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THE
THIRD & FOURTH BOOKS OF THE ELEMENTS OF ALGEBRA.

Compiled by
JOHN KERSEY.

——— *Si quid novisti rectius istis,*
Candidus imperti; si non, his utere mecum.

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The CONTENTS of the Third Book.

THE *Third Book is an* Analysis *in* Species *of the choicest of* Diophantus's *much admired Questions concerning Squares, Cubes, and right-angled Triangles in rational numbers; with other Questions of like nature. To which is also added a brief Exposition upon Monsieur* Fermat's *Analytical Invention, prefix'd to Monsieur* Bachet's *Comment upon* Diophantus, *Printed at* Toloze, *Ann.* 1670. *The number of Questions in the Third Book is* 130.

The CONTENTS of the Fourth Book.

THE *Fourth Book is an Introduction to Mathematical Resolution and Composition; where, the excellent use of the Algebraical or Analytical Art is evidently shewn, both in finding out the Solutions of Geometrical Plane Problems,* (viz. *of such whose Delineations require only the drawing of Right and Circular lines;) as also in discovering Synthetical Demonstrations from the Analytical steps, to prove the truth of those Solutions by a* Series *of Consequences deduced altogether from things really given or granted. All which are clearly expounded, and copiously exemplified both Geometrically and Arithmetically, by a choice Collection of useful and delightfull Problems. The Fourth Book is divided into Ten Chapters, the Contents whereof are these,* viz.

 Faults

Faults to be Corrected in the Third and Fourth Books, before they be read.

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T H E

THE

ELEMENTS

OF THE

ALGEBRAICAL ART.

Book III.

AMong all the Writers upon the *Algebraical Art*, there hath none been hitherto known more antient, nor any more famous for shewing the admirable force of that Art in solving Arithmetical Problems, than *Diophantus* of *Alexandria*, who lived, (as Authors compute,) above thirteen hundred years ago. He wrote thirteen Books of *Arithmetick*, and one concerning *Multangular numbers*; but of those thirteen, six only are extant, which contain two hundred and eight Questions, many of which are so knotty and abstruse, that they can hardly be solved without the help of *Diophantus* his peculiar Method; whose Speculations are so sublime, that where there seems to be an impossibility of finding out a single Answer to a Question, he shews how to find out innumerable Answers in rational (or ordinary) numbers. Now to give the ingenious Reader of these *Elements* a delightful Prospect of the rare Inventions of that Prince of Arithmeticians, I have with no small labour framed this Third Book, and therein explain'd the Resolutions of the hardest and choicest of his Questions, in the second, third, fourth, fifth and sixth Books of his Arithmetick before mentioned; as also of divers other subtil Questions invented upon his grounds by *Vieta*, *Bachet*, (the best Commentator hitherto upon *Diophantus*,) *Fermat* and others; among which also divers Questions of my own are inserted, to wit, those which have no citation referring to any Author.

Note, That $\triangle$ stands for the word *Triangle*, and $\square$ for a *Square number*; but as to the rest of the Characters used in this third Book, they have already been explain'd in *Chap.* 1. *Book* 1. of these Elements.

QUEST. 1. (This is the 9*th* of the second Book of *Diophantus*.)

To divide a given square number into two Squares.

RESOLUTION 1.

1. Let the square number given to be divided be . . . . . $\succ$ 16
2. The Root or side thereof is . . . . $\succ$ 4
3. For the Root or side of the first of the two Squares sought put $\succ$ a
4. Therefore the first Square is . . . . $\succ$ aa
5. And consequently, (by the first and fourth steps,) the second Square must be equal to . . . . $\Big\}$ $16 - aa$
6. Now let the side of the second Square be feigned to be . . . $\succ$ $2a - 4$, or $4 - 2a$
7. Therefore the Square of the said feigned side is . . . . . $\succ$ $4aa - 16a + 16$
8. Which Square must be equal to $16 - aa$ in the fifth step, hence this following Equation ariseth, *viz.*

$$4aa - 16a + 16 = 16 - aa;$$

9. From

9. From which Equation, after due Reduction, the side of the first Square will be made known, *viz.* . . . . $\Big\}\ a = \frac{16}{5}$

10. And by the ninth and sixth steps, the side of the second Square will likewise be discovered, *viz.* . . . . $\Big\}\ 2a - 4 = \frac{12}{5}$

So the sides of the two Squares sought are found $\frac{16}{5}$ and $\frac{12}{5}$; which will solve the Question; for the Square of $\frac{16}{5}$ is $\frac{256}{25}$, and the Square of $\frac{12}{5}$ is $\frac{144}{25}$, both which Squares added together make $\frac{400}{25}$, that is, 16; as was required.

Note. That which is most remarkable in the foregoing Resolution of *Diophantus*, is, his ingenious and peculiar way of feigning the side of a Square to be equated to $16 - aa$ in such manner, that after the Equation is duely reduced, the number represented by a will necessarily be Rational. Now because he makes great use of the like manner of feigning the sides of Squares to be equal to Algebraical quantities, in resolving divers hard Questions, (as will copiously appear in this third Book,) the Learner must endeavour to be very well acquainted with the said Method; for his ease therefore I shall explain it in the following Observations.

Observations upon Quest. 1.

1. Concerning the feigned side $2a - 4$ in the sixth step of the foregoing Resolution, it may be asked, why -4, and not -5, -6, or some other number? to this I answer, There is a necessity that this number be alwayes the side of the Square given to be divided into two Squares; so *Diophantus* feigns the second side to be $2a - 4$, (4 being the side or square Root of 16 the given Square,) to the end that in the Square of $2a - 4$ there may be found the Absolute number 16, to wit, the Square given to be divided; for then the Square of $2a - 4$ being equated to $16 - aa$, (as in the eighth step of the Resolution,) there will be found $+ 16$ on each part of the Equation, whence by subtracting 16 from each part, there ariseth $5aa = 16a$; and consequently each part of this Equation being divided by a, the Quotients give $5a = 16$, wherefore $a = \frac{16}{5}$.

2. One of the parts of the said feigned side of the second Square must (in this *Quest.* 1.) necessarily have the sign $-$ prefixt to it; so *Diophantus* feigns the said second side to be $2a - 4$, for if it were $2a + 4$, all the parts or terms of its Square would be Affirmative, and consequently no possible Equation would arise; as will easily appear by comparing the Square of $2a + 4$, that is, $4aa + 16a + 16$ to $16 - aa$, whence $5aa + 16a = 0$.

3. The Learner may also demand, why in the sixth step of the foregoing Resolution, the side of the second Square sought is feigned to be $2a - 4$, or $4 - 2a$, and not $a - 4$, or $4 - a$, *viz.* why is 2, and not 1 or some other number prefixt to a? to this I answer, If the side of the first Square sought be assumed or supposed to be a or $1a$, (as it is in the third step of the Resolution,) then the side of the second Square cannot be $a - 4$, or $4 - a$, as will be evident by a due process upon that supposition; for the Square of $a - 4$, or $4 - a$, that is, $aa - 8a + 16$ being equated to $aa - 16$, there will arise, after due Reduction, $a = 4$, and consequently, $a - 4$, or $4 - a$, which was put for the second Square, will be equal to nothing: The like absurdity will follow as often as the numbers prefixt to a in the feigned sides of the two Squares sought are equal to one another, *viz.* if the first side be feigned to be $5a$, and the second $5a - 4$, or $4 - 5a$; or if the first side be $8a$, and the second $8a - 4$, or $4 - 8a$, &c. from such suppositions a fruitless Equation will ensue, for the side of the second Square will be found equal to nothing. Now for the avoiding of such absurdity, the Learner may take this for a Rule, (the reason whereof will hereafter appear by *Observat.* 1. of the following *Resolution* 2. of this *Quest.* 1.) Let the numbers prefixt to a in the feigned sides of the two Squares sought be any two unequal numbers, *viz.* if a or $1a$ be put for the first side, the second may be $2a - 4$, or $3a - 4$, &c. Again, if we put $3a$ for the first side, the Square thereof will be $9aa$; and consequently because the Square given to be divided into two Squares is 16, the second Square shall be equal to $16 - 9aa$, whose side we may feign to be $2a - 4$, or $4 - 2a$, the Square whereof is $4aa - 16a + 16$; this æquated to the said $16 - 9aa$, gives after due Reduction $a = \frac{16}{13}$, therefore $3a$ which was put for the side of the first Square shall be $\frac{48}{13}$, and $4 - 2a$ which was put for the side of the second Square will be found $\frac{20}{13}$, and consequently, the two Squares sought shall be $\frac{2304}{169}$ and $\frac{400}{169}$, whose summ makes 16, as the Question requires.

In which last Example (which is worthy of the Learner's observation) it happens, that in resolving the Positions, the second side is expounded by $4 - 2a$, not by $2a - 4$, although the

the Refolution be juftly formed from either of them; for the Square $4aa - 16a + 16$, whiles a is unknown, may have for its fide either $2a - 4$, or $4 - 2a$, and which fo ever of thefe fides be feigned, the fame Equation will arife to find out the number a, which, after it is difcovered, is to be compared to fuch of thofe two feigned fides as will produce a number greater than nothing; fo the number a being before found out to be $\frac{16}{13}$, it is manifeft that $2a - 4$ is lefs than nothing, but $4 - 2a$ gives $\frac{20}{13}$, which is the true fide of the fecond Square. So likewife in the Example of *Diophantus*, the fide of the fecond Square cannot be expounded by $4 - 2a$, (although the value of a may be rightly found out from that fuppofition, as well as from $2a - 4$;) for a being found equal to $\frac{16}{5}$, the faid $4 - 2a$ is lefs than nothing.

4. Here I fhall recommend to the Learner one general Obfervation, *viz.* The principal fcope in feigning the fide of a Square to be equated to fome Algebraick Quantity wherein the higheft unknown Power is aa, muft be to feign the faid fide in fuch manner, that after due Reduction, either the Abfolute numbers may vanifh, and confequently an Equation remain between fome number of aa, and fome number of a, whence by Divifion the number a will neceffarily be Rational; or elfe, (as hereafter will fully appear in this Book,) that aa may vanifh out of each part, and confequently an Equation remain between fome number of a, and fome Abfolute number; hence alfo the number a will be Rational.

Having explain'd *Diophantus* his Refolution of *Queft.* 1. by Numeral *Algebra*, I fhall in the next place refolve the fame by Literal *Algebra*, whence divers ufeful Canons will be brought to light.

RESOLUTION 2. of *Queft.* 1. which is here repeated, *viz.*

To divide a given fquare number into two Squares.

1. For the fide of the given Square put	d
2. Therefore the faid Square is	dd
3. Take any two unequal numbers, fuppofe s the greater, and r the leffer, (which s and r are to be ufed inftead of the numbers prefixt to a in the foregoing Refolution,) then for the fide of the firft Square fought put	ra
4. And for the fide of the fecond Square fought, put	$sa - d$, or, $d - sa$
5. Therefore, from the third ftep, the firft Square is	$rraa$
6. And from the fourth ftep, the fecond Square is . .	$ssaa - 2sda + dd$
7. Therefore the fumm of thofe Squares is . . .	$ssaa + rraa - 2sda + dd$
8. Which fumm muft be equal to the given Square dd, hence this Equation arifeth, *viz.*	$ssaa + rraa - 2sda + dd = dd$
9. Which Equation, after due Reduction, gives . .	$a = \frac{2sd}{ss + rr}$
10. Therefore out of the ninth and third fteps, the fide of the firft Square fought is now made known, for it is equal to	$\frac{2rsd}{ss + rr}$
11. And from the ninth and fourth fteps the fide of the fecond Square is alfo known, for it's equal to . .	$\frac{ssd - rrd}{ss + rr}$

Obfervations upon the preceding Refolution 2. *of* Queft. 1.

1. The eleventh ftep of the faid Refolution difcovers that the known numbers s and r muft be unequal, to the end the difference of their Squares may be greater than nothing.

2. After the number a is made known, (as in the ninth ftep,) it will be manifeft that the fide of the fecond Square is to be expounded by $sa - d$, not by $d - sa$; for fince a is found equal to $\frac{2sd}{ss + rr}$, (as appears by the ninth ftep of *Refolut.* 2.) it follows that $sa = \frac{2ssd}{ss + rr}$, and $d - sa = \frac{rrd - ssd}{ss + rr}$, which is lefs than nothing, for s is greater than r by fuppofition: But whiles a is unknown, the fide of the fecond Square may be feigned $d - sa$ as well as $sa - d$, for each of thefe produceth the fame Square $ssaa - 2sda + dd$.

3. The fide of the firft Square may be feigned sa, and the fide of the fecond $ra - d$, or $d - ra$, for from thefe Pofitions the true fides of the two Squares fought will be found

the ſame as before are expreſt in the tenth and eleventh ſteps of *Reſolut.* 2. but in this latter way of feigning the ſides, the ſide of the ſecond Square will be expounded by $d - ra$, not by $ra - d$, for this will be found leſs than nothing.

4. The tenth and eleventh ſteps of the foregoing *Reſolution* 2: give this

CANON.

Take any two unequal numbers; multiply ſeverally the double of the Product of their multiplication, and the difference of their Squares by the ſide of the given Square; laſtly, divide thoſe Products ſeverally by the ſumm of the Squares of the two numbers firſt taken, and the Quotients ſhall be the ſides of the two Squares ſought.

An Example in Numbers.

Let the ſide of the given Square be 4, then take two unequal numbers at pleaſure, as 1 and 2; the double Product of their multiplication is 4, the difference of their Squares is 3, then by multiplying the ſaid 4 and 3 ſeverally by 4, (the ſide of the given Square,) the Products are 16 and 12; theſe divided ſeverally by 5, (that is, by the ſumm of the Squares of 1 and 2 the numbers firſt taken,) give the Quotients $\frac{16}{5}$ and $\frac{12}{5}$, which are the ſides of the two Squares ſought; for the Squares of $\frac{16}{5}$ and $\frac{12}{5}$ added together make 16, which was given to be divided into two Squares.

5. For as much as (by *Prop.* 47. *Elem.* 1. Euclid.) when a Square is equal to two Squares, the ſides of thoſe three Squares will make a right-angled Triangle, the preceding *Queſt.* 1. may be thus ſtated, *viz.*

A Rational number being given for the *Hypothenuſal* of a right-angled Triangle, to find Rational numbers to expreſs the Baſe and Perpendicular, *viz.* the ſides about the right-angle.

This may be ſolved by the preceding Canon; for if d be put to repreſent the given Hypothenuſal, and s and r any two unequal numbers, r being the leſſer, theſe three following numbers will conſtitute a right-angled Triangle having d for its Hypothenuſal, *viz.*

Hypothenuſal,	Baſe,	Perpendicular.
d ,	$\frac{ssd - rrd}{ss + rr}$,	$\frac{2rsd}{ss + rr}$.

6. Moreover, if thoſe three ſides of a right-angled Triangle be ſeverally multiplied by the Denominator $ss + rr$, the Products ſhall alſo be the ſides of a right-angled Triangle, to wit, theſe following;

Hypothen.	Baſe,	Perp.
$ssd + rrd$,	$ssd - rrd$,	$2rsd$.

7. And by dividing every one of the three ſides laſt expreſt, by their common Factor d, the Quotients will give theſe three following ſides of a right-angled Triangle, *viz.*

Hypothen.	Baſe,	Perp.
$ss + rr$,	$ss - rr$,	$2rs$.

8. Which three ſides laſt above expreſt are in words the following uſeful Canon, to form a right-angled Triangle in numbers by the help of any two unequal numbers given.

CANON.

Take any two unequal numbers, (ſuppoſe s the greater, and r the leſſer,) then the ſumm of their Squares ſhall be the Hypothenuſal, the difference of the ſame Squares ſhall be one of the ſides about the right-angle, and the double Product of the multiplication of the ſaid two numbers, the other ſide.

The Proof of this Canon.

The Square of $ss + rr$ is . . . . $s^4 + 2ssrr + r^4$,
The Square of $ss - rr$ is . . . . $s^4 - 2ssrr + r^4$,
The Square of $2rs$ is . . . . $+ 4ssrr$.

The firſt of thoſe three Squares is manifeſtly equal to the ſumm of the other two, and therefore the ſides of thoſe three Squares, if they be expreſt by numbers, ſhall be the meaſures of the ſides of a right-angled Triangle.

An Example of the ſaid Canon *in Numbers.*

Take two unequal numbers at pleaſure, as 1 and 2; then the ſumm of their Squares is 5 for the Hypothenuſal, the difference of the Squares of the ſame two numbers is 3 for the Baſe, (that is, either of the ſides about the right-angle,) and the double Product of the two

numbers

numbers is 4 for the Perpendicular ; but that the numbers 5, 3, 4 may be taken for the meaſures of the ſides of a right-angled Triangle is evident, for the Square of the firſt is equal to the Squares of the two latter.

9. *Three* Corollaries *deduced from the laſt preceding* Canon:

Firſt, in every right-angled Triangle in ſuch whole numbers which are Prime between themſelves, the ſumm of the Hypothenuſal ($ss+rr$) and ($2rs$) one of the ſides about the right-angle is a ſquare number, to wit, ($ss+rr+2rs$) the Square of the ſumm of (s and r) the two numbers by which the ſaid Triangle may be formed according to the laſt preceding Canon.

Secondly, the ſumm of the Hypothenuſal ($ss+rr$) and ($ss-rr$) the other of the ſides about the right-angle is the double of a ſquare number, to wit, the double of (ss) the Square of (s) the greater of the two numbers by which the Triangle may be formed ; And the exceſs of the Hypothenuſal ($ss+rr$) above the ſaid ſide ($ss-rr$) is the double of the Square of (r) the leſſer of the ſame two numbers ; therefore,

Thirdly, the three ſides of any right-angled Triangle in ſuch Rational whole numbers as are Prime between themſelves being ſeverally given, we may find two whole numbers by which the ſaid Triangle may be formed according to the Canon in *Obſervat.* 8. As, for example, to find two numbers to form theſe three ſides of a right-angled Triangle, to wit, 65, 33, 56, (which are Prime between themſelves, for they have no common Diviſor but Unity,) I add the Hypothenuſal 65 to 33 and 56 ſeverally, and it makes 98 and 121; which latter ſumm is a Square, and therefore (*per Coroll.* 1.) its Root 11 is the ſumm of the two numbers ſought, and the firſt ſumm 98 is the double of the Square 49, whoſe Root 7 ſhall be the greater of the two numbers ſought, (*per Coroll.* 2.) laſtly, by ſubtracting 7 from 11, the Remainder 4 is the leſſer number ſought ; whence I conclude, that the right-angled Triangle propoſed may be formed out of 7 and 4 ; for the ſumm of their Squares makes the Hypothenuſal 65 , the difference of the ſame Squares is 33 one of the ſides about the right-angle, and the double Product of 7 and 4, to wit, 56 is the other ſide.

10. From the two preceding Canons (in *Obſervat.* 4, and 8.) another may be deduced to ſolve *Queſt.* 1. *viz.* to divide a given ſquare number into two Squares ; or a Rational number being given for the Hypothenuſal of a right-angled Triangle, to find the Baſe and Perpendicular in Rational numbers.

CANON.

By the foregoing Canon in *Obſervat.* 8. let a right-angled Triangle be formed out of any two unequal numbers, and call this Triangle the firſt ; then it ſhall be, as the Hypothenuſal of the ſaid firſt Triangle is to its Baſe, ſo is the given Hypothenuſal of a ſecond Triangle deſired to its Baſe ; and as the Hypothenuſal of the firſt Triangle is to its Perpendicular, ſo is the Hypothenuſal of the ſecond to its Perpendicular.

An Example in Numbers.

Let it be required to find the Baſe and Perpendicular of a right-angled Triangle in numbers whoſe Hypothenuſal ſhall be 7.

First, by the Canon in *Obſervat.* 8. I form a right-angled Triangle in numbers, as, ... } 5 , 3 , 4

Then, by the Rule of Three, I find $4\frac{1}{5}$ for the deſired Baſe, thus, ... } 5 . 3 :: 7 . $4\frac{1}{5}$

Likewiſe $5\frac{3}{5}$ for the deſired Perpendicular , thus, ... } 5 . 4 :: 7 . $5\frac{3}{5}$

Therefore 7, $4\frac{1}{5}$, $5\frac{3}{5}$ will conſtitute a right-angled Triangle whoſe Hypothenuſal is 7, as was deſired.

11. After the ſame manner , as many right-angled Triangles in numbers as ſhall be deſired may be found out, which ſhall have one common Hypothenuſal given : As, for example, if three right-angled Triangles in Rational numbers be deſired, that 2 may be a Hypothenuſal to every one of them, they may be found out thus ;

First, by the Canon in the foregoing *Obſervat.* 8. let three right-angled Triangles be formed, ſuppoſe theſe, ... }

5	3	4
13	5	12
17	15	8

Then,

Then by the Rule of Three, the Bafes and Perpendiculars of the three right-angled Triangles fought may be found out thus;

$$\text{I.}\ \begin{cases} 5\ .\ 3\ ::\ 2\ .\ 1\frac{1}{5} & (\textit{Bafe.}) \\ 5\ .\ 4\ ::\ 2\ .\ 1\frac{3}{5} & (\textit{Perp.}) \end{cases}$$

$$\text{II.}\ \begin{cases} 13\ .\ 5\ ::\ 2\ .\ \frac{10}{13} & (\textit{Bafe.}) \\ 13\ .\ 12\ ::\ 2\ .\ 1\frac{11}{13} & (\textit{Perp.}) \end{cases}$$

$$\text{III.}\ \begin{cases} 17\ .\ 15\ ::\ 2\ .\ 1\frac{13}{17} & (\textit{Bafe.}) \\ 17\ .\ 8\ ::\ 2\ .\ \frac{16}{17} & (\textit{Perp.}) \end{cases}$$

Whence the defired fides of the three right-angled Triangles having 2 for a common Hypothenufal are found to be thefe, *viz.*

Hypoth.	*Bafes.*	*Perp.*
2	$1\frac{1}{5}$	$1\frac{3}{5}$
2	$\frac{10}{13}$	$1\frac{11}{13}$
2	$1\frac{13}{17}$	$\frac{16}{17}$

12. But note well, that in the fearch of the Triangles laft mentioned, the preparatory right-angled Triangles firft found out by the Canon in the preceding *Obfervat.* 8. muft not be like, (that is, fuch as have proportional Sides,) for it will not be difficult to apprehend, that if from them, other Triangles be deduced by the Rule of Three in fuch manner as before hath been fhewn, there will be but one right-angled Triangle found out, when many are defired to have a common Hypothenufal: That your labour therefore may not be in vain, the preparatory right-angled Triangles muft be unlike; to which end they muft be formed from pairs of numbers expreffing different Reafons, and fuch, that the two numbers by which any one of the preparatory Triangles is formed, muft not be in fuch proportion to one another as the fumm is to the difference of two numbers by which any other of the preparatory Triangles is formed. As, for example, if a right-angled Triangle be formed from 1 and 2, then another right-angled Triangle muft not be formed from 2 and 4, 3 and 6, *&c.* becaufe each of thefe pairs of numbers expreffing the fame Reafon as 1 and 2 will produce a right-angled Triangle like to the firft; nor from 3 and 1, 6 and 2, *&c.* becaufe 3 having fuch proportion to 1, likewife 6 to 2, as the fumm of 1 and 2 to their difference, thofe pairs alfo will produce right-angled Triangles like to the firft. But that two right-angled Triangles formed from pairs of numbers expreffing the fame Reafon, or from two fuch pairs, that one number of the one pair hath fuch proportion to its yoak-fellow, as the fumm of the two numbers of the other pair hath to their difference, are like, I prove thus;

	Hyp.	Bafe,	Perp.
Firft, let a right-angled Triangle be formed from two numbers s and r, fo the three fides will be thefe, *viz.*	$ss + rr$,	$ss - rr$,	$2sr$
Then let a fecond right-angled Triangle be formed from ds and dr, which have the fame proportion to one another as s and r; fo the three fides will be thefe, *viz.*	$ddss + ddrr$,	$ddss - ddrr$,	$2ddsr$
Again, let a third right-angled Triangle be formed from $s + r$ and $s - r$, *viz.* the fumm and difference of the two numbers by which the firft Triangle was formed, fo the three fides will be thefe, *viz.* . . .	$2ss + 2rr$,	$4sr$,	$2ss - 2rr$

Now I fay, that the fecond Triangle is like to the firft, for the fides of the fecond are the Products of the fides of the firft multiplied by the common Factor dd. The third Triangle is alfo like to the firft, for the fides of the third are the doubles of the fides of the firft, and confequently Proportionals to them, but in this order, *viz.* As the Hypothenufal of the firft is to its Bafe, fo is the Hypothenufal of the third to its Perpendicular; and, As the Hypothenufal of the firft is to its Perpendicular, fo is the Hypothenufal of the third to its Bafe.

13. By the help of the preceding Canon in *Obfervat.* 8. as many right-angled Triangles in whole numbers as fhall be defired, and which fhall have a common Hypothenufal, may be found out in manner following, *viz.*

Let it be required to find out three right-angled Triangles in whole numbers, which fhall have one common Hypothenufal.

First,

Firſt, by the Canon in the foregoing *Obſervat.* 8. with reſpect alſo to the *Note* in the laſt preceding Obſervation, let three unlike right-angled Triangles be formed, ſuppoſe theſe, ...	5 , 3 , 4 13 , 5 , 12 17 , 15 , 8
Secondly, multiply ſeverally the three ſides of the firſt Triangle 5, 3, 4, by 221, that is, the Product of the ſecond and third Hypothenuſals 13 and 17; ſo the three Products ſhall be the ſides of a right-angled Triangle, to wit, ...	1105 , 663 , 884
Thirdly, multiply ſeverally the three ſides of the ſecond right-angled Triangle 13, 5, 12, by 85, that is, the Product of the firſt and third Hypothenuſals 5 and 17; ſo the three Products ſhall be the ſides of this right-angled Triangle, *viz.*	1105 , 425 , 1020
Laſtly, multiply ſeverally the three ſides of the third right-angled Triangle 17, 15, 8, by 65, that is, the Product of the firſt and ſecond Hypothenuſals 5 and 13; ſo the three Products ſhall be alſo the ſides of a right-angled Triangle, to wit, ...	1105 , 975 , 520

From the premiſſes it is manifeſt that three right-angled Triangles are found out in whole numbers, having 1105 for a common Hypothenuſal, and by the ſame Method you may find out as many as you pleaſe.

14. But the ſmaller the numbers are that expreſs the ſides of thoſe preparatory right-angled Triangles the better, and therefore I think it not amiſs in this place to ſhew, how to find out all the unlike right-angled Triangles in whole numbers orderly enumerated, according as their Hypothenuſals increaſe in greatneſs, ſo, as that the greateſt Hypothenuſal may not exceed a given number, ſuppoſe 180: To which end,

Firſt, I extract the ſquare Root of 180, and find it falls between 13 and 14, and conſequently a right-angled Triangle formed from 14 and 1, will have its Hypothenuſal greater than 180; therefore all the pairs of whole numbers, which have the greater number of each part, either 14 or greater than 14, will be unfit for our preſent ſearch.

Secondly, I ſubtract 169 the Square of the ſaid 13 from 180 the given limit, and the Remainder is 11, whoſe ſquare Root falls between 3 and 4; whence 'tis evident that a right-angled Triangle formed from 13 and 4 will have a Hypothenuſal greater than 180, but 13 and 3 will give an Hypothenuſal leſs than 180; and therefore I proceed to make an orderly choice of pairs of whole numbers, from the firſt pair 2 and 1, until I come to 13 and 3 incluſive, and no farther, in this manner, *viz.*

Thirdly, I write in the firſt Column of the following Table a Series of whole numbers proceeding from 1, according to the natural order of numbers, as, 1, 2, 3, 4, 5, &c. then at the top of the ſecond Column I ſet 2 and 1 for the firſt pair, that done, I combine every number following or ſtanding underneath 2 in the firſt Column, with every one of the numbers that ſtands above ſuch following number, except in theſe two Caſes, *viz.* Firſt, when two numbers ſo combined are ſuch, that their ſumm and difference have the ſame proportion to one another as the two numbers of any pair already ſet in Column 2. then the two numbers ſo combined are to be caſt out of Column 2. As, for example, becauſe the ſumm of 3 and 1, to wit, 4, is to their difference 2, as 2 to 1, which 2 and 1 make the firſt pair already ſet in Column 2; I omit the writing of the pair 3 and 1 in the ſecond Column: And for the ſame Reaſon the pair 5 and 1 is not inſerted in the ſecond Column; for the ſumm of 5 and 1, to wit, 6, is to their difference 4, as 3 to 2, which 3 and 2 made the ſecond pair before written in Column 2. and in like manner all other pairs cauſing that effect are to be excluded out of the ſecond Column. Again, when two numbers combined as aforeſaid happen to be in the ſame proportion as the two numbers of any pair already ſet in the ſecond Column, then alſo the two numbers ſo combined are to be excluded out of the ſaid Column 2; ſo 4 and 2 having the ſame Reaſon as the firſt pair 2 and 1, are not inſerted in the ſecond Column: the reaſon of excluding all pairs in thoſe two Caſes is, for that they would produce right-angled Triangles like to others before produced, which is contrary to the import of the Propoſition. So at length I find only thirty-two pairs of numbers that are fit to be inſerted in the ſaid ſecond Column.

Fourthly, from every one of thoſe thirty-two pairs of numbers in the ſecond Column, (the laſt of which pairs is 13 and 2,) I form a right-angled Triangle (by the Canon in the foregoing *Obſervat.* 8.) and inſert thoſe Triangles into Column 3, among which I find five, to wit, thoſe formed from the pairs 10 and 9; 11 and 8; 11 and 10;

12 and 7;

12 and 7; and 12 and 11, whose Hypothenusals exceed 180 the prescribed limit, and therefore I cast away those five Triangles, and transferr the rest, which are 27 in multitude, into the fourth Column, in such order as the Hypothenusals do increase in greatness. So 27 unlike right-angled Triangles are found out, which are all that can be given in whole numbers, so as that the greatest Hypothenusal may not exceed 180, as was required. But for further illustration of the premisses view the following Table.

A Table whose fourth Column contains 27 unlike right-angled Triangles in numbers, orderly enumerated according as their Hypothenusals *increase in greatness.*

		H. B. P	H. B. P	H. B. P	
1	2, 1	5. 3. 4	5. 3. 4	5. 3. 4	8
2	3, 2	13. 5. 12	13. 5. 12	13. 5. 12	12
3	4, 1	17. 15. 8	17. 15. 8		
4	4, 3	25. 7. 24	25. 7. 24	25. 7. 24	16
5	5, 2	29. 21. 20	29. 21. 20		
6	5, 4	41. 9. 40	37. 35. 12	41. 9. 40	20
7	6, 1	37. 35. 12	41. 9. 40		
8	6, 5	61. 11. 60	53. 45. 28	61. 11. 60	24
9	7, 2	53. 45. 28	61. 11. 60		
10	7, 4	65. 33. 56	65. 33. 56		
11	7, 6	85. 13. 84	65. 63. 16	85. 13. 84	28
12	8, 1	65. 63. 16	73. 55. 48		
13	8, 3	73. 55. 48	85. 13. 84		
14	8, 5	89. 39. 80	85. 77. 36		
15	8, 7	113. 15. 112	89. 39. 80	113. 15. 112	32
16	9, 2	85. 77. 36	97. 65. 72		
17	9, 4	97. 65. 72	101. 99. 20		
18	9, 8	145. 17. 144	109. 91. 60	145. 17. 144	36
19	10, 1	101. 99. 20	113. 15. 112		
20	10, 3	109. 91. 60	125. 117. 44		
21	10, 7	149. 51. 140	137. 105. 88		
22	10, 9	181. 19. 180	145. 17. 144	181. 19. 180	40
23	11, 2	125. 117. 44	145. 143. 24		
24	11, 4	137. 105. 88	149. 51. 140		
25	11, 6	157. 85. 132	157. 85. 132		
26	11, 8	185. 57. 176	169. 119. 120		
27	11, 10	221. 21. 220	173. 165. 52	221. 21. 220	44
28	12, 1	145. 143. 24			
29	12, 5	169. 119. 120			
30	12, 7	193. 95. 168			
31	12, 11	265. 23. 264		265. 23. 264	
32	13, 2	173. 165. 52			

15. By inspecting the preceding Table we may perceive that the unlike right-angled Triangles in Column 5. which are formed from 2 and 1; 3 and 2; 4 and 3; 5 and 4; *&c.* *viz.* from pairs of such whole numbers as differ by Unity, have these properties, namely;

First, their Bases 3, 5, 7, 9, 11, 13, *&c.* which you see standing under *B* in the fifth Column are in an Arithmetical Progression proceeding from the Base 3 of the Primitive right-angled Triangle 5, 3, 4 by the common difference 2.

Secondly, if an Arithmetical Progression be formed from 8 as the first and least Term, and the common difference of the Terms be 4; as this Progression 8, 12, 16, 20, *&c.* (which is placed in the last Columel of the Table,) then 8 the first Term added to the first Hypothenusal 5, makes the second Hypothenusal 13 standing under *H* in the fifth Column; also 12 the second Term of the same Progression added to 13 the second Hypothenusal, gives 25 the third Hypothenusal in the same Column; and 16 the third Term added to 25 the third Hypothenusal, gives 41 the fourth Hypothenusal, and so forwards continually,

continually. In like manner, 8 the firſt Term of the ſame Progreſſion added to 4 the firſt Perpendicular, gives 12 the ſecond Perpendicular, ſtanding under *P* in the ſaid fifth Column; alſo 12 the ſecond Term added to 12 the ſecond Perpendicular, gives 24 the third Perpendicular; and 16 the third Term added to 24 the third Perpendicular, makes 40 the fourth Perpendicular; and ſo forwards perpetually. So that by the help of the Primitive right-angled Triangle 5, 3, 4, and the ſaid Progreſſion 8, 12, 16, 20, 24, &c. innumerable unlike right-angled Triangles may be found out by Addition only.

Thirdly, the difference between the Hypothenuſal and Perpendicular of every one of the ſaid Triangles in Column 5. is Unity.

Fourthly, the Baſe is equal to the ſumm of the two numbers forming the Triangle.

Fifthly, the ſumm of the Hypothenuſal and Perpendicular is a Square, whoſe ſide is equal to the Baſe, or ſumm of the two numbers forming the Triangle; therefore,

Sixthly, if the ſumm of the Hypothenuſal and Perpendicular be multiplied into the Baſe, the Product ſhall be a Cube, whoſe ſide is equal to the Baſe.

Seventhly, the difference of the Hypothenuſal and Baſe is equal to the double of the Square of the leſſer of the two numbers forming the Triangle.

The certainty of all the ſaid Properties will be apparent, if you form right-angled Triangles from theſe following pairs of numbers, and compare thoſe Triangles to one another, according to the import of the ſaid Properties.

a, $a+1$	the firſt pair;	$a+2$, $a+3$,	the third pair;
$a+1$, $a+2$	the ſecond pair;	$a+3$, $a+4$,	the fourth pair, &c.

QUEST. 2. (Quæſt. 10. Lib. 2. *Diophant.*)

To divide (13) a number given, which is compos'd of two Squares, (9 and 4,) into two other Squares.

RESOLUTION 1.

1. The ſide or ſquare Root of 9 the greater Square given is	3
2. The ſide of the leſſer Square 4 is	2
3. Let the ſide of the firſt of the two Squares ſought be aſſumed to be	$a+2$
4. And let the ſide of the ſecond Square ſought be feigned	$2a-3$; or, $3-2a$
5. Therefore from the third ſtep the firſt Square deſired is	$aa+4a+4$
6. And from the fourth ſtep the ſecond Square ſought is . .	$4aa-12a+9$
7. Therefore the ſumm of the two Squares ſought is . .	$5aa-8a+13$
8. Which ſumm laſt expreſt muſt be equal to the given number 13, hence this Equation ariſeth, *viz.*	$5aa-8a+13=13$
9. And that Equation, after due Reduction, gives . . .	$a=\frac{8}{5}$
10. Therefore from the ninth and third ſteps, the ſide of the firſt Square ſought is made known, *viz.*	. . $\frac{18}{5}$
11. And from the ninth and fourth ſteps, the ſide of the ſecond Square ſought is likewiſe diſcovered, *viz.* . . .	. . $\frac{1}{5}$

So the ſides of the two Squares ſought are found $\frac{18}{5}$ and $\frac{1}{5}$; for $\frac{324}{25}$ the Square of $\frac{18}{5}$, added to $\frac{1}{25}$ the Square of $\frac{1}{5}$, makes $\frac{325}{25}$, that is, 13; as was required.

This Queſtion is of the ſame nature with the foregoing, and deſerves to be ranked among the moſt excellent Problems; for it affords divers admirable Canons concerning the conſtruction of Right-angled Triangles, and is of great uſe for the underſtanding of many of *Diophantus*'s Queſtions, eſpecially in his fifth Book; I ſhall therefore firſt explain the preceding Numeral Reſolution of the Queſtion, and afterwards reſolve the ſame by Literal *Algebra.*

Obſervations upon Queſt. 2.

1. It is evident by the foregoing Reſolution of *Diophantus*, That after $a+2$ and $2a-3$, or $3-2a$ are feigned to be the ſides of the two Squares ſought, the ſumm of thoſe Squares, that is, $5aa-8a+13$, is equated to the given number 13. *viz.* $5aa-8a+13=13$; which Equation, if there were not the ſame Abſolute number 13 in each part, could not be reduced to an Equation between ſome number of aa and ſome number of a, and conſequently the number a would not be Rational, unleſs by meer chance: Whence

Whence then comes it to pass, that the same Absolute number 13 is found in each part of the said Equation? If the Operation be well examin'd, it will appear that the numbers 2 and 3 in the feigned sides of the two Squares sought are the sides of the two given Squares 4 and 9; which 2 and 3 are the only numbers that can be used in the said feigned sides, to cause the number 13 to be found in the summ of their Squares.

2. As to the Signs to be prefixt before the given sides 2 and 3 in the feigned sides of the two Squares sought, they must necessarily be either both —, or one of them +, and the other —, to the end that in the summ of the feigned Squares there may be some number of a with the sign — prefixt; whence it will follow that the said number of a may be transferr'd to the other part of the Equation with the sign +, and then the Absolute numbers vanishing by Subtraction, because they are one and the same number as hath been shewn in the preceding *Observat.* 1. there will remain an Equation between some number of aa and some number of a; whence by due Division the number a will be Rational.

3. The numbers to be prefixt before a in the feigned sides of the two Squares sought, may be variously chosen according to divers particular Rules that might be given, among which I shall recommend but two to the Learner's practice: The first Rule is this;

Let two unequal numbers be taken to be prefixt before a in the feigned sides, but with this Caution, *viz.* That the greater of the two numbers taken may not have the same proportion to the lesser as the summ of the sides of the two Squares given in the Question hath to their difference: As, if the two Squares given be 4 and 9, whose sides are 2 and 3, the greater of the two numbers taken must not be to the lesser as 5 to 1, because 5 is the summ, and 1 the difference of the said 2 and 3. Suppose therefore that 5 and 3 be taken; then let the first feigned side be $3a + 2$, (3 being the lesser of the said two numbers taken, and 2 the lesser of the sides of the two Squares given,) and let the second feigned side be $5a - 3$, or $3 - 5a$, (5 being the greater of the two numbers taken, and 3 the side of the greater Square given:) Now if from those feigned sides the Operation be prosecuted like as in the preceding Resolution of *Quest.* 2. an Equation will rightly ensue to find out two Squares different from those given, but such as being added together shall make the same summ as those given.

The second Rule is this; Let two unequal numbers be taken with this Caution, *viz.* That they be not in the same Reason (or Proportion) as the sides of the two Squares given: As, if the two Squares given in the Question be 9 and 4, whose sides are 3 and 2, then the two numbers taken must not be 3 and 2, 6 and 4, 9 and 6, nor any numbers in the same Reason: Suppose therefore that 5 and 4 be chosen; then for the side of the first Square sought put $4a - 2$, or $2 - 4a$, (4 being the lesser of the said two numbers chosen, and 2 the lesser of the sides of the two Squares given,) and for the side of the second Square sought put $5a - 3$, or $3 - 5a$, (5 being the greater of the two numbers before chosen, and 3 the greater of the sides of the two Squares given;) then if from the said feigned sides the Operation be prosecuted like as in the foregoing Resolution of *Quest.* 2. an Equation will ensue, to find out two Squares different from those given, but such as being added together shall make the same summ as those given. The reason of these two Cautions will hereafter appear.

The preceding Observations may suffice for explication of the Resolution of *Quest.* 2. by Numeral *Algebra*; I shall in the next place shew how to resolve the same by Literal *Algebra*, and among various ways that might be used, I shall chuse but two, which correspond with the Rules before given in *Observat.* 3. and do produce divers excellent Canons.

RESOLUTION 2. of *Quest.* 2. which is here repeated, *viz.*

To divide a number given which is compos'd of two known Squares, into two other Squares.

1. For the side of the greater Square given, put . . . > d
2. And for the side of the lesser Square given, put . . . > b
3. Therefore the greater Square is . . . . > dd
4. And the lesser Square is . . . . > bb
5. Take two unequal numbers, s the greater, and r the lesser, with this Caution, *viz.* that s be not in such proportion to r, as $d + b$ to $d - b$; which numbers s and r are to be used instead of the numbers that were prefixt before the unknown number a in the

the foregoing Numeral Reſolution of this Queſtion, and the reaſon of the Caution will be ſhewn in the ſixteenth ſtep of this Reſolution.

6. For the ſide of the firſt of the two Squares ſought, put . . . . $ra + b$
7. And for the ſide of the ſecond Square ſought put $sa - d$, or, $d - sa$
8. Therefore from the ſixth ſtep the firſt Square ſought is . . . . $rraa + 2rba + bb$
9. And from the ſeventh ſtep the ſecond Square ſought is . . . . $ssaa - 2sda + dd$
10. Therefore the ſumm of thoſe two Squares is $ssaa + rraa + 2rba - 2sda + bb + dd$
11. But the ſaid ſumm muſt be equal to $dd + bb$ the ſumm of the two Squares given in the Queſtion, and before expreſt in the third and fourth ſteps; hence the following Equation ariſeth, *viz.*

$$ssaa + rraa + 2rba - 2sda + bb + dd = bb + dd.$$

12. Which Equation, after due Reduction, gives $a = \dfrac{2sd - 2rb}{ss + rr}$
13. Therefore from the twelfth and ſixth ſteps, the ſide of the firſt Square ſought is now made known, and found equal to this following Quantity,

$$\frac{2rsd + ssb - rrb}{ss + rr}.$$

14. And from the twelfth and ſeventh ſteps, the ſide of the ſecond Square ſought is likewiſe known, and found equal to

$$\frac{ssd - rrd - 2rsb}{ss + rr}, \quad \text{or}, \quad \frac{2rsb + rrd - ssd}{ss + rr};$$

That is to ſay, The former of thoſe two Quantities expreſt Fraction-wiſe ſhall be the ſide of the ſecond Square when $ssd - rrd$ is greater than $2rsb$, but the latter of thoſe Quantities ſhall be the ſaid ſide when $ssd - rrd$ is leſs than $2rsb$. For if $ssd - rrd$ be greater than $2rsb$, then by ſubtracting $2rsb$ from $ssd - rrd$, the Remainder is the ſame with the Numerator of the firſt of the two Fractions above expreſt; but if $2rsb$ be greater than $ssd - rrd$, then by ſubtracting $ssd - rrd$ from $2rsb$, the Remainder is the ſame with the Numerator of the latter of the ſaid Fractions; therefore the ſide of the ſecond Square may be expreſt thus,

$$\frac{\overline{ssd - rrd} \backsim 2rsb}{ss + rr},$$

That is to ſay, *If the difference* between $ssd - rrd$ and $2rsb$ be divided by $ss + rr$, the Quotient ſhall be the ſide of the ſecond Square ſought.

From the premiſſes ariſeth this following

CANON 1.

15. Take two unequal numbers, with this Caution, *viz.* That the greater may not have the ſame proportion to the leſſer, as the ſumm of the ſides of the two Squares given hath to the difference of the ſame ſides: Multiply the double Product of the multiplication of thoſe two numbers firſt taken by each of the ſaid two ſides given, and reſerve the Products; multiply alſo the difference of the Squares of the ſaid two numbers firſt taken by each of the ſaid two ſides given, and reſerve theſe Products; then add the greater of the two firſt reſerved Products to the leſſer of the two latter, and reſerve the ſumm for a Dividend; take alſo the difference between the leſſer of the two firſt Products and the greater of the two latter for a ſecond Dividend; laſtly, divide ſeverally the ſaid Dividends by the ſumm of the Squares of the two numbers firſt taken, ſo ſhall the Quotients be the ſides of the two Squares ſought.

Example 1. *Where the number given is compos'd of two unequal Squares.*

Let it be required to divide 13 which is compos'd of two Squares, 9 and 4, into two other Squares.

The ſide of the greater Square given is 3
The ſide of the leſſer Square given is 2
Take two unequal numbers, with reſpect to the Caution in the Canon, as, theſe, ... 2 and 1
Then by uſing thoſe four numbers as the Canon doth direct, the ſides of the two Squares ſought will be found theſe, ... $\frac{18}{5}$ and $\frac{1}{5}$.

The

The Squares of which sides being added together make 13, as was required.

Example 2.

Let it again be required to divide 13, which is compos'd of two Squares, 9 and 4, into two other Squares different from those found out in *Example* 1.

The sides of the two Squares given are . . . . > 3 and 2

Take two unequal numbers with respect to the Caution in the foregoing *Canon* 1. as these, ... } 4 and 3

Then by using the four numbers last before exprest, as the said Canon doth direct, the sides of the two Squares sought will be found these, . . } $\frac{86}{25}$ and $\frac{27}{25}$

The Proof.

The Square of $\frac{86}{25}$ is . . . . $\frac{7396}{625}$,

The Square of $\frac{27}{25}$ is . . . . $\frac{729}{625}$,

The summ of those Squares is . . . . $\frac{8125}{625}$, or 13.

Example 3. Where the given number is compos'd of two equal Squares.

Let it be required to divide 2, which is compos'd of two equal Squares, 1 and 1, into two unequal Squares.

The side of either of the Squares given is . . . . > 1

Take in this Case any two unequal numbers, as . . . . > 1 and 2

Then by working with those three numbers according to the direction of *Canon* 1. the sides of the two Squares sought will be found these, . . } $\frac{7}{5}$ and $\frac{1}{5}$

The Squares of which sides being added together make 2, as may easily be proved.

16. Now that the necessity of the Caution prescribed in the foregoing *Canon* 1. about chusing the unequal numbers s and r may appear, I shall prove, That if s the greater of them hath the same proportion to r the lesser, as $d+b$ the summ of the sides of the two unequal Squares given in *Quest.* 2. hath to $d-b$ the difference of the same sides, then the said Canon will produce the same sides d and b for the sides of the two Squares sought, and consequently the Operation in such Case will be in vain. First, it is manifest by the thirteenth step, that one of the sides found out by the Canon is $\frac{2rsd+ssb-rrb}{ss+rr}$; so that if we prove this side to be equal to d the side of the greater of the two Squares given, then consequently the other side found out by the Canon, that is, the side exprest by the fourteenth step, shall be equal to the side of the lesser of the two Squares given; for the summ of the Squares found out is equal to the summ of those given.

17. Let it therefore be supposed that . . . . > $d+b \,.\, d-b :: s \,.\, r$

18. And then, we are to demonstrate that . . . . . } $\frac{2rsd+ssb-rrb}{ss+rr} = d$

Demonstration.

19. By supposition in the seventeenth step, ... > $d+b \,.\, d-b :: s \,.\, r$

20. Therefore by comparing the Rectangle of the extremes to the Rectangle of the means, ... } $rd+rb = sd-sb$

21. And by adding sb to each part of the last Equation, this ariseth, ... } $sb+rd+rb = sd$

22. And by subtracting rd from each part, it makes . . > $sb+rb = sd-rd$

23. And by resolving the last Equation into Proportionals, this Analogy ariseth, *viz.* . . . . } $s+r \,.\, s-r :: d \,.\, b$

24. And by drawing $s-r$ as a common Factor into the two first Terms of that Analogy, this ariseth,

$$ss-rr \,.\, ss+rr-2rs :: d \,.\, b.$$

25. Therefore, by comparing the Product of the extremes in the last Analogy to the Product of the means, this Equation ariseth, *viz.*

$$ssb-rrb = ssd+rrd-2rsd.$$

26. Whence by equal Addition of $2rsd$, this Equation ariseth, *viz.*

$$2rsd+ssb-rrb = ssd+rrd.$$

27. Wherefore by dividing each part of the last Equation by $ss+rr$, this ariseth, *viz.*

$$\frac{2rsd+ssb-rrb}{ss+rr} = d.$$ Which was to be demonstrated.

Resolution 3.

RESOLUTION 3. of *Quest.* 2. which is here repeated, *viz.*

To divide a given number which is compos'd of two known Squares, into two other Squares.

1. For the side of the greater Square given, put . . . . . . d
2. And for the side of the lesser Square given, put . . . . b
3. Therefore the greater of those Squares shall be . . . . dd
4. And the lesser . . . . bb
5. Take two unequal numbers, s the greater, and r the lesser, with this Caution, *viz.* That s may not be in such proportion to r, as d to b; which s and r do represent the numbers to be prefixt to the unknown number a, according to the second Rule before mentioned in *Observat.* 3. *Resolut.* 1. *Quest.* 2. and the reason of the Caution will be shewn in the sixteenth step of this Resolution.
6. Then for the side of the first Square sought, put . . . . $ra - b$, or, $b - ra$
7. And for the side of the second Square sought, put . . . $sa - d$, or, $d - sa$
8. Therefore from the sixth step the first Square sought is . . $rraa - 2rba + bb$
9. And from the seventh step the second Square sought is . . . $ssaa - 2sda + dd$
10. Therefore the summ of the Squares in the eighth and ninth steps is
$$ssaa + rraa - 2rba - 2sda + bb + dd.$$
11. But the said summ must be equal to the two Squares given, to wit, dd and bb, hence therefore ariseth the following Equation, *viz.*
$$ssaa + rraa - 2rba - 2sda + bb + dd = bb + dd.$$
12. Which Equation, after due Reduction, gives . . . . . $a = \frac{2rb + 2sd}{ss + rr}$
13. Therefore from the twelfth and sixth steps the side of the first Square sought is now made known, and equal to one of these two Quantities, to wit,
$$\frac{2rsd + rrb - ssb}{ss + rr}, \text{ or, } \frac{ssb - rrb - 2rsd}{ss + rr};$$
That is to say, the former of those two Quantities exprest Fraction-wise shall be the side of the first Square sought, when $ssb - rrb$ is less than $2rsd$; but the latter shall be the said side when $ssb - rrb$ is greater than $2rsd$. For if $ssb - rrb$ be less than $2rsd$, then by subtracting $ssb - rrb$ from $2rsd$, the Remainder will be the same with the Numerator of the first of the two Quantities above exprest Fraction-wise; but if $ssb - rrb$ be greater than $2rsd$, then by subtracting $2rsd$ from $ssb - rrb$, the Remainder will be the same with the Numerator of the latter of the said Quantities: Therefore the side of the first Square sought may be exprest thus,
$$\frac{\overline{ssb - rrb} \backsim 2rsd}{ss + rr};$$
That is to say, If the difference between $\overline{ssb - rrb}$ and $2rsd$, be divided by $ss + rr$ the Quotient shall be the side of the first Square sought.
14. But from the twelfth and seventh steps the side of the second Square will be found equal to this known Quantity, *viz.* $\frac{2rsb + ssd - rrd}{ss + rr}$.

From the premisses ariseth this following

CANON 2.

15. Take two unequal numbers, with this Caution, *viz.* That the greater may not have the same proportion to the lesser, as the side of the greater of the two Squares given hath to the lesser side: Multiply the double Product of the multiplication of the two unequal numbers first taken by each of the said two sides given, and reserve the Products; multiply also the difference of the Squares of the two numbers first taken, by each of the said two sides given, and reserve these Products; then take the difference between the greater of the two first reserved Products and the lesser of the two latter for a Dividend; take also the summ of the lesser of the two first Products and the greater of the two latter for a second Dividend; lastly, divide each of those Dividends by the summ of the Squares of the two numbers first taken, so shall the Quotients be the sides of the two Squares sought.

Example 1.

Example 1. Where the number given to be divided is compos'd of two unequal Squares.

Let it be required to divide 13, which is compos'd of two Squares, 9 and 4, into two other Squares.

The side of the greater Square given is 3

The side of the lesser Square given is 2

Take two unequal numbers, with respect to the Caution in *Canon* 2. as these, ... 2 and 1

Then by using those four numbers according to the direction of *Canon* 2. the sides of the two Squares sought will be found these, *viz.* . . . . $\frac{6}{5}$ and $\frac{17}{5}$

The Squares of which sides $\frac{6}{5}$ and $\frac{17}{5}$ being added together make $\frac{325}{25}$ or 13, as was required.

Example 2.

Let it be again required to divide 13, which is compos'd of 9 and 4 into two other Squares different from those found out in *Example* 1.

The sides of the two given Squares, 9 and 4, are . . . . 3 and 2

Take two unequal numbers with respect to the Caution in *Canon* 2. as these, ... 4 and 1

Then by working with those four numbers as the said *Canon* 2. doth direct, the sides of the two Squares sought will be found these, ... $\frac{6}{17}$ and $\frac{61}{17}$

The Squares of which sides are $\frac{36}{289}$ and $\frac{3721}{289}$, whose summ makes $\frac{3757}{289}$, that is, 13, as was required.

Example 3. Where the number given to be divided is compos'd of two equal Squares.

Let it be required to divide 18, which is compos'd of two equal Squares, 9 and 9, into two unequal Squares.

The side of either of the Squares given is 3

Take in this Case any two unequal numbers, as . . . . 1 and 2

Then by using those three numbers according to the direction of the foregoing *Canon* 2. the sides of the two Squares sought will be found these, *viz.* . . . . $\frac{3}{5}$ and $\frac{21}{5}$

The Squares of which sides are $\frac{9}{25}$ and $\frac{441}{25}$, whose summ makes $\frac{450}{25}$, that is 18, as was required.

16. Now that the necessity of the Caution prescribed in the foregoing *Canon* 2. about chusing the unequal numbers s and r may appear, I shall prove, That if s the greater of them hath the same proportion to r the lesser, as d the side of the greater of the two Squares given in *Quest.* 2. hath to b the side of the lesser of the same Squares, then the said Canon will produce the same sides d and b for the sides of the two Squares sought, and consequently the Operation in such Case will be in vain: First, it is manifest by the fourteenth step, that one of the sides found out by the Canon is $\frac{ssd - rrd + 2rsb}{ss + rr}$; so that if we prove this side to be equal to d the side of the greater of the two Squares given, then consequently the other side found out by the Canon, that is, the side exprest in the thirteenth step shall be equal to the side of the lesser of the two Squares given, for the summ of the Squares found out is equal to the summ of those given.

17. Let it therefore be supposed that . . . . $d . b :: s . r$

18. And then we are to demonstrate that . . . . $\frac{ssd - rrd + 2rsb}{ss + rr} = d$

Demonstration.

19. By supposition in the sixteenth step, ... $d . b :: s . r$

20. Therefore by comparing the Rectangle of the means to the Rectangle of the extremes, ... $sb = rd$

21. And by drawing $2r$ into each part of the last Equation, this ariseth, *viz.* . . . . $2rsb = 2rrd$

22. And by adding ssd to each part of the last Equation, it gives . . . . $ssd + 2rsb = ssd + 2rrd$

23. And by subtracting rrd from each part of the last Equation, there remains . . . . $ssd - rrd + 2rsb = ssd + rrd$

24. Where-

24. Wherefore by dividing each part of the laſt Equation by $ss + rr$, there ariſeth . . . . $\left.\right\}\ \frac{ssd - rrd + 2rsb}{ss + rr} = d$

Which was to be proved.

Obſervations upon the preceding Reſolutions 2, *and* 3. *of* Queſt. 2. *by Literal* Algebra.

1. If z be put equal to $\sqrt{:\overline{bb + dd}:}$ that is, the ſquare Root of the number compos'd of two Squares given in *Queſt.* 2. that Queſtion may be ſtated thus, *viz.*

Two Rational numbers, b and d, being given for the Baſe and Perpendicular of a right-angled Triangle whoſe Hypothenuſal is z, Rational or Irrational; to find out other Rational numbers to expreſs the Baſe and Perpendicular of a ſecond right-angled Triangle whoſe Hypothenuſal ſhall be z likewiſe.

The Baſe and Perpendicular of the Triangle ſought ſhall be given either by *Canon* 1. in the fifteenth ſtep of *Reſolution* 2. of *Queſt.* 2. or by *Canon* 2. in the fifteenth ſtep of *Reſolution* 3. and may be expreſt by Letters, as before in the thirteenth and fourteenth ſteps of *Reſolution* 2. or by the thirteenth and fourteenth ſteps of *Reſolution* 3. *viz.*

By *Canon* 1. | By *Canon* 2.

$$\triangle K.\left\{\begin{array}{ll} \text{Hyp.} & z, (\text{or}, \sqrt{:\overline{bb + dd}:}) \\ \text{Perp.} & \frac{2rsd + ssb - rrb}{ss + rr} \\ \text{Baſe,} & \frac{ssd - rrd \backsim 2rsb}{ss + rr} \end{array}\right. \quad\Bigg|\Bigg|\quad \left.\begin{array}{ll} \text{Hyp.} & z, (\text{or}, \sqrt{:\overline{bb + dd}:}) \\ \text{Perp.} & \frac{ssb - rrb \backsim 2rsd}{ss + rr} \\ \text{Baſe,} & \frac{2rsb + ssd - rrd}{ss + rr} \end{array}\right\}\triangle L.$$

2. If the Baſes and Perpendiculars of thoſe two right-angled Triangles above-expreſt, which I call $\triangle$K and $\triangle$L, be well examined, another way will be diſcovered to find out the ſame Baſes and Perpendiculars by the help of the Baſes and Perpendiculars of two like right-angled Triangles whoſe Hypothenuſals are b and d. For,

Firſt, it is manifeſt by *Obſervat.* 5. *Reſolut.* 2. *Queſt.* 1. of this Book, that theſe three following numbers will conſtitute a right-angled Triangle, which hath b for an Hypothenuſal, *viz.*

$$\begin{array}{ccccc} \text{Hyp.} & & \text{Baſe,} & & \text{Perp.} \\ b & . & \frac{ssb - rrb}{ss + rr} & . & \frac{2rsb}{ss + rr} \end{array} \quad (\triangle\text{M.})$$

Likewiſe theſe three following numbers will conſtitute a right-angled Triangle, having d for an Hypothenuſal, *viz.*

$$\begin{array}{ccccc} \text{Hyp.} & & \text{Baſe,} & & \text{Perp.} \\ d & . & \frac{ssd - rrd}{ss + rr} & . & \frac{2rsd}{ss + rr} \end{array} \quad (\triangle\text{N.})$$

Which two Triangles laſt before expreſt, to wit, $\triangle$M and $\triangle$N, are like, for each of them is like to a right-angled Triangle whoſe three ſides are $ss + rr$, $ss - rr$, and $2rs$; Now I ſay, if the Perpendiculars and Baſes of the two right-angled Triangles K and L before expreſt in *Obſervat.* 1. be well viewed, it will be evident, that they are deduced from the two like right-angled Triangles M and N before exprreſt in this *Obſervat* 2. which have b and d for Hypothenuſals. For, firſt, the Perpendicular of $\triangle$K is compos'd of the Baſe of $\triangle$M and the Perpendicular of $\triangle$N; ſecondly, the Baſe of $\triangle$K is equal to the difference between the Perpendicular of $\triangle$M and the Baſe of $\triangle$N; thirdly, the Perpendicular of $\triangle$L is equal to the difference between the Baſe of $\triangle$M and the Perpendicular of $\triangle$N; laſtly, the Baſe of $\triangle$L is compos'd of the Perpendicular of $\triangle$M and the Baſe of $\triangle$N.

3. Hence therefore another Canon comes to light, to ſolve as well the preceding *Queſt.* 2. as alſo the following Propoſition, (which is *Prop.* 47. in *pag.* 35. of *Vieta's* Works,) *viz.*

From two right-angled Triangles given to deduce a third right-angled Triangle, ſuch, that the Square of the Hypothenuſal of the third may be equal to the Squares of the Hypothenuſals of the firſt and ſecond.

This Propoſition may be ſolved by the following

CANON 3.

CANON 3.

First, (by the Canon in *Observat.* 10. *Resolut.* 2. of the preceding *Quest.* 1.) find out two like right angled Triangles in numbers, such, that their Hypothenusals may be the sides of the two Squares given in the foregoing *Quest.* 2. then, for the Perpendicular of the third right-angled Triangle sought, take the summ of the Base of the first and Perpendicular of the second; for the Base of the third, take the difference between the Perpendicular of the first and Base of the second; and the Hypothenusal of the third shall be the square Root of the summ of the Squares of the Hypothenusals of the first and second.

Or thus:

For the Perpendicular of the third right-angled Triangle sought, take the difference between the Base of the first and Perpendicular of the second, for the Base of the third, take the summ of the Perpendicular of the first and Base of the second; and the Hypothenusal of the third shall be the same as before is exprest.

Example 1. *in Numbers.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two other Squares.

The sides of 4 and 9 the two Squares given are . . .	2 and 3		
	Hyp.	*Base.*	*Perp.*
Find a first right-angled Triangle in numbers whose Hypothenusal shall be 2, the side of 4 the lesser of the two Squares given, as, ...	2 ,	$\frac{6}{5}$,	$\frac{8}{5}$
Find likewise a second right-angled Triangle like to the first, and such, that its Hypothenusal may be 3 the side of the greater Square given, as, ...	3 ,	$\frac{9}{5}$,	$\frac{12}{5}$
Then by using the Bases and Perpendiculars of those two like right-angled Triangles as *Canon* 3. doth direct, this third right-angled Triangle will be found out, whose Hypothenusal is equal to $\sqrt{:4+9}$: that is, $\sqrt{13}$, and consequently the Base and Perpendicular are the sides of the two Squares sought,	$\sqrt{13}$,	$\frac{1}{5}$,	$\frac{18}{5}$
Or, according to the latter part of *Canon* 3. the sides of the two Squares sought will be found the Base and Perpendicular of this third right-angled Triangle whose Hypothenusal is $\sqrt{13}$, that is, $\sqrt{:4+9}$: as before;	$\sqrt{13}$,	$\frac{17}{5}$,	$\frac{6}{5}$

Example 2.

Let it be required to divide 25, which is composed of two Squares, 9 and 16, into two other Squares.

The sides of 9 and 16 the two given Squares are . . .	3 and 4		
	Hyp.	*Base,*	*Perp.*
Find a first right-angled Triangle in numbers, whose Hypothenusal shall be 3 the side of the lesser Square given, as,	3 ,	$\frac{36}{13}$,	$\frac{15}{13}$
Find likewise a second right-angled Triangle like to the first, and such, that its Hypothenusal shall be 4 the side of the greater Square given, as, ...	4 ,	$\frac{48}{13}$,	$\frac{20}{13}$
Then by using the Bases and Perpendiculars of the two like right-angled Triangles last found out, according to the direction of *Canon* 3. this third right-angled Triangle will be dicovered, whose Hypothenusal is 5, that is, $\sqrt{:9+16}$: and consequently the Base and Perpendicular are, the sides of the two Squares sought, ...	5 ,	$\frac{33}{13}$,	$\frac{56}{13}$
Or, according to the latter part of *Canon* 3. the sides of the two Squares sought will be found the Base and Perpendicular of this third right-angled Triangle, whose Hypothenusal is 5, that is, $\sqrt{:9+16}$: as before, ...	5 ,	$\frac{63}{13}$,	$\frac{16}{13}$

4. If every one of the three sides of the two right-angled Triangles K and L before exprest in *Observat.* 1. having z for a common Hypothenusal, Rational or Irrational, be multiplied by $ss+rr$, the Products shall be also the sides of two right-angled Triangles like to the two former respectively; which Products or sides shall be these, *viz.*

Hyp.

Hyp.	$zss + zrr$	Hyp.	$zss + zrr$
Perp.	$2rsd + ssb - rrb$	Perp.	$ssb - rrb \backsim 2rsd$
Baſe,	$ssd - rrd \backsim 2rsb$	Baſe,	$2rsb + ssd - rrd$

Now if the two right-angled Triangles laſt expreſt be well examined, it will appear, that each of them may be deduced from two right-angled Triangles, one of which hath for its Hypothenuſal $ss + rr$, Baſe $ss - rr$, and Perpendicular $2rs$, (or $2rs$ may be called the Baſe, and $ss - rr$ the Perpendicular ;) but of the other the Hypothenuſal is $z = \sqrt{bb + dd}$: Rational or Irrational, the Baſe is b, and the Perpendicular is d, (or d may be called the Baſe, and b the Perpendicular ;) I ſay, from theſe two laſt mentioned Triangles each of the two former may be deduced in ſuch manner as is directed in the following *Canon*, which is the ſame with that raiſed by *Vieta* in ſolving *Prop.* 46. in *pag.* 34. of his Works, *viz.*

From two right-angled Triangles given, to form a third right-angled Triangle.

CANON.

For the Hypothenuſal of the third right-angled Triangle, take the Product of the multiplication of the Hypothenuſals of the two right-angled Triangles given : for the Perpendicular, the ſumm of the Product of the Baſe of the firſt into the Perpendicular of the ſecond, and the Product of the Baſe of the ſecond into the Perpendicular of the firſt : and for the Baſe, take the difference between the Product of the Baſes of the firſt and ſecond, and the Product of their Perpendiculars.

Or thus :

For the Hypothenuſal of the third right-angled Triangle, take (as before) the Product of the multiplication of the Hypothenuſals of the firſt and ſecond right-angled Triangles given : for the Perpendicular, the difference between the Product of the Baſe of the firſt into the Perpendicular of the ſecond, and the Product of the Baſe of the ſecond into the Perpendicular of the firſt : laſtly, for the Baſe, take the ſumm of the Product of the Baſes of the firſt and ſecond, and the Product of their Perpendiculars.

An Example in Numbers.	*Hyp.*	*Baſe,*	*Perp.*
Let there be two right-angled Triangles given in numbers, ſuppoſe theſe, ...	5 ;	3 ,	4
	13 ,	5 ,	12
Then from thoſe Triangles, theſe two are deduced by the two Canons laſt before expreſt, *viz.*	65 ,	33 ,	56
	65 ,	63 ;	16

Note 1. If the two right-angled Triangles given be unlike, then either of thoſe Canons will form a third right-angled Triangle ; but if like, then the firſt only will take place : for when the two right-angled Triangles given are like, then the difference of the Products mentioned in the latter *Canon* are equal to nothing, as will be evident to every diligent Reader.

Note 2. If from any right-angled Triangle taken twice, ſuppoſe from theſe two, ...	H .	B .	P
	H .	B .	P
A third right-angled Triangle be deduced according to the firſt Canon, as this, ...	HH .	$BB \backsim PP$.	2BP

Then the angle at the Baſe of the third right-angled Triangle ſo deduced, *viz.* the angle oppoſite to the ſide 2BP ſhall be equal to the double of the angle at the Baſe of the firſt right-angled Triangle, *viz.* of the angle oppoſite to the ſide P ; or elſe equal to the Complement of the ſaid double angle unto two right-angles, when the ſaid double exceeds a right-angle.

Likewiſe, if from two unlike right-angled Triangles, ſuppoſe from theſe, ...	H .	B .	P
	h .	b .	p
A third right-angled Triangle be deduced according to the firſt Canon, as theſe, ...	Hh .	B$b \backsim$ Pp .	Bp + Pb

Then the angle at the Baſe of this third right-angled Triangle, *viz.* the angle oppoſite to the ſide Bp + Pb ſhall be equal to the ſumm of the angles at the Baſes of the firſt and ſecond right-angled Triangles, *viz.* of the angles oppoſite to the ſides P and p ; or elſe equal to the Complement of the ſaid ſumm unto two right-angles, when that ſumm exceeds a right-angle.

The converse of this rare Speculation is demonstrated by *Andersonus*, in *Theorem.* 2. of *Vieta's* mysterious Doctrine of Angular Sections; and likewise by *Herigonius* at the latter end of the First Tome of his *Cursus Mathemat.*

QUEST. 3.

To divide a given square number into two such Squares, that one of them may consist within given limits.

Let it be required to divide 16 into two such Squares, that one of them may be greater than 10, but less than 11.

Or thus:

A Rational number 4 being given for the Hypothenusal of a right-angled Triangle, to find the Base and Perpendicular in such Rational numbers, that one of them may be greater than $\sqrt{10}$, but less than $\sqrt{11}$.

RESOLUTION.

1. For the given Hypothenusal 4, (which is the side of the given Square 16) put . . . . d
2. For $\sqrt{10}$ the lesser of the prescribed limits, put . . f
3. For $\sqrt{11}$ the greater of the prescribed limits, put . . g
4. Let two unequal numbers be represented by . . . s and r
5. Then the sides about the right-angle of a right-angled Triangle whose Hypothenusal is d, will be found equal to these Quantities, (by the Canon in *Observat.* 5. *Resolut.* 2. *Quest.* 1.) *viz.* . . . . $\frac{2rsd}{ss+rr}$ and $\frac{ssd \backsim rrd}{ss+rr}$
6. But since this Question requires that one of those sides, suppose $\frac{2rsd}{ss+rr}$, may be greater than f, yet less than g, the said numbers s and r cannot be any two unequal numbers; and therefore I shall here shew a way to chuse them, so as that they may cause the said side to agree with the said limits: To which end, first, a number at pleasure may be taken for one of the said numbers s and r, as, $1 = r$; and then to search out limits for the chusing of s, I proceed in this manner, *viz.* I put a instead of s while it is unknown, and then since $1 = r$, the before-mentioned side $\frac{2rsd}{ss+rr}$ will be exprest thus, $\frac{2da}{aa+1}$, where the number a only is unknown: Now,
7. Let it be supposed (according to the import of the Question) that . . . . $\frac{2da}{aa+1} \sqsubset f$
8. Let it also be supposed that . . . . $\frac{2da}{aa+1} \sqsupset g$
9. Then by multiplying each part of the supposition in the seventh step by $aa+1$, it follows that . . . . $2da \sqsubset faa+f$
10. Therefore by comparing the latter part of the ninth step to the former, ... $faa+f \sqsupset 2da$
11. And by dividing each part in the last step by f, it follows, that . . . . $aa+1 \sqsupset \frac{2da}{f}$
12. And by subtracting 1 from each part, ... $aa \sqsupset \frac{2da}{f} - 1$
13. Likewise by equal subtraction of $\frac{2da}{f}$ it follows, that $aa - \frac{2da}{f} \sqsupset -1$
14. And by adding the Square of half the Coefficient $\frac{2d}{f}$, to each part in the thirteenth step, ... $aa - \frac{2da}{f} + \frac{dd}{ff} \sqsupset \frac{dd}{ff} - 1$
15. And by extracting the square Root out of each part in the fourteenth step, ... $a - \frac{d}{f} \sqsupset \sqrt{:}\frac{dd-ff}{ff}:$
16. Wherefore by adding $\frac{d}{f}$ to each part in the fifteenth step, it follows that . . . . $a \sqsupset \frac{d}{f} + \sqrt{:}\frac{dd-ff}{ff}:$

Again, 17.

17. Again, because $\frac{d}{f} - a$, (as well as $a - \frac{d}{f}$,) may be the side of the Square in the first part of the fourteenth step, it thence follows that . . . . $\frac{d}{f} - a \sqsupset \sqrt{:}\frac{dd-ff}{ff}:$

18. And by adding a to each part in the seventeenth step, $\frac{d}{f} \sqsupset a + \sqrt{:}\frac{dd-ff}{ff}:$

19. And by equal subtraction of $\sqrt{:}\frac{dd-ff}{ff}:$ it follows that $\frac{d}{f} - \sqrt{:}\frac{dd-ff}{ff}: \sqsupset a$

20. Wherefore by comparing the latter part of the nineteenth step to the first, it's evident that . . . . . $a \sqsubset \frac{d}{f} - \sqrt{:}\frac{dd-ff}{ff}:$

21. Again, by supposition in the eighth step, ... $\frac{2da}{aa+1} \sqsupset g$

22. And consequently, ... $g \sqsubset \frac{2da}{aa+1}$

23. Whence by arguing in like manner as before, with f, from the ninth step to the sixteenth, it will appear that . . . . $a \sqsubset \frac{d}{g} + \sqrt{:}\frac{dd-gg}{gg}:$

24. Again, by arguing with g in like manner as before with f, from the seventeenth step to the twentyeth, it will be evident that . . . . $a \sqsupset \frac{d}{g} - \sqrt{:}\frac{dd-gg}{gg}:$

25. Now because a was put instead of s, the sixteenth and twenty-third steps give a Canon for limiting the number s, when $r = 1$; *viz.*

CANON 1.

$$s \sqsupset \frac{d + \sqrt{:}\overline{dd-ff}:}{f} \quad (2.039, \&c.)$$

$$s \sqsubset \frac{d + \sqrt{:}\overline{dd-gg}:}{g} \quad (1.880, \&c.)$$

26. Again, the twentyeth and twenty-fourth steps give another Canon for limiting the number s, (which was represented by a in the preceding argumentation,) when $r = 1$; *viz.*

CANON 2.

$$s \sqsubset \frac{d - \sqrt{:}\overline{dd-ff}:}{f} \quad (0.490, \&c.)$$

$$s \sqsupset \frac{d - \sqrt{:}\overline{dd-gg}:}{g} \quad (0.531, \&c.)$$

27. Therefore, if 1 be put for r, and there be given (as before in the first, second and third steps,) $4 = d$; $\sqrt{10} = f$; and $\sqrt{11} = g$; then by *Canon* 1. s may be any number less than $2\frac{39}{1000}$, but greater than $1\frac{88}{100}$; and consequently, if $1 = r$, and $s = 2$, (which is within the last mentioned limits of s,) then the sides of the two Squares sought (being expounded according to the two Quantities in the fifth step of the Resolution of this *Quest.* 3.) shall be $\frac{16}{5}$ and $\frac{12}{5}$, *viz.*

$$\frac{16}{5} = \frac{2rsd}{ss+rr}; \text{ and, } \frac{12}{5} = \frac{ssd-rrd}{ss+rr}.$$

Therefore the two Squares sought are $\frac{256}{25}$ and $\frac{144}{25}$, whose summ is $\frac{400}{25}$; that is, 16; and one of those Squares, to wit, $\frac{256}{25}$, or $10\frac{6}{25}$, is greater than 10, but less than 11; as was required.

Again, if $1 = r$; $4 = d$; $\sqrt{10} = f$; $\sqrt{11} = g$; (as before,) then by *Canon* 2. s may be any Fraction greater than $\frac{490}{1000}$, but less than $\frac{531}{1000}$: and consequently, if $1 = r$. and $s = \frac{1}{2}$, (which is within the last mentioned limits of s,) then the sides of the two Squares sought will be found the same as before, *viz.*

$$\frac{16}{5} = \frac{2rsd}{rr+ss}; \text{ and, } \frac{12}{5} = \frac{rrd-ssd}{rr+ss}.$$

Again, if $1 = r$, and $s = \frac{13}{25}$, (which is also within the limits of s discovered by *Canon* 2.) then the sides of the two Squares sought will be found these, to wit, $\frac{1300}{397}$ and $\frac{912}{397}$; whose

whose Squares are $\frac{1690000}{157609}$ and $\frac{831744}{157609}$, which added together make 16; and the first of those Squares is greater than 10, but less than 11; as was required.

Note, That the manner of searching out limits in this and divers following Questions, is agreeable to the method of resolving Quadratick Equations in *Sect.* 5, 7, 9. *Chap.* 15. *Book* 1.

QUEST. 4.

(This is the fifth of the fourth Book of Vieta's *Zeteticks; 'tis also resolved by* Bachet *in his Comment upon the twelfth of the fifth Book of* Diophantus; *but I shall wave their ways of Resolution, and deduce one from* Canon 1. *in the fifteenth step of* Resolut. 2. *of the preceding second Question of this Book.)*

To divide a given number which is compos'd of two Squares, into two other Squares, that one of the Squares sought may consist within given limits.

Preparation.

Because the following Resolution of this Question presupposeth each of the prescribed limits to be greater than the lesser of the two Squares given, I shall in the first place shew how from the given limits, when they are not qualified as aforesaid, to inferr others, each of which shall be greater than the lesser of the two Squares given, and then the following Resolution will solve the Question propos'd according to any possible limits whatever.

Case 1. *When the lesser of the given limits is equal to the lesser of the given Squares.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 4, but less than 5: Here instead of 4 the lesser limit, (which is equal to the lesser Square given,) we may take $4\frac{1}{2}$ or any number between 4 and 5, (5 being the greater limit given:) then since $4\frac{1}{2}$ and 5 are each of them greater than 4, (the lesser Square given,) the following Resolution will find out two Squares whose summ shall be 13; and one of them shall be greater than $4\frac{1}{2}$, but less than 5, and consequently greater than 4, but less than 5; as was required.

Case 2. *When the lesser limit is less than the lesser Square given, but the greater limit exceeds the same; viz. When the lesser Square given falls between the given limits.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 3, but less than 5: Here instead of 3 we may take $4\frac{1}{2}$, or any number between 4 the lesser Square given, and 5 the greater limit: then since $4\frac{1}{2}$ and 5 are each of them greater than 4, (the lesser Square given,) the following Resolution will find out two Squares whose summ shall be 13; and one of them shall be greater than $4\frac{1}{2}$, but less than 5, and consequently greater than 3, but less than 5; as was required.

Case 3. *When the greater of the two limits given is equal to the lesser of the two Squares given.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 3, but less than 4: First, subtract 3 and 4 severally from 13, so each of the Remainders 10 and 9 is greater than 4 the lesser Square given, and therefore by the following Resolution two Squares may be found out whose summ shall be 13; and one of them less than 10, but greater than 9, and consequently the other Square shall be greater than 3, but less than 4; as was required.

Case 4. *When each of the two limits given is less than the lesser Square given.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 1, but less than 2: First, subtract the said limits 1 and 2 severally from 13 the number given to be divided, so each of the Remainders 12 and 11 is greater than 4 the lesser Square given; and therefore by the following Resolution two Squares may be found out whose summ shall be 13; and one of them less than 12, but greater than 11, and consequently the other Square shall be greater than 1, but less than 2; as was required.

Now let it be desired to divide 13, which is compos'd of two Squares, 4 and 9, into two such other Squares, that one of them may be greater than 6, but less than 7.

RESO-

RESOLUTION.

1. For 2, the ſide or ſquare Root of 4 the leſſer of the two Squares given, put > b
2. For 3, the ſide of 9 the greater Square given, put . . . . . . > d
3. For $\sqrt{6}$, that is, the ſquare Root of the leſſer of the two limits given, put > f
4. For $\sqrt{7}$, that is, the ſquare Root of the greater of the two limits, put > g
5. Let two unequal numbers be repreſented by . . . . > s and r
6. Now if s be greater than r, and be to r in any Reaſon (or Proportion) except that which $d+b$ hath to $d-b$, then by *Canon* 1. in the fifteenth ſtep of *Reſolut.* 2. of the preceding *Queſt.* 2. the ſides of two Squares different from dd and bb, but ſuch, whoſe ſumm is equal to $bb+dd$, ſhall be equal to theſe two following Quantities, (which are expreſt alſo in the thirteenth and fourteenth ſteps of the ſaid *Reſolut.* 2. *Queſt.* 2.) *viz.*

$$\frac{2rsd+ssb-rrb}{ss+rr} \text{ and } \frac{ssd-rrd \backsim 2rsb}{ss+rr}.$$

7. But becauſe it's deſired that one of thoſe two Quantities or ſides laſt above expreſt, ſuppoſe, $\frac{2rsd+ssb-rrb}{ss+rr}$ may be greater than f, but leſs than g, the two unequal numbers s and r muſt be choſen ſo as that they may cauſe the ſaid ſide to agree with the ſaid limits. To which end, firſt, a number at pleaſure may be taken for one of the ſaid numbers s and r, as $1=r$, and then to ſearch out limits for the chuſing of s, I proceed in this manner, *viz.* I put a inſtead of s while it is unknown, and then ſince $1=r$, the before-mentioned ſide $\frac{2rsd+ssb-rrb}{ss+rr}$ will ſtand thus, $\frac{2da+baa-b}{aa+1}$, where the number a only is unknown: Now,

8. Let it be ſuppoſed that . . . . . . > $\frac{2da+baa-b}{aa+1} \sqsubset f$
9. Let it alſo be ſuppoſed that . . . . . > $\frac{2da+baa-b}{aa+1} \sqsupset g$
10. Then by multiplying each part in the eighth ſtep by $aa+1$, it follows that . . . . } $2da+baa-b \sqsubset faa+f$
11. And by adding b to each part in the tenth ſtep, > $2da+baa \sqsubset faa+f+b$
12. And by ſubtracting baa from each part in the eleventh ſtep, it follows that . . . . } $2da \sqsubset faa-baa+f+b$
13. By ſuppoſition in the firſt and third ſteps, (agreeable to the *Preparation* to the Reſolution of this Queſtion,) f is greater than b, ſuppoſe therefore . . . . } $c=f-b$
14. Then from the twelfth and thirteenth ſteps it follows, that . . . . } $2da \sqsubset caa+f+b$
15. And by ſubtracting $f+b$ from each part in the fourteenth ſtep, it's manifeſt that . . } $2da-f-b \sqsubset caa$
16. And by dividing each part of the fifteenth ſtep by c, ... } $\frac{2d}{c}a-\frac{f+b}{c} \sqsubset aa$
17. And by ſubtracting $\frac{2d}{c}a$ from each part of the ſixteenth ſtep, ... } $-\frac{f+b}{c} \sqsubset aa-\frac{2d}{c}a$
18. And by adding the Square of half the Coefficient $\frac{2d}{c}$ to each part of the ſeventeenth ſtep, it follows that . . . . } $\frac{dd}{cc}-\frac{f+b}{c} \sqsubset aa-\frac{2d}{c}a+\frac{dd}{cc}$
19. And by extracting the ſquare Root out of each part of the eighteenth ſtep, . . . } $\sqrt{:}\frac{dd-fc-bc}{cc}: \sqsubset a-\frac{d}{c}$
20. And by adding $\frac{d}{c}$ to each part of the 19*th* ſtep, it follows, that . . . . } $\frac{d}{c}+\sqrt{\frac{dd-fc-bc}{cc}} \sqsubset a$
21. Wherefore by comparing the quantity a in the latter part of the 20*th* ſtep to the ſumm of the quantities in the firſt part, it is found that } $a \sqsupset \frac{d}{c}+\sqrt{\frac{dd-fc-bc}{cc}}$

22. Again,

22. Again, because $\frac{d}{c} - a$ (as well as $a - \frac{d}{c}$ in the latter part of the nineteenth step,) may be the square Root of the Square which is the latter part of the eighteenth step, it follows, that

$$\sqrt{\frac{dd - fc - bc}{cc}} \sqsubset \frac{d}{c} - a$$

23. And by adding a to each part of the twenty-second step, ...

$$a + \sqrt{\frac{dd - fc - bc}{cc}} \sqsubset \frac{d}{c}$$

24. Wherefore by subtracting $\sqrt{\frac{dd - fc - bc}{cc}}$ from each part of the twenty-third step, it's evident that . . . .

$$a \sqsubset \frac{d}{c} - \sqrt{\frac{dd - fc - bc}{cc}}$$

25. Again, by supposition in the ninth step,

$$\frac{2da + baa - b}{aa + 1} \sqsupset g$$

26. Whence by multiplying each part by $aa+1$,

$$2da + baa - b \sqsupset gaa + g$$

27. And by subtracting baa from each part of the twenty-sixth step, ...

$$2da - b \sqsupset gaa - baa + g$$

28. And by adding b to each part of the twenty-seventh step, ...

$$2da \sqsupset gaa - baa + g + b$$

29. By supposition in the first and fourth steps, (agreeable to the *Preparation* to the Resolution of this Question,) g is greater than b; suppose therefore, ...

$$n = g - b$$

30. Then from the twenty-eighth and twenty-ninth steps it follows, that . . . .

$$2da \sqsupset naa + g + b$$

31. Whence, by arguing in like manner as before from the fourteenth step to the twenty-first, it will appear that . . . .

$$a \sqsubset \frac{d}{n} + \sqrt{\frac{dd - gn - bn}{nn}}$$

32. Again, by arguing in like manner as before from the twenty-second step to the twenty-fourth, it will be evident that . . . . .

$$a \sqsupset \frac{d}{n} - \sqrt{\frac{dd - gn - bn}{nn}}$$

33. Then out of the twenty-first and thirty-first steps, after a, c and n are exchanged for s, $f - b$ and $g - b$, for these are equal to those, as appears by the Positions in the seventh, thirteenth and twenty-ninth steps, the following *Canon* 1. ariseth for limiting the number s, when $r = 1$; *viz.*

CANON 1.

$$s \sqsupset \frac{d + \sqrt{:dd + bb - ff:}}{f - b} \quad (12.562, \&c.)$$

$$s \sqsubset \frac{d + \sqrt{:dd + bb - gg:}}{g - b} \quad (8.439, \&c.)$$

Again, out of the twenty-fourth and thirty-second steps, after a, c and n are exchanged for s, $f - b$ and $g - b$, (as before,) another *Canon* ariseth for limiting the number s, when $r = 1$; *viz.*

CANON 2.

$$s \sqsubset \frac{d - \sqrt{:dd + bb - ff:}}{f - b} \quad (0.788, \&c.)$$

$$s \sqsupset \frac{d - \sqrt{:dd + bb - gg:}}{g - b} \quad (0.852, \&c.)$$

Therefore if 1 be taken for the value of r, and there be given, $2 = b$; $3 = d$; $\sqrt{6} = f$; and $\sqrt{7} = g$; (as before in the first, second, third and fourth steps,) then by *Canon* 1. above-exprest, s may be any number between $12\frac{562}{1000}$ and $8\frac{439}{1000}$; and consequently, if $r = 1$ and $s = 9$, (which value of s is within the limits last before-mentioned, then the sides of the two Squares sought (being expounded according to the two Quantities in the sixth step of the Resolution of this *Quest.* 4.) shall be $\frac{123}{41}$ and $\frac{282}{41}$, *viz.*

2rsd +

$$\frac{2rsd + ssb - rrb}{ss + rr} = \frac{107}{41}, \text{ and } \frac{ssd - rrd \backsim 2rsb}{ss + rr} = \frac{102}{41}.$$

Therefore the two Squares ſought are $\frac{11449}{1681}$ and $\frac{10404}{1681}$, whoſe ſumm is $\frac{21853}{1681}$, that is, 13; and the firſt of thoſe Squares is greater than 6, but leſs than 7, as was required.

Again, if $1 = r$; $2 = b$, $3 = d$; $\sqrt{6} = f$; $\sqrt{7} = g$; (as before,) then by *Canon* 2. s may be any Fraction between $\frac{282}{1000}$ and $\frac{934}{1000}$; and conſequently, if $1 = r$ and $\frac{22}{25} = s$, (which value of s is within the laſt mentioned limits,) then the ſides of the two Squares ſought will be found $\frac{1391}{533}$ and $\frac{1326}{533}$, *viz.*

$$\frac{2rsd + ssb - rrb}{ss + rr} = \frac{1391}{533}, \text{ and } \frac{ssd - rrd \backsim 2rsb}{ss + rr} = \frac{1326}{533}.$$

Therefore the two Squares ſought are $\frac{1934881}{284089}$ and $\frac{1758276}{284089}$, whoſe ſumm is $\frac{3693157}{284089}$, *viz.* 13; and the firſt of thoſe Squares is greater than 6, but leſs than 7, as was required.

Again, if $1 = r$, and $s = \frac{4}{5}$, (which value of s is alſo within the limits diſcovered by *Canon* 2.) then the ſides of the two Squares ſought being expounded as before, will be found $\frac{107}{41}$ and $\frac{102}{41}$; which are the ſame with thoſe before-found in the Example of *Canon* 1.

Note. If 1 be put equal to r, and the number s be taken by *Canon* 2. then becauſe in this caſe s is leſs than r, the *Algebraical* Rules of + and − in adding, ſubtracting, &c. muſt be obſerved to reſolve the aforeſaid literal values of the ſides of the Squares ſought into numbers, as in the two laſt Examples.

QUEST. 5. (Quæſt. 11. Lib. 2. *Diophant.*)

To find two ſquare numbers whoſe difference ſhall be equal to a given number, ſuppoſe 60, (or d.)

RESOLUTION.

1. To the given difference 60, that is,	d
2. Let ſome number whoſe Square is leſs than the given difference be repreſented by	b
3. For the ſide of the leſſer Square ſought put	a
4. And for the ſide of the greater Square ſought put	$a + b$
5. Therefore the leſſer Square is	aa
6. And the greater Square is	$aa + 2ba + bb$
7. And the difference of thoſe Squares is	$2ba + bb$
8. But the ſaid difference muſt be equal to the given difference d, therefore	$2ba + bb = d$
9. Which Equation, after due Reduction, makes known the value of the ſide of the leſſer Square, *viz.*	$a = \frac{d - bb}{2b}$
10. And from the ninth and fourth ſteps, the value of the ſide of the greater Square is alſo diſcovered, *viz.*	$a + b = \frac{d + bb}{2b}$

The two laſt ſteps give the following

CANON 1.

Take any ſquare number leſs than the given difference, and ſubtract it from the ſaid difference; then divide the Remainder by the double of the ſide of the Square firſt taken, and the Quotient ſhall be the ſide of the leſſer of the two Squares ſought; laſtly, this ſide added to the ſide of the Square firſt taken, gives the ſide of the other Square ſought.

So if two Squares be deſired whoſe difference ſhall be 60, I take a ſquare number leſs than 60, as 36; this ſubtracted from that leaves 24, which divided by 12 the double of the ſquare Root of 36, gives the Quotient 2, which ſhall be the ſide of the leſſer Square ſought; and then by adding 6 the ſquare Root of the ſaid 36, to the ſide 2, the ſumm 8 is the ſide of the greater Square ſought; laſtly, the Squares of the ſaid ſides 2 and 8, to wit, 4 and 64 will ſolve the Queſtion, for their difference is 60, as was required.

Obſervations upon Queſt. 5.

1. It is evident by the two laſt ſteps of the preceding *Reſolution*, that the values of the ſides of the two Squares ſought are $\frac{d - bb}{2b}$ and $\frac{d + bb}{2b}$

Now

Now if we ſuppoſe $bc = d$, then thoſe ſides will be converted into theſe, *viz.* . . . . $\frac{bc + bb}{2b}$ and $\frac{bc - bb}{2b}$

Which laſt mentioned ſides or Quotients, after the common Factor b is caſt away, will be reduced to theſe, to wit, . . . $\frac{1}{2}c + \frac{1}{2}b$ and $\frac{1}{2}c - \frac{1}{2}b$

Hence ariſeth this elegant Canon, often uſed by *Diophantus* to find out two Squares in a given difference, *viz.*

CANON 2.

Take two ſuch unequal numbers that the Product of their multiplication may be equal to the given difference; then half the ſumm and half the difference of thoſe two numbers ſhall be the ſides of the two Squares ſought.

As, for example, if two Squares be deſired whoſe difference ſhall be 60, I take two ſuch numbers (10 and 6) which being mutually multiplied make 60; then half the ſumm of 10 and 6 and half their difference are 8 and 2 the ſides of the two Squares ſought, and conſequently the Squares themſelves are 64 and 4, whoſe difference is 60; as was required.

Again, inſtead of 10 and 6 taken as before, we may take 30 and 2, for the Product of theſe is equal to the given difference 60; then half the ſumm of 30 and 2, and half their difference, give 16 and 14, whoſe Squares 256 and 196 have 60 for their difference; as was required.

After the ſame manner, Fractions being admitted, innumerable pairs of Squares may be found out, ſuch, that the difference of each pair ſhall be equal to one and the ſame number given: For if the given number be divided by a number taken at pleaſure, half the ſumm, and half the difference of the Diviſor and Quotient ſhall be the ſides of two Squares whoſe difference is equal to the given number.

2. But for farther illuſtration of the truth of the preceding *Canon* 2. let c and b repreſent two unequal numbers, and ſuppoſe c to be the greater; then

The Square of $\frac{1}{2}c + \frac{1}{2}b$ is . . . . $\frac{1}{4}cc + \frac{1}{2}cb + \frac{1}{4}bb$,
The Square of $\frac{1}{2}c - \frac{1}{2}b$ is. . . . . $\frac{1}{4}cc - \frac{1}{2}cb + \frac{1}{4}bb$,
The difference of thoſe Squares is . . . . $+ cb$.

Whence it is manifeſt, That the Product of the multiplication of any two unequal numbers is equal to the difference of two Squares, the greater of which is the Square of half the ſumm of the ſaid two numbers, and the leſſer is the Square of half their difference. Wherefore the truth of the foregoing *Canon* 2. doth evidently appear.

3. *Vieta* uſeth the following Canon (which differs but little from the preceding *Canon* 2.) to find out two Squares in a given difference.

CANON 3.

Take two ſuch unequal numbers, that the Product of their multiplication may be equal to a quarter of the given difference of two Squares ſought; then the ſumm and difference of thoſe two numbers firſt taken ſhall be the ſides of the deſired Squares.

As, for example, if it be deſired to find out two Squares whoſe difference ſhall be 60; firſt, I take $\frac{1}{4}$ of the ſaid 60, to wit, 15; then I chuſe two ſuch unequal numbers that the Product of their multiplication may make 15, as 5 and 3; laſtly, the ſumm, and difference of 5 and 3, give 8 and 2 for the ſides of two Squares whoſe difference is 60.

The truth of this *Canon* 3. may be demonſtrated thus; let c and b repreſent two unequal numbers, and ſuppoſe c to be the greater, then

The Square of $c + b$ is . . . . $cc + 2cb + bb$,
The Square of $c - b$ is . . . . $cc - 2cb + bb$,
The difference of thoſe Squares is . . . . . . $+ 4cb$.

Whence it is manifeſt, That the quadruple of the Product of the multiplication of any two unequal numbers is equal to the difference of two Squares, the greater of which is the Square of the ſumm of thoſe numbers, and the leſſer Square is the Square of the difference of the ſame two numbers. Wherefore the truth of *Canon* 3. is evident.

4. If a Square be equal to two Squares, then (by *prop.* 47. *Elem.* 1. Euclid.) the ſides of thoſe three Squares will conſtitute a right-angled Triangle, *viz.* the greateſt ſide ſhall be the Hypothenuſal, and the other two the ſides about the right-angle; whence it follows, that the Square of one of the ſides about the right-angle is equal to the difference of the Squares of the other two ſides: And therefore if any Rational number be given

for

for] one of the ſides about the right-angle of a right-angled Triangle, the other ſide about the right-angle and the Hypothenuſal ſhall be given alſo in Rational numbers by the help of any of the three preceding Canons: As, for example, if 4 be given for the Baſe, the Square thereof is 16, then by any of the ſaid Canons find out two Squares whoſe difference may be 16, ſuch are 25 and 9, (and innumerable other pairs of Squares;) therefore their ſquare Roots or ſides, *viz.* 5 and 3, ſhall be the deſired Hypothenuſal and Perpendicular. Whence it is evident, that by the like Operation innumerable right-angled Triangles may be found out in Rational numbers, which ſhall have one common Baſe (or Perpendicular) preſcribed.

QUEST. 6.

To find two ſuch numbers, that the Product of their multiplication may be equal to a given number, ſuppoſe d, and that the Square of half the ſumm of the ſaid numbers may be greater than a number given, ſuppoſe b.

RESOLUTION.

1. For one of the numbers ſought put . . . . a
2. Then by dividing the given Product d by a, the Quotient ſhall be the other number ſought, to wit, ... $\frac{d}{a}$
3. Therefore half the ſumm of thoſe two numbers is . . . $\frac{aa+d}{2a}$
4. Now ſuppoſe it be deſired that the Square of the ſaid half ſumm may be greater than the given number b, then it neceſſarily follows, that the ſumm it ſelf muſt be greater than the ſquare Root of b, viz. . . . . $\frac{aa+d}{2a} \sqsubset \sqrt{b}$
5. Therefore from the fourth ſtep, by multiplying each part by $2a$, it follows, that . . . . $aa+d \sqsubset 2a\sqrt{b}$
6. That is, (becauſe $a\sqrt{4b} = 2a\sqrt{b}$,) . . . . $aa+d \sqsubset a\sqrt{4b}$
7. And by ſubtracting d from each part of the ſixth ſtep, it follows that . . . . $aa \sqsubset a\sqrt{4b}-d$
8. And by ſubtracting $a\sqrt{4b}$ from each part of the ſeventh ſtep, $aa - a\sqrt{4b} \sqsubset -d$
9. And by adding a quarter of the Square of the known Coefficient $\sqrt{4b}$, to wit, b, to each part of the eighth ſtep, it follows that . . . . $aa-a\sqrt{4b}+b \sqsubset b-d$
10. And by extracting the ſquare Root out of each part of the ninth ſtep, ... $a-\sqrt{b} \sqsubset \sqrt{:\overline{b-d}:}$
11. Wherefore by adding $\sqrt{b}$ to each part of the tenth ſtep, $a \sqsubset \sqrt{b}+\sqrt{:\overline{b-d}:}$
12. Again, becauſe $-a+\sqrt{b}$ (as well as $a-\sqrt{b}$) may be the ſide of the Square $aa-a\sqrt{4b}+b$ in the firſt part of ninth ſtep, it thence follows that . . . . $-a+\sqrt{b} \sqsubset \sqrt{:\overline{b-d}:}$
13. And by adding a to each part of the twelfth ſtep, . . $\sqrt{b} \sqsubset a+\sqrt{:\overline{b-d}:}$
14. And by ſubtracting $\sqrt{:\overline{b-d}:}$ from each part of the thirteenth ſtep, ... $\sqrt{b}-\sqrt{:\overline{b-d}:} \sqsubset a$
15. Wherefore from the fourteenth ſtep, by comparing the latter part to the former, ... $a \sqsupset \sqrt{b}-\sqrt{:\overline{b-d}:}$
16. The eleventh and fifteenth ſteps give limits for the choice of (a) one of the two numbers ſought by this ſixth Queſtion, when it requires that the Square of half the ſumm of the ſame numbers may be greater than a given number; and the premiſſes afford this following

CANON 1.

For one of the numbers ſought take any number greater than $\sqrt{b}+\sqrt{:\overline{b-d}:}$ or leſs than $\sqrt{b}-\sqrt{:\overline{b-d}:}$ then divide d the given Product of the multiplication of the two numbers ſought, by the number firſt taken, ſo ſhall the Quotient be the other number ſought.

An Example in Numbers.

Suppoſe . . . . $d = 128$, and $b = 192$;
Thence it follows, that . . . $\sqrt{b}+\sqrt{:\overline{b-d}:} = 21.856$, &c.
Alſo, ... $\sqrt{b}-\sqrt{:\overline{b-d}:} = 5.856$, &c.

Therefore according to the direction of the preceding Canon, I take for one of the two numbers sought some number greater than $21\frac{856}{1000}$, or less than $5\frac{856}{1000}$, as the number 2; by this I divide the given number $128 = d$, and the Quotient gives 64 for the other number sought; which two numbers, 2 and 64, will solve the Question, as will be evident by

The Proof.

The Product of the multiplication of 2 and 64 makes the given number 128 (or d,) and the Square of half the summ of 2 and 64, *viz.* the Square of 33 is 1089, which is greater than 192, (or b,) as was required. But to the end there may be a possibility of solving the Question proposed, the Canon above-exprest doth shew there is a necessity that the number d must not exceed the number b.

17. The preceding Resolution of *Quest.* 6. presupposeth it to be desired that the Square of half the summ of the two numbers sought may be greater than a number given; but if it were desired that the said Square might be less than a number given, then $\sqsupset$ being used instead of $\sqsubset$ in the said Resolution, there would at length arise this following *Canon* to solve the said *Question* in the latter Case.

CANON 2.

For one of the numbers sought take any number less than $\sqrt{b}+\sqrt{:\overline{b-d}}:$ but greater than $\sqrt{b}-\sqrt{:\overline{b-d}}:$ then divide d the given Product of the multiplication of the two numbers sought, by the number first taken, and the Quotient shall be the other number sought.

An Example of this Canon.

Suppose (as before) . . . $d = 128$, and $b = 192$,
Thence it follows, that . . . $\sqrt{b}+\sqrt{:\overline{b-d}}: = 21.856$, &c.
Also, ... $\sqrt{b}-\sqrt{:\overline{b-d}}: = 5.856$, &c.

Therefore (according to the latter Canon) I take for one of the two numbers sought some number between $5\frac{856}{1000}$ and $21\frac{856}{1000}$, as 16; by this I divide the given number 128 (or d,) and the Quotient gives 8 for the other number sought; which two numbers, 16 and 8, will solve the Question when it requires that the Square of half their summ may be less than the given number 192, as may easily be proved: For the Product of the said 16 and 8 makes the given number 128, and the Square of half the summ of 16 and 8, *viz.* the Square of 12, is 144, which is less than the given number 192; as was required.

QUEST. 7.

To find two square numbers in a given difference, and that one of those Squares may be greater or less than a given number.

1. Let it be required to find two such square numbers, that their difference may be equal to a given number, suppose d; and that the greater Square may exceed a given number, suppose b.

RESOLUTION.

It is manifest by *Canon* 2. of the foregoing *Quest.* 5. That if two numbers be taken, such, that the Product of their multiplication is equal to the given difference of two Squares sought, then half the summ and half the difference of the numbers so taken shall be the sides of those Squares: Therefore if two numbers be found out, such, that their Product is equal to the given difference d, and that the Square of half the summ of the same numbers is greater than the given number b, then that Square shall be the greater of the two Squares required, and the Square of half the difference of the said numbers shall be the lesser Square required: But two such numbers may be found out by the first Canon of the preceding sixth Question, and consequently this seventh Question may be solved by the following

CANON 1.

Take some number greater than $\sqrt{b}+\sqrt{:\overline{b-d}}:$ or less than $\sqrt{b}-\sqrt{:\overline{b-d}}:$ then divide d the given difference of the two Squares sought by the number first taken, and reserve the Quotient; lastly, half the summ and half the difference of the said Quotient and number first taken shall be the sides of the two Squares sought.

An

An Example in Numbers.

Suppose . . . . $d = 128$, and $b = 192$,
Thence it follows that . . . . . $\surd b + \surd : \overline{b - d} : = 21.856$, &c.
Also, ... $\surd b - \surd : \overline{b - d} : = 5.856$, &c.

Therefore according to the direction of the Canon, I take some number greater than $21\frac{856}{1000}$, or less than $5\frac{856}{1000}$, as 2; then by this 2 I divide 128, (to wit, d) and the Quotient is 64; lastly, half the summ of the said 2 and 64 is 33, and half their difference is 31, which 33 and 31 are the sides of two Squares that will solve the Question proposed, as will be evident by

The Proof.

The Squares of 33 and 31 are 1089 and 961; the difference of these is equal to the given difference 128, (to wit, d;) and the greater Square 1089 is greater than 192, (or b;) as was required.

2. In like manner, if it were required to find out two Squares whose difference shall be equal to a given number d, and the greater Square less than a given number b; the sides of the said Squares may be found out by this following

CANON 2.

Take some number less than $\surd b + \surd : \overline{b - d} :$ but greater than $\surd b - \surd : \overline{b - d} :$ then divide d the given difference of the Squares sought, by the number so taken, and reserve the Quotient; lastly, half the summ and half the difference of the said Quotient and number first taken shall be the sides of the two Squares sought.

An Example in Numbers.

Suppose . . . . $d = 128$, and $b = 192$;
Thence it follows, that . . . . . $\surd b + \surd : \overline{b - d} : = 21.856$, &c.
Also, ... $\surd b - \surd : \overline{b - d} : = 5.856$, &c.

Therefore according to the direction of the last preceding Canon, I take some number between $5\frac{856}{1000}$ and $21\frac{856}{1000}$, as 16; then by this I divide the given number 128, (to wit, d,) and the Quotient is 8; lastly, half the summ of the said 16 and 8 is 12, but half their difference is 4; which 12 and 4 are the sides of two Squares that will solve the Question, as will be evident by

The Proof.

The Squares of 12 and 4 are 144 and 16, the difference of these is equal to the given difference 128, (or d;) and the greater Square 144 is less than 192, (or b;) as was required.

3. But if it were required to find out two Squares in a given difference d, and that the lesser Square might be greater than a given number g; they may be discovered by the help of the preceding Canons of this seventh Question, in this manner, *viz.*

Let it be required to find two Squares whose difference shall be 24 (or d,) and that the lesser Square may be greater than 12, (or g.) Here the scope must be to find out two such Squares that their difference may be 24, and that the greater Square may exceed 36, that is, $24 + 12$, and then the lesser Square will consequently exceed 12. Therefore,

Suppose . . . . $d = 24$, and $g = 12$,
Suppose also . . . . $b = 36 = d + g$,
Thence it follows, that . . . . $\surd b + \surd : \overline{b - d} : = 9.46$, &c.
Also, ... $\surd b - \surd : b - d : = 2.53$, &c.

Then (according to the first Canon of this seventh Question) I take some number greater than $9\frac{46}{100}$, or less than $2\frac{53}{100}$, as 2; by this I divide 24, (to wit, d,) and the Quotient is 12; then half the summ of 2 and 12 is 7, and half their difference is 5; which 7 and 5 are the sides of two Squares 49 and 25, whose difference is 24, (to wit, d,) and the lesser Square 25 is greater than 12, (or g;) as was required. But for the greater evidence, let ff be put for the lesser Square found out, and hh for the greater; then

By Construction, ... $hh = d + ff$,
Also by Construction, ... $hh \sqsubset d + g \; (= b,)$
Therefore . . . . $ff \sqsubset g$. Which was to be proved.

 4. Lastly,

4. Laſtly, if it were deſired to find out two Squares in a given difference d, and that the leſſer Square might be leſs than a given number g; let the ſumm of thoſe two given numbers, (to wit, $d + g$) be called b (as before,) and then by the latter of the two preceding Canons of this ſeventh Queſtion find out two Squares that their difference may be equal to the given difference d, and that the greater Square may be leſs than the ſumm b, ſo ſhall the leſſer Square be leſs than the given number g.

QUEST. 8.

[*This is the twelfth of the ſecond Book of* Diophantus, *and the ſeventh of the fourth Book of* Vieta's *Zeteticks.*]

Two numbers being given, ſuppoſe 192 and 128, to find a third, which added to each of thoſe given may make each ſumm to be a Square.

RESOLUTION 1.

1. For the number ſought put . . . . a
2. Then the Queſtion requires that each of theſe two ſumms may be a ſquare number, *viz.* . . . . $192 + a = \square$, $128 + a = \square$
3. Now that Duplicate Equality (for ſo *Diophantus* calls it) may be reſolved thus, *viz.* Firſt, ſubtract $128 + a$ from $192 + a$, and the Remainder 64 is the difference both of the given numbers and likewiſe of the two Squares ſought; then (by the preceding ſeventh Queſtion) find two ſuch ſquare numbers that their difference may be 64, and that the greater Square may exceed 192 the greater number given; ſuch are the Squares 289 and 225, whoſe ſides are 17 and 15.
4. Then equate $192 + a$ to 289, thus, ... $192 + a = 289$
5. Or equate $128 + a$ to 225, thus, ... $128 + a = 225$
6. Laſtly, from either of thoſe Equations in the fourth and fifth ſteps, the number a ſought will be alſo made known, *viz.* . . . . $a = 97$

I ſay 97 will ſolve the Queſtion; for if it be added to 192 and 128 ſeverally, the ſumms 289 and 225 are Squares, as was required: And out of the premiſſes well examined, reſpect alſo being had to the preceding ſeventh Queſtion, there will ariſe this following Canon to find out innumerable Anſwers to the Queſtion propoſed.

CANON.

7. Take any number greater than the ſumm, or leſs than the difference of the ſquare Roots of the two numbers given; divide the difference of the two numbers given, by the number firſt taken, and reſerve the Quotient; then from the Square of half the ſumm of that Quotient and the number taken, ſubtract the greater of the two numbers given; or, from the Square of half the difference between the ſaid Quotient and the number firſt taken, ſubtract the leſſer number given; ſo ſhall either of the Remainders (for they are equal to one another) be the numbers ſought.

An Example of the Canon.

8. Let there be two numbers given to find a third, according to *Queſt.* 8. as, ... 96, 8
9. Their difference is 88
10. Alſo . . . . $\sqrt{96} + \sqrt{8} = 12.626$, &c.
11. And . . . . $\sqrt{96} - \sqrt{8} = 6.969$, &c.
12. Let a number be taken, either greater than $12\frac{626}{1000}$, or leſs than $6\frac{969}{1000}$, ſuch is 4
13. Divide 88 in the ninth ſtep, by 4 in the twelfth, and the Quotient is 22
14. The half of the ſumm of 4 and 22 in the twelfth and thirteenth ſteps is 13
15. The Square of the ſaid 13 is 169
16. From that Square ſubtract the greater of the two numbers given in the eighth ſtep, to wit, ... 96
17. So the Remainder is the number ſought, to wit, ... 73

The Proof.

$96 + 73 = 169$, whoſe $\sqrt{}$ is 13:
$8 + 73 = 81$, whoſe $\sqrt{}$ is 9.

In

In like manner, if it were deſired to find out ſome number ſignified by (a) that $10a + 54$ may make a Square, alſo that $10a + 6$ may make a Square, the preceding Canon will give innumerable values of a, among which 1 will be found a true value; for if $a = 1$, then $10a + 54 = 64$, and $10a + 6 = 16$.

Obſervations upon Queſt. 8.

1. *Diophantus* in reſolving this Queſtion makes an entrance into one of his peculiar ſubtilties, which he calls a Duplicate Equality, (an ingenious Invention variouſly uſed by him, as divers knotty Queſtions in this Book will make manifeſt;) the principal hinge whereof depends on the finding of two Squares whoſe difference ſhall be equal to the difference of two *Algebraick* Quantities, each of which is propos'd to be found equal to ſome known ſquare number: As, in the preceding Reſolution of this *Queſt.* 8, two Squares, to wit, 289 and 225 are found out; whoſe difference 64 is equal to the difference of the two *Algebraick* Quantities $192 + a$ and $128 + a$, each of which, according to the import of the Queſtion, is to be found equal to ſome ſquare number, and therefore the number a ſought muſt be ſuch as will cauſe that effect.

2. But that the reaſon of the Operation in reſolving the Duplicate Equality in this eighth Queſtion may clearly appear, two things are to be proved, *viz.*

Firſt, That the greater of the two ſquare numbers found out in the third ſtep of the Reſolution muſt neceſſarily exceed the greater of the two numbers given in the Queſtion, and the leſſer Square exceed the leſſer number given. To prove this, firſt, it is intended that the deſired number a ſhould be affirmative, that is, greater than nothing; but if any ſquare number not greater than 192 were ſet in the place of 289 in the Equation in the fourth ſtep, *viz.* $192 + a = 289$; or any ſquare number not greater than 128, in the place of 225 in the Equation in the fifth ſtep, *viz.* $128 + a = 225$; the value of a would be leſs than nothing: and therefore the neceſſity of the before-mentioned qualification of the ſaid Squares is apparent.

Secondly, That when two ſquare numbers are found out, ſuch, that their difference is equal to the difference of the two numbers given, and that the greater Square exceeds the greater of thoſe given numbers; then conſequently the leſſer Square ſhall exceed the leſſer number, and theſe two laſt mentioned exceſſes or differences ſhall be equal to one another. The truth of this conſequence will be evident by the following

THEOREM.

If two ſquare numbers, ſuppoſe dd the greater and ff the leſſer, have the ſame difference as two other numbers, ſuppoſe b the greater and c the leſſer, and that the greater Square dd exceeds the greater number b; then the leſſer Square ff ſhall exceed the leſſer number c, and the former exceſs $dd - b$ ſhall be equal to the latter $ff - c$. For,

By ſuppoſition	$dd - ff = b - c \sqsubset 0$
Therefore by adding ff to each part, it follows, that	$dd = ff + b - c$
But by ſuppoſition	$dd \sqsubset b$
Wherefore, by ſubtracting b from each part of the Equation in the laſt ſtep but one, that which the *Theorem* affirms is manifeſt, *viz.*	$dd - b = ff - c$

And conſequently either of thoſe equal exceſſes or differences (in the laſt Equation) ſhall be a number to ſolve the Queſtion propoſed; for it is manifeſt that if the former exceſs $dd - b$ be added to b, and the latter exceſs $ff - c$ to c, the ſumms will be Squares, to wit, dd and ff.

Another manner of reſolving the foregoing Queſt. 8. *which is here repeated*, viz.

To find a number which added ſeverally to 128 and 192, may make the ſumms to be Squares.

RESOLUTION 2.

1. For the number ſought put $aa - 128$ whereby part of the Queſtion is ſatisfied; for if $aa - 128$ be added to 128, it makes a Square, to wit, aa; let therefore the number ſought be feigned to be	$aa - 128$

2. But

2. But the Question requires also, that if the number sought be added to 192, the summ may be a Square; add therefore $aa - 128$ to 192, so the summ $aa - 128 + 192$, that is, $aa + 64$ must be equal to a square number, *viz.* . . . . $aa + 64 = \square$

3. It remains therefore to equate the said $aa + 64$ to a Square, whose side (to the end the value of aa may be greater than 128, as the number $aa - 128$ assumed in the first step requires) may be feigned to be $a +$ any Absolute number less than $2\frac{544}{1000}$, or $-a +$ any Absolute number greater than $25\frac{171}{1000}$, (which limits are discovered by the following third way of resolving this Question;) let therefore the said side be feigned to be $a + 2$, and then the Square of this side being equated to $aa + 64$ as the second step requires, this Equation ariseth, *viz.*

$$aa + 4a + 4 = aa + 64.$$

4. Which Equation, after due Reduction, makes known the value of a, viz. . . . . $a = 15$

Therefore by the first and fourth steps the number sought will be found 97, which will solve the Question; for if 97 be added to 128 and 192 severally, the summs are Squares, to wit, 225 and 289; and the limits in the third step for feigning the side of one of the Squares sought do shew that the Question is capable of innumerable Answers.

A third manner of resolving the preceding Quest. 8. *which is here repeated*, viz.

To find a number, which added first to a given number (f,) and then to a greater given number (b,) may make the summs to be Squares.

RESOLUTION 3.

1. For the difference of the two given numbers b and f put d, viz. suppose . . . . $d = b - f$

2. And for the number sought put $aa - f$, (a representing a number unknown,) whereby the first part of the Question is satisfied; for if $aa - f$ be added to f it makes a Square, to wit, aa; let therefore the number sought be feigned to be . . . . $aa - f$

3. But the Question requires also, that if the number sought be added to b it may make a Square; add therefore $aa - f$ to b, and it makes $aa + b - f$, that is, $aa + d$, (for d was put equal to $b - f$,) which must be equal to a Square, *viz.* . . . . $aa + d = \square$

4. It remains then to equate $aa + d$ to some Square, whose side may be feigned to be either $a + e$, or $-a + u$, (which numbers, a, e and u are all yet unknown;) First, let the side of the said Square be feigned to be $a + e$, so its Square being equated to $aa + d$, this Equation ariseth, *viz.*

$$aa + 2ae + ee = aa + d.$$

5. Which Equation, after it is duly reduced to find what a is equal to, gives . . . . $a = \frac{d - ee}{2e}$

6. But by the second step . . . . $a \sqsubset \sqrt{f}$

7. Therefore by the fifth and sixth steps . . . . . . $\frac{d - ee}{2e} \sqsubset \sqrt{f}$

8. And by multiplying each part of the seventh step by $2e$, it follows, that . . . . $d - ee \sqsubset 2e\sqrt{f}$, or $e\sqrt{4f}$

9. And by adding ee to each part of the eighth step, it gives $d \sqsubset ee + e\sqrt{4f}$

10. And by adding f, that is, $\frac{1}{4}$ of the Square of the known Coefficient $\sqrt{4f}$ in the ninth step, to each part thereof, it follows, that . . . . $d + f \sqsubset ee + e\sqrt{4f} + f$

11. And by extracting the square Root out of each part of the tenth step, ... $\sqrt{:d + f:} \sqsubset e + \sqrt{f}$

12. And by subtracting $\sqrt{f}$ from each part of the eleventh step, ... $\sqrt{:d + f:} - \sqrt{f} \sqsubset e$

13. And by comparing the latter part of the twelfth step to the first part, it's manifest that . . . . $e \sqsupset \sqrt{:d + f:} - \sqrt{f}$

14. But by the first step, ... $b = d + f$

15. And consequently . . . . $\sqrt{b} = \sqrt{:d + f:}$

16. Where-

16. Wherefore, by ſetting $\sqrt{b}$ in the place of $\sqrt{:d+f:}$ in the thirteenth ſtep, it's evident that . . . . $e \sqsupset \sqrt{b} - \sqrt{f}$

17. And becauſe by the fifth ſtep, ... $ee \sqsupset d$

18. Therefore . . . . $e \sqsupset \sqrt{d}$

19. Again, foraſmuch as the ſide of the Square mentioned in the fourth ſtep may be feigned to be $-a+u$, let the Square of $-a+u$ be equated to $aa+d$, as the third ſtep requires, ſo this Equation ariſeth, *viz.*

$$aa+d = aa - 2ua + uu.$$

20. Which Equation, after due Reduction, to find out what a is equal to, gives . . . . . . $a = \frac{uu-d}{2u}$

21. But by the ſecond ſtep, ... $a \sqsubset \sqrt{f}$

22. Therefore from the twentyeth and twenty-firſt ſteps, ... $\frac{uu-d}{2u} \sqsubset \sqrt{f}$

23. And by arguing to find out limits for u, in like manner as before for e from the ſeventh to the ſixteenth ſtep *incluſive*, it will at length appear, that . $u \sqsubset \sqrt{b}+\sqrt{f}$

24. And becauſe by the twentyeth ſtep . . . . $uu \sqsubset d$

25. Therefore . . . . $u \sqsubset \sqrt{d}$

26. Now ſuppoſe . . . . $192 = b$; $128 = f$

27. And conſequently . . . . $64 = b - f = d$

28. And from the ſixteenth and twenty-ſixth ſteps it follows that . . . . $e \sqsupset 2\frac{542}{1000}$, &c. $(\sqrt{b}-\sqrt{f})$

29. And from the twenty-third and twenty-ſixth ſteps, ... $u \sqsubset 25\frac{170}{1000}$, &c. $(\sqrt{b}+\sqrt{f})$

30. Likewiſe from the eighteenth and twenty-ſixth ſteps, ... $e \sqsupset 8$ $(\sqrt{d})$

31. And from the twenty-fifth, and twenty-ſeventh ſteps . . . . $u \sqsubset 8$

But of the limits found out in the four laſt preceding ſteps, the two former only are neceſſary for chuſing the numbers e and u, for the two latter not being ſo ſtrict as the two former are uſeleſs. Then after the number e or u is duely choſen according to the ſaid limits in the twenty-eighth and twenty-ninth ſteps, the number a will be diſcovered either by the fifth, or by the twentieth ſtep; and laſtly, the number ſought by the ſecond and twenty-ſixth ſteps. All which will be farther illuſtrated by the following Canon and Examples.

CANON.

32. Take any number leſs than the difference, or greater than the ſumm of the ſquare Roots of the two numbers given in the Queſtion: then divide the difference between the Square of the number firſt taken, and the difference of the two numbers given, by the double of the number taken; and from the Square of the Quotient ſubtract the leſſer of the two numbers given, ſo the Remainder ſhall be the number ſought.

Example 1.

Let there be two numbers given to find a third, according to *Queſt.* 8. as, ... $b = 192$, and $f = 128$

Thence it follows, that . . . . $b - f = 64 = d$

Alſo, ... $\sqrt{b} - \sqrt{f} = 2.542$, &c.

And . . . . $\sqrt{b} + \sqrt{f} = 25.170$, &c.

Now according to the *Canon*, take ſome number leſs than $2\frac{542}{1000}$, and call it e, as . . . . $e = 1$

Thence it follows, that . . . . $\frac{d-ee}{2e} = 31\frac{1}{2} = a$

And laſtly, ... $aa - f = 864\frac{1}{4}$ ſought.

The Proof.

$192 + 864\frac{1}{4} = 1056\frac{1}{4}$, whoſe $\sqrt{}$ extracted is $32\frac{1}{2}$.

$128 + 864\frac{1}{4} = 992\frac{1}{4}$, whoſe $\sqrt{}$ extracted is $31\frac{1}{2}$.

Example 2.

Example 2.

Again, the ſame things being given as in *Example* 1. take ſome number greater than $25\frac{128}{1000}$, and call it n, as . . $n = 32$

Thence it follows that . . . . $\frac{nn - d}{2n} = 15 = a$

And laſtly, ... $aa - f = 97$ ſought.

The Proof.

$$192 + 97 = 289, \text{ whoſe } \surd \text{ is } 17.$$
$$128 + 97 = 225, \text{ whoſe } \surd \text{ is } 15.$$

Note. In divers of *Diophantus*'s Queſtions, where Algebraick Quantities are to be equated to Squares, there is great uſe of finding out Limits, (after the manner delivered in the laſt preceding *Reſolution*,) to direct how to feign the ſides of the ſaid Squares, ſo, as that their values in numbers may be greater than nothing; and therefore for the more ample Illuſtration of that Method I have framed the five Queſtions next following, in the Reſolutions whereof, the induſtrious Learner will meet with no difficulty, if he be well exercis'd in the manner of reſolving Quadratick Equations according to *Sect.* 5, 7, 9. *Chap.* 15. *Book* 1. as alſo in the *Obſervations* upon the firſt Queſtion of this third Book.

QUEST. 9.

To find a number, call it a, that ſhall be leſs than 3, and cauſe $aa + 12$ to be a ſquare number.

RESOLUTION.

1. Put Letters for the given numbers, *viz.* . . . . $d = 12$, $f = 3$

2. Then the Queſtion requires that $aa + d$ may be equal to a Square, but its Side (or Root) muſt be ſo feigned that the value of a may be leſs than f, and greater than nothing; to which end the ſaid Side may be feigned to be either $a + e$, or $-a + n$; (which a, e and n do repreſent numbers yet unknown ;) Firſt therefore ſuppoſing the ſaid Side to be $a + e$, the Square thereof is $aa + 2ae + ee$, which muſt be equated to $aa + d$ above-mentioned; hence this following Equation ariſeth, *viz.*
$$aa + 2ae + ee = aa + d.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . $a = \frac{d - ee}{2e}$

4. And ſince the Queſtion requires that a may be leſs than 3, *viz.* . . . . $a \sqsupset f$

5. Therefore from the third and fourth ſteps . . . . $\frac{d - ee}{2e} \sqsupset f$

6. And by multiplying each part of the fifth ſtep by $2e$, it follows that . . . . $d - ee \sqsupset 2fe$

7. And by adding ee to each part of the ſixth ſtep, . . $d \sqsupset ee + 2fe$

8. And by adding the Square of half the Coefficient $2f$ to each part of the ſeventh ſtep, it gives . . . . . $d + ff \sqsupset ee + 2fe + ff$

9. And by extracting the ſquare Root out of each part of the eighth ſtep, ... $\surd : \overline{d + ff} : \sqsupset e + f$

10. And by ſubtracting f from each part of the ninth ſtep, $\surd : \overline{d + ff} : - f \sqsupset e$

11. Wherefore from the tenth ſtep, by comparing the latter part to the firſt, ... $e \sqsubset \surd : \overline{d + ff} : - f$

12. And ſince by the third ſtep $ee \sqsupset d$, therefore, . . $e \sqsupset \surd d$

13. Again, for as much as the Side of the Square mentioned in the ſecond ſtep may be feigned to be $-a + n$, let the Square of $-a + n$ be equated to $aa + d$ as the Queſtion requires, ſo this Equation ariſeth, *viz.*
$$aa + d = aa - 2na + nn.$$

14. Which Equation, after due Reduction to find out the value of a, gives . . . . $a = \frac{nn - d}{2n}$

15. And

ssssssss

$$\tfrac{2}{3}ab - \tfrac{1}{2}aa \Big)\ \tfrac{4}{9}aabbbb + \tfrac{13}{12}a^4b - a^6 \Big(\tfrac{2}{3}abb + \tfrac{13}{8}aaa + \tfrac{1}{2}aab + \tfrac{3}{4}aaa$$

$$\tfrac{4}{9}aabbb - \tfrac{1}{3}aaabb$$

$$\tfrac{1}{2}\ \tfrac{2}{3} \qquad 0 \quad 1 \quad 0 + \tfrac{13}{12}a^4b + \tfrac{1}{3}aaabb - a^6$$

$$\tfrac{13}{12}a^4b - \tfrac{13}{16}a^6$$

$$\tfrac{3}{6}\ 4\ \overset{2}{3}\Big)\tfrac{13}{12}\Big(\tfrac{39}{24}\ \ \tfrac{13}{8}$$

$$0 \quad + \tfrac{1}{3}aaabb - \tfrac{3}{16}a^6$$

$$\tfrac{1}{2}\ \tfrac{13}{8}\ \overset{2}{3}\Big)\tfrac{1}{3}\Big(\tfrac{3}{6}$$

$$+ \tfrac{1}{3}aaabb - \tfrac{1}{4}a^4b$$

$$0 - \tfrac{3}{16}a^6 + \tfrac{1}{4}a^4b$$

$$-\tfrac{3}{16}a^6 + \tfrac{1}{4}a^4b$$

$\frac{aab}{aa}$ $\frac{ddf}{df}$ $\frac{abc}{b}$ $\frac{a^3}{a^3}$ $\frac{ab+ac-a}{a}$

$\frac{2b-2a}{3x}$ $b+c-1$

$\frac{b-2}{3}$

$2abd+3bd$
$3b-b$

$\frac{2ba^3+caa-3aa}{baa-daa+aa}$

$\frac{45d}{2c}$ $\frac{2cd}{1c}$

$2ad+3d$
$3b-1$

$2ba+c-3$
$b-d+1$

$\frac{27ab}{9ad}$ $\frac{3ab}{4ad}$

$\frac{16gh}{8gh}$ 2

$\frac{18a^4}{16aa}$ $8aa$

b
$\frac{70b^5c^4dd}{5bbccd}$ 7 $30b^3c^4$

$\frac{28bbc+16bbd}{28bb}$ 4

$2baa-3caa$
$2b-3c$

$2b-3c$
$\frac{aa}{2baa-3caa}$

$c+d\,)\,ac+ad+bc+bd\,(\,a+b$
$ac+ad$
$+bc+bd$
$bc+bd$

$a+b\,)\,aa-bb\,(\,a-b$
$aa+ab$
$-ab-bb$
$-ab-bb$

15. And because the Question requires a to be less than 3, viz. . . . . $a \sqsupset f$

16. Therefore from the fourteenth and fifteenth steps it follows that . . . . $\frac{uu - d}{2u} \sqsupset f$

17. And by continuing the Process to find out Limits for u, in like manner as before for e from the fifth step to the eleventh inclusive, it will at length appear that . . . . $u \sqsupset \sqrt{:d + ff:} + f$

18. And since by the sixteenth step, $uu \sqsubset d$, therefore $u \sqsubset \sqrt{d}$

From the first, eleventh, twelfth, seventeenth, eighteenth, third, and fourteenth steps the following Canon is deduced, by the help whereof innumerable Answers may be found out to the Question proposed.

CANON.

19. Take any number (e) between $\sqrt{:d + ff:} - f$ and $\sqrt{d}$, that is, between $1\frac{582}{1000}$, &c. and $3\frac{464}{1000}$, &c. Or any number (u) between $\sqrt{d}$ and $\sqrt{:d + ff:} + f$, that is, between $3\frac{464}{1000}$ &c. and $7\frac{582}{1000}$, &c. Then divide the difference between the Square of the number taken and d, or 12, by the double of the number taken, so the Quotient shall be the number a sought.

Example 1.

Let there be two numbers given in such manner as before is supposed in this *Quest.* 9. viz. . . . . $d = 12$, and $f = 3$

Then according to the first limits in the Canon take some number between $1\frac{582}{1000}$ and $3\frac{464}{1000}$ as 2, and call this e, viz. suppose . . . . $e = 2$

And then by the latter part of the Canon, ... $\frac{d - ee}{2e} = 2 = a$ sought.

Which number 2, to wit, a, will solve the Question, for it is less than 3; and $aa + 12$, that is, 16, is a Square, as was required.

Example 2.

Again, the same things being given as in *Example* 1. take some number between $3\frac{464}{1000}$ and $7\frac{582}{1000}$, (according to the latter limits in the Canon,) as 4, and call this u, viz. suppose $u = 4$

And then you will find . . . . $\frac{uu - d}{2u} = \frac{1}{2} = a$

Which Fraction $\frac{1}{2}$, to wit, a, will solve the Question, for it is less than 3; and $aa + 12$, that is, $12\frac{1}{4}$, is a Square as was required.

QUEST. 10.

To find out a number, call it a, that $aa - 60$ may be greater than $5a$, but less than $8a$.

RESOLUTION.

1. Put Letters for the given numbers, as . . . . $b = 60$, $c = 5$, $d = 8$

2. Then the Question requires that $aa - b$ may be greater than ca, yet less than da; first then let us suppose $aa - b \sqsubset ca$

3. Thence it follows, by adding b to each part, that . . $aa \sqsubset ca + b$

4. And by subtracting ca from each part in the third step $aa - ca \sqsubset b$

5. And by adding the Square of half the known Coefficient c to each part of the fourth step, it follows that $aa - ca + \frac{1}{4}cc \sqsubset b + \frac{1}{4}cc$

6. And by extracting the square Root out of each part of the fifth step, it gives . . . . $a - \frac{1}{2}c \sqsubset \sqrt{:b + \frac{1}{4}cc:}$

7. Wherefore by adding $\frac{1}{2}c$ to each part of the sixth step, $a \sqsubset \frac{1}{2}c + \sqrt{:b + \frac{1}{4}cc:}$

8. Again, let us suppose, as the Question also requires, that $aa - b \sqsupset da$

9. Whence by arguing in like manner as before from the second step to the seventh inclusive, saving that instead of $\sqsubset$ there, $\sqsupset$ is to be used in this latter argumentation, it will at length appear that . . . . . $a \sqsupset \frac{1}{2}d + \sqrt{:b + \frac{1}{4}dd:}$

10. Thus,

10. Thus, (by the seventh and eighth steps) limits are discovered, within which any number may be taken for the value of a the number sought, *viz.*

$$a \sqsubset \tfrac{1}{2}c + \sqrt{:b + \tfrac{1}{4}cc}: \quad (\ 10\tfrac{6394102}{10000000},\ \&c.\ )$$

$$a \sqsupset \tfrac{1}{2}d + \sqrt{:b + \tfrac{1}{4}dd}: \quad (\ 12\tfrac{7177978}{10000000},\ \&c.\ )$$

As, for example, if $a = 12$, which is within the said Limits, then $aa - 60 = 84$; also $5a = 60$, and $8a = 96$: But 84 (that is, $aa - 60$) is greater than 60, (that is, $5a$,) and less than 96, (that is, $8a$;) and therefore the number 12, (that is, a,) doth manifestly solve the Question proposed.

QUEST. 11.

To find out a number, call it a, that shall be greater than $10\frac{6394102}{10000000}$, but less than $12\frac{7177978}{10000000}$, and cause $aa - 60$ to be equal to some square number.

RESOLUTION.

1. Put Letters for the given numbers, as, ... $\left\{ \begin{array}{l} b = 60, \\ f = 10\frac{6394102}{10000000}, \\ d = 12\frac{7177978}{10000000}. \end{array} \right.$

2. Then, (according to the import of the Question,) $aa - b$ must be equal to some Square, but the side thereof must be so feigned that the value of a may be greater than f, but less than d; to which purpose, the said side may be feigned to be $a - e$, or $e - a$, (which a and e do represent numbers unknown,) and then the Square of the said $a - e$, or $e - a$ being equated to $aa - b$ above mentioned, gives this Equation, *viz.*

$$aa - b = aa - 2ae + ee.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . $a = \frac{ee + b}{2e}$

4. But according to the Question, ... $a \sqsubset f$

5. Therefore from the third and fourth steps, ... $\frac{ee + b}{2e} \sqsubset f$

6. And by multiplying each part of the fifth step by $2e$, it follows that . . . . $ee + b \sqsubset 2fe$

7. And by subtracting b from each part of the sixth step, . . $ee \sqsubset 2fe - b$

8. And by equal subtraction of $2fe$ from each part of the seventh step, ... $ee - 2fe \sqsubset -b$

9. And by adding ff, that is, the Square of half the known Coefficient $2f$ in the eighth step, to each part, it follows that $ee - 2fe + ff \sqsubset ff - b$

10. And by extracting the square Root out of each part of the ninth step, ... $e - f \sqsubset \sqrt{:ff - b}:$

11. Wherefore by adding f to each part of the tenth step, it's evident that . . . . $e \sqsubset f + \sqrt{:ff - b}:$

12. Again, because $f - e$, (as well as $e - f$,) may be the side of the Square $ee - 2fe + ff$ in the first part of the ninth step, it thence follows, that . . . . $f - e \sqsubset \sqrt{:ff - b}:$

13. And by adding e to each part of the twelfth step, . . . $f \sqsubset e + \sqrt{:ff - b}:$

14. And by subtracting $\sqrt{:ff - b}:$ from each part of the thirteenth step, ... $f - \sqrt{:ff - b}: \sqsubset e$

15. Wherefore from the fourteenth step, by comparing the latter part to the former, 'tis manifest that . . . . $e \sqsupset f - \sqrt{:ff - b}:$

16. Again, because the Question requires . . . . . . $a \sqsupset d$

17. It follows from the third and sixteenth steps, that . . . $\frac{ee + b}{2e} \sqsupset d$

18. Whence by arguing in like manner as before from the fifth step to the fifteenth *inclusivè*, saving that d is to be used here, instead of f there, and $\sqsupset$ instead of $\sqsubset$, it will at length appear that . . . . $\left\{ \begin{array}{l} e \sqsupset d + \sqrt{:dd - b}: \\ e \sqsubset d - \sqrt{:dd - b}: \end{array} \right.$

From the eleventh, eighteenth, fifteenth, first and third steps the following Canon ariseth, which will find out innumerable Answers to the Question proposed.

