\ . \ . \ e = 4$

7. Then I set 12 (which by the sixth Equation is the value of $3e$) in the place of $3e$ in the first, and this Equation arises . . $\Big\}\ 4a + 12 = 36$

8. Lastly, the seventh Equation duly reduced discovers the number a, *viz.* . . . . $\Big\}\ . \ . \ a = 6$

From the 8th. and 6th. Equations the two numbers sought are found 6 and 4, which will solve the Question; for four times 6 with thrice 4 make 36; and thrice 6, to wit 18, lessened by twice 4 gives 10, as was required.

QUESTION 6.

1. If . . . . $2a + 3e - 2y = 50$
2. And . . . . $5a - 2e + 5y = 240$
3. And . . . . $a + 5e - 3y = 10$

What are the numbers a, e, and y? ||

RE-

RESOLUTION.

4. The first Equation multiplied by 5, which is prefix'd to a in the second, produces . . . . $10a+15e-10y=250$

5. Likewise the second Equation multiplied by 2, which is prefix'd to a in the first, makes . . . . . . $10a-4e+10y=480$

6. Then (according to *Rule* 2.) by subtracting the fourth Equation from the fifth, the quantity $10a$ vanishes, and this Equation arises . . . . $\cdot -19e+20y=230$

7. Again, the third Equation multiplied by 5, which is prefix'd to a in the second produces . . . . . $-5a+25e-15y=50$

8. And the second Equation multiplied by 1, which is suppos'd to be prefix'd to a in the third, gives the same second Equation without alteration, *viz.* . . . . . . $+5a-2e+5y=240$

9. Then because $+5a$ and $-5a$ by Addition will destroy one another, therefore (according to *Rule* 2.) I add the seventh and eight Equations together, so the letter a vanishes, and this Equation arises, ... $\cdot +23e-10y=290$

10. Again, I proceed with the sixth and ninth Equations according to *Rule* 3. *viz.* I multiply the sixth Equation by 23, (which is prefix'd to e in the ninth) and it makes $-437e+460y=5290$

11. Also the ninth Equation multiplied by 19 (which is prefix'd to e in the sixth) produces . . . . $+437e-190y=5510$

12. Then (according to *Rule* 2.) by adding the tenth and eleventh Equations together, the Letter e vanishes, and this Equation arises, *viz.* . . . . $\cdot\ \cdot +270y=10800$

13. And by dividing each part of the twelfth Equation by 270, the number y is discovered, *viz.* . . . . . $y=40$.

14. Then instead of $10y$ in the ninth Equation taking ten times 40, that is 400, (which by the thirteenth Equation is equal to $10y$) the ninth will be reduced to this, $+23e-400=290$

15. And from the fourteenth Equation, after due Reduction, the number e will be discovered, *viz.* . . . . . . $e=30$

16. Then instead of $3e-2y$ in the first Equation, I take $90-80$, (which by the fifteenth and thirteenth Equations will be found equal to $3e-2y$) so the first Equation will be converted into this, *viz.* . . . . . . $2a+90-80=50$

17. Lastly, the sixteenth Equation duly reduced discovers the number a, *viz.* . . . . $a=20$

From the 17th. 15th. and 13th. Equations the 3 desired numbers a, e, y, are 20, 30, and 40, which will constitute the 3 Equations first proposed, as may easily be proved.

QUESTION 7.

Three Men discourse of their Moneys in this manner; the first saith to the other two, if you give me 100 Pounds, my Money will be made equal to both your remaining Moneys: the second saith to the other two, if ye give me 100 Pounds, my Money will be made equal to the double of both your remaining Moneys: lastly, the third saith to the other two, if ye give me 100 Pounds, my Money will be equal to the triple of both your remaining Moneys. I demand how many Pounds each Man had?

Let a Letter be assumed to represent each Mans Money, as a for the first, e for the second, and y for the third; then the Question may be stated thus, *viz.*

1. If . . . . $a+100=e+y-100$
2. And . . . . $e+100=2a+2y-200$
3. And . . . . $y+100=3a+3e-300$

What are the numbers a, e, and y? ||

RESOLUTION.

4. The first Equation by transposition will be reduced to this . . . . $-a+e+y=200$

5. Like-

5. Likewise the second Equation by transposition gives . . . . $+2a-e+2y=300$

6. And the 3*d* Equation by transposition produces $+3a+3e-y=400$

7. Then I proceed with the fourth and fifth Equations according to *Rule* 3. *viz.* I multiply the fourth Equation by 2, (which is prefix'd to *a* in the fifth,) and it produces . . $-2a+2e+2y=400$

8. The Sum of the fifth and seventh Equations gives . . $e+4y=700$

9. Again, I proceed with the fifth and sixth Equations according to *Rule* 3. *viz.* multiplying the fifth Equation by 3, (which is prefix'd to *a* in the sixth;) it gives . . . . $6a-3e+6y=900$

10. Also the sixth Equation multiplied by 2, (which is prefix'd to *a* in the fifth) produces $6a+6e-2y=800$

11. Then by subtracting the tenth Equation from the ninth, the Remainder is . . . $-9e+8y=100$

12. Again, I proceed with the eighth and eleventh Equations according to *Rule* 3. *viz.* multiplying the eighth Equation by 9, (which is prefix'd to *e* in the eleventh,) it makes . . $+9e+36y=6300$

13. Then (according to *Rule* 2.) the eleventh and twelfth Equations added together make . . $44y=6400$

14. And by dividing the thirteenth Equation by 44, the number *y* is made known, *viz.* . . . $y=145\frac{5}{11}$

15. From the eighth and fourteenth, by exchange of equal Quantities, this arises, *viz.* $e+581\frac{9}{11}=700$

16. And from the fifteenth, by subtraction of $581\frac{9}{11}$ from each part, the number *e* is discovered, *viz.* . . . . $e=118\frac{2}{11}$

17. From the first, fourteenth and sixteenth Equations, by exchange of equal Quantities, this Equation arises, *viz.* . . . . . . $a+100=118\frac{2}{11}+145\frac{5}{11}-100$

18. Lastly, the seventeenth Equation, after due Reduction, discovers the number *a*, *viz.* . . . . $a=63\frac{7}{11}$

Thus, by the 18*th*, 16*th* and 14*th* Equations it is found that the first Man had $63\frac{7}{11}$ *l.* the second $118\frac{2}{11}$ *l.* and the third $145\frac{5}{11}$ *l.* which three Numbers will satisfie the Question, as may easily be proved.

QUESTION. 8.

1. If . . . . $a+\frac{2}{3}e+\frac{2}{3}y+\frac{2}{3}u=112$
2. And . . . . $e+\frac{3}{4}a+\frac{3}{4}y+\frac{3}{4}u=114$
3. And . . . . $y+\frac{4}{5}a+\frac{4}{5}e+\frac{4}{5}u=125\frac{3}{5}$
4. And . . . . $u+\frac{5}{6}a+\frac{5}{6}e+\frac{5}{6}y=133\frac{1}{3}$

What are the numbers *a*, *e*, *y* and *u*? ||

RESOLUTION.

5. The first Equation multiplied by 3, (the Denominator of the Fraction $\frac{2}{3}$) produces this Equation in Integers, to wit, ... $3a+2e+2y+2u=336$

6. Likewise the second Equation multiplied by 4, produces . . . . $3a+4e+3y+3u=456$

7. And the third Equation multiplied by 5 gives $4a+4e+5y+4u=628$

8. Also the fourth Equation multiplied by 6 produces . . . . $5a+5e+5y+6u=800$

9. Forasmuch as $3a$ is found in the fifth, and also in the sixth Equation, I subtract the lesser from the greater, so $3a$ quite vanishes, and this Equation arises, ... $2e+y+u=120$

10. Then I proceed with the fifth and seventh Equations according to *Rule* 3. *viz.* I multiply the fifth Equation by 4, (which is prefix'd to a in the seventh,) and there comes forth	$12a+8e+8y+8u=1344$
11. Also I multiply the seventh Equation by 3, (which is prefix'd to a in the fifth,) and it produces	$12a+12e+15y+12u=1884$
12. Then by subtracting the tenth Equation from the eleventh, the quantity $12a$ quite vanishes, and this Equation arises, to wit, ...	. . $4e+7y+4u=540$
13. The ninth Equation multiplied by 2, produces .	. . $4e+2y+2u=240$
14. Then by subtracting the thirteenth Equation from the twelfth, this arises to wit, ...	. . . $5y+2u=300$
15. Again, I proceed with the fifth and eighth Equations according to *Rule* 3. *viz.* I multiply the fifth Equation by 5, (which is prefix'd to a in the eighth,) and it produces	$15a+10e+10y+10u=1680$
16 Likewise the eighth Equation multiplied by 3, (which is prefix'd to a in the fifth,) produces .	$15a+15e+15y+18u=2400$
17. Then by subtracting the fifteenth Equation from the sixteenth, this arises, *viz.*	. . $5e+5y+8u=720$
18. Again, I proceed with the ninth and seventeenth Equations according to *Rule* 3. *viz.* I multiply the ninth Equation by 5, (which is prefix'd to e in the seventeenth,) and it produces	. . $10e+5y+5u=600$
19. And the seventeenth Equation multiplied by 2, (which is prefix'd to e in the ninth,) produces .	. $10e+10y+16u=1440$
20. Then by subtracting the eighteenth Equation from the nineteenth, there remains	. . . $5y+11u=840$
21. And by subtracting the 14*th* Equation from the 20*th*, (for since $5y$ is found in each of those Equations, they need no Reduction according to *Rule* 3.) there remains	. . . . . . $9u=540$
22. Which twenty first Equation divided by 9 discovers the number u, *viz.*	. . . . $u=60$
23. From the 20*th* and 22*d* Equations, by setting eleven times 60, to wit, 660 in the place of $11u$ in the 20*th*, there arises	. . . $5y+660=840$
24. Therefore from the twenty third Equation, after due Reduction, the number y is discovered, *viz.*	. . . . . $y=36$
25. And from the 9*th*, 24*th*, and 22*d* Equations, this arises,	. . $2e+36+60=120$
26. The 25*th* duly reduced discovers the number e, *viz.*	. . . $e=12$
27. From the 5*th*, 26*th*, 24*th*, and 22*d* Equations, by exchange of equal Quantities, this Equation arises,	$3a+24+72+120=336$
28. Lastly, from the 27*th*, after due Reduction, the number a is discovered, *viz.*	. . . . $a=40$

Thus by the 28*th*, 26*th*, 24*th* and 22*d* Equations the four numbers sought, (to wit, a,e,y,u,) are found 40, 12, 36 and 60, which will constitute the four Equations in *Quest.* 8.

QUEST. 9.

A Maid being at the Market is offer'd 10 Apples for a penny, and 25 Pears for two pence; now if at those rates she would lay out $9\frac{1}{2}$ pence to buy 100 Apples and Pears together, how many Apples, and how many Pears ought she to have?

1. For the number of Apples sought put	a
2. And for the number of Pears sought put	e
3. Then search out the cost of the number of Apples in the first step, and say, If $10 . 1 :: a . (\frac{a}{10}$. so the cost of the number of Apples sought is	. . $\frac{a}{10}$

4. Search

4. Search out also the cost of the number of Pears in the second step, and say, If 25 . 2 :: e . ($\frac{2e}{25}$. so the cost of the number of Pears sought is found . . . . $\frac{2e}{25}$

5. Then (according to the Question) the Money laid out for all the Apples and Pears sought must be equal to $9\frac{1}{2}$ Pence; hence this Equation, ... $\frac{a}{10}+\frac{2e}{25}=9\frac{1}{2}$

6. But the number of Apples, together with the number of Pears bought must make 100, therefore . . . . $a+e=100$

7. Then the Equation in the fifth step, after due Reduction, will give this Equation in Integers, to wit, ... $50a+40e=4750$

8. And the Equation in the sixth step being multiplied by 50 produces . . . . $50a+50e=5000$

9. Then by subtracting the Equation in the seventh step from that in the eighth, there arises . . . . $10e=250$

10. And the Equation in the ninth step divided by 10, discovers the number e, *viz.* . . . . $e=25$

11. Lastly, from the sixth and tenth steps, the number a is also made known, *viz.* . . . . $a=75$

By the first, second, eleventh and tenth steps it appears that there might be bought 75 Apples, and 25 Pears; which numbers will solve the Question, as may easily be proved.

QUESTION 10.

To divide 90 into four such Numbers, that if the first be increased with 2; the second lessened by 2; the third multiplied by 2; and the fourth divided by 2; the Sum, Remainder, Product and Quotient may be equal between themselves.

Let b and d be put for the two given Numbers, 90 and 2; also a, e, y and u for the four numbers sought, then the Question may be stated thus;

1. If . . . . $a+e+y+u=b$
2. And . . . . $a+d=e-d$
3. And . . . . $a+d=dy$
4. And . . . . $a+d=\frac{u}{d}$

What are the numbers a, e, y and u? ||

RESOLUTION.

5. The first Number sought is equal to it self, *viz.* . . . $a=a$
6. From the second Equation, by transposition of $-d$ this arises, ... $a+2d=e$
7. And by dividing each part of the third Equation by d, this arises . . . . $\frac{a+d}{d}=y$
8. And the fourth Equation multiplied by d produces $da+dd=u$
9. The Sum of the four last Equations gives

$$2a+2d+\frac{a+d}{d}+da+dd=a+e+y+u=b$$

10. Which last Equation, after due Reduction, gives . $a=\frac{bd-ddd-2dd-d}{dd+2d+1}$
11. Then from the tenth and sixth Equations, by exchange of equal Quantities, ... $e=\frac{bd+ddd+2dd+d}{dd+2d+1}$
12. And from the tenth and seventh Equations . . . $y=\frac{b}{dd+2d+1}$
13. And from the tenth and eighth Equations, ... $u=\frac{ddd}{dd+2d+1}$

The four last Equations give a Canon to find out the four numbers sought, which are 18, 22, 10 and 40, which will solve the Question. For, first, their sum is 90; then if the first number 18 be increased with the given number 2, it makes 20; and if the second number 22 be lessened by 2, the Remainder is also 20: Moreover, if the third number 10 be multiplied by 2, it likewise produces 20: Lastly, if the fourth number 40 be divided by 2, the Quotient is also 20. Therefore the conditions in the Question are satisfied.

But the Numerator of the Fraction in the latter part of the tenth Equation shews, That the Numbers b and d must not be given at random, but so, that $ddd+2dd+d$ may be subtracted from bd and leave a Remainder greater than nothing; therefore bd must be greater than $ddd+2dd+d$, and consequently b must be greater than $dd+2d+1$. Therefore, to the end the Question may be possible, the numbers given must be subject to this,

Determination.

The number given to be divided (b) must be greater than the Square of $(d+1)$ the sum of the other number given and Unity.

QUEST. 11.

There are two numbers whose Sum is equal to the difference of their Squares; and if the Sum of the Squares of those two numbers be subtracted from the Square of their Sum, the Remainder will be 60: what are the two numbers?

Put b for the given number 60, also a for the greater number sought, and e for the lesser; then the Question may be stated thus, *viz.*

1. If . . . . $aa-ee = a+e$
2. And . . . . $aa+ee+2ae-aa-ee = b$

What are the numbers a and e?

RESOLUTION.

3. The second Equation after its first part is duly contracted is . . . . $2ae = b$
4. And the third Equation divided by 2 gives . . . $ae = \frac{1}{2}b$
5. And if each part of the first Equation be divided by $a+e$ it will give . . . . $a-e = \frac{a+e}{a+e} = 1$
6. From the fifth Equation, by transposition of e, there arises . . . . $a = e+1$
7. The sixth Equation multiplied by e produces . $ae = ee+e$
8. From the fourth and seventh Equations, by exchanging equal Quantities . . . . . . $ee+e = \frac{1}{2}b$
9. Then the eighth Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. the lesser number sought will be made known, *viz.* . $e = \sqrt{:\frac{1}{4}+\frac{1}{2}b:}-\frac{1}{2}=5$
10. And from the ninth and sixth Equations the greater number sought will also be made known *viz.* . . . . $a = \sqrt{:\frac{1}{4}+\frac{1}{2}b:}+\frac{1}{2}=6$

The two last Equations give a Canon to find out the two numbers sought, which are 6 and 5; as may easily be proved.

QUEST. 12.

There are two numbers, such, that if their Sum be subtracted from the Sum of their Squares, the Remainder is 42; but if the Sum of the said two numbers be added to the Product of their Multiplication, it makes 34: what are the numbers?

Let a and e represent the two numbers sought, then the Question may be stated thus, *viz.*

1. If . . . . $aa+ee-a-e = 42$
2. And . . . . $aa+a+e = 34$

What are the Numbers a and e?

RESOLUTION.

3. By adding the first and second Equations together, the Sum will be . . . . $aa+ee+ae = 76$
4. And by adding the second Equation to the third, the sum will be . . . . $aa+ee+2ae+a+e = 110$
5. Suppose . . . . $y = a+e$

6. Then

6. Then by squaring each part of the fifth Equation, this arises . . . . $yy = aa + ee + 2ae$
7. The Sum of the two last Equations makes . $yy + y = aa + ee + 2ae + a + e$
8. And from the seventh and fourth Equations, by exchange of equal quantities, this Equation arises, $yy + y = 110$
9. Which eighth Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. the number y, to wit, $a+e$ will be made known, *viz.* $y\ (= a+e) = 10$
10. Then by setting 10 (the value of $a+e$) in the place of $a+e$ in the second Equation, there arises $ae + 10 = 34$
11. And by subtracting 10 from each part of the tenth Equation, there remains . . . . . $ae = 24$
12. And from the ninth Equation, by transposition of a, there arises . . . . . . $e = 10 - a$
13. And if a in the eleventh be multiplied by $10 - a$ instead of e, the said eleventh Equation will be reduced to this, ... $10a - aa = 24$
14. Wherefore the last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers sought will be discovered, *viz.* $\begin{cases} a=6 \\ e=4 \end{cases}$

Thus 6 and 4 are found out, which will solve the Question proposed, as will be evident by the Proof.

QUEST. 13.

There are two numbers, such, that the Sum of their Squares make 100, and if the Sum of the two numbers be added to the Product of their Multiplication, it makes 62; what are the numbers?

Let a and e be put for the two numbers sought, then the Question may be stated thus, *viz.*

1. If . . . . $aa + ee = 100$
2. And . . . . $ae + a + e = 62$

What are the Numbers a and e?

RESOLUTION.

3. The second Equation multiplied by 2 produces . $2ae + 2a + 2e = 124$
4. The sum of the first and third Equations gives $aa + ee + 2ae + 2a + 2e = 224$
5. Suppose . . . . $y = a + e$
6. Then by squaring each part of the fifth Equation this is produced, *viz.* . . . . . $yy = aa + ee + 2ae$
7. And by adding the double of the fifth Equation to the sixth, it gives . . . . $yy + 2y = aa + ee + 2ae + 2a + 2e$
8. And from the seventh and fourth Equations, by exchange of equal quantities, this Equation arises $yy + 2y = 224$
9. Which last Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. the number y, to wit $a+e$, will be made known, *viz.* . . $y = a + e = 14$
10. Then from the ninth and second Equations, by taking 14 instead of $a+e$, the second Equation will be reduced to this, *viz.* . . . . $ae + 14 = 62$
11. Which last Equation, by equal subtraction of 14, gives . . . . $ae = 48$
12. The ninth Equation by transposition of a gives $e = 14 - a$
13. Then by multiplying a in the eleventh Equation by $14 - a$ instead of e, this Equation is produced, to wit, ... $14a - aa = 48$
14. Wherefore the last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers sought will be discovered, *viz.* . $\begin{cases} a=8 \\ e=6 \end{cases}$

So the numbers sought are found 8 and 6, which will solve the Question, as will appear by the Proof.

QUEST. 14.

QUESTION 14.

There are two numbers, such, that their sum is equal to the Product of their multiplication; and if the Product or sum of the said Numbers be added to the sum of their Squares, it makes $15\frac{3}{4}$: What are the Numbers?

Let a and e be put for the two numbers sought, then the Quest. may be stated thus, *viz.*

1. If . . . . $ae = a+e$
2. And . . . . $aa+ee+ae = 15\frac{3}{4}$

What are the Numbers a and e?

RESOLUTION.

3. The Sum of the first and second Equations is . . . $aa+ee+2ae=a+e+15\frac{3}{4}$
4. And from the third Equation, by transposition of $a+e$, there arises . . . . $aa+ee+2ae-a-e=15\frac{3}{4}$
5. Suppose . . . . $y = a+e$
6. Then by squaring each part of the fifth Equation . . $yy = aa+ee+2ae$
7. And by subtracting the fifth Equation from the sixth, there remains . . . . $yy-y = aa+ee+2ae-a-e$
8. And from the fourth and seventh Equations, by exchange of equal Quantities, there will arise . . $yy-y = 15\frac{3}{4}$
9. Which last Equation being resolved by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1. the number y, to wit, $a+e$ will be made known, *viz.* . . . . . $y = a+e = 4\frac{1}{2}$
10. Therefore from the first and ninth Equations, . . $a+e = ae = 4\frac{1}{2}$
11. From the ninth Equation by transposition of a . . . . . $e = 4\frac{1}{2}-a$
12. The eleventh Equation multiplied by a, produces . . . . $ae = 4\frac{1}{2}a-aa$
13. And from the tenth and twelfth Equations, by exchange of equal Quantities, ... $4\frac{1}{2}a-aa = 4\frac{1}{2}$
14. Wherefore the last Equation being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers sought will be discovered, *viz.* . . $\begin{cases} a = 3 \\ e = 1\frac{1}{2} \end{cases}$

So the numbers sought are found 3 and $1\frac{1}{2}$, which will solve the Question; for their Sum is equal to the Product of their Multiplication, and if their Sum $4\frac{1}{2}$ be added to $11\frac{1}{4}$ the Sum of their Squares, it makes $15\frac{3}{4}$, as the Question requires.

QUESTION. 15.

There are two Numbers, such, that the Square of their difference is equal to the Product of their Multiplication, and the Sum of their Squares makes 20: what are the Numbers?

Let a and e be put for the two Numbers sought, and let a be the greater; then the Question may be stated thus, *viz.*

1. If . . . . $aa-2ae+ee = ae$
2. And . . . . $aa+ee = 20$

What are the Numbers a and e?

RESOLUTION.

3. From the first Equation by transposition of $-2ae$, this arises . . . . $aa+ee = 3ae$
4. Therefore from the second and third Equations . $3ae = 20$
5. And the third Equation divided by 3, gives . . . $ae = \frac{20}{3}$
6. And by adding the double of the fifth Equation to the second, it makes . . . . . . $aa+ee+2ae = \frac{100}{3}$
7. Therefore by extracting the Square Root of each part of the sixth Equation, the sum of the two numbers sought will be made known, *viz.* $a+e = \sqrt{\frac{100}{3}}$
8. From the seventh Equation, by transposition of a, this arises . . . . $e = \sqrt{\frac{100}{3}}-a$
9. The eighth Equation multiplied by a, produces . . $ae = \sqrt{\frac{100}{3}}\times a, -aa$

10. And

10. And from the fifth and ninth Equations this ariſes, . $\sqrt{\frac{100}{3}} \times a, - aa = \frac{20}{3}$
11. Wherefore the laſt Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. *Book.* 1. the two numbers ſought will be diſcovered, *viz.* . . . . $\begin{cases} a = \sqrt{8\frac{1}{3}} + \sqrt{1\frac{2}{3}} \\ e = \sqrt{8\frac{1}{3}} - \sqrt{1\frac{2}{3}} \end{cases}$

The Proof.

The difference of the two numbers in the eleventh ſtep is . $\sqrt{1\frac{2}{3}} + \sqrt{1\frac{2}{3}} = \sqrt{\frac{20}{3}}$
The ſquare of the ſaid difference is . . . . $\frac{20}{3}$
And (by the laſt of the three Rules in *Sect.* 10. *Chap.* 9. of this *Book*) the Product of the Multiplication of the ſame two numbers is alſo . . . . $\frac{20}{3}$
Laſtly, (by the firſt and ſecond of the ſaid three Rules) the ſum of the Squares of the ſaid two numbers is 20

QUEST. 16.

There are two numbers, ſuch, that if their ſum be multiplied by their difference, the Product is 21; but if the ſum of the Squares of thoſe two numbers be multiplied by the difference of their Squares, the Product is 609: what are the numbers?

Let *a* and *e* be put for the two numbers ſought, and let *a* repreſent the greater; then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $\overline{a+e} \times \overline{a-e}$, that is, $aa - ee, = 21$
2. And . . . . $\overline{aa+ee} \times \overline{aa-ee}$, that is, $aaaa - eeee, = 609$

What are the numbers *a* and *e*? ||

RESOLUTION.

3. By ſuppoſition in the firſt Equation, ... $aa - ee = 21$
4. Therefore (by tranſpoſition of $-ee$) . . . . . . $aa = ee + 21$
5. And by ſquaring each part of the fourth Equation this ariſes, ... $aaaa = eeee + 42ee + 441$
6. And by taking the latter part of the fifth Equation inſtead of *aaaa* in the ſecond, the ſaid ſecond Equation will be reduced to this, ... $eeee + 42ee + 441 - eeee = 609$
7. The ſixth Equation, after due Reduction, gives . . . $ee = 4$
8. Therefore by extracting the ſquare Root out of each part of the ſeventh Equation, the leſſer number ſought is diſcovered, *viz.* . . . . $e = 2$
9. Then from the fourth and ſeventh Equations this ariſes, ... $aa = 4 + 21 = 25$
10. Therefore by extracting the ſquare Root out of each part of the laſt Equation, the greater number ſought is alſo made known, *viz.* . . . . . $a = 5$

So the numbers ſought are found 5 and 2, which will ſolve the Queſtion, as will be evident by the Proof.

QUEST. 17.

There are two numbers, ſuch, that if their ſum be multiplied by the ſum of their Squares, the Product is 272; but if the difference of the ſame two numbers be multiplied by the difference of their Squares the Product is 32: what are the numbers?