CANON.

CANON.

19. Take any number (e) greater than $f + \sqrt{:\overline{ff-b}}:$ but leſs than $d + \sqrt{:\overline{dd-b}}:$ (that is, any number between $17\frac{921}{1000}$, &c. and $22\frac{824}{1000}$, &c.) or any number greater than $d - \sqrt{:\overline{dd-b}}:$ but leſs than $f - \sqrt{:\overline{ff-b}}:$ (that is, any number between $2\frac{621}{1000}$, &c. and $3\frac{341}{1000}$, &c.) Then $\frac{ee+b}{2e}$ ſhall be equal to (a) the number ſought.

Examples.

Firſt, for the number e take 22 which is within the former limits in the Canon; then $\frac{ee+b}{2e}$ gives $12\frac{4}{11}$ for the number a ſought by the Queſtion: For if from the Square of $12\frac{4}{11}$, to wit, $\frac{18496}{121}$, you ſubtract the given number 60, (or b,) the Remainder $\frac{11236}{121}$ is a Square whoſe ſide is $\frac{106}{11}$; and the ſaid $12\frac{4}{11}$ (that is, a,) is greater than $10\frac{63}{100}$, &c. but leſs than $12\frac{72}{100}$, &c. as the Queſtion requires.

Again, for the number e take 3 which is within the latter limits in the Canon; then $\frac{ee+b}{2e}$ gives $11\frac{1}{2}$ (the number a,) which will likewiſe ſolve the Queſtion propoſed: For if from the Square of $11\frac{1}{2}$ you ſubtract 60, there will remain a Square, to wit, $\frac{289}{4}$, whoſe ſide is $\frac{17}{2}$; and the ſaid $11\frac{1}{2}$ (that is, a) is greater than $10\frac{63}{100}$, &c. but leſs than $12\frac{72}{100}$, &c. as was required.

QUEST. 12.

To find out a number, call it a, that ſhall be greater than $2\frac{1}{3}$, (a number given,) and cauſe $aa + 4a + 2$ to be equal to ſome ſquare number.

RESOLUTION.

1. Put letters for the given numbers, as, ... $\begin{cases} b = 4; \\ d = 2\frac{1}{3}, \\ f = 2. \end{cases}$

2. Then the Queſtion requires that $aa + ba + f$ may make a ſquare number, but its ſide muſt be ſo feigned that the value of a may be greater than d. Now to cauſe thoſe effects, the ſaid ſide may be feigned to be $a - e$, or $e - a$, (which e and a do repreſent numbers yet unknown,) and then the Square of $a - e$ or $e - a$, that is, $aa - 2ae + ee$, being equated to $aa + ba + f$, gives this Equation, *viz.*

$$aa + ba + f = aa - 2ae + ee.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . $a = \frac{ee-f}{2e+b}$
4. And becauſe the Queſtion requires . . . . . . $a \sqsubset d$
5. It follows from the third and fourth ſteps, that . . $\frac{ee-f}{2e+b} \sqsubset d$
6. And by multiplying each part of the fifth ſtep, by $2e+b$, $ee - f \sqsubset 2de + db$
7. And by adding f to each part of the ſixth ſtep, . . $ee \sqsubset 2de + db + f$
8. And by ſubtracting $2de$ from each part of the ſeventh ſtep, ... $ee - 2de \sqsubset db + f$
9. And by adding the Square of half the known Coefficient $2d$ in the eighth ſtep, to each part, it's manifeſt that $ee - 2de + dd \sqsubset db + f + dd$
10. And by extracting the ſquare Root out of each part of the ninth ſtep, ... $e - d \sqsubset \sqrt{:\overline{db+f+dd}}:$
11. Wherefore by adding d to each part of the tenth ſtep, it's evident that . . . . $e \sqsubset \sqrt{:\overline{db+f+dd}}: + d$
12. And conſequently by reſolving the latter part of the eleventh ſtep into numbers, according to the Poſitions in the firſt, ... $e \sqsubset 6\frac{428}{1000}$, &c.
13. The third ſtep alſo ſhews that $ee \sqsubset f$, and conſequently . . . . $e \sqsubset \sqrt{f}$, ($1\frac{414}{1000}$, &c.)

But this latter limit for the chusing of e is uselefs, for if e be greater than $6\frac{772}{1000}$, &c. as appears by the twelfth ftep, it is evidently greater than $1\frac{414}{1000}$, &c.

14. Laftly, from the eleventh, twelfth, third and firft fteps the following *Canon* arifeth, which will find innumerable Anfwers to the Queftion propofed.

CANON 1.

Take any number greater than $\sqrt{:\overline{db+f+dd}:}+d$, (*viz.* greater than $6\frac{772}{1000}$, &c.) and call the number taken e. Then $\frac{ee-f}{2e+b}$ fhall be equal to the number a fought.

15. But if it were defired to find a number a that might be lefs than $2\frac{1}{2}$, and greater than nothing, and make $aa+4a+2$ to be a fquare number, then the fame Pofitions and Procefs being made as before, faving that $\sqsupset$ is to be ufed inftead of $\sqsubset$ from the fourth ftep to the twelfth inclufive, at length there would arife this following

CANON 2.

Take any number (e) greater than $\sqrt{f}$, but lefs than $\sqrt{:\overline{db+f+dd}:}+d$: (*viz.* any number between $1\frac{414}{1000}$, &c. and $6\frac{772}{1000}$, &c.) Then $\frac{ee-f}{2e+b}$ will give the number a fought.

An Example of the firft Canon.

For the number e take 8 which exceeds $6\frac{772}{1000}$, &c. as the firft Canon doth direct. Then $\frac{ee-f}{2e+b}$ gives $3\frac{1}{10}$ for the number a fought; for 'tis greater than $2\frac{1}{2}$ (or d,) and $aa+4a+2$ makes a Square, to wit, $\frac{2401}{100}$, whofe fide is $\frac{49}{10}$, as was required.

Note, That $a+u$ might be feigned to be the fide of the Square mentioned in the fecond ftep, and thence limits would be difcovered to chufe the number u, by which the number a would confequently be made known; but I leave the fearch of thefe latter limits as an exercife for the Learner.

QUEST. 13.

To find out a number, call it a; that fhall be greater than 1, but lefs than 4, and make $121+45a-9aa$ to be a fquare number.

RESOLUTION.

1. Firft put Confonants to reprefent the numbers given in the Queftion, as, ... $b=1$, $d=4$, $f=11$, $ff=121$, $g=45$, $h=9$

2. Then the Queftion requires that $ff+ga-haa$ may make a fquare number, whofe fide muft be fo feigned that the value of a may be greater than b, but lefs than d: To which purpofe the faid fide may be feigned to be $f+ea$, or $f-ua$; (where a, e, u do reprefent numbers unknown:) Firft then let the faid fide be feigned $f+ea$, and let its Square $ff+2fea+eeaa$ be equated to $ff+ga-haa$ above-mentioned, fo this following Equation arifeth, *viz.*

$$ff+2fea+eeaa=ff+ga-haa.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . $a=\frac{g-2fe}{h+ee}$

4. And becaufe the Queftion requires . . . . . $a \sqsubset b$

5. It follows from the third and fourth fteps, that $\frac{g-2fe}{h+ee} \sqsubset b$

6. And by multiplying each part in the fifth ftep, by the Denominator $h+ee$, it follows, that . . $g-2fe \sqsubset bh+bee$

7. And by fubtracting bh from each part in the fixth ftep, ... $g-2fe-bh \sqsubset bee$

8. And

8. And by adding $2fe$ to each part in the feventh ftep $\}$ $g - bh \sqsubset bee + 2fe$

9. And by dividing every quantity in the eighth ftep by b, that bee may be freed from its Coefficient b, it follows that . . . . $\}$ $\frac{g - bh}{b} \sqsubset ee + \frac{2f}{b}e$

10. And by adding $\frac{ff}{bb}$, that is, the Square of half the Coefficient $\frac{2f}{b}$ to each part of the ninth ftep, it gives . . . . $\}$ $\frac{ff + bg - bbh}{bb} \sqsubset ee + \frac{2f}{b}e + \frac{ff}{bb}$

11. And by extracting the fquare Root out of each part of the tenth ftep, ... $\}$ $\sqrt{\frac{ff + bg - bbh}{bb}} \sqsubset e + \frac{f}{b}$

12. And by fubtracting $\frac{f}{b}$ from each part of the eleventh ftep, ... $\}$ $\sqrt{:\frac{ff + bg - bbh}{bb}:} - \frac{f}{b} \sqsubset e$

13. Therefore from the twelfth ftep, by comparing the latter part to the firft, it's manifeft that . . $\}$ $e \sqsupset \sqrt{:\frac{ff + bg - bbh}{bb}:} - \frac{f}{b}$

14. Again, becaufe the Queftion requires . . . $\}$ $a \sqsupset d$

15. And by the third ftep, ... $\}$ $a = \frac{g - 2fe}{h + ee}$

16. It follows from the fourteenth and fifteenth fteps, that . . . . $\}$ $\frac{g - 2fe}{h + ee} \sqsupset d$

17. Whence by arguing in like manner as before from the fifth ftep to the thirteenth inclufive, it will at length appear that . . . . $\}$ $e \sqsubset \sqrt{:\frac{ff + dg - ddh}{dd}:} - \frac{f}{d}$

18. Again, let the fide of the Square mentioned in the fecond ftep be feigned to be $f - ua$, and then the Square of $f - ua$ being equated to $ff + ga - haa$, (as the Queftion requires) this following Equation arifeth, *viz.*

$$ff - 2fua + uuaa = ff + ga - haa.$$

19. Which Equation gives this value of a, viz. . . $\}$ $a = \frac{g + 2fu}{h + uu}$

20. But the value of a laft mentioned muft (as the Queftion requires) be greater than b, and lefs than d; and if the Procefs be continued from the laft preceding ftep, to find out limits for u in like manner as before for e from the third ftep to the feventeenth inclufive, it will at length appear that

$$u \sqsupset \frac{f}{b} + \sqrt{\frac{ff + bg - bbh}{bb}},$$

$$u \sqsubset \frac{f}{d} + \sqrt{\frac{ff + dg - ddh}{dd}}.$$

Now after any number is taken for the value of e within the limits in the thirteenth and feventeenth fteps, the number a required by the Queftion will be difcovered by the third and firft fteps. Or, after any number is taken for the value of u within the limits in the preceding twentieth ftep, the number a fought will be made known by the nineteenth and firft fteps. All which will be made manifeft by the following Canon and Examples.

CANON.

21. Take any number lefs than $\sqrt{:\frac{ff + bg - bbh}{bb}:} - \frac{f}{b}$; but greater than $\sqrt{:\frac{ff + dg - ddh}{dd}:} - \frac{f}{d}$, (that is, any number between $1\frac{129}{1000}$, *&c.* and $\frac{182}{1000}$, *&c.*) and call the number taken e; then $\frac{g - 2fe}{h + ee}$ fhall be the number a fought. Or take any number lefs than $\frac{f}{b} + \sqrt{\frac{ff + bg - bbh}{bb}}$, but greater than $\frac{f}{d} + \sqrt{\frac{ff + dg - ddh}{dd}}$, (that is, any number between $23\frac{129}{1000}$, *&c.* and $5\frac{882}{1000}$, *&c.*) and call the number taken u; then $\frac{g + 2fu}{h + uu}$ will give the number a fought.

Examples.

Examples.

Suppose $e = 1$, which is within the first limits in the Canon, then $\frac{g - 2fe}{h + ee} = \frac{23}{10} = a$ the number sought: For $\frac{23}{10}$ is greater than 1, but less than 4; and if $a = \frac{23}{10}$, then $121 + 45a - 9aa = \frac{17689}{100}$, which is a Square, (for its side is $\frac{133}{10}$) as the Question requires.

Again, suppose $e = \frac{1}{2}$, (which is likewise within the first limits,) then $\frac{g - 2fe}{h + ee} = \frac{16}{15} = a$ the number sought: For $\frac{16}{15}$ is greater than 1, but less than 4; and if $a = \frac{16}{15}$, then also $121 + 45a - 9aa$ makes a Square, to wit, $\frac{35721}{225}$, whose side is $\frac{189}{15}$.

Again, suppose $u = 18$, which is within the latter limits in the Canon, then $\frac{g + 2fu}{h + uu} = \frac{49}{37} = a$ the number sought: For if $a = \frac{49}{37}$, then $121 + 45a - 9aa$ makes a Square, to wit, $\frac{225625}{1369}$, whose side is $\frac{475}{37}$; and $\frac{49}{37}$ (or a) is greater than 1, but less than 4, as the Question requires.

QUEST. 14. (Quæst. 13. Lib. 2. *Diophant.*)

To find a number, that if it be subtracted first from 192, and then from 64, each Remainder may be a Square.

RESOLUTION.

1. For the number sought put . . . . a
2. Which number must be such, that each of these Quantities (or Remainders) may make a Square, *viz.* . . . $\begin{cases} 192 - a = \square \\ 64 - a = \square \end{cases}$
3. Now to resolve that Duplicate Equality; first, (by *Canon* 2. *Quest.* 7. of this *Book* 3.) find out two such square numbers that their difference may be equal to 128, that is, the difference of the two given numbers 192 and 64, or the difference between the two Algebraick Quantities $192 - a$ and $64 - a$, and that the greater Square may be less than 192, (the greater of the two numbers given in the Question;) but two such Squares are 144 and 16.
4. Then from either of these Equations, ... $\begin{cases} 192 - a = 144 \\ 64 - a = 16 \end{cases}$
5. One and the same value of a, that is, the number sought will be discovered, *viz.* . . . . $a = 48$
7. I say 48 will solve the Question, as will be evident by

The Proof.

$$\left.\begin{aligned} 192 - 48 &= 144, \\ 64 - 48 &= 16, \end{aligned}\right\} \text{ which are Squares, as was required.}$$

The premisses give this following

CANON.

8. First, (by the second Canon of the seventh Question of this third Book,) find out two square numbers in the same difference with the numbers given, and that the greater Square may be less than the greater number given; whence consequently, (as will appear by the following *Theorem*,) the lesser Square shall be less than the lesser number given: Then from the greater number given subtract the greater Square, or from the lesser number subtract the lesser Square, so shall either of those Remainders (for they are equal to one another) be the number sought.

But the certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in the Question, will be evident by this following

THEOREM.

9. If two Square numbers have the same difference as two other numbers, and that the greater number exceeds the greater Square, then the lesser number shall exceed the lesser Square, and the excess of the greater number above the greater Square shall be equal to the excess of the lesser number above the lesser Square. To make this manifest, let dd and gg represent two square numbers, whereof dd is the greater; also b the greater, and c the lesser of two other numbers; Then, (according to the import of the *Theorem*,)

10. Sup-

10. Suppoſe . . . . $b - c = dd - gg \sqsupset 0$
11. And . . . . $b \sqsupset dd$
12. Then by adding c to each part of the Equation in the tenth ſtep, it follows that . . . . $b = c + dd - gg$
13. Therefore by ſubtracting dd from each part of the laſt Equation, (which ſubtraction appears to be poſſible by the eleventh and tenth ſteps,) that which the Theorem affirms is manifeſt, *viz.* . . . . $b - dd = c - gg$
14. And conſequently either of thoſe two equal Exceſſes or Remainders which make the laſt Equation, ſhall be a number that will ſolve the Queſtion propoſed; for it is manifeſt, that if the firſt Exceſs $b - dd$ be ſubtracted from b, and the latter Exceſs $c - gg$ from c, the Remainders will be Squares, to wit, dd and gg.

To ſolve the foregoing 14th Queſtion *after another manner*, viz.

To find a number, that if it be ſubtracted firſt from 9, and then from 21, each Remainder may be a Square.

RESOLUTION.

1. It is evident, that if $9 - aa$ be ſubtracted from 9, there will remain a Square, to wit, aa, therefore for the number ſought put . . . $9 - aa$
2. And then if $9 - aa$ be ſubtracted from 21, the Remainder muſt likewiſe make a Square, therefore . . . . $aa + 12 = \square$
3. It remains to equate $aa + 12$ to ſome Square, whoſe ſide muſt be ſo feigned that the value of a may be leſs than 3, (for by the firſt ſtep $aa \sqsubset 9$, and conſequently $a \sqsubset 3$;) But to cauſe that effect, the ſaid ſide may be feigned to be either $a +$ any abſolute number between $1\frac{582}{1000}$, &c. and $3\frac{464}{1000}$, &c. or elſe $-a +$ any abſolute number between $3\frac{464}{1000}$, &c. and $7\frac{582}{1000}$, &c. (which limits are found out by the ninth Queſtion of this third Book;) let therefore the ſaid ſide be feigned $a + 2$, and then by equating the Square of $a + 2$ to $aa + 12$ before-mentioned in the ſecond ſtep, this Equation ariſeth, *viz.*

$$aa + 4a + 4 = aa + 12.$$

4. Which Equation gives . . . . $a = 2$.

Therefore from the fourth and firſt ſteps the number ſought is 5; for if it be ſubtracted from 9 and 21 ſeverally, it leaves the Squares 4 and 16.

From this latter Reſolution of *Queſt.* 14. (reſpect being had to the foregoing ninth Queſtion,) a Canon may be deduced to find out innumerable Anſwers to the ſaid fourteenth Queſtion; but I leave it to the Learners exerciſe.

QUEST. 15. (Quæſt. 14. Lib. 2. *Diophant.*)

To find a number, from which if 27 and 15 (two numbers given) be ſeverally ſubtracted, each Remainder may be a Square.

RESOLUTION 1.

1. For the number ſought put . . . . a
2. Which number muſt be ſuch that each of theſe Quantities or Remainders may make a Square, *viz.* . . . . $a - 27 = \square$, $a - 15 = \square$
3. Now to reſolve that Duplicate Equality, find out (by *Canon* 2. of the preceding *Queſt.* 5.) two ſquare numbers whoſe difference may be 12, that is, the difference of the two given numbers 27 and 15; but here is no need of limiting either of the ſaid Squares: Suppoſe then the ſaid Squares are found 16 and 4,
4. Then from either of theſe Equations, ... $a - 27 = 4$, $a - 15 = 16$
5. The number a ſought will be diſcovered, *viz.* . . . . $a = 31$

I ſay 31 will ſolve the Queſtion; for if from 31 you ſubtract 27 and 15 ſeverally, the Remainders are Squares, to wit, 4 and 16.

From the premiſſes there ariſeth this following

CANON.

CANON.

6. Firſt, (by the preceding fifth Queſtion,) find two ſquare numbers that ſhall have the ſame difference as the two numbers given; then add the leſſer Square to the greater number, or the greater Square to the leſſer number; ſo ſhall either of thoſe ſumms (for they are equal to one another) be the number ſought.

The truth of which Canon, and conſequently of the Reſolution of the Duplicate Equality in the Queſtion, will be evident by the following

THEOREM.

7. If two ſquare numbers, ſuppoſe dd the greater and gg the leſſer, have the ſame difference as two other numbers, ſuppoſe b the greater and c the leſſer; then the ſumm of the leſſer Square and the greater number ſhall be equal to the ſumm of the greater Square and the leſſer number: For,

By ſuppoſition, ... $b - c = dd - gg$

And by adding gg to each part, ... $b - c + gg = dd$

Wherefore by adding c to each part of the laſt Equation, that which the Theorem affirms is manifeſt, *viz.* . . . . $b + gg = c + dd$

8. And conſequently either of thoſe two equal ſumms in the laſt Equation ſhall be a number to ſolve the Queſtion propoſed: For it is evident, that if b be ſubtracted from the firſt ſumm $b + gg$, and c from the latter ſumm $c + dd$, the Remainders are Squares, to wit, gg and dd.

To ſolve the foregoing Queſt. 15. *after another manner*, viz.

To find a number, from which if 27 and 15 be ſeverally ſubtracted, each Remainder may be a Square.

RESOLUTION 2.

1. It is evident that if 27 be ſubtracted from $aa + 27$, the Remainder will be a Square, to wit, aa; therefore for the number ſought put . . . . $aa + 27$

2. But then 15 being ſubtracted from the ſaid $aa + 27$, the Remainder muſt likewiſe be equal to a Square, therefore . . $aa + 12 = \square$

3. It remains to equate $aa + 12$ to ſome Square, whoſe ſide may be feigned either $a +$ any abſolute number leſs than $\sqrt{12}$, or $3\frac{464}{1000}$, *&c.* or elſe $-a +$ any abſolute number greater than the ſaid $3\frac{464}{1000}$, *&c.* (which limits are found out in like manner as in the foregoing *Queſt* 9.) Let therefore the ſaid ſide be feigned $a + 3$, and then by equating the Square of $a + 3$, that is, $aa + 6a + 9$ to $aa + 12$, the value of a will thence be found $\frac{1}{2}$, and conſequently the number ſought, (which in the firſt ſtep was put $aa + 27$) ſhall be $27\frac{1}{4}$, which will ſolve the Queſtion: For if from $27\frac{1}{4}$ the given numbers 27 and 15 be ſeverally ſubtracted, the Remainders will be Squares, to wit, $\frac{1}{4}$ and $\frac{49}{4}$.

But if this ſecond manner of reſolving *Queſt.* 15. be formed by Literal *Algebra*, (like to the third manner of reſolving the preceding *Queſt.* 8.) there will ariſe this

CANON.

Take any number greater than the ſumm, or leſs than the difference of the ſquare Roots of the two numbers given; then divide the difference between the Square of the number taken and the difference of the given numbers by the double of the number taken; laſtly, to the Square of that Quotient add the greater of the numbers given, ſo ſhall the ſumm be the number ſought.

A third way of ſolving the preceding Queſt. 15.

Let the Poſitions in the firſt and ſecond ſteps of the preceding *Reſolut.* 2. be reſumed; then ſince $aa + 12$ muſt be equal to a Square, 'tis evident that 12 is the difference between that Square and aa; therefore by the preceding fifth Queſtion find two Squares whoſe difference may be 12; ſuch are 16 and 4, the leſſer of which ſhall be the value of aa; therefore $aa + 27$ which was put for the number ſought will be found 31, as before in the firſt Reſolution of this Queſtion.

QUEST. 16.

QUEST. 16.

To find a number, that if 12 be added to it; and 8 subtracted from the same; as well the Summ as the Remainder may be a Square.

RESOLUTION.

1. For the number sought put . . . . a
2. Then each of these Quantities must make a Square, *viz.* . $\left\{ \begin{array}{l} a + 12 = \square \\ a - 8 = \square \end{array} \right.$
3. Now to resolve that Duplicate Equality, first, subtract $a - 8$ from $a + 12$, and the Remainder is 20; this is equal as well to the summ of the given numbers as to the difference of the two Squares sought: Then (by the second *Canon* of the fifth *Question* of this third *Book*) find two Squares whose difference shall be 20; such are 36 and 16.
4. Then from either of these Equations . . . . $\left\{ \begin{array}{l} a + 12 = 36 \\ a - 8 = 16 \end{array} \right.$
5. The number a sought will be made known, *viz.* . . . . . $a = 24$

Which number found out, to wit, 24, will solve the Question; for if it be increased with 12, and lessened by 8, the Summ and Remainder are Squares, to wit, 36 and 16.

The substance of the Resolution is contain'd in this following

CANON.

6. First, (by the second *Canon* of the fifth *Question* aforegoing) find out two square numbers whose difference shall be equal to the summ of the two numbers given; then subtract the number given to be added (whether it be the greater or the lesser of those given) from the greater Square, or add the number given to be subtracted to the lesser Square; so as well the Remainder as the Summ (for they are equal to one another) shall be the number sought.

The certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in the Question, will be evident by this following

THEOREM.

7. If there be two square numbers, suppose dd the greater, and gg the lesser, whose difference is equal to the summ of two other numbers, b and c; then the excess of the greater Square above either of the two numbers, shall be equal to the summ of the lesser Square and the other number.
8. For, by supposition . . . . $dd - gg = b + c$
9. Whence by adding gg to each part, it follows that . . . $dd = gg + b + c$
10. Therefore by subtracting b from each part of the last Equation, this ariseth, *viz.* . . . . $dd - b = gg + c$
11. Likewise, by subtracting c from each part of the Equation in the ninth step, there remains . . . . $dd - c = gg + b$

Wherefore from the two last Equations, the truth of the Theorem, and consequently of the Canon is manifest.

QUEST. 17.

To find a number, that if it be added to, and subtracted from a given square number, suppose 4, the Summ and Remainder may be Squares.

RESOLUTION.

1. For the number sought put . . . . a
2. Which number must be such, that if it be added to and subtracted from 4, as well the Summ as the Remainder may make a Square, *viz.* . . . . $\left\{ \begin{array}{l} 4 + a = \square \\ 4 - a = \square \end{array} \right.$
3. To resolve that Duplicate Equality, first, (after the manner of *Example* 3. *Canon* 1. *Resolut.* 2. *Quest.* 2. of this *Book*,) divide 8 the double of the given Square 4 into two unequal Squares, $\frac{196}{25}$ and $\frac{4}{25}$.

4. Then

4. Then from either of these Equations, ... $\begin{cases} 4 + a = \frac{196}{25} \\ 4 - a = \frac{4}{25} \end{cases}$

5. The number sought will be discovered, *viz.* . . . . . . $a = \frac{96}{25}$

Which number $\frac{96}{25}$ will solve the Question proposed; for if it be added to, and subtracted from 4, the Summ and Remainder are Squares, to wit, $\frac{196}{25}$ and $\frac{4}{25}$, whose sides are $\frac{14}{5}$ and $\frac{2}{5}$. By the like Operation (the substance whereof is contained in the following Canon,) you may find out innumerable Answers to this 17th Question.

CANON.

6. Divide the double of the given Square into two unequal Squares, (by the preceding *Quest.* 2.) then from the greater of the two Squares found out subtract the given Square, or from this subtract the lesser of those two, so shall either of the Remainders, (for they are equal to one another) be the number sought.

But the certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in this 17th Question, will be evident by the following

THEOREM.

7. If two square numbers, suppose cc the greater, and dd the lesser, be equal to the double of a square number, as $bb + bb$, or $2bb$; then the excess of the greater of those unequal Squares above one of the equal Squares shall be equal to the excess of the other of the equal Squares above the lesser of the unequal Squares.

8. For by supposition . . . . $cc + dd = bb + bb$
9. Also by supposition . . . . $cc \sqsubset dd$
10. Therefore . . . . $cc \sqsubset bb$
11. And . . . . $dd \sqsupset bb$
12. And by subtraction of bb from each part of the Equation in the eighth step, ... $cc + dd - bb = bb$
13. Wherefore by subtracting dd from each part of the last Equation, (which subtraction the 10th and 11th steps do shew to be possible,) that which the Theorem asserts is manifest, *viz.* $cc - bb = bb - dd$
14. And consequently either of the Excesses (or Remainders) which make the last preceding Equation shall be the number a sought: For if the first Excess $cc - bb$ be added to bb, and the latter Excess $bb - dd$ subtracted from bb, the Summ and Remainder are Squares, to wit, cc and dd.

QUEST. 18.

To find a number, that if a given Square 9 be added to that number, and from another given Square 4 the same number sought be subtracted, the Summ and Remainder may be Squares.

RESOLUTION.

1. For the number sought put . . . . a

2. Then the Question requires that . . . . $\begin{cases} 9 + a = \square \\ 4 - a = \square \end{cases}$

3. To resolve that Duplicate Equality, first (by the preceding *Quest.* 4.) divide 13 the summ of the given Squares 9 and 4 into two such other Squares that one of these found may exceed 9 the Square given to be added; but two such Squares are $\frac{324}{25}$ and $\frac{1}{25}$, whose summ is 13, and the greater of them exceeds 9.

4. Then from either of these Equations . . . . $\begin{cases} 9 + a = \frac{324}{25} \\ 4 - a = \frac{1}{25} \end{cases}$

5. The number (a) sought is discovered, *viz.* . . . . . $a = \frac{99}{25}$

Which number found out, to wit, $\frac{99}{25}$, will solve the Question; for if it be added to 9, and subtracted from 4, the Summ and Remainder are Squares, to wit, $\frac{324}{25}$ and $\frac{1}{25}$, whose sides are $\frac{18}{5}$ and $\frac{1}{5}$. By the like Operation (the substance whereof is exprest in the following Canon,) you may find out innumerable Answers to this 18th Question, because (by *Quest.* 4. of this *Book*) the summ of two Squares may be divided into as many pairs of Squares as you please, such, that one of each pair shall consist within given limits.

CANON.

CANON.

7. First, (by *Quest.* 4. of this *Book*,) divide the summ of the two Squares given into two such Squares, that the greater of these found out may exceed the Square given to be added; then from the greater of the two Squares found out subtract the Square given to be added; or, from the other Square given subtract the other Square found out; so shall either of the Remainders (for they are equal to one another) be the number sought.

The certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in the Question proposed will be manifest by this following

THEOREM.

8. If the summ of two square numbers, suppose bb the greater, and cc the lesser, be found equal to the summ of two other unequal Squares, dd and ff, and that the greater of of the two former exceeds either of the two latter, then the other of the two latter shall exceed the lesser of the two former, and one excess shall be equal to the other. For,

9. By supposition . . . . $bb + cc = dd + ff$
10. By supposition also . . . . $bb \sqsubset dd$
11. Therefore . . . . $cc \sqsupset ff$
12. And by subtracting dd from each part of the Equation in the ninth step, this ariseth, *viz.* . . . . $bb + cc - dd = ff$
13. Wherefore by subtracting cc from each part of the last Equation, (which subtraction the tenth and eleventh steps do shew to be possible) that which the Theorem affirms is manifest, *viz.* . . . . $bb - dd = ff - cc$
14. And consequently the truth of the Canon and Resolution of the Duplicate Equality in this 18th Question is evident.

QUEST. 19.

To find a square number, that if it be increased or lessened by its side, may make a Square.

RESOLUTION.

1. For the side of the Square sought put . . . . a
2. Therefore the Square it self is . . . . aa
3. Then the Question requires . . . . $\begin{cases} aa + a = \square \\ aa - a = \square \end{cases}$
4. Which Duplicate Equality differs but little from that in the foregoing seventeenth Question, and may be resolved thus: First, (after the manner of *Example* 3. *Canon* 1. *Resolut.* 2. *Quest.* 2. of this *Book*, divide 2, which is compos'd of two Squares, 1 and 1, suppos'd to be prefixt to aa and aa in the Duplicate Equality in the third step, into two unequal Squares, $\frac{49}{25}$ and $\frac{1}{25}$; then multiply each of these by aa, and equate the greater Product $\frac{49}{25}aa$ to $aa + a$, or the lesser Product $\frac{1}{25}aa$ to $aa - a$, so from either of those Equations one and the same value of a will be discovered, *viz.* $a = \frac{25}{24}$, which is the side of the Square sought; for if $\frac{25}{24}$ be added to and subtracted from its Square $\frac{625}{576}$, the Summ and Remainder are Squares, to wit, $\frac{1225}{576}$ and $\frac{25}{576}$, whose sides are $\frac{35}{24}$ and $\frac{5}{24}$. It is also easie to be perceived that this Question is capable of innumerable Answers.

QUEST. 20.

To find two numbers in a given Reason, suppose the greater to the lesser as 2 to 1; and that the Square of the summ of the two numbers being added to each of them may make square numbers.

RESOLUTION.

1. For the lesser number sought put . . . . a
2. Then the greater, to the end it may be to the lesser as 2 to 1, shall be . . . . $2a$
3. Therefore the summ of the two numbers sought is . . . $3a$
4. And the Square of their summ is . . . . $9aa$

5. To which Square if the two numbers $2a$ and a be ſeverally added, each ſumm muſt be a Square, therefore . . . . $\left.\begin{matrix} 9aa + 2a = \square \\ 9aa + a = \square \end{matrix}\right.$

6. Which Duplicate Equality, (according to *Diophantus*'s Method, before explain'd and demonſtrated in the Obſervations upon the firſt manner of ſolving the eighth Queſtion of this Book,) may be reſolved thus; *viz.* Firſt, ſubtract $9aa + a$ from $9aa + 2a$, and the Remainder a is the difference of the two Squares that are to be equated to thoſe Algebraick Quantities; then find two Squares whoſe difference may be equal to the ſaid difference a, but with this Caution, that in each of thoſe Squares there may be found $9aa$, to the end that when the greater Square is equated to $9aa + 2a$, or the leſſer to $9aa + a$, the ſaid $9aa$, after due Reduction, may vaniſh, and an Equation remain between ſome number of a and ſome known Rational number, whence the value of a will be expreſſible by ſome known number either Affirmative or Negative. Now to find out two ſuch Squares, let two numbers be taken, ſuch, that (according to *Canon* 2. *Queſt.* 5. of this *Book*,) the Product of their Multiplication may make a, and that the half of their ſumm may conſiſt of $3a +$ ſome abſolute number, and the half of their difference of $3a -$ ſome abſolute number, (for then it will follow that as well in the Square of the ſaid half ſumm, as in the Square of the ſaid half difference there will be found $9aa$;) whence we may inferr, that the ſumm of the ſaid two numbers muſt conſiſt of $a +$ ſome abſolute number, and their difference of $6a -$ ſome abſolute number; to which purpoſe, divide 1 by 6, (1 becauſe the difference is $1a$, and 6 becauſe it is the double of the ſquare Root of 9 which is prefixt to aa,) ſo the Quotient is $\frac{1}{6}$, and conſequently $6a$ multiplied by $\frac{1}{6}$ makes a; whence 'tis evident that the only two numbers fit for the purpoſe aforeſaid are $6a$ and $\frac{1}{6}$, whoſe half ſumm is $3a + \frac{1}{12}$, and half difference $3a - \frac{1}{12}$, (or $-3a + \frac{1}{12}$,) therefore the Squares of the ſaid half ſumm and half difference are $9aa + \frac{1}{2}a + \frac{1}{144}$ and $9aa - \frac{1}{2}a + \frac{1}{144}$; now let the greater of thoſe Squares be equated to $9aa + 2a$, and the leſſer to $9aa + a$, ſo theſe two following Equations will ariſe,

$$viz. \left\{\begin{matrix} 9aa + 2a = 9aa + \frac{1}{2}a + \frac{1}{144}, \\ 9aa + a = 9aa - \frac{1}{2}a + \frac{1}{144}. \end{matrix}\right.$$

7. Then from either of thoſe Equations, after due Reduction, the value of a will be found to be $\frac{1}{216}$, and conſequently, (from the firſt and ſecond ſteps,) the numbers ſought are $\frac{2}{216}$ and $\frac{1}{216}$, which will ſolve the Queſtion propoſed: For firſt, the greater hath ſuch proportion to the leſſer as 2 to 1; and if the Square of the ſumm of $\frac{2}{216}$ and $\frac{1}{216}$ be added to them ſeverally, the two ſumms will be Squares, to wit $\frac{49}{5184}$ and $\frac{25}{5184}$, whoſe ſides are $\frac{7}{72}$ and $\frac{5}{72}$.

8. In like manner, to reſolve this Duplicate Equality, *viz.*

$$\text{If} \left\{\begin{matrix} 4aa + 3a - 1 = \square \\ 4aa - a - 1 = \square \end{matrix}\right\} \text{what is } a = ?$$

Firſt, I find the difference of thoſe two Algebraick Quantities to be $4a$; then I ſearch out two Quantities that being mutually multiplied may make $4a$, and that as well in half their ſumm as in half their difference there may be found $2a$, (that is, the ſquare Root of $4aa$;) ſo by working as before is directed, I find $4a$ and 1 to be the only two Quantities agreeing with thoſe conditions: Then the Square of half the ſumm of $4a$ and 1, *viz.* the Square of $2a + \frac{1}{2}$ being equated to $4aa + 3a - 1$ will give $a = \frac{5}{4}$; or, the Square of half the difference of $4a$ and 1, *viz.* the Square of $2a - \frac{1}{2}$ being equated to $4aa - a - 1$, gives $a = \frac{5}{4}$, as before.

QUEST. 21.

To find two numbers in a given Reaſon, ſuppoſe the greater to the leſs as 3 to 2, and that the ſumm of the numbers being added to each of their Squares, may make Squares.

RESOLUTION.

1. For the leſſer number put . . . . $2a$
2. Then the greater (to the end that both numbers may be in the Reaſon preſcribed,) ſhall be . . . . $3a$
3. Therefore their ſumm is . . . . $5a$

4. Which

4. Which ſumm added to the Square of each of the ſaid two numbers $2a$ and $3a$, muſt (as the Queſtion requires) make a Square, therefore . . . . $\left\{\begin{array}{l} 4aa + 5a = \square \\ 9aa + 5a = \square \end{array}\right.$

5. Now in order to reſolve that Duplicate equality, it muſt firſt be reduced to another, wherein there may be equal ſquare numbers prefixt to aa, which may be done thus; Divide 9 the greater of the two Squares that are prefixt to aa by the leſſer 4, and then by the Quotient $\frac{9}{4}$ multiply that Algebraick Quantity where the ſaid Diviſor 4 is prefixt to aa, viz. $4aa + 5a$ by $\frac{9}{4}$, and it produceth $9aa + \frac{45}{4}a$ to be equated to a Square; ſo now inſtead of the Duplicate equality in the fourth ſtep, this ariſeth,

$$viz.\ \left\{\begin{array}{l} 9aa + \frac{45}{4}a = \square \\ 9aa + 5a = \square \end{array}\right.$$

6. In which Duplicate equality laſt above expreſt, there are equal Squares, to wit, 9 and 9 prefixt to aa, and therefore it may be reſolved after the manner before ſhewn in the 20th Queſtion, and when the value of a is diſcovered in this latter duplicate equality, it will neceſſarily conſtitute the former duplicate equality in the fourth ſtep; for as a Square multiplied by a Square produceth a Square, ſo converſly, a Square divided by a Square gives the Quotient a Square. In order then to reſolve the ſaid duplicate equality in the fifth ſtep, ſubtract $9aa + 5a$ from $9aa + \frac{45}{4}a$, and the Remainder is $\frac{25}{4}a$, this muſt be eſteem'd the Product made by the mutual multiplication of two quantities to be taken with ſuch Caution, that as well in half their ſumm as in half their difference there may be found $3a$, (becauſe the ſquare Root of $9aa$ in each of the two quantities to be equated to a Square is $3a$;) ſo by conſidering well that Caution, and what hath been ſaid to the like purpoſe in the ſixth ſtep of the Reſolution of the foregoing 20th Queſtion, you will find that $6a$ and $\frac{25}{24}$ are the only two numbers, that being mutually multiplied make $\frac{25}{4}a$, and have $3a$ as well in half their ſumm as in half their difference: Therefore let the Square of half the ſumm of the ſaid $6a$ and $\frac{25}{24}$, *viz.* the Square $9aa + \frac{25}{24}a + \frac{625}{2304}$ be equated to $9aa + \frac{45}{4}a$; or, let the Square of half the difference of the ſaid $6a$ and $\frac{25}{24}$, *viz.* the Square $9aa - \frac{25}{24}a + \frac{625}{2304}$, be equated to $9aa + 5a$; ſo from either of thoſe Equations, one and the ſame value of a will be diſcovered, *viz.* $a = \frac{125}{3744}$, and conſequently $2a$ and $3a$, which in the firſt and ſecond ſteps were put for the numbers ſought, will be diſcovered to be $\frac{125}{1872}$ and $\frac{125}{1248}$, which will ſolve the Queſtion: For, firſt, the greater is in proportion to the leſſer as 3 to 2; and if their ſumm be added to their Squares ſeverally, the two ſumms made by ſuch addition will be Squares, to wit, $\frac{600625}{3504384}$ and $\frac{275625}{1557504}$, whoſe ſides are $\frac{775}{1872}$ and $\frac{525}{1248}$.

7. But the Duplicate equality in the fourth ſtep may be reduced to another wherein there ſhall be equal ſquare numbers prefixt to aa by this following Operation, which differs from that in the ſixth ſtep, *viz.* becauſe 9 times 4 makes the ſame Product as 4 times 9, and becauſe a Square multiplied by a Square produceth a Square, let $4aa + 5a$ (in the fourth ſtep) be multiplied by 9, and $9aa + 5a$ by 4; ſo there will neceſſarily be found $9aa$ in each Product, and this following duplicate equality comes now to be reſolved inſtead of that in the fourth ſtep,

$$viz.\ \left\{\begin{array}{l} 36aa + 45a = \square \\ 36aa + 20a = \square \end{array}\right.$$

8. Laſtly, this Duplicate equality having equal ſquare numbers prefixt to aa, may be reſolved like that in the preceding fifth ſtep, and at length the value of a will be found $\frac{125}{3744}$, as before.

QUEST. 22.

To find two ſuch ſquare numbers, that if to the Product of their multiplication a given number (d) be added, the ſumm may be a Square.

RESOLUTION.

1. For one of the Squares ſought take any known ſquare number which may be repreſented by . . . . bb
2. And for the other Square ſought put . . . . aa
3. Then the Product of their multiplication is . . . . $bbaa$
4. To which Product the given number d being added, the ſumm is $bbaa + d$

5. Which

5. Which summ must be equal to a Square, the side whereof may be feigned to be ba — any known number greater than $\sqrt{d}$, suppose $ba - c$; then the Square of $ba - c$, that is, $bbaa - 2bca + cc$ being equated to $bbaa + d$, this Equation ariseth, *viz.*

$$bbaa + d = bbaa - 2bca + cc.$$

6. Whence, after due Reduction, ... $a = \frac{cc - d}{2bc}$

From the premisses ariseth this following

CANON.

For one of the Squares sought take any square number; then from any square number subtract the given number, and divide the Remainder by the double of the Product made by the multiplication of the sides of those two Squares; so the Quotient shall be the side of the other Square sought.

An Example in Numbers.

Let the number given be . . . . $12 = d$
For one of the Squares sought take any square number, as . . . $4 = bb$
Take also some other square number greater than 12, (or d,) as $36 = cc$
Then (by the Canon) the side of the other Square sought shall be $1 = \frac{cc - d}{2bc}$

I say, 4 and 1 are two Squares, which will solve the Question when the number given is 12; for if to 4, the Product of 4 and 1, you add 12, the summ makes a Square, to wit, 16. By the like Operation you may find out innumerable Answers to the Question without varying the given number; and 'tis easie also to find out other Canons to solve the same.

QUEST. 23. (Quæst. 15. Lib. 2. *Diophant.*)

To divide a given number (b) into two parts, and to find a square number, which if it be increased with each of those parts, may make a Square.

RESOLUTION.

1. Take two such numbers, that the summ of their Squares may be less than the giuen number b, suppose these, . . c and d
2. Then for the side of the Square sought put . . . . a
3. The Square thereof is . . . . aa
4. To the side a add severally c and d, and assume the summs to be the sides of two Squares, so the first side will be $a + c$
5. And the other side . . . . $a + d$
6. The Square of $a + c$ (in the fourth step) is . . . . $aa + 2ca + cc$
7. The Square of $a + d$ (in the fifth step) is . . . . $aa + 2da + dd$
8. Then for one of the desired parts of (b) put $2ca + cc$, (for it's evident, that if the Square aa in the third step be increased with $2ca + cc$, it makes the Square $aa + 2ca + cc$ in the sixth step,) . . . . $2ca + cc$
9. And for the other part of b put $2da + dd$, for this added to aa makes a Square, to wit, that in the seventh step . . $2da + dd$
10. But the summ of the parts in the eighth and ninth steps must be equal to b, therefore . . . . $2ca + cc + 2da + dd = b$
11. Which Equation, after due Reduction, gives . . . . $a = \frac{b - cc - dd}{2c + 2d}$

The premisses well examined, afford this following

CANON.

12. Take two numbers, with this Caution, that the summ of their Squares may be less than the number given to be divided; then subtract the summ of those Squares from the given number, and divide the Remainder by the double summ of the numbers taken, so the Quotient shall be the side of the Square sought; then multiply the double of the said side severally by the numbers first taken, and to the Products add severally the respective Squares of the numbers taken; so the summs made by those additions shall be the desired parts of the number given.

An Example in Numbers.

Let the number given to be divided be . . . . $33 = b$

Let two numbers be taken, such, that the summ of their Squares may be less than 33, as . . . . $2 = c$, $3 = d$

Then (by the Canon) the side of the Square sought, (which side is represented by a in the Resolution,) shall be . . . $2 = \frac{b - cc - dd}{2c + 2d}$

Also, one of the desired parts of 33 is . . . . $12 = 2ca + cc$

And the other is . . . . $21 = 2da + dd$

The Proof.

$4 = aa$ the Square sought;

$12 + 21 = 33$ the number given;

$4 + 12 = 16$, $4 + 21 = 25$, which are Squares; as was required.

QUEST. 24.

To find two such numbers, that their summ may make a Square: Also, that each number being added to the Square of the other number, may make a Square.

RESOLUTION.

1. For the summ of the two numbers sought assume some Square, as . . . ee
2. And for the first of the two desired numbers put . . . . . a
3. Therefore the other shall be . . . . $ee - a$
4. Which added to the Square of the first number a, makes the summ $aa - a + ee$
5. Which summ last exprest, the Question requires to be a Square, and such it will be, if we suppose $e = \frac{1}{2}$, for then the said $aa - a + ee$ (in the fourth step) will be equal to $aa - a + \frac{1}{4}$, which is the Square of $a - \frac{1}{2}$, or $\frac{1}{2} - a$: Now therefore let the Resolution be renewed thus,
6. For the summ of the two numbers sought, (instead of ee) put . . . $\frac{1}{4}$
7. And for the first number put (as before) . . . . a
8. Therefore the other shall be . . . . $\frac{1}{4} - a$
9. Which latter number added to the Square of the first, makes the summ $aa - a + \frac{1}{4}$

But the summ last exprest is manifestly a Square, whose side is $a - \frac{1}{2}$, or $\frac{1}{2} - a$; therefore two of the conditions in the Question are satisfied.

10. Again, if to the Square of the second number $\frac{1}{4} - a$, viz. to $aa - \frac{1}{2}a + \frac{1}{16}$, the first number a be added, the summ is $aa + \frac{1}{2}a + \frac{1}{16}$, which the Question likewise requires to be a Square, and so it is, for 'tis the Square of $a + \frac{1}{4}$; but if the last mentioned summ had not happened to have been a Square, then a Square might have been feigned equal to it, according to the method in divers preceding Questions of this *Book* 3.

The premisses discover this following

THEOREM.

11. If the Fraction $\frac{1}{4}$ be divided into any two parts, each part increased with the Square of the other part shall make a Square.

By the help therefore of this Theorem, innumerable Answers to the Question proposed may be found out.

An Example.

Let two Fractions be taken whose summ makes $\frac{1}{4}$, as $\frac{1}{6}$ and $\frac{1}{12}$; I say these will solve the Question: For first, their summ is a Square; secondly, the first Fraction $\frac{1}{6}$ increased with $\frac{1}{144}$, (the Square of the latter Fraction $\frac{1}{12}$) makes the Square $\frac{25}{144}$; likewise the latter Fraction $\frac{1}{12}$ increased with $\frac{1}{36}$, the Square of the first Fraction $\frac{1}{6}$) makes a Square, to wit, $\frac{1}{9}$.

Moreover, by the help of the said Theorem, this following Question may be solved, *viz.*

12. To find two numbers in a given Reason, suppose the greater to the lesser as 3 to 2; and that each number being added to the Square of the other number, may make a Square.

Divide the Fraction $\frac{1}{4}$ into two such parts, that the greater may be to the lesser as 3 to 2, so you will find $\frac{3}{20}$ and $\frac{2}{20}$, which will solve the Question: For first, by

by Construction they are in the Reason prescribed; secondly, $\frac{3}{20}$ with $\frac{1}{100}$ (the Square of $\frac{1}{10}$) makes the Square $\frac{4}{25}$; and lastly, $\frac{1}{10}$ with $\frac{9}{400}$ (the Square of $\frac{3}{20}$) makes the Square $\frac{49}{400}$.

13. Another manner of solving the Question last proposed may be this, *viz.*

For the two numbers sought in the given Reason of 3 to 2, put $>$ $3a$ and $2a$

Then, since the Question requires, that each number being added to the Square of the other number may make a Square, this Duplicate equality ariseth, *viz.* . . . . $\left\{ \begin{array}{l} 9aa + 2a = \square \\ 4aa + 3a = \square \end{array} \right.$

Which Duplicate equality may be resolved like that in the foregoing twenty-first Question; so the value of a will be found $\frac{361}{10080}$, and consequently $3a$ and $2a$ give $\frac{1083}{10080}$ and $\frac{722}{10080}$ for the numbers sought: For first, they are in proportion as 3 to 2; secondly, if to the Square of the first you add the second number, it makes the Square $\frac{8450649}{101606400}$, whose side is $\frac{2907}{10080}$; lastly, if to the Square of the second number you add the first, it makes the Square $\frac{11437924}{101606400}$, whose side is $\frac{3382}{10080}$.

QUEST. 25. (Quæst. 29. Lib. 2. *Diophant.*)

To find two such square numbers, that each being increased with the Product of their multiplication may make a Square.

RESOLUTION.

1. First, (by the preceding *Quest.* 5.) find out two Squares that may differ by unity, such are $\frac{9}{16}$ and $\frac{25}{16}$, and take the lesser for one of the Squares sought, as . . . . $\}$ $\frac{9}{16}$
2. Then for the other Square sought assume . . . . $>$ aa
3. Therefore the Product of their multiplication is . . . . $>$ $\frac{9}{16}aa$

Which Product $\frac{9}{16}aa$ being added to the second Square aa doth manifestly make a Square, to wit, $\frac{25}{16}aa$; but if the said Product $\frac{9}{16}aa$ be added to the first Square $\frac{9}{16}$ it must also make a Square, therefore $\frac{9}{16}aa + \frac{9}{16}$ not being a Square, must be equated to a Square, the side whereof may be variously feigned, let it be $\frac{3}{4}a - 1$; then the Square of $\frac{3}{4}a - 1$ being equated to $\frac{9}{16}aa + \frac{9}{16}$ gives $a = \frac{7}{24}$, and consequently $aa = \frac{49}{576}$ is the second Square sought. I say, $\frac{9}{16}$ and $\frac{49}{576}$ are two Squares, which will solve the Question, as will appear by

The Proof.

The two Squares found out are . . . . $>$. $\frac{9}{16}$ and $\frac{49}{576}$
The Product of their multiplication is . . . . $>$ . . $\frac{441}{9216}$
Which Product added severally to $\frac{9}{16}$ and $\frac{49}{576}$, makes these Squares, ... $\}$ $\frac{5625}{9216}$ and $\frac{1225}{9216}$
The sides of which last mentioned Squares are . . . . $>$. $\frac{75}{96}$ and $\frac{35}{96}$

After the same manner you may easily find out two such Squares, that if the Product of their multiplication be subtracted from them severally, the Remainders may be Squares.

QUEST. 26. (Quæst. 35. Lib. 2. *Diophant.*)

To find three such numbers, that the Square of every one of them being added to the summ of the three numbers, may make a Square.

RESOLUTION.

1. If to the Product of the multiplication of any two unequal numbers the Square of half their difference be added, the summ shall be a Square, to wit, the Square of half the summ of the two numbers multiplied: Therefore by the help of this Theorem numbers proper for the Resolution of the Question proposed may be taken, *viz.* Take some number at pleasure, as 12, and divide it thrice into two such numbers that the Product of the two numbers of each pair may make 12; such are these three pairs of numbers, *viz.* 1, 12. | 2, 6. | 3, 4; then take the half-difference of the two numbers of each pair, so you will find the three half-differences to be these, $\frac{11}{2}$, 2 and $\frac{1}{2}$. Now by the Theorem above-mentioned, if to the Product 12 the Squares of the said three half-differences be severally added, every one of the summs will be a Square; therefore

2. For

2. For the ſumm of the three numbers ſought put . . . . $12aa$
3. And for the firſt number . . . . $\frac{11}{2}a$
4. And for the ſecond number . . . . $2a$
5. And for the third number . . . . $\frac{1}{2}a$
6. Therefore the ſumm of thoſe three numbers is . . . . $8a$
7. Which ſumm muſt be equal to $12aa$ in the ſecond ſtep, therefore . $8a = 12aa$
8. Which Equation, after due Reduction, gives . . . . $a = \frac{2}{3}$
9. Therefore from the eighth, third, fourth and fifth ſteps the three numbers ſought are theſe, *viz.* . . . . $\frac{11}{3}, \frac{4}{3}, \frac{1}{3}$
10. Which three numbers will ſolve the Queſtion, for if their Squares be ſeverally added to their ſumm, the three ſumms will be Squares, to wit, $\frac{169}{9}$, $\frac{64}{9}$ and $\frac{49}{9}$. It is alſo evident, that the Queſtion may be extended to four, five, or as many numbers as ſhall be deſired, by this following

CANON.

11. Take ſome number at pleaſure, and divide it into two numbers as many times as there be numbers deſired by the Queſtion, but ſo, as that the Product of the two numbers of each pair may make the number firſt taken; then divide the ſumm of the half-differences of the two numbers of each pair by the number taken, and multiply the Quotient by the ſaid half-differences ſeverally; ſo the Products ſhall be the numbers ſought.

The certainty of this Canon depends upon the Theorem aſſumed in the Reſolution; which Theorem may be demonſtrated thus;

Let two numbers be aſſumed, as, ... $\begin{cases} a \text{ the greater,} \\ e \text{ the leſſer.} \end{cases}$

The Product of their multiplication is . . . . ae
The half-difference of the ſaid two numbers is . . . . . $\frac{1}{2}a - \frac{1}{2}e$
The Square of the ſaid half-difference is . . . . $\frac{1}{4}aa - \frac{1}{2}ae + \frac{1}{4}ee$
To which Square if ae the Product above-mentioned be added, the ſumm is . . . . $\frac{1}{4}aa + \frac{1}{2}ae + \frac{1}{4}ee$

Which ſumm is the Square of $\frac{1}{2}a + \frac{1}{2}e$, to wit, half the ſumm of the two numbers a and e firſt taken, as the Theorem affirmed.

QUEST. 27. (pars Quæſt. 9. Lib. 3. *Diophant.*)

To find three Squares in Arithmetical proportion, and ſuch, that the half of their ſumm may exceed the greateſt of the three Squares.

RESOLUTION.

1. For the leaſt of the three Squares ſought put . . . . . . aa
2. And to the end the mean Square may exceed the leaſt, let the ſide of the mean Square be $a + 1$, therefore the mean Square is $aa + 2a + 1$
3. Therefore the exceſs of the mean Square above the leaſt is . . $2a + 1$
4. And by adding the ſaid exceſs to the mean Square, the ſumm muſt be equal to the greateſt, which ſumm is . . . . $aa + 4a + 2$
5. Therefore $aa + 4a + 2$ muſt be equated to ſome Square, with this condition, that $\frac{1}{2}aa + 3a + \frac{3}{2}$, which is the half-ſumm of the three Squares in the firſt, ſecond and fourth ſteps, muſt (according to the Queſtion) exceed the greateſt of the ſaid three Squares; ſuppoſing therefore $\frac{1}{2}aa + 3a + \frac{3}{2} \sqsubset aa + 4a + 2$, it will thence follow (by arguing after the manner of ſearching out limits in divers of the foregoing Queſtions) that $a \sqsubset 1 + \sqrt{2}$, that is, $a \sqsubset 2.414$, &c.

Therefore $aa + 4a + 2$ (in the fourth ſtep) muſt be ſo equated to a Square, that the value of a may be greater than $2\frac{414}{1000}$, &c. Now to cauſe that effect, innumerable values of a may be found out by the firſt Canon of the twelfth Queſtion of this Book. Suppoſe therefore a be found equal to $\frac{31}{10}$, (as in the Example of that Canon) then from the firſt, ſecond and fourth ſteps, theſe three Squares will be diſcovered, to wit, $\frac{961}{100}$, $\frac{1681}{100}$ and $\frac{2401}{100}$, which will ſolve the Queſtion.

And becauſe a Square multiplied by a Square produces a Square, by multiplying ſeverally the ſaid three Squares found out by the common Denominator 100, the Products

961, 1681 and 2401 will be Squares, whoſe ſides are 31, 41 and 49; alſo the difference of the firſt and ſecond is equal to the difference of the ſecond and third, (each difference being 720;) therefore the ſaid three Squares are in Arithmetical proportion, and the half of their ſumm is manifeſtly greater than every one of them: Therefore all the conditions in the Queſtion are ſatisfied.

QUEST. 28. (Quæſt. 9. Lib. 3. *Diophant.*)

To find three numbers in Arithmetical proportion, and ſuch, that the ſumm of every two of them may make a Square.

RESOLUTION.

1. By the preceding twenty-ſeventh Queſtion find three Squares in Arithmetical proportion, and ſuch, that the half of their ſumm may exceed every one of them, (the reaſon of which condition will be evident by the following eighth, ninth and tenth ſteps,) ſuch are theſe three Squares, whoſe ſides are 31, 41 and 49, . . .	961; 1681; 2401
2. For the three numbers ſought put	a; e; y
3. Then equate the ſumm of the firſt and ſecond numbers to the leaſt of the three Squares before found, *viz.* ſuppoſe	$a + e = 961$
4. Likewiſe equate the ſumm of the firſt and third numbers to the mean Square, and it makes	$a + y = 1681$
5. Equate alſo the ſumm of the ſecond and third numbers to the greateſt Square, and it gives	$e + y = 2401$
6. The ſumm of the three laſt Equations is	$2a + 2e + 2y = 5043$
7. The half of the ſaid ſumm is	$a + e + y = 2521\frac{1}{2}$
8. Then by ſubtracting the third Equation from the ſeventh, there remains	$y = 1560\frac{1}{2}$
9. And by ſubtracting the fourth Equation from the ſeventh, there remains	$e = 840\frac{1}{2}$
10. And by ſubtracting the fifth Equation from the ſeventh, there remains	$a = 120\frac{1}{2}$

I ſay theſe three numbers, $120\frac{1}{2}$, $840\frac{1}{2}$, $1560\frac{1}{2}$ will ſolve the Queſtion; for the difference between the firſt and ſecond, to wit, 720, is equal to the difference between the ſecond and third; therefore they are in Arithmetical proportion, and the ſumm of every two of them makes a Square,

$$viz. \begin{cases} 120\frac{1}{2} + 840\frac{1}{2} = 961, & \text{whoſe } \surd \text{ is } 31; \\ 120\frac{1}{2} + 1560\frac{1}{2} = 1681, & \text{whoſe } \surd \text{ is } 41; \\ 840\frac{1}{2} + 1560\frac{1}{2} = 2401, & \text{whoſe } \surd \text{ is } 49. \end{cases}$$

QUEST. 29. (Quæſt. 12. Lib. 3. *Diophant.*)

To find three ſuch numbers, that if to the Product of the multiplication of every two of them, a number given, ſuppoſe 12, be added, the three ſumms ſhall be Squares.

This Queſtion may be reſolved divers ways; I ſhall here ſhew three of my own, and if the curious Reader deſire to ſee more variety, he may conſult Vieta's *Zetet.* 7. lib. 5. *alſo* Bachet's *Comment, and the Solution of* Sluſius *in* pag. 177. *of his* Meſolabum, *printed in* 1668.

RESOLUTION 1.

1. Firſt, by *Queſt.* 22. of this *Book*) I ſeek two ſuch Squares, that if to the Product of their multiplication the given number 12 be added, the ſumm ſhall be a Square; ſuch are theſe Squares 1 and 4, which may be taken for two of the three numbers ſought, ...	1 and 4
2. Then for the third number ſought I put	a

3. Now (according to the Queſtion) the Product of the multiplication of the firſt and third numbers being increaſed with 12 muſt make a Square; alſo, the Product of the

the second and third numbers together with 12 must make a Square; it remains therefore to resolve this Duplicate equality,

$$viz.\ \begin{cases} 1a + 12 = \square \\ 4a + 12 = \square \end{cases}$$

4. But before the said Duplicate equality can be resolved, it must be reduced to another that shall have equal numbers of a, to which purpose I multiply the first of the two quantities to be equated, to wit, $1a + 12$, by 4, (which is prefixt to the latter of those two quantities) and it produceth $4a + 48$ to be equated to a Square, so this Duplicate equality ariseth, (instead of the former,)

$$viz.\ \begin{cases} 4a + 48 = \square \\ 4a + 12 = \square \end{cases}$$

5. Now to resolve this latter Duplicate equality, (and consequently the former,) I proceed according to the first manner of solving the preceding *Quest.* 8. *viz.* First, the difference between $4a + 48$ and $4a + 12$ is 36; then I seek two such Squares that their difference may be 36, and that the greater of them may exceed 48; but two such Squares are 100 and 64, (found out by *Canon* 1. of the preceding *Quest.* 7.)

6. Then from either of these Equations, ... $\begin{cases} 4a + 48 = 100 \\ 4a + 12 = 64 \end{cases}$

7. The third number sought is discovered, *viz.* . . . . $a = 13$

I say, the numbers 1, 4 and 13 will solve the Question proposed; for if 12 be added to the Product of the first and second, likewise to the Product of the first and third, and lastly to the Product of the second and third, the three summs will be Squares, to wit, 16, 25 and 64. The premisses shew how to solve the Question by innumerable Answers.

Another way of resolving Quest. 29. *which is here repeated*, viz.

To find three such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 12, be added, the three summs shall be Squares.

RESOLUTION 2.

1. For the given number 12 put . . . . b
2. For the three numbers sought put . . . . $a,\ e,\ u$
3. Then supposing b to be the difference of two Squares, search out those Squares (by the fifth Question of this Book) and let cc represent the greater, and xx the lesser, therefore . . . . $cc - b = xx$
4. Now the Question requires, that if b be added to the Product of the multiplication of a and e (the first and second numbers sought) the summ must be a Square, therefore by supposing that Square to be cc above found, this Equation arises, *viz.* . . . . $ae + b = cc$
5. Therefore from the last Equation by equal subtraction of b, . $ae = cc - b$
6. And because by Construction in the third step . . . . . . $xx = cc - b$
7. Therefore from the fifth and sixth steps . . . . $ae = xx$
8. Therefore by dividing each part of the last Equation by a, it gives . . . . $e = \frac{xx}{a}$
9. Again, (by the fifth Question) find two other Squares whose difference shall be equal to b, suppose dd the greater Square, and zz the lesser, therefore . . . . $dd - b = zz$
10. Then the Product of the multiplication of the first and third numbers sought, with b added, makes $au + b$, which must be equal to some Square, let it be dd before found, therefore . . $au + b = dd$
11. Therefore by equal subtraction of b from the last Equation, . $au = dd - b$
12. And because by Construction in the ninth step . . . . $zz = dd - b$
13. Therefore from the two last Equations . . . . $au = zz$
14. Therefore by dividing each part of the last Equation by a, it gives $u = \frac{zz}{a}$
15. Now since by the second, eighth, fifth, fourteenth and eleventh steps, the positions for the three numbers sought are a, $\frac{xx}{a}$, (or $\frac{cc - b}{a}$) $\frac{zz}{a}$, (or $\frac{dd - b}{a}$) it is

 evident,

evident, that if b be added to the Product of the multiplication of the first and second numbers, the summ is a Square, to wit, cc. Likewise, if b be added to the Product of the first and third numbers the summ is a Square, to wit, dd; but if b be added to the Product of the second and third numbers the summ must also be a Square; therefore $\frac{xxzz}{aa} + b$ must be equal to a Square whose side we may feign to be $\frac{xz}{a} + t$, or $\frac{xz}{a} - t$, and consequently a will be found equal to $\frac{2xzt}{b \backsim tt}$; But x, z, b and t are known numbers, therefore a the first number sought is known also; and from the eighth and fourteenth steps the second and third numbers will be discovered.

From this second Resolution of *Quest.* 29. it will not be difficult to deduce the following

CANON.

First, supposing the given number (b) to be the difference of two Squares, find out (by the second Canon of the foregoing fifth Question) two pair of Squares in that difference, and let the side of the lesser Square of the one pair be called (x,) and the side of the lesser Square of the other pair, (z;) then take some square number whose side may be called (t,) and let the difference between (tt and b) be called (g;) then divide the double of the solid Product of the three sides x, z, t, viz. $2xzt$, by (g,) and the Quotient shall be one of the three numbers sought; lastly, multiply severally the sides x and z by g, and divide the first Product by $2zt$, and the latter by $2xt$, so the Quotients shall be the two other numbers sought. Compare this Canon with the two following Examples.

$b =$	12	12	given in the Question.
$x =$	2	2	found out according to the directions in the Canon.
$z =$	$\frac{11}{10}$	$\frac{1}{2}$	
$t =$	$\frac{1}{2}$	5	
$g =$	$\frac{11}{4}$	13	
$\frac{2xzt}{g} =$	$\frac{4}{5}$	$\frac{10}{13}$	found out by the Canon to solve the Question.
$\frac{gx}{2zt} =$	5	$\frac{26}{5}$	
$\frac{gz}{2xt} =$	$\frac{121}{80}$	$\frac{13}{40}$	

I say the number given being 12, the Question may be solved by these three numbers $\frac{4}{5}$, 5, $\frac{121}{80}$; likewise by these, $\frac{10}{13}$, $\frac{26}{5}$, $\frac{13}{40}$, as may easily be proved.

A third way of resolving Quest. 29. *which is here repeated*, viz.

To find three such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 12, be added, the three summs shall be Squares.

RESOLUTION 3.

1. For the given number 12 put . . . . b
2. For the three numbers sought put . . . . a, e, u
3. Then, according to the Question, $ae + b$ must be equal to a Square, let it be some known Square cc, therefore . . . . $ae + b = cc$
4. Therefore from the last Equation by equal subtraction of b, . . $ae = cc - b$
5. And by dividing each part of the last Equation by a, it gives . $e = \frac{cc - b}{a}$
6. Thus, the first of the three numbers sought being a, and the second $\frac{cc - b}{a}$, it is evident, that if b be added to the Product of their multiplication, the summ is a Square, to wit, cc.
7. Again, according to the conditions in the Question, $au + b$ must be equal to a Square, let it be some known Square dd, therefore $au + b = dd$
8. Therefore by equal subtraction of b, ... $au = dd - b$

9. And

9. And by dividing each part of the last Equation by a, ... $\succ u = \frac{dd - b}{a}$

10. Thus, the first of the three numbers sought being a, and the third $\frac{dd - b}{a}$, it is evident, that if b be added to the Product of their multiplication the summ is a Square; to wit, dd. But the Question requires also, that if to the Product of the second and third numbers the given number b be added, the summ must be a Square, therefore let the second and third numbers, to wit, $\frac{cc - b}{a}$ and $\frac{dd - b}{a}$ be mutually multiplied, and to the Product add b, so the summ will be $\frac{baa + ddcc + bb - bdd - bcc}{aa}$, which must be equal to a Square, the side whereof may be feigned to be either $\frac{dc + b}{a}$ or $\frac{dc \backsim b}{a}$; first let the side be $\frac{dc + b}{a}$, then its Square being equated to the summ above-mentioned, after due Reduction of that Equation it will appear that $a = c + d$.

11. Therefore, the Equation last exprest, to wit, $a = c + d$ being compared with the Quantities in the second, fifth and ninth steps, the three numbers sought, to wit, a, e, u will be found equal to these known Quantities, *viz.*

$$c + d \quad , \quad \frac{cc - b}{c + d} \quad , \quad \frac{dd - b}{c + d}.$$

12. Again, for as much as the side of the Square to be equated to $\frac{baa + ddcc + bb - bdd - bcc}{aa}$ (the summ above-mentioned in the tenth step) may be feigned to be $\frac{dc \backsim b}{a}$, (as well as $\frac{dc + b}{a}$,) let the Square of $\frac{dc \backsim b}{a}$ be equated to the said summ, then by proceeding as before, three other numbers capable of solving the Question will be found equal to these, *viz.*

$$c \backsim d \quad , \quad \frac{cc - b}{c \backsim d} \quad , \quad \frac{dd - b}{c \backsim d}.$$

From the premisses two excellent Canons are deducible to solve the forgoing *Quest.* 29.

CANON 1.

13. Subtract the given number from two Squares severally, then divide each of the Remainders by the summ of the sides of the same Squares, so shall the two Quotients and the said summ of the sides be three numbers which will solve the Question.

For example, let the given number be 12, subtract it from the Squares 36 and 64, the Remainders are 24 and 52, which being severally divided by 14, (the summ of 6 and 8, which are the sides of the said Squares 36 and 64,) the Quotients $1\frac{5}{7}$ and $3\frac{5}{7}$, with the said 14, are three numbers to solve the Question, as will be evident by

The Proof.

$$\left.\begin{array}{l} 14 \times 1\tfrac{5}{7}, + 12 = 36 \\ 14 \times 3\tfrac{5}{7}, + 12 = 64 \\ 1\tfrac{5}{7} \times 3\tfrac{5}{7}, + 12 = \tfrac{900}{49} \end{array}\right\} \text{Which are Squares, as was required.}$$

CANON 2.

14. Subtract the given number from two Squares severally, then divide each of the Remainders by the difference of the sides of the same Squares, so shall the two Quotients and the said difference be three numbers, which will solve the Question.

For example, let the given number be 12, subtract it from the Squares 36 and 64, and divide each of the Remainders 24 and 52, by 2, (which is the difference of 6 and 8, the sides of the said Squares 36 and 64,) so the Quotients 12 and 26, with the said 2, are three numbers that will solve the Question, as will appear by

The Proof.

$$\left.\begin{array}{l} 2 \times 12, + 12 = 36 \\ 2 \times 26, + 12 = 64 \\ 12 \times 26, + 12 = 324 \end{array}\right\} \text{Which are Squares, as was required.}$$

Bachet

Bachet in his Comment upon *Quest.* 12. *Lib.* 3. *Dioph.* (which is the ſame with the preceding 29th Queſtion) delivers two Canons, one of which is the ſame with *Canon* 2. above expreſt, and the other is this following

CANON 3.

15. Subtract the given number from two Squares ſeverally, divide the Remainders ſeverally by the difference of the ſides of the ſame Squares; then ſhall the two Quotients and their double ſumm leſſened by the aforeſaid difference be three numbers which will ſolve the Queſtion propounded.

For example, let the given number be 12, ſubtract it from two Squares, ſuppoſe 36 and 64, the Remainders 24 and 52 being divided ſeverally by 2, (the difference of the ſides of the ſaid Squares 36 and 64) give the Quotients 12 and 26, which are two of the three numbers ſought; then from 76 (the double ſumm of the ſaid Quotients) ſubtracting 2 (the before-mentioned difference,) the Remainder 74 ſhall be the third number ſought. By this Operation it is evident that the two firſt numbers are the ſame with thoſe found out by *Canon* 2. but the third numbers are different: I ſay the three numbers 12, 26 and 74 will ſolve the Queſtion, as may eaſily be proved.

But to manifeſt the certainty of *Canon* 3. both its Operation and Demonſtration may be ſymbolically expreſt in this manner, *viz.*

Operation.

16. Take two unequal numbers, as	d the greater; c the leſſer.
17. Their difference is	$d-c$
18. Take any number leſs than the Square of c, as, . . .	b
19. Subtract the number b from the Squares of c and d ſeverally, ſo the Remainders are	$cc-b$ and $dd-b$
20. Divide each of thoſe Remainders by $d-c$, and the Quotients are	$\frac{cc-b}{d-c}$ and $\frac{dd-b}{d-c}$
21. The double ſumm of thoſe Quotients is	$\frac{2cc+2dd-4b}{d-c}$
22. From that double ſumm ſubtract $d-c$, and the Remainder is	$\frac{cc+dd+2cd-4b}{d-c}$

23. Thus, the three numbers found out by *Canon* 3. laſt afore-going to ſolve *Quest.* 29. are equal to theſe, *viz.*

$$\frac{cc-b}{d-c}, \quad \frac{dd-b}{d-c}; \quad \frac{cc+dd+2cd-4b}{d-c}.$$

Now I ſay, if every two of thoſe three numbers be mutually multiplied, and to the Products ſeverally the number b be added, the three ſumms ſhall be Squares.

Demonſtration.

24. By *Canon* 1. *Reſolut.* 3. *Quest.* 29. if the Product of the multiplication of the two firſt numbers, to wit, of $\frac{cc-b}{d-c}$ and $\frac{dd-b}{d-c}$ be increaſed with b, the ſumm will be a Square; it remains to prove, that if the Product of the multiplication of the firſt and third numbers be increaſed with b, the ſumm ſhall alſo be a Square; likewiſe that the Product of the multiplication of the ſecond and third numbers increaſed with the number b, makes a Square. But if the number b be added to the Product of the multiplication of the firſt and third numbers, the ſumm is $\frac{cccc+ccdd+2cccd-4ccb-4cdb+4bb}{dd+cc-2cd}$, which is a Square, whoſe ſide is $\frac{cc+cd-2b}{d-c}$; And if b be added to the Product of the multiplication of the ſecond and third numbers, the ſumm is . . . . . : . $\frac{dddd+ddcc+2dddc-4ddb-4cdb+4bb}{dd+cc-2cd}$, which is a Square whoſe ſide is $\frac{dd+cd-2b}{d-c}$. Therefore the truth of *Canon* 3. is evident.

By the help of the ſecond and third Canons laſt before expreſt, *Bachet* extends the preceding *Quest.* 29. to four numbers, as I ſhall ſhew in the next Queſtion.

QUEST. 30.

QUEST. 30. (*Bachet* in Quæſt. 12. Lib. 3. *Diophant.*)

To find four ſuch numbers, that if to the Product of the multiplication of every two of them, a number given, ſuppoſe 3, be added, the four ſumms ſhall be Squares.

RESOLUTION.

1. The number given in the Queſtion is . . . . > 3
2. Let two Squares be feigned from $a +$ two ſuch known numbers that their difference may be a Square, and that each of them may exceed the given number 3. To which end let the ſide of one of thoſe Squares be feigned $a + 2$, and let the other ſide be $a + 6$, for the difference of 2 and 6 is 4, (a ſquare number;) then will the Squares of the ſaid ſides $a + 2$ and $a + 6$ be theſe,
   viz. $\begin{cases} aa + 4a + 4, & \text{(the Square of } a+2.) \\ aa + 12a + 36, & \text{(the Square of } a+6.) \end{cases}$
3. From each of thoſe Squares ſubtract the given number 3, ſo the Remainders will be theſe,
   viz. $\begin{cases} aa + 4a + 1 \\ aa + 12a + 33. \end{cases}$
4. Divide the ſaid Remainders ſeverally by 4, (the difference of the aforeſaid ſides $a + 2$ and $a + 6$) and let the Quotients be aſſumed for two of the four numbers ſought,
   viz. $\begin{cases} \frac{1}{4}aa + a + \frac{1}{4}, & \text{(the firſt number.)} \\ \frac{1}{4}aa + 3a + 8\frac{1}{4}, & \text{(the ſecond number.)} \end{cases}$
5. The double ſumm of the ſaid Quotients (aſſumed in the laſt ſtep for the firſt and ſecond numbers ſought) is $aa + 8a + 17$, from which ſubtract 4, (the difference before-mentioned) and let the Remainder be aſſumed for the third number,
   viz. $<\ aa + 8a + 13$, (the third number.)
6. If the Conſtruction hitherto be compared with the third Canon in the third Reſolution of the foregoing 29th Queſtion, it will thence be evident, that if to the Product made by the multiplication of every two of thoſe three numbers before aſſumed in the fourth and fifth ſteps, there be added the given number 3, the three ſumms will be Squares.
7. For the fourth number ſought aſſume the difference of the ſides $a+2$ and $a+6$ before-mentioned, to wit, $\}$ 4, (the fourth number.)
8. Then by comparing the Conſtruction in the ſecond, third, fourth and ſeventh ſteps with *Canon* 2. in the third Reſolution of the foregoing twenty-ninth Queſtion, it will be manifeſt, that if to the Product made by the multiplication of every two of theſe three numbers, to wit, the firſt, ſecond and fourth numbers before aſſumed in the fourth and ſeventh ſteps, there be added the given number 3, the three ſumms will be Squares.
9. It remains to make the Product of the multiplication of the third and fourth numbers, the given number 3 being added, equal to a Square; but the third number $aa + 8a + 13$, multiplied by the fourth number 4, gives the Product $4aa + 32a + 52$, to which adding 2, the ſumm is $4aa + 32a + 55$, which muſt be equal to a Square, the ſide whereof may be feigned to be $2a -$ any known number whoſe Square exceeds 55; therefore let the ſaid ſide be $2a - 10$, then the Square thereof being equated to $4aa + 32a + 55$, the value of a will thence be found equal to $\frac{5}{8}$, by the help whereof, recourſe being had to the fourth, fifth and ſeventh ſteps, the four numbers ſought will be found theſe, *viz.* $\frac{249}{256}$, $\frac{2617}{256}$, $\frac{1177}{64}$ and 4, which will ſolve the Queſtion. For if the firſt be multiplied by every one of the other three, and the Products be ſeverally increaſed with the given number 3, the three ſumms will be theſe Squares, to wit, $\frac{848241}{65536}$, $\frac{342225}{16384}$, $\frac{441}{64}$, whoſe ſides are $\frac{921}{256}$, $\frac{585}{128}$ and $\frac{21}{8}$. Alſo, if the ſecond be multiplied by the third and fourth numbers ſeverally, and each Product be increaſed with 3, theſe two Squares will ariſe, to wit, $\frac{3129361}{16384}$ and $\frac{2809}{64}$, whoſe ſides are $\frac{1769}{128}$ and $\frac{53}{8}$; laſtly, the Product of the third number multiplied by the fourth being added to 3 makes the Square $\frac{1225}{16}$, whoſe ſide is $\frac{35}{4}$.

QUEST. 31. (Quæſt. 15. Lib. 3. *Diophant.*)

To find three ſuch numbers, that if the Product of the multiplication of every two of them be leſſened by the third, the three Remainders ſhall be Squares.

RESOLUTION.

1. For the firſt number put . . . . > a

2. And

2. And for the second number put $a+$ some known square number, as . . . . $a+4$
3. Then the Product of the multiplication of the first and second numbers is . . . . $aa+4a$
4. From which Product if $4a$ be subtracted, it is evident the Remainder will be a Square, to wit, aa, therefore for the third number we may put $4a$, (and so one of the conditions in the Question will be satisfied) . . . . $4a$
5. Then (from the second and fourth steps,) the Product of the multiplication of the second and third number is $4aa+16a$, from which the first number a being subtracted, there remains $4aa+15a$, which (according to the Question) must be equal to a Square, *viz.* . . . . $4aa+15a=\square$
6. Also (from the first and fourth steps,) the Product of the multiplication of the first and third numbers is $4aa$, from which subtracting the second number $a+4$, there remains $4aa-a-4$, which (according to the Question) must be equal to a Square, *viz.* . . . . $4aa-a-4=\square$
7. So in the two last preceding steps we are faln upon a Duplicate equality, which (upon the grounds before demonstrated) may be resolved thus, *viz.* First, the difference of the two Algebraical quantities to be equated to two Squares, (by subtracting the lesser from the greater) is manifestly $16a+4$; then two Squares must be found, such, that their difference may be $16a+4$, and that $4aa$ may be in each of those Squares: Therefore (agreeable to *Canon* 2. of *Quest.* 5. of this *Book*,) two numbers are to be taken, that being mutually multiplied will produce $16a+4$; moreover, that $2a$ may be found as well in the half-summ as in the half-difference of the numbers taken; but the only two numbers that will agree with those conditions are $4a+1$ and 4, whose half-summ is $2a+\frac{5}{2}$, and their half-difference is $2a-\frac{3}{2}$.
8. Then by equating the Square of the said $2a+\frac{5}{2}$ to $4aa+15a$ (in the fifth step,) or the Square of $2a-\frac{3}{2}$ to $4aa-a-4$ (in the sixth step,) from either of those Equations the value of a will be found $\frac{5}{4}$, which is the first of the three numbers sought, and consequently from the second and fourth steps, the second and third numbers are $\frac{21}{4}$ and 5, which three numbers will solve the Question; for if from the Product of every two of them the other be subtracted, the three Remainders are Squares, to wit, $\frac{25}{16}$, 25 and 1.

QUEST. 32.

To find three numbers, such, that if to the Square of every one of them the summ of the other two be added, the three summs may be Squares.

RESOLUTION.

1. Take any number of $a+$ some known number, as $a+1$, for the side of a Square, then the Square of $a+1$ is $aa+2a+1$; now if for the first number sought we put a, for the second $2a$, and for the third 1, then it is evident that the Square of the first number, together with the summ of the second and third, makes a Square, to wit, $aa+2a+1$, whereby one of the conditions in the Question is satisfied; therefore,

   For the three numbers sought put . . . . a, $2a$, 1
2. Then (according to the import of the Question,) the Square of the second number, together with the summ of the first and third must make a Square, therefore . . . . . $4aa+a+1=\square$
3. Likewise the Square of the third number, together with the summ of the first and second must make a Square, therefore $+3a+1=\square$
4. So we are faln upon a Duplicate equality, which differs from any of the preceding Forms, but (upon the same foundation by which those have been resolved) this may be resolved thus; First, supposing the former of the two quantities to be equated to a Square to exceed the latter, (for here we may indifferently take either of them for the greater,) their difference, by subtracting $3a+1$ from $4aa+a+1$, is manifestly

$4aa-$

$4aa - 2a$; this difference, (as in moſt of the Duplicate equalities hitherto) muſt be eſteem'd the Product made by the mutual multiplication of two quantities, or factors; but here theſe two factors muſt be ſuch, that as well in half their ſumm as in half their difference there may be found 1, that is, the ſide of the known Square, (or abſolute number) in each of the two quantities to be equated, to the end that when the Square of the ſaid half-ſumm is equated to $4aa + a + 1$; or the Square of the ſaid half-difference to $3a + 1$, the ſquare number 1, by due Reduction of either of thoſe Equations, may vaniſh. Now to find out two factors qualified as aforeſaid, firſt, take 2, the double of the ſide of the known Square 1 in each of the two quantities to be equated, with $-$ prefixt, (becauſe $-$ is prefixt to $2a$ in the difference $4aa - 2a$ abovementioned,) for part of the firſt of the two deſired factors; then divide $2a$, (which is part of the ſaid difference,) by 2 the double of the ſide of the ſaid known Square 1, and take the Quotient a for the latter factor; then divide $4aa$, (the other part of the ſaid difference $4aa - 2a$) by the ſaid latter factor a, and the Quotient $4a$ connected with -2 firſt taken, ſhall be the compleat firſt factor: So two factors or quantities to agree with the conditions above mentioned are found to be $4a - 2$ and a; for the Product of their multiplication is $4aa - 2a$, and 1 is found both in half their ſumm and half their difference: Then by equating the Square of half the ſumm of the ſaid factors $4a - 2$ and a, *viz.* the Square of $\frac{5}{2}a - 1$, to $4aa + a + 1$; or by equating the Square of half the difference of the ſame factors, *viz.* the Square of $\frac{3}{2}a - 1$, to $3a + 1$, the value of a will be diſcovered, *viz.*

5. From either of theſe Equations, after due Reduction, ... $\left\{ \begin{array}{l} \frac{25}{4}aa - 5a + 1 = 4aa + a + 1 \\ \frac{9}{4}aa - 3a + 1 = 3a + 1 \end{array} \right.$
6. The ſame value of a will be diſcovered, *viz.* $> a = \frac{8}{3}$

Therefore by the ſixth and firſt ſteps, theſe three numbers are found out, to wit, $\frac{8}{3}$, $\frac{16}{3}$ and 1, which will ſolve the Queſtion propoſed: For $\frac{64}{9}$ the Square of the firſt number, together with $\frac{57}{9}$ the ſumm of the ſecond and third, makes a Square, to wit, $\frac{121}{9}$; alſo $\frac{256}{9}$ the Square of the ſecond, together with $\frac{33}{9}$ the ſumm of the firſt and third, makes a Square, to wit, $\frac{289}{9}$; and laſtly, 1 the Square of the third, with $\frac{24}{3}$ the ſumm of the firſt and ſecond makes a Square, to wit, 9. By what hath been ſaid in the firſt ſtep of the Reſolution this Queſtion is capable of innumerable Anſwers.

QUEST. 33.

To find a number leſs than 2 a number given, and ſuch, that if it be multiplied by two given numbers ſeverally, ſuppoſe by 8 and 6, and if to each of the Products a given ſquare number, ſuppoſe 4, be added, the ſumms may be ſquare numbers.

RESOLUTION.

1. For the number ſought put . . . . $> a$
2. Then if that poſition be proſecuted according to the conditions in the Queſtion, this Duplicate equality will ariſe, to wit, . . . $\left\{ \begin{array}{l} 8a + 4 = \square \\ 6a + 4 = \square \end{array} \right.$
3. Which kind of Duplicate equality *Diophantus* uſeth in divers Queſtions, and becauſe the Reſolution thereof is a very ſubtil invention, I have framed this Queſtion purpoſely to explain it.

Firſt, obſerve well theſe three numbers, ... $> 8a + 4$, $6a + 4$ and 4.

Then ſeek what proportion the exceſs of the greateſt of thoſe three numbers above the mean hath to the exceſs of the mean above the leaſt; ſo you will find that the former exceſs is to the latter as 1 to 3. For the exceſs of $8a + 4$ above $6a + 4$ is $2a$, and the exceſs of $6a + 4$ above 4 is $6a$; but $2a$ is to $6a$ as 1 to 3. Therefore the former exceſs is to the latter as 1 to 3, and conſequently the former exceſs is one third part of the latter.

4. Now the principal ſcope in reſolving the ſaid Duplicate equality is to find out two ſquare numbers with this condition, that the exceſs of the greater above the leſs may have ſuch proportion to the exceſs of the leſſer above 4 (the Square given in the Queſtion) as 1 to 3; to wit, as the difference of the numbers 8 and 6, which are prefixt to a in the Duplicate equality, is to 6 the leſſer number prefixt: For when two ſuch Squares are found out, then if the greater be equated to $8a + 4$, or the leſſer to $6a + 4$,

 one

one and the same value of a will come forth. But to find out the said two Squares I proceed thus:

5. The least of the three Squares above mentioned, to wit, that given in the Question, by the help whereof the other two are to be found out, is 4
6. And to the end the mean Square may exceed the least, let the side of the mean Square be feigned $e + 2$, (2 being the side of the given Square 4;) therefore the mean Square it self is $ee + 4e + 4$
7. Therefore the excess of the mean Square above the least is manifestly . . . . $ee + 4e$
8. But by what hath been said before, the excess of the greatest Square above the mean must be $\frac{1}{3}$ part of the excess of the mean Square above the least; therefore (from the last step) the excess of the greatest Square above the mean shall be . . . . . $\frac{1}{3}ee + \frac{4}{3}e$
9. Therefore by adding the last mentioned excess, to wit, $\frac{1}{3}ee + \frac{4}{3}e$ to the mean Square in the sixth step, the summ will be the greatest of the said three Squares, to wit, ... $\frac{4}{3}ee + \frac{16}{3}e + 4$
10. Which $\frac{4}{3}ee + \frac{16}{3}e + 4$ must be equated to a Square, but the value of e must be subject to a Determination thus found out; *viz.* Forasmuch as the two greatest of the three Squares above mentioned must be such, that when the greatest is equated to $8a + 4$, or the mean to $6a + 4$, the value of a may be less than 2, (according to the conditions in the Question;) therefore such a square number must be found out equal to the said $\frac{4}{3}ee + \frac{16}{3}e + 4$, that when 4 is subtracted from the said square number, $\frac{1}{8}$ part of the Remainder may be less than 2. Therefore from $\frac{4}{3}ee + \frac{16}{3}e + 4$ subtract 4, and the Remainder is $\frac{4}{3}ee + \frac{16}{3}e$, whereof $\frac{1}{8}$ is $\frac{1}{6}ee + \frac{2}{3}e$, which must be less than 2; therefore
11. Suppose . . . . $\frac{1}{6}ee + \frac{2}{3}e \sqsupset 2$
12. Thence, by multiplying all by 6, it follows, that . . . $ee + 4e \sqsupset 12$
13. And by adding the Square of half the Coefficient 4 to each part, there ariseth . . . . $ee + 4e + 4 \sqsupset 16$
14. And by extracting the square Root out of each part of the last step, ... $e + 2 \sqsupset 4$
15. Therefore by equal subtraction of 2, it is manifest that . . . . . $e \sqsupset 2$
16. Thus we have found that $\frac{4}{3}ee + \frac{16}{3}e + 4$ must be equated to a Square, with this condition, that the value of e may be less than 2. Now to cause that effect, the side of the said Square may be feigned $- 2 +$ any number of e greater than $3\frac{237}{1000}e$, therefore let the said side be feigned $3\frac{1}{3}e - 2$, then the Square of $3\frac{1}{3}e - 2$ being equated to the said $\frac{4}{3}ee + \frac{16}{3}e + 4$, the value of e will thence be found $\frac{21}{11}$.
17. Now if . . . . $e = \frac{21}{11}$
18. Then consequently the Square of $3\frac{1}{3}e - 2$, that is, the greater of the two Squares sought, will be . . . . $\frac{2304}{121}$
19. And the Square of $e + 2$ (which in the sixth step was put for the side of the lesser of the two Squares sought,) will be $\frac{1849}{121}$
20. Which two Squares, to wit, $\frac{2304}{121}$ and $\frac{1849}{121}$, (whose sides are $\frac{48}{11}$ and $\frac{43}{11}$,) together with 4, (the square number given in the Question,) are such, that the excess of the greatest above the mean is $\frac{1}{3}$ part of the excess of the mean above the least, (according to the scope designed in the fourth step.) Now if the greater Square $\frac{2304}{121}$ be equated to $8a + 4$, or the lesser Square $\frac{1849}{121}$ to $6a + 4$, from either of those Equations the value of a, to wit, the number sought by the Question will be found $\frac{455}{242}$: For first, it is less than 2, also eight times that number, together with 4, makes the Square $\frac{2304}{121}$; and six times the same number $\frac{455}{242}$, together with 4, makes the Square $\frac{1849}{121}$. It is also evident by the sixteenth step, that as many numbers as one will may be found out to solve the Question proposed.
21. But for the greater evidence of the infallibility of the method of resolving this Duplicate equality, I shall demonstrate the same in manner following, *viz.*

Suppose

Suppose $\left\{\begin{array}{l} r = 8 \\ s = 6 \\ c = 4 \\ d = \frac{2304}{121} \\ f = \frac{1849}{121} \\ a = \frac{455}{242} \end{array}\right.$ $\begin{array}{l} \text{two Multiplicators given in the Question,} \\ \\ \text{a square number given in the Question,} \\ \text{two square numbers found out according to the direction} \\ \text{in the fourth step of the Resolution,} \\ \text{the number sought.} \end{array}$

22. Suppose also, according to the Construction in the Resolution, that the excess of d above f hath such proportion to the excess of f above c, as the excess of r above s hath to s, *viz.* as, $d - f \,.\, f - c :: r - s \,.\, s$.

23. Then according to the Construction in the twentieth step of the Resolution, let these two Equations be instituted, *viz.* $ra + c = d$
$sa + c = f$.

24. Now since the Conclusion of the Resolution (in the said twentieth step) takes it for granted, that one and the same value of a, (to wit, the number sought) will be given by either of those two Equations, we must prove that these two Quotients are equal to one another, *viz.* $\frac{d-c}{r} = \frac{f-c}{s}$.

Demonstration.

25. Forasmuch as by Construction in the 22d step, . . $d - f \,.\, f - c :: r - s \,.\, s$
26. Therefore by Composition of Reason, ... $d - c \,.\, f - c :: r \,.\, s$
27. Therefore alternately . . . . $d - c \,.\, r :: f - c \,.\, s$
28. But if four numbers be Proportionals, the Reason of the first to the second is equal to the Reason of the third to the fourth, therefore . . . . $\frac{d-c}{r} = \frac{f-c}{s}$

Which was to be demonstrated.

Observat. 1. *upon* Quest. 33.

In the Duplicate-equality used in the preceding *Quest.* 33. both the numbers of a are affirmative; but if they were both negative, or one of them affirmative and the other negative, the Resolution would differ very little from the former, as will appear by the two following Questions.

QUEST. 1.

1. Let it be required to find out the number signified by a in this Duplicate equality, *viz.* . . . . $\begin{cases} 4 - 2a = \square \\ 4 - 3a = \square \end{cases}$

RESOLUTION.

2. First, these three numbers are to be considered . . . . . $4, 4 - 2a, 4 - 3a$

Then because the excess of 4 above $4 - 2a$, hath such proportion to the excess of $4 - 2a$ above $4 - 3a$, as 2 to 1, let 4 be considered as the greatest of three Squares, and find the other two, with this condition, that the excess of 4 above the mean may be the double of the excess of the mean above the least; to which end,

3. Let the greatest of the said three Squares be 4
4. And to the end the mean Square may be less than the greatest, let the side of the mean Square be $2 - e$, therefore the mean Square shall be . . . . $ee - 4e + 4$
5. Therefore the excess of the greatest Square above the mean is $4e - ee$
6. Therefore, according to the condition prescribed in the second step, the half of the excess in the fifth step shall be the excess of the mean Square above the least, to wit, ... $2e - \frac{1}{2}ee$
7. Which last excess, to wit, $2e - \frac{1}{2}ee$ being subtracted from the mean Square in the fourth step, the Remainder shall be equal to the least Square, to wit, ... $\frac{3}{2}ee - 6e + 4$
8. Therefore $\frac{3}{2}ee - 6e + 4$ must be equated to a Square, but the value of e must be subject to a Determination thus found out, *viz.* Forasmuch as the least of the three Squares above mentioned must be such, that when it is equated to $4 - 3a$ (in the first step) the value of a may be greater than nothing, it is evident the said least Square must necessarily be less than 4; Supposing therefore $\frac{3}{2}ee - 6e + 4 \sqsupset 4$ from this suppo-sition,

ſition, (by arguing in like manner as before from the eleventh ſtep to the ſixteenth,) we ſhall find $e \sqsupset 4$; therefore $\frac{3}{2}ee - 6e + 4$ muſt be equated to a Square, ſo, as the value of e may be leſs than 4. Now to cauſe that effect, the ſide of the ſaid Square may be feigned to be $2 -$ any number of e greater than $\frac{3}{2}e$, let therefore the ſaid ſide be feigned $2 - 2e$, then its Square being equated to $\frac{3}{2}ee - 6e + 4$ will give $e = \frac{4}{5}$, and conſequently the mean and leaſt Squares ſought will be $\frac{36}{25}$ and $\frac{4}{25}$, the former of which being equated to $4 - 2a$, or the latter to $4 - 3a$, from either of thoſe Equations the value of a, or the number ſought, will be found $\frac{32}{25}$; which will ſolve *Queſt.* 1. before propounded, as will be evident by

The Proof.

If $\frac{32}{25}$ (or a) be multiplied by 2 and 3 ſeverally, and if the Products be ſeverally ſubtracted from 4, the two Remainders will be $\frac{36}{25}$ and $\frac{4}{25}$, which are Squares, as was required.

QUEST. 2.

1. Let it be required to find out the number ſignified by a, in this Duplicate equality, *viz.* . . . . $\left\{\begin{array}{l} 4 + 2a = \square \\ 4 - 3a = \square \end{array}\right.$

RESOLUTION.

2. Firſt, theſe three numbers are to be conſidered, . . $4 + 2a$, 4, and $4 - 3a$

Then, becauſe the exceſs of $4 + 2a$ above 4, hath ſuch proportion to the exceſs of 4 above $4 - 3a$, as 2 to 3, let 4 be aſſumed to be the mean of three Squares, and find out the other two, with this condition, that the exceſs of the greateſt above the mean may be $\frac{2}{3}$ of the exceſs of the mean above the leaſt; to which end,

3. Let the mean Square be 4
4. And to the end the leaſt of the three Squares may be leſs than the mean, let the ſide of the leaſt Square be $2 - e$, therefore the leaſt Square ſhall be . . $ee - 4e + 4$
5. Therefore the exceſs of the mean Square above the leaſt is . . . . $-ee + 4e$
6. Therefore according to the condition preſcribed in the ſecond ſtep, $\frac{2}{3}$ of the laſt mentioned exceſs ſhall be equal to the exceſs of the greateſt Square above the mean, to wit, $-\frac{2}{3}ee + \frac{8}{3}e$
7. Which laſt exceſs added to the mean Square 4, will give the greateſt Square, to wit, ... $-\frac{2}{3}ee + \frac{8}{3}e + 4$

Therefore $-\frac{2}{3}ee + \frac{8}{3}e + 4$ muſt be equated to a Square, but the value of e muſt be ſubject to a Determination thus found out; Foraſmuch as the greateſt of the three Squares required muſt be ſuch, that when it is equated to $4 + 2a$ (in the firſt ſtep) the value of a may be greater than nothing, it is evident that the ſaid greateſt Square muſt be greater than 4.

Suppoſe therefore $-\frac{2}{3}ee + \frac{8}{3}e + 4 \sqsubset 4$, thence it will follow, that $e \sqsupset 4$; therefore the ſaid $-\frac{2}{3}ee + \frac{8}{3}e + 4$ muſt be equated to a Square, with this caution, that the value of e may be leſs than 4. Now to find out ſuch a Square, the value of e may be $\frac{12}{7}$, (found out by the Canon of *Queſt.* 13. of this Book, *mutatis mutandis*,) whence the greateſt and leaſt Squares ſought will be $\frac{324}{49}$ and $\frac{4}{49}$; the former of which being equated to $4 + 2a$, or the latter to $4 - 3a$, from either of thoſe Equations the number a will be found $\frac{64}{49}$, which will ſolve the Queſtion, as will be manifeſt by

The Proof.

If $\frac{64}{49}$ (or a) be multiplied firſt by 2, and then by 3; alſo, if the firſt Product be added to 4, and the latter Product be ſubtracted from 4, the Summ and Remainder will be $\frac{324}{49}$ and $\frac{4}{49}$, which are Squares, as was required.

Obſervat. 2. *upon the preceding* Queſt. 33.

By the ſame artifice that hath been uſed in ſolving the ſaid *Queſt.* 33. this following Duplicate equality may be reſolved, *viz.*

1. Let it be required to find a number, call it a, that ſhall make . . . . $\left\{\begin{array}{l} 10a + 9 = \square \\ 5a + 4 = \square \end{array}\right.$

2. First, by multiplying $5a + 4$ by 9, the Product is $45a + 36$; likewise $10a + 9$ multiplied by 4 produceth $40a + 36$; so the Duplicate equality propounded is reduced to this,

$$viz. \left\{ \begin{array}{l} 45a + 36 = \square \\ 40a + 36 = \square \end{array} \right.$$

3. Which latter Duplicate equality being of the same kind with that in the foregoing *Quest.* 33. may be solved by innumerable Answers; but for the greater evidence, the search may be made as before, *viz.*

4. Let these three numbers be considered, ... $45a + 36$, $40a + 36$, 36

Then because the excess of the greatest of those three numbers above the mean, hath such proportion to the excess of the mean above the least as 1 to 8, let 36 be assumed for the least of three Squares, and search out the other two, with this caution, That the excess of the greatest above the mean may be $\frac{1}{8}$ of the excess of the mean above the least; to which end,

5. Let the least of those three Squares be 36
6. And to the end the mean Square may be greater than the least, let the side of the mean Square be $e + 6$, therefore the mean Square it self shall be . . . . $ee + 12e + 36$
7. Therefore the excess of the mean Square above the least is $ee + 12e$
8. Therefore $\frac{1}{8}$ of that excess (which according to the Caution given as above, must be the excess of the greatest Square above the mean) shall be . . . . $\frac{1}{8}ee + \frac{3}{2}e$
9. Which last excess, to wit, $\frac{1}{8}ee + \frac{3}{2}e$, being added to the mean Square in the sixth step, the summ is equal to the greatest Square, to wit, ... $\frac{9}{8}ee + 13\frac{1}{2}e + 36$
10. Therefore $\frac{9}{8}ee + 13\frac{1}{2}e + 36$ must be equated to a Square, which square number when 'tis found out must be equated to $45a + 36$, and therefore the said Square must be greater than 36; but from any affirmative value of e whatsoever, the said $\frac{9}{8}ee + 13\frac{1}{2}e + 36$ will be manifestly greater than 36. Therefore here being no need of any limit for the value of e, the side of the said Square may be variously feigned; let then the said side be $\frac{3}{2}e - 6$, the Square thereof will be $\frac{9}{4}ee - 18e + 36$, which being equated to $\frac{9}{8}ee + 13\frac{1}{2}e + 36$, the value of e will thence be found 28; and consequently (from the ninth and sixth steps) the greatest and mean Squares sought will be 1296 and 1156, whose sides are 36 and 34: Then,
11. From this Equation, ... $45a + 36 = 1296$
12. Or from this Equation, ... $40a + 36 = 1156$
13. The number a will be discovered, to wit, ... $a = 28$

I say the number 28 will solve the Question, as will be evident by

The Proof.

If 28 (or a) be multiplied by 10 and 5 severally; also, if to the former Product 9 be added, and to the latter Product 4, the two summs will be 289 and 144, which are Squares, as was required.

Observat. 3. *upon* Quest. 33.

1. In the Duplicate equalities used in the preceding *Quest.* 33. and the two Observations thereon, the two Algebraick Quantities given to be equated to two Squares, do consist of two unequal numbers of a and of two known square numbers, either equal or unequal; which kind of Duplicate equality you have seen exactly resolved by Rational numbers: But neither *Diophantus*, nor any Author that I have met with, delivers a Rule to resolve in all cases a Duplicate equality consisting of unequal numbers of a, and of absolute numbers which are not Squares. Monsieur *Bachet* indeed (in his Comment upon *Quest.* 45. *Book* 4. of *Diophantus*) shews how to resolve the last mentioned Duplicate equality in two Cases, which I shall here explain.
2. The first Case is, when in a Duplicate equality of the kind last mentioned, the difference of the two Algebraick Quantities propos'd to be equated is such, that if it be multiplied or divided by some known number, and if the Product or Quotient be subtracted from the lesser of the two Quantities, the Remainder is a square number. As, for example,

3. Let

3. Let it be required to find out the number signified by a in this Duplicate equality, *viz.* . . . . $\left\{\begin{array}{l} 3a + 13 = \square \\ a + 7 = \square \end{array}\right.$

4. Because $2a + 6$, the difference of the two Algebraick Quantities proposed, being divided by 2 gives the Quotient $a + 3$, which subtracted from $a + 7$ leaves a Square, to wit, 4; the Duplicate equality propos'd is explicable by the method used in the preceding *Quest.* 33. as I shall here make manifest.

First, let these three numbers be considered, to wit, . . $3a + 13$, $a + 7$, 4

Then, by continuing the search in such manner as hath been shewn in solving the foregoing *Quest.* 33. you may find out (among innumerable values of a) $a = 29$, which will solve the Duplicate equality proposed; for if 29 be multiplied by 3, and 13 be added to that Product, the summ makes a Square, to wit, 100; also, if 7 be added to 29, it makes the Square 36.

5. The second Case is, when the difference of the two Algebraick Quantities which are to be equated to two Squares is such, that if it be multiplied or divided by some known number, and the Product or Quotient be subtracted from the lesser of those two Algebraick Quantities, there remains a negative known number, which taken affirmatively hath such proportion to the said Multiplicator or Divisor, as a square number to a square number. For example,

6. Let it be required to find a number, suppose it to be a, that will make . . . . $\left\{\begin{array}{l} 6a + 25 = \square \\ 2a + 3 = \square \end{array}\right.$

7. Because $4a + 22$, the difference of those two Algebraick Quantities, being divided by 2 gives the Quotient $2a + 11$, which subtracted from the lesser of the said two Quantities, leaves -8; and the number 8 to the Divisor 2, is as 4 to 1, that is, as a square number to a square number; the Duplicate equality propounded may be resolved thus, *viz.*

8. Let these three numbers be considered, to wit, . . $6a + 25$, $2a + 3$, and -8

Then forasmuch as $4a + 22$, to wit, the difference of the two greater of those three numbers, is the double of $2a + 11$ which is the difference of the two lesser of the said three numbers, seek two Squares, that the excess of the greater above the lesser may be the double of the excess of the lesser above -8; to which end you may proceed thus, *viz.*

9. For the lesser of the two Squares put . . . . aa
10. Then the excess of aa above -8 is . . . . $aa + 8$
11. The double whereof is . . . . $2aa + 16$
12. Which added to the lesser Square makes a summ equal to the greater, to wit, ... $3aa + 16$

Now the said $3aa + 16$ must be equated to a Square, with this caution, That the square number found out must exceed 25, to the end that when the said Square is equated to $6a + 25$, the value of a may be greater than nothing: But to cause that effect, innumerable Squares may be found out, (by the method before-delivered in divers Questions of this Book,) such are 64 and 16; for whether you equate 64 to $6a + 25$, or 16 to $2a + 3$, one and the same value of a, to wit, $6\frac{1}{2}$, will be discovered to solve the Duplicate equality proposed in the sixth step.

QUEST. 34. (Quæst. 17. Lib. 3. *Diophant.*)

To find three such numbers, that the Product of the multiplication of every two of them, with the summ of the same two numbers, may make a Square.

RESOLUTION.

1. Forasmuch as the Product of the multiplication of two Squares whose sides differ by unity, being added to the summ of the said Squares, will make a Square, (by the *Theorem* at the end of the Resolution of this Question,) let there be put for the first and second numbers . . . . 4 and 9

So is one of the conditions in the Question satisfied; for 4 multiplied by 9 produceth 36, to which if 13 (the summ of 4 and 9) be added, the summ is a Square, to wit, 49.

2. Let the third number be . . . . a

3. Then

3. Then the Product of the multiplication of the ſecond and third numbers is $9a$, to which adding their ſumm $a + 9$, it makes $10a + 9$, which (according to the Queſtion) muſt be equal to a Square, *viz.* . . . . $10a + 9 = \square$

4. Likewiſe the Product of the multiplication of the firſt and third numbers is $4a$, which with their ſumm $a + 4$ makes $5a + 4$, which (according to the Queſtion) muſt be equal to a Square, *viz.* $5a + 4 = \square$

5. So in the two laſt ſteps we are faln into a Duplicate equality, which may be ſolved by innumerable values of a, as hath been ſhewn in the ſecond Obſervation upon the foregoing *Queſt.* 33. of this *Chapt.* For example, take that value of a there found, to wit, $28\ (= a)$ for the third number ſought by this Queſtion; I ſay 4, 9 and 28 will ſolve this 34[th] Queſtion, as will be evident by

The Proof.

The three numbers found out are . . . . $4, 9, 28$.

Now according to the Queſtion,

I.	$4 \times 9, + 9 + 4 = 49$	Which are Squares, as was required.
II.	$9 \times 28, + 9 + 28 = 289$	
III.	$4 \times 28, + 4 + 28 = 144$	

The firſt ſtep of the Reſolution of this 34[th] Queſtion is grounded upon this

THEOREM.

6. If two numbers differ by unity, the Product made by the multiplication of their Squares, together with the ſumm of their Squares ſhall be a Square.

The truth of this Theorem may be demonſtrated thus,

7. Let there be two numbers which differ by 1 (or unity,) as . . . . a, and $a + 1$

8. Then their Squares are . . . . aa; $aa + 2a + 1$

9. The ſumm of thoſe Squares is . . . . $2aa + 2a + 1$

10. The Product of the multiplication of the ſaid Squares is $aaaa + 2aaa + aa$

11. The ſumm of the ſaid Summ and Product in the ninth and tenth ſteps is . . . . $aaaa + 2aaa + 3aa + 2a + 1$

12. Which Aggregate is a Square whoſe ſide is . . . . $aa + a + 1$

As will eaſily appear by multiplying the ſaid ſide by it ſelf. Therefore the truth of the Theorem is manifeſt.

QUEST. 35. (Quæſt. 18. Lib. 3. *Diophant.*)

This Queſtion is the ſame with the foregoing 34[th], *which is here repeated, and ſolved after another manner.*

To find three ſuch numbers, that the Product of the multiplication of every two of them, being added to the ſumm of the ſame two numbers, may make a Square.

RESOLUTION.

1. Let the firſt number be . . . . a
2. And let the ſecond number be 3
3. Then the Product of their multiplication added to their ſumm, makes . . . . $4a + 3$
4. Which $4a + 3$ muſt (according to the Queſtion) be equal to ſome Square, let it be 25, therefore . . . . $4a + 3 = 25$
5. Therefore from that Equation, ... $a = 5\frac{1}{2}$
6. So we have found two numbers, to wit, $5\frac{1}{2}$ and 3, which will ſatisfie one of the conditions in the Queſtion, for the Product of their multiplication with their ſumm makes 25, which is a Square. It remains to find a third number, which muſt be ſuch, that the Product of the ſecond and third numbers being added to their ſumm may make a Square; alſo, that the Product of the firſt and third numbers being added to their ſumm, may make a Square: Now to find out the ſaid third number, *Diophantus* begins again, thus,
7. Let the firſt number be (as before it was found) . . . . $5\frac{1}{2}$

8. And

8. And let the fecond number be (as before it was affumed ,) . . 3

9. Then for the third number put . . . . a

10. And fince (according to the Queftion) the Product of the multiplication of the fecond and third numbers , with their fumm, muft make a Square ; therefore , from the eighth and ninth fteps , $4a + 3 = \square$

11. Alfo the Product of the firft and third numbers with their fumm muft be a Square ; therefore , from the feventh and ninth fteps , . $6\frac{1}{2}a + 5\frac{1}{2} = \square$

12. So in the two laft fteps we are faln into a Duplicate equality , but 'tis not refolvable by any of the preceding Rules of *Diophantus* ; he frames therefore the Pofitions a-new , wherein his fcope is to find fuch numbers of a in the two Algebraick Quantities to be equated to two Squares , that fhall be in proportion one to the other as a fquare number to a fquare number , and then he fhews how to refolve this new kind of Duplicate equality, which hath not hitherto happened. Firft , if we examine whence 4 and $6\frac{1}{2}$ (to wit , the numbers prefixt before a in the tenth and eleventh fteps) do proceed , we fhall find that they arife from the addition of unity to each of the numbers 3 and $5\frac{1}{2}$ firft found ; (for by multiplying a into thofe numbers feverally , and by adding a to each Product , there arifeth $4a$ and $6\frac{1}{2}a$ above expreft.) Therefore the next fearch muft be to find two fuch numbers , that being feverally increafed with unity , the one fumm may be to the other as a fquare number to a fquare number : And becaufe (by the Theorem in the following firft Obfervation upon this Queftion ,) if we add unity to each of two numbers whereof the greater exceeds the quadruple of the leffer by 3 , the two fumms will be in the Reafon of a Square to a Square ; therefore,

13. Let the firft of the three numbers fought be . . . . . a

14. Then (by the faid Theorem) the fecond number fhall be . . $4a + 3$

15. Now according to the Queftion , the Product of the firft and fecond numbers together with their fumm muft be a Square, therefore from the two laft preceding fteps . . . . . . $4aa + 8a + 3 = \square$

16. The fide of which Square may be feigned $2a$ — any known number whofe Square is greater than 3 , let therefore the faid fide be $2a - 3$, then its Square being equated to $4aa + 8a + 3$, the value of a will be found $\frac{3}{10}$ for the firft number , and confequently $\frac{42}{10}$ ($= 4a + 3$) fhall be the fecond number.

So we have found two numbers which will anfwer the firft part of the Queftion , and moreover they are fit to raife a Duplicate equality that will be explicable by a Rational number : Therefore now an effectual Refolution may be formed thus ;

17. Let the firft number be . . . . $\frac{3}{10}$

18. And the fecond number . . . . $\frac{42}{10}$

19. And let the third number be . . . . a

20. Then according to the Queftion , the Product of the multiplication of the fecond and third numbers , together with their fumm, muft be equal to a Square , therefore from the 18th and 19th fteps, $\frac{52}{10}a + \frac{42}{10} = \square$

21. Alfo according to the Queftion , the Product of the firft and third numbers , with their fumm , muft be a Square; therefore , from the 17th and 19th fteps , ... $\frac{13}{10}a + \frac{3}{10} = \square$

22. Now becaufe the numbers drawn into a , in the Duplicate equality expreft in the two laft preceding fteps , are (by Conftruction) in the Reafon of a Square to a Square , for $\frac{52}{10} . \frac{13}{10} :: 4 . 1$, and confequently (by *Prop.* 19. *Elem.* 7. Euclid.) $\frac{52}{10} \times 1 = \frac{13}{10} \times 4$. Therefore by multiplying the Algebraick quantity in the twentieth ftep by 1, and that in the twenty-firft ftep by 4 , the numbers of a in the Products will be equal to one another , for the firft Product will be $\frac{52}{10}a + \frac{42}{10}$, and the latter $\frac{52}{10}a + \frac{12}{10}$; hence a new Duplicate equality is formed ,

$$viz. \begin{cases} \frac{52}{10}a + \frac{42}{10} = \square \\ \frac{52}{10}a + \frac{12}{10} = \square \end{cases}$$

23. Which Duplicate equality being of the fame kind with that explained in the preceding eighth Queftion , may be folved by innumerable Anfwers.

But I fhall exhibit only one Anfwer for an Example. Firft then , becaufe the difference of the two Algebraick quantities in the Duplicate equality laft before expreft is 3 , let two Squares be found out (by *Canon* 1. *Queft.* 7. of this *Book* ,) whofe difference fhall be

be 3, and that the greater Square may exceed $\frac{42}{10}$; ſuch are the Squares $\frac{121}{100}$ and $\frac{784}{100}$, whoſe ſides are $\frac{11}{10}$ and $\frac{28}{10}$: Then,

24. From either of theſe Equations, ... $\left\{ \begin{array}{l} \frac{13}{10}a + \frac{3}{10} = \frac{121}{100} \\ \frac{52}{10}a + \frac{42}{10} = \frac{784}{100} \end{array} \right.$

25. The ſame value of a will be diſcovered for the third number ſought, *viz.* . . . . $a = \frac{7}{10}$

Thus three numbers are found out, to wit, $\frac{3}{10}$, $\frac{42}{10}$ and $\frac{7}{10}$, which will ſolve the Queſtion, as will be evident by

The Proof.

$$\left.\begin{array}{l} \frac{3}{10} \times \frac{42}{10}, + \frac{3}{10} + \frac{42}{10} = \frac{576}{100} \\ \frac{42}{10} \times \frac{7}{10}, + \frac{42}{10} + \frac{7}{10} = \frac{784}{100} \\ \frac{3}{10} \times \frac{7}{10}, + \frac{3}{10} + \frac{7}{10} = \frac{121}{100} \end{array}\right\} \text{Which are Squares, as the Queſtion requires.}$$

Obſervations upon Queſt. 35.

1. If the Reſolution of this Queſtion be well examined, it will appear, that the forming of the Duplicate equality in the twentieth and twenty-firſt ſteps, where the numbers prefixt to a have ſuch Reaſon to one another as a ſquare number to a ſquare number; agreeable to the Scope before-mentioned in the twelfth ſtep, doth depend upon this following

THEOREM.

If there be two ſuch numbers, that the greater exceeds the quadruple of the leſſer by 3, and if unity be added to each number, the ſumms ſhall have ſuch Reaſon between themſelves as a Square to a Square, *viz.* the greater ſumm ſhall be to the leſſer as 4 to 1.

Which Theorem may be eaſily demonſtrated, thus,

$$\text{Suppoſe} \left\{ \begin{array}{ll} \left.\begin{array}{l} a \\ 4a + 3 \end{array}\right\} & \text{Two numbers, whereof the greater exceeds the quadruple of the leſſer by 3.} \\ a + 1 = & \text{The firſt number increaſed with unity.} \\ 4a + 4 = & \text{The ſecond number increaſed with unity.} \end{array} \right.$$

I ſay $4a + 4$ hath ſuch proportion to $a + 1$, as a Square to a Square, for,

$$4a + 4 \,.\, a + 1 :: 4 \,.\, 1.$$

In like manner, if there be two numbers whereof the greater exceeds nine times the leſſer by 8, as 17 and 1, then if you add 1 to each number, the ſumms ſhall be to one another as a ſquare number to a ſquare number, *viz.* the greater ſumm ſhall be to the leſſer as 9 to 1; the like is to be underſtood of other Squares.

2. After the two numbers prefixt before a in the Duplicate equality formed in the twentieth and twenty-firſt ſteps of this Queſtion, are found ſuch, that they have ſuch Reaſon one to the other as a Square to a Square; then may any two ſquare numbers in that Reaſon be uſed as is directed in the twenty-ſecond ſtep: So inſtead of 4 and 1 there taken, we may take 100 and 25, which have the ſame Reaſon between themſelves as $\frac{52}{10}$ and $\frac{13}{10}$; For, $\frac{52}{10} \,.\, \frac{13}{10} :: 100 \,.\, 25$,

Therefore, $\frac{52}{10} \times 25 = \frac{13}{10} \times 100$.

Then by multiplying the Algebraick quantity in the twentieth ſtep of the Reſolution by 25, and that in the twenty-firſt ſtep by 100, the following Duplicate equality (being that which *Diophantus* uſeth in ſolving this Queſtion) will ariſe,

$$viz. \left\{ \begin{array}{l} 130a + 105 = \square \\ 130a + 30 = \square \end{array} \right.$$

Hence, by the Canon in the ſeventh ſtep of *Reſolut.* 1. *Queſt.* 8. (among innumerable values of a that might be found out,) you may find $a = \frac{7}{10}$ (as before) for the third number ſought.

QUEST. 36. (Quæſt. 20. Lib. 3. *Diophant.*)

To find two numbers, that the Product of their multiplication increaſed ſeverally with each of them, and alſo with their ſumm, may make three Squares.

RESOLUTION.

1. For one of the numbers put . . . . a
2. And for the other, ... $4a - 1$
3. Then their Product is . . . . $4aa - a$

Whence it is evident, that if the first number a be added to the said Product, the summ is a Square, to wit, $4aa$.

4. It remains, that the second number and the summ of both being severally added to the said Product may make a Square; but the second number added to the Product makes $4aa + 3a - 1$, and the summ of both numbers, together with their Product, makes $4aa + 4a - 1$; therefore . . . . .

$$\left.\begin{array}{l} 4aa + 3a - 1 = \square \\ 4aa + 4a - 1 = \square \end{array}\right.$$

5. Which Duplicate equality may be resolved by the method before explained in the preceding twentieth and twenty-first Questions. For the difference of those two Algebraick quantities which are to be equated to Squares is a, which is to be divided into two such quantities that the Product of their multiplication may make a, and that both in the half-summ and in the half-difference of those two quantities there may be found $2a$; but such are the quantities $4a$ and $\frac{1}{4}$, whose Product is a; also the half of their summ is $2a + \frac{1}{8}$, and the half-difference is $2a - \frac{1}{8}$: then by equating the Square of $2a + \frac{1}{8}$ to $4aa + 4a - 1$, or the Square of $2a - \frac{1}{8}$ to $4aa + 3a - 1$, from either of those Equations the value of a will be found $\frac{65}{224}$. Therefore the first number shall be $\frac{65}{224}$, and the second $\frac{36}{224}$; which numbers will solve the Question, as may easily be proved.

QUEST. 37. (Quæst. 22. Lib. 3. *Diophant.*)

To find four such numbers, that every one of them being added to, and subtracted from the Square of the summ of them all, as well the four summs as the four remainders shall be Squares.

RESOLUTION.

1. In every right-angled Triangle, if the Square of the Hypothenusal be increased or lessened by the quadruple of the Area, that is, by the double Product of the multiplication of the two sides about the right-angle, it makes a Square, (which Theorem is made manifest at the end of the Resolution.) Therefore the chief scope is to find four right-angled Triangles in numbers having equal Hypothenusals: But those may be found out thus;

First, (by the Canon in *Observat.* 8. *Resolut.* 2. of *Quest.* 1. of this *Book*,) find out two unlike right-angled Triangles in numbers, such are these,

5 , 4 , 3
13 , 12 , 5

2. Then multiply the three sides of the first Triangle by the Hypothenusal of the second, also multiply the three sides of the latter Triangle by the Hypothenusal of the first; so the Products will give these two right-angled Triangles having equal Hypothenusals,

viz. 65 , 52 , 39
65 , 60 , 25

3. By the help of the two unlike right-angled Triangles first found, to wit, 5, 4, 3 and 13, 12, 5, the Canon in *Observat.* 4. upon *Resolut.* 2, and 3. of *Quest.* 2. of this *Book* will give two other right-angled Triangles unlike to those in the second step, but having the same Hypothenusal 65, to wit, these;

65 , 56 , 33
65 , 16 , 63

4. Then assume a to represent a number unknown, and let it be multiplied by every one of the sides of those four Triangles having 65 for a common Hypothenusal, so the Products will be these,

$65a$, $52a$, $39a$
$65a$, $60a$, $25a$
$65a$, $56a$, $33a$
$65a$, $16a$, $63a$

5. Now for the summ of the four numbers sought by the Question put . $65a$
6. Therefore the Square of the said summ is . . . . $4225aa$
7. Then for the first number sought, take the quadruple of the Area of the first of the four Triangles in the fourth step, *viz.* multiplying $52a$ by $39a$, take the double of that Product for the first number, to wit, $4056aa$

8. In

8. In like manner for the second number take the double Product of $60a$ by $25a$, that is, ... $\Big\}$ $3000aa$
9. And for the third number take the double Product of $56a$ by $33a$, that is, ... $\Big\}$ $3696aa$
10. And for the fourth number take the double Product of $63a$ by $16a$, that is, ... $\Big\}$ $2016aa$
11. The summ of the four numbers exprest in the seventh, eighth, ninth and tenth steps is . . . . $\Big\}$ $12768aa$
12. Which summ must be equal to $65a$, which in the fifth step was assumed for the summ of the four numbers sought, hence this Equation, ... $\Big\}$ $12768aa = 65a$
13. Which Equation reduced, gives . . . . $a = \frac{65}{12768}$
14. Therefore from the thirteenth, seventh, eighth, ninth and tenth steps the four numbers required will be found these, *viz.* $\frac{17136600}{163021824}$, $\frac{12675000}{163021824}$, $\frac{15615600}{163021824}$, $\frac{8517600}{163021824}$, which four numbers will solve the Question, as will be evident to him that will take the pains of forming the Proof.

But because the Resolution of this Question is chiefly grounded upon a Theorem taken for granted in the first step, I shall here demonstrate the same

THEOREM.

15. If the Square of the Hypothenusal of a right-angled Triangle be increased or lessened by the quadruple of the Area, (that is, the double Product of the multiplication of the sides about the right-angle,) the summ, as also the remainder shall be a Square. For,

If a and e = > the sides about the right-angle of a right-angled Triangle,
Then $2ae$ = > the double Product of those sides,
Also aa and ee = > the Squares of those sides,
And $aa + ee$ = > the Square of the Hypothenusal, (*per* 47. *Prop.* 1. *Elem.* Euclid.)

Hence that which the Theorem asserts is manifest,

viz. $\begin{cases} aa + ee + 2ae = \square, & \text{whose side is } a + e, \\ aa + ee - 2ae = \square, & \text{whose side is } a - e. \end{cases}$

It is also evident from the premisses, that this 37th Question may be extended to five, six, or as many numbers as shall be desired; but first of all, so many numbers as are required, so many right-angled Triangles in numbers must be found out having equal Hypothenusals; which Triangles in whole numbers may be readily discovered by the method delivered in *Observat.* 13. upon *Resolut.* 2. of *Quest.* 1. of this *Book.*

QUEST. 38.

[*This is* Quest. 20. *of the fourth Book of* Vieta's *Zeteticks*, *and the same with* Quest. 3. *in* Bachet's *Comment upon the fourth Book of* Diophantus.]

Two cube-numbers being given, such, that the double of the lesser exceeds the greater; to find two other cube-numbers whose difference shall be equal to the difference of the given Cubes. (But how to perform this when the double of the lesser Cube is less than the greater, I shall hereafter shew in *Quest.* 42.)

RESOLUTION.

1. Let the sides of the given Cubes be . . $\Big\{$ d the greater; b the lesser.
2. Then the Cubes of those sides are . . . > d^3 and b^3
3. And the difference of the said Cubes is > $d^3 - b^3$
4. For the side of the lesser Cube sought put > $a - d$
5. And for the side of the greater Cube sought put . . . . $\Big\}$ $\frac{dd}{bb}a - b$
6. Therefore the greater Cube sought is . . $\Big\}$ $\frac{d^6}{b^6}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$
7. And the lesser Cube sought is . . . . > $aaa - 3daa + 3dda - d^3$
8. Therefore the difference of the two Cubes sought is . . . . $\Big\}$ $\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3$

9. Which

9. Which difference must be equal to the difference of the given Cubes, therefore,

$$\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3 = d^3 - b^3.$$

10. From that Equation, after due Reduction, this ariseth, viz. . . . . $a = \frac{3d^4b^3 - 3db^6}{d^6 - b^6}$

11. And by reducing the Fraction in the latter part of the last preceding Equation into its least Terms, by the common Divisor $d^3 - b^3$, it gives . . . . $a = \frac{3db^3}{d^3 + b^3}$

12. Therefore from the first, fourth, fifth and eleventh steps, the sides of the Cubes sought will be found equal to these known quantities,

viz. $\begin{cases} \frac{2db^3 - d^4}{d^3 + b^3} = \text{the lesser side,} \\ \frac{2bd^3 - b^4}{d^3 + b^3} = \text{the greater side.} \end{cases}$

13. The same sides will be produced, if you put $a - b$ for the side of the greater of the Cubes sought, and $\frac{bb}{dd}a - d$ for the lesser side, (instead of the Positions in the fifth and fourth steps;) and it's evident that each of the sides found out in the twelfth step will be greater than nothing, if $2b^3$ exceeds d^3, (that is, if the double of the lesser of the two Cubes given exceeds the greater, as the Question presupposeth.

The twelfth step affords this following

CANON.

14. Multiply the excess of the double of the lesser of the two Cubes given above the greater, by the side of the greater; multiply also the excess of the double of the greater Cube above the lesser, by the side of the lesser: then divide each of those Products by the summ of the given Cubes, and the Quotients shall be the sides of the Cubes sought.

Examples in Numbers.

15. Let two Cubes be given, such, that the double of the lesser exceeds the greater, as, ... $125 = d^3$, and $64 = b^3$

16. The sides of which Cubes are . . . . $5 = d$, and $4 = b$

17. Then by the Canon, $\left\{ \begin{array}{l} \frac{2db^3 - d^4}{d^3 + b^3} = \frac{5}{63} \\ \frac{2bd^3 - b^4}{d^3 + b^3} = \frac{248}{63} \end{array} \right\}$ the sides of the Cubes sought.

18. The Cubes of those sides $\frac{5}{63}$ and $\frac{248}{63}$ are $\frac{125}{250047}$ and $\frac{15252992}{250047}$, whose difference is 61, which is equal to the difference of the two given Cubes 125 and 64.

19. In like manner, if these two Cubes be given, to wit, 1728 and 1000, whose difference is 728, the foregoing Canon will give $\frac{408}{341}$ and $\frac{3070}{341}$, the sides of two Cubes whose difference is 728.

Observations upon Quest. 38.

First, the chief scope in the Resolution of this Question is, to raise an Equation between some number of aaa and some number of aa, that a may be found equal to a Rational number; to which purpose, the side of one of the Cubes sought may be feigned to be a — one of the sides of the given Cubes, and the other side sought some number of a — the other side given; but this latter number of a must be such as will cause equal numbers of a to arise in the Cubes of those feigned sides, that when the lesser of the feigned Cubes is subtracted from the greater, the numbers of a may vanish, and then the Remainder being equated to the difference of the given Cubes, this difference will likewise vanish, (because 'tis also found in the difference of the feigned Cubes,) and an Equation remain between some number of aaa and some number of aa. Now to cause that effect, supposing (as before in the Resolution) d to represent the side of the greater Cube given, and b the side of the lesser, we may put $a - d$ for the side of the lesser of the Cubes sought, and then the greater side must necessarily be $\frac{dd}{bb}a - b$; or $a - b$ may be put for the greater

side

ſide ſought, and then the leſſer muſt be $\frac{bb}{dd}a - d$; from either of which ways of framing the Poſitions there will ariſe, after due Reduction, an Equation between aaa and aa, whence a will be found equal to a Rational number. All which will be manifeſt to him that diligently examines the preceding Reſolution.

Secondly, if two pairs of Cubes which ſhall have equal differences be deſired in whole numbers, they may eaſily be found out by the help of the foregoing Canon, in this manner, *viz.* let d and b repreſent the ſides of two ſuch Cubes, that the double of the Cube of the leſſer ſide b exceeds the Cube of the greater ſide d, then the ſaid Canon gives this Equation,

$$\left.\begin{cases} + \text{ Cube of } \frac{2bd^3 - b^4}{d^3 + b^3} \\ - \text{ Cube of } \frac{2db^3 - d^4}{d^3 + b^3} \end{cases}\right\} = \text{Cube of } d - \text{Cube of } b.$$

Now to contract that Equation, ſuppoſe f, b and g to be equal to the Numerators and common Denominator, ſo that Equation will be converted into this, *viz.*

$$\frac{f^3}{g^3} - \frac{b^3}{g^3} = d^3 - b^3.$$

Whence, by multiplying every Term by the Denominator g^3, this Equation is produced, *viz.*

$$f^3 - b^3 = g^3d^3 - g^3b^3.$$

That is, $\left.\begin{cases} + \text{ Cube of } f \\ - \text{ Cube of } b \end{cases}\right\} = \text{Cube of } gd - \text{Cube of } gb.$

In which laſt Equation, if inſtead of f, b and g, you take $2bd^3 - b^4$, $2db^3 - d^4$, and $d^3 + b^3$, which were before ſuppoſed equal to f, b and g reſpectively, this following Equation will ariſe, *viz.*

$$\left.\begin{cases} + \text{ Cube of } \overline{2bd^3 - b^4} \\ - \text{ Cube of } \overline{2db^3 - d^4} \end{cases}\right\} = \begin{cases} + \text{ Cube of } \overline{d^4 + db^3}. \\ - \text{ Cube of } \overline{bd^3 + b^4}. \end{cases}$$

Which laſt Equation gives this following

CANON.

Firſt, take two ſuch Cubes in whole numbers that the double of the leſſer may exceed the greater, and multiply the exceſs of the double of the greater above the leſs by the ſide of the leſſer Cube; ſecondly, multiply the exceſs of the double of the leſſer Cube above the greater by the ſide of the greater Cube; thirdly, multiply the ſumm of the ſame Cubes by the ſide of the greater; fourthly, multiply the ſumm of the ſaid Cubes by the ſide of the leſſer: then the difference of the Cubes of the firſt and ſecond Products ſhall be equal to the difference of the Cubes of the third and fourth Products.

An Example in Numbers.

Let two ſuch Cubes be taken, that the double of the leſſer exceeds the greater, as, ...	$125 = ddd$ $64 = bbb$
The ſides of which Cubes are	$5 = d$ $4 = b$
Then by working according to the directions of the laſt preceding Canon, the four Products, that is, the ſides of the four Cubes ſought, in their leaſt terms will be found theſe, to wit,	248, 5, 315, 252

Which four numbers will ſatisfie the Propoſition; for the difference between the Cubes of 248 and 5 is equal to the difference between the Cubes of 315 and 252, as may eaſily be proved.

Hence it is eaſie to find four Cubes in whole numbers, ſuch, that the ſumm of two of them ſhall be equal to the ſumm of the other two; for if two pairs of Cubes be found out by the laſt preceding Canon, ſuch, that the firſt pair hath the ſame difference as the latter, then the ſumm of the greater Cube of the firſt pair and the leſſer of the latter, ſhall be equal to the ſumm of the leſſer Cube of the firſt pair and the greater of the latter.

Thirdly, *Albert Girard* (in his Comment in *Simon Stevin*'s Arithmetick, upon the 19th of the fifth Book of *Diophantus*,) obſerves, but doth not demonſtrate, that the Cubes found out by the preceding *Queſt.* 38. are always leſs than the Cubes given; which property, ſince it will be uſeful in the following *Queſt.* 39. I ſhall here demonſtrate.

Suppoſe

$$\text{Suppoſe}\left\{\begin{array}{rcll} \ldots d &=& 5 & \multirow{2}{*}{\} the ſides of two Cubes, ſuch, that the double of the leſſer exceeds the greater.} \\ \ldots b &=& 4 & \\ \ldots d^3 &=& 125 & \} \text{ the Cubes of thoſe ſides.} \\ \ldots b^3 &=& 64 & \\ d^3-b^3 &=& 61 & > \text{the difference of the ſame Cubes.} \\ \ldots 2b^3 &\sqsubset& d^3 & \end{array}\right.$$

Then by the Canon in *Sect.* 14. *Queſt.* 38. the ſides of two Cubes whoſe difference is equal to the difference of the given Cubes, whoſe ſides are $d\ (=5)$ and $b\ (=4)$ will be found theſe that follow, to wit,

$$\frac{2db^3-d^4}{d^3+b^3}=\frac{5}{63};\ \text{and}\ \frac{2bd^3-b^4}{d^3+b^3}=\frac{248}{63}.$$

Now becauſe the Cubes given and found out have equal differences, if it be proved that the greater Cube found out is leſs than the greater Cube given, then conſequently the leſſer Cube found out ſhall be leſs than the leſſer Cube given: But that the ſide of the greater Cube found out, is leſs than the ſide of the greater Cube given, (and by conſequence the greater Cube found out leſs than the greater Cube given,) I prove thus;

The greater ſide found out (as before) is . . . . . $\left\{\dfrac{2bd^3-b^4}{d^3+b^3}\right.$

Therefore we muſt demonſtrate that . . . . $\left\{\dfrac{2bd^3-b^4}{d^3+b^3}\sqsupset d\right.$

Demonſtration.