Put *a* for the greater number ſought, and *e* for the leſſer; then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $\overline{a+e} \times \overline{aa+ee} = 272$
2. And . . . . $\overline{a-e} \times \overline{aa-ee} = 32$

What are the numbers *a* and *e*? ||

RESOLUTION.

3. By multiplying $a+e$ into $aa+ee$, the firſt Equation will be reduced to this, . . . $aaa + aae + aee + eee = 272$

4. Likewiſe,

4. Likewise by multiplying $a-e$ into $aa-ee$, the second Equation will be reduced to this, . . . $aaa-aae-aee+eee = 32$
5. The sum of the third and fourth Equations gives $2aaa \quad +2eee = 304$
6. The half of the fifth Equation is, ... $aaa \quad + eee = 152$
7. The fourth Equation subtracted from the third leaves . . . . $2aae+2aee = 240$
8. The half of the seventh Equation is . . . . $aae+ aee = 120$
9. The Sum of the seventh and eighth Equations is . . $3aae+3aee = 360$
10. The sum of the sixth and ninth Equations is $aaa+3aae+3aee+eee = 512$
11. The Cubic Root of the tenth being extracted, there arises . . . . $a+e = 8$
12. By dividing each part of the first Equation by the respective part of the eleventh, there will arise . . $aa+ee = 34$

By the two last Equations, the sum of the two numbers sought is found 8, and the sum of their Squares 34; therefore by the Canon of *Quest.* 7. *Chap.* 16. *Book* 1. the numbers themselves will be found 5 and 3, which will solve the Question, as may easily be proved.

QUEST. 18.

To divide a given number 14 (or b) into three continual Proportionals, such, that if the said given number be divided severally by every one of the said three Proportionals, the sum of the three Quotients may be equal to $12\frac{1}{4}$ (or d) a number given.

RESOLUTION.

1. For the first (or least) of the three Proportionals sought put . . . . e
2. For the second (or mean) Proportional put . a
3. Then the square of the mean Proportional being divided by the first gives the third, to wit, $\frac{aa}{e}$
4. Therefore the Sum of the three Proportionals is $e+a+\frac{aa}{e}$
5. Which sum must be equal to the given number 14, (or b,) whence this Equation arises, *viz.* $e+a+\frac{aa}{e} = b$
6. Then by reducing that Equation to Integers, this arises $ee+ae+aa = be$
7. Again, (according to the Question) let the given number b be divided by every one of the three Proportionals in the fourth step, so the three Quotients added together will give . . $\frac{b}{e}+\frac{b}{a}+\frac{be}{aa}$
8. But the sum of the three Quotients in the seventh step must be equal to the given sum $12\frac{1}{4}$, (or d,) hence this Equation arises, . . . $\frac{b}{e}+\frac{b}{a}+\frac{be}{aa} = d$
9. Which last Equation reduced to Integers will produce . . . . $baa+bae+bee = daae$
10. And by dividing every Term of the Equation in the ninth step by a, this arises . . . $baa+bae+bee = daae$
11. The sixth Equation multiplied by b, produces $baa+bae+bee = bbe$
12. And from the tenth and eleventh Equations, (where each of two Quantities is found equal to a common third) this arises, *viz.* . . . . $daae = bbe$
13. The twelfth Equation divided by e gives . . : $daa = bb$
14. And the thirteenth Equation divided by d gives . . $aa = \frac{bb}{d}$
15. Therefore by extracting the square Root out of each part of the fourteenth Equation, the mean Proportional sought will be made known, *viz.* $a = \sqrt{\frac{bb}{d}} = 4$
16. And because a is now known, to wit, 4; and $b = 14$; therefore the Equation in the sixth step may be reduced into this, *viz.* . . . $ee+4e+16 = 14e$

17. Which

17. Which last Equation, after due Reduction, will give . $10e - ee = 16$
18. Lastly, the Equation in the seventeenth step being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the first and third Proportionals will be discovered, *viz.* . . $e = \begin{cases} 2 \\ 8 \end{cases}$

Thus the three Proportionals sought are found 2, 4, 8, which will satisfie the conditions in the Question: For first, 2, 4 and 8 are manifestly in continual proportion; secondly, their sum is 14; thirdly, if 14 be divided by 2, 4 and 8 severally, the sum of the Quotients 7, $3\frac{1}{2}$ and $1\frac{3}{4}$ is $12\frac{1}{4}$; as was prescribed in the Question.

It may also be observed, that those three Quotients are continual Proportionals, as will be manifest from the seventh step of the Resolution, where they are represented by $\frac{b}{e}$, $\frac{b}{a}$ and $\frac{be}{aa}$; for the Product made by the Multiplication of the two extremes, to wit, the Product $\frac{bbe}{aae}$, that is, $\frac{bb}{aa}$, is equal to the Square of the mean Proportional $\frac{b}{a}$.

QUEST 19.

To find three numbers in Arithmetical Progression, such that if the first be multiplied by 1, the second by 2, the third by 3, the sum of the Products may be 62; and that the sum of the squares of the three numbers may make 275.

Let the three numbers sought be represented by a, e, y, and suppose a to be the smallest and first Term, then the Question may be stated thus, *viz.*

1. If . . . . $e - a = y - e$
2. And . . . . $a + 2e + 3y = 62$
3. And . . . . $aa + ee + yy = 275$

What are the numbers a, e, y? ||

RESOLUTION.

4. By supposition in the first step . . . . $e - a = y - e$
5. Therefore by Transposition of $-a$ and $-e$, there arises . . . . $a + y = 2e$
6. And by dividing each part of the last Equation by 2, it gives . . . . $\frac{1}{2}a + \frac{1}{2}y = e$
7. And by squaring the Equation in the sixth step there comes forth . . . . $\frac{1}{4}aa + \frac{1}{2}ay + \frac{1}{4}yy = ee$
8. Then if instead of $2e$ in the second Equation, there be taken the first part of the fifth, the second will be converted into this, *viz.* . . $a + a + y + 3y = 62$
9. That is, ... $2a + 4y = 62$
10. The half of the last Equation is . . . . $a + 2y = 31$
11. And by transposition of Quantities in the tenth Equation this arises, *viz.* . . . $31 - 2y = a$
12. And by squaring the eleventh Equation, there comes forth . . . . $961 - 124y + 4yy = aa$
13. From the seventh, eleventh and twelfth Equations this arises, ... $\frac{961}{4} - \frac{31}{2}y + \frac{1}{4}yy = ee$
14. It is evident that . . . . $yy = yy$
15. And by adding the twelfth, thirteenth and fourteenth Equations into one sum, it makes $\frac{21}{4}yy - \frac{279}{2}y + \frac{4805}{4} = aa + ee + yy$
16. But by supposition in the third step, ... $275 = aa + ee + yy$
17. Therefore from the fifteenth and sixteenth Equations, by Exchange of equal Quantities, $\frac{21}{4}yy - \frac{279}{2}y + \frac{4805}{4} = 275$
18. And after due Reduction the Equation in the seventeenth step gives . . . . . $y\frac{184}{7} - yy = \frac{1235}{7}$
19. Therefore by resolving the Equation in the 18 step, (according to the Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) two values of y will be discovered, *viz.* . . . . $y = 13$, or $13\frac{4}{7}$
20. And from the 19*th* and 11*th* Equations . . . . $a = 5$, or $3\frac{6}{7}$
21. Lastly, from the 20*th*, 19*th* and 6*th* Equations . . . . . $e = 9$, or $8\frac{5}{7}$

From

From the three last Equations 'tis evident, that the three desired Numbers a, e, y may be either 5, 9, 13, or $3\frac{6}{7}$, $8\frac{5}{7}$, and $13\frac{4}{7}$: For first, 5, 9, 13 are in Arithmetical Progression; and if 5 be multiplied by 1, 9 by 2, and 13 by 3, the sum of the three Products is 62; moreover, the sum of the Squares of 5, 9, 13 makes 275, as was required. The like may be proved by $3\frac{6}{7}$, $8\frac{5}{7}$ and $13\frac{4}{7}$.

QUEST. 20.

To find three such numbers, that the Square of the first being added to the Product of the first multiplied into the second may make the sum 48; also, that the Square of the first being subtracted from the Product of the first multiplied into the third the Remainder may be 32; and that the Sum of the Squares of the first and third, may have the same Proportion to the square of the second as 5 to 2.

Let the three Numbers sought be represented by a, e, y; and then the Question may be stated thus, *viz.*

1. If . . . . $aa+ae=48$
2. And . . . . $ay-aa=32$
3. And . . . . $aa+yy \,.\, ee :: 5 \,.\, 2$

What are the numbers a, e, y?

RESOLUTION.

4. From the first Equation by transposition of aa, this arises, *viz.* . . . . $ae=48-aa$
5. And by dividing each part of the last Equation by a, it gives . . . . $e=\dfrac{48-aa}{a}$
6. And by transposition of $-aa$ in the second Equation, it makes . . . . $ay=aa+32$
7. And by dividing the sixth Equation by a, there arises . . . . $y=\dfrac{aa+32}{a}$
8. From the Analogy in the third step, by comparing the Product of the extremes to the Product of the means, this Equation arises . . $5ee=2aa+2yy$
9. The Square of the seventh Equation is . . . . $yy=\dfrac{1024+64aa+a^4}{aa}$
10. The double of the ninth Equation is . . . . $2yy=\dfrac{2048+128aa+2a^4}{aa}$
11. If instead of $2yy$ in the later part of the eighth Equation there be taken the later part of the tenth, the eighth will be converted into this, *viz.* . . . . . . $5ee=\dfrac{2048+128aa+4a^4}{aa}$
12. The Square of the fifth Equation is . . . . $ee=\dfrac{2304-96aa+a^4}{aa}$
13. The twelfth Equation multiplied by 5 gives . . . $5ee=\dfrac{11520-480aa+5a^4}{aa}$
14. From the eleventh and thirteenth Equations, by comparing their later parts one to the other, and reducing the Equation thereby resulting, this Equation arises, *viz.* . . $608aa-a^4=9472$
15. Which Equation in the 14*th* step being resolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. will discover two values of a, *viz.* . . $a=\sqrt{592}$, or 4
16. But the lesser of those two values of a, to wit, 4, is the first number sought by the Question, for the Square of the greater value $\sqrt{592}$ exceeds 48, but according to the supposition in the first step it ought to be less than 48; supposing then $a=4$, it follows from the fifth step, that . . . . . $e=8$
17. Lastly, from the 15*th* and 7*th* Equations, ... $y=12$

So three numbers are found out, to wit, 4, 8 and 12; which will satisfie the Question, as may easily be proved.

QUEST. 18.

QUEST 21.

To find three such numbers, that the square of the first, together with the Product of the first multiplied by the second may make 10; also, that the Square of the second with the Product of the second into the third may make 21; and lastly, that the Square of the third, with the Product of the third into the first may make 24.

Let the three numbers sought be represented by a, e, y, and then the Question may be stated thus;

1. If . . . . $aa+ae=10$
2. And . . . . $ee+ey=21$
3. And . . . . $yy+ya=24$

What are the Numbers a, e, y?

RESOLUTION.

4. By transposition of aa in the first Equation this arises, ... $ae=10-aa$
5. And by dividing each part of the fourth Equation a, it gives . . $e=\frac{10-aa}{a}$
6. And by squaring the fifth Equation it makes . . . . $ee=\frac{a^4-20aa+100}{aa}$
7. And from the second, fifth and sixth Equations this arises, . . $\frac{a^4-20aa+100}{aa}+\frac{10-aa}{a}y=21$
8. And by subtracting $\frac{a^4-20aa+100}{aa}$ from each part of the seventh Equation this remains . . . . $\frac{10-aa}{a}y=\frac{41aa-100-a^4}{aa}$
9. And by dividing each part of the 8*th* Equation by $\frac{10-aa}{a}$ this arises $y=\frac{41aa-100-a^4}{10a-aaa}$
10. And by squaring the ninth Equation it makes . . . . $yy=\frac{a^8-82a^6+1881a^4-8200aa+10000}{100aa-20a^4+a^6}$
11. And by multiplying the ninth Equation by a, it produces . . $ya=\frac{41aaa-100a-a^5}{10a-aaa}$
12. And by adding the eleventh Equation to the tenth, the sum makes
$$yy+ya=\frac{2a^8-133a^6+2391a^4-9300aa+10000}{100aa-20a^4+a^6}$$
13. Therefore from the third and twelfth Equations this arises,
$$\frac{2a^8-133a^6+2391a^4-9200aa+10000}{100aa-20a^4+a^6}=24$$
14. Which last preceding Equation, after due Reduction, gives this that follows, *viz.*
$$-a^8+78\tfrac{1}{2}a^6-1435\tfrac{1}{2}a^4+5800aa=5000.$$
15. That is, after Transposition of 5000,
$$-a^8+78\tfrac{1}{2}a^6-1435\tfrac{1}{2}a^4+5800aa-5000=0.$$
16. Then by supposing $u=2a$, and proceeding according to the Rule in *Sect.* 7. *Ch.* 11. of this second *Book*, the Equation last above written will be reduced to this following Equation in Integers, *viz.*
$$-u^8+314u^6-22968u^4+371200uu-1280000=0$$
17. And by supposing $x=uu$ we may instead of $-u^8$ in the last preceding Equation write $-x^4$, and instead of $+314u^6$ we may set $314x^3$, also $-22968xx$ in the place of $-22968u^4$, and $+371200x$ instead of $+371200uu$, and last of all the Absolute number -1280000: whence this following Equation arises; and then after x is made known, its square Root be the number u; (for by supposition $x=uu$,)
$$-x^4+314x^3-22968xx+371200x-1280000=0.$$
18. Now because the last Term -1280000 in the Equation last above written has many Divisors which will be useless in the finding of the value of x, it will be convenient before they be found out, to search out limits, within which such a value of the Root x doth fall as will produce a value of a capable of solving the Question proposed; to which end I proceed thus, *viz.*

19. By

19. By the latter part of the fourth Equation it's manifest that $a \sqsupset \sqrt{10}$

20. And by the second Equation, after transposition of ee, it will likewise appear that . . . . $e \sqsupset \sqrt{21}$

21. Now suppose . . . . $e = \sqrt{21}$

22. Then by multiplying $\sqrt{21}$ instead of e by a in the first Equation, it will be reduced to this, *viz.* . . . . . . $aa + \sqrt{21} \times a = 10$

23. Which last Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. gives . . . . $a = 1\frac{61}{100}$, *&c.*

24. And because when e is supposed to be equal to $\sqrt{21}$, the Equation in the twenty second step gives $a = 1\frac{61}{100}$, it may easily be conceived that when e is less than $\sqrt{21}$, (as it ought to be) then the first Equation, to wit $aa + ea = 10$ will necessarily give . . . . $a \sqsubset 1\frac{61}{100}$, *&c.*

25. Therefore by doubling each part of the nineteenth and twenty fourth steps, it is manifest that . . . . $2a \begin{cases} \sqsupset \sqrt{40} \\ \sqsubset 3\frac{22}{100}, \text{ &c.} \end{cases}$

26. And by squaring each part in the twenty fifth step, it follows that . . . . $4aa \begin{cases} \sqsupset 40 \\ \sqsubset 10\frac{36}{100}, \text{ &c.} \end{cases}$

27. But by supposition in the sixteenth step $u = 2a$, and consequently . . . . $uu = 4aa$

28. Therefore from the two last precedent steps it's evident that $uu \begin{cases} \sqsupset 40 \\ \sqsubset 10\frac{36}{100}, \text{ &c.} \end{cases}$

29. And because by supposition in the seventeenth step, . . $x = uu$

30. Therefore from the twenty eighth and twenty ninth steps it follows that . . . . $x \begin{cases} \sqsupset 40 \\ \sqsubset 10\frac{36}{100}, \text{ &c.} \end{cases}$

31. Having found that such a value of x in the Equation in the seventeenth step as is capable of producing a true value of the desired first number a, must be less than 40, but greater than $10\frac{36}{100}$; it is manifest that among the Divisors of 1280000, the last Term of that Equation, these three only, to wit, 16, 20, 32, are necessary to make tryals in finding out the said value of x, and consequently of a; and therefore (according to the *Rule* in *Sect.* 9. *Chap.* 11, of this *Book*) I first divide the said Equation in the seventeenth step, to wit, $-x^4 + 314x^3 - 22968xx + 371200x - 1280000 = 0$ by $a - 16$, and the Quotient is exactly $-x^3 + 298xx - 18200x + 80000$, wherefore 16 shall be a true value of x in that Equation: And because by supposition $x = uu = 4aa$, it follows that $\sqrt{16}$ (that is, $\sqrt{x}) = u = 2a$, and consequently $2 = a$ the first number sought.

32. Now since 2 is found equal to a, the first Equation, to wit, $aa + ae = 10$ will be reduced to this, *viz.* . . . . $4 + 2e = 10$

33. Whence the second number e is discovered, *viz.* . . . . $e = 3$

34. And consequently the second Equation will be reduced to this . . . . $9 + 3y = 21$

35. Whence the third number y is discovered, *viz.* . . . . . $y = 4$

Thus the three numbers sought (to wit, a, e, y, are found 2, 3, 4, which will solve the Question: For the Square of the first with the Product of the first and second makes 10; also the Square of the second with the Product of the second and third makes 21; and the Square of the third with the Product of the third and first makes 24, as was required.

Note, That the Quotient found out in the thirty first step, to wit, the Equation $-x^3 + 298xx - 18200x + 80000 = 0$ has three Affirmative Roots, whose values (by the *Rule* in *Sect.* 9. *C.* 11. of this second *Book*) will be found very near equal to $4\frac{76}{100}$, $78\frac{11}{100}$, and $215\frac{13}{100}$; but these are without the limits of x discovered in the thirtieth step, and therefore although the Equation in the fifteenth step may be expounded by four Affirmative values of a, yet only one of them, to wit, 2, is capable of solving the Question proposed.

Note also, That if none of those Divisors which were discovered to be within the limits for the finding of a due value of x had produced an exact Quotient without a Remainder, and consequently in such case the number a had been Irrational, yet a Rational number, near the true value of x, and consequently of a might be found out by the help of the General Method in *Chap.* 10. of this second *Book*.

CHAP.

CHAP. XIII.

Concerning the Reſolution of ſuch Arithmetical Queſtions as are capable of innumerable Anſwers.

I. AFter a Queſtion is ſtated by Equations in ſuch manner as has been ſhewn in the foregoing twelfth *Chapter*, if thoſe Equations be equal in multitude to the Quantities ſought, then the Queſtion has a certain determinable number of Anſwers; but whenſoever a Queſtion affords not as many given Equations, not mutually depending upon one another, as there be Quantities required, it is capable of innumerable Anſwers. Queſtions of this latter kind are very pleaſant and delightful, but oftentimes exceeding hard to be reſolved, eſpecially when all the Anſwers in whole numbers that a Queſtion is capable of are defired; and therefore I ſuppoſe it will not be unacceptable to the Learner, if in this *Chapter* I give him a taſte of that vaſt Skill, by expounding three Propoſitions found out by Monſieur *Bachet*; the two firſt of which contain the ſubſtance of the eighteenth and twenty firſt in his ingenious little Book, entituled *Problemes plaiſans & delectables, qui ſe font par les Nombres*, (Printed at *Lyons* in 1624;) but his Method of ſolving and demonſtrating the ſame being very tedious and obſcure, I ſhall wave it, and deliver two ways of my own finding out, which are both intelligible and demonſtrative. The third Propoſition (which is handled by the ſame Author in his Comment upon 41 *Prop.* of the Fourth Book of *Diophantus*,) I ſhall alſo explain at large by various Queſtions.

PROP. I.

Two whole numbers prime between themſelves being given, to find out two others, ſuppoſe a and b; that if a be multiplied by the greater of the two given numbers, and to the Product there be added a given whole number, the ſum ſhall be equal to the Product of b multiplied by the leſſer of the two numbers firſt given. Moreover to find out all the whole numbers a and b that are capable of producing the ſame effect.

Explication.

1. Numbers prime between themſelves are ſuch as have only Unity for their common Diviſor; (*per Defin.* 12. *Elem.* 7. *Euclid.*) ſo 12 and 5 are ſaid to be Prime between themſelves, becauſe they have no common Diviſor but 1, to divide them ſeverally, ſo as to leave no Remainder; the like may be ſaid of 20 and 21, 7 and 3, *&c.*
2. I call a number the *Multiple* of another when it exactly contains that other twice, thrice, or more times, without any Remainder: As, 6 is a Multiple of 3, becauſe it contains 3 exactly twice; likewiſe 18 is a Multiple of 6, becauſe it contains 6 juſt thrice without any Remainder. Moreover I take the Liberty to call a number the Multiple of it ſelf, becauſe it contains it ſelf juſt once. Theſe things premiſed, I ſhall proceed to ſhew two ways of ſolving the preceding *Prop.* 1. and explain the ſame by Queſtions.

Sect. II. *The firſt Method of ſolving the foregoing* Prop. 1.

QUEST. 1.

To find out all the values of a and b in whole numbers that may make $9a+6=7b$, *viz.* that nine times the whole number a with 6 added may make ſeven times the whole number b.

The Equation propoſed . . $9a+6=7b$,

The Reſolution, . . .

1	15	14	2
2	24	21	3
3	33	28	4
4	42	35	5
5	51	42	6
6	60		
7	69		

Expli-

Explication.

1. To the number 9 prefixt to a I add 6, (to wit, $+6$ which follows $9a$) and it makes 15, to this I add again 9 and the sum is 24, to which I add again 9, and it gives 33: and in like manner I continue the addition of 9 to every next preceding sum until I have found out these seven numbers, 15, 24, 33, 42, 51, 60, 69, which stand (as you see in the Example) under $9a$, and on the left hand of those numbers I set 1, 2, 3, 4, 5, 6, 7. These two Columels of numbers do shew that if 1 be taken for the value of a, then $9a+6$ makes 15; but if $2=a$, then $9a+6=24$; if $3=a$, then $9a+6=33$; and so of the rest. The addition aforesaid is in this Example continued only to the seventh sum inclusive, because (as hereafter will appear) the smallest whole number that can express the value of a, never exceeds the number prefix'd to b in the Equation propos'd.
2. Then under $7b$ I set the Multiples of 7 orderly one under another, *viz.* 14. (to wit twice 7,) 21, 28, &c. until I have found out a number equal to one of the seven numbers 15, 24, 33, &c. so at length among the Multiples of 7, I find 42, that is, six times 7, to be equal to 42 that stands among the numbers in the second Columel, which later 42 (by the construction aforesaid) is compos'd of 6 and four times 9. Whence 'tis manifest that if 4 be taken for the value of a, and 6 for the value of b, then $9a+6=7b$ $(=42)$ *viz.* nine times 4 together with 6 is equal to seven times 6, and therefore one Answer to the Question is discovered.

Note 1. When the given whole number prefix'd to b in the Equation propos'd is a single figure, or some small number of two places, then this first Method will readily discover the smallest values of a and b in whole numbers; for the smallest whole number a never exceeds the given number prefix'd to b, as hereafter will be made manifest: But if the number prefix'd to b be large, then the work by this first Method will be intolerably tedious, especially in the solving of *Prop.* 2.

Note 2. If the two given whole numbers which are prefix'd to a and b in the Equation propos'd be not prime between themselves, then it will sometimes be impossible to find out any whole numbers for the values of a and b, to solve the Proposition: as, if two whole numbers a and b be desired that may make $6a+3=2b$, it may easily be shewn that 'tis impossible to find out two such whole numbers; for the whole number a must be either even or odd, but whither it be even or odd, if it be multiplied by the even number 6 the Product shall be even; (by *Prop.* 21, & 28 *Elem.* 9. *Euclid*) to which adding 3 the sum will be odd, (for odd, added to even makes odd,) which sum must be equal to $2b$, and consequently the half of that sum is the number b; but the half of an odd number cannot be a whole Number, and therefore b in the Equation propos'd cannot be a whole number: But if the given whole numbers which are prefix'd to a and b be Prime to one another, then whatever whole number be given to be added to the desired Multiple of a, innumerable whole numbers may be found out for the values of a and b, as hereafter will be shewn.

3. After the two smallest whole numbers are found out for the values of a and b to constitute the Equation proposed, all other pairs of whole numbers that are capable of producing the same effect, may be orderly enumerated into two Arithmetical Progressions thus formed; *viz.* Having found 4 for the smallest whole number a, and 6 for the smallest whole number b to constitute the Equation before proposed, to wit, $9a+6=7b$, let the said 4 be made the first Term, and 7, which is prefix'd to b, the common difference of the Terms of the first Progression; then let 6, the smallest whole number b, be the first Term. and 9 which is prefix'd to a in the said Equation, the common difference of the Terms of the latter Progression, so the Terms of those Progressions will be these, *viz.*

   Values of a; 4, 11, 18, 25, 32, 39, 46, 53, &c.
   Values of b; 6, 15, 24, 33, 42, 51, 60, 69, &c.

4. Now out of the first of those Progressions you may take any Term for the value of a, as 11, (the second Term,) and then the correspondent Term in the latter Progression, to wit, 15, shall be the value of b; by which two numbers 11 and 15 the Equation $9a+6=7b$ may be expounded, *viz.* nine times 11 with 6 added is equal to seven times 15. Likewise 18 and 24, also 25 and 23, and every pair of correspondent Terms in those two Progressions will cause the same effect, as I shall now demonstrate.

Prepa-

Preparation.

5. Let c and n repreſent two whole numbers Prime between themſelves, and a, b, d, three other whole numbers, ſuch that all five will make this Equation, *viz.* . . . . . $ca+d=nb$

6. Let an Arithmetical Progreſſion be ſo formed that a may be the firſt and leaſt Term, and n the common difference of the Terms, as . . . . $a,\ a+n,\ a+2n,$ &c.

7. Let another Arithmetical Progreſſion be formed from b the firſt and leaſt Term, and c the common difference of the Terms, as, ... $b,\ b+c,\ b+2c,$ &c.

8. I ſay, if you multiply c by $a+n$ (the ſecond Term of the firſt Progreſſion,) inſtead of a in the Equation in the fifth ſtep, and to the Product add d, the ſum ſhall be equal to a Multiple of n, to wit, the Product of n multiplied into $b+c$, (the ſecond Term of the later Progreſſion;) and the like may be affirmed of every following Term in each Progreſſion.

Demonſtration.

9. By ſuppoſition in the fifth ſtep, ... $ca+d=nb$

10. And by adding cn to each part of that Equation, this ariſes, ... $ca+cn+d=nb+cn$

11. Therefore from the laſt Equation, ... $c\times\overline{a+n},\ +d=n\times\overline{b+c}$
Which was to be ſhewn.

12. Again, if to each part of the Equation firſt granted in the ninth ſtep you add $2cn$, it makes . . . $ca+2cn+d=nb+2cn$

13. That is, ... $c\times\overline{a+2n},\ +d=n\times\overline{b+2c}$

14. After the ſame manner it may be ſhewn that . . $c\times\overline{a+3n},\ +d=n\times\overline{b+3c}$
And ſo forwards. Which was to be proved.

15. Now ſuppoſing a and b to expreſs the ſmalleſt whole numbers that are capable of conſtituting the Equation in the fifth ſtep, to wit, $ca+d=nb$, I muſt demonſtrate that no other whole numbers beſides the Terms which follow a and b in the two Progreſſions formed in the ſixth and ſeventh ſteps, can be taken inſtead of a and b to produce the ſame effect: If it be poſſible, let $a+$ ſome whole number f, viz. $a+f$ be taken inſtead of a; and let $b+$ ſome whole number g, viz. $b+g$ be taken inſtead of b; then c multiplied by $a+f$ makes $ca+cf$, to which adding d, the ſum is $ca+cf+d$, which muſt be equal to the Product of n multiplied by $b+g$, to wit, $nb+ng$, whence . . . . $ca+cf+d=nb+ng$

16. And by ſuppoſition in the fifth ſtep, ... $ca+d=nb$

17. Therefore by ſubtracting the laſt Equation from the laſt but one, this remains, ... $cf=ng$

18. And by reſolving the laſt Equation into Proportionals, this Analogy ariſes, *viz.* . . . $n\ .\ c\ ::\ f\ .\ g$

19. Whence it is manifeſt that the whole numbers f and g are in the ſame Reaſon (or Proportion) as the whole numbers n and c, and conſequently, ſince n and c are by ſuppoſition whole Numbers Prime between themſelves, f muſt neceſſarily be equal either to n, or $2n$, or $3n$, &c. and g muſt be equal to c, or $2c$, or $3c$, &c. Wherefore $a+n$, $a+2n$, $a+3n$, &c. *viz.* the Terms which follow a in the Progreſſion in the ſixth ſtep, and $b+c$, $b+2c$, $b+3c$, &c. *viz.* the Terms which follow b in the Progreſſion in the ſeventh ſtep, are the only whole numbers that can be taken inſtead of a and b, the leaſt whole numbers to conſtitute the Equation propoſed, to wit, $ca+d=nb$. Which was to be ſhewn.

20. If there be two whole numbers a, and b, given or found out, which will conſtitute the Equation before, propoſed or ſuch like, and thoſe two numbers be not the ſmalleſt values of a and b, you may by the help of thôſe given find out the ſmalleſt, by this Rule; *viz.* Divide the given whole number a, by the given number which is prefixt to b in the Equation propoſed, then after the Diviſion is finiſh'd there will remain either a number or nothing; if a number remain, it ſhall be the ſmalleſt value of a but if o remain, then the number prefixt to b is the ſmalleſt value of a, and conſequently the correſpondent value of b is eaſily diſcovered by the Equation. The reaſon of this *Rule* is manifeſt by *S.9.C.17.B.1.* For if any Term greater than the leaſt of an Arithmetical

Progreſſion

Progression be given, as also the common Difference, the least Term shall be given also, either by a continual subtraction of the common Difference, or by the *Rule* above exprest.

As for Example, If in the former of the two Arithmetical Progressions in the third step, which express values of a and b to constitute the Equation $9a+6+7b$, there be given 32 for the value of a, I divide 32 by 7 which is prefix'd to b, and find 7 contain'd four times in 32, and there remains 4; now this Remainder 4 is the smallest value of a, whence the correspondent whole number b, is easily discovered; for if $a=4$, then $9a+6=42=7b$; Therefore 42 divided by 7 gives 6 for the whole number b.

Again, if $a=20$, and $b=26$, then this will be a true Equation, *viz.* $5a+4=4b$; now if you desire the smallest whole numbers a and b to constitute that Equation, divide 20 the given value of a by 4 which is prefix'd to b, and there remains 0, therefore (according to the *Rule* before given) the said 4 shall be the smallest value of a; whence $5a+4=24=4b$, and consequently $6=b$.

Lastly, from what has been said in the third step, all the values of a and b in whole numbers that are capable of constituting the said Equation $5a+4=4b$ are the Terms of these two Arithmetical Progressions, *viz.*

Values of a; 4, 8, 12, 16, 20, 24, 28, 32, &c.
Values of b; 6, 11, 16, 21, 26, 31, 36, 41, &c.

Sect. III *Another way of solving the foregoing* Prop. 1.

In this later Method there are four principal Cases, which I shall first explain by Questions, and then shew how the Resolution of the Proposition will always run into one of those four Cases.

QUEST. 2.

To find all the whole numbers a and b that are capable of constituting this Equation *viz.* $8a+97=5b$.

The Equation proposed, ...	1	$8a+97=5b$
The Resolution	2	$8+97=105$
	3	$\frac{105}{5}=21=b$
	4	$1=a$

Explication.

First I add 97 (to wit, +97 in the Equation proposed) to 8, which is prefix'd to a, and it makes 105, this I divide by 5 the number prefix'd to b; and because the Quotient 21 happens to be exactly a whole number without any remainder, it shall be the smallest whole number b sought, and the whole number a in this case is always 1. The Reason is evident, for if $a=1$, then $8a+97=8+97$; and if this sum happens to be a Multiple of the given number prefix'd to b, then b is necessarily a whole number. This is the first of the four Cases above mentioned.

Then after 1 and 21, the smallest whole numbers a and b to constitute the Equation propos'd, are found out, all the other values of a and b in whole numbers will be found in these two following Arithmetical Progressions formed according to the *Rule* in the third step of the foregoing *Sect.* 2. *viz.*

Values of a; 1, 6, 11, 16, 21, 26, &c.
Values of b; 21, 29, 37, 45, 53, 61, &c.

I say every two correspondent numbers in those Progressions may be taken for values of a and b in this Equation, $8a+97=5b$; as for Example, if 11 be taken for a, and 37 for b, then eight times 11, with 97 added shall be equal to five times 37, *viz.* 185 = 185. And so of the rest.

QUEST. 3.

To find all the whole numbers a and b that are capable of constituting this Equation, *viz.* $49a+6=13b$

The

The Equation proposed, . .	1	$49a+6$	$= 13b$
The Resolution, ...	2	55	$= 65-10$
	3	49	$= 39+10$
	4	104	$= 104$
	5	$\frac{104}{13}$	$= 8 = b$
	6	$\frac{104-6}{46}$	$= 2 = a$

Explication.

First, I add 6 (to wit, +6 in the Equation proposed) to 49 which is prefix'd to a, and it makes 55; now if this 55 were exactly divisible by 13 which is prefix'd to b, the Quotient would be the whole number b sought, and 1 the number a, (as in *Quest.* 2.) But 55 not being a Multiple of 13, I proceed thus, *viz.* I seek the Multiple of 13 which is next greater than 55, by dividing 55 by 13, so I find that four times 13 is less than 55, but five times 13, that is, 65, exceeds 55, by 10; therefore 55 is equal to 65 wanting 10, *viz.* $55=65-10$. This is the second Equation in the Example.

2. Then I divide 49 which is prefix'd to a, by 13 which is prefix'd to b, so I find that three times 13, that is, 39, is the greatest Multiple of 13 contained in 49, and there remains 10; therefore $49=39+10$: which is the third Equation.

3. Now because +10 is found in the third Equation, and −10 in the second, I add those Equations together, so the said 10 vanishes, and there arises $104=104$; which is the fourth Equation.

4. Then I divide 104, that is, either part of the fourth Equation, by 13 which is prefix'd to b in the Equation propos'd, and the Quotient 8 is the whole number b sought.

5. Then from the said 104 in the fourth Equation, I subtract 6, (to wit, +6 in the Equation propos'd) and divide the Remainder 98 by 49 which is prefix'd to a, so the Quotient gives 2 for the whole number a sought.

I say $2=a$ and $8=b$ will make $49a+6=13b$, as was required in *Quest.* 3. and all the values of a and b in whole numbers that are capable of producing the same effect, are the Terms of these two following Arithmetical Progressions whose construction has been shewn before.

Values of a;	2,	15,	28,	41,	54,	67,	&c.
Values of b;	8,	57,	106,	155,	204,	253,	&c.

Note, That the manner of forming the second and third Equations in the foregoing Resolution of *Quest.* 3. must be diligently observed, because the like work is constantly used in the following fourth, fifth, sixth, seventh, eighth and ninth Questions: But it's by accident, that the same number 10 follows the Signs − and + in the said second and third Equations, and therefore the adding them together to produce the fourth Equation, is an Operation peculiar only to this and the like accident, which I call the second of the four Cases before mentioned.

But that in this second Case, the Resolution infallibly produces whole Numbers for the values of a and b, I prove thus: First by Construction, $65-10$ (the later part of the second Equation) wants 10 of a Multiple of 13, and $39+10$ (the later part of the third Equation) exceeds a Multiple of 13 by 10; therefore the Sum of the said $65-10$ and $39+10$, to wit, 104 (the later part of the fourth Equation) shall be a Multiple of 13; and consequently 104 divided by 13 will exactly give a whole Number, to wit, 8, for the value of b. Secondly, because 104 (the first part of the fourth Equation) is by construction compos'd of a Multiple of 49 together with 6; by subtracting 6 from 104, the Remainder 98 shall be a Multiple of 49, and consequently 98 divided by 49 will give the Quotient an exact whole number, to wit, 2, for the value of a. Whence it is manifest, that if after the second and third Equations are formed out of the first (to wit, the Equation proposed) according to the preceding Directions for solving *Quest.* 3. it happens that the number following + in the later part of the third Equation, is the same with the Number following − in the later part of the second, there will certainly arise two whole Numbers for the values of a and b.

QUEST. 4.

To find all the whole Numbers a and b that may make $82a+66=13b$.

The Equation propos'd, . .	1	$82a+66 = 13b$
The Resolution, ...	2	$148 = 156-8$
	3	$82 = 78+4$
	4	$164 = 156+8$
	5	$312 = 312$
	6	$\frac{312}{13} = 24 = b$
	7	$\frac{312-66}{82} = 3 = a$

Explication.

1. The second and third Equations are formed out of the first in such manner as before has been explain'd in the Resolution of *Quest.* 3.

2. Because the Number 4 which follows the Sign + in the later part of the third Equation, happens to be an Aliquot Part, to wit, $\frac{1}{2}$ of 8 which follows the Sign — in the later part of the second Equation, I multiply each part of the third Equation by 2 (the Denominator of the said Aliquot Part,) to the end there may be +8 in the Equation made by that Multiplication; so there is produced $164=156+8$, which is the fourth Equation.

3. Now since +8 is found in the fourth Equation, and —8 in the second, I add those Equations together, so the said 8 vanishes, and there arises $312=312$; which is the fifth Equation.

4. Then I divide 312, (to wit, either part of the fifth Equation) by 13 which is prefix'd to b in the Equation proposed, and the Quotient 24 is the whole number b sought.

5. Lastly, from the said 312 (in the fifth Equation) I subtract 66, to wit, +66 in the Equation propos'd, and divide the Remainder 246 by the given number 82, (which is prefix'd to a;) so the Quotient 3 is the whole Number a sought.

I say, $3=a$ and $24=b$ will make $82a+66=13b$, as was required in *Quest.* 4. and all the values of a and b in whole Numbers that are capable of producing that Equation, are the Terms of these two Arithmetical Progressions, (whose Construction has been shewn before in the third step of *Sect.* 2.) *viz.*

Values of a; 3, 16, 29, 42, 55, 68, &c.
Values of b; 24, 106, 188, 270, 352, 434, &c.

Note, That it was by meer chance that the number following the Sign + in the third Equation happened to be an Aliquot Part of the number following the Sign — second, and therefore the multiplying of the third Equation by the Denominator of the Aliquot Part, is an Operation peculiar only to that and the like accident, which is the third of the four Cases before mentioned. The Reason of the Operation in this fourth Question (or third Case,) may be easily discerned by the Demonstration before given in *Quest.* 3. but for further illustration I shall add another Example of *Case* 3.

QUEST. 5.

To find all the whole Numbers that may be values of a and b in this Equation, *viz* $601a+9=200b$.

The Equation proposed . . .	1	$601a+9 = 200b$
The Resolution, ...	2	$610 = 800-190$
	3	$601 = 600+1$
	4	$114190 = 114009+190$
	5	$114800 = 114800$
	6	$\frac{114800}{200} = 574 = b$
	7	$\frac{114800-9}{601} = 191 = a$

Explica-

Explication.

The Resolution of this Question is like that in the foregoing *Quest.* 4. for since + 1 in the later part of the third Equation happens to be an Aliquot part of 190 which follows — in the second Equation, I multiply each part of the third by 190, to the end that + 190 may be found in the Product, as you see in the fourth Equation; then by adding the fourth Equation to the second, the Sum makes the fifth, which is free from the Signs + and —; lastly, from the fifth Equation the whole numbers 574 and 191 expressing the values of *b* and *a* are discovered, in like manner as in the preceding third and fourth Questions; which numbers will constitute the Equation proposed: For 601 times 191 together with 9 is equal to 200 times 574, that is, 114800; and all the rest of the values of *a* and *b* in whole Numbers to make that Equation will be found in these two following Arithmetical Progressions formed by the Rule before given in the third step of *Sect.* 2.

Values of *a*; 191, 391, 591, 791, 991, &c.
Values of *b*; 574, 1175, 1776, 2377, 2978, &c.

QUEST. 6.

If	1	$121a + 5$	$= 93b$,	What are *a* and *b* in whole Numbers?
Out of 1.	2	126	$= 186 - 60$	
	3	121	$= 93 + 28$	
Suppose	4	$93c + 60$	$= 28d$	$c = ?\ d = ?$
Out of 4.	5	153	$= 168 - 15$	
	6	93	$= 84 + 9$	
Suppose	7	$28e + 15$	$= 9f$	$e = ?\ f = ?$
Out of 7.	8	43	$= 45 - 2$	
	9	28	$= 27 + 1$	
Eq. $9 \times \overline{2}$.	10	56	$= 54 + 2$	
Eq. $8 + 10$.	11	99	$= 99$	
Out of 11 and 7.	12	$\frac{99}{9}$	$= 11 = f$	Here the Regressive work begins.
12, 6 and 5.	13	$11 \times 93 + 153$	$= 1176$	
13 and 4.	14	$\frac{1176}{28} = 42$	$= d$	
14, 3 and 2.	15	$42 \times 121 + 126$	$= 5208$	
15 and 1.	16	$\frac{5208}{93} = 56$	$= b$	
15 and 1.	17	$\frac{5208 - 5}{121} = 43$	$= a$	

Explication.

1. The second and third Equations are formed out of the first in like manner as before in the Explication of *Quest.* 3.

2. But because 28 which follows + in the third Equation, is not equal to, nor an Aliquot part of 60 which follows — in the second, the process cannot be made like that in the third, fourth and fifth Questions; so that now a fourth Case takes rise, and the scope of a new search is to find out a number *d*, such, that if it multiply the said + 28, the Product may exceed a Multiple of 93 (which is prefix'd to *b*) by 60; for then it will be evident, that if the third Equation be multiplied by that number *d*, an Equation will be produced whose first part shall be a Multiple of 121, and the latter part shall exceed a Mutiple of 93 by 60, and then the rest of the work will be like that in Case 2. in *Quest.* 3. In the search therefore of the number *d*, the fourth Equation is assumed, to wit, $93c + 60 = 28d$.

3. The fifth and sixth Equations are formed out of the fourth, in like manner as the second and third out of the first.

4. Because 9 which follows + in the sixth Equation, is neither equal to, nor an Aliquot part of 15 which follows the Sign — in the fifth, the next scope (for the like reason before

given concerning the number d) is to find out a number f, such, that if it multiply the said $+9$, the Product may exceed a Multiple of 28 which is prefix'd to d, by the said 15; to which end the seventh Equation is assumed, to wit, $28e+15=9f$.

5. The eighth and ninth Equations are formed out of the seventh, in like manner as the second and third out of the first.

6. Because 1 which follows + in the ninth Equation, is an Aliquot Part of 2 which stands next after — in the eighth, the ninth is multiplied by 2 the Denominator of the said part; (according to the Rule in Case 3. *Quest.* 3.) whence the tenth Equation is produced, to wit, $56=54+2$.

7. The eleventh Equation, to wit, $99=99$ is the Sum of the eighth and tenth; and since the said eleventh is free from the Signs + and —, a Regressive work now begins, to find out the whole numbers f, d, b and a; in this manner, *viz.*

8. By dividing either part of the eleventh Equation, to wit, 99, by 9 which is prefix'd to f in the seventh, there arises $11=f$, as in the twelfth Equation.

9. Then multiplying the number f, to wit, 11, by 93, that is, either part of the sixth Equation, and to the Product adding 153, that is, either part of the fifth Equation, the Sum makes 1176, (as you see in the thirteenth Equation) which 1176 is a Multiple of 28, to wit, that which is represented by $28\,d$ in the fourth Equation; Therefore,

10. By dividing the said 1176 by 28, the Quotient 42 is the number d, as in the fourteenth Eqnation.

11. Then multiplying the number d, to wit, 42, by 121, that is, either part of the third Equation, and to the Product adding 126, that is, either part of the second Equation, the Sum makes 5208, as you see in the fifteenth Equation, which 5208 is a Multiple of 93, to wit, that which is represented by $93\,b$ in the first Equation; Therefore,

12. By dividing either part of the fifteenth Equation, to wit, 5208 by 93, the Quotient 56 is the number b sought.

13. Then from the said 5208 subtracting 5, to wit, $+5$ in the first Equation, and dividing the Remainder 5203 by 121 which is prefix'd to a in the first Equation, the Quotient gives 43 for the number a sought, as in the seventeenth and last Equation. Therefore, if 43 be for a, and 56 for b, then $121a+5=93b$, which is the Equation proposed in *Quest.* 6. and all the values of a and b in whole Numbers that are capable of constituting that Equation are the Terms of these two following Arithmetical Progressions, whose Construction has been shewn before in the third step of *Sect.* 2.

Values of a;	43,	136,	229,	322,	415,	508,	&c.
Values of b;	56,	177,	298,	419,	540,	661,	&c.

14. After the Numbers f and d in the foregoing Resolution of *Quest.* 6. are known, the Numbers e and c in the seventh and fourth Equations, may easily be discovered; but there is no need of their help in the finding out of the desired Numbers a and b.

15. But methinks I hear the Reader make this Objection, *viz.* How does it appear, that from every three whole numbers given in such sort as before is declared in *Prop.* 1. there may infallibly be found out two whole numbers a and b to solve the said Proposition, by the Operation before explained in the four Cases before mentioned: For Answer to this Objection, I shall here shew how far the Process need be continued at the farthest, to find out an Equation having $+1$ in its later part; for when such Equation arises, 'tis manifest by the Operation in the third Case explain'd in *Quest.* 4, and 5. that two whole numbers a and b will infallibly be discovered to satisfie the Proposition, and consequently innumerable other pairs of whole numbers to produce the same effect. First, then in the foregoing *Quest.* 6. the given number 121 which is prefix'd to a, being divided by the given number 93 which is prefix'd to b, after the Division is finish'd there remains 28, to wit $+28$ in the later part of the third Equation: Secondly, the said Divisor 93 being divided by the said Remainder 28, after the Division is ended there remains 9, to wit, $+9$ in the later part of the sixth Equation: Again, the last Divisor 28 being divided by the last Remainder 9, after this Division is ended there remains 1, that is, $+1$ in the later part of the ninth Equation, which Remainder 1 you will always infallibly come unto by a continued Division in that manner, because the two given Numbers prefix'd to a and b are (as the Proposition requires) Prime between themselves; and that continued Division is nothing else but the Method of finding out the greatest common Divisor unto two Numbers;

Numbers; so that you may at first (if you please) discover unto what Letter at the farthest, the process need be continued before you return backward according to the Operation explain'd in *Quest.* 6. But oftentimes before you come to the said Remainder 1, the Resolution will run into one of the three Cases explain'd in *Quest.* 2, 3, 4, and 5. as will appear by the following seventh, eighth, and ninth Questions.

QUEST. 7.

If	1	$97a + 1$	$= 26b,$	What are a and b in whole Numbers?
Out of 1.	2	98	$= 104 - 6$	
	3	97	$= 78 + 19$	
Suppose	4	$26c + 6$	$= 19d$	$c = ?\ d = ?$
Out of 4.	5	32	$= 38 - 6$	
	6	26	$= 19 + 7$	
Suppose	7	$19e + 6$	$= 7f$	$e = ?\ f = ?$
Out of 7.	8	25	$= 28 - 3$	
	9	19	$= 14 + 5$	
Suppose	10	$7g + 3$	$= 5h$	$g = ?\ h = ?$
Out of 10.	11	$7 + 3$	$= 10$	
Out of 10 and 11.	12	$\frac{10}{5}$	$= 2 - h$	Here the Regressive work begins.
Out of 12, 9, 8.	13	$2 \times 19, + 25$	$= 63$	
13, and 7.	14	$\frac{63}{7} = 9$	$= f$	
14, 6 and 5.	15	$9 \times 26, + 32$	$= 266$	
15 and 4.	16	$\frac{266}{19} = 14$	$= d$	
16, 3 and 2.	17	$14 \times 97, + 98$	$= 1456$	
17 and 1.	18	$\frac{1456}{26} = 56$	$= b$	
17 and 1.	19	$\frac{1456 - 1}{97} = 15$	$= a$	

Explication.

In this seventh Question the process is formed like that in the foregoing sixth; and the last Letter in the work is h, whose value is discovered in the twelfth Equation by the help of the tenth and eleventh, according to the Operation in *Quest.* 2. and then by the help of the Number h, the Work returns backward to find out the Numbers f, d, b and a, in like manner as in *Quest.* 6. But in this seventh Question the last Letter in the Process, to wit, h, is made known before an Equation arises which has $+1$ in its later Part; aud the like effect happens in the following eighth and ninth Questions.

Now in Answer to this seventh Question, all the values of a and b in whole Numbers that are capable of constituting the Equation proposed, to wit, $97a + 1 = 26b$, are the Terms of the two following Arithmetical Progressions, which are deduced from the two smallest values of a and b, (to wit, 15 and 56 found out as above,) according to the Rule in the third step of *Sect.* 2.

Values of a; 15, 41, 67, 93, 119, 145, &c.
Values of b; 56, 153, 250, 347, 444, 541, &c.

QUEST. 8.

QUEST. 8.

If	1	$119a+6$	$=57b$,	What are the whole numbers a and b?
Out of 1.	2	125	$=171-46$	
	3	119	$=114+5$	
Suppose	4	$57c+46$	$=5d$	$c=?$ $d=?$
Out of 4.	5	103	$=105-2$	
	6	57	$=55+2$	
5 + 6.	7	160	$=160$	
7, 4.	8	$\frac{160}{5}=32$	$=d$	Regress.
8, 3, 2.	9	$32\times119,+125$	$=3933$	
9, 1.	10	$\frac{3933}{57}$	$=69=b$	
9, 1.	11	$\frac{3933-6}{119}$	$=33=a$	

Values of a; 33, 90, 147, 204, 261, 318, &c.
Values of b; 69, 188, 307, 426, 545, 664, &c.

In which Progressions, every two correspondent Terms may be taken for values of a and b to constitute the Equation in *Quest.* 8.

QUEST. 9.

If	1	$173a+1$	$=71b$,	What are the whole numbers a and b?
Out of 1.	2	174	$=213-39$	
	3	173	$=142+31$	
Suppose	4	$71c+39$	$=31d$	$c=?$ $d=?$
Out of 4.	5	110	$=124-14$	
	6	71	$=62+9$	
Suppose	7	$31e+14$	$=9f$	$e=?$ $f=?$
Out of 7.	8	$31+14$	$=45$	
8, and 7.	9	$\frac{45}{9}$	$=5=f$	Regress.
9, 6, 5.	10	$5\times71,+110$	$=465$	
10, 4.	11	$\frac{565}{31}=15$	$=d$	
11, 3, 2.	12	$15\times173,+174$	$=2769$	
12, 1.	13	$\frac{2769}{71}$	$=39=b$	
12, 1.	14	$\frac{2769-1}{173}$	$=16=a$	

Values of a; 16, 87, 158, 229, 300, 371, &c.
Values of b; 39, 212, 385, 558, 731, 904, &c.

Sect. 4. *PROP.* II.

Two whole numbers Prime between themselves being given, to find out two others, suppose a and b, that if a be multiplied by the lesser of those two numbers given, and to the Product there be added a whole number given, the sum shall be equal to the Product of b multiplied by the greater of the two numbers first given. Moreover, to discover all the whole numbers a and b that are capable of producing the same effect.

When each of the two given numbers which are Prime between themselves is a single Figure, or some small number consisting of two Characters, then the first of the two ways of solving the foregoing *Prop.* 1. will readily solve this second; but waving that Method I shall shew two other ways by the help of the later of those two Methods.

The

The first Method of solving Prop. 2.

QUEST. 10.

If	1	$71a+3 = 173b,$	What are a and b in whole Numbers?
Out of 1.	2	$145 = 173-28$	
By *Prop.* 1.	3	$2769 = 2768+1$	
Eq. 3 × 28.	4	$77532 = 77504+28$	
2 + 4.	5	$77677 = 77677$	
Out of 5, 1.	6	$\frac{77677}{173} = 449 = b$	true Values.
5, 1.	7	$\frac{77677-3}{71} = 1094 = a$	true Values.
By the Rule in	8	$56 = a$	the least Values.
Sect. 2. *Num.* 20.	9	$23 = b$	the least Values.