By ſuppoſition, ... > $d \sqsubset b$

Therefore by multiplying d and b ſeverally by bb, it follows, that . . . . } $dbb \sqsubset b^3$

And by adding b^3 to each part, ... > $b^3+dbb \sqsubset 2b^3$

By ſuppoſition . . . . > $d^3 \sqsupset 2b^3$

Therefore from the two laſt preceding ſteps, ... > $d^3 \sqsupset b^3+dbb$

And by multiplying each part in the laſt ſtep by b, . . > $bd^3 \sqsupset b^4+db^3$

But by multiplying each number in the firſt ſtep of this Demonſtration by d^3, ... } $bd^3 \sqsupset d^4$

Therefore by comparing the ſumm of the numbers in the firſt parts of the two laſt preceding ſteps, to the ſumm of thoſe in the latter parts, ... } $2bd^3 \sqsupset d^4+b^4+db^3$

And by ſubtracting b^4 from each part of the laſt preceding ſtep, ... } $2bd^3-b^4 \sqsupset d^4+db^3$

Wherefore by dividing each part of the laſt ſtep by d^3+b^3, it's manifeſt that . . . . } $\dfrac{2bd^3-b^4}{d^3+b^3}\sqsupset d$

Which was to be demonſtrated.

Having proved that the greater of the two ſides found out by the Canon before diſcovered for reſolving *Queſt.* 38. is leſs than the greater of the two ſides given, it follows, that the Cube of that ſide found out is leſs than the Cube of the greater ſide given, and that the leſſer Cube found out is leſs than the leſſer Cube given, (becauſe by Conſtruction the two Cubes found out have the ſame difference as the Cubes given.) Therefore the truth of the property before affirmed is manifeſt.

Fourthly, if two pairs of numbers have equal differences, the leſſer number of the leſſer pair ſhall have leſſer Reaſon (or Proportion) to the greater number of the ſame pair, than the leſſer number of the greater pair hath to the greater number of this pair. To make this manifeſt,

$$\text{Suppoſe}\left\{\begin{array}{rcll} b &=& 8 & \} \text{ two unequal numbers taken at pleaſure.} \\ c &=& 6 & \\ b-c &=& 2 & > \text{the difference of thoſe numbers.} \\ d &=& 5 & > \text{a number leſs than } c\,(=6) \text{ the leſſer of the two numbers firſt taken.} \\ b-d &=& 3 & \} \text{ two numbers whoſe difference is equal to the difference of the} \\ c-d &=& 1 & \quad \text{numbers firſt taken.} \end{array}\right.$$

Now I ſay that the Reaſon (or Proportion) of $c-d$ to $b-d$ is leſs than that of c to b; therefore,

The Propoſition to be demonſtrated, is, that . . . . . $\left\{\dfrac{c-d}{b-d}\sqsupset\dfrac{c}{b}\right.$

Demon-

Demonſtration.

By ſuppoſition, ...	$c \sqsupset b$
Therefore by multiplying c and b ſeverally by d, it follows that	$dc \sqsupset bd$
And by adding bc to each part in the laſt ſtep, ...	$bc + dc \sqsupset bc + bd$
And by ſubtracting bd from each part, ...	$bc + dc - bd \sqsupset bc$
And by ſubtracting dc from each part in the laſt preceding ſtep, ...	$bc - bd \sqsupset bc - dc$
And by dividing each part of the laſt ſtep by $b - d$, . .	$\frac{bc - bd}{b - d} \sqsupset c$
Wherefore by dividing each part of the laſt preceding ſtep by b, ...	$\frac{c - d}{b - d} \sqsupset \frac{c}{b}$

Which was to be demonſtrated.

Fifthly and laſtly, from the preceding 3[d] and 4[th] Obſervations we may deduce this

COROLLARY.

If two cube-numbers be given, ſuch, that the double of the leſſer exceeds the greater, then (by the help of the preceding Canon in *Sect.* 14. *Queſt.* 38.) two cube-numbers may be found out, whoſe difference ſhall be equal to the difference of the given Cubes, and the double of the leſſer of the Cubes found out ſhall be leſs than the greater of them.

For if two given Cubes, (which I ſhall call the firſt pair) be ſuch that the double of the leſſer exceeds the greater, we may by the ſaid Canon find out a ſecond pair of Cubes, whoſe difference ſhall be equal to the difference of the firſt pair, and (by *Obſervat.* 3.) the Cubes of the ſecond pair ſhall be leſs than thoſe of the firſt pair, (*viz.* the greater Cube of the ſecond pair ſhall be leſs than the greater Cube of the firſt pair, and the leſſer Cube leſs than the leſſer;) and the leſſer Cube of the ſecond pair ſhall have leſs Reaſon or proportion to the greater Cube of the ſame pair, than the leſſer Cube of the firſt pair hath to the greater of the ſame pair, (by *Obſervat.* 4.) But if the double of the leſſer Cube of the ſecond pair doth yet happen to exceed the greater of the ſame pair, then by the help of the ſecond pair of Cubes and the ſaid Canon, we may find a third pair of Cubes, whoſe difference ſhall be equal to the common difference of the firſt and ſecond pairs; and by proceeding in like manner, the double of the leſſer of the two Cubes found out will at length neceſſarily be leſs than the greater, becauſe (as before hath been proved,) the leſſer Cube of each pair found out hath leſs proportion to the greater of the ſame pair, than the leſſer of the next precedent pair (by which the latter were found out) hath to the greater.

QUEST. 39.

Two cube-numbers being given, ſuch, that the double of the leſſer is either greater or leſs than the greater, to divide the difference of the given Cubes into two Rational cube-numbers.

Preparation.

1. When the double of the leſſer of the given Cubes exceeds the greater, two others muſt be found out, (according to the directions following the Corollary in *Obſervat.* 5. *Queſt.* 38.) ſuch, that the difference of theſe Cubes may be equal to the difference of thoſe given, and that the double of the leſſer of the Cubes found out may be leſs than the greater. Then two cube-numbers being given or found out, ſuch, that the double of the leſſer is leſs than the greater, their difference may be divided into two Rational cube-numbers by the following Reſolution, (which is the ſame in ſubſtance with that of the 18[th] of the 4[th] Book of *Vieta's Zeteticks*, and of the firſt Queſtion of *Bachet* in his Comment upon the fourth Book of *Diophantus.*)

RESOLUTION.

2. Let the ſides of the given Cubes (qualified as above is ſuppoſed) be	d the greater, and b the leſſer.
3. Therefore the Cubes of thoſe ſides are	d^3 and b^3
4. And the difference of the ſaid Cubes is . . .	$d^3 - b^3$
5. For the ſide of one of the Cubes ſought put . .	$d - a$

6. And

6. And for the side of the other Cube sought put . . . . . . $\frac{dd}{bb}a - b$

7. Therefore the first Cube is . . . . . $-a^3 + 3daa - 3dda + d^3$

8. And the latter Cube is . . . . . . $\frac{d^6}{b^6}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$

9. Therefore the summ of those Cubes is . . $\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3$

10. Which summ must be equal to the difference of the given Cubes; therefore,

$$\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3 = d^3 - b^3.$$

11. Which Equation, after due Reduction, gives $a = \frac{3db^3}{d^3 + b^3}$

12. Therefore from the first, fifth, sixth and eleventh steps, the sides of the two Cubes sought will be found equal to these known Quantities,

Viz. $\frac{d^4 - 2db^3}{d^3 + b^3}$ = the first side, $\frac{2bd^3 - b^4}{d^3 + b^3}$ = the other side.

13. The same sides will be produced, if instead of the Positions in the fifth and sixth steps, there be put $a - b$ and $d - \frac{bb}{dd}a$ for the sides of the Cubes sought. And 'tis evident that each of the sides found out in the twelfth step will be greater than nothing if $2b^3$ be less than d^3, that is, if the double of the lesser of the two Cubes given be less than the greater, as is supposed in the Preparation to the Resolution of this Question. The sides in the twelfth step, being exprest by words, will give this

CANON.

14. Multiply the excess of the greater of the two given Cubes above the double of the lesser by the side of the greater; multiply also the excess of the double of the greater Cube above the lesser by the side of the lesser; then divide each of those Products by the summ of the said Cubes, and the Quotients shall be the sides of the Cubes sought.

Example 1. *in Numbers.*

15. Let two such Cubes be given, that the double of the lesser is less than the greater, as, ... $8 = d^3$ and $1 = b^3$

16. The sides of those Cubes are . . . . $2 = d$ and $1 = b$

17. Then by the Canon, $\frac{d^4 - 2db^3}{d^3 + b^3} = \frac{4}{3}$, $\frac{2bd^3 - b^4}{d^3 + b^3} = \frac{5}{3}$ the sides of the Cubes sought.

The Cubes of those sides $\frac{4}{3}$ and $\frac{5}{3}$, are $\frac{64}{27}$ and $\frac{125}{27}$, whose summ is 7, which is equal to the difference of the two given Cubes 8 and 1, as was required by the Question.

Example 2.

18. Let it be required to divide 61, which is the difference of these Cubes 125 and 64, into two Rational cube-numbers.

Here, because the double of the lesser Cube exceeds the greater, the Canon above exprest in *Sect.* 14. is of no force; therefore by the help of the given Cubes, (according to the directions following the Corollary in *Observat.* 5. *Quest.* 38.) two other Cubes must be found out, such, that their difference may be equal to 61, to wit, the difference of the given Cubes 125 and 64, and that the double of the lesser of the two Cubes found out may be less than the greater of them: But two such Cubes are these, *viz.* $\frac{15252992}{250047}$ and $\frac{125}{250047}$, whose sides are $\frac{248}{63}$ and $\frac{5}{63}$; then using these Cubes as the Canon in the preceding *Sect.* 14. of this *Quest.* 39. doth direct, you will find $\frac{152529295}{960946371}$ and $\frac{3782680016}{960946371}$ for the sides of the Cubes required; for the summm of the Cubes of the said sides is 61, which is equal to the difference of the given Cubes, 125 and 64.

19. Hence

19. Hence it is eaſie to find four cube numbers, the greateſt of which ſhall be equal to the ſumm of the other three; for when the difference of two given Cubes is divided into two rational Cubes, theſe two, together with the leſſer of the two given Cubes ſhall make the greater Cube given. But if four ſuch Cubes be deſired in whole numbers, they may be readily found out by the following Canon, which is raiſed by the like manner of arguing as was before uſed in the ſecond Obſervation upon the preceding *Queſt.* 38.

CANON.

20. Firſt, take two Cubes in whole numbers, with this caution, That the double of the leſſer may be leſs than the greater, and multiply the exceſs of the greater Cube above the double of the leſſer by the ſide of the greater; ſecondly, multiply the exceſs of the double of the greater Cube above the leſs by the ſide of the leſſer; thirdly, multiply the ſumm of the ſaid Cubes by the ſide of the greater; fourthly, multiply the ſumm of the ſame Cubes by the ſide of the leſſer: Then the ſumm of the Cubes of the firſt, ſecond and fourth Products ſhall be equal to the Cube of the third Product.

An example in Numbers.

21. Let two ſuch Cubes be taken, that the double of the leſſer is leſs than the greater, as, ...	$8 = ddd$ $1 = bbb$
22. The ſides of which Cubes are	$2 = d$ $1 = b$
23. Then by the laſt preceding Canon, the ſides of the four Cubes ſought, in their leaſt terms will be found theſe, *viz.*	4, 5, 6, 3
24. I ſay the ſumm of the Cubes of 4, 5 and 3 is equal to the Cube of 6, *viz.*	$64 + 125 + 27 = 216$

QUEST. 40.

To divide any cube-number, ſuppoſe 8, into three Rational cube-numbers.

RESOLUTION.

Take any cube-number leſs than the given Cube 8, as 1; then (by the preceding *Queſt.* 39.) divide 7 the difference of thoſe Cubes into two Cubes, ſuppoſe into theſe, $\frac{64}{27}$ and $\frac{125}{27}$, whoſe ſumm 7 is equal to the difference of the given Cubes 8 and 1.

$$\text{Therefore} \begin{cases} \frac{64}{27} + \frac{125}{27} = 8 - 1, \\ \frac{64}{27} + \frac{125}{27} + 1 = 8. \end{cases}$$

Whereby 'tis manifeſt that three Cubes, to wit, $\frac{64}{27}$, $\frac{125}{27}$ and 1 are found out, whoſe ſumm is equal to the given Cube 8, as was required.

Hence you may eaſily perceive a way to divide a given Cube into any odd number of Cubes; as to divide a Cube into five Cubes, firſt divide the given Cube into three Cubes, and then divide one of thoſe three into three Cubes, ſo the other two, with the three Cubes laſt found out are five Cubes, whoſe ſumm is equal to the Cube firſt given; in like manner you may divide a Cube into 7, 9, 11, 13, &c. Cubes. But there is not any Rational cube-number that can be divided into two Rational Cube numbers; which negative Propoſition the Learned Dr *Wallis* hath demonſtrated.

QUEST. 41.

[*This is the* 19th *of the* 4th *Book of* Vieta's *Zeteticks, and the ſame with* Queſt. 2. *of* Bachet *in his Comment upon* Queſt. 2. *of the* 4th *Book of* Diophantus.]

Two cube-numbers being given, to find two other cube-numbers whoſe difference ſhall be equal to the ſumm of the given Cubes.

RESOLUTION.

1. Let the ſides of the given Cubes be	d the greater, b the leſſer.
2. Therefore the Cubes of thoſe ſides are	d^3 and b^3
3. For the ſide of the greater Cube ſought put	$a + d$
4. And for the ſide of the leſſer Cube ſought put . . .	$\frac{dd}{bb}a - b$

 5. There-

5. Therefore the greater Cube sought is . . $a^3 + 3daa + 3dda + d^3$

6. And the lesser Cube sought is . . . . . $\frac{d^6}{b^6}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$

7. Therefore the difference of the Cubes sought is . . . . $\frac{b^6a^3 - d^6a^3}{b^6} + \frac{3d^4}{b^3}aa + 3daa + d^3 + b^3$

8. Which difference must be equal to the summ of the given Cubes, therefore;

$$\frac{b^6a^3 - d^6a^3}{b^6} + \frac{3d^4}{b^3}aa + 3daa + d^3 + b^3 = d^3 + b^3.$$

9. Which Equation, after due Reduction, gives $a = \frac{3db^3}{d^3 - b^3}$

10. Therefore from the first, third, fourth and tenth steps, the sides of the two Cubes sought will be found equal to these known quantities,

Viz. $\frac{2db^3 + d^4}{d^3 - b^3}$ = the greater side,

$\frac{2bd^3 + b^4}{d^3 - b^3}$ = the lesser side.

The same sides will be produced, if instead of the Positions in the third and fourth steps there be put $d + \frac{bb}{dd}a$ and $a - b$; and those sides above-exprest by Letters give this

CANON.

11. Add the greater of the two Cubes given to the double of the lesser, and multiply the summ by the side of the greater Cube; add also the lesser Cube to the double of the greater, and multiply this summ by the side of the lesser Cube; lastly, divide each of those Products by the difference of the given Cubes, and the Quotients shall be the sides of the Cubes sought.

An Example in Numbers.

Let two Cubes be given, as, ... $8 = ddd$, and $1 = bbb$

The sides whereof are . . . . $2 = d$, and $1 = b$

Then by the Canon, $\frac{2db^3 + d^4}{d^3 - b^3} = \frac{20}{7}$, $\frac{2bd^3 + b^4}{d^3 - b^3} = \frac{17}{7}$ the sides of the Cubes sought.

The Cubes of those sides $\frac{20}{7}$ and $\frac{17}{7}$ are $\frac{8000}{343}$ and $\frac{4913}{343}$, whose difference 9 is equal to the summ of the given Cubes 8 and 1.

12. Hence 'tis easie to find out four cube-numbers, the greatest of which shall be equal to the summ of the other three: For when by this Question two Cubes are found out, having their difference equal to the summ of two given Cubes, the lesser of the two Cubes found out, together with the two given Cubes shall be equal to the greater of the Cubes found out. But if four such Cubes be desired in whole numbers, they may be readily found out by the following *Canon*, which is raised by the like manner of arguing as was before used in *Observat.* 2. *Quest.* 38.

CANON.

13. First take any two Cubes in whole numbers, add the greater to the double of the lesser and multiply the summ by the side of the greater; secondly, add the lesser Cube to the double of the greater, and multiply this summ by the side of the lesser; thirdly, multiply the difference of those Cubes by the side of the greater; fourthly, multiply the said difference by the side of the lesser Cube: Then the summ of the Cubes of the three latter Products shall be equal to the Cube of the first Product.

An Example in Numbers.

Let two Cubes in whole numbers be taken at pleasure, as $8 = ddd$, and $1 = bbb$

The sides of those Cubes are . . . . $2 = d$, and $1 = b$

Then by the *Canon* last afore-going, the sides of the four Cubes sought will be found these, to wit, . . . 20, 17, 14, 7

I say the Cube of 20 is equal to the summ of the Cubes of 17, 14 and 7, *viz.* . . . . $8000 = 4913 + 2744 + 343$

QUEST. 42.

QUEST. 42.

Two cube-numbers being given, such, that the double of the lesser is less than the greater, to find out two other Cubes whose difference shall be equal to the difference of the given Cubes. (But how this is to be done when the double of the lesser Cube exceeds the greater, hath already been shewn in *Quest.* 38.)

RESOLUTION.

1. Let there be two Cubes given, to wit, ... $\left\{ \begin{array}{l} ddd = 8 \\ bbb = 1 \end{array} \right.$

2. By the Canon in *Sect.* 14. of the foregoing *Quest.* 39. find out two Cubes whose summ shall be equal to the difference of the given Cubes, such are these, ... $\left\{ \begin{array}{l} ccc = \frac{64}{27} \\ ggg = \frac{125}{27} \end{array} \right.$

   Therefore by that Construction, . . $\succ$ $ddd - bbb = ccc + ggg = 8 - 1 = 7.$

3. By the Canon in *Sect.* 11. of the foregoing *Quest.* 41. find two Cubes whose difference shall be equal to the summ of the Cubes *ccc* and *ggg*, (found out in the preceding second step,) such are these,

   $$\left\{ \begin{array}{l} kkk = \frac{2024284625}{6128487}, \text{ whose side is } \frac{1265}{183}. \\ lll = \frac{1981385216}{6128487}, \text{ whose side is } \frac{1256}{183}. \end{array} \right.$$

   Therefore by this Construction, . . . $\succ$ $kkk - lll = ccc + ggg = 7.$

4. Therefore from the second and third steps, (*per* 1. *Axiom.* 1. *Elem.* Euclid.)

   $$kkk - lll = ddd - bbb = 7.$$

I say *kkk* and *lll*, that is, $\frac{2024284625}{6128487}$ and $\frac{1981385216}{6128487}$, (whose sides are $\frac{1265}{183}$ and $\frac{1256}{183}$,) will solve the Question proposed; for their difference 7 is equal to the difference of the given Cubes 8 and 1.

Note. Although by the preceding Resolutions of this and *Quest.* 38. innumerable pairs of cube-numbers may be found out, such, that the difference of each pair shall be equal to the difference of two Cubes given, yet neither of those Resolutions will find out all the pairs of Cubes that have the same difference with two given Cubes; for example, if the Cubes 1728 and 1000 be given, whose difference is 728, the Canon in the 14th step of the foregoing *Quest.* 38. will not find out the Cubes 729 and 1, whose difference is 728; although that Canon, with the help of the Resolution of this *Quest.* 42. will find out innumerable pairs of Cubes, such, that the difference of each pair shall be 728.

QUEST. 43.

To divide a given number 28 compos'd of two cube numbers 27 and 1, into two other Rational cube-numbers.

[*This Question was propounded in* 1657. *by Mons.* Fermat, (*as appears by an Epistolical Commerce printed at* Oxford *in* 1658.) *but his way of solving it came not to light, till it was publish'd* (*after his death*) *among other his Analytical Inventions, by way of Supplement to Mons.* Bachet's *Comment upon* Diophantus, *printed at* Tholose *in* 1670. *yet the very same way of solving this Question was found out long before by our Learned* Dr John Wallis, (*though, it seems, not timely enough to have been inserted in the little Book above-mentioned,*) *and likewise by my self, before I had seen or heard of any Solution to the said Question, in such manner as here follows.*]

RESOLUTION.

1. Let the Cubes 27 and 1, whose summ makes the given number 28, be represented by *ddd* and *bbb*, viz. . . . . $\left\{ \begin{array}{l} ddd = 27 \\ bbb = 1 \end{array} \right.$

2. By the Canon in *Sect.* 11. of the foregoing *Quest.* 41. find two cube-numbers whose difference may be equal to 28 the summ of the given Cubes 27 and 1, that is, *ddd* and *bbb*;) such are these Cubes,

   $$Viz. \left\{ \begin{array}{l} ggg = \frac{658503}{17576}, \text{ whose side is } \frac{87}{26}. \\ ccc = \frac{166375}{17576}, \text{ whose side is } \frac{55}{26}. \end{array} \right.$$

   Therefore, ... $\succ$ $ggg - ccc = ddd + bbb = 27 + 1 = 28.$

3. By

3. By the foregoing *Quest.* 39. find out two Cubes whose summ shall be equal to 28 the difference of the two Cubes ggg and ccc, such are these,

$$kkk = \frac{253452325273412980702625}{9864820937041015055552}.$$

$$lll = \frac{22762660963735440852831}{9864820937041015055552}.$$

The sides of which Cubes are these, to wit, ... $\begin{cases} k = \frac{63284735}{21446828}. \\ l = \frac{28342111}{21446828}. \end{cases}$

Therefore, $ggg - ccc = kkk + lll$.

4. But by Construction in the 2[d] step, $ggg - ccc = ddd + bbb = 27 + 1 = 28.$

5. Therefore from the two last Equations, (*per* 1. *Axiom.* 1. *Elem.* Euclid.) $kkk + lll = ddd + bbb = 27 + 1 = 28.$

Whence it is manifest that the two Cubes found out, to wit, kkk and lll, (which with their sides are before severally exprest by numbers in the third step,) will solve the Question, for their summ makes 28, which is the summ of the given Cubes 27 and 1. And because by the help of the known Cubes ggg and ccc in the second step, divers pairs of Cubes having the same difference with the said ggg and ccc may be found out, (by the 38[th] or 42[d] Question aforegoing:) Therefore by the help of any of the pairs of Cubes so found out, their difference may be divided into two Cubes whose summ shall be equal to the summ of the given Cubes 27 and 1.

Another Example.

Let it be required to divide 9, which is compos'd of the Cubes 8 and 1, into two other Cubes.

RESOLUTION.

1. Let the Cubes 8 and 1, whose summ makes the given number 9, be represented by ddd and bbb, viz. . . . . $\begin{cases} ddd = 8 \\ bbb = 1 \end{cases}$

2. By the Canon in *Sect.* 11. *Quest.* 41. of this *Book*, find out two Cubes whose difference may be equal to 9, the summ of the given Cubes 8 and 1; such are these Cubes,

$$Viz. \begin{cases} ggg = \frac{8000}{343}, \text{ whose side is } \frac{20}{7}. \\ ccc = \frac{4913}{343}, \text{ whose side is } \frac{17}{7}. \end{cases}$$

Therefore, $ggg - ccc = ddd + bbb = 8 + 1 = 9.$

3. Then by the preceding 39[th] Question divide the difference of the Cubes ggg and ccc into two rational Cubes, *viz.* divide 9 the difference of the Cubes $\frac{8000}{343}$ and $\frac{4913}{343}$ into two Cubes: But here because the double of the lesser Cube $\frac{4913}{343}$ exceeds the greater $\frac{8000}{343}$, two Cubes must first be found out, (by the help of the foregoing *Quest.* 38.) that the difference of these may be equal to the difference of those, and that the double of the lesser of the Cubes found out may be less than the greater; such are these Cubes,

$$Viz. \begin{cases} mmm = \frac{6695590842626239}{738542637646471}, \text{ whose side is } \frac{188479}{90391}. \\ nnn = \frac{487071038080000}{738542637646471}, \text{ whose side is } \frac{36520}{90391}. \end{cases}$$

4. Now forasmuch as the double of the lesser of the two Cubes last found out is less than the greater, we may by the help of the preceding *Quest.* 39. divide 9 the difference of those Cubes mmm and nnn, (and likewise of ggg and ccc) into two rational Cubes, whose sides will be found these,

$$Viz. \begin{cases} k = \frac{1243612733990094836481}{609623835676137297449}, \\ l = \frac{487267171714352336560}{609623835676137297449}. \end{cases}$$

5. Therefore by Construstion in the two last preceding steps, ... $ggg - ccc = kkk + lll$

6. But by Construction in the second step, $ggg - ccc = ddd + bbb = 8 + 1 = 9$

7. Therefore from the two last Equations, . . $kkk + lll = ddd + bbb = 8 + 1 = 9$

Thus two Cubes (whose sides k and l are above exprest in numbers) are found out, which added together make 9, the summ of the given Cubes 8 and 1, as was required.

QUEST. 44.

QUEST. 44.

To divide the double of any given cube-number into four cube-numbers.

For example, let it be required to divide 54 the double of the Cube 27, into four cube-numbers.

RESOLUTION.

1. For the given Cube 27 put ddd, *viz.* ſuppoſe . . . . $ddd = 27$
2. Therefore the double of that Cube is . . . . $2ddd = 54$
3. Take any cube-number leſs than the given Cube 27, as 1, for which put bbb, *viz.* ſuppoſe . . . . $bbb = 1$
4. By the foregoing *Queſt.* 43. find two Cubes whoſe ſumm may be equal to $ddd + bbb$, (to wit, $27 + 1$,) ſuppoſe thoſe which ſolved the ſaid *Queſt.* 43. in *Example* 1.

$$Viz. \begin{cases} kkk = \dfrac{253452325273412980702625}{9864820937041015055552}, \\ lll = \dfrac{22762660963735440852831}{9864820937041015055552}. \end{cases}$$

The ſides of which Cubes are theſe, to wit, ... $\begin{cases} k = \dfrac{63284725}{21446828}, \\ l = \dfrac{28340511}{21446828}. \end{cases}$

Therefore by that Conſtruction, ... $ddd + bbb = kkk + lll = 28$

5. By *Queſt.* 39. of this *Book* divide 26 the difference of the Cubes 27 and 1, to wit, $ddd - bbb$ into two Cubes, ſuppoſe into theſe,

$$Viz. \begin{cases} rrr = \dfrac{421875}{21952}, & \text{whoſe ſide is } \dfrac{75}{28}. \\ sss = \dfrac{148877}{21952}, & \text{whoſe ſide is } \dfrac{53}{28}. \end{cases}$$

Therefore by this Conſtruction, ... $ddd - bbb = rrr + sss = 26$

6. Therefore by adding together the Equations in the fourth and fifth ſteps, this will ariſe, *viz.* $2ddd = kkk + lll + rrr + sss = 54$

Therefore four Cubes are found out, to wit, kkk, lll, rrr and sss, which with their ſides are before expreſt in numbers in the fourth and fifth ſteps, and the ſumm of thoſe Cubes makes 54, which is equal to the double of the Cube 27 firſt given, as was required by the Queſtion.

QUEST. 45. (Quæſt. 17. Lib. 4. *Diophant.*)

To find out three numbers whoſe ſumm may make a Square; and that the ſecond number added to the Square of the firſt may make a Square; alſo, that the third number added to the Square of the ſecond may make a Square; and laſtly, that the firſt number added to the Square of the third may make a Square.

RESOLUTION.

1. For the firſt number ſought put a — any known number, as, . . . $a - 1$
2. The Square thereof is . . . . $aa - 2a + 1$
3. To which Square if $+ 4a$ be added, (to wit, the double of $- 2a$, but with the contrary ſign $+$,) it makes a Square, to wit, . . . $aa + 2a + 1$
4. Therefore for the ſecond number put . . . . $4a$

   Whereby one of the conditions in the Queſtion is ſatisfied; for the ſecond number $4a$ added to the Square of the firſt number $a - 1$ makes the Square $aa + 2a + 1$, whoſe Root is $a + 1$.
5. Then form a Square from $4a + 1$, (which is the ſumm of the ſecond number $4a$ and the known number 1 in the firſt aſſumed number $a - 1$, but with the contrary ſign $+$,) ſo the Square of $4a + 1$ will be $16aa + 8a + 1$; from which ſubtract the Square of the ſecond number $4a$, to wit $16aa$, and put the Remainder $8a + 1$ for the third number: Whence it is evident, that if this third number be added to the Square of the ſecond, the ſumm is a Square, whereby another of the conditions in the Queſtion is ſatisfied.
6. From the firſt, fourth and fifth ſteps the ſumm of the three numbers ſought is $13a$, which according to the Queſtion muſt be a Square, let it therefore be equated to ſome Square, *viz.* ſuppoſe $13a = 169aa$, whence $a = 13aa$; now according to this

value

value of a, the first number which was put $a - 1$ will be $13aa - 1$, the second number which was put $4a$ will be $52aa$, and lastly, the third number which was assumed $8a + 1$ will be $104aa + 1$. It remains that the Square of the third number $104aa + 1$, to wit, $10816aaaa + 208aa + 1$, added to the first number $13aa - 1$ may make a Square; but it makes $10816aaaa + 221aa$, this therefore must be equated to a Square, or the same divided by aa gives $10816aa + 221$ to be equated to a Square, whose side, to the end that a may be greater than $\sqrt{\frac{1}{13}}$, and consequently $13aa$ greater than 1, may be feigned to be $104a +$ any known number less than $4\frac{1}{10}$, or $104a -$ any known number greater than $60\frac{8}{10}$; let therefore the side of the said Square be feigned $104a + 1$, whence the Square it self is $10816aa + 208a + 1$, which being equated to the aforesaid $10816aa + 221$, will give $a = \frac{55}{52}$. Therefore the positions being resolved, the first number will be $\frac{36621}{2704}$, the second $\frac{157300}{2704}$, the third $\frac{317304}{2704}$, which three numbers will solve the Question, for their summ is $\frac{511225}{2704}$ the Square of the side $\frac{715}{52}$; also the Square of the first number, to wit, $\frac{1341097641}{7311616}$ added to the second makes the Square $\frac{1766436841}{7311616}$ from the side $\frac{42029}{2704}$; moreover the Square of the second, to wit, $\frac{24743290000}{7311616}$ added to the third makes the Square $\frac{25601280016}{7311616}$, from the side $\frac{160004}{2704}$; lastly, the Square of the third, to wit, $\frac{100681828416}{7311616}$ added to the first makes the Square $\frac{100780851600}{7311616}$, from the side $\frac{317460}{2704}$.

QUEST. 46.

To find three numbers, that as well the summ of every two, as of all three, may make a Square.

RESOLUTION.

1. Let b represent any known number, and a some number unknown, then from $a +$ some even number of b, (for avoiding Fractions) as from $a + 2b$ form a Square, which will be . .	$aa + 4ba + 4bb$
2. Then for the first number sought put the two first terms of the said Square, as	$aa + 4ba$
3. Then take the half of the said $4ba$, to wit, $2ba$, and prefixing the sign $-$ to it, it makes $-2ba$, to which add bb the Square of half the Coefficient $2b$, and take the summ for the second number sought, to wit, ...	$-2ba + bb$
4. Subtract bb in the said second number from $4bb$ part of the Square first formed, and add the Remainder $3bb$ to $+2ba$, to wit, the same multitude of ba as is in the second number, but with a contrary sign, and put this summ for the third number sought, to wit, ...	$+2ba + 3bb$
5. Then from the premisses it necessarily follows, that the summ of the first and second numbers (in the second and third steps) makes a Square, to wit, ...	$aa + 2ba + bb$
6. And the summ of the second and third numbers (in the third and fourth steps) is a Square, to wit, ...	. . . . $+4bb$
7. Also the summ of all the three numbers is a Square, to wit, .	$aa + 4ba + 4bb$

8. It remains that the summ of the first and third numbers make a Square, but it makes $aa + 6ba + 3bb$, which must be equated to a Square, yet so as the value of a may be less than $\frac{1}{2}b$, to the end that the second number $-2ba + bb$ may be greater than nothing. Now to cause that effect, the side of the said Square may be feigned $-a +$ any number between $\sqrt{3bb}$ and $3b$, (as may be collected from the Canon in *Sect.* 15. *Quest.* 12. of this *Book*:) Let therefore the said side be $-a + 2b$, and then its Square $aa - 4ba + 4bb$ being equated to $aa + 6ba + 3bb$ (the summ of the first and third numbers) this Equation ariseth, to wit,

$$aa + 6ba + 3bb = aa - 4ba + 4bb.$$

9. Whence after due Reduction, the value of a is made known, *viz.* $a = \frac{1}{10}b$

10. Therefore, the positions in the second, third and fourth steps being resolved according to that value of a, the three numbers sought are discovered, to wit,

$$\tfrac{41}{100}bb \;,\; \tfrac{4}{5}bb \text{ and } \tfrac{16}{5}bb.$$

Hence this

CANON.

CANON.

Take any ſquare number, then $\frac{41}{100}$ of that Square, alſo $\frac{8}{10}$ of the ſame Square, and $\frac{32}{10}$ thereof, will give three numbers to ſolve the Queſtion.

As, for example, if 10 be taken for the ſide of a Square, then theſe three numbers will be found out by the Canon, to wit, 41, 80 and 320, which will ſolve the Queſtion: For the ſumm of 41 and 80 makes the Square 121, whoſe ſide is 11; alſo the ſumm of 80 and 320 makes the Square 400, whoſe ſide is 20; and the ſumm of 320 and 41 makes the Square 361, whoſe ſide is 19; laſtly, the ſumm of all the three numbers 41, 80 and 320 makes the Square 441, whoſe ſide is 21. In like manner you may find out as many Anſwers in whole numbers as you pleaſe, by taking 20, 30, 40 or 50, *&c.* for the ſide of a Square, and then taking ſuch parts thereof as the Canon directs.

QUEST. 47. (Quæſt. 23. Lib. 4. *Diophant.*)

To find three numbers, that if they be ſeverally added to the Solid produced by their continual multiplication, every one of the three ſumms may be a Square.

(I ſhall wave Diophantus's *Reſolution, and uſe that of* Fermat *in his Obſervation upon this Queſtion, which is much eaſier.)*

RESOLUTION.

1. Let a Square be formed from a — any known number, as from $a - 1$, whoſe Square is . . . . . $aa - 2a + 1$
2. Then for the Solid of the three numbers ſought put the two firſt terms of that Square, to wit, ... $aa - 2a$
3. And for the firſt number ſought put the laſt term of the ſaid Square, to wit, ... 1

Whence one of the conditions is ſatisfied; for if the ſaid firſt number 1 be added to $aa - 2a$, (that is, the Solid of all the three numbers,) the ſumm is a Square, to wit, that firſt formed.

5. For the ſecond number put . . . . . $2a$

This added to the ſaid Solid $aa - 2a$ makes the Square aa, whereby another of the conditions in the Queſtion is ſatisfied.

5. Then divide $aa - 2a$, (the Solid of all the three numbers) by $2a$ the Product of the firſt and ſecond, ſo the Quotient is
$\frac{1}{2}a - 1$, (the third number.)
6. Which third number added to the Solid of all the three muſt alſo make a Square, but it makes $aa - \frac{3}{2}a - 1$.
7. Therefore $aa - \frac{3}{2}a - 1$ muſt be equated to a Square, yet ſo, as the value of a may be greater than 2, to the end that the third number $\frac{1}{2}a - 1$ may be greater than nothing: But to cauſe that effect, the ſide of the ſaid Square may be feigned a — any number leſs than 2, but greater than $\frac{3}{4}$; or a — any number greater than 2; let then the ſaid ſide be feigned $a - 3$, whoſe Square equated to $aa - \frac{3}{2}a - 1$, will give $a = \frac{20}{9}$. According to which value, the Poſitions being reſolved, the firſt number ſought is 1, the ſecond $\frac{40}{9}$, and the third $\frac{1}{9}$, which will ſolve the Queſtion: For the ſolid Product of their multiplication one into another, to wit, $\frac{40}{81}$, taking to it ſeverally the ſaid three numbers, makes the Squares $\frac{121}{81}$, $\frac{400}{81}$ and $\frac{49}{81}$.

QUEST. 48. (Quæſt. 31. Lib. 4. *Diophant.*)

To find four ſquare numbers, whoſe ſumm added to the ſumm of their ſides may make a number given, ſuppoſe 12.

RESOLUTION.

Foraſmuch as (by the firſt *Propoſition* in the following *Obſervation* upon this *Queſt.*) every Square increaſed with his ſide and $\frac{1}{4}$ of unity makes a Square, whoſe ſide leſſened by $\frac{1}{2}$ of unity gives the ſide of the former Square; therefore the ſumm of the four Squares ſought together with four times $\frac{1}{4}$ will make four Squares; but the given number 12 increaſed with four times $\frac{1}{4}$, to wit, 1, makes 13. Therefore we muſt divide 13 into four Squares; then if from every one of their ſides we ſubtract $\frac{1}{2}$, there will remain the ſides of the four Squares ſought. But 13 is compos'd of two Squares 4 and 9; therefore

therefore (by the first Question of this Book) each of these may be divided into two Squares, *viz.* 4 into $\frac{64}{25}$ and $\frac{36}{25}$, and 9 into $\frac{144}{25}$ and $\frac{81}{25}$: now the four Roots of those Squares are $\frac{8}{5}$, $\frac{6}{5}$, $\frac{12}{5}$ and $\frac{9}{5}$, from each of which Roots if $\frac{1}{2}$ be subtracted there will remain the sides of the four Squares sought, to wit, the sides $\frac{11}{10}$, $\frac{7}{10}$, $\frac{19}{10}$ and $\frac{13}{10}$, whose Squares $\frac{121}{100}$, $\frac{49}{100}$, $\frac{361}{100}$ and $\frac{169}{100}$ will solve the Question: For if their summ 7 be added to 5 the summ of their sides, it makes the given number 12; which was required.

Observations upon Quest. 48.

The preceding Resolution depends upon two Propositions, *viz.*

First, if any square number be increased with its side and $\frac{1}{4}$ of unity the summ will be a Square, whose side lessened by $\frac{1}{2}$ of unity gives the side of the former Square. This may be demonstrated thus;

Let there be a Square . . . . > aa
Then to that Square add its side and $\frac{1}{4}$ of unitty, to wit, ... > $a + \frac{1}{4}$
So the summ makes this Square, ... > $aa + a + \frac{1}{4}$
Whose Root is . . . . > $a + \frac{1}{2}$
From which Root if you subtract . . . . > . $+ \frac{1}{2}$
The Remainder is the side of the first Square, to wit, ... > a

Therefore the Proposition is manifest.

Secondly, the said Resolution takes this Proposition for granted, *viz.* That any given whole number increased with 1 may be divided into four Squares; how this may be generally done *Diophantus* doth not shew: But 'tis evident, that if a given square number increased with 1 makes a Square, or a number composed of two Squares, then the summ may easily be divided into four Squares, or into as many as you please, (by the first or second Question of this Book.) But if 13 be given, how shall 13 increased with 1, that is 14, which is neither a Square nor composed of two Squares, be divided into four Squares? This at first sight seems to be a very hard Task, but if the matter be narrowly considered, the difficulty will soon vanish; for 14 is compos'd of three Squares, to wit, 1, 4, 9; wherefore if one of these be divided into two Squares, then consequently 14 is divided into four Squares. But *Fermat* in his Observation upon this *Quest.* 31. of the fourth Book of *Diophantus*, affirms that every whole number is either a Square, or else compos'd of two, three or four Squares, and he there promiseth to give the Demonstration of this and other abstruse Mysteries in Numbers in a particular Treatise. *Bachet* confesseth he could not demonstrate the same, but gives Examples of the certainty thereof in all whole numbers from 1 to 120, and saith he had made experiment of all whole numbers to 325. If this *Prop.* be granted, then the Question may be easily extended to five, six, or as many Squares as you please, without any Determination. But if two Squares only be desired, whose summ with the summ of their sides may make a Square, then after 2 is added to the quadruple of the given number, the summ must be compos'd of two Squares: And if three Squares be sought, then after 3 is added to the quadruple of the given number the summ must be compos'd of three Squares, which conditions are manifest from the first Proposition above exprest.

QUEST. 49.

[*This is the* 34^{th} *of the fourth Book of* Diophantus, *and the* 13^{th} *of the fifth Book of* Vieta's *Zeteticks.*]

To divide a given number x into two such parts, that if the first part be increased with a given number b, and the other part with a given number d; the Product made by the multiplication of the two summs one into the other may be a square number.

RESOLUTION.

1. For the first part sought put . . . . > $a - b$
2. Therefore the other part, to the end the summ of the parts may make x, shall be . . . . > $x - a + b$
3. Then (according to the Question) adding b to the first part, the summ is . . . . > a
4. And adding d to the second part, the summ is . . . . . . > $x - a + b + d$

5. There-

5. Therefore the Product made by the multiplication of thoſe two ſumms one into the other will be . . . . $xa - aa + ba + da$
6. That is . . . . $\overline{x+b+d}$ into a, $-aa$
7. Which Product muſt be equated to a Square, whoſe ſide may be feigned sa, and then the Square of sa, to wit, $ssaa$, being equated to the Product in the ſixth ſtep, this Equation ariſeth, *viz.*

$$\overline{x+b+d} \text{ into } a,\ -aa = ssaa.$$

8. Which Equation, after due Reduction, gives . . . . . $a = \dfrac{x+b+d}{ss+1}$
9. Therefore from the eighth and firſt ſteps, the firſt part ſought will be made known, to wit, ... $\dfrac{x+d-ssb}{ss+1}$
10. And from the eighth and ſecond ſteps the ſecond part ſought will be diſcovered, to wit, ... $\dfrac{ssx+ssb-d}{ss+1}$

But to the end that each of the two parts in the ninth and tenth ſteps may be greater than nothing, the number s cannot be taken at pleaſure, but within the limits hereafter diſcovered, *viz.*

11. Foraſmuch as the numerator in the ninth ſtep requires that . . $x+d \sqsubset ssb$
12. Therefore by dividing each part in the laſt ſtep by b, . . $\dfrac{x+d}{b} \sqsubset ss$
13. That is, ... $ss \sqsupset \dfrac{x+d}{b}$
14. Therefore by extracting the ſquare Root out of each part in the laſt ſtep, ... $s \sqsupset \sqrt{\dfrac{x+d}{b}}$
15. Again, the Numerator in the tenth ſtep ſhews, that . . . $ssx + ssb \sqsubset d$
16. Therefore by dividing each part by $x+b$, it follows that $ss \sqsubset \dfrac{d}{x+b}$
17. Therefore by extracting the ſquare Root out of each part in the laſt ſtep, ... $s \sqsubset \sqrt{\dfrac{d}{x+b}}$
18. Thus in the fourteenth and ſeventeenth ſteps it is diſcovered, that for the number s we may take any number between $\sqrt{\dfrac{x+d}{b}}$ and $\sqrt{\dfrac{d}{x+b}}$, and then the two deſired parts whoſe ſumm is equal to x the number to be divided will be ſuch as are before expreſt in the ninth and tenth ſteps.

An Example in Numbers.

Suppoſe $\left\{\begin{array}{l} 4 = x \\ 12 = b \\ 20 = d \end{array}\right\}$ numbers given in the Queſtion;

Whence $\frac{4}{3} = s$ a number choſen within the limits in the eighteenth ſtep.

Then by the help of thoſe known numbers, the ninth and tenth ſteps will give $\frac{24}{25}$ and $\frac{76}{25}$ the two parts ſought, whoſe ſumm is 4, (or x;) the former of which parts increaſed with 12 (or b,) and the latter with 20 (or d,) make the two ſumms $\frac{324}{25}$ and $\frac{576}{25}$; theſe multiplied one by the other produce a Square whoſe ſide is $\frac{432}{25}$. Therefore the Queſtion is ſolved, and manifeſtly capable of innumerable Anſwers.

QUEST. 50. (Quæſt. 35. Lib. 4. *Diophant.*)

To divide a given number into three numbers, ſuch, that if the Product of the multiplication of the firſt into the ſecond be increaſed and leſſened by the third, as well the ſumm as the remainder may be a ſquare number.

RESOLUTION.

1. Let the given number be 6
2. For the third number ſought put . . . . a
3. For the ſecond number ſought put ſome known number leſs than the given number 6, as 2
4. Therefore, becauſe all the three numbers ſought muſt make 6, the firſt number ſhall be . . . . $4-a$

5. Then (according to the Question) the Product of the first and second numbers sought, together with the third, must make a Square, *viz.* $\Big\}$ $8 - a = \square$

6. Also the same Product lessened by the third number must make a Square, *viz.* . . . . $\Big\}$ $8 - 3a = \square$

7. So in the two last steps we are faln upon a Duplicate equality, but 'tis inexplicable; *Diophantus* therefore seeks out another Duplicate equality wherein the numbers prefixt to a may have such proportion to one another as a square number hath to a square number, for then it will be resolvable like that Duplicate equality which hath been already explain'd in *Quest.* 35. of this *Book.* First, then instead of 2 which was assumed for the second number sought, some other number less than 6 must be taken, such, that if it be increased and lessened by unity, the summ may be to the remainder as a square number to a square number; (for if the rise of 3 and 1, which are prefixt to a in the Duplicate equality above exprest, be examined, it will appear that 3 ariseth by adding 1 to 2, and 1 ariseth by subtracting 1 from the same number 2.) Therefore let e represent some number to be taken instead of 2 for the second number sought; then $e+1$ must be to $e-1$, as a Square to a Square, suppose as bb to dd.
Therefore, as . . . . $e+1 \,.\, e-1 :: bb \,.\, dd$

8. Therefore by comparing the Product of the extremes to the Product of the means, ... $\Big\}$ $dde+dd = bbe-bb$

9. Whence after due Reduction, ... $e = \frac{bb+dd}{bb-dd}$

10. But the value of e must be less than 6, which I shall call f, therefore . . . . $\Big\}$ $\frac{bb+dd}{bb-dd} \sqsupset f$

11. Therefore by multiplying each part by f, ... $bb+dd \sqsupset f \times \overline{bb-dd}$:
Hence this Canon to find out the number e, *viz.*

12. Take any two square numbers whose summ may be less than the Product of their difference multiplied into the number given in the Question; then divide the summ of the said Squares by their difference, so the Quotient shall be the number to be put for the second number sought by the Question proposed, for it shall be less than the number given in the Question; and if it be increased with 1 and lessened by 1, the summ shall be to the remainder as the greater of the Squares taken is to the lesser. Therefore I take the Squares 4 and 1 and divide their summ 5 by their difference 3, so there ariseth $\frac{5}{3}$ for the second number. Now let the Positions be renewed thus, *viz.*

13. The number given to be divided into three numbers is . . 6

14. For the second number put . . . . $\frac{5}{3}$

15. And for the third number . . . . a

16. Then the summ of the second and third numbers subtracted from 6 (the summ of all three) leaves the first number . . $\Big\}$ $\frac{13}{3} - a$

17. Now (according to the Question) the Product of the first and second numbers together with the third, must make a Square, *viz.* . . . . $\Big\}$ $\frac{65}{9} - \frac{2}{3}a = \square$

18. And the same Product lessened by the third number must leave a Square, *viz.* . . . . $\Big\}$ $\frac{65}{9} - \frac{8}{3}a = \square$

19. So in the two last steps there is a Duplicate equality wherein the numbers prefixt to a have such proportion one to another as a Square to a Square, for $\frac{2}{3}$ is to $\frac{8}{3}$ as 1 to 4; and therefore this Duplicate equality may be resolved like that in the preceding *Quest.* 35. in this manner, *viz.* Forasmuch as a square number multiplied by a Square produceth a Square, therefore to take away the Fractions, multiply all the Quantities in the 17th and 18th steps by the Denominator 9, so this Duplicate equality ariseth,

$$viz. \begin{cases} 65 - 6a = \square \\ 65 - 24a = \square. \end{cases}$$

20. Then the former of those two Equations multiplied by 4 produceth $260 - 24a = \square$, so at length this Duplicate equality remains to be resolved,

$$viz. \begin{cases} 260 - 24a = \square \\ 65 - 24a = \square. \end{cases}$$

21. Now the difference of these two Equations being 195, I seek by *Canon* 2. *Quest.* 7. of this *Book* two such square numbers that their difference may be 195, and that the greater Square may be less than 260. But the only pair of Squares in whole numbers

so

ſo qualified are 196 and 1, the greater of which being equated to $260 - 24a$, or the leſſer to $65 - 24a$, will give $a = \frac{8}{3}$ for the third number ſought, and conſequently, by the Poſitions in the fourteenth and ſixteenth ſteps, the firſt and ſecond numbers are $\frac{5}{3}$ and $\frac{5}{3}$, which three numbers will ſolve the Queſtion, as is evident by the Proof, for their ſumm is 6; alſo if $\frac{25}{9}$ the Product of the firſt and ſecond be increaſed with the third number $\frac{8}{3}$ it makes the Square $\frac{49}{9}$; but if the ſame Product be leſſened by the ſaid $\frac{8}{3}$ it leaves the Square $\frac{1}{9}$.

QUEST. 51. (Quæſt. 36. Lib. 4. *Diophant.*)

To find three numbers, whereof the third may be ſuch a Fraction of unity, that if the firſt number takes from the ſecond ſuch part or parts as the Fraction expreſſeth, the ſumm may be to the remainder in a given Reaſon, ſuppoſe as b to d. Alſo, that the ſecond number taking the ſame part or parts from the firſt, the ſumm may be to the remainder in a given Reaſon, ſuppoſe as f to g. But the Product made by the mutual multiplication of the firſt term of each Reaſon muſt exceed the Product of the latter terms one into the other, *viz.* bf muſt be greater than dg.

Preparation.

Let a and e repreſent the firſt and ſecond numbers, and u the third, or Fraction ſought; then becauſe to take any part or parts of a number, the number muſt be multiplied by the Fraction expreſſing the parts, the Queſtion may be ſtated thus, *viz.*

1. If . . . . $a + ue \,.\, e - ue :: b \,.\, d$,
2. And . . . . $e + ua \,.\, a - ua :: f \,.\, g$.

What are the numbers, a, e, u?

RESOLUTION.

3. By comparing the Product of the extremes to the Product of the means in the firſt Analogy, this Equation is produced, to wit,
$$da + due = be - bue.$$
4. Likewiſe from the latter Analogy this Equation is produced, *viz.*
$$ge + gua = fa - fua.$$
5. From the Equation in the third ſtep by tranſpoſition of due, this ariſeth,
$$da = be - bue - due.$$
6. And by dividing each part of the laſt Equation by d, this ariſeth,
$$a = \frac{be - bue - due}{d}.$$
7. Then by exchanging a in the fourth ſtep, for that which is equal to a in the laſt Equation, and multiplying all into d, the Equation in the fourth ſtep will be converted into this, *viz.*
$$\left.\begin{matrix} dge + bgue - bguue \\ - dguue \end{matrix}\right\} = \left\{\begin{matrix} bfe - 2bfue - dfue \\ + bfuue + dfuue \end{matrix}\right.$$
8. From which Equation, after due Reduction, this following Equation ariſeth, wherein u only is unknown, *viz.*
$$\frac{bg + 2bf + df}{bg + dg + bf + df}u - uu = \frac{bf - dg}{bg + dg + bf + df}.$$
9. Which laſt Equation may be reſolved (by the Canon in *Sect.* 10. *Chap* 15. *Book* 1.) in this manner, *viz.* half the Coefficient drawn into u in the ſaid Equation is
$$\frac{\frac{1}{2}bg + bf + \frac{1}{2}df}{bg + dg + bf + df}.$$
10. The Square of the ſaid half Coefficient is
$$\frac{\frac{1}{4}bbgg + bbff + \frac{1}{4}ddff + bbfg + bdff + \frac{1}{2}bdfg}{bg + dg + bf + df \text{ into } bg + dg + bf + df}.$$
11. Then to reduce the known Abſolute quantity which ſolely poſſeſſeth the latter part of the Equation in the eighth ſtep to the ſame Denominator with the Square in the tenth ſtep, I multiply as well the Numerator as the Denominator of the ſaid Abſolute quantity, by its Denominator $bg + dg + bf + df$, and it makes
$$\frac{bbfg + bbff + bdff - bdgg - ddgg - ddfg}{bg + dg + bf + df \text{ into } bg + dg + bf + df}.$$

12. Which

12. Which Fraction last above exprest being subtracted from the Fraction in the tenth step will leave this that follows, to wit,

$$\frac{\frac{1}{4}bbgg + \frac{1}{4}ddff + ddgg + \frac{1}{2}bdfg + bdgg + ddfg}{bg + dg + bf + af \text{ into } bg + dg + bf + df}.$$

13. The square Root of the Fraction in the twelfth step is

$$\frac{\frac{1}{2}bg + \frac{1}{2}df + dg}{bg + dg + bf + df}.$$

14. Which square Root being added to and subtracted from the half-Coefficient in the ninth step, the summ and remainder shall be the two values of u in the Equation in the eighth step, *viz.*

$$u = 1; \text{ also, } u = \frac{bf - dg}{bg + dg + bf + df};$$

The latter of which values of u, to wit, $\frac{bf - dg}{bg + dg + bf + df}$ is the Fraction sought by the Question.

15. Then according to the said lesser value of u, the compound quantity $\frac{b - bu - du}{d}$ into which e is multiplied in the latter part of the Equation in the sixth step will be reduced into this fractional quantity, to wit, $\frac{bbg + bdg + ddg}{bdg + ddg + bdf + ddf}$, which multiplied into any number taken at pleasure for the value of e, will give the number a; and therefore to find out a and e in whole numbers we may take the Numerator of that fractional quantity for a, and the Denominator for e, or any two whole numbers in the same proportion with the said Numerator and Denominator, and the lesser of the two values of u before found for the Fraction sought.

An Example in Numbers.

16. Let there be given . . . . $\left\{\begin{array}{l} 3 = b \\ 1 = d \\ 5 = f \\ 1 = g \end{array}\right.$

17. Then the three numbers sought will be these, *viz.* $\left\{\begin{array}{l} 16 = bbg + 2bdg + ddg \\ 24 = bdg + ddg + bdf + ddf \\ \frac{7}{12} = \frac{bf - dg}{bg + dg + bf + df} \end{array}\right.$

18. Which three numbers, to wit, 16, 24 and $\frac{7}{12}$ will solve the Question, as will be manifest by

The Proof.

19. If the first number 16 receive $\frac{7}{12}$ of the second number 24, that is, 14, the summ 30 will be to the remainder 10 as 3 to 1, (*viz.* as b to d;) Again, if the second number 24 take $\frac{7}{12}$ parts of the first number 16, to wit, $9\frac{1}{3}$, the summ $\frac{100}{3}$ will be to the remainder $\frac{20}{3}$ as 5 to 1, (to wit, as f to g:) Or instead of 16 and 24 you may take 2 and 3, or any two numbers in the same Proportion.

20. Again, if $b = 3$. $d = 2$. $f = 4$. and $g = 5$. then the literal quantities in the preceding 17th step will give these three numbers, 125, 90 and $\frac{2}{45}$ to solve the Question; or instead of 125 and 90 you may take 25 and 18, or any two numbers in the same Proportion.

Note. If in reducing the Equation in the seventh step, bf were supposed either equal to dg or less than dg, there would come forth an Equation wherein the value of u would be either unity or greater than unity; but according to the import of the Question it ought to be less than unity, and such is the lesser value of u in the Equation in the eighth step, where bf is supposed greater than dg, as the Determination annexed to the Question requires.

QUEST. 52. (Quæst. 41. Lib. 4. *Diophant.*)

To find two numbers, that the Product of their multiplication may be to their summ in a given Reason (or Proportion,) suppose as r to s.

RESO-

RESOLUTION.

1. For the first number put . . . . a
2. And for the second . . . . e
3. Then their summ is . . . . $a + e$
4. And the Product of their multiplication is . . . . ae
5. But according to the Question the Product must be to the summ as r to s, therefore . . . . $ae \,.\, a+e :: r \,.\, s$
6. Therefore by comparing the Product of the extremes to the Product of the means this Equation ariseth, ... $sae = ra + re$
7. And by subtracting re from each part of that Equation, this ariseth, *viz.* . . . . $sae - re = ra$
8. And by dividing each part of the last Equation by $sa - r$, this ariseth, *viz.* . . . . $\therefore e = \frac{ra}{sa - r}$

Which Equation gives this

CANON.

9. For the first number sought take any number greater than the Quotient that ariseth by dividing the former term of the given Reason by the latter; then multiply the first number so taken by the latter term, and from the Product subtract the former term; lastly, by the Remainder divide the Product made by the multiplication of the first number into the former term, and the Quotient shall be the second number sought. For example, if two numbers be desired that their Product may be to their summ as 3 to 2, (that is, as r to s in the Resolution,) you may take any number greater than $\frac{3}{2}$, as 2, for the first number; then (by the Canon) the other number will be found 6; which numbers 2 and 6 are such, that their Product 12 is to their Summ 8, as 3 to 2, as was desired. After the same manner you may find out innumerable Answers to the Question.

10. But if it were desired to find out two numbers that the Product of their multiplication might be equal to their summ, and that the summ or Product might be a square number; the numbers may be found out thus, *viz.*

For the first number sought put . . . . a
And for the second number . . . . e
Then according to the Question, ... $ae = a + e$
Therefore by transposition of e, ... $\therefore ae - e = a$
Therefore each part of the last Equation being divided by $a - 1$, there will arise . . . . $e = \frac{a}{a - 1}$
Which last Equation multiplied by a gives . . . . . . $ae = \frac{aa}{a - 1}$ $(= a + e)$

Which $\frac{aa}{a - 1}$ must (according to the Question) be a square number. But the Numerator aa is a Square; it remains then to equate the Denominator $a - 1$ to some square number, let it be dd, *viz.* suppose $a - 1 = dd$, whence $a = dd + 1$; according to which value of a, the number e which was before found equal to $\frac{a}{a - 1}$, will be $\frac{dd + 1}{dd}$. Therefore for the two numbers sought take $dd + 1$ and $\frac{dd + 1}{dd}$, which in words give this

CANON.

For one of the numbers sought take any square number increased with unity; then divide that summ by the square number taken, and the Quotient shall be the other number sought.

As, for example, you may take $4 + 1$, that is, 5 for one of the numbers sought; then dividing 5 by 4 (the Square taken,) the Quotient $\frac{5}{4}$ shall be the other number sought. So 5 and $\frac{5}{4}$ will solve the Question last proposed; for their Product is $\frac{25}{4}$, their Summ also is $\frac{25}{4}$, which is a square number as was required.

QUEST. 53.

To find two numbers, that their difference may be equal to the difference of their Squares, and that the summ of the Squares of the two numbers may be a Square.

RESO-

RESOLUTION.

1. For the greater number put . . . . a
2. And for the lesser . . . . e
3. Then their difference is . . . . $a - e$
4. And the difference of their Squares is . . . . $aa - ee$
5. Therefore according to the Question, ... $aa - ee = a - e$
6. Therefore by dividing each part of that Equation by $a - e$, the Quotient gives . . . . $a + e = 1$
7. Therefore by transposition of a, ... $e = 1 - a$
8. Now taking $1 - a$ instead of e, the two numbers sought are . . a and $1 - a$
9. The difference of which numbers is either $2a - 1$ or $1 - 2a$, and the same is the difference of their Squares: But the summ of their Squares must make a Square; therefore $2aa - 2a + 1$ must be equated to a Square, yet so, as the value of a may be less than 1. Now to cause that effect the said side may be feigned $-1 +$ any multitude of a greater than $2a$; let therefore the said side be feigned $3a - 1$, then the Square of $3a - 1$ being equated to the said $2aa - 2a + 1$ the value of a will be found $\frac{4}{7}$, which subtracted from 1 leaves $\frac{3}{7}$; therefore $\frac{4}{7}$ and $\frac{3}{7}$ are the numbers sought. For as well their difference as the difference of their Squares is $\frac{1}{7}$; and the summ of the Squares of $\frac{4}{7}$ and $\frac{3}{7}$ makes a Square, to wit, $\frac{25}{49}$.

From the premisses ariseth this

CANON.

10. Take any number greater than 2; then divide the excess of that number above unity, by the excess of half the Square of the same number above unity, and the Quotient shall be one of the numbers sought, which subtracted from unity leaves the other number sought. As, for example, take the number 3; then dividing the excess of 3 above 1, that is, 2, by the excess of half the Square of 3 above 1, that is, by $\frac{7}{2}$; the Quotient $\frac{4}{7}$ is one of the numbers sought, which subtracted from 1 leaves $\frac{3}{7}$ for the other number.
11. The same Question may be propounded thus, *viz.* To find a right-angled Triangle in Rational numbers, that the difference of the sides about the right-angle may be equal to the difference of the Squares of the same sides. For solving this Question, take any two numbers found out by the said Canon, as $\frac{4}{7}$ and $\frac{3}{7}$ for the sides about the right-angle, whence the Hypothenusal (to wit, the square Root of the summ of the Squares of $\frac{4}{7}$ and $\frac{3}{7}$) is $\frac{5}{7}$.

Moreover, from the foregoing Resolution of *Quest.* 53. we may deduce this

THEOREM.

12. If unity be divided into any two parts, the difference of the parts is equal to the difference of the Squares of the same parts: And if the Product made by the mutual multiplication of the parts be subtracted from each of them, each Remainder will be a Square: Also the excess of the greater part above its Square is equal to the excess of the lesser part above its Square, and each excess is equal to the Product of the parts.

This will easily be manifested by these two numbers a and $1 - a$, whose summ is unity.

QUEST. 54. (Quæst. 45. Lib. 4. *Diophant.*)

To find three numbers, that the excess of the greatest above the mean may be to the excess of the mean above the least in a given Reason, suppose as 3 to 1; and that the summ of every two of the three numbers may be a Square.

RESOLUTION.

1. Forasmuch as the summ of the mean and least of the three numbers must be a Square, let it be 4
2. Therefore the mean is greater than 2; for if we should put it 2, the least would be also 2, which is absurd; therefore for the mean let there be put $a + 2$
3. Therefore from the said positions the least number is . . . . . $2 - a$
4. And the excess of the mean above the least is . . . . $2a$
5. But the excess of the greatest above the mean must be the triple of the excess of the mean above the least; therefore from the last step the excess of the greatest above the mean is . . . . $6a$

6. Which last mentioned excess, to wit, $6a$, being added to the mean $a + 2$, gives the greatest of the three numbers, to wit, . . . $\Big\}$ $7a + 2$
7. But the Question requires two things more, to wit, that the greatest with the mean may make a Square; and that the greatest with the least may make a Square; hence ariseth this Duplicate equality to be resolved, to wit, ... $\left\{ \begin{array}{l} 8a + 4 = \square \\ 6a + 4 = \square \end{array} \right.$
8. Which Duplicate equality hath already been resolved in the preceding *Quest.* 33. of this *Book*, where the number $\frac{455}{242}$, (among innumerable other numbers that might be discovered, and each to be less than 2, as the third step of the preceding Resolution requires,) was found out for the value of a. Therefore $\frac{455}{242}$ being taken for the value of a, and the positions in the sixth, second and third steps resolved accordingly, the three numbers sought will be found these, to wit, $\frac{7338}{484}$, $\frac{1878}{484}$ and $\frac{58}{484}$, which will solve the Question. For first, the excess of the greatest above the mean, to wit, $\frac{5460}{484}$, is the triple of $\frac{1820}{484}$ the excess of the mean above the least; secondly, the summ of the greatest and mean is the Square $\frac{9216}{484}$, whose side is $\frac{96}{22}$; thirdly, the summ of the greatest and least is the Square $\frac{7396}{484}$, whose side is $\frac{86}{22}$; lastly, the summ of the mean and least is the Square $\frac{1936}{484}$, whose side is $\frac{44}{22}$, that is, 2.

QUEST. 55. (Quæst. 2. Lib. 5. *Diophant.*)

To find three numbers in Geometrical proportion, that every one of them increased with a given number d may make a Square.

RESOLUTION.

1. First, seek a Square which added to the given number d may make a Square, and whose $\frac{1}{4}$ part may exceed d; suppose it be found bb, let this be put for one of the extreme Proportionals sought, to wit, ... $\Big\}$ bb
2. For the other extreme put . . . . $\}$ aa
3. Then because the Product of the extremes is equal to the Square of the mean, therefore the Square of the mean is $bbaa$, whose Root is the mean it self, to wit, ... $\Big\}$ ba
4. By Construction in the first step the first extreme bb increased with the given number d makes a Square; but according to the Question the other extreme and the mean being severally increased with the same given number d must also make a Square, whence this Duplicate equality ariseth,

$$viz. \left\{ \begin{array}{l} aa + d = \square. \\ ba + d = \square. \end{array} \right.$$

5. Now to resolve this Duplicate equality I proceed as in former Questions, *viz.* First, the difference of those two Equations is $aa - ba$, which is equal to the Product of a into $a - b$; then if the Square of half the summ of a and $a - b$ be equated to $aa + d$, or the Square of half the difference of a and $a - b$ to $ba + d$, from either of those Equations the value of a will be made known: But half the difference of the said a and $a - b$ is $\frac{1}{2}b$, whose Square is $\frac{1}{4}bb$; let this be equated to $ba + d$, and it will be

$$ba + d = \tfrac{1}{4}bb.$$

6. Whence, after due Reduction, the value of a will be discovered, *viz.* . . . . $\Big\}$ $a = \dfrac{\frac{1}{4}bb - d}{b}$
7. And by multiplying each part of the last Equation by b, it gives $\}$ $ba = \frac{1}{4}bb - d$

From the premisses ariseth the following Canon to find out the three Proportionals sought.

CANON.

8. Take for one of the extreme Proportionals sought such a square number that if it be increased with the given number it may make a Square, and that a quarter of the first Square may exceed the given number; then from a quarter of the first Square subtract the given number and there will remain the mean Proportional; lastly, divide the mean by the side of the Square first taken, and the Square of the Quotient shall be the other extreme.

An Example in Numbers.

Suppose $19 = d$ the number given in the Question; then find a Square that if it be increased with 19 may make a Square, and that a quarter of the Square found out may exceed

exceed the ſaid 19, (or that the ſide of the ſaid Square may exceed $\surd 76$.) But ſuch is the Square 81, (found out after the manner of reſolving the ninth Queſtion of this Book,) for 81 increaſed with 19 makes the Square 100, alſo $\frac{1}{4}$ of 81 is greater than 19; therefore 81 ſhall be the firſt of the three Proportionals ſought. Then by the Canon above expreſt, the other two will be found $\frac{5}{4}$ and $\frac{25}{1296}$. I ſay, 81, $\frac{5}{4}$ and $\frac{25}{1296}$ will ſolve the Queſtion: For firſt they are continual Proportionals, in regard the Product of the extremes is equal to the Square of the mean; ſecondly, the firſt Proportional 81 increaſed with the given number 19 makes the Square 100; thirdly, the mean Proportional $\frac{5}{4}$ increaſed with the ſaid 19 makes the Square $\frac{81}{4}$; laſtly, the third Proportional $\frac{25}{1296}$ increaſed with the ſaid 19 makes the Square $\frac{24649}{1296}$, whoſe ſide is $\frac{157}{36}$. Therefore the Queſtion is ſolved, and manifeſtly capable of innumerable Anſwers.

QUEST. 56. (Quæſt. 7. Lib. 5. *Diophant.*)

To find two numbers, that the Product of their multiplication added to the ſumm of their Squares may make a Square.

RESOLUTION.

1. For one of the numbers ſought take any known number, as	b
2. For the other number put	a
3. The Square of the firſt is	bb
4. The Square of the ſecond is	aa
5. The Product of the multiplication of the two numbers is	ba
6. Therefore the ſumm of the ſaid Squares and Product is	$aa + ba + bb$
7. Which ſumm muſt be equal to a Square, the ſide whereof may be feigned to be $a -$ any known number greater than b, let it be $a - d$, and then the Square of $a - d$ being equated to the ſaid ſumm, this Equation ariſeth, *viz.*	$aa + ba + bb = aa - 2da + dd$
8. Which Equation after due Reduction gives . . .	$a = \frac{dd - bb}{2d + b}$
9. Therefore from the firſt, ſecond and eighth ſteps the two numbers ſought are equal to theſe known numbers, *viz.*	b and $\frac{dd - bb}{2d + b}$
10. But to avoid Fractions multiply thoſe two numbers ſeverally by the Denominator $2d + b$, and take the Products for two numbers to ſolve the Queſtion, *viz.*	$2db + bb$ and $dd - bb$

The Proof.

11. The Square of $dd - bb$ is	$dddd - 2ddbb + bbbb$
12. The Square of $2db + bb$ is	$4ddbb + 4dbbb + bbbb$
13. The Product of $dd - bb$ into $2db + bb$ is . .	$2bddd - 2dbbb + bbdd - bbbb$
14. The ſumm of the ſaid Squares and Product is . .	$d^4 + b^4 + 3ddbb + 2db^3 + 2bd^3$
15. Which ſumm is a Square whoſe Root is	$dd + db + bb$

From the tenth ſtep ariſeth

CANON 1.

16. Take any two unequal ſquare numbers, then their difference ſhall be one of the numbers ſought; and the leſſer Square increaſed with the double Product of the multiplication of the ſides of thoſe Squares ſhall be the other number ſought.

Moreover, becauſe the Product of the multiplication of the ſumm of any two numbers into their difference, is equal to the difference of their Squares, therefore from the preceding Canon ariſeth

CANON 2.

17. Take any two unequal numbers, then the Product of their ſumm multiplied into their difference ſhall be one of the two numbers ſought, and the double Product made by the mutual multiplication of the two numbers firſt taken, together with the Square of the leſſer number ſhall be the other number ſought.

From

From the premiffes 'tis evident that the Queftion is capable of innumerable Anfwers in whole numbers, of which (for the Learners exercife) I fhall exhibit fix, with their Proofs, in the following Table.

	d, b	s, r	ss	rr	sr	qq	q
1	2, 1	5, 3	25	9	15	49	7
2	3, 1	8, 7	64	49	56	169	13
3	3, 2	16, 5	256	25	80	361	19
4	4, 3	33, 7	1089	49	231	1369	37
5	5, 1	24, 11	576	121	264	961	31
6	5, 3	39, 16	1521	256	624	2401	49

18. The numbers under d and b in this Table are fix pairs of numbers taken at pleafure, by which the latter of the two preceding Canons gives fix pairs of numbers under s and r to folve *Queft.* 56. As, for example, if 2 and 1 be taken, then *Canon* 2. gives 5 and 3, (that is, s and r) to folve the Queftion: For 25 and 9, the Squares of 5 and 3, together with 15 the Product of 5 and 3, (that is, $ss + rr + sr$) make the Square 49, (that is qq,) whofe fide is 7, (to wit, q.)

But for a further Proof, you may obferve from the fourteenth and fifteenth fteps of the Refolution, that every number ftanding under q is equal to its refpective $dd + db + bb$; fo 7 in the Columel of q is equal to the Squares of 2 and 1 ftanding under d, b, together with the Product of 2 into 1. The like is to be underftood of the other five Anfwers in the Table.

Obfervat. 1. *upon the foregoing* Queft. 56.

Whereas it is taken for granted in the tenth ftep of the preceding Refolution of *Queft.* 56. that two numbers in the fame Reafon (or Proportion) with thofe found out to folve the faid Queftion will likewife fatisfie the fame, I fhall here demonftrate the truth thereof.

Suppofitions.

1. Let two numbers capable of folving *Queft.* 56. fuppofe 5 and 3, be reprefented by . . . . a and b
2. And let their Squares be fignified by . . . . . aa and bb
3. Then according to the import of *Queft.* 56. . . . $aa + ba + bb = \square$
4. Let two other numbers having the fame Proportion to one another as a to b be reprefented by d and c, viz. fuppofe . . . . $a \,.\, b :: d \,.\, c$
5. Then add cd the Product of c and d to cc and dd the Squares of c and d, fo the fumm is . . . . $dd + cd + cc$
6. Now we muft demonftrate that . . . . $dd + cd + cc = \square$ (a Square.)

Demonftration.

7. By *Prop.* 17. *Elem.* 7. Euclid. . . . . . . $a \,.\, b :: aa \,.\, ba$
8. And by *Prop.* 11. *Elem.* 8. . . . . . . $aa \,.\, ba :: ba \,.\, bb$
9. Therefore out of the two laft preceding Analogies, ... $a \,.\, b :: aa \,.\, ba :: ba \,.\, bb$
10. And by the like argumentation, ... $d \,.\, c :: dd \,.\, cd :: cd \,.\, cc$
11. And becaufe by fuppofition in the fourth ftep, $a \,.\, b :: d \,.\, c$
12. Therefore out of the ninth, tenth and eleventh fteps, (*per Prop.* 11. *Elem.* 5.) . . $aa \,.\, ba :: ba \,.\, bb :: dd \,.\, cd :: cd \,.\, cc$
13. Therefore alternately (*per Prop.* 13. *Elem.* 7.) $aa \,.\, dd :: ba \,.\, cd :: bb \,.\, cc$
14. Therefore (*per Prop.* 12. *Elem.* 7.) . . . $aa + ba + bb \,.\, dd + cd + cc :: aa \,.\, dd$
15. And becaufe by fuppofition . . . . . aa and dd are Squares.
16. And by fuppofition in the third ftep, . . $aa + ba + bb = \square$
17. Therefore from the fourteenth, fifteenth and fixteenth fteps, (*per Prop.* 24. *Elem.* 8.) . . $dd + cd + cc = \square$

Which was to be proved.

Observat. 2. *upon* Quest. 56.

Albert Girard in *pag.* 618. of *Simon Stevin*'s Arithmetick printed in the *French* Tongue at *Leyden*, in 1625. doth from the said seventh Question of the fifth Book of *Diophantus* deduce this following

THEOREM.

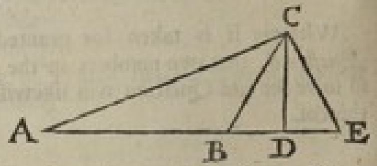
If a plain Triangle be made of three such sides, that the summ of the Squares of two of those sides, together with the Rectangle (or Product of the multiplication) of the same two sides, is equal to the Square of the third side; then the angle opposite to such third side hath for its measure exactly 120 degrees. But if the said Rectangle (or Product) be subtracted from the summ of the said Squares, and the Remainder be equal to the Square of the third side, then the angle opposite to such third side shall have for its measure infallibly 60 degrees.

This may easily be demonstrated by *Prop.* 12, & 13. *Elem.* 2. *Euclid.* but waving the Demonstration, I shall explain the Theorem by Numbers.

First then, if three numbers be desired to express the measures of the sides of a plain Triangle that shall have one angle whose measure is 120 degrees, the preceding Table will furnish you with six such Triangles; for in every rank of numbers in that Table, the three numbers which answer to s, r and q will constitute the Triangle desired. As, for example, 5, 3 and 7; likewise, 8, 7, 13 | 16, 5, 19, &c. will express the Quantities of the sides of Triangles, in every one of which, the measure of the angle opposite to the greatest side is exactly 120 degrees.

Otherwise, without the help of the said Table, if two unequal numbers be taken at pleasure, as $2 = d$, and $1 = b$, then these three following numbers shall express the measures of the sides of a Triangle having an angle of 120 degrees, *viz.*

$$\left\{\begin{array}{l} 2db + bb = 5 = AB, \\ dd - bb = 3 = BC = BE = EC, \\ dd + db + bb = 7 = AC. \end{array}\right.$$



Which three numbers, (as is manifest by the preceding Resolution of *Quest.* 56.) have this property, *viz.* the summ of the Squares of the two lesser numbers together with the Product of their multiplication is equal to the Square of the third or greatest number.

Moreover, if three unequal numbers be desired to express the Quantities of the sides of a plain Triangle that shall have for the measure of one of its angles exactly 60 degrees, you may readily find them out by the help of the preceding Table: For the numbers answering to $s + r$, r, and q are the three numbers desired; so from the first rank of numbers in the Table, you may take

$$\left\{\begin{array}{l} 8 = s + r = AE, \\ 3 = \quad r = EC = EB = BC, \\ 7 = \quad q = AC. \end{array}\right.$$

Which three numbers 8, 3, 7 are the measures of the sides of a Triangle having one of its angles, (to wit, that oposite to 7 or q,) exactly 60 degrees.

Otherwise, without the help of the said Table, if two unequal numbers be taken at pleasure, as $2 = d$, and $1 = b$, then these three following numbers shall be the measures of the sides of a Triangle having one angle of 60 degrees, to wit, that opposite to the side $dd + db + bb$.

$$\left\{\begin{array}{l} 2db + dd = 8 = AE, \\ dd - bb = 3 = EC, \\ dd + db + bb = 7 = AC. \end{array}\right.$$

In which Triangle the Square of the side $dd + db + bb$ is equal to the summ of the Squares of the other two sides $2db + dd$ and $dd - bb$, less by the Rectangle (or Product) of the same two sides, as is evident by the following

Proof.

The Square of $2db + dd$ is . . . $>$ $4ddbb + 4bddd + dddd$
The Square of $dd - bb$ is . . . . $>$ $dddd - 2ddbb + bbbb$

The

The summ of those Squares is	$2ddbb + 4bddd + 2dddd + bbbb$
The Product of $2db + dd$ into $dd - bb$ is	$2bddd + dddd - 2dbbb - ddbb$
Which Product being subtracted from the said summ of the Squares, leaves . . .	$dddd + bbbb + 3ddbb + 2dbbb + 2bddd$

Which Remainder is the Square of $dd + db + bb$, as was affirmed.

QUEST. 57. (Quæst. 8. Lib. 5. *Diophant.*)

To find three right-angled Triangles in Rational numbers that shall have equal Area's.

RESOLUTION.

1. First, by either of the Canons in *Sect.* 16, 17. of the foregoing *Quest.* 56. find out two numbers capable of solving that Question, suppose . . . s the greater, r the lesser.
2. Therefore, (according to the import of the said *Quest.* 56.) the Squares of s and r, with the Product of s into r is equal to some Rational square number, let it be qq, whence . . . . . . $ss + sr + rr = qq$
3. Therefore by Construction in the first and second steps, ... $\sqrt{:ss + sr + rr:} = q$, (Rational.)
4. By the Canon in *Observat.* 8. *Resolut.* 2. *Quest.* 1. form a right-angled Triangle from $\sqrt{:ss + sr + rr:}$ (that is, q) and r, so the three sides will be expressible by these Rational numbers,

$$viz. \begin{cases} ss + sr + 2rr = \text{Hypothenusal}, \\ ss + sr = \text{Base}, \\ 2qr = \text{Perpendicular}. \end{cases}$$

5. In like manner form a second right-angled Triangle from $\sqrt{:ss + sr + rr:}$ (that is, q) and s, so the three sides will be expressible by these Rational numbers,

$$viz. \begin{cases} 2ss + sr + rr = \text{Hypothenusal}, \\ sr + rr = \text{Base}, \\ 2qs = \text{Perpendicular}. \end{cases}$$

6. Likewise, form a third right-angled Triangle from $\sqrt{:ss + sr + rr:}$ (that is, q) and $s + r$, so the three sides will be expressible by these Rational numbers,

$$viz. \begin{cases} 2ss + 3sr + 2rr = \text{Hypothenusal}, \\ sr = \text{Base}, \\ 2qs + 2qr = \text{Perpendicular}. \end{cases}$$

7. I say those three Triangles will solve this 57th Question: For first, by Construction they are right-angled Triangles; secondly, all the sides are expressible by Rational numbers, for s, r and q are Rational numbers by Construction; thirdly, if in every one of those three Triangles the Base be multiplied by the Perpendicular, every one of the three Products will manifestly be equal to $2qrss + 2qsrr$; therefore the halves of those Products, that is, the Area's of those three right-angled Triangles are equal to one another, as was required.
8. And since the foregoing *Quest.* 56. gives innumerable whole numbers answering to s, r and q, you may find out as many Ternions of right-angled Triangles in whole numbers as shall be desired to solve this 57th Question. But to have the nine sides of every Ternion expressible by whole numbers in the least Terms, every pair of the said numbers s, r must be the least Terms of a Reason, that is, two such numbers as have no common Divisor besides unity.
9. The premisses give the following Canon to find out innumerable Ternions of right-angled Triangles in whole numbers to solve the Question proposed, after the Rational whole numbers represented by s, r and q are first found out by either of the Canons of *Quest.* 56.

CANON.

$$\left.\begin{array}{l} ss + sr + 2rr = \text{h} = \text{Hypoth.} \\ ss + sr = \text{b} = \text{Base,} \\ 2qr = \text{p} = \text{Perp.} \end{array}\right\} \text{of Triangle I,}$$

$$\left.\begin{array}{r}2ss + sr + rr = h = \text{Hypoth.}\\ sr + rr = b = \text{Base,}\\ 2qs = p = \text{Perp.}\end{array}\right\}\text{ of Triangle II.}$$

$$\left.\begin{array}{r}2ss + 3sr + 2rr = \mathrm{H} = \text{Hypoth.}\\ sr = \mathrm{B} = \text{Base,}\\ 2qs + 2qr = \mathrm{P} = \text{Perp.}\end{array}\right\}\text{ of Triangle III.}$$

Observations upon the Canon *last aforegoing.*

Divers properties, besides the equality of Area's, in the three right-angled Triangles found out by the said Canon, do present themselves to your view, and are worthy of Observation. The principal Properties are these four, *viz.*

$$\begin{array}{l|l}1. & \mathrm{h} + \mathrm{b} = h + b.\\ 2. & \mathrm{H} + \mathrm{B} = 2\mathrm{b} + 2b.\\ 3. & \mathrm{H} - \mathrm{B} = h + b = \mathrm{h} + \mathrm{b}.\\ 4. & \mathrm{P} = \mathrm{p} + p.\end{array}$$

That is to say, in words,

1. The summ of the Hypothenusal and Base of the first Triangle, is equal to the summ of the Hypothenusal and Base of the second.

2. The summ of the Hypothenusal and Base of the third Triangle, is equal to the double-summ of the Bases of the first and second.

3. The excess of the Hypothenusal above the Base of the third Triangle is equal to the summ of the Hypothenusal and Base of the second, and likewise to the summ of the Hypothenusal and Base of the first.

4. The Perpendicular of the third Triangle is equal to the summ of the Perpendiculars of the first and second.

By the first of those three Triangles is meant that which hath the shortest Hypothenusal; by the second, that whose Hypothenusal is next greater than the shortest; and by the third, that which hath the longest Hypothenusal; in which order they are set in the Table. But the better to explain the Canon and Properties, I shall resume the Table belonging to *Quest.* 56. and call it *Table* I. whence six Answers in whole numbers to this *Quest.* 57. are deduced, and inserted in the following *Table* II.

Table I. *brought from* Quest. 56.

	d, b	*s, r*	*ss*	*rr*	*sr*	*qq*	*q*
1	2, 1	5, 3	25	9	15	49	7
2	3, 1	8, 7	64	49	56	169	13
3	3, 2	16, 5	256	25	80	361	19
4	4, 3	33, 7	1089	49	231	1369	37
5	5, 1	24, 11	576	121	264	961	31
6	5, 3	39, 16	1521	256	624	2401	49

Table II. *deduced from* Tab. I. *by the Canon in* Sect. 9. Quest. 57.

	h.	b.	p.	*h.*	*b*	*p.*	H.	B.	P.
1	58	40	42	74	24	70	113	15	112
2	218	120	182	233	105	208	394	56	390
3	386	336	190	617	105	608	802	80	798
4	1418	1320	518	2458	280	2442	2969	231	2960
5	1082	840	682	1537	385	1488	2186	264	2170
6	2657	2145	1568	3922	880	3822	5426	624	5390

The Construction of *Table* I. hath already been exprest in *Quest.* 56. whence the latter *Table* is deduced according to the Canon in *Sect.* 9. of this 57th Question, and contains six Answers to it; the first of which is to be understood thus,

In

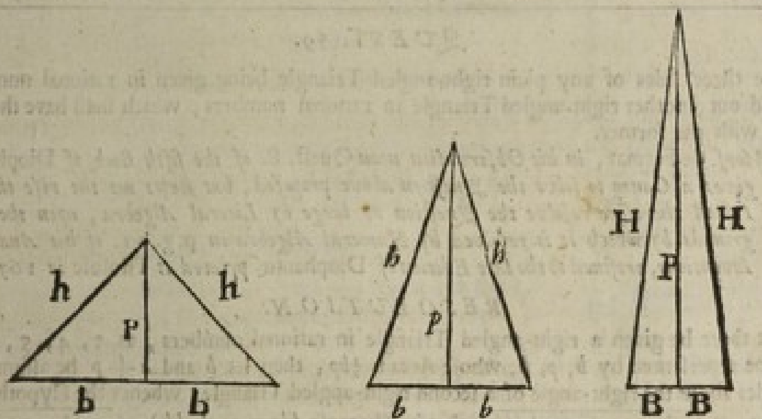
In the first rank, under . .	h , b , p.	*h* , *b* , *p*.	H , B , P.
you will find . . .	58 , 40 , 42.	74 , 24 , 70.	113 , 15 , 112.

Which three Triangles have equal Area's, and such other properties as before have been declared, *viz.* h + b = *h* + *b*, &c. as will easily appear by comparing the numbers answering to those Equations. The like is to be understood of the other five Answers.

QUEST. 58.

[*This is* Probl. 29. *in* pag. 131. *of the Introduction to* Algebra, *translated out of High* Dutch *into* English *in* 1668. *by* Tho. Brancker, *M. A.*]

To find three equicrural Triangles equal to one another in Area; and that the Perimeters of two of those Triangles may be equal to one another; and that the sides and Perpendiculars of every one of those three Triangles may be expressible by rational numbers.



RESOLUTION.

1. By the Canon in the ninth step of the Resolution of the foregoing *Quest.* 57. find out three right-angled Triangles in rational numbers, and equal to one another in Area; such are the three Triangles in any one of the six ranks of numbers in *Table* II. belonging to the said *Quest.* 57. for example, take those in the first rank,

viz.	h , b , p.	*h* , *b* , *p*.	H , B , P.
	58 , 40 , 42.	74 , 24 , 70.	113 , 15 , 112.

2. Then (as is evident by the Diagram belonging to this Question) the sides of the three equicrural Triangles desired shall be these following,

viz.	h , h , 2b.	*h* , *h* , 2*b*.	H , H , 2B.
	58 , 58 , 80.	74 , 74 , 48.	113 , 113 , 30.

3. And the Perpendiculars falling upon the Bases, *viz.* upon 2b, 2*b*, 2B of those three equicrural Triangles, are these, . . . p , *p* , P. — 42 , 70 , 112.

Which three equicrural Triangles in rational numbers above exprest in the second step will solve the Question, as will be evident by

The Proof.

By Construction in the first step the three right-angled Triangles h, b, p. | *h*, *b*, *p*. | H, B, P are equal to one another in Area, therefore their double Area's are equal to one another; but the said double Area's are the Area's of the three equicrural Triangles h, h, 2b. | *h*, *h*, 2*b*. | H, H, 2B, and therefore the Area's of those three equicrural Triangles are equal between themselves,

$$viz. \begin{cases} bp = bp = BP, \\ 40 \times 42 = 24 \times 70 = 15 \times 112 = 1680. \end{cases}$$

5. Moreover, by the first property in the Observations upon the Canon for resolving *Quest.* 57. this Equation is manifest,

$$viz. \begin{cases} h + b = h + b, \\ 58 + 40 = 74 + 24 = 98, \end{cases}$$

6. There-

6. Therefore the double of the first part of that Equation shall be equal to the double of the latter part,

$$viz. \begin{cases} 2\mathrm{h} + 2\mathrm{b} = 2h + 2b, \\ 116 + 80 = 148 + 48 = 196. \end{cases}$$

7. But the said double summs (if you view the Diagram belonging to this Question) are manifestly equal to the Perimeters of the two equicrural Triangles h, h, 2b and $h, h, 2b$, therefore those two equicrural Triangles are equal to one another in their Perimeters as well as in their Area's, and each Area is equal to the Area of the third equicrural Triangle H, H, 2B; also all their sides and Perpendiculars are exprest by rational numbers, as the Question required. In like manner five Answers more to this 58th Question may be collected from *Table* II. in *Quest.* 57. and 'tis evident from the premisses, that innumerable Ternions of equicrural Triangles in rational whole numbers may be found out to solve the said *Quest.* 58.

QUEST. 59.

The three sides of any plain right-angled Triangle being given in rational numbers; to find out another right-angled Triangle in rational numbers, which shall have the same Area with the former.

[*Monf.* de Fermat, *in his Observation upon* Quest. 8. *of the fifth Book of* Diophantus, *gives a Canon to solve the Question above proposed, but shews not the rise thereof; I shall therefore resolve the Question at large by Literal Algebra, upon the same grounds by which it is resolved by Numeral Algebra in* pag. 11. *of his Analytical Inventions, prefixed to the late Edition of* Diophantus *printed at* Tholose *in* 1670.]

RESOLUTION.

Let there be given a right-angled Triangle in rational numbers, as 3, 4, 5, which may be represented by b, p, h, whose Area is $\frac{1}{2}bp$; then let b and $a+p$ be assumed for the sides about the right-angle of a second right-angled Triangle, whence the Hypothenusal will be $\sqrt{:aa+2pa+pp+bb}:$ that is, (because $hh = pp+bb$) $\sqrt{:aa+2pa+hh}:$ and the Area of this latter Triangle is $\frac{1}{2}ba+\frac{1}{2}bp$: Now if this latter Area be divided by the former Area $\frac{1}{2}bp$; and if by the square Root of the Quotient $\frac{a}{p}+1$, *viz.* by $\sqrt{:\frac{a}{p}+1}:$ the three sides of the second Triangle be severally divided, the Quotients shall be the three sides of a third right-angled Triangle, whose Area is equal to the Area of the right-angled Triangle first given: For if the sides of the second right-angled Triangle, to wit, b, $a+p$ and $\sqrt{:aa+2pa+hh}:$ be severally divided by $\sqrt{:\frac{a}{p}+1}:$ the Quotients $\dfrac{b}{\sqrt{:\frac{a}{p}+1}:}$; $\dfrac{a+p}{\sqrt{:\frac{a}{p}+1}:}$ and $\dfrac{\sqrt{:aa+2pa+hh}:}{\sqrt{:\frac{a}{p}+1}:}$ are the sides of a third right-angled Trianlge whose Area $\dfrac{ba+bp}{\frac{2a}{p}+2}$ is equal to $\frac{1}{2}bp$ the Area of the first right-angled Triangle; (for $\frac{2a}{p}+2$ into $\frac{1}{2}bp$ makes $ba+bp$:) So that if $aa+2pa+hh$ and $\frac{a}{p}+1$ were square numbers, then the Question were solved: But how to make those two Algebraick quantities to be square numbers, the following Resolution shews.

1. Let there be given, as before, the three sides of a right-angled Triangle, as	3 : 4 : 5 b : p : h
2. Then for one of the sides about the right-angle of a second Triangle put	b
3. And for the other side about the right-angle, ...	$a+p$
4. Then because the summ of the Squares of those two sides must be equal to the Square of the Hypothenusal, the Square of the Hypothenusal of the second right-angled Triangle shall be	$aa+2pa+pp+bb$
5. That is, (because $hh = pp+bb$,)	$aa+2pa+hh$

6. From

6. From the ſecond and third ſteps the Area of the ſecond right-angled Triangle is . . . . $\frac{1}{2}ba + \frac{1}{2}bp$

7. Which Area divided by $\frac{1}{2}bp$, the Area of the firſt right-angled Triangle, gives the Quotient . . . . $\frac{a}{p} + 1$

8. Now according to the ſcope before-mentioned, each of the quantities in the fifth and ſeventh ſteps muſt be equated to a Square, ſo we are fallen upon this Duplicate equality, *viz.* $aa + 2pa + bb = \square$ . . . $\frac{a}{p} + 1 = \square$

9. In order to reſolve that Duplicate equality, I multiply the ſaid $\frac{a}{p} + 1$ by the Square bb, to the end there may be the ſame known Square bb in each Equation, ſo it produceth . . $\frac{bb}{p}a + bb = \square$

10. Then the difference of the ſaid $aa + 2pa + bb$ and $\frac{bb}{p}a + bb$, *viz.* the difference of the two Squares ſought is . . $aa + \frac{2pp - bb}{p}a$

11. Which difference is equal to the Product of the multiplication of theſe two quantities, to wit, ... $a + \frac{2pp - bb}{p}$ and a

12. The half-ſumm of the two quantities laſt mentioned is . . $a + \frac{2pp - bb}{2p}$

13. Then the Square of the ſaid half-ſumm being equated to $aa + 2pa + bb$, (aſſumed to be the greater of the two quantities in the eighth ſtep) gives this Equation, *viz.*

$$aa + 2pa + bb = aa + \frac{2pp - bb}{p}a + \frac{4pppp + bbbb - 4bbpp}{4pp}.$$

14. Which Equation, after due Reduction, gives . . . $a = \frac{pppp + \frac{1}{4}bbbb - 2bbpp}{bbp}$

15. Therefore from the ſecond, third and fourteenth ſteps the two ſides about the right-angle of the ſecond right-angled Triangle ſought are . . . . b and $\frac{pppp + \frac{1}{4}bbbb - bbpp}{bbp}$

16. The ſumm of the Squares of thoſe two ſides, by taking $bb + pp$ inſtead of the Factor bb, and $bbbb + 2bbpp + pppp$ inſtead of the Factor $bbbb$, will be found

$$\frac{\frac{1}{16}b^8 + \frac{1}{16}p^8 + \frac{12}{8}b^4p^4 + \frac{1}{4}b^6pp + \frac{1}{4}bbp^6}{b^4pp + 2bbp^4 + p^6}.$$

17. The ſquare Root of the ſumm of the Squares in the laſt ſtep is the Hypothenuſal of the ſecond right-angled Triangle ſought, to wit, ... $\frac{\frac{1}{4}bbbb + \frac{1}{4}pppp + \frac{1}{2}bbpp}{bbp + ppp}$

18. Which Hypothenuſal, by taking bb inſtead of $bb + pp$; and $bbbb$ inſtead of $bbbb + 2bbpp + pppp$, may be expreſt thus, ... $\frac{\frac{1}{4}bbbb + bbpp}{bbp}$

19. Therefore from the fifteenth and eighteenth ſteps, the three ſides of the ſecond right-angled Triangle ſought are theſe, to wit,

$$b\,,\ \frac{pppp + \frac{1}{4}bbbb - bbpp}{bbp} \text{ and } \frac{\frac{1}{4}bbbb + bbpp}{bbp}.$$

20. Therefore the Area of the ſaid ſecond right-angled Triangle is . . . . $\frac{ppppb + \frac{1}{4}bbbbb - bbppb}{2bbp}$

21. Which Area divided by $\frac{bp}{2}$ the Area of the right-angled Triangle firſt given, gives the Quotient . . $\frac{pppp + \frac{1}{4}bbbb - bbpp}{bbpp}$

22. The ſquare Root of that Quotient is . . . . $\frac{pp - \frac{1}{2}bb}{bp}$

23. By which ſquare Root, if the three ſides of the ſecond right-angled Triangle before expreſt in the nineteenth ſtep be ſeverally divided, the Quotients will be the three ſides of the third right-angled Triangle ſought, to wit,

1. $\frac{bbpb}{ppb - \frac{1}{2}bbb}$, (or $\frac{bpb}{pp - \frac{1}{2}bb}$.)
2. $\frac{pppp + \frac{1}{4}bbbb - bbpp}{ppb - \frac{1}{2}bbb}$.
3. $\frac{\frac{1}{4}bbbb + bbpp}{ppb - \frac{1}{2}bbb}$.

24. But

24. But $bbpp + pppp = hhpp$, therefore instead of subtracting $hhpp$ in the Numerator of the second of the three sides last before exprest, we may subtract $bbpp + pppp$, whence that side may be exprest thus $\frac{\frac{1}{2}hhhh - bbpp}{pph - \frac{1}{2}hhh}$, and consequently the three sides of the third right-angled Triangle sought shall be these, to wit,

$$\frac{hhpb}{pph - \frac{1}{2}hhh}, \quad \frac{\frac{1}{2}hhhh - bbpp}{pph - \frac{1}{2}hhh}, \quad \frac{\frac{1}{2}hhhh + bbpp}{pph - \frac{1}{2}hhh}.$$

25. Or, (to avoid Fractions) we may multiply the Numerator and Denominator of every one of the three sides last above exprest, by 4, so these three following sides (which are of the same value with those) will be produced for the third right-angled Triangle sought, to wit,

$$\frac{4hhpb}{4pph - 2hhh}, \quad \frac{hhhh - 4bbpp}{4pph - 2hhh}, \quad \frac{hhhh + 4bbpp}{4pph - 2hhh}$$

Hence ariseth,

CANON 1.

26. Let b, p, h represent the three sides of any right-angled Triangle given in rational numbers, whereof the Hypothenusal is h, and p the greater of the two sides about the right-angle; then from hh and $2bp$ form a right-angled Triangle, (by the Canon in *Observat.* 8. upon the Resolution by literal Algebra of the first Question of this Book;) that done, divide severally the three sides found out by $4pph - 2hhh$, so there will arise the three sides of a right-angled Triangle whose Area shall be equal to the Area of the given right-angled Triangle b, p, h.

27. Again, by supposition, $bb + pp = hh + 2pp$, and $p \sqsubset b$, therefore $pp - bb = 2pp - hh$; whence by multiplying each part into $2h$ it follows that $2pph - 2bbh = 4pph - 2hhh$: Therefore instead of the Divisor $4pph - 2hhh$ in *Canon* 1. we may take $2pph - 2bbh$, and so there will arise the following Canon, (which is the same with Monf. *Fermat*'s in the place before cited.)

CANON 2.

$$\frac{4hhpb}{2pph - 2bbh}, \quad \frac{hhhh - 4bbpp}{2pph - 2bbh}, \quad \frac{hhhh + 4bbpp}{2pph - 2bbh}.$$

In words thus,

28. Let b, p, h represent the three sides of any right-angled Triangle given in rational numbers, *viz.* h the Hypothenusal, p the greater side about the right-angle, and b the lesser; then from hh and $2bp$ form a right-angled Triangle, and divide severally the three sides found out by $2pph - 2bbh$, so the Quotients shall be the three sides of a right-angled Triangle whose Area is equal to the Area of the given right-angled Triangle b, p, h.

An Example in Numbers.

Let there be given the three sides of a right-angled Triangle in rational numbers, to wit, ...	b . p . h 3 . 4 . 5
Then by either of the Canons these three sides of a right-angled Triangle will be found out, to wit, ...	$\frac{49}{70}$, $\frac{1200}{70}$, $\frac{1201}{70}$

Which latter Triangle hath the same Area with the former, to wit, 6.

Again,

Let there be given this right-angled Triangle in numbers, to wit, ...	b . p . h 5 . 12 . 13
Then by either of the preceding Canons this right-angled Triangle will be found out, to wit, ...	$\frac{14161}{3094}$, $\frac{40560}{3094}$, $\frac{42961}{3094}$

Which latter Triangle hath the same Area with the former, to wit, 30.

From the premisses it is evident, that from any right-angled Triangle given in rational numbers another of the same Area may be found out; and from the second a third; and from the third a fourth, &c. all which right-angled Triangles shall have one common Area, to wit, that of the given right-angled Triangle, and all their sides expressible by rational numbers.

29. Moreover, from the nineteenth step of the preceding Resolution, several Canons may be deduced, to shew how by the help of any right-angled Triangle given in rational numbers,

numbers, to find out another, whose Area divided by the Area of the given Triangle will give the Quotient a square number. But I shall exhibit only this following

CANON.

Let h represent the Hypothenusal, p the greater side about the right-angle, and b the lesser in any right-angled Triangle given in rational numbers; then from hh and $2bp$ form a right-angled Triangle, and divide severally the three sides found out by $4hbp$; so there will come forth the three sides of a right-angled Triangle, whose Area divided by the Area of the given Triangle, will give the Quotient a square number whose Root is $\frac{pp - \frac{1}{2}bb}{hp}$ or $\frac{2pp - hh}{2hp}$.

An Example in Numbers.

Let there be given the three sides of a right-angled Triangle in numbers, to wit,	b . 3	p . 4	h . 5
Then by the last Canon, this right-angled Triangle will be found out, to wit,	3	$\frac{49}{400}$	$\frac{1201}{400}$
The Area of the latter Triangle is		$\frac{147}{800}$	
Which Area divided by 6 the Area of the given right-angled Triangle 3, 4, 5 gives a square number, to wit,		$\frac{49}{1600}$	
The square Root whereof is		$\frac{7}{40}$	

QUEST. 60. (Quæst. 12. Lib. 5. *Diophant.*)

To divide unity into two such parts, that if to each part a given number, suppose 6, be added, the two summs may be square numbers. But the summ of the double of the given number and unity must either be a square number, or else composed of two Squares.

RESOLUTION.

Let AB be 1, and AD = BE be the given number 6; therefore DE = 13. Now we must divide AB, to wit, unity, into two parts, suppose in the point G, that GD and GE may be square numbers; so that in effect we must divide DE, that is, 13 into two such Squares that one of them may be greater than 6, but less than 7. But that may be done by the fourth Question of this Book, where 13 is divided into the Squares $\frac{11449}{1681}$ and $\frac{10404}{1681}$, whose sides are $\frac{107}{41}$ and $\frac{102}{41}$, and each of those Squares is greater than 6, but less than 7; therefore taking each Square from 7, the remainders $\frac{318}{1681}$ and $\frac{1363}{1681}$, (that is GA and GB,) are the desired parts of unity: For if to each of those parts the given number 6 be added, the two summs will be the Squares $\frac{11449}{1681}$ and $\frac{10404}{1681}$.

It is also evident, that instead of unity, any number may be given to be divided; provided that the summ of this number and of the double of the other number given make a Square, or else a number composed of two Squares.

QUEST. 61. (Quæst. 13. Lib. 5. *Diophant.*)

To divide unity into two such parts, that to the one adding 2, (a number given,) and to the other 6, (another number given,) the summs may be square numbers. But the summ of the given numbers with unity must either make a square number, or else a number compos'd of two Squares.

RESOLUTION.

Let AB be 1, AD = 2. BE = 6; therefore DE = 9. Now we must divide AB, to wit, 1 into two parts, suppose in the point G, that GD and GE may be square numbers: So that in effect we must divide DE, that is, 9 into two such Squares that one may be greater than 2, but less than 3; or that one Square may be greater than 6, but less than 7. But the third Question of this Book shews how to find out two Squares so qualified, as the Square $\frac{7056}{2809}$ for GD, and the Square $\frac{18225}{2809}$ for GE; the sides of which

which Squares are $\frac{84}{53}$ and $\frac{135}{53}$: then ſubtracting 2 from the Square $\frac{7056}{2809}$, and 6 from the Square $\frac{18225}{2809}$, the remainders $\frac{1438}{2809}$ and $\frac{1371}{2809}$ are the deſired parts GA, GB of unity. For if the former part be added to 2, and the latter to 6, the ſumms are Squares, to wit, $\frac{7056}{2809}$ and $\frac{18225}{2809}$, as was required.

QUEST. 62. (Quæſt. 14. Lib. 5. *Diophant.*)

To divide unity into three ſuch parts, that if to every one of them a given number 3 be added, the three ſumms may be Squares. But the ſumm of the triple of the given number and unity muſt either be a ſquare number, or elſe a number compoſed of two or three Squares.

RESOLUTION.

It is eaſie to apprehend that the ſumm of the three Squares ſought makes 10, and that the ſcope of the ſearch muſt be to find out three Squares, every one of which may fall between 3 and 4, and that their ſumm may be 10; for then the three exceſſes of thoſe Squares above 3 will be the three deſired parts of unity. Firſt then, foraſmuch as 10 is compos'd of two Squares 9 and 1, I divide 10 into two other Squares whereof one may be greater than 3, but leſs than 4; ſuch are the Squares $3\frac{934}{1369}$ and $6\frac{435}{1369}$, whoſe ſides are $\frac{71}{37}$ and $\frac{93}{37}$; for the ſumm of thoſe Squares is 10, and the firſt Square $3\frac{934}{1369}$ is between 3 and 4: Then I take the Fraction $\frac{934}{1369}$, (to wit, the exceſs of the firſt Square above 3) for the firſt of the three deſired parts of unity; for if to that Fraction the given number 3 be added, it makes the Square $3\frac{934}{1369}$.

Then I divide the latter of the Squares firſt found, to wit, $6\frac{435}{1369}$, whoſe ſide is $\frac{93}{37}$, into two ſuch Squares that one may be greater than 3, but leſs than $3\frac{435}{1369}$, or leſs than $3\frac{1}{4}$; for if ſuch Square be leſs than $3\frac{1}{4}$, it will neceſſarily be leſs than $3\frac{435}{1369}$, becauſe $\frac{1}{4}$ is leſs than $\frac{435}{1369}$. But the ſides of two ſuch Squares are $\frac{1953}{1073}$ and $\frac{1860}{1073}$, therefore the Squares themſelves are $3\frac{360222}{1151329}$ and $3\frac{5613}{1151329}$; from each of which if 3 be ſubtracted, the remainders $\frac{360222}{1151329}$ and $\frac{5613}{1151329}$, with the Fraction $\frac{934}{1369}$ firſt found, that is, (in the ſame Denominator with the former) $\frac{785494}{1151329}$ ſhall be the three deſired parts of unity to ſolve the Queſtion.

After the ſame manner any number given inſtead of unity may be divided into three ſuch parts, that a number given being added to every one of them may make three Squares. But the ſumm of the number given to be divided, and the triple of the number to be added, muſt be either a Square, or a number compos'd of two or three Squares.

As, if it were deſired to divide 2 into three parts, that each part increaſed with 4 may make a Square: Firſt, foraſmuch as 14 the ſumm of 2 and the triple of 4 is compos'd of three Squares 1, 4 and 9; let 10 the ſumm of 9 and 1 (two of thoſe three Squares) be divided into two other Squares that the firſt may exceed 4 the number given to be added, but be leſs than 6; to the end the exceſs may be leſs than 2 the number given to be divided; then add the other of the two Squares found out to 4, (the other of the three Squares before mentioned, whoſe ſumm made 14,) and divide the ſumm into two ſuch Squares that each may be greater than 4; laſtly, from each of theſe two Squares laſt found out, as alſo from the firſt Square before found, ſubtract 4, ſo the Remainders ſhall be the deſired parts of 2. But the Operation I leave to the Learner's exerciſe.

QUEST. 63. (Quæſt. 15. Lib. 5. *Diophant.*)

To divide unity into three ſuch parts, that if the firſt be increaſed with 2, the ſecond with 3, and the third with 4, the three ſumms may be ſquare numbers. But the ſumm of the three numbers given and unity muſt either be a Square, or compos'd of two or three Squares.

RESOLUTION.

Firſt we muſt divide 10 (the ſumm of the three numbers given with unity) that the firſt may exceed 2, the ſecond 3, and the third 4. To which end, firſt, (by *Queſt.* 4. of this *Book*) divide 10 into two Squares that one may fall between 2 and 3; ſuch are the Squares $\frac{1849}{841}$ and $\frac{6561}{841}$, whoſe Roots are $\frac{43}{29}$ and $\frac{81}{29}$; then from the firſt Square $\frac{1849}{841}$ ſubtract 2, and take the Remainder $\frac{167}{841}$ for one of the deſired parts of unity.

It remains to divide the other Square $\frac{6561}{841}$ into two other Squares, that one may fall between

between 3 and 4: But two such Squares will be found $\frac{2893401}{707281}$ and $\frac{2624400}{707281}$; whose sides are $\frac{1701}{841}$ and $\frac{1620}{841}$; then from the latter of the said Squares subtract 3, and from the former, 4; so the two Remainders $\frac{502557}{707281}$ and $\frac{64277}{707281}$ with $\frac{167}{841}$ before found, that is, (in the same Denominator with the two former) $\frac{140447}{707281}$ shall be the three desired numbers, which will solve the Question. For first, their summ makes unity; moreover if 2 be added to $\frac{140447}{707281}$, that is, $\frac{167}{841}$, (the first number) the summ is the Square $\frac{1849}{841}$; and if 3 be added to the second number $\frac{502557}{707281}$, it makes the Square $\frac{2624400}{707281}$; lastly, if 4 be added to the third number $\frac{64277}{707281}$, it makes the Square $\frac{2893401}{707281}$.

QUEST. 64. (Quæst. 16. Lib. 5. *Diophant.*)

To divide a given number 10 into three numbers, that the summ of every two may be a Square; but the double of the given number must be either a Square, or else compos'd of two or three Squares.

RESOLUTION.

Forasmuch as the three numbers sought are to be such that the summ of the first and second must make a Square, also the summ of the first and third must make a Square, likewise the summ of the second and third must make a Square; therefore these three Squares are equal to the said three numbers twice taken. And because the summ of the three numbers is 10, therefore twice their summ, to wit, 20 shall be the summ of the three Squares. We must therefore divide 20 into three Squares, each of which may be less than 10; (for every one of these Squares must be equal to two of the three numbers, and consequently less than 10 the summ of all the three numbers.) But 20 is compos'd of two Squares 16 and 4; therefore we may take 4 (which is less than 10) for one of the three Squares sought, and then divide 16 into two Squares, one of which may fall between 10 and 6, for then the other will also be less than 10, (because both must make 16.) But (by *Quest.* 4. of this *Book*) the sides of two such Squares will be found $\frac{480}{169}$ and $\frac{476}{169}$; wherefore the three Squares sought are 4, $\frac{230400}{28561}$ and $\frac{226576}{28561}$, which subtracted severally from 10 leave three Remainders, 6, $\frac{55210}{28561}$ and $\frac{59034}{28561}$ for the three desired numbers, whose summ is 10, and every two of them added together makes a Square, as was desired.

But to make it more clearly evident that three numbers so found out will solve the Question, let bb, cc, dd represent three square numbers, then

Suppose . . . . $bb + cc + dd = 20$

And consequently . . . . $\frac{1}{2}bb + \frac{1}{2}cc + \frac{1}{2}dd = 10$

Then from 10, that is, from $\frac{1}{2}bb + \frac{1}{2}cc + \frac{1}{2}dd$ subtract bb, cc and dd severally, so the three Remainders are . . .
$$\left\{\begin{array}{l} \frac{1}{2}cc + \frac{1}{2}dd - \frac{1}{2}bb \\ \frac{1}{2}bb + \frac{1}{2}dd - \frac{1}{2}cc \\ \frac{1}{2}bb + \frac{1}{2}cc - \frac{1}{2}dd \end{array}\right.$$

I say those three Remainders shall be three numbers to solve the Question; for the summ of the first and second makes the Square dd, the summ of the second and third makes the Square bb, and the summ of the first and third makes the Square cc.

QUEST. 65. (Quæst. 17. Lib. 5. *Diophant.*)

To divide a given number, suppose 10, into four numbers, that the summ of every three may make a Square.

RESOLUTION.

Forasmuch as the summ of every three of the four numbers sought must make a Square, therefore the four Squares sought are equal to the four desired numbers thrice taken. But the four numbers thrice taken make 30, therefore 30 must be divided into four such Squares that every one of them may be less than 10, (for every one of the four Squares must be equal to three of the numbers sought, and consequently be less than 10 the summ of all four.) But 30 is compos'd of four Squares, 16, 9, 4 and 1; two of which, to wit, 9 and 4 may be taken for two of the Squares sought, and then 17 (the summ of the other two Squares 16 and 1) must be divided into two Squares that one may be less than 10, but greater than 7, and then the other will be also less than 10; but the sides of two such Squares will (by *Quest* 4. of this *Book*) be found $\frac{52}{17}$ and $\frac{47}{17}$, and the Squares themselves are $\frac{2704}{289}$ and $\frac{2209}{289}$, each of which is less than 10. Therefore the four Squares sought

fought are 9, 4, $\frac{2704}{289}$ and $\frac{2209}{289}$, which fubtracted feverally from 10, leave the Remainders 1, 6, $\frac{186}{289}$ and $\frac{681}{289}$ for the four numbers fought, whofe fumm makes 10, and every three of them added together make the Squares $\frac{2209}{289}$, 9, 4 and $\frac{2704}{289}$, whofe fides are $\frac{47}{17}$, 3, 2 and $\frac{52}{17}$.

Note. If the quadruple of the number given be a whole number, this Queftion may be extended to five numbers, or as many as you pleafe: for every whole number is compos'd of four Squares, which may be divided into any multitude of Squares within any poffible limits, by the help of the third Queftion of this Book.

QUEST. 66. (*A* Lemma, *ufed in the following* Queft. 67.)

To find three fuch Cube-numbers, that if from every one of them a number given, fuppofe 1, be fubtracted, the fumm of the Remainders may be a Square.

RESOLUTION.

1. For the fide of the firft Cube put $a +$ any abfolute number, as, . . . $a + 1$
2. Then take fome fquare number, as 9, and from its $\frac{1}{3}$, to wit, from 3 fubtract 1 the abfolute number in the fide of the firft Cube, and let the Remainder 2 be connected by $+$ with $- a$ for the fide of the fecond Cube; the reafon whereof will appear in *Obfervation* 1. upon this Queftion, ... $- a + 2$
3. Let the fide of the third Cube be any known number whofe Cube exceeds the number given in the Queftion, to wit, the fide . . . 2
4. Then the Cubes of thofe three fides are thefe, to wit,

1.	$aaa + 3aa + 3a + 1,$
2.	$- aaa + 6aa - 12a + 8,$
3.	. . . . $+ 8.$

5. From every one of which Cubes fubtract 1 the number given in the Queftion, and add the Remainders together, fo the fumm will be $9aa - 9a + 14$, which muft be equated to a Square; but the fide thereof muft be fo feigned that the value of a may be lefs than 2, to the end that $- a + 2$, or $2 - a$ the fide of the fecond Cube may be greater than nothing. Now to caufe that effect, the faid fide may be feigned $3a -$ any abfolute number between $\sqrt{14}$ and $\sqrt{32} + 6$, (which limits are found out after the method before delivered in divers Queftions of this Book;) let therefore the fide of the faid Square be feigned $3a - 4$, and then the Square of $3a - 4$ being equated to $9aa - 9a + 14$ above mentioned, this Equation arifeth, to wit,
$$9aa - 9a + 14 = 9aa - 24a + 16.$$
6. From which Equation, the value of a will be made known, *viz.* . . . $a = \frac{2}{15}$
7. Therefore from the fixth, firft, fecond and third fteps the fides of the three Cubes fought are $\frac{17}{15}$, $\frac{28}{15}$ and 2; wherefore the Cubes themfelves are $\frac{4913}{3375}$, $\frac{21952}{3375}$ and 8, which will folve the Queftion. For if from every one of thofe Cubes the given number 1 be fubtracted, the fumm of the three Remainders in its leaft terms is $\frac{324}{25}$, which is a Square, as was required.

Obfervations upon Queft. 66.

1. Two things are remarkable in the preceding pofitions for the fides of the Cubes fought; Firft, in the firft fide $a + 1$ there is put $+ a$, and in the fecond fide $- a + 2$ there is put $- a$, to the end that in adding together the Remainders mentioned in the fifth ftep, $+ aaa$ and $- aaa$ may deftroy one another: Secondly, the unities in the fecond fide muft be chofen with fuch Caution that the number prefixt to aa in the fumm of the faid three Remainders may be a Square, for if $9aa$ had not been a Square, then $9aa - 9a + 14$ could not have been equated to a Square. Therefore that the number prefixt to aa in the faid fumm of the Remainders may infallibly come forth a fquare number, the number of unities to be connected with $- a$ by $+$ in the fide of the fecond Cube, muft be the excefs of one third part of fome fquare number above the unit or unities in the fide of the firft Cube: The reafon whereof may be thus manifefted, Suppofe $a + b$, and $- a + d$, and 2, to be the fides of the three Cubes fought; then from the Cubes of thofe fides fubtracting the given number 1 feverally, the fumm of the Remainders is $3baa + 3daa + 3bba - 3dda + bbb + ddd + 5$; which fumm cannot be equated to a Square, unlefs the number fignified by $3b + 3d$, which is multiplied into aa be a fquare number;

number; ſuppoſe therefore $3b + 3d = ff$, whence by due Reduction, $d = \frac{1}{3}ff - b$; which ſhews that d the number of unities in $-a + d$ the ſide of the ſecond Cube, muſt be the exceſs of one third part of ſome ſquare number, above b the unit or unities in $a + b$ the feigned ſide of the firſt Cube, as was directed in the ſecond ſtep of the Reſolution.

2. It is eaſie to apprehend that this Queſtion may be extended to as many numbers as you pleaſe: For having put (as before) $a + 1$ for the ſide of the firſt Cube, and (for the reaſons before given) $-a + 2$ for the ſide of the ſecond Cube, you may put 2 for the ſide of a third Cube, 3 for the ſide of a fourth, &c. provided that the Cubes of theſe abſolute numbers 2, 3, &c. do every one of them exceed the number given to be ſubtracted; then from the ſumm of all thoſe Cubes ſubtracting the given number, the ſumm of the Remainders may be equated to a Square, becauſe by Conſtruction the number prefixt to aa is a Square.

QUEST. 67. (Queſt. 18. Lib. 5. *Diophant.*)

To find three numbers, that if they be ſeverally added to the Cube of their ſumm, the three ſumms made by thoſe additions may be Cubes.

RESOLUTION.

1. For the ſumm of the three numbers ſought put . . . a
2. Then the Cube of their ſumm is . . . . aaa
3. For the three numbers ſought put . . . . $7aaa$, $26aaa$ and $63aaa$
4. Whence it is manifeſt, that if each of the three laſt mentioned quantities aſſumed for the three numbers ſought, be increaſed with aaa which was put for the Cube of their ſumm, there will come forth the Cubes $8aaa$, $27aaa$ and $64aaa$; but the ſumm of theſe three quantities $7aaa$, $26aaa$ and $63aaa$ muſt be equal to a; therefore $96aaa = a$, and conſequently $96aa = 1$: where if 96 were a ſquare number, the value of a would be expreſſible by a rational number, and conſequently the Queſtion ſolved. Whence therefore comes 96? examine the Poſitions and you will find that by ſubtracting 1 (to wit, unity) from the three Cubes 8, 27 and 64, the remainders 7, 26 and 63 added together make 96. Therefore we muſt ſeek three ſuch Cubes that if 1 be ſubtracted from every one of them, the ſumm of the three remainders may be a Square: But the ſides of three ſuch Cubes are $\frac{17}{15}$, $\frac{28}{15}$ and 2, (found out by the preceding *Queſt.* 66.) and the Cubes themſelves are $\frac{4913}{3375}$, $\frac{21952}{3375}$ and 8, from every one of which if 1 be ſubtracted, the three remainders will be $\frac{1538}{3375}$, $\frac{18577}{3375}$ and 7, whoſe ſumm in its leaſt terms is the Square $\frac{324}{25}$, whoſe ſide is $\frac{18}{5}$. Now by the help of thoſe remainders, let the work be renewed thus, *viz.*
5. For the ſumm of the three numbers ſought put . . . a
6. Whence the Cube of their ſumm is . . . . aaa
7. Then for the firſt number put . . . . $\frac{1538}{3375}aaa$
8. For the ſecond . . . . $\frac{18577}{3375}aaa$
9. And for the third . . . . $7aaa$
10. Then the ſumm of the three numbers laſt expreſt is . . . $\frac{324}{25}aaa$
11. Which ſumm muſt be equal to a, which in the fifth ſtep was put for the ſumm of the three numbers ſought, therefore . $\frac{324}{25}aaa = a$
12. Which Equation, after due Reduction, diſcovers the ſumm of the three numbers ſought, *viz.* . . . . $a = \frac{5}{18}$

Therefore from the twelfth, ſeventh, eighth and ninth ſteps, the three numbers ſought will be $\frac{1538}{157464}$, $\frac{18577}{157464}$, $\frac{23625}{157464}$, whoſe ſumm is $\frac{5}{18}$, the Cube whereof is $\frac{125}{5832}$; which being added to every one of the ſaid three numbers, the ſumms will be Cubes, to wit, $\frac{4913}{157464}$, $\frac{21952}{157464}$, $\frac{27000}{157464}$, whoſe ſides are $\frac{17}{54}$, $\frac{28}{54}$ and $\frac{30}{54}$; therefore the Queſtion is ſatisfied, and by the help of the preceding *Queſt.* 66. may be extended to four, five or as many numbers as ſhall be deſired.

QUEST. 68.

To find two Cube-numbers, that if their difference be increaſed with a given number, ſuppoſe 2, it may make a Square, and that the ſide of the greater Cube may be leſs than 1 a number given.

RESO-

RESOLUTION.

1. For the ſide of the leſſer Cube ſought put . . . . a
2. For the ſide of the greater Cube put $a +$ one third part of ſome ſquare number; but ſuch third part muſt be leſs than 1 the preſcribed limit, therefore let the ſide of the greater Cube be . . $a + \frac{1}{3}$
3. Therefore the greater Cube is . . . . $aaa + aa + \frac{1}{3}a + \frac{1}{27}$
4. And the leſſer Cube is . . . . aaa
5. Therefore the difference of the ſaid Cubes is . . . . $aa + \frac{1}{3}a + \frac{1}{27}$
6. To which difference add 2 the number firſt given in the Queſtion, and the ſumm is . . . . $aa + \frac{1}{3}a + \frac{55}{27}$
7. Which ſumm muſt be equal to a Square, the ſide whereof muſt be ſo feigned that the the value of a may be leſs than $\frac{2}{3}$, for then $a + \frac{1}{3}$ (the ſide of the greater Cube) will be leſs than 1, as the Queſtion requires. Now to cauſe that effect, the ſide of the ſaid Square may be feigned to be either $a +$ any known number between $\frac{978}{1000}$ and $1\frac{427}{1000}$, or elſe $- a +$ any known number between $1\frac{428}{1000}$ and $2\frac{31}{100}$, (which limits may be found out by the method directed in the preceding *Queſt.* 12.) let therefore the ſaid ſide be feigned $a + \frac{4}{3}$, and then the Square of $a + \frac{4}{3}$ being equated to the ſumm in the ſixth ſtep, this Equation ariſeth, *viz.*

$$aa + \frac{8}{3}a + \frac{16}{9} = aa + \frac{1}{3}a + \frac{55}{27}.$$

8. Which Equation after due Reduction gives . . . . $a = \frac{1}{9}$
9. Therefore from the eighth, firſt and ſecond ſteps the ſides of the two Cubes ſought are . . . . $\frac{1}{9}$ and $\frac{4}{9}$
10. The Cubes of which ſides $\frac{1}{9}$ and $\frac{4}{9}$, *viz.* $\frac{1}{729}$ and $\frac{64}{729}$ will ſolve the Queſtion; for if to their difference $\frac{63}{729}$ you add 2, the ſumm $\frac{169}{81}$ is a Square, and $\frac{4}{9}$ the ſide of the greater Cube is leſs than 1, as was required.

Example. 2.

Let it be required to find two Cube-numbers whoſe difference added to 1458 (a given number) may make a Square, and that the ſide of the greater Cube may be leſs than 9 (a number given.)

Reſolution.

1. For the ſide of the leſſer Cube put . . . . a
2. For the ſide of the greater Cube put $a +$ one third part of ſome ſquare number, but ſuch third part muſt be leſs than 9 the limit above preſcribed; therefore let the ſaid ſide be . . $a + 3$
3. Therefore the greater Cube is . . . . $aaa + 9aa + 27a + 27$
4. And the leſſer Cube is . . . . aaa
5. Therefore the difference of thoſe Cubes is . . . . $+ 9aa + 27a + 27$
6. To which difference adding the number firſt given in this *Example* 2. to wit, ... 1458
7. The ſumm will be . . . . $+ 9aa + 27a + 1485$
8. Which ſumm muſt be equal to a Square, the ſide whereof muſt be ſo feigned that the value of a may be leſs than 6, for then $a + 3$ (which in the ſecond ſtep was aſſumed for the ſide of the greater Cube ſought) will be leſs than the preſcribed limit 9; Now to cauſe that effect, the ſide of the ſaid Square may be feigned to be $3a +$ any number between 26 and 39, or $- 3a +$ any number between 38 and 63; ſuppoſe therefore the ſaid ſide be feigned $3a + 36$, then the Square of $3a + 36$ being equated to the ſumm in the ſeventh ſtep, this Equation ariſeth, *viz.*

$$9aa + 216a + 1296 = 9aa + 27a + 1485.$$

9. Which Equation after due Reduction gives . . . . . . $a = 1$
10. Therefore from the ninth, firſt and ſecond ſteps of this ſecond Example, the ſides of the Cubes ſought are . . . 1 and 4
11. The Cubes of which ſides 1 and 4, *viz.* 1 and 64 will ſolve the Queſtion; for if their difference 63 be added to 1458 the number given in *Example* 2. it makes the Square 1521, whoſe ſide is 39; and 4 the ſide of the greater Cube is leſs than 9, as was required.

Example 3.

Example 3.

Again, the ſame numbers 1458 and 9 being given as before in *Example* 2. the ſide of the Square mentioned in the eighth ſtep may be feigned to be $-3a+48$, (which is within the limits there expreſt,) and then the Square of $-3a+48$ being equated to $9aa+27a+1485$, (before expreſt in the ſeventh ſtep,) after due Reduction the ſides of two Cubes to ſolve *Queſt.* 68. as it is before propos'd in *Example* 2. will be found $\frac{28}{5}$ and $\frac{13}{5}$; therefore the Cubes themſelves are $\frac{21952}{125}$ and $\frac{2197}{125}$, whoſe difference $\frac{19755}{125}$ added to the given number 1458 makes a Square, to wit, $\frac{40401}{25}$, whoſe ſide is $\frac{201}{5}$; and $\frac{28}{5}$ the ſide of the greater Cube is leſs than the preſcribed number 9.

QUEST. 69.

To find two ſuch cube-numbers, that if each of them be ſubtracted from a given ſquared cube-number, the ſumm of the remainders may be a Square.

RESOLUTION.

1. Let the given ſquared cube-number be . . . . $ddddddd$ or d^6.
2. The Root or ſide whereof is . . . . d
3. For the ſide of the firſt Cube ſought put . . . a
4. The Cube thereof is . . . . aaa
5. For the ſide of the other Cube ſought put $-a+$ the cube-root of the given ſquared Cube, *viz.* . . $-a+dd$
6. The Cube thereof is . . . . $-aaa+3ddaa-3d^4a+d^6$
7. Then by ſubtracting the Cube in the fourth ſtep from the given ſquared Cube in the firſt, there will remain $+d^6-aaa$
8. And by ſubtracting the Cube in the ſixth ſtep from the given ſquared Cube in the firſt, there will remain $+aaa-3ddaa+3d^4a$
9. The ſumm of thoſe remainders (in the ſeventh and eighth ſteps) is . . . . $-3ddaa+3d^4a+d^6$
10. Which ſumm muſt be equal to a Square, the ſide whereof (in regard d^6 is a Square) we may feign to be either $ea+ddd$, or $ea-ddd$, (where e repreſents a number yet unknown, and to be choſen according to the limit hereafter diſcovered:) Firſt then let the ſaid ſide be feigned $ea+ddd$, and then the Square of $ea+ddd$ being equated to the ſumm in the ninth ſtep, will give

$$-3ddaa+3d^4a+d^6 = eeaa+2ed^3a+d^6.$$

11. Which Equation after due Reduction gives . . $a = \frac{3d^4-2ed^3}{3dd+ee}$
12. Therefore from the eleventh, firſt, ſecond, third and fifth ſteps the ſides of the two Cubes ſought will be found equal to theſe quantities, *viz.* . . $\frac{3d^4-2ed^3}{3dd+ee}$ and $\frac{eedd+2ed^3}{3dd+ee}$
13. Again, foraſmuch as the ſide of the Square mentioned in the tenth ſtep may be feigned to be $ea-ddd$ (as well as $ea+ddd$,) let the ſaid ſide be $ea-ddd$, and then its Square being equated to the ſumm in the ninth ſtep, this Equation ariſeth, *viz.*

$$-3ddaa+3d^4a+d^6 = eeaa-2ed^3a+d^6.$$

14. Which Equation after due Reduction gives . . $a = \frac{3d^4+2ed^3}{3dd+ee}$
15. Therefore from the fourteenth, firſt, ſecond, third and fifth ſteps the ſides of the two Cubes ſought will be found equal to theſe quantities, *viz.* . . . $\frac{3d^4+2ed^3}{3dd+ee}$ and $\frac{ddee-2ed^3}{3dd+ee}$

The two quantities expreſt by letters in the twelfth ſtep will give

CANON 1.

16. Suppoſing d to be the ſide or Root of the ſquared cube-number given, take ſome known number, with this Caution, That its double may be leſs than the triple of d, and call the number taken e, then the ſides of the two Cubes ſought ſhall be theſe, *viz.*

$$\frac{3d^4-2ed^3}{3dd+ee} \text{ and } \frac{eedd+2ed^3}{3dd+ee}.$$

An

An Example in Numbers.

Let there be given any squared cube-number, as . . . . $\}$ $64 = d^6$
The Root or side whereof is . . . . $\}$ $2 = d$
Then take a number for e, according to the Caution in the *Canon*, as . $\}$ $2 = e$

Then by the *Canon* you will find $\left\{\begin{array}{l} \dfrac{3d^4 - 2ed^3}{3dd + ee} = 1 \\[2ex] \dfrac{eedd + 2ed^3}{3dd + ee} = 3 \end{array}\right\}$ the sides of the Cubes sought.

The Cubes of which sides 1 and 3, *viz.* 1 and 27 will solve the Question proposed; for if each of those Cubes be subtracted from the given squared Cube 64, the summ of the remainders 63 and 37 makes a Square, to wit, 100.

The two quantities exprest by letters in the fifteenth step will give

CANON 2.

17. Supposing d to be the side or Root of a given squared cube-number, take some known number with this Caution, That it be greater than the double of d, and call the number taken e, then the sides of the two Cubes sought shall be these, *viz.*

$$\frac{3d^4 + 2ed^3}{3dd + ee} \quad \text{and} \quad \frac{ddee - 2ed^3}{3dd + ee}.$$

An Example of Canon 2. *in Numbers.*

Let there be given any squared Cube, as . . . . $\}$ $1 = d^6$
The side or Root whereof is . . . . $\}$ $1 = d$
Then take a number for e, according to the Caution in *Canon* 2. as . $\}$ $4 = e$

Then by *Canon* 2. you will find $\left\{\begin{array}{l} \dfrac{3d^4 + 2ed^3}{3dd + ee} = \frac{11}{19} \\[2ex] \dfrac{ddee - 2ed^3}{3dd + ee} = \frac{8}{19} \end{array}\right\}$ the sides of the Cubes sought.

The Cubes of which sides $\frac{11}{19}$ and $\frac{8}{19}$, *viz.* $\frac{1331}{6859}$ and $\frac{512}{6859}$ will solve *Quest.* 69. for if each of those Cubes be subtracted from 1 the given squared Cube, the summ of the remainders makes a Square, to wit, $\frac{625}{361}$, whose side is $\frac{25}{19}$.

QUEST. 70.

To find three such cube-numbers, that if every one of them be subtracted from a given Cube, suppose 1, the summ of the three remainders may be a Square.

RESOLUTION.

1. First, by the foregoing *Quest.* 68. find two such cube-numbers, that their difference being added to 2, (the double of the Cube given in this Question,) the summ may be a Square, and that the greater of those two Cubes may be less than the given Cube 1. But two such Cubes are $\frac{64}{729}$ and $\frac{1}{729}$, whose sides are $\frac{4}{9}$ and $\frac{1}{9}$, (found out in the first *Example* of *Quest.* 68.) for if the difference of the said Cubes be added to 2, the summ is $\frac{169}{81}$, which is a square number whose side is $\frac{13}{9}$.
2. Then for the side of the first of the three Cubes sought put $a - \frac{4}{9}$, ($\frac{4}{9}$ being the side of the greater of the two Cubes found out in the first step,) . . . . $\}$ $a - \frac{4}{9}$
3. For the side of the second Cube sought put $-a + 1$, (1 being the side of the Cube given in the Question,) . . . $\}$ $-a + 1$
4. And let the side of the third Cube be the side of the lesser of the two Cubes found out in the first step, to wit, . . . $\}$ $\frac{1}{9}$
5. Then from the second step the first Cube will be . . . . $\}$ $aaa - \frac{12}{9}aa + \frac{16}{27}a - \frac{64}{729}$
6. And from the third step, the second Cube will be . . . . $\}$ $-aaa + 3aa - 3a + 1$
7. And from the fourth step the third Cube will be . . . . $\}$ $\frac{1}{729}$
8. Then according to the Question subtract those three Cubes severally from the given Cube 1, so the three remainders shall be these, to wit,

1.	$-aaa + \frac{12}{9}aa - \frac{16}{27}a + \frac{793}{729}$,
2.	$+aaa - 3aa + 3a$,
3.	$+\frac{728}{729}$.