Explication.

1. I multiply 71 which is prefix'd to a in the Equation proposed, by such a Number, that when 3, to wit, $+3$ in the same Equation is added to the Product, the Sum may be either equal to, or less than some Multiple of 173; so multiplying 71 by 2, the Product 142 increased with 3 makes 145, which is equal to 173 wanting 28, *viz.* $145=173-28$, which is the second Equation.

2. Then by *Prop.* 1. of this *Chap.* I seek two such Numbers a and b, that if a be multiplied by 173, and the Product increased with $+1$, the Sum may be equal to the Product of b multiplied by 71; *viz.* Supposing $173a+1=71b$, and proceeding according to the foregoing *Quest.* 9. I find 16 for the value of a, and 39 for b; therefore $\overline{173\times16},+1=71\times39$; or $71\times39=\overline{173\times16},+1$; that is, $2769=2768+1$, which is the third Equation.

3. Because $+1$ in the later part of the third Equation is an Aliquot Part of 28 in the second, I multiply the third Equation by 28 the Denominator of the said Part, and it makes the fourth Equation, to wit, $77532=77504+28$.

4. Then by adding the fourth Equation to the second the Sum gives the fifth, which is free from the Signs + and —; and from the fifth Equation the whole Numbers 449 and 1094 are discovered for values of b and a, in like manner as in *Quest.* 4, and 5. and by the help of those the smallest values of a and b, to wit, 56 and 23 are found out by the Rule in the twentieth step of *Sect.* 2.

5. Lastly, by the help of the two smallest values of a and b, and the Rule in the third step of *Sect.* 2, all that are capable of solving *Quest.* 10. will be found in the two following Arithmetical Progressions, which may be continued as far as you please.

Values of a; 56, 229, 402, 575, 748, 921, 1094, &c.
Values of b; 23, 94, 165, 236, 307, 378, 449, &c.

QUEST. 11.

If	1	$22a+5000 = 65b,$	What are a and b in whole Numbers?
Out of 1.	2	$5022 = 5070-48$	
By *Prop.* 1.	3	$66 = 65+1$	
Eq. 3 × 48.	4	$3168 = 3120+48$	
2 + 4.	5	$8190 = 8190$	
Out of 5, and 1.	6	$\frac{8190}{65} = 126 = b$ . . .	true Values.
5, 1.	7	$\frac{8190-5000}{22} = 145 = a$	true Values.
By the Rule in	8	$15 = a$	the least Values.
Sect. 2. *Num.* 20.	9	$82 = b$	the least Values.

Expli-

Explication.

1. I add 22 to 5000 and it makes 5022, which is not exactly divisible by 65, for 77 times 65 is less than 5022, but 78 times 65, that is, 5070, exceeds 5022 by 48; therefore $5022 = 5070 - 48$, which is the second Equation.

2. Then by *Prop.* 1. of this *Chap.* I seek two such whole numbers a and b, that if a be multiplied by 65, and to the Product there be added 1, the Sum may be equal to the Product of b multiplied by 22; *viz.* Supposing $65a + 1 = 22b$, and proceding according to the later Method of resolving the foregoing *Prop.* 1. I find 1 and 3 to be values of a and b; therefore, $\overline{65 \times 1}, + 1 = 22 \times 3$; or $22 \times 3 = \overline{65 \times 1}, + 1$; that is, $66 = 65 + 1$, which is the third Equation.

3. By prosecuting the Work as before in the Explication of *Quest.* 10. all the desired values of a and b in whole numbers that are capable of constituting the Equation first proposed in this eleventh Question will be found to be the Terms of these two following Arithmetical Progressions, *viz.*

Values of a;	15,	80,	145,	210,	275,	340,	&c.
Values of b;	82,	104,	126,	148,	170,	192,	&c.

Another way of solving Prop. 2.

QUEST. 12.

If	1	$71a + 3 = 173b$,	What are a and b in whole numbers?
Out of 1.	2	$145 = 173 - 28$	
	3	$213 = 173 + 40$	
Suppose	4	$173c + 28 = 40d$	$c = ?\ d = ?$
Out of 4.	5	$201 = 240 - 39$	
	6	$173 = 160 + 13$	
$6 \times \overline{3}$.	7	$519 = 480 + 39$	
$5 + 7$.	8	$720 = 720$	
8, 4.	9	$\frac{720}{40} = 18 = d$	Regress.
9, 3, 2.	10	$18 \times 213, + 145 = 3979$	
10, 1.	11	$\frac{3979}{173} = 23 = b$	
10, 1.	12	$\frac{3979 - 3}{71} = 56 = a$	

Explication.

1. In this Question, which is the same with the foregoing tenth, the second Equation is formed as is there directed.

2. The third Equation is thus formed: Forasmuch as the given number 71 is less than 173 which is prefix'd to b, I multiply 71 by such a Number that the Product may exceed 173, and be also Prime to it; so multiplying 71 by 3, the Product 213 exceeds 173, also 213 and 173 are Prime to one another; then I divide the same 213 by 173, and find that 213 contains 173 once, and 40 over and above; therefore $213 = 173 + 40$, which is the third Equation.

3. The fourth, fifth, and sixth Equations here, are formed like the fourth, fifth, and sixth Equations in the foregoing *Quest.* 6.

4. Then because 13 which follows + in the sixth Equation is an Aliquot part of 39 which follows — in the fifth, I multiply the sixth Equation by 3 the Denominator of the said Part, (for 13 is $\frac{1}{3}$ of 39,) and it produces the seventh Equation, to wit, $519 = 480 + 39$.

5. The eighth Equation is the Sum of the fifth and seventh, (according to the Operation in Case 2.) and then in the ninth Equation the Regressive Work begins, to find out the values of d, b and a in such manner as has been shewn in divers preceding Questions of this *Chap.* So at length all the values of a and b in whole numbers to solve this twelfth Question will by this later Method be found the same as before in *Quest.* 10.

Sect. 5. *PROP.* III.

To divide a given number into three or more numbers, such, that if every one of them be multiplied by a different number given, the sum of the Products may be equal to a given number. But the sum of those Products must fall between the two Products made by multiplying the given Dividend into the greatest and least of the given Multiplicators.

The solution of this Problem is explain'd by the following Questions of this Chapter, and oftentimes requires the help of the two preceding Propositions, as will partly appear by the fifteenth Question.

QUEST. 13.

To divide 24 into three such whole numbers, that if the first be multiplied by 36, the second by 24, and the third by 8, the sum of the three Products may make 516.

Let the numbers sought be represented by a, e & y, then the Question may be stated thus:

1. If . . . . $a + e + y = 24$
2. And . . . . $36a + 24e + 8y = 516$

What are the whole numbers a, e and y? ||

RESOLUTION.

3. The first Equation multiplied by 36, which is prefix'd to a in the second, produces . . . $36a + 36e + 36y = 864$
4. The 2*d* Equation subtracted from the third, leaves . . $12e + 28y = 348$
5. The 4*th* Equation by transposition of $+28y$, gives . . $12e = 348 - 28y$
6. The fifth Equation divided by 12 gives . . . . . . $e = 29 - \frac{7y}{3}$
7. If instead of e in the first Equation there be taken the later part of the sixth, this arises . . $a + 29 - \frac{7y}{3} + y = 24$
8. That is, ... $a + 29 - \frac{4y}{3} = 24$
9. From the eighth Equation by transposition of $29 - \frac{4y}{3}$ this arises, ... $a = 24 - 29 + \frac{4y}{3}$
10. That is, ... $a = \frac{4y}{3} - 5$
11. By the later part of the tenth Equation 'tis evident that . . . . $\frac{4y}{3} \sqsubset 5$
12. Therefore by multiplying each part in the eleventh step by 3, it follows that . . . . . $4y \sqsubset 15$
13. And by dividing each part in the 12*th* step by 4, $y \sqsubset 3\frac{3}{4}$
14. And from the later part of the sixth Equation, by arguing in like manner as in the eleventh, twelfth and thirteeth steps, it will be manifest that $y \sqsupset 12\frac{3}{7}$
15. Now if Fractions or mixt numbers were admitted to be the values of a, e, and y, then by the thirteenth, fourteenth, tenth and sixth steps 'tis evident that

$y =$ any number between $3\frac{3}{4}$ and $12\frac{3}{7}$;

$a = \frac{4y}{3} - 5$;

$e = 29 - \frac{7y}{3}$

16. But to find out whole numbers to solve the Question, the limits in the thirteenth & fourteenth steps do shew that y must be some whole number greater than 3, but not greater than 12, yet every whole number within those limits will not serve the turn, for the values of a and e before discovered will not be whole numbers unless $\frac{4y}{3}$ and $\frac{7y}{3}$ be whole numbers; but $\frac{4y}{3}$ and $\frac{7y}{3}$ cannot be whole numbers unless y be 3, or some Multiple of 3, and because 3 is without the limits, y may be 6, or 9, or 12, and consequently from

a	e	y
3	15	6
7	8	9
11	1	12

from the fifteenth step a shall be 3, or 7, or 11; and e, 15, or 8, or 1. Now in answer to the Question, 3, 15 and 6, (to wit, a, e and y) are three such whole numbers, that their sum is 24, and if the first be multiplied by 36, the second by 24, and the third by 8, the sum of the three Products makes 516, as was required. The like may be said of each of the two other Answers. But if Fractions or mixt numbers were admitted, innumerable Answers might be given to the Question, as before has been shewn in the fifteenth step.

Note. When one part of an Equation consists of an Affirmative letter and some Negative Absolute number, a limit may thence be inferr'd, above which the number signified by that letter ought to be taken. But if one part of an Equation consists of a Negative letter and of an Affirmative absolute number, it will give a limit beneath which the number represented by that letter must be chosen. Sometimes also two limits will be discovered, (as in this thirteenth Question for the choice of the number y;) and sometimes but one, (as in divers of the following Questions.)

QUEST. 14.

To find three such whole numbers that their sum may make 100; and that if the first be multiplied by 4, the second by 3, and the third by $1\frac{4}{5}$, the sum of the three Products may make 300.

For the three numbers sought put a, e and y, then the Question may be stated thus;

1. If . . . . $a+e+y=100$
2. And . . . . $4a+3e+1\frac{4}{5}y=300$

What are the whole numbers a, e and y?

RESOLUTION.

3. The first Equation multiplied by 4, (which is prefix'd to a in the second Equation,) produces . . . . $4a+4e+4y=400$
4. The second Equation subtracted from the third, leaves . . $e+\frac{11y}{5}=100$
5. The fourth Equation by transposition of $+\frac{11y}{5}$ gives . . $e=100-\frac{11y}{5}$
6. If instead of e in the first Equation there be taken the later part of the fifth, this will arise, ... $a+100-\frac{11y}{5}+y=100$
7. That is, after due Reduction, ... $a=\frac{6y}{5}$
8. From the later part of the fifth Equation it's manifest that . . . . $\frac{11y}{5}\sqsupset 100$
9. And consequently by multiplying each part in the eighth step by 5 . . . . $11y\sqsupset 500$
10. And by dividing each part in the ninth step by 11, it follows that . . . . $y\sqsupset 45\frac{5}{11}$

Whence 'tis manifest, that if the three numbers sought were not restrained to whole numbers, any number less than $45\frac{5}{11}$ might be taken for the number y, and then the numbers a and e would be discovered from the seventh and fifth steps. But to have the Question solved by whole numbers, the number y must be some whole number not greater than 45, and such as may cause $\frac{11y}{5}$ and $\frac{6y}{5}$ to be whole Numbers, for otherwise the values of e and a in the fifth and seventh steps will not be expressible by whole Numbers; but $\frac{11y}{5}$ and $\frac{6y}{5}$ cannot be whole Numbers unless y be 5, or some Multiple of 5, and therefore y may be 5, or 10, or 15, or any of the rest of the numbers in the third Columel of this Table; and consequently, from the fifth and seventh steps of the Resolution, the whole numbers e and a will be such as stand under e and a. Thus you see that the Question receives nine Answers in whole Numbers, which are all that it's capable of: So that if you take 6 for a; 89 for e; and 5 for y, their sum is 100; and if 6 be multiplied

a	e	y
6	89	5
12	78	10
18	67	15
24	56	20
30	45	25
36	34	30
42	23	35
48	12	40
54	1	45

by 4;

by 4; 89 by 3; and 5 by $1\frac{1}{4}$, the fum of the three Products makes 300, as the Queftion requires. The like may be proved of every one of the other eight Anfwers.

Note. When three numbers are fought by a Queftion of this nature that is capable of many Anfwers in whole numbers, all the values of every one of the letters in whole numbers are in Arithmetical Progreffion, and therefore when two of thofe Anfwers are found out, all the reft within the limits difcovered by the Refolution are confequently given by Addition or Subtraction of the common difference in each Rank, as may eafily be perceived by the values of *a*, *e*, *y* in the Table above-written. But when four numbers are fought, the values of a letter are oftentimes found in feveral Arithmetical Progreffions, as in the following *Queft.* 20.

QUEST. 15.

To divide 1533 into three whole numbers, fuch, that $\frac{1}{8}$ of the firft, together with $\frac{3}{8}$ of the fecond and $\frac{2}{113}$ of the third may make 167.

For the three whole numbers fought put *a*, *e*, and *y*, then the Queftion may be ftated thus?

1. If . . . . $a + e + y = 1533$
2. And . . . . $\frac{1}{8}a + \frac{3}{8}e + \frac{2}{113}y = 167$

What are the whole numbers *a*, *e*, and *y*? ||

RESOLUTION.

3. The firft Equation multiplied by $\frac{1}{8}$, produces . $\frac{1}{8}a + \frac{1}{8}e + \frac{1}{8}y = \frac{1533}{8}$
4. The fourth Equation fubtracted from the third, leaves . . . . $-\frac{1}{4}e + \frac{97}{904}y = \frac{197}{8}$
5. The fecond Equation by tranfpofition of $-\frac{1}{4}e$ gives . . . . $\frac{97}{904}y = \frac{197}{8} + \frac{1}{4}e$
6. The fifth Equation divided by $\frac{97}{904}$, gives . $y = \frac{22261}{97} + \frac{226e}{97}$
7. If inftead of *y* in the firft Equation there be taken the later part of the fixth this arifes, $a + e + \frac{22261}{97} + \frac{226e}{97} = 1533$
8. The feventh Equation, after due Reduction, gives . . . . $a = \frac{126440}{97} - \frac{323e}{97}$
9. By the eighth Equation it's manifeft that . $323e \sqsupset 126440$
10. And confequently by dividing each part of the laft ftep by 323, ... $e \sqsupset 391\frac{147}{323}$
11. Now to find out the values of *a*, *e* and *y* in whole numbers, (if there be a poffibility) I multiply the fixth Equation by the Denominator 97, and it makes . . . . $97y = 22261 + 226e$
12. That is, ... $226e + 22261 = 97y$
13. Then by the foregoing *Prop.* 1. of this Chapter, I fearch out all fuch whole numbers as may be values of *e* and *y* to conftitute the laft Equation, that is, $226e + 22261 = 97y$; but with this Condition, *viz.* That the greateft whole number among thofe that are found out for the values of *e* may not exceed 391, as the preceding tenth ftep requires; fo I find four values of *e*, to wit, 47, 144, 241, 338; and four values of *y*, to wit, 339, 565, 791 and 1017: Then the Sum of every two correfpondent values of *e* and *y* being fubtracted from 1533 the Number firft given to be divided, the Remainders fhall be the defired values of *a*, to wit, 1147, 824, 501 and 178; fo there are only four Anfwers to the Queftion in whole Numbers, to wit, thofe inferted in the Table in the Margin.

a	*e*	*y*
1147	47	339
824	144	565
501	241	791
178	338	1017

The Proof of the firft Anfwer.

The Sum of 1147, and 47 339 is 1533,
$\frac{1}{8}$ of 1147 is . . . . $143\frac{3}{8}$,
$\frac{3}{8}$ of 47 is . . . . $17\frac{5}{8}$,
$\frac{2}{113}$ of 339 is 6,
Laftly, the fum of thofe three Products is . . . 167,

Therefore

Therefore all the Conditions in the Question are satisfied, and the like may be proved by every one of the other three Answers in whole Numbers; but if Fractions were admitted, innumerable Answers might be given by the tenth, eighth, and sixth steps of the Resolution.

QUEST. 16.

To find the three Numbers, that their Sum may make 300; and that if the first be multiplied by 6, the second by 5, and the third by $2\frac{7}{300}$, the Sum of the three Products may make 1496.

Let a, e, y be put for the three Numbers sought; then by forming the resolution in like manner as in the preceding thirteenth, fourteenth and fifteenth Quest. it will appear that

$y =$ any Number between $1\frac{307}{893}$ and $76\frac{532}{1193}$;

$$e = 304 - \frac{1193y}{300};$$

$$a = \frac{893y}{300} - 4.$$

Whence 'tis evident, that there cannot be three whole numbers found out to solve this Question, for 300 is the smallest whole Number that can be taken for y to cause $\frac{1193y}{300}$ and $\frac{893y}{300}$ to be whole Numbers; but 300 exceeds the greater of the two limits above discovered for chusing of the number y.

QUEST. 17.

If one would lay out 98 pence to buy 40 Birds, suppose Patridges, Larks and Quails; how many of each kind may be bought when Patridges are at 3 pence a piece, Larks at an half penny a piece, and Quails at 4 pence a piece?

Let a represent the number of Patridges, e the number of Larks, and y the number of Quails; then according to the Question, $a+e+y=40$; and because the number of all the Patridges multiplied by the price of one of them produces the full cost of all, it's manifest that $3a$ is the full cost of all the Patridges; and for the like reason $\frac{1}{2}e$ signifies the full cost of all the Larks; likewise $4y$ the full cost of the Quails: But those three particular Sums of Money must be equal to 98 pence, therefore $3a+\frac{1}{2}e+4y = 98$; so that the Question may be stated thus;

1. If . . . . $a+e+y = 40$
2. And . . . . $3a+\frac{1}{2}e+4y=98$

What are the whole Numbers a, e and y?

RESOLUTION.

3. The first Equation multiplied by 3 (which is prefix'd to a in the second,) produces . . . . $3a+3e+3y=120$
4. The second Equation subtracted from the third, leaves . $\frac{5e}{2} - y = 22$
5. From the fourth Equation, after due Transposition, this arises . . . . $y = \frac{5e}{2} - 22$
6. Then instead of y in the first Equation, if there be set the later part of the fifth, the first will be reduced to this, $a+e+\frac{5e}{2}-22=40$
7. The sixth Equation, after due Reduction, gives . . $a = 62 - \frac{7e}{2}$
8. By the later part of the fifth Equation it's evident that $\frac{5e}{2} \sqsubset 22$
9. And consequently by multiplying each part in the eighth step by 2, ... $5e \sqsubset 44$
10. Whence by dividing each part by 5, it follows that $e \sqsubset 8\frac{4}{5}$
11. Again, from the later part of the seventh Equation, by arguing in like manner as in the eighth, ninth and tenth steps, it will appear that . . . . $e \sqsupset 17\frac{5}{7}$

12. Now

12. Now ſince the nature of this Queſtion requires that the deſired value of a, e and y be whole numbers, it's evident from the fifth and ſeventh ſteps that e muſt be an even number, otherwiſe $\frac{5e}{2}$ and $\frac{7e}{2}$ will not be whole numbers; for if e be an odd number, the Dividends $5e$ and $7e$ will be odd, (for odd multiplied by odd produces odd) and therefore their halves cannôt be whole numbers. Since then e muſt be an even number, it's manifeſt by the tenth and eleventh ſteps, that e may be 10, or 12, or 14, or 16, but no other even number whatever; and conſequently from the fifth ſtep y ſhall be 3, or 8, or 13, or 18; and from the ſeventh ſtep, a ſhall be 27, or 20, or 13, or 6. Thus it appears that the Queſtion may be ſolved by four ſeveral Anſwers (and not more) in whole numbers, *viz.* Firſt, 27 Patridges, 10 Larks, and 3 Quails, which are in multitude 40, may be bought for 98 pence at their reſpective prices given in the Queſtion; or 20 Partridges, 12 Larks, and 8 Quails, which are likewiſe 40 in Multitude, and the like may be affirmed of the other two Anſwers inſerted in the Table in the Margin.

Partr.	Larks.	Quails.
a	e	y
27	10	3
20	12	8
13	14	13
6	16	18

But if a Queſtion of the ſame nature be deſired that has but one anſwer in whole numbers, the following Epigram (cited by Monſieur *Bachet* in his Comment upon the one and fortieth Queſtion of the fourth Book of *Diophantus*,) will be ſatisfactory.

QUEST. 18.

Ut tot emantur aves, bis denis utere nummis;
Perdix, Anſer, Anas empta vocetur avis.
Sit ſimplex obolus pretium Perdicis, ematur
Sex obolis Anſer, biſque duobus Anas.
Ut tua procedat in lucem quæſtio, mentem
Conſule, ſic loquiter pectoris arca mihi.
Sint Anates tres atque duæ, ſimplex erit Anſer,
Accippe Perdices quatuor atque decem.

The ſence is this: If the price of a Patridge be an half penny, a Gooſe 3 pence, and a Duck 2 pence; how many of each kind may be bought at thoſe rates, if it be deſired that all the Birds bought may be 20 in number, and coſt 20 pence?

Let a repreſent the number of Patridges, e the number of Geeſe, and y the number of Ducks, then this Queſtion (like the preceding ſeventeenth) may be ſtated thus:

1. If . . . . $a+e+y=20$
2. And . . . . $\frac{1}{2}a+3e+2y=20$

What are the whole Numbers a, e and y?

RESOLUTION.

3. The firſt Equation multiplied by $\frac{1}{2}$, produces . . . $\frac{1}{2}a+\frac{1}{2}e+\frac{1}{2}y=10$
4. The third Equation ſubtracted from the ſecond, leaves . $\frac{5e}{2}+\frac{3y}{2}=10$
5. By tranſpoſition of $\frac{3y}{2}$ in the fourth Equation, this ariſes . $\frac{5e}{2}=10-\frac{3y}{2}$
6. The fifth Equation divided by $\frac{5}{2}$, gives . . . . . . $e=4-\frac{3y}{5}$
7. By ſetting the later part of the fixth Equation in the place of e in the firſt, this ariſes . . . . $a+4-\frac{3y}{5}+y=20$
8. Which laſt Equation, after due Reduction, gives . $a=16-\frac{2y}{5}$
9. From the later part of the fixth Equation it may be inferr'd, (in like manner as in divers of the preceding Queſtions) that . . . . $y \sqsupset 6\frac{2}{3}$
10. But the fixth and eight ſteps do ſhew, that to the end the values of e and a may be whole numbers, as the nature of this Queſtion requires, it is requiſite that $\frac{3y}{5}$ and $\frac{2y}{5}$

be

be whole numbers; by $\frac{3y}{5}$ and $\frac{2y}{5}$ cannot be whole numbers, unless y be 5 or some Multiple of 5; and by the ninth step y must be less than $6\frac{2}{3}$, therefore 5 is the only whole number that can be taken for y, or the number of Ducks; and consequently the sixth step gives 1 for the value of e, that is, 1 Goose; and by the eighth step, the value of a is 14, that is, 14 Partridges; which three numbers will solve the Question, as may easily be proved.

The Resolutions of the following nineteenth and twentieth Questions do shew how to find out innumerable Answers to any Question belonging to the Rule of Alligation alternate *in vulgar Arithmetic, when three or more things are to be mixed together, according to the import of that Rule.*

QUEST. 19.

A Vintner having three sorts of Wines, the prices whereof *per* Gallon are 24 pence, 22 pence, and 18 pence, desires to make a Mixture out of them that may contain 60 Gallons, in such manner, that the total Mixture being sold at some mean price *per* Gallon between 24 pence and 18 pence, suppose at 20 pence, may make the same sum of Money, as all the particular quantities of Wine in the Mixture at their own prices. The Question is, to find what quantity of each sort of Wine may be taken to make that Mixture.

For the desired number of Gallons of the first sort of Wine to make the Mixture, put a; for the number of the second sort e; and of the third y: Then $a+e+y=60$, (the total number of the Gallons in the Mixture;) and because every Gallon of the mix'd quantity must be sold for 20 pence, the 60 Gallons mix'd are worth 1200 pence, and so much also must all the Products of the particular Quantities of each sort of Wine multiplied by their peculiar prices amount unto; therefore $24a+22e+18y=1200=60\times20$. So that the Question may be stated thus:

1. If . . . . $a+e+y=60$
2. And . . . . $24a+22e+18y=1200\ (=60\times20)$

What are the numbers a, e, y?

RESOLUTION.

3. The first Equation multiplied by 24 (which is prefix'd to a in the second Equation) produces . . . . $24a+24e+24y=1440$
4. The second Equation subtracted from the third, leaves . . . . $2e+6y=240$
5. The fourth Equation by transposition of $6y$, gives . . . . $2e=140-6y$
6. The fifth Equation divided by 2, gives . . $e=120-3y$
7. By taking the later part of the sixth Equation instead of e in the first, this arises, $a+120-3y+y=60$
8. The seventh Equation, after due Reduction, discovers the value of a, viz. $a=2y-60$
9. From the 8*th* Equation it's evident that $y \sqsubset 30$
10. And from the sixth Equation, . . $y \sqsupset 40$
11. By the 10*th*, 9*th*, 8*th* and 6*th* steps it's manifest that innumerable Answers may be given to the Question proposed; for since Fractions are not here excluded from being Answers, you may esteem . $y =$ any number between 30 and 40;
$a = 2y-60$;
$e = 120-3y$.
12. Whence nine Answers in whole numbers are discovered, to wit, those exprest in this Table. But the Rule of Alligation in Vulgar Arithmetic finds out only one Answer to this Question, to wit, the sixth. And because innumerable Numbers may be taken between 30 and 40 for values of y, you may find out as many Answers as you please in Fractions, (which are not excluded in Questions of this Nature;) so if for y you take $30\frac{1}{2}$, then $a=1$, $(=2y-60,)$ and $e=28\frac{1}{2}$, $(=120-3y.)$

a	e	y
2	27	31
4	24	32
6	21	33
8	18	34
10	15	35
12	12	36
14	9	37
16	6	38
18	3	39

The

The Proof of the firſt Anſwer.