9. The

9. The summ of the said three remainders is . . . . . $-\frac{5}{3}aa + \frac{65}{27}a + \frac{169}{81}$

10. Which summ must be equal to a Square, whose side, to the end the value of a may be greater than $\frac{4}{9}$, but less than 1, (as the Positions in the second and third steps do require) may be feigned to be either $\frac{13}{9}$ + any number of a between $\frac{237}{1000}a$ and $\frac{533}{1000}a$, or else $\frac{13}{9}$ — any number of a greater than $3\frac{126}{1000}a$, (which limits may be found out by the method before delivered in *Quest.* 13. of this *Book*;) suppose therefore the said side to be $\frac{1}{3}a + \frac{13}{9}$, then the Square of $\frac{1}{3}a + \frac{13}{9}$ being equated to the summ of the three remainders in the ninth step, this Equation ariseth, *viz.*

$$-\frac{5}{3}aa + \frac{65}{27}a + \frac{169}{81} = \frac{1}{9}aa + \frac{26}{27}a + \frac{169}{81}.$$

11. Which Equation after due Reduction will give . . $a = \frac{13}{16}$

12. Therefore from the eleventh, second, third and fourth steps, the sides of the three Cubes sought will be these, $\frac{53}{144}$, $\frac{27}{144}$, and $\frac{16}{144}$

13. The Cubes of which sides are these . . . . . $\frac{148877}{2985984}$, $\frac{19683}{2985984}$, $\frac{4096}{2985984}$

Which three Cubes will solve the Question, as will appear by

The Proof.

By subtracting severally the said three Cubes found out from 1, (the Cube given in the Question,) the three remainders will be these,

$$\frac{2837107}{2985984},\quad \frac{2966301}{2985984},\quad \frac{2981888}{2985984}.$$

The summ of those remainders is . . . . $\frac{8785296}{2985984}$

Which summ reduced to its least terms by the common Divisor 144, will be . . . . $\frac{61009}{20736}$

Which $\frac{61009}{20736}$ is a Square, whose side is $\frac{247}{144}$; therefore the Question is solved, and (as is evident by the tenth step) capable of innumerable Answers, the positions in the second, third and fourth steps standing unaltred.

Observations upon the preceding Resolution *of* Quest. 70.

1. The chief scope in the said Resolution is, to form the positions for the sides of the three Cubes sought in such manner, that when the said Cubes are severally subtracted from the Cube given in the Question, there may be a possibility of equating the summ of the three remainders to a Square, which summ (as you see in the ninth step) is $-\frac{5}{3}aa + \frac{65}{27}a + \frac{169}{81}$, which could not be equated to a Square if $\frac{169}{81}$ were not an affirmative square number; I shall therefore shew how the said $\frac{169}{81}$ doth necessarily become an affirmative square number by the preceding Operation.

2. If the subtraction of every one of the three feigned Cubes in the fifth, sixth and seventh steps from the given Cube 1, as also the adding of the remainders together be well examined, it will appear, that by adding the Cube $\frac{64}{729}$ to 1, and by subtracting the Cube $\frac{1}{729}$ from 1, and then by adding that summ and remainder together, their summ is $\frac{169}{81}$, which (in regard by Construction the greater of the said Cubes, to wit, $\frac{64}{729}$ is added to 1, and the lesser $\frac{1}{729}$ is subtracted from 1) is the same with the summ that will arise by adding the difference of those Cubes unto 2, (the double of 1.) For if the greater of two numbers be added unto, and the lesser be subtracted from a third number, the summ and remainder added together will make the same summ that ariseth by adding the difference of those two numbers to the double of the said third number: But by Construction in the first step of the Resolution, the said Cubes $\frac{64}{729}$ and $\frac{1}{729}$ are found such that their difference added to 2 makes a Square, to wit, $\frac{169}{81}$. Whence it is manifest that the Algebraick quantity $-\frac{5}{3}aa + \frac{65}{27}a + \frac{169}{81}$ is capable of being equated to a Square, and that variously, as you see in the tenth step of the Resolution.

Example 2.

Let it be required to find three such Cube-numbers, that if every one of them be subtracted from a given Cube, suppose 729, the summ of the three remainders may be a square number.

Resolution.

1. First, by *Quest.* 68. find two such Cubes that if their difference be added to 1458, to wit, the double of the given Cube 729, the summ may be a Square, and that the side of the greater of those two Cubes may be less than 9 the side of the given Cube 729: But two such Cubes are 64 and 1, (found out in the second *Example* of *Quest.* 68.)

for if their difference 63 be added to the prescribed number 1458, the summ 1521 is a Square whose side is 39.

2. Then for the side of the first of the three Cubes sought let there be put $a - 4$, (4 being the side of the Cube 64, the greater of the two Cubes found out in the first step,) . . . $a - 4$
3. For the side of the second Cube sought put $-a + 9$, (9 being the side of the given Cube 729,) . . . $-a + 9$
4. Let the side of the third Cube be 1, to wit, the side of the lesser of the two Cubes found out in the first step, ... 1
5. Then (according to the Question) subtract severally the Cubes of those three sides (assumed in the three last steps) from the given Cube 729, and add the three remainders together, so the summ will be

$$-15aa + 195a + 1521.$$

6. Which summ must be equal to a Square, whose side, to the end the value of a may be greater than 4, but less than 9, as the second and third steps require, may be feigned to be either $39 +$ any number of a between $\frac{218}{1000}a$ and $1\frac{222}{1000}a$, or else $39 -$ any number of a between $9\frac{122}{1000}a$ and $21\frac{33}{1000}a$, (which limits may be found out by the method delivered in *Quest.* 13. of this Book.) Suppose therefore the said side be feigned $39 + a$, then the Square thereof being equated to the summ in the fifth step, this Equation will arise, to wit,

$$aa + 78a + 1521 = -15aa + 195a + 1521.$$

7. Which Equation after due Reduction gives . . . . $a = \frac{117}{16}$
8. Therefore from the seventh, second, third and fourth steps the sides of the three Cubes sought are these, . . $\frac{53}{16}$, $\frac{27}{16}$ and 1 (or $\frac{16}{16}$)
9. And consequently the Cubes themselves are . . . $\frac{148877}{4096}$, $\frac{19683}{4096}$, $\frac{4096}{4096}$ (or 1)

Which three Cubes will solve the Question, as will be evident by

The Proof.

By subtracting every one of the said three Cubes in the ninth step from the given Cube 729, the three remainders will be these,

$$\frac{2837107}{4096},\quad \frac{2966301}{4096},\quad \frac{2981888}{4096}.$$

The summ of those remainders is . . . . $\frac{8785296}{4096}$

Which summ being reduced to its least terms by the common Divisor 16, will be . . . . $\frac{549081}{4096}$

Which is a Square, whose side is . . . . $\frac{741}{16}$

Example 3.

1. Again, the same things remaining as before in the second Example from the first to the sixth step, we may feign the side of the Square mentioned in the said sixth step to be $39 - 10a$, and then the Square of $39 - 10a$ being equated to $-15aa + 195a + 1521$ will give $a = \frac{195}{23}$.
2. Therefore from the second, third and fourth steps of *Example* 2. the sides of three other Cubes to solve the Question propounded in the said second Example will be found these, to wit, $\frac{103}{23}$, $\frac{12}{23}$, and 1 (or $\frac{23}{23}$.)
3. And consequently the Cubes themselves are

$$\frac{1092727}{12167},\quad \frac{1728}{12167},\quad 1 \ (\text{or } \frac{12167}{12167}.)$$

4. Which three Cubes being severally subtracted from the given Cube 729, the summ of the three remainders in its least terms will be $\frac{1108809}{529}$, which is a Square, whose side is $\frac{1053}{23}$, as was required in *Example* 2.

QUEST. 71.

[*Another way of solving the preceding* Quest. 70. *when the given Cube is a squared Cube, or the Cube of a Square.*]

Let it be required to find three Cube-numbers, such, that if every one of them be subtracted from a given squared Cube-number, suppose 64, the summ of the three remainders may be a Square.

RESO-

RESOLUTION.

1. Firſt, by the foregoing *Queſt.* 69. find two ſuch cube-numbers, that if each of them be ſubtracted from the given ſquared Cube 64, the ſumm of the remainders may be a Square; ſuch are the Cubes 1 and 27, (found out in the Example of *Canon* 1. of the ſaid *Queſt.* 69.) for if each of them be ſubtracted from 64, the ſumm of the remainders makes the Square 100.
2. Then for the ſide of the firſt of the three Cubes ſought put $a +$ either of the ſides of the two Cubes found out in the firſt ſtep, *viz.* . . . . $a + 1$
3. For the ſide of the ſecond Cube put $-a + 4$, (4 being the ſide of the given ſquared Cube 64,) . . . . . $-a + 4$
4. Let the ſide of the third Cube, be the ſide of the other of the two Cubes found out in the firſt ſtep, to wit, . . 3
5. Therefore from the ſecond ſtep the firſt Cube is . . . $aaa + 3aa + 3a + 1$
6. And from the third ſtep, the ſecond Cube is . . . . $-aaa + 12aa - 48a + 64$
7. And from the fourth ſtep, the third Cube is . . . . $+27$
8. Then (according to the Queſtion) ſubtract thoſe three Cubes ſeverally from the given ſquared Cube 64, ſo the three remainders will be theſe, to wit,

$$\begin{array}{l|l} 1. & -aaa - 3aa - 3a + 63, \\ 2. & +aaa - 12aa + 48a, \\ 3. & \quad\quad\quad\quad\quad + 37. \end{array}$$

9. The ſumm of which remainders is $-15aa + 45a + 100$.
10. Which ſumm is to be equated to a Square, but the ſide thereof muſt be ſo feigned that the value of a may be leſs than 3, to the end the ſide $a + 1$ in the ſecond ſtep may be leſs than 4, becauſe the Cube of $a + 1$ muſt be ſubtracted from the Cube of 4. Now to cauſe that effect, the ſide of the ſaid Square may be feigned $10 +$ any number of a leſs than $2\frac{1}{4}a$, or $10 -$ any number of a greater than $6\frac{2}{3}a$; (which limits may be found out by the method before delivered in *Queſt.* 13. of this Book:) Let therefore the ſaid ſide be feigned $a + 10$, then the Square thereof, to wit, $aa + 20a + 100$ being equated to the ſumm of the remainders in the ninth ſtep, this Equation ariſeth, *viz.*

$$aa + 20a + 100 = -15aa + 45a + 100.$$

11. Which Equation, after due Reduction, gives . . . $a = \frac{25}{16}$
12. Thererefore from the eleventh, ſecond, third and fourth ſteps, the ſides of the three Cubes ſought will be found theſe, to wit, ... $\frac{41}{16}$, $\frac{39}{16}$ and 3
13. Therefore the Cubes themſelves are . . . . $\frac{68921}{4096}$, $\frac{59319}{4096}$ and 27

Which three Cubes will ſolve the Queſtion, as will be evident by

The Proof.

By ſubtracting the ſaid three Cubes ſeverally from 64, (the ſquared Cube given in the Queſtion,) the three remainders will be theſe,

$$\frac{193223}{4096},\quad \frac{202825}{4096},\quad \frac{151552}{4096}.$$

The ſumm of thoſe remainders is . . . . $\frac{547600}{4096}$

Which ſumm being reduced to its leaſt terms by the common Diviſor 16, will be . . . . $\frac{34225}{256}$.

Which $\frac{34225}{256}$ is a Square whoſe ſide is $\frac{185}{16}$, therefore the Queſtion is ſolved.

Example. 2.

Let it be required to find three ſuch Cube-numbers, that if every one of them be ſubtracted from 1, the ſumm of the three remainders may be a Square.

Reſolution.

1. Firſt, by the preceding *Queſt.* 69. find two ſuch Cube-numbers that if each of them be ſubtracted from 1, to wit, the given ſquared Cube, the ſumm of the three remainders may be a Square; ſuch are the Cubes $\frac{1}{64}$ and $\frac{27}{64}$, for if each of them be ſubtracted from 1, the ſumm of the remainders will be $\frac{25}{16}$, which is a Square.

2. Then

2. Then for the side of the first of the three Cubes sought put $a +$ the side of one of the two Cubes found out in the first step, to wit, . . . $a + \frac{1}{4}$

3. And for the side of the second Cube put $-a + 1$, (1 being the side of the given squared Cube,) . . . . $-a + 1$

4. And let the side of the third Cube be the side of the other of the two Cubes found out in the first step, to wit, ... $\frac{3}{4}$

5. Then (according to the Question) subtract severally the Cubes of those three sides assumed in the three last foregoing steps from 1, (the given squared Cube,) and add the three remainders together, so the summ will be

$$-\tfrac{15}{4}aa + \tfrac{45}{16}a + \tfrac{100}{64} \text{ (or } \tfrac{25}{16}.)$$

6. Which summ must be equal to a Square, whose side must be so feigned that the value of a may be less than $\frac{3}{4}$, to the end the side $a + \frac{1}{4}$ in the second step of this second Example may be less than 1, for then every one of the three remainders of the subtraction mentioned in the fifth step will be greater than nothing. Now to cause that effect, the side of the said Square may be feigned $\frac{5}{4} +$ any number of a less than $\frac{9}{8}a$, or else $\frac{5}{4} -$ any number of a greater than $\frac{10}{3}a$; therefore let the said side be feigned $\frac{1}{2}a + \frac{5}{4}$, then the Square thereof being equated to the summ of the three remainders in the fifth step, from that Equation you will find

$$a = \tfrac{25}{64}.$$

7. Therefore from the sixth, second, third and fourth steps, the sides of the three Cubes sought will be $\frac{41}{64}$, $\frac{39}{64}$ and $\frac{48}{64}$.

8. And consequently the Cubes themselves are

$$\tfrac{68921}{262144}, \quad \tfrac{59319}{262144}, \quad \tfrac{110592}{262144}.$$

Which three Cubes will solve the Question before propounded in *Example* 2. as will be manifest by

The Proof.

By subtracting every one of the said three Cubes in the eighth step from 1, (the given squared Cube, the three remainders will be these, to wit,

$$\tfrac{193223}{262144}, \quad \tfrac{202825}{262144}, \quad \tfrac{151552}{262144}.$$

The summ of those remainders is . . . . $\frac{547600}{262144}$

Which summ being reduced to its least terms by the common Divisor 16, will be . . . . $\frac{34225}{16384}$

Which is a Square, whose side is $\frac{185}{128}$; therefore the Question is solved.

Example 3.

1. The same things being supposed as in the first step of *Example* 2. we may vary the sides assumed in the second, third and fourth steps, in this manner, *viz.*

2. Instead of . . . . $\left\{ \begin{array}{c} a + \frac{1}{4}, \\ -a + 1, \\ \frac{3}{4}, \end{array} \right\}$ we may assume $\left\{ \begin{array}{c} a + \frac{3}{4}. \\ -a + 1. \\ \frac{1}{4}. \end{array} \right.$

3. Then by subtracting severally the Cubes of the three sides above-exprest on the right hand from 1, and adding the three remainders together, the summ will be

$$-\tfrac{21}{4}aa + \tfrac{21}{16}a + \tfrac{25}{16}.$$

4. Which summ must be equal to a Square, whose side must be so feigned that the value of a may be less than $\frac{1}{4}$, to the end the side $a + \frac{3}{4}$ may be less than 1. Now to cause that effect we may feign the said side to be $\frac{5}{4} +$ any number of a less than $\frac{21}{40}a$, or $\frac{5}{4} -$ any number of a greater than $10a$; suppose therefore the said side be feigned $\frac{1}{2}a + \frac{5}{4}$, then the Square thereof being equated to the summ of the three remainders before mentioned in the third step, from that Equation you will find

$$a = \tfrac{1}{88}.$$

5. Therefore from those three assumed sides which are placed on the right hand in the second step of this third Example, the sides of the three Cubes sought will be these, to wit,

$$\tfrac{67}{88}, \quad \tfrac{87}{88} \text{ and } \tfrac{22}{88}.$$

6. And consequently the Cubes themselves are

$$\tfrac{300763}{681472}, \quad \tfrac{658503}{681472}, \quad \tfrac{10648}{681472}.$$

Which three Cubes will solve the Question before-propos'd in *Example* 2. for if every one of them be subtracted from the given squared Cube 1, the summ of the three remainders in its least terms will be $\frac{48841}{30976}$, which is a Square whose side is $\frac{221}{176}$.

QUEST. 72.

QUEST. 72.

To find four cube-numbers, ſuch, that if they be ſeverally ſubtracted from a given ſquared cube-number, ſuppoſe 64, the ſumm of the four remainders may be a ſquare number.

RESOLUTION.

1. By the foregoing *Queſt.* 71. find three ſuch Cubes, that if they be ſeverally ſubtracted from the given ſquared Cube 64, the ſumm of the three remainders may be a Square; ſuch are the Cubes $\frac{68921}{4096}$, $\frac{59319}{4096}$ and 27, whoſe ſides are $\frac{41}{16}$, $\frac{39}{16}$ and 3, (found out in *Example* 1. of *Queſt.* 71.) for if the ſaid Cubes be ſeverally ſubtracted from 64, the ſumm of the three remainders in its leaſt terms will be $\frac{34225}{256}$, which is a Square, whoſe ſide is $\frac{185}{16}$.
2. Then for the ſide of the firſt of the four Cubes ſought put $a+3$, (which 3 is the ſide of one of the three Cubes found out in the firſt ſtep,) . . . . $a+3$
3. For the ſide of the ſecond Cube put $-a+4$, (4 being the Cube-root of the given ſquared Cube 64,) . . $-a+4$
4. Let the ſide of the third Cube be another of the ſides of the ſaid three Cubes found out in the firſt ſtep, *viz.* . . . . $\frac{41}{16}$
5. And let the ſide of the fourth Cube be the ſide of the remaining Cube in the firſt ſtep, to wit, ... $\frac{39}{16}$
6. Therefore (from the ſecond ſtep) the firſt Cube ſought is $aaa+9aa+27a+27$
7. The ſecond Cube is . . . . $-aaa+12aa-48a+64$
8. The third Cube is . . . . $\frac{68921}{4096}$
9. And the fourth Cube is . . . . $\frac{59319}{4096}$
10. Then thoſe four Cubes being ſeverally ſubtracted from the given ſquared Cube 64, the four remainders will be theſe, to wit,

1.	$-aaa-9aa-27a+\frac{151552}{4096}$,
2.	$+aaa-12aa+48a$,
3.	. . . . $\frac{193223}{4096}$,
4.	. . . . $\frac{202825}{4096}$.

11. The ſumm of thoſe remainders is $-21aa+21a+\frac{34225}{256}$.
12. Which ſumm muſt be equated to a Square, the ſide whereof muſt be ſo feigned that the value of a may be leſs than 1, to the end the ſide $a+3$ in the firſt ſtep may be leſs than 4, (becauſe the Cube of the ſaid $a+3$ muſt be ſubtracted from the Cube of 4,) and if the value of a be leſs than 1, it will be much leſs than 4, as the third ſtep requires. Now to cauſe a to be leſsthan 1, the ſide of the feigned Square may be $\frac{185}{16}+$ any number of a leſs than $\frac{168}{185}a$, or elſe $\frac{185}{16}-$ any number of a greater than $23\frac{1}{8}a$, (which limits may be diſcovered by the method in *Queſt.* 13. of this Book,) therefore we may feign the ſaid ſide to be $\frac{1}{2}a+\frac{185}{16}$, whoſe Square being equated to the ſumm of the four remainders in the eleventh ſtep, this Equation ariſeth, *viz.*
$$\tfrac{1}{4}aa+\tfrac{185}{16}a+\tfrac{34225}{256}=-21aa+21a+\tfrac{34225}{256}.$$
13. Which Equation duly reduced will give . . . . . $a=\frac{151}{340}$
14. Therefore from the thirteenth, ſecond, third, fourth and fifth ſteps, the ſides of the four Cubes ſought are diſcovered to be theſe, to wit,
$$\frac{4684}{1360},\quad \frac{4836}{1360},\quad \frac{3485}{1360},\quad \frac{3315}{1360}.$$
15. And conſequently the four Cubes are theſe, to wit,
$$\frac{102766285504}{2515456000},\quad \frac{113099029056}{2515456000},\quad \frac{42326109125}{2515456000},\quad \frac{36429280875}{2515456000}.$$

Which four Cubes will ſolve the Queſtion propos'd, as will be manifeſt by

The Proof.

By ſubtracting thoſe four Cubes ſeverally from the given ſquared Cube 64 (or $\frac{160989184000}{2515456000}$,) the four remainders will be
$$\frac{58222898496}{2515456000},\quad \frac{47890154944}{2515456000},\quad \frac{118663074875}{2515456000},\quad \frac{124559903125}{2515456000}.$$

The ſumm of thoſe four remainders is . . . . $\frac{349336031440}{2515456000}$

Which ſumm being reduced to its leaſt terms by the common Diviſor 1360, will be . . . . $\frac{256864729}{1849600}$

Which is a Square, whoſe ſide is . . . . $\frac{16027}{1360}$

Therefore

Therefore the Queſtion is ſolved ; and if the method of reſolving this and the preceding *Queſt.* 71. be well examined, it will not be difficult to apprehend how to find out as many Cubes as ſhall be deſired, which being ſeverally ſubtracted from a given ſquared cube-number, or from the Cube of a given Square, the ſumm of the remainders may be a Square.

QUEST. 73.

To find two numbers, that if each of them be ſubtracted from the Cube of their ſumm, the remainders may be Cubes.

RESOLUTION.

Firſt, making choice of ſome ſquare number as 4, I put $4a$ for the ſumm of the two numbers ſought, the Cube whereof will be $64aaa$; then for the firſt number I put $56aaa$, and for the other number $37aaa$, for theſe two being ſeverally ſubtracted from $64aaa$, the remainders will be $8aaa$ and $27aaa$, which are manifeſtly Cubes, whereby one part of the Queſtion is ſatisfied : It remains that the ſumm of the ſaid aſſumed numbers $56aaa$ and $37aaa$ be equal to $4a$, *viz.* $93aaa = 4a$, whence by dividing each part by a, there ariſeth $93aa = 4$. Now if the ſaid 93 were a ſquare number, then the value of a would be a rational number, and conſequently the Queſtion ſolved.

But 93 not being a Square, we muſt enquire whence it ariſeth, and by examining the Operation it will appear, that the two Cubes 8 and 27 having been ſubtracted ſeverally from 64, the remainders are 56 and 37, the ſumm whereof makes 93 before mentioned. So that our firſt ſcope muſt be to find two ſuch Cubes that if each of them be ſubtracted from the ſquared Cube 64, the ſumm of the remainders may be a Square : But ſuch are the Cubes 1 and 27, (found out by *Canon* 1. of the foregoing *Queſt.* 69.) for if each of them be ſubtracted from 64, the ſumm of the remainders 63 and 37 makes the Square 100 : therefore I begin the Reſolution a-new thus ; *viz.*

1. For the ſumm of the two numbers ſought I put . . . . $4a$
2. The Cube thereof is . . . . $64aaa$
3. Then for one of the two numbers ſought I put . . . . $63aaa$
4. And for the other number ſought I put . . . . $37aaa$
5. Which two numbers being ſeverally ſubtracted from $64aaa$ the remainders will be Cubes, to wit, ... aaa and $27aaa$
6. But the ſumm of the numbers aſſumed in the third and fourth ſteps muſt be equal to $4a$ in the firſt ſtep, therefore . . . . . . $100aaa = 4a$
7. Which Equation duly reduced, gives . . . . $a = \frac{1}{5}$
8. Therefore from the ſeventh, third and fourth ſteps the two numbers ſought are . . . . $\frac{63}{125}$ and $\frac{37}{125}$

Which numbers will ſolve the Queſtion propos'd, as will be manifeſt by

The Proof.

9. The ſumm of the two numbers found out, to wit, $\frac{63}{125}$ and $\frac{37}{125}$, is . $\frac{4}{5}$
10. Therefore the Cube of their ſumm is . . . . $\frac{64}{125}$
11. From which $\frac{64}{125}$ ſubtract each of the numbers $\frac{63}{125}$ and $\frac{37}{125}$, ſo the remainders are Cubes, to wit, ... $\frac{1}{125}$ and $\frac{27}{125}$

Another Example.

1. Firſt, I take ſome ſquare number, as 1, then I ſearch out two ſuch Cubes that each of them being ſubtracted from 1, (the Cube of the Square firſt taken,) the ſumm of the remainders may be a Square ; ſuch are the Cubes $\frac{1331}{6859}$ and $\frac{512}{6859}$, whoſe ſides are $\frac{11}{19}$ and $\frac{8}{19}$, found out in the Example of *Canon* 2. *Queſt.* 69.) for if each of thoſe Cubes be ſubtracted from 1, the ſumm of the remainders $\frac{5528}{6859}$ and $\frac{6347}{6859}$ will be $\frac{11875}{6859}$, or in its leaſt terms $\frac{625}{361}$, which is a Square whoſe ſide is $\frac{25}{19}$; then by the help of thoſe Cubes and remainders I form the Reſolution as before in the firſt Example, *viz.*
2. For the ſumm of the two numbers ſought I put a or $1a$, (1 being the ſquare number firſt taken,) . . . . $1a$
3. The Cube of the ſaid ſumm is . . . . $1aaa$
4. Then for one of the two numbers ſought I put . . . . $\frac{5528}{6859}aaa$
5. And for the other number ſought, ... $\frac{6347}{6859}aaa$
6. Which two numbers being ſeverally ſubtracted from $1aaa$, the remainders will be Cubes, to wit, ... $\frac{1331}{6859}aaa$ and $\frac{512}{6859}aaa$

7. But

7. But the ſumm of the two numbers in the fourth and fifth ſteps muſt be equal to $1a$ in the ſecond ſtep, therefore . . . . . $\frac{625}{361}aaa = 1a$
8. Which Equation duly reduced gives . . . . $a = \frac{19}{25}$
9. Therefore from the eighth, fourth and fifth ſteps, the two numbers ſought, in their leaſt terms, will be found . . . . $\frac{5528}{15625}$ and $\frac{6347}{15625}$

Which numbers will ſolve the Queſtion propos'd, as will be evident by

The Proof.

The ſumm of the two numbers found out in the ninth ſtep, to wit, of $\frac{5528}{15625}$ and $\frac{6347}{15625}$ is in the leaſt terms . . . . $\frac{19}{25}$
The Cube of that ſumm is . . . . $\frac{6859}{15625}$
From which Cube if you ſubtract ſeverally the ſaid numbers found out, the remainders will be the Cubes of theſe ſides, to wit, . . . $\frac{11}{25}$ and $\frac{8}{25}$

QUEST. 74. (Quæſt. 19. Lib. 5. *Diophant.*)

To find three ſuch numbers, that if every one of them be ſubtracted from the Cube of their ſumm, the three remainders may be Cubes.

[*The text of* Diophantus *in the Reſolution of this Queſtion is ſo obſcure, that it affords not any ſatisfactory Anſwer; I ſhall therefore ſhew how to ſolve it by two different ways of my own, by the latter of which this Queſtion may be extended to four, five, or as many numbers as ſhall be deſired.*]

RESOLUTION.

1. Firſt, take any ſquare number, as 1, then ſearch out three ſuch Cubes that if they be ſeverally ſubtracted from the Cube of the ſaid ſquare number 1, the ſumm of the three remainders may be a Square: But three ſuch Cubes are $\frac{148877}{2985984}$, $\frac{19683}{2985984}$ and $\frac{4096}{2985984}$, whoſe ſides are $\frac{53}{144}$, $\frac{27}{144}$ and $\frac{16}{144}$; (found out in *Example* 1. of the preceding *Queſt.* 70.) for if thoſe Cubes be ſeverally ſubtracted from 1 (or unity,) the ſumm of the three remainders $\frac{2837107}{2985984}$, $\frac{2966301}{2985984}$ and $\frac{2981888}{2985984}$ will be $\frac{61009}{20736}$, which is a Square, whoſe ſide is $\frac{247}{144}$. Now by the help of thoſe three preparatory Cubes and remainders, I proceed thus,
2. For the ſumm of the three numbers ſought I put a, or $1a$, (1 being the ſquare number firſt taken,) . . . . $1a$
3. Therefore the Cube of the ſaid ſumm is . . . . $1aaa$
4. Then for the firſt of the three numbers ſought I put $\frac{2837107}{2985984}aaa$, (the ſaid $\frac{2837107}{2985984}$ being one of the three remainders before mentioned in the firſt ſtep,) . . . . $\frac{2837107}{2985984}aaa$
5. In like manner having multiplied the ſecond remainder into aaa, I put the Product for the ſecond number ſought, to wit, ... $\frac{2966301}{2985984}aaa$
6. Likewiſe multiplying the third remainder into aaa, I put the Product for the third number ſought, to wit, ... $\frac{2981888}{2985984}aaa$
7. Which three numbers in the three laſt ſteps being ſeverally ſubtracted from $1aaa$, the three remainders will (by the Conſtruction in the firſt ſtep) be Cubes, to wit,
$$\frac{148877}{2985984}aaa,\quad \frac{19683}{2985984}aaa,\quad \frac{4096}{2985984}aaa.$$
8. But the ſumm of the three numbers in the fourth, fifth and ſixth ſteps muſt be equal to $1a$ in the ſecond ſtep, whence this Equation ariſeth,
$$\frac{61009}{20736}aaa = 1a.$$
9. Which Equation duly reduced gives . . . . $a = \frac{144}{247}$
10. Therefore from the ninth, ſecond, fourth, fifth and ſixth ſteps, the three numbers ſought will be made known, to wit, theſe,
$$\frac{2837107}{15069223},\quad \frac{2966301}{15069223},\quad \frac{2981888}{15069223}.$$

Which three numbers will ſolve the Queſtion, as will be evident by

The Proof.

The ſumm of the ſaid three numbers is $\frac{8785296}{15069223}$, which reduced to it ſmalleſt terms by the common Diviſor 61009, makes . . . . . . $\frac{144}{247}$
The Cube of the ſaid ſumm is . . . . $\frac{2985984}{15069223}$

From

From which Cube ſubtracting ſeverally the three numbers found out in the tenth ſtep, the remainders will be theſe three Cubes, to wit,

$$\frac{148877}{15069223},\quad \frac{19683}{15069223},\quad \frac{4096}{15069223}.$$

The ſides of which Cubes are theſe, *viz.*

$$\frac{53}{247},\quad \frac{27}{247},\quad \frac{16}{247}.$$ Therefore the Queſtion is ſolved.

But becauſe the operation in finding out the three numbers, as alſo the three Cubes with their ſides as aforeſaid will be exceeding laborious by reaſon of long fractions, unleſs ſome Compendiums be uſed, I ſhall give a Canon deducible from the premiſſes to leſſen the work, reſpect being firſt had to theſe following

Preparatory Directions.

1. Firſt, the Rules for multiplying and dividing Fractions in *Sect.* 22, 26. of *Chap.* 6. *Book* 1. muſt be diligently obſerved, that the Products and Quotients may come out in the ſmalleſt terms.

2. Secondly, when one, two or more numbers are to be ſeverally multiplied by ſome number, and the Products are to be ſeverally divided by the ſame number, that multiplication and diviſion may be quite omitted, for the numbers firſt propos'd to be multiplied will be the ſame with the Quotients that ariſe by the ſaid multiplication and diviſion. Moreover, when one, two or more numbers are to be ſeverally multiplied by ſome number, and the Products are to be divided by ſome number greater or leſs than that multiplying number, reduce the ſaid Multiplicator and Diviſor into the leaſt terms (when they are not ſuch already) by their greateſt common Diviſor, and take the Quotients for a new Multiplicator and Diviſor inſtead of thoſe firſt preſcribed: As, if 41, 39 and 48 be to be ſeverally multiplied by 32, and the Products be to be ſeverally divided by 16, I firſt reduce the ſaid 32 and 16 to the ſmalleſt terms in the ſame Reaſon by the common Diviſor 16, ſo the Quotients or new terms will be 2 and 1; then multiplying 41, 39 and 48 ſeverally by 2, (inſtead of 32,) and dividing the Products ſeverally by 1, (inſtead of 16,) the Quotients will be 82, 78 and 96, which are found out much ſpeedier and in ſmaller terms than thoſe that would be found out by multiplying the ſaid 41, 39 and 48 by 32, and dividing the Products by 16 as was firſt preſcribed. This Rule will oftentimes be very uſeful in the fourth branch of the following Canon.

3. Thirdly, let the ſquare number firſt taken in the firſt ſtep of the foregoing Reſolution of this *Queſt.* 74. be called bb, and its ſide b.

4. Fourthly, let the other ſquare number which is equal to the ſumm of the three remainders found out in the ſaid firſt ſtep of the Reſolution be called cc, and its ſide c.

Theſe things premiſed, I proceed to the

CANON.

1. Divide the known number b by the known number c, and call the Quotient a, which is now a known number.

2. Divide the Cube of b by c, and let the Quotient be called d, which known number is the ſumm of the numbers ſought by the Queſtion.

3. Reduce the numbers a and d to their ſmalleſt common Denominator.

4. Reduce likewiſe the ſides of the preparatory Cubes (found out in the firſt ſtep of the Reſolution) to their ſmalleſt common Denominator, then multiply ſeverally the Numerators of thoſe ſides by the Numerator of the number a, and divide the Products ſeverally by the ſaid common Denominator of the ſides of the ſaid preparatory Cubes, and reſerve the Quotients for Dividends.

5. Divide ſeverally thoſe Dividends reſerved, by the Denominator of a or d, (for theſe were above reduced to a common Denominator,) ſo ſhall the Quotients be the ſides of the Cubes ſought.

6. Laſtly, by ſubtracting ſeverally the Cubes of the ſides laſt found out, from the Cube of the ſumm of the numbers ſought, (which ſumm was above found by the ſecond ſtep of the Canon,) the remainders ſhall be the numbers ſought, and the ſmalleſt that have a common Denominator with the Cubes found out in the fifth ſtep of the Canon.

This Canon with the preceding preparatory Directions may be practically illuſtrated by the Examples of the preceding *Queſt.* 73. and of this and the following 75 and 76 Queſtions.

Example 2.

Example 2.

Let it be required to find three such numbers, that if every one of them be subtracted from the Cube of their summ, the three remainders may be Cubes.

1. First take some square number, as 9, then find three such Cubes, that if they be severally subtracted from 729 (the Cube of the said Square 9) the summ of the three remainders may be a Square: But three such Cubes are $\frac{148877}{4096}$, $\frac{19683}{4096}$ and $\frac{4096}{4096}$, (or 1,) whose sides are $\frac{53}{16}$, $\frac{27}{16}$ and 1, (found out in the second Example of *Quest.* 70.) for if those Cubes be severally subtracted from 729, (or $\frac{2985984}{4096}$,) the summ of the three remainders $\frac{2837107}{4096}$, $\frac{2966301}{4096}$ and $\frac{2981888}{4096}$ being reduced to its least terms will be $\frac{549081}{256}$, which is a square number whose side is $\frac{741}{16}$.

2. Then by proceeding according to the foregoing preparatory Directions and Canon, the numbers and Cubes sought will be found to be the same as were before found out in the first Example of this 74^{th} Question.

Example 3.

1. Taking again the same square number 9 as in the second Example, I seek three other Cubes, that every one of them being subtracted from 729 (the Cube of the said Square 9) the summ of the three remainders may be a Square: But three such Cubes are [illegible], [illegible] and 1, whose sides are [illegible], [illegible] and 1, (or $\frac{23}{23}$,) found out in the third Example of *Quest.* 70. for if those three Cubes be severally subtracted from the said 729, the summ of the three remainders in its least terms will be [illegible], which is a Square, whose side is [illegible].

2. Then by proceeding according to the preparatory Directions and the Canon, (which follow the first Example of this 74^{th} Question,) the sides of the three Cubes sought will be found [illegible], [illegible] and [illegible], and the three numbers sought are these, to wit, [illegible], [illegible] and [illegible], whose summ in its least terms is [illegible], from the Cube whereof if the said three numbers be severally subtracted, the three remainders will be Cubes, whose sides are those above found out. Therefore the Question is solved.

QUEST. 75. (*Another way of solving the preceding* Quest. 74.)

To find three such cube-numbers, that if every one of them be subtracted from the Cube of their summ, the remainders may be Cubes.

RESOLUTION.

1. First take some square number, as 4, then find three such Cubes that if they be severally subtracted from 64, (the Cube of the said Square 4,) the summ of the three remainders may be a Square: But three such Cubes are $\frac{68921}{4096}$, $\frac{59319}{4096}$ and 27, whose sides are $\frac{41}{16}$, $\frac{39}{16}$ and 3, (found out in the first Example of *Quest.* 71.) for if those Cubes be severally subtracted from the said 64, the three remainders will be these, to wit, $\frac{193223}{4096}$, $\frac{202825}{4096}$ and $\frac{151552}{4096}$, whose summ in its least terms is $\frac{34225}{256}$, which is a Square whose side is $\frac{185}{16}$.

2. Then by proceeding according to the preparatory Directions and Canon which follow the first Example of the preceding *Quest.* 74. the sides of the three Cubes sought will be found these, to wit,

$$\frac{82}{185}, \quad \frac{78}{185}, \quad \frac{96}{185}.$$

3. And consequently the Cubes themselves are

$$\frac{551368}{6331625}, \quad \frac{474552}{6331625}, \quad \frac{884736}{6331625}.$$

4. And the three numbers sought are these,

$$\frac{1545784}{6331625}, \quad \frac{1622600}{6331625}, \quad \frac{1212416}{6331625}.$$

Which will solve the Question, as will be manifest by

The Proof.

5. The summ of the said three numbers is $\frac{4380800}{6331625}$, which reduced to its smallest terms by the common Divisor 34225, gives . . . . $\frac{128}{185}$.

6. The Cube of the said summ is . . . . $\frac{2097152}{6331625}$

From which Cube if you subtract severally the three numbers before found out in the fourth step, the remainders will be the three Cubes above exprest in the third step.

Another Example.

1. First take some square number, as 1, then find three such Cubes that if they be severally subtracted from 1 (the Cube of the Square first taken) the summ of the three remainders may be a Square: But three such Cubes are $\frac{300763}{681472}$, $\frac{658503}{681472}$ and $\frac{10648}{681472}$, whose sides are $\frac{67}{88}$, $\frac{87}{88}$ and $\frac{22}{88}$, (found out in the third Example of *Quest.* 71.) for if those Cubes be severally subtracted from 1, (the Cube of the Square first taken,) the summ of the three remainders $\frac{380709}{681472}$, $\frac{22969}{681472}$ and $\frac{670824}{681472}$ being reduced to its least terms will be $\frac{48841}{30976}$, which is a Square whose side is $\frac{221}{176}$.

2. Then by proceeding according to the preparatory Directions and Canon (which follow the first Example of the preceding *Quest.* 74.) the sides of the three Cubes sought will be found these, to wit, $\frac{[illegible]}{221}$, $\frac{[illegible]}{221}$ and $\frac{[illegible]}{221}$; and the three numbers sought will be these, to wit, $\frac{[illegible]}{10793861}$, $\frac{[illegible]}{10793861}$ and $\frac{[illegible]}{10793861}$, which will solve the said 75th Question, as will appear by the Proof.

QUEST. 76.

To find four such numbers, that if every one of them be subtracted from the Cube of their summ, the four remainders may be Cubes.

RESOLUTION.

1. First take some square number, as 4, then find four such Cubes that if they be severally subtracted from 64 (the Cube of the said Square 4) the summ of the four remainders may be a Square: But four such Cubes are these $\frac{[illegible]}{2515456000}$, $\frac{[illegible]}{2515456000}$, $\frac{[illegible]}{2515456000}$ and $\frac{[illegible]}{2515456000}$, whose sides are $\frac{[illegible]}{1360}$, $\frac{[illegible]}{1360}$, $\frac{[illegible]}{1360}$ and $\frac{[illegible]}{1360}$, (found out in the Example of the said fifth Question;) for if those Cubes be severally subtracted from the said 64, the four remainders will be these, to wit, $\frac{[illegible]}{2515456000}$, $\frac{[illegible]}{2515456000}$, $\frac{[illegible]}{2515456000}$ and $\frac{[illegible]}{2515456000}$; which remainders being added together, the summ in its least terms is $\frac{256704484}{1849600}$, which is a Square whose side is $\frac{16022}{1360}$.
2. Then by proceeding according to the preparatory Directions and Canon, (which follow the first Example of the foregoing *Quest.* 74.) the sides of the four Cubes sought will be found these, to wit,

$$\frac{9368}{16027},\ \frac{9672}{16027},\ \frac{6970}{16027} \text{ and } \frac{6630}{16027}.$$

3. Therefore the four Cubes sought are these,

$$\frac{822130284032}{4116771011683},\ \frac{904792232448}{4116771011683},\ \frac{338608873000}{4116771011683},\ \frac{291434247000}{4116771011683}.$$

4. Which Cubes being severally subtracted from the Cube of $\frac{10880}{16027}$, (which by the second branch of the Canon will be found out for the summ of the four numbers sought,) the remainders will be the numbers sought, to wit, these,

$$\frac{465783187968}{4116771011683},\ \frac{949304599000}{4116771011683},\ \frac{383121239552}{4116771011683},\ \frac{996479225000}{4116771011683}.$$

Which four numbers solve the Question, as will appear by the Proof.

From the manner of solving this and the preceding 75th Question, it is easie to apprehend, how five, six or as many numbers as shall be desired, may be found out, which being severally subtracted from the Cube of their summ, may leave as many Cubes: But so many numbers as are desired, so many preparatory Cubes must be first found out, such, that if they be severally subtracted from the Cube of some square number chosen at pleasure, the summ of the remainders may be a Square; which preparatory Cubes may be found out by the method before delivered in *Quest.* 71, and 72.

QUEST. 77. (Quæst. 20. Lib. 5. *Diophant.*)

To find three such numbers that if the Cube of their summ be subtracted from every one of them, the remainders may be Cubes.

RESOLUTION.

1. For the summ of the three numbers sought put . . . . . a
2. And let the three numbers be . . . . $2aaa$, $9aaa$, $28aaa$
3. It remains that their summ $39aaa$ be equated to a, whence $39aa = 1$; where if 39 were a Square the Question would be solved by Rational numbers. But 39 is not

not a Square, whence therefore is it produced? Examine the Poſitions, and you will find that 1 being added ſeverally to the three Cubes 1, 8 and 27, the ſumm of thoſe three additions makes 39. We muſt therefore ſearch out three Cubes whoſe ſumm increaſed with 3 may make a Square, to which end

4. For the ſides of the three Cubes put . . . . e; $-e+3$; and 1
5. Then the ſumm of the Cubes of thoſe three ſides increaſed with 3, makes . . . . $9ee-27e+31$
6. Which ſumm is to be equated to a Square, but the ſide thereof muſt be ſo feigned that the value of e may be leſs than 3; now to cauſe that effect the ſide may be variouſly feigned within limits eaſie to be diſcovered from the method in divers preceding Queſtions of this Book, let it be $3e-7$, then the Square of $3e-7$ being equated to $9ee-27e+31$ will give $e=\frac{6}{5}$; therefore the ſides of the three Cubes are $\frac{6}{5}$, $\frac{9}{5}$ and 1, and the Cubes themſelves are $\frac{216}{125}$, $\frac{729}{125}$ and 1, by the help whereof the work is to be renewed thus, *viz.*
7. Add 1 to every one of the three Cubes before found, and the ſumms will be $\frac{341}{125}$, $\frac{854}{125}$ and $\frac{250}{125}$; then inſtead of $2aaa$, $9aaa$ and $28aaa$ (in the ſecond ſtep) put for the three numbers ſought,

$$\tfrac{341}{125}aaa\ ,\quad \tfrac{854}{125}aaa\ ,\quad \tfrac{250}{125}aaa.$$

8. Then the ſumm of thoſe three numbers being equated to a, (which in the firſt ſtep was put for the ſumm of the three numbers ſought,) gives this Equation, to wit, ... $\frac{1445}{125}aaa=a$
9. Whence, after due Reduction, ... $a=\frac{5}{17}$

Therefore from the ninth and ſeventh ſteps the three numbers ſought are $\frac{341}{4913}$, $\frac{854}{4913}$ and $\frac{250}{4913}$; for if from every one of them, the Cube of their ſumm, to wit, $\frac{125}{4913}$ be ſubtracted, there will remain the Cubes $\frac{216}{4913}$, $\frac{729}{4913}$, $\frac{125}{4913}$, whoſe ſides are $\frac{6}{17}$, $\frac{9}{17}$, $\frac{5}{17}$.

From the premiſſes it is evident that the Queſtion is capable of innumerable Anſwers, and may eaſily be extended to four, five, or as many numbers as you pleaſe.

QUEST. 78. (Quæſt. 21. Lib. 5. *Diophant.*)

To find three ſuch numbers that their ſumm may be a Square, and that if to the Cube of the ſaid ſumm the three numbers be ſeverally added, the three ſumms may be ſquare numbers.

RESOLUTION.

1. For the ſumm of the three numbers ſought, that it may be a Square, put . . . . aa
2. Then for the firſt number put . . . . $3aaaaaa$
3. For the ſecond, ... $8aaaaaa$
4. And for the third, ... $15aaaaaa$
5. Whence it is evident that every one of them added to the Cube of their ſumm makes a Square; But the ſumm of the three numbers (in the ſecond, third and fourth ſteps) muſt be equal to aa which was firſt put for their ſumm; therefore $26aaaaaa=aa$, and conſequently, (by dividing each part by aa,) $26aaaa=1$. In which laſt Equation if 26 were a ſquared ſquare number, the value of a would be a rational number: Whence therefore comes 26? Examine the Poſitions, and you will find that 'tis the ſumm of the three numbers 3, 8 and 15, every one of which increaſed with 1 makes a Square; therefore the ſcope of our ſearch muſt be to find three numbers, every one of which increaſed with 1 may make a Square, and that the ſumm of the ſaid three numbers may be a Biquadrate: To which end let the three numbers be $aaaa-2aa$; $aa+2a$ and $aa-2a$; for every one of theſe increaſed with 1 makes a Square, and their ſumm makes a Biquadrate, to wit, $aaaa$; and 'tis evident the value of a may be any number greater than 2, (for the third number $aa-2a$ ſhews that aa muſt be greater than $2a$, and conſequently a greater than 2.) Suppoſe therefore $a=3$, whence the three numbers $aaaa-2aa$; $aa+2a$; $aa-2a$, will be 63, 15 and 3; now with the help of theſe three numbers the work may be renewed thus, *viz.*
6. Let the ſumm of the three numbers ſought be (as before) . aa
7. And the firſt number . . . . $63aaaaaa$
8. The ſecond . . . . $15aaaaaa$
9. And the third . . . . $3aaaaaa$

 10. There-

10. Therefore the summ of the three numbers is . . . . . > $81aaaaaa$
11. Which summ must be equal to aa, *viz.* . . . . . . > $81aaaaaa = aa$
12. Which Equation after due Reduction gives . . . . . > . . . $a = \frac{1}{3}$

Therefore from the twelfth, seventh, eighth and ninth steps the three numbers sought are $\frac{63}{729}$, $\frac{15}{729}$ and $\frac{3}{729}$, which will solve the Question; for their summ is a Square, to wit, $\frac{1}{9}$. Also the Cube of the said summ is $\frac{1}{729}$, to which if the three numbers be severally added there will come forth three square numbers, $\frac{64}{729}$, $\frac{16}{729}$ and $\frac{4}{729}$, for their sides are $\frac{8}{27}$, $\frac{4}{27}$ and $\frac{2}{27}$.

QUEST. 79. (Quæst. 22. Lib. 5. *Diophant.*)

To divide a given number, suppose 2, into three such numbers, that every one of them subtracted from the Cube of their summ, *viz.* from 8, may leave a Square.

RESOLUTION.

Forasmuch as every one of the three numbers sought being less than 2 is to be subtracted from 8, each remainder shall be greater than 6, but less than 8; and the summ of the three remainders, to wit, the three Squares sought makes 22: for the summ of the three desired numbers, to wit, 2, subtracted from three times 8 leaves 22; we must therefore divide 22 into three such Squares that every one of them may be greater than 6, but less than 8. But 22 is composed of three Squares 9, 9 and 4; therefore, first, (by *Quest* 4. of this Book) let 13 (the summ of 9 and 4) be divided into two Squares, that one may be between 6 and 8, such are the Squares $\frac{4761}{625}$ and $\frac{3364}{625}$, whose sides are $\frac{69}{25}$ and $\frac{58}{25}$; then the lesser of those Squares, to wit, $\frac{3364}{625}$ added to 9 makes $\frac{8989}{625}$, which must also be divided into two Squares that each may be between 6 and 8; but by the said *Quest.* 4. the sides of two such Squares will be found $\frac{12398}{4525}$ and $\frac{11865}{4525}$, which with $\frac{69}{25}$, (that is, $\frac{12489}{4525}$) before found are the sides of the three Squares sought; therefore the three Squares themselves are $\frac{153710404}{20475625}$, $\frac{140778225}{20475625}$ and $\frac{155975121}{20475625}$, whose summ makes 22, and every one of them is greater than 6, but less than 8; therefore those three Squares severally subtracted from 8, leave the three desired parts of 2, to wit, $\frac{10094596}{20475625}$, $\frac{23026775}{20475625}$ and $\frac{7829879}{20475625}$.

QUEST. 80. [*This is the* 12th *of the* 4th *Book of* Vieta's *Zetericks.*]

To find three right-angled Triangles in rational numbers, that the Solid of the Perpendiculars may be to the Solid of the Bases as a square number to a square number.

Note. By the Solid of three numbers is meant the Product made by their multiplication one into another, as, the Solid of 2, 3 and 4 is 24, that is, $2 \times 3 \times 4$.

RESOLUTION.

1. Let h, b, p represent the Hypothenusal, Base and Perpendicular of any right-angled Triangle in numbers given or found out by the Canon in *Observat.* 8. *Resolut.* 2. *Quest.* 2. of this Book,

viz. { Hypoth. Base, Perp. — h . b . p }

2. Then from h and b (by the Canon above-mentioned) form a second right-angled Triangle, and let $2hb$ be called the Base, so the three sides will be these,

viz. { Hypoth. Base, Perp. — $hh + bb$. $2hb$. $hh - bb$ }

3. Again, from h and p form a third right-angled Triangle, and let $2hp$ be called the Base, so the three sides will be these,

viz. { Hypoth. Base, Perp. — $hh + pp$. $2hp$. $hh - pp$ }

4. I say the Solid of the Perpendiculars of those three right-angled Triangles is to the Solid of their Bases as a Square to a Square, to wit, as pp to $4hh$; which I prove thus,

The Perpendicular of the first right-angled Triangle is p; the second Perpendicular is $hh - bb$, that is, pp, (for by Construction in the first step, $hh = bb + pp$, whence $hh - bb = pp$;) and the third Perpendicular is $hh - pp$, that is, bb. So that the three Perpendiculars are p, pp and bb, which multiplied one into another will produce

$pppbb$ = the Solid of the three Perpendiculars.

Again,

Again, the Bases of the same three right-angled Triangles are b, $2bb$, $2bp$, which multiplied one into another will produce

$4bbbbp$ = the Solid of the three Bases.

Now because bbp is a common Factor in those two Solids, they shall be in such proportion one to another as the Quotients that arise by dividing the said Solids by bbp, viz.

$pppbb \; . \; 4bbbbp \; :: \; pp \; . \; 4bb.$

Which was to be proved.

An Example in Numbers.

Let the first right-angled Triangle be . . . . $\left\{ \begin{matrix} h ; & b ; & p \\ 5, & 3, & 4 \end{matrix} \right.$

Then by Construction in the second step, the second right-angled Triangle shall be 34, 30, 16

And by Construction in the third step the third right-angled Triangle is 41, 40, 9

Which three Triangles will solve the Question; for the Solid of the Perpendiculars 4, 16, 9 hath such proportion to the Solid of the Bases 3, 30, 40 as the Square of 4 to the Square of 10.

Note. Instead of any one of the Triangles thus found out, you may take another like Triangle, as instead of 34, 30, 16, you may take 17, 15, 8, which with the other two Triangles will solve the Question.

QUEST. 81. (Quæst. 13. Lib. 4. Zetet. *Viet.*)

To find two right-angled Triangles in rational numbers, that the Product made by the mutual multiplication of the Perpendiculars, less by the Product of the Bases, may be a Square.

RESOLUTION.

1. Let h, b, p represent the Hypothenusal, Base and Perpendicular of a right-angled Triangle, in numbers so given or found out that $2p$ may be greater than b,

viz. $\left\{ \begin{matrix} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ h & . \; b & . \; p \end{matrix} \right.$

2. Then from $2p$ and b let a second right-angled Triangle be formed, and let $4pb$ be called the Perpendicular, so the three sides will be these,

viz. $\left\{ \begin{matrix} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ 4pp + bb & . \; 4pp - bb & . \; 4pb \end{matrix} \right.$

3. Then divide the sides of the said second right-angled Triangle severally by b the Base of the first, so there will arise these following sides of a third right-angled Triangle,

viz. $\left\{ \begin{matrix} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ \frac{4pp + bb}{b} & . \; \frac{4pp - bb}{b} & . \; 4p \end{matrix} \right.$

I say the first and third right-angled Triangles will solve the Question; for if the Product of their Perpendiculars p and $4p$, to wit, $4pp$ be lessened by the Product of their Bases b and $\frac{4pp - bb}{b}$, that is, by $4pp - bb$, the remainder will be a Square, to wit, bb; which was required.

An Example in Numbers.

Let the first right-angled Triangle be . . . . $\left\{ \begin{matrix} h ; & b ; & p \\ 5, & 4, & 3 \end{matrix} \right.$

Then the other shall be 13, 5, 12

Which Triangles will solve the Question; for the Product of the Perpendiculars 3 and 12, to wit, 36, exceeds 20 the Product of the Bases, by the Square 16.

Note. If the two Triangles found out by this Question be severally multiplied or divided by the same number, they will produce two other Triangles to perform the same effect.

QUEST. 82. (Quæst. 14. Lib. 4. Zetet. *Viet.*)

To find two right-angled Triangles in rational numbers, that the Product made by the mutual multiplication of the Perpendiculars, together with the Product of the Bases, may make a Square.

RESO-

RESOLUTION.

1. Let h, b, p represent the Hypothenusal, Base and Perpendicular of a right-angled Triangle, in numbers so given or found out that p may exceed $2b$,

viz. { Hypoth. h . Base, b . Perp. p

2. Then from p and $2b$ form a second right-angled Triangle, and let $4bp$ be called the Base; so the three sides will be these,

viz. { Hypoth. $pp + 4bb$. Base, $4pb$. Perp. $pp - 4bb$

3. Then divide the sides of the said second right-angled Triangle severally by p the Perpendicular of the first, so there will arise these following sides of a third right-angled Triangle,

viz. { Hpyoth. $\frac{pp+4bb}{p}$. Base, $4b$. Perp. $\frac{pp-4bb}{p}$

4. I say the first and third right-angled Triangles will solve the Question; for if to $pp - 4bb$ the Product of the Perpendiculars p and $\frac{pp - 4bb}{p}$, you add $4bb$, to wit, the Product of the Bases b and $4b$, the summ will be the Square pp. Which was required.

An Example in Numbers.

The first right-angled Triangle may be . . . . $13, 5, 12$
Then the other Triangle sought shall be . . . . $\frac{61}{3}, 20, \frac{11}{3}$

Which Triangles will solve the Question; for the Product of the Perpendiculars 12 and $\frac{11}{3}$, to wit, 44 increased with 100 the Product of the Bases 5 and 20, makes the Square 144.

Note. If the two Triangles found out by this Question be multiplied or divided by the same number, they will produce two other Triangles performing the same effect. So, if $13, 5, 12$ and $\frac{61}{3}, 20, \frac{11}{3}$ be multiplied severally by 3, there will be produced $39, 15, 36$ and $61, 60, 11$; where 396 the Product of the Perpendiculars, with 900 the Product of the Bases makes the Square 1296, whose Root is 36.

QUEST. 83. (Quæst. 15. Lib. 4. Zetet. *Viet.*)

To find three right-angled Triangles in rational numbers, that the Solid of the Hypothenusals may be to the Solid of the Bases, as a square number to a square number.

RESOLUTION.

1. Let h, b, p represent the Hypothenusal, Base and Perpendicular of a right-angled Triangle, in numbers so given or found out that $2b$ may exceed p,

viz. { Hypoth. h . Base, b . Perp. p

2. Then from $2b$ and p form a second right-angled Triangle, and let $4bp$ be called the Base, so the three sides will be these,

viz. { Hypoth. $4bb + pp$. Base, $4bp$. Perp. $4bb - pp$

3. Then by the help of those two right-angled Triangles, find out a third, by the Canon in *Observat.* 4. upon *Resolut.* 2, and 3. of *Quest.* 2. *viz.* for the Hypothenusal of the third right-angled Triangle take the Product of the Hypothenusals of the first and second; the Base shall be the Product of the Bases of the first and second, less by the Product of the Perpendiculars; and the Perpendicular shall be equal to the summ of the Product of the Base of the first into the Perpendicular of the second, and the Product of the Perpendicular of the first into the Base of the second; so the three sides of the third right-angled Triangle will be these,

viz. { Hypoth. $4bbh + pph$. Base, ppp . Perp. $4bbb + 3bpp$

I say those three right-angled Triangles will solve the Question; for the Solid of the Hypothenusals is to the Solid of the Bases as the Square of $4bbh + pph$ is to the Square of $2bpp$.

An

An Example in Numbers.

The first right-angled Triangle may be . . . . > 5 , 3 , 4
Then the second in its least terms will be found . . . . . > 13 , 12 , 5
And the third in its least terms is . . . . > 65 , 16 , 63

Which three Triangles will solve the Question; for the Solid of the Hypothenusals is to the Solid of the Bases, as the Square of 65 to the Square of 24.

Otherwise thus;

1. Let h, b, p represent the Hypothenusal, Base and Perpendicular of a right-angled Triangle, so given or found out that p may exceed $2b$,

$$viz.\ \begin{cases} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ h & .\ b & .\ p \end{cases}$$

2. Then from p and $2b$ form a second right-angled Triangle, and let $4bp$ be called the Base, so the three sides will be these,

$$viz.\ \begin{cases} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ pp + 4bb & .\ 4pb & .\ pp - 4bb \end{cases}$$

3. Then by the help of those two right-angled Triangles find out a third by the latter Canon in *Observat.* 4. before mentioned, *viz.* for the Hypothenusal of the third right-angled Triangle take the Product of the multiplication of the Hypothenusals of the first and second; for the Base, take the summ of the Product of the Bases and the Product of the Perpendiculars; and for the Perpendicular take the difference of these two Products, to wit, the Product of the Base of the first into the Perpendicular of the second, and the Product of the Perpendicular of the first into the Base of the second. So the three sides of the third right-angled Triangle will be found these,

$$viz.\ \begin{cases} \text{Hypoth.} & \text{Base,} & \text{Perp.} \\ pph + 4bbh & .\ ppp & .\ 3bpp + 4bbb \end{cases}$$

An Example in Numbers.

Let the sides of the first right-angled Triangle be . . . . > 13 , 5 , 12
Then the second in its least terms will be found . . . . . > 61 , 60 , 11
And the third in its least terms shall be . . . . > 793 , 432 , 665

Which three Triangles will solve the Question; for the Solid of the Hypothenusals is to the Solid of the Bases, as the Square of 793 to the Square of 360.

QUEST. 84. (Quæst. 24. Lib. 5. *Diophant.*)

To find three square numbers, that if they be severally added to the solid Product made by their continual multiplication, the three summs may be Squares.

RESOLUTION.

1. For the Solid of the three Squares sought put . . . . . . > aa
2. Then find three Squares, every one of which increased with unity may make a Square; which Squares may easily be found out by the help of three unlike right-angled Triangles; for if the Square of one of the sides about the right-angle be divided by the Square of the other side, the Quotient will be a Square, which increased with 1 will make a Square: As, if there be exposed these three unlike right-angled Triangles, to wit, 3, 4, 5. | 5, 12, 13. | 8, 15, 17; then in the first Triangle the Square of the Base 3 divided by the Square of the Perpendicular 4 gives the Square $\frac{9}{16}$, which increased with 1, that is, $\frac{16}{16}$, makes the Square $\frac{25}{16}$; the reason whereof is manifest, for by Construction the Numerators 9 and 16 added together make a Square, wherefore the whole Fraction $\frac{25}{16}$ shall be a Square. In like manner in the second Triangle the Square of the Base 5 divided by the Square of the Perpendicular 12 gives the Square $\frac{25}{144}$, which increased with 1, that is, $\frac{144}{144}$, makes the Square $\frac{169}{144}$. And in the third Triangle the Square of the Base 8 divided by the Square of the Base 15, gives the Square $\frac{64}{225}$, which increased with 1, makes the Square $\frac{289}{225}$. Thus three Squares are found out, to wit, $\frac{9}{16}$, $\frac{25}{144}$ and $\frac{64}{225}$, every one of which increased with 1 makes a Square. Now multiply those Squares severally by aa, and take the Products $\frac{9}{16}aa$, $\frac{25}{144}aa$ and $\frac{64}{225}aa$ for the three Squares sought; for every one of them added to aa, (which was first put for the Solid made by their continual multiplication,) makes a Square: It remains that the Solid of the said $\frac{9}{16}aa$, $\frac{25}{144}aa$ and $\frac{64}{225}aa$ be equal to aa; But their Solid by continual multiplication is $\frac{14400}{518400}aaaaaa$,

$\frac{14400}{518400}aaaaaa$, this equated to aa gives $\frac{14400}{518400}aaaaaa = aa$; whence by dividing each part by aa there ariseth $\frac{14400}{518400}aaaa = 1$, and by extracting the square Root out of each part it gives $\frac{120}{720}aa = 1$; in which last Equation if $\frac{120}{720}$ were a Square, then the value of a would be expressible by a rational number, and consequently the Question were solved. Whence therefore comes $\frac{120}{720}$? Examine the work, and you will find that the Numerator 120 is the Solid of the Bases 3, 5 and 8 of the three Triangles first exposed; (for 14400 is the Solid of the three Squares 9, 25 and 64, and therefore the square Root of 14400, to wit, 120 is the Solid of the sides of those three Squares;) and the Denominator 720 is the Solid of the three Perpendiculars 4, 12 and 15; (for 518400 is the Solid of the three Squares 16, 144 and 125, and therefore the square Root of 518400, to wit, 720 is the Solid of the three Roots of those Squares.) We must therefore find three such right-angled Triangles, that the Solid of their Perpendiculars may be to the Solid of their Bases as a Square to a Square. But by the precedent *Quest.* 80. three such Triangles may be found out, as these, 4, 3, 5. | 8, 15, 17. | 9, 40, 41; the Solid of whose Bases 4, 8, 9, to wit, 288, is to the Solid of their Perpendiculars 3, 15, 40, to wit, 1800, as 144 to 900; that is, as a Square to a Square; then with these Triangles let the work be renewed as before, *viz.*

3. For the Solid of the three Squares sought put . . . . . . aa
4. Then divide the Squares of the Bases of the three right-angled Triangles last found out, by the Squares of the Perpendiculars, and multiplying the Quotients severally by aa, put the Products for the three Squares sought, to wit, ... $\frac{16}{9}aa$, $\frac{64}{225}aa$, $\frac{81}{1600}aa$
5. The Solid of those three Squares equated to aa, gives . . $\frac{82944}{3240000}aaaaaa = aa$
6. Which Equation, after due Reduction, gives . . . . $a = \frac{5}{2}$
7. Wherefore from the sixth and fourth steps the three Squares sought are $\frac{100}{9}$, $\frac{16}{9}$ and $\frac{81}{256}$, which mutually multiplied one into another make the Solid $\frac{25}{4}$, to which if the three Squares themselves be severally added, the summs will also be Squares, to wit, $\frac{625}{36}$, $\frac{289}{36}$ and $\frac{1681}{256}$; for their sides are $\frac{25}{6}$, $\frac{17}{6}$ and $\frac{41}{16}$. Therefore the Question is solved, and manifestly capable of innumerable Answers.

QUEST. 85. (Quæst. 25. Lib. 5. *Diophant.*)

To find three such Squares, that if they be severally subtracted from the solid Product made by their continual multiplication, the three remainders may be Squares.

RESOLUTION.

1. For the solid Product of the three Squares sought put . . aa
2. Then search out three Squares, every one of which subtracted from unity may leave a Square; but three such Squares may be found out by the help of three unlike right-angled Triangles; for if the Square of one of the sides about the right-angle be divided by the Square of the Hypothenusal, the Quotient shall be a Square, which subtracted from 1 will leave a Square: Let therefore three unlike right-angled Triangles be exposed, as 3, 4, 5. | 12, 5, 13. | 15, 8, 17; then by dividing the Squares of 4, 5 and 8, which I shall here call Bases, (for it matters not which of the sides about the right-angle be called the Base,) by the Squares of the Hypothenusals 5, 13 and 17, the Quotients will be the Squares $\frac{16}{25}$, $\frac{25}{169}$ and $\frac{64}{289}$, every one of which subtracted from 1 leaves a Square. Then multiply every one of these Squares by aa and assume the Products to be the three Squares sought, to wit, $\frac{16}{25}aa$, $\frac{25}{169}aa$ and $\frac{64}{289}aa$; for every one of these subtracted from aa (which was first put for the Solid of the three desired Squares) leaves a Square. It remains that the solid Product of the said $\frac{16}{25}aa$, $\frac{25}{169}aa$ and $\frac{64}{289}aa$ be equal to aa; but the said Solid by continual multiplication will be found $\frac{25600}{1221025}aaaaaa$, therefore $\frac{25600}{1221025}aaaaaa = aa$, whence after due Reduction there will arise $\frac{160}{1105}aa = 1$; in which last Equation if $\frac{160}{1105}$ were a Square, then the value of a would be expressible by a rational number. Whence therefore comes $\frac{160}{1105}$? Examine the work, and you will find that the Numerator 160 is the solid Product of the Perpendiculars 4, 5 and 8 of the three Triangles first exposed; and the Denominator 1105 is the Solid of the Hypothenusals 5, 13 and 17. We must therefore find three such right-angled Triangles that the Solid of the Hypothenusals may be to the Solid of their Bases as a Square to a Square: But three such right-angled Triangles may be found out by the preceding *Quest.* 83. suppose these, 5, 3, 4. | 13, 12, 5. | 65, 16, 63; here I shall call 3, 12 and 16 the Bases, by the help whereof the work may be renewed thus, *viz.*

3. For

3. For the Solid of the three Squares sought put . . . . . aa

4. Then divide the Squares of the Bases of the three Triangles last found out, by the Squares of the Hypothenusals, and multiply the Quotients severally by aa, and put the Products for the three Squares sought, to wit, ... $\frac{9}{25}aa$, $\frac{144}{169}aa$, $\frac{256}{4225}aa$

5. The Solid of those three Squares being equated to aa, gives $\frac{331776}{17850625}aaaaaa = aa$
Which Equation, after due Reduction, will give . . . . . $a = \frac{65}{24}$

6. According to which value of a, the Positions in the fourth step being resolved, the three Squares sought will be found $\frac{169}{64}$, $\frac{25}{4}$, $\frac{4}{9}$; for these mutually multiplied make the Solid $\frac{4225}{576}$, from which if every one of the said three Squares be subtracted, the remainders will be Squares, to wit, $\frac{169}{36}$, $\frac{625}{576}$, $\frac{441}{64}$, whose sides are $\frac{13}{6}$, $\frac{25}{24}$, $\frac{21}{8}$.

QUEST. 86. (Quæst. 26. Lib. 5. *Diophant.*)

To find three Squares, that the Solid or Product made by their continual multiplication being subtracted from every one of them, the three remainders may be Squares.

RESOLUTION.

The Resolution of this Question depends upon the *Lemma* used in the last preceding Question, for as there, so here, three right-angled Triangles are first to be found out, that the Solid of their Hypothenusals may be to the Solid of their Bases as a Square to a Square. Then instead of $\frac{9}{25}aa$, $\frac{144}{169}aa$ and $\frac{256}{4225}aa$, assumed in the fourth step of the preceding *Quest* 85. to be the three Squares sought, let $\frac{25}{9}aa$, $\frac{169}{144}aa$ and $\frac{4225}{256}aa$ be put for the three Squares sought in this 86[th] Question; where observe, that the Numerators of the three latter Squares which are multiplied into aa, are the same with the Denominators of the former, and the Denominators of the latter the same with the Numerators of the former; by which conversion it will follow, that if from every one of the latter Squares, *viz.* from $\frac{25}{9}aa$, $\frac{169}{144}aa$ and $\frac{4225}{256}aa$ you subtract aa, which (as before in *Quest.* 85.) is to be taken for the Solid of the three Squares sought, the three remainders will be Squares: Then the Solid of the three assumed Squares, that is, of $\frac{25}{9}aa$, $\frac{169}{144}aa$ and $\frac{4225}{256}aa$, being equated to aa, gives $\frac{17850625}{331776}aaaaaa = aa$; whence, after due Reduction, $a = \frac{24}{65}$, according to which value of a, the Positions being resolved will give $\frac{64}{169}$, $\frac{4}{25}$, $\frac{9}{4}$ for the three Squares sought; for the Solid made by their continual multiplication is $\frac{576}{4225}$, which subtracted from every one of the three Squares, leaves the Squares $\frac{1024}{4225}$, $\frac{4}{169}$ and $\frac{35721}{16900}$, whose sides are $\frac{32}{65}$, $\frac{2}{13}$, $\frac{189}{130}$.

QUEST. 87. (Quæst. 30. Lib. 5. *Diophant.*)

To find three Squares, that if to the summ of every two of them, a given number, suppose 15, be added, the three summs may be Squares.

RESOLUTION.

1. For one of the Squares sought take any square number at pleasure, as 9

2. Then we must find two other Squares, such, that each of them added to 24 may make a Square; for since 9 one of the three Squares added to the given number 15 makes 24, it will not be difficult to conceive from the tenor of the Question, that each of the other two Squares taking to it 24 must make a Square. Now to find out those two Squares, divide 24 by each of two sides about the right-angle of some right-angled Triangle, as 3 and 4; so the Quotients 8 and 6 shall also be the sides about the right-angle of a right-angled Triangle, because they are in the same proportion with the former; for by Construction $8 \times 3 = 4 \times 6 = 24$, therefore $3 . 4 :: 6 . 8$; and consequently the halves of 3 and 4, to wit, $\frac{3}{2}$ and 2; and the halves of 8 and 6, to wit, 4 and 3 shall be also the sides about the right-angle of a right-angled Triangle. These things premised,

3. Let the side of one of the two Squares be the difference between $3a$ and $\frac{2}{a}$, to wit, ... $3a - \frac{2}{a}$, or $\frac{2}{a} - 3a$

4. And the side of the other Square the difference between $4a$ and $\frac{3}{a}$, to wit, ... $4a - \frac{3}{a}$, or $\frac{3}{a} - 4a$

Q 5. There-

5. Therefore the Square of the side in the third step is . > $9aa - 12 + \frac{4}{aa}$

6. And the Square of the side in the fourth step is . . > $16aa - 12 + \frac{\frac{9}{4}}{aa}$

7. The former of those two Squares increased with 24 makes a Square, to wit, ... > $9aa + 12 + \frac{4}{aa}$

8. And the latter Square in the sixth step increased with 24 makes a Square, to wit, ... > $16aa + 12 + \frac{\frac{9}{4}}{aa}$

9. The sides of which two Squares in the two last steps are > $3a + \frac{2}{a}$ and $4a + \frac{\frac{3}{2}}{a}$

10. It remains that the summ of the Squares in the fifth and sixth steps, together with the given number 15 may make a Square, but it makes . . . . > $25aa - 9 + \frac{\frac{25}{4}}{aa}$

11. Which summ must be equated to a Square, *viz.* either to $25aa$, or to $\frac{\frac{25}{4}}{aa}$, let it first be equated to $25aa$, *viz.* suppose . . . . > $25aa - 9 + \frac{\frac{25}{4}}{aa} = 25aa$

12. Which Equation, after due Reduction, gives . . . > $a = \frac{5}{6}$

13. Therefore from the twelfth, third and fourth steps the sides of the second and third Squares sought are $\frac{1}{10}$ and $\frac{23}{15}$, and consequently 9, $\frac{1}{100}$ and $\frac{529}{225}$ are the three Squares sought; for every two with 15 make Squares, to wit, $\frac{2401}{100}$, $\frac{5929}{225}$, $\frac{625}{36}$, whose sides are $\frac{49}{10}$, $\frac{77}{15}$, $\frac{25}{6}$.

But if $25aa - 9 + \frac{\frac{25}{4}}{aa}$ be equated to $\frac{\frac{25}{4}}{aa}$, then the value of a will be $\frac{3}{5}$, yet the same Solution will be found as before, because here we must conceive $\frac{2}{a} - 3a$ and $\frac{\frac{3}{2}}{a} - 4a$ to be the sides of the two Squares, which sides being resolved according to the latter value of a, to wit, $a = \frac{3}{5}$, there will come forth (as before) $\frac{23}{15}$ and $\frac{1}{10}$.

This Question is capable of innumerable Answers upon a double ground; for first, the first square may be any known square number at pleasure; then the sides of the second and third Squares may be variously feigned from divers numbers, which may be the sides about the right-angle of unlike right-angled Triangles; as instead of 3 and 4 we may take 8 and 15, 5 and 12, and innumerable others.

QUEST. 88. (Quæst. 32. Lib. 5. *Diophant.*)

To find three Squares that the summ of their Squares may make a Square; or, (which is the same thing) to find three numbers that the summ of their squared Squares may be a Square.

RESOLUTION.

1. For one of the square numbers sought put . . . . > aa
2. And for the two others put . . . . > bb and cc
3. Then the summ of the Squares of those three Squares is > $aaaa + bbbb + cccc$
4. Which summ must be equated to a Square, let its side be $aa - dd$, the Square whereof equated to the said summ gives

$$aaaa + bbbb + cccc = aaaa - 2ddaa + dddd.$$

5. Which Equation, after due Reduction, gives . . . > $aa = \frac{dddd - bbbb - cccc}{2dd}$
6. In which last Equation, because the Numerator and Denominator are not perfect Squares, the value of a is not expressible by a rational number; but to cause it to be rational, we may discover by the said Fraction that a certain number and two Squares must be found out, such, that if from the Square of the number the summ of the two Squares be subtracted, the remainder may be to the double of the said number as a square number to a square number: Now to find out such a number and two Squares, let $rr + ss$ be put for the number, and rr for one Square, and ss for the other; then from the Square of $rr + ss$, that is, from $rrrr + 2rrss + ssss$ subtract the summ of the Squares of rr and ss, that is, $rrrr + ssss$, and the remainder $2rrss$ must be to the double of the number $rr + ss$, that is, to $2rr + 2ss$, as a Square to a Square; therefore also the

the halves of $2rrss$ and $2rr + 2ss$, that is, $rrss$ and $rr + ss$ ſhall be as a Square to a Square, for the Proportion is not changed: And ſince $rrss$ is a Square, whoſe ſide is rs, it remains only to make $rr + ss$ a Square; but ſuch it will be if r and s repreſent the ſides about the right-angle of a right-angled Triangle whoſe three ſides are expreſſible by rational numbers, for then $rr + ss$ will be the Square of the Hypothenuſal. Therefore from the premiſſes the following Canon is deducible to ſolve the Queſtion propoſed.

CANON.

Take for two of the three Squares ſought the Squares of the ſides about the right-angle of ſome right-angled Triangle in rational numbers; then divide the Product made by the mutual multiplication of thoſe two Squares, by the Square of the Hypothenuſal, and there will come forth the third Square ſought.

As, for example, let there be expoſed the right-angled Triangle 3, 4, 5, then two of the Squares ſought ſhall be 9 and 16, (to wit, the Squares of 3 and 4 the ſides about the right-angle;) and if the Product of 9 into 16, that is, 144, be divided by 25 (the Square of the Hypothenuſal,) the Quotient $\frac{144}{25}$ ſhall be the third Square ſought. I ſay 9, 16 and $\frac{144}{25}$ are three Squares, which will ſolve the Queſtion, for the ſumm of their Squares makes $\frac{231361}{625}$, which is a Square, whoſe ſide is $\frac{481}{25}$.

QUEST. 89. (Quæſt. 33. Lib. 5. *Diophant.*)

A certain Vintner made a mixture of two ſorts of Wines, whereof one coſt eight pence the quart, and the other five pence; at which prices the whole mixture was worth a ſquare number of pence, unto which 60 being added the ſumm would alſo make a ſquare number, whoſe ſide was the number of quarts contained in the mixture. The Queſtion is, to find the number of quarts of each ſort of Wine in the mixture.

RESOLUTION.

1. For the price of the whole mixture put . . . . . . $aa - 60$
2. Then if 60 be added to that total price the ſumm will be the Square aa, whoſe Root (as the Queſtion requires) muſt be equal to the number of quarts of both ſorts of Wine in the mixture, to wit, ... a
3. From the premiſſes we may rightly inferr, that $aa - 60$ (the total coſt of the mixture) is greater than $5a$, but leſs than $8a$; (that is, greater than the Product of the multiplication of the total number of mixed quarts by the price of the worſer ſort of Wine in the mixture, but leſs than the Product of the ſame number of quarts multiplied into the dearer ſort of Wine:) But if $aa - 60$ be greater than $5a$, and leſs than $8a$, then the value of a (by *Queſt* 10. of this *Book*) is greater than $10\frac{[illegible]}{100000000}$, &c. but leſs than $12\frac{[illegible]}{100000000}$, &c. Therefore $aa - 60$ (which the Queſtion requires to be a Square) muſt be equated to ſome Square whoſe ſide muſt be ſo feigned that the value of a may be within thoſe limits: Now to cauſe that effect, the ſide of the ſaid Square may be feigned $- a +$ any abſolute number between $17\frac{211}{1000}$, &c. and $22\frac{[illegible]}{1000}$, &c. or $a -$ any abſolute number between $2\frac{[illegible]}{1000}$, &c. and $3\frac{[illegible]}{1000}$, &c. (as hath been ſhewn in *Queſt*. 11. of this *Book*.) Let then the ſaid ſide be feigned $- a + 22$, whoſe Square $aa - 44a + 484$ equated to $aa - 60$ will give $a = 12\frac{4}{11}$ for the deſired number of quarts in the whole mixture.
4. Then from $\frac{18496}{121}$, the Square of the ſaid $12\frac{4}{11}$, ſubtract 60, and the remainder $\frac{11236}{121}$ is the price of the whole mixture, which is a Square whoſe ſide is $\frac{106}{11}$; and becauſe $\frac{11236}{121}$ is the number of pence expreſſing the value of the mixture, it muſt be equal to the Product of 8 multiplied by a certain part of $12\frac{4}{11}$, (the number of quarts in the mixture,) together with the Product of 5 multiplied by the remaining part of $12\frac{4}{11}$; we muſt therefore divide $12\frac{4}{11}$ into two ſuch parts, that if the one be multiplied by 5, and the other by 8, the ſumm of the two Products may make $\frac{11236}{121}$; but that may be done thus,
5. For one of the deſired parts of $12\frac{4}{11}$ put . . . . e
6. Then the other ſhall be . . . . $12\frac{4}{11} - e$
7. And if the former part be multiplied by 5, and the latter by 8, the ſumm of the Products will be . . . . $98\frac{10}{11} - 3e$
8. Which ſumm muſt be equated to $\frac{11236}{121}$, viz. . . . . $98\frac{10}{11} - 3e = \frac{11236}{121}$

9. Which Equation, after due Reduction, makes known one of the desired parts, to wit, ... $e = \frac{244}{121}$

10. Which subtracted from $12\frac{4}{11}$, leaves the other part, to wit, . . $\frac{1252}{121}$

11. I say the total mixture of Wine might be composed of $\frac{244}{121}$ quarts of five pence the quart, and $\frac{1252}{121}$ quarts of eight pence the quart; whence the value of the whole mixt quantity, to wit, of $12\frac{4}{11}$ quarts is $\frac{11236}{121}$ pence, which is a square number whose side is $\frac{106}{11}$; and if to the said Square $\frac{11236}{121}$ you add 60, the summ is also a Square, to wit, $\frac{18496}{121}$, whose side $12\frac{4}{11}$ is the number of quarts in the mixture.

12. But because the side of the Square to be equated to $aa - 60$ may be feigned $a -$ any absolute number between $2\frac{631}{1000}$, &c. and $3\frac{345}{1000}$, &c. let the said side be $a - 3$, the Square whereof equated to $aa - 60$ will give $a = 11\frac{1}{2}$ for the number of quarts in the mixture; then the Square of $11\frac{1}{2}$ is $\frac{529}{4}$, from which subtracting 60, the remainder $\frac{289}{4}$ is the square number of pence expressing the value of the mixture. Now the said $11\frac{1}{2}$ is to be divided into two such numbers that if one of them be multiplied by 5, and the other by 8, the summ of the Products may make the Square $\frac{289}{4}$; but two such numbers (by working as before) will be found $\frac{79}{12}$ and $\frac{59}{12}$.

I say again, the mixture may be compos'd of $\frac{79}{12}$ quarts of five pence the quart, and $\frac{59}{12}$ quarts of eight pence the quart; whence the value of the whole mixt quantity, to wit, of $11\frac{1}{2}$ quarts, is $\frac{289}{4}$ pence, which is a Square, to which if you add 60 the summ is also a Square, to wit, $\frac{529}{4}$, whose side $11\frac{1}{2}$ is the number of quarts in the whole mixture. From the premisses 'tis evident that the Question is capable of innumerable Answers in rational numbers.

QUEST. 90.

To find a right-angled Triangle in rational numbers, that one of the sides about the right-angle may be to the Area in a given Reason, suppose as r to s.

RESOLUTION.

1. For the Triangle sought let a right-angled Triangle be formed from two numbers, *viz.* a the greater and e the lesser, so the three sides will be these, to wit, ... $aa + ee;\ aa - ee;\ 2ae$

2. The Area of that Triangle is . . . . $aaae - aeee$

3. Then (according to the Question) let these four quantities be supposed to be Proportionals, *viz.* $r \,.\, s :: aa - ee \,.\, aaae - aeee.$

4. And because if the two latter terms of that Analogy be severally divided by $aa - ee$, the Quotients are 1 and ae, therefore that Analogy may be reduced to this, *viz.*
$r \,.\, s :: 1 \,.\, ae.$

5. And by comparing the Product of the extremes to the Product of the means, this Equation ariseth, *viz.* . . . . . . $rae = s$

6. And by dividing each part of the last Equation by ra, this ariseth . $e = \frac{s}{ra}$
Hence this

CANON.

7. Take any number at pleasure, which may be called a, then divide s the latter term of the given Reason, by the Product of the first term r multiplied into the number a, and call the Quotient the number e; lastly, from the said numbers a and e form a right-angled Triangle, and it shall be that which is sought.

An Example in Numbers.

Let it be required to find a right-angled Triangle, such, that one of the sides about the right-angle may be to the Area as 1 to 10.

Suppose, . . . $\left\{ \begin{array}{l} r = 1 \\ s = 10 \end{array} \right\}$ the terms of the given Reason;

Then by the Canon, $\left\{ \begin{array}{l} a = 2 \text{ taken at pleasure,} \\ e = 5 = \frac{s}{ra}. \end{array} \right.$

Lastly, from 5 and 2 form a right-angled Triangle, and the three sides will be 29, 21 and 20, which Triangle will solve the Question; for 21, one of the sides about the right-angle, is to the Area 210, as 1 to 10. Which was required.

Likewise

Likewiſe by the Canon, this right-angled Triangle, to wit, 101, 99 and 20 will be found to ſolve the Queſtion, for 99 is to the Area 990, as 1 to 10; and innumerable right-angled Triangles in Fractions may be found to perform the ſame effect.

QUEST. 91.

To find a right-angled Triangle in rational numbers, that the Hypothenuſal may be to the Area in a given Reaſon, ſuppoſe as r to s.

RESOLUTION.

1. For one of the ſides about the right-angle put . . . . . . a
2. And for the other ſide about the right-angle put . . . . . e
3. Then the ſquare Root of the ſumm of the Squares of thoſe ſides ſhall be the Hypothenuſal, to wit, ... $\sqrt{:aa+ee:}$
4. And the Area of the ſaid Triangle is . . . . $\frac{1}{2}ae$
5. Now according to the Queſtion, the Hypothenuſal muſt be to the Area as r to s, therefore from the third and fourth ſteps this Analogy ariſeth, *viz.*
$$r \;.\; s \;::\; \sqrt{:aa+ee:} \;.\; \tfrac{1}{2}ae.$$
6. But the Squares of thoſe Proportionals are alſo Proportionals, therefore
$$rr \;.\; ss \;::\; aa+ee \;.\; \tfrac{1}{4}aaee.$$
7. And from the laſt Analogy, by comparing the Product of the multiplication of the extremes to the Product of the means, this Equation ariſeth, *viz.*
$$\tfrac{1}{4}rraaee = ssaa + ssee.$$
8. From which Equation, by tranſpoſition of $ssee$, this ariſeth,
$$\tfrac{1}{4}rraaee - ssee = ssaa.$$
9. And by dividing each part of the laſt Equation by $\frac{1}{4}rraa - ss$, there will ariſe . . . . $ee = \frac{ssaa}{\frac{1}{4}rraa - ss}$
10. In which laſt Equation the Numerator $ssaa$ is a Square whoſe ſide is sa, and if the Denominator were a Square, then the whole Fraction would be alſo a Square, and conſequently the ſide thereof, to wit, the number e would be rational; it remains therefore to equate the Denominator $\frac{1}{4}rraa - ss$ to a Square, to which end, let the ſide thereof be feigned $\frac{1}{2}ra - b$; then the Square of $\frac{1}{2}ra - b$ being equated to $\frac{1}{4}rraa - ss$, this Equation ariſeth, *viz.*
$$\tfrac{1}{4}rraa - ss = \tfrac{1}{4}rraa - rba + bb.$$
11. Whence, after due Reduction, you will find . . . . . $a = \frac{ss+bb}{rb}$
12. Now if we ſuppoſe r, s and b to repreſent known rational numbers, then a, e and $\sqrt{:aa+ee:}$ which in the three firſt ſteps were put for the three ſides of the right-angled Triangle ſought, will alſo (from the eleventh, ninth and tenth ſteps,) be expreſſible by rational numbers, to wit, theſe,
$$\frac{ss+bb}{rb}, \quad \frac{2sss+2sbb}{rss-rbb}, \quad \frac{ssss+2ssbb+bbbb}{rssb-rbbb}.$$
13. Or the two firſt of the three ſides laſt expreſt may be reduced to the ſame Denominator with the third, and then the three ſides of the right-angled Triangle ſought will be theſe, to wit,
$$\frac{ssss-bbbb}{rssb-rbbb}, \quad \frac{2sssb+2sbbb}{rssb-rbbb}, \quad \frac{ssss+2ssbb+bbbb}{rssb-rbbb}.$$
Which three ſides, if they be expreſt by words, will give this

CANON.

14. Take for b any number leſs than s the latter term of the given Reaſon; then from the numbers s and b form a right-angled Triangle, and multiply the three ſides ſeverally by the Hypothenuſal; laſtly, divide thoſe three Products ſeverally by the Product made by the multiplication of the difference of the Squares of the two numbers s and b, (which formed the ſaid Triangle,) into the Product of b the leſſer of the ſame two numbers and r the firſt term of the given Reaſon; ſo ſhall the Quotients be the three ſides of a right-angled Triangle, which will ſolve the Queſtion propoſed.

An

An Example in Numbers.

Let it be required to find out a right-angled Triangle whose Hypothenusal may be to the Area as 2 to 3.

Suppose . . . . . . $\left\{\begin{array}{l} r = 2 \\ s = 3 \\ b = 1 \end{array}\right.$ $\begin{array}{l} \left.\right\} \text{the Terms of the given Reason,} \\ \text{less than 3, (or } s.) \end{array}$

Then form a right-angled Triangle from 3 and 1, (to wit, s and b,) and the three sides will be 10, 8 and 6; these multiplied severally by the Hypothenusal 10 will produce 100, 80 and 60, which divided severally by the Product which answers to $ss - bb$ into rb, that is, by 16, will give $\frac{25}{4}$, $\frac{20}{4}$ and $\frac{15}{4}$ for the Triangle sought; for the Hypothenusal $\frac{25}{4}$, or $\frac{100}{16}$, is to the Area $\frac{150}{16}$, as 2 to 3. Which was required.

In like manner, if it were desired to find a right-angled Triangle whose Hypothenusal might be to the Area as 3 to 2; then by the Canon, the three sides will be found $\frac{25}{9}$, $\frac{20}{9}$ and $\frac{15}{9}$.

QUEST. 92.

To find a right-angled Triangle in rational numbers, that the summ of all the three sides may be to the Area in a given Reason, suppose as r to s.

RESOLUTION.

1. For the right-angled Triangle sought let a Triangle be formed from any two numbers, suppose from a the greater and e the lesser, so the three sides will be these, . . . $aa + ee$, $aa - ee$, $2ae$
2. Then the Area will be . . . . $aaae - aeee$
3. And the summ of the three sides is . . . . $2aa + 2ae$
4. Now (according to the Question) the summ of the three sides must be to the Area, as r to s, therefore,

   As $r \,.\, s :: 2aa + 2ae \,.\, aaae - aeee$.
5. Or, by dividing each of the two latter terms of that Analogy by a, this ariseth; *viz.*

   As $r \,.\, s :: 2a + 2e \,.\, aae - eee$.
6. Whence, by comparing the Product of the extremes to the Product of the means, this Equation ariseth, *viz.* . . . $reaa - reee = 2sa + 2se$
7. Therefore by due transposition, ... $reaa - 2sa = reee + 2se$
8. And by dividing all in the last Equation by re, this ariseth, ... $aa - \frac{2s}{re}a = \frac{ree + 2s}{r}$
9. Which last Equation being resolved by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1. the value of a will be discovered, *viz.* . . . . $\therefore a = e + \frac{2s}{re}$

Hence this

CANON.

10. Take any number at pleasure, which may be called e; then to the number e add the Quotient that ariseth by dividing the double of the latter term of the given Reason, by the Product of the first term multiplied into the number e, and call the summ the number a; lastly, from the said numbers a and e form a right-angled Triangle, and it shall be that which is sought.

An Example in Numbers.

Let it be required to find out a right-angled Triangle, that the summ of all the three sides may be to the Area as 1 to 5.

Suppose . . . $\left\{\begin{array}{l} r = 1 \\ s = 5 \end{array}\right\}$ the Terms of the given Reason, then

By the Canon, . . $\left\{\begin{array}{l} e = 2 \;\text{ taken at pleasure,} \\ a = 7 = e + \frac{2s}{re}. \end{array}\right.$

Then form a right-angled Triangle from 7 and 2, and the three sides will be 53, 45, 28, which will solve the Question, for the summ of all the three sides, to wit, 126, is to the Area 630, as 1 to 5. Which was required to be done.

Likewise if a right-angled Triangle be formed from 11 and 1, (to wit, a and e, found out by the Canon,) the three sides will be 122, 120, 22, whose summ 264 is to the Area 1320, as 1 to 5.

Again,

Again, if a right-angled Triangle be formed from 11 and 10, the three sides will be 221, 21 and 220, whose summ 462 is to the Area 2310, as 1 to 5.

Lastly, you may find out as many right-angled Triangles in Fractions as you please to solve the Question. Also, upon the same ground it will not be difficult to find out innumerable Isosceles-Triangles, in every one of which the Perimeter shall be to the Area in a given Reason.

QUEST. 93.

To find a right-angled Triangle in rational numbers, that as well the Hypothenusal as the difference of the sides about the right-angle may be a Square.

RESOLUTION.

1. For the Triangle sought let a right-angled Triangle be formed from two numbers, suppose from a and e, and let a be the greater, so the three sides will be these, *viz.* — $aa+ee$, $aa-ee$; $2ae$
2. Now (according to the Question) as well the Hypothenusal as the difference of the sides about the right-angle must be a Square, so we are faln upon this Duplicate equality, *viz.* — $aa+ee=\square$, $2ae-aa+ee=\square$
3. The difference of those two quantities is — $2aa-2ae$
4. Which difference is equal to the Product of these two quantities, to wit, — $2a-2e$ and a
5. The half-summ of those two Factors in the last step is — $\frac{3}{2}a-e$
6. Then the Square of the said half-summ being equated to the greater of the two quantities in the second step, this Equation ariseth, *viz.* — $aa+ee=\frac{9}{4}aa-3ae+ee$
7. Whence after due Reduction there will arise — $12e=5a$
8. And by reducing the last Equation into Proportionals, it shall be
   As $12 . 5 :: a . e.$

Hence this

CANON.

9. If from 12 and 5, or any two numbers in that proportion, a right-angled Triangle be formed, it will solve the Question.

As, for example, in the right-angled Triangle 169, 119 and 120, which is formed from 12 and 5, the Hypothenusal 169 is a Square; also the difference of the sides about the right-angle, to wit, 1 is a Square. The same effect will be produced in a right-angled Triangle formed from any two numbers which have such proportion one to another as 12 to 5.

QUEST. 94.

To find a right-angled Triangle, that one of the sides about the right-angle may be a Square, which added to a given multiple, suppose the triple, of the Square of the difference of the sides about the right-angle may make a Square.

RESOLUTION.

1. For one of the sides about the right-angle, that it may be a Square, put — 4
2. And for the other put — a
3. The summ of their Squares must make a Square, to wit, the Square of the Hypothenusal, therefore — $aa+16=\square$
4. The difference of the sides about the right-angle is — $a \backsim 4$
5. The triple of the Square of that difference is — $3aa-24a+48$
6. To which (according to the Question) add 4, the square number first assumed for one of the sides about the right-angle, and the summ must be equal to a Square, *viz.* — $3aa-24a+52=\square$
7. So in the third and sixth steps we are faln upon a Duplicate equality, but the numbers prefixt to aa in the quantities to be equated, are not Squares, neither are the two known numbers in the same quantities both Squares, for 52 is not a Square, whereby the said Duplicate equality is inexplicable; but if the said 52 were a Square, then the Duplicate equality

equality might be resolved; therefore instead of the square number 4 which in the first step was assumed for one of the sides about the right-angle, we must seek such a Square that if it be added to the triple of its Square the summ may be a Square. Suppose therefore that Square sought to be ee, this added to the triple of its Square makes $3eeee + ee$ to be equated to a Square, the side whereof may be variously feigned, let it be $ee + e$, then the Square of $ee + e$, to wit, $eeee + 2eee + ee$ being equated to $3eeee + ee$, after due Reduction the value of e will be found 1, and ee is also 1. So we have found a Square, to wit, 1, which added to the triple of its Square makes the Square 4; therefore now the Resolution may be renewed thus, *viz.*

8. For one of the sides about the right-angle put the Square . . . 1
9. And for the other side . . . . a
10. The summ of their Squares must be equal to a Square, *viz.* $aa + 1 = \square$
11. The difference of the sides about the right-angle is . . . $a \backsim 1$
12. The triple of the Square of that difference is . . . . . $3aa - 6a + 3$
13. To which adding the side 1 in the eighth step, the summ must be equal to a Square, *viz.* . . . . $3aa - 6a + 4 = \square$
14. Also from the tenth step, ... $aa + 1 = \square$
15. So in the two last steps we have a new Duplicate equality which may be resolved thus; first, to the end there may be one and the same known square number in each of the two quantities to be equated to Squares, I multiply the quantity in the fourteenth step, to wit, $aa + 1$ by 4, and it makes $4aa + 4$; now each of these quantities is to be equated to a Square, *viz.* $3aa - 6a + 4 = \square$, $4aa + 4 = \square$
16. The difference of those two quantities is . . . . . . $aa + 6a$
17. Which difference is equal to the Product of these two Factors, to wit, ... $\frac{2}{3}a + 4$, and $\frac{3}{2}a$
18. Half the summ of those Factors is . . . . $\frac{13}{12}a + 2$
19. The Square of the said half-summ is . . . . $\frac{169}{144}aa + \frac{13}{3}a + 4$
20. Which Square equated to $4aa + 4$, (the greater of the two quantities in the fifteenth step,) will after due Reduction give $a = \frac{624}{407}$.
21. Therefore from the twentieth, ninth and eighth steps the sides about the right-angle are $\frac{624}{407}$ and 1, the summ of whose Squares is $\frac{555025}{165649}$, whose square Root $\frac{745}{407}$ is the Hypothenusal sought.

I say $\frac{624}{407}$, 1 and $\frac{745}{407}$ are the sides of a right-angled Triangle, which will solve the Question; for one of the sides about the right-angle is a Square, to wit, 1, and if this be added to the triple of the the Square of the difference of the sides about the right-angle, it makes the Square $\frac{306916}{165649}$, whose side is $\frac{554}{407}$. From the premisses it is evident that innumerable right-angled Triangles may be found to solve the Question.

QUEST. 95.

To find out a right-angled Triangle in rational numbers, that the Square of one of the sides about the right-angle may be equal to the other of the same sides.

[*This is* Problem. 15. *in* pag. 80. *of the Introduction to* Algebra *before cited in* Quest. 58. *but I shall resolve it after another manner.*]

RESOLUTION.

1. For one of the sides about the right-angle put ra, (r representing some known number, and a some number unknown,) . ra
2. Then (according to the Question) the Square of that side must be the other of the sides about the right-angle, to wit, . . $rraa$
3. The summ of the Squares of those sides is . . . . $rrrraaaa + rraa$
4. Which summ must be equal to the Square of the Hypothenusal, and therefore it remains to equate the said $rrrraaaa + rraa$ to a Square, to which end take some known number s greater than r, and then the side of the said Square may be feigned $sa - rraa$, the Square whereof being equated to $rrrraaaa + rraa$, and due Reduction made, you will find

$$a = \frac{ss - rr}{2srr}.$$

5. But

5. But s and r were assumed to represent two known numbers whereof s is the greater, therefore from the premisses the three sides of the Triangles sought shall be known also, and may be exprest thus,

$$\left.\begin{matrix} \dfrac{ss-rr}{2sr}, \text{ or } \dfrac{2sssr-2srrr}{4ssrr}, \\ \dfrac{ssss-2ssrr+rrrr}{4ssrr}, \end{matrix}\right\} \text{The sides about the right-angle.}$$

$$\dfrac{ssss-rrrr}{4ssrr}, \quad \text{The Hypothenusal.}$$

6. Moreover, because the Square of $\dfrac{2sr}{ss-rr}$ is $\dfrac{4ssrr}{ssss-2ssrr+rrrr}$; and the summ of the Squares of the two last Fractions is equal to the Square of $\dfrac{2sssr+2srrr}{ssss-2ssrr+rrrr}$, (as will easily appear by Multiplication and Addition;) therefore the three quantities last exprest shall also be the sides of a right-angled Triangle to solve the Question proposed, *viz.*

$$\left.\begin{matrix} \dfrac{2sr}{ss-rr}, \text{ or } \dfrac{2sssr-2srrr}{ssss-2ssrr+rrrr}, \\ \dfrac{4ssrr}{ssss-2ssrr+rrrr}, \end{matrix}\right\} \text{The sides about the right-angle.}$$

$$\dfrac{2sssr+2srrr}{ssss-2ssrr+rrrr}, \quad \text{The Hypothenusal.}$$

7. If the three sides of the Triangle sought, as they be above exprest in the fifth and sixth steps be compared together, it will be easie to deduce from thence this following

CANON.

First, form a right-angled Triangle from any two unequal numbers; then multiply the three sides of that Triangle severally by either of the sides about the right-angle; lastly, divide severally those three Products by the Square of the other of the sides about the right-angle; so the three Quotients shall be the sides of the right-angled Triangle sought.

Examples in Numbers.

First, find out three numbers to express the sides of a right-angled Triangle, suppose these, ... $3, \; 4, \; 5$

Then multiply those three sides severally by 4, the greater of the sides about the right-angle, and the three Products will be these, *viz.* $12, \; 16, \; 20$

Lastly, divide those three Products severally by 9, the Square of the lesser of the sides about the right-angle; so the Quotients will be the sides of a right-angled Triangle to solve the Question proposed, to wit, these, ... $\frac{4}{3}, \; \frac{16}{9}, \; \frac{20}{9}$

Again,

Let the right-angled Triangle first found out be here repeated, to wit, ... $4, \; 3, \; 5$

Then multiply those three sides severally by 3, the lesser of the sides about the right-angle, and the Products will be these, . $12, \; 9, \; 15$

Lastly, divide those three Products severally by 16, the Square of the greater of the sides about the right-angle; so the Quotients will give these three sides of a right angled Triangle to solve the Question, to wit, ... $\frac{3}{4}, \; \frac{9}{16}, \; \frac{15}{16}$

QUEST. 96. (Quæst. 1. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the sides about the right-angle being severally subtracted from the Hypothenusal may leave cube-numbers.

RESOLUTION.

1. First, form a right-angled Triangle from two unequal numbers, suppose from a and e, and let a be the greater, so the three sides will be these, *viz.* . . . . . . Hypoth. Base; Perp. $aa+ee$. $aa-ee$. $2ae$

 2. Then

2. Then by subtracting the Base from the Hypothenusal, the remainder is $2ee$, which should be a Cube, but it is not; yet it shews that e (to wit, one of the numbers from which the desired right-angled Triangle is to be formed,) must be such that the double of its Square may make a Cube. Now that we may chuse the number e with that condition, let $2ee$ be equated to the Cube $bbbeee$, *viz.* suppose $2ee = bbbeee$, (where bbb may represent any known cube-number,) whence after due Reduction the value of e will be made known, *viz.*

$$e = \frac{2}{bbb}.$$

3. It remains, that the Perpendicular subtracted from the Hypothenusal may leave a Cube; but the remainder is the Square $aa + ee - 2ae$, which is not a Cube, but if its Root $a - e$ were a Cube, then that Square would be a Cube; (for a Cube multiplied into a Cube produceth a Cube, by *Prop.* 3, & 4. *Elem.* 9. *Euclid.*) Let therefore the said Root $a - e$ be equated to some Cube, *viz.* suppose $a - e = ddd$; hence, and from the third step it follows, that

$$a = e + ddd = \frac{2}{bbb} + ddd.$$

Now, if from $\frac{2}{bbb} + ddd$ and $\frac{2}{bbb}$ (the values of a and e) a right-angled Triangle be formed, it will solve the Question. Hence this

CANON.

4. Divide 2 by any cube-number, and reserve the Quotient; then to the said Quotient add any cube-number; lastly, from the summ and Quotient form a right-angled Triangle and it shall solve the Question proposed.

As, for example, I divide 2 by the Cube 8, and reserve the Quotient $\frac{1}{4}$ for one of the numbers by which the right-angled Triangle is to be formed; then to the Quotient $\frac{1}{4}$ I add some Cube, as 1, and the summ is $\frac{5}{4}$; lastly, from $\frac{5}{4}$ and $\frac{1}{4}$ I form a right-angled Triangle and find the sides $\frac{13}{8}$, $\frac{12}{8}$ and $\frac{5}{8}$, which will solve the Question; for the sides about the right-angle, to wit, $\frac{12}{8}$ and $\frac{5}{8}$ being severally subtracted from the Hypothenusal $\frac{13}{8}$, the remainders are the Cubes $\frac{1}{8}$ and 1.

But after one right-angled Triangle is found out to solve the Question, if you multiply or divide every one of the sides thereof by one and the same cube-number, the Products or Quotients will give another right-angled Triangle to solve the Question: As, if the three sides before found out, to wit, $\frac{13}{8}$, $\frac{12}{8}$ and $\frac{5}{8}$ be severally multiplied by the Denominator 8, they will produce 13, 12 and 5, which will solve the Question. Likewise, if 13, 12 and 5 be severally multiplied by 8, there will be produced the right-angled Triangle 104, 96 and 40, where the differences between the Hypothenusal and the other two sides are Cubes, to wit, 8 and 64. The reason is evident; for,

First, by Construction $\frac{13}{8}$, $\frac{12}{8}$, $\frac{5}{8}$ are the sides of a right-angled Triangle that will solve the Question, *viz.* . . . $\left\{\begin{array}{l} \frac{13}{8} - \frac{12}{8} = \text{a Cube,} \\ \frac{13}{8} - \frac{5}{8} = \text{a Cube.} \end{array}\right.$

And because a Cube multiplied by a Cube produceth a Cube, those two differences or Cubes multiplied by the Cube 8 shall necessarily produce these differences or Cubes, *viz.* $\left\{\begin{array}{l} 13 - 12 = \text{a Cube,} \\ 13 - 5 = \text{a Cube.} \end{array}\right.$

Wherefore the right-angled Triangle 13, 12, 5 shall necessarily solve the Question, as well as $\frac{13}{8}$, $\frac{12}{8}$, $\frac{5}{8}$.

QUEST. 97. (This is a *Lemma,* used by *Dioph.* in resolving the following *Quest.* 98.)

To find a right-angled Triangle and a Square in rational numbers, that if the Area of the Triangle be subtracted from the Square, the remainder multiplied by a given number (d) may produce a square number.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and $\frac{1}{a}$; so the three sides will be . . . . $\left\{\begin{array}{l} aa + \frac{1}{aa} = \text{the Hypoth.} \\ aa - \frac{1}{aa} = \text{the Perpend.} \\ 2 = \text{the Base.} \end{array}\right.$

2. Therefore the Area, (by multiplying the Perpendicular into half the Base,) is . . . . $\}$ $aa - \frac{1}{aa}$

3. Then

3. Then feign the ſide of the Square ſought to be . . . . . $a + \frac{2d}{a}$

4. Therefore the Square it ſelf is . . . . $aa + 4d + \frac{4dd}{aa}$

5. From which Square, the Area above expreſt being ſubtracted, there will remain . . . . $4d + \frac{4dd + 1}{aa}$

6. That is, ... $\frac{4daa + 4dd + 1}{aa}$

7. Then the ſaid remainder being multiplied into the given number (d) produceth . . . . $\frac{4ddaa + 4ddd + d}{aa}$

8. Which Product muſt (according to the Queſtion) be a Square: But the Denominator aa is a Square, it remains therefore to equate the Numerator to a Square, *viz.* $4ddaa + 4ddd + d$ muſt be equated to a Square, the ſide whereof may be variouſly feigned, let it be $2da + d$; and then the Square of $2da + d$ being equated to the ſaid $4ddaa + 4ddd + d$, this Equation ariſeth, to wit,

$$4ddaa + 4dda + dd = 4ddaa + 4ddd + d.$$

9. Which Equation after due Reduction gives . . . . . . $a = \frac{4dd + 1 - d}{4d}$

From the ninth, firſt and third ſteps ariſeth this

CANON.

10. From $\frac{4dd + 1 - d}{4d}$ and $\frac{4d}{4dd + 1 - d}$ form a right-angled Triangle, which ſhall be that ſought by the Queſtion; and the ſide of the Square ſought ſhall be $\frac{4dd + 1 - d}{4d} + \frac{8dd}{4dd + 1 - d}$.

An Example in Numbers.

Suppoſe $5 = d$ the number given; then form a right-angled Triangle from $\frac{24}{5}$ and $\frac{5}{24}$, ſo the Hypothenuſal will be $\frac{332401}{14400}$, the Perpendicular $\frac{331151}{14400}$, and the Baſe 2; moreover, the ſide of the Square ſought will be $\frac{413}{60}$, and the Square it ſelf $\frac{170569}{3600}$, or (in the ſame Denominator with the ſaid Hypothenuſal and Perpendicular) $\frac{682276}{14400}$: which Square and Triangle will ſolve the Queſtion; for if the Area of the Triangle, to wit, $\frac{331151}{14400}$ be ſubtracted from the ſaid Square $\frac{682276}{14400}$, the remainder $\frac{351125}{14400}$, that is, $\frac{14045}{576}$ multiplied into the given number 5, produceth the Square $\frac{70225}{576}$, whoſe Root is $\frac{265}{24}$.

Another Example.

Suppoſe $3 = d$ the number given in the Queſtion; then let a right-angled Triangle be formed from $\frac{17}{6}$ and $\frac{6}{17}$, (which numbers are diſcovered by the preceding Canon,) ſo the Hypothenuſal will be $\frac{84817}{10404}$, the Perpendicular $\frac{82225}{10404}$, and the Baſe 2: moreover, the ſide of the Square ſought will be found $\frac{505}{102}$, and the Square it ſelf $\frac{255025}{10404}$; from which Square ſubtracting the Area of the Triangle, to wit, $\frac{82225}{10404}$, the remainder $\frac{172800}{10404}$, or in its leaſt terms $\frac{4800}{289}$, multiplied into the given number 3, produceth the Square $\frac{14400}{289}$, whoſe ſide is $\frac{120}{17}$.

Note. Inſtead of 2 which is prefixt to d in the Numerator of the Fraction $\frac{2d}{a}$ in the third ſtep of the preceding Reſolution, you may take the half of any ſquare number and prefix it to d for a Numerator, over the Denominator a; as, $4\frac{1}{2}d$, $8d$, $12\frac{1}{2}d$, &c. and then by proſecuting the work as before from the third ſtep to the end of the Reſolution, various Anſwers to the Queſtion from one and the ſame given number will be diſcovered.

QUEST. 98. (Quæſt. 3. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the Area thereof increaſed with a given number, ſuppoſe 5, may make a Square.

RESOLUTION.

1. Let the ſides of ſome known right-angled Triangle, as 5, 4 and 3 be ſeverally multiplied by a, and take the Products to repreſent the ſides of the Triangle ſought, to wit, ... $5a$, $4a$, $3a$

2. Then the Area thereof increased with 5 (the number given in the Question) makes . . . . $6aa + 5$
3. Which summ must be equal to a Square, suppose it be $9aa$, therefore . . . . $6aa + 5 = 9aa$
4. Whence by equal subtraction of $6aa$, there remains . . . . $5 = 3aa$
5. Let each part of the last Equation be multiplied by 5, and it produceth . . . . $25 = 15aa$
6. Now if 15 which is prefix'd to aa were a Square, then the value of a would be rational: Whence therefore comes 15? Examin the work, and you will find, that by subtracting 6 the Area of the Triangle 5, 4, 3 from the square number 9, and then by multiplying the remainder 3 into the given number 5, there is produced 15; whereby it is manifest that the scope of our search must be to find a right-angled Triangle and a square number, that the Area of the Triangle being subtracted from the Square and the remainder multiplied by the given number 5 may make a Square. But the preceding 97th Question shews how to find out such a Triangle and Square; take if you please those there found in the first Example, to wit, the right-angled Triangle whose Hypothenusal is $\frac{332401}{14400}$, Perpendicular $\frac{331151}{14400}$, and Base 2; and the Square $\frac{682276}{14400}$, whose side is $\frac{826}{120}$: Then renew the Resolution thus,
7. For the three sides of the Triangle sought put . . $\frac{332401}{14400}a$, $\frac{331151}{14400}a$, $2a$
8. The Area thereof increased with 5 makes . . . $\frac{331151}{14400}aa + 5$
9. Which summ must be equated to the Product of $\frac{682276}{14400}$ (the Square before found) multiplied into aa, so this Equation ariseth, ... $\frac{331151}{14400}aa + 5 = \frac{682276}{14400}aa$
10. Therefore by subtracting $\frac{331151}{14400}aa$ from each part of that Equation this remains, to wit, ... $5 = \frac{351125}{14400}aa = \frac{14045}{576}aa$
11. And by multiplying each part of the last Equation into 5 it produceth . . . . $25 = \frac{70225}{576}aa$
12. And by extracting the square Root out of each part of the last Equation, there will arise . . . $5 = \frac{265}{24}a$
13. Whence by dividing each part by $\frac{265}{24}$, the value of a will be made known, *viz.* . . . . . . $a = \frac{24}{53}$
14. Wherefore from the thirteenth and seventh steps the three sides of the Triangle sought will be found these, to wit, $\frac{332401}{31800}$, $\frac{331151}{31800}$, $\frac{48}{53}$, the Area whereof is $\frac{331151}{70225}$, to which adding 5, the summ will be the Square $\frac{682276}{70225}$, whose Root is $\frac{826}{265}$. Therefore the Question is solved.

Vieta, in the 9th of the 5th Book of his *Zeteticks*, shews how to find out a right-angled Triangle whose Area increased with a number compos'd of two Squares may make a Square; whereby 'tis probable he thought this Question to be applicable only to a number compos'd of two Squares; because *Diophantus* propos'd the given number 5, which is compos'd of two Squares; but 'tis evident from the precedent Resolution that the Question may be extended to any given number whatsoever: And for greater illustration, let it be required to find out a right-angled Triangle whose Area increased with 3 may make a Square.

First a right-angled Triangle is to be sought, and also a Square, that the Area of the right-angled Triangle being subtracted from the Square and the remainder multiplied by the given number 3, the Product may be a square number: But by the latter Example of the preceding 97th Question such a Triangle and Square are found out, *viz.* the Triangle whose Hypothenusal is $\frac{84817}{10404}$, the Perpendicular $\frac{82225}{10404}$, and the Base 2; and the Square $\frac{255025}{10404}$, whose side is $\frac{505}{102}$: then for the three sides of the right-angled Triangle sought let there be put $\frac{84817}{10404}a$, $\frac{82225}{10404}a$ and $2a$, the Area whereof is $\frac{82225}{10404}aa$, to which adding 3 it makes $\frac{82225}{10404}aa + 3$; which summ equated to $\frac{255025}{10404}aa$, ($\frac{255025}{10404}$ being the Square above found,) will give $\frac{17}{40}$ for the value of a, by which $\frac{17}{40}$ the three sides above put being resolved, you will find the three sides of the right-angled Triangle sought to be these, to wit, $\frac{84817}{24480}$, $\frac{16445}{4896}$ and $\frac{34}{40}$; the Area of which Triangle is $\frac{3289}{2304}$, to which adding 3 it makes the Square $\frac{10201}{2304}$, whose side is $\frac{101}{48}$.

QUEST. 99.

QUEST. 99. (Quæſt. 6. Lib. 6. *Diophant.*)

To find a right-angled Triangle, that the Area thereof increaſed with one of the ſides about the right-angle may make a given number, ſuppoſe n.

RESOLUTION.

1. For the ſides about the right-angle of the Triangle ſought put . . . a and e
2. Then is the Area . . . . $\frac{1}{2}ae$
3. Which increaſed with one of the ſides about the right-angle muſt make the given number n, hence this Equation, ... $\frac{1}{2}ae + a = n$
4. From which Equation, after due Reduction, you will find . . $e = \frac{n - a}{\frac{1}{2}a}$
5. And becauſe $aa + ee$ muſt be equal to the Square of the Hypothenuſal, it follows from the firſt and fourth ſteps, that the Square of the Hypothenuſal muſt be equal to the ſumm of the Squares of a and $\frac{n - a}{\frac{1}{2}a}$; that is,
$$\frac{\frac{1}{4}aaaa + aa - 2na + nn}{\frac{1}{4}aa}.$$
6. Of which fractional quantity the Denominator $\frac{1}{4}aa$ is a Square, whoſe ſide is $\frac{1}{2}a$; if therefore the Numerator were a Square the whole Fraction would be a Square: It remains then to equate the Numerator $\frac{1}{4}aaaa + aa - 2na + nn$ to ſome Square, to which end, let its ſide be feigned $\frac{1}{2}aa - n$, and then the Square of $\frac{1}{2}aa - n$ being equated to the ſaid Numerator, this Equation ariſeth, *viz.*
$$\tfrac{1}{4}aaaa + aa - 2na + nn = \tfrac{1}{4}aaaa - naa + nn.$$
7. Whence the value of a will be made known, *viz.* . . . . . $a = \frac{2n}{n + 1}$
8. And according to that value of a, the value of e will alſo be be made known by the fourth ſtep, *viz.* . . . . $e = n - 1$
9. And becauſe the ſquare Root of $aa + ee$ is equal to the Hypothenuſal, therefore the ſquare Root of the ſumm of the Squares of the two quantities in the latter parts of the Equations in the ſeventh and eighth ſteps, will give for the Hypothenuſal ſought $\frac{nn + 1}{n + 1}$
10. Therefore from the ſeventh, eighth and ninth ſteps the three ſides of the right-angled Triangle ſought are theſe, to wit,
$$\frac{2n}{n + 1},\ n - 1,\ \left(\text{or } \frac{nn - 1}{n + 1},\right) \text{ and } \frac{nn + 1}{n + 1}.$$

Which three ſides if they be expreſt by words will give the following Canon, which is the ſame with that delivered by *Fermat* in his Obſervation upon the ſixth Queſtion of the ſixth Book of *Diophantus*.

CANON.

11. When the given number is greater than unity let a right-angled Triangle be formed from thoſe two numbers, and then divide the three ſides ſeverally by the ſumm of the given number and unity; ſo ſhall the Quotients be the three ſides of the right-angled Triangle ſought.

An Example in Numbers.

Let it be required to find a right-angled Triangle whoſe Area increaſed with one of the ſides about the right-angle may make 7.

Firſt (as the Canon directs) let a right-angled Triangle be formed from the given number 7 and 1, ſo the three ſides will be theſe, . . . 50, 48, 14

Then divide thoſe three ſides ſeverally by 8, (to wit, 7 + 1,) ſo the Quotients ſhall be the ſides of the right-angled Triangle ſought, to wit, ... $6\frac{1}{4}$, 6, & $1\frac{3}{4}$

I ſay $6\frac{1}{4}$, 6, and $1\frac{3}{4}$ are the ſides of a right-angled Triangle, which will ſolve the Queſtion; for the Area $\frac{21}{4}$ increaſed with $\frac{7}{4}$ (one of the ſides about the right-angle) makes $\frac{28}{4}$, that is, 7, as was required.

But becauſe the ſaid Canon takes not place unleſs the given number exceed unity, I ſhall in the next place explain *Diophantus*'s Reſolution of this Queſtion, by which way, whatever the given number be, a right-angled Triangle may be found out to ſolve the Queſtion propoſed.

Another

Another way of resolving the foregoing 99^{th} Question.

1. Let h, b, p represent the Hypothenusal, Base and Perpendicular of some right-angled Triangle known in numbers, then multiply those three sides severally by a, (which represents a number unknown,) and take the Products for the three sides of the Triangle sought, to wit, ... ha, ba; pa
2. The Area of which Triangle increased with one of the sides about the right-angle, suppose with ba, must be equal to the given number n, therefore . . . . $\frac{1}{2}bpaa + ba = n$
3. Which Equation divided by $\frac{1}{2}bp$ gives . . . . $aa + \frac{b}{\frac{1}{2}bp}a = \frac{n}{\frac{1}{2}bp}$
4. Now to the end that the value of a in the last Equation may be a rational number, the Square of half the Coefficient which is drawn into a, together with the absolute quantity which possesseth the latter part of the Equation must make a Square, (as is evident by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1.) therefore . . . . $\frac{\frac{1}{4}bb + \frac{1}{2}bpn}{\frac{1}{4}bbpp} = \square$
5. And because the Denominator $\frac{1}{4}bbpp$ is a Square, it remains only to equate the Numerator to a Square, *viz.* . . . $\frac{1}{4}bb + \frac{1}{2}bpn = \square$
6. Or, to avoid Fractions, let the said $\frac{1}{4}bb + \frac{1}{2}bpn$ be multiplied by 4, and then . . . . $bb + 2bpn = \square$
7. Which last Equation shews, that in order to the solving of the Question proposed, a right-angled Triangle must first be found, such, that the Square of one of the sides about the right-angle, together with the Product of the quadruple Area multiplied by the given number n, may make a Square. Now to find out such a Triangle,
8. For one of the sides about the right-angle put . . . . . e
9. And for the other side put some square number, as, . . . 1
10. Then the quadruple of the Area is . . . . $2e$
11. Which multiplied by the given number n, suppose by $\frac{1}{2}$, makes e
12. To which Product add the Square of one of the sides about the right-angle, to wit, the Square of 1, which is also 1, and the summ must be equal to a Square, *viz.* . . . . . . $e + 1 = \square$
13. Also the summ of the Squares of the sides about the right-angle must be equal to a Square, to wit, the Square of the Hypothenusal, therefore . . . . $ee + 1 = \square$
14. Now in order to resolve the Duplicate equality in the two last steps, first the difference of the two quantities which are to be equated to Squares is . . . . $ee - e$
15. Which difference is equal to the Product of the multiplication of these two quantities, or Factors, to wit, . . . $2e - 2$ and $\frac{1}{2}e$
16. The half-summ of those two Factors is . . . . . . $\frac{5}{4}e - 1$
17. The Square of which half-summ being equated to $ee + 1$, will give . . . . $e = \frac{40}{9}$
18. And because e and 1 were put for the sides about the right-angle, therefore the square Root of the summ of the Squares of $\frac{40}{9}$ (that is e) and 1, shall be the Hypothenusal, to wit, $\frac{41}{9}$; so we have found out a right-angled Triangle whose three sides are $\frac{41}{9}$, $\frac{40}{9}$ and 1, which are fit for renewing the search of the right-angled Triangle sought by the Question, in this manner, *viz.*
19. For the three sides of the right-angled Triangle sought put $\frac{41}{9}a$, $\frac{40}{9}a$, and a
20. The Area of which Triangle increased with one of the sides about the right-angle, suppose with a, must be equal to the given number $\frac{1}{2}$, therefore . . . . $\frac{20}{9}aa + a = \frac{1}{2}$
21. Which Equation being resolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. will give . . . . $\therefore a = \frac{3}{10}$
22. According to which value of a, the three sides in the nineteenth step being resolved, there will be produced the three sides of the right-angled Triangle sought, to wit, . . . $\frac{41}{30}$, $\frac{40}{30}$, $\frac{9}{30}$

I say $\frac{41}{30}$, $\frac{40}{30}$ and $\frac{9}{30}$ are the three sides of a right-angled Triangle which will solve the Question;

Question; for the Area $\frac{6}{30}$ increased with $\frac{9}{30}$, (one of the sides about the right-angle,) makes $\frac{15}{30}$, that is, $\frac{1}{2}$; as was required.

QUEST. 100.

To find a right-angled Triangle in rational numbers, that the Area subtracted from one of the sides about the right-angle may leave a given number; let the given number be n.

RESOLUTION.

1. For the sides about the right-angle of the Triangle sought put . . . . a and e
2. Then is the Area . . . . $\frac{1}{2}ae$
3. Which subtracted from one of the sides about the right-angle, suppose from a, leaves . . . . $a - \frac{1}{2}ae = n$
4. Whence, after due Reduction, ... $e = \dfrac{a - n}{\frac{1}{2}a}$
5. And because $aa + ee$ must be equal to the Square of the Hypothenusal, it follows from the first and fourth steps, that the Square of the Hypothenusal must be equal to the summ of the Squares of a and $\dfrac{a-n}{\frac{1}{2}a}$, that is, $\dfrac{\frac{1}{4}aaaa + aa - 2na + nn}{\frac{1}{4}aa}$
6. And since the Denominator $\frac{1}{4}aa$ is a Square, whose side is $\frac{1}{2}a$, it remains only to equate the Numerator $\frac{1}{4}aaaa + aa - 2na + nn$ to a Square, whose side may be feigned either $\frac{1}{2}aa - n$ or $\frac{1}{2}aa + n$; first then, let the side of the said Square be feigned $\frac{1}{2}aa - n$, and then the Square of $\frac{1}{2}aa - n$ being equated to the said $\frac{1}{4}aaaa + aa - 2na + nn$, this Equation ariseth, *viz.*
$$\tfrac{1}{4}aaaa + aa - 2na + nn = \tfrac{1}{4}aaaa - naa + nn.$$
7. From which Equation the value of a will be made known, *viz.* . . . . $a = \dfrac{2n}{1+n}$
8. According to which value of a, the number e will be discovered from the fourth step, *viz.* . . . . $e = 1 - n$
9. And the square Root of the summ of the Squares of $\dfrac{2n}{1+n}$ and $1 - n$ shall be the Hypothenusal, to wit, ... $\dfrac{1+nn}{1+n}$
10. Therefore from the three last preceding steps, the three sides of the right-angled Triangle sought are these, to wit,
$$\frac{2n}{1+n},\ 1-n,\ \left(\text{or } \frac{1-nn}{1+n},\right) \text{ and } \frac{1+nn}{1+n}.$$
11. But if instead of $\frac{1}{2}aa - n$, which in the sixth step was feigned for the side of a Square, we assume $\frac{1}{2}aa + n$, and equate the Square of this side to the before-mentioned $\frac{1}{4}aaaa + aa - 2na + nn$, there will arise $a = \dfrac{2n}{1-n}$; according to which value, the sides of the Triangle sought will be found these, to wit,
$$\frac{2n}{1-n},\ \frac{1-nn}{1-n},\ \frac{1+nn}{1-n}.$$

The two last steps give this

CANON.

12. When the given number is less than unity, let a right-angled Triangle be formed from unity and the given number; then divide the three sides severally by the summ or difference of unity and the given number, so shall the Quotients be the sides of the Triangle sought.

As, for example, if it be desired to find out a right-angled Triangle, that the Area subtracted from one of the sides about the right-angle may leave $\frac{4}{7}$, the Canon will discover the sides of two Triangles, to wit, $\frac{65}{77}$, $\frac{33}{77}$, $\frac{56}{77}$ and $\frac{65}{21}$, $\frac{33}{21}$, $\frac{56}{21}$, each of which Triangles will satisfie your desire; for in the first Triangle the Area $\frac{12}{77}$ subtracted from $\frac{56}{77}$, (one of the sides about the right-angle,) leaves the given number $\frac{4}{7}$; likewise in the latter Triangle, the Area $\frac{44}{21}$ subtracted from $\frac{56}{21}$ leaves $\frac{4}{7}$.

But how to solve this Question when the given number is any number whatever, I shall hereafter shew by *Fermat*'s method, in *Quest.* 130. of this Book.

QUEST. 101.

QUEST. 101. (Quæst. 12. Lib. 6. *Diophant.*)

To find a right-angled Triangle, that as well the difference of the sides about the right-angle as the greater of the same sides may be a Square; and that the Area, with the lesser of the sides about the right-angle may make a Square.

The Resolution of this Question depends upon three *Lemma's*, which I shall first explain.

LEMMA 1.

1. If a right-angled Triangle be formed from two numbers whereof the greater is the double of the lesser, as well the difference of the sides about the right-angle as the greater of the same sides shall be a Square. Moreover, if the Area of the said Triangle be multiplied by the Square of a Fraction having unity for its Numerator, and the lesser of the two numbers by which the said Triangle was formed for a Denominator, the Product increased with the lesser of the sides about the right-angle, will make a Square containing nine times the Square of the lesser of the two numbers by which the said Triangle was formed.

To make this manifest, let a right-angled Triangle be formed from a and $2a$, so the three sides will be these, to wit, . . } $5aa$; $3aa$, $4aa$

Whence 'tis evident, first, that the difference of the sides about the right-angle is a Square, to wit, aa; secondly, that the greater of the sides about the right-angle is a Square, to wit, $4aa$; and lastly, if the Area $6aaaa$ be multiplied by the Square of $\frac{1}{a}$, that is, by $\frac{1}{aa}$, the Product $6aa$ increased with $3aa$, (that is, the lesser of the sides about the right-angle,) makes the Square $9aa$, which contains nine times the Square of the lesser of the two numbers by which the said right-angled Triangle was formed. Which was to be shewn.

LEMMA 2.

2. Two numbers being given whose summ is a Square, to find innumerable Squares, every one of which being multiplied by one of the given numbers, and taking to the Product the other number, may make a Square.

Let there be two given numbers 6 and 3, and let it be desired to find a Square, such that if it be multiplied by 6, and 3 be added to the Product, the summ may make a Square.

Let the side of the Square sought be	$a+1$
Whence the Square it self is	$aa+2a+1$
Which multiplied by 6 produceth	$6aa+12a+6$
To which Product add 3 and it makes	$6aa+12a+9$

Which $6aa+12a+9$ is to be equated to a Square, whose side, (because the absolute number 9 is a Square,) may be variously feigned; let it then be $3a-3$, the Square whereof, to wit, $9aa-18a+9$ being equated to the said $6aa+12a+9$ will give $a=10$; wherefore $a+1$ the side of the Square sought is 11, and the Square it self is 121, which multiplied by 6 produceth 726, to which adding 3 it makes 729, which is a Square whose side is 27. It is also evident, that instead of 121 innumerable other Squares may be found out to perform the like effect, because the side of the Square to be equated to $6aa+12a+9$ may be feigned infinitely.

In like manner if it were desired to find a Square which multiplied into 3, and taking 6 to the Product, may make a Square; let the Square sought be feigned $aa+2a+1$, this multiplied by 3 and the Product increased with 6, gives $3aa+6a+9$ to be equated to a Square whose side may be feigned $3a-3$, whence $a=4$, and therefore $a+1$ the side of the Square sought is 5.

LEMMA 3.

3. If two numbers b and c whose summ makes not a Square be given, such, that by multiplying one of them, suppose b, by a given Square dd, and by adding to the Product the other number c, the summ $bdd+c$ makes a Square; we may find innumerable other Squares instead of the given Square dd to produce the like effect.

$$\text{Suppose . . . } \left\{ \begin{array}{l} b = 2, \\ c = 8, \\ d = 2, \\ dd = 4; \end{array} \right\} \text{ whence } bdd+c = 16.$$

Now

Now it is required to find another Square inſtead of *dd*, or 4, that if the Square found out be multiplied by 2, to wit, *b*; and the Product be increaſed with 8, to wit, *c*, the the ſumm may be a Square.

For the ſide of the Square ſought put *a* + the ſide of the given Square *dd*, viz. . . . . $a + 2$

Whence the Square ſought is . . . . $aa + 4a + 4$

Which multiplied by 2, (to wit, *b*,) produceth . . . . . $2aa + 8a + 8$

To which Product add 8, (to wit, *c*,) and the ſumm is . . $2aa + 8a + 16$

Which ſumm is to be equated to a Square, the ſide whereof (in regard 16 is a Square) may be variouſly feigned, let it then be $2a - 4$, the Square whereof being equated to the ſaid $2aa + 8a + 16$ will give $a = 12$; wherefore $a + 2 = 14$ the ſide of the Square ſought, and the Square it ſelf is 196, which multiplied by 2, and taking 8 to the Product makes the Square 400, whoſe ſide is 20. I ſhall now proceed to

The RESOLUTION of the preceding QUEST. 101.

4. Let a right-angled Triangle be formed from two ſuch numbers, that the greater may be the double of the leſſer, as from *a* and 2*a*, ſo the three ſides will be theſe, to wit, ... $5aa$, $3aa$, $4aa$

Whence it is evident that the two firſt parts of the Queſtion are ſatisfied, for as well the difference of the ſides about the right-angle as the greater of them is a Square. It remains that we ſee whether the Area of the Triangle, with the leſſer of the ſides about the right-angle makes a Square: But it makes $6aaaa + 3aa$, and by dividing all by *aa* there ariſeth $6aa + 3$ to be equated to a Square, which is poſſible to be done, becauſe the ſumm of 6 and 3 is a Square; for if *a* be 1, then *aa* is alſo 1, and conſequently $6aa + 3$ makes the ſame Square as $6 + 3$, to wit, 9. Since then *a* is found 1, a right-angled Triangle formed from 1 and 2 (that is, *a* and 2*a*) will ſolve the Queſtion. Wherefore the ſides of the Triangles ſought are 5, 3, 4.

5. Having found out one Square for the value of *aa*, to wit, 1, we may by the help of the preceding *Lemma* 2. find out innumerable other Squares to perform the ſame effect; ſo inſtead of 1, the Square 121 is found out in the Example of the ſaid *Lemma*. And therefore let a Triangle be formed from 11 and 22, and the ſides will be 605, 363, 484, which will ſolve the Queſtion; for the difference of the ſides about the right-angle is the Square 121, and the greater of the ſides about the right-angle, to wit, 484, is a Square whoſe ſide is 22; alſo the Area added to the leſſer ſide makes 88209, which is a Square whoſe ſide is 297.

6. But for removing an Obſtruction which the Learner may meet with, let a right-angled Triangle be formed from 2*a* and 4*a*, whence the ſides are $20aa$, $12aa$ and $16aa$; where as well the difference of the ſides about the right-angle, as the greater of them is a Square. But the Area increaſed with the leſſer ſide makes $96aaaa + 12aa$, and by dividing all by *aa* it makes $96aa + 12$, which muſt be equated to a Square. But how ſhall that be done, ſince 96 which is multiplied into *aa* is not a Square, neither is the abſolute number 12 a Square, nor is the ſumm of 96 and 12 a Square? This knot, though it ſeems to be a very hard one, may yet by the help of the laſt clauſe of the preceding *Lemma* 1. be eaſily untied: For ſince 96 is the Area of a right-angled Triangle formed from 2 and 4, and 12 is the leſſer of the ſides about the right-angle, it will appear by *Lemma* 1. that if 96 be multiplied by $\frac{1}{4}$, and the Product 24 be increaſed with 12 it ſhall neceſſarily make a Square, to wit, 36; and conſequently by *Lemma* 3. we may find out innumerable Squares inſtead of $\frac{1}{4}$, every one of which being multiplied by 96, and taking 12 to the Product will make a Square. As, for example,

For the ſide of a Square to be found out inſtead of $\frac{1}{4}$, let there be put *a* — the ſquare Root of $\frac{1}{4}$, *viz.* . . . . $a - \frac{1}{2}$

Then the Square of $a - \frac{1}{2}$ is . . . . $aa - a + \frac{1}{4}$

Which multiplied by 96 produceth . . . . $96aa - 96a + 24$

To which Product adding 12 it makes . . . . $96aa - 96a + 36$

Which $96aa - 96a + 36$ muſt be equated to a Square, (the ſide whereof in regard 36 is a Square,) may be variouſly feigned, let it then be $4a - 6$, the Square whereof being equated to $96aa - 96a + 36$ will give $a = \frac{3}{5}$; therefore $a - \frac{1}{2}$, (that is, $\frac{6}{10} - \frac{5}{10}$) will give $\frac{1}{10}$ for the ſide of the Square ſought and the Square it ſelf is $\frac{1}{100}$, by which if you

S multiply

multiply 96, and to the Product add 12, it makes the Square $\frac{324}{25}$, whose side is $\frac{18}{5}$. Now forasmuch as the said side $\frac{1}{10}$ is to be taken for the value of a in the $2a$ and $4a$ by which the Triangle was first formed, let a Triangle be now formed from $\frac{1}{5}$ and $\frac{2}{5}$, and the three sides are $\frac{1}{5}$, $\frac{4}{25}$, $\frac{3}{25}$; where 'tis evident that the difference of the sides about the right-angle, as also the greater of them is a Square: But the Area is $\frac{6}{625}$, to which if you add the lesser side $\frac{3}{25}$, it makes the Square $\frac{81}{625}$, whose side is $\frac{9}{25}$.

QUEST. 102.

To find a right-angled Triangle, and a square number, such, that if the Square be multiplied by the lesser of the sides about the right-angle, and to that Product there be added the Product made by the multiplication of the Area of the said Triangle into the difference of the sides about the right-angle, the summ may be a Square. Moreover, that the greater of the sides about the right-angle may be a Square.

CANON.

1. By the preceding *Quest.* 101. find a right-angled Triangle, that as well the difference of the sides about the right-angle, as the greater of the same sides, may be a Square: Moreover, that the Area increased with the lesser of the said sides may make a Square; so shall such Triangle be that which is sought by this *Quest.* 102. and the difference of the sides about the right-angle shall be the first Square sought.

But that this Canon will solve the Question proposed, I demonstrate thus;

2. Suppose . . . . $\left\{\begin{array}{l} h = \text{the Hypothenusal}; \\ b = \text{the greater side}, \\ p = \text{the lesser side}, \end{array}\right. \left.\begin{array}{l} \\ \\ \end{array}\right\}$ about the right-angle.

3. Suppose also that by the preceding *Quest.* 101. h, b, p are found such, that . . . . . $\left\{\begin{array}{l} b-p, \\ b, \\ \frac{1}{2}bp+p, \end{array}\right\}$ are three square numbers.

4. I say $b-p$ is such a square number, that if it be multiplied by p the lesser of the sides about the right-angle, and to the Product there be added the Product made by the multiplication of the Area $\frac{1}{2}bp$, into $b-p$ the difference of the sides about the right-angle, the summ shall be a Square: For,

5. The Product of $b-p$ into p is . . . . $bp-pp$

6. To which if you add the Product of the Area into the difference of the sides about the right-angle, to wit, . . $\frac{1}{2}bbp-\frac{1}{2}bpp$

7. The summ is . . . . $bp-pp+\frac{1}{2}bbp-\frac{1}{2}bpp$

8. Which summ is a square number, for it is equal to the Product of $b-p$ multiplied into $\frac{1}{2}bp+p$: But each of these Factors $b-p$ and $\frac{1}{2}bp+p$ is a Square by Construction; wherefore the Product of their multiplication, to wit, $bp-pp+\frac{1}{2}bbp-\frac{1}{2}bpp$ is a Square.

An Example in Numbers.

9. Take any right-angled Triangle which will solve the preceding *Quest.* 101. as, ... 5 ; 3 ; 4

In which Triangle the difference of the sides about the right-angle, to wit, 1, is such a Square that if it be multiplied by 3 the lesser side, and the Product be increased with 6, to wit, the Product of the Area multiplied by the difference of the sides about the right-angle it makes the Square 9. Moreover, the greater side about the right-angle, to wit, 4, is a Square; as was required.

10. But the same right-angled Triangle 5, 3, 4 being retained, we may instead of the Square 1, (to wit, the difference of the sides about the right angle) find out innumerable Squares, (by the help of *Lemma* 2. in *Quest.* 101.) every one of which shall solve this Question, and be within given limits if need require. So if it were desired to find out a Square greater than 6, and such as together with the said right-angled Triangle 5, 3, 4 may solve this 102^d Question, the said *Lemma* 2. will discover the Squares 25 and 361, (among innumerable others,) which are such, that if each of them be multiplied by 3, (the lesser of the sides about the right-angle of the said Triangle,) and the Products 75 and 1083 be severally increased with 6, to wit, the Product of the Area multiplied into the difference of the sides about the right-angle, it makes the Squares 81 and 1089, whose sides are 9 and 33.

11. And

11. And in like manner, by the help of any other right-angled Triangle which will solve the preceding *Quest.* 101. as the Triangle 605, 363, 484, we may find out innumerable Answers to this *Quest.* 102. For, first, the difference of the sides about the right-angle, to wit, 121 is a Square, and such, that if it be multiplied by the lesser side 363, and to the Product 43923 there be added 10629366, to wit, the Product of the Area multiplied into the difference of the sides about the right-angle, it makes the Square 10673289, whose side is 3267. And therefore by the help of the third *Lemma* which belongs to the Resolution of the preceding *Quest.* 101. you may instead of the Square 121 find out infinite other Squares to perform the same thing, and each Square shall be within given limits if need require.

QUEST. 103. (Quæst. 13. Lib. 6. *Diophant.*)

To find a right-angled Triangle, that the Area thereof being increased severally with each of the sides about the right-angle may make Squares.

RESOLUTION.

1. Let the Hypothenusal, Base and Perpendicular of some right-angled Triangle in numbers be represented by . .	$h\ ;\ b\ ;\ p$
2. Then multiply those sides severally by a, and suppose the Products to be the sides of the right-angled Triangle sought, to wit, ...	$ha\ ,\ ba\ ,\ pa$
3. Then, (according to the Question,) the Area of the Triangle in the last step being increased with each of the sides about the right-angle must make a Square, hence this Duplicate equality ariseth to be resolved, *viz.* . .	$\frac{1}{2}bpaa + ba = \square$ $\frac{1}{2}bpaa + pa = \square$
4. Now suppose	$\frac{1}{2}bpaa + ba = eeaa$
5. Whence, after due Reduction, you will find	$a = \dfrac{b}{ee - \frac{1}{2}bp}$
6. According to which value of a, the latter of the two quantities in the third step being resolved, instead of that quantity this that follows ariseth to be equated to a Square, *viz.*	$\dfrac{bpee + \frac{1}{2}bbbp - \frac{1}{2}bbpp}{eeee - bpee + \frac{1}{4}bbpp} = \square$

7. But because the Denominator of the Fraction last exprest is a Square, whose side is $ee - \frac{1}{2}bp$, it remains only to equate the Numerator to a Square: And because a Square divided by a Square gives the Quotient a Square, therefore if we suppose b in the said Numerator $bpee + \frac{1}{2}bbbp - \frac{1}{2}bbpp$ to be a square number, then the said Numerator divided by b gives $pee + \frac{1}{2}bbp - \frac{1}{2}bpp$, that is, $pee, + \frac{1}{2}bp \times \overline{b - p}$ to be equated to a Square. So that the matter is reduced to this, we must find out a right-angled Triangle, such, that the greater of the sides about the right-angle, suppose b, may be a square number; and we must also find another square number, suppose ee, that may be greater than the Area of the said Triangle, (as may be inferr'd from the Denominator of the Fraction in the fifth step,) and such, that if it be multiplied by p the lesser of the sides about the right-angle, and to the Product there be added the Product of the Area multiplied into the difference of the sides about the right-angle, the summ may make a Square: But such a right-angled Triangle and Square, the preceding *Quest.* 102. shews how to find out. Suppose then the Hypothenusal, Base and Perpendicular of the said Triangle so found out to be h, b, p; and the Square to be ee, all which being known in numbers, the number represented by a shall consequently be known from the fifth step: And lastly, if you multiply the numbers h, b, p severally by the number a, the Products shall be the three sides of the Triangle sought. From the premisses there ariseth the following

CANON.

First, by the Canon of the preceding *Quest.* 102. find out a right-angled Triangle, (whose three sides in the Resolution of this *Quest.* 103. are represented by h, b, p;) and besides, a Square, call it ee, that may be greater than the Area of the said Triangle, and such that if it be multiplied by the lesser of the sides about the right-angle, and to the Product there be added the Product of the Area multiplied into the difference of the same sides, the summ may be a Square; then divide the greater of the sides about the right-angle of the said Triangle, by the excess of the said Square ee above the Area, and by the Quotient multiply severally the three sides h, b, p, so shall the Products be the three sides of the Triangle sought.

An Example in Numbers.

It hath been ſhewn in *Sect.* 10. of the preceding *Queſt.* 102. that the right-angled Triangle 5, 3, 4 and the Square 25 will ſolve that Queſtion; and beſides, the ſaid Square 25 is greater than 6 the Area of the ſaid Triangle; therefore according to the directions of the Canon above-delivered, I divide 4 the greater of the ſides about the right-angle, by 19 the exceſs of the ſaid Square 25 above 6 the Area of the ſaid Triangle, and the Quotient is $\frac{4}{19}$, by which I multiply ſeverally 5, 3, 4 (the ſides of the Triangle firſt found) and there comes forth $\frac{20}{19}$, $\frac{12}{19}$, $\frac{16}{19}$, which ſhall be the ſides of the Triangle ſought; for the Area is $\frac{96}{361}$; to which if we add ſeverally $\frac{12}{19}$ and $\frac{16}{19}$, (the ſides about the right-angle,) there will be made the Squares $\frac{324}{361}$ and $\frac{400}{361}$, whoſe ſides are $\frac{18}{19}$ and $\frac{20}{19}$.

After the ſame manner, the ſame right-angled Triangle 5, 3, 4 and the Square 361, (found out alſo by the preceding *Queſt.* 102.) will diſcover $\frac{20}{355}$, $\frac{12}{355}$ and $\frac{16}{355}$ for the three ſides of another right-angled Triangle to ſolve this Queſtion, as may eaſily be proved. And becauſe innumerable right-angled Triangles and Squares may be found out to ſolve the ſaid *Queſt.* 102. infinite Anſwers may be given to this.

QUEST. 104. (Quæſt. 15. Lib. 6. *Diophant.*)

To find a right-angled Triangle, ſuch, that if from its Area the Hypothenuſal and one of the ſides about the right-angle be ſeverally ſubtracted, each remainder may be a Square.

The Reſolution of this and the following 105th Queſtion depends upon a *Lemma*, which I ſhall here premiſe and demonſtrate.

LEMMA.

1. If a right-angled Triangle be formed from two ſquare numbers, or from two numbers in proportion one to another as a ſquare number to a ſquare number; I ſay firſt, the Square of the difference of thoſe two ſquare numbers being multiplied by the Product of their multiplication will produce a Square leſs than the Area of the ſaid Triangle; ſecondly, if from the ſolid Product made by the multiplication of theſe three numbers, to wit, the ſquare number above produced leſs than the Area, the Hypothenuſal, and that ſide about the right-angle which is the double Product of the multiplication of the two ſquare numbers forming the Triangle, there be ſubtracted the ſolid Product made of theſe three numbers, to wit, the Area, the ſaid ſide about the right-angle, and the exceſs by which the Hypothenuſal exceeds the ſame ſide, the remainder ſhall be a ſquare number; thirdly and laſtly, the ſumm of thoſe two ſolid Products ſhall alſo be a ſquare number.

Demonſtration.

2. Let a right-angled Triangle be formed from two ſquare numbers, ſuppoſe bb the greater, and cc the leſſer, whoſe ſides are b and c, ſo the three ſides of the ſaid Triangle will be theſe,

$$\text{viz.}\left\{\begin{array}{rcl} bbbb + cccc & = & \text{the Hypothenuſal,} \\ bbbb - cccc & = & \text{the Baſe,} \\ 2bbcc & = & \text{the Perpendicular.} \end{array}\right.$$

3. The Area of the ſaid Triangle is . . . . . $b^6cc - bbc^6$
4. The Product of the multiplication of bb and cc, to wit, $bbcc$, being multiplied by the Square of $bb - cc$, produceth this Square, to wit, . . $b^6cc - 2b^4c^4 + bbc^6$.
5. Which Square is leſs than the Area . . . . . $b^6cc - bbc^6$
6. For it is evident that . . . . $bbbc - bccc ⊐ bbbc + bccc$
7. Therefore by multiplying each part into $bbbc - bccc$, it neceſſarily follows that . . . . $b^6cc - 2b^4c^4 + bbc^6 ⊐ b^6cc - bbc^6$

Which was affirmed in the firſt part of the *Lemma.*

8. In the next place, if theſe three following quantities be multiplied one into another,

$$\text{viz.}\left\{\begin{array}{rl} b^6cc - 2b^4c^4 + bbc^6, & (\text{the Square in the fourth ſtep,}) \\ b^4 + c^4, & (\text{the Hypothenuſal,}) \\ 2bbcc, & (\text{the Perpendicular;}) \end{array}\right.$$

The ſolid Product of their multiplication will be . . . . $2b^{12}c^4 - 4b^{10}c^6 + 4b^8c^8 - 4b^6c^{10} + 2b^4c^{12}$.

9. And

9. And if these three following quantities be multiplied one into another, to wit,

$b^6cc - bbc^6$, (the Area,)
$+ 2bbcc$, (the Perpendicular,)
$bbbb + cccc - 2bbcc$, (the excess of the Hypothenusal above the Perpendicular;)

The solid Product of their multiplication will be $\rangle$ $2b^{12}c^4 - 4b^{10}c^6 - 2b^4c^{12} + 4b^6c^{10}$.

10. Which latter solid Product subtracted from the former, leaves this Square, to wit, ... $4b^8c^8 - 8b^6c^{12} + 4b^4c^{12}$.
The side whereof is . . . . $\rangle$ $2b^4c^4 - 2bbc^6$.
Wherefore the second part of the *Lemma* is manifest.

11. Lastly, the summ of the two solid Products in the eighth and ninth steps makes this Square, to wit, . . . $4b^{12}c^4 - 8b^{10}c^6 + 4b^8c^8$
The side whereof is . . . . $\rangle$ $2b^6cc - 2b^4c^4$.
Wherefore the *Lemma* is every way proved.

An Example in Numbers.

Suppose $4 = bb$, and $1 = cc$; then let a right-angled Triangle be formed from 4 and 1, so the three sides will be 17, 15, 8. Now I say, first, that if 9 the Square of the difference between the said 4 and 1, be multiplied by 4 the Product of 4 and 1, there will be produced the square number 36, which is less than 60 the Area of the said Triangle.

Secondly, if from 4896, which is the solid Product made by the multiplication of these three numbers, to wit, 36 the Square before found, the Hypothenusal 17, and 8 the double Product of 4 and 1; there be subtracted 4320, which is the solid Product of these three numbers, to wit, the Area 60; the said double Product 8, and 9 the excess of the Hypothenusal above the said 8; there will remain the Square 576, whose side is 24.

Thirdly and lastly, the summ of the said solid Products 4896 and 4320 makes the Square 9216, whose side is 96.

Now followeth the RESOLUTION of QUEST. 104: before proposed.

1. Let the Hypothenusal, Base and Perpendicular of some right-angled Triangle in numbers be represented by . . $\}$ h , b , p
2. Then multiply those sides severally by a, and suppose the Products to be the sides of the Triangle sought, to wit, $\}$ ha , ba , pa
3. Then, (according to the Question,) if the Area of the Triangle in the last step be lessened by the Hypothenusal and one of the sides about the right-angle severally, each remainder must be a Square; hence this Duplicate equality ariseth to be resolved, *viz.* . . . . $\}$ $\frac{1}{2}bpaa - ha = \square$, $\frac{1}{2}bpaa - pa = \square$
4. Now in order to resolve that Duplicate equality, suppose $\rangle$ $\frac{1}{2}bpaa - pa = eeaa$
5. Whence after due Reduction you will find . . . . $\rangle$ . . . $a = \dfrac{p}{\frac{1}{2}bp - ee}$
6. According to which value of a, if the former of the two quantities in the third step be resolved, instead of that quantity, (to wit, $\frac{1}{2}bpaa - ha$) this that follows ariseth to be equated to a Square, *viz.* . . . . . $\}$ $\dfrac{hpee, - \frac{1}{2}bpp \times \overline{h - p}}{\frac{1}{4}bbpp - bpee + eeee} = \square$
7. But because the Denominator of the Fraction last exprest is a Square, for its side is $\frac{1}{2}bp - ee$, it remains only to equate the Numerator to a Square. We must therefore inquire into the rise of the Numerator, so we shall find that it imports the search of a right-angled Triangle h, b, p in numbers, and a square number ee less than the Area of the said Triangle; (for the Denominator of the Fraction in the fifth step of the Resolution shews that $\frac{1}{2}bp$ must exceed ee.) Moreover, the said Triangle and Square must be such, that if from the solid Product made by the multiplication of the said Square, the Hypothenusal, and one of the sides about the right-angle, there be subtracted the solid Product made by the multiplication of the Area, the said side about the right-angle, and the excess of the Hypothenusal above the same side, the remainder may be a Square. But the last preceding *Lemma* shews how to find out such a Triangle and Square. Suppose then the Hypothenusal, Base and Perpendicular of the said Triangle so found out, to wit, h, b, p, and the Square ee, to be all known in numbers, then the number represented by a shall consequently be known from the fifth step of the Resolution; and lastly, if you multiply the numbers h, b, p severally by the number a, the Products shall be the three sides of the Triangle sought. From the premisses there ariseth the following

CANON.

CANON.

8. Firſt, let a right-angled Triangle be formed from two ſquare numbers, or from two numbers which have ſuch proportion to one another as a ſquare number to a ſquare number; then multiply the Square of the difference of thoſe two numbers by the Product of their multiplication and reſerve the Product, which is a ſquare number, and may be called *ee*; that done, divide that ſide about the right-angle which is the double Product of the two numbers that formed the ſaid Triangle, by the exceſs of its Area above the Square *ee* before reſerved, and by the Quotient multiply ſeverally the three ſides of the ſame Triangle, ſo ſhall the Products be the three ſides of the right-angled Triangle ſought.

An Example in Numbers.

Firſt, I form a right-angled Triangle from the ſquare numbers 4 and 1, ſo the three ſides are 17, 15, 8; then I multiply 9 (the Square of the difference between 4 and 1) by 4, the Product of 4 into 1, and there comes forth the Square 36, (to wit, *ee*;) then I divide 8, (to wit, that ſide about the right-angle of the ſaid Triangle which is the double Product of 4 into 1,) by 24, which is the exceſs of 60 the Area of the ſaid Triangle above the Square 36, (to wit, *ee*,) and the Quotient is $\frac{1}{3}$, by which I multiply ſeverally 17, 15, 8, (the ſides of the right-angled Triangle firſt found,) and the Products $\frac{17}{3}$, $\frac{15}{3}$ and $\frac{8}{3}$ are the ſides of the right-angled Triangle ſought: For, if from the Area $\frac{20}{3}$, the Hypothenuſal $\frac{17}{3}$, and the ſide $\frac{8}{3}$ be ſeverally ſubtracted, there will remain the Squares 1 and 4.

This Queſtion is capable of innumerable Anſwers in a double reſpect; for firſt, inſtead of 4 and 1 we may take any two ſquare numbers, or any two numbers which are in ſuch proportion to one another as a ſquare number to a ſquare number, for the forming of a right-angled Triangle as the Canon directs: ſecondly, the ſame right-angled Triangle 17, 15, 8 being retained, we may inſtead of the Square 36, to wit, *ee*, find infinite others, every one of which ſhall be leſs than the Area 60; and ſuch, that if it be multiplied into 136, (to wit, *hp*,) and from the Product there be ſubtracted 4320, (to wit, $\frac{1}{2}bpp \times \overline{h-p}$,) the remainder ſhall be a ſquare number. (The finding out of which Squares may eaſily be deduced from *Lemma* 3. in the preceding *Queſt.* 101.)

QUEST. 105.

To find a right-angled Triangle, that if its Area be ſubtracted from the Hypothenuſal, and from one of the ſides about the right-angle, each remainder may be a Square.

RESOLUTION.

1. Let the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle in numbers be repreſented by . . . h ; b , p
2. Then multiply thoſe ſides ſeverally by a, and aſſume the Products to be the ſides of the right-angled Triangle ſought, to wit, ... ha ; ba , pa
3. Then, (according to the Queſtion,) each of theſe two quantities muſt be equated to a Square, *viz.* . . $ha - \frac{1}{2}bpaa = \square$, $pa - \frac{1}{2}bpaa = \square$
4. Now in order to reſolve that Duplicate equality, ſuppoſe $pa - \frac{1}{2}bpaa = eeaa$
5. Whence after due Reduction you will find . . . . $a = \dfrac{p}{ee + \frac{1}{2}bp}$
6. According to which value of a, if the former of the two quantities in the third ſtep be reſolved, inſtead of that quantity, (to wit, $ha - \frac{1}{2}bpaa$,) this that follows ariſeth to be equated to a Square, *viz.* . . . . $\dfrac{hpee, + \frac{1}{2}bpp \times \overline{h-p}}{eeee + bpee + \frac{1}{4}bbpp} = \square$
7. But becauſe the Denominator of the Fraction laſt expreſt is a Square, for its ſide is $ee + \frac{1}{2}bp$, it remains only to equate the Numerator to a Square; and the Numerator well examined, ſhews that we muſt find a right-angled Triangle h, b, p in numbers, and a ſquare number ee, ſuch, that if to the ſolid Product made by the multiplication of the ſaid Square, the Hypothenuſal and one of the ſides about the right-angle, there be added the ſolid Product made by the multiplication of the Area, the ſaid ſide about the right-angle, and the exceſs of the Hypothenuſal above the ſame ſide, the ſumm may be a Square: But the *Lemma* prefixt to the Reſolution of the preceding *Queſt.* 104. ſhews how to find out ſuch a Triangle and Square: Suppoſe then the Hypothenuſal,

Baſe

Baſe and Perpendicular of the ſaid Triangle ſo found out, to wit, h, b, p, and the Square ee to be all known in numbers, then conſequently the number a ſhall be known alſo from the fifth ſtep of the Reſolution: And laſtly, the numbers h, b, p being multiplied ſeverally by the number a will give the three ſides of the Triangle ſought. From the premiſſes there ariſeth the following

CANON.

8. Firſt, let a right-angled Triangle be formed from two ſquare numbers, or from two numbers which have ſuch proportion to one another as a ſquare number to a ſquare number; then multiply the Square of the difference of thoſe two numbers by the Product of their multiplication, and reſerve the Product, which is a ſquare number, and may be called ee; then divide that ſide about the right-angle which is the double Product of the two numbers that formed the ſaid Triangle, by the ſumm of its Area and the Square ee before reſerved; laſtly, by the Quotient multiply ſeverally the three ſides of the Triangle firſt formed: So ſhall the Products be the three ſides of the right-angled Triangle ſought.

An Example in Numbers.

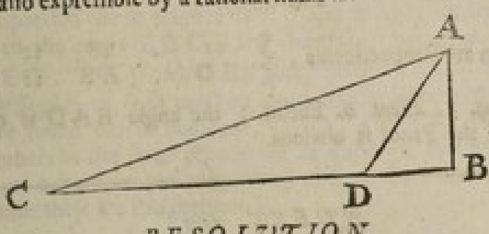
Firſt, a right-angled Triangle being formed from the ſquare numbers 4 and 1, the three ſides will be 17, 15, 8; then I multiply 9 the Square of the difference between 4 and 1, by 4 the Product of 4 into 1, and it produceth the Square 36, (to wit, ee;) then I divide the ſide 8, by 96 which is the ſumm of the Area 60 and the Square 36, (to wit, ee,) ſo the Quotient is $\frac{1}{12}$; laſtly, the ſides 17, 15, 8 being multiplied ſeverally by $\frac{1}{12}$ will give $\frac{17}{12}$, $\frac{15}{12}$ and $\frac{8}{12}$ for the right-angled Triangle ſought. For if from the Hypothenuſal $\frac{17}{12}$, and from the ſide $\frac{8}{12}$, the Area $\frac{5}{12}$ be ſubtracted, there will remain the Squares 1 and $\frac{1}{4}$.

After the ſame manner, among innumerable right-angled Triangles that might be found out by the ſaid Canon to ſolve this Queſtion, theſe three will be diſcovered, to wit, $\frac{41}{36}, \frac{40}{36}, \frac{9}{36}$ | $\frac{97}{45}, \frac{65}{45}, \frac{72}{45}$ | $\frac{257}{240}, \frac{255}{240}, \frac{32}{240}$; every one of which Triangles beſides that firſt found, to wit, $\frac{17}{12}, \frac{15}{12}, \frac{8}{12}$ is expreſt by ſmaller numbers than that Triangle found out by *Fermat*'s method, in the following *Queſt.* 127.

This Queſtion is alſo capable of innumerable Anſwers upon another ground, as may eaſily be collected from what hath been ſaid at the latter end of the preceding *Queſt.* 104.

QUEST. 106. (Quæſt. 18. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, ſuppoſe A B C right-angled at B, that one of its acute-angles B A C being cut into two equal parts by the line A D, the ſaid line A D may be alſo expreſſible by a rational number.



RESOLUTION.

1. Let the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle known in numbers be repreſented by } h ; b ; p

2. Then multiply thoſe ſides ſeverally by a, (which repreſents a number unknown,) and put . . . . . { ha = A D; ba = B D; pa = A B; $b - ba$ = D C

3. Now foraſmuch as (*per* 3. *Prop.* 6. *Elem.* Euclid.) theſe lines in the preceding Diagram are Proportionals, *viz.* } BD . AB :: DC . CA

4. Therefore from the premiſſes theſe numbers ſhall be alſo Proportionals, *viz.* . . . . } $ba \,.\, pa :: b - ba \,.\, p - pa$

5. And becauſe (*per* 47. *Prop.* 1. *Elem.* Euclid.) the Square of A B together with the Square of B C is equal to the Square of C A; therefore the Square of pa together with the

the Square of b, (for $ba + b - ba = b$,) shall be equal to the Square of $p - pa$, whence this Equation ariseth, to wit,

$$ppaa + bb = ppaa - 2ppa + pp.$$

6. Which Equation, after due Reduction, gives . . . . . $a = \frac{pp - bb}{2pp}$

7. Therefore from the first, second and sixth steps, all the lines sought shall now be known in rational numbers, *viz.* . .

$$\left\{\begin{array}{l} \frac{ppp - pbb}{2pp} = AB \\ \frac{2bpp}{2pp} = BC \\ \frac{ppp + pbb}{2pp} = CA \\ \frac{bpp - bbb}{2pp} = AD \\ \frac{bpp - bbb}{2pp} = BD \\ \frac{bpp + bbb}{2pp} = DC \end{array}\right.$$

8. And by multiplying all the Fractions in the last step by the common Denominator $2pp$, the Products will give these numbers which will also solve the Question, and may serve as a Canon for that purpose, *viz.* . . . .

$$\left\{\begin{array}{rl} ppp - pbb &= AB \\ 2bpp &= BC \\ ppp + pbb &= CA \\ bpp - bbb &= AD \\ bpp - bbb &= BD \\ bpp + bbb &= DC \end{array}\right.$$

An Example in Numbers.

Take any right-angled Triangle in numbers, as 5, 3, 4, then by putting $5 = h$, $3 = b$ and $4 = p$, (the greater of the sides about the right-angle,) you will find

$$\left\{\begin{array}{rcl} ppp - pbb = & 28 = & AB \text{ the Perpendicular,} \\ 2bpp = & 96 = & BC \text{ the Base,} \\ ppp + pbb = & 100 = & CA \text{ the Hypothenusal,} \\ bpp - bbb = & 35 = & AD \text{ the line bisecting the acute-angle opposite to the Base,} \\ bpp - bbb = & 21 = & BD \\ bpp + bbb = & 75 = & DC \end{array}\right\} \text{the segments of the Base.}$$

The Proof.

First, these numbers are Proportionals, $\left\{\begin{array}{cccc} 21 & . \; 28 & :: \; 75 & . \; 100 \\ BD & . \; AB & :: \; DC & . \; CA \end{array}\right.$

Therefore (*per Prop.* 3. *Elem.* 6. Euclid.) the angle BAD is equal to the angle DAC. The rest of the Proof is obvious.

QUEST. 107.

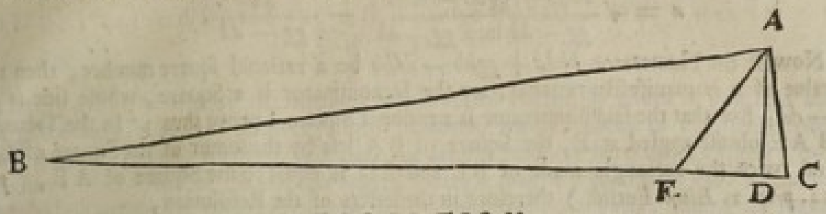
To find out an oblique-angled Triangle, whose three sides, as also the Perpendicular, and a line cutting one of the angles into two equal parts may be exprest severally by rational numbers.

[Jac. de Billy *in* probl. 5. cap. 4. *of the latter part of his* Diophant. redivi. *printed at* Lyons *in* 1670. *resolves this Question briefly by Numeral* Algebra; *but to the end a Canon may be raised to solve the Question proposed, I shall form the Resolution thereof by Literal* Algebra.]

Preparation.

Let there be an oblique-angled Triangle ABC; then supposing AD to be perpendicular to the Base BC, and the line AE to cut the angle BAC into two equal angles EAB

E A B and E A C, let it be required to find out rational numbers to expreſs the quantities of the ſides A B, B C, A C; as alſo of A D, A E, BE, E D, D C.



RESOLUTION.

1. Firſt, let b, p, h repreſent the Baſe, Perpendicular and Hypothenuſal of any right-angled Triangle known in numbers, and ſuppoſe the Perpendicular p to be greater than the Baſe b, then

$$\text{Put} \ldots \left\{ \begin{array}{l} b = \text{DE the Baſe,} \\ p = \text{AD the Perp.} \\ h = \text{AE the Hyp.} \end{array} \right\} \text{of } \triangle \text{ ADE.}$$

2. Secondly, making p, (that is, A D) to be the Perpendicular of a ſecond right-angled Triangle, ſuppoſe A D B, find out the Baſe D B and the Hypothenuſal B A in rational numbers, which may be done thus, *viz.* Foraſmuch as the Square of the Perpendicular A D is equal to the difference of the Squares of the Hypothenuſal B A and the Baſe D B, let the Square of the given number p be eſteem'd to be the difference of two ſquare numbers, and find out the Squares themſelves, then put the ſide of the leſſer Square for D B, and the ſide of the greater for B A; but the ſaid Squares muſt be found out with this Caution, that D B the ſide of the leſſer Square may have greater proportion to D E, (that is, b) than $2pp$ hath to $pp - bb$; as may eaſily be inferr'd from the Canon of the preceding *Queſt.* 106. where it appears, that when the angle D A C in the Diagram belonging to that Queſtion is equal to the angle D A B, then the Baſe B C hath ſuch proportion to B D, as $2pp$ to $pp - bb$; but in the Diagram of of this Queſtion the angle E A B muſt be greater than the angle E A D, in regard by ſuppoſition the angle E A B is equal to the angle E A C. Now ſince by *Queſt.* 7. of this *Book*, innumerable pairs of Squares having one common difference may be found out, ſuch, that the ſide of one Square of each pair ſhall be greater or leſs than a given number, let us ſuppoſe the ſides of two ſquare numbers to be found out agreeable to the ſaid Caution,

$$viz. \left\{ \begin{array}{l} b + d = \text{DE} + \text{EB} = \text{DB,} \\ \quad\;\; d = \text{EB,} \\ \quad\;\; g = \text{BA.} \end{array} \right.$$

3. Thirdly, the next ſcope is to find out E C and A C in rational numbers, which muſt have the ſame proportion one to another as E B and B A; (for by ſuppoſition the angle E A B is equal to the angle E A C, and therefore (*per Prop.* 3. *Elem.* 6. Euclid.) EB . BA :: EC . AC.) Moreover, the Square of $\overline{\text{EC} - \text{ED}}$, that is, of D C, together with the Square of A D muſt be equal to the Square of A C. But E B and B A were before found out in numbers, to wit, $d = $ E B, and $g = $ B A; now to find out two numbers in the ſame proportion as d and g, multiply theſe ſeverally by a (which repreſents a number yet unknown,) and put the Products da and ga for E C and A C; whence theſe are Proportionals,

$$viz. \left\{ \begin{array}{ccccccc} d & . & g & :: & da & . & ga, \\ \text{EB} & . & \text{BA} & :: & \text{EC} & . & \text{AC.} \end{array} \right.$$

4. And becauſe the Square of $\overline{\text{EC} - \text{ED}}$, that is, the Square of D C, together with the Square of A D makes the Square of A C, therefore in the letters of the Reſolution, the Square of $da - b$ together with the Square of p muſt be equal to the Square of ga; hence this Equation ariſeth, *viz.*

$$ddaa - 2bda + bb + pp = ggaa.$$

5. And becauſe (by the firſt ſtep of the Reſolution,) . . . . . $hh = bb + pp$
6. Therefore . . . . $ddaa - 2bda + hh = ggaa$
7. Whence, after due Reduction, this following Equation ariſeth, *viz.* . . . . $aa + \dfrac{2bd}{gg - dd}a = \dfrac{hh}{gg - dd}$

 8. Which

8. Which last Equation being resolved by the Canon in *Sect. 6, Chap. 15. Book* 1. the value of a will be made known, *viz.*

$$a = \sqrt{\frac{bbdd + gghh - ddhh}{gg - dd \text{ into } gg - dd}} : - \frac{bd}{gg - dd}.$$

9. Now if the Numerator $bbdd + gghh - ddhh$ be a rational square number, then the value of a is manifestly rational, for the Denominator is a Square, whose side is $gg - dd$; but that the said Numerator is a rational Square I prove thus: In the Triangle B A E obtuse-angled at E, the Square of B A less by the summ of the Square of B E and twice the rectangle made of B E and E D is equal to the Square of A E, (*per* 12. *prop.* 2. *Elem.* Euclid.) therefore in the letters of the Resolution,

$$gg - dd - 2db = hh.$$

10. But if $gg - dd - 2db$ instead of hh be multiplied into $gg - dd$, and to the Product there be added $bbdd$, then instead of the aforesaid Numerator $bbdd + gghh - ddhh$, this following Square will arise, to wit,

$$gggg - 2ggdd - 2ggdb + dddd + 2dddb + ddbb;$$

Whose side is . . . . $gg - dd - db$.

11. Therefore from the eighth and tenth steps, the value of a is expressible by a rational number, *viz.* $\Big\}\ a = \dfrac{gg - dd - 2db}{gg - dd}$

12. Therefore from the third and eleventh steps, . . $\Big\{\ \dfrac{dgg - ddd - 2ddb}{gg - dd} = \text{EC}$

13. And from the third and eleventh steps, . . . $\Big\{\ \dfrac{ggg - gdd - 2gdb}{gg - dd} = \text{AC}$

14. And by subtracting b from the quantity in the twelfth step, (*viz.* E D from E C,) it gives . $\Big\}\ \dfrac{dgg - ddd - bgg - bdd}{gg - dd} = \text{DC}$

15. Lastly, if the three Fractions in the three last preceding steps, as also b, p, h, d, g, be severally multiplied by the Denominator $gg - dd$, there will come forth the following quantities in Integers, which may serve as a Canon to solve the Question proposed; provided that the numbers b, p, h, d, g be first found out agreeable to the Caution before prescribed.

CANON.

$$\left.\begin{array}{rcl} ggd - ddd - 2ddb & = & \text{EC} \\ ggd - ddd & = & \text{EB} \\ 2ggd - 2ddd - 2ddb & = & \text{BC} \\ ggg - gdd & = & \text{BA} \\ ggg - gdd - 2gdb & = & \text{AC} \\ ggp - ddp & = & \text{AD} \\ ggh - ddh & = & \text{AE} \\ ggb - ddb & = & \text{DE} \\ ggd - ddd - ggb - bdd & = & \text{DC} \\ ggd + ggb - ddd - ddb & = & \text{DB} \end{array}\right\} \text{in the Diagram belonging to this Question.}$$

An Example in Numbers.

First, take any right-angled Triangle in whole numbers, as the Triangle 18, 24, 30; then

$$\text{Put} \ldots \left\{\begin{array}{l} b = 18 \ \text{ED}, \\ p = 24 \ \text{AD}, \\ h = 30 \ \text{AE}. \end{array}\right.$$

Secondly, making 24, (to wit, $p = \text{AD}$) to be the Perpendicular of a second right-angled Triangle, as well as of the first A D E, find out the Base D B, and the Hypothenusal A B in rational numbers; but for the reason before given, the number of the Base D B must have greater proportion to the number of the Base D E, than $2pp$ to $pp - bb$, *viz.* in this Example, greater proportion than 32 to 7, and consequently D B must exceed D E taken $4\frac{4}{7}$ times: But by supposition D E = 18; therefore D B must exceed $82\frac{2}{7}$. Now because the Square of the Perpendicular A D is equal to the difference of the Squares of A B and D B, therefore 576 the Square of the Perpendicular 24 (= A D) being taken for the difference of two Squares, find out the Squares severally, but with this condition, that the side of the lesser of them may exceed $82\frac{2}{7}$: But two such Squares (among innumerable other pairs of Squares that may be found out by the seventh Question

of

of this Book,) are 20449 and 21025, whoſe ſides are 143 and 145; therefore 143 $=$ DB, | 145 $=$ AB, and 125 $=$ EB, (that is, DB $-$ DE;) then

Put . . . . $\left\{ \begin{array}{l} d = 125 = EB, \\ g = 145 = BA. \end{array} \right.$

Laſtly, by the help of the numbers before found out for the values of b, p, h, d and g, the preceding Canon will diſcover rational numbers, which reduced to their leaſt terms by their greateſt common Diviſor, will give the whole numbers here-under expreſt, for the meaſures of the ſides of the oblique-angled Triangle ſought; as alſo of the Perpendicular, and the line cutting the angle oppoſite to the Baſe into two equal parts, and of the ſegments of the Baſe made as well by the Perpendicular as by the line biſecting the ſaid angle, *viz.*

$$\left. \begin{array}{l} EC = 125 \\ EB = 750 \\ BC = 875 \\ BA = 870 \\ AC = 145 \\ AD = 144 \\ AE = 180 \\ DE = 108 \\ DC = 17 \\ DB = 858 \end{array} \right\} \text{agreeable to the Diagram and Canon belonging to this Queſtion.}$$

The Proof.

Firſt, theſe numbers are Proportionals, *viz* $\left\{ \begin{array}{l} 750 \;.\; 870 :: 125 \;.\; 145 \\ EB \;.\; BA :: EC \;.\; AC \end{array} \right.$

Therefore, (*per* 3. *prop.* 6. *Elem.* Euclid.) the angle EAB is equal to the angle EAC. The reſt of the Proof is obvious.

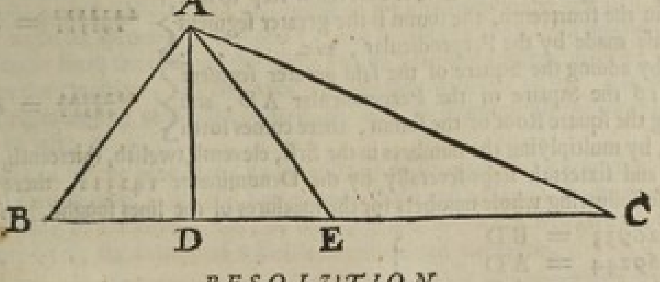
QUEST. 108.

To find out an oblique-angled Triangle, whoſe three ſides, as alſo the Perpendicular, and a line drawn from the angle oppoſite to the Baſe, and cutting the Baſe into two equal parts, may be expreſt ſeverally by rational numbers.

[Jac. de Billy *in his Appendix to the Problem cited in the preceding* 107th *Queſtion reſolves this alſo, but very briefly; I ſhall therefore form the Reſolution thereof at large by Numeral Algebra, by the ſteps whereof the more curious Analyſt may eaſily reſolve it by Specious Algebra, but the Canon thence ariſing will be exceeding tedious.*]

Preparation.

Let there be an oblique-angled Triangle ABC; then ſuppoſing AD to be perpendicular to the Baſe BC, and the line AE to cut the Baſe BC into two equal parts in the point E, let it be required to find out rational numbers to expreſs the quantities of the ſides AB, BC, AC; as alſo of the lines AD, AE, BD, DE.



RESOLUTION.

1. Firſt, take any right-angled Triangle in numbers, as 3, 4, 5; then,

Put . . . $\left\{ \begin{array}{l} 3 = BD \text{ the Baſe,} \\ 4 = AD \text{ the Perpendicular,} \\ 5 = AB \text{ the Hypothenuſal,} \end{array} \right\}$ of the right-angled Triangle BDA.

2. Then for the diſtance between the foot of the Perpendicular and the middle of the Baſe BC, put a, *viz.* ſuppoſe . . $\}$ $a =$ DE.

3. And

3. And becauſe by ſuppoſition, BE = EC, if to a you add 3, (to wit, BD,) the ſumm ſhall be equal to half the Baſe, *viz.* $a + 3 = BE = EC$

4. And becauſe $DE + EC = DC$, the ſumm of the two Equations in the ſecond and third ſteps ſhall be equal to the greater ſegment of the Baſe made by the Perpendicular, *viz.* $2a + 3 = DC$

5. And becauſe the Square of D C together with the Square of A D is equal to the Square of A C, the Square of $2a + 3$ together with the Square of 4 muſt make a Square, *viz.*

$$4aa + 12a + 25 = \square.$$

6. And becauſe the Square of D E together with the Square of A D is equal to the Square of A E, the Square of a together with the Square of 4 muſt be equated to a Square, *viz.*

$$aa + 16 = \square.$$

7. So in the two laſt ſteps we are faln upon a Duplicate equality, which, in regard the Squares 25 and 16 are unequal, I reduce to another that ſhall have equal known Squares, *viz.* (after the manner uſed in divers preceding Queſtions of this Book,) I divide the greater Square 25 by the leſſer 16, and by the Quotient $\frac{25}{16}$ I multiply the quantity in the ſixth ſtep, to wit, $aa + 16$, ſo the Product $\frac{25}{16}aa + 25$ is to be equated to a Square, and therefore this Duplicate equality comes now to be reſolved,

$$viz.\ \begin{cases} 4aa + 12a + 25 = \square \\ \frac{25}{16}aa + 25 = \square. \end{cases}$$

8. Now in order to reſolve the Duplicate equality laſt expreſt, firſt, by ſubtracting the leſſer quantity from the greater, I find their difference to be

$$\tfrac{39}{16}aa + 12a.$$

9. Then I ſearch out two quantities, the Product of whoſe multiplication may make the ſaid difference $\frac{39}{16}aa + 12a$; and that as well in the difference as in the ſumm of the ſame quantities there may be found 10, (to wit, the double of the ſide of the Square 25,) ſo I find thoſe two quantities to be

$$\tfrac{65}{32}a + 10 \text{ and } \tfrac{6}{5}a.$$

10. Then the Square of half the difference of the two quantities laſt expreſt, *viz.* the Square of $\frac{133}{320}a + 5$ being equated to $\frac{25}{16}aa + 25$, gives

$$\tfrac{25}{16}aa + 25 = \tfrac{17689}{102400}aa + \tfrac{133}{32}a + 25.$$

11. Which Equation, after due Reduction, will diſcover the number a, *viz.* . . . . $a = \frac{425600}{142311} = DE$

12. Then by adding the Square of the ſaid number a to 16, and extracting the ſquare Root of the ſumm, there ariſeth $\frac{710756}{142311} = AE$

13. And by adding 3 (to wit, B D) to the number a, (to wit, D E,) the ſumm ſhall be the meaſure of half the Baſe BC, *viz.* . . . . $\frac{852533}{142311} = BE = EC$

14. Therefore the double of the number in the laſt ſtep is the meaſure of the Baſe B C, *viz.* . . . . . . $\frac{1705066}{142311} = BC$

15. And by adding the number a in the eleventh ſtep to half the Baſe in the fourteenth, the ſumm is the greater ſegment of the Baſe made by the Perpendicular, *viz.* . . . $\frac{1278133}{142311} = DC$

16. Then by adding the Square of the ſaid greater ſegment D C to 16 the Square of the Perpendicular A D, and extracting the ſquare Root of the ſumm, there comes forth $\frac{1399165}{142311} = AC$

17. Laſtly, by multiplying the numbers in the firſt, eleventh, twelfth, thirteenth, fourteenth, fifteenth and ſixteenth ſteps ſeverally by the Denominator 142311, there will come forth theſe following whole numbers for the meaſures of the lines ſought, *viz.*

$$\left.\begin{array}{l} 426933 = BD \\ 569244 = AD \\ 711555 = AB \\ 425600 = DE \\ 710756 = AE \\ 852533 = BE = EC \\ 1705066 = BC \\ 1278133 = DC \\ 1399165 = AC \end{array}\right\}\ \text{in the Diagram belonging to this Queſtion.}$$

The

The Proof is eaſie to be made, by comparing the ſumm of the Squares of the numbers anſwering to the ſides about the right-angle of every right-angled Triangle in the Diagram, to the number anſwering to the Square of the Hypothenuſal of ſuch Triangle reſpectively.

QUEST. 109. (Quæſt. 21. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the Area thereof increaſed with one of the ſides about the right-angle may make a Square; and that the ſumm of all the three ſides may be a Cube.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and $a+1$; then divide the three ſides ſeverally by $a+1$, and take the Quotients for the ſides of the Triangle ſought, *viz.* . . . . $\left\{ \begin{array}{l} \frac{2aa+2a+1}{a+1} = \text{the Hypoth.} \\ \frac{2aa+2a}{a+1} = \text{the Baſe;} \\ \frac{2a+1}{a+1} = \text{the Perpend.} \end{array} \right.$
2. The ſumm of thoſe three ſides is . . . . . . $\frac{4aa+6a+2}{a+1}$
3. Which ſumm reduced to its leaſt terms (by dividing the Numerator by the Denominator according to the general method of Diviſion in *Sect.* 9. *Chap.* 5. *Book* 1.) will be . . . . $4a+2$

Therefore (according to the Queſtion) the ſaid $4a+2$ muſt be equal to a Cube; which in the following ninth ſtep I ſhall ſhew how to find out.

4. Moreover, the Area together with one of the ſides about the right-angle of the Triangle in the firſt ſtep muſt make a Square: But the Area (by multiplying half the Baſe into the Perpendicular) will be found $\frac{2aaa+3aa+a}{aa+2a+1}$
5. And one of the ſides about the right-angle (to wit, the Perpendicular) is . . . . $\frac{2a+1}{a+1}$
6. Which ſide reduced to the ſame Denominator with the Area, (by multiplying as well the Numerator $2a+1$, as the Denominator $a+1$, by $a+1$,) will be $\frac{2aa+3a+1}{aa+2a+1}$
7. Now if the ſide in the laſt ſtep be added to the Area in the fourth, the ſumm will be . . . . $\frac{2aaa+5aa+4a+1}{aa+2a+1}$
8. Which ſumm reduced to its leaſt terms, (by dividing the Numerator by the Denominator) will be . . $2a+1$
9. Therefore (according to the Queſtion) $2a+1$ muſt be equated to a Square; and before in the third ſtep it was found that $4a+2$ muſt be equated to a Cube; now becauſe $4a+2$ is the double of $2a+1$, we muſt find out a Cube that may be the double of a Square; but the Cube 8 is the double of the Square 4, therefore let $4a+2$ be equated to 8, or $2a+1$ to 4; whence you will find $a = \frac{3}{2}$, and conſequently $a+1 = \frac{5}{2}$. Wherefore according to the Poſitions in the firſt ſtep, let a right-angled Triangle be formed from $\frac{3}{2}$ and $\frac{5}{2}$, and divide the three ſides ſeverally by $\frac{5}{2}$, ſo there will come forth the ſides of the right-angled Triangle ſought, to wit, $\frac{17}{5}$, $\frac{15}{5}$, $\frac{8}{5}$, which ſolve the Queſtion; for the Area $\frac{12}{5}$ together with $\frac{8}{5}$ (to wit, one of the ſides about the right-angle) makes the Square 4; and the ſumm of all the three ſides makes a Cube, to wit, 8.

It is alſo evident by the premiſſes, that the Queſtion is capable of innumerable Anſwers; in regard you may find out as often as you pleaſe a Cube and a Square, ſuch, that the former ſhall be the double of the latter: As, by equating aaa to $8aa$, it will give the Cube 512, and the Square 256, the former of which is the double of the latter.

QUEST. 110. (Quæſt. 23. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the ſumm of all the three ſides may be a Square, and that the ſaid ſumm increaſed with the Area may make a Cube.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and 1, ſo the three ſides will be theſe, to wit, ... $aa+1$, $aa-1$, $2a$

2. The

2. The summ of all the three sides is $2aa + 2a$, which must be equal to a Square, let it therefore be equated to $eeaa$, *viz.* suppose . . . . . $2aa + 2a = eeaa$

3. Whence, after due Reduction, you will find . . . . $a = \frac{2}{ee - 2}$

4. Therefore by squaring the last Equation, there ariseth . $aa = \frac{4}{eeee - 4ee + 4}$

5. Which last Equation doubled, is . . . . $2aa = \frac{8}{eeee - 4ee + 4}$

6. And by adding the double of the Equation in the third step, to wit, $2a = \frac{4}{ee - 2}$ to the Equation in the fifth step, the summ gives this Equation, to wit,

$$2aa + 2a = \frac{4ee}{eeee - 4ee + 4}.$$

7. The latter part of which Equation is manifestly a Square, and by Construction 'tis equal to the summ of all the three sides of the Triangle in the first step. It remains that the said summ with the Area make a Cube; but from the first step the Area is $aaa - a$, and according to the value of a in the third step, the Area $aaa - a$ will be reduced to this, to wit, $\frac{8ee - 2eeee}{eeeeee - 6eeee + 12ee - 8}$.

8. To which add the summ of all the three sides, to wit, the latter part of the Equation in the sixth step, that is, $\frac{4ee}{eeee - 4ee + 4}$, (which, by multiplying the Numerator and Denominator severally by $ee - 2$, will be reduced to $\frac{4eeee - 8ee}{eeeeee - 6eeee + 12ee - 8}$,) and there will come forth this summ, to wit, $\frac{2eeee}{eeeeee - 6eeee + 12ee - 8}$.

9. Which summ last produced must be equal to a Cube; and because by Construction the Denominator is a Cube, to wit, the Cube of $ee - 2$, it remains that we equate the Numerator $2eeee$ to some Cube; or by dividing $2eeee$ by eee it gives $2e$ to be equated to a Cube, which is easie to be done, for we may put $2e$ equal to any known cube-number, as ddd;

10. Suppose therefore . . . . $2e = ddd$

11. Then because the Denominator of the Fraction in the third step shews that . . . . $e \sqsubset \sqrt{2}$

12. And consequently by doubling each part, ... $2e \sqsubset \sqrt{8}$

13. It follows from the tenth and twelfth steps, that . . $ddd \sqsubset \sqrt{8}$

14. Again, one of the sides about the right-angle is by Construction in the first step $aa - 1$, therefore . . $aa \sqsubset 1$

15. But by the fourth step, ... $\frac{4}{eeee - 4ee + 4} = aa$

16. Therefore from the two last steps, ... $\frac{4}{eeee - 4ee + 4} \sqsubset 1$

17. And by multiplying each part in the sixteenth step by the Denominator $eeee - 4ee + 4$, 'tis manifest that . . $4 \sqsubset eeee - 4ee + 4$

18. Whence, by adding $4ee$ to each part, ... $4 + 4ee \sqsubset eeee + 4$

19. And by subtracting 4 from each part in the last step . . $4ee \sqsubset eeee$

20. And by dividing each part in the nineteenth step by ee, . $4 \sqsubset ee$

21. And by extracting the square Root out of each part in the 20th step, 'tis evident that . . . . $2 \sqsubset e$

22. And by doubling each part in the twenty-first step, . . . $4 \sqsubset 2e$

23. But by supposition in the tenth step, ... $ddd = 2e$

24. Therefore from the two last steps, ... $4 \sqsubset ddd$

25. And consequently, ... $ddd \sqsupset 4$

26. Thus in the 13th and 25th steps it is found that the cube-number represented by ddd must be greater than the square Root of 8, but less than 4; and then the half of the cube-number taken within those limits shall be the number e, which being known, the number a will also be discovered by the third step: Lastly, a right-angled Triangle formed from the number a and unity, shall be that which is sought. From the premisses ariseth this

CANON.

CANON.

27. First, take any cube-number greater than the square Root of 8, (*viz.* greater than $2\frac{828}{1000}$, *&c.*) but less than 4; then take the half of that cube-number, and call it e; that done, divide 2 by the excess of the Square of the number e above 2, and call the Quotient a; lastly, let a right-angled Triangle be formed from the number a and 1, so shall this Triangle be that which is sought.

An Example in Numbers.

First, I take some cube-number within the limits prescribed in the Canon, as $\frac{27}{8}$; whose half, to wit, $\frac{27}{16}$ is the number e; then I divide 2 by the excess of the Square of $\frac{27}{16}$ above 2, so the Quotient $\frac{512}{217}$ is the number a; lastly, from $\frac{512}{217}$ and 1 I form a right-angled Triangle and find the three sides to be these, to wit, $\frac{309233}{47089}$, $\frac{215055}{47089}$ and $\frac{222208}{47089}$, which solve the Question: For the summ of those three sides make the Square $\frac{746496}{47089}$, whose side is $\frac{864}{217}$; and the said summ of the sides together with the Area $\frac{110108160}{10218313}$ makes the Cube $\frac{272097792}{10218313}$, whose side is $\frac{648}{217}$.

QUEST. 111. (Quæst. 24. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the summ of all the three sides may be a Cube; and that the said summ increased with the Area may make a Square.

RESOLUTION.

1. For the summ of all the three sides of the Triangle sought put . . . s
2. And for the Area . . . . a
3. Then because the double of the Area is equal to the Product made by the mutual multiplication of the sides about the right-angle, the said Product is . . . . $2a$
4. For one of the sides about the right-angle put . . . . . $\frac{1}{e}$
5. Then because the Product of the said sides is $2a$, 'tis manifest that by dividing $2a$ by $\frac{1}{e}$, the Quotient shall be the other side, to wit, $2ae$
6. Therefore from the fourth and fifth steps the summ of the sides about the right-angle is . . . . $2ae + \frac{1}{e}$
7. Which summ subtracted from s the summ of all the three sides, leaves the Hypothenusal, to wit, ... $s - 2ae - \frac{1}{e}$
8. The Square of the said Hypothenusal is

$$ss + 4aaee + \frac{1}{ee} + 4a - 4sae - \frac{2s}{e}.$$

9. And the summ of the Squares of the sides about the right-angle, to wit, of the Squares of $\frac{1}{e}$ and $2ae$ is $\frac{1}{ee} + 4aaee.$

10. But the Square of the Hypothenusal is equal to the summ of the Squares of the sides about the right-angle, therefore from the eighth and ninth steps this Equation ariseth, *viz.*

$$ss + 4aaee + \frac{1}{ee} + 4a - 4sae - \frac{2s}{e} = \frac{1}{ee} + 4aaee.$$

11. From which Equation, after due Reduction in order to find out the value of e, by the letters s and a, there will arise this following Equation, *viz.*

$$\frac{ss + 4a}{4sa}e - ee = \frac{2s}{4sa}.$$

12. And by resolving the last Equation according to the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two values of e will be found these, to wit,

$$e = \frac{\frac{1}{2}ss + 2a}{4sa} + \sqrt{:\ \frac{\frac{1}{4}ssss + 4aa - 6ssa}{16ssaa}}\ ;$$

$$e = \frac{\frac{1}{2}ss + 2a}{4sa} - \sqrt{:\ \frac{\frac{1}{4}ssss + 4aa - 6ssa}{16ssaa}}\ ;$$

13. But 'tis evident that those values of e will not be rational unless the Numerator $\frac{1}{4}ssss + 4aa - 6ssa$ be equal to a Square. Moreover, the Question requires that the

the ſumm of all the three ſides with the Area, to wit, $s + a$ may be equal to a Square; ſo we are faln upon this Duplicate equality,

$$viz. \begin{cases} \frac{1}{4}ssss + 4aa - 6ssa = \square \\ s + a = \square \end{cases}$$

14. Now if in that Duplicate equality any known ſquare number be taken for the value of s, then we may diſcover the number a But the Queſtion requires s, (that is, the ſumm of all the three ſides of the Triangle ſought,) to be a Cube, therefore ſome number which is both a Square and a Cube muſt be taken for the value of s; let therefore the ſquared Cube 64 be put equal to s, and then the ſaid Duplicate equality will will be reſolved into this,

$$viz. \begin{cases} 4aa - 24576a + 4194304 = \square \\ a + 64 = \square \end{cases}$$

15. Or you may divide the firſt of thoſe two quantities by 4, then inſtead of that firſt quantity the Quotient is to be equated to a Square, and ſo this following Duplicate equality ariſeth,

$$viz. \begin{cases} aa - 6144a + 1048576 = \square \\ a + 64 = \square \end{cases}$$

16. But becauſe in the Duplicate equality laſt expreſt, the ſquare numbers 1048576 and 64 are unequal, I divide the greater of them by the leſs, and by the Quotient 16384 I multiply $a + 64$, and then the Product $16384a + 1048576$ is to be equated to a Square inſtead of $a + 64$; ſo at length this Duplicate equality remains to be reſolved

$$viz. \begin{cases} aa - 6144a + 1048576 = \square \\ 16384a + 1048576 = \square \end{cases}$$

17. Now in order to reſolve that Duplicate equality, firſt, ſuppoſing the former of the two quantities to be equated to be the greater, their difference is $aa - 22528a$, then according to the method uſed in divers preceding Queſtions of this Book, two ſuch numbers are to be found out that the Product of their multiplication may make the ſaid difference $aa - 22528a$, and that as well in the half of the ſumm as in the half of the difference of the ſaid two numbers there may be found 1024, which is the ſide of the Square 1048576. But two ſuch numbers are $11a$ and $\frac{1}{11}a - 2048$; the half-difference of theſe is $\frac{60}{11}a + 1024$, and the Square of $\frac{60}{11}a + 1024$ being equated to $16384a + 1048576$ will give $a = \frac{39424}{225}$, to wit, the Area of the Triangle ſought.

Since then the value of a is found to be $\frac{39424}{225}$, and s was before aſſumed to be 64, according to theſe values of a and s, the twelfth ſtep will diſcover the two values of e to be $\frac{-33}{448}$ and $\frac{-2}{176}$; by either of which if you reſolve the poſitions in the fourth and fifth ſteps, you will find the ſides about the right-angle to be $\frac{4032}{225}$ and $\frac{4400}{225}$; therefore the Hypothenuſal is $\frac{5968}{225}$, and the ſumm of all the three ſides is the Cube 64, to which if you add the Area $\frac{39424}{225}$ it makes the Square $\frac{53824}{225}$, whoſe ſide is $\frac{232}{15}$.

QUEST. 112. (Quæſt. 25. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the Square of the Hypothenuſal may be compoſed of ſome Square and its ſide; and that the Square of the ſaid Hypothenuſal being divided by one of the ſides about the right-angle may give a number compoſed of a Cube and its ſide.

RESOLUTION.

1. For one of the ſides about the right-angle put . . . . a
2. And for the other, ... aa
3. The ſumm of their Squares is equal to the Square of the Hypothenuſal, to wit, ... $aaaa + aa$
4. Now 'tis evident that $aaaa + aa$ is compos'd of the Square $aaaa$ and its ſide aa; but if the ſaid $aaaa + aa$ be divided by a, (which was put for one of the ſides about the right-angle,) it gives the Quotient $aaa + a$, which is compos'd of a Cube with its ſide; ſo that it remains only to equate $aaaa + aa$ to a Square, that is, to find out a right-angled Triangle that one of the ſides about the right-angle may be equal to the Square of the other of the ſame ſides: But the preceding 95^{th} Queſtion ſhews how to find out ſuch a Triangle; take if you pleaſe that in the firſt Example of the ſaid Queſtion, to wit, $\frac{4}{3}$, $\frac{16}{9}$, $\frac{20}{9}$, which will ſolve this 112^{th} Queſtion; for the Square of the Hypothenuſal $\frac{20}{9}$, *viz.* $\frac{400}{81}$, is compos'd of the Square $\frac{256}{81}$ and its ſide $\frac{16}{9}$. Moreover,

if the

if the said $\frac{448}{81}$ be divided by $\frac{4}{3}$, (one of the sides about the right-angle,) it gives the Quotient $\frac{112}{27}$, which is compos'd of the Cube $\frac{64}{27}$ and its side $\frac{4}{3}$.

A Prospect of all Diophantus's *kinds of Duplicate Equality, shewing also at first sight in which of the preceding Questions they are resolved.*

I. The first kind of Duplicate Equality is, when each of two Quantities to be equated to square numbers consists of an unknown Root or number a, with some absolute or known number, and the numbers prefixt to the Root a are equal to one another; as in the five following Examples.

1.	If . . . . $a + 192 = \square$ And . . . $a + 128 = \square$ What is the number a?	Resolved in *Quest.* 8. of this *Book* 3.
2.	If . . . . $10a + 54 = \square$ And . . . $10a + 6 = \square$ What is the number a?	Resolved in *Quest.* 8.
3.	If . . . . $192 - a = \square$ And . . . . $64 - a = \square$ What is the number a?	Resolved in *Quest.* 14. See the like in *Quest.* 50.
4.	If . . . . $a - 27 = \square$ And . . . $a - 15 = \square$ What is the number a?	Resolved in *Quest.* 15.
5.	If . . . . $a + 12 = \square$ And . . . $a - 8 = \square$ What is the number a?	Resolved in *Quest.* 16.

II. The second kind of Duplicate Equality is, when each of two Quantities to be equated to Squares consists of an unknown Root a, and of one and the same known square number. Also, when the numbers prefixt to a in both Quantities are unequal, and the absolute numbers are unequal Squares; as in the six following Examples.

1.	If . . . . . $4 + a = \square$ And . . . . . $4 - a = \square$ What is the number a?	Resolved in *Quest.* 17.
2.	If . . . . . $9 + a = \square$ And . . . . $4 - a = \square$ What is the number a?	Resolved in *Quest.* 18.
3.	If . . . . $8a + 4 = \square$ And . . . . $6a + 4 = \square$ What is the number a?	Resolved in *Quest.* 33.
4.	If . . . . $4 - 2a = \square$ And . . . . $4 - 3a = \square$ What is the number a?	Resolved in *Observat.* 1. *Quest.* 33.
5.	If . . . . $4 + 2a = \square$ And . . . . $4 - 3a = \square$ What is the number a?	Resolved in *Observat.* 1. *Quest.* 33.
6.	If . . . . $10a + 9 = \square$ And . . . $5a + 4 = \square$ What is the number a?	Resolved in *Observat.* 2. *Quest.* 33.

III. The third kind of Duplicate Equality is, when each of two Quantities to be equated to Squares consists of some number of a, and an absolute number not a Square, but the numbers prefixt to a have such proportion to one another as a square number to a square number; as in the three following Examples.

V 1. If

1.	If . . . . $a + 12 = \square$ And . . . $4a + 12 = \square$ What is the number a?	Resolved in *Quest.* 29.
2.	If . . . . $\frac{22}{10}a + \frac{42}{10} = \square$ And . . . $\frac{42}{10}a + \frac{12}{10} = \square$ What is the number a?	Resolved in *Quest.* 35.
3.	If . . . . $\frac{22}{9} - \frac{2}{3}a = \square$ And . . . $\frac{22}{9} - \frac{8}{3}a = \square$ What is the number a?	Resolved in *Quest.* 50.

IV, The fourth kind of Duplicate Equality is, when two Quantities to be equated to Squares are diversly compos'd of aa and a, or of aa, a and absolute numbers; in which cases, to the end the Duplicate Equality propos'd may be explicable by rational numbers, these two things are requisite to be found therein; *viz.* First, either the numbers prefixt to aa, or the absolute numbers must be rational Squares: Secondly, the difference of the Quantities proposed must be either some sole number of a, or compos'd of some number of a and an absolute number, or of some numbers of aa and a; as in the following Examples.

1.	If . . . . $aa + a = \square$ And . . . . $aa - a = \square$ What is the number a?	Resolved in *Quest.* 19.
2.	If . . . $9aa + 2a = \square$ And . . . $9aa + a = \square$ What is the nnmber a?	Resolved in *Quest.* 20.
3.	If . . . $4aa + 5a = \square$ And . . . $9aa + 5a = \square$ What is the number a?	Resolved in *Quest.* 21. See the like in *Quest.* 22.
4.	If . . $4aa + 3a - 1 = \square$ And . . $4aa - a - 1 = \square$ What is the number a?	Resolved in *Quest.* 20.
5.	If . . . $4aa + 15a = \square$ And . . $4aa - a - 4 = \square$ What is the number a?	Resolved in *Quest.* 31.
6.	If . . $4aa + a + 1 = \square$ And . . . $3a + 1 = \square$ What is the number a?	Resolved in *Quest.* 32.
7.	If . . . . $aa + d = \square$ And . . . . $ba + d = \square$ What is the number a?	Resolved in *Quest.* 55.
8.	If . . $aa + 2pa + bb = \square$ And . . . $\frac{a}{p} + 1 = \square$ What is the number a?	Resolved in *Quest.* 59.
9.	If . . $3aa - 6a + 4 = \square$ And . . . $4aa + 4 = \square$ What is the number a?	Resolved in *Quest.* 94. See the like in *Quest.* 93.
10.	If . . . . $ee + 1 = \square$ And . . . . $e + 1 = \square$ What is the number e?	Resolved in *Quest.* 99.
11.	If $aa - 6144a + 1048576 = \square$ And . . . $a + 64 = \square$ What is the number a?	Resolved in *Quest.* 111.

A

A brief Expoſition upon Monſieur Fermat's *Analytical Invention, inſerted in the late Edition of Monſieur* Bachet's *Comment upon* Diophantus, *printed at* Toloze, *Anno* 1670.

JAC. *de* BILLY, who collected the ſaid Invention out of *Fermat's* Letters, divides it into three Parts. The firſt is an Improvement of one of *Diophantus's* kinds of Duplicate Equality; (to wit, the fourth and laſt kind in my preceding Proſpect, but the ſixth with *Bachet* in his Comment upon *Queſt.* 24. *Book* VI. *Diophant.*) for whereas *Diophantus's* method of reſolving that kind of Duplicate Equality, finds out but one value or two at the moſt of the Root ſought, and ſometimes the value found out is Negative, *viz.* leſs than nothing, *Fermat's* method finds out innumerable affirmative values.

The ſecond part ſhews how to reſolve two kinds of Triplicate Equality, by reducing them to a Duplicate Equality of the ſame kind with that above mentioned.

The third part ſhews how to equate a Quantity compos'd of five Terms to a Square; likewiſe a Quantity of four Terms to a Square or Cube, and for the moſt part to find out innumerable affirmative values of the Root ſought.

Theſe three parts I ſhall explain in order, and according to my uſual method put *a*, (inſtead of N by *Diophantus*) for a Root or number unknown, *aa* for the Square of that Root, *aaa* for the Cube, *&c.*

Concerning the Firſt part of Fermat's *Analytical Invention.*

Fermat's Rule to find out innumerable affirmative values of the Root ſought, in a Duplicate equality of the kind before mentioned, is this,

Firſt, by the vulgar method of *Diophantus*, (explain'd in divers preceding Queſtions of this Book,) find out one value of the ſought Root, (*a*,) it matters not whether it be affirmative or negative; then to *a* joyn that value with its own ſign, whether it be + or −, and take the ſumm for a new Root inſtead of *a*; then according to the ſaid new Root let a new Duplicate equality be deduced from the firſt, and find out the value of *a* in the new Duplicate equality by the vulgar method. That done, connect this latter value to the firſt before found, with their own ſigns, and it will give a ſecond value of the Root ſought in the firſt Duplicate equality. In like manner by the help of this ſecond value you may find out a third, and from the third a fourth, and ſo infinitely.

Note. After one value of the Root ſought is found out in the vulgar way, there will always certainly ariſe a known Square in each of the two new Algebraick quantities to be equated to Squares, in the ſecond, third, or any following Duplicate equality deduced from the firſt as the Rule doth direct, the reaſon whereof will be evident to him that diligently examines the Operation. But when thoſe two known Squares are unequal, the leſſer muſt be reduced to the greater, (in ſuch manner as before hath been ſhewn in *Queſt* 21, and 111. of this Book,) to the end there may be one and the ſame known ſquare number in each of the two quantities to be equated, and then the difference of the ſaid quantities will be compoſed of ſome number of *aa* and ſome number of *a*; which kind of difference is abſolutely neceſſary in the uſe of *Fermat's* Rule before expreſt. All which will be made manifeſt by the following Queſtions 113, 114, 115, 116.

QUEST. 113.

To find a number, that if to the Product of its Square multiplied by 32 you add unity, and from the ſumm ſubtract eight times the number ſought, the remainder may be a Square. Alſo, that eight times the number ſought, together with unity, may make a Square.

Let a be put for the number fought, and then the Queftion may be ftated thus, *viz.*

1. If . . . . $32aa - 8a + 1 = \square$ } what is the number a?
2. And . . . . $8a + 1 = \square$ }

RESOLUTION.

3. To refolve that Duplicate equality, firft, (according to the vulgar method before explained in *Queft.* 32. of this Book,) I take the difference of the two quantities to be equated, and becaufe in this Example, either of them may be taken for the greater, I fuppofe the firft to be the greater, fo their difference is $32aa - 16a$; then I feek two fuch quantities that the Product of their multiplication may make $32aa - 16a$, and that as well in half the fumm as in half the difference of the fame two quantities, there may be found the abfolute number 2, (to wit, the double of the fquare Root of 1, the known Square in each of the two quantities to be equated;) fo I find $4a - 2$ and $8a$, whofe Product is $32aa - 16a$, and their difference is $4a + 2$; the half of this difference is $2a + 1$, whofe Square $4aa + 4a + 1$ being equated to $8a + 1$, (which was fuppofed to be the leffer of the two quantities in the Duplicate equality,) will give, after due Reduction, $a = 1$, according to which value, the firft of the two quantities propos'd to be equated, to wit, $32aa - 8a + 1$, makes the Square 25, and the other quantity $8a + 1$ makes the Square 9; wherefore the Queftion is folved, and the faid value of a, to wit, 1, is the only value either affirmative or negative that can be found out by the common method. But *Fermat's* Rule, by the help of the faid firft value finds out a fecond, and from the fecond a third, &c. As, for example, to find out another number, or value of a befides 1 to folve the Queftion, let $a + 1$ be taken for a new Root inftead of a, ($+1$, becaufe the firft Root was found $+1$, but if it had been found -1, then $a - 1$ ought to be put for the new Root;) then let a new Duplicate equality be deduced from that before refolved in this manner, *viz.*

4. Inftead of $32aa$ take 32 times the Square of the new Root $a + 1$, that is, ... } $32aa + 64a + 32$
5. To that Product add unity, and it makes this fumm, . > $32aa + 64a + 33$
6. From that fumm fubtract eight times $a + 1$, (inftead of $8a$ in the firft Duplicate equality,) that is, . . . } . . $+ 8a + 8$
7. And the remainder muft be equal to a Square, *viz.* . > $32aa + 56a + 25 = \square$
8. Again, inftead of $+ 8a$ the later quantity of the firft Duplicate equality, take eight times $a + 1$, that is, . } . . $+ 8a + 8$
9. And to that Product add unity, fo this fumm muft be equal to a Square, *viz.* . . . . } . . $+ 8a + 9 = \square$
10. the feventh and ninth fteps give a new Duplicate equality, *viz.* . . . . } $32aa + 56a + 25 = \square$; $+ 8a + 9 = \square$
11. But becaufe in this Duplicate equality the Squares 25 and 9 are unequal, and *Fermat's* Rule takes place only in a Duplicate equality which hath one and the fame known fquare number in each quantity to be equated, let 25 be divided by 9, and by the Quotient $\frac{25}{9}$ multiply $8a + 9$; fo this quantity comes forth (inftead of $8a + 9$) to be equated to a Square, *viz.* . . . . } . . $+ \frac{200}{9}a + 25 = \square$
12. Thus at length, from the Duplicate equality before expreft in the firft and fecond fteps, another is deduced, and qualified as *Fermat's* Rule requires, *viz.* . . . . } $32aa + 56a + 25 = \square$; $+ \frac{200}{9}a + 25 = \square$
13. Then by refolving this laft Duplicate equality in like manner as the firft, you will find . . . . } $a = -\frac{1509075}{1697809}$
14. To which negative value of a if you add 1, becaufe $a + 1$ was taken for the new Root, you will find $\frac{188734}{1697809}$ for a fecond Root or value of a in the Duplicate equality in the firft and fecond fteps, *viz.* . . } $a + 1 = \frac{188734}{1697809}$

I fay $\frac{188734}{1697809}$, befides 1 firft found, will folve the Queftion; for, if you multiply the Square of $\frac{188734}{1697809}$ by 32, and add 1 to the Product, and from this fumm fubtract the Product of the faid $\frac{188734}{1697809}$ into 8, there will remain a Square whofe fide is $\frac{1207865}{1697809}$. Moreover, if the Product of the faid $\frac{188734}{1697809}$ into 8, be increafed with 1, it makes a Square, whofe fide is $\frac{1791}{1303}$.

In

In like manner, by taking or ſubſtituting $a + \frac{1488744}{1697809}$ for a new Root inſtead of a, and proceeding as before, you may find out a third number, and from the third a fourth, &c. to ſolve the Queſtion propoſed; but the operation in finding out Anſwers by this method is ſo exceſſively laborious, that for the moſt part he that hath found out two Anſwers, will hardly have patience to find out a third.

QUEST. 114.

To find a number, that if to the double of its Square you add unity, and from that ſumm ſubtract four times the number ſought, the remainder may be a Square. Alſo, that if from unity you ſubtract the double of the number ſought, the remainder may be a Square.

Let a be put for the number ſought, and then the Queſtion may be ſtated thus, *viz.*

1. If	$2aa - 4a + 1 = \square$	what is the number a?
2. And	$-2a + 1 = \square$	

RESOLUTION.

3. If that Duplicate equality be reſolved by the vulgar method, the only value of a will be found -4; but this being leſs than nothing, I ſearch out an affirmative value of a by *Fermat*'s Rule, thus,

4. Inſtead of a I take for a new Root	$a - 4$
5. Then inſtead of $2aa$ I take the double of the Square of the new Root $a - 4$, *viz.*	$2aa - 16a + 32$
6. To which double Square I add unity, (as the Queſtion requires,) and it makes this ſumm, ...	$2aa - 16a + 33$
7. Then from that ſumm I ſubtract four times the new Root $a - 4$, (inſtead of $4a$ in the Duplicate equality in the firſt ſtep,) *viz.* I ſubtract	$+ 4a - 16$
8. So this remainder muſt be equal to a Square, *viz.* . .	$2aa - 20a + 49 = \square$
9. Again, if from unity, the double of the new Root $a - 4$ be ſubtracted, the remainder muſt be equal to a Square, *viz.*	$-2a + 9 = \square$
10. So in the eighth and ninth ſteps there is a new Duplicate equality deduced from that in the firſt and ſecond ſteps; but becauſe in the new Duplicate equality the known ſquare numbers 49 and 9 are unequal, I divide 49 by 9, and by the Quotient $\frac{49}{9}$ I multiply $-2a + 9$ in the ninth ſtep, and it gives to be equated to a Square . . .	$-\frac{98}{9}a + 49 = \square$
11. The eighth and tenth ſteps give this new Duplicate equality to be reſolved, in which there is the ſame known ſquare number, (to wit, 49, as *Fermat*'s Rule requires,) *viz.*	$2aa - 20a + 49 = \square$ $-\frac{98}{9}a + 49 = \square$
12. Now to reſolve the laſt preceding Duplicate equality, according to the vulgar method, I take the difference of the two quantities to be equated, which, by ſuppoſing the firſt quantity to be the greater, is	$2aa - \frac{82}{9}a$
13. Then I ſearch out two ſuch quantities, that being mutually multiplied may make the ſaid difference, and that as well in half the ſumm as in half the difference of the ſaid two quantities there may be found 7, (the ſquare Root of 49,) ſo I find the two quantities to be . .	$\frac{126}{41}a - 14$ and $\frac{41}{63}a$
14. Half the difference of thoſe two quantities is . . .	$\frac{6257}{5166}a - 7$
15. Then the Square of the ſaid half-difference being equated to $-\frac{98}{9}a + 49$ (in the eleventh ſtep) will give . . .	$a = \frac{161933436}{39150049}$
16. From which value of a I ſubtract 4, becauſe the new Root was aſſumed to be $a - 4$, and the remainder is .	$\frac{5333240}{39150049}$

I ſay $\frac{5333240}{39150049}$ ſhall be a value of the Root a in the Duplicate equality in the firſt and ſecond ſteps, and therefore will ſolve the Queſtion propoſed; for if $a = \frac{5333240}{39150049}$, then $2aa - 4a + 1$ makes a Square whoſe ſide is $\frac{27466831}{39150049}$; alſo, $1 - 2a$ makes a Square whoſe ſide is $\frac{5337}{6257}$. Thus although a number leſs than nothing, to wit, -4 was found out for the firſt value of the Root a, yet by the help thereof an affirmative Root or number is found out to ſolve the Queſtion propoſed; and from that ſecond Root you may find out a third.

QUEST. 115.

QUEST. 115.

To find a right-angled Triangle in rational numbers, that the fumm of the fides about the right-angle may make a Square. Alfo, that either of the fame fides being added to the Square of the other of them may make a Square: And that one of the faid fides about the right-angle, together with a given multiple (fuppofe the triple) of the other of them may make a Square.

RESOLUTION.

1. It is manifeft by the Theorem in *Queft.* 24. of this *Book*, that if the Fraction $\frac{1}{4}$ be divided into any two parts, then either part added to the Square of the other makes a Square; therefore, to folve the firft part of the Queftion infinitely, put for the fides about the right-angle . . . a and $\frac{1}{4} - a$
2. The fumm of the Squares of thofe fides fhall be equal to the Square of the Hypothenufal, therefore . . . . $2aa - \frac{1}{2}a + \frac{1}{16} = \square$
3. And becaufe the Queftion requires that one of the fides about the right-angle, together with the triple of the other, may make a Square, let $\frac{1}{4} - a$ be added to $3a$, and then the fumm muft be equal to a Square, *viz.* . . . . . . $+ 2a + \frac{1}{4} = \square$
4. So in the fecond and third fteps there is a Duplicate equality, which, (by the method in divers preceding Queftions) may be reduced to this having equal abfolute Squares, *viz.* $\left\{\begin{array}{l} 32aa - 8a + 1 = \square \\ \quad\;\; + 8a + 1 = \square \end{array}\right.$
5. Which Duplicate equality laft expreft being refolved by the vulgar method gives but one fingle value of a, to wit, unity, (as hath been fhewn in the preceding *Queft.* 113.) But according to the Pofitions in the firft ftep of the Refolution of this Queftion, the number fignified by a ought to be lefs than $\frac{1}{4}$, and confequently $1 = a$ will not folve this Queftion by affirmative numbers; for if a be 1, then $\frac{1}{4} - a$, (which was put for one of the fides about the right-angle) will be lefs than nothing. Therefore to fearch out another value of a, I proceed according to *Fermat*'s Rule thus, *viz.* Inftead of a, I take for a new Root $a + 1$, by which the Duplicate equality in the fourth ftep will be reduced to this that follows, (as before hath been fhewn in *Queft.* 113.)

$$viz. \left\{\begin{array}{l} 32aa + 56a + 25 = \square \\ \qquad\;\; + \frac{200}{9}a + 25 = \square \end{array}\right.$$

6. Now if this Duplicate equality be refolved in the vulgar way, it will give $a = -\frac{1509075}{1697809}$, therefore $\frac{188734}{1697809}$ $(= a + 1)$ fhall be a value of a, (befides 1,) in the Duplicate equality in the fourth ftep; and becaufe the faid $\frac{188734}{1697809}$ is lefs than $\frac{1}{4}$, it fhall be the firft of the fides about the right-angle of the Triangle fought; then fince $\frac{1}{4} - a$ was put for the other fide, this latter, by fubtracting the former from $\frac{1}{4}$, fhall be $\frac{942873}{6791236}$, and by reducing the firft fide to the fame Denominator with the latter, the two defired fides about the right-angle are $\frac{754936}{6791236}$ and $\frac{942873}{6791236}$, therefore the Hypothenufal is $\frac{1207865}{6791236}$; which right-angled Triangle will folve the Queftion: For, firft, the fumm of the fides about the right-angle makes a Square, to wit, $\frac{1}{4}$; fecondly, if $\frac{754936}{6791236}$ the firft fide about the right-angle, be added to the Square of the latter fide $\frac{942873}{6791236}$, it makes a Square whofe fide is $\frac{2452745}{6791236}$; thirdly, if $\frac{942873}{6791236}$ the latter fide about the right-angle, be added to the Square of the firft fide $\frac{754936}{6791236}$, it makes a Square, whofe fide is $\frac{2640682}{6791236}$; and laftly, if $\frac{942873}{6791236}$ the latter fide about the right-angle, be added to the triple of the firft fide $\frac{754936}{6791236}$, it makes a Square, whofe fide is $\frac{1791}{2606}$.

QUEST. 116.

To find a right-angled Triangle, that the Square of the Area being added to the fumm of the fides about the right-angle may make a Square.

RESOLUTION.

1. For the fides about the right-angle put . . . . a and 1
2. Then the fumm of the Squares of thofe fides muft be equal to a Square, to wit, the Square of the Hypothenufal, therefore . . $aa + 1 = \square$
3. And the Square of the Area being added to the fumm of the fides about the right-angle muft make a Square, *viz.* . . . $\frac{1}{4}aa + a + 1 = \square$

4. So

4. So in the two last steps there is a Duplicate equality qualified as *Fermat*'s Rule presupposeth, but if it be resolved in the ordinary way, it gives no value of a either affirmative or negative, I therefore reduce that Duplicate equality to another that shall have equal numbers of aa, by multiplying $\frac{1}{4}aa + a + 1$ in the third step by the Square 4, so it produceth $aa + 4a + 4$, and now this following Duplicate equality comes to be resolved,

$$viz. \left\{ \begin{aligned} aa + 1 &= \square \\ aa + 4a + 4 &= \square. \end{aligned} \right.$$

5. Which Duplicate equality last exprest being resolved in the vulgar way, (explain'd in *Quest.* 20, 31. of this *Book*,) will give this negative value of a, viz. $a = -1\frac{7}{8}$

6. But that value of a being less than nothing, I renew the work according to *Fermat*'s method, *viz.* instead of a I take for a new Root . . . . . $a - 1\frac{7}{8}$

7. By which new Root $a - 1\frac{7}{8}$ I form this Duplicate equality out of that in the second and third steps, (after the manner before explain'd in *Quest.* 113.) *viz.* . . . . $\left\{ \begin{aligned} aa - 3\tfrac{3}{4}a + \tfrac{1}{64} &= \square \\ aa - 3\tfrac{3}{4}a + \tfrac{289}{64} &= \square \end{aligned} \right.$

8. Then I reduce that new Duplicate equality to another that shall have in each quantity one and the same absolute square number, (as *Fermat*'s Rule requires,) so this ariseth, *viz.* . . . . . $\left\{ \begin{aligned} 64aa - 240a + 289 &= \square \\ 18496aa - 69360a + 289 &= \square \end{aligned} \right.$

9. Now if the Duplicate equality last exprest be resolved in the vulgar way, the value of a will again be found negative; and if I should assume a third Root and proceed as before, the work would be intollerably tedious, and perhaps produce another negative value. This difficulty leads me to another way by which *Fermat* hath solved divers knotty Problems, but it seems to me to depend more upon chance than Art; however, I shall try whether it will find out an affirmative value of a to solve the Duplicative equality in the eighth step or not.

First then, supposing the latter of the two quantities in the eighth step to be the greater, their difference is $18432aa - 69120a$, then I seek two such numbers that being mutually multiplied may make 18432, and that their summ may make 272, (to wit, the double of 136 the side of the Square 18496 which is prefixt to aa in the eighth step,) so I find 144 and 128, which are very luckily rational, and therefore fit for the present purpose; then these two quantities $144a - 540$ and $128a$ being mutually multiplied will produce the above-mentioned difference, and in half the summ of those Factors there will be found $136a$ the side of the Square $18496aa$ in the eighth step: Then by equating the Square of half the summ of the said Factors $144a - 540$ and $128a$, *viz.* the Square of $136a - 270$, to $18496aa - 69360a + 289$ (in the eighth step,) there will thence arise $\frac{72611}{4080}$ for the value of a; wherefore $a - 1\frac{7}{8}$, (which was assumed for the new Root in the sixth step instead of a,) gives $\frac{64961}{4080}$ for the value of a in the first Duplicate equality in the second and third steps, that is, one of the sides about the right-angle of the Triangle sought; but the other of the said sides was assumed to be 1, therefore the Hypothenusal shall be $\frac{65089}{4080}$; and the three sides of the right-angled Triangle sought are $\frac{64961}{4080}$, 1, $\frac{65089}{4080}$; for if the summ of the sides about the right-angle, that is, of $\frac{64961}{4080}$ and 1 be added to the Square of the Area it makes a Square, whose side is $\frac{73121}{8160}$. And if you desire a second right-angled Triangle to solve the Question, you may put $a + \frac{64961}{4080}$ for a new Root, and proceed as before.

Concerning the second Part of Fermat's *Analytical Invention.*

Diophantus doth not so much as mention a Triplicate Equality, but *Fermat* shews how to solve two kinds thereof. The first is, when three quantities to be equated to Squares are such, that every one of them is compos'd of some number of a, either affirmative or negative, and one and the same known affirmative square number. The second kind of Triplicate equality is, when three quantities to be equated to Squares are such, that every one of them is compos'd of the same affirmative number of aa, and some number of a either affirmative or negative, without any absolute number. How each of these kinds of Triplicate equality may be resolved the following Questions will make manifest.

QUEST. 117.

QUEST. 117.

To find a number, that if it be multiplied by three given numbers ſeverally, ſuppoſe by 1, 2, 5, and to every one of the Products one and the ſame given Square, ſuppoſe 1, be added, the three ſumms may be Squares.

RESOLUTION.

1. For the number ſought put . . . . a

2. Then the Queſtion requires that every one of theſe three ſumms may make a Square, *viz.* . . . . $\left\{\begin{array}{l} a+1=\square \\ 2a+1=\square \\ 5a+1=\square \end{array}\right.$

3. Now to reſolve that Triplicate equality, firſt form a Square from any number of a + the ſide of the given Square 1, as from $a+1$, whoſe Square is $aa+2a+1$; then divide $aa+2a$ (the two firſt terms of that Square) by any one of the three numbers prefixt to a in the ſaid Triplicate equality, as by 1, which is tacitly prefixt to a in the firſt quantity $a+1$, and take the Quotient $aa+2a$ for a new Root inſtead of a, (which was put for the number ſought,) whereby the firſt part of the Queſtion is ſolved indefinitely; for if $aa+2a$ be put for the number ſought, then unity added to it makes a Square, to wit, $aa+2a+1$. Then multiply the ſaid $aa+2a$ by 2 which is prefixt to a in the ſecond quantity $2a+1$, and it produceth $2aa+4a$, to which add the given Square 1, and it makes $2aa+4a+1$ to be equated to a Square. Again, multiply $aa+2a$ by 5 which is prefixt to a in the third quantity $5a+4$, and it makes $5aa+10a$; to this add unity and it makes $5aa+10a+1$, to be equated to a Square; hence the following Duplicate equality ariſeth,

$$viz.\ \left\{\begin{array}{l} 2aa+4a+1=\square \\ 5aa+10a+1=\square \end{array}\right.$$

4. This Duplicate equality being reſolved by the vulgar way, will give $a=-6$, by which value of a if $aa+2a$ be expounded, it makes 24; (for the Square of -6 is $+36$, to which if you add -12 the double of -6, it makes $+24$;) wherefore 24 is the number ſought, and will ſolve the Queſtion: For if unity be added firſt to 24, then to 48 the double of 24, and laſtly, to 120 the quintuple of 24, the three ſumms are Squares, to wit, 25, 49, 121.

If you deſire another number beſides 24 to ſolve the Queſtion propoſed, you may aſſume $a-6$ for a new Root of the Duplicate equality laſt reſolved, and thence (by the method before explained in the firſt Part) find out a ſecond number to ſolve that Duplicate equality, and conſequently the Queſtion.

Note. When in a Triplicate equality of the firſt kind before defined, the greateſt of the three numbers of a is equal to the ſumm of the other two, then in ſuch caſe that Triplicate equality, although it may be poſſible in it ſelf, is inexplicable by the method of reſolving the preceding *Queſt.* 117. As, for example, if theſe three quantities be propoſed to be equated to as many Squares, *viz.* $5a+1$; $16a+1$; $21a+1$; where the greateſt number of a is equal to the ſumm of the other two, (for $21=16+5$,) and the value of the Root a is 3, according to which, thoſe three quantities being expounded will give theſe three Squares, 16, 49, 64; it will be in vain to ſeek out any Anſwer to that Triplicate equality by *Fermat*'s Rule, for it will produce an abſurd or fruitleſs Equation, as you will find upon tryal.

QUEST. 118.

To find a number, that if it be multiplied by three given numbers, ſuppoſe by 3, 4, 5, and the Products be ſeverally ſubtracted from 1 a given Square, the three remainders may be Squares.

RESOLUTION.

1. For the number ſought put . . . . a

2. Then the Queſtion requires that theſe three Remainders may be Squares, *viz.* . . . . $\left\{\begin{array}{l} 1-3a=\square \\ 1-4a=\square \\ 1-5a=\square \end{array}\right.$

3. This Triplicate equality may be reſolved like that in the foregoing *Queſt.* 117. For firſt, I form a Square from 1 — any number of a, as from $1-3a$, whoſe Square is $9aa-6a+1$; then I divide $9aa-6a$ (the two firſt terms of that Square,) by 3 which

which is prefixt to a in the firſt quantity $1 - 3a$, and the Quotient is $3aa - 2a$, this with its contrary ſigns is $-3aa + 2a$, which I put for a new Root inſtead of a (the number ſought,) whereby the firſt part of the Queſtion is ſatisfied infinitely; for if the triple of $-3aa + 2a$ be ſubtracted from the given Square 1, the remainder is a Square, to wit, $9aa - 6a + 1$. Then I multiply the ſaid new Root $-3aa + 2a$ by 4 and 5 ſeverally, (which are prefixt to a in the ſecond and third quantities of the Triplicate equality in the ſecond ſtep,) and ſubtracting the Products $-12aa + 8a$ and $-15aa + 10a$ ſeverally from the given Square 1, the remainders $12aa - 8a + 1$ and $15aa - 10a + 1$ are to be equated to Squares, ſo it remains only to ſolve this following Duplicate equality,

$$viz. \left\{ \begin{array}{l} 12aa - 8a + 1 = \square \\ 15aa - 10a + 1 = \square \end{array} \right.$$

4. This Duplicate equality being ſolved in the ordinary way, gives $a = \frac{6}{11}$, by which value of a, the new Root $-3aa + 2a$ being expounded will give $\frac{24}{121}$ for the value of a in the Triplicate equality in the ſecond ſtep. Wherefore $\frac{24}{121}$ will ſolve the Queſtion; for if its triple, quadruple and quintuple be ſeverally ſubtracted from unity, the three remainders are Squares, to wit, $\frac{49}{121}$, $\frac{25}{121}$, $\frac{1}{121}$.

QUEST. 119.

To find a number, as alſo four other numbers in Geometrical proportion continued, that if from theſe Proportionals ſeverally the firſt number be ſubtracted, the three remainders may be Squares.

RESOLUTION.

1. For the firſt number ſought put . . . . $a - 1$
2. Then multiply a into any four known numbers continually proportional, as into 1, 2, 4, 8, and aſſume the Products to be the four Proportionals ſought, *viz.* . . . . $a,\ 2a,\ 4a,\ 8a$
3. Then ſubtract the number in the firſt ſtep from thoſe four Proportionals ſeverally, and every one of the remainders muſt make a Square; but the firſt remainder is manifeſtly the Square 1, it remains therefore to reſolve this Triplicate equality, *viz.* . . $\left\{ \begin{array}{l} a + 1 = \square \\ 3a + 1 = \square \\ 7a + 1 = \square \end{array} \right.$

Now to reſolve that Triplicate equality you may take $aa + 2a$ for a new Root inſtead of a, whereby the firſt part of the Triplicate equality will be ſolved indefinitely, for if $aa + 2a$ be increaſed with 1 it makes a Square, to wit, $aa + 2a + 1$; then the two other parts of the ſaid Triplicate equality (by the like Operation as was uſed in *Queſt.* 117.) will be converted into this following Duplicate equality,

$$viz. \left\{ \begin{array}{l} 3aa + 6a + 1 = \square \\ 7aa + 14a + 1 = \square \end{array} \right.$$

4. Which Duplicate equality being reſolved in the vulgar way gives $a = -12$, whence $aa + 2a$ (the new Root) will be found 120; (for the Square of -12 is $+144$, to which if you add -24, that is, $2a$, it makes 120.) Therefore the firſt number ſought by the Queſtion is 119, (that is, $a - 1$,) and the four numbers required to be in continual proportion are 120, 240, 480, 960; (which anſwer to a, $2a$, $4a$, $8a$, in the ſecond ſtep:) for if 119 be ſubtracted from thoſe four Proportionals ſeverally, the remainders are the Squares 1, 121, 361, 841.

QUEST. 120.

Three ſquare numbers Geometrically proportional being given, to find a number, that if it be added to thoſe Proportionals ſeverally, the three ſumms may be Squares.

RESOLUTION.

1. Let the three given Squares in continual proportion be . . . $1,\ 4,\ 16 \div\!\div$
2. For the number ſought put . . . . a
3. Then the Queſtion requires . . . . $\left\{ \begin{array}{l} a + 1 = \square \\ a + 4 = \square \\ a + 16 = \square \end{array} \right.$

4. Now

4. Now in regard the three known Squares in that Triplicate equality are unequal, they muft be reduced to the fame Square, which may be done thus, *viz.* Becaufe a Square multiplied by a Square produceth a Square, multiply the firft quantity $a + 1$ by 64, (the Product of 4 into 16,) and it makes $64a + 64$ to be equated to a Square. Again, multiply the fecond quantity $a + 4$ by 16, (the Product of 1 into 16,) and it gives $16a + 64$ to be equated to a Square. Likewife, multiply the third quantity $a + 16$ by 4, (the Product of 1 into 4,) and it produceth $4a + 64$ to be equated to a Square. So this Triplicate equality comes forth to be refolved,

$$viz. \left\{ \begin{array}{l} 64a + 64 = \square \\ 16a + 64 = \square . \\ 4a + 64 = \square \end{array} \right.$$

5. This Triplicate equality having in every one of its three quantities the fame Square 64 may be refolved (like that in the foregoing Queftions 117, 119.) thus, *viz.* Firft, form a Square from any number of a + the fide of the known Square 64, as from $2a + 8$, the Square whereof is $4aa + 32a + 64$; then divide $4aa + 32a$ by 4, which is prefixt to a in the third quantity $4a + 64$, and take the Quotient $aa + 8a$ inftead of the Root a, (which was put for the number fought;) whence the laft part of the Triplicate equality in the fourth ftep is folved indefinitely, for four times $aa + 8a$ together with 64 makes a Square, to wit, $4aa + 32a + 64$. Again, by taking the faid $aa + 8a$ inftead of the Root a, the firft and fecond parts of the faid Triplicate equality will be reduced to this Duplicate equality,

$$viz. \left\{ \begin{array}{l} 64aa + 512a + 64 = \square \\ 16aa + 128a + 64 = \square . \end{array} \right.$$

6. Which Duplicate equality being refolved in the vulgar way gives $a = \frac{304}{105}$, whence $aa + 8a$ (the new Root) will be found $\frac{347776}{11025}$, which is the number fought by the Queftion; for if 1, 4, 16 be feverally added to the faid $\frac{347776}{11025}$, the three fumms will be the Squares of thefe fides $\frac{599}{105}$, $\frac{626}{105}$, $\frac{724}{105}$.

I fhall now proceed to the fecond kind of Triplicate equality, and explain it by Queftions.

QUEST. 121. (Probl. 4. cap. 1. part. 2. *Dioph.* redivivi.)

To find a number, that if it be multiplied by every one of three given numbers Geometrically proportional, fuppofe by 1, 2, 4, and the Products be added feverally to the Square of the number fought, the three fumms may be Squares.

RESOLUTION.

1. For the number fought put . . . . a
2. Then multiply that number a by the three given Proportionals 1, 2, 4, and to the Products feverally add the Square of the faid number a, fo (according to the import of the Queftion) thefe three fumms muft make Squares, *viz.* . . . . $\left\{ \begin{array}{l} aa + a = \square \\ aa + 2a = \square \\ aa + 4a = \square \end{array} \right.$
3. Now that Triplicate equality being of the fecond kind before defined, muft be reduced to a Triplicate equality of the firft kind thus, *viz.* Take $\frac{1}{a}$ inftead of the Root a, (the number fought,) whence the firft quantity $aa + a$ will be converted into $\frac{1}{aa} + \frac{1}{a}$; the fecond quantity $aa + 2a$ will be reduced to $\frac{1}{aa} + \frac{2}{a}$; and the third quantity $aa + 4a$ to $\frac{1}{aa} + \frac{4}{a}$. Then becaufe a Square multiplied by a Square produceth a Square, if every one of the three new terms $\frac{1}{aa} + \frac{1}{a}$; $\frac{1}{aa} + \frac{2}{a}$ and $\frac{1}{aa} + \frac{4}{a}$, which are to be equated to Squares, be multiplied by the Square aa, that Triplicate equality will be reduced to this,

$$viz. \left\{ \begin{array}{l} 1 + a = \square \\ 1 + 2a = \square . \\ 1 + 4a = \square \end{array} \right.$$

4. This Triplicate equality falling under the firft form may be refolved like that in the preceding *Queft.* 117. thus, *viz.* Inftead of a I take for a new Root $aa + 2a$, which increafed with 1 makes a Square, to wit, $aa + 2a + 1$, whereby the firft part of the

Triplicate

Triplicate equality is folved indefinitely; then by multiplying $aa + 2a$ by 2 and 4 feverally, and adding 1 to each Product, this Duplicate equality arifeth,

$$viz. \left\{ \begin{array}{l} 2aa + 4a + 1 = \square \\ 4aa + 8a + 1 = \square \end{array} \right.$$

5. Whence $a = -\frac{20}{7}$, by which if $aa + 2a$ (the new Root) be refolved it makes $\frac{120}{49}$; this divided by 1, becaufe $\frac{1}{a}$ was taken inftead of a in the firft Triplicate equality, gives $\frac{49}{120}$, which will folve the Queftion; for if $\frac{49}{120}$ be multiplied feverally by 1, 2, 4, and the Products be added feverally to the Square of the faid $\frac{49}{120}$, the three fumms will be Squares, whofe fides are $\frac{91}{120}$, $\frac{119}{120}$, $\frac{161}{120}$.

QUEST. 122.

To find three fquare numbers, whofe fumm added to their three fides feverally, may make Squares.

RESOLUTION.

1. Divide fome fquare number, fuppofe 121, into three fuch Squares, that the greateft may exceed the fumm of the other two, fuch are 4, 36, 81; then for the three Squares fought put $\Big\}$ $4aa$, $36aa$, $81aa$
2. The fides of thofe Squares are . . . . $2a$, $6a$, $9a$
3. And the fumm of the fame Squares is . . . . $121aa$
4. But if the faid fumm $121aa$ be added to the three fides in the fecond ftep feverally, every one of the three fumms muft make a Square; hence this Triplicate equality arifeth, *viz.* . . $\left\{ \begin{array}{l} 121aa + 2a = \square \\ 121aa + 6a = \square \\ 121aa + 9a = \square \end{array} \right.$
5. Which Triplicate equality; (by the method delivered in the preceding *Queft.* 121.) will be reduced to this Triplicate equality, *viz.* . . . . $\left\{ \begin{array}{l} 121 + 2a = \square \\ 121 + 6a = \square \\ 121 + 9a = \square \end{array} \right.$
6. Now to refolve the Triplicate equality laft expreft, I take (according to the Rule before given) $2aa + 22a$ for a new Root inftead of a, whence the firft part of the faid Triplicate equality will be folved indefinitely, and the fecond and third parts will be converted into this Duplicate equality,

$$viz. \left\{ \begin{array}{l} 12aa + 132a + 121 = \square \\ 18aa + 198a + 121 = \square \end{array} \right.$$

7. Which Duplicate equality being refolved in the ordinary way will give $a = -\frac{121}{47}$; according to which, the new Root $2aa + 22a$ will be found $\frac{61920}{2209}$, and this divided by 1, (becaufe $\frac{1}{a}$ was put inftead of a, in reducing the Triplicate equality in the fourth ftep to that in the fifth,) gives $\frac{2209}{61920}$ for the value of a in the Triplicate equality in the fourth ftep; according to which laft mentioned value, the three fides in the fecond ftep are $\frac{4418}{61920}$, $\frac{13254}{61920}$, $\frac{19881}{61920}$, whofe Squares will folve the Queftion; for if thofe fides be feverally added to the fumm of their Squares, the three fumms will be Squares, whofe fides are $\frac{32469}{61920}$, $\frac{37241}{61920}$, $\frac{43311}{61920}$.

Again, if this Triplicate equality were propofed to be refolved, *viz.* . . . . $\left\{ \begin{array}{l} aa + 2a = \square \\ 4aa + 3a = \square \\ 16aa + 6a = \square \end{array} \right.$

It may be reduced to this by putting $\frac{1}{a}$ for a, . . . $\left\{ \begin{array}{l} 1 + 2a = \square \\ 4 + 3a = \square \\ 16 + 6a = \square \end{array} \right.$

And that laft expreft, (by what hath been faid in *Queft.* 120.) may be reduced to this, ... $\left\{ \begin{array}{l} 64 + 128a = \square \\ 64 + 48a = \square \\ 64 + 24a = \square \end{array} \right.$

And this Triplicate equality being refolved like thofe in *Queft.* 117, 120. will give $\frac{112615}{529}$ for the value of a in either of the two laft preceding Triplicate equalities; and laftly, by dividing 1 by $\frac{112615}{529}$, it gives $\frac{529}{112615}$ for the value of a in the Triplicate equality propofed in the eighth ftep.

QUEST. 123. (Probl. 9. in cap. 1. part. 2. *Dioph. redivivi.*)

To find a number, that if it be multiplied by three given numbers in Arithmetical Progreſſion, and the Products be ſubtracted from the Square of the number ſought, the three remainders may be Squares. But the following Reſolution preſuppoſeth, (for the reaſon before given in the *Note* at the end of *Queſt.* 117.) that the greateſt of the three numbers given is not equal to the ſumm of the other two. Let therefore the given numbers be 3, 4, 5.

RESOLUTION.

1. For the number ſought put . . . . a
2. Then (according to the import of the Queſtion) this Triplicate equality muſt be reſolved, *viz.* . . . . $\left\{\begin{array}{l} aa - 3a = \square \\ aa - 4a = \square \\ aa - 5a = \square \end{array}\right.$
3. But that (according to the *Rule* in *Queſt.* 121.) will be reduced to this, *viz.* . . . . $\left\{\begin{array}{l} 1 - 3a = \square \\ 1 - 4a = \square \\ 1 - 5a = \square \end{array}\right.$

In which Triplicate equality laſt expreſt the value of the Root a was by *Queſt.* 118. found $\frac{24}{121}$, but becauſe in the reduction of the Triplicate equality in the ſecond ſtep, $\frac{1}{a}$ was taken for a new Root inſtead of a, we muſt divide 1 by the ſaid $\frac{24}{121}$, ſo there ariſeth $\frac{121}{24}$ for the number ſought by the Queſtion; for if $\frac{121}{24}$ be multiplied by 3, 4, 5 ſeverally, and the Products be ſeverally ſubtracted from the Square of $\frac{121}{24}$, the three remainders will be the Squares of $\frac{77}{24}$, $\frac{55}{24}$ and $\frac{11}{24}$.

Note 1. Sometimes, when four, five or more quantities are to be equated to Squares, they may be reſolved by the method before explain'd: As, for example,

If this Quadruplicate equality be propos'd to be reſolved, . $\left\{\begin{array}{r} 20a + 64 = \square \\ 12a + 16 = \square \\ 8a + 4 = \square \\ 2a + 1 = \square \end{array}\right.$

You may (by the method in the preceding *Queſt.* 120.) reduce that four-fold equality to this, *viz.* . . . . $\left\{\begin{array}{r} 20a + 64 = \square \\ 48a + 64 = \square \\ 128a + 64 = \square \\ 128a + 64 = \square \end{array}\right.$

Which laſt Quadruplicate equality is in effect but a Triplicate equality, for there are two Terms which happen to be the ſame, to wit, $128a + 64$; and therefore you may reſolve that Triplicate equality by the method before delivered.

Note 2. Sometimes alſo by the preceding method of reſolving a Triplicate equality of the firſt kind, you may reſolve one of *Diophantus's* kinds of Duplicate equality, (to wit, that explain'd in the preceding *Queſt.* 33.) more eaſily than by his method, as will appear by the following *Queſt.* 124, 125, 126.

QUEST. 124. (The ſame with the foregoing *Queſt.* 33.)

To find a number leſs than 2, and ſuch, that if it be multiplied ſeverally by two numbers given, ſuppoſe by 6 and 8, and to each of the Products there be added the ſame given Square 4, the two ſumms may be Squares.

RESOLUTION.

1. For the number ſought put . . . . a
2. Then the Queſtion requires this Duplicate equality to be reſolved, ... $\left\{\begin{array}{l} 6a + 4 = \square \\ 8a + 4 = \square \end{array}\right.$
3. To which end you may proceed thus; Firſt, (according to the method of reſolving a Triplicate equality before delivered,) form a Square from $a + 2$, (2 being the ſide of the given Square 4,) ſo that Square will be $aa + 4a + 4$; then take $\frac{1}{6}$ part of $aa + 4a$, ($\frac{1}{6}$ part, becauſe 6 is prefixt to a in the firſt part of the given Duplicate equality,) and it is $\frac{1}{6}aa + \frac{2}{3}a$, which is to be aſſumed for a new Root inſtead of a. Whence 'tis evident, that if the given Square 4 be added to ſix times $\frac{1}{6}aa + \frac{2}{3}a$, it makes a Square, to wit, $aa + 4a + 4$, whereby the firſt part of the Queſtion is ſolved indefinitely. Then multiply the new Root $\frac{1}{6}aa + \frac{2}{3}a$ by 8, and to the Product add 4, ſo

ſo there comes forth $\frac{4}{3}aa + \frac{16}{3}a + 4$ to be equated to a Square, the ſide whereof muſt be ſo feigned that $\frac{1}{6}aa + \frac{2}{3}a$ (the new Root) may be leſs than 2; but if $\frac{1}{6}aa + \frac{2}{3}a$ be leſs than 2, it will follow that a is leſs than 2. Wherefore let $\frac{4}{3}aa + \frac{16}{3}a + 4$ be equated to a Square, ſo, as that a may be leſs than 2; to which end the ſide of the ſaid Square may be feigned $-\,2 +$ any number of a greater than $3\frac{24}{100}a$; let therefore the ſaid ſide be feigned $4a - 2$, and then the Square of $4a - 2$ being equated to $\frac{4}{3}aa + \frac{16}{3}a + 4$, will give $a = \frac{16}{11}$, by which, if the new Root $\frac{1}{6}aa + \frac{2}{3}a$ be reſolved, it makes $\frac{160}{121}$ for the number ſought by the Queſtion propoſed: For firſt, it is leſs than 2; ſecondly, ſix times $\frac{160}{121}$ together with 4 makes a Square, to wit, $\frac{1444}{121}$, whoſe ſide is $\frac{38}{11}$; and laſtly, eight times $\frac{160}{121}$ together with 4 makes the Square $\frac{1764}{121}$, whoſe ſide is $\frac{42}{11}$.

Again, if this Duplicate equality were propoſed, ... $\left\{\begin{array}{l} 4 - 6a = \square \\ 4 - 8a = \square \end{array}\right.$

You may put $\frac{2}{3}a - \frac{1}{6}aa$ for a new Root inſtead of a, and then proceed as before.

Again, if this Duplicate equality were propoſed, ... $\left\{\begin{array}{l} 4 + 6a = \square \\ 4 - 8a = \square \end{array}\right.$

You may take $\frac{1}{6}aa + \frac{2}{3}a$ for a new Root inſtead of a, and then proceed as before.

QUEST. 125. (Probl. 10. in cap. 1. part. 2. *Dioph. redivivi.*)

To find a number, as alſo three other numbers in Geometrical proportion, that if from theſe Proportionals ſeverally the firſt number be ſubtracted, the three remainders may be Squares.

RESOLUTION.

1. For the firſt number ſought put . . . . $2a - 1$
2. And for the three continual Proportionals ſought put . . $a,\ 2a,\ 4a$
3. From which if you ſubtract the firſt number ſought, the remainders are . . . . $1 - a,\ 1,\ 1 + 2a$
4. Among which remainders the mean is a Square, wherefore it remains to equate each of the two extremes to a Square, *viz.* $\left\{\begin{array}{l} 1 - a = \square \\ 1 + 2a = \square \end{array}\right.$

Now to reſolve that Duplicate equality, you may take $2aa + 2a$ for a new Root inſtead of a, whence $1 + 2a$ will be converted into the Square $4aa + 4a + 1$, and $1 - a$ into $1 - 2a - 2aa$, which muſt be equated to a Square, but with this Caution, That the new Root $2aa + 2a$ may be leſs than 1, and that $4aa + 4a$ may exceed 1, and conſequently that the value of a may fall between $\frac{21}{100}$ and $\frac{36}{100}$; to which end, the ſide of the ſaid Square may be feigned $1 - 2a$, whoſe Square $4aa - 4a + 1$ being equated to $1 - 2a - 2aa$, gives $a = \frac{1}{3}$; therefore the new Root $2aa + 2a$ is $\frac{8}{9}$, and the firſt number ſought, which was repreſented by $2a - 1$, is conſequently $\frac{7}{9}$, and the three deſired Proportionals are $\frac{8}{9}$, $\frac{16}{9}$, $\frac{32}{9}$; which will ſolve the Queſtion, as may eaſily be proved.

QUEST. 126.

To find two numbers, ſuch, that if their ſumm be increaſed and leſſened, as well by their difference as the difference of their Squares, the ſumms and remainders may be Squares.

RESOLUTION.

1. If unity be divided into any two parts their difference is equal to the difference of their Squares, (as hath been ſhewn in *Queſt.* 53. of this Book,) therefore for the two numbers ſought put . . . . $\frac{1}{2} + a$ and $\frac{1}{2} - a$
2. Whence their difference, as alſo the difference of their Squares is $2a$
3. It remains therefore that the ſumm of the two numbers in the firſt ſtep, to wit, unity, being increaſed and leſſened by $2a$, the ſumm and remainder may be Squares; hence this Duplicate equality ariſeth, *viz.* . . . . $\left\{\begin{array}{l} 1 + 2a = \square \\ 1 - 2a = \square \end{array}\right.$
4. Now to reſolve that Duplicate equality, you may take $\frac{1}{2}aa + a$ for a new Root inſtead of a, whence $1 + 2a$ will be converted into the Square $aa + 2a + 1$, and $1 - 2a$ into $1 - 2a - aa$; which muſt be equated to a Square, but with this Caution, That the new Root $\frac{1}{2}aa + a$ may be leſs than $\frac{1}{2}$, and conſequently a leſs than $\sqrt{2} - 1$,

that

that is, lefs than $\frac{414}{1000}$; but to caufe that effect, the fide of the faid Square may be feigned to be $1 -$ any number of a greater than a, as $1 - 3a$, the Square whereof being equated to the faid $1 - 2a - aa$ will give $a = \frac{2}{5}$, therefore the new Root $\frac{1}{2}aa + a$ is $\frac{12}{25}$, according to which, the Pofitions in the firft ftep being expounded, will give $\frac{49}{50}$ and $\frac{1}{50}$ for the numbers fought: For if to their fumm, which is 1, you you add and fubtract their difference $\frac{48}{50}$, the fumm and remainder are Squares, to wit, $\frac{49}{25}$ and $\frac{1}{25}$; and thefe will alfo come forth when the difference of the Squares of the two numbers $\frac{49}{50}$ and $\frac{1}{50}$ is added to and fubtracted from their fumm 1, becaufe the difference of the two numbers is equal to the difference of their Squares.

Concerning the third Part of Fermat's *Analytical Invention.*

I. The Scope of this third Part is chiefly to fhew how to equate a quantity compos'd of five Terms, *viz.* of fome numbers of $aaaa$, aaa, aa, a with an abfolute (or known) number, to a Square; as alfo to equate a quantity compos'd of four Terms to a Square or Cube; and that in fuch manner, that for the moft part innumerable values of the unknown Root a may be found out.

II. In equating a quantity compos'd of five Terms to a Square, one of thefe two things is abfolutely neceffary, *viz.* either the firft Term muft be a Biquadrate, or elfe the laft Term, to wit, the abfolute number, muft be a rational Square. Likewife, in equating a quantity confifting of four Terms to a Square, one of the extremes muft be a Square. And laftly, in equating a quantity of four Terms to a Cube, one of the extremes muft be a Cube.

III. When a quantity compos'd of five Terms is given to be equated to a Square, and the firft Term is a Biquadrate, but the abfolute number, that is, the laft Term, hath not a rational fquare Root, then the fide of the Square muft be feigned fo, as that in the Square it felf there may be found the fame numbers of $aaaa$, aaa and aa as are in the quantity given to be equated, to the end that thofe three firft Terms may by Reduction deftroy one another, and confequently an Equation remain between fome number of a and an abfolute number, whence the value of the Root a, if it hath any poffible value, may be expreffible by fome rational number either affirmative or negative; but how to feign the faid fide fo as to caufe that effect, the following Propofition and Canon will fhew; where to evidence the certainty thereof, I fhall affume b, c, d to ftand for Coefficients or known numbers prefixt to aaa, aa and a, and n for the abfolute number, (or laft Term,) which in this cafe is fuppofed to have no rational fquare Root.

PROP.

1. Let this quantity be given to be equated to a Square, *viz.* . . . . $\}$ $aaaa + baaa + caa + da + n = \square$

CANON.

2. When $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, then let the fide of the Square fought be feigned . . . . $\}$ $aa + \frac{1}{2}ba + \frac{1}{2}c - \frac{1}{8}bb$
3. But if $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, then let the fide of the Square be feigned . . . . $\}$ $\frac{1}{8}bb - \frac{1}{2}c - \frac{1}{2}ba - aa$

Examples in Numbers.

4. Let this quantity be given to be equated to a Square, *viz.* . . . . $\}$ $aaaa + 4aaa + 6aa + 2a + 7 = \square$
5. Here, becaufe $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, *viz.* half 6 exceeds $\frac{1}{8}$ of the Square of 4, the firft part of the Canon gives this feigned fide of a Square, *viz.* $\}$ $aa + 2a + 1$
6. Then by equating the Square of the faid fide $aa + 2a + 1$ to the given quantity $aaaa + 4aaa + 6aa + 2a + 7$, the value of a, after due Reduction, will be found 3, by which if the given quantity be refolved, it makes the Square 256, whofe fide is 16.
7. Again, let this quantity be given to be equated to a Square, ... $\}$ $aaaa + 4aaa + 2aa - 6a + 11 = \square$
8. Here, becaufe $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, *viz.* $\frac{1}{8}$ of the Square of 4 exceeds the half of 2, the latter part of the Canon gives this feigned fide of a Square, *viz.* $\}$ $1 - 2a - aa$

9. Then

9. Then by equating the Square of the ſaid feigned ſide $1 - 2a - aa$ to the given quantity $aaaa + 4aaa + 2aa - 6a + 11$, the value of the Root a will be found 5, by which if the given quantity be reſolved it makes the Square 1156, whoſe ſide is 34.

IV. When a quantity compos'd of five Terms is to be equated to a Square, and the the firſt Term is not a perfect Biquadrate, but the laſt Term, that is, the abſolute number is a Square, then the ſide of a Square muſt be feigned ſo, as that in the Square it ſelf there may be found the ſame numbers of aa, a and abſolute number, as are in the quantity given to be equated; to the end that thoſe three laſt Terms by due Reduction may vaniſh out of each part, and an Equation remain between ſome numbers of $aaaa$ and aaa, whence the value of the Root a, if it hath any poſſible value, may be expreſſible by ſome rational number either affirmative or negative. But how to feign the ſaid ſide ſo as to cauſe ſuch an effect, the following Propoſition and Canon will ſhew; where to evidence the certainty thereof, I put b, c, d to ſtand for the Coefficients or known numbers prefixt to aaa, aa, a; alſo, rr (whoſe ſide is r) for the laſt Term, which in this Caſe is a rational ſquare number, and f, which is prefixt to $aaaa$, ſtands for a number not a Square.

PROP.

1. Let this quantity be given to be equated to a Square, *viz.* . . . . $faaaa + baaa + caa + da + rr = \square$

CANON.

2. When $4crr$ exceeds dd, then let the ſide of the Square ſought be feigned . . . . . . $\frac{4crr - dd}{8rrr}aa + \frac{d}{2r}a + r$

But if dd exceeds $4crr$, then let the ſide of the Square be feigned . . . . $r + \frac{d}{2r}a - \frac{dd - 4crr}{8rrr}aa$

Examples in Numbers.

3. Let this quantity be given to be equated to a Square, *viz.* . . . . $10aaaa + 4aaa + 19aa + 6a + 9 = \square$

4. Here, becauſe $4crr$ exceeds dd, *viz.* four times $19 \times 9 \times 9$ exceeds the Square of 6, the firſt part of the Canon gives this feigned ſide of a Square, *viz.* . . . . . . $3aa + 3a + 3$

5. Then by equating the Square of the ſaid ſide $3aa + 3a + 3$ to the given quantity $10aaaa + 4aaa + 19aa + 6a + 9$, the value of the Root a will be found 2, according to which the ſaid given quantity being expounded makes the Square 289, whoſe ſide is 17.

6. Again, let this quantity be given to be equated to a Square, *viz.* . . . . $2aaaa + 3aaa + 3aa + 6a + 1 = \square$

7. Here, becauſe dd exceeds $4crr$, *viz.* the Square of 6 exceeds four times $3 \times 1 \times 1$, the latter part of the Canon gives this feigned ſide of a Square, *viz.* . . . . . $1 + 3a - 3aa$

8. Then by equating the Square of the ſaid ſide $1 + 3a - 3aa$, to the given quantity $2aaaa + 3aaa + 3aa + 6a + 1$, the value of the Root a will be found 3, according to which, the ſaid given quantity being expounded makes the Square 289.

V. When a Quantity compos'd of five Terms given to be equated to a Square is ſuch, that the firſt Term is a Biquadrate, and the laſt Term (that is, the abſolute number) hath a rational ſquare Root, then the ſide of a Square to be equated to ſuch Quantity may be varied ſix ſeveral ways, (including into this number the two laſt preceding Canons,) by every one of which the value of the Root a may oftentimes be found out, and expreſt by a rational number either affirmative or negative. To evidence this, I ſhall (as before) put b, c, d to ſtand for the Coefficients or known numbers prefixt to aaa, aa and a; and rr (whoſe ſide is r,) for the rational ſquare number which is the laſt Term. As, for example,

Let this Quantity be propos'd to be equated to a Square, *viz.* . . . . $aaaa + baaa + caa + da + rr$

1. Then to the end that $aaaa + da + rr$ may be found in a Square to be equated to the quantity propoſed, the ſide of that Square may be feigned $aa + \frac{d}{2r}a + r$

2. Or,

2. Or, to caufe the fame effect, the fide of the faid Square may be feigned . . . . $r + \frac{d}{2r}a - aa$

3. Again, that $caa + da + rr$ may be found in a Square to be equated to the quantity propofed, the fide of fuch Square, when $4crr$ exceeds dd, may (agreeable to the Canon in the preceding *Sect.* 4.) be feigned . . $\frac{4crr - dd}{8rrr}aa + \frac{d}{2r}a + r$

4. But to caufe the fame effect, if dd exceeds $4crr$, then then let the fide of the Square be feigned . . . . $r + \frac{d}{2r}a - \frac{dd - 4crr}{8rrr}aa$

5. Again, that $aaaa + baaa + rr$ may be found in a Square to be equated to the quantity propofed, let the fide of the Square be feigned . . . . $aa + \frac{1}{2}ba + r$

6. Or, to caufe the fame effect, the fide of the faid Square may be feigned . . . . $aa + \frac{1}{2}ba - r$

7. Again, that $aaaa + baaa + caa$ may be found in a Square to be equated to the quantity propofed, the fide of that Square, when $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, may (agreeable to the Canon in *Sect.* 3.) be feigned . . . . $aa + \frac{1}{2}ba + \frac{1}{2}c - \frac{1}{8}bb$

8. But, to caufe the fame effect, if $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, let the fide of the faid Square be feigned . . . . . . $\frac{1}{8}bb - \frac{1}{2}c - \frac{1}{2}ba - aa$

Examples in Numbers of the preceding fides of Squares expreft by Letters.

Let this quantity be propofed to be equated to a Square, *viz.* . . . . $aaaa + 4aaa + 10aa + 20a + 1 = \square$

1. Then fuppofing b, c, d, r, to ftand for 4, 10, 20, 1, the firft of the preceding literal fides will give . . . . $aa + 10a + 1$

The Square of which fide $aa + 10a + 1$ being equated to the propofed quantity $aaaa + 4aaa + 10aa + 20a + 1$, there will arife, after due Reduction, $16aaa = -92aa$; whence by dividing each part by $16aa$, there comes forth $-\frac{23}{4}$ for the value of the Root a, according to which, the propofed quantity being expounded will give the Square $\frac{140625}{256}$, whofe fide is $\frac{375}{16}$.

2. Again, the third literal fide gives . . . $1 + 10a - aa$

The Square of which fide $1 + 10a - aa$ being equated to the propofed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = \frac{11}{3}$, according to which, the fame quantity being refolved, makes the Square of $\frac{218}{9}$.

3. Again, becaufe in the quantity propofed, dd exceeds $4crr$, the fourth literal fide gives . $1 + 10a - 45aa$

The Square of which fide $1 + 10a - 45aa$ being equated to the propofed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = \frac{113}{253}$, according to which, the fame quantity being expounded makes the Square of the fide $\frac{224706}{64009}$.

4. Again, the fifth literal fide gives . . . $aa + 2a + 1$

The Square of which fide $aa + 2a + 1$ being equated to the propofed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = -4$, according to which, the fame quantity being expounded makes the Square 81.

5. Again, the fixth literal fide gives . . . $aa + 2a - 1$

The Square of which fide $aa + 2a - 1$ being equated to the propofed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = -3$, according to which the faid quantity being refolved makes the Square 4.

6. Laftly, becaufe $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, the feventh literal fide gives . . . . $aa + 2a + 3$

The Square of which fide $aa + 2a + 3$ being equated to the propofed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = 1$, according to which the fame quantity being refolved makes the Square 36.

VI. Some-

VI. Sometimes, when a quantity compos'd of five Terms is equated to a Square formed from one of the eight literal sides before exprest in *Sect.* V. no value of the Root a either affirmative or negative can thence be discovered. As, for example,

If this quantity be propos'd to be equated to a Square, *viz.*	$aaaa + 18aaa + 120aa + 351a + 1519 = \square$
The Canon in *Sect.* III. (or the seventh literal side in *Sect.* VI.) will give this feigned side of a Square, *viz.* .	$aa + 9a + 19\frac{1}{2}$
The Square of which side $aa + 9a + 19\frac{1}{2}$ is	$aaaa + 18aaa + 120aa + 351a + 380\frac{1}{4}$
Which Square being equated to the quantity proposed will give this fruitless and absurd Equation, *viz.*	$1138\frac{3}{4} = 0$

VII. When negative Terms are intermingled with affirmative, in a quantity compos'd of five Terms given to be equated to a Square, the side of the Square, (when such an Equation is possible,) shall be one of the eight literal sides before exprest in *Sect.* V. saving that one, and sometimes two of its signs $+$ must be changed into $-$. As, for example,

If $aaaa - 8aaa + 28aa - 40a + 4$ be given to be equated to a Square, it may be variously done, in regard the extremes $aaaa$ and 4 are Squares. First then, I imagine all the Terms of the proposed quantity to be affirmative, so it will be $aaaa + 8aaa + 28aa + 40a + 4$; now to feign the side of a Square, that $aaaa + 8aaa + 4$ may by due Reduction vanish out of each part, the fifth literal side in *Sect.* V. being resolved into numbers will give $aa + 4a + 2$ for the feigned side; but here two of its signs $+$ must be changed into $-$, that in its Square there may be found $aaaa - 8aaa + 4$ to destroy $aaaa - 8aaa + 4$ in the quantity given to be equated, to which end, the said side $aa + 4a + 2$ must be converted into $aa - 4a - 2$, and then the Square of this side being equated to the proposed quantity $aaaa - 8aaa + 28aa - 40a + 4$ will give $a = \frac{7}{2}$.

Again, to feign the side of a Square to be equated to the same given quantity $aaaa - 8aaa + 28aa - 40a + 4$, so, as to destroy the first, second and last Terms in each part, the first of the literal sides in *Sect.* V. being resolved into numbers, gives $aa + 10a + 2$ for the feigned side, if all the Terms of the given quantity were affirmative; but that $aaaa - 8aaa + 4$ may vanish out of each part, the said side $aa + 10a + 2$ must be changed into $aa - 10a + 2$, and then the Square of this side being equated to the given quantity $aaaa - 8aaa + 28aa - 40a + 4$, will give $a = \frac{19}{3}$.

Again, to feign the side of a Square to be equated to the same given quantity $aaaa - 8aaa + 28aa - 40a + 4$, in such manner that $aaaa - 40a + 4$ may vanish out of each part, the first of the eight literal sides in *Sect.* V. being resolved into numbers, gives $aa + 10a + 2$ for the feigned side, if all the Terms of the given quantity were affirmative; but that $aaaa - 40a + 4$ may be expunged out of each part, the said side $aa + 10a + 2$ must be changed into $2 - 10a - aa$, and then the Square of this side being equated to the given quantity $aaaa - 8aaa + 28aa - 40a + 4$, will give $a = -\frac{17}{7}$. Other sides might likewise be feigned, as is evident by the foregoing *Sect.* V. and to him that is a little exercis'd in this Method it will not be difficult to change the Signs in any possible case.

VIII. When a quantity consisting of five Terms is to be equated to a Square, and one or more values of the Root a are found out, either affirmative or negative, by the Rules before given, you may from every one of those Primitive Roots or values, find out other values of the Root a, even as many as you please, which latter, *Fermat* calls Derivative Roots of the first, second, third, *&c.* degree. As, for example,

To find a Derivative Root of the first degree out of the quantity $aaaa + 4aaa + 10aa + 20a + 1$, before proposed in *Sect.* V. to be equated to a Square, take one of its Primitive Roots there found out, to wit, -3, and connect it to a, so it makes $a - 3$, then instead of a take $a - 3$ for a new Root, according to which the proposed quantity $aaaa + 4aaa + 10aa + 20a + 1$ will be converted into $aaaa - 8aaa + 28aa - 40a + 4$, as here you see;

$aaaa$	$aaaa - 12aaa + 54aa - 108a + 81$
$4aaa$	$+ 4aaa - 36aa + 108a - 108$
$10aa$	$+ 10aa - 60a + 90$
$20a$	$+ 20a - 60$
1	$+ 1$
Summ,	$aaaa - 8aaa + 28aa - 40a + 4$

This fumm muft be equated to a Square, whofe fide (as before hath been fhewn in *Sect.* 7.) may be feigned $aa - 4a - 2$, the Square whereof being equated to the faid fumm will give $a = \frac{7}{2}$; but becaufe the new Root was put $a - 3$, out of $\frac{7}{2}$ fubtract 3, and there will remain $\frac{1}{2}$ for a fecond value of the Root a in the propofed quantity $aaaa + 4aaa + 10aa + 20a + 1$, which fecond value may be called a Derivative Root of the firft degree.

In like manner by the help of the faid fecond value $\frac{1}{2}$ you may find out a third, by joyning $+\frac{1}{2}$ to a, and taking $a + \frac{1}{2}$ for a new Root, according to which, the given quantity $aaaa + 4aaa + 10aa + 20a + 1$ will be converted into $aaaa + 6aaa + \frac{35}{2}aa + \frac{67}{2}a + \frac{225}{16}$ to be equated to a Square, the fide whereof may be feigned $aa + 3a + \frac{15}{4}$, (agreeable to the fifth literal fide in *Sect.* 5.) the Square whereof being equated to the given quantity, there will thence arife $a = -11$, therefore the new Root $a + \frac{1}{2}$ gives $a = -\frac{21}{2}$ for a third value of the Root a in the given quantity, that is, a Derivative Root of the fecond degree.

Nor will the Operation be otherwife to find out a fourth value, or derivative Root of the third degree, by putting for a new Root $a - \frac{21}{2}$, for according to this, every member of the propofed quantity $aaaa + 4aaa + 10aa + 20a + 1$ being refolved, there will come forth $aaaa - 38aaa + \frac{1091}{2}aa - \frac{6995}{2}a + \frac{134689}{16}$, this fumm is to be equated to a Square, whofe fide may be feigned $aa - 19a - \frac{367}{4}$, and the Square of this fide being equated to the faid fumm will give $a = \frac{873}{46}$, from which if you fubtract $\frac{21}{2}$, (becaufe the new Root was put $a - \frac{21}{2}$,) there will remain $\frac{195}{23}$ for a fourth value of the Root a, that is, a derivative Root of the third degree, out of the quantity firft propofed to be equated to a Square.

Laftly, as by the help of one of the primitive Roots of the propofed quantity $aaaa + 4aaa + 10aa + 20a + 1$ other Roots have been derived, fo by the help of any one of the reft of the primitive Roots of the fame quantity, found out in the Examples of *Sect.* 5. you may proceed to find out other derivative Roots, but fometimes you will meet with fruitlefs Equations.

IX. A quantity compos'd of four Terms may be equated to a Square, when either the abfolute number, that is, the laft Term is a Square, or the firft Term a Biquadrate.

Firft, let $20aaa + 5aa + 40a + 16$ be given to be equated to a Square. Feign the fide fo, that $40a + 16$ may vanifh out of each part, to which purpofe, let the fide be $5a + 4$, (5 being the Quotient that arifeth by dividing 40 in the $4a$, by 8 the double of the fide of the given Square 16;) then by equating the Square of $5a + 4$ to the given quantity $20aaa + 5aa + 40a + 16$ you will find $a = 1$, according to which, that quantity being refolved makes the Square 81. Now to find a fecond value of the Root a, you may put for a new Root $a + 1$, according to which, the given quantity $20aaa + 5aa + 40a + 16$ will be converted into $20aaa + 65aa + 110a + 81$ to be equated to a Square, the fide whereof, (that $110a + 81$ may vanifh out of each part,) may be feigned $\frac{55}{9}a + 9$, whence after due Reduction, there will arife $a = -\frac{112}{81}$; therefore $a + 1$ (which was put for the new Root) gives $1 - \frac{112}{81}$, that is, $\frac{31}{81}$ for a fecond value of the Root a, (or a derivative Root of the firft degree,) and by putting $a + \frac{31}{81}$ for a new Root you may find out a third value, and fo infinitely.

Secondly, an Example where the firft Term is $aaaa$ may be this, *viz.* Let $aaaa + 4aaa - 3aa + 2a$ be given to be equated to a Square. That $aaaa + 4aaa$ may vanifh out of each part, feign the fide of a Square to be $aa + 2a$, (2 in the $2a$ being the half of 4 prefixt to aaa in the given quantity;) then the Square of $aa + 2a$ being equated to $aaaa + 4aaa - 3aa + 2a$ will give $a = \frac{2}{7}$; and to find out a fecond value of a you may put $a + \frac{2}{7}$ for a new Root, and proceed as in former Examples.

Thirdly,

Thirdly, although some intermediate Term be omitted in a quantity compos'd of four Terms, such quantity may be equated to a Square: As, to equate $5aaaa + 16aaa + 24aa + 16$ to a Square, you may feign its side to be $3aa + 4$, (3 in the $3aa$ being the Quotient that ariseth by dividing 24 which is prefixt to aa in the given quantity, by 8 the double of the side of the given Square 16,) and thence the value of a will be found 4; then you may put $a + 4$ for a new Root to find out a second value of a.

In like manner, if $aaaa + 60aa + 80a + 500$ be proposed to be equated to a Square, you may feign its side to be $aa + 30$, (30 being the half of 60 which is prefixt to aa in the proposed quantity,) whence you will find $a = 5$, and for a derivative Root you may put $a + 5$.

X. A Quantity compos'd of four Terms may be equated to a Cube, when either the absolute number, (that is, the last Term,) or the first Term is a perfect Cube.

First, let $2aaa + aa + 3a + 1$ be given to be equated to a Cube. That the two last Terms may vanish out of each part, feign the side of a Cube to be $a + 1$, (1 being the side of the given Cube 1, and a being the Quotient that ariseth by dividing $3a$ in the given quantity by 3 the triple Square of 1, the cubick Root of the given Cube 1,) then the Cube of $a + 1$, that is, $aaa + 3aa + 3a + 1$ being equated to the given quantity $2aaa + aa + 3a + 1$, will give $a = 2$; then to find out a second value of a you may put $a + 2$ for a new Root.

Secondly, but if the first Term be a rational Cube, as, if $8aaa + 24aa + 2a + 48$ be given to be equated to a Cube; that the first and second Terms may vanish out of each part, feign the side of a Cube to be $2a + 2$, ($2a$ being the side of the Cube $8aaa$, and 2 being the Quotient that ariseth by dividing 24 which is prefixt to aa, by 12 the triple Square of 2 the cubick Root of 8 in $8aaa$;) then the Cube of $2a + 2$ being equated to the given quantity $8aaa + 24aa + 2a + 48$, will give $a = \frac{16}{11}$; whence you may find out derivative Roots as before.