Two Gallons of Wine at 24 pence *per* Gallon, together with 27 Gallons at 22 pence *per* Gallon, and 31 Gallons at 18 pence *per* Gallón, amount to 1200 pence; which is alſo the value of 60 Gallons at 20 pence *per* Gallon.

QUEST. 20.

A Vintner having four ſorts of Wines, whoſe prices *per* Quart are 16 pence, 10 pence, 8 pence, and 6 pence, deſires to make a Mixture out of them that may contain 100 Quarts, ſo as this mixt quantity being ſold at ſome mean price *per* Quart between 16 pence and 6 pence, ſuppoſe at 12 pence, may produce the ſame ſum of money, as all the particular quantities of Wine in the Mixture if they were ſold at their own prices. The Queſtion is, to find what quantity of Wine of each ſort may be taken to make that Mixture?

Let a, e, y and u be put for the unknown quantities of Wine that are ſought to make the Mixture; then $a+e+y+u=100$, (the total number of Quarts in the Mixture,) and by multiplying thoſe Quantities ſeverally into their peculiar prices, the ſum of the Products is $16a+10e+8y+6u$; which ſum muſt be equal to the Product of 100 multiplied into 12, that is, 1200 pence; So that the Queſtion may be ſtated thus;

1. If . . . . $a+e+y+u=100$
2. And . . . . $16a+10e+8y+6u=1200$

What are the Numbers a, e, y and u?

The given Equations being fewer in multitude than the numbers ſought, it's a ſign that the Queſtion is capable of innumerable Anſwers; now that you may find out as many of them as you pleaſe, the firſt ſcope in the Reſolution muſt be to diſcover limits to direct your choice of ſome one of the numbers ſought, and accordingly, the drift in the eight Equations next following is to ſearch out limits for the firſt number a.

RESOLUTION.

3. From the firſt Equation by tranſpoſition of a, this ariſes, ... $e+y+u=100-a$
4. And from the ſecond Equation by tranſpoſition of $16a$, this ariſes, ... $10e+8y+6u=1200-16a$
5. The third Equation multiplied by 6, to wit, the leaſt of the known numbers which are prefix'd to the letters in the firſt part of the fourth Eqation, produces . . . . $6e+6y+6u=600-6a$
6. Again, the third Equation multiplied by 10, that is, the greateſt of the known numbers which are prefix'd to the letters in the firſt part of the fourth Equation produces . . . . $10e+10y+10u=1000-10a$
7. It is manifeſt that the firſt part of the fifth Equation is leſs than the firſt part of the fourth, therefore alſo the later part of the fifth ſhall be leſs than the later part of the fourth, *viz.* . . $600-6a \sqsupset 1200-16a$
8. Therefore from the ſeventh ſtep, after due Reduction, it follows, that . . . . . . $a \sqsupset 60$
9. Again, for as much as the firſt part of the ſixth Equation is greater than the firſt part of the 4*th*, therefore alſo the later part of the ſixth ſhall be greater than the later part of the fourth, *viz.* . . $1000-10a \sqsubset 1200-16a$
10. Therefore from the ninth ſtep, after due Reduction, it follows, that . . . . $a \sqsubset 35\frac{1}{3}$

Now ſince it is found by the eighth and tenth ſteps, that a the number of Quarts ſought of the firſt ſort of Wine to make the Mixture muſt be leſs than 60, but greater than $33\frac{1}{3}$, let ſome number within thoſe limits be taken for the value of a, viz.

11. Suppoſe

11. Suppose . . . . $47=a$

12. Then by setting 47 in the place of a in the first Equation, this arises . . . . . . $47+e+y+u=100$

13. Whence by equal subtraction of 47 there remains . . . . $e+y+u=53$

14. And by multiplying the Equation in the eleventh step by 16, (the number prefix'd to a in the second,) it gives . . . . . . $752=16a$

15. Then by setting 752 in the place of $16a$ in the second Equation, this arises . . $752+10e+8y+6u=1200$

16. And by subtracting 752 from each part of the Equation in the fifteenth step, this remains, *viz.* . . . . $10e+8y+6u=448$

17. The Equation in the thirteenth step multiplied by 10, (which is prefix'd to e in the sixteenth,) produces . . . . $10e+10y+10u=530$

18. Then by subtracting the Equation in the sixteenth step from that in the seventeenth, the letter e vanishes, and this Equation remains, *viz.* . . . . $2y+4u=82$

19. From the eighteenth step by transposition of $+4u$, this Equation arises, ... $2y=82-4u$

20. And by dividing each part of the Equation in the nineteenth step by 2, it gives . . . $y=41-2u$

21. Then by seting the latter part of the Equation in the twentieth step in the place of y in the thirteenth step, it makes . . . . . $e+41-2u+u=53$

22. Whence, after due Reduction, ... $e=u+12$

23. By the later part of the Equation in the twentieth step, it's evident that $2u \sqsupset 41$, therefore . . . . $u \sqsupset 20\frac{1}{2}$

And because the known number 12 which follows $+u$ in the twenty second step, (expressing the value of e) is Affirmative, there is not any limit to shew above which the number u ought to be taken; and therefore, according to the three and twentieth step, u may be any number less than $20\frac{1}{2}$: Therefore,

24. Suppose . . . . $u=20$

25. Then from the twentieth and twenty fourth steps it follows, that . . . . $y=1, (=41-2u)$

26. And from the twenty second and twenty fourth steps, ... $e=32, (=u+12)$

Thus by the eleventh, twenty sixth, twenty fifth and twenty fourth steps; four whole numbers are discovered, to wit, 47, 32, 1 and 20 for the values of a, e, y, and u, which numbers will solve the Question. For if 42 Quarts of the first sort of Wine, 37 Quarts of the second, 1 quart of the third, and 20 of the fourth be mixed together, the sum makes 100 quarts, which at 12 pence *per* quart yields 1200 pence; and the same number of pence will be produced by selling 47 quarts at 16 pence *per* Quart, 32 quarts at 10 pence, 1 quart at 8 pence, and 20 quarts at 6 pence; which was required.

But because (by the twenty third step) u may be any whole number less than $20\frac{1}{2}$, nineteen Answers more in whole numbers may be found out by repeating the Process in the twenty fourth, twenty fifth and twenty sixth steps; so that 47 being taking for a, there will be twenty Answers in whole numbers, which are inserted in the following Table. And by putting a equal to every whole number severally between $33\frac{1}{3}$ and 60, which are the limits discovered in the eighth and tenth steps, for the chusing of the number a, after a due repetition of the Process with every one of those whole numbers, in like manner as before with 47 from the eleventh step to the end of the Resolution, two hundred ninety four Answers more in whole numbers will be discovered, which with those twenty in the Table make three hundred and fourteen Answers in whole numbers to this twentieth Question,

Queſtion, to which the Rule of *Alligation* in Vulgar Arithmetic gives only one Anſwer, which conſiſts partly of Fractions too; but by the Method above deliver'd, innumerable Anſwers may be found out in Fractions. The Table follows.

a	e	y	u
47	32	1	20
47	31	3	19
47	30	5	18
47	29	7	17
47	28	9	16
47	27	11	15
47	26	13	14
47	25	15	13
47	24	17	12
47	23	19	11
47	22	21	10
47	21	23	9
47	20	25	8
47	19	27	7
47	18	29	6
47	17	31	5
47	16	33	4
47	15	35	3
47	14	37	2
47	13	39	1

QUEST. 21.

Forty-one perſons conſiſting of Men, Women and Children, ſpent in the whole at a Feaſt 40 Shillings; whereof every Man paid 4 Shillings, every Woman 3 Shillings, and every Child 4 pence, or $\frac{1}{3}$ of a Shilling: It's deſired to find the number of Men, likewiſe of the Women and Children.

The Nature of this Queſtion not admitting Fractions in the Anſwer, the ſcope of the Reſolution muſt be to divide 41 into three ſuch whole Numbers, that if the firſt be multiplied by 4, the ſecond by three, and the third by $\frac{1}{3}$, the Sum of the three Products may make 40: To which purpoſe, let a, e and y be put for the deſired numbers of Men, Women and Children, and then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $a + e + y = 41$
2. And . . . . $4a + 3e + \frac{1}{3}y = 40$

What are the whole numbers a, e, y? ||

RESOLUTION.

3. By forming the Reſolution in like manner as in the foregoing thirteenth, fourteenth and fifteenth Queſtions it will appear, that . . . .

$$\begin{cases} y \sqsubset 31\frac{1}{8}, \\ y \sqsupset 33\frac{9}{11}, \\ e = 124 - \dfrac{11y}{3}, \\ a = \dfrac{8y}{3} - 83. \end{cases}$$

Whence 'tis manifeſt that 32 and 33 are the only whole Numbers within the Limits for the chuſing of the Number y, but this muſt neceſſarily be a Multiple of 3, otherwiſe $\frac{11y}{3}$ and $\frac{8y}{3}$ will not be whole Numbers, and conſequently the values of e and a above expreſs'd cannot be whole Numbers; therefore 33 is the ſole whole Number that can be taken for the value of y, to wit, the number of Children, and conſequently the values of e and a above expreſs'd will give 3 for the number of Women, and 5 for the number of Men: which three numbers 5, 3 and 33 will ſolve the Queſtion, for their ſum is 41; and if the firſt be multiplied by 4, the ſecond by 3, and the third by $\frac{1}{3}$, the ſum of the three Products is 40, as was required.

QUEST. 22.

Twenty persons, consisting of Men, Women, Boys and Girls spent at a Feast in the whole 94 Shillings; whereof every Man paid 6 Shillings, every Woman 4 Shillings, every Boy 3 Shillings, and every Girl 1 Shilling: It's desired to find out the number of Men, likewise of Women, Boys and Girls.

The scope of this Question is to find out four such whole numbers that their sum may make 20; and that if the first be multiplied by 6, the second by 4, the third by 3, and the fourth by 1, the sum of the four Products may make 94; therefore by putting a, e, y, u, to represent those four whole numbers, the Questions may be stated thus;

1. If . . . . $a+e+y+u=20$
2. And . . . . $6a+4e+3y+u=94$

What are the whole Numbers a, e, y, u?

RESOLUTION.

The first Scope is to search out Limits for the Number a in like manner as before in the twentieth Question, *viz.*

3. By transposition of a in the first Equation, this arises, $e+y+u=20-a$
4. Likewise by transposition of $6a$ in the second Equation, there comes forth . . . . $4e+3y+u=94-6a$
5. The third Equation multiplied by 1, (to wit, the smallest of the Numbers prefix'd to the Letters in the first part of the fourth Equation, where 1 is supposed to be prefix'd to u,) does produce the same third, *viz.* $e+y+u=20-a$
6. Again, the third Equation multiplied by 4, to wit, the greatest of the Numbers prefix'd to the Letters in the first part of the fourth Equation, does produce $4e+4y+4u=80-4a$
7. It is manifest that the first part of the fifth Equation is less than the first part of the fourth, therefore also the later part of the fifth shall be less than the later part of the fourth, *viz.* . . . . $20-a \sqsupset 94-6a$
8. Therefore from the seventh step, after due Reduction, it follows that . . . . $a \sqsupset 14\frac{4}{5}$
9. Again, forasmuch as the first part of the sixth Equation is greater than the first part of the fourth, therefore also the later part of the sixth shall be greater than the later part of the fourth, *viz.* . . . . . . $80-4a \sqsubset 94-6a$
10. Therefore from the ninth step, after due Reduction, it follows, that . . . . $a \sqsubset 7$

Now since 'tis found by the tenth and eigth steps, that a, (or the number of Men) is greater than 7, but less than $14\frac{4}{5}$, let some whole number within those Limits be taken for the value of a, *viz.*

11. Suppose . . . . $12=a$
12. Then by setting 12 in the place of a in the first Equation, this arises, ... $12+e+y+u=20$
13. Whence by equal subtraction of 12, there remains . . $e+y+u=8$
14. And by multiplying the Equation in the eleventh step by 6, it makes . . . . $72=6a$
15. Then by setting 72 in the place of $6a$ in the second Equation, it gives . . . . $72+4e+3y+u=94$
16. And by subtracting 72 from each part of the last Equation, the Remainder is . . . . $4e+3y+u=22$
17. The Equation in the thirteenth step being multiplied by 4, (which is prefix'd to e in the sixteenth) gives . . $4e+4y+4u=32$
18. Then by subtracting the Equation in the sixteenth step from that in the seventeenth, the Letter e vanishes, and this Equation remains, ... $y+3u=10$

19. Whence

19. Whence by tranſpoſition of $3u$, this Equation ariſes, ... $y = 10 - 3u$

20 Then by ſetting the later part of the Equation in the nineteenth ſtep in the place of y in the thirteenth, this ariſes, ... $e + 10 - 3u + u = 8$

21. Whence, after due Reduction, this Equation ariſes, $e = 2u - 2$

22. From the later part of the nineteenth Equation, it may be infer'd that . . . . $u \sqsupset 3\frac{1}{3}$

23. And from the later part of the twenty firſt Equation, $u \sqsubset 1$

Now ſince by the twenty ſecond and twenty third ſteps, u (or the number of Girls) is found to fall between 1 and $3\frac{1}{3}$ let 2 be taken for the value of u, *viz.*

24. Suppoſe . . . . $u = 2$

25. Then from the nineteenth and twenty-fourth ſteps, $y = 4$ $(= 10 - 3u)$

26. And from the twenty firſt and twenty fourth ſteps, $e = 2$ $(= 2u - 2)$

Thus by the eleventh, twenty ſixth, twenty fifth and twenty fourth ſteps, four whole numbers are diſcovered, to wit, 12, 2, 4 and 2, for the values of a, e, y and u.

Again, by taking 3 for the value of u, (which is within the Limits before diſcovered) the nineteenth and twenty firſt ſteps will diſcover 1 and 4 for the values of y and e, (a being 12, as before. Wherefore two Anſwers to the Queſtion are found out; for the number of Men being put 12, the number of Women will be 2, the number of Boys 4, and the number of Girls 2; or the number of Men being 12 as before, there will be four Women, 1 Boy and 3 Girls. Again, if 11 be put equal to a, (or the number of Men,) and the proceſs be repeated from the eleventh ſtep to the end of the Reſolution, there will be found two Anſwers more in whole numbers. In like manner, if 9, 10 and 13 be ſeverally be put equal to a, three Anſwers more will be diſcovered; But if 8 and 14 be ſeverally put equal to a, altho they be within the Limits in the eighth and tenth ſteps, yet the work being repeated as before will not ſucceed to find e, y and u in whole numbers; ſo that there are only ſeven Anſwers, to wit, thoſe inſerted in the Table; but that every one of them will ſolve the Queſtion may eaſily be proved.

a	e	y	u
9	9	1	1
10	6	3	1
11	5	2	2
11	3	5	1
12	2	4	2
12	4	1	3
13	1	3	3

If a Queſtion of this nature be deſired that has but one Anſwer in whole numbers, let the number of perſons be 60, and 100 the number of Shillings ſpent; alſo let every Man ſpend 2 Shillings, every Woman $\frac{2}{3}$ of a Shilling, every Boy $\frac{3}{5}$ of a Shilling, and every Girl $\frac{1}{2}$ of a Shilling; then by forming the Reſolution as before, the number of Men will be found 46, the number of Women 3, the number of Boys 5, and the number of Girls 6.

QUEST. 23.

To divide 200 into five ſuch whole numbers, that if the firſt be multiplied by 12, the ſecond by 3, the thitd by 1, the fourth by $\frac{1}{2}$, and the fifth by $\frac{1}{3}$, the Sum of the Products may alſo make 200.

This Queſtion may be reſolved like the foregoing twentieth and twenty ſecond, but I ſhall leave it as an exerciſe to the induſtrious Analyſt, who (if he thinks it to be worth his pains,) may find out 6639 Anſwers to it in whole numbers, (as Monſieur *Bachet*, in the two laſt Pages of his little Book before cited in *Sect.* 1. of this Chapter, does affirm.

Nicholas Tartaglia handling this Queſtion, (which is the laſt of the ſeventeenth Book of the Firſt Part of his Arithmetic,) thought it a great matter that he had found out one ſingle Anſwer to it in theſe five whole numbers, to wit, 6, 12, 34, 52, 96, and aſſerted, That Queſtions of this ſort could not be perfectly ſolved, either by the *Algebraical Art*, or any certain *Rule*; but the Contents of this Chapter do manifeſtly ſhew, that the Imperfection was in the Artiſt, and not in the Art.

The End of the First Volume.

APPENDIX.

Lectures read in the

School of Geometry

IN

OXFORD,

CONCERNING

The *Geometrical* Construction of Algebraical Equations; And the *Numerical* Resolution of the same by the *Compendium* of Logarithms.

LECTURE I.

I Have oftentimes experienc'd, on several Occasions, how difficult a thing it is to Discourse, especially of Mathematical Matters, so as to please the Learned therein, and at the same time to Instruct such as yet want to be taught: The former require nothing but what is New and Curious, nor are pleas'd but with Elegant Demonstrations, made Concise by Art and Pains: The later demand Explications drawn out in Words at length, least any part of the Reasoning not being clearly apprehended, shou'd hinder the Evidence of the whole Argument; whilst those already vers'd in Mathematics cannot endure such Prolixity.

But seeing, according to the Intent of the Noble Sir *Henry Savil*, the Mathematic Studies of the Junior Academics are committed to the Care of his Professor of Geometry; I thought it fit to consult, not so much my own Reputation, as the Profit of the Auditory: Omitting therefore what might make a shew of deeper Learning, the Geometric Construction of Analytic Equations shall be the Subject of these Lectures: 'Tis, indeed, a common one, and treated of by Authors of great Note; and on that Account, perhaps, I may seem to do no more than the same thing over again. But having some Grounds to think I have added something of my own, whereby these Constructions may be perform'd with all possible Facility, and having likewise extended them to Equations of Six Dimensions, without any Reduction; I don't doubt but that, as it will be of Advantage to Studious Learners, so it may not be unacceptable to Mathematicians of a higher Class.

For our Method needs no Preparation of the Equation, requiring only the Bisection of the given Co-efficients: Whereas the Construction of Equations of five or six Dimensions, that Mr. *Des Cartes* gives at the end of his Geometry, requires the labour of an intollerable *Calculus*; and contrary to the Tenor of his own Rules, he makes use of a Curve-Line, than which there is scarce another that is more Compounded, among all those of the second Kind, (lately enumerated by the great Sir *If. Newton*) which from its Tricuspid Form is by him called *Tridens*.

What serves our purpose is only one Invariable Curve, and that also the most simple of its kind, *viz.* a *Cubic Paraboloid*, or that wherein the Cubes of the *Ordinates* are to one another as their respective *Abscissa's*: which Curve being once described may serve instead of an Instrument for the Construction of any such Equation; and the Roots will be had by means of the Intersections of this Curve and a

 Conic-

Conic-Section, whose Position is readily defin'd by the Co-efficients of the given Equation, and thence easy to be describ'd.

They are undoubtedly in the right, who require in *Geometric* Problems, a Geometrical Construction by Lines, such as we are about to shew; and in *Arithmetical* ones, an Arithmetical Effection, *i. e.* by Numbers or Calculation. But these Sciences being very near a-kin, give mutual Assistance to one another; so that whenever 'tis requir'd, that any thing in *Geometry* shou'd be more accurately determin'd, no Mathematician will undertake to do it by a Rule and Compass (because of the defect of Instruments, and of our Senses, whereby the Intersections of Lines imperfectly drawn, are yet more imperfect) but he will give a Solution as near the Truth as you please, by an Arithmetic *Calculus*, according to an Equation determining the Nature of the Problem.

To this end I have formerly, (in *Philos. Transact.* Numb. 210) Publish'd a general Method of Calculation, which is sufficiently Compendious: But that *Calculus* seems to be something Defective in higher Equations, explicable by many Roots, and those not bounded within narrow Limits: For this way we come at the true quantities of the Roots only by Trial, and Correcting of Errors, much after the manner of the Rule of false Position. On the contrary, a Geometric Construction rightly manag'd lays open the whole Mystery in a short view, and at once shews directly as well the Number and Quantities of the Roots, as their Signs, *viz.* whether they be Affirmative or Negative: And then the Measure of any Root being taken out of the Scheme, as not much differing from the Truth, may presently be verified by the help of the aforementioned *Calculus*, to what Number of Places you please: And this is one Notable Use (if not the chief) of these Constructions.

That these Constructions, therefore, might be perform'd with the greatest Facility and Ease, we must consider, that all Problems determin'd by *Simple Equations*, and which may be resolv'd by the common Rules of Arithmetic, *viz* Addition, Subduction, Multiplication, and Division, or by any Operations any way Compounded of them, require only *Right-Lines* to Construct them.

But *Plane Equations*, *viz.* such as involve the Square of the Quantity sought, and are solv'd Arithmetically by extracting the Square Root, require, besides *Right-Lines*, some Curve of the *Conic-Sections*, to Construct them: Among which Curves, the *Circle*, for the Facility of its Description, is look'd on as the most simple; and next it the *Parabola*, which, indeed, from the Nature of its Equation, is more simple than the Circle it self: But seeing it cannot be describ'd but by Points, and the uncertain Motion of the Hand, the Antients hardly admitted it into their Geometry; and would scarce allow that to be Geometrically effected, which could not be describ'd by the help of the Compasses: Whence that Famous Disquisition, concerning the Duplication of the Cube Geometrically came to nothing: Seeing they attempted to solve a *Solid Problem* by the Geometry of Planes.

But the Modern Mathematicians, in this Business, exclude no Curves, provided it be certain that the Thing propos'd cannot be done without them, or by more simple ones: And 'tis a Fault, if, without necessity require it, you make use of a *Parabola* instead of a *Circle*, or an *Ellipse* or *Hyperbola* instead of a *Parabola*; consequently, a *Circle* only can have place in the Construction of *Plane Problems.*

But if there are three or four Dimensions of the Quantity sought in the Equation; besides a Circle, a *Parabolic* Curve is most commodiously made use of: which, together with the Circle, will construct all Cubic and Biquadratic Equations, with the greatest ease imaginable.

And, admitting the *Parabola* described, nothing is more facil than, *The Duplication of the Cube*, *Trisection of an Angle*, and *the finding of Two or Three Mean Proportionals*, &c. nor as yet is there any need of an *Ellipse* or *Hyperbola*, unless, in the Problem to be solved, that Conic-Section be given; But any *Parabola* once accurately describ'd, and cut in Brass, or the like, will serve instead of an Instrument for the Construction of Solid Equations; which is a *Compendium* by no means to be slighted.

If

If there be five or six Dimensions of the Quantity sought in the Equation, the Conic-Sections alone are not sufficient, therefore the Assistance of some Curve of the *Second Kind* must be had, of which, as I said before the *Cubic Paraboloid* is the most simple; This Curve, combined with some one of the Conic Sections, will Construct all *Sursolid* (as they are called) and *Quadrato-Cubic* Equations, however affected. And this *Paraboloid* once rightly describ'd, and cut in Brass, will be ready at hand for the Solving of all such Equations of five or six Dimensions.

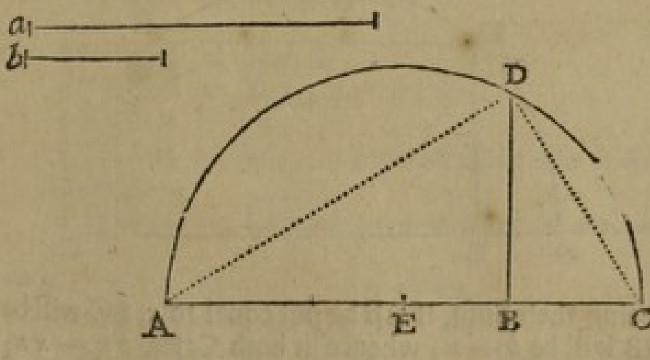
But if the Equation propos'd be of a higher Degree, suppose 7, 8, or 9 Dimensions, there will be need of some other of those seventy two Curves of the *Second Kind*, enumerated by the Illustrious Sir *Is. Newton*; but which of them it must be, and in what Situation or Position to be applied, will depend on the Co-efficients of the given Equation: and the Intersections of that Curve with the *Cubic Paraboloid* (whereof there may be nine) will design all the Roots of the Equation. But seeing we have not as yet thorowly attain'd to all the Properties and Descriptions of these new Invented Curves, we shall at present content our selves with constructing all Equations under those of seven Dimensions in as clear a Method as may be.

These things being premis'd in general, let us come to the thing it self: And first of all, as to *Simple Equations*, that are constructed by *Right-Lines* only; These require no more than the first Rudiments of *Geometry*, namely, to exhibit the Sum or Difference of given Right-Lines: To find a fourth Proportional to three given Right-Lines: To cut a given Right-Line in a given *Ratio*, and the like: Which, as they contain no manner of difficulty to any tho' never so little vers'd in the Elements of *Euclid*, I shall therefore leave, as more proper, to each Person's private Study and Exercise, and shall take no farther notice of them.

But *Plane Equations*, or (as they are now commonly called) *Quadratics*, *viz.* such as contain the Square of the Line sought, require a Circle, as was said before, to construct them: And after a due Reduction, will all be in some one of these Forms, *viz.*

1. $xx = ab$
2. $xx + bx = aa$
3. $xx - bx = aa$
4. $bx - xx = aa$

In the First, where the Square of the unknown Quantity x is equal to the Rectangle ab, the *Quadratic Equation* is said to be *Pure*, and x the Quantity sought, is a Mean Proportional between a and b; and consequently, is constructed by 13: El. 6. of *Euclid*, thus,



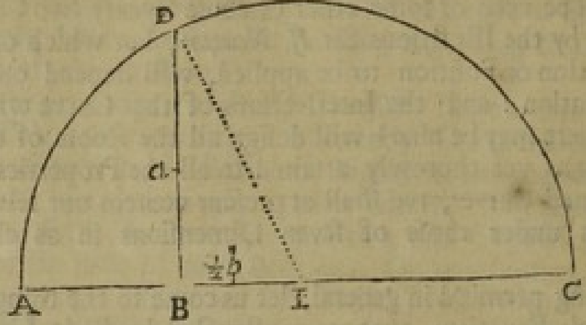
Make the Right Lines AB, BC, equal to the Lines or Quantities a, b; Bisect AC in E: from E, as a Centre, with the distance AE or CE, describe a Semicircle ADC. Then on the Point B, erect BD Perpendicular to AC, which will Intersect the Semicircle in D: I say, BD is the mean Proportional sought or x.

For the Triangles ADB, DBC, are similar by 31 El. 3 *Euclid*. Consequently AB : BD : : BD : BC, wherefore the Square of BD or xx is equal to the Rectangle AB×BC or ab, by 17 El. 6 *Euclid*. Which was to be done.

The

The three other *Quadratic Equations* are called *Affected Equations*; of which the ſecond and third Forms have the ſame way of Conſtruction; For whether $xx + bx$, or $xx - bx$ be equal to the Square of a, the Quantity b is every where the difference of the two Extremes, between which a is a mean Proportional; ſince x is to a, as a to $x + b$ in the ſecond Form, or $x - b$ in the third Form, by 17. El. 6. *Euclid.* Hence ariſes the Conſtruction.

Make $BE = \frac{1}{2}b$, and erect the Perpendicular DB, which make equal to a.

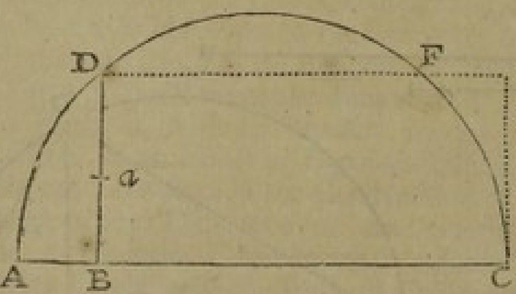


On E as a Centre, with the Radius DE, deſcribe a Semicircle ADC, interſecting the right Line BE, produc'd both ways, in the Points A and C; I ſay that the right Line AB is the Affirmative Root of the Equation $xx + bx = aa$, and BC that of the Equation $xx - bx = aa$; But BC is the Negative Root of the former, as AB is that of the later.

For ſeeing BE is half the Difference of the right Lines AB and BC, if AB be put for the Quantity x, BC will be $x + b$, and therefore the Rectangle $xx + bx$, or $AB \times BC$, will be equal to the Square of DB or a: In like manner, if BC be equal x, AB will be $x - b$, and conſequently, their Rectangle $xx - bx$ will be equal to Square of a. Wherefore the Conſtruction is right.

In the fourth Form, *viz.* $bx - xx = aa$, a is a mean Proportional between the extremes x and $b - x$; wherefore b is the Sum of the Extremes: Hence the Conſtruction may be perform'd after this manner.

Deſcribe a Semicircle, whoſe Diameter AC let be equal to b; draw DF a Parallel to AC, at the Diſtance $DB = a$: Which Parallel, if the Equation be poſſible, will interſect the Circle in the Points D and F; from the Point of interſection D, let fall the Perpendicular DB to the Diameter AC; I ſay, that both AB, and BC are Affirmative Roots of the Equation.



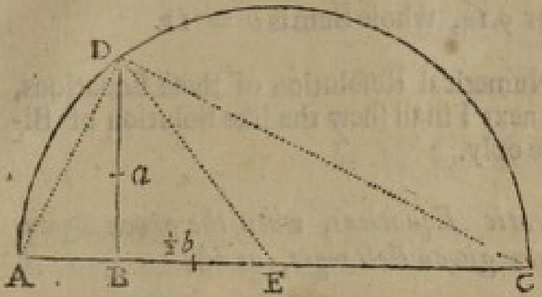
For AC or b being their Sum, if AB be put equal to x, BC will be equal to $b - x$; or if BC be x, AB will be $b - x$; whence in both Caſes, $bx - xx$, or the Rectangle $AB \times BC$, will be equal to aa, or the Square of DB. Which was to be done.

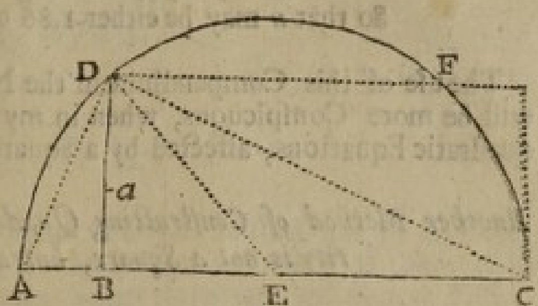
This laſt Equation ſometimes becomes Impoſſible, *viz.* when a is ſo great as that the Parallel DF does neither cut nor touch the Circle ADC, that is, when a is greater than $\frac{1}{2}b$: For a ought to be a Geometrical mean Proportional between the Parts of b, and conſequently leſs than an Arithmetical Mean, or $\frac{1}{2}b$; nor are they equal, except in the Caſe of Contact, where likewiſe x and a become equal.

Hence

Hence is discover'd a New, and, for its Facility, not unuseful Method for Resolving these sort of Equations, by the help of the Tables of Logarithms.

For the Second and Third Form. For the Fourth Form.





For, by (*Euclid El.* III. *Prop.* 20) the Angle DEA is double the Angle DCA or ADB; and if DB or a be made Radius, the Roots AB and BC will be Tangents of the Arcs that answer the Angles ADB, CDB, which together are equal to a Right-Angle, because in a Semicircle, (by *Euclid El.* III. *Prop.* 31;) Consequently, if, in the second and third Form, you make, as the half of b to a so the Radius to a Tangent; or in the fourth Form, so Radius to a Sine, the Arc answering thereto measures the Angle DEA, which having Bisected, you have the Angle ADB, whose Complement to a Quadrant is the Angle CDB; so that the Logarithmic Tangent of half the Arc AD Added to, and Subducted from, the Logarithm of BD or a, will give the Logarithms of both Roots.

Examples of the Praxis in Numbers.

Let the Roots of the Equation $xx \pm bx = aa$, (expounding b by 15 and aa by 175) be required.

Then $7\frac{1}{2} : \sqrt{175} :: $ Radius : Tang. 60° .27'
its $\frac{1}{2}$ = 30 .13½

Log. 175 = 2.243038

Log. $\sqrt{175}$ = 1.121519
Log. $7\frac{1}{2}$ = 0.875061

10.246458 = Tang. 60° .27'

And 1.121519 = Log. a.
9.765366 = Tang. 30° .13' ½.

Sum 0.886885 = Log. 7.7070 = Root of $xx + 15x = 175$.

Diff. 1.356153 = Log. 22.7070 = Root of $xx - 15x = 175$.

Again, let $bx - xx = aa$, be $11x - xx = 17$.
Then $5\frac{1}{2} : \sqrt{17} :: $ Radius : Sine of 48° .33' 40"
its $\frac{1}{2}$ = 24 .16.50

Log. 17 = 1.230449

Log. $\sqrt{17}$ = 0.615224
Log. $5\frac{1}{2}$ = 0.740363

9.874861 = Log. Sine 48° .33' 40"

And $9.654281 = \text{Log. Tang. } 24^\circ .16' .50''$
$0.615224 = \text{Log. } \surd 17.$

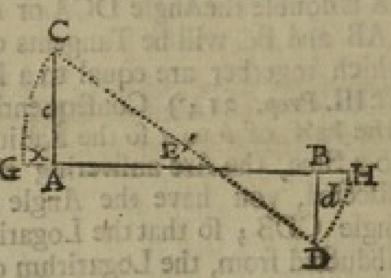
$$\left.\begin{array}{l}\text{Sum } 0.269505 = \text{Log. } 1.86 \textit{fere} = x \\ \text{Diff. } 0.960943 = \text{Log. } 9.14 \textit{ fere} = x\end{array}\right\} \text{fought.}$$

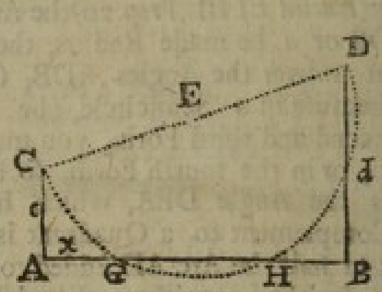
So that x may be either 1.86 or 9.14, whose Sum is $b = 11$.

The use of this Compendium in the Numerical Resolution of these Equations, will be more Conspicuous, when in my next I shall shew the like Solution of Biquadratic Equations, affected by a Square only.

Another Method of Constructing Quadratic Equations, when the given Quantity is not a Square, but any given Rectangle, as cd.

Case 1. $xx \pm bx = cd$. Case 2. $bx - xx = cd$.

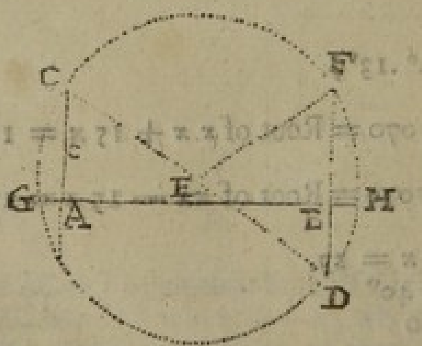


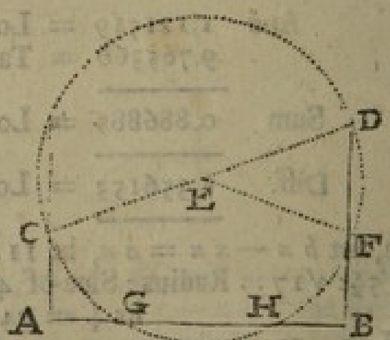


Let AB be made equal to b. On A and B erect the Perpendiculars AC and BD, make AC$=c$, BD$=d$, which, in Case 1, place the contrary ways, but, in Case 2, the same way with the Line AB: Joyn CD, and bisect it in E. With the Centre E, and Radius EC or ED, describe an Arc cutting the Line AB (produc'd in Case 1) in G and H; I say that AH and AG are the Roots of the propos'd Equation, *viz.*

In Case 1, AG is the Affirmative, AH the Negative Root of the Equation $xx+bx=cd$; but AH the Affirmative, and AG the Negative Root of the Equation $xx-bx=cd$; and in Case 2, AH and AG are the two Affirmative Roots of the Equation $bx-xx=cd$; where 'tis to be Noted, That if the Semicircle whose Diameter is CD, neither cut nor touch the Line AB, the Equation propos'd is impossible.

For since CA or c, and DB or d are at right Angles to the Right Line AB, and





the Centre E is equally distant from them, (by *Euclid.* III. 14.) the Right Line BF is equal to AC; therefore, the Rectangle cd, that is BD×CA, is equal to BD × BF, which (from the 35 and 36 III. Elem. *Euclid,*) is equal to the Rectangle BH × BG or AG × AH.

But by Construction AB $= b$ is equal (in Case 1) to the Difference of AG and AH, as (in Case 2) to their Sum; Wherefore cd is equal to $xx \pm bx$, in Case 1, and cd is equal to $bx - xx$ in Case 2. *Q. E. D.*

This,

This, 'tis probable, is the Method the Antients used, when by their *Analysis* they had a given Rectangle, the Sum or Difference of whose sides was known, and it was required to find the sides; which they called applying a Rectangle exceeding or deficient by a Square to a given right Line: Being but one particular Case of the more general Construction deliver'd by *Euclid*, Elem. VI. Prop. 28, 29.

Octob. 25, 1704.

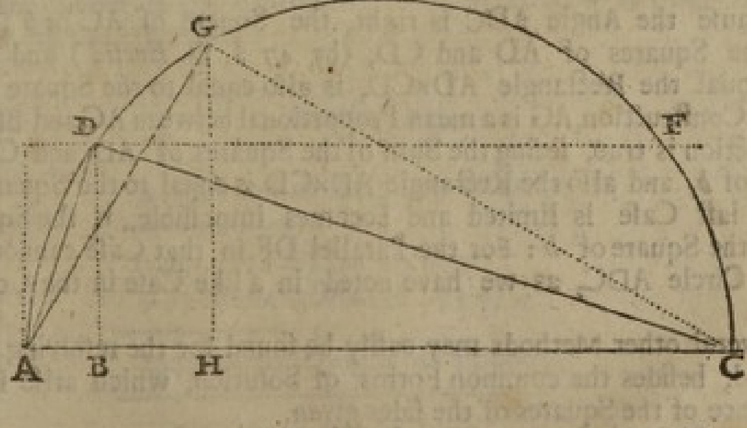
LECTURE II.

IN my last Lecture, I endeavour'd to shew you the Construction of all Equations of the Quadratic Form, and that by a Method which I think to be concise enough, *viz.* by finding the Extremes, when the Mean, and Sum or Difference of the extremes, of three continual Proportionals are given: And this is done agreeable to the Mind of the Antients, as you may see in the 84 and 85 Prop. of *Euclid's Data.*

At the same time I shew'd that those Equations might be resolv'd by a Logarithmic *Calculus*, *viz.* by the Bisection of an Angle. But before I pass to Cubic Equations, there occurs that *Species* of Biquadratics affected with a Square; which in its own Nature is really Quadratic, but whose Roots are not Lines, but Squares; and the Square being given, the Root is also given.

Now the Construction of any of these is as easy as that of simple Quadratics, on consideration that in the Equation where $x^4 \pm x^2 b^2 = d^4$, dd is a Mean Proportional between xx and $x^2 \pm b^2$: consequently $\overline{bb}$ is the given Difference between the two Extremes. But in the Equation where $b^2 x^2 - x^4 = d^4$, bb will be the Sum of the Extremes; wherefore, the Business comes to the same, as if the Problem were thus propos'd, *The Sum or Difference of two Squares, and the Rectangle of the sides being given, to find the sides.* Whence arises this Construction.

In the first Case, where bb is the Difference of the Squares; Describe a Semicircle, whose Diameter let be $AC = \sqrt{\sqrt{4d^4 + b^4}}$; in this Semicircle inscribe the Chord AG, which let be equal to d: Let fall the Perpendicular GH upon the Diameter AC; then AG or d will be a mean Proportional between AC and AH, because of the similar Triangles ACG and AGH: At the Distance BD, which let be equal to AH, draw DF Parallel to AC, cutting the Circle in the Point D: I say the Construction is finish'd; and that the Chords AD, CD, are the Roots x of the



Equation propos'd, namely, AD the Affirmative and CD the Negative, if the Product $bbxx$, in the Equation, be Affirmative, that is, if it be $+bbxx$; and on the contrary, CD will be the Affirmative Root and AD the Negative, if the said Product $bbxx$, in the Equation, be mark'd with the Sign $-$.

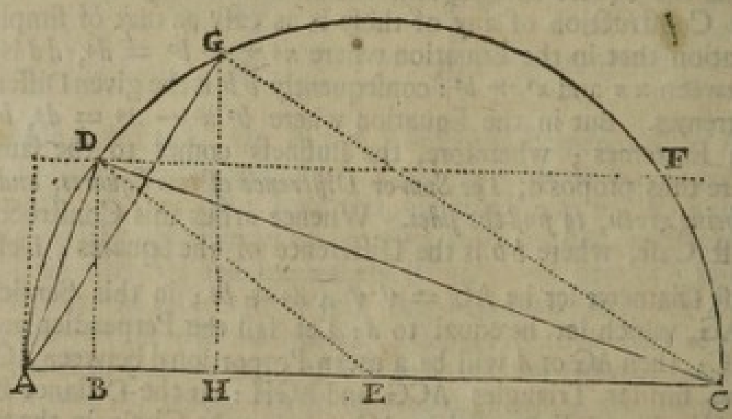
The

The Demonſtration is evident,becauſe in any two Quantities, the Square of the Sum exceeds the Square of the Difference, by four times the Rectangle of the Parts; and conſequently if to the Biquadrate of b, you add four times the Biquadrate of d, the Sum will be the ſquare of the Sum of thoſe ſquares of which bb is the Difference; therefore, the ſide of this ſquare, *viz.* $\sqrt{4d^4+b^4}$, will be the ſum of the ſquares of the Roots ſought, equal to the Square of AC.

Hence(by 47. I. El. *Euclid*) the Roots AD, CD will be the ſides of a Right angled Triangle, whoſe Hypotenuſe is AC, and conſequently are in the Semicircle ADC, (by 31. III. El. *Euclid*,) And ſeeing d is a mean Proportional between the Diameter AC, and the Perpendicular BD, the Rectangle AC×BD will be equal to the Square of d; but AC is to AD, as CD is to BD, becauſe of the ſimilar Triangles ACD, DCB, therefore the Rectangle AD×CD, equal to the Rectangle AC×BD, will be alſo equal to the Square of d; and the difference of the Squares of AD and CD being equal to bb, the Chords AD, CD are the Roots of the propos'd Equation: Conſequently the Conſtruction holds.

And that the Antients handled this Matter in a Method not much different from this, may be ſeen in the 87 Propoſition of *Euclid*'s *Data*.

But in the other Caſe, *viz.* where $d^4 = bbxx - x^4$, the Conſtruction is ſomewhat readier; becauſe bb is now become the Sum of the Squares of the ſides of which dd is the Rectangle; Conſequently, on AC, which let be equal to b, as a Diameter, deſcribe the Semicircle ADC. Let the Chord AG be equal to d. From G let fall the Perpendicular GH upon the Diameter AC: and at the Diſtance



BD, equal to AH, draw DF Parallel to AC, cutting the Circle in the Point D: I ſay the Chords AD, CD, exhibit, even in this Caſe, the Roots of the Equation propos'd, and that they are both Affirmative.

For, becauſe the Angle ADC is right, the Square of AC or b is equal to the Sum of the Squares of AD and CD, (by 47 I. El. *Euclid*) and the Rectangle AC×BD, equal the Rectangle AD×CD, is alſo equal to the Square of AG or d; becauſe by Conſtruction,AG is a mean Proportional between AC and BD. Wherefore the Conſtruction is true, ſeeing the Sum of the Squares of AD and CD is equal to the Square of b, and alſo the Rectangle AD×CD is equal to the Square of d.

But this laſt Caſe is limited and becomes impoſſible, if the Square of d exceeds half the Square of b: For the Parallel DF in that Caſe cannot ſo much as touch the Circle ADC, as we have noted in a like Caſe in the Conſtruction of Quadratics.

Hence ſeveral other Methods may eaſily be found for the reſolving of Equations of this Kind, beſides the common Forms of Solution, which ariſe from the Sum and Difference of the Squares of the ſides given.

In the ſecond Caſe, there is one which will certainly appear new, and no leſs fit for Practice; for becauſe bb is the ſum of the Squares, and dd the Rectangle of their ſides, $bb + 2dd$ will be the Square of the Sum of the Roots, and $bb - 2dd$ will be the Square of their Difference, by the 4th and 7th of the II. El. of *Euclid*, and conſequently half the Sum and half the Difference of the ſides of theſe Squares

will

will be the Roots of the Equation sought; both of which will be had by two Extractions of the Square Root; which is somewhat more compendious than the common Method.

The Bisection of an Angle gives us also two different Solutions, both of them commodious enough, and to be perform'd very easily by the Logarithms.

For if you make it, as half the Co-efficient bb to the Square of d, so the Radius, to the Tangent of the Angle DEA, in the first Case; or to its Sine, in the second Case: Bisect the Angle DEA, and you'll have the Angle DCA (by the 20th of III Elem. of *Euclid*) equal to the Angle ADB, and their Complements to a Quadrant will be equal to the Angles DAB, BDC. Consequently, if the Logarithm of the Square of d be increased and diminished by the Logarithm of the Tangent of the Angle ADB, the Sum and Difference will be Logarithms of the Squares of the Roots sought; Whence the halves of the said Logarithms will be the Logarithms of the Roots.

All these things clearly follow from what I have demonstrated in my former Lecture concerning Quadratics.

But the same may be obtain'd another way, by the Sines of the same Angle, and of its Complement to a Quadrant: For if you put the Diameter AC for the Radius of a Circle, the Roots AD, CD will be the Sines of the Angles DCA, DAC; and consequently are had by adding the Logarithms of those Sines to the Logarithm of $\sqrt{\sqrt{4dddd+bbbb}}$, in the First Case; or to the Logarithm of b, in the second Case. And I cannot easily believe, that Equations of this Power may be Constructed by fewer Lines, or resolved by an easier Arithmetic Operation.

Example 1. Let $x^4 \pm bbxx = d^4$ be $x^4 \pm 7xx = 145$.

Then $3\frac{1}{2} : \sqrt{145} :: \text{Radius} : \text{Tang. } 73°.47'.35''$
its half $= 36°.53'.47\frac{1}{2}''$

For Log. 145 2.161368

Log. $\sqrt{145}$ 1.080684
Log. $3\frac{1}{2}$ 0.544068

10.536616 = Tang. $73°.47'.35''$

Then, 1.080684 = Log. $\sqrt{145}$
9.875482 = Log. T. $36°.53'.47''\frac{1}{2}$

2) Sum = 0.956166 (0.478083 = Log. 3.00665 } The Roots sought.
2) Diff. = 1.205202 (0.602601 = Log. 4.00499 }

Whereof the Lesser is the Affirmative Root, if it be $+bb$; but the Greater, if it be $-bb$, in the Equation.

Another way.

2.7986506 = Log. 629 = $4d^4 + b^4$

0.6996626 = Log. $\sqrt{\sqrt{629}}$
9.7784204 = Sine $36°.53'47''\frac{1}{2}$

0.4780830 = Log. x = 3.00665

0.699663 = Log. $\sqrt{\sqrt{629}}$
9.902938 = Co-Sine, $36°.53'.47''\frac{1}{2}$

0.602601 = Log. x = 4.00499

Example 2. $7xx - x^4 = 10$.

Then $3\frac{1}{2} : \sqrt{10}$ :: Radius : Sine $64°.37'.23''$
its half $= 32.18.41\frac{1}{2}$

For Log. $\sqrt{10} = 0.500000$
Log. $3\frac{1}{2} = 0.544068$

$9.955932 =$ Sine $64°37'23''$

Then $9.727966 =$ Sine $32°.18'.41''\frac{1}{2}$
0.422549 Log. $\sqrt{7} = b$.

Sum $= 0.150515 =$ Log. $x = 1.41421 = \sqrt{2}$.

And $9.926936 =$ Co-Sine $32°.18'41''\frac{1}{2}$.
$0.422549 =$ Log. $\sqrt{7} = b$.

Diff. $0.349485 =$ Log. $x = 2.23607 = \sqrt{5}$.

Being about to ſhew the Conſtruction of Cubics and Biquadratics, in the next Lecture, 'twill be neceſſary that the young Student ſhould acquaint himſelf with ſuch Properties of the *Parabola*, as are deliver'd in the firſt Book of *Apollonius*'s Conics; and likewiſe conſult what is to be found of this matter in *Des Cartes*'s third Book of Geometry: the Inveſtigation of all which, I ſhall endeavour to deliver in ſuch a Method as may render expedite the Conſtructing all Solid Problems, even of thoſe in which there is a ſecond Term; which is wanting in *Des Cartes*'s Method.

Novemb. 8, 1704.

LECTURE III.

HItherto we have been Treating of thoſe Equations whereby Plane Problems are reſolv'd; which the Antients made the limits of their Geometry, as not caring in their Conſtructions to make uſe of Curves to be deſcrib'd by Points, but rather contenting themſelves with Circles only. Wherefore they deny'd that Solid Equations could be Geometrically effected, that is, by Rule and Compaſſes: But the modern Geometry allowing it ſelf a greater Freedom, in its Conſtructions rejects no Curve that it knows how to deſcribe or find the Points of, provided it be certain that the thing propos'd cannot be effected without it, or by ſome more ſimple Curve.

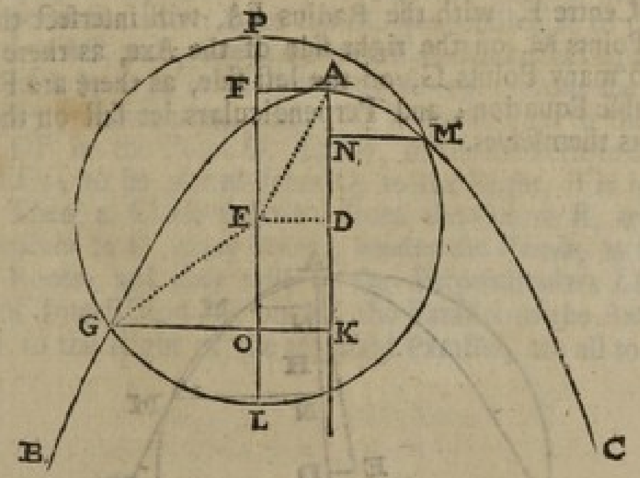
Now the moſt ſimple Curve, in reſpect of its Equation, is the *Parabola*, *viz.* That, the Squares of whoſe Ordinates are to one another, as the *Abſciſſa*: which is evident from the 11th of the 1ſt Book of *Apollonius*'s Conics. And any *Parabola* once deſcrib'd, is ſufficient to Conſtruct any Cubic or Biquadratic Equation, by letting fall Perpendiculars on the Axis of the *Parabola*, from its Interſections with a Circle, to be deſcrib'd according to the direction of the Signs and Quantities of the given Co-efficients of the ſeveral Terms of the Equation. And indeed, we are very much oblig'd to *Des Cartes*, for his ſhewing, not only that the *Parabola* would do the buſineſs, but that his Method comprehended all Equations of three or four Dimenſions, whoſe ſecond Term was wanting, by a very elegant and eaſy Conſtruction;

as

as may be ſeen in the third Book of his Geometry: But ſince *Des Cartes* requires the taking away the ſecond Term of the Equation, if there be any; and beſides, he having no where delivered the Inveſtigation of his Method: we ſhall therefore, in the firſt place, ſhew you the Inveſtigation of the Method; and then the Conſtruction, even where there is a ſecond Term preſent.