XI. If the first Term of a Quantity compos'd of four Terms given to be equated to a Cube be a rational Cube, and the last Term, to wit, the absolute number be also a Cube, then that given Quantity may be equated to a Cube in a threefold manner.

As, for example, if $aaa + 2aa + 4a + 1$ be proposed to be equated to a Cube; first, that the first and last Terms may vanish out of each part, feign the side of the Cube to be $a + 1$, (which is compos'd of the cubick Roots of aaa and 1; then the Cube of $a + 1$ being equated to the quantity proposed will give $a = 1$. Secondly, that the first and second Terms may vanish out of each part, you may feign the side to be $a + \frac{2}{3}$, (a being the side of the Cube aaa, and $\frac{2}{3}$ being the Quotient that ariseth by dividing 2 which is prefixt to aa, by 3 the triple Square of the cubick Root of 1 which is prefixt to aaa,) whence $a = -\frac{19}{72}$. And lastly, that the third and fourth Terms may vanish, you may feign the side to be $\frac{4}{3}a + 1$, (1 being the side of the given Cube 1, and $\frac{4}{3}a$ being the Quotient that ariseth by dividing $4a$ by 3 the triple Square of the cubick Root of the given Cube 1,) whence $a = \frac{90}{37}$; and by the help of those three primitive Roots you may find out derivatives, in like manner as before.

XII. Sometimes when a Quantity compos'd of four Terms, whereof one or both the extremes are Cubes, is to be equated to a Cube, no value of the Root a either affirmative or negative can be found out by any of the Rules before delivered.

As, if $4aaa + 3aa + 3a + 1$ be given to be equated to a Cube, its side can only be feigned $a + 1$, the Cube whereof being equated to the given quantity will give $3aaa = 0$; and therefore the given quantity cannot be equated to a Cube.

Also, if $aaa + 2aa + 3a + 1$ be to be equated to a Cube, there can but one primitive Root be found out, although there be a threefold way of feigning the side of the Cube according to *Sect.* XI. which primitive Root will be discovered from the feigned side $a + \frac{2}{3}$, but neither of the other two ways will prove effectual.

I shall now add a few Questions to illustrate the foregoing Third Part of *Fermat's* Invention, and so conclude this Book.

QUEST. 127.

(*The fame with the foregoing* Queft. 105. *but refolved after another manner.*)

To find a right-angled Triangle , that the Area being fubtracted as well from the Hypothenufal as from one of the fides about the right-angle, each remainder may be a Square.

RESOLUTION.

1. Let h, b, p reprefent the Hypothenufal, Bafe and Perpendicular of a right-angled Triangle in numbers. Divide thofe three fides feverally by a, and put the Quotients for the three fides of the Triangle fought, *viz.* . . . . . .
$$\frac{h}{a},\ \frac{b}{a},\ \frac{p}{a}$$

2. Then by fubtracting the Area $\frac{\frac{1}{2}bp}{aa}$, as well from $\frac{b}{a}$, (one of the fides about the right-angle, as from the Hypothenufal $\frac{h}{a}$, each remainder muft be equal to a Square, and by multiplying each remainder by the Denominator aa, this Duplicate equality arifeth, *viz.* . . . .
$$ba - \tfrac{1}{2}bp = \square$$
$$ha - \tfrac{1}{2}bp = \square$$

3. Let the firft of thofe two quantities be equated to fome Square, *viz.* fuppofe . . . .
$$ba - \tfrac{1}{2}bp = bb$$

4. Whence, after due Reduction, ...
$$a = b + \tfrac{1}{2}p$$

5. Therefore by multiplying h into $b + \frac{1}{2}p$, (inftead of a,) the latter of the two quantities in the fecond ftep, will be converted into this quantity, which muft be equated to a Square, *viz.* . . . .
$$hb + \tfrac{1}{2}hp - \tfrac{1}{2}bp = \square$$

6. Now fince h, b, p, were put for the Hypothenufal, Bafe and Perpendicular of a right-angled Triangle, the quantity in the fifth ftep fhews that a right-angled Triangle muft be found out fuch, that if the Hypothenufal be multiplied into the fumm of one of the fides about the right-angle and half the other fide, and the Product be leffened by the Area, the remainder muft be a Square : But fuch a right-angled Triangle, by *Fermat*'s method, (before explained,) may be found out thus, *viz.*

7. Form a right-angled Triangle from two numbers taken at pleafure, as from $a+1$ and 1, fo the three fides will be thefe, *viz.* . . .
$$aa + 2a + 2 = \text{Hypoth.}$$
$$aa + 2a = \text{Bafe,}$$
$$+ 2a + 2 = \text{Perpend.}$$

8. The Product of the Hypothenufal into the fumm of the Bafe and half the Perpendicular is .
$$aaaa + 5aaa + 9aa + 8a + 2$$

9. The Area is . . . .
$$\ldots + aaa + 3aa + 2a$$

10. Which fubtracted from the faid Product leaves this quantity to be equated to a Square , *viz.*
$$aaaa + 4aaa + 6aa + 6a + 2 = \square$$

11. Feign the fide of that Square, (according to the 7th literal fide in the foregoing *Sect.* 5. *Part* 3.)
$$aa + 2a + 1$$

12. Then the Square of the faid fide $aa + 2a + 1$ being equated to the quantity in the tenth ftep will give $a = -\frac{1}{2}$, therefore $a+1$ and 1 the numbers forming the Triangle fhall be $\frac{1}{2}$ and 1, or (in Integers in the fame Reafon) 1 and 2 ; by which, it a right-angled Triangle be formed, one of the fides about the right-angle will be lefs than nothing, to wit, -3; (for the Square of 2 is to be fubtracted from the Square of 1, becaufe 1 and 2 anfwer to $a+1$ and 1 the numbers that formed the Triangle in the feventh ftep;) to caufe therefore all the fides to be affirmative, the work muft be renewed in manner following, *viz.*

13. Let a right-angled Triangle be formed from $a+1$ and 2, fo the three fides will be thefe, *viz.* . . . .
$$aa + 2a + 5 = \text{Hypoth.}$$
$$aa + 2a - 3 = \text{Bafe,}$$
$$+ 4a + 4 = \text{Perpend.}$$

14. The Product of the Hypothenufal into the fumm of the Bafe and half the Perpendicular is
$$aaaa + 6aaa + 12aa + 18a - 5$$

15. The Area is . . . .
$$\ldots + 2aaa + 6aa - 2a - 6$$

16. Which being fubtracted from the faid Product, leaves this quantity to be equated to a Square, *viz.*
$$aaaa + 4aaa + 6aa + 20a + 1 = \square$$

17. The

17. The ſide of that Square may be variouſly feigned, according to the preceding *Sect.* 5.) let it be the ſecond literal ſide in that *Sect.* viz. $\Big\}$ $1 + 10a - aa$

18. Then the Square of the ſaid ſide $1 + 10a - aa$ being equated to the quantity in the ſixteenth ſtep will give $a = \frac{23}{6}$; therefore $a + 1$ and 2 ſhall be $\frac{29}{6}$ and 2, that is, in Integers in the ſame Reaſon, 29 and 12, by which if you form a right-angled Triangle, the three ſides will be 985, 697, 696, that is, h, b, p; then according to the Poſitions in the firſt ſtep, divide every one of thoſe ſides by 1045, that is, by $b + \frac{1}{2}p = a$, (as appears by the fourth ſtep,) ſo the Quotients $\frac{985}{1045}$, $\frac{697}{1045}$, $\frac{696}{1045}$ ſhall be the ſides of a right-angled Triangle to ſolve the Queſtion: For if the Area be ſubtracted from the Hypothenuſal $\frac{985}{1045}$ and the Baſe $\frac{697}{1045}$ ſeverally, the remainders will be the Squares of theſe ſides $\frac{887}{1045}$ and $\frac{697}{1045}$.

And becauſe the quantity in the ſixteenth ſtep is capable of being equated to innumerable Squares, (according to the Method before explained,) the Queſtion is alſo capable of innumerable Anſwers; but in larger numbers than thoſe, that may be found out by the foregoing *Queſt.* 105. which is the ſame with this.

QUEST. 128. (Probl. 1. in cap. 1. part. 1. *Dioph.* redivivi.)

To find a right-angled Triangle, that if the double of its Area be ſubtracted from every one of the three ſides, the remainders may be Squares.

RESOLUTION.

1. Let b, p, h repreſent the Baſe, Perpendicular, and Hyothenu- of a right-angled Triangle. Divide thoſe ſides ſeverally by a, and aſſume the Quotients to be the three ſides of the Triangle ſought, *viz.* . . . . $\frac{b}{a}$, $\frac{p}{a}$, $\frac{h}{a}$

2. Then by ſubtracting the double Area $\frac{bp}{aa}$ from every one of thoſe three ſides, the remainders muſt be Squares, and multiplying the remainders ſeverally by the Denominator aa, this Triplicate equality ariſeth to be reſolved, *viz.* . . . .
$$ba - bp = \square$$
$$pa - bp = \square$$
$$ha - bp = \square$$

3. Now in order to reſolve that Duplicate equality, let the firſt of its three quantities be equated to ſome Square, *viz.* ſuppoſe $\Big\}$ $ba - bp = bb$

4. Whence, after due Reduction to find out the value of a, you will diſcover . . . . $a = b + p$

5. Then by multiplying $b + p$, inſtead of a, into p, the ſecond of the three quantities in the ſecond ſtep (to wit, $pa - bp$,) will be converted into this quantity, which is manifeſtly a Square, *viz.* . . . . $bp + pp - bp = \square = pp$

6. And by multiplying $b + p$, inſtead of a, into h, the third quantity in the ſecond ſtep will be converted into this quantity to be equated to a Square, *viz.* . . . . $hb + hp - bp = \square$

7. Thus the Triplicate equality in the ſecond ſtep is reduced to a Duplicate equality in the fifth and ſixth ſteps, and becauſe the firſt of the two quantities in that Duplicate equality happens to be a Square, to wit, pp, it remains only to equate $hb + hp - bp$ (in the ſixth ſtep,) to a Square, which diſcovers the Scope of our ſearch muſt be this, *viz.* to find a right-angled Triangle, ſuch, that if the Hypothenuſal be multiplied by the ſumm of the ſides about the right-angle, and the Product be leſſened by the double of the Area, the remainder muſt be a Square: But ſuch a right-angled Triangle may be found out thus, *viz.*

8. Form a right-angled Triangle from two numbers taken at pleaſure, as from $a + 1$ and 2, ſo the three ſides will be theſe, *viz.* . . . .
$$aa + 2a + 5 = \text{Hyp.}$$
$$aa + 2a - 3 = \text{Baſe,}$$
$$+ 4a + 4 = \text{Perp.}$$

9. The Product of the Hypothenuſal into the ſumm of the Baſe and Perpendicular is . . $\Big\}$ $aaaa + 8aaa + 18aa + 32a + 5$

10. The double Area is . . . . $\Big\}$ . . $+ 4aaa + 12aa - 4a$

11. Which ſubtracted from the ſaid Product, leaves this quantity to be equated to a Square, *viz.* . . . . $aaaa + 4aaa + 6aa + 36a + 17 = \square$

12. Feign

12. Feign the ſide of that Square according to the Canon in the preceding *Sect.* 3. *Part* 3. and it will be . . . . $\Big\}$ $aa + 2a + 1$

13. Then the Square of the ſaid ſide $aa + 2a + 1$ being equated to the quantity in the eleventh ſtep, will give $a = -\frac{3}{2}$; therefore $a + 1$ and 2, the numbers forming the Triangle in the eighth ſtep, ſhall be $\frac{1}{2}$ and 2, or, (in Integers in the ſame Reaſon,) 1 and 4, by which if a right-angled Triangle be formed, one of the ſides about the right-angle will be leſs than nothing, to wit, -15; (for the Square of 4 is to be ſubtracted from the Square of 1, becauſe 1 and 4 anſwer to $a + 1$ and 2, the numbers that formed the Triangle in the eighth ſtep, where $a + 1$ was ſuppoſed to exceed 2.) To cauſe therefore all the ſides to be affirmative, the work muſt be renewed thus, *viz.*

14. Form a right-angled Triangle from $a + 1$ and 4, ſo the three ſides will be theſe, *viz.* $\left\{\begin{array}{l} aa + 2a + 17 = \text{Hyp.} \\ aa + 2a - 15 = \text{Baſe,} \\ \quad + 8a + 8 = \text{Perp.} \end{array}\right.$

15. The Product of the Hypothenuſal into the ſumm of the Baſe and Perpendicular is . . $\Big\}$ $aaaa + 12aaa + 30aa + 156a - 119$

16. The double Area is . . . . $\Big\}$ . . $+ 8aaa + 24aa - 104a - 120$

17. Which ſubtracted from the ſaid Product, leaves this quantity to be equated to a Square, *viz* . . . . $\Big\}$ $aaaa + 4aaa + 6aa + 260a + 1 = \square$

18. The ſide of that Square may be variouſly feigned, (according to the preceding *Sect.* V.) let it be the ſecond literal ſide in that *Sect.* *viz.* . . . . $\Big\}$ $1 + 130a - aa$

19. The Square of the ſaid ſide $1 + 130a - aa$ being equated to the quantity in the ſeventeenth ſtep, will give $a = \frac{4223}{66}$, therefore $a + 1$ and 4 ſhall be $\frac{4289}{66}$ and 4, or, (in Integers in the ſame Reaſon,) 4289 and 4223; from which, a right-angled Triangle being formed, the three ſides will be 18465217, 18325825, 2264592, that is, h, b, p: Then according to the Poſitions in the firſt ſtep, divide thoſe three ſides ſeverally by 20590417, that is, by $b + p = a$, (as is evident by the fourth ſtep;) ſo the Quotients $\frac{18465217}{20590417}$, $\frac{18325825}{20590417}$, $\frac{2264592}{20590417}$ ſhall be the ſides of a right-angled Triangle to ſolve the Queſtion, as may eaſily be proved.

Note 1. Although the Queſtion be truly ſolved, yet 'tis evident that it was by chance that the Triplicate equality in the ſecond ſtep came to be reduced to a ſingle equality; for if the quantity to be equated to a Square in the fifth ſtep, had not happened to have been a Square, there would have been an inexplicable Duplicate equality.

Note 2. It is eaſie to perceive by the ſecond, third and fourth ſteps, that inſtead of bp the double Area, the Product of bp multiplied by any ſquare number may be given in the Queſtion: As, if it were required to find out a right-angled Triangle, that $4bp$, that is, eight times the Area being ſubtracted from every one of the three ſides may leave Squares, you need only to multiply the Denominator 20590417 of the three ſides before found by 4, without altering the Numerators; or, if $9bp$, that is, eighteen times the Area, were preſcribed, then to multiply the Denominator by 9.

QUEST. 129. (Probl. 1. in cap. 2. part. 1. *Dioph.* redivivi.)

To find a right-angled Triangle, that the Product of the Hypothenuſal into one of the ſides about the right-angle, being ſubtracted from every one of the three ſides, may leave Squares.

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, and for the three ſides of the Triangle ſought put . . . . $\Big\}$ $\frac{h}{a}$, $\frac{b}{a}$, $\frac{p}{a}$

2. Then from $\frac{hp}{aa}$, (the Product of the Hypothenuſal into the Perpendicular,) ſubtract every one of the three ſides, and the remainders muſt be Squares; therefore alſo thoſe remainders multiplied into the Denominator aa muſt make Squares; hence this Triplicate equality ariſeth, *viz.* . . . . $\left\{\begin{array}{l} ha - hp = \square \\ ba - hp = \square \\ pa - hp = \square \end{array}\right.$

3. Now

3. Now in order to reſolve that Triplicate equality, let the firſt of its three quantities be equated to ſome Square, *viz.* ſuppoſe $ha - hp = hh$

4. Whence, after due Reduction to find out the value of a, you will diſcover . . . . $a = h + p$

5. Then by multiplying $h + p$, inſtead of a, into b, the ſecond of the three quantities in the ſecond ſtep will be reduced to this to be equated to a Square, *viz.* . . . . $hb + bp - hp = \square$

6. Likewiſe by multiplying $h + p$, inſtead of a, into p, the third quantity in the ſecond ſtep will be reduced to this quantity, which is manifeſtly a Square, *viz.* . . . . . . $hp + pp - hp = \square = pp$

7. Thus the Triplicate equality in the ſecond ſtep is reduced to a Duplicate equality in the fifth and ſixth ſteps; and becauſe the latter quantity in that Duplicate equality happens to be a Square, to wit, pp, it remains only to equate the former, that is, $hb + bp - hp$, to a Square; which ſhews that a right-angled Triangle muſt be found, ſuch, that if the ſumm of the Hypothenuſal and Perpendicular be multiplied by the Baſe, and from the Product you ſubtract the Product of the Hypothenuſal into the Perpendicular, the remainder may be a Square: But ſuch a right-angled Triangle may be found out thus, *viz.*

8. Form a right-angled Triangle from $a + 2$ and 1, (2 and 1 being numbers taken at pleaſure,) ſo the three ſides will be theſe, *viz.*
$aa + 4a + 5 =$ Hyp.
$aa + 4a + 3 =$ Baſe,
$+ 2a + 4 =$ Perp.

9. The ſumm of the Hypothenuſal and Perpendicular being multiplied by the Baſe, produceth $aaaa + 10aaa + 36aa + 54a + 27$

10. The Product of the Hypothenuſal and Perpendicular is . . . . $2aaa + 12aa + 26a + 20$

11. Which latter Product being ſubtracted from the former, leaves to be equated to a Square, $aaaa + 8aaa + 24aa + 28a + 7 = \square$

12. Feign the ſide of that Square according to the Canon in the preceding *Sect.* III. *Part* 3. and it will be . . . . $aa + 4a + 4$

13. Then the Square of the ſaid ſide being equated to the quantity in the eleventh ſtep will give $a = -\frac{9}{4}$, and therefore $a + 2$ and 1, which were the numbers forming the Triangle in the eighth ſtep, ſhall be -1 and 4, but one of theſe being negative, the work muſt be renewed; and now a right-angled Triangle may be confidently formed from $a - 1$ and 4, which Triangle being uſed like the former in the ninth, tenth and eleventh ſteps, at length there will remain $aaaa - 4aaa + 6aa - 260a + 1$ to be equated to a Square, the ſide whereof may be variouſly feigned, let it be $1 - 130a + aa$, then the Square of this ſide being equated to the ſaid $aaaa - 4aaa + 6aa - 260a + 1$, will give $a = 66$; therefore $a - 1$ and 4 the numbers forming the Triangle ſhall be 65 and 4, by which if you form a right-angled Triangle, the three ſides will be found 4241, 4209, 520, that is, h, b, p. Then according to the Poſitions in the firſt ſtep, divide thoſe three numbers ſeverally by 4761, that is, by $h + p = a$, as appears by the fourth ſtep, and the Quotients $\frac{4241}{4761}$, $\frac{4209}{4761}$, $\frac{520}{4761}$ ſhall be the ſides of a right-angled Triangle to ſolve the Queſtion, as may eaſily be proved.

QUEST. 130. (Probl. 39. in cap. 1. part. 1. *Dioph.* redivivi.)

To find a right-angled Triangle, whoſe Area ſubtracted from one of the ſides about the right-angle, may leave a given number, ſuppoſe 2, (or n.)

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle; then multiply thoſe ſides ſeverally by a, and put the Products for the ſides of the Triangle ſought, *viz.* . . . . ha; ba; pa

2. The Area of which Triangle being ſubtracted from one of its ſides about the right-angle, ſuppoſe from ba, muſt leave a remainder equal to the given number 2, (or n,) therefore . . . . $ba - \frac{1}{2}bpaa = n = 2$

3. Which Equation divided by $\frac{1}{2}bp$, gives . . . . . $\frac{b}{\frac{1}{2}bp}a - aa = \frac{n}{\frac{1}{2}bp}$

4. Now

4. Now that the value of a in the laſt Equation may be expreſſible by a rational number, it is evident by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. that if the abſolute quantity in the latter part of that Equation be ſubtracted from the Square of half the Coefficient which is drawn into a, the remainder muſt have a rational ſquare Root, therefore . . . . $\dfrac{\frac{1}{4}bb - \frac{1}{2}bpn}{\frac{1}{4}bbpp} = \square$

5. And becauſe the Denominator $\frac{1}{4}bbpp$ is a Square, it remains only to equate the Numerator to a Square, therefore . . . . $\frac{1}{4}bb - \frac{1}{2}bpn = \square$

6. Or, to avoid Fractions, let the ſaid $\frac{1}{4}bb - \frac{1}{2}bpn$ be multiplied by 4, and then it remains to equate . . . . $bb - 2bpn = \square$

7. Whence it appears, that in order to ſolve the Queſtion propoſed, a right-angled Triangle muſt firſt be found, ſuch, that if from the Square of one of its ſides about the right-angle, the Product of the quadruple of the Area multiplied by the given number n be ſubtracted, the remainder may be a Square: But ſuch a right-angled Triangle may be found out thus, *viz.*

8. Form a right-angled Triangle from two numbers taken at pleaſure, as from $a - 1$ and 4, ſo the three ſides will be theſe, *viz.* . . . .
$aa - 2a + 17 =$ Hyp.
$aa - 2a - 15 =$ Baſe,
$+ 8a - 8 =$ Perp.

9. The Square of the Baſe is . . . . . $aaaa - 4aaa - 26aa + 60a + 225$

10. The Product of the quadruple of the Area into the given number 2 is . . . $+ 32aaa - 96aa - 416a + 480$

11. Which Product being ſubtracted from the ſaid Square of the Baſe, leaves this quantity to be equated to a Square, *viz.* $aaaa - 36aaa + 70aa + 476a - 255$

12. Feign the ſide of the deſired Square to be $aa - 18a - 127$

13. Then the Square of the ſaid ſide being equated to the quantity in the eleventh ſtep, will give $a = -4$. But becauſe this value is negative, let $a - 4$ be put for a new Root, and according to that let all the members of the quantity in the eleventh ſtep be reſolved; ſo this new quantity $aaaa - 52aaa + 598aa - 2068a + 1521$ comes forth to be equated to a Square, the ſide whereof may be feigned $39 - \frac{1224}{39}a + aa$, whence $a = \frac{278236}{1560}$; therefore $a - 4$ (the new Root) ſhall be $\frac{67999}{390}$, and therefore $a - 1$ and 4 the numbers that formed the Triangle in the eighth ſtep ſhall be $\frac{67609}{390}$ and 4; or in Integers in the ſame proportion, 67609 and 1560. Wherefore the preparatory Triangle formed from thoſe numbers, and agreeable to the Scope mentioned in the ſeventh ſtep, ſhall be 4573410481, 4568543281, 210940080; For if its quadruple Area be multiplied by the given number 2, and the Product be ſubtracted from the Square of the ſecond ſide, there will remain a Square, whoſe ſide is 4125146321.

14. Now let thoſe three ſides of the preparatory Triangle be taken for the values of h, b, p, and the given number 2 for n; in the Equation in the third ſtep, then that Equation being reſolved will give $a = \frac{5542462}{12046111064720031}$; and if this number be multiplied into the three ſides of the preparatory Triangle it will give the Triangle ſought, whoſe three ſides conſiſt of theſe three Numerators, 25347953801344222, 25320977530297822, 1169127377676960, having 12046111064720031, for a common Denominator.

Note. This Queſtion is very eaſily ſolved when the given number is leſs than unity, by the Canon of the preceding *Queſt.* 100. of this *Book*.

The End of the Third BOOK.

THE ELEMENTS OF THE ALGEBRAICAL ART.

BOOK IV.

CHAP. I.

Concerning the Scope of this fourth Book, and the Signification of Characters, Abbreviations and Citations used therein.

THE Design of this Fourth Book is, to shew the excellent Use of the *Algebraical* Art in the Resolution and Composition of Plane Problems, to wit, such as may be solved or effected by drawing only Right (or straight) and Circular Lines. In pursuance of that Design I have divided this Book into Ten Chapters, whereof the first Six are Preparatory to the rest, which contain Four *Classes* or Forms of Examples, shewing how to find out as well Theorems, as Geometrical Effections of Plane Problems, with their Demonstrations, by the Steps of Algebraical Resolution. All which I have endeavour'd to render clear and intelligible to such Readers as are competently exercis'd in the first Six Books of *Euclid*'s Elements, and in the First and Second Books of these *Algebraical Elements*.

The Explication of the Signs *or* Characters.

$+$	Signifies	*More.*
$-$		*Less.*
$\times$		*Into, or By.*
$::$		*Proportionals.*
$\div\div$		*Continual Proportionals.*
$\surd$		*The Square Root, or Side of a Square.*
$=$		*Equal*
$\sqsubset$		*Greater.*
$\sqsupset$		*Lesser.*
$\parallel$		*Parallel.*
$\angle$		*A plain Angle in general.*
$\llcorner$		*A Right-angle.*
$\perp$		*Perpendicular.*
$\odot$		*A Circle.*
$\square$		*A Square.*
$\square$		*A long Square.*
$\triangle$		*A plain Triangle in general.*

 Examples,

Examples, shewing more at large the Signification of the foregoing Characters.

$a + b$	Signifies the summ of the right lines or numbers represented by *a* and *b*.
$a - b$	The excess by which the right line or number *a* exceeds the right line or number *b*; or, it imports that the latter quantity *b* is subtracted, or to be subtracted from the former Quantity *a*.
$a \times b$, or ab	The Rectangle or Product made by the multiplication of the right line or number *a*, by the right line or number *b*.
aa	The Square of the right line or number signified by *a*.
$\frac{aa}{c}$	The right line arising by the Application of the Square of the right line *a*, to the right line *c*; or, the Quotient arising by the Division of the Square of the number *a*, by the number *c*.
$\frac{ab}{c}$	The right line or number arising by the Application or Division of the Rectangle or Product of *a* into *b*, by *c*.
$a \,.\, b :: c \,.\, d$,	*viz.* As *a* is to *b*; so *c* to *d*.
$a, b, c \div$,	*viz.* As *a* is to *b*; so *b* to *c*.
$\sqrt{ab}$	The square Root of the Product of *a* into *b*; or, the side of a Square equal to the Rectangle *ab*.
$\sqrt{:aa + bb:}$	The square Root universal of $aa + bb$; or, the side of a Square equal to the summ of the Squares *aa* and *bb*.
$\sqrt{:aa - bb:}$	The square Root universal of $aa - bb$; or, the side of a Square equal to the excess of the Square *aa* above the Square *bb*.
$a = b$	The line or number *a* is equal to the line or number *b*.
$a = 5c$	The line or number *a* is equal to five times the line or number *c*.
$a = \frac{1}{2}d$	The line or number *a* is equal to half the line or number *d*.
$a \sqsubset f$	The line or number *a* is greater than the line or number *f*.
$a \sqsupset g$	The line or number *a* is less than the line or number *g*.
AB ∥ CD	The line AB is parallel to the line CD.
∠ABC	The angle ABC. Observe here, that when an angle is exprest by three letters, the middle letter stands at the angular point.
∠A	The angle A.
∠ABC is ⊥	The angle ABC is a right-angle.
AB ⊥ BC	The right-line AB is perpendicular to the right-line BC.
ABCD is ⊙	ABCD is a Circle. Observe here, that the first letter towards the left hand is usually set at the Center.
□ AD	Signifies either the Square AD when the letters A and D stand at the opposite angles of the Square; or else, the Square of the right-line AD, when A and D stand at the ends of the side of the Square.
□ A	The Square of the right-line A.
▭ AB, BC	The long Square, or Rectangle, made of the right-lines AB and BC.
▭ AB, C	The Rectangle of the right-lines AB and C.
▭ A, B	The Rectangle of the right-lines A and B.
△ ABC	The Triangle ABC.

Explication of Abbreviations *and* Citations.

Probl.	Problem.
Suppos.	Suppositions.
Req.	It is (or, let it be) required.
Prepar.	Preparation.
Constr.	Construction.
Req. demonstr.	It is (or, let it be) required to Demonstrate.
Conclus.	Conclusion.
Coroll.	Corollary.
Annot.	Annotation.
Explicat.	Explication.
Per Prop. 11. *Elem.* 5.	By the 11th Proposition of the fifth Book of *Euclid's* Elements.

Per

Per Defin. 29. *Elem.* 1. } By the 29th Definition of the first Book of *Euclid's* Elements.

Per Ax. 1. *Chap.* 2. } By the first Axiom of the second Chapter of this fourth Book.

Note. In the handling of every Proposition, whether it be a Theorem or Problem, I proceed from the beginning to the end by Steps numbred in the Margin by 1, 2, 3, 4, 5, *&c.* that by referring to preceding Steps, the rise of the following may be apparent: So when 'tis said, [*Therefore out of*, or, *From* 3°. *and* 4°.] it imports, that the thing asserted or inferr'd is manifest by the third and fourth Steps of the Proposition in hand. If any other Abbreviations occurr, their meaning will be obvious to every intelligent Reader.

CHAP. II.

The explication of Axioms, or common notions, upon which the force of Inferences or Conclusions, about the Equality, Majority and Minority of Quantities compared to one another, doth chiefly depend.

Axiom 1.

1. IF each of two Quantities be equal to a third, those two are equal between themselves.

Explicat.

If $AB = EF$,
And $CD = EF$,
Then $AB = CD$; *per Ax.* 1.

A —— B
C —— D
E —— F

That is to say,

If A B be equal to E F, and C D be equal to E F, then A B is equal to C D, by the first *Axiom* of *Chap.* 2.

Axiom 2.

2. Quantities which are equal to equal Quantities, are also equal between themselves.

Explicat.

If $\begin{cases} C = D, \\ A = C, \\ B = D, \end{cases}$
Then $A = B$; *per Ax.* 2.

A ——
B ——
C ——
D ——

Axiom 3.

3. That which is greater or less than one of two equal Quantities, is also greater or less than the other.

Explicat.

If $B = C$,
And $A \sqsubset B$,
Then $A \sqsubset C$; *per Ax.* 3.

A ——
B ——
C ——

That is to say,

If B be equal to C, and A be greater than B, then A is greater than C, by *Ax.* 3.

Axiom 4.

4. If one of two equal Quantities be greater or less than a third, the other of those two shall be also greater or less than the same third.

Explicat.

If $A = B$,
And $A \sqsubset C$,
Then $B \sqsubset C$; *per Ax.* 4.

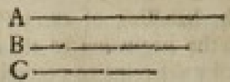
A ——
B ——
C ——

Axiom 5.

5. That which is greater than the greater of two Quantities, is also greater than the lesser; and that which is less than the lesser of two Quantities, is also less than the greater.

Explicat.

If $B \sqsubset C$,
And $A \sqsubset B$,
Then $A \sqsubset C$; *per Ax.* 5.

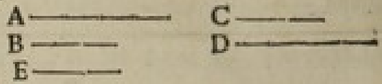


Axiom 6.

6. The exchanging of equal Quantities doth not alter equality.

Explicat.

If $A + B = C + D$,
And $E = B$,
Then $A + E = C + D$; *per Ax.* 6.



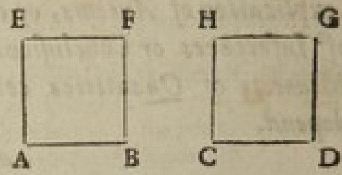
Axiom 7.

7. Interpretation doth not change equality.

Explicat.

Suppos. $\begin{cases} \square AF = \square CG, \\ AF \text{ is } \square AB, \\ CG \text{ is } \square CD, \end{cases}$

$\square AB = \square CD$; *per Ax.* 7.



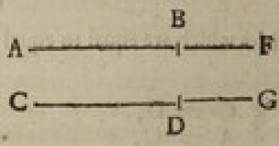
That is to say,
The Square AF is equal to the Square CG, by supposition.
AF is the Square of the side AB, by supposition.
CG is the Square of the side CD, by supposition.
The Square of the side AB is equal to the Square of the side CD, by the seventh *Axiom* of *Chap.* 2.

Axiom 8.

8. If to equal Quantities you add equal Quantities, or one and the same Quantity, the wholes shall be equal.

Explicat.

If $AB = CD$,
And $BF = DG$,
Then $AB + BF = CD + DG$,
That is, $AF = CG$; *per Ax.* 8.

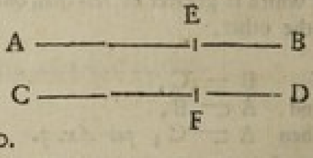


Axiom 9.

9. If from equal Quantities you take away equal Quantities, or one and the same Quantity, the Quantities remaining shall be equal to one another.

Explicat.

If $AB = CD$,
And $AE = CF$,
Then $AB - AE = CD - CF$,
That is, $EB = FD$; *per Ax.* 9.

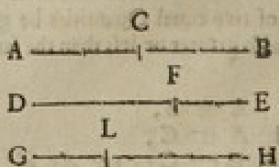


Axiom 10.

10. If from a whole the half be taken away, half will remain; and if more than half be taken away, less than half will remain; but if one third part be taken away, two thirds will remain, &c.

Explicat.

If $AC = \frac{1}{2}AB$,
Then $CB = \frac{1}{2}AB$, *per Ax.* 10.
If $DF \sqsubset \frac{1}{2}DE$,
Then $FE \sqsupset \frac{1}{2}DE$, *per Ax.* 10.
If $GL = \frac{1}{3}GH$,
Then $LH = \frac{2}{3}GH$, *per Ax.* 10.

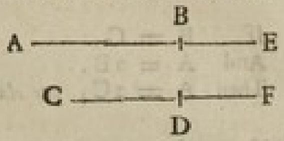


Axiom 11.

Axiom 11.

11. If to unequal quantities equal quantities be added, the wholes are unequal.

Explicat.

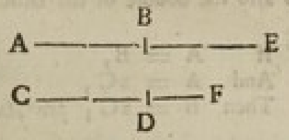


If $AB \sqsubset CD$,
And $BE = DF$,
Then $AE \sqsubset CF$; *per Ax.* 11.

Axiom 12.

12. If to equal quantities you add unequal quantities, the wholes are unequal.

Explicat.

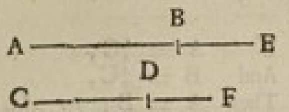


If $AB = CD$,
And $BE \sqsubset DF$,
Then $AE \sqsubset CF$; *per Ax.* 12.

Axiom 13.

13. If to unequal quantities unequal quantities be added, the greater to the greater; and the less to the less, the wholes are unequal, to wit, the former the greater, and the latter the lesser.

Explicat.

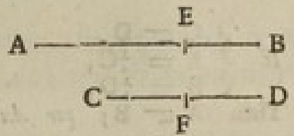


If $AB \sqsubset CD$,
And $BE \sqsubset DF$,
Then $AE \sqsubset CF$; *per Ax.* 13.

Axiom 14.

14. If from unequal quantities equal quantities or one and the same quantity be taken away, the remainders will be unequal.

Explicat.

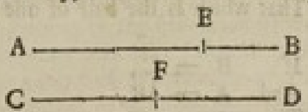


If $AB \sqsubset CD$,
And $EB = FD$,
Then $AE \sqsubset CF$; *per Ax.* 14.

Axiom 15.

15. If from equal quantities unequal quantities be taken away, the remainders are unequal.

Explicat.

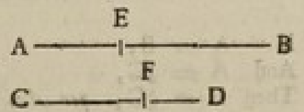


If $AB = CD$,
And $AE \sqsubset CF$,
Then $EB \sqsupset FD$; *per Ax.* 15.

Axiom 16.

16. If from unequal quantities unequal quantities be taken away, from the greater the lesser, and from the lesser the greater, the remainders are unequal; to wit, the former the greater, and the latter the lesser.

Explicat.

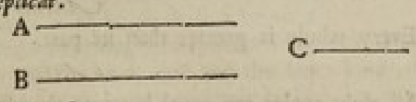


If $AB \sqsubset CD$,
And $CF \sqsubset AE$,
Then $EB \sqsubset FD$; *per Ax.* 16.

Axiom 17.

17. Quantities which are the doubles of one and the same quantity, or of equal quantities, are equal between themselves. Conceive the same of triples, quadruples, &c.

Explicat.



If $A = 2C$,
And $B = 2C$,
Then $A = B$; *per Ax.* 17.

Axiom 18.

18. The double of the greater of two quantities is greater than the double of the lesser.

Explicat.

If $\begin{cases} C \sqsubset D, \\ A = 2C, \\ B = 2D, \end{cases}$
Then $A \sqsubset B$; *per Ax.* 18.

Axiom 19.

Axiom 19.

19. That which is the double of one of two equal quantities, is also the double of the other.

Explicat.

If $B = C$,
And $A = 2B$,
Then $A = 2C$; *per Ax.* 19.

Axiom 20.

20. If one of two equal quantities be the double of a third, the other of those two is also the double of the same third.

Explicat.

If $A = B$,
And $A = 2C$,
Then $B = 2C$; *per Ax.* 20.

Axiom 21.

21. Quantities which are the halves of one and the same quantity, or of equal quantities, are equal between themselves. Understand the same of thirds, fourths, *&c.*

Explicat.

If $A = \frac{1}{2}C$,
And $B = \frac{1}{2}C$,
Then $A = B$; *per Ax.* 21.

Axiom 22.

22. The half of the greater of two quantities is greater than the half of the lesser.

Explicat.

If $\left\{ \begin{array}{l} C \sqsubset D, \\ A = \frac{1}{2}C, \\ B = \frac{1}{2}D, \end{array} \right.$
Then $A \sqsubset B$; *per Ax.* 22.

Axiom 23.

23. That which is the half of one of two equal quantities is also the half of the other.

Explicat.

If $B = C$,
And $A = \frac{1}{2}B$,
Then $A = \frac{1}{2}C$; *per Ax.* 23.

Axiom 24.

24. If one of two equal quantities be the half of a third, the other of those two shall be the half of the same third.

Explicat.

If $A = B$,
And $A = \frac{1}{2}C$,
Then $B = \frac{1}{2}C$; *per Ax.* 24.

What hath been said in the eight last preceding *Axioms* concerning the double and the half, may be also understood of the triple, quadruple, quintuple, *&c.* and of thirds, fourths, fifths, *&c.*

Axiom 25.

25. Every whole is greater than its part.

Axiom 26.

26. All right angles are equal between themselves.

Explicat.

If $\angle A$ be $\lrcorner$,
And $\angle B$ be $\lrcorner$,
Then $\angle A = \angle B$; *per Ax.* 26.

A $\lrcorner$ B $\lrcorner$

That is to say, if the angle A be a right-angle, and the angle B be a right-angle; then the angle A is equal to the angle B.

Axiom 27.

Axiom 27.

27. If one of two or more equal angles be a right-angle, every one of the rest of those equal angles is also a right-angle.

Explicat.

If $\angle A = \angle B = \angle C$,
And $\angle A$ be $\lrcorner$,
Then $\angle B$ and $\angle C$ are $\lrcorner$; *per Ax.* 27.

A $\lrcorner$, B $\lrcorner$, C $\lrcorner$.

That is to say, If the angles A, B and C be equal to one another, and the angle A be a right-angle, then the angles B and C are also right-angles. *Per Axiom.* 24. *Chap.* 2.

Axiom 28.

28. Every whole is equal to all its parts taken together.

Axiom 29.

29. If a quantity be neither greater nor less than another quantity, those quantities are equal between themselves.

Explicat.

If A be neither $\sqsubset$ nor $\sqsupset$ than B,
Then $A = B$; *per Ax.* 29.

A ————
B ————

Chap. III.

The explication of Definitions, concerning the ways of arguing used by Mathematicians, to inferr one Analogie from another.

THE ways of arguing about *Reasons*, or *Proportions*, are principally six, which are explain'd in this Chapter in such order as they are exprest by the 12th, 13th, 14th, 15th, 16th and 17th Definitions at the beginning of the fifth Book of *Euclid's* Elements; to which six ways of reasoning, six others are also here inserted as Annotations, being the *Scholies* of *Clavius* and *Herigonius* upon such Propositions of *Euclid's* Elements as are hereafter cited. All which are very useful in Mathematical Resolution and Composition, as will appear in the following 7th, 8th, 9th and 10th Chapters.

Definition I.

1. *Alternate Reason* is the comparing of the Antecedent to the Antecedent; and the Consequent to the Consequent.

Explicat.

If this Analogy be proposed, ... $\begin{cases} a \;.\; b :: c \;.\; d \\ 6 \;.\; 4 :: 12 \;.\; 8 \end{cases}$

Then alternately, or by permutation, . . $\begin{cases} a \;.\; c :: b \;.\; d \\ 6 \;.\; 12 :: 4 \;.\; 8 \end{cases}$ *Per Prop.* 16. *Elem.* 5.

That is to say, If a hath such Reason (or Proportion) to b, as c to d; then alternately, or by permutation of Reason, as a is to c, so shall b be to d.

But note diligently, that in this first way of arguing, all the four Proportionals in the Analogy propounded must necessarily be Quantities of one and the same kind, that is, either all Lines, or all Planes, &c. For although it may properly be said, as the line a is to the line b, so is the Plane c to the Plane d; yet it cannot be thence inferr'd by Alternate Reason, that the line a is to the Plane c, as the line b to the Plane d, because there is no Proportion between a Line and a Plane, which are quantities of different kinds: But in all the following ways of arguing, the two first Proportionals may be of one kind, and the two latter of another, as is manifest by the Demonstrations in the fifth Book of *Euclid's* Elements.

Defini-

Definit. II.

2. *Inverse Reason* is the taking of the Consequent as the Antecedent, to compare it to the Antecedent as if it were the Consequent.

Explicat

$$\text{If} \ldots\ldots\ldots\ldots \left\{\begin{array}{ccccccc} a & . & b & :: & c & . & d \\ 6 & . & 4 & :: & 12 & . & 8 \end{array}\right.$$

Per Coroll. prop. 4. *Elem.* 5.

$$\text{Then inversly,} \ldots\ldots\ldots \left\{\begin{array}{ccccccc} b & . & a & :: & d & . & c \\ 4 & . & 6 & :: & 8 & . & 12 \end{array}\right.$$

Definit. III.

3. *Composition of Reason* is the taking of the Antecedent and Consequent both as one, to compare it to the same Consequent.

Explicat.

$$\text{If} \ldots\ldots\ldots\ldots \left\{\begin{array}{ccccccc} a & . & b & :: & c & . & d \\ 6 & . & 12 & :: & 4 & . & 8 \end{array}\right.$$

Per prop. 18. *Elem.* 5.

$$\text{Then by Composition,} \ldots\ldots \left\{\begin{array}{ccccccc} a+b & . & b & :: & c+d & . & d \\ 18 & . & 12 & :: & 12 & . & 8 \end{array}\right.$$

Annot. 1.

4. *Composition of Reason converse* is the taking of the Antecedent and Consequent both as one, to compare it to the same Antecedent.

Explicat.

$$\text{If} \ldots\ldots\ldots\ldots \left\{\begin{array}{ccccccc} a & . & b & :: & c & . & d \\ 6 & . & 4 & :: & 12 & . & 8 \end{array}\right.$$

Per Schol. 1. Clavii *in prop.* 18. *Elem.* 5.

$$\text{Then by Compos. converse,} \ldots\ldots \left\{\begin{array}{ccccccc} a+b & . & a & :: & c+d & . & c \\ 10 & . & 6 & :: & 20 & . & 12 \end{array}\right.$$

Annot. 2.

5. *Composition of Reason contrary* is the comparing of the Antecedent to the Antecedent and Consequent taken both as one.

Explicat.

$$\text{If} \ldots\ldots\ldots\ldots \left\{\begin{array}{ccccccc} a & . & b & :: & c & . & d \\ 6 & . & 4 & :: & 12 & . & 8 \end{array}\right.$$

Per Schol. 2. Clavii *in prop.* 18. *Elem.* 5.

$$\text{Then by Compos. contrary,} \ldots\ldots \left\{\begin{array}{ccccccc} a & . & a+b & :: & c & . & c+d \\ 6 & . & 10 & :: & 12 & . & 20 \end{array}\right.$$

Annot. 3.

6. *Composition of Reason inversly contrary* is the comparing of the Consequent to the Antecedent and Consequent taken both as one.

Explicat.

$$\text{If} \ldots\ldots\ldots\ldots \left\{\begin{array}{ccccccc} a & . & b & :: & c & . & d \\ 6 & . & 4 & :: & 12 & . & 8 \end{array}\right.$$

Per Schol. 2. Herigon. *in prop.* 18. *Elem.* 5.

$$\text{Then by Composition inversly contrary,} \ldots \left\{\begin{array}{ccccccc} b & . & a+b & :: & d & . & c+d \\ 4 & . & 10 & :: & 8 & . & 20 \end{array}\right.$$

Definit. IV.

7. *Division of Reason* is the comparing of the excess whereby the Antecedent exceeds the Consequent, to the same Consequent.

Explicat.

$$\text{If} \ldots\ldots\ldots\ldots \left\{\begin{array}{ccccccc} a & . & b & :: & c & . & d \\ 6 & . & 4 & :: & 12 & . & 8 \end{array}\right.$$

Per prop. 17. *Elem.* 5.

$$\text{Then by Division,} \ldots\ldots \left\{\begin{array}{ccccccc} a-b & . & b & :: & c-d & . & d \\ 2 & . & 4 & :: & 4 & . & 8 \end{array}\right.$$

But in this way of arguing by Division of Reason, 'tis manifest that the Antecedent must necessarily be greater than the Consequent.

Annot. 1.

Annot. 1.

8. *Division of Reason converse* is the comparing of the Consequent to the excess whereby the Antecedent exceeds the Consequent.

Explicat.

If . . . . $\left\{\begin{array}{ccccccc} a & . & b & :: & c & . & d \\ 9 & . & 4 & :: & 18 & . & 8 \end{array}\right.$

Then by Division converse, ... $\left\{\begin{array}{ccccccc} b & . & a-b & :: & d & . & c-d \\ 4 & . & 5 & :: & 8 & . & 10 \end{array}\right.$ *Per Schol.* 1. Clavii *in prop.*17 *Elem.* 5.

Annot. 2.

9. *Division of Reason contrary* is the comparing of the Antecedent, to the excess whereby the Consequent exceeds the Antecedent.

Explicat.

If . . . . $\left\{\begin{array}{ccccccc} a & . & b & :: & c & . & d \\ 4 & . & 6 & :: & 8 & . & 12 \end{array}\right.$

Then by Division contrary, ... $\left\{\begin{array}{ccccccc} a & . & b-a & :: & c & . & d-c \\ 4 & . & 2 & :: & 8 & . & 4 \end{array}\right.$ *Per Schol.* 2. Clavii *in prop.*17. *Elem.* 5.

But here 'tis manifest that the Consequent must be greater than the Antecedent.

Annot. 3.

10. *Division of Reason inversly contrary* is the comparing of the excess whereby the Consequent exceeds the Antecedent, to the same Antecedent.

Explicat.

If . . . . $\left\{\begin{array}{ccccccc} a & . & b & :: & c & . & d \\ 4 & . & 6 & :: & 8 & . & 12 \end{array}\right.$

Then by Division inversly contrary, . . $\left\{\begin{array}{ccccccc} b-a & . & a & :: & d-c & . & c \\ 2 & . & 4 & :: & 4 & . & 8 \end{array}\right.$ *Per Schol.* 2. Herigon. *in prop.* 17. *Elem.* 5.

Definit. V.

11. *Conversion of Reason* is the comparing of the Antecedent to the excess by which the Antecedent exceeds the Consequent.

Explicat.

If . . . . $\left\{\begin{array}{ccccccc} a & . & b & :: & c & . & d \\ 6 & . & 4 & :: & 12 & . & 8 \end{array}\right.$

Then by converse Reason, ... $\left\{\begin{array}{ccccccc} a & . & a-b & :: & c & . & c-d \\ 6 & . & 2 & :: & 12 & . & 4 \end{array}\right.$ *Per Coroll.* *prop.* 19. *Elem.* 5.

Definit. VI.

12. *Reason of equality* is, when more than two quantities in one Rank, and as many in another are such, that if two to two be compared, they are in the same Reason; and it also happens, that as the first is to the last in the first rank of Quantities, so is the first to the last in the latter rank. Or otherwise, 'tis a comparison of the extremes to one another, the mean quantities being taken away.

But there are two ways of arguing by Reason of equality, to wit, one when the Proportion is Ordinate, the other when the Proportion is Inordinate or Disturbed; both which are explain'd in the two following Definitions.

Definit. VII.

13. *Ordinate proportion* is, when in the first rank of quantities, as the Antecedent is to the Consequent; so in the latter rank is the Antecedent to the Consequent: and when in the first rank as the Consequent is to some other, so in the latter rank is the Consequent to some other.

Explicat.

If to these quantities propounded,	A,4	.	B,6	.	C,12	.	D,8	
	E,10	.	F,15	.	G,30	.	H,20	
These Analogies do happen, . .	A,4	.	B,6	::	E,10	.	F,15	
	B,6	.	C,12	::	F,15	.	G,30	
	C,12	.	D,8	::	G,30	.	H,20	
Then by Reason of equality, . .	A,4	.	D,8	::	E,10	.	H,20	

Per prop. 22. Elem. 5.

That is to say, when the proportion in both ranks of quantities propounded is ordinate, (according to *Defin.* 7.) then by Reason of equality, as the first is to the last in the first rank of quantities, A, B, C, D; so shall the first be to the last in the second rank of quantities E, F, G, H.

Definit. VIII.

14. *Inordinate proportion* is, when three quantities standing in one rank and three in another do afford these Analogies, *viz.* as the first quantity in the first rank is to the second in the same rank; so is the second quantity in the second rank to the third in the same rank: and as the second quantity in the first rank, is to the third in the same rank; so is the first quantity in the second rank to the second in the same rank.

Explicat.

If to these quantities propounded,	A,4	.	B,6	.	C,3		
	D,20	.	E,10	.	F,15		
These Analogies do happen; . .	A,4	.	B,6	::	E,10	.	F,15
	B,6	.	C,3	::	D,20	.	E,10
Then by reason of Equality, . .	A,4	.	C,3	::	D,20	.	F,15

Per prop: 23. Elem. 5.

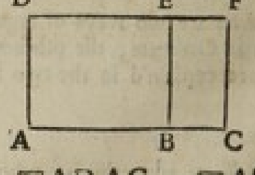
That is to say, when the proportion in both ranks of quantities propounded is inordinate, (according to *Defin.* 8.) then by Reason of equality, as the first is to the last in the first rank of quantities, A, B, C; so shall the first be to the last in the latter rank of quantities, D, E, F.

Chap. IV.

Various fundamental Theorems frequently used in Mathematical Resolution and Composition.

Theorem I.

A Rectangle (or right-angled Parallelogram) comprehended under any right-line and the difference of any two right-lines, is equal to the difference of two Rectangles comprehended under the first line and each of the two latter.



Suppos.

1. A D is a right-line,
2. A C and B C are right-lines,
3. A B C is a right-line,
4. AB = AC—BC.

5. . . *Req. demonstr.* . . . : : ▭AD,AB = ▭AD,AC — ▭AD,BC.

Preparat.

6. Make ▭ AF to be contain'd under AD and AC,
7. Make BE ⊥ AC,

Demon-

Demonstration.

8. By Constr. in 6°, and 7°. (and *per prop.* 1. *Elem* 2.)	$\square AE + \square BF = \square AF$
9. Therefore, by subtracting $\square BF$ from each part of that Equation,	$\square AE = \square AF - \square BF$
10. That is, (*per Ax.* 7. *Chap.* 2.)	$\square AD,AB = \square AD,AC - \square AD,BC$ BE

Which was to be demonstrated.

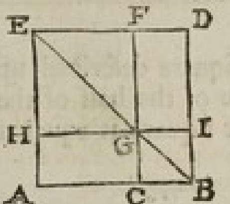
Illustration Algebraical.

Let three right-lines be represented by	a . . .	3
	b . . .	6
	c . . .	2
Suppose also	$b \sqsubset c$	
Then if the first line be multiplied by the difference of the second and third, that is, $a \times \overline{b - c}$, the Product will be	$ab - ac$	$18 - 6 = 12$

Which Product is manifestly the difference between the Product of the first line a into the second b, and the Product of the first line a into the third c, according to the tenour of *Theorem* 1.

Theorem II.

If a right-line be cut into any two parts, the Square described upon the whole line is equal to the Squares described upon the parts, and to twice the Rectangle comprehended under the parts.



Suppos.

1. AB is a right-line,
2. AC and CB are parts of AB,
3. $AC + CB = AB$.
4. . . *Req. demonstr.* . . . $\square AB = \square AC + \square CB + 2\square AC,CB$.

Prepar.

5. Upon AB describe the $\square AD$, (*per prop.* 46. *Elem.* 1.)
6. Draw the Diameter EB
7. Draw CF ∥ AE (or BD,) and cutting EB in G, (*per prop.* 31. *Elem.* 1.)
8. By the point G draw HGI ∥ AB, (or ED.)

Demonstration.

9. By Constr. in 5°, ...	AD is $\square AB$.
10. Therefore, (*per* 19. *Defin.* 1. *Elem.*)	$\angle A, \angle AED, \angle D, \angle DBA$ are ⅃.
11. And out of 7°, 8°, and 10°; (*per* 29. *prop.* 1. *Elem.*)	$\angle EHG, \angle EFG, \angle HGF$ are ⅃.
12. And out of 5°, (*per* 29. *defin* 1. *Elem.*)	$AE = AB = BD = DE$.
13. Therefore out of 10°, and 12°; (*per prop.* 5, & 32. *Elem.* 1.)	$\angle AEB$ is $\frac{1}{2}$ ⅃.
14. Likewise, out of 10°, and 12°, . .	$\angle DEB$ is $\frac{1}{2}$ ⅃.
15. Likewise out of 11°, and 13°; . .	$\angle HGE$ is $\frac{1}{2}$ ⅃.
16. Likewise out of 11°, and 14°; . .	$\angle FGE$ is $\frac{1}{2}$ ⅃.
17. Therefore out of 13°, and 15°; (*per prop.* 6. *Elem.* 1.)	$HE = HG$.
18. Likewise out of 14°, and 16°; . .	$EF = FG$.
19. And from 7°, and 8°; (*per prop.* 34. *Elem.* 1.)	$EF = HG$.
20. Wherefore out of 11°, 17°, 18°, 19°; (*per* 29. *defin.* 1. *Elem.*)	HF is $\square HG$, or $\square AC$.
21. And in the same respect, ...	CI is $\square CB$.
22. Therefore from 21°, ...	$CG = CB$.
23. And from 22°, (*per* 36. *prop.* 1. *Elem.*)	$\square AG$, (or $\square AC,CG$,) $= \square AC,CB$.

 24. But

24. But (*per* 43. *prop.* 1. *Elem.*) . > $\square AG = \square GD$.
25. Therefore out of 23°, and 24°; (*per Ax.* 1. *Chap.* 2.) . . . > $\square GD = \square AC, CB$.
26. But (*per Ax.* 28. *Chap.* 2.) . > $\square AD = \square HF + \square CI + \square AG + \square GD$.
27. Wherefore, out of 5°, 20°, 21°, 24°, 25°, 26°; (*per Ax.* 7. *Chap.* 2.) > $\square AB = \square AC + \square CB + 2\,\square AC, CB$.
Which was to be dem.

Coroll. 1.

Hence it is manifest that the Parallelograms which are about the Diameter of a Square, are also Squares themselves.

Coroll. 2.

It also appears that the Diameter of any Square divides its angles into two equal parts.

Illustration Algebraical.

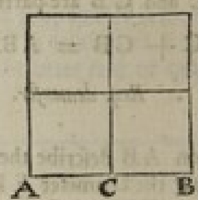
Suppose a right-line to be cut into two parts, to wit, >	a and b . .	6 and 2
Then the summ of the parts is . . . . >	$a + b$ . . .	8
Which summ multiplied by it self produceth the Square of the whole line, to wit, ... >	$aa + 2ba + bb$	64

Which Product or Square doth manifestly consist of the Squares of the parts a and b, and twice the Product (or Rectangle) of the same parts; according to the tenour of the preceding *Theor.* 2.

Theorem III.

A Square described upon any right-line is equal to four times the Square of the half of the same line; and consequently, a quarter of the former Square is equal to the latter.

Suppos.



1. AB is a right-line;
2. $AC = CB$, therefore
3. $AB = AC + CB = 2AC$ or $2CB$.
4. . . . *Req. demonstr.* . . . . $\square AB = 4\square AC$, (or $4\square CB$;) *Also,* $\frac{1}{4}\square AB = \square AC$, or $\square CB$.

Demonstration.

5. By supposition, ... > . . $AC = CB$.
6. Therefore, (*per Sch.* of *prop.* 46. *El.* 1.) > $\square AC = \square CB$.
7. And out of 5°, (*per prop.* 36. *Elem.* 1.) > $\square AC = \square AC, CB$.
8. And out of 7°, (*per Ax.* 17. *Chap.* 2.) > $2\square AC = 2\square AC, CB$.
9. And out of 6°, (*per Ax.* 8. *Chap* 2.) > $2\square AC = \square AC + \square CB$.
10. And out of 8°, and 9°; (*per Ax.* 8.) > $4\square AC = \square AC + \square CB + 2\square AC, CB$.
11. But *per Theor.* 2. of this *Chap.* . > $\square AB = \square AC + \square CB + 2\square AC, CB$.

Conclus.
12. Wherefore out of 10°, and 11°; (*per Ax.* 1.) . . . . > $\square AB = 4\square AC$, (or $4\square CB$.)
13. And out of 12°, (*per Ax.* 21. *Chap.* 2.) . . . . > $\frac{1}{4}\square AB = \square AC$, (or $\square CB$.)
Which was to be dem.

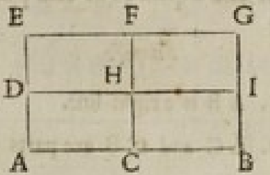
Illustration Algebraical.

Let a right-line be represented by . . . . >	$2a$	10
The half thereof is . . . . >	a	5
The Square of the whole line $2a$ is . . . . >	$4aa$	100
The Square of the half, to wit, of a is . . . . >	aa	25

The first of those Squares is evidently equal to four times the latter, and consequently a quarter of the former is equal to the latter; as is affirmed by *Theor.* 3.

Theorem IV.

A Rectangle (or long Square) comprehended under any two unequal right-lines is equal to four times the Rectangle comprehended under the half of each of those lines; and consequently a quarter of the first Rectangle is equal to the latter.



Suppos.

1. AB and AE are right-lines,
2. $AC = CB = \frac{1}{2}AB$,
3. $AD = DE = \frac{1}{2}AE$.

4. . . . *Req demonstr.* . . $\left\{ \begin{array}{l} \square AB.AE = 4\square AC,AD. \text{ Also,} \\ \frac{1}{4}\square AB,AE = \square AC,AD. \end{array} \right.$

Prepar.

5. Make $\square$ AG to be contain'd under AB and AE.
6. Draw CF $\parallel$ AE, (or BG.) Likewise, DI $\parallel$ EG, (or AB.)

Demonstration.

7. By Constr. in 5°, . . .	AG is $\square$AB,AE.
8. Therefore (*per* 30. *defin.* 1. *Elem.*)	$\angle A, \angle E, \angle G, \angle B$ are $\perp$
9. And because by Constr. in 6°, ...	CF $\parallel$ AE $\parallel$ BG.
10. Likewise by Constr. in 6°,	DI $\parallel$ EG $\parallel$ AB.
11. Therefore out of 8°, 9°, and 10°; (*per prop.* 29. *Elem.* 1.)	AH, DF, HG, CI are $\square$.
12. And because by Suppos. in 2° and 3°, ...	AC = CB. Also AD = DE.
13. Therefore out of 11° and 12°, (*per prop.* 36. *Elem.* 1.)	$\square AH = \square DF = \square HG = \square CI$.
14. And from 13°, (*per Ax.* 8. *Chap.* 2.)	$\square AH + \square DF + \square HG + \square CI = 4\square AH$.
15. But (*per Ax.* 18°. *Ch.* 2.)	$\square AH + \square DF + \square HG + \square CI = \square AG$.
16. Therefore out of 14° and 15°, (*per Ax.* 1.) . .	$\square AG = 4\square AH$.
17. That is, (*per Ax.* 7. *Chap.* 2.)	$\square AB,AE = 4\square AC,AD$. *Conclus.*
18. And consequently from 17°, (*per Ax.* 21.) . .	$\frac{1}{4}\square AB,AE = \square AC,AD$. *Conclus.*

Which was to be dem.

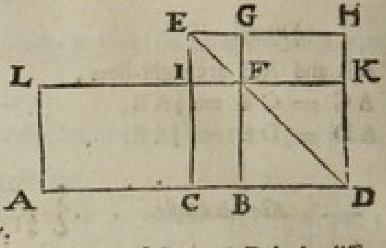
Illustration Alegebraical.

Let a right-line be represented by	$2a$	6
And another right-line by	$2b$	4
The half of the former line is	a	3
And the half of the latter is	b	2
The Product or Rectangle of the two whole lines is	$4ab$	24
The Product of the half of each line is	ab	6

The first of those Products is evidently equal to four times the latter, and consequently a quarter of the former is equal to the latter, according to the tenour of *Theor.* 4.

Theorem V.

If a right-line be cut into any two unequal parts, the Square of the difference of the parts is equal to the Squares of the parts, less by twice the Rectangle (or long Square) comprehended under the parts: Also, the Square of half the difference of the said parts is equal to a quarter of each of the Squares of the parts, less by half the Rectangle of the parts.



Suppos.

1. A B is a right-line.
2. A C and C B are parts of A B.
3. $AC \sqsubset CB$.

Prepar.

4. Produce A B to D, so, that A C = C D, then 'tis manifest that B D is the difference of the parts A C and C B; for C D (or A C) − C B = B D.
5. Upon C D describe the □CH; (*per prop.* 46. *Elem* 1.)
6. Draw the Diameter E D.
7. By the point B draw B G ∥ C E (or D H,) and cutting E D in F.
8. By the point F draw L F K ∥ A D, or E H.
9. By the point A draw A L ∥ C E.
10. . . . *Req. demonstr.* . . . □BD = □AC + □CB − 2▭AC,CB. Also, □$\frac{1}{2}$BD = $\frac{1}{4}$□AC + $\frac{1}{4}$□CB − $\frac{1}{2}$▭AC,CB.

Demonstration.

11. By Constr. in 4° and 5°,	CH is □CD or □AC.
12. Therefore (*per Coroll.* 1. *Theor.* 2. of this *Chapt.*	BK is □BD.
13. Likewise	IG is □EG or □CB.
14. And (*per prop.* 43. *Elem.* 1.)	▭CF = ▭FH.
15. Therefore (*per Ax.* 8. *Chap.* 2.) by adding □IG to each part of the Equation in 14°, ...	▭CG = ▭IH.
16. Again, (by Constr. in 4° and 5°, ...	CE = AC.
17. And 'tis evident that .	CB = CB.
18. Therefore from 16° and 17°, (*per prop.* 35. *Elem.* 1.)	▭CE,CB (or ▭CG) = ▭AC,CB.
19. Therefore out of 15° and 18°, (*per Ax.* 6, & 8 *Ch* 2.)	▭CG + ▭IH = 2▭AC,CB.
20. And because (*per Ax.* 28, & 8. *Chap.* 2.) .	□BK + ▭CG + ▭IH = □CH + □IG.
21. Therefore out of 19° and 20°, (*per Ax.* 6. *Ch.* 2.)	□BK + 2▭AC,CB = □CH + □IG.
22. Therefore from 21°, (*per Ax.* 9. *Chap.* 2.) .	□BK = □CH + □IG − 2▭AC,CB.
23. Therefore out of 22°, 12°, 11°, 13°; (*per Ax.* 7. *Chap.* 2.	□BD = □AC + □CB − 2▭AC,CB.

Conclus. 1. Which was to be dem.

24. Moreover, out of 23°, (*per Ax.* 21. *Chap.* 2.) .	$\frac{1}{4}$□BD = $\frac{1}{4}$□AC + $\frac{1}{4}$□CB − $\frac{1}{2}$▭AC,CB.
25. And *per The.* 3. of this *Ch.*	$\frac{1}{4}$□BD = □$\frac{1}{2}$BD
26. Therefore out of 24° and 25°, (*per Ax.* 1.)	□$\frac{1}{2}$BD = $\frac{1}{4}$□AC + $\frac{1}{4}$□CB − $\frac{1}{2}$▭AC,CB.

Conclus. 2. Which was also to be dem.

Illustra-

Illustration Algebraical.

Suppose a right-line to be cut into two unequal parts, to wit,	a and b . . .	16 and 10
Suppose also	$a \sqsubset b$ . . .	
Then the difference of the parts is	$a - b$ . . .	6
And half the difference of the parts is . . .	$\frac{1}{2}a - \frac{1}{2}b$ . . .	3
The Square of the whole difference is . . .	$aa + bb - 2ba$	36
The Square of the half difference is	$\frac{1}{4}aa + \frac{1}{4}bb - \frac{1}{2}ba$	9

Which two Squares do manifestly prove the certainty of what is affirmed in the foregoing *Theor.* 5.

Theorem VI.

If a right-line be cut into any two unequal parts, the Square of the whole line together with the Square of the difference of the parts is equal to twice the Squares of the parts; and consequently half the Square of the whole line together with half the Square of the difference of the parts is equal to the summ of the Squares of the parts.

Suppos.

1. A B is a right-line,
2. A C and C B are parts of A B,
3. A C $\sqsubset$ C B.

Prepar.

4. From C A cut off C D = C B, thence it follows that A D (= A C — C B) is the difference of the parts A C and C B.
5. . . *Req. demonstr.* . . $\begin{cases} \square AB + \square AD = 2\,\square AC + 2\,\square CB. \ \textit{Also,} \\ \frac{1}{2}\square AB + \frac{1}{2}\square AD = \square AC + \square CB. \end{cases}$

Demonstration.

6. By *Theor.* 2. of this *Chapt.* . > $\square AB = \square AC + \square CB + 2\,▭\, AC,CB.$
7. And by *Theor.* 5. of this *Chapt.* > $\square AD = \square AC + \square CB - 2\,▭\, AC,CB.$
8. Therefore out of 6° and 7°, (*per Ax.* 8. *Chap.* 2.) . . . . > $\square AB + \square AD = 2\,\square AC + 2\,\square CB.$ } *Conclus.*
9. And consequently, (*per Ax.* 21. *Chap.* 2.) . . . . . . > $\frac{1}{2}\square AB + \frac{1}{2}\square AD = \square AC + \square CB.$

Which was to be dem.

Illustration Alegebraical.

Suppose a right-line to be cut into two unequal parts, to wit, ...	a and b . . .	6 & 4
Suppose also, ...	$a \sqsubset b$	
Then the summ of the parts is	$a + b$ . . .	10
And the difference of the parts is	$a - b$ . . .	2
The Square of the whole line, that is, the Square of the summ of the parts, is	$aa + bb + 2ba$	100
The Square of the difference of the parts is . . .	$aa + bb - 2ba$	4
The summ of those Squares is	$2aa + 2bb$ . .	104

Which summ, (according to the tenour of the preceding *Theor.* 6.) is manifestly equal to twice the summ of the Squares of the parts.

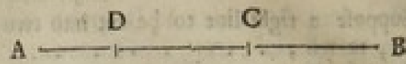
Theorem VII.

If a right-line be cut into any two unequal parts, the Square of the whole line is equal to four times the Rectangle (or long Square) comprehended under the parts, together with the Square of the difference of the parts: Also, the Square of half the said right-line, (or of half the summ of the parts,) is equal to the Rectangle of the parts together with a quarter of the Square of the difference of the parts.

Suppos.

Suppos.

1. A B is a right-line,
2. A C and C B are parts of A B,
3. A C $\sqsubset$ C B.



Prepar.

4. From C A cut off $CD = CB$, whence $AD\ (= AC - CB)$ is the difference of the parts A C and C B.

5. . . . . *Req. demonstr.* . . $\begin{cases} \square AB = 4\,\text{▭}AC,CB + \square AD. \text{ Also,} \\ \square \tfrac{1}{2}AB = \text{▭}AC,CB + \tfrac{1}{4}\square AD. \end{cases}$

Demonstration.

6. By *Theor.* 5. of this *Chapt.* . . $\square AD = \square AC + \square CB - 2\,\text{▭}AC,CB$.
7. Therefore by adding $4\,\text{▭}AC$, C B to each part of that Equation this ariseth, (*per ax.* 8. *ch.* 2.) $4\,\text{▭}AC,CB + \square AD = \square AC + \square CB + 2\,\text{▭}AC,CB$;
8. But *per Theor.* 2. of this *Chap.* . $\square AB = \square AC + \square CB + 2\,\text{▭}AC,CB$.
9. Therefore from 7° and 8°, (*per Ax.* 1. *Chap.* 2.) . . $\square AB = 4\,\text{▭}AC,CB + \square AD$.
   Which was to be dem.
10. Moreover, from 9°, (*per Ax.* 21. *Chap.* 2.) . . . $\tfrac{1}{4}\square AB = \text{▭}AC,CB + \tfrac{1}{4}\square AD$.
11. And by *Theor.* 3. of this *Cha.* . $\tfrac{1}{4}\square AB = \square \tfrac{1}{2}AB$.
12. Therefore from 10° and 11°, (*per Ax.* 1.) . . . . . $\square \tfrac{1}{2}AB = \text{▭}AC,CB + \tfrac{1}{4}\square AD$.
    Which was also to be dem.

Illustration Algebraical.

Suppose a right-line to be cut into two parts, to wit,	a and b . . .	6 and 4
Suppose also	$a \sqsubset b$	
Then the summ of the parts is	$a + b$ . . .	10
And the difference of the parts is	$a - b$ . . .	2
The Square of the summ of the parts, that is, the Square of the whole line is	$aa + bb + 2ba$	100
And the Square of the difference of the parts is .	$aa + bb - 2ba$	4
To which Square of the difference if you add the quadruple Product of the parts, to wit, . . .	. . . $+ 4ba$	96

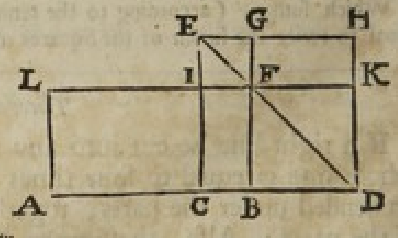
The summ (according to the tenour of *Theor.* 7.) makes the Square of the whole line, to wit, $aa + bb + 2ba$.

Theorem VIII.

If a right-line be cut into any two unequal parts, the Rectangle (or long Square) comprehended under the whole line and the difference of the parts is equal to the difference of the Squares of the parts. Also, the Rectangle under half the said right-line, (that is, half the summ of the parts) and half the difference of the parts, is equal to a quarter of the difference of the Squares of the parts.

Suppos.

1. A B is a right-line.
2. A C and C B are parts of A B.
3. A C $\sqsubset$ C B.



Prepar.

4. Produce A B to D, so, that $AC = CD$, thence it follows that B D is the difference of the parts A C and C B; for CD (or AC) $- CB = BD$.

5. Upon

5. Upon C D describe the □ CH.
6. Draw the Diameter E D.
7. By the point B draw BG ∥ CE (or D H,) and cutting E D in F.
8. By the point F draw L F K ∥ A D, or E H.
9. By the point A draw A L ∥ C E.

10. . . *Req. demonstr.* . .	$\square AB,BD = \square AC - \square CB$. Also; $\square \frac{1}{2}AB, \frac{1}{2}BD = \frac{1}{4}\square AC - \frac{1}{4}\square CB$.	

Demonstration.

11. By Constr. in 5°, ...	CH is □ C D.	
12. Therefore out of 6° and 11°, (*per Cor.* 1. *The.* 2. of this *Cha.*)	IG = □ IF, and BK = □ BD.	
13 And from 12°, (*per* 29. *defin.* 1. *Elem.*)	BF = BD.	
14. By Constr. in 7° and 8°, . .	CF is ▭.	
15. Therefore from 14°, (*per* 34. *prop.* 1. *Elem.*)	BF = CI; and IF = CB.	
16. And from 13° and 15°, (*per Ax.* 1.)	BD = CI.	
17. By Constr. in 4° and 5°, .	DH = CA:	
18. Therefore out of 16° and 17°, (*per* 36. *prop.* 1. *Elem.*) . . .	▭ BD, DH = ▭ CI, CA.	
19. That is, (*per Ax.* 7. *Chap.* 2.)	▭ B H = ▭ AI.	
20. Therefore by adding ▭ CF to each part of the Equation in 19°,	*Gnomon*, ICDHG = AF = ▭ AB,BD (BF.)	
21. But 'tis manifest (*per Ax.* 9. *Chap.* 2.) that	*Gnomon*, ICDHG = □ CH − □ IG.	
22. Therefore from 20° and 21°, (*per Ax.* 1.)	▭ AB, BD = □ CH − □ IG.	
23. And because from 4°, 5°, 11°, and 15°;	□ CD (or □ AC) − □ CB (□ IG) = □ CH − □ IG.	
24. Therefore from 22° and 23°, (*per Ax.* 1.)	▭ A B, B D = □ AC − □ C B.	*Conclus.* 1.
Which was to be dem.		
25. Moreover, from 24°, (*per Ax.* 21. *Chap.* 2.)	$\frac{1}{4}$ ▭ AB, BD = $\frac{1}{4}$ □ AC − $\frac{1}{4}$ □ CB.	
26. And by *Theor.* 4. of this *Chapt.*	$\frac{1}{4}$ ▭ AB, BD = ▭ $\frac{1}{2}$ AB, $\frac{1}{2}$ BD.	
27. Therefore out of 25° and 26°, (*per Ax.* 1.)	▭ $\frac{1}{2}$ AB, $\frac{1}{2}$ BD = $\frac{1}{4}$ □ AC − $\frac{1}{4}$ □ CB.	*Conclus.* 2.
Which was also to be dem.		

Illustration Algebraical.

Suppose a right-line to be cut into two parts, to wit,	a and b	6 and 4
Suppose also	$a \sqsubset b$	
Then the summ of the parts is	$a + b$	10
And the difference of the parts is	$a - b$	2
Therefore the Rectangle (or Product) of the summ and difference of the parts is	$aa - bb$	20

Which Rectangle (according to the tenour of *Theor.* 8.) is manifestly equal to the difference of the Squares of the parts a and b. And by multiplying $\frac{1}{2}a + \frac{1}{2}b$ into $\frac{1}{2}a - \frac{1}{2}b$, the latter part of the said Theorem will be also manifest.

Theorem IX.

If a right-line be cut into any two unequal parts, the greater part shall be equal to half the whole line, together with half the difference of the parts: And, the lesser part shall be equal to half the whole line less by half the difference of the parts.

Suppos.

Suppos.

1. A B is a right-line,
2. A E and E B are parts of A B,
3. A E $\sqsubset$ E B.

Prepar.

4. From A B cut off $AD = EB$, thence it follows that D E is the difference of the parts A E and E B; for $DE = AE - AD$ (E B.)
5. Divide D E into two equal parts in C; therefore $DC = CE = \frac{1}{2}DE$.
6. . . . *Req. demonstr.* . . $\left\{\begin{array}{l} AE = \frac{1}{2}AB + \frac{1}{2}DE. \\ EB = \frac{1}{2}AB - \frac{1}{2}DE. \end{array}\right.$

Demonstration.

7. Because by Constr. in 4°, . . $AD = EB$.
8. And by Constr. in 5°, . . . $DC = CE = \frac{1}{2}DE$.
9. Therefore the summ of the Equations in 7° and 8°, gives (*per Ax.* 8. *Chap.* 2.) . . . . $AC = CB = \frac{1}{2}AB$.
10. And the summ of the Equations in 8° and 9° gives . . . . $AE = \frac{1}{2}AB + \frac{1}{2}DE$.
11. And the Equation in 8° subtracted from the Equation in 9°, gives $EB = \frac{1}{2}AB - \frac{1}{2}DE$.

Which was to be Dem.

Illustration Algebraical.

Suppose a right-line to be cut into two unequal parts, to wit, ...	a and b	6 and 4
Suppose also	$a \sqsubset b$	
Then half the whole line, that is, half the summ of the parts is	$\frac{1}{2}a + \frac{1}{2}b$	5
And half the difference of the parts is	$\frac{1}{2}a - \frac{1}{2}b$	1

Now (according to the import of *Theor.* 9.) the summ of the said half summ and half difference doth manifestly make a the greater part: And the excess of the said half summ above the said difference is manifestly equal to b the lesser part.

CHAP. V.

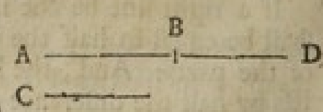
A Collection of Canonical Geometrical Effections, frequently used in the Construction of Plane Problems; more especially of those whose Solutions are found out by the Algebraical Art.

AS all Arithmetical Operations are compris'd under five kinds, to wit, *Addition, Subtraction, Multiplication, Division* and the *Extraction of Roots*, so all those Geometrical Constructions which are formed according to Canons deduced from the Algebraical Resolutions of Problems, do principally depend upon the like kinds of Operations, or Effections; but how these Geometrical Effections, (or the Arithmetick of Geometry) may be perform'd, so far as is necessary to the Construction of Plane Problems, to wit, such as may be solved by drawing only right-lines and describing the Circumferences of Circles, I shall shew in this Chapter, the Contents whereof are extracted out of the first six Books of *Euclid*'s Elements, wherein I presuppose the Reader to be competently versed.

Problem I.

To add a given right-line to a right-line given.

Let A B and C be two right-lines given to be added together, *viz.* let it be required to find out a right-line which shall be equal to both the given right-lines taken together as one right-line, ...



Con-

Construction.

By *prop.* 2. *Elem.* 1. produce (or continue) the given line AB to D, that BD may be equal to C, so is AD the right-line sought; for by Construction $AD = AB + BD$, and $BD = C$, therefore (*per Ax.* 6. *Chap.* 2.) $AD = AB + C$, as was required.

Probl. II.

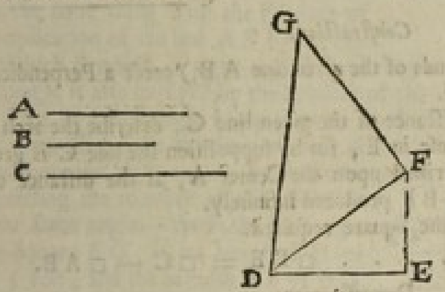
Two or more Squares being given, to find a Square equal to them all.

Suppos.

1. A, B, C are the sides of three Squares given.

Req. to find

2. DG a right-line, such, that $\square DG = \square A + \square B + \square C$.



$DE = 16 = A$

$EF = 12 = B$

$DF = 20$

$FG = 21 = C$

$DG = 29$

Constr.

3. Make $DE = A$.
4. Make $EF \perp DE$.
5. Make $EF = B$.
6. Draw DF.
7. Make $FG \perp DF$.
8. Make $FG = C$.
9. Draw DG, which shall be the side of the Square required.
10. . . . *Req. demonstr.* . $\square GD = \square A + \square B + \square C$.

Demonstration.

11. Because by Constr. in 7° and 4°, $\angle DFG = \llcorner = \angle DEF$.
12. Therefore, (*per prop.* 47. *Elem.* 1.) $\square DG = \square FG + \square FD$.
13. Likewise . . . . . . $\square FD = \square DE + \square EF$.
14. Therefore from 12° and 13°, (*per Ax.* 6. *Chap.* 2.) . . . $\square DG = \square DE + \square EF + \square FG = \square A + \square B + \square C$.

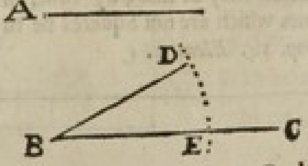
Which was to be done.

After the same manner of Construction, as many Squares as one will may be added into one. But if Planes of any other kind, as Long-Squares, Rhombs, Rhomboids, Triangles, &c. be given to be added, they must first be transformed into Squares, which may be done by *Prop.* 14. *Elem.* 2. or by various ways delivered in the practical Geometry of divers Mathematicians, and then they may be added together as before.

Probl. III.

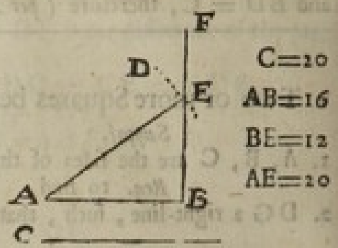
To subtract or cut off a right-line given from a greater right-line given.

The subtracting or cutting off one right-line from another, to wit, a lesser from a greater, is perform'd by *Prop.* 3. *Elem.* 1. For, if two unequal right-lines be given, suppose BC the greater, and A the lesser, then by describing a Circle from B as a Center, with the distance or Semidiameter BD equal to the lesser line A, the right-line $BE = BD$ or A will be cut off from the greater line BC, and consequently, EC is the excess whereby BC exceeds A or BE.



Probl. IV.

Two unequal Squares being given, to find a Square equal to the excess whereby the greater exceeds the less.



Suppos.

1. C and A B are the sides of two Squares given.
2. C $\sqsubset$ AB.

Req. to find

3. BE a right-line, such, that $\square$BE $= \square$C $-$ $\square$AB.

Construction.

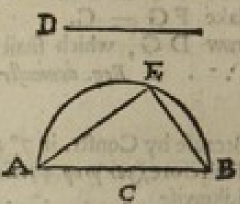
4. Upon the point B, (one of the ends of the given line A B,) erect a Perpendicular, and draw it forth at length, as BF.
5. From A as a Center, at the distance of the given line C, describe the arch D E, to cut the Perpendicular BF, suppose in E; for by supposition the line C is greater than A B, and therefore a Circle described upon the Center A, at the distance of C shall necessarily cut the Perpendicular BF produced infinitely.
6. I say B E shall be the side of the Square required.
7. . . . *Req. demonst.* . . . . . . $\square$BE $= \square$C $-$ $\square$AB.

Demonstration.

8. By Constr. in 4° and 5°, ...$>$ $\angle$ABE is $\lrcorner$, and C $=$ AE.
9. Therefore (*per prop.* 47. *Elem.* 1.) .$>$ $\square$AB $+$ $\square$BE $= \square$AE $= \square$C.
10. Therefore (*per Ax.* 9. *Chap.* 2.) . .$>$ . . . $\square$BE $= \square$C $-$ $\square$AB.

Which was to be done.

Another Construction of Probl. 4.



Suppos.

11. AB and D are the sides of two Squares given.
12. AB $\sqsubset$ D.

Req. to find

13. E B a right-line, such, that $\square$EB $= \square$AB $-$ $\square$D.

Construction.

14. Upon the given line A B as a Diameter, describe the Semicircle C A E B, and inscribe A E $=$ D, which is possible to be done, for by supposition A B $\sqsubset$ D. Lastly, draw E B which shall be the side of the Square required.
15. . . *Req. demonstr.* . . . $\square$EB $= \square$AB $-$ $\square$D.

Demonstration.

16. By Constr. in 14°, and *per prop.* 31. *Elem.* 3. . . . .$>$ $\angle$AEB is $\lrcorner$, and AE $=$ D.
17. Therefore, *per prop.* 47. *Elem.* 1. .$>$ $\square$AE (or $\square$D) $+$ $\square$EB $= \square$AB.
18. Therefore *per Ax.* 9. *Chap.* 2. . .$>$ $\square$EB $= \square$AB $-$ $\square$D.

Which was to be done.

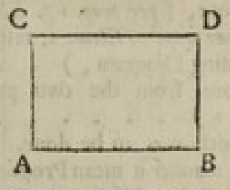
Note. If many Squares be given to be subtracted from a Square given, those to be subtracted must first be added together, by the preceding *Probl.* 2. of this *Chapt.* and then subtraction may be made by either of the two foregoing Constructions of *Probl.* 4. But if Planes which are not Squares be to be subtracted, they must first be reduced to Squares, by *Prop.* 14. *Elem.* 2.

Probl. V.

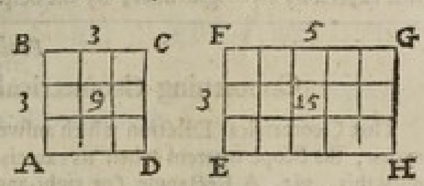
Probl. V.

Concerning Geometrical Multiplication.

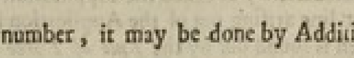
1. A right-line is said to be multiplied by a right-line, when a right-angled Parallelogram, whether it be a Square or a Long-square, is comprehended under one of the said right-lines as a length, and the other as a breadth. As, if the right-line A C be conceived to be moved along the line A B, so, as that A C always makes a right-angle with the line AB, until the point C be come to the point D, and the point A to the point B; then the right-angled Parallelogram A C D B is described by such moving of the line A C, and imports the same thing with the Product of the multiplication of the line A B by the line A C. Which Product, or right-angled Parallelogram, is also usually called a Rectangle.



2. A Rectangle is also implyed by the Product of the multiplication of any two numbers; for the Area of a Rectangle is equal to the Product made by the multiplication of the number expressing the measure of one of the sides about the right-angle, by the number expressing the measure of the other side about the same angle. As in the Rectangle or long-Square E G, if its length E H or FG be 5 feet, and the breadth E F or H G, 3 feet, then the Product of the multiplication of 5 by 3, to wit, 15, imports the Area, or number of square feet contain'd in the said Rectangle. Likewise, if A B or A D the side of the Square A C be 3 feet, the Area is 9 square feet. All which evidently appears by the Diagrams here in view.

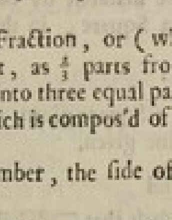


3. If a right-line be to be multiplied by a whole number, it may be done by Addition, (by *Probl.* 1. of this *Chapt.*) As, if the line A B be to be multiplied by 3, it implies only the producing or continuing of the said line to such a point D, that the whole line A D may be equal to the triple of the given line A B.



4. But if a right-line A D be to be multiplied by a Fraction, or (which is of the same import,) if it be required to cut off some segment, as $\frac{2}{3}$ parts from A D; first, (*per Schol.* of *Prop.* 10. *Elem.* 6.) divide the line A D into three equal parts, which suppose to be A B, B C, C D, then the segment A C which is compos'd of two of those three parts is manifestly $\frac{2}{3}$ of the line A D.

5. If a Square be to be multiplied by a whole number, the side of the Square sought may be found out by addition of Squares, as before in *Probl.* 2. So if the Square of the right-line A B, or B C be to be doubled, or multiplied by 2, it implies only the finding of the right-line A C, (by *Probl.* 2. of this *Chapt.*) whose Square is equal to the double of the Square of AB or BC. For if $AB = BC$, and $BC \perp AB$, then (*per prop.* 47. *Elem.* 1.) $\square AC = \square AB + \square BC = 2\square AB$ or $2\square BC$. The same thing also may be done by the way delivered in the following *Sect.* 6.

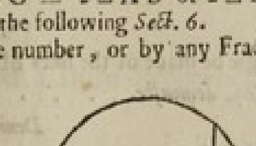


$AB = 3$

$BC = 3$

$AC = \sqrt{18}$

6. If a Square be to be multiplied by a whole number, or by any Fraction whatever, whether it be a proper or improper Fraction, that is, if it be required to find the side of a Square that shall be equal to any prescribed Multiple, or to any part or parts of a given Square, it may be done thus: Let the right-line A be the side of a Square given, and let it be required to find a Square which shall contain five times the given Square whose



$A = 2$

$5A = 10$

$5\square A = 20$

$M = \sqrt{20}$

side

ſide is A. To effect this, find (*per Probl.* 9. of this *Chapt.*) a mean Proportional between the given ſide A, and a right-line equal to five times A; which mean ſuppoſe to be the right-line M: I ſay the line M is the Square required; which I prove thus,

Req. demonſtr. . . . . □M = 5□A.

Demonſtration.

By Conſtr. theſe are Proportionals, *viz.* . . . . A . M :: M . 5 A

Therefore, (*per prop.* 17. *Elem* 6.) . . . . □A, 5 A = □M

But (*per prop.* 1. *Elem.* 2. reſpect being had to the laſt preceding Diagram,) . . . . □A, 5 A = 5□A

Therefore from the two preceding Equations, (*per Ax.* 1.) . . . . □M = 5□A.

Which was to be done.

In like manner a mean Proportional between A and $\frac{4}{3}$A ſhall be the ſide of a Square equal to $\frac{4}{3}$□A. Alſo a mean Proportional between $3\frac{1}{2}$A (or $\frac{7}{2}$A) and A ſhall be the ſide of a Square equal to $\frac{7}{2}$□A, or $3\frac{1}{2}$□A. The ſame thing may be effected by *Probl.* 11. of this *Chapt.* the proportion of the Square given to the Square ſought being firſt expreſt by two right-lines, by the help of the foregoing *Sect.* 3, or 4. of this *Probl.* 5.

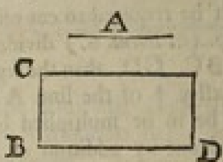
Probl. VI.

Concerning Geometrical Diviſion, or Application.

That Geometrical Effection which anſwers to *Diviſion* in Arithmetick is called *Application*, the Scope whereof when 'tis exercis'd about the Conſtruction of Plane Problems is only this, *viz.* A Rectangle, (or right-angled Parallelogram) being given, as alſo a right-line, to find out another right-line, ſuch, that a Rectangle contain'd under the line found out, and the line given ſhall be equal to the Rectangle firſt given; which Effection (or Conſtruction) is called the Application of a given Rectangle to a right-line given, and the right-line ariſing out of the Application is called the *Parabola*, or the Geometrical Quotient, which is found out by the *Rule of Three* in right-lines by the following 7th or 8th *Problems* of this *Chapter*: For as the line given is to either of the ſides about the right-angle of the given Rectangle, ſo is the other ſide about the ſame angle, to the line ſought, to wit, the Geometrical Quotient.

This will be made manifeſt by the two following Examples, in the firſt whereof the Rectangle given is a Square, in the latter a long-Square.

Suppoſ. 1.



A = 6
BC = 4
BD = 9

1. A is the ſide of a Square given,
2. B C is a right-line given.

Req. to find

3. BD, a right-line, ſuch, that ▭BD, BC = □A.

Conſtruction.

4. By *Probl.* 7. of this *Chap.* let it be made as B C to A, ſo A to a third Proportional, which ſuppoſe to be the line B D, therefore, ... BC . A :: A . BD.
5. I ſay B D is the right-line ſought by the *Probl.* propounded; it remains then to prove that a Rectangle contain'd under the right lines B D and B C is equal to the Square of line A.

Prepar.

6. Let a Rectangle be made of the lines BD and BC, as ▭CD, (*per prop* 46. *Elem.* 1.) Then
7. . . . *Req. demonſtr.* . . . . ▭CD = □A.

Demonſtration.

8. By Conſtr. in 4°, ... BC . A :: A . BD.
9. Therefore, (*per prop.* 17. *Elem.* 6.) . . . □A = ▭BC, BD.
10. But by Conſtr. in 6°, ... ▭CD = ▭BC, BD.
11. Therefore (*per Ax.* 1. *Chap.* 2.) . . . . ▭CD = □A.

Which was to be done.

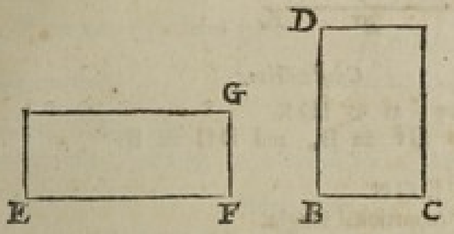
Suppoſ. 2.

Suppos. 2.

12. EG is a □ whose sides EF, FG are given severally.
13. BC is a right-line given.

Req. to find

14. BD a right-line, such, that . . . . □BC,BD = □EG.



EF = 12
FG = 5
BC = 6
BD = 10

Construction.

15. By *Probl.* 8. of this *Chapt.* let it be made as BC to EF, so FG to a fourth Proportional, which suppose to be the line BD, therefore, } BC . EF :: FG . BD.
16. I say BD is the right-line sought; it remains therefore to prove that a Rectangle contain'd under BC and BD is equal to the given Rectangle EG.

Prepar.

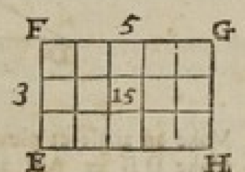
17. Let a Rectangle be made of the lines BC and BD, as □CD, (*per prop.* 46. *Elem.* 1.) Then
18. . . . *Req. demonstr.* . . . . □CD = □EG.

Demonstration.

19. By *Constr.* in 15°, ... } BC . EF :: FG . BD.
20. Therefore, (*per prop.* 16. *Elem.* 6.) . . } □EF,FG = □BC,BD.
21. But by *Constr.* in 17°, ... } □CD = □BC,BD.
22. Therefore from the two last preceding Equations, (*per Ax.* 1. *Chap.* 2.) . . . . } □CD = □EF,FG, or □EG.

Which was to be done.

23. From the premisses 'tis evident that Geometrical Application answers to Division in Arithmetick, for the Rectangle applied is correspondent to the Dividend, and the right-line to which the Rectangle is applied answers to the Divisor, and the right-line arising out of the Application, the Quotient: Therefore, if the Area of a Rectangle and either of its sides be given, the other side is also given; for if the Area be divided by the given side, the Quotient is the other side. So if FG, or EH, one of the sides of the Rectangle EG, be 5, and FE, or GH, the other side, 3, the Area 15 divided by 5, (to wit, FG,) gives 3 for FE. Likewise the Area 15 divided by 3, (to wit, FE,) gives 5 for FG, or EH.



Probl. VII.

Unto two right-lines given to find a third Proportional.

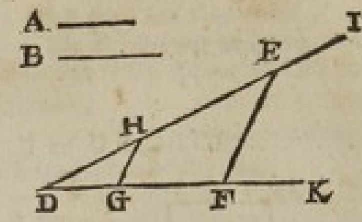
Suppos.

1. A and B are two right-lines given.

Req. to find

2. HE a right-line, such, that . . . . . . A . B :: B . HE.

A —



A = 4
B = 6
HE = 9

Construction.

3. Make an angle at pleasure; as $\angle$ IDK.
4. Make DG = A. Also GF = B, and DH = B.
5. Draw the right-line HG.
6. By the point F draw FE $\parallel$ GH.
7. I say, HE is the third Proportional sought.
8. . . . *Req. demonstr.* . . . . A . B :: B . HE.

Demonstration.

9. Because by *Constr.* in 6°, ... HG $\parallel$ EF (in $\triangle$DEF.)
10. Therefore, (*per prop. 2. Elem. 6.*) . . . . DG . GF :: DH . HE.
11. Therefore out of 4° and 10°, by exchanging equal right-lines, ... A . B :: B . HE.

Which was to be done.

Probl. VIII.

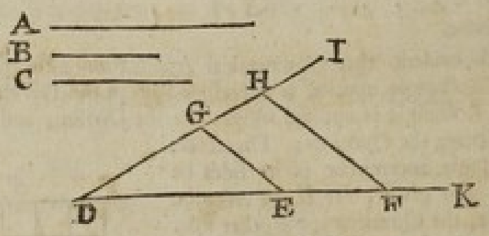
Unto three right-lines given to find a fourth Proportional.

Suppos.

1. A, B, C are three right-lines given.

Req. to find

2. GH a right-line, such, that . . . . . . A . B :: C . GH.



A = 27
B = 18
C = 24
GH = 16

Construction.

3. Make an angle at pleasure, as $\angle$ IDK.
4. Make DE = A, also EF = B, and DG = C.
5. Draw the right-line EG.
6. By the point F draw FH $\parallel$ EG.
7. I say, GH is the fourth Proportional required.
8. . . . *Req. demonstr.* . . . . A . B :: C . GH.

Demonstration.

9. Because by *Constr.* in 6°, ... EG $\parallel$ FH (in $\triangle$ DHF.)
10. Therefore, (*per prop. 2. Elem. 6.*) . . . DE . EF :: DG . GH.
11. Therefore out of 4° and 10°, by exchanging equal right-lines, ... A . B :: C . GH.

Which was to be done.

Con-

Concerning the extraction of the Square Root.

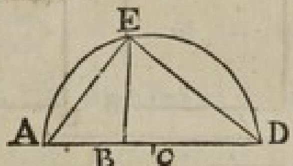
Probl. IX.

Between two right-lines given to find a mean Proportional; or, To transform a Long-square into a Square.

Suppos.

1. A B and B D are two right-lines given.

Req. to find



AB = 9
BD = 16
BE = 12
AD = 25
AC = $12\frac{1}{2}$

2. B E a right-line, such, that . . . . . > AB . BE :: BE . BD.
3. Whence consequently, (*per prop.* 17. *Elem.* 6.) > □BE = ▭AB,BD.

Construction.

4. Joyn the given lines A B and B D so together in the point B, that they may make one right-line, as A D.
5. Upon A D as a Diameter, describe the Semicircle C A E D.
6. Upon the point B where the given lines A B and B D are joyned together, erect a Perpendicular and extend it to the Circumference of the Semicircle C A E D, as B E, which shall be the mean Proportional required.
7. . . . *Req. demonstr.* . . . . AB . BE :: BE . BD.

Prepar.

8. Draw the right-lines, A E and D E.

Demonstration.

9. By *prop.* 31. *Elem.* 3. . . . . > ∠AED is ⅃.
10. And by *Constr.* in 6°, ... > BE ⊥ AD.
11. Therefore from 9° and 10°, (*per Coroll.* of *prop.* 8. *Elem.* 6.) . . . . > AB . BE :: BE . BD.

Which was to be done.

Corollary.

12. Hence 'tis manifest, that a right-line drawn in a Circle from any point of the Diameter perpendicularly, and extended to the Circumference, is a mean proportional between the two segments of the Diameter which are made by the same Perpendicular.
13. Moreover, because from the preceding eleventh step, (*per prop.* 17. *Elem.* 6.) □BE = ▭AB,BD; therefore, if A B and B D be the sides of a Rectangle, a mean proportional B E between those sides shall be the side of a Square equal to that Rectangle or Long-square.
14. Here also by the way, the Learner may take notice, that a mean proportional number between two given numbers, is found out by extracting the square Root of the Product made by the mutual multiplication of the two numbers given: As, if it be required to find out a Mean between 16 (or BD,) and 9 (or AB,) the Rectangle or Product of 16 into 9 is 144, whose square Root 12, (or B E,) is the mean proportional sought; for 16 . 12 :: 12 . 9.

Probl. X.

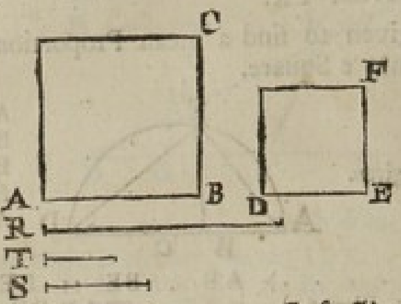
To find a right-line, to which a given right-line may be in the proportion of two Squares given.

Suppos.

1. A C is a Square given, whose side is A B.
2. DF is a Square given, whose side is DE.
3. R is a right-line given.

Req. to find

4. S a right-line, such, that . . : . □AC . □DF :: R . S.



AB = 6
□AB = 36
DE = 4
□DE = 16
R = 9
T = $2\frac{2}{3}$
S = 4

Construction.

5. By *Probl.* 7. of this *Chapt.* let it be made as AB to DE, so DE to a third proportional, which suppose to be the line T, therefore . . . } : AB . DE :: DE : T.
6. Again, (by the foregoing *Probl.* 8.) let it be made as AB to T, so R to a fourth proportional, which suppose to be the line S, therefore . . . } : AB : T :: R : S.

I say the line S is that required by the *Problem*, therefore

7. . . . *Req. demonstr.* . . . . □AC . □DF :: R . S.

Demonstration.

8. Because by *Construction* in 5°, . . } . AB : DE :: DE : T.
9. Therefore, (*per Coroll. Prop.* 20. *Elem.* 6.) . . . . } . AB : T :: □AB . □DE.
10. And because by *Constr.* in 6°, . . } : AB : T :: R . S.
11. Therefore from 9° and 10°, (*per Prop.* 11. *Elem.* 5.) . . . . . . } □AB . □DE :: R . S.
12. That is, (*per Ax.* 7. *Chap.* 2.) . } □AC : □DF :: R . S.

Which was to be done.

Probl. XI.

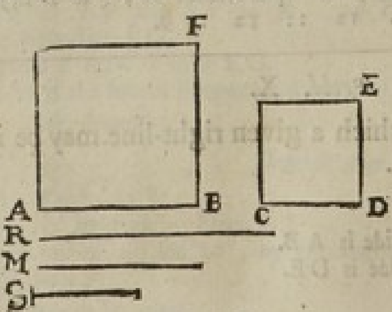
A Square being given, to find out another Square greater or less than that given, according to a Proportion given.

Suppos.

1. AF is a Square given, whose side is AB.
2. R and S are two right-lines expressing the given Proportion.

Req. to make

3. CE a Square, such, that . . . . R : S :: □AF . □CE.



AB = 6
□AB = 36
R = 9
S = 4
M = 6
CD = 4
□CD = 16

Con-

Construction.

4. By *Probl.* 9. of this *Chapt.* find a mean proportional between the given lines R and S, which mean suppose to be the right-line M, therefore . . . . R . M :: M . S.

5. Again, (by *Probl.* 8. of this *Chapt.*) let it be made as R to M, so A B (the side of the given Square,) to a fourth proportional, which suppose to be the right-line C D, therefore R . M :: AB . CD.

6. Upon the line C D describe a Square, as C E, which shall be that required by the *Problem*, therefore

7. . . . *Req. demonstr.* . . . . R . S :: □AF . □CE.

Demonstration.

8. Because by *Constr.* in 4°, ... R . M :: M . S.
9. Therefore, (*per Coroll. prop.* 20. *Elem.* 6.) □R . □M :: R . S.
10. And because by *Constr.* in 5°, ... R . M :: AB . CD.
11. Therefore, (*per prop.* 22. *Elem.* 6.) . . □R . □M :: □AB . □CD.
12. Therefore out of 9° and 11°, (*per prop.* 11. *Elem.* 5.) . . . . R . S :: □AB . □CD.
13. That is, (*per Ax.* 7. *Chap.* 2.) . . . R . S :: □AF . □CE.

Which was to be done.

Probl. XII.

The mean of three proportional right-lines being given; as also the difference of the extremes, to find the extremes.

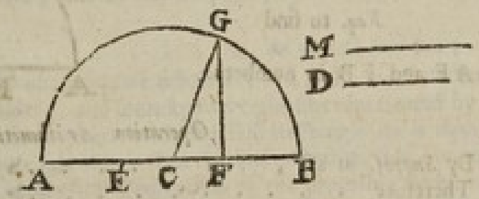
Suppos.

1. M = the mean of three proportionals is given.
2. D = the difference of the extremes is given.

Req. to find

3. A F and F B two right-lines, such, that A F — F B = D; Also, that
4. A F . M :: M . F B.

M = 12	AF = 18
D = 10	FB = 8
EF = 10	AE = 8
FG = 12	CA = 13
EC = 5	CG = 13
CF = 5	CB = 13



Construction.

5. Make E F = D (the given difference.)
6. Upon the point F erect FG ⊥ F E.
7. Make F G = M (the given mean.)
8. Divide E F into two equal parts in C.
9. Draw the right-line C G.
10. From C as a Center, at the distance of C G describe the Semicircle C A G B
11. Draw the Diameter A E C F B.
12. I say A F and F B are the extreme proportionals required.
13. . . . *Req. demonstr.* . . . . AF — FB = D. AF . M :: M . FB.

Demonstration.

14. Because by *Construction* in 10°, CAGB is a Semicircle whose Center is C; therefore, (*per defin.* 15. *Elem.* 1.) . . . . . . $CA = CB = CG.$

15. And because by *Constr.* in 8°, ... $CE = CF.$

16. Therefore the Equation in 15° being subtracted from that in 14°, gives (as is evident by the Diagram) . . . . $EA = FB.$

17. It is also evident by the Diagram that . . $AF - EA = EF.$

18. Therefore out of 16° and 17°, (*per Ax.* 6. *Chap.* 2.) . . . . $AF - FB = EF.$

19. But by *Constr.* in 5°, ... $D = EF.$

Conclus. 1. 20. Therefore from 18° and 19°, (*per Ax.* 1.) $AF - FB = D.$

Which was to be dem.

21. Again, because (*per Coroll. Probl.* 9. of this *Chapt.*) . . . . $AF \,.\, FG :: FG \,.\, FB.$

22. And by *Constr.* in 7°, ... $M = FG.$

Conclus. 2. 23. Therefore from 21° and 22°, by exchanging equal right-lines, ... $AF \,.\, M :: M \,.\, FB.$

Which was also to be dem.

Therefore the *Problem* propounded is satisfied; and out of the premisses the following *Theorem* is deducible, for the solving of the same *Probl.* 12. Arithmetically.

Theorem.

24. If half the difference of the extremes of three Proportionals, be added to the side (or square Root) of that Square which is equal to the Square of half the difference of the extremes together with the Square of the mean, the summ shall be the greater extreme: But if the said half difference be subtracted from the said side, the remainder shall be the lesser extreme. Hence,

The Arithmetical solution of Probl. 12.

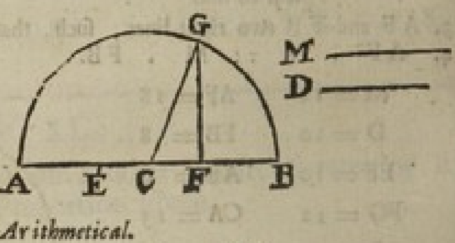
Suppos.

25. AF, FG, FB are $\div\!\div$, *viz.* $AF \,.\, FG :: FG \,.\, FB.$

26. AF $\sqsubset$ FB.

27. $FG = 12.$ . . . . } Given.

28. $EF = 10 = AF - FB.$ } Given.



Req. to find

29. AF and FB in numbers.

Operation Arithmetical.

30. By *Suppos.* in 28°, ... $EF = 10.$

31. Therefore . . . . $\frac{1}{2}EF = CF = 5 = CE.$

32. And consequently, ... $\square CF = 25.$

33. Also from 27°, ... $\square FG = 144.$

34. The summ of the Equations in 32° and 33°, gives (*per prop.* 47. *Elem.* 1.) . . . . . $\square CG = 169.$

35. The square Root of each part of the Equation in 34° gives . . . . $CG = 13 = CA.$

36. Therefore the summ of the Equations in 31° and 35° gives . . . . $AF = 18.$ } Sought.

37. And by subtracting the Equation in 31° from that in 35°, ... $EA = FB = 8.$ } Sought.

The Proof.

38. It is manifest that these are Proportionals, for the Product of the extremes is equal to the Square of the mean, *viz.* . . . . $18 \,.\, 12 :: 12 \,.\, 8.$ / $AF \,.\, FG :: FG \,.\, FB.$

39. Alſo the mean proportional is 12, and the difference of the extremes is 10, as was preſcribed.

Coroll. 1.

40. From the premiſſes it may be inferr'd, That if the Algebraical Reſolution of a Geometrical Problem diſcovers an Equation ſo conſtituted, that the Square of an unknown right-line leſs by a Rectangle contain'd under that unknown line and ſome known right-line, is equal to a known Plane; the ſaid unknown line ſhall be given by the preceding Geometrical *Conſtruction* of *Probl.* 12. But here is to be noted, that if the ſaid known Plane be not a Square, it muſt firſt be reduced to a Square, by *Probl.* 9. of this *Chapt.* or by *prop.* 14. *Elem.* 2.

41. As, in this Equation . . . . $aa - da = mm$;
If we ſuppoſe aa to repreſent an unknown Square whoſe ſide is a, alſo da a Rectangle contain'd under the ſaid unknown ſide a and ſome known right-line repreſented by d, and that the ſaid unknown Square aa leſs by the ſaid Rectangle da is equal to ſome known Square, as mm, whoſe ſide is m; then the ſaid unknown ſide or right-line a may be found out by the Geometrical *Conſtruction* of the preceding *Probl.* 12.

42. For (by *prop.* 14. *Elem.* 6.) the Equation above propos'd in 41° may be reſolved into theſe Proportionals,

$$a - d \ . \ m \ :: \ m \ . \ a;$$

Of which three Proportionals the mean m is known by ſuppoſition, as alſo d the difference of the extremes a and $a - d$, (for a exceeds $a - d$ by d,) therefore by the *Conſtruction* of the foregoing *Probl.* 12. the extremes ſhall be given ſeverally, the greater whereof is the line a ſought.

43. Morover, if in the Equation above propounded, to wit, $aa - da = mm$, we ſuppoſe $m = 12$, and $d = 10$, then the quantity of the line a ſhall be alſo given in number; for by the firſt part of the preceding *Theorem* in 24°,

$$a = \tfrac{1}{2}d + \sqrt{:\tfrac{1}{4}dd + mm:} = 18.$$

Coroll. 2.

44. It alſo follows from the preceding *Conſtruction* of *Probl.* 12. that if by the Algebraical Reſolution of a Geometrical *Problem*, an Equation be found out, ſuch, that the Square of a right-line ſought, together with a Rectangle contain'd under that unknown line and ſome known right-line, is equal to a known Plane; the ſaid unknown line ſhall be given by the Geometrical *Conſtruction* of the ſaid *Probl.* 12. But if (as before hath been ſaid) the ſaid known Plane be not a Square, it muſt firſt be reduced to a Square.

45. As, in this Equation, ... $aa + da = mm$;
If we ſuppoſe aa to repreſent an unknown Square whoſe ſide is a, alſo da a Rectangle contain'd under the ſaid unknown ſide a, and ſome known right-line repreſented by d; and that the ſaid unknown Square aa together with the ſaid Rectangle da is equal to ſome known Square, as mm, whoſe ſide is m; then the ſaid unknown ſide, or right-line a, may be found out by the Geometrical *Conſtruction* of the preceding *Probl.* 12.

46. For the Equation above propounded in 45° may be reſolved into theſe Proportionals, *viz.*

$$a + d \ . \ m \ :: \ m \ . \ a;$$

Of which three Proportionals the mean m is known by ſuppoſition, as alſo d the difference of the extremes $a + d$ and a, (for $a + d$ exceeds a by d,) therefore by the *Conſtruction* of the foregoing *Probl.* 12. the extremes ſhall be given ſeverally, the leſſer whereof is the line a ſought.

47. Laſtly, if in the Equation above propounded in 45°, to wit, $aa + da = mm$, we ſuppoſe $m = 12$ and $d = 10$, then the quantity of the line a ſhall be alſo given in number; for by the latter part of the preceding *Theorem* in 24°,

$$a = \sqrt{:\tfrac{1}{4}dd + mm:} - \tfrac{1}{2}d = 8.$$

Probl. XIII.

Probl. XIII.

The mean of three proportional right-lines being given, as alſo the ſumm of the extremes, to find the extremes. But the given mean muſt not exceed half the given ſumm of the extremes.

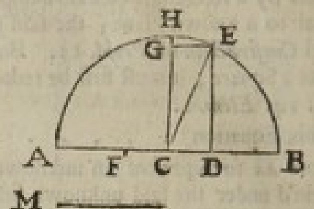
$M = 12$
$AB = 26$
$AC = 13$
$CE = 13$
$CB = 13$
$AD = 18$
$DB = 8$
$DE = 12$
$CD = 5$
$CF = 5$

Suppoſ.

1. $M =$ the mean of three Proportionals is given.
2. $AB =$ the ſumm of the extremes is given. Alſo M not $\sqsubset \frac{1}{2} AB$.

Req. to find

3. AD and DB two right-lines, ſuch, that $AD + DB = AB$. Alſo, that
4. $AD \,.\, M :: M \,.\, DB$.



Conſtruction.

5. Divide AB into two equal parts in C.
6. From C as a Center, with the diſtance CA, or CB, deſcribe the Semicircle $CAHB$.
7. Upon the Center C erect $CH \perp AB$.
8. From CH cut off $CG = M$, (the given mean Proportional,) which is poſſible to be done, for by the Determination annex'd to the *Problem* propounded, M not $\sqsubset CH$, or $\frac{1}{2} AB$.
9. By the point G draw $GE \parallel AB$, which GE will either touch the Semicircle in H, when $M = CH = \frac{1}{2} AB$; or elſe cut the Semicircle, when $M \sqsupset CH$, (or $\frac{1}{2}AB$,) ſuppoſe then in this Example that $M \sqsupset CH$, and conſequently that GE cuts the Circumference, as in E.
10. From the point E let fall $ED \perp AB$, then ſhall AD and DB be the extreme Proportionals required; for firſt their ſumm, in regard ADB is a right-line, (to wit, the Diameter of the Semicircle $CAHB$,) is equal to AB the given ſumm; it remains to prove that as AD is to M, ſo M to DB, therefore
11. . . . *Req. demonſtr.* . . . . $AD \,.\, M :: M \,.\, DB$.

Demonſtration.

12. Becauſe by *Conſtruction* in 5° and 6°, ... $CAEB$ is a Semicircle.
13. And by *Conſtr.* in 10°, ... $DE \perp AB$.
14. Therefore from 12° and 13°, (*per Coroll.* in 12° of *Probl.* 9. *Chap.* 5.) . . . . $AD \,.\, DE :: DE \,.\, DB$.
15. And becauſe by *Conſtr.* in 7°, 9°, 10°, . . DG is $\square$.
16. Therefore (*per prop.* 34. *Elem.* 1.) . . . $DE = CG$.
17. But by *Conſtr.* in 8°, ... $M = CG$.
18. Therefore out of 16° and 17°, (*per Ax.* 1.) $DE = M$.
19. Wherefore from 14° and 18°, (by taking M inſtead of DE,) . . . . $AD \,.\, M :: M \,.\, DB$.

Which was to be done.

20. The reaſon of the Determination annex'd to the *Problem*, to wit, that the line preſcribed for the mean muſt not exceed half the line given for the ſumm of the extremes, will be evident by this that follows. Firſt, if the right-line M be leſs than CH, or $\frac{1}{2} AB$, and at the diſtance of M a line be drawn parallel to the Diameter AB, as the parallel GE, it will neceſſarily cut the Semicircle, as in E, in which caſe the extreme Proportionals, to wit, the ſegments of the Diameter which are made by the falling of the Perpendicular ED, will always be unequal. Secondly, if the line M be equal to CH, or $\frac{1}{2} AB$, and at the diſtance of M or CH a line be drawn parallel to the Diameter AB, ſuch parallel will touch the Semicircle in H, and conſequently HC which is perpendicular to AB will be a mean between AC and CB, in which caſe, the three Proportionals AC, CH and CB are equal to one another, for each of them is the Semidiameter of the Semicircle. Laſtly, if the line M be greater than CH, or $\frac{1}{2} AB$, then 'tis eaſie to perceive, that a right-line drawn parallel to the Diameter AB at the diſtance of ſuch line M cannot poſſibly either touch or cut the Semicircle, but will lye altogether without the ſame, and conſequently ſuch line M cannot be a mean pro-

proportional between any two segments of the Diameter; for a mean proportional line between two extremes is a right-line within a Circle, drawn perpendicularly to the Diameter, and extended only unto (not beyond) the Circumference: And therefore that there may be a possibility of solving this 13th *Problem*, the line given for the mean Proportional must not be longer than half the line given for the summ of the extremes; which is the Determination annex'd to the *Problem.*

The premisses being well understood, it will not be difficult to apprehend how the following *Theorem* is thence deducible, for the solving of *Probl.* 13. Arithmetically.

Theorem.

21. If half the summ of the extremes of three Proportionals be increased with the side (or square Root) of that Square which is equal to the excess whereby the Square of the said half summ exceeds the Square of the mean; the said half summ so increased shall be equal to the greater extreme: But if the said half summ be lessened by the side (or square Root) aforesaid, the remainder shall be the lesser extreme. Hence,

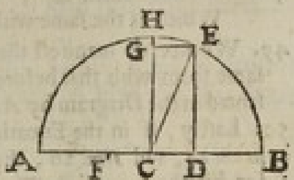
The Arithmetical solution of Probl. 13.

22. *Suppos.* AD, DE, DB are $\div\!\div$, *viz.* AD . DE :: DE . DB.

23. DE = 12. . . . . } Given.
24. AB = 26 = AD + DB. }

Req. to find

25. AD and BD in numbers.



Operation Arithmetical. M

26. By *Suppos.* in 24°, ... AB = 26.
27. Therefore, ... $\frac{1}{2}$AB = CB = 13 = CE = CA.
28. And consequently, ... □CE = 169.
29. And from 23°, ... □DE = 144.
30. Therefore by subtracting the Equation in 29°, from that in 28°, there will remain (*per prop.* 47. *Elem.* 1.) . . . . □CD = 25.
31. And consequently, by extracting the square Root out of each part of the last Equation, . . . CD = 5.
32. Therefore by adding together the Equations in 27° and 31°, ... AD = 18. } Sought.
33. And by subtracting the Equation in 31° from that in 27°, ... DB = 8. }

Coroll.

34. From the premisses it may be inferr'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Equation so constituted, that a Rectangle contain'd under some known right-line and a right-line sought, less by the Square of the same line sought, is equal to a known Plane; then the said right-line sought shall be given by the preceding Geometrical *Construction* of *Probl* 13. But here is to be noted, that if the said known Plane be not a Square, it must first be reduced to a Square, by *Probl.* 9. of this *Chapt.* or by *Prop.* 14. *Elem.* 2.

35. As, if this Equation be proposed, ... $sa - aa = mm$;
We may suppose a in that Equation to represent a right-line unknown, aa the Square of that line, s a right line known, and that sa the Rectangle contain'd under those lines, less by the said unknown Square aa, is equal to some known Square, as mm, whose side is m; then shall the said unknown right line a be given by the preceding *Construction* of *Probl.* 13.

36. For the Equation before propos'd in 35°, may be resolved into these Proportionals, *viz.*

$$s - a \;.\; m \;::\; m \;.\; a;$$

Of which three Proportionals the mean m is known by supposition, as also s the summ of

of the extremes $s - a$ and a; therefore, by the foregoing *Construction* of *Probl.* 13. the extremes shall be given severally, either of which may be taken for the line a sought.

37. For (viewing the Diagram belonging to *Probl.* 13.) let us suppose that . . . . AD . DE :: DE . DB.
38. Then putting . . . . $m =$ DE.
39. Also . . . . $s =$ AB.
40. And . . . . $a =$ AD.
41. It follows that . . . . $s - a =$ DB $=$ AB $-$ AD.
42. And . . . . $a \,.\, m :: m \,.\, s - a$.
43. Wherefore, by comparing the Rectangle of the extremes to the Square of the mean, this Equation is produced, ... $sa - aa = mm$.

Which is the same with the Equation before propos'd in 35°.

44. Again, the same Equation will arise if we put $a =$ DB, the *Suppositions* in 37°, 38° and 39° remaining unalter'd; for,
45. Then it will follow that . . . . $s - a =$ AD.
46. And . . . . $s - a \,.\, m :: m \,.\, a$.
47. That is, ... AD . DE :: DE . DB.
48. Therefore from 46°, ... $sa - aa = mm$.

Which is the same with the Equation before produced in 43°.

49. Whence 'tis manifest that the right line a sought in all Quadratick Equations of the same form with that before propos'd in 35°, may be either of two right lines, represented in the Diagram by AD and DB, for which cause such Equation is called *Ambiguous*.
50. Lastly, if in the Equation above propounded, to wit, $sa - aa = mm$, we suppose $m = 12$, and $s = 26$, then the quantity of the line a shall be also given in number; for by the preceding *Theorem* in 21°,

$$a = \tfrac{1}{2}s + \sqrt{:\overline{\tfrac{1}{4}ss - mm}:} = 18;$$

$$\text{Or,}\quad a = \tfrac{1}{2}s - \sqrt{:\overline{\tfrac{1}{4}ss - mm}:} = 8.$$

Probl. XIV.

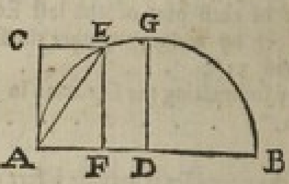
To divide a right-line given into two parts, such, that another right-line may be a mean Proportional between the parts. But the line given for the mean must not exceed half the line given to be divided.

Suppos.

1. AB is a right line given to be cut into two parts.
2. AC is a right line given to be a mean Proportional between the parts.
3. AC not $\sqsubset \frac{1}{2}$ AB.

Req. to find

4. AF and FB such parts of AB that AF $+$ FB $=$ AB; Also, that
5. AF . AC :: AC . FB.



AB $= 26$
AC $= 12$
AD $= 13$
DB $= 13$
AF $= 8$
FE $= 12$
FB $= 18$

Construction.

6. Upon A one of the ends of AB, erect AC $\perp$ AB.
7. Divide AB into two equal parts in D.
8. From D as a Center, with the distance DA, or DB, describe the Semicircle DAEB.
9. By the point C draw CE $\parallel$ AB, so shall CE (by what hath been said in 20° of *Probl.* 13.) either touch or cut the Semicircle DAEB, for by *Suppos.* AC not $\sqsubset \frac{1}{2}$AB, (or DG;) suppose then in this Example that AC $\sqsupset$ DG, or DA, and consequently that CE cuts the Semicircle as in E.
10. From the point E let fall EF $\perp$ AB, then shall AF and FB be the desired parts of AB, between which parts, AC is a mean Proportional, as I shall in the next place demonstrate.
11. . . . *Req. demonstr.* . . . . . . AF . AC :: AC . FB.

Preparat.

12. Draw the right lines AE and EB.

Demon-

Demonstration.

13. Because by *Constr.* in 8°, DAEB is a Semicircle, therefore (*per prop.* 31. *Elem.* 3.) . . . . . } $\angle$AEB is ⅃.
14. And because by *Constr.* in 10°, ... > EF ⊥ AB.
15. Therefore from 13° and 14°, (*per Coroll. prop.* 8. *Elem.* 6.) . . . . } AF . FE :: FE . FB.
16. And because by *Constr.* in 6°, 9° and 10°, . . > CF is □.
17. Therefore, (*per prop.* 34. *Elem.* 1.) . . . > AC = FE.
18. Therefore from 17° and 15°, by taking AC instead of FE, ... } AF . AC :: AC . FB.

Which was to be done.

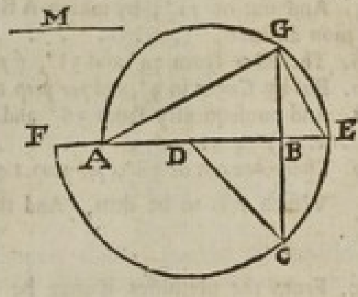
Note. This Problem may be solved Arithmetically in the same manner as the preceding *Probl.* 13.

Probl. XV.

The mean of three Squares in continual proportion being given, as also a Square equal to the difference of the extremes, to find out the extremes.

Or thus;

The Base of a right-angled Triangle being given, as also a mean proportional between the Hypothenusal and Perpendicular, to find the Triangle.



Suppos.

1. AB = the Base of a right-angled Triangle is given.
2. M = a mean proportional between the Hypothenusal and Perpendicular is given.

Req. to find out the Triangle.

3. Making the given Base AB the first of three Proportionals, and the given Mean M the second, find out a third, (*per Probl.* 7. of this *Chapt.*) which suppose to be BC, therefore, } AB : M :: M : BC.
4. Upon B one of the ends of the given Base AB, make BC ⊥ AB, whence $\angle$ABC is ⅃.
5. Divide AB into two equal parts in D.
6. Draw the right-line DC.
7. From D as a Center, at the distance of DC, describe the Semicircle DFCE having FADBE for its Diameter.
8. Upon AE as a Diameter describe the Semicircle AGE.
9. Continue CB to the Circumference in G.
10. Draw the right lines AG and EG.
11. I say ABG is the right-angled Triangle required; now we must shew that it will satisfie the *Problem* propounded. First then, by *Supposition* AB is equal to the given Base, but that the angle ABG is a right-angle, and that the given line M is a mean proportional between the Hypothenusal AG and the Perpendicular BG, the following *Demonstration* will make manifest.
12. . . . *Req. demonstr.* : . . . { $\angle$ABG is ⅃; Also, AG . M :: M . BG.

Demonstration.

13. Because by *Constr.* in the fourth step, . > $\angle$ABC is ⅃,
14. And by *Constr.* in 9°, ... > CBG is a right-line;
15. Therefore, (*per Coroll. Prop.* 13. *Elem.* 1.) > $\angle$ABG is ⅃. Which was to be Dem.
16. Again, because by *Constr.* in 7°, . . > DF = DE = DC,
17. And by *Constr.* in 5°, ... > DA = DB;
18. Therefore by subtracting the Equation in 17° from that in 16°, ... } AF = BE.

19. And by adding A B to each part of the last preceding Equation, ...	. . FB = AE.
20. Again, out of 4° and 7° (*per Coroll. Probl* 9. of this *Chapt.*	FB . BC :: BC . BE.
21. Therefore from 20°, (*per prop.* 17. *Elem* 6.)	□FB, BE = □BC.
22. And from 19°, (*per prop.* 1. *Elem.* 6.) by taking B E as a common altitude, . . .	□FB, BE = □AE, BE.
23. Therefore out of 21° and 22° (*per Ax.* 1. *Chap.* 2.)	. . □BC = □AE, BE.
24. Again, because by *Constr.* in 8°, . . .	AGE is a Semicircle.
25. Therefore, (*per prop.* 31. *Elem.* 3.) . .	$\angle$AGE is ⊥.
26. And because by what hath been proved in 15°,	GB ⊥ BA.
27. Therefore from 25° and 26°, (*per Coroll.* 2. *prop.* 8. *Elem.* 6.)	AE . GE :: GE . BE.
28. Therefore out of 27°. (*per prop.* 17. *Elem.* 6.)	□GE = □AE, BE.
29. But it hath been proved in 23°, that . .	□BC = □AE, BE.
30. Therefore from 28° and 29°, (*per Ax.* 1.)	□BC = □GE.
31. And consequently, ...	BC = GE.
32. Again, out of 25° and 26°, (*per prop.* 8. *El.* 6.)	△ABG and △GBE are like.
33. Therefore from 32°, (*per prop.* 4. *Elem.* 6.)	AB . AG :: BG . GE.
34. And from 33°, (*per prop.* 16. *Elem.* 6.)	□AG, BG = □AB, GE.
35. And out of 31°, by taking A B for a common altitude, ...	□AB, BC = □AB, GE.
36. Therefore from 34° and 35°, (*per Ax.* 1.)	□AB, BC = □AG, BG.
37. But by *Constr* in 3°, and *per prop.* 17. *Elem.* 6.	□AB, BC = □M.
38. And consequently from 36° and 37°, (*per Ax.* 1.)	□AG, BG = □M.
39. Therefore out of 38°, (*per prop.* 14. *Eeml.* 6.)	AG . M :: M . BG.

Which was to be dem. And therefore the *Problem* is satisfied.

Coroll. 1.

40. From the premisses it may be inferr'd, That if the Algebraical Resolution of a Geometrical *Problem* discovers an Analogy consisting of three Planes in continued proportion, such, that the greater extreme is a Square unknown, the mean a known Square, and the lesser extreme the excess whereby that unknown Square exceeds some known Square; then the side of the said unknown Square shall be given by the Geometrical *Construction* of the preceding *Probl.* 15.

As, for example, let there be propos'd the following Analogy, where aa represents a Square unknown, and a its side; also mm and bb two known Squares, whose sides are m and b;

$$aa - bb \;.\; mm :: mm \;.\; aa.$$

41. Then, (because, by *prop.* 22. *Elem.* 6. the sides of proportional Squares are proportionals also,) from the Analogy propos'd this ariseth, *viz.*

$$\sqrt{:\overline{aa - bb}:} \;.\; m :: m \;.\; a.$$

42. Which three Proportionals last exprest being well consider'd, it will be manifest that the greater extreme, to wit, a, may be esteem'd the Hypothenusal of a right-angled Triangle whose Base is b, and Perpendicular $\sqrt{:\overline{aa - bb}:}$ the lesser extreme: Now in that right-angled Triangle the Base b is given by supposition, as also m a right line which is a mean proportional between the said Hypothenusal and Perpendicular, and therefore the Hypothenusal and Perpendicular shall be given severally by the preceding *Construction* of *Probl.* 15. which Hypothenusal shall be the line represented by a in the Analogy propos'd.

43. It is also manifest, that if the Terms of the Analogy propos'd for an Example in 40°, be suppos'd to represent numbers, then by comparing the Rectangle of the the extremes to the Rectangle of the means, this following Biquadratick Equation will thence be produced, *viz.*

$$aaaa - bbaa = mmmm;$$

Where,

where, if we ſuppoſe $bb = 6$; alſo $mm = \sqrt{27}$, and a to ſtand for ſome number unknown, that Biquadratick Equation may be expreſt thus, *viz.*

$$aaaa - 6aa = 27.$$

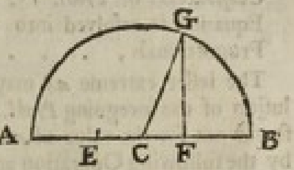
44. In which laſt Equation, the unknown number a may be found out, either by an Arithmetical Operation deducible from the preceding Geometrical *Conſtruction* of *Probl.* 15. or elſe after the ſaid Equation is reſolved into theſe three continual Proportionals,

$$aa - 6 \;.\; \sqrt{27} :: \sqrt{27} \;.\; aa,$$

The greater extreme aa may firſt be made known after the manner of the Arithmetical Solution of the foregoing *Probl.* 12. and then the ſquare Root of that number found out for the value of aa ſhall be the number a ſought. All which will be manifeſt by the following Operation and Diagram.

Suppoſ.

45. AF, FG, FB $\div\!\div$; *viz.* AF . FG :: FG . FB.
46. AF ⊏ FB.
47. □FG = 27. . . . . } given.
48. AF — FB = 6 = EF. } given.



Req. to find out

49. AF the greater extreme, ſignified by aa in the Analogy before expreſt in 44°.

Operation Arithmetical.

50. By *Suppoſition* in 48°, ... EF = 6.
51. Therefore . . . . $\tfrac{1}{2}$EF = CF = 3 = CE.
52. And conſequently, ... □CF = 9.
53. By *Suppoſ.* in 47°, ... □FG = 27.
54. The ſumm of the two laſt Equations gives (*per prop.* 47. *Elem.* 1.) . . . . □CG = 36.
55. The ſquare Root of the laſt Equation gives . . . . CG = 6 = CA.
56. The ſumm of the Equations in 51° and 55°, gives . . . . AF = 9 = aa.
57. Wherefore, ... $\sqrt{9} = 3 = a$.
58. Moreover, out of the premiſſes ariſeth the following *Canon*, for the Arithmetical Reſolution of all Biquadratick Equations falling under the ſame form with that before expreſt in 43°, and here-under repeated, where a repreſents a number ſought, but b and m two numbers given; *viz.*

If . . . . $aaaa - bbaa = mmmm$,

Then . . . . . . $a = \sqrt{:\tfrac{1}{2}bb + \sqrt{\tfrac{1}{4}bbbb + mmmm}:}$

Coroll. 2.

59. From the premiſſes alſo it may be inferr'd, That if the Algebraical Reſolution of a Geometrical *Problem* diſcovers an Analogy conſiſting of three Planes in continual proportion, ſuch, that the leſſer extreme is a Square unknown, the mean a known Square, and the greater extreme is compoſed of the ſaid unknown Square and ſome known Square; then the ſide of the ſaid unknown Square ſhall be given by the Geometrical *Conſtruction* of the foregoing *Probl.* 15. As, for example,

If this Analogy be propos'd, where aa repreſents a Square unknown, and a its ſide; alſo, mm and bb two known Squares, whoſe ſides are m and b, ... $aa + bb \;.\; mm :: mm \;.\; aa.$

60. Then (*per prop.* 22. *Elem.* 6.) theſe alſo ſhall be Proportionals, ... $\sqrt{:aa + bb} \;.\; m :: m \;.\; a.$

61. Which three laſt preceding Proportionals being well examined, it will be manifeſt that the greater extreme, to wit, $\sqrt{:aa + bb:}$ may be eſteem'd the Hypothenuſal of a right-angled Triangle whoſe Baſe is b, and Perpendicular a, (the leſſer extreme.) Now in that right-angled Triangle the Baſe b is given by ſuppoſition, as alſo m a right-line which is a mean Proportional between the ſaid Hypothenuſal and Perpendicular, and therefore the Hypothenuſal and Perpendicular ſhall be given ſeverally by the preceding *Conſtruction* of *Probl.* 15. which Perpendicular ſhall be the line repreſented by a in the Analogy propos'd in 59°.

62. It is alſo manifeſt, that if the Terms of the Analogy in 59° be ſuppoſed to repreſent numbers, then by comparing the Rectangle of the extremes to the Rectangle of the means, this following Biquadratick Equation will thence be produced, *viz.*

$$aaaa + bbaa = mmmm.$$

Where, if we ſuppoſe $bb = 6$, alſo $mm = \sqrt{135}$, and a to repreſent ſome number unknown; that Biquadratick Equation may be expreſt thus, $\Big\}$ $aaaa + 6aa = 135$.

63. In which laſt Equation the unknown number a may be found out either by an Arithmetical Operation, deducible out of the preceding Geometrical *Conſtruction* of *Probl.* 15. or elſe after the ſaid Equation is reſolved into theſe three continual Proportionals, ... $\Big\}$ $aa + 6 \,.\, \sqrt{135} :: \sqrt{135} \,.\, aa.$

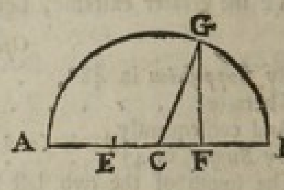
The leſſer extreme aa may firſt be diſcovered after the manner of the Arithmetical Solution of the foregoing *Probl.* 12. of this *Chapt.* and then the ſquare Root of that number found out for the value of aa ſhall be the number a ſought. All which will be manifeſt by the following Operation and Diagram.

Suppoſ.

64. AF, FG, FB, $\div\!\div$ *viz.* AF . FG :: FG . FB.
65. AF $\sqsubset$ FB.
66. $\square$FG $= 135$.
67. AF $-$ FB $= 6 =$ EF.