Since, from Arithmetical Principles, 'tis certain that ſome Cubic Equations may be expounded by three different Roots, as Biquadratics by four; which is the number of Interſections of a *Circle* with a Conic-Section; 'tis evident, that theſe Roots may be Analogous to thoſe Interſections, and conſequently may be diſcover'd by a *Circle* given in Poſition (that is, to be deſcrib'd according to the known Quantities in the Equation) applied to a given *Parabola*. Now a *Circle* is ſaid to be given in Poſition, when the Radius and Poſition of the Centre is given, which Poſition cannot generally be defined without two given Lines beſides the Radius.

Wherefore to the *Parabola* ABC, whoſe *Latus Rectum* is a, let there be applied a *Circle*, whoſe Radius EP or EL call r, and let the Centre be E, whoſe Diſtance AD or FE, below or above the Vertex of the *Parabola*, let be b, and the Diſtance AF or DE, of the ſame Centre from the Axis of the *Parabola* call c. Let this Circle croſs or touch the *Parabola* in the Points G, M; and from G, M, let fall the



Ordinates GK, MN on the *Axis*: and call AK, the Abſciſſe on the Axe of the *Parabola*, y and the correſponding Ordinate GK x. Then (by the 11th of the 1ſt of *Apollonius*,) the Rectangle ay is equal to xx; and if D be above the Vertex of the *Parabola*, DK or EO is the Sum of AD and AK, or $y+b$; but if it be below, it will be the Difference of them, or $y-b$: Whoſe Square Subtracted from the Square of the Radius of the Circle, leaves the Square of (GO) the Ordinate in the Circle, becauſe of the Right Angled Triangle GEO (by 47th *Euclid.* 1ſt.) Wherefore, the Square of GO will be equal to $rr-bb-yy\pm 2by$; But ſeeing y (becauſe of the *Parabola*) is equal to $\frac{xx}{a}$; let this value be put for y, and its ſquare inſtead of yy, then you will have $rr-bb-\frac{x^4}{aa}\pm\frac{2bxx}{a}$ equal to the ſquare of GO, or the Square of GK$\pm$ED, that is, the Square of $x\pm c$ or $xx\pm 2cx+cc$: Which Equation by Reduction becomes

$$\left.\begin{array}{l} x^4 * \pm 2abxx \pm 2aacx - aarr \\ \quad\quad + aaxx \quad\quad\quad + aabb \\ \quad\quad\quad\quad\quad\quad\quad\quad\; + aacc \end{array}\right\} = 0$$

Let $x^4 * \pm adxx \pm aapx \pm aaaq = 0$, be the Equation to be Conſtructed: And mutually comparing the Co-efficients of the correſponding Terms, $a\pm d$ becomes equal to $2b$; conſequently, if it be $-d$ in the Equation, then the half Sum, but if it be $+d$, then the half Difference of a and d, becomes b, that is, the Line AD, which is to be uſed in the Conſtruction: By the like reaſon c or (ED) the Diſtance of the Centre from the Axe, will be equal to $\frac{1}{2}p$. And the Radius

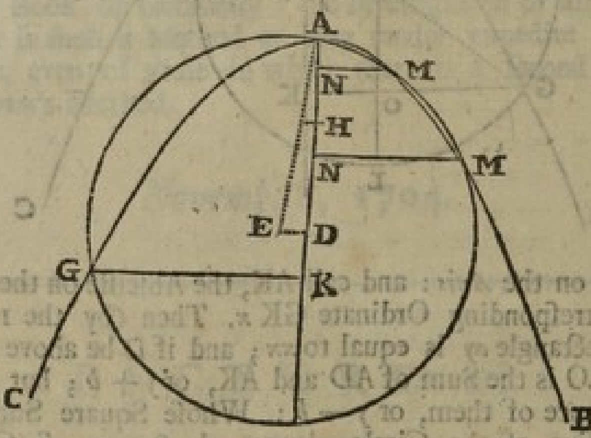
dius of the Circle (r) is had by comparing the last Terms; for the Sum of the Squares of b and c, that is, the Square of AE + or − the Rectangle aq, is found equal to (rr) the Square of the Radius; Wherefore if the Square of the Line AE be encreas'd by the Rectangle aq, if it be $-q$, or diminish'd by the same, if $+q$, the Square of the Radius of the Circle sought will be had.

But if the Quantity q be wanting in the Equation, then (each of the Terms being to be divided by x) it becomes a Cubic; to be Constructed the same way, only here the Rectangle aq vanishing, the Radius of the Circle becomes then AE, and it passes through the Vertex of the *Parabola*.

Whence arises the following general Construction of all Equations of these Forms, where the second Term is wanting, *viz.*

$$1.\ x^3 \ * \pm abx \pm aap = 0$$
$$2.\ x^4 \ * \pm abxx \pm aapx \pm aaaq = 0$$

Any Parabola (BAC) being describ'd, on the Axis AK, its *Latus Rectum* call a; make AH equal to half the *Latus Rectum*; and from the Point H, below towards K, if in the Equation it be $-b$, or above, if it be $+b$; let HD be made equal to half b: Erect DE Perpendicular to the Axe, to the right side of it, if it be $-p$, but to the left, if $+p$, and make it equal to half p: The Circle describ'd on the Centre E, with the Radius EA, will intersect the Parabola in so many different Points M, on the right side of the Axe, as there are Affirmative Roots; and in so many Points G, on the left side, as there are False or Negative Roots in the Cubic Equation; and Perpendiculars let fall on the Axis, as MN, GK, are the Roots themselves.



But if it be a Biquadratic Equation, you must take a mean Proportional between a and q; whose Square, or the Rectangle aq, is to be Added to the Square of AE, if it be $-q$, or Subducted, if it be $+q$, to have the Radius of the Circle requir'd to perform the Construction. And this is *Cartes*'s own Construction; which we have not only demonstrated, but have also shewn the Method of Investigation; whose further use will be evident by what follows, in finding the Position of the Conic-Section to be applied to a *Cubic Paraboloid*, in the Construction of Quadrato-Cubic Equations: Nor have we any thing to add to this of *Cartes*, only that in our Constructions the Affirmative Roots are always on the Right, and the Negative always on the Left side of the Axis; which he places sometimes on the Right, sometimes on the Left, not without some hazard of mistaking.

But *Des Cartes*, as we said before, first of all orders the second Term, if present, to be destroyed, in these Equations; and if it be present, his Constructions will not do; we shall therefore take care to supply this Defect; and shew how the *Parabola* it self performs the Office of taking away the second Term.

Let

$$3.\ x^3 \pm bxx \pm apx + aaq = 0$$
$$4.\ x^4 \pm bx^3 \pm apx^2 \pm aaqx \pm r = 0$$

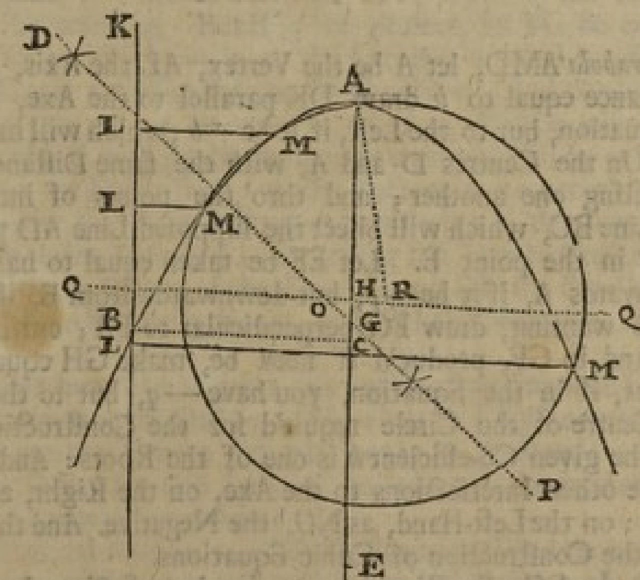
Which comprehends all the Equations of these Forms that can be imagined.

Now all Cubics may be Constructed various ways by different Circles and a given *Parabola*; three of which I shall here exhibit: But in Biquadratics the business can be done but by one only Circle.

The Demonstration of all which, requiring an Algebraic *Calculus*, I shall leave as an Exercise for the Studious *Tyro.* (*Vide Philos. Transact. No.* 188; *and* 190.)

The first Construction of Cubic Equations arises from the consideration of the taking away of the second Term, by putting, after the common way, y equal to $x+$ or $-$ the third part of the Co-efficient of the second Term, whence the following Rule may easily be Demonstrated, *viz.*

The *Parabola* BAM, the Axe AE, and the *Latus Rectum* (a,) being given, let the Equation be reduced to the foregoing Forms; Then at the Distance BC equal to the third part of b, draw BK parallel to the Axe, to the Right-Hand, if it be $+b$, otherwise to the left, intersecting the *Parabola* in B: draw the indefinite right Line DP, perpendicular to and bisecting the suppos'd Line AB, and cutting the Axis in the point G: From B let fall BC perpendicular to the Axe, and make GE always equal to AC, and place it downwards; Make EH equal to half p, to be placed upwards if it be $+p$, but downwards if $-p$. From the Point H, or from E if the Quantity p be wanting, erect HQ Perpendicular to the Axe, cutting the indefinite Line DP in the Point O. Lastly, in the indetermin'd Line HQ, make OR equal to half q; to be placed from O, to the Right, if it be $-q$, but to the Left, if $+q$: Then a Circle describ'd from the Centre R, with RA as Radius, will cut the *Parabola* in so many Points, besides the *Vertex*, as the Equation propos'd has true Roots; and they will be the Perpendiculars LM, demitted from several Points of Intersection M, on BK the Parallel to the Axe: which, in this Figure being all to the Right of the aforesaid Parallel, are all to be look'd upon as Affirmative.

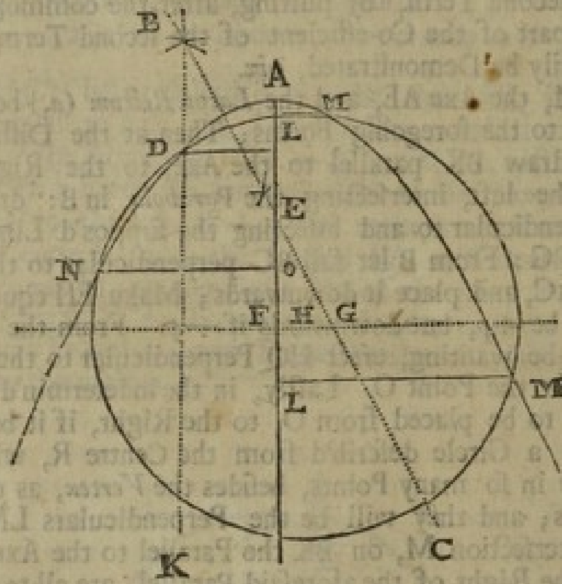


The Construction of the Equation
$$x^3 - bxx + apx - aaq = 0.$$

 The

The Usefulness of this Construction consists in this, that 'tis perform'd by a Circle passing through the Vertex of the *Parabola*, as well as if the second Term were wanting; and therefore seems fittest for determining the Number of Roots in those Cubic Equations where all the Terms are present.

The Second Construction of Cubics is deriv'd from the Cubic Equation's being reducible to a Biquadratic, in which the second Term is wanting, by multiplying the Equation propos'd into $x - b = 0$, if it be $+ b$ in the Equation, or into $x + b = 0$, if it be $-b$: Whence arises a Biquadratic wanting the second Term, which will have the same Roots as the Cubic, and one more equal to $+b$, if it $+b$ in the Equation, or equal to $-b$, if it be $-b$. The Construction is thus:



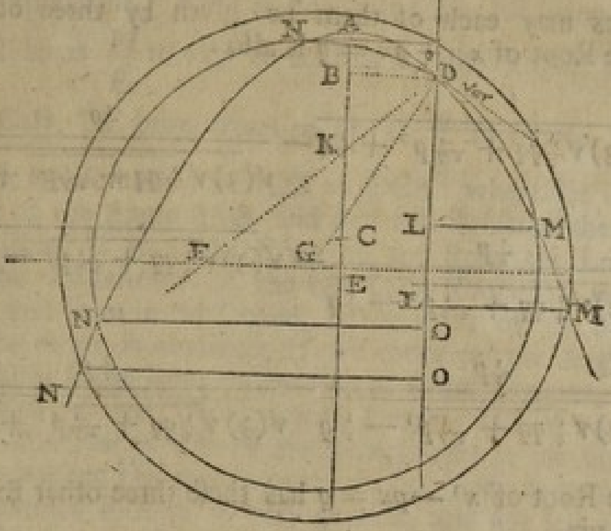
The Construction of the Equation
$x^3 - bxx - apx + aaq = 0$

Of the given *Parabola* AMD, let A be the Vertex, AL the Axis, and *a* the *Latus Rectum*. At a distance equal to *b* draw DK parallel to the Axe, to the Right, if it be $+b$ in the Equation, but to the Left, if it be $-b$; which will meet the *Parabola* in the Point D. On the Centres D and A, with the same Distance, describe occult Arcs intersecting one another; and thro' the points of intersection, draw the indetermin'd Line BC, which will bisect the suppos'd Line AD perpendicularly, and cut the Axe in the point E. Let EF be taken equal to half p, and set upwards from E towards A, if it be $+p$, but downwards from E, if $-p$. Thro' F, or thro' E, if p be wanting, draw FG perpendicular to FA; cutting the Line BC in the Point G; And in GF, produc'd if need be, make GH equal to half q; set it off to the Right, if in the Equation, you have $-q$, but to the Left, if $+q$. I say, H is the Centre of the Circle requir'd for the Construction, and HD its Radius, because the given Co-efficient b is one of the Roots: And Perpendiculars demitted from the other Intersections to the Axe, on the Right, as LM, shew the Affirmative Roots: on the Left-Hand, as NO, the Negative. And this Method is the most eligible for the Construction of Cubic Equations.

The third Method of Construction is properly that of Biquadratics, but which agrees also with Cubics, a Cubic being to be rais'd to a Biquadratic, by multiplying the Equation equal to nothing into x; whence the Cubic may be consider'd as a Biquadratic having the fifth Term (r) wanting.

This Construction is deriv'd from hence, that in Biquadratics, the second Term is taken away by putting $y - \frac{1}{4}b$ equal to the Root x, if it be $+b$ in the Equation, and the contrary: whence y the Roots of the new Equation will always differ from the Roots x by a fourth part of b: Hence the following Construction is evident.

The



The Construction of the Equations.

$$x^3 + bxx - apx - aaq = 0$$
$$\text{Or } x^4 + bx^3 - apx^2 - aaqx - aaar = 0$$

The *Parabola* NAM being given; whose *Latus Rectum* let be a; at the Distance BD, equal to the fourth part of b, draw the Line DL parallel to the Axe AC, to the Left if it be $-b$, but to the Right if $+b$, meeting the *Parabola* in the point D: From D let fall DB perpendicular to the Axe; make BK, in the Axe, equal to half the *Latus Rectum*; draw the indefinite right Line DK; make KC equal to the double of AB, in the Axe always continued beyond K; and set off CE equal to half p, towards the same part, if it be $-p$, but towards the contrary part, if $+p$: upon the Point E, erect GE perpendicular to the Axe, cutting the right Line DK, produced if there be occasion, in F, which is the Centre of the Circle requir'd, if q be wanting. But if q be present, let FG be equal to half q, and place it to the Right if it be $-q$, to the Left if $+q$; and the Point G will be the Centre of the Circle requisite for the Construction. And the Line GD will be the Radius, if the Quantity r be wanting, that is, if it be only a Cubic Equation; But the Square of GD in Biquadratics is to be encreas'd, if it be $-r$, or lessen'd if $+r$ by the Addition, or Subduction of the Rectangle ar contained under r and the *Latus Rectum*: after the same manner as was shewn in the *Cartesian* Constructions. Thus the Circle being describ'd; by letting fall Perpendiculars from the several Intersections with the Curve of the *Parabola*, on DL the Parallel to the Axe, you will have LM the Affirmative Roots, and NO the Negative ones, under the same Law as before.

I might exhibit here several other ways of Constructing such Equations, different from these, namely to be effected by an *Hyperbola* or *Ellipse* combined with a Circle; but seeing they are much more difficult, nor to be perform'd without more Lines, I thought fit to supersede this Labour, according to the received Maxim, *Frustra fit per plura quod fieri potest per pauciora.*

As to the Numerical Resolution of Cubic Equations I had thoughts to refer wholly to *Cardan*'s Rules, which are delivered in the last Section of the XIth Chap. of Mr. *Kersey*'s Algebra, and elsewhere: But on recollection concluded the following Additions might not be unacceptable, *viz.* that whereas the Root of $x^3 + px = q$, there is shewn to be

$$\sqrt{(3)}\overline{\sqrt{\tfrac{1}{4}qq + \tfrac{1}{27}ppp} + \tfrac{1}{2}q} - \sqrt{(3)}\overline{\sqrt{\tfrac{1}{4}qq + \tfrac{1}{27}p^3} - \tfrac{1}{2}q} = x.$$

And the Root of the Equation $x^3 - px = q$, to be

$\sqrt{(3)}$

$$\sqrt{(3)\,\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}ppp}}+\sqrt{(3)\,\tfrac{1}{2}q-\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}ppp}}=x.$$

The same Roots may each of them be given by three other different Expressions, *viz.* the Root of $x^3+px=q$ is also

$$\sqrt{(3)\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}+\tfrac{1}{2}q}-\frac{\tfrac{1}{3}p}{\sqrt{(3)\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}+\tfrac{1}{2}q}}=x$$

Or, $$\frac{\tfrac{1}{3}p}{\sqrt{(3)\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}-\tfrac{1}{2}q}}-\sqrt{(3)\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}-\tfrac{1}{2}q}=x$$

Or lastly, $$\frac{\tfrac{1}{3}p}{\sqrt{(3)\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}-\tfrac{1}{2}q}}-\frac{\tfrac{1}{3}p}{\sqrt{(3)\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}+\tfrac{1}{2}q}}=x$$

So likewise the Root of $x^3-px=q$ has these three other Expressions, besides those of *Cardan*, *viz.*

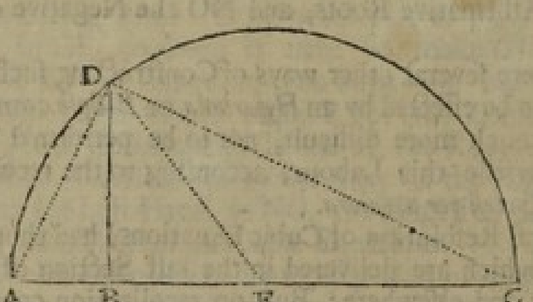
$$\sqrt{(3)\,\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}+\frac{\tfrac{1}{3}p}{\sqrt{(3)\,\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}}=x$$

$$\sqrt{(3)\,\tfrac{1}{2}q-\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}+\frac{\tfrac{1}{3}p}{\sqrt{(3)\,\tfrac{1}{2}q-\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}}=x$$

$$\frac{\tfrac{1}{3}p}{\sqrt{(3)\,\tfrac{1}{2}q-\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p}}}+\frac{\tfrac{1}{3}q}{\sqrt{(3)\,\tfrac{1}{2}q+\sqrt{\tfrac{1}{4}qq-\tfrac{1}{27}p^3}}}=x$$

Now the two first of these in both Cases are evidently simpler than *Cardan's* Rules, in as much as *Division* is an easier Operation than the *Extraction* of the *Cube-Root*, and they arise from the following Considerations.

If BE be made equal to $\tfrac{1}{2}q$, and BD $=\sqrt{\tfrac{1}{27}p^3}$ that is $=\tfrac{1}{3}p\sqrt{\tfrac{1}{3}p}$, the Angle DBE being right, the Hypotenuse DE will be $\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}$: And describing the Circle ADC, AB will be equal to $\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}-\tfrac{1}{2}q$, and BC $=\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}+\tfrac{1}{2}q$. Now AB, BD and BC being continual Proportionals (by reason of the Circle,) their Cube Roots will be so likewise: That is,



$\sqrt{(3)\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}-\tfrac{1}{2}q}$, $\sqrt{\tfrac{1}{3}p}$ and $\sqrt{(3)\sqrt{\tfrac{1}{4}qq+\tfrac{1}{27}p^3}+\tfrac{1}{2}q}$ are continual Proportionals; and $\sqrt{\tfrac{1}{3}p}$ is a Geometrical Mean between the two Cube Roots. Whence if its Square, *viz.* $\tfrac{1}{3}p$ be divided by either of those Roots, the Quote will be the other of them. And the like may be Demonstrated in the other Case, where

'tis

'tis $-p$; putting DE $= \frac{1}{2}q$, and BD $= \sqrt{\frac{1}{27}p^3}$; whence BE will be $\sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$, and AB $= \frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$, *&c.* Hence evidently follow all the foregoing Expreſſions. Now in the firſt Caſe, BE being $= \frac{1}{2}q$, and BD $= \frac{1}{3}p\sqrt{\frac{1}{3}p}$, it will be as $\frac{3q}{p}$ to $\sqrt{\frac{4}{3}p}$ ſo Radius to the Tangent of the Angle DEA; or in the ſecond Caſe, DE being equal to $\frac{1}{2}q$, as $\frac{3q}{p}$ to $\sqrt{\frac{4}{3}p}$ ſo Radius to the Sine of the angle DEA, whoſe half is $=$ ADB $=$ DCB: wherefore if you take the Logarithm Tangent of the Angle ADB, and Add and Subſtract the third part thereof, that is the Logarithm of its Cube Root, to and from the Logarithm of $\sqrt{\frac{1}{3}p}$; you will have the Logarithms of the two Cube Roots, of which the difference in the firſt Caſe, and Sum in the ſecond, is the Root of the Equation ſought.

But the entire Root is obtained, if we conceive the Angle ADB to be that whoſe Tangent is the Cube Root of the former found Tangent, and doubling that Angle, we ſhall have a new DEA, whoſe Tangent DB is to the Radius BE as $\sqrt{\frac{1}{3}p}$ to half the Root, or as $\sqrt{\frac{4}{3}p}$ to the Root ſought in the firſt Caſe: or whoſe Sine DB is to the Radius DE as $\sqrt{\frac{4}{3}p}$ to the Root, in the Second Caſe.

From theſe Premiſes follows a very general, and not leſs elegant Solution of all *Cubic* Equations, by the ***Logarithm* 'Sines** and ***Tangents***, analogous to what has been ſhewn before in the Quadratics.

Say then: As $\frac{3q}{p}$ to $\sqrt{\frac{4}{3}p}$ ſo Radius to a Tangent, if it be $+p$; or to a Sine, if $-p$: and look the Log. Tangent of half the Arc correſponding to that Tangent or Sine, and take its Third, that is, the Logarithm of its Cube Root: Then in the Table of Tangents ſeek the Arc anſwering to that Cube Root, and double it. I ſay that the Tangent, if it be $+p$; or the Sine, if it be $-p$, of that doubled Arc is to the Radius as $\sqrt{\frac{4}{3}p}$ to x the Root of the Equation ſought. The *Praxis* will perhaps be better underſtood by an Example or two: Nor will it be much Trouble to verify your Work by the exact Agreement of theſe two Proceſſes.

Example 1.

Let $xxx + px = q$ be $xxx + 27x = 64$, as in the 21ſt Step of the aforeſaid Section of Mr. *Kerſey*.

Say then as $\frac{3q}{p}$ to $\sqrt{\frac{4}{3}p}$, that is, $\frac{192}{27} : \sqrt{36} = 6$ :: Rad. Tang. 40°. 9′. 21″ $\frac{1}{2}$
Its Half. 20 . 4 . 41 fere.

For Log. $\sqrt{\frac{4}{3}p}$ or 6 is 0.7781513
Log. p 27 is 1.4313638

Sum 2.2095151
Log. 192 2.2833012

9.9262139 T. 40°. 9. 21″ $\frac{1}{2}$
Log. T. 20°. 4′. 41″ 29.5629037
Its Third 9.8543012 = Log. Tang. 35°. 33′. 52″
Its Double 71 . 7. 44

Then Log. $\sqrt{\frac{4}{3}p} =$ 0.778151
T. 71. 7. 44 = 10.466211

Log. x ſought 0.311940
Therefore x is 2,05088

Or Log. $\sqrt{\frac{1}{3}p}$ 0.4771213
$\frac{1}{3}$ Log. T. 20°. 4′. 41″ 9.8543012

Sum 0.3314225 L. 2.14497
Diff. 0,6228201 L. 4 19585

Diff. 2.05088 $= x$

Example 2.

Let $xxx - px = q$ be $xxx - 12x = 18$, as in the 41st Step of the same Section.

As $\frac{3q}{p} : \sqrt{\frac{4}{3}p}$, that is, $4\frac{1}{2} : (\sqrt{16})\ 4$ :: Rad : S. 62°. 44'. 2''$\frac{1}{2}$

its half is = 31. 22. 1$\frac{1}{4}$

For L. $\sqrt{\frac{4}{3}p}$ = L. 4 = 0.6020600
L. $4\frac{1}{2}$ = 0.6532125

9.9488475 = S. 62°. 44'. 2''$\frac{1}{2}$

T. 31°. 22' 1''$\frac{1}{4}$ = 29.7850539
Its third = 9.9283513 = T. 40°. 17'. 42''

Its Double = 80. 35. 24

Then L. $\sqrt{\frac{4}{3}p}$ is = 0.6020600
Log. S. 80°. 35'. 24'' = 9.9941163

Log. x = 0.6079437

Therefore x = 4.05456

Or 0.3010300 = $\sqrt{\frac{1}{3}p}$
9.9283513 = $\frac{1}{3}$Log. T. 31°. 22'. 1''$\frac{1}{4}$

Sum = 0.2293813 = L. 1.69583
Diff. = 0.3726787 = L. 2.35873

Sum = 4.05456 = x

Now tho' this may appear to be as much work as to extract the Cubic Roots in the aforesaid Rules; yet when p and q are great Numbers, or Decimal Fractions, I am assured our Method will be much more eligible.

Example 3.