Req. to find out

68. FB the leſſer extreme, which anſwers to aa in the Analogy before expreſt in 63°.



Operation Arithmetical.

69. By *Suppoſ.* in 67°, ... EF $=$ 6.
70. Therefore . . . . $\frac{1}{2}$EF $=$ CF $=$ $3 =$ CE,
71. And conſequently, ... $\square$CF $=$ 9.
72. By *Suppoſ.* in 66°, ... $\square$FG $= 135$.
73. The ſumm of the two laſt preceding Equations gives (*per prop.* 47. *Elem.* 1.) . . . . . $\square$CG $= 144$.
74. The ſquare Root of the Equation in 73° gives . . . . CG $=$ $12 =$ CB.
75. And by ſubtracting the Equation in 70°, from that in 74°, ... FB $=$ $9 = aa$.
76. Wherefore . . . . $\sqrt{9} =$ $3 = a$.
77. Moreover, out of the premiſſes ariſeth the following *Canon*, for the Arithmetical Reſolution of all Biquadratick Equations falling under the ſame Form with that before expreſt in 62°, and here-under repeated; where a repreſents a number ſought, but b and m two given numbers; *viz.*

If . . . . $aaaa + bbaa = mmmm$,

Then . . . . . . $a = \sqrt{:\sqrt{\tfrac{1}{4}bbbb + mmmm} - \tfrac{1}{2}bb:}$

Coroll. 3.

78. From the premiſſes alſo it may be inferr'd, That if the Algebraical Reſolution of a Geometrical *Problem* diſcovers an Analogy conſiſting of three Planes in continual proportion, ſuch, that the mean is a Square known, and in each of the extremes one and the ſame unknown Square is found Affirmative, (that is, with the Sign $+$ prefixt to it,) together with ſome known Square having either $+$ or $-$ prefixt to it, or when in each of the extremes one and the ſame unknown Square is found negative, (that is, with the ſign $-$ ſet before it,) together with ſome known Square having the ſign $+$ prefixt to it; then in either of thoſe Caſes, the ſide of that unknown Square may be found out Geometrically by the preceding *Conſtruction* of *Probl.* 15. as will be manifeſt by the four following Examples.

Example 1.

79. . . $\begin{cases} \text{Suppose} \;\ldots\; aa + bb \;.\; mm :: mm \;.\; aa + cc; \\ \text{And consequently}, \; \sqrt{:aa + bb:} \;.\; m :: m \;.\; \sqrt{:aa + cc:} \end{cases}$

In

In which Analogies, let aa represent an unknown Square whose side is a, but bb, mm and cc three known Squares, whose sides are b, m and c; suppose also that b is greater than c, and consequently $aa + bb \sqsubset aa + cc$. I say then, that the unknown side a shall be given by the preceding *Construction* of *Probl.* 15. For the difference of the extremes of the three Proportionals in the first Analogy above propos'd is $bb - cc$, which by *Suppos.* is given, therefore the side of a Square equal to that difference, to wit, $\sqrt{:\overline{bb - cc}}:$ may be esteem'd the Base of a right-angled Triangle, by the help of which given Base, and of the right-line m, which by *Supposition* is a given mean Proportional between the Hypothenusal $\sqrt{:\overline{aa + bb}}:$ and the Perpendicular $\sqrt{:\overline{aa + cc}}:$ the Hypothenusal and Perpendicular shall be given severally (by the said *Probl.* 15.) and may be represented by h and p, whose Squares are hh and pp; then by equating hh to the greater extreme $aa + bb$, or pp, to $aa + cc$, the side a will be found equal to $\sqrt{:\overline{hh - bb}}:$ or $\sqrt{:\overline{pp - cc}}:$ which Roots are given, and equal to one another.

Example 2.

80. . . . $\left\{\begin{array}{l} \text{Suppose} \; . \; . \; . \; . \; aa - bb \; . \; mm \; :: \; mm \; . \; aa - cc, \\ \text{And consequently}, \; \sqrt{:\overline{aa - bb}}: \; . \; m \; :: \; m \; . \; \sqrt{:\overline{aa - cc}}: \end{array}\right.$

In which Analogies, if (as before in *Example* 1.) we suppose aa to represent an unknown Square whose side is a; also bb, mm and cc three known Squares, whose sides are b, m and c, and b to be greater than c, whence consequently $aa - cc \sqsubset aa - bb$; then the unknown side a shall be given by the preceding *Construction* of *Probl.* 15. For the difference of the extremes of the three Proportionals in the first Analogy above propos'd is $bb - cc$, which by *Supposition* is given; then the side of a Square equal to that difference, to wit, $\sqrt{:\overline{bb - cc}}:$ may be esteem'd the Base of a right-angled Triangle, by the help of which given Base and of the right-line m, which by *Supposition* is a given mean Proportional between the Hypothenusal $\sqrt{:\overline{aa - cc}}:$ and the Perpendicular $\sqrt{:\overline{aa - bb}}:$ the Hypothenusal and Perpendicular shall be given severally by the said *Probl.* 15. and may be represented by h and p, whose Squares are hh and pp; then by equating hh to $aa - cc$ (the greater extreme,) or pp to $aa - bb$; the unknown side or right-line a will be found equal to $\sqrt{:\overline{hh + cc}}:$ or $\sqrt{:\overline{pp + bb}}:$ which Roots are given, and equal to one another.

Example 3.

81. . . . $\left\{\begin{array}{l} \text{Suppose} \; . \; . \; . \; . \; aa + bb \; . \; mm \; :: \; mm \; . \; aa - cc, \\ \text{And consequently}, \; \sqrt{:\overline{aa + bb}}: \; . \; m \; :: \; m \; . \; \sqrt{:\overline{aa - cc}}: \end{array}\right.$

Example 4.

82. . . . $\left\{\begin{array}{l} \text{Suppose} \; . \; . \; . \; . \; bb - aa \; . \; mm \; :: \; mm \; . \; cc - aa, \\ \text{And consequently}, \; \sqrt{:\overline{bb - aa}}: \; . \; m \; :: \; m \; . \; \sqrt{:\overline{cc - aa}}: \end{array}\right.$

By what hath been said in the Explication of *Examples* 1. and 2. it will not be difficult to conceive how to find out in like manner the unknown Root or right-line represented by a in the third and fourth *Examples*, where b, m and c are suppos'd to be right-lines severally given.

83. *Note.* When more than one known Square or Rectangle is found in any one of the three continual Proportionals mentioned in any of the three preceding *Corollaries* of *Probl.* 15. such known Squares or Rectangles must first of all be converted into a simple Square, *per Probl.* 2. *Chap.* 4.

Probl. XVI.

The mean of three Squares in Continual proportion being given, as also a Square equal to the summ of the extremes, to find out the extremes.

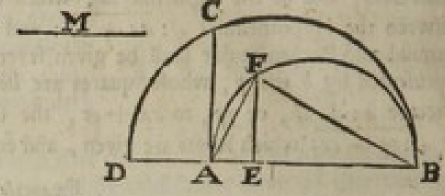
Or

Or thus,

The Hypothenusal of a right-angled Triangle being given, as also a mean Proportional between the Base and Perpendicular, to find out the Triangle. But a right-angle arising out of the Application of the Square of the given Mean to the given Hypothenusal must not exceed half the Hypothenusal.

Suppos.

1. A B = the Hypothenusal of a right-angled Triangle is given.
2. M = a mean proportional between the Base and Perpendicular is given.
3. $\frac{\Box M}{AB}$ not $\sqsubset \frac{1}{2}$ A B.



Req. to find out the Triangle.

Construction.

4. Upon the point A, (one of the ends of the given Hypothenusal AB,) erect AC $\perp$ AB, and make A C = M the given mean Proportional.
5. Let it be made (*per Probl.* 7. of this *Chapt.*) as AB to AC, so AC to a third Proportional, which suppose to be found AD, therefore, } AB . AC :: AC . AD.
That is, (because M = AC,) . . . . } AB . M :: M . AD.
6. Upon A B as a Diameter describe the Semicircle A F B.
7. Upon D B as a Diameter compos'd of A B and A D as one right-line, describe the Semicircle D C B.
8. By *Probl.* 14. of this *Chapt.* divide the given Hypothenusal AB into two such parts, that the line A D may be a mean proportional between the parts; which is possible to be done, for by *Construction* in 5°, A D = $\frac{\Box M}{AB}$, and by *Supposition* in 3°, agreeable to the Determination annex'd to the *Problem*, $\frac{\Box M}{AB}$ or A D not $\sqsubset \frac{1}{2}$ A B; suppose then the line A B to be cut in E, into two such parts A E and E B, that E F is a mean proportional between A E and E B, and that E F is equal to A D, therefore,
AE . EF (or AD,) :: EF (or AD,) . EB.
9. Draw the lines A F and B F, then shall A F B be the Triangle sought. Now we must shew that it will satisfie the *Problem*. First then, by supposition A B is equal to the given Hypothenusal; but that the angle A F B is a right-angle, and that the given right-line M is a mean proportional between A F and B F, (to wit, the Base and Perpendicular,) the following *Demonstration* will make manifest.
10. . . *Req. demonstr.* . . . . } $\angle$AFB is $\lrcorner$; also, AF . M :: M . FB.

Demonstration.

11. Because by *Constr.* in 6°, ... } A F B is a Semicircle.
12. Therefore, (*per prop.* 31. *Elem.* 3.) . . } $\angle$AFB is $\lrcorner$. Which was to be Dem.
13. Again, because by *Constr.* in 8°, . . . } EF = AD.
14. Therefore, by taking in A B as a common altitude, it follows from 13°, (*per prop.* 1. *Elem.* 6.) that . . . . } $\Box$EF, AB = $\Box$AD, AB.
15. And because by *Constr.* in 6° and 8°, . } FE $\perp$ AB.
16. And it hath been proved in 12°, . . . } $\angle$A F B is $\lrcorner$.
17. Therefore out of 15° and 16° (*per prop.* 8. *Elem.* 6.) . . . . } $\triangle$ A E F and $\triangle$ A F B are like.
18. And consequently, (*per prop.* 4. *Elem.* 6.) } EF . AF :: FB . AB.
19. And from 18°, (*per prop.* 17. *Elem.* 6.) } $\Box$EF, AB = $\Box$AF, FB.
20. Therefore from 14° and 19°, (*per Ax.* 1.) } $\Box$AD, AB = $\Box$AF, FB.
21. And because by *Constr.* in 5°, ... } AB . M :: M . AD.
22. And consequently (*per prop.* 17. *Elem.* 6.) } $\Box$A D, A B = $\Box$M.
23. Therefore out of 20° and 22°, (*per Ax.* 1.) } $\Box$AF, FB = $\Box$M.

24. There-

24. Therefore from 23°, (*per prop.* 14. *Elem.* 6.) . . > $AF \,.\, M :: M \,.\, FB.$

Which was to be Dem. And therefore the *Problem* is satisfied.

Coroll. 1.

25. From the premisses it may be inferr'd, That if the Algebraical Resolution of a Geometrical Problem discovers an Analogy consisting of three Planes in Continual proportion, such, that one of the extremes is an unknown Square, the mean a known Square, and the other extreme the excess whereby some known Square exceeds the said unknown Square; then the side of the said unknown Square shall be given by the preceding Geometrical *Construction* of *Probl.* 16.

As, for example, supposing aa to represent an unknown Square, whose side is a; also mm and bb two known Squares, whose sides are m and b, let this Analogy be propos'd, *viz.* } $bb - aa \,.\, mm :: mm \,.\, aa.$

26. Then (*per prop.* 22. *Elem.* 6.) the sides of proportional Squares are Proportionals also, therefore . . . . } $\sqrt{:bb - aa:} \,.\, m :: m \,.\, a.$

27. Which three continual Proportionals last exprest being well observed, it will be manifest that the extremes, to wit, $\sqrt{:bb - aa:}$ and a may be esteem'd the Base and Perpendicular of a right-angled Triangle, whose Hypothenusal is b. Now in that right-angled Triangle, the Hypothenusal b is given by *Supposition*, as also m a right-line, which is a mean Proportional between the Base and Perpendicular; and therefore the Base and Perpendicular shall be given severally by the preceding *Construction* of *Probl.* 16. either of which right lines, *viz.* either the Base or Perpendicular so found out may be taken for the line represented by a in the Analogy propos'd in 25°. For, viewing the Diagram belonging to *Probl.* 16.

28. Suppose . . . . > $\angle AFB = \llcorner$ in $\triangle AFB$.
29. Suppose also . . . . > $FA \,.\, M :: M \,.\, FB.$
30. And consequently (*per prop.* 22. *Elem.* 6.) > $\square FA \,.\, \square M :: \square M \,.\, \square FB.$
31. Then put . . . . > $mm = \square M.$
32. Also, ... > $bb = \square AB.$
33. And . . . . > $aa = \square FB.$
34. Then from 28°, 32° and 33°, it will be manifest (*per prop.* 47. *Elem.* 1.) that . } $bb - aa = \square FA \, (= \square AB - \square FB.)$
35. Also from 30°, 31°, 33°, 34°, . . . > $bb - aa \,.\, mm :: mm \,.\, aa.$
36. Which Analogy is the same with that propos'd in 25°, and it will also be produced by putting $aa = \square FA$, (the Positions in 31° and 32° remaining unalter'd;)

For then it follows that . . . . . > $bb - aa = \square FB \, (= \square AB - \square FA.)$

37. And because by *Suppos.* in 30°, . . . > $\square FB \,.\, \square M :: \square M \,.\, \square FA.$
38. Therefore from the premisses, ... > $bb - aa \,.\, mm :: mm \,.\, aa.$

Thus it appears, that from either of the two right-lines found out by *Probl* 16. one and the same Analogy may be constituted, in which respect 'tis said to be Ambiguous.

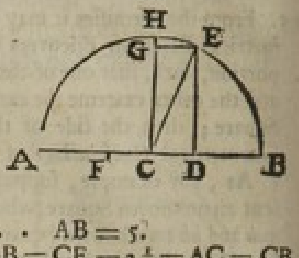
39. It is also manifest, that if the Terms of the Analogy in 25° be suppos'd to represent numbers, then by comparing the Rectangle of the extremes to the Rectangle of the means, this Biquadratick Equation will be produced, *viz.* } $bbaa - aaaa = mmmm.$

Where, if we suppose $bb = 5$, also $mm = 2$, and a to stand for some number unknown, that Biquadratick Equation may be exprest thus, *viz.* . . . . } $5aa - aaaa = 4$

40. In which last Equation the unknown number a may be found out either by an Arithmetical Operation deducible from the preceding Geometrical *Construction* of *Probl.* 16. or else, after the said Equation is resolved into these three Continual Proportionals, . . } $5 - aa \,.\, 2 :: 2 \,.\, aa.$

The two values of aa may be found out after the manner of the Arithmetical Solution of the foregoing *Probl.* 13. of this *Chapt.* and consequently the square Root of each number

number found out for the value of aa being extracted, there will arise two numbers, each of which may be taken for the number a in the Equation in 39°. All which will be evident by the following Operation and Diagram.



Suppos.

41. AD, DE, DB $\div\!\div$, *viz.* AD . DE :: DE . DB.
42. DE $= 2$, whence $\square$DE $= 4$.
43. AD + DB $= 5 =$ AB.

Req. to find

44. AD and DB in numbers, (signified by aa in the Equation in 39°.)

Operation Arithmetical.

45. By *Suppos.* in 43°, ... > . . . AB $= 5$.
46. Therefore, ... > $\frac{1}{2}$AB $=$ CE $= 2\frac{1}{2} =$ AC $=$ CB.
47. And consequently, ... > . . . $\square$CE $= 6\frac{1}{4}$.
48. By *Supposition* in 42°, ... > . . . $\square$DE $= 4$.
49. And by subtracting the Equation in 48° from that in 47°, ... } . . . $\square$CD $= 2\frac{1}{4}$.
50. The square Root of the last Equation gives . > . . . CD $= 1\frac{1}{2}$.
51. The summ of the Equations in 46° and 50°, gives > . . . AD $= 4 = aa$.
52. And consequently, ... > . . . $\sqrt{4} = 2 = a$.
53. Again, by subtracting the Equation in 50, from that in 46°, ... } . . . DB $= 1 = aa$.
54. And consequently, ... > . . . $\sqrt{1} = 1 = a$.

Whence 'tis manifest, that the number signified by a in the Analogy in 40° (or in the Equation in 39°,) may be either 2 or 1.

55. Moreover, out of the premisses ariseth the following *Canon*, for the Arithmetical Resolution of all Biquadratick Equations falling under the same Form with that before propos'd in 39°, and here-under repeated; where a represents a number sought, but h and m two numbers given, and subject to the Determination annex'd to *Probl.* 16. *viz.* $\frac{mm}{h}$ must not be greater than $\frac{1}{2}h$, and consequently mm not greater than $\frac{1}{2}hh$.

If . . . $hhaa - aaaa = mmmm$.

Then, . . $a = \sqrt{:\tfrac{1}{2}hh + \sqrt{\tfrac{1}{4}hhhh - mmmm}:}$

Or, . . $a = \sqrt{:\tfrac{1}{2}hh - \sqrt{\tfrac{1}{4}hhhh - mmmm}:}$

Coroll. 2.

56. From the premisses it may be also inferr'd, That if the Algebraical Resolution of a Geometrical *Problem* discovers an Analogy consisting of three Planes in Continual proportion, such, that the mean is a known Square, and also one of the extremes is the excess of some unknown Square above a known Square, and the other extreme is the excess of some known Square above the said unknown Square; or, when one of the extremes is the excess of a known Square above an unknown Square, and the other is the summ of the same unknown Square and a known Square; in each of those Cases, the side of that unknown Square shall be given by the preceding *Construction* of *Probl.* 16.

Example 1.

57. . . { Suppose . . . . $aa - dd$. $4mm$ :: $4mm$. $cc - aa$;
And consequently, $\sqrt{:aa - dd:}$. $2m$ :: $2m$. $\sqrt{:cc - aa:}$

In which Analogies, if we suppose aa to represent an unknown Square whose side is a; but dd, $4mm$ and cc three known Squares, whose sides are d, $2m$ and c; the unknown side or right line a shall be given by the preceding *Construction* of *Probl.* 16. For the summ of the extremes in the first of those Analogies is $cc - dd$, which is given by *Supposition*; then the side of a Square equal to that summ, to wit, $\sqrt{:cc - dd:}$ may be esteem'd the Hypothenusal of a right-angled Triangle, by the help of which given Hypothenusal and the right-line $2m$, which by *Supposition* is a given mean Proportional between the Base and Perpendicular, (represented by $\sqrt{:aa - dd:}$ and $\sqrt{:cc - aa:}$) the Base and Perpendicular shall be given severally by the said *Probl.* 16. and may be represented by

f and g,

f and g, whose Squares are ff and gg; then by equating ff to $aa - dd$, or gg to $cc - aa$, one value of the right-line a sought shall be given; again, by equating gg to $aa - dd$, or ff to $cc - aa$, another value of the right-line a shall be given: So two lines are found out, either of which may be taken for the line a sought, and therefore the Analogy above-propos'd in *Example* 1. and all others of the same kind are said to be Ambiguous.

Example 2.

58. . . $\left\{\begin{array}{l} \text{Suppose} \;.\;.\;.\;.\; bb - aa \;.\; mm :: mm \;.\; aa + cc, \\ \text{And consequently}, \sqrt{:bb - aa:} \;.\; m :: m \;.\; \sqrt{:aa + cc:} \end{array}\right.$

By what hath been said in the Explication of *Example* 1. 'tis very easie to conceive how to find out in like manner the unknown Root or right-line represented by a in *Example* 2. where b, m and c are suppos'd to be right-lines severally given.

83. *Note.* When more than one known Square or Rectangle is found in any one of the three continual Proportionals mentioned in either of the three preceding *Corollaries* of *Probl.* 16. such known Squares or Rectangles must first of all be converted into a simple Square, *per Probl.* 2. *Chap.* 4.

CHAP. VI.

The manner of finding out such Right-lines and Squares as are represented by Algebraical Fractions, the Quantities constituting those Fractions being given severally.

Probl. I.

Suppos.

1. $a = \frac{bb}{c}$. An Equation propounded.
2. b and c are right lines given severally.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into these Proportionals, *viz.* . . . . $\}$ $c : b :: b \;.\; a$.

In which Analogy, the two first Terms are right lines given by *Supposition*, therefore a third Proportional to them shall be given also, (*per Probl.* 7. *Chap.* 5.) which third Proportional is the line a sought.

Probl. II.

Suppos.

1. $a = \frac{bd}{c}$. An Equation propos'd.
2. b, c and d are right lines given severally.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, *viz.* . . . . $\}$ $c : b :: d : a$.

In which Analogy, the three first Terms are right lines given by *Supposition*, therefore (*per Probl.* 8. *Chap.* 5.) the fourth Proportional, to wit, the line a shall be given also.

Probl. III.

Suppos.

1. $a = \frac{bb + bd}{c - f}$. An Equation propos'd.
2. b, c, d and f are right lines given severally.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, ... $c - f \;.\; b + d \;::\; b \;.\; a.$

In which Analogy the three first Terms are given; for first, c and f being given by *Supposition*, a right line equal to $c - f$ shall be given also, (*per Probl.* 3. *Chap.* 5.) secondly, b and d being given by *Supposition*, a right line equal to $b + d$ shall be given, (*per Probl.* 1. *Chap.* 4.) and lastly, the fourth Proportional line a sought shall be given, *per Probl.* 8. *Chap.* 5.

Probl. IV.

Suppos.

1. $a = \dfrac{bb - dd}{c + f}$. An Equation propos'd.
2. b, c, d and f are right lines given.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, ... $c + f \;.\; b + d \;::\; b - d \;.\; a.$
4. In which Analogy the three first Terms are given, and therefore the fourth proportional line a sought shall be given also by *Probl.* 8. *Chap.* 5.
5. In like manner, if . . . . $a = \dfrac{bb + 2bc + cc}{d + f}$;
6. Then this Analogy will discover the line a sought, *viz.* . . . . $d + f \;.\; b + c \;::\; b + c \;.\; a.$
7. Again, if . . . . $a = \dfrac{bb - 2bc + cc}{d + f}$.
8. Then supposing $b \sqsubset c$, the line a shall be given by this Analogy, *viz.* . . . . $d + f \;.\; b - c \;::\; b - c \;:\; a.$

Probl. V.

Suppos.

1. $a = \dfrac{bc + df}{g}$. An Equation propos'd.
2. b, c, d, f, g are right lines given.

Req. to find the line a.

Construction.

3. First reduce df to a Rectangle, which shall have b for one of its sides, *viz.* let it be made (*per Probl.* 8. *Chap.* 5.) as b to d, so f to a fourth Proportional, which may be called h; therefore . . . $b \;.\; d \;::\; f \;.\; h.$
4. Therefore by comparing the Rectangle of the extremes to the Rectangle of the means, . . . $bh = df.$
5. Then by setting bh in the place of df in the Equation propos'd, it will be converted into this, *viz.* $a = \dfrac{bc + bh}{g}$.
6. Which last Equation may be resolved into this Analogy, ... $g \;.\; b \;::\; c + h \;.\; a.$

But in the Analogy last exprest, the three first Terms are given by *Supposition* and *Construction*, therefore (*per Probl.* 8. *Chap.* 5.) the fourth proportional line a sought shall be given also. The same line a may be found out divers other ways, as the industrious Learner will easily perceive.

Probl. VI.

Suppos.

1. $a = \dfrac{bcdf}{gcd - gbk}$. An Equation propos'd.
2. b, c, d, f, g, h and k are right lines given.

Req. to find the line a.

Constru-

Construction.

3. First, reduce bc to a Rectangle which shall have g for one of its sides, *viz.* let it be made (*per Probl.* 8. *Chap.* 5.) as g to b, so c to a fourth Proportional, call it l, therefore, ... $g \,.\, b :: c \,.\, l$.
4. Therefore, by comparing the Rectangle of the extremes to the Rectangle of the means, . . . $gl = bc$.
5. Therefore from 1° and 4°, by setting gl in the place of bc in the Equation propos'd, it gives . . . $a = \frac{gldf}{gcd - gbk} = \frac{ldf}{cd - bk}$.
6. Again, reduce bk to a Rectangle that shall have d for one of its sides, *viz.* let it be made as d to b, so k to a fourth, which may be called m, therefore $dm = bk$.
7. Therefore from 5° and 6°, by setting dm in the place of bk in the Fraction in the latter part of the 5th step, it gives . . . . $a = \frac{ldf}{cd - dm} = \frac{lf}{c - m}$.
8. Therefore by resolving the latter Fraction in the 7th step into Proportionals, this Analogy ariseth, *viz.* . . . . $c - m \,.\, l :: f \,.\, a$.

In which Analogy the three first Terms are given by *Supposition* and *Construction*, therefore the fourth Proportional, to wit, the line a sought shall be given also, by *Probl.* 8. *Chap.* 5. of this Book.

Probl. VII.

Suppos.

1. $a = \frac{dbb}{cc}$. An Equation propos'd.
2. b, c, d are right lines given.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, *viz.* . . . . $cc \,.\, bb :: d \,.\, a$.

In which Analogy, the three first Terms are given by *Supposition*, and qualified according to the tenour of *Probl.* 10. *Chap.* 5. therefore the fourth Term, that is, the line a sought shall be given also by that *Problem*.

Probl. VIII.

Suppos.

1. $a = \frac{bcd}{ff}$. An Equation propos'd.
2. b, c, d, f, are right lines given.

Req. to find the line a.

Construction.

3. First, the Equation propos'd may be resolved into this Analogy, ... $ff \,.\, bc :: d \,.\, a$.
4. Then let ff be reduced to a Rectangle that shall have b for one of its sides; *viz.* let it be made (*per probl.* 7. *Chap.* 5.) as b to f, so f to a third Proportional, which may be called g, therefore $bg = ff$.
5. Then by taking bg instead of ff for the first Term of the Analogy in the third step, that Analogy will be converted into this, ... $bg \,.\, bc :: d \,.\, a$.
6. Whence, by casting away the common altitude b, this Analogy ariseth, ... $g \,.\, c :: d \,.\, a$.

In which last Analogy the three first Terms are right lines given by *Supposition* and *Construction*, therefore the fourth proportional line a shall be given also.

Probl. IX.

Suppos.

1. $a = \frac{2bbc}{bb - cc}$. An Equation propos'd.
2. b and c are right lines given.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, ... $bb - cc \,.\, bb :: 2c \,.\, a.$
4. Then (by *Probl.* 4. *Chap.* 5.) find a Square equal to $bb - cc$, which Square may be called dd; this being set in the place of $bb - cc$, (the first Term of the preceding Analogy,) will give these Proportionals, *viz.* $dd : bb :: 2c \,.\, a.$
5. Reduce dd to a Rectangle that shall have b for one of its sides, *viz.* let it be made as b to d, so d to a third Proportional, which may be called f, therefore . $bf = dd.$
6. Then by taking bf instead of dd, the Analogy in the 4th step will be converted into this, ... $bf \,.\, bb :: 2c \,.\, a.$
7. Whence, by rejecting the common altitude b, this Analogy ariseth, ... $f \,.\, b :: 2c \,.\, a.$

In which last Analogy the three first Terms are right lines given by *Supposition* and *Construction*; therefore the fourth Proportional line a shall be given also, *per Probl.* 8. *Ch.* 5.

Probl. X.

Suppos.

1. $a = \frac{bbcc - ddff}{dff - bcc}$. An Equation propos'd.
2. b, c, d, f, are right lines given.

Req. to find the line a.

Construction.

3. Let it be made (*per Probl.* 8. *Chap.* 5.) as c to f, so d to a fourth proportional line, call it g, therefore, $cg = df.$
4. Then setting cg in the place of df in the Equation propos'd, and expunging c out of the Numerator and Denominator, it gives . . . . $a = \frac{bbc - gdf}{gf - bc}.$
5. Again, let it be made as c to f, so g to a fourth proportional line, call it k, therefore, ... $ck = gf.$
6. Then setting ck in the place of gf in the Fraction in the fourth step, and casting c out of the Numerator and Denominator, there ariseth . . . . $a = \frac{bb - kd}{k - b}.$
7. Again, let it be made as k to b, so b to a third proportional line, call it m, therefore, ... $km = bb.$
8. Then setting km in the place of bb in the Numerator of the Fraction in the 6th step, there will thence arise . . $a = \frac{km - kd}{k - b}.$
9. Again, let it be made as $k - b$ to $m - d$, so k to a fourth Proportional, which shall be the line a sought, *viz.* . $k - b \,.\, m - d :: k \,.\, a.$
10. From the preceding *Construction* 'tis evident that the desired line a may be found out by these four following Rules of Three, *viz.*

$$\left\{\begin{array}{lccccccc} 1. & c & . & f & :: & d & . & g. \\ 2. & c & . & f & :: & g & . & k. \\ 3. & k & . & b & :: & b & . & m. \\ 4. & k - b & . & m - d & :: & k & . & a. \end{array}\right.$$

11. But the Numerator and Denominator of the Algebraical Fraction in the Equation propos'd in this *Problem* do manifestly shew, that the lines b, c, d, f must be given with this Caution or Determination, *viz.*

$$f \sqsupset \frac{bc}{d}; \text{ but } f \sqsubset \sqrt{\frac{bcc}{d}}.$$

That

That is to say, if it be made as d to b, so c to a fourth Proportional, the line f must be greater than that fourth Proportional; but if (by *Probl.* 11. *Chap.* 5.) it be made as d to b, so cc to another Square, then the line f must be greater than the side of that latter Square.

Now if the line f be given within those limits, then k will be greater than b, and m greater than d, as the last Analogy in the 10th step requires. The same limits of f may be easily inferr'd also from the four Analogies in the 10th step.

An Example in Numbers.

If $\left\{\begin{array}{l} b=40, \\ c=24, \\ d=16, \\ f=48; \end{array}\right\}$ then by the Analogies in the 10th step you will find $\left\{\begin{array}{l} g=32, \\ k=64, \\ m=25, \\ a=24. \end{array}\right.$

Thence it follows that $a = 24 = \dfrac{bbcc - ddff}{dff - bcc}$, the Equation propos'd.

Probl. XI.

Suppos.

1. $aa = \dfrac{bbd}{c}$. An Equation propos'd.
2. b, c, d are right lines given.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, ... $c \ . \ d \ :: \ bb \ . \ aa.$

In which Analogy, the three first Terms are given, and qualified according to the tenour of *Probl.* 11. *Chap.* 5. therefore the line a shall be given also by that *Problem*.

Probl. XII.

Suppos.

1. $aa = \dfrac{bdf}{c}$. An Equation propos'd.
2. b, c, d and f are right lines given.

Req. to find the line a.

3. The Fraction $\dfrac{bdf}{c}$ signifies a Rectangle contain'd under a right line equal to $\dfrac{bd}{c}$ and the line f, therefore first, according to the Construction of *Probl.* 2. of this Chapter, find a right line equal to $\dfrac{bd}{c}$, which line may be called g, therefore,

$$aa = fg = \frac{bdf}{c}.$$

4. Then by *Probl.* 9. *Chap.* 5. reduce the Rectangle fg to a Square, which may be called mm, whose side is m, therefore,

$aa = mm \ (= fg.)$ And consequently, $m = a$ the line sought

Probl. XIII.

Suppos.

1. $aa = \dfrac{bbcc}{dd}$. An Equation propos'd.
2. b, c and d are right lines given.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, ... $dd \ . \ bb \ :: \ cc \ . \ aa.$
4. But the sides of proportional Squares are also Proportionals, therefore from 3°, ... $d \ . \ b \ :: \ c \ . \ a.$

In which last Analogy the three first Terms are right lines given, therefore the fourth proportional line a sought shall be given also.

Probl. XIV.

Suppos. *Probl.* XIV.

1. $aa = \frac{2bcdf}{gg}$. An Equation propos'd.
2. b, c, d, f, g are right lines given.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, ...	$gg \cdot 2bc :: df \cdot aa$.
4. By *Probl.* 9. *Chap.* 5. reduce $2bc$ to a Square, which may be called hh; reduce likewise df to a Square, which you may call kk, then the preceding Analogy will be converted into this, ...	$gg \cdot hh :: kk \cdot aa$.
5. But the sides of proportional Squares are also Proportionals, therefore, from 4°, ...	$g \cdot h :: k \cdot a$.

In which last Analogy the three first Terms are right lines given, and therefore the fourth proportional line a sought shall be given also.

Many other ways might be shewn to construct (or effect) most of the preceding *Problems* of this Chapter; but for brevity sake, I leave them to be found out by the industrious Learner, who by the help of those before deliver'd will also easily perceive how to solve other *Problems* of like nature: And now having explain'd all such things as are materially necessary by way of Preparation to the Resolution and Composition of *Plane Problems*, I shall proceed to *Examples*, which I have divided into four *Classes* or Forms, contain'd in the four following Chapters.

CHAP. VII.

The first Classis *of Examples of the Resolution and Composition of Plane Problems, to wit, such whose Construction may be perform'd by drawing only Right and Circular lines.*

IN which Examples, the Resolution ends either in an Analogy whose three first Terms are right lines known, and the fourth gives the right line sought; or else it ends in a simple Equation between the right line sought, and one or more right lines known. What is meant by Mathematical *Resolution* and *Composition*, I have hinted by Definitions in the beginning of *Chap.* 1. *Book* I. of this Treatise, and now I come to expound and illustrate the same by Examples, after I have recommended a few things by way of Caution and Direction to Learners.

First, Let the Analyst take care to understand the import and meaning of a Problem propounded, lest by too much hast he lose his labour, or be too forward in censuring the Proposer, when the fault is in himself; for many undertake to be Correctors of others, when they themselves have indeed more need of correction.

Secondly, Forasmuch as the most part of Problems propos'd in Geometrical Figures have need of Preparation, let the Analyst endeavour, before he begins the Algebraical Resolution, to find out as much as he can by the Synthetical Method, which proceeds by a Series of Consequences deduced altogether from known Quantities; and sometimes it will be convenient to premise one or more preparatory Propositions to render the Resolution of a Problem propos'd, the more simple and intelligible.

Thirdly, When the Resolution of a Geometrical Problem is begun, the like care must be taken to keep every step thereof in the simplest Terms, for avoiding Equations of higher Powers than the nature of the Problem requires, especially such as exceed Geometrical Dimensions; for example, in the Resolution of a Plane Problem, no Term of any Analogy or Equation ought to exceed two Dimensions, *viz.* every Term must be either a right Line or a Plane, for 'tis improper to introduce Solids in the Resolution or Composition of a Plane Problem.

Fourthly,

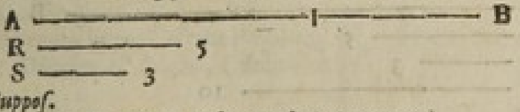
Fourthly, The Scope or aim of the Analyſt in ſolving a Problem muſt be, firſt, to find out a Canon to direct how the Conſtruction of the Problem may be effected by the Quantities given, and then the Conſtruction being finiſh'd to form a Demonſtration Synthetically, that may clearly prove the Problem to be fully ſatisfied. But although a Canon rightly found out by the Algebraick Art bids that only to be done which is poſſible, yet oftentimes in the Conſtruction even of a Plane Problem, ſuch objections will ſtart up againſt the poſſibility of the Conſtruction, as cannot be ſolved by any thing apparent either in the Canon, or in the propoſition of the Problem: As, if a plain Triangle be to be made of three right lines, whereof one is rightly found out by *Conſtruction* according to the direction of the Canon, and the other two are alſo diſcovered by the help of Quantities given and found out, yet before a Triangle can be made of thoſe three right lines, it muſt be proved, that every two of them being joyned together as one right line, are longer than the third; which Proof may happen to be a more difficult work than the invention of the Canon: And therefore when any doubt ariſeth concerning the poſſibility of any particular Conſtruction, and it doth not clearly appear whether ſuch Conſtruction can be done or not, an induſtrious Enquiry muſt be made to diſcover what is abſolutely neceſſary to be given or granted to make that poſſible to be done, which the Problem requires, or the Canon bids to be done: Upon which Search, oftentimes one or more Determinations or Cautions will be found neceſſary to limit the Quantities given in the Problem, that its Conſtruction may meet with no Impediment. Examples of *Determinations* will appear in divers Problems in this and the following Chapters.

Fifthly, After all neceſſary Determinations are premis'd, and the Conſtruction of a Problem is finiſh'd, it remains to demonſtrate that the Quantity or Quantities found out by the *Conſtruction* will ſatisfie the Problem: But the Demonſtration of the Solution of a Plane Problem, if its Conſtruction be Algebraically found out in ſuch manner that no Term of any Analogy or Equation in the Reſolution exceeds Geometrical dimenſions, may be formed by a repetition of the ſteps of the Reſolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Reſolution; and the Demonſtration of a Theorem may be formed by the ſteps of the Algebraick Reſolution in a direct order, that is, by proceeding forward from the beginning to the end of the Reſolution. All which will be copiouſly illuſtrated by the *Reſolutions* and *Compoſitions* of Problems in this and the following Chapters.

Sixthly and laſtly, I deſire the Reader to take notice that in the *Reſolution* of a Problem, I uſe the ſmall *Italian* letters, *a*, *b*, *c*, *d*, &c. aſſuming always ſome Vowel, as *a*, or *e*, &c. to repreſent a line ſought; and Conſonants, as *b*, *c*, *d*, &c. to ſignifie lines given or known: But in the *Compoſition* of a Problem, that is, in its Conſtruction and Demonſtration, I uſe the *Roman* Capital letters, A, B, C, D, &c. to expreſs lines known, that the *Reſolution* and *Compoſition* may be compared to one another without Confuſion.

Problem I.

To divide a given right line into two parts which ſhall be in a given Reaſon, that is, one part to the other as two right lines given.



Suppoſ.

1. b = AB a right line given to be cut into two parts.

2. $\left.\begin{matrix} r = \mathrm{R} \\ s = \mathrm{S} \end{matrix}\right\}$ the Terms of the given Reaſon of the parts ſought.

Req. to find

3. AC and CB ſuch parts of AB, that AC + CB = AB; alſo,
4. AC . CB :: R . S.

Reſolution.

5. For one of the parts ſought put . . . . . . a.
6. Therefore from 1° and 5° the other part is . . . $b - a$.
7. And according to the tenour of the Problem propounded, ... $r . s :: a . b - a$.

8. There-

8. Therefore by *Composition of Reason converse*, (defin'd in *Sect.* 4. *Chap.* 3.) . . . . $r+s \;.\; r :: b \;.\; a.$

Which last Analogy gives this

CANON.

9. As the summ of the Terms of the given Reason is to the first Term; (that is, which of the two you please;) so is the line given to be divided, to one of the parts desired, which part subtracted from the line given to be divided leaves the other part.

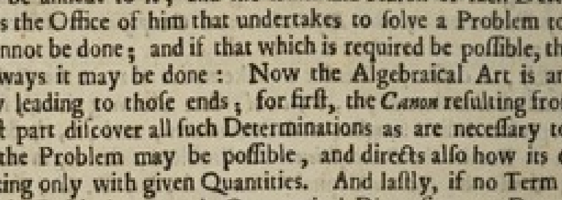
10. *Note.* Although the Analogy in the seventh step may be converted into an Equation, (by comparing the Rectangle of the extremes to the Rectangle of the means,) from which, after due Reduction, the Analogy in the eighth step will arise, yet in Geometrical Demonstrations, which require a Contemplation upon Schemes or Figures, an Analogy in right lines is more simple, and easier to be understood than an Equation between Planes, or Solids; and therefore 'tis more usual with Geometricians in their Argumentations, to proceed as much as is possible from one Analogy to another, by Composition, Division, and other ways of arguing about Proportions, (defin'd in *Chap.* 3. of this *Book*,) that at length an Analogy may arise, when there is a possibility, wherein the three first Terms are given to find a fourth Proportional, which gives the Quantity sought; but there will be very often a necessity of converting an Analogy into an Equation, when known Quantities cannot be otherwise separated from unknown, as will hereafter appear by variety of Examples.

Concerning the Composition of a Geometrical Problem.

11. The *Composition* of a *Problem* consists of two parts, to wit, *Construction*, (or *Delineation*,) and *Demonstration*; the former finds out that which is required to be done or found out, and the latter proves that that which is done or found out will satisfie the *Problem* propounded.

But before the *Construction* be begun, if the Problem be not universal, such Determinations (or Cautions) as are needful to limit the given Quantities, that the Problem may be possible must be annext to it, and the truth and reason of such Determinations made manifest; for 'tis the Office of him that undertakes to solve a Problem to determine what can, and what cannot be done; and if that which is required be possible, then to shew how, and how many ways it may be done: Now the Algebraical Art is an excellent Guide to shew the way leading to those ends; for first, the *Canon* resulting from the Resolution doth for the most part discover all such Determinations as are necessary to limit the given Quantities that the Problem may be possible, and directs also how its *Construction* may be made by working only with given Quantities. And lastly, if no Term of any Analogy or Equation in the Resolution exceeds Geometrical Dimensions, a Demonstration of the Solution of the Problem may be form'd out of the steps of the Resolution in a retrograde order, that is, by returning backwards from the end of the Resolution to its beginning. But these things will best appear by Examples, and therefore I shall proceed to

The Composition of Probl. 1.



Suppos.

12. A B is a right line given to be cut into two parts.

13. R and S are the Terms of the given Reason of the parts sought.

Req. to find

14. AC and CB such parts of AB, that $AC + CB = AB$. Also,

15. AC . CB :: R . S.

Construction.

16. Let it be made (*per Probl.* 8. *Chap.* 5.) as R+S to R, so AB to a fourth Proportional line, which may be called L, therefore

R+S . R :: AB . L.

17. From AB cut off $AC = L$, which is possible to be done if AB be greater than L,

but

but R + S the first Term of the last preceding Analogy is evidently greater than R the second Term, therefore (*per Schol Prop.* 14. *Elem.* 5.) the third Term AB shall be greater than the fourth L; and consequently AC = L may be cut off from AB. That done, the given line AB is divided in the point C into two parts AC, CB, which will satisfie the Problem For first, AC + CB = AB, and that AC is to CB as R to S, I shall demonstrate by a retrograde repetition of the steps of the Resolution in manner following.

18. . . . *Req. demonstr.* . . . . R . S :: AC . CB.

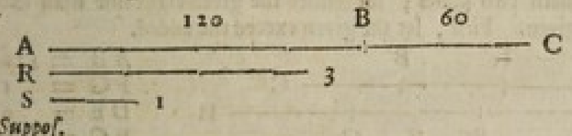
Demonstration

19. Because by *Constr.* in 16°, ... > R + S . R :: AB . L.
20. And by *Constr.* in 17°, ... > AC = L.
21. Therefore from 19°, by taking AC instead of L, > R + S . R :: AB . AC.
That is, in 8°, (the last step of the Resolution,) > $r + s$. r :: b . a.
22. Therefore from the Analogy in 21°, by *Division of Reason converse*, (defin'd in *Sect.* 8. *Ch.* 3.) } . R . S :: AC . CB. *Conclus.*
That is, in 7°, ... > . r . s :: a . $b - a$.
Which was to be done.

Note. In forming a *Demonstration* by a repetition of the steps of the Resolution in a backward order, it must be observed as a perpetual Rule, That when in the Resolution you pass forward from one step to another by *Composition of Reason*, in the Demonstration you are to return backward by *Division of Reason*; and when you pass by *Division of Reason* in the Resolution, you are to return by *Composition of Reason* in the Demonstration: also, *Addition* in the one, answers to *Subtraction* in the other. All which will be evident in the following *Problems*.

Probl. II.

To a given right line to add another right line, that the given with the added may have a given Reason to the line added. But the first Term of the Reason must be greater than the latter.



Suppos.

1. b = AB a right line given to be increased.
2. $\left\{\begin{matrix} r = R \\ s = S \end{matrix}\right\}$ the Terms of the given Reason.
3. R ⊏ S.

Req. to find

4. BC a right line, such, that AB + BC . BC :: R . S.

Resolution.

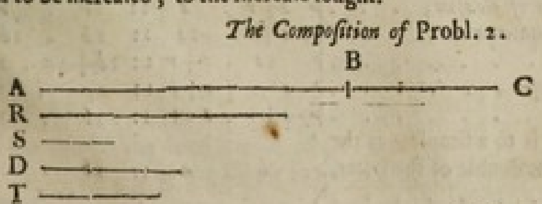
5. For the line sought put . . . . > a.
6. Which added to the given line b makes . . > $b + a$.
7. Then according to the import of the Problem, > r . s :: $b + a$. a.
8. Therefore by *Division of Reason*, (defin'd in *Sect.* 7. *Chap.* 3.) . . . . } $r - s$. s :: b . a.

Hence this

Canon.

9. As the difference of the Terms of the given Reason is to the lesser Term, so is the line given to be increased, to the increase sought.

The Composition of Probl. 2.



AB = 120
R = 3
S = 1
D = 2
T = 60
BC = 60

Suppos.

Suppos.

10. A B is a right line given.
11. R and S are the Terms of a given Reason.
12. $R \sqsubset S$.
13. $D = R - S$.

Req. to find

14. B C a right line, such, that . . . $AB + BC \,.\, BC :: R \,.\, S$.

Construction.

15. Let it be made (*per Probl.* 8. *Chap.* 5.) as D (or $R - S$) to S, so A B to a fourth proportional line, call it T, therefore, . . . $D \,.\, S :: AB \,.\, T$.
16. Let A B be continued to C, so, that . . $BC = T$.

Now the line BC (or T) being found out by the help of the given lines, A B, R and S, according to the direction of the *Canon*, we must shew that it will satisfie the *Probl.* therefore,

17. . . . *Req. demonstr.* . . . . . $R \,.\, S :: AC \,.\, (AB + BC.)\ BC$.

Demonstration.

18. Because by *Constr.* in 15° and 16°, . . $R - S \,.\, S :: AB \,.\, BC$.
19. Therefore by *Compos. of Reason*, . . . $R \,.\, S :: AC \,.\, BC$.

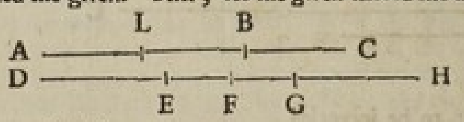
Which was to be done.

Note. In passing from the first step of this Demonstration, (which is the last step in the Resolution,) to the second; the Argumentation is made by *Composition of Reason*, because in passing to the last step of the Resolution from the last but one, it was argued by *Division of Reason*; agreeable to the *Note* at the end of the preceding *Probl.* 1.

Probl. III.

To a given right line to add another right line, that the Difference of the given and added may have a given Reason to their Summ. But the first Term of the Reason must be less than the latter Term.

This Problem hath two Cases; for either the given right line shall exceed the added, or the added the given. First, let the given exceed the added.



$AB = 120$
$FG = 1$
$DF = 3$
$BC = 60$

Suppos.

1. $b = AB$ a right line given.
2. $\left.\begin{matrix} r = FG \\ s = DF \end{matrix}\right\}$ the Terms of the given Reason.
3. $r \sqsupset s$.

Req. to find

4. B C a right line, such, that $AB - BC \,.\, AB + BC :: FG \,.\, DF$.

Resolution.

5. For the line sought to be added put . . . a.
6. Therefore the excess of the given line b above the line sought shall be . . . . $b - a$.
7. And the summ of the given line and the line sought shall be . . . . $b + a$.
8. Therefore, according to the tenour of the Problem propounded, this Analogy ariseth, . . $r \,.\, s :: b - a \,.\, b + a$.
9. Therefore by *Composition of Reason*, . . . $s + r \,.\, s :: 2b \,.\, b + a$.
10. And by doubling the Consequents, . . . $s + r \,.\, 2s :: 2b \,.\, 2b + 2a$.
11. And inversly, ... $2s \,.\, s + r :: 2b + 2a \,.\, 2b$.
12. And by *Division of Reason*, ... $s - r \,.\, s + r :: 2a \,.\, 2b$.
13. And inversly, ... $s + r \,.\, s - r :: 2b \,.\, 2a$.
14. But a simple quantity is to a simple, as the double of the former to the double of the latter, therefore . . . . $b \,.\, a :: 2b \,.\, 2a$.

15. And

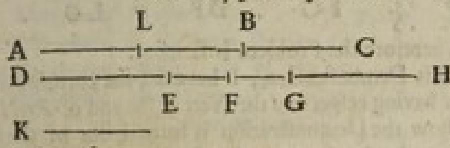
15. And out of 13° and 14°, (*per prop.* 11. *Elem.* 5.) > $s + r \,.\, s - r :: b \,.\, a$.

Hence this *CANON.*

16. As the summ of the Terms of the given Reason is to their difference, so is the given line to the line sought. Therefore the line required to be added to the given line, is given also.

17. *Note.* The line sought may easily be discovered by the Analogy in the ninth step, where the three first Terms being known, the fourth is known by Consequence; and since that fourth Term is evidently compos'd of the given line and the line sought, the given line subtracted from that known fourth Proportional shall necessarily give the line sought; whence 'tis manifest, that the Argumentation continued from the ninth step to the end of the Resolution is not of necessity, but only to shew how the line sought may be purely the fourth Proportional of an Analogy whose three first Terms are known, and consequently the line sought is known also: Which way of arguing by Analogies is more proper, (when it may be used,) than that by Equations, as hath before been hinted in *Sect.* 10. *Probl.* 1. of this Chapter.

The Composition of Case 1. Probl. 3.



$AB = 120$
$FG = 1$
$DF = 3$
$K = 60$
$BC = 60$

Suppos.

18. AB is a right line given.
19. FG and DF are the Terms of a given Reason.
20. FG ⊐ DF.

Req. to find

21. BC a right line, such, that . . . $AB - BC \,.\, AB + BC :: FG \,.\, DF$.

Construction.

22. Let it be made as DF+FG (that is, DG,) to DF — FG, so AB to a fourth proportional line, call it K, therefore, ... } $DF + FG \,.\, DF - FG :: AB \,.\, K$.
23. Let AB be continued to C, so, that . > $BC = K$.
24. Now the line BC, or K being found out by the help of the given lines AB, DF, FG, (according to the direction of the *Canon,*) we must shew that it will satisfie the Problem, *viz.* that the difference of AB and BC is to their summ, as FG to DF; but this Analogy, (after I have premis'd a few things to contract the *Demonstration,*) I shall make manifest by a repetition of the steps of the Resolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Resolution.

Prepar.

25. From AB cut off AL = BC = K, which is possible to be done, for the first Term of the Analogy in 22° is evidently greater than the second, and therefore (*per Schol. Prop.* 14. *Elem.* 5.) the third Term AB shall be greater than the fourth K, or BC; suppose therefore } $AL = BC$.
26. Thence it follows that . . . . . . > $LB = AB - BC$.
27. Let DF be continued to H, so, that . > $DF = FH$.
28. From DF cut off FE=FG, which is possible to be done, for by *Supposition* in 20°, DF ⊏ FG; suppose therefore . . . . } $FE = FG$.
29. Therefore by subtracting the last Equation from that in 27°, ... } $DE = GH = DF - FG$.
30. . . . . *Req. demonstr.* . . $FG \,.\, DF :: AB - BC \,.\, AB + BC :: LB \,.\, AC$.

Demonstration.

31. By *Constr.* in 22° and 23°, ... > $DF + FG \,.\, DF - FG :: AB \,.\, BC$.
That is in 15,° (the last step of the Resolution,) > $s + r \,.\, s - r :: b \,.\, a$.
32. But there is the same Reason of the double to the double, as of the simple to the simple, therefore, ... } $2AB \,.\, 2BC :: AB \,.\, BC$.

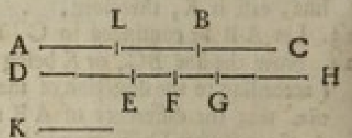
That

That is, in the 14th step, . . .	. 2*b* . 2*a* :: *b* . *a*.
33. Therefore out of 31° and 32°, (*per Prop.* 11. *Elem.* 5.) . .	DF+FG . DF−FG :: 2AB . 2BC.
That is, in the 13th step, . . .	*s*+*r* . *s*−*r* :: 2*b* . 2*a*.
34. Therefore inversly, ...	DF−FG . DF+FG :: 2BC . 2AB.
That is, in the 12th step, . . .	. *s*−*r* . *s*+*r* :: 2*a* . 2*b*.
35. Therefore by *Compos. of Reason*,	. 2DF . DF+FG :: 2AB+2BC . 2AB.
That is, in the 11th step, . . .	. 2*s* . *s*+*r* :: 2*b*+2*a* . 2*b*.
36. Therefore inversly, ...	DF+FG . 2DF :: 2AB . 2AB+2BC.
That is, in the 10th step, . . .	*s*+*r* . 2*s* :: 2*b* . 2*b*+2*a*.
27. Therefore by halving the Consequents in the 36th step, . . .	DF+FG . DF :: 2AB . AB+BC.
That is, in the 9th step, . . .	*s*+*r* . *s* :: 2*b* . *b*+*a*.
38. Therefore by *Division of Reason*,	. FG . DF :: AB−BC . AB+BC.
That is, in the 8th step, . . .	. . *r* . *s* :: *b*−*a* . *b*+*a*.
39. Therefore from 38° and 36°, (*per Ax.* 6. *Chap.* 2.)	FG . DF :: LB . AC.

Which was to be Dem. Therefore the Problem is satisfied.

30. *Note.* Under every step of this Demonstration, I have set the correspondent step of the Resolution, that the Learner having respect to the *Note* at the end of *Probl.* 1. of this Chapter, may clearly perceive how the Demonstration is form'd out of the steps of the Resolution in a Retrograde order, that is, by returning backwards from the end to the beginning of the Resolution; for the first step in the Demonstration answers to the last in the Resolution, the second in the Demonstration, to the last but one in the Resolution, and so backwards in the Resolution, until the Analogy that was first assumed in the Resolution be positively and infallibly proved to be true. But after the Demonstration is in that manner discovered, the Algebraical steps must be omitted: So when the foregoing Demonstration beginning at the 31th step, is freed from the Analogies exprest by the small *Italian* letters belonging to the Resolution, and contracted by the help of the preparatory Equations in the 26th and 29th steps, respect also being had to the Diagram, there will arise this following

Demonstration.



41. Because by *Constr.* in 22° and 23°,	DG . DE :: AB . BC.
42. And by *prop.* 15. *Elem.* 1. . .	2AB . 2BC :: AB . BC.
43. Therefore, *per prop.* 11. *Elem.* 5.	DG . DE :: 2AB . 2BC.
44. And inversly, ...	DE . DG :: 2BC . 2AB.
45. And by *Composition*, ...	2DF . DG :: 2AC . 2AB.
46. And inversly, ...	DG . 2DF :: 2AB . 2AC.
47. And by halving the Consequents,	DG . DF :: 2AB . AC.
48. Wherefore by *Division of Reason*,	FG . DF :: LB . AC.

Which in 30° was *Req. dem.*

49. Thus you have seen the first Case of *Probl.* 3. effected and demonstrated Synthetically, or by way of Composition, which argues altogether with known quantities; but the substance of the Composition, to wit, the *Construction* and *Demonstration*, was found out Analytically, or by way of Resolution, which from an Assumption of the quantity sought as if it were known or granted, together with the help of one or more known quantities, proceeds by Consequences, until in Conclusion the quantity so assumed or feigned to be known, is found equal to some quantity certainly known, and is therefore known also.

But it may be objected, that *Demonstrations* formed by the steps of Algebraical Resolution are for the most part rude and prolix; this I grant, but experience shews, that a *Demonstration* so found out may oftentimes be easily contracted, or, at least, give light to find out others more succinct and elegant. And since my purpose is, to shew the Learner a general and ready way of forming the *Demonstrations* of such *Theorems*, and *Solutions* of *Problems* as he finds out by ALGEBRA, when no Term of any Analogy or Equation

Equation in any step of the Resolution exceeds Geometrical Dimensions, I shall very seldom digress from the steps of the Resolution.

The Resolution of Case 2. Probl. 3.

In this *Case* the line sought is suppos'd to exceed the given line A B.

50. For the given line A B put (as before) b.
51. And for the line sought, ... a.
52. Therefore the excess of the line sought above the given is . . . . $a - b$.
53. And the summ of both lines is . . $a + b$.
54. Therefore, according to the tenour of *Probl.* 3. this Analogy ariseth, . . . $r\ .\ s :: a - b\ .\ a + b$.
55. Therefore by *Composition of Reason*, . $s + r\ .\ s :: 2a\ .\ a + b$.
56. And by doubling the Consequents, . $s + r\ .\ 2s :: 2a\ .\ 2a + 2b$.
57. And inversly, ... $2s\ .\ s + r :: 2a + 2b\ .\ 2a$.
58. And by *Division of Reason*, ... $s - r\ .\ s + r :: 2b\ .\ 2a$.
59. And because there is the same Reason of the simple to the simple, as of the double to the double, therefore . . . $b\ .\ a :: 2b\ .\ 2a$.
60. And out of 58° and 59°, *(per prop.* 11. *Elem.* 5.) . . . . $s - r\ .\ s + r :: b\ .\ a$.

Hence this

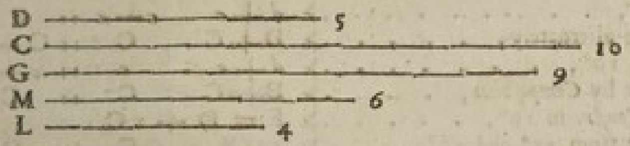
CANON.

61. As the Difference of the Terms of the given Reason is to their Summ; so is the given line to the line sought. Therefore the line required to be added to the given line, is given also.

The Composition of this latter *Case* differing but little from the former, I shall leave it as an exercise to the Learner.

Probl. IV.

The difference of the extremes of three proportional right lines being given, as also the summ of the mean and lesser extreme; to find the Proportionals.



Suppos.

1. $G, M, L \div\!\div$, *viz.* $G\ .\ M :: M\ .\ L$.
2. $G \sqsubset L$.
3. $d = G - L$ is given.
4. $c = M + L$ is given.

Req. to find G, M, L.

Resolution.

5. For the lesser extreme Proportional put . . . a.
6. Therefore out of 3° and 5°, the greater extreme is $d + a$.
7. And from 4° and 5°, the mean is . . . . $c - a$.
8. Therefore according to the tenour of the Problem, $d + a\ .\ c - a :: c - a\ .\ a$.
9. Therefore by *Composition of Reason*, ... $d + c\ .\ c - a :: c\ .\ a$.
10. And alternately, ... $d + c\ .\ c :: c - a\ .\ a$.
11. Wherefore by *Composition*, ... $d + 2c\ .\ c :: c\ .\ a$.

Hence this

CANON.

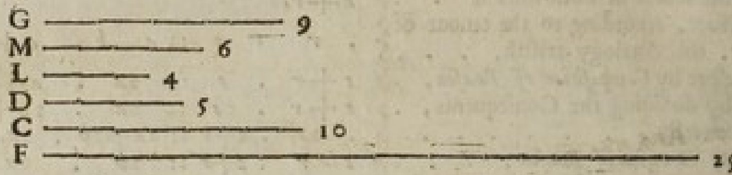
12. As the summ of the difference and the extremes of the double summ of the mean and lesser extreme, is to the summ of the mean and lesser extreme; so is the last mentioned summ to the lesser extreme. Therefore the lesser extreme sought is given.

13. After

13. After a Canon is found out by the Algebraical Art, it may be propounded in the form of a Theorem, whose Demonstration may be made by a repetition of the steps of the Resolution in a direct order, (not in a retrograde,) *viz.* by proceeding from the beginning to the end of the Resolution; as for example, the last preceding Canon may be propos'd in the form of a Theorem, and demonstrated thus.

THEOREM 1.

14. If three right lines be Proportionals, the summ of the mean and lesser extreme, shall be a mean proportional between the lesser extreme, and the summ of the difference of the extremes and the double summ of the mean and lesser extreme.



Suppos.

15. $G, M, L \div\div$; *viz.* $G \,.\, M :: M \,.\, L$.
16. $G \sqsubset L$.

Prepar.

17. Make $D = G - L$, therefore $D + L = G$.
18. Make $C = M + L$, therefore $C - L = M$.
19. Make $F = D + 2C = G + 2M + L$.
20. . . . *Req. demonstr.* . . . . $F \,.\, C :: C \,.\, L$.

Demonstration.

21. By *Suppos.* in 15°, ... > $G \,.\, M :: M \,.\, L$.
22. Therefore out of 17°, 18° and 21°, by exchanging equal right lines, ... > $D+L \,.\, C-L :: C-L \,.\, L$.
That is, in 8°, (the first Analogy in the Resolution,) . . . . > $d+a \,.\, c-a :: c-a \,.\, a$.
23. Therefore from 22°, by *Composition of Reason*, ... > $D+C \,.\, C-L :: C \,.\, L$.
That is, in 9°, ... > $d+c \,.\, c-a :: c \,.\, a$.
24. Therefore alternately, ... > $D+C \,.\, C :: C-L \,.\, L$.
That is, in 10°, ... > $d+c \,.\, c :: c-a \,.\, a$.
25. Therefore by *Composition*, ... > $D+2C \,.\, C :: C \,.\, L$.
26. But by *Constr.* in 19°, ... > $F = D + 2C$.
27. Therefore from 25° and 26°, . . . > $F \,.\, C :: C \,.\, L$.
Which was to be Dem.

But that Demonstration, after the letters of the Resolution are cast away, may be compendiously reduced unto this that follows, respect being had to the *Suppositions* and *Preparation* in 15°, 16°, 17°, 18°, 19°.

28. . . . *Req. demonstr.* . . . . . . $G+2M+L \,.\, M+L :: M+L \,.\, L$.

Demonstration.

29. By *Suppos.* in 15°, ... > $G \,.\, M :: M \,.\, L$.
30. Therefore by *Composition*, ... > $G+M \,.\, M :: M+L \,.\, L$.
31. And alternately, ... > $G+M \,.\, M+L :: M \,.\, L$.
32. Wherefore by *Compos.* . . . . . > $G+2M+L \,.\, M+L :: M+L \,.\, L$.
Which was to be Dem.

Hence ariseth

THEOREM 2.

33. If three right lines be Proportionals, the summ of the mean and lesser extreme shall be a mean Proportional between the lesser extreme, and the aggregate of the summ of the extremes and double summ of the mean.

The

The Composition of Probl. 4.

D ——————— 5
C ——————————————— 10
L ———————— 4
M ——————————— 6
G ——————————————— 9

Suppos.

34. $D =$ the difference of the extremes of three Proportionals is given.

35. $C =$ the summ of the mean and lesser extreme is given.

Req. to find the Proportionals.

Construction.

36. Let it be made (by *Probl.* 7. *Chap.* 5.) as $D+2C$ to C, so C to a third proportional line, which suppose to be found L, therefore, $\Big\}$ $D+2C \,.\, C :: C \,.\, L.$

37. By which Analogy, (and *per Schol. Prop.* 14. *Elem.* 5.) the mean C is greater than L, therefore find a right line M equal to $C-L$, thence it follows that . . . . $\Big\}$ $M+L=C.$

38. Find a right line G equal to $D+L$, therefore, $\}$ $G-L=D.$

39. So by the help of the given lines D and C, according to the direction of the Canon in the preceding twelfth step, three right lines are found out, to wit, L, M and G, which shall be the three Proportionals required. Now we must shew that they will satisfie the Problem. First then, 'tis manifest by *Construction* in 38°, that the difference of the extremes G and L is equal to the given difference D. Secondly, by *Construction* in 37°, the summ of the mean M and the lesser extreme L is equal to the given summ C. It remains only to prove that the said G, M and L are Proportionals, *viz.* that as G is to M, so M to L; but this Analogy may be made manifest by a Repetition of the steps of the preceding Resolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Resolution, in manner following.

40. . . . *Req. demonstr.* . . . . $G \,.\, M :: M \,.\, L.$

Demonstration.

41. Because by *Constr.* in 36°, ... $\}$ $D+2C : C :: C : L.$
That is, in 11°, (the last step of the Resolution,) $\}$ $d+2c \,.\, c :: c \,.\, a.$

42. Therefore by *Division of Reason*, ... $\}$ $D+C \,.\, C :: C-L \,.\, L.$
That is, in the tenth step, ... $\}$ $d+c \,.\, c :: c-a \,.\, a.$

43. Therefore alternately, ... $\}$ $D+C \,.\, C-L :: C \,.\, L.$
That is, in the ninth step, ... $\}$ $d+c \,.\, c-a :: c \,.\, a.$

44. Therefore, by *Division of Reason*, . . . $\}$ $D+L \,.\, C-L :: C-L \,.\, L.$
That is, in the eighth step, ... $\}$ $d+a \,.\, c-a :: c-a \,.\, a.$

45. And because by *Constr.* in 38° and 37°, . . $\}$ $G=D+L.$ Also, $M=C-L.$

46. Therefore out of 44° and 45°, by exchanging equal right lines, ... $\Big\}$ $.\ G : M :: M : L.$

Which was to be Dem. And therefore the Problem is satisfied.

Another way of resolving the foregoing Probl. 4.

47. The same things being supposed and given as before in 1°, 2°, 3° and 4° of this *Probl.* put a for the mean proportional sought; *viz.* suppose . . . . . . $\Big\}$ $a=M.$

48. Therefore out of 4° and 47° the lesser extreme shall be $\}$ $c-a$ $(=L.)$

49. And by adding d the given difference of the extremes to the said lesser extreme $c-a$, the greater extreme shall be . . . . $\Big\}$ $d+c-a$ $(=G.)$

50. Therefore according to the tenour of the *Probl.* this Analogy ariseth out of 47°, 48°, 49°, ... $\Big\}$ $d+c-a \,.\, a :: a : c-a.$

51. Therefore by *Composition of Reason*, ... $\}$ $d+c \,.\, a :: c \,.\, c-a.$

52. And alternately, ... $\}$ $d+c \,.\, c :: a : c-a.$

53. And inversly, ... $\}$ $c \,.\, d+c :: c-a \,.\, a.$

54. Therefore by *Composition of Reason*, ... $\}$ $d+2c \,.\, d+c :: c \,.\, a.$

Which

Which last Analogy gives this *CANON.*

55. As the aggregate of the difference of the extremes and the double summ of the mean and lesser extreme, is to the aggregate of the difference of the extremes and the summ of the mean and lesser extreme; so is the summ of the mean and lesser extreme to the mean.

Therefore the difference of the extremes of three Proportionals being given, as also the summ of the mean and lesser extreme, the mean shall be also given by the Canon last exprest. The Demonstration whereof, and the *Composition* of the Problem according to this latter way of Resolution being very easie, I shall leave the same to the Learners exercise.

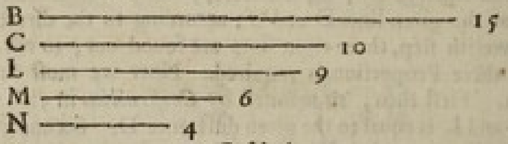
Probl. V.

The summ of the first and second of three Proportionals being given, as also the summ of the second and third, to find the Proportionals.

Suppos.

1. L, M, N are $\div\div$; *viz.* L . M :: M . N.
2. $b =$ L + M is given.
3. $c =$ M + N is given.

Req. to find L, M, N.



Resolution.

4. Put a for the first Proportional sought, *viz.* suppose . . . . $a =$ L.
5. Therefore out of 2° and 4°, the mean is . . . $b - a$ (= M.)
6. And by subtracting the said mean $b - a$ from the given summ c, the remainder gives the third Proportional, to wit, ... $c + a - b$ (= N.)
7. Therefore (according to the *Probl.*) these must be Proportionals, *viz.* . . . . . $a \,.\, b-a :: b-a \,.\, c+a-b.$
8. Therefore inversly, ... $b-a \,.\, a :: c+a-b \,.\, b-a.$
9. And by *Composition of Reason*, ... $b \,.\, a :: c \,.\, b-a.$
10. And alternately, ... $b \,.\, c :: a \,.\, b-a.$
11. And inversly, ... $c \,.\, b :: b-a \,.\, a.$
12. Therefore by *Composition of Reason*, . . . $b+c \,.\, b :: b \,.\, a.$

Which last Analogy gives this *CANON.*

13. As the aggregate of the summ of the first and second Proportionals and summ of the second and third, is to the summ of the first and second; so is the last mentioned summ to the first Proportional.

Therefore if the summ of the first and second of three Proportionals be given, as also the summ of the second and third, the mean shall be also given by the said Canon; whence also this *THEOREM.*

14. If three right lines be Proportionals, the summ of the first and second is a mean Proportional, between the first, and the aggregate of the summ of the first and second, and summ of the second and third.

Which Theorem may easily be demonstrated by a repetition of the steps of the Resolution in a direct order, after the manner of demonstrating the Theorem in 14° of the foregoing *Probl.* 4. but for brevity sake I shall leave the Demonstration to the Learners practice, and proceed to the *Composition* of *Probl.* 5.

The Composition of Probl. 5.

Suppos.

15. B = the summ of the first and second of three Proportionals is given.
16. C = the summ of the second and third Proportionals is given.

Req.

Req. to find the Proportionals. [*In the preceding Diagram.*]

17. By *Probl.* 7. *Chap.* 5. let it be made as $B + C$ to B, so B to a third Proportional, suppose it be L, therefore

$$B + C \;.\; B :: B \;.\; L.$$

18. Make $M = B - L$, whence $L + M = B$; but that $B \sqsubset L$, as that effection requires, is manifest by *Construction* in 17°; for $B + C$ the first Term of the Analogy in 17° is greater than B in the second, and therefore (*per Schol. Prop.* 14. *Elem.* 5.) the third Term, which is also B, shall be greater than L the fourth, therefore 'tis possible to cut off from B the right line L, and a right line will remain, which may be called M.

19. Make $N = C + L - B$, which is possible to be done if $C + L \sqsubset B$; but that $C + L \sqsubset B$, I prove thus,

By *Constr.* in 17°, ... $B + C \;.\; B :: B \;.\; L.$
Therefore (*per Schol. Prop.* 25. *Elem.* 5.) $B + C + L \sqsubset 2B.$
And consequently, by equal subtraction of B, . $C + L \sqsubset B.$

Which was to be proved. Therefore 'tis possible from the summ of the right lines C and L to cut off the right line B, and a right line will remain, which may be called N.

20. I say L, M and N are the three Proportionals required. Now we must shew that they will satisfie the Problem.

21. First then, the summ of the right lines L and M is (by *Construction* in 18°,) equal to the given summ B.

22. Secondly, that the summ of the right lines M and N is equal to the given summ C, I prove thus,

By *Constr.* in 18°, ... $M = B - L.$
And by *Constr.* in 19°, ... $N = C + L - B.$
Therefore by adding the two last Equations together, $M + N = C.$

Which was to be proved. It remains to shew that the said three right lines L, M and N are Proportionals, but that will be made manifest by the following Demonstration, which is formed out of the preceding Resolution by a repetition of the steps thereof in a retrograde order, *viz.* by returning backwards from the end to the beginning of the Resolution.

23. . . . *Req. demonstr.* . . . . $L \;.\; M :: M \;.\; N.$

Demonstration.

24. Forasmuch as by *Construction* in 17°, . . } $B + C \;.\; B :: B \;.\; L.$
That is, in 12°, (the last step of the Resolut.) } $b + c \;.\; b :: b \;.\; a.$
25. Therefore by *Division of Reason*, . . . } $C \;.\; B :: B - L \;.\; L.$
That is, in 11°, ... } $c \;.\; b :: b - a \;.\; a.$
26. Therefore inversly, ... } $B \;.\; C :: L \;.\; B - L.$
That is, in 10°, ... } $b \;.\; c :: a \;.\; b - a.$
27. Therefore alternately, ... } $B \;.\; L :: C \;.\; B - L.$
That is, in 9°, ... } $b \;.\; a :: c \;.\; b - a.$
28. And because by *Construction* in 18°, . . } $M = B - L.$
29. Therefore out of 27° and 28°, ... } $B \;.\; L :: C \;.\; M.$
30. And because it hath been proved in 18°, that } $B \sqsubset L.$
31. Therefore out of 29°, by *Division of Reason*, } $B - L \;.\; L :: C - M \;.\; M.$
That is, in 8°, ... } $b - a \;.\; a :: c + a - b \;.\; b - a.$
32. Therefore inversly, ... } $L \;.\; B - L :: M \;.\; C - M.$
That is, in 7°, ... } $a \;.\; b - a :: b - a \;.\; c + a - b.$
33. But by *Constr.* in 18°, ... } $M = B - L.$
34. And by what hath been proved in 22°, . } $N = C - M.$
35. Therefore out of 32°, 33° and 34°, . . } $L \;.\; M :: M \;.\; N.$

Which was to be Dem. and therefore the Problem is satisfied.

Another way of resolving the foregoing Probl. 5.

36. The same things being given and supposed, as before in 1°, 2°, 3°, put a for the mean Proportional sought, *viz.* . . } $a = M.$
37. Therefore out of 2° and 36°, the first Proportional shall be } $b - a$ ($= L.$)
38. And out of 3° and 36°, the third Proportional shall be . } $c - a$ ($= N.$)

39. Therefore out of 36°, 37° and 38°, according to the tenour of *Probl.* 5. . . . . } b—a . a :: a . c—a.
40. Therefore inversly, ... > a . b—a :: c—a . a.
41. And by *Composition of Reason*, ... > b . b—a :: c . a.
42. And alternly, ... > b . c :: b—a . a.
43. Wherefore by *Composition of Reason*, ... > b+c . c :: b . a.

Which last Analogy gives this

CANON.

44. As the aggregate of the given summ of the first and second Proportionals, and the given summ of the second and third, is to the summ of the second and third; so is the summ of the first and second, to the mean Proportional sought.

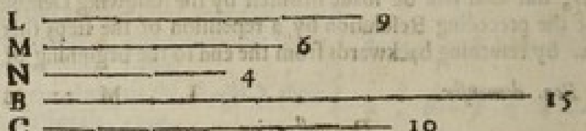
Which Canon, if it be propounded in the form of a Theorem, may be demonstrated by a repetition of the steps of the Resolution in a direct order. But leaving that and the Composition of *Probl.* 5. according to the latter Resolution, to the Learners exercise, I shall demonstrate the following Theorem by a repetition of the steps of the latter Resolution in a retrograde order.

THEOREM.

45. If three right lines be such, that the aggregate of the summ of the first and second and summ of the second and third, is to the summ of the second and third; as the summ of the first and second, to the second: those three lines shall be Proportionals, *viz.* As the first is to the second, so is the second to the third.

Suppos.

46. L, M, N, are three right lines.
47. B = L + M, whence B — M = L.
48. C = M + N, whence C — M = N.
49. B + C . C :: B . M.



50. ∴ *Req. demonstr.* L, M, N are ÷÷, *viz.* L . M :: M . N.

Demonstration.

51. Because by *Suppos.* in 49°, ... > B+C . C :: B . M.
52. Therefore by *Division of Reason*, ... > B . C :: B—M . M.
53. And alternately, ... > B . B—M :: C . M.
54. Therefore by *Division of Reason*, ... > M . B—M :: C—M . M.
55. And inversly, ... > B—M . M :: M . C—M.
56. But by *Suppos.* in 47°, ... > L = B — M.
57. And by *Suppos.* in 48°, ... > N = C — M.
58. Therefore out of 55°, 56° and 57°, ... > L . M :: M . N.

Which was to be dem.

Probl. VI.

The difference of the greater extreme and mean of three Proportionals being given, as also the difference of the mean and lesser extreme, to find the Proportionals. But the first difference must be greater than the latter.

Suppos.

1. Q, R, S are ÷÷, *viz.* Q . R :: R . S.
2. Q ⊏ R.
3. b = Q — R is given.
4. c = R — S is given.

Req. to find Q, R, S.

Q—

Q ———————————————— 25
R ——————————————— 20
S ————————————— 16
B ———— 5
C ——— 4

Resolution.

5. Put a for the mean Proportional sought, *viz.* . . } $a = R.$
6. Therefore out of 3° and 5°, the greater extreme shall be } $a + b \; (= Q.)$
7. And out of 4° and 5°, the lesser extreme shall be . . } $a - c \; (= S.)$
8. Therefore out of 6°, 5° and 7°, this Analogy will arise, (according to the import of the Problem,) *viz.* . } $a+b \,.\, a :: a \,.\, a-c.$
9. Therefore by *Division of Reason*, ... } $b \,.\, a :: c \,.\, a-c.$
10. And alternately, ... } $b \,.\, c :: a \,.\, a-c.$
11. And by *Conversion of Reason*, ... } $b \,.\, b-c :: a \,.\, c.$
12. And inversly, ... } $b-c \,.\, b :: c \,.\, a.$

Which last Analogy gives this

CANON.

13. As the excess by which the given difference of the greater extreme and mean exceeds the given difference of the mean and lesser extreme, is to the difference of the greater extreme and mean; so is the difference of the mean and lesser extreme, to the mean Proportional sought, whence the extremes will be easily discovered.

Which Canon, if it be propounded in the form of a Theorem, may be easily demonstrated by a repetition of the steps of the Resolution in a direct order; but leaving that to the Learner's practice, I shall demonstrate the following Theorem by a retrograde repetition of the steps of the Resolution.

THEOREM 1.

14. If three right lines be such, that the excess by which the excess of the first above the second exceeds the excess of the second above the third, be to the excess of the second above the third, as the excess of the first above the second is to the second; then those three right lines shall be Proportionals, *viz.* As the first is to the second, so the second to the third.

Suppos.

15. Q, R, S are three right lines.
16. $Q \sqsubset R.$
17. $R \sqsubset S.$
18. $B = Q - R$, whence $Q = B + R.$
19. $C = R - S$, whence $S = R - C.$
20. $B - C \,.\, C :: B \,.\, R.$
21. . . *Req. demonstr.* Q, R, S are $\div\!\div$, *viz.* $Q \,.\, R :: R \,.\, S.$

Demonstration.

22. Because by *Suppos.* in 20°, ... } $B-C \,.\, C :: B \,.\, R.$
23. Therefore alternately, ... } $B-C \,.\, B :: C \,.\, R.$
24. And inversly, ... } $B \,.\, B-C :: R \,.\, C.$
25. And by *Conversion of Reason*, ... } $B \,.\, C :: R \,.\, R-C.$
26. And alternately, ... } $B \,.\, R :: C \,.\, R-C.$
27. And by *Compos. of Reason*, ... } $B+R \,.\, R :: R \,.\, R-C.$
28. But by *Supposition* in 18°, ... } $Q = B + R.$
29. Also by *Suppos.* in 19°, ... } $S = R - C.$
30. Therefore out of 27°, 28° and 29°, by exchange of equal right lines, ... } $Q \,.\, R :: R \,.\, S.$

Which was to be Dem.

31. The Determination annex'd to *Probl. 6.* to wit, That the given difference of the greater extreme and mean must be greater than the given difference of the mean and lesser extreme, is discovered by the preceding Canon in 13°, and is necessarily to be prescribed for limiting the said differences, that they may be capable of constructing the Problem; as will be manifest by the subsequent

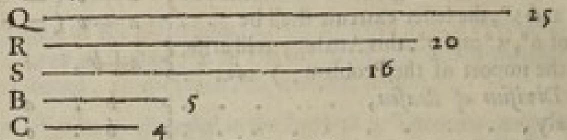
 LEMMA.

LEMMA.

32. If three right lines be Proportionals, the difference of the greater extreme and mean, is greater than the difference of the mean and lesser extreme.

Suppos.

33. Q, R, S are $\div\!\div$; *viz.* $Q \,.\, R :: R \,.\, S$.
34. $Q \sqsubset R$, and consequently, $R \sqsubset S$.
35. $B = Q - R$, whence $B + R = Q$.
36. $C = R - S$, whence $R - C = S$.



37. . . . *Req. demonstr.* . . . $B \sqsubset C$.

Demonstration.

38. Because by *Suppos.* in 33°, ... $Q \,.\, R :: R \,.\, S$.
39. And by *Suppos.* in 35° and 36°, ... $B + R = Q$. And $R - C = S$.
40. Therefore out of 38° and 39°, ... $B + R \,.\, R :: R \,.\, R - C$.
41. And by *Division of Reason*, ... $B \,.\, R :: C \,.\, R - C$.
42. But $R \sqsubset R - C$, therefore from 41°, (*per Schol. prop.* 14. *Elem.* 5.) . . . . $B \sqsubset C$.

Which was to be Demonstr.

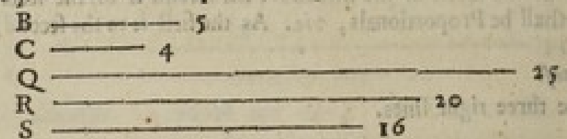
The Determination being demonstrated, I shall proceed to

The Composition of Probl. 6.

Suppos.

43. B = the difference of the greater extreme and mean of three Proportionals is given.
44. C = the difference of the mean and lesser extreme is given.
45. $B \sqsubset C$. (Determination.)

Req. to find the Proportionals.



Construction.

46. By *Probl.* 8. *Chap.* 5. let it be made as $B - C$ to B, so C to a fourth Proportional which may be called R, therefore,

$$B - C \,.\, B :: C \,.\, R.$$

which fourth Proportional R shall be greater than the third C, because the second B is greater than the first $B - C$.

47. Make $Q = R + B$; whence, $Q - R = B$.
48. Make $S = R - C$; whence, $R - S = C$, which effection is possible, for by the Analogy in 46°, it is manifest that $R \sqsubset C$.
49. I say Q, R and S are the three Proportionals required: Now we must shew that they will satisfie the Problem; First then, by *Construction* in 47°, the excess of Q above R is equal to the given difference B; secondly, by *Constr.* in 48°, the excess of R above S is equal to the given difference C. So it remains only to prove that Q, R and S are Proportionals, in this order, *viz.* $Q \,.\, R :: R \,.\, S$, but that is made manifest by the subsequent Demonstration, which is form'd out of the preceding Resolution, by a repetition of the steps thereof in a retrograde (not in a direct) order.

50. . . . *Req. demonstr.* . . . . $Q \,.\, R :: R \,.\, S$.

Demonstration.

51. Because by *Constr.* in 46°, which answers to the last step of the Resolution, to wit, 12°, ... $B - C \,.\, B :: C \,.\, R$.
52. Therefore inversly, ... $B \,.\, B - C :: R \,.\, C$.
53. And by *Conversion of Reason*, ... $B \,.\, C :: R \,.\, R - C$.
54. And alternately, ... $B \,.\, R :: C \,.\, R - C$.

55. There-

55. Therefore by *Compofition of Reafon*, ... $\succ$ B + R . R :: R . R − C.
56. But by *Conftr.* in 47°, ... $\succ$ . . . Q = B + R.
57. And by *Conftr.* in 48°, ... $\succ$ . . . S = R − C.
58. Therefore out of 55°, 56°, 57°, ... $\succ$. Q . R :: R . S.
Which was to be dem. therefore that is done which was reqnired by *Probl. 6.*

Another way of refolving the foregoing Probl. 6.

59. The fame things being given and fuppofed as before in 1°, 2°, 3°, 4° of *Probl. 6.* for the leffer extreme of the three Proportionals fought, put . . . . $\Big\}$ a.
60. To which leffer extreme if you add the given difference c, it makes the mean, to wit, ... $\Big\}$ $a + c$.
61. And by adding the given difference b to the mean Proportional, it gives the greater extreme, to wit, $\Big\}$ $a + c + b$.
62. Therefore according to the import of the Problem, thefe muft be Proportionals, *viz.* . . . . . $\Big\}$ $a \,.\, a+c :: a+c \,.\, a+c+b$.
63. Therefore inverfly, ... $\succ$ $a+c \,.\, a :: a+c+b \,.\, a+c$.
64. And by *Divifion of Reafon*, ... $\succ$ $c \,.\, a :: b \,.\, a+c$.
65. And by *altern* and *inverfe Reafon*, ... $\succ$ $b \,.\, c :: a+c \,.\, a$.
66. Wherefore by *Divifion of Reafon*, ... $\succ$ $b-c \,.\, c :: c \,.\, a$.
Which laft Analogy gives this

CANON.

67. As the excefs whereby the given difference of the greater extreme and mean exceeds the given difference of the mean and leffer extreme, is to the difference of the mean and leffer extreme; fo is the difference laft mentioned, to the leffer extreme fought: whence the mean and greater extreme are eafily difcovered.

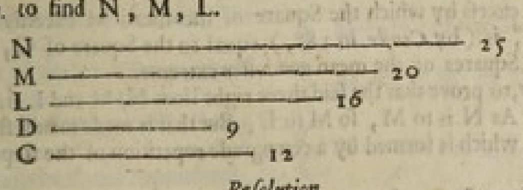
Probl. VII.

The difference of the extremes of three Proportionals being given; as alfo a right line whofe Square is equal to the difference between the Square of the mean and the Square of one of the extremes, to find the Proportionals.

Suppof.

1. N, M, L are $\div\!\!\div$; *viz.* N . M :: M . L.
2. N $\sqsubset$ L.
3. d = N − L is given.
4. $c = \sqrt{:\square M - \square L:}$ is given; and confequently,
5. $cc = \square M - \square L$ is given.

Req. to find N, M, L.



Refolution.

6. Put a for the leffer extreme fought, *viz.* . . . . . . $\succ$ a = L.
7. Therefore out of 3° and 6°, the greater extreme fhall be . . $\succ$ $a + d$ (= N.)
8. And out of 6° and 7°, the Rectangle contained under the extremes, or the Square of the mean, is equal to . . . . $\Big\}$ $aa + da$.
9. And out of 6°, the Square of the leffer extreme is . . . $\succ$ aa.
10. Which Square aa being fubtracted from $aa + da$, (to wit, from the Square of the mean,) the remainder fhall be the difference of the faid Squares, *viz.* . . . . $\Big\}$ da.
11. But the difference laft mentioned muft be equal to the given difference cc, therefore . . . . $\Big\}$ $da = cc$.
12. Which Equation may be refolved into this Analogy, . . $\succ$ $d \,.\, c :: c \,.\, a$.

Hence

Hence this *CANON.*

13. As the given difference of the extremes, is to the given right line whose Square is equal to the difference of the Squares of the mean and lesser extreme; so is the same right line to the lesser extreme sought.

Or thus,

As the given difference of the extremes, is to the given right line whose Square is equal to the difference of the Squares of the mean and greater extreme; so is the same right line to the greater extreme sought.

Hence this *THEOREM.*

14. If three right lines be Proportionals, the difference of the Squares of the mean and lesser extreme is equal to the Rectangle contained under the difference of the extremes and the lesser extreme.

Or thus,

The difference of the Squares of the mean and greater extreme, is equal to the Rectangle contained under the difference of the extremes and greater extreme.

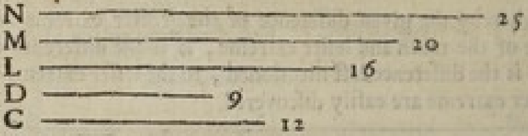
The Composition of Probl. 7.

Suppos.

15. D = the difference of the extremes of three Proportionals is given.

16. C = a right line, whose Square is equal to the difference of the Squares of the mean and lesser extreme is given.

Req. to find the Proportionals.



Construction.

17. By *Probl.* 7. *Chap.* 5. let it be made as D to C, so C to a third Proportional, which suppose to be the right line L, therefore,

D . C :: C . L.

18. By *Probl.* 2. *Chap.* 5. find a right line, as M, such, that its Square may be equal to the Square of L together with the Square of C, therefore,

$\square M = \square L + \square C$;

And consequently, $\square M - \square L = \square C$.

19. Make $N = L + D$; whence, $N - L = D$.

20. I say N, M and L are the three Proportionals required. Now we must shew that they will satisfie the Problem.

21. First then, by *Construction* in 19°, the excess of N above L is equal to D the given difference of the extremes.

22. Secondly, the excess by which the Square of the mean M exceeds the Square of the lesser extreme L, is (by *Constr.* in 18°,) equal to the Square of C, to wit, the given difference of the Squares of the mean and lesser extreme.

23. It remains only, to prove that the said three right lines N, M and L are Proportionals, in this order, *viz.* As N is to M, so M to L; But that is made manifest by the subsequent Demonstration, which is formed by a retrograde repetition of the steps of the preceding Resolution.

18. . . *Req. demonstr.* . . . . N . M :: M . L.

Demonstration.

25. Because by *Construction* in 17°, ... D . C :: C . L.

That is, in 12°, (the last step of the *Resolut.*) . . . *d . c :: c . a.*

26. Therefore, *per* 17. *prop.* 6. *Elem.* . . . . . . $\square D, L = \square C$.

That is, in 11°, ... $da = cc$.

27. Again, because by *Constr.* in 19°, ... $N = L + D$.

28. Therefore (*per prop.* 1. *Elem.* 6.) by drawing L as a common altitude into each part of the Equation in 27°, . . . $\square NL = \square L + \square D, L$.

29. And consequently out of 28° and 26°, by exchange of equal Rectangles, ... $\square NL = \square L + \square C$.

30. But

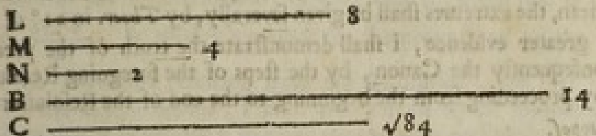
30. But by *Constr.* in 18°, ... $\square M = \square L + \square C$.
31. Therefore out of 29° and 30°, *per* 1. *Ax. Chap.* 2. $\square NL = \square M$.
32. Therefore out of 31°, *per* 14. *prop.* 6. *Elem.* $N . M :: M . L$.

Which was to be Demonstr. therefore the Problem propounded is satisfied.

Having by the preceding Examples of Resolution and Composition given the Learner a tast of the manner of arguing by Analogies, which is the best way when the nature of a Problem will admit the same, I shall now proceed to Examples of arguing partly by Equations, and partly by Analogies. But it must be remembred, that when in the Resolution you pass from one step to another by Addition, in the Demonstration of the Problem you must return by Subtraction: For Addition in the Resolution requires Subtraction in the Composition, and Subtraction in the one, Addition in the other; also Composition of Reason in the one requires Division of Reason in the other, as before hath been said in the *Note* at the end of *Probl.* 1. of this Chapter.

Probl. VIII.

A right line equal to the summ of three proportional right lines being given, as also a right line whose Square is equal to the summ of the Squares of all the said Proportionals, to find out the Proportionals severally. But the first of those lines given must be greater than the latter, yet not greater than the right line arising out of the Application of the triple Square of the said latter line to the first.



Suppos.

1. $L, M, N \div\!\div$, *viz.* $L . M :: M . N$.
2. $b = L + M + N$ is given.
3. $c = \sqrt{:\square L + \square M + \square N:}$ is given; therefore,
4. $cc = \square L + \square M + \square N$ is given also.

Req. to find L, M, N.

Resolution.

5. For the mean Proportional sought put a, viz. suppose . . . . $a = M$.
6. Therefore from 2° and 5°, the summ of the extremes shall be . . . . $b - a = L + N$.
7. Therefore the Square of the summ of the extremes is . . . . $bb + aa - 2ba = \square L + \square N + 2\square LN$.
8. And the Square of the mean Proportional, or the Rectangle of the extremes is . . . . $aa = \square M = \square LN$.
9. Which Square or Rectangle in 8°, being subtracted from the Square in 7°, leaves the summ of the Squares of all the three Proportionals, *viz.* $bb - 2ba = \square L + \square M + \square N$.
10. But by *Supposition* in 4°, ... $cc = \square L + \square M + \square N$.
11. Therefore from 9° and 10°, (*per Ax.* 1. *Chap.* 2.) . . . . $bb - 2ba = cc$.
12. And by adding $2ba$ to each part of the Equation in 11°, this ariseth, ... $bb = cc + 2ba$.
13. And by subtracting cc from each part of the Equation in 12°, ... $bb - cc = 2ba$.
14. And because (*per Theor.* 8. *Chap.* 4.) . . $bb - cc = \overline{b + c} \times \overline{b - c}$.
15. Therefore from 13° and 14°, (*per Ax.* 1. *Chap.* 2.) . . . . $2ba = \overline{b + c} \times \overline{b - c}$.
16. Therefore by resolving the last preceding Equation into Proportionals, it shall be . . $2b . b + c :: b - c . a$.

From

From the ninth step ariseth

THEOREM 1.

17. If three right lines be Proportionals, the excess whereby the Square of their summ exceeds twice the Rectangle made of that summ and the mean proportional, shall be equal to the summ of the Squares of all the three proportionals.

From the last step of the Resolution ariseth

THEOREM 2.

18. If three right lines be Proportionals, then this Analogy will attend them, *viz.* As the double summ of all the three Proportionals is to the simple summ increased with the side of a Square equal to the summ of the Squares of the three Proportionals; so is the excess whereby the summ of the three Proportionals exceeds the said side, to the mean Proportional.

Therefore the summ of three Proportionals being given, as also the summ of their Squares, the mean Proportional shall be given also by the preceding *Theor.* 2. whence the summ of the extremes is consequently given. And lastly, the summ of the extremes being given, as also the mean, the extremes shall be given severally, by *Probl.* 13. *Chap.* 5.

But to solve this *Probl.* 8. Arithmetically, the following Canon, (deducible from the 13th step,) will be more ready than *Theor.* 2.

CANON.

19. From the Square of the given summ of three Proportionals, subtract the given summ of their Squares, and divide the remainder by the double of the first given summ; so shall the Quotient be the mean Proportional; which subtracted from the summ of all three, leaves the summ of the extremes. And lastly, the summ of the extremes being given, as also the mean, the extremes shall be given severally, by *Theor.* in 21° of *Probl.* 13. *Ch.* 5.

But for the greater evidence, I shall demonstrate the truth of the preceding *Theor.* 1, and 2. and consequently the Canon, by the steps of the foregoing Resolution in a direct order, *viz.* by proceeding from the beginning to the end of the Resolution.

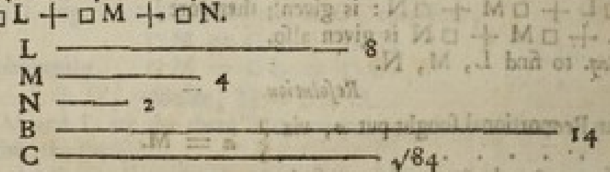
Suppos.

20. $L, M, N \div\!\div$, *viz.* $L \,.\, M :: M \,.\, N.$

21. $B = L + M + N.$

22. $C = \sqrt{:\square L + \square M + \square N:}$

23. $\square C = \square L + \square M + \square N.$



24. . . *Req. demonstr.* . . . . $\begin{cases} \textit{Theor. } 1. \quad \square B - 2\,\Box BM = \square C. \\ \textit{Theor. } 2. \quad 2B \,.\, B + C :: B - C \,.\, M. \end{cases}$

Demonstration.

25. By *Supposition* in 21°, ... $B = L + M + N.$

26. Therefore by subtracting M from each part, ... $B - M = L + N.$

27. And by squaring each part in 26°, this Equation will arise, (*per Theor.* 5, and 2. of *Chap.* 4.) . . . . $\square B + \square M - 2\,\Box BM = \square L + \square N + 2\,\Box LN.$

28. And from 20°, (*per prop.* 17. *Elem.* 6.) $\square M = \Box LN.$

29. Therefore by subtracting the Equation in 28° from that in 27°, this will manifest, (*per Ax.* 9, & 6. *Chap.* 2.) . . $\square B - 2\,\Box BM = \square L + \square M + \square N.$

30. But by *Suppos.* in 23°, ... $\square C = \square L + \square M + \square N.$

31. Therefore from 29° and 30°, (*per Ax.* 1. *Chap.* 2.) . . . . $\square B - 2\,\Box BM = \square C.$

Which was *Theor.* 1. to be demonstr.

32. Again, by adding $2\,\Box BM$ to each part of the Equation in 31°, this ariseth, . $\square B = \square C + 2\,\Box BM.$

33. And

33. And by subtracting $\square C$ from each part of the Equation in 32°, ... $\Big\}$ $\square B - \square C = 2\,\Box BM.$
34. But by *Theor.* 8. *Chap.* 4. . . . . . . $>$ $\square B - \square C = \Box : \overline{B + C} \times \overline{B - C} :$
35. Therefore from 33° and 34°, (*per Ax.* 1.) . $>$ $2\,\Box BM = \Box : \overline{B + C} \times \overline{B - C} :$
36. Therefore, (*per prop.* 14. *Elem.* 6.) . . . $>$ $2B \,.\, B + C :: B - C \,.\, M.$

Which was *Theor.* 2. to be dem. Therefore the truth of both the preceding Theorems is made manifest; also the Canon in 19° is evident from the Equation in 33°, by Application of each part thereof unto 2 B.

37. I shall in the next place, in order to the Composition of *Probl.* 8. before propounded, demonstrate the Determination annex'd to it for limiting the lines given, that they may be capable of effecting the Problem.

Determination. $\left\{ \begin{array}{l} B \sqsubset C, \\ B \text{ not } \sqsubset \dfrac{3 \square C}{B}. \end{array} \right.$

That is, the line given for the summ of three Proportionals must be greater than that given right line whose Square is equal to the summ of the Squares of all the three Proportionals, yet not greater than the right line arising out of the Application of the triple of the said Square, to the right line given for the summ of the three Proportionals.

38. The Scope of the Determination is, to remove two Objections that may be brought against the Construction of the Problem (in the following 76th and 87th steps,) unless the given lines be limited as the Determination prescribes, whose first part, to wit, that $B \sqsubset C$ is discovered by the last step of the Resolution, and already demonstrated. The latter part of the Determination, to wit, that the given line B ought not to be greater than $\dfrac{3 \square C}{B}$, is neither apparent in the proposition of the Problem, nor in either of the Theorems resulting from the Resolution; but that it is a property adherent to three Proportionals, I shall demonstrate by the following *Lemma*, and afterwards shew that it is necessary to make the Problem possible.

LEMMA.

39. If three right lines be Proportionals, their summ shall sometimes be equal to the right line arising out of the Application of the triple summ of all their Squares to their said summ; sometimes less, but never greater than the right line arising by the said Application.
40. Two Cases are to be demonstrated to prove this *Lemma*, for three Proportionals are either equal between themselves, or unequal. In the first Case, 'tis easie to perceive that the Square of the summ of three Proportionals is equal to the triple summ of their Squares; for supposing N, N, N to represent three Proportionals, their summ is 3N, the Square whereof is $9\square N$, which is manifestly equal to $3\square N + 3\square N + 3\square N$, to wit, the triple summ of the Squares of the said three Proportionals; and therefore if $9\square N$, the triple summ of the Squares of the said three equal Proportionals N, N, N, be applied to (or divided by) 3 N the summ of the same Proportionals, the line (or Quotient) arising by that Application shall necessarily be 3 N, (the summ of the said Proportionals.) Therefore the first Case of the *Lemma* is manifest.

Suppos. in *Case* 2.

41. $L, M, N \div\!\div$, *viz.* $L \,.\, M :: M \,.\, N.$
42. $L \sqsubset M$; and consequently $M \sqsubset N$.

Prepar.

43. By *Probl.* 1. *Chap.* 5. make $B = L + M + N.$
44. By *Probl.* 2. *Chap.* 5. make $C = \sqrt{:} \overline{\square L + \square M + \square N} :$
45. Thence it follows, that $\square C = \square L + \square M + \square N.$
46. By *Probl.* 8. *Chap.* 5. let it be made, $B \,.\, C :: 3C$ (to a fourth) T.
47. Thence it follows, that $T = \dfrac{3 \square C}{B}.$

L ———————— 8
M ———— 4
N ——— 2
B ———————————————— 14
G ———————————— $\sqrt{84}$
T ———————————————————— 18

48. . . *Req. demonstr.* . . . . $B \sqsupset T$, or $\frac{3\square C}{B}$.

Demonstration.

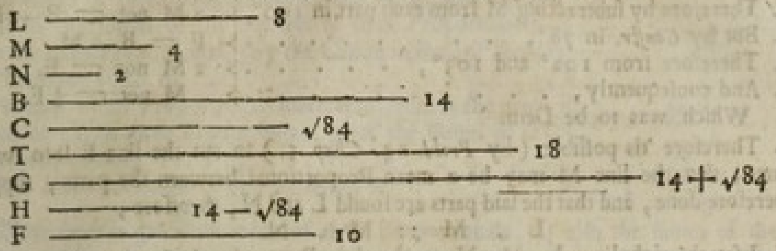
49. Because by *Suppos.* in 41°, . . .	$L \,.\, M :: M \,.\, N.$
50. Therefore, (*per prop.* 1. *Elem.* 6.) by reason of the common altitude L,	$\square L \,.\, \Box LM :: L \,.\, M :: M \,.\, N.$
51. Likewise by reason of the common altitude N, ...	$\Box MN \,.\, \square N :: M \,.\, N :: L \,.\, M.$
52. Therefore out of 50° and 51°, (*per prop.* 11. *Elem.* 5.) . . .	$\Box MN \,.\, \square N :: \square L \,.\, \Box LM.$
53. And out of 42° and 52°, by *Division of Reason*, ...	$\Box MN - \square N \,.\, \square N :: \square L - \Box LM \,.\, \Box LM.$
54. And inversly, ...	$\square N \,.\, \Box MN - \square N :: \Box LM \,.\, \square L - \Box LM.$
55. But from 41° and 42°, . . .	$\square N \sqsupset \Box LM.$
56. Therefore from 54° and 55°, (*per prop.* 14. *Elem.* 5.)	$\Box MN - \square N \sqsupset \square L - \Box LM.$
57. And by adding $\Box LM$, to each part in 56°, ...	$\Box LM + \Box MN - \square N \sqsupset \square L.$
58. And by adding $\square N$ to each part in 57°, ...	$\Box LM + \Box MN \sqsupset \square L + \square N.$
59. And by adding $\square M$ to each part in 58°, ...	$\Box LM + \square M + \Box MN \sqsupset \square L + \square M + \square N.$
60. But by *Constr.* in 45°, . . .	$\square C = \square L + \square M + \square N.$
61. Therefore from 59° and 60°, (*per Ax.* 3. *Chap.* 2.)	$\Box LM + \square M + \Box MN \sqsupset \square C.$
62. And because by *Constr.* in 43°,	$L + M + N = B.$
63. Therefore from 62°, by reason of the common altitude M, . . .	$\Box LM + \square M + \Box MN = \Box BM.$
64. Therefore from 61° and 63°, (*per Ax.* 4. *Chap.* 2.) . . .	$\Box BM \sqsupset \square C.$
65. And consequently, ...	$2 \Box BM \sqsupset 2 \square C.$
66. And by adding $\square C$ to each part in 65°, ...	$\square C + 2 \Box BM \sqsupset 3 \square C.$
67. But it hath been proved in 32°, (in the preceding Demonstration of *Theor.* 1, and 2.) that . .	$\square C + 2 \Box BM = \square B.$
68. Therefore out of 66° and 67°, (*per Ax.* 4. *Chap.* 2.)	$\square B \sqsupset 3 \square C.$
69. Therefore by Application of each part in 68° to B, ...	$B \sqsupset \frac{3\square C}{B}.$
70. But by *Constr.* in 46°, . . .	$T = \frac{3\square C}{B}.$
Therefore from 69° and 70°, (*per Ax.* 3. *Chap.* 2.)	$B \sqsupset T$ or $\frac{3\square C}{B}.$

Which was to be Demonstr.

71. Now because every three Proportionals whatever, are either equal or unequal between themselves, and it hath been shewn, that when they be equal to one another, their summ is equal to the right line arising out of the Application of the triple summ of their Squares to their summ; but when unequal, the summ of the three Proportionals is less than the right line arising by the said Application; it is manifest that the said summ can never be greater than the said right line: And therefore the truth of the preceding *Lemma*, and consequently the reason of the latter part of the Determination annext to *Probl.* 8. are evident.

The

The Composition of the foregoing Probl. 8.



Suppos.

72. $B =$ the summ of three proportional right lines is given.
73. $C =$ a right line, whose Square is equal to the summ of the Squares of the said three Proportionals, is given.
74. $B \sqsubset C$; yet B not $\sqsubset \frac{3 \square C}{B}$. (*Determination.*)

Req. to find out the three Proportionals.

Construction.

75. By *Probl.* 1. *Chap.* 5. find a right line, as G, equal to the summ of the given right lines B and C, therefore . . . . $G = B + C$.
76. By *Probl.* 3. *Chap.* 5. find a right line, as H, equal to the excess of B above C; which effection is possible, for by *Suppos.* in 74°, $B \sqsubset C$, therefore, $H = B - C$.
77. By *Probl.* 8. *Chap.* 5. let it be made as 2B to G; so H to a fourth proportional line, call it M, therefore $2B \,.\, G :: H \,.\, M$.
78. Find a right line $F = B - M$, which is possible to be done if B exceeds M, but that $B \sqsubset M$ I prove thus,
79. By *Construction* in 75°, ... $B + C = G$.
80. And by *Constr.* in 76°, ... $B - C = H$.
81. Therefore (*per Theor.* 8. *Chap.* 4.) by comparing the Rectangles made of the two last Equations, . $\square B - \square C = \Box GH$.
82. And by the first part of the last Equation 'tis manifest that . . . . $2 \square B \sqsubset \square B - \square C$.
83. Therefore from 81° and 82°, (*per Ax.* 3. *Chap.* 2.) $2 \square B \sqsubset \Box GH$.
84. But from 77°, (*per prop.* 16. *Elem.* 6.) . . . . $2 \Box BM = \Box GH$.
85. Therefore from 83° and 84°, (*per Ax.* 3. *Chap.* 2.) $2 \square B \sqsubset 2 \Box BM$.
86. Therefore by Application of each part in 85° to 2 B, ... $B \sqsubset M$.

Which was to be shewn.

87. By *Probl.* 14. *Chap* 5. divide the line F (before found in 78°) into two such parts, that the line M may be a mean Proportional between the parts; which is possible to be done if M be not greater than $\frac{1}{2}$ F, but that M is not greater than $\frac{1}{2}$ F, I prove thus,
88. By *Supposition* in 74°, ... B not $\sqsubset \frac{3 \square C}{B}$.
89. Therefore by drawing B into each part, . . $\square B$ not $\sqsubset 3 \square C$.
90. And by adding $2 \square B$ to each part in 89°, . . $3 \square B$ not $\sqsubset 2 \square B + 3 \square C$.
91. And because by *Suppos.* in 74°, ... $B \sqsubset C$.
92. And consequently, ... $3 \square B \sqsubset 3 \square C$.
93. Therefore by subtracting $3 \square C$ from each part in 90°, ... $3 \square B - 3 \square C$ not $\sqsubset 2 \square B$.
94. It hath been proved in 81°, that . . . . . $\square B - \square C = \Box GH$.
95. And in 84°, that . . . . $2 \Box BM = \Box GH$.
96. Therefore from the two last preceding Equations, $\square B - \square C = 2 \Box BM$.
97. And consequently, ... $3 \square B - 3 \square C = 6 \Box BM$.
98. Therefore from 93° and 97°, (*per Ax.* 4. *Chap.* 2.) $6 \Box BM$ not $\sqsubset 2 \square B$.
99. And consequently, ... $3 \Box BM$ not $\sqsubset \square B$.

 100. There-

100. Therefore by Application of each part in 99° to B, . . . $3\,M$ not $\sqsubset B$.
101. It hath been proved in 86°, that . . . . $B \sqsubset M$.
102. Therefore by subtracting M from each part in 100°, . . . $2\,M$ not $\sqsubset B - M$.
103. But by *Constr.* in 78°, ... $F = B - M$.
104. Therefore from 102° and 103°, ... $2\,M$ not $\sqsubset F$.
105. And consequently, ... M not $\sqsubset \frac{1}{2} F$.
Which was to be Dem.

106. Therefore 'tis possible (by *Probl.* 14. *Chap.* 5.) to cut the line F into two such parts, that the line M may be a mean Proportional between the parts; suppose it therefore done, and that the said parts are found L and N, therefore,

$$L \,.\, M :: M \,.\, N.$$

107. I say the right lines L, M, N are the three Proportionals sought: Now we must shew that they will satisfie the Problem propounded; First then, by Construction in the last preceding step they are Proportionals. Secondly, that the summ of the said L, M, N is equal to the given summ B, I prove thus,

108. Because by *Constr.* in 106°, ... $L + N = F$.
109. Therefore by adding M to each part, ... $L + M + N = F + M$.
110. But by *Constr.* in 78°, ... $B - M = F$.
111. And consequently, ... $B = F + M$.
112. Therefore from 109° and 111°, (*per Ax.* 1. *Chap.* 2.) $L + M + N = B$.
Which was to be proved.

113. It remains only to shew, that the summ of the Squares of the said right lines L, M and N is equal to the Square of the given line C; but that is made manifest by the following Demonstration, which is formed out of the steps of the preceding Resolution, in a retrograde (not in a direct) order.

114. . . *Req. demonstr.* . . . . $\square L + \square M + \square N = \square C$.

Demonstration.

115. Because by *Constr.* in 77°, ... $2B \,.\, G :: H \,.\, M$.
That is, in 16°, (the last step of the Resolution,) $2b \,.\, b+c :: b-c \,.\, a$.
116. Therefore from 115°, (*per prop.* 16. *Elem.* 6.) $2\,\square BM = \square GH$.
That is, in 15°, ... $2ba = \overline{b+c} \times \overline{b-c}$.
117. But from 81°, ... $\square B - \square C = \square GH$.
That is, in 14°, ... $bb - cc = \overline{b+c} \times \overline{b-c}$.
118. Therefore from 116° and 117°, (*per Ax.* 1. *Chap.* 2.) . . . . $\square B - \square C = 2\,\square BM$.
That is, in 13°, ... $bb - cc = 2ba$.
119. And by addition of $\square C$ to each part in 118°, $\square B = \square C + 2\,\square BM$.
That is, in 12°, ... $bb = cc + 2ba$.
120. And by subtraction of $2\,\square BM$ from each part in 119°, ... $\square B - 2\,\square BM = \square C$.
That is, in 11°, ... $bb - 2ba = cc$.
121. By *Constr.* in 106°, ... $L \,.\, M :: M \,.\, N$.
122. And it hath been proved in 112°, that . . . $L + M + N = B$.
123. Therefore out of 121° and 122° (*per Theor.* 1. in 17° of this *Probl.* 8.) . . . . $\square B - 2\,\square BM = \square L + \square M + \square N$.
124. Therefore from 120° and 123°, (*per Ax.* 1. *Chap.* 2.) . . . . $\square L + \square M + \square N = \square C$.
Which was to be Dem.

Thus you have seen the preceding *Probl.* 8. effected and demonstrated Synthetically, that is, by working and arguing with given Quantities only; but the substance of the Effection and Demonstration was first found out Analytically.

125. But me-thinks I hear the Learner object, That although the preceding Resolution of *Probl.* 8. be clear and easie, yet the prescribed Determination, especially its latter part, and the reason thereof seems hard to be found out: For answer to this, I say, That the first part of the Determination is apparent in the last step of the Resolution, in which also the latter part of the Determination is tacitly implied, and may thence be easily inferr'd thus;

126. Suppose (as before in the Resolution,) b to represent the given summ of three Proportionals, and cc a given Square equal to the summ of the Squares of those Proportionals, then the mean Proportional, (by the Canon in the 19th step,) will be found equal to . . . . $\frac{bb - cc}{2b}$.

127. Which mean Proportional subtracted from b the given summ of all the three Proportionals, gives the summ of the extremes, to wit, ... $\frac{bb + cc}{2b}$.

Therefore half the summ of the extremes is . . . . . . $\frac{bb + cc}{4b}$.

128. Now there is given the mean of three Proportionals, as also the summ of the extremes, to find out the extremes severally, which may be done by *Probl.* 13. *Chap.* 5. of this Book, provided that the mean doth not exceed half the summ of the extremes; that such Effection therefore may be possible, this following Determination is necessary for limiting the given lines (or numbers) b and c; viz.

$$\frac{bb - cc}{2b}, \text{ or } \frac{2bb - 2cc}{4b} \text{ not } \sqsubset \frac{bb + cc}{4b}.$$

129. Therefore, each part being multiplied by $4b$, . . . $2bb - 2cc$ not $\sqsubset bb + cc$.

130. And by adding $2cc$ to each part in 129°, ... $2bb$ not $\sqsubset bb + 3cc$.

131. And by subtracting bb from each part in 130°, . . . bb not $\sqsubset 3cc$.

132. And by Application of b to each part in 131°, . . . b not $\sqsubset \frac{3cc}{b}$.

Which is the latter part of the Determination added to *Probl.* 8. for limiting the lines given, that they may be capable of effecting or solving the Problem. If therefore any one neglecting this Determination, should rashly give $10 = b$ for the summ of the three continual Proportionals, and $32 = cc$ for the summ of their Squares, the said Determination shews that 'tis impossible to find out three such Proportionals: For although $10 \sqsubset \sqrt{32}$, that is, $b \sqsubset c$, as the first first part of the Determination requires, yet $100 \sqsubset 96$, that is, bb is greater than $3cc$, which contradicts the latter part of the Determination: But the better to shew the impossibility, let the Canon in the forgoing 19th step be used to find out a mean Proportional by the help of 10 given for the summ of three continual Proportionals, and of 32 given for the summ of their Squares, so you will find $3\frac{2}{5}$ for the mean Proportional, which subtracted from 10 the summ of all the three Proportionals, leaves $6\frac{3}{5}$ for the summ of the extremes; but 'tis manifest by *Probl.* 14. *Chap.* 5. that $6\frac{3}{5}$ cannot possibly be divided into two such parts that $3\frac{2}{5}$ may be a mean Proportional between the parts, because the mean $3\frac{2}{5}$ exceeds the half of $6\frac{3}{5}$ the summ of the extremes. This may suffice to shew that Determinations are necessary to be added to such Problems as are not universal, that is, such as cannot be solved or effected by numbers or right lines given at random, but with due restrictions or limitations.

Probl. IX.

The Base and leggs of a plain Triangle which hath unequal acute angles at the Base being severally given, to find the segments of the Base which are made by the falling of the Perpendicular from the angle opposite to the Base.

Note. The Geometrical effection of this Problem is very easie, for if (by *prop.* 12. *Elem.* 1.) a Perpendicular be let fall upon the Base from the opposite angle, it will divide the Base into the two segments required; but the scope of the Problem is to find out a Theorem whereby the said segments, and consequently the Perpendicular and Area of the Triangle may be discovered Arithmetically.

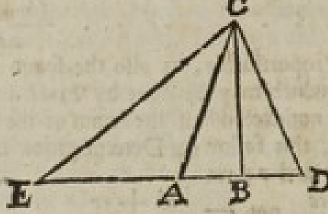
Suppos.

1. E D C is a $\triangle$, whose angles E and D are acute.
2. $\angle D \sqsubset \angle E$, therefore,
3. E C $\sqsubset$ D C, (*per prop.* 19. *Elem.* 1.)
4. C B $\perp$ E D.
5. B E and B D are the segments of the Base E D.
6. BE $\sqsubset$ B D.

7. BA = BD.

7. $BA = BD$.
8. $AE = BE - BD\ (BA)$ the difference of the segments.
9. $b = ED$ the Base is given.
10. $c = EC$ the greater legg is given.
11. $d = DC$ the lesser legg is given.

Req. to find BE and BD.



$ED = 63$
$EC = 60$
$DC = 39$
$EA = 33$
$BA = 15 = BD$
$BE = 48$
$BC = 36$
$\triangle\ EDC = 1134$

Resolution.

12. Put a for the difference of the segments of the Base, *viz.* $a = AE$.
13. Then (*per Theor.* 9. *Chap.* 4.) the greater segment of the Base shall be . . . . $\frac{1}{2}b + \frac{1}{2}a = BE$.
14. Likewise by the same *Theor.* the lesser segment shall be $\frac{1}{2}b - \frac{1}{2}a = BD$.
15. Therefore the Square of the greater segment shall be $\frac{1}{4}bb + \frac{1}{2}ba + \frac{1}{4}aa\ (= \square BE.)$
16. And the Square of the lesser segment shall be . . $\frac{1}{4}bb - \frac{1}{2}ba + \frac{1}{4}aa\ (= \square BD.)$
17. By 47. *prop.* 1. *Elem.* . . . . $\square EC = \square BE + \square BC$.
18. Therefore by equal subtraction of $\square BE$, . . . $\square EC - \square BE = \square BC$.
19. That is, in the letters belonging to the Resolution, $cc - \frac{1}{4}bb - \frac{1}{2}ba - \frac{1}{4}aa = \square BC$.
20. In like manner, (*per* 47. *prop.* 1. *Elem.*) . . . $\square DC - \square DB = \square BC$.
21. That is, in the letters of the Resolution, ... $dd - \frac{1}{4}bb + \frac{1}{2}ba - \frac{1}{4}aa = \square BC$.
22. Therefore out of 19° and 21°, (*per* 1. *Axiom.* 2. *Chap.*)

$$cc - \tfrac{1}{4}bb - \tfrac{1}{2}ba - \tfrac{1}{4}aa = dd - \tfrac{1}{4}bb + \tfrac{1}{2}ba - \tfrac{1}{4}aa.$$

23. Threfore from 22°, by equal addition and subtraction, $cc - dd = ba$.
24. Therefore from 23°, by Application of each part of that Equation to b, ... $\dfrac{cc - dd}{b} = a$.
25. Moreover, because by *Theor.* 8. *Chap.* 4. . . . . $cc - dd = \overline{c + d} \times \overline{c - d}$.
26. Therefore out of 23° and 25°, ... $\overline{c + d} \times \overline{c - d} = ba$.
27. Which last Equation may be resolved into this Analogy, ... $b\ .\ c + d :: c - d\ .\ a$.

Out of 24° ariseth

THEOR. 1.

28. If the difference of the Squares of the leggs of a plain Triangle which hath unequal acute angles at the Base, be applied to, (or divided by) the Base; the line or number arising shall be equal to the difference of the segments of the Base which are made by the falling of the Perpendicular from the angle opposite to the Base.

Out of 27° ariseth

THEOR. 2.

29. As the Base of a plane Triangle which hath unequal acute angles at the Base, is to the summ of the leggs; so is the difference of the leggs, to the difference of the segments of the Base which are made by the falling of the Perpendicular from the angle opposite to the Base.

30. Then the Base being given, as also the difference of the segments of the Base, the segments themselves shall be given severally; for (by *Theor.* 9. *Chap* 4.) half the Base, together with half the difference of the segments shall be equal to the greater segment; and half the Base less by half the difference of the segments shall be equal to the lesser segment. Lastly, the square Root of the excess of the Square of the greater legg above the Square of the greater segment of the Base; or the square Root of the excess of the Square of the lesser legg above the Square of the lesser segment, will give the Perpendicular, which multiplied into half the Base, gives the Area of the Triangle.

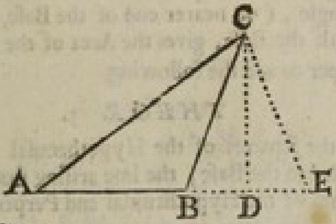
Probl. X.

Probl. X.

The Base and leggs of a plane Triangle obtusangled at the Base being severally given, to find out Arithmetically the distance from the foot of the Perpendicular falling without the Triangle, to each end of the Base: And consequently the Perpendicular and Area of the Triangle.

Suppos.

1. ABC is a $\triangle$ obtusangled at B, therefore
2. $AC \sqsubset BC$.
3. ABE is a right line.
4. $CD \perp AE$.
5. $DE = DB$.
6. $AE = AB + 2BD$, ($2DE$.)



$AB = 33$
$AC = 60$
$BC = 39$
$AE = 63$
$BD = 15 = DE$
$DC = 36$
$\triangle ABC = 594$

7. $b = AB$ the Base is given.
8. $c = AC$ the greater legg is given.
9. $d = BC$ the lesser legg is given.

Req. to find DB, and DA.

Resolution.

10. Suppose . . . . $a = AE$.
11. Then because . . . . $BE = AE - AB$.
12. Therefore the halves of all in 11° being taken, . . $\frac{1}{2}BE$ (or BD) $= \frac{1}{2}AE - \frac{1}{2}AB$.
13. That is, in the letters belonging to the Resolution, . $\frac{1}{2}a - \frac{1}{2}b = BD$.
14. And because . . . . $AB + BD = AD$.
15. Therefore in the letters of the Resolution, . . . $b + \frac{1}{2}a - \frac{1}{2}b = AD$.
16. That is, ... $\frac{1}{2}a + \frac{1}{2}b = AD$.
17. The Square whereof is . . . . $\frac{1}{4}aa + \frac{1}{2}ba + \frac{1}{4}bb = \square AD$.
18. And the Square of the Equation in 13° is . . . $\frac{1}{4}aa - \frac{1}{2}ba + \frac{1}{4}bb = \square BD$.
19. By 47. *prop.* 1. *Elem.* . . . . $\square AC = \square AD + \square DC$.
20. Therefore by equal subtraction of $\square AD$, . . . $\square AC - \square AD = \square DC$.
21. That is, in the letters of the Resolution, . . . $cc - \frac{1}{4}aa - \frac{1}{2}ba - \frac{1}{4}bb = \square DC$.
22. In like manner, by 47. *prop.* 1. *Elem.* . . . . $\square BC - \square BD = \square DC$.
23. That is, in the letters of the Resolut. . . . . $dd - \frac{1}{4}aa + \frac{1}{2}ba - \frac{1}{4}bb = \square DC$.
24. Therefore out of 21° and 23°, (*per* 1. *Axiom.* 2. *Chap.*)

$$cc - \tfrac{1}{4}aa - \tfrac{1}{2}ba - \tfrac{1}{4}bb = dd - \tfrac{1}{4}aa + \tfrac{1}{2}ba - \tfrac{1}{4}bb.$$

25. Therefore out of 24°, by equal addition and subtraction, ... $cc - dd = ba$.
26. Therefore each part of the last Equation being applied to b, ... $\dfrac{cc - dd}{b} = a$.
27. Moreover, because by *Theor.* 8. *Chap.* 4. . . . $cc - dd = \overline{b + c} \times \overline{b - c}$.
28. Therefore out of 25° and 27°, ... $\overline{c + d} \times \overline{c - d} = ba$.
29. Which last Equation may be resolved into this Analogy, ... $b \,.\, c + d :: c - d \,.\, a$.

Out of 26° ariseth

THEOR. 1.

30. If the difference of the Squares of the leggs of a plane Triangle obtusangled at the Base, be applied to (or divided by) the Base, the line (or number) arising shall be equal to the summ of the Base and the double of the distance from the obtuse angle to the foot of the Perpendicular, falling upon the Base produced without the Triangle.

Out of 29° ariseth

THEOR. 2.

31. As the Base of a plane Triangle obtusangled at the Base, is to the summ of the leggs; so is the difference of the leggs, to a fourth Proportional, which is compos'd of the Base and the double of the distance from the obtuse angle to the foot of the Perpendicular, falling without the Triangle upon the Base prolonged.

32. Then

31. Then the Base, and the line compos'd of the Base and the double distance from the obtuse angle to the foot of the Perpendicular being severally given by either of the said Theorems, the distance from the foot of the Perpendicular to each end of the Base shall be given also; for out of 13° and 16° of the preceding Resolution it is manifest, that half the said compos'd line together with half the Base, is equal to the distance from the foot of the Perpendicular to the remoter end of the Base; and half the said compos'd line less by half the Base, is equal to the distance from the foot of the Perpendicular to the nearer end of the Base. Lastly, the square Root of the excess of the Square of the lesser legg above the Square of the distance from the foot of the Perpendicular to the obtuse angle, (or nearer end of the Base,) will give the Perpendicular, which multiplied into half the Base, gives the Area of the Triangle.

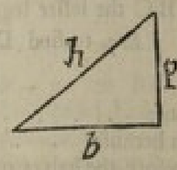
Here it will not be improper to add the following

THEOR. 3.

33. If the difference of the Squares of the Hypothenusal and Perpendicular of a right-angled Triangle be applied to the Base, the line arising shall be equal to the Base. Or, As the Base is to the summ of the Hypothenusal and Perpendicular, so is their difference to the Base it self.

This may be demonstrated in manner following.

Suppos.



34. $h =$ the Hypothenusal of a right-angled Triangle.
35. $p =$ the Perpendicular.
36. $b =$ the Base.

Req. demonstr.

37. $\frac{hh - pp}{b} = b$, that is,

38. $b \,.\, h + p :: h - p \,.\, b$.

Demonstration.

39. By 47. *prop.* 1. *Elem.* . . . . $hh = bb + pp$.
40. Therefore, ... $hh - pp = bb$.
41. Therefore by Application of each part of the last Equation to b, ... $\frac{hh - pp}{b} = b$.

Which was to be dem.

42. Moreover, because by *Theor.* 8. *Chap.* 4. . . $hh - pp = \square \begin{cases} h + p, \\ h - p. \end{cases}$
43. Therefore out of 40° and 42°, (*per Ax.* 1. *Chap.* 2.) . . . . $bb = \square \begin{cases} h + p, \\ h - p. \end{cases}$
44. Therefore, from the last Equation this Analogy ariseth, ... $b \,.\, h + p :: h - p \,.\, b$.

Which was to be dem.

From *Theor.* 1. in 28° of the foregoing *Probl.* 9. and from *Theor.* 1, and 3. in 30° and 33° of the preceding *Probl.* 10. we may deduce the following

COROLLARY.

45. If the three sides of any plane Triangle be severally given in numbers, the kind of every one of the three angles is given also: For supposing any one of the three sides to be called the Base, and the other two the leggs: First, if the leggs be equal to one another, the angles at the Base are equal acute angles; secondly, if the leggs be unequal, and the Quotient that ariseth by dividing the difference of the Squares of the leggs by the Base be less than the Base, then there will be unequal acute angles at the Base; thirdly, if the said Quotient be greater than the Base, then that angle at the Base which is opposite to the greater legg shall be obtuse, and consequently the other two angles acute; and lastly, if the said Quotient be equal to the Base, then the angle at the Base opposite to the greater legg is a right angle.

Or more easily, thus;

If the Square of one of the three sides of a plain Triangle be equal to the summ of the Squares of the other two, then the angle opposite to that side is a right angle; if greater, obtuse; if lesser, acute; as is manifest by *prop.* 48. *Elem.* 1. and by *prop.* 12, and 13. *Elem.* 2.

A Lemma,

A Lemma, *uſeful in the following* Probl. 11, *and* 12.

If within a plane Triangle a right line be drawn parallel to the Baſe, the Perpendicular falling upon the Baſe from the oppoſite angle, ſhall be to the Baſe, as that ſegment of the Perpendicular which is intercepted between the ſaid parallel line and the ſaid angle, is to the ſaid parallel line.

Suppoſ.

1. CEG is a $\triangle$.
2. NM $\parallel$ CE.
3. GL $\perp$ CE.
4. . . *Req. demonſtr.* . . . . . LG . CE :: OG . NM.

Demonſtration.

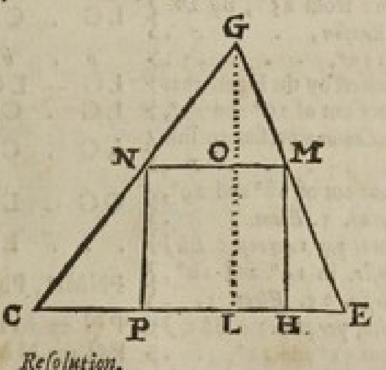
5. By *Suppoſition* in 2°, ... NM $\parallel$ CE.
6. Therefore (*per* 29. *prop.* 1. *Elem.*) . . $\angle$ GNM $=$ $\angle$ GCE.
7. Likewiſe . . . . $\angle$ GMN $=$ $\angle$ GEC.
8. Therefore, (*per Coroll. prop.* 32. *Elem.* 1.) $\triangle$ CLG and $\triangle$ NOG are equiangular.
9. Likewiſe . . . . $\triangle$ ELG and $\triangle$ MOG are equiangular.
10. Therefore (*per prop.* 4. *Elem.* 6.) . . LC . LG :: ON . OG.
11. Therefore alternately, ... LC . ON :: LG . OG.
12. And by the like argumentation, . . . LE . OM :: LG . OG.
13. Therefore out of 11° and 12°, (*per* 11. *prop.* 5. *Elem.*) . . . . LC . ON :: LE . OM.
14. Therefore (*per* 12. *prop.* 5. *Elem.*) . . LC . ON :: LC+LE . ON+OM.
15. But it hath above been proved in 11°, that LC . ON :: LG . OG.
16. Therefore out of 14° and 15°, (*per* 11. *prop.* 5. *Elem*) . . . . LC+LE . ON+OM :: LG . OG.
17. That is, (viewing the *Fig.*) . . . . CE . NM :: LG . OG.
18. Therefore alternly, ... CE . LG :: NM . OG.
19. Therefore inverſly, ... LG . CE :: OG . NM.

Which was to be Dem.

Probl. XI.

To inſcribe a Square in a Triangle given.

Note. If the given Triangle be obtuſangled, one of the ſides of the Square to be inſcribed muſt neceſſarily be a ſegment of that ſide of the Triangle which is oppoſite to the obtuſe angle, for otherwiſe, (as may eaſily be perceived) all the angular points of the Square cannot lye in the ſides of the Triangle; but if the Triangle given be right-angled or acute-angled, the Square may ſtand upon any one of the three ſides as a Baſe.



CE $=$ 182
CG $=$ 195
EG $=$ 169
GL $=$ 156
CL $=$ 117
LE $=$ 65
NP $=$ 84
NM $=$ 84

Suppoſ.

1. CEG is a $\triangle$ given.
2. $\angle$ GCE is not obtuſe.
3. $\angle$ GEC is not obtuſe.
4. GL $\perp$ CE.
5. $b =$ CE the Baſe is given.
6. $p =$ LG the Perpendic. is given.
7. . . *Req.* to inſcribe $\square$ PNMH in the $\triangle$ CEG.

Reſolution.

8. Suppoſing PNMH to be the Square required, put a to repreſent its ſide, *viz.* . . . . . $a =$ LO, or NM, or NP.
9. It is manifeſt by the foregoing *Lemma*, that . LG . CE :: OG . NM.
10. That is, in the letters belonging to the *Reſolut.* $p \,.\, b :: p-a \,.\, a$.
11. Therefore by *Compoſition of Reaſon*, . . . $p+b \,.\, b :: p \,.\, a$.

Hence this

CANON.

12. As the summ of the Perpendicular and Base is to the Base, so is the Perpendicular to the side of the inscribed Square sought.

The Composition of Probl. 11.

Suppos.

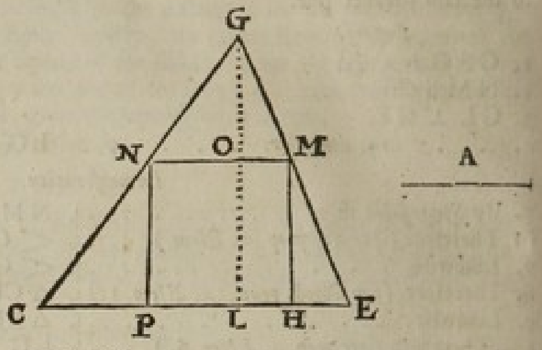
13. CEG is a △ given.

14. ∠ GCE is not obtuse.

15. ∠ GEC is not obtuse.

Req.

To inscribe □ PNMH in △ CEG.



Construction.

16. First, make GL ⊥ CE.
17. Then making LG + CE the first of four Proportionals, CE the second, and LG the third; find (*per Probl.* 8. *Chap.* 5.) a fourth Proportional, which suppose to be the line A, therefore,

LG + CE . CE :: LG . A.

18. From LG cut off LO = A, which is possible to be done, for in the Analogy in 17° the first Proportional is manifestly greater than the second, therefore the third Proportional shall be greater than the fourth, (*per Schol. prop.* 14. *Elem.* 5.) that is, LG ⊏ A; therefore 'tis possible to cut off from LG a segment equal to A, as LO.
19. By the point O draw NOM ∥ CE.
20. Also draw NP and MH ∥ OL.
21. I say PNMH is the Square required.
22. . . *Req. demonstr.* . . . . : that PNMH is a □;

Demonstration.

23. By *Constr.* in 17°, . . .	LG + CE . CE :: LG . A.
24. And by *Constr.* in 18°, . .	. . . . LO = A.
25. Therefore out of 23° and 24°,	LG + CE . CE :: LG . LO.
That is, in 11°, ...	. $p + b$. b :: p . a.
26. Therefore from 25°, by *Division of Reason*, ...	LG . CE :: LG − LO . LO.
That is, in 10°, ...	p . b :: $p - a$. a.
27. It is manifest by the Figure that	LG − LO = OG.
28. Therefore out of 26° and 27°,	LG . CE :: OG . LO.
29. By the *Lemma* prefixt to this Problem, ...	LG . CE :: OG . NM.
30. Therefore out of 28° and 29°, (*per* 11. *prop.* 5. *Elem.*) . .	OG . LO :: OG . NM.
31. Therefore (*per* 14. *prop.* 5. *El.*)	. . . LO = NM.
32. By *Constr.* in 19° and 20°, and by *defin.* 35. *Elem.* 1. .	PNMH, PNOL and LOMH are Parallelograms.
33. Therefore (*per prop.* 34. *El.* 1.)	PN = MH = LO. Also, NM = PH.
34. And from 31° and 33°, . .	PN = NM = MH = PH.
35. And because by *Constr.* in 16°,	∠GLC = ⅃ = ∠GLE.
36. Therefore from 32° and 35°, (*per prop.* 29, & 34. *Elem.* 1.)	∠PNM = ⅃ = ∠NMH = ∠MHP = ∠HPN.
37. Therefore from 34° and 36°, (*per defin.* 29. *Elem.* 1.) . .	PNMH is a □.

Which was to be Dem.

Probl. XII.

Probl. XII.