Let $xxx - px = q$ be $xxx - 17.3577x = 782.41$

As $\frac{3q}{p} : \sqrt{\frac{4}{3}p}$, that is, $\frac{234.723}{17.3577} : \sqrt{23.1436}$ :: Rad : S. 20°. 50'. 23''

its $\frac{1}{2}$ = 10. 25. 11$\frac{1}{2}$

For L. $\sqrt{\frac{4}{3}p}$ = 0.6822155
L. p = 1.2394922

Sum = 1.9217077
L. $3q$ = 2.3705556

9.5511521 = Log. S. 20°. 50'. 23''

Log. T. 10°. 25' 11''$\frac{1}{2}$ = 29.2645644
Its Third = 9.7548548 = Log. T. 29°. 37'. 31''

Its double = 59. 15. 02

Then Log. $\sqrt{\frac{4}{3}p}$ = 0.6822155
S. 59°. 15'. 02'' = 9.9342010

L. x = 0.7480145

Therefore x = 5.59776

Or 0.3811855 = L. $\sqrt{\frac{1}{3}p}$
9.7548548 = $\frac{1}{3}$T. 10°. 25'. 11''$\frac{1}{2}$

Sum = 0.1360403 = L. 1.36786
Diff. = 0.6263307 = L 4.22991

Sum = 5.59777 = x

But

But if in the Equation where 'tis $-p$, q be either Negative or so small, that $\sqrt{\frac{4}{3}p}$ exceed $\frac{3q}{p}$; such an Equation has three Roots: And if q be Affirmative, the greater of the three is Affirmative, and the two lesser Negative: But if it be $-q$, or $px - xxx = +q$, the two lesser Roots are Affirmative, and the greater Negative; all which are very easily obtained by the Trisection of an Angle, thus:

Let the Equation $xxx - px = q$ be $x^3 - 12x = 10$.

Here $\frac{30}{12}$ or $2\frac{1}{2} = \frac{3q}{p}$ is less than $\sqrt{\frac{4}{3}p}$ or 4. Say then as $\sqrt{\frac{4}{3}p}$ to $\frac{3q}{p}$, so Rad. to the *Sine* of an Arc. Take the third part of the Arc answering thereto, and add it to, and substract it from the Arc of 60 Degrees. Then seek the Logarithm Sines of those three Arcs, and to them add severally the Log. of $\sqrt{\frac{4}{3}p}$. Those three Sums shall be the Logarithms of the Roots of the Equation sought.

Wherefore in the aforesaid Equation $x^3 - 12x = 10$ say,

As $\sqrt{\frac{4}{3}p}$ to $\frac{3q}{p}$; that is, as 4 to $2\frac{1}{2}$ so Rad. to $0.625 = $ Sin. 38°: 40′ 56″

Arc	38°. 40′. 56″		
Its Third	12 . 53 . 38⅔	Log. Sine	9.3485955
— 60	47 . 6 . 21⅓		9.8648747
+ 60	72 . 53 . 38⅔		9.9803500
		Log. $\sqrt{\frac{4}{3}p}$	0.6020600

1st Sum	9.9506555	Log. 0.89260	Neg. Roots.
2d Sum	0.4669347	Log. 2.93045	Neg. Roots.
3d Sum	0.5824100	Log. 3.82305	Aff. Root.

But if the Equation had been $12x - x^3 = 10$, the two former had been Affirmative, and the latter and greater Root Negative.

And this may suffice for the exact Solution of Cubick Equations wanting the second Term; but if it be present, you are shewn by Mr. *Kersey*, in his said Chapter, how to take it away, and then you may resolve them as above.

Novemb. 15. 1704.

LECTURE IV.

IN my foregoing Lecture I endeavoured to shew how Solid and Quadrato-Quadratic Equations might be constructed, and that after a very easy manner, *viz.* by a given Parabola and a Circle; and as to Solid or Cubic Equations I have effected their Construction by three different Ways, being the readiest and most simple of infinite others whereby the same may be done: I say of infinite others, because in the Reduction of the proposed Cubic Equation to a Biquadratic, any other Root x may be supposed. But Biquadratics are constructed by one only Circle in a given Parabola, that is to say, by a given Circle also; whereas Cubics are to be effected by infinite Circles, or which may pass through any given Point in the Parabola.

Let us now come to Sursolids and Quadrato Cubics, or Equation of five or six Dimensions, whose Construction by a general Method has not hitherto been shewn by any one except *Des Cartes*; who, tho' he prefers the Circle, because of the Readiness of its Description, yet for the sake thereof, he lays aside that Simplicity which he every where professes in his Writings, and combines with it one of the most compounded of those Seventy two *Curves* of the *Second Kind*, wherewith the most renown'd Sir *Isaac Newton* has lately enriched the Science of Geometry. And if any one inspect the Tediousness of the *Algebraic Calculus*, and the Preparation his Method requires, it will be very evident that he was not arrived at the Thing propounded, to wit, the nearest and best Construction; but rather hath fallen into very intricate and laborious *Ambages*.

We

We have hinted before, that of all the *Curves* of the *Second Kind* the *Cubic Paraboloid*, or that whose *Abſciſſes* are as the Cubes of the *Ordinates*, was the moſt ſimple; and that this Curve, combined with ſome one of the *Conic-Sections*, would exhibit the Roots of all Equations of five or ſix Dimenſions: How this may be done we ſhall endeavour to ſhew in the preſent Lecture.

Since then this *Parabaloid* is to be combined with a *Conic-Section*, it will be neceſſary to add ſomething about the Nature and Properties of the Curve; eſpecially ſince they have not been treated of by the Ancients, and the Geometers of the preſent Age have diſcovered ſeveral of them, *viz.*

1. That it hath a *double Flexure*, and is therefore of that Kind which Sir *Iſaac Newton* from the Form calls *Anguineous Curves*.

2. That the Point of *Contrary Flexure* is in the Beginning of the Curve, or where the *Negative* Part joins to the *Affirmative*.

3. That the *Subtangents* are triple of the *Abſciſſes*, as in the *Quadratic* or *Apollonian Parabola*, they are double of them.

4. That its Area is three Fourths of the circumſcribed Parallelogram, which in the common Parabola is only two Thirds thereof.

5. That in the Point of Contrary Flexure, it goes off, as it were, into a Right Line; at leaſt the *Radius* of *Concavity* becomes infinite: Nor can any Circular Arc, tho' of never ſo great a Circle, be drawn between the Curve and its Tangent, which ſhall not cut the Paraboloid before it come to the Point of Contact.

But 'tis ſufficient for our preſent Purpoſe, that in this Curve the Cubes of the *Ordinates* (which we will call x) are always equal to the Solids, whoſe Altitudes are the *Abſciſſes* y, and Baſe the Square of a given Line a, that is $aay = xxx$.

Suppoſe therefore the Curve NAM, the Paraboloid we are ſpeaking of, to be deſcribed, and let its lower Part to the Right Hand, as AM, be the Affirmative; and the upper Part to the Left, as AN, be Negative: That is, let the Affirmative y encreaſe downwards, and the Affirmative x encreaſe towards the Right Side of the *Axe* AO; and the contrary as to the Negative. To this Curve let the *Conic-Section* MXLNW be to be apply'd, and the Poſition thereof will be thus obtained.

Put AB equal to b, and BC equal to c; and erecting CD from C perpendicular to DB, let AZ,CD be made equal to the *Latus Rectum* of the Paraboloid, which call a. Produce the Right Line BD both ways, on which let be the Poſition of the Diameter of the Conic Section, and let its Center be K. Let the *Ratio* of its Diameter to its *Latus Rectum* be as $2r$ to p; and let BK, the Diſtance of the Center K from the Point B, be equal to f; and put r for KL the Semidiameter of the Section, if it be the Ellipſis or Hyperbola: But if it be the Parabola, let BL be named f, L being the Vertex of the Section; and the *Latus Rectum* of the Parabola call p. Laſtly, Let AO in the *Axis* of the Paraboloid be equal to y, and MO, its correſponding Ordinate, be x.

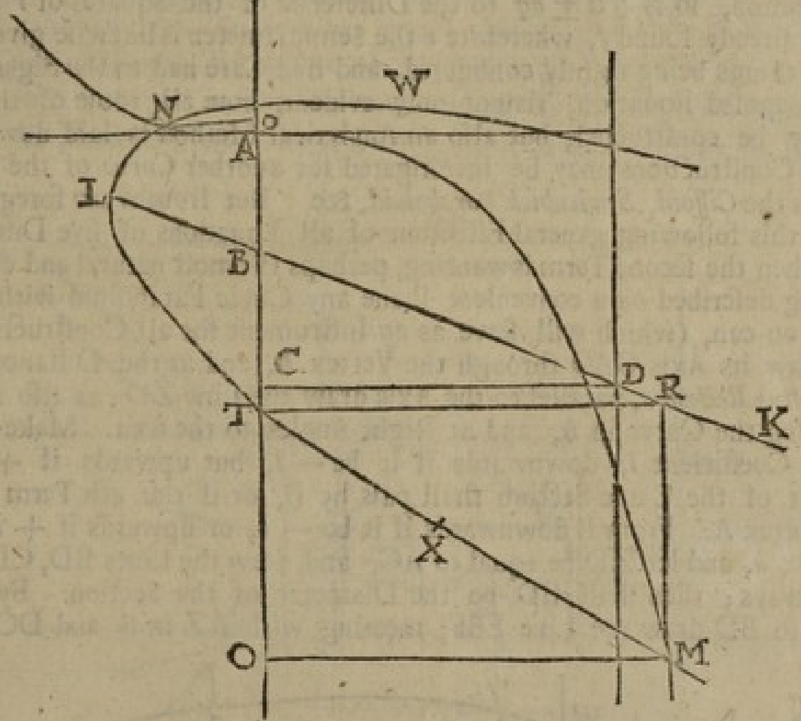
Theſe things being ſuppoſed, 'tis evident that any Ordinate in the Conic-Section, as MR, may be expreſs'd two different ways: For, Firſt, as CD to CB, ſo is MO = TR to BT; ſo that MR will be = AO $+$ BT $\pm$ AB, that is, MR $= y \pm \frac{cx}{a} \pm b$, and MR ſquar'd will be $= yy \pm \frac{2cxy}{a} \pm 2by + \frac{ccxx}{aa} \pm \frac{2bcx}{a} + bb$. which ſame Square is obtained another way on account of the Conic-Section. For putting d for the Line BD, 'twill be as a to d ſo x to $\frac{dx}{a} =$ RB; and the Difference between RB and KB, that is RK $= f \backsim \frac{xd}{a}$, will alſo be the Difference between the Semi-diameter LK and LR: Conſequently the Rectangle contained under the Sum and Difference of LK and KR, or $LK^q - KR^q$, if it be an Ellipſe; or the Rectangle of the Sum of RB and KB, that is, KR $+$ LK into the Exceſs of KR above KL, if it be an Hyperbola, will be to the Square of the Ordinate MR, as the Diameter of the Conic-Section to the *Latus Rectum*, or as $2r$ to p: Hence the Square of MR will be equal to

$\pm$

$$\begin{matrix}+\\-\end{matrix}\frac{pddxx}{2raa} + \text{vel} - \frac{pfdx}{ra}\begin{matrix}-\\+\end{matrix}\overline{rr \pm ff} \times \frac{p}{2r}\begin{matrix}\text{if an Hyperbola.}\\ \text{if an Ellipse.}\end{matrix}$$

$$\pm\frac{p\,dx}{a} \pm pf \qquad \text{if a Parabola.}$$

and taking the one Equation out of the other, 'tis obvious that the Remainder will be equal to nothing; and putting instead of y the Cube of x apply'd to the Square of a the *Latus Rectum*, (that is putting $\frac{xxx}{aa}$ for y) and multiplying all the Terms by a^4, we shall have an Equation of six Dimensions, to be compared with any given Equation of the same Form. Whence the Manner of the Construction we desire will be readily discovered.



Let the Equations stand so, that each Member of the same Dimension of x be directly under its Correlative. Thus,

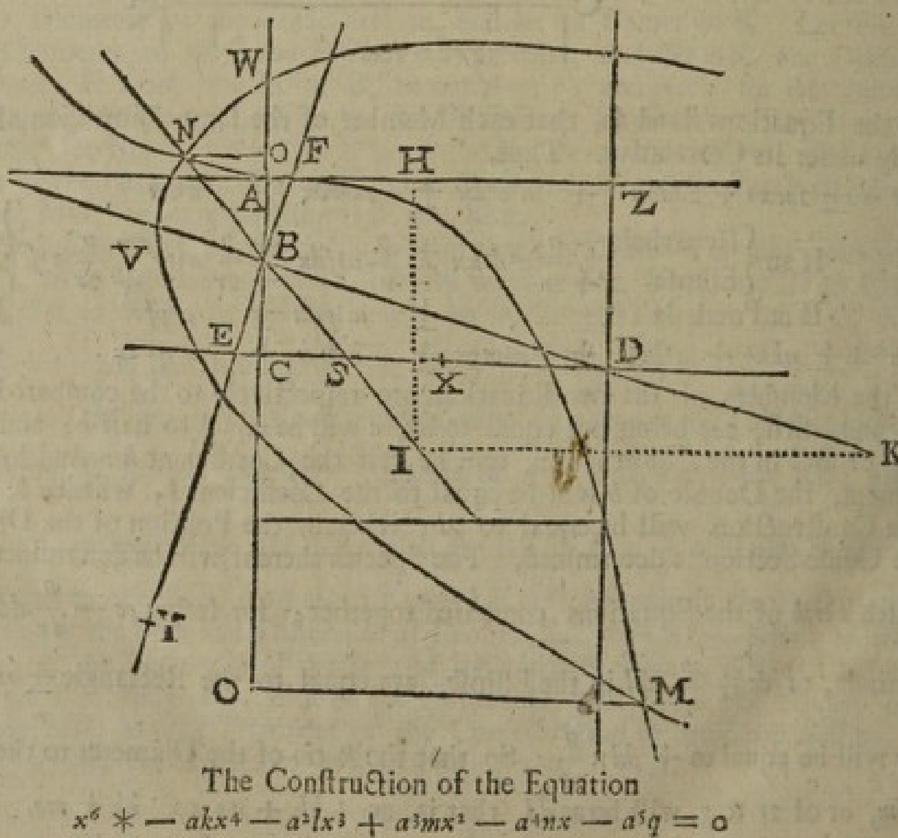
$$\begin{array}{llllll}
x^6 * \pm 2acx^4 \pm 2aabx^3 & + \; a^2c^2xx & \pm \; 2a^3bcx & + \; a^4bb & & \\
\text{If an}\begin{cases}\text{Hyperbola} \\ \text{Ellipse}\end{cases} & \begin{matrix}-\\+\end{matrix}\frac{p}{2r}a^2d^2xx & \pm\frac{p}{r}a^3fdx & \begin{matrix}+\\-\end{matrix}\frac{p}{2r}a^4rr & \begin{matrix}-\\+\end{matrix}\frac{p}{2r}a^4ff & \left.\right\} \circ \\
\text{If a Parabola} & & \pm \; a^3pdx & \pm \; a^4pf & & \\
= x^6 * \pm akx^4 \pm a^2lx^3 & \pm \; a^3mxx & \pm \; a^4nx & \pm \; a^5q = & & \circ
\end{array}$$

Then the Members of the two Equations are respectively to be compared together; and, first, $2ac$ being put equal to ak, c will be equal to half k; and therefore c, or BC in the Construction, will be half the Coefficient k: And by a like Argument, the Double of b will be equal to the Coefficient l; whence b, or AB in the Construction, will be equal to $\frac{1}{2}l$; whereby the Position of the Diameter of the Conic-Section is determined. The Species thereof will be determined from the fifth *Term* of the Equations compared together; for seeing $cc - \frac{p}{2r}dd$ in the Hyperbola, or $cc + \frac{p}{2r}dd$ in the Ellipse, are equal to the Rectangle $\pm am$, $\frac{1}{4}kk \pm am$ will be equal to $\pm dd \times \frac{p}{2r}$: So that the *Ratio* of the Diameter to the *Latus Rectum*, or of $2r$ to p, will be as dd, that is, as $\frac{1}{4}kk + aa$ to $\frac{1}{4}kk \pm ma$. But if it it be $+ ma$ in the Equation, and it be equal to $\frac{1}{4} kk$, the Conic-Section will be a Parabola; if $+ ma$ be greater than $\frac{1}{4} kk$, 'twill be an Ellipse; if less or Negative, then an Hyperbola. The Species therefore of the Conic-Section to be described is given, whose Center will be discovered by Help of the Sixth Term; $\pm bc \pm \frac{pdf}{2r}$ being equal to $\pm \frac{1}{2} an$; whence $f = \frac{\pm bc \pm \frac{1}{2} an}{d} \times \frac{2r}{p} = \frac{2r}{p} \times \frac{\frac{1}{2} kl \pm \frac{1}{2} an}{d}$ $=$ BK in the Construction. But in the Case of the Parabola $2bc \pm an = \pm pd$; whence $\frac{\frac{1}{2} kl \pm an}{d}$ becomes equal to the *Latus Rectum* of the Parabola sought.

Lastly, The Semidiameter r of the Conic-Section is concluded from the seventh and last Term; for since $bb \pm aq$ is equal to the Difference of the Squares of r and f (that is of KB and KL) into $\frac{p}{2r}$, therefore as the *Latus Rectum* to the Dia- of the Section, so is $\frac{1}{4} ll \pm aq$ to the Difference of the Squares of r and f. But we have already found f, wherefore r the Semidiameter is likewise given.

These things being rightly considered, and due Care had to the Signs $+$ and $-$ in the proposed Equation, 'tis not only evident, how all those of these Dimensions may be constructed, but also an Analytical Method is laid down, whereby the like Constructions may be investigated for another *Curve* of the *Second Kind* given, as the *Cissoid*, *Semicubick Paraboloid*, &c. But from what foregoes we have deduced this following general Effection of all Equations of five Dimensions, or of six, when the second Term is wanting, perhaps the most natural and easy possible.

Having described on a convenient Plane any Cubic Paraboloid with all the Accuracy you can, (which will serve as an Instrument for all Constructions of this Sort) draw its Axis OAO through the Vertex A, and at the Distance AZ equal to the *Latus Rectum* a, parallel to the Axis draw the Line ZD; as also AZ touching and cutting the Curve in A, and at Right Angles to the Axis. Make AB equal to half the Coefficient l, downwards if it be $-l$, but upwards if $+l$, and the Diameter of the Conic-Section shall pass by B, or if the 4th Term be wanting, by the Vertex A. From B downwards if it be $-k$, or upwards if $+k$, make BC equal to $\frac{1}{2} k$, and let ZD be equal to AC, and draw the Lines BD, CD indefinitely both ways; then shall BD be the Diameter of the Section. By B at Right Angles to BD draw the Line EBF, meeting with AZ in F and DC in E; and

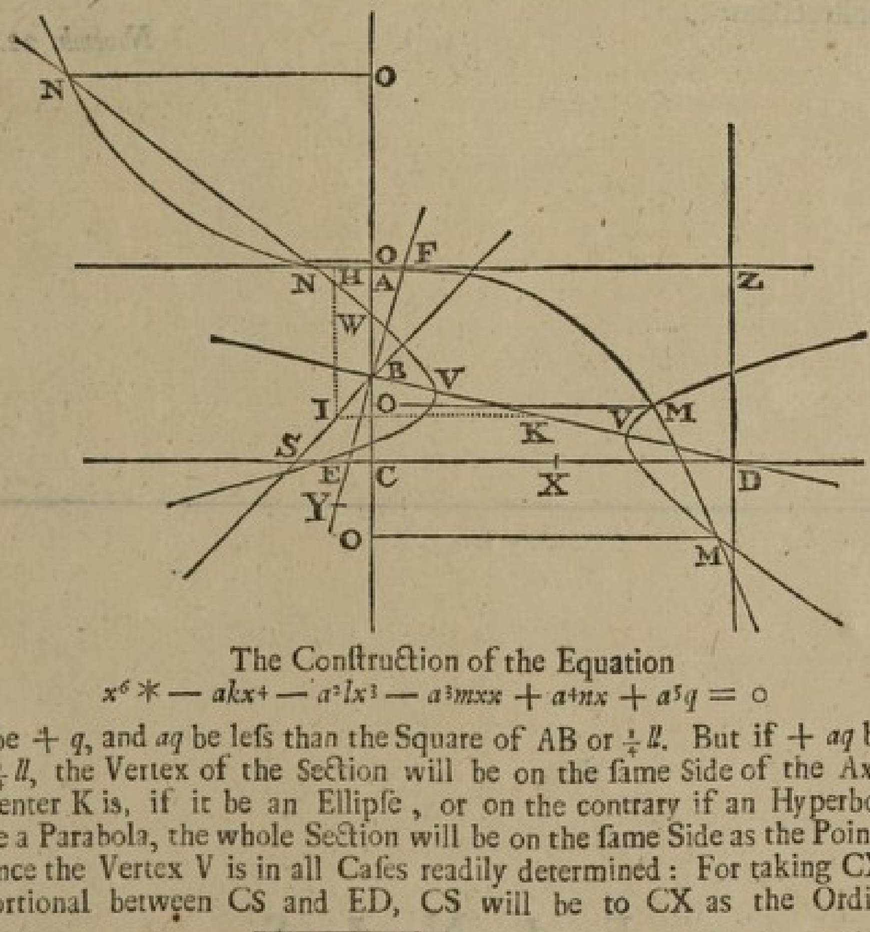


The Construction of the Equation
$x^6 * - akx^4 - a^2lx^3 + a^3mx^2 - a^4nx - a^5q = 0$

from

from E towards D, on the Line ED, make ES equal to m, if it be $+m$, or the contrary way if $-m$; and if S fall between C and D, or beyond D, the Section will be an Ellipsis; but if between C and E, or it be $-m$, an Hyperbola. And in either Case the *Ratio* of ED to CS will be that of the Diameter to the *Latus Rectum* of the Section. But if $+m$ be equal to EC, it will be a Parabola. Draw BS, and continue it both ways; and on the Line AZ make FH equal to $\frac{1}{2}n$, to be laid to the Right of F, if it be $-n$, or to the Left, if $+n$. By H, parallel to the Axe AO, draw the Line HI meeting with BS in I, and the Line IK parallel to AZ, shall intersect BD the Diameter of the Section, in the Point K the Center thereof, if it have a Center. But if it be a Parabola, the *Latus Rectum* thereof will be to 2AH as CD to DB; or equal to $2FH = n$, if the Term k be wanting; and the Diameter of the Parabola will extend it self infinitely, on the same Side of the Axis of the Paraboloid, on which the Point H is found.

Lastly, If the Term q be wanting, that is, if the Equation be but of five Dimensions, the Section, be it what it will, passes by the Vertex of the Paraboloid A, and consequently BA is one of its Ordinates. But if it be $-aq$, $BW = \sqrt{AB^2 + aq}$ will be equal to the Ordinate passing by the same Point of the Diameter B: As likewise $\sqrt{AB^2 - aq}$ will be equal to a like Ordinate of the Section,



The Construction of the Equation
$x^6 * - akx^4 - a^2lx^3 - a^3mxx + a^4nx + a^5q = 0$

if it be $+q$, and aq be less than the Square of AB or $\frac{1}{4}ll$. But if $+aq$ be greater than $\frac{1}{4}ll$, the Vertex of the Section will be on the same Side of the Axis AO as the Center K is, if it be an Ellipse, or on the contrary if an Hyperbola: And if it be a Parabola, the whole Section will be on the same Side as the Point H.

Hence the Vertex V is in all Cases readily determined: For taking CX a mean Proportional between CS and ED, CS will be to CX as the Ordinate BW $= \sqrt{\frac{1}{4}ll \pm aq}$ to $BY = \sqrt{\frac{2r}{p} \times \frac{1}{4}ll \pm aq} = \sqrt{rr - ff}$ or $\sqrt{ff - rr}$. Wherefore in the Case of the Ellipse, place BY on the Line FBE, and $KY = KV$ shall be the Semidiameter of the Section required, and V the Vertex thereof. But in the Hyperbola, in the Semicircle whose Diameter is KB inscribe the Line BY, and make $KV = KY$, and V shall be the Vertex, and KV the Semidiameter sought. But when $+aq$ is greater than $\frac{1}{4}ll$, then the said Line $BY = \sqrt{\frac{2r}{p}aq - \frac{1}{4}ll}$, if it be an Hyperbola, must be placed on the Line FBE as before, and $KV = KY$ will be the Semidiameter of the Section, whose Vertex V will be on the other Side of the Axis AO. But in the Ellipsis, BY being inscribed in the Semicircle whose Diameter is KB, $KV = KY$ shall be the Semidiameter of the Section, which shall fall wholly on the same Side the Axis on which is its Center K. So like-

likewiſe in the Parabola, the Rectangle of BV into the *Latus Rectum* p before found, will be equal to $\frac{1}{4}ll \pm aq$; or $BV \times p = aq - \frac{1}{4}ll$, when $\frac{1}{4}ll$ is leſs than aq: In which Caſe the Vertex V falls on the ſame Side of the Axis AO on which is the Point H. From theſe *data* the *Conic-Section* will be readily deſcribed, and its Interſections with the Paraboloid ſhew the Quantity and Number of the poſſible Roots of the Equation ſo conſtructed; the Affirmative on the Right Side of the Axis, as OM, the Negative NO on the Left, as has been ſaid before. And I have been the more particular, not to leave any Difficulties in the Way of thoſe that are deſirous to reſolve theſe high Equations.

If in an Equation of five Dimenſions all Termes be preſent, the ſecond Term muſt be taken away after the ſame manner as we did in our ſecond Conſtruction of the *Cubics*, Pag. 14. by aſſuming another Root equal to the Coefficient of the ſecond Term, under a contrary Sign; whereby it will be reduced to a Quadrato-Cubic wanting the ſecond Term, and may be conſtructed as ſuch with very little more Trouble: And the Roots be all the ſame as in that of five Dimenſions.

To prevent your Conic-Section from excurring beyond your Plane, it may be proper to divide your Equation, ſo as the Ordinates of the Section may be pretty near at Right Angles with its Diameter, the Convenience of which Caution will be obvious to thoſe that ſhall go about to put in Practice the Rules of theſe Conſtructions.

Novemb. 22. 1704.

F I N I S.

Alegria
A