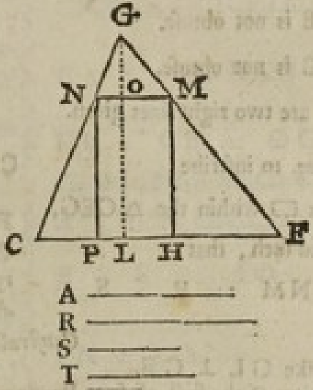
Within a given Triangle to inscribe a long Square, whose sides shall be in a given Reason.

Suppos.

1. CEG is a Triangle given.
2. $\angle$ GCE is not obtuse.
3. $\angle$ GEC is not obtuse.
4. GL $\perp$ CE.
5. b = CE the Base is given.
6. p = LG the Perpendicular is given.
7. r and s are the Terms of a given Reason.

Req. to inscribe

8. PNMH a $\square$ within the $\triangle$ CEG, and such, that
9. PN . NM :: R . S.



CE = 140
CG = 130
EG = 150
GL = 120
LE = 90
LC = 50
PN = 84
NM = 42
R = 2
S = 1
T = 60
A = 84

Resolution.

10. Supposing PNMH to be the Rectangle sought, put . . . . . . a = PN, or LO.
11. Therefore out of 6°, and 10°, ... $p - a$ = OG.
12. It is manifest by the *Lemma* prefixt to the foregoing *Probl.* 11. that . . . . . LG . CE :: OG . NM.
13. That is, in the letters belonging to the Resolution, ... $p \;.\; b \;::\; p-a \;.\; \frac{bp-ba}{p}$.
14. It is required in 9°, that . . . . . R . S :: PN . NM.
15. That is, ... $r \;.\; s \;::\; a \;.\; \frac{bp-ba}{p}$;
16. Therefore the two latter terms of the last Analogy being drawn into p, this Analogy will arise, . . $r \;.\; s \;::\; pa \;.\; bp-ba$.
17. Now to avoid an Equation between Solids, (for 'tis improper to introduce Solids in arguing about a plane Problem,) let it be made as r to s, so p to a fourth Proportional, which suppose to be t, therefore . $r \;.\; s \;::\; p \;.\; t$.
18. Therefore from the Analogies in 16° and 17°, this will arise, (*per* 11. *prop.* 5. *Elem.*) . . . $p \;.\; t \;::\; pa \;.\; bp-ba$.
19. And by drawing a as a common altitude into p and t severally, this Analogy will be manifest, (*per* 1. *prop.* 6. *Elem.*) . . . . . $p \;.\; t \;::\; pa \;.\; ta$.
20. Therefore from 18° and 19° this Analogy will arise, (*per* 11. *prop.* 5. *Elem.*) . . . . . $pa \;.\; bp-ba \;::\; pa \;.\; ta$.
21. Therefore out of 20°, (*per* 14. *prop.* 5. *Elem.*) $bp - ba = ta$.
22. Therefore out of 21°, by addition of ba to each part, $bp = ba + ta$.
23. Which last Equation gives this following Analogy, (*per* 14. *prop.* 6. *Elem.*) . . . . . $b+t \;.\; b \;::\; p \;.\; a$.

Out of 17° and 23° ariseth this *CANON.*

24. Let it be made as R to S, so the Perpendicular GL to a fourth Proportional T; also let it be made as the summ of the Base CE and the fourth Proportional T to the Base CE; so the Perpendicular GL to a fourth Proportional A, which shall be equal to the altitude PN of the Rectangle required to be inscribed. Lastly, as R to S, so A to NM, to wit, that side of the Rectangle which must be parallel to the Base CE.

The Composition of Probl. 12.

Suppos.

25. CEG is a △ given.

26. < GCE is not obtuse.

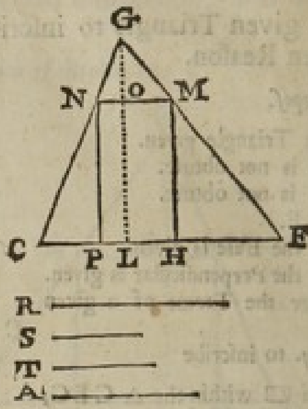
27. < GEC is not obtuse.

28. R and S are two right lines given.

Req. to inscribe

29. PNMH a ▭ within the △ CEG, and such, that

30. PN . NM :: R . S.



CE = 140
CG = 130
EG = 150
GL = 120
LE = 90
LC = 50
PN = 84
NM = 42
R = 2
S = 1
T = 60
A = 84

Construction.

31. First, make GL ⊥ CE.

32. Then making R the first of four Proportionals, S the second, and GL the third, find a fourth Proportional, which suppose to be T, therefore

R . S :: GL . T.

33. Again, making CE + T the first of four Proportionals, CE the second, and GL the third, find a fourth Proportional, let it be A, therefore

CE + T . CE :: GL . A.

34. From GL cut off LO = A, which is possible to be done; for in the Analogy in 33° the first Proportional is manifestly greater than the second, therefore (*per Schol.* of 14. *prop.* 5. *Elem.*) the third shall be greater than the fourth, therefore GL ⊏ A.

35. By the point O draw NOM ∥ CE, also by the points N and M draw NP and MH ∥ OL.

36. I say PNMH is the right-angled Parallelogram required to be inscribed. Now we must shew that PN is to NM as R to S; also that the angles NPH, MHP, NMH and PNM are right angles. First, that the side PN is to the side NM, as the line R to the line S, I shall make manifest by a retrograde repetition of the steps of the foregoing Resolution of *Probl.* 12.

37. . . *Req. demonstr.* . . . . R . S :: PN . NM.

Demonstration.

38. Because by *Construction* in 33°, . . . } CE + T . CE :: GL . A.
That is, in 23°, (the last step of the *Resolut.*) } . *b* + *t* . *b* :: *p* . *a*.
39. And by *Constr.* in 34°, ... } . . . . LO = A.
40. Therefore out of 38° and 39°, . . . } CE + T . CE :: GL . LO.
41. Therefore (*per* 16. *prop.* 6. *Elem.*) . } □ GE,GL = □ CE,LO + □ T,LO.
That is, in 22°, ... } . . *bp* = *ba* + *ta*.
42. Therefore out of 41°, by equal subtraction of □ CE,LO, ... } □ CE,GL − □ CE,LO = □ T,LO.
That is, in 21°, ... } . . *bp* − *ba* = *ta*.

43. From 42°, (*per* 7. *prop.* 5. *Elem.*) this subsequent Analogy will be manifest,

□ GL,LO . □ CE,GL − □ CE,LO :: □ GL,LO . □ T,LO.

That is, in 20°, *pa* . *bp* − *ba* :: *pa* . *ta*.

44. And by reason of the common altitude LO this Analogy is manifest, (*per* 1. *prop.* 6. *El.*)

GL . T :: □ GL,LO . □ T,LO.

That is, in 19°, *p* . *t* :: *pa* . *ta*.

45. Therefore from the Analogies in 43° and 44°, (*per* 11. *prop.* 5. *Elem.*)

GL . T :: □ GL,LO . □ CE,GL − □ CE,LO.

That is, in 18°, *p* . *t* :: *pa* . *bp* − *ba*.

46. By *Constr.* in 32°, ... GL . T :: R . S.
That is, in 17°, ... *p* . *t* :: *r* . *s*.

47. There-

47. Therefore from 45° and 46°, (*per* 11. *prop.* 5. *Elem.*)
R . S :: ▭GL,LO . ▭CE,GL − ▭CE,LO.
That is, in 16°, *r* . *s* :: *pa* . *bp* − *ba*.

48. It is manifest by the Figure, that . . } GL − LO = OG.

49. Therefore by taking CE as a common altitude, ... } ▭CE,GL − ▭CE,LO = ▭CE,OG.

50. Therefore out of 47° and 49°, by exchanging equal Rectangles, this Analogy is manifest, ... } R . S :: ▭GL,LO . ▭CE,OG.

51. By the *Lemma* placed next before *Probl.* 11. it's manifest that . . . . } GL . CE :: OG . NM.

52. And consequently, (*per* 16 *prop.* 6. *El.*) } ▭GL,NM = ▭CE,OG.

53. Therefore out of 50° and 52°, by exchanging equal Rectangles, ... } R . S :: ▭GL,LO . ▭GL,NM.

54. Therefore, the common altitude GL being cast away, ... } R . S :: LO . NM.

55. And because by *Constr.* in 35°, (and by 35. *defin.* 1. *Elem.*) PNOL is a Parallelogram, therefore (*per* 34. *prop.* 1. *Elem.*) } PN = LO.

56. Therefore out of 54° and 55°, by setting PN in the place of LO, ... } R . S :: PN . NM.

Which was to be Demonstr.

But that the angles of the inscribed Parallelogram PNMH are right angles, is easie to prove out of the Construction, after the same manner as in the preceding *Probl.* 11. from 31° to 37°.

Probl. XIII.

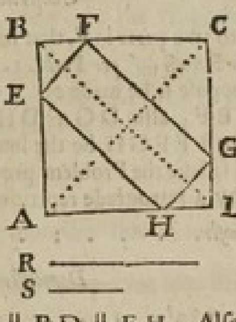
Within a given Square to inscribe a long Square, whose sides shall be in a given Reason, and parallel to the Diameters of the Square.

Suppos.

1. ABCD is a □ given.
2. *b* = BC (or BA) is given.
3. *r* and *s* are the Terms of a given Reason.

Req. to inscribe

4. EFGH a ▭ within the □ ABCD, so, as that the points E, F, G and H may lye in the sides of the □ABCD; also, that



R = 2
S = 5
AB = 28 = BC
BF = 8 = BE
DG = 8 = DH
FC = 20 = CG
AE = 20 = AH
EF = 4√8 = HG
FG = 10√8 = EH

5. EF ∥ AC ∥ HG; likewise, FG ∥ BD ∥ EH. Also;
6. EF . FG :: *r* . *s*.

Preparat.

7. Suppose EFGH to be the ▭ required, then according to the tenour of the Problem, . . } . . . EF ∥ AC.
8. Therefore (*per* 29. *prop.* 1. *Elem.*) . . . } . $\angle$BEF = $\angle$BAC.
9. Likewise, ... } . $\angle$BFE = $\angle$BCA.
10. Therefore equiangular are . . . . . . } . △BEF and △BAC.
11. Therefore (*per* 4. *prop.* 6. *Elem.*) . . . } BA . BC :: BE . BF.
12. Therefore alternately, ... } BA . BE :: BC . BF.
13. But by *Supposition* in 1°, ... } . . . BA = BC.
14. Therefore (*per* 14. *prop.* 5. *Elem.*) . . . } . . . BE = BF.
15. And because by *Suppos.* in 1°, ... } . $\angle$EBF = ⌐.
16. Therefore out of 14° and 15°, (*per* 47. *prop.* 1. *Elem.*) . . . . } . . 2□BF = □EF.
17. And by the like argumentation, ... } . . 2□FC = □FG.

Resolu-

Resolution.

18. Suppose . . . . $a = BF$, or BE.
19. Therefore out of 2° and 18°, ... $b - a = FC$, or CG.
20. The Square of the segment in 18° is . . . $aa = \square BF$.
21. The Square of the segment in 19° is . . . $bb - 2ba + aa = \square FC$.
22. Therefore out of 20° and 16°, ... $2aa = \square EF$.
23. Also out of 21° and 17°, ... $2bb - 4ba + 2aa = \square FG$.
24. Therefore out of 22°, ... $\sqrt{2aa} = EF$.
25. And out of 23°, ... $\sqrt{2bb - 4ba + 2aa} = FG$.
26. Therefore according to the condition in the Problem,
$$r \,.\, s :: \sqrt{2aa} \,.\, \sqrt{2bb - 4ba + 2aa}.$$
27. But their Squares also are Proportionals, therefore
$$rr \,.\, ss :: 2aa \,.\, 2bb - 4ba + 2aa.$$
28. And because the two latter Terms of the last Analogy are in the same Reason as their halves, therefore
$$rr \,.\, ss :: aa \,.\, bb - 2ba + aa.$$
29. But the Roots of the four last Proportionals are also Proportionals, therefore . . . . $r \,.\, s :: a \,.\, b - a$.
30. Therefore inversly, ... $s \,.\, r :: b - a \,.\, a$.
31. Therefore by *Composition of Reason*, ... $s + r \,.\, r :: b \,.\, a$ (BF.)
32. Also out of 29°, by *Compos. of Reason*, . . $r + s \,.\, s :: b \,.\, FC$.

From the premisses ariseth this

CANON.

33. By *Probl.* 1. of this *Chapt.* divide the side of the given Square into two segments that may be one to the other in the given Reason; then find out the side of a Square equal to the double Square of the greater segment, also the side of a Square equal to the double Square of the lesser segment; so shall those sides be the length and bredth of the desired Rectangle, (or long Square,) which may be inscribed within the given Square by this following

Construction.

34. First divide B C into two parts in F, that may be one to the other as R to S, (*per Probl.* 1. of this *Chapt.*) *viz.* suppose it be made as . . $R \,.\, S :: BF \,.\, FC$.
35. Then let $BE = BF$; also $DG = DH = BF$, and draw the right lines EF, FG, GH and HE, so shall EFGH be the long Square given to be inscribed: Now we must shew that it will satisfie the Problem propounded; first, that EF is to FG as R to S, I shall demonstrate by a retrograde repetition of the steps of the preceding Resolution.
36. . . *Req. demonstr.* . . . . $R \,.\, S :: EF \,.\, EG$.

Demonstration.

37. Because by *Constr.* in 34°, ... $R \,.\, S :: BF \,.\, FC$.
38. Therefore (*per prop.* 22. *Elem.* 6.) . . $\square R \,.\, \square S :: \square BF \,.\, \square FC$.
39. And by taking the doubles of the two latter Terms of the last preceding Analogy, . . $\square R \,.\, \square S :: 2\square BF \,.\, 2\square FC$.
40. By *Constr.* in 35°, ... $BE = BF$.
41. And by *Suppos.* . . . . $\angle EBF = \lrcorner$.
42. Therefore from 40° and 41°, (*per prop.* 47. *Elem.* 1.) . . . . $\square EF = 2\square BF$.
43. Likewise, ... $\square FG = 2\square FC$.
44. Therefore out of 39°, 42° and 43°, by exchange of equal quantities, ... $\square R \,.\, \square S :: \square EF \,.\, \square FG$.
45. Therefore from 44°, (*per prop.* 22. *Elem.* 6.) $R \,.\, S :: EF \,.\, FG$.

Which was to be Dem.

It remains to demonstrate that the quadrilateral Figure E F G H is right-angled at E, F, G and H.

46. . . *Req. demonstr.* . $\angle EFG = \lrcorner = \angle FGH = \angle GHE = \angle HEF$.

Demon-

Demonstration.

47. Becaufe by *Conftr.* in 35°, ... $BF = BE$.
48. And by *Suppof.* . . . . $\angle EBF = \perp$.
49. Therefore (*per Coroll. prop.* 32. *Elem.* 1.) . . . $\angle BFE = \frac{1}{2}\perp$.
50. Likewife . . . . $\angle CFG = \frac{1}{2}\perp$.
51. Therefore from 49° and 50°, (*per Coroll. prop.* 13. *Elem.* 1.) . . . . $\angle EFG = \perp$.
52. And by the like argumentation, ... $\angle FGH, \angle GHE, \angle HEF$ are $\perp$.
Which was to be Dem. Therefore the Problem is fatisfied.

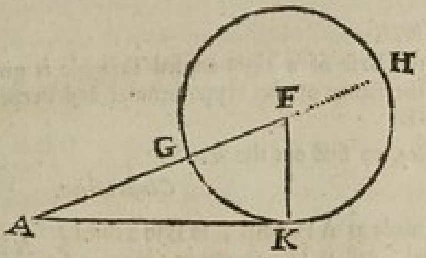
Probl. XIV.

In a right-angled Triangle, one of the fides about the right angle being given, as alfo the fumm of the other fide about the right angle and the Hypothenufal, to find out the Triangle. But the faid fumm muft be greater than the given fide, as is manifeft by *prop.* 22. *Elem.* 1.

Suppof.

1. $\triangle$ AKF is right-angled at K.
2. $b = AK$ is given.
3. $c = AF + FK$ is given.
4. $c \sqsubset b$.

Req. to find $\triangle$ AKF.



$AK = 12$
$KF = 5$
$FA = 13$

Refolution.

5. For the difference between the Hypothenufal AF and the Perpendicular FK, put . . . . a.
6. Then from 3° and 5°, (*per Theor.* 9. *Chap.* 4.) the Hypothenufal fhall be . . . . $\frac{1}{2}c + \frac{1}{2}a$.
7. And by the fame Theorem, the Perpendicular KF fhall be . $\frac{1}{2}c - \frac{1}{2}a$.
8. Therefore from 6° the Square of the Hypothenufal fhall be . $\frac{1}{4}cc + \frac{1}{4}aa + \frac{1}{2}ca$.
9. And from 7° the Square of the Perpendicular is . . . . $\frac{1}{4}cc + \frac{1}{4}aa - \frac{1}{2}ca$.
10. The difference of thofe Squares is . . . . ca.
11. But (*per prop.* 47. *Elem.* 1.) the difference of the Square of the Hypothenufal and the Square of one of the fides about the right angle is equal to the Square of the other fide about the right angle, therefore from 10° and 2°, ... $ca = bb$.
12. Therefore, by refolving the laft preceding Equation into Proportionals, ... $c \,.\, b :: b \,.\, a$.
13. Therefore out of 12° and 6°, the Hypothenufal is now given, *viz.* . . . . $\frac{1}{2}c + \frac{bb}{2c} = AF$.
14. And from 12° and 7°, the defired fide about the right angle is alfo given, *viz.* . . . . $\frac{1}{2}c - \frac{bb}{2c} = FK$.

Out of 12° arifeth this

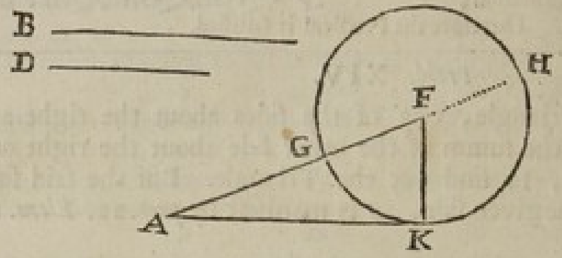
THEOREM.

15. One of the fides about the right angle of a right-angled Triangle is a mean Proportional between the fumm and difference of the Hypothenufal and the other fide about the right-angle. And confequently (*per prop.* 17. *Elem.* 1.) the Rectangle of the faid fumm and difference is equal to the Square of the firft mentioned fide, or mean Proportional.
16. Therefore, the Bafe of a right-angled Triangle, (that is, which of the two fides about the right angle you pleafe,) being given, as alfo the fumm of the Hypothenufal and Perpendicular, the difference of the Hypothenufal and Perpendicular fhall be given alfo. For by the Theorem above expreft, (which is the fame with *Prop.* 36. *Elem.* 3. when the

the line there mentioned to cut the Circle passeth through the Center,) the said difference is a third Proportional proceeding from the said Base and Summ.

Therefore also, the said Hypothenusal and Perpendicular shall be given severally, (*per Theor.* 9. *Chap.* 4.) for they shall be equal to the given Quantities before exprest in 13° and 14° of this Problem.

The Composition of Probl. 14.



$AK = 12 = B$
$KF = 5 = FH$
$FA = 13$
$AH = 18$
$D = 8 = AG$

Suppos.

17. $B =$ the Base of a right-angled Triangle is given.
18. $AH =$ the summ of the Hypothenusal and Perpendicular is given.
19. $AH \sqsupset B$.

Req. to find out the $\triangle$.

Construction.

20. Let it be made as AH to B, so B to a third Proportional, call it D, therefore . . . $AH \,.\, B :: B \,.\, D$.
21. From AH cut off $AG = D$, which is possible to be done if $AH \sqsupset D$; but by *Supposition* $AH \sqsupset B$, therefore from the Analogy in 20°, $AH \sqsupset D$.
22. Divide GH into two equal parts in F, therefore $FG = FH$.
23. From F as a Center, at the distance of FG or FH describe the Circle FGKH; and (*per prop.* 17. *Elem.* 3.) draw the right line AK to touch the said Circle in K; draw also the Semidiameter FK; then shall AKF be the Triangle sought. Now we must shew that it will satisfie the Problem.
24. First then, by *Construction* in 23°, $FH = FK$, therefore $AF + FK = AH$ the given summ of the Hypothenusal and Perpendicular. It remains to shew that AK is equal to the given Base B, and that the angle AKF is a right angle; therefore,
25. . . *Req. demonstr.* . . . . . . $AK = B$. Also, $\angle AKF = \llcorner$.

Demonstration.

26. Because by *Constr.* in 20° and 21°, . . > $AH \,.\, B :: B \,.\, D\,(AG.)$
27. Therefore, (*per prop.* 17. *Elem.* 6.) . . > $\square AH, AG = \square B$.
28. And because by *Constr.* in 23°, AK toucheth ⊙FGKH in K, therefore (*per prop.* 36. *Elem.* 3.) $\square AH, AG = \square AK$.
29. Therefore out of 27° and 28°, (*per Ax.* 1. *Chap.* 2.) . . . . $\square AK = \square B$.
30. Therefore, (*per Schol. prop.* 48. *Elem.* 1.) > . $AK = B$. Which was to be dem.
31. And because by *Constr.* in 23°, AK toucheth ⊙FGKH in K, and FK is Radius, therefore, (*per prop.* 18. *Elem.* 3.) . . . . . . $FK \perp KA$. Therefore, $\angle AKF = \llcorner$.

Which was also to be Dem. Therefore the Problem is satisfied.

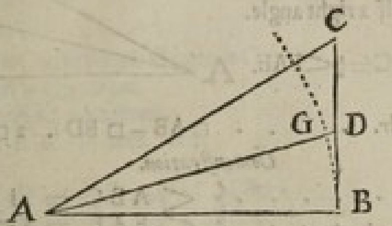
Probl. XV.

The Radius (or Semidiameter) of a Circle being given, as also the Tangent of an arch, to find the Tangent of the double of that arch. But the given Tangent must be less than the Radius.

Suppos.

Suppos.

1. $\triangle$ ABC is right-angled at B.
2. $< BAD = < DAC = \frac{1}{2} < CAB$.
3. r = AB the Radius of a Circle, whose Center is A, is given.
4. t = BD the Tangent of the arch BG, or angle DAB, is given.
5. BD $\sqsupset$ AB.



Req. to find BC the Tangent of the angle CAB, that is, the Tangent of $2 <$ DAB.

Resolution.

6. Put . . . . } a = BC.
7. Therefore out of 4° and 6°, ... } $a - t$ = DC.
8. And out of 1°, 3° and 6°, (*per* 47. *prop.* 1. *Elem.*) . . . . } $\sqrt{: rr + aa :}$ = AC.
9. And because by *Supposition* $<$ BAD $= <$ DAC, therefore (*per* 3. *prop.* 6. *Elem.*) } DC . BD :: AC . AB.
10. That is, in the letters of the Resolution, } $a - t \;.\; t \;::\; \sqrt{: rr + aa :} \;.\; r$.
11. But their Squares also shall be Proportionals, (*per* 22. *prop.* 6. *Elem.*) *viz.* . . } $aa - 2ta + tt \;.\; tt \;::\; rr + aa \;.\; rr$.
12. Therefore from the Analogy in the 11th step, by *Division of Reason*, ... } $aa - 2ta \;.\; tt \;::\; aa \;.\; rr$.
13. Therefore alternly, ... } $aa - 2ta \;.\; aa \;::\; tt \;.\; rr$.
14. And by reason of the common altitude a, in the two first Terms of this Analogy, it is manifest (*per* 1. *prop.* 6. *Elem.*) that . . } $aa - 2ta \;.\; aa \;::\; a - 2t \;.\; a$.
15. Therefore from the two last preceding Analogies, (*per* 11. *prop.* 5. *Elem.*) . . } $a - 2t \;.\; a \;::\; tt \;.\; rr$.
16. Therefore inversly, ... } $rr \;.\; tt \;::\; a \;.\; a - 2t$.
17. And conversly, ... } $rr \;.\; rr - tt \;::\; a \;.\; 2t$.
18. And by doubling the Antecedents, (*per Schol.* Clavii *in prop.* 22. *Elem.* 5.) . . } $2rr \;.\; rr - tt \;::\; 2a \;.\; 2t$.
19. And inversly, ... } $rr - tt \;.\; 2rr \;::\; 2t \;.\; 2a$.
20. But (*per* 15. *prop.* 5. *Elem.*) . . . . } $t \;.\; a \;::\; 2t \;.\; 2a$.
21. Therefore out of 19° and 20°, (*per* 11. *prop.* 5. *Elem.*) . . . . } $rr - tt \;.\; 2rr \;::\; t \;.\; a$.

Hence this

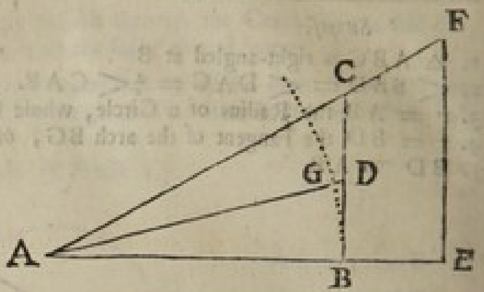
THEOREM.

22. As the excess whereby the square of the Radius exceeds the square of the Tangent of an arch less than half a Quadrant, is to the double square of the Radius; so is the said Tangent, to the Tangent of the double of that arch.

Therefore the Radius being given, as also the Tangent of an arch less than 45. degrees, the Tangent of the double of that arch is given also, without extracting any Root.

This Theorem was found out by the Learned Dr. *John Pell*, who thereby hath clearly shew'd the falsity of *Longomontanus*'s pretended quadrature of the Circle, as appears by the Controversie between them, (printed at *Amsterdam* in 1647.) where the Demonstrations of divers eminent Mathematicians to manifest the truth of the Theorem are inserted, which may also be easily demonstrated by the steps of the preceding Resolution, by proceeding in a direct order from the beginning to the end thereof; but leaving that Demonstration to the Learner's practice, I shall give another which here follows.

Suppos.



23. <ABC and<AEF are right angles.

24. ACF and ABE are right lines.

25. < DAB less than half a right angle.

26. < DAB = < DAC = ½ < FAE.

27. . . *Req. demonstr.* . . . . . □AB − □BD . 2□AB :: BD . BC.

Demonstration.

28. By *Supposition*, ...	< ABC = ⌋ = < AEF.
29. By *Suppos.* also, ...	< FAE = 2< DAB.
30. Therefore, (*per Theor.* 2. *angular. sect.* Viet.)	AE . EF :: □AB−□BD . 2▭AB,BD.
31. And because from 23° and 24° △ AEF and △ ABC are like, it's manifest (*per* 4. *prop.* 6. *Elem.*) that .	AE . EF :: AB . BC.
32. Therefore from the two last preceding Analogies, (*per* 11.*prop.* 5.*Elem*)	□AB−□BD . 2▭AB,BD :: AB . BC.
33. But by reason of the common altitude 2 AB, this Analogy is manifest, (*per* 1.*prop.* 6. *Elem.*) *viz.*	2▭AB,BD . 2□AB :: BD . AB.
34. Wherefore from the two last preceding Analogies, (*per* 23.*prop.* 5.*Elem* agreeable to *Defin.* 8. of Inordinate proportion in the foregoing *Chap.* 3.)	□AB−□BD . 2□AB :: BD . BC. Which was to be dem.

Illustration by Numbers.

35. Let the Radius AB be	. . . . . 100000.
36. Then supposing the arch BG or angle DAB to be 15. degrees, its Tangent BD (by the Trigonometrical Canon) is	. . . . . 26795.
37. Now to find out BC, the Tangent of 30. degrees, (the double of the given angle,) according to the preceding Theorem, multiply the given Tangent 26795, by the double Square of the Radius, *viz.* by	. . 20000000000.
38. The Product is to be reserved for a Dividend, to wit, .	535900000000000.
39. Then from the Square of the Radius, *viz.* from . . .	. . . 10000000000.
40. Subtract the Square of the given Tangent 26795, *viz.* .	. . . . 717972025.
41. The remainder is a Divisor, to wit, ...	. . . 9282027975.
42. Lastly, divide the Dividend in the 38th step, by the Divisor in the 41th, and the Quotient, neglecting the remainder, is	. . . . . 57735.

Which 57735 is the desired Tangent of 30. degrees, the double of the given angle or arch 15. degrees, and exactly agrees with the Trigonometrical Canon, the Radius being assumed 100000.

Probl. XVI.

The Radius of a Circle, and the Tangents of two arches being given severally, to find the Tangents of the summ and difference of those arches. But when the Tangent of the summ is required, the Rectangle of the given Tangents must be less than the Square of the Radius.

In finding out the Demonstration of the Theorem in the 22th step of the foregoing *Probl.* 15. I discovered the two following Theorems, to solve this *Probl.* 16. which I have not met with in any Author.

THEOR. 1.

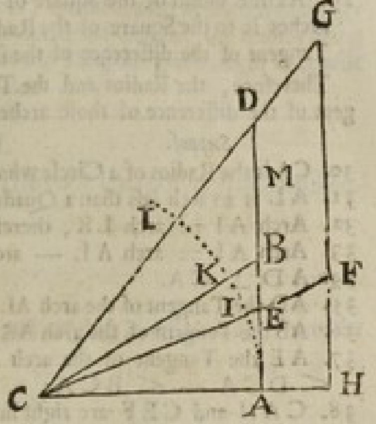
THEOR. 1.

As the excess whereby the Square of the Radius exceeds the Rectangle of the Tangents of two arches, is to the Square of the Radius; so is the summ of those Tangents, to the Tangent of the summ of the said arches.

Therefore the Radius of a Circle, and the Tangents of two arches whose summ is less than a Quadrant, being given severally, the Tangent of the summ of those arches shall be given also.

Suppos.

1. CA is the Radius of a Circle whose Center is C.
2. AL an arch less than a Quadrant.
3. Arch LK = arch AI, therefore
4. Arch AL = arch AK + arch AI.
5. AD ⊥ CA.
6. AB the Tangent of the arch AK, or $\angle$ BCA.
7. AE the Tangent of the arch AI, or $\angle$ ECA.
8. AD the Tangent of the arch AL, or of $\angle$ BCA + $\angle$ ECA.
9. CAH and CDG are right lines.
10. HG ⊥ CAH.
11. BM = AE, therefore AM = BA + AE.



Req. demonstr.

12. $\square CA - \square BA,AE \;.\; \square CA :: \underset{BA+AE}{AM} \;.\; AD.$

Demonstration.

13. Because by *Supposition*, ... } $\angle CAD = \perp = \angle CHG$.
14. Also by *Suppos.* in the right-angled Triangles GCH, BCA and ECA, ... } $\angle GCH = \angle BCA + \angle ECA$.
15. Therefore, (*per Theor.* 2. *Angular. section.* Viet.)

$$CH \;.\; HG :: \square CA - \square BA,AE \;.\; \square AM,CA.$$

16. And because by *Suppos.* in 5°, 9° and 10°, △CAD and △CHG are equiangular, therefore, (*per* 4. *prop.* 6. *Elem.*)

$$CH \;.\; HG :: CA \;.\; AD.$$

17. Therefore from the two last preceding Analogies, (*per* 11. *prop.* 5. *Elem.*)

$$\square CA - \square BA,AE \;.\; \square AM,CA :: CA \;.\; AD.$$

18. But by reason of the common altitude CA this following Analogy is manifest, (*per* 1 *prop.* 6. *Elem.*)

$$\square AM,CA \;.\; \square CA :: AM \;.\; CA.$$

19. Therefore from the two last preceding Analogies, (*per* 23. *prop.* 5. *Elem.* agreeable to *Defin.* 8. of Inordinate proportion, in the foregoing *Chap.* 3.)

$$\square CA - \square BA,AE \;.\; \square CA :: \underset{BA+AE}{AM} \;.\; AD.$$

Which was to be dem.

Illustration by Numbers.

20. Let the Radius CA be . . . . } . . . . . 100000.
21. Then supposing the arch AK to be 30. degrees, its Tangent AB by the Trigonometrical Canon is . . . . } . . . . . 57735.
22. Likewise supposing the arch AI to be 20. degrees, its Tangent AE is . . . . } . . . . . . 36397.

Now to find out AD the Tangent of the arch AL, 50. degrees, *viz.* the Tangent of the summ of the given arches AK, 30. degrees, and AI, (or KL,) 20. degrees, the Operation, according to the preceding *Theor.* 1. will be as it here follows, *viz.*

23. Multiply the summ of the given Tangents 57735 and 36397, to wit, ... } . . . . . . 94132.
24. By the Square of the Radius, *viz.* by . . . . . . } . . . 10000000000.
22. The Product shall be a Dividend, to wit, ... } 941320000000000.
26. Then from the said Square of the Radius in the 24th step, subtract the Product of the given Tangents, to wit, . . } . . . 2101380795.

27. The

27. The remainder is a Divisor, to wit, ... 7898619205.
28. Lastly, divide the Dividend in the 25th step, by the Divisor in the 27th and the Quotient, (neglecting the remainder,) is . . . . . . 119175.

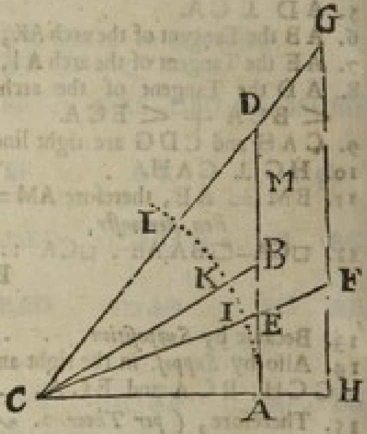
Which 119175 is the desired Tangent of 50. degrees, the summ of the given arches 30, and 20. degrees, and exactly agrees with the Trigonometrical Canon, the Radius being assumed 100000.

THEOR. 2.

29. As the summ of the Square of the Radius and the Rectangle of the Tangents of two arches is to the Square of the Radius; so is the difference of those Tangents, to the Tangent of the difference of the said arches.

Therefore, the Radius and the Tangents of two arches being severally given; the Tangent of the difference of those arches shall be given also.

Suppos.



30. CA is the Radius of a Circle whose Center is C
31. AL is an arch less than a Quadrant.
32. Arch AI = arch LK, therefore
33. Arch AI = arch AL — arch AK.
34. AD $\perp$ CA.
35. AD the Tangent of the arch AL or $\angle$ DCA.
36. AB the Tangent of the arch AK or $\angle$ BCA.
37. AE the Tangent of the arch AI, *viz.* of $\angle$ DCA — $\angle$ BCA.
38. CAH and CEF are right lines.
39. HF $\perp$ CAH.

Req. demonstr.

40. $\square$CA + $\boxed{\text{AD,AB}}$. $\square$CA :: BD . AE.
DA−AB

Demonstration.

41. Because by *Supposition* . . . . $\angle$ CAD = $\llcorner$ = $\angle$ CHG.
42. Also by *Suppos.* in the right-angled Triangles FCH, DCA, BCA, ... $\angle$FCH = $\angle$DCA − $\angle$BCA.
43. Therefore, (*per Theor.* 1. *Angular. section.* Viet.)
CH . HF :: $\square$ CA + $\boxed{\text{AD,AB}}$. $\boxed{\text{BD,CA}}$; (that is, $\overline{DA-AB}\times CA$.)
44. And because by *Suppos.* in 34°, 38° and 39°, $\triangle$ CAE and $\triangle$ CHF are equiangular, therefore, (*per* 4. *prop.* 6. *Elem.*)
CH . HF :: CA . AE.
45. Therefore from the two last preceding Analogies, (*per* 11. *prop.* 5. *Elem.*)
$\square$ CA + $\boxed{\text{AD, AB}}$. $\boxed{\text{BD, CA}}$ :: CA . AE.
46. But by reason of the common altitude CA, this following Analogy is manifest; (*per* 1. *prop.* 6. *Elem.*)
$\boxed{\text{BD, CA}}$. $\square$ CA :: BD . CA.
47. Therefore from the two last preceding Analogies, (*per* 23. *prop.* 5. *Elem.* agreeable to *Defin.* 8. of Inordinate proportion, in the foregoing *Chap.* 3.)
$\square$ CA + $\boxed{\text{AD,AB}}$. $\square$ CA :: BD . AE.
Which was to be dem. DA−AB

Illustration by Numbers.

48. Let the Radius CA be . . . . 100000.
49. Then supposing the arch AL to be 50. degrees, its Tangent AD by the Trigonometrical Canon is . . . . 119175.
50. Likewise, supposing the arch AK to be 30. degrees, its Tangent is . . . . 57735.

Now to find out AE the Tangent of the arch AI, 20. degrees, *viz.* the Tangent of the difference of the given arches AL, 50. degrees, and AK, 30. degrees, the Operation, according to the preceding *Theor.* 2. will be as followeth, *viz.*

51. Multiply the difference of the given Tangents 119175 and 57735, to wit, ... 61440.

52. By

52. By the Square of the Radius, *viz.* by . . . . 10000000000.
53. The Product shall be a Dividend, to wit, ... 6144000000000000.
54. Then to the said Square of the Radius, add the Product of the given Tangents 119175 and 57735, *viz.* . . . . . . 6880568625.
55. The summ is a Divisor, to wit, ... 16880568625.
56. Lastly, divide the Dividend in the 53^{th} step by the Divisor in the 55^{th}, and the Quotient, neglecting the remainder, will be . . . . 36396.

Which 36396 is the desired Tangent of 20. degrees, the difference of the given arches 50, and 30. degrees, and nearly agrees with the Trigonometrical Canon, the Radius being assumed 100000. But for more exactness, the Radius and Tangents given ought to consist of more places than those before exprest in 48°, 49° and 50°.

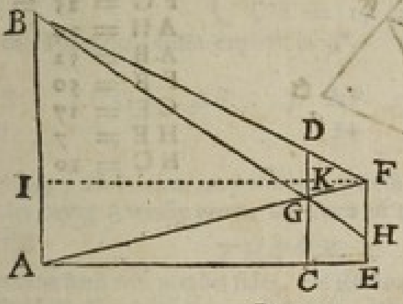
Probl. XVII.

To find the altitude of a Tower, when the foot thereof is inaccessible.

Suppos.

1. EA = AC + CE is an horizontal line.
2. CA part of EA is accessible only at C.
3. AB the height of a Tower.
4. CD and EF two staves perpendicular to EA, and at such a distance one from the other, that FDB is a right line.
5. FA a right line cutting CD in G.
6. BH a right line cutting CD in G.
7. FI = and || EA, and cuts CD in K.
8. b = DG,
9. d = CE, } are given severally.
10. f = FH;

Req. to find AC and AB.



DG = 24
CE = 30
FH = 25
AC = 720
AB = 600

Preparat.

11. Because in △ BFH, DG || FH, therefore, △ BFH and △BDG are like, therefore (*per prop.4. Elem.6.*) BF . BD :: FH . DG.
12. Likewise because in △ FBI, DK || BI, therefore BF . BD :: IF . IK.
13. Therefore out of 11° and 12°, (*per pro.11. Ele 5.*) FH . DG :: IF . IK.
14. That is, (because AE = IF, and AC = IK,) FH . DG :: AE . AC.

Resolution.

15. Put . . . . a = AC.
16. Therefore from 9° and 15°, ... $a + d$ = AE.
17. Then from the 8^{th}, 10^{th}, 14^{th}, 15^{th} and 16^{th} steps this Analogy ariseth, *viz.* . . . . $f \,.\, b :: a+d \,.\, a$.
18. Therefore by *Division of Reason*, ... $f-b \,.\, b :: d \,.\, a$.
19. Again, in order to find AB, add d, that is, CE, to $\frac{db}{f-b}$, (which by the 18^{th} step is manifestly equal to a,) that is, AC, and it makes . . . $\frac{fd}{f-b}$ = AE.
20. Then because △ FBA and △ FDG are like, (for DG || BA,) this Analogy will be manifest (by the *Lemma* prefixt to *Probl.* 11. of this *Chapt.*) *viz.* FK . GD :: FI . AB.

21. That

21. That is, by exchange of equal right lines, $\succ$ CE . GD :: AE . AB.

22. That is, in the letters belonging to the Resolution, ... $d \cdot b :: \frac{fd}{f-b} \cdot \frac{fb}{f-b}$ (AB.)

From the 18th, 21th and 22th steps ariseth this

CANON.

23. As the excess whereby FH exceeds DG, is to DG, so is CE to AC; and so is FH to AB.

The Demonstration of this Canon being very easie to be deduced from the premisses, I shall only illustrate it by

An Example in Numbers.

Suppose . . . $\left\{ \begin{array}{l} DG = b = 24 \\ CE = d = 30 \\ FH = f = 25 \end{array} \right\}$ Inches.

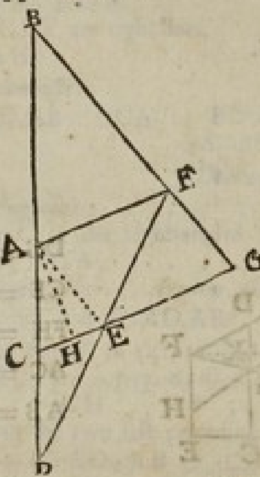
Then by the Canon,

$$\text{I.} \left\{ \begin{array}{ccccccc} FH-DG & . & DG & :: & CE & . & AC. \\ 1 & . & 24 & :: & 30 & : & 720. \end{array} \right.$$

$$\text{II.} \left\{ \begin{array}{ccccccc} FH-DG & . & DG & :: & FH & . & AB. \\ 1 & . & 24 & :: & 25 & . & 600. \end{array} \right.$$

Probl. XVIII.

To find the length of a right line AB, when there is access to one of its ends, suppose to A.



GC = 51
CA = 26
AF = 34 = GE
FG = 25 = AE
AH = 24
AB = 52
FB = 50
CE = 17
HE = 7
HC = 10

Preparat.

1. From the given point A measure two equal distances, as AC = CD, and so, as DCAB may be a right (or straight) line; measure likewise DE and EF, so as DE may be equal to EF and DEF a straght line, making an angle ADF at pleasure: then measure FA, which shall be parallel to a line drawn from C to E, (*per 2.prop.6. Elem.*) for DC . CA :: DE . EF; then measure CEG, that is, the parallel CE produced to such a Station G, that GFB may be a straight line, measure also GF. Now of the Trapezium CAFG the two sides AF and CG being parallels, and every one of the four sides CA, AF, FG, GC, given by measure as above, in Yards or what equal parts you please, the length of AB, also of FB may be found out in the same kind of parts, by the following Resolution.

Suppos.

2. . . . . $\left\{ \begin{array}{l} CAB \text{ and } GFB \text{ are right lines.} \\ AF \parallel CG. \\ b = CG \\ c = CA \\ d = AF \\ f = FG \end{array} \right.$ given severally.

Req. to find AB and FB.

Resolu-

Resolution.

3. Put . . . . $a = AB$.
4. Then because $\triangle$ BCG and $\triangle$ BAF are equiangular, for AF $\parallel$ CG, this Analogy is manifest, (*per 4. prop. 6. Elem.*) . . . . . CG . CB :: AF . AB.
5. That is, in the letters of the Resolution, . . . $b \;.\; c+a :: d \;.\; a$.
6. Therefore alternately, ... $b \;.\; d :: c+a \;.\; a$.
7. Therefore by *Division of Reason*, ... $b-d \;.\; d :: c \;.\; a$.
8. And by the like argumentation this Analogy will arise, ... $b-d \;.\; d :: f \;.\;$ FB.

The Analogies in the seventh and eighth steps afford this

THEOR. 1.

9. In a Trapezium CAFG having two parallel sides AF, CG; As the difference of those two parallel sides is to the lesser of them; so is each of the two other sides not parallels, to its continuation to the point of their concurrence, *viz.* As, CE . AF :: CA . AB :: FG . FB.

10. The Demonstration whereof is manifest; for drawing AE $\parallel$ FG, the Triangles CAE and ABF shall be like; and because AE $\parallel$ FG and AF $\parallel$ EG, it follows (*per prop. 34. Elem. 1.*) that AE = FG, also AF = EG, and CE = CG − AF. Wherefore CE . AF :: CA . AB :: FG . FB, as the Theorem affirms.

An Example in Numbers.

11. Suppose . . . . $\begin{cases} AF \parallel CG \text{ in the Trapezium CAFG; also,} \\ CG = 51. \\ CA = 26. \\ AF = 34 = GE. \\ FG = 25 = EA. \\ CE = 17 = CG - AF. \end{cases}$

12. Then by the Theorem before exprest in 9°,

$$\begin{matrix} CE & . & (CG - AF\;.) & AF \\ 17 & & & 34 \end{matrix} :: \begin{cases} CA \;.\; AB. \\ 26 \quad\; 52 \\ GF \;.\; FB. \\ 25 \quad\; 50 \end{cases}$$

Here by the way, the Learner may observe, that the four sides of a Trapezium having two parallel sides being severally given, the Area of that Trapezium shall be given also by this following

THEOR. 2.

13. If a Trapezium hath two parallel sides, the Rectangle or Product made by the multiplication of half the summ of the parallel sides, into the shortest distance between the said Parallels, shall be equal to the Area of the Trapezium.

14. . . *Suppos.* . . . . . $\begin{cases} AF \parallel CG \text{ in the Trapezium CAFG.} \\ CG \sqsubset AF. \\ AC \text{ not } \parallel FG. \end{cases}$

Prepar.

15. Make AH $\perp$ CG; also GE = AF, and draw AE.

16. . . *Req. demonstr.* . . $\begin{cases} AH \times \overline{\tfrac{1}{2}CG + \tfrac{1}{2}AF} = \text{Trapez. CAFG.} \\ \text{That is, } AH \times \overline{EG + \tfrac{1}{2}CE} = \text{Trapez. CAFG.} \end{cases}$

Demonstration.

17. By *Supposition* in 14°, ... AF $\parallel$ CG.
18. And by *Constr.* in 15°, ... GE = AF and AH $\perp$ HG.
19. Therefore out of 17° and 18°, (*per 33, & 35. prop. 1. Elem.*) . . . . $AH \times EG$ = Parallelogr. EAFG.
20. And (*per Schol. 41. prop. 1. Elem.*) . . . . $AH \times \tfrac{1}{2}CE = \triangle$ CAE.
21. Therefore from the two last Equations,

$$AH \times \overline{EG + \tfrac{1}{2}CE} = EAFG + \triangle CAE = \text{Trapez. CAFG.}$$

Which was to be Demonstr.

An

An Example in Numbers.

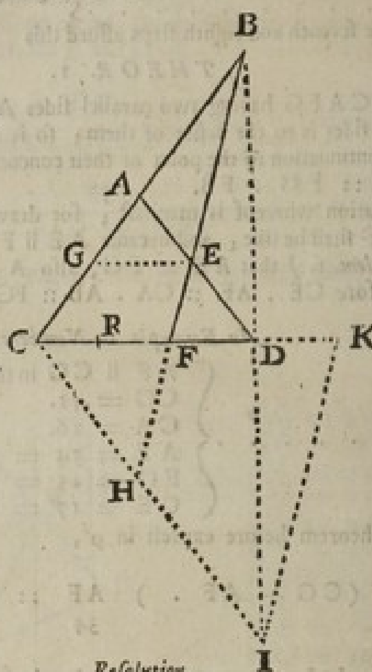
Let the same things be given before as in the 11th step, whence the three sides of the Triangle C E A will be 17, 26, 25, and by the help of these, the Perpendicular A H will be found 24, (by *Theor.* in 29° and 30° of *Probl.* 9. *Chap.* 7.) which multiplied by $42\frac{1}{2}$, that is, half the summ of the parallel sides A F, C G, gives 1020 for the Area of the Trapezium C A F G.

Probl. XIX.

Another way to measure a distance, when we can come to one end thereof.

Suppos.

1. A B a distance accessible only at A.
   - C A B a right line.
   - b = CA, c = AD, d = DC, f = AE, g = CF — given severally.
   - GE ∥ CD.
   - B E F a right line.



CA = 210
AD = 210
CD = 252
AE = 90
CF = 152
CK = $354\frac{6}{7}$
AB = $204\frac{6}{11}$

Req. to find A B.

Resolution.

2. Put . . . . a = A B.
3. Then because △ ADC and △ AEG are equiangular, (for GE ∥ CD,) it shall be, (*per prop.* 4. *Elem.* 6.) . . . . . AD . AE :: DC . GE.
4. That is, in the letters of the Resolution, . c . f :: d . $\frac{df}{c}$.
5. Likewise in the same △ADC and △AEG, AD . AE :: CA . GA.
6. That is, in the letters of the Resolution, c . f :: b . $\frac{bf}{c}$.
7. Again, in the equiangular Triangles CBF and G B E, ... CF . CA + AB :: GE . GA + AB.
8. That is, ... g . $b + a$ :: $\frac{df}{c}$. $\frac{bf}{c} + a$.
9. Which Analogy gives this Equation, *viz.* $\frac{bfg}{c} + ga = \frac{bdf}{c} + \frac{dfa}{c}$.
10. Therefore by subtracting $\frac{dfa}{c}$ from each part, (for g, or CF, is greater than $\frac{df}{c}$ or EG,) . . . . $\frac{bfg}{c} + ga - \frac{dfa}{c} = \frac{bdf}{c}$.
11. And by subtracting $\frac{bfg}{c}$ from each part of the last Equation, this ariseth, . . . $\overline{g - \frac{df}{c}}$ into $a = \frac{bdf - bfg}{c}$.
12. That is, ... $\overline{g - \frac{df}{c}}$ into $a = \frac{df - fg}{c}$ into b.

13. There-

13. Therefore by resolving the last preceding Equation into Proportionals, this Analogy ariseth, *viz.* . . . $\left.\right\}$ $g - \frac{df}{c} \;.\; \frac{df - fg}{c} :: b \;.\; a.$

14. And by multiplying each of the two first Terms of that Analogy by c, this ariseth, *viz.* . . . $\left.\right\}$ $gc - df \;.\; df - fg :: b \;.\; a.$

15. Let fm be made equal to gc, *viz.* As f to c, so g to a fourth, which may be called m, then setting fm in the place of gc in the first Term of the Analogy in the 14th step, this ariseth, *viz.* . . . . . . $\left.\right\}$ $fm - df \;.\; df - fg :: b \;.\; a.$

16. Therefore by dividing the first and second Terms of the last preceding Analogy by f, this ariseth, *viz.* $\left.\right\}$ $m - d \;.\; d - g :: b \;.\; a.$

Which Analogy last before exprest affords this

CANON.

17. Let it be made as f to c, so g to a fourth, which may be called m; (that is, in the Diagram, as AE to AD, or as CH to CI, so CF to CK;) then let it be made, as $m - d$ to $d - g$, so b to a the distance sought; (that is, in the Diagram, as KD to DF, or as ID to DB, so CA to AB.)

The Demonstration whereof, (in regard CI is parallel to AD, and IK to HFB,) is manifest (*per prop.* 2, & 4. *Elem.* 6.) the Proportionals in the Canon being compared to the respective lines in the Diagram belonging to this Problem. Which Canon may be demonstrated also by proceeding according to the steps of the Resolution, in a direct order, *viz.* from the beginning to the end; and although the Demonstration in this way be prolix, yet since 'tis certain, and may serve to improve the younger Analyst's skill in the Resolution of Plane Problems, I shall here form it at large.

Prepar.

18. Let it be made as . . . . . . > AD . AE :: CF . CR.

That is, ... > $c \;.\; f :: g \;.\; \frac{fg}{c}.$

Then the same things being supposed and made as in the first and seventeenth steps,

19. . . *Req. demonstr.* . . . . . KD . DF :: CA . AB.

Demonstration.

20. Forasmuch as △ ADC and △ AEG are like, it shall be (*per prop.* 4. *Ele.* 6.) $\left.\right\}$ AD . AE :: DC . GE.

That is, in the fourth step, . . . > $c \;;\; f :: d \;;\; \frac{df}{c}.$

21. Again, in the same △ ADC and △ AEG, ... $\left.\right\}$ AD . AE :: CA . GA.

That is, in 6°, ... > $c \;.\; f :: b \;.\; \frac{bf}{c}.$

22. And because △ CBF and △ GBE are like, it shall be . . . . . $\left.\right\}$ CF . CA+AB :: GE . GA+AB.

That is, in 8°, ... > $g \;.\; b + a :: \frac{df}{c} \;.\; \frac{bf}{c} + a.$

23. Therefore from the 22th step, this Analogy is manifest, (*per* 16. *prop. Elem.* 6.) *viz.*

$$\square GA,CF + \square CF,AB = \square CA,GE + \square GE,AB.$$

That is, in 9°, $\left.\right\}$ $\frac{bfg}{c} + ga = \frac{bdf}{c} + \frac{dfa}{c}.$

24. And by subtracting □ GE, AB from each part of the Equation in the 23th step, this remains, *viz.*

$$\square GA,CF + \square CF,AB - \square GE,AB = \square CA,GE.$$

That is, in 10°, $\left.\right\}$ $\frac{bfg}{c} + ga - \frac{dfa}{c} = \frac{bdf}{c}.$

25. And by subtracting □ GA, CF from each part of the Equation in the 24th step, this remains, *viz.*

$$\square CF,AB - \square GE,AB = \square CA,GE - \square GA,CF.$$

That is, in 11°, $\left.\right\}$ $g - \frac{df}{c}$ into $a = \frac{bdf - bfg}{c}.$

That is, in 12°, $\left.\right\}$ $g - \frac{df}{c}$ into $a = \frac{df - fg}{c}$ into $b.$

Now that $\square CR,CA$ may be set in the place of $\square GA,CF$ in the latter part of the Equation in the 25th step, I shall by the four following steps prove that $\square CR, CA = \square GA, CF$. First then,

26. By the 21th step, ... } $AD \,.\, AE :: CA \,.\, GA.$
27. And by the 18th step, ... } $AD \,.\, AE :: CF \,.\, CR.$
28. Therefore from the 26th and 27th steps, (*per* 11. *prop.* 5. *Elem.*) . . . . } $CA \,.\, GA :: CF \,.\, CR.$
29. Therefore from the 28th step, (*per* 16. *prop. El.* 6.) } $\square CR, CA = \square GA, CF.$
30. Therefore by setting $\square CR, CA$ in the place of $\square GA, CF$ in the latter part of the Equation in the 25th step, that Equation will be converted into this, *viz.*

$$\overline{\square CF,AB - \square GE,AB} = \square CA,GE - \square CR,CA.$$

That is, in 12°, $\quad g - \frac{df}{c}$ into $a = \frac{df - fg}{c}$ into b.

31. Therefore by resolving the Equation in the 30th step into Proportionals, this Analogy ariseth, *viz.*

$$CF - GE \,.\, GE - CR :: CA \,.\, AB.$$

That is, in 13°, $\quad g - \frac{df}{c} \,.\, \frac{df - fg}{c} :: b \,.\, a.$

32. And by drawing AD into the first and second Terms of the Analogy in 31°, this will be manifest, (*per* 1. *prop.* 6. *Elem.*)

$$\square CF,AD - \square GE,AD \,.\, \square GE,AD - \square CR,AD :: CF - GE \,.\, GE - CR.$$

33. Therefore out of 31° and 32°, (*per* 11. *prop.* 5. *Elem.*)

$$\square CF,AD - \square GE,AD \,.\, \square GE,AD - \square CR,AD :: CA \,.\, AB.$$

That is, in 14°, $\quad gc - df \,.\, df - fg :: b \,.\, a.$

34. And because from the *Construction* in 17°, . . } $\square CK,AE = \square CF,AD.$
35. And out of 20°, ... } $\square DC,AE = \square GE,AD.$
36. And out of 18°, ... } $\square CF,AE = \square CR,AD.$
37. Therefore out of 33°, 34°, 35° 36°, by exchange of equal Rectangles, this Analogy ariseth, *viz.*

$$\square CK,AE - \square DC,AE \,.\, \square DC,AE - \square CF,AE :: CA \,.\, AB.$$

That is, in 15°, $\quad fm - df \,.\, df - gf :: b \,.\, a.$

38. And this following Analogy, by reason of the common altitude AE, will be manifest by *prop.* 1. *Elem.* 6.

$$\square CK,AE - \square DC,AE \,.\, \square DC,AE - \square CF,AE :: CK - DC \,.\, DC - CF.$$

39. Therefore out of 37° and 38°, (*per* 11. *prop.* 5. *Elem.*)

$$CK - DC \,.\, DC - CF :: CA \,.\, AB.$$

That is, in 16°, $\quad m - d \,.\, d - g :: b \,.\, a.$

40. Therefore from 39°, respect being had to the Diagram,

$$KD \,.\, DF :: CA \,.\, AB.$$

Which was to be Dem.

Divers other Canons might be deduced from the premisses, which for brevity sake I shall pass over; but to satisfie such Artists as delight in Problems of this kind, because they may be practically applied to measure distances in the field without any Mathematical Instrument to observe angles, I shall illustrate the preceding Canon (in 17°,) by

An Example in Numbers.

$$\text{Suppose} \;\ldots\; \left\{ \begin{array}{l} CA = b = 210 \\ AD = c = 210 \\ CD = d = 252 \\ AE = f = 90 \\ CF = g = 152 \end{array} \right\} \text{Feet.}$$

Then by the Canon in 17° of this *Probl.*

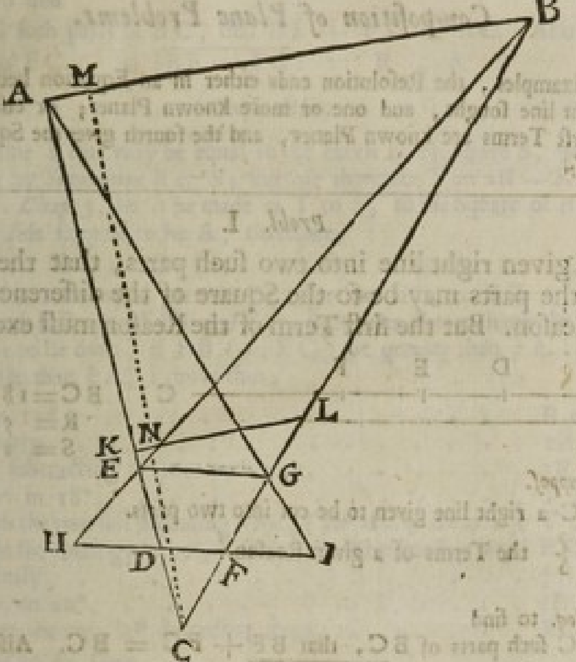
$$\text{I.} \left\{ \begin{array}{ccccccc} AE & . & AD & :: & CF & . & CK. \\ 90 & . & 210 & :: & 152 & . & 354\tfrac{2}{3}. \end{array} \right.$$

$$\text{II.} \left\{ \begin{array}{ccccccc} CK - CD & . & DC - CF & :: & CA & . & AB. \\ \text{or, } KD & . & DF & :: & CA & . & AB. \\ 102\tfrac{2}{3} & . & 100 & :: & 210 & . & 204\tfrac{6}{11}. \end{array} \right.$$

Probl. XX.

Probl. XX.

To find the length of a right line A B, when neither of its ends A or B is acceſſible.



Make choice of ſome ſtation C, where you may both ſee, and have liberty to meaſure towards A and B; then meaſure two equal lines C D and D E, in Yards or Feet, *&c.* ſo as C D E A may be a right (or ſtraight) line; likewiſe meaſure C F and F G equal to one another, and ſo as C F G B may be a ſtraight line. Meaſure alſo E G, which ſhall be parallel to a line drawn from D to F, (*per* 2. *prop.* 6. *Elem.*) for C D . D E :: C F . F G. Then ſuppoſing D F to be produced to H and I, to wit, ſuch ſtations that H E B and I G A may be right lines, meaſure H E and G I; alſo H F and D I. Now there are given ſeverally the four ſides of the Trapezium H E G F, having two parallel ſides E G and H F, therefore the length of G B ſhall be given alſo by the preceding *Probl.* 18. In like manner, the four ſides of the Trapezium D E G I having two parallel ſides E G, D I, are given ſeverally, therefore the line E A ſhall be given alſo by the ſaid *Probl.* 18. Then E A and G B, alſo C E and C G being given (as before,) the lines C A and C B ſhall be given alſo, by Addition. Then take ſome Aliquot part of C A, ſuppoſe one third part, and meaſure it from C to K; meaſure likewiſe one third part of C B from C to L; ſo ſhall K L be parallel to the inacceſſible line A B ſought, (*per prop.* 2. *Elem.* 6.) Then meaſure K L. Laſtly, as C K to K L, ſo is C A to A B; or as C L to L K, ſo is C B to B A; but C K, K L, C A, C L, C B are given in Yards or Feet, *&c.* (as before,) and therefore A B the diſtance ſought is given alſo in the ſame kind of meaſure.

If you deſire to find the ſhorteſt diſtance from C to A B, to wit, the Perpendicular C M, it may be found by the help of the three ſides of the Triangle C A B, whoſe meaſures were before diſcovered. Or if you would draw a line in the field from C towards the inacceſſible line A B, that ſhall fall perpendicularly upon the ſaid A B produced infinitely, you may by the three ſides of the Triangle C K L, and by the *Coroll.* in 45° of *Probl.* 10. of this *Chapt.* diſcover whether a Perpendicular from C will fall within or without the Triangle CKL, and alſo find the diſtance from either end of the Baſe K L to the foot of the Perpendicular, which diſtance being meaſured in the field accordingly, the Perpendicular C N may be drawn, which being produced, ſhall be alſo perpendicular to the inacceſſible line AB. But how this Problem may be ſolved by meaſuring only five right lines, I ſhall hereafter ſhew in *Probl.* 20. *Chap.* 9.

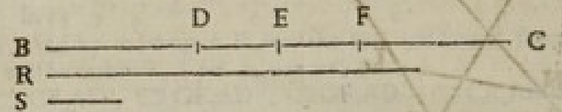
CHAP. VIII.

The second Classis *of Examples of the Resolution and Composition of Plane Problems.*

IN which Examples, the Resolution ends either in an Equation between the Square of the right line sought, and one or more known Planes; or else in an Analogy whose three first Terms are known Planes, and the fourth gives the Square of the right line sought.

Probl. I.

To cut a given right line into two such parts, that the summ of the Squares of the parts may be to the Square of the difference of the parts in a given Reason. But the first Term of the Reason must exceed the latter.



BC = 18	DF = 6
R = 5	BF = 12
S = 1	FC = 6

Suppos.

1. $b = BC$ a right line given to be cut into two parts.
2. $\left\{ \begin{array}{l} r = R \\ s = S \end{array} \right\}$ the Terms of a given Reason.
3. $r \sqsubset s$.

Req. to find

4. BF and FC such parts of BC, that $BF + FC = BC$. Also;
5. $\square BF + \square FC \,.\, \square :\overline{BF - FC}: \;::\; R \,.\, S$.

Resolution.

6. Put a for the difference of the parts sought, *viz.* . . . $a = BF - FC$.
7. Therefore the Square of the difference of the parts is . . . aa.
8. And from 1° and 6°, the summ of the Squares of the parts (*per Theor.* 6. *Chap.* 4.) is . . . . $\frac{1}{2}bb + \frac{1}{2}aa$.
9. Therefore according to the tenour of the Problem these must be Proportionals, *viz.* . . . . $r \,.\, s :: \frac{1}{2}bb + \frac{1}{2}aa \,.\, aa$.
10. Whence, by doubling the Antecedents, this Analogy ariseth, ... $2r \,.\, s :: bb + aa \,.\, aa$.
11. Therefore by *Division of Reason*, ... $2r - s \,.\, s :: bb \,.\, aa$.

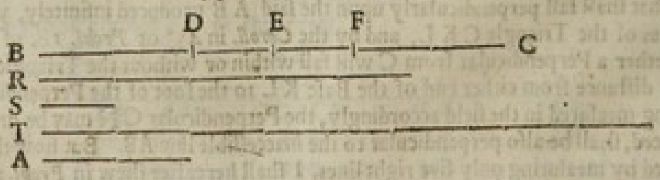
Which last Analogy gives this

CANON.

12. As the excess whereby the double of the first Term of the given Reason exceeds the latter Term, is to the latter Term; so is the Square of the line given to be cut into two parts, to the Square of the difference of the parts. Therefore the difference of the parts is given, and consequently the parts are given severally by *Theor.* 9. *Chap.* 4.

The reason of the Determination annex'd to the Problem is evident by *Theor.* 5. *Chap.* 4. which shews, that if a right line be divided into two unequal parts, the summ of the Squares of the parts is greater than the Square of the difference of the parts, by the double Rectangle of the parts.

The Composition of the foregoing Probl. 1.



Suppos.

Suppos.

13. B C is a right line given to be cut into two parts.
14. R and S are the Terms of a given Reason.
15. $R \sqsubset S$.

Req. to find

16. B F and F C such parts of B C, that $BF + FC = BC$. Also,
17. $\square BF + \square FC \;.\; \square : \overline{BF - FC} : \;::\; R \;.\; S$.

Construction.

18. Find a right line T that may be equal to the excess of 2R above S, which is possible to be done, for by *Supposition* $R \sqsubset S$; suppose therefore $T = 2R - S$.
19. By *Probl.* 11. *Chap.* 5. let it be made as T to S, so the Square of BC to another Square, whose side suppose to be A, therefore,

$$T \;.\; S \;::\; \square BC \;.\; \square A.$$

20. Divide B C into two equal parts in E, therefore $EB = EC$.
21. From E C and E B cut off E F and E D, such parts, that each may be equal to $\frac{1}{2}A$; which is possible to be done, if EB ($= EC$) be greater than $\frac{1}{2}A$. But that E B or E C is greater than $\frac{1}{2}A$, I prove thus;

By *Suppos.* in 15°, ...	$R \sqsubset S$.
And consequently, ...	$2R \sqsubset 2S$.
Therefore by subtracting S from each part, ...	$2R - S \sqsubset S$.
But by *Constr.* in 18°, ...	$T = 2R - S$.
Therefore from the two last preceding steps, (*per Ax.* 4. *Chap.* 2.)	$T \sqsubset S$.
Therefore from the Analogy in 19°, and from the last preceding step,	$BC \sqsubset A$.
And consequently, ...	$\frac{1}{2}BC \sqsubset \frac{1}{2}A$.
But by *Constr.* in 20°, ...	$\frac{1}{2}BC = EB = EC$.
Therefore from the two last preceding steps, ...	EB or $EC \sqsubset \frac{1}{2}A$.

Which was to be Dem.

22. I say B F and F C are the desired parts of B C. For first, their summ is manifestly equal to B C; and by *Constr.* in 20° and 21° the difference between the said parts B F and FC, that is, $BF - FC$ (BD) is equal to D E. But that the summ of the Squares of the parts B F and F C, is to the Square of their difference D F, as R to S, I shall demonstrate by a repetition of the steps of the foregoing Resolution in a backward order.

23. . . . *Req. demonstr.* . . . . . . $R \;.\; S \;::\; \square BF + \square FC \;.\; \square DF$.

Demonstration.

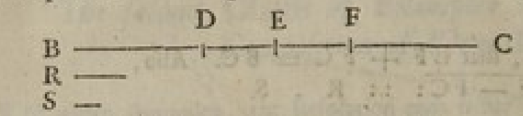
24. By *Constr.* in 19°, ...	$T \;.\; S \;::\; \square BC \;.\; \square A$.
25. And by *Constr.* in 18° and 21°, .	$2R - S = T$. And $DF = A$.
26. Therefore from 24° and 25°, by exchange of equal quantities, . . .	$2R - S \;.\; S \;::\; \square BC \;.\; \square DF$.
That is, in 11°, ...	$2r - s \;.\; s \;::\; bb \;.\; aa$.
27. Therefore from 26°, by *Composition of Reason*, ...	$2R \;.\; S \;::\; \square BC + \square DF \;.\; \square DF$.
That is, in 10°, ...	$2r \;.\; s \;::\; bb + aa \;.\; aa$.
28. Therefore from 27°, by halving the Antecedents, ...	$R \;.\; S \;::\; \frac{1}{2}\square BC + \frac{1}{2}\square DF \;.\; \square DF$.
That is, in 9°, ...	$r \;.\; s \;::\; \frac{1}{2}bb + \frac{1}{2}aa \;.\; aa$.
29. By *Constr.* in 20° and 21°, BC is the summ, and D F the difference of the parts B F and F C, therefore (*per Theor.* 6. *Chap.* 4.)	$\square BF + \square FC = \frac{1}{2}\square BC + \frac{1}{2}\square DF$.
30. Therefore from 28° and 29°, by exchanging equal quantities, . . .	$R \;.\; S \;::\; \square BF + \square FC \;.\; \square DF$.

Which was to be Demonstr. Therefore that is done which was required by the Problem.

Probl. II.

Probl. II.

To cut a given right line into two such parts, that the Rectangle of the parts, to the Square of their difference, may have a given Reason.



$BC = 15$	$DF = 5$
$R = 2$	$BF = 10$
$S = 1$	$FC = 5$

Suppos.

1. $b = BC$ a right line given to be cut into two parts.
2. $\left\{ \begin{matrix} r = R \\ s = S \end{matrix} \right\}$ the Terms of a given Reason.

Req. to find

3. BF and FC such parts of BC, that $BF + FC = BC$. Also,
4. $\square\, BF, FC \;.\; \square : \overline{BF - FC} : \;::\; R \;.\; S.$

Resolution.

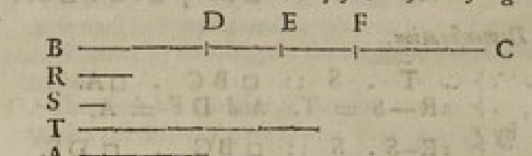
5. Put a for the difference of the parts sought, *viz.* $a = BF - FC$.
6. Therefore from 1° and 5°, (*per Theor. 9. Chap. 4.*) the greater part shall be . . . . $\frac{1}{2}b + \frac{1}{2}a$ ($= BF$.)
7. And by the same Theorem, the lesser part shall be $\frac{1}{2}b - \frac{1}{2}a$ ($= FC$.)
8. Therefore from 6° and 7°, the Rectangle (or Product) of the parts is . . . . $\frac{1}{4}bb - \frac{1}{4}aa$.
9. And from 5°, the Square of the difference of the parts is . . . . aa.
10. Therefore from 4°, 8° and 9°, according to the tenor of the Problem, this Analogy ariseth, *viz.* . $r \;.\; s \;::\; \frac{1}{4}bb - \frac{1}{4}aa \;.\; aa.$
11. Whence, by quadrupling the Antecedents, . . $4r \;.\; s \;::\; bb - aa \;.\; aa.$
11. Therefore by *Composition of Reason*, ... $4r + s \;.\; s \;::\; bb \;.\; aa.$

Which last Analogy affords this *CANON.*

13. As the summ of the second Term of the given Reason and the quadruple of the first is to the second Term; so is the Square of the line given to be divided into two parts, to the Square of the difference of the parts.

Thereefore the difference of the parts is given, and consequently the parts severally, by *Theor. 9. Chap. 4.*

The Composition of the foregoing Probl. 2.



$BC = 15$	$A = 5$
$R = 2$	$BF = 10$
$S = 1$	$FC = 5$
$T = 9$	$DF = A$

Suppos.

14. BC is a right line given to be cut into two parts.
15. R and S are the Terms of a given Reason.

Req. to find

16. BF and FC such parts of BC, that $BF + FC = BC$. Also,
17. $\square\, BF, FC \;.\; \square : \overline{BF - FC} : \;::\; R \;.\; S.$

Construction.

18. Find a right line $T = 4R + S$.
19. By *Probl. 11. Chap. 5.* let it be made as T to S, so the Square of BC to another Square, whose side suppose to be A, therefore,

$$T \;.\; S \;::\; \square BC \;.\; \square A.$$

20. Divide BC into two equal parts in E, therefore $EC = EB$.
21. From EC and EB cut off EF and ED, such parts, that each may be equal to $\frac{1}{2}A$, which may be done, if EC ($= EB$) be greater than $\frac{1}{2}A$; but that EC or EB is greater than $\frac{1}{2}A$, I prove thus;

By

By *Construction* in 18°, ... $T \sqsubset S$.
Therefore from 19°, ... $BC \sqsubset A$.
And consequently, ... $\frac{1}{2}BC \sqsubset \frac{1}{2}A$.
But by *Constr.* in 20°, ... $\frac{1}{2}BC = EC = EB$.
Therefore from the two last preceding steps, ... EC or $EB \sqsubset \frac{1}{2}A$.

Which was to be Dem.

22. I say BF and FC are such parts of BC as will satisfie the Problem. For first; $BF + FC = BC$; and by *Construction* in 20° and 21°, the difference between the said parts BF and FC, that is, $BF - FC$ (BD) is equal to DF. But that the Rectangle of the said parts BF and FC is to the Square of their difference DF as R to S; the following Demonstration, form'd out of the preceding Resolution by a repetition of its steps in a backward order will make manifest.

23. . . . *Req. demonstr.* . . . . . . $R . S :: \square BF, FC . \square DF$.

Demonstration.

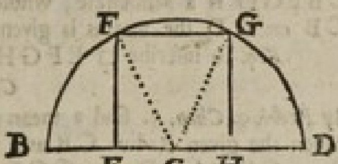
24. By *Constr.* in 19°, ... $T . S :: \square BC . \square A$.
25. And by *Constr.* in 18° and 21°, . . $4R + S = T$. And $DF = A$.
26. Therefore from 24° and 25°, by exchange of equal quantities, ... $4R + S . S :: \square BC . \square DF$.
27. Therefore by *Division of Reason*, . . $4R . S :: \square BC - \square DF . \square DF$.
28. And by taking $\frac{1}{4}$ of the Antecedents in 27°, $R . S :: \frac{1}{4}\square BC - \frac{1}{4}\square DF . \square DF$.
29. By *Constr.* in 20° and 21°, BC is the summ, and DF the difference of the parts BF and FC; therefore by *Theor.* 7. *Chap.* 4. $\square BF, FC = \frac{1}{4}\square BC - \frac{1}{4}\square DF$.
30. Therefore from 28° and 29°, by exchange of equal quantities, ... $R . S :: \square BF, FC . \square DF$.

Which was to be Dem. Therefore that is done which was required by the Problem.

A LEMMA, *leading to the following* Probl. 3.

If a Square, or long Square be inscribed in a Semicircle, the Center of the Semicircle is in the middle of the Base of the Square, or long Square.

Suppos.



1. CBFGD a Semicircle, whose Center is C.
2. EFGH is a Square.
3. . . . *Req. demonstr.* . $CE = CH$.

Prepar.

4. Draw the right lines CF and CG.

Demonstration.

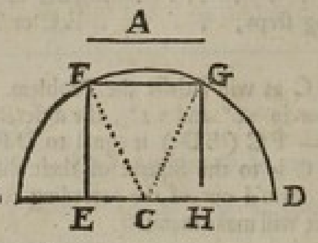
5. By *Defin.* 15. *Elem.* 1. . . . . $CF = CG$.
6. And by *Suppos.* in 2°, ... $\angle FEC = \perp = \angle GHC$.
7. Therefore (*per prop.* 47. *Elem.* 1.) . . $\square EF + \square CE = \square CF = \square CG$.
8. Likewise, ... $\square HG + \square CH = \square CG = \square CF$.
9. Therefore from 7° and 8°, (*per Ax.* 1. *Chap.* 2.) . . . . $\square EF + \square CE = \square HG + \square CH$.
10. But by *Suppos.* . . . . $EF = HG$.
11. And consequently, ... $\square EF = \square HG$.
12. Therefore by subtracting $\square EF$ or $\square HG$ from each part of the Equation in 9°, the remainders will be equal, *viz.* . . . $\square CE = \square CH$.
13. Therefore . . . . $CE = CH$.

Which was to be Dem. The same Demonstration may be made when a long Square is inscribed in a Semicircle.

Probl. III.

Probl. III.

To inscribe a Square in a given Semicircle.



$CB = CD = 10$

$EF = EH = \sqrt{80}$

$CE = CH = \sqrt{20}$

$EB = HD = 10 - \sqrt{20}$

Suppos.

1. CBFGD is a Semicircle ; whose Center is C.
2. $r = CB = CD$ is given.

Req. to inscribe $\square$ EFGH.

Resolution.

3. Put a for the side of the Square required, *viz.* . . . $a = EF = EH$.
4. Therefore by the preceding *Lemma*, ... $\frac{1}{2}a = CE = CH$.
5. And because (*per prop.* 47. *Elem.* 1.) . . . . . $\square EF + \square CE = \square CF$;
6. Therefore in the letters of the Resolution, . . . $aa + \frac{1}{4}aa = rr$.
7. Therefore, by multiplying the last Equation by 4, . . $4aa + aa = 4rr$.
8. That is, ... $5aa = 4rr$.
9. Therefore by taking $\frac{1}{5}$ of the last Equation, . . . $aa = \frac{4}{5}rr$.
10. Therefore, by extracting the square Root out of each part of the last preceding Equation, ... $a = \sqrt{:\frac{4}{5}rr}$.

Hence this

CANON.

11. The square Root of $\frac{4}{5}$ parts of the Square of the Semidiameter is equal to the side of the Square inscribed in the Semicircle.

The Composition of the foregoing Probl. 3.

Suppos.

12. CBFGD is a Semicircle, whose Center is C.
13. CB = CD the Radius is given.

Req. to inscribe $\square$ EFGH.

Construction.

14. By *Probl.* 9. *Chap.* 5. find a mean proportional line A between the given Radius CB and $\frac{4}{5}$ CB, therefore $CB . A :: A . \frac{4}{5}CB$.
15. From CB and CD cut off CE and CH, such segments, that as well CE as CH may be equal to $\frac{1}{2}$ A, and consequently $EH = A$; which Effection is possible, for by *Construction* in 14° CB is the greatest of three Proportionals, whereof A is the mean, therefore $CB \sqsubset A$, and because $CD = CB$, therefore also $CD \sqsubset A$; and consequently a segment equal to $\frac{1}{2}$A, as CE or CH may be cut off from CB or CD.
16. Make EF and $HG \perp BD$, and draw FG, so is EFGH the Square required to be inscribed, as will be evident by the following Demonstration, form'd out of the preceding Resolution, by a repetition of its steps in a backward (not direct) order.
17. . . *Req. demonstr.* . . . . EFGH is a $\square$.

Preparat.

18. Draw the right lines CF and CG.

Demonstration.

19. By *Constr.* in 14°, ... $CB . A :: A . \frac{4}{5}CB$.
20. And by *Constr.* in 15°, ... $EH = A$.
21. Therefore from 19° and 20°, by exchanging equal right lines, ... $CB . EH :: EH . \frac{4}{5}CB$.
22. And from 21°, (*per prop.* 17. *Elem.* 6.) . . . $\square EH = \frac{4}{5} \square CB$.
23. But the quintuples of equal quantities are also equal, therefore from 22°, ... $5 \square EH = 4 \square CB$.

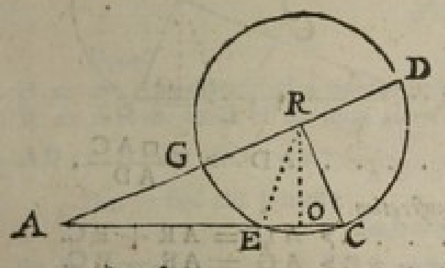
24. That

24. That is . . . . $4\square EH + \square EH = 4\square CB.$
25. And by taking $\frac{1}{4}$ of all in 24°, ... $\square EH + \frac{1}{4}\square EH = \square CB = \square CF.$
26. And because by *Constr.* in 15°, ... $CE = \frac{1}{2}EH = CH.$
27. And consequently, (*per Theor.* 3. *Chap.* 4.) $\square CE = \frac{1}{4}\square EH.$
28. Therefore from 25° and 27°, by exchanging equal quantities, ... $\square EH + \square CE = \square CF.$
29. And because by *Constr.* in 16°, ... $\angle FEH = \perp = \angle GHE.$
30. And consequently, (*per prop.* 47. *Elem.* 1.) $\square EF + \square CE = \square CF.$
31. Therefore from 28° and 30°, (*per Ax.*1.*Ch.*2.) $\square EH + \square CE = \square EF + \square CE.$
32. Therefore from 31°, by subtracting $\square$ CE from from each part, ... $\square EH = \square EF.$
33. But the sides of equal Squares are also equal, therefore from 32°, ... $EH = EF.$
34. And by arguing in like manner as in the six last preceding steps, it will be manifest that . . . $EH = HG.$
35. And because from 33°, 34° and 29°, (*per Ax.* 1. & *prop.* 28. *Elem.* 1.) . . . . . EF = and ∥ HG.
36. Therefore from 35°, (*per prop.* 33. *Elem* 1.) EH = and ∥ FG.
37. Therefore from 33°, 34° and 36°, . . . EFGH is equilateral.
38. And from 29°, and *Coroll. prop.* 29. *Elem* 1. EFGH is right-angled.
39. Therefore from 37° and 38°, (*per def.*29.*El.*1.) EFGH is a $\square$.

Which was to be Demonstr. Therefore the Problem is satisfied.

Probl. IV.

The Hypothenusal of a right-angled Triangle being given, as also the summ of the leggs containing the right angle, to find the Triangle. But the summ of the leggs must be greater than the Hypothenusal, yet not greater than the right line arising by application of the double Square of the Hypothenusal to the summ of the leggs.



AC = 169	AE = 119
AR = 156	EC = 50
RC = 65	OE = 25
AD = 221	OC = 25
AG = 91	RO = 60

Suppos.

1. ARC is a $\triangle$ right-angled at R.
2. b = AC the Hypothenusal is given.
3. b = AR + RC, the summ of the leggs is given.

Req. to find $\triangle$ ARC.

Resolution.

4. Supposing the leggs about the right angle to be unequal, to wit, AR ⊏ RC, put a for their difference, *viz.* . . $a = AR - RC = AG.$
5. Therefore from 3° and 4°, the summ of the Squares of the leggs (*per Theor.* 6. *Chap.* 4.) is . . . . $\frac{1}{2}bb + \frac{1}{2}aa.$
6. Therefore from 5° and 2°, (*per prop.* 47. *Elem.* 1.) this Equation ariseth, ... $\frac{1}{2}bb + \frac{1}{2}aa = bb.$
7. And by doubling each part of that Equation, ... $bb + aa = 2bb.$
8. And by subtracting bb from each part of the last Equation, $aa = 2bb - bb.$
9. Therefore by extracting the square Root out of each part of the last preceding Equation, ... $a = \sqrt{:2bb - bb:}$

Hence this

CANON.

10. The difference of the leggs about the right angle is equal to the square Root of the excess whereby the double Square of the Hypothenusal exceeds the Square of the summ of the leggs.

Therefore the difference of the leggs is given, and consequently by the given summ and difference of the leggs, the leggs shall be given severally, *per Theor. 9. Chap. 4.*

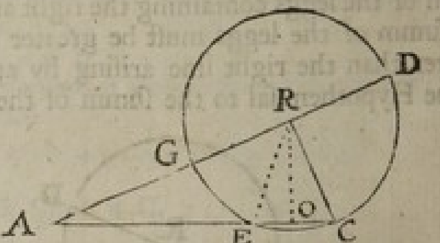
11. But in order to the Geometrical Effection of the Problem propounded, the truth and reason of the Determination annex'd to it must be made manifest. First then, the reason of the first part of the Determination, to wit, that the right line given for the summ of the leggs about the right angle must be longer than the line given for the Hypothenusal, is evident by *prop. 22. Elem. 1.* which shews that the summ of every two sides of a plain Triangle is greater (or longer) than the third. The latter part of the Determination is discovered by the Canon, which requires that bb be subtracted from $2hh$, and therefore bb must not exceed $2hh$, and consequently, by dividing as well bb as $2hh$ by b, 'tis manifest that b must not be greater than $\frac{2hh}{b}$; that is, the right line given for the summ of the leggs about the right angle, must not be longer than the right line arising by the Application of the double Square of the Hypothenusal to the summ of the leggs. The truth of this Determination will more fully appear by the following

THEOREM.

12. In a right-angled plain Triangle, the summ of the leggs about the right angle is sometimes less than the right line arising by the Application of the double Square of the Hypothenusal to the summ of the leggs, and sometimes equal to, but never greater than the said right line.

The leggs about the right angle are either unequal, or else equal between themselves; I shall begin with the first Case.

Suppos. in *Case* 1.



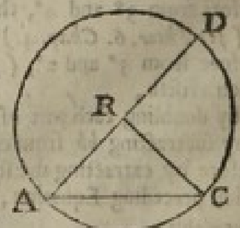
13. ARC is a $\triangle$.
14. $\angle ARC$ is $\perp$.
15. $AR \sqsubset RC$.
16. $RCGD$ is a $\odot$, whence
17. $AD = AR + RC (RD.)$
18. $AG = AR - RC (RG.)$

19. . . *Req. demonstr.* . . . . $AD \sqsupset \frac{2\square AC}{AD}$.

Demonstration.

20. By *Supposition* in 17°, ... $AD = AR + RC$.
21. And by *Suppos.* in 18°, ... $AG = AR - RC$.
22. Therefore from 20° and 21, (*per Theor. 6. Ch. 4.*) $\frac{1}{2}\square AD + \frac{1}{2}\square AG = \square AR + \square RC$.
23. And because by *Suppos.* in 14°, $\angle ARC$ is $\perp$, therefore (*per prop. 47. Elem. 1.*) . . . . $\square AC = \square AR + \square RC$.
24. Therefore from 22° and 23°, (*per Ax. 1. Ch. 2.*) $\frac{1}{2}\square AD + \frac{1}{2}\square AG = \square AC$.
25. And by doubling the last Equation, ... $\square AD + \square AG = 2\square AC$.
26. Therefore from 25°, ... $\square AD \sqsupset 2\square AC$.
27. Therefore by Application of each part to AD, . $AD \sqsupset \frac{2\square AC}{AD}$.
Which was *Case* 1. to be Dem.

Suppos. in *Case* 2.



28. ARC is a $\triangle$.
29. $\angle ARC$ is $\perp$.
30. $AR = RC = RD$.
31. $AD = AR + RC$.

32. . . *Req. demonstr.* . $AD = \frac{2\square AC}{AD}$.

Demon-

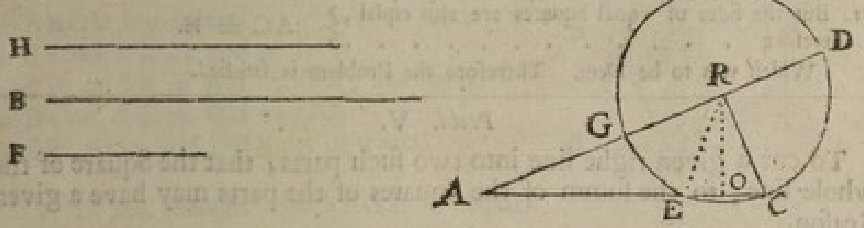
Demonstration.

33. Because by *Suppos.* in 30° and 31°, ... } $AR = RC = \frac{1}{2} AD$.
34. Therefore their Squares are also equal, *viz.* . . } $\square AR = \frac{1}{4} \square AD$.
35. Likewise from 33°, ... } $\square RC = \frac{1}{4} \square AD$.
36. The summ of the Equations in 34° and 35°, gives, } $\square AR + \square RC = \frac{1}{2} \square AD$.
37. And because by *Suppos.* < ARC is ⊥, therefore (*per prop.* 47. *Elem.* 1.) . . . . } $\square AR + \square RC = \square AC$.
38. Therefore from 36° and 37°, (*per Ax.* 1. *Chap.* 2.) } $\frac{1}{2} \square AD = \square AC$.
39. And by doubling the last Equation, ... } $\square AD = 2 \square AC$.
40. Therefore by Application of each part to AD, . . } $AD = \frac{2 \square AC}{AD}$.

Which was *Case* 2. to be Demonstr.

41. Now because in every right-angled Triangle, the sides about the right angle are either unequal or equal between themselves, and it hath been demonstrated, that when the said sides are unequal, their summ is less than the right line arising by the Application of the double Square of the Hypothenusal to the said summ; but when the said sides are equal to one another, their summ is equal to the said right line; it is evident that the summ of the sides about the right angle can never be greater than the right line arising by the said Application. Therefore the truth of the Theorem is manifest, and consequently the Hypothenusal and the summ of the leggs about the right angle must be given with due Caution, according to the import of the Determination annex'd to the Problem, that its Solution may be possible.

The Composition of the foregoing Probl. 4.



Suppos.

42. $H =$ the Hypothenusal of a right-angled Triangle is given.
43. $B = AD$ the summ of the leggs about the right angle is given.
44. $AD \sqsubset H$, but AD not $\sqsubset \frac{2 \square H}{AD}$. (*Determination.*)

Req. to find the Triangle.

Construction.

45. By the *Determination* in 44° AD is not greater than $\frac{2 \square H}{AD}$, suppose then it be granted, or discovered by H and B given in numbers, that AD is less than $\frac{2 \square H}{AD}$, and consequently, (by multiplying each part by AD,) that $\square AD \sqsupset 2 \square H$; then it evidently follows, that 'tis possible (*per Probl.* 4. *Chap.* 5.) to find out a right line F, such, that its Square shall be equal to $\overline{2 \square H - \square AD}$; suppose therefore

$$F = \sqrt{:2 \square H - \square AD:}$$

46. From AD cut off $AG = F$, which may be done, for that AD is greater than F, I prove thus;

By *Suppos.* in 44°, ... $AD \sqsubset H$.
Therefore . . . . $\square AD \sqsubset \square H$.
And by doubling each part, ... $2\square AD \sqsubset 2\square H$.
And by subtracting $\square AD$ from each part, ... $\square AD \sqsubset 2\square H - \square AD$.
But by *Constr.* in 45°, ... $\square F = 2\square H - \square AD$.
Therefore from the two last preceding steps, (*per Ax.* 3. *Ch.* 2.) $\square AD \sqsubset \square F$.
Therefore . . . . $AD \sqsubset F$.

Which was to be Dem.

47. Divide GD into two equal parts in R, therefore $RG = RD$.

 48. Make

48. Make $RC \perp AR$; also $RC = RD$ or RG, and draw AC.
49. I say ARC is the Triangle sought. Now we must shew that it will satisfie the Problem; first then by *Construction* in 48°, $RC \perp AR$, therefore the angle ARC is a right angle; secondly, the summ of the leggs AR and RC about the right angle, is equal to AD ($= B$) the given summ of the leggs. It remains only to prove that AC is equal to the given Hypothenusal H; but that will be made manifest by the following Demonstration, form'd out of the preceding Resolution by a repetition of its steps in a backward (not direct) order.

50. . . *Req. demonstr.* . . . . $AC = H$.

Demonstration.

51. By *Constr.* in 45°, ... $\sqrt{:2\square H - \square AD:} = F$.
52. And by *Constr.* in 46°, ... $AG = F$.
53. Therefore from 51° and 52°, (*per Ax.* 1. *Chap.* 2.) $AG = \sqrt{:2\square H - \square AD:}$
54. But the Squares of equal right lines are also equal, therefore from 53°, ... $\square AG = 2\square H - \square AD$.
55. Therefore from 54°, by adding $\square AD$ to each part, $\square AD + \square AG = 2\square H$.
56. And by taking the halves of all in 55°, . . . $\frac{1}{2}\square AD + \frac{1}{2}\square AG = \square H$.
57. And because by *Constr.* in 47° and 48°, AD is the summ, and AG the difference of the parts AR and RC, (or RD,) therefore (*per Theor.* 6. *Chap.* 4.) $\frac{1}{2}\square AD + \frac{1}{2}\square AG = \square AR + \square RC$.
58. Therefore from 56° and 57°, (*per Ax.* 1. *Cha.* 2.) $\square H = \square AR + \square RC$.
59. But by *Constr.* in 48°, $\angle ARC$ is $\lrcorner$, and consequently, (*per prop.* 47. *Elem.* 1.) . . . . $\square AC = \square AR + \square RC$.
60. Therefore from 58° and 59°, (*per Ax.* 1. *Chap.* 2.) $\square AC = \square H$.
61. But the sides of equal Squares are also equal, therefore . . . . $AC = H$.

Which was to be Dem. Therefore the Problem is satisfied.

Probl. V.

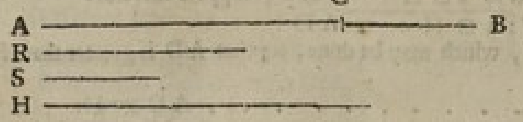
To cut a given right line into two such parts, that the Square of the whole line, to the summ of the Squares of the parts may have a given Reason.

Or thus,

In a right-angled Triangle, the summ of the leggs about the right angle being given, as also the Proportion which the Square of the said summ hath to the Square of the Hypothenusal, to find the Triangle. But the given quantities must be liable to this

Determination.

The first Term of the given Reason must be greater than the latter Term, yet not greater than the double of the latter Term. For in a right-angled Triangle, the Square of the summ of the leggs about the right angle is always greater than the Square of the Hypothenusal, but never greater than the double Square of the Hypothenusal, as hath been demonstrated in the preceding *Probl.* 4.



$AB = 34$	$H = 26$
$R = 289$	$AC = 24$
$S = 169$	$CB = 10$

Suppos.

1. AB is a right line given to be cut into two parts.
2. R and S are the Terms of a given Reason.
3. $R \sqsubset S$; but R not $\sqsubset 2S$.

Req. to find

4. AC and CB such parts of AB, that $AC + CB = AB$. Also,
5. $\square AB . \square AC + \square CB :: R . S$.

Con-

Construction.

6. By *Probl.* 11. *Chap.* 5. let it be made, as R to S, so the Square of AB to another Square, whose side suppose to be H, therefore

R . S :: □ AB . □ H.

7. Then supposing H to be the Hypothenusal of a right-angled Triangle, and AB the summ of the sides about the right-angle, find out the said sides and Triangle by the foregoing *Probl.* 4. For if R be greater than S, but not greater than 2S, according to the import of the Determination added to that Problem, 'tis possible to find out such a right-angled Triangle, and then the sides about the right-angle shall be equal to AC and CB, the parts sought by this *Probl.* 5. Therefore,

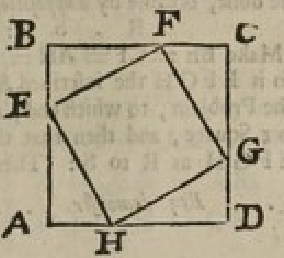
8. . . *Req. demonstr.* . . . . R . S :: □AB . □AC + □CB.

Demonstration.

9. Because by *Constr.* in 6°, ... R . S :: □AB . □H.
10. And by *Constr.* in 7°, ... □ AC + □ CB = □ H.
11. Therefore from 9° and 10°, by exchanging equal quantities, ... R . S :: □ AB . □ AC + □ CB.

Which was to be Dem. Therefore the Problem is satisfied.

A LEMMA, *leading to the following* Probl. 6.



Suppos.

1. ABCD is a Square.
2. BF and FC are parts of BC.
3. CF = BE = AH = DG; therefore,
4. BF = AE = DH = CG.
5. EF, FG, GH, HE are right lines.

6. . . *Req. demonstr.* that EFGH is a Square.

Demonstration.

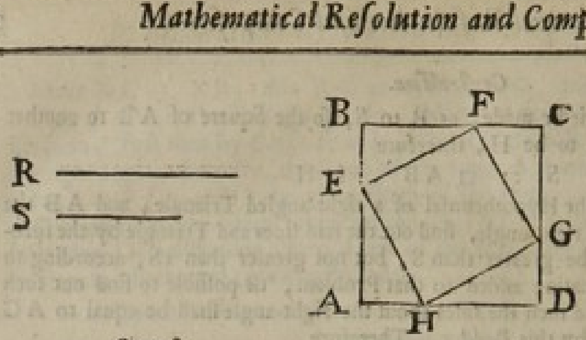
7. Because by *Suppos.* in 3° and 4°, ... CF = BE; and BF = CG.
8. And by *Suppos.* in 1°, ... $\angle$ EBF = $\angle$ FCG.
9. Therefore from 7° and 8°, (*per prop.* 4. *El.* 1.) EF = FG; and $\angle$BEF = $\angle$GFC.
10. And in like manner, ... EF = EH = HG = FG.
11. Again, because by *Suppos.* in 1°, . . . $\angle$ EBF is ⅃.
12. Therefore (*per Coroll. prop.* 32. *Elem.* 1.) $\angle$ BFE + $\angle$ BEF = ⅃.
13. But it hath been proved in 9°, that . . . $\angle$ GFC = $\angle$ BEF.
14. Therefore from 12° and 13°, (*per Ax.* 6. *Chap.* 2.) . . . . $\angle$ BFE + $\angle$ GFC = ⅃.
15. And from 14°, (*per Coroll. prop.* 13. *Elem.* 1.) $\angle$ EFG is ⅃.
16. And by arguing in like manner as in the five last preceding steps, it will be evident that . $\angle$FGH = ⅃ = $\angle$GHE = $\angle$GEF.
17. Therefore from 10°, 15° and 16°, (*per Defin.* 29. *Elem.* 1.) . . . . EFGH is a Square.

Which was to be Dem.

Probl. VI.

In a given Square to inscribe another Square whose angular points may lye in the sides of the given Square; and that the Square given to the Square inscribed may be in a given Reason, suppose as R to S. But R must be greater than S, yet not greater than 2 S, as may easily be inferr'd from the preceding *Probl.* 5.

R ——



BC = 34 = BA
R = 289
S = 169
BF = 24
FC = 10 = BE
EF = 26 = FG

Suppos.

1. ABCD is a Square; whose side AB or BC is given.
2. R and S are the Terms of a given Reason.
3. R ⊏ S; but R not ⊏ 2 S.

Req. to inscribe

4. □ EFGH in the □ ABCD. Also, that
5. □ ABCD . □ EFGH :: R . S.

Construction.

6. By the foregoing *Probl.* 5. of this *Chapt.* cut BC the side of the given Square into two such parts in F, that the Square of BC, that is, □ ABCD, may be in such proportion to the summ of the Squares of the parts BF, FC, as R to S; which may be done, because by *Supposition* R ⊏ S, yet R not ⊏ 2S; suppose therefore
R . S :: □ BC . □ BF + □ FC.
7. Make BE = CF = AH = DG, and draw the right lines EF, FG, GH and HE; so is EFGH the inscribed Square required. Now we must shew that it will satisfie the Problem, to which end, two things are to be proved, *viz.* First, that EFGH is a Square, and then that the Square ABCD hath such proportion to the Square EFGH as R to S: Therefore,

8. . . *Req. demonstr.* . . . . { EFGH is a □. Also, that R . S :: □ABCD . □EFGH.

Demonstration.

9. By *Suppos.* in 1°, ... } ABCD is a □.
10. Therefore . . . . } BC = BA = AD = DC.
11. By *Constr.* in 7°, ... } CF = BE = AH = DG.
12. Therefore by subtracting the Equations in 11° from those in 10°, there will remain . . . } BF = AE = DH = CG.
13. Therefore out of 9°, 10°, 11° and 12°, by the *Lemma* prefixt before this Problem,) . . } EFGH is a □.
14. Again by *Constr.* in 6°, ... } R . S :: □BC . □BF + □FC.
15. And because by *Suppos.* in 1°, ... } ∠ EBF is ⌋.
16. And consequently, (*per prop.* 47. *Elem.* 1.) } □EF = □BF + □BE (□FC.)
17. Therefore from 14° and 16°, by exchanging equal quantities,
R . S :: □ BC (or □ ABCD) . □ EF (or □ EFGH.)

Which was to be Demonstr. Therefore the Problem is satisfied.

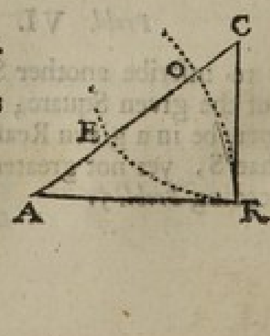
Probl. VII.

In a right-angled Triangle, the difference between the Hypothenusal and each of the sides about the right angle being given, to find the Triangle.

Suppos.

1. ARC is a △ right-angled at R.
2. b = OC = AC − AR is given.
3. d = AE = CA − CR is given; whence,
4. $g = d + b$ is given. And,
5. $kk = dd + bb$ is given.

Req. to find △ ARC.



AR = 4
RC = 3
AC = 5
OC = 1 = AC − AR (AO)
AE = 2 = CA − CR (CE)

Resolu-

Resolution.

6. Put a for the excess of the Hypothenusal above the summ of the given differences, *viz.*	$a = EO.$
7. Therefore from the premisses the Hypothenusal shall be	$a + g \ (= AC.)$
8. And the Base	$a + d \ (= AR.)$
9. And the Perpendicular	$a + b \ (= CR.)$
10. Therefore from 7° the Square of the Hypothenusal is	$aa + 2ga + gg.$
11. And from 8° the Square of the Base is	$aa + 2da + dd.$
12. And from 9° the Square of the Perpendicular is .	$aa + 2ba + bb.$
13. Therefore the summ of all in 11° and 12°, gives the summ of the Squares of the Base and Perpendicular, *viz.*	$2aa + 2da + 2ba + dd + bb.$
14. That is, (because by *Suppos.* in 5°, $kk = dd + bb$, and from 4°, $2g = 2d + 2b$,)	$2aa + 2ga + kk.$
15. Therefore from 10° and 14°, this Equation ariseth, (*per prop.* 47. *Elem.* 1.)	$2aa + 2ga + kk = aa + 2ga + gg.$
16. Therefore, by subtracting $aa + 2ga$ from each part of that Equation, this ariseth, ...	$aa + kk = gg.$
17. And by subtracting kk from each part of the last Equation, ...	$aa = gg - kk.$
18. But from 4°, ...	$dd + bb + 2db = gg.$
19. And from 5°, ...	$dd + bb = kk.$
20. And by subtracting the Equation in 19° from that in 18°, this remains, ...	$2db = gg - kk.$
21. Therefore from 17° and 20°, (*per Ax.* 1. *Chap.* 2.)	$aa = 2db.$
22. Therefore by extracting the square Root out of each part of the last Equation, it gives	$a = \sqrt{2db}.$
23. Therefore out of 22°, 6°, 7° and 4°, the Hypothenusal is given, to wit, ...	$d + b + \sqrt{2db} = AC.$
24. And from 22°, 6°, 8° and 3°, the Base is also given, *viz.*	$d + \sqrt{2db} = AR.$
25. And from 22°, 6°, 9° and 3°, the Perpendicular is also discovered, *viz.*	$b + \sqrt{2db} = CR.$

Which three last preceding steps give this

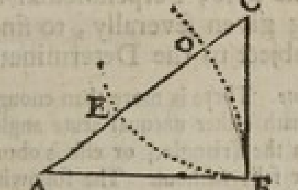
CANON.

26. To the summ of the given differences add the square Root of their double Rectangle, so shall the summ of that Addition be the Hypothenusal sought. Then add that square Root to the given differences severally, and these two summs shall be the desired sides about the right angle.

The Composition of Probl. 7.

B —— 1
D ———— 2
M ———— 2
F ——————— 3

$AR = 4$
$RC = 3$
$AC = 5$



Suppos.

27. B = the excess whereby the Hypothenusal of a right-angled Triangle exceeds one of the sides about the right angle is given.

28. D = the excess of the Hypothenusal above the other side about the right angle is given also.

Req. to find out the Triangle.

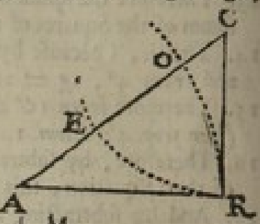
Construction.

29. By *Probl.* 9. *Chap.* 5. find a mean Proportional line, as M, between 2 D and B, therefore, ...	2D . M :: M . B.

30. Then

30. Then let a Triangle, as ARC, be made of three right lines equal to these given, to wit, D+B+M, | D+M, | B+M; which may be done, (*per probl.* 22. *Elem.*1.) for the summ of every two of those three lines is manifestly greater than the third; suppose therefore . . . . $AC = D+B+M$, $AR = D+M$, $RC = B+M$.

31. I say ARC is the Triangle required. Now we must shew that it will satisfie the Problem. First then, 'tis manifest that the difference between AC, that is, D+B+M, and AR, that is, D+M, is equal to the given difference B; also the difference between AC, that is, D+B+M, and RC, that is, B+M, is manifestly equal to the given difference D. So it remains only to prove that the angle ARC is a right angle, which will be made manifest by the following Demonstration.



Prepar.

32. Make $F = B + M$, therefore
33. From 30° and 32°, $AC = D + F$. Also $RC = F$.
34. . . *Req. demonstr.* . . . . . $\angle ARC = \lrcorner$.

Demonstration.

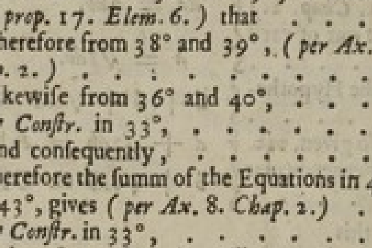
35. Because by *Constr.* in 30°, ... $AR = D + M$.
36. Therefore, (*per Theor.* 2. *Chap.* 4.) . . $\square AR = \square D + \square M + 2\square DM$.
37. By *Constr.* in 32°, ... $F = B + M$.
38. Therefore from 37°, by drawing 2 D into each part, (*per prop.* 1. *Elem.* 6.) . . . $2\square DF = 2\square DB + 2\square DM$.
39. But from the *Constr.* in 29°, it follows (*per prop.* 17. *Elem.* 6.) that . . . . $\square M = 2\square DB$.
40. Therefore from 38° and 39°, (*per Ax.* 6. *Chap.* 2.) . . . . $2\square DF = \square M + 2\square DM$.
41. Likewise from 36° and 40°, ... $\square AR = \square D + 2\square DF$.
42. By *Constr.* in 33°, ... $RC = F$.
43. And consequently, ... $\square RC = \square F$.
44. Therefore the summ of the Equations in 41° and 43°, gives (*per Ax.* 8. *Chap.* 2.) . . $\square AR + \square RC = \square D + 2\square DF + \square F$.
45. By *Constr.* in 33°, ... $AC = D + F$.
46. And consequently, (*per Theor.* 2. *Chap.* 4.) $\square AC = \square D + 2\square DF + \square F$.
47. Therefore from 44° and 46°, (*per Ax.* 1. *Chap.* 2.) . . . . $\square AC = \square AR + \square RC$.
48. Therefore, (*per prop.* 48. *Elem.* 1.) . . $\angle ARC = \lrcorner$.

Which was to be Dem. Therefore the Problem is satisfied.

Probl. VIII.

The Base, Perpendicular and summ of the leggs of a plain Triangle being given severally, to find the Triangle. But the lines given must be subject to the Determination hereafter declared.

Note. There is more than enough given in this Problem, unless it requires a Triangle that hath either unequal acute angles at the Base, in which Case the Perpendicular falls within the Triangle; or else is obtusangled at the Base, in which latter Case the Perpendicular falls without. The following Resolution handles the first Case, but with a little alteration it may be applied to the latter, as will hereafter appear.



AR = 21	FR = 6
AO = 17	OF = 8
RO = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Preparat.

1. Suppose A R O to be the Triangle sought, having unequal acute angles A and R at the ends of the Base A R; then from the Center O, at the distance of the lesser legg O R, describe the Circle O R G D cutting the greater legg O A in G; so shall A G be the difference of the leggs O A and O R, for O G = O R.
2. Produce A O to the Circumference in D, then is A D equal to the summ of the leggs A O and O R; for O D = O R.
3. Draw the Semidiameter O E, and let fall O F ⊥ E R, so will O F cut E R into two equal parts in F, (*per prop.* 3. *Elem.* 3.) These things premised, the Resolution of the Problem propos'd may be formed in manner following.

Suppos.

4. b = A R, the Base of △ A R O is given.
5. p = O F the Perpendicular is given.
6. c = A D = A O + O R, the summ of the leggs is given.

Req. to find the Triangle.

Resolution.

7. Put a for the unknown difference of the leggs, *viz.* assume . . . . a = A G (= A O − O R.)
8. Then the difference of the segments of the Base made by the falling of the Perpendicular shall be $\frac{ca}{b}$, for, (by *Theor.* 2. *Probl.* 9. *Chap.* 7.) . . $b \,.\, c :: a \,.\, \frac{ca}{b}$ (A E.)
9. Therefore from 4° and 8°, (by *Theor.* 9. *Chap.* 4.) the greater segment of the Base shall be . . . $\frac{1}{2}b + \frac{ca}{2b}$ (= A F.)
10. And (by the Theorem last mentioned,) the lesser segment shall be $\frac{1}{2}b - \frac{ca}{2b}$ (= F R.)
11. Therefore the Square of the greater segment (in 9°) shall be . . . $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca.$
12. And the Square of the lesser segment (in 10°) shall be . . . $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca$
13. By *prop.* 47. *Elem.* 1. . . . $\square$ A F + $\square$ F O = $\square$ A O.
14. Therefore out of 11°, 13° and 5°, the Square of the greater legg A O shall be . . . . $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca + pp.$
15. And out of 5° and 12°, the Square of the lesser legg O R shall be . . $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca + pp.$
16. Therefore from 14° and 15°, the summ of the Squares of the leggs shall be . . . . $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp.$

Now because by *Theor.* 2. *Chapt.* 4.

$\square$ A O + $\square$ R O + 2 ▭ A O, R O = $\square : \overline{AO + OR} : = \square$ A D;

Therefore to the end the quantities in 16° may be made a compleat Square of A D, let the double Rectangle of the leggs A O and R O be found out in this manner, *viz.*

17. From 6° and 7°, (by *Theor.* 9. *Chap.* 4.) the greater legg is . . $\frac{1}{2}c + \frac{1}{2}a$ (= A O.)
18. And (by the Theorem last mentioned,) the lesser legg is . . . $\frac{1}{2}c - \frac{1}{2}a$ (= R O.)
19. Therefore the Rectang. of the legs is $\frac{1}{4}cc - \frac{1}{4}aa$ (= ▭ A O, R O)
20. And the double Rectangle of the leggs is . . . . $\frac{1}{2}cc - \frac{1}{2}aa$ (= 2 ▭ A O, R O.)
21. Therefore the summ of all in 16° and 20°, gives the Square of the summ of the leggs, *viz.* . . . $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \frac{1}{2}cc - \frac{1}{2}aa$ (= $\square$ A D)

22. Which quantities in 21° must be equal to cc the Square of the given summ of the leggs; hence this Equation,

$$\tfrac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \tfrac{1}{2}cc - \tfrac{1}{2}aa = cc.$$

23. Therefore by subtracting $\tfrac{1}{2}cc$ from each part of that Equation, this will arise,

$$\tfrac{1}{2}bb + \frac{ccaa}{2bb} + 2pp - \tfrac{1}{2}aa = \tfrac{1}{2}cc.$$

24. And by doubling all in 23°, $bb + \frac{ccaa}{bb} + 4pp - aa = cc$.

25. Now to the end that known quantities may be separated from unknown, enquiry must be made, whether $\frac{ccaa}{bb}$ be greater or less than aa. But because (as appears in 8°,) these are Proportionals, *viz.* $b \,.\, a :: c \,.\, \frac{ca}{b}$, and $c \sqsubset b$, for the summ of the leggs of any plain Triangle is longer than the Base, therefore (*per prop.* 14. *Elem.* 5.) $\frac{ca}{b}$ shall be greater than a, and consequently the Square of the former greater than the Square of the latter, *viz.* $\frac{ccaa}{bb} \sqsubset aa$, therefore aa may be subtracted from $\frac{ccaa}{bb}$, and there will remain a quantity greater than nothing. From the premisses therefore it is manifest that $bb + 4pp$ may be subtracted from each part of the Equation above exprest in 24°, and the quantity remaining on each part will be greater than nothing, and the Equation arising by that subtraction will be this,

$$\frac{ccaa}{bb} - aa = cc - bb - 4pp.$$

26. That is, by reducing $\frac{ccaa}{bb} - aa$ into the form of a Fraction,

$$\frac{ccaa - bbaa}{bb} = cc - bb - 4pp.$$

27. Which last Equation may be resolved into this Analogy,

$$bb \,.\, cc - bb :: aa \,.\, cc - bb - 4pp.$$

28. Therefore by Inverse and Altern Reason,

$$cc - bb \,.\, cc - bb - 4pp :: bb \,.\, aa.$$

29. But the sides of proportional Squares are also Proportionals, therefore from the last preceding Analogy,

$$\sqrt{:\overline{cc - bb}} : \,.\, \sqrt{:\overline{cc - bb - 4pp}} : :: b \,.\, a.$$

Of which last Analogy the three first Terms are given, therefore the fourth Term, which is the difference of the leggs sought, is given also. Moreover, by the same quantities first given, the line A E, which is the difference of the segments of the Base made by the falling of the Perpendicular, shall be given also; for,

30. By *Theor.* 2. *Probl.* 9. *Chap.* 7. $c \,.\, AE :: b \,.\, a$.

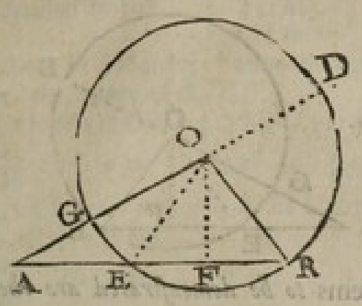
31. Therefore out of 29° and 30°, (by *prop.* 11. *Elem.* 5.)

$$\sqrt{:\overline{cc - bb}} : \,.\, \sqrt{:\overline{cc - bb - 4pp}} : :: c \,.\, AE.$$

32. *Note.* In this Resolution, the Perpendicular is supposed to fall within the Triangle sought, and $\frac{ca}{b}$ (the fourth Proportional of the Analogy in 8°,) represents A E the difference of the segments of the Base made by the falling of the Perpendicular; but when the Perpendicular falls without, as in △ A E O, where A E is the Base, the said $\frac{ca}{b}$ represents the line A R, which is compos'd of the Base A E, and E R the double of the distance F E from F the foot of the Perpendicular to the obtuse angle at E. But whether the Perpendicular falls within or without the Triangle, the Resolution runs into that Equation before exprest in 22°. Which things being well observed, the Theorems hereafter exprest, (which will be very useful in the following Problems,) will clearly arise out of the preceding Resolution; *viz.*

The Equation in 25° gives

THEOR. 1.



THEOR. 1.

33. In a plain Triangle whose leggs are unequal, if the Perpendicular falls within, the Square of the difference of the segments of the Base made by the falling of the Perpendicular, is greater than the Square of the difference of the leggs, by the excess whereby the Square of the summ of the leggs exceeds the summ of the Square of the Base and the Square of the double Perpendicular. But when the Perpendicular falls without the Triangle upon the Base increased, then the Square of the line compos'd of the Base and double distance between the foot of the Perpendicular and the obtuse angle, is greater than the Square of the difference of the leggs, by the excess above-mentioned.

The Analogy in 29° gives

THEOR. 2.

34. As the right line whose Square is equal to the excess whereby the Square of the summ of the leggs of a plain Triangle exceeds the Square of the Base, is to the right line whose Square is equal to the excess of the said Square of the summ of the leggs above the summ of the Square of the Base and the Square of the double Perpendicular; so is the Base to the difference of the leggs.

Or thus, which is more convenient for Arithmetical practice.

As the excess of the Square of the summ of the leggs above the Square of the Base, is to the excess of the Square of the summ of the leggs above the summ of the Square of the Base and the Square of the double Perpendicular; so is the Square of the Base to the Square of the difference of the leggs.

Therefore, if the Base, Perpendicular, and summ of the leggs of a plain Triangle whose leggs are unequal, be given severally, the difference of the leggs shall be given also; and consequently the leggs severally, by *Theor.* 9. *Chap.* 4.

The Analogy in 31° gives

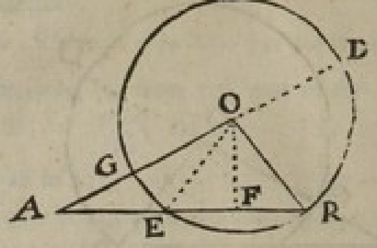
THEOR. 3.

35. As the right line whose Square is equal to the excess whereby the Square of the summ of the unequal leggs of a plain Triangle exceeds the Square of the Base, is to the right line whose Square is equal to the excess of the Square of the summ of the leggs above the summ of the Square of the Base and the Square of the double Perpendicular; so is the summ of the leggs to a fourth Proportional, which is less than the Base, when the Perpendicular falls within, for then it is the difference of the segments of the Base made by the Perpendicular. But when the Perpendicular falls without, the said fourth Proportional exceeds the Base, and is compos'd of the Base and double distance between the foot of the Perpendicular and the obtuse angle at the nearer end of the Base. Lastly, when the said fourth Proportional is equal to the Base, the Perpendicular falls upon the end thereof.

Therefore, if the quantities of the lines given in this *Probl.* 8. be exprest by numbers, we may discover by *Theor* 3. above exprest, whether the Triangle sought be acute-angled, or obtuse-angled, or right-angled at the Base, *viz.* of what kind the angles at the Base are.

36. The truth of the three preceding Theorems, when the angles at the Base are acute and unequal, may be demonstrated by the steps of the foregoing Resolution, by proceeding in a direct order from the beginning to the end of the Resolution, in manner following.

Let there be a Triangle propos'd, as A R O, having unequal acute angles at the ends of the Base A R, and let the same things be supposed as before in 1°, 2° and 3° of this Problem.



The three Theorems *to be demonstrated are these following*, viz.

37. $. \ \square AE - \square AG = \square AD - \square AR - 4\square OF.$

38. $\sqrt{:}\overline{\square AD - \square AR}: . \sqrt{:}\overline{\square AD - \square AR - 4\square FO}: :: AR . AG.$

39. $\sqrt{:}\overline{\square AD - \square AR}: . \sqrt{:}\overline{\square AD - \square AR - 4\square FO}: :: AD . AE.$

Demonstration.

40. By *Theor.* 2. in 29° of *Probl.* 9. *Chap.* 7. this Analogy is manifest, ... $AR . AD :: AG . AE.$

That is, in 8°, ... $b . c :: a . \frac{ca}{b}.$

41. And because A R is divided into two unequal parts in F, and A E is the difference of those parts, therefore (*per Theor.* 9. *Chap.* 4.) . . . . $\frac{1}{2}AR + \frac{1}{2}AE = AF.$

That is, in 9°, ... $. \frac{1}{2}b + \frac{ca}{2b} = AF.$

42. And by the same Theorem, ... $\frac{1}{2}AR - \frac{1}{2}AE = FR.$

That is, in 10°, ... $. \frac{1}{2}b - \frac{ca}{2b}$ $(= FR)$

43. And (by *Theor.* 2. *Chap.* 4.) the Square of the Equation in 41° gives

$$\tfrac{1}{4}\square AR + \tfrac{1}{4}\square AE + \tfrac{1}{2}\square AR, AE = \square AF.$$

That is, in 11°, . . $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca$ $(= \square AF.)$

44. And (by *Theor.* 5. *Chap.* 4.) the Square of the Equation in 42° gives

$$\tfrac{1}{4}\square AR + \tfrac{1}{4}\square AE - \tfrac{1}{2}\square AR, AE = \square FR.$$

That is, in 12°, . . . $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca$ $(= \square FR.)$

45. And by adding $\square OF$ to each part of the Equation in 43°, it makes (*per prop* 47. *Ele.* 1.)

$$\tfrac{1}{4}\square AR + \tfrac{1}{4}\square AE + \tfrac{1}{2}\square AR, AE + \square OF = \square AO.$$

That is, in 14°, . . $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca + pp$ $(= \square AO.)$

46. And by adding $\square OF$ to each part of the Equation in 44°, this ariseth, (*per pro.* 47. *El.* 1.)

$$\tfrac{1}{4}\square AR + \tfrac{1}{4}\square AE - \tfrac{1}{2}\square AR, AE + \square OF = \square RO.$$

That is, in 15°, . . $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca + pp$ $(= \square RO.)$

47. And the summ of the Equations in 45° and 46° gives

$$\tfrac{1}{2}\square AR + \tfrac{1}{2}\square AE + 2\square OF = \square AO + \square RO.$$

That is, in 16°, . . . $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp$ $(= \square AO + \square RO.)$

It is manifest (*per Theor.* 1. *Chap.* 4.) that the Equation in 47° wants only $2\square AO, RO$ to compleat the Square of $AO + RO$, that is, the Square of AD; therefore in order to fill up that Square, I proceed as in the five steps next following.

48. By *Theor.* 9. *Chap.* 4. . . . . $\frac{1}{2}AD + \frac{1}{2}AG = AO.$

That is, in 17°, ... $. \frac{1}{2}c + \frac{1}{2}a = AO.$

49. And by the same Theorem, ... $\frac{1}{2}AD - \frac{1}{2}AG = RO.$

That is, in 18°, ... $. \frac{1}{2}c - \frac{1}{2}a = RO.$

50. And by *Theor.* 8. *Chap.* 4. the Rectangle of the Equations in 48° and 49° is . . . . . $\frac{1}{4}\square AD - \frac{1}{4}\square AG = \square AO, RO.$

That is, in 19°, ... $. \frac{1}{4}cc - \frac{1}{4}aa = \square AO, RO.$

51. And

51. And by doubling the Equation in 50°, $\cdot\} \ \frac{1}{2}\square AD - \frac{1}{2}\square AG = 2\square AO,RO.$
That is, in 20°, ... $\cdot\} \cdot \ \frac{1}{2}cc - \frac{1}{2}aa = 2\square AO,RO.$

52. The summ of the Equations in 47° and 51° makes the Square of the summ of the leggs AO and RO, *viz.* the Square of AD,

$$\tfrac{1}{2}\square AR + \tfrac{1}{2}\square AE + 2\square OF + \tfrac{1}{2}\square AD - \tfrac{1}{2}\square AG = \square AD.$$

That is, in 22°, $\Big\}\cdot \ \frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \frac{1}{2}cc - \frac{1}{2}aa = cc.$

53. And by subtracting $\frac{1}{2}\square AD$ from each part of the Equation in 52°, this remains,

$$\tfrac{1}{2}\square AR + \tfrac{1}{2}\square AE + 2\square OF - \tfrac{1}{2}\square AG = \tfrac{1}{2}\square AD.$$

That is, in 23°, $\Big\}\cdot \ \frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp - \frac{1}{2}aa = \frac{1}{2}cc.$

54. And by doubling the Equation in 53°,

$$\square AR + \square AE + 4\square OF - \square AG = \square AD.$$

That is, in 24°, $\Big\}\cdot \ \cdot \ bb + \frac{ccaa}{bb} + 4pp - aa = cc.$

55. And because (by *prop.* 8. *Elem.* 3.) $AE \sqsubset AG$, and consequently $\square AE \sqsubset \square AG$, therefore $\square AE - \square AG \sqsubset O$; whence it is manifest, that if $\square AR + 4\square OF$ be subtracted from each part of the Equation in 54°, there will remain on each part a quantity greater than nothing, and the Equation arising by that subtraction will be this that follows, *viz.*

$$\square AE - \square AG = \square AD - \square AR - 4\square OF.$$

That is, in 25°, $\Big\}\cdot \ \cdot \ \frac{ccaa}{bb} - aa = cc - bb - 4pp.$

And in 26°, $\Big\}\cdot \ \cdot \ \frac{ccaa - bbaa}{bb} = cc - bb - 4pp.$

Which was *Theor.* 1. to be demonstr.

Now to pass from the 26th step to the 27th of the preceding Algebraical Resolution, by the lines of the Diagram, some Analogies, not exprest in the Resolution, must be introduced; and in order to their discovery, the Learner may observe, that the Algebraical Fraction $\frac{ccaa - bbaa}{bb}$ in 26° denotes a Plane, which is the fourth Term of an Analogy whose three first Terms are these three Planes, to wit, bb, $cc - bb$ and aa, which answer to these three Planes, (in the lines of the Diagram,) to wit, $\square AR$, $\square AD - \square AR$ and $\square AG$; therefore, $\square AE - \square AG$ which is correspondent to the said Algebraical Fraction $\frac{ccaa - bbaa}{bb}$, must likewise be the fourth Term of an Analogy whose three first Terms are the said Planes $\square AR$, $\square AD - \square AR$ and $\square AG$; but how the said Analogy is brought to light, the four steps next following will shew.

56. Because, (as hath been shewn in 40°,)

$$AD \ . \ AR :: AE \ . \ AG.$$

57. Therefore, (*per prop.* 22. *Elem.* 6.)

$$\square AD \ . \ \square AR :: \square AE \ . \ \square AG.$$

58. Therefore by *Division of Reason*,

$$\square AD - \square AR \ . \ \square AR :: \square AE - \square AG \ . \ \square AG.$$

59. Therefore inversly,

$$\square AR \ . \ \square AD - \square AR :: \square AG \ . \ \square AE - \square AG.$$

60. But it hath been shewn in 55°, that

$$\square AE - \square AG = \square AD - \square AR - 4\square OF.$$

61. Therefore from 59° and 60° (by exchanging equal quantities) this Analogy ariseth,

$$\square AR \ . \ \square AD - \square AR :: \square AG \ . \ \square AD - \square AR - 4\square OF.$$

That is, in 27°,

$$bb \ . \ cc - bb :: aa \ . \ cc - bb - 4pp.$$

62. Therefore from 61°, by *inverse* and *altern Reason*,

$$\square AD - \square AR \ . \ \square AD - \square AR - 4\square OF :: \square AR \ . \ \square AG.$$

That is, in 28°,

$$cc - bb \ . \ cc - bb - 4pp :: bb \ . \ aa.$$

63. But

63. But the sides of proportional Squares are also Proportionals, therefore from 62° this Analogy ariseth,

$$\sqrt{:\square AD - \square AR} \;.\; \sqrt{:\square AD - \square AR - 4\square OF} :: AR \;.\; AG.$$

That is, in 29°,

$$\sqrt{:cc - bb} \;.\; \sqrt{:cc - bb - 4pp} :: b \;.\; a.$$

Which was *Theor.* 2. to be Demonstr.

64. Again, because by *Theor.* 2. in 29° of *Probl.* 9. *Chapt.* 7.

$$AD \;.\; AE :: AR \;.\; AG.$$

65. Therefore from 63° and 64°, (*per prop.* 11. *Elem.* 5.)

$$\sqrt{:\square AD - \square AR} \;.\; \sqrt{:\square AD - \square AR - 4\square OF} :: AD \;.\; AE.$$

Which was *Theor.* 3. to be Demonstr.

66. Again, it hath been shewn in 62°, that

$$\square AR \;.\; \square AG :: \square AD - \square AR \;.\; \square AD - \square AR - 4\square OF.$$

67. Therefore by *converse Reason*,

$$\square AR \;.\; \square AR - \square AG :: \square AD - \square AR \;.\; 4\square OF.$$

Which last Analogy affords

THEOR. 4.

68. As the Square of the Base of any plain Triangle whose leggs are unequal, is to the excess whereby the Square of the Base exceeds the Square of the difference of the leggs; so is the excess whereby the Square of the summ of the leggs exceeds the Square of the Base, to the Square of the double Perpendicular.

Therefore, the Base and leggs of any plain Triangle whose leggs are unequal, being severally given in numbers, the Perpendicular falling upon that Base within the Triangle, or without upon the Base increased, shall be given also in numbers.

From the said *Theor.* 4. and *prop.* 41. *Elem.* 1. 'tis easie to deduce this following

THEOR. 5.

69. The Rectangle made of these two right lines, to wit, the right line whose Square is equal to the excess whereby a quarter of the Square of the Base of a plain Triangle exceeds a quarter of the Square of the difference of the leggs; and the right line whose Square is equal to the excess of a quarter of the Square of the summ of the leggs above a quarter of the Square of the Base, shall be equal to the Triangle.

To make this manifest, let the $\triangle ARO$ be taken as before in the Resolution, then

70. . . *Req. demonstr.* $\Box$ of $\sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG:} \times \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR:} = \triangle ARO.$

Demonstration.

71. By *Theor.* 4. in 68° of this Problem,

$$\square AR \;.\; \square AR - \square AG :: \square AD - \square AR \;.\; 4\square OF.$$

72. And by taking $\tfrac{1}{4}$ of every Term of that Analogy,

$$\tfrac{1}{4}\square AR \;.\; \tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG :: \tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR \;.\; \square OF.$$

73. But the sides of proportional Squares are also Proportionals, therefore from the last Analogy,

$$\tfrac{1}{2}AR \;.\; \sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG} :: \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR} \;.\; OF.$$

74. Therefore, (*per prop.* 16. *Elem.* 6.)

$$\tfrac{1}{2}\Box\, AR, OF = \Box \text{ of } \sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG:} \times \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR}.$$

75. But (*per prop.* 41. *Elem.* 1.)

$$\tfrac{1}{2}\Box\, AR, OF = \triangle ARO.$$

76. Therefore from 74° and 75°, (*per Ax.* 1. *Chapt.* 2.)

$$\Box \text{ of } \sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG:} \times \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR} = \triangle ARO.$$

Which was to be Dem.

Hence the following Canons are deducible, to find out the Area of a plain Triangle Arithmetically, without the help of the Perpendicular, the Base and leggs being severally given in numbers, and the leggs unequal between themselves.

CANON 1.

77. From a quarter of the Square of the Base subtract a quarter of the Square of the difference of the leggs, and reserve the remainder; then from a quarter of the Square of the summ of the leggs subtract a quarter of the Square of the Base, and reserve the remainder;

remainder ; that done, multiply the first remainder by the second, and extract the Square Root of the Product, so shall that square Root be the Area of the Triangle.

78. Again, because by *Theor*. 8. *Chapt*. 4.

$$\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG = \square \text{ of } \overline{\tfrac{1}{2}AR + \tfrac{1}{2}AG} \times \overline{\tfrac{1}{2}AR - \tfrac{1}{2}AG}.$$

79. Likewise by the same Theorem,

$$\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR = \square \text{ of } \overline{\tfrac{1}{2}AD + \tfrac{1}{2}AR} \times \overline{\tfrac{1}{2}AD - \tfrac{1}{2}AR}.$$

Therefore from 77°, 78° and 79°, by exchanging equal Factors, there will arise

CANON 2.

80. Multiply these four numbers one into another, to wit,

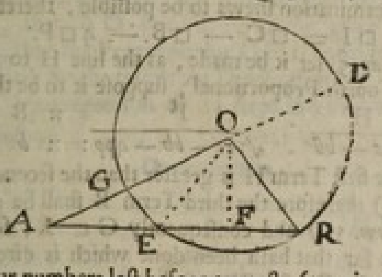
1. The summ of half the Base, and half the difference of the leggs.
2. The excess of half the Base above half the difference of the leggs.
3. The summ of half the summ of the leggs, and half the Base.
4. The excess of half the summ of the leggs above half the Base.

Then extract the square Root of the Product made by the continual multiplication of those four numbers, so shall that square Root be the Area of the Triangle.

81. Again, if we suppose $\left\{\begin{array}{l} B = \text{the Base}, \\ A = \text{the greater legg}, \\ E = \text{the lesser legg} \end{array}\right\}$ of a plain Triangle.

Then the four numbers above mentioned in *Canon* 2. may be exprest thus, *viz*.

$$\begin{array}{r|l} 1. & \tfrac{1}{2}B + \tfrac{1}{2}A - \tfrac{1}{2}E, \\ 2. & \tfrac{1}{2}B - \tfrac{1}{2}A + \tfrac{1}{2}E, \\ 3. & \tfrac{1}{2}A + \tfrac{1}{2}E + \tfrac{1}{2}B, \\ 4. & \tfrac{1}{2}A + \tfrac{1}{2}E - \tfrac{1}{2}B. \end{array} \Bigg\} \text{ Or thus, } \left\{\begin{array}{l} \tfrac{1}{2}B + \tfrac{1}{2}A + \tfrac{1}{2}E - E, \\ \tfrac{1}{2}B + \tfrac{1}{2}A + \tfrac{1}{2}E - A, \\ \tfrac{1}{2}A + \tfrac{1}{2}E + \tfrac{1}{2}B, \\ \tfrac{1}{2}A + \tfrac{1}{2}E + \tfrac{1}{2}B - B. \end{array}\right.$$



82. Therefore, if the four numbers last before exprest, (to wit, those standing on the right hand,) be multiplied one into another continually, the Product shall be equal to the Square of the Area of the Triangle whose three sides are represented by B, A, E. But if those four numbers be well observed, it will be evident that the number third in order is the half summ of the three sides of the Triangle, and the other three numbers are the Remainders arising by the subtraction of the three sides severally from their half summ; Hence therefore ariseth the vulgar Canon, to find out the Area of any plain Triangle whose three sides are severally given in numbers, *viz*.

CANON 3.

83. From half the summ of the three sides of any plain Triangle subtract the three sides severally ; then multiply the said half summ and the three remainders one into another, according to the Rule of Continual Multiplication, and extract the square Root of the last Product, so shall that square Root be the Area of the Triangle.

Divers other Canons might be raised from the premisses, to find out the Area of a plain Triangle; but 'tis now time to proceed to the Composition of the Problem in hand, and that its Construction may be possible, the lines given must be subject to this

Determination.

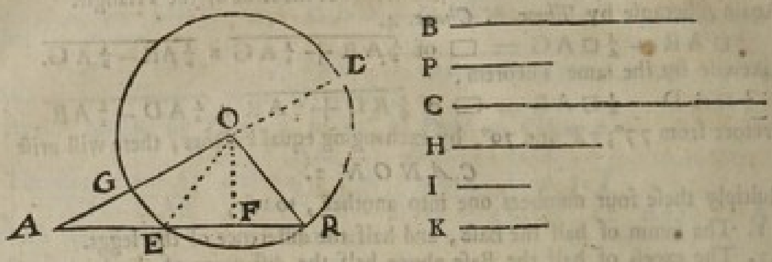
84. . . . . $c \sqsubset \sqrt{:\overline{bb + 4pp}:}$ that is, in words,

The given summ of the leggs must be longer than that right line whose Square is equal to the summ of the Square of the Base and the Square of the double of the Perpendicular.

This Determination doth openly shew it self in *Theor*. 2. in the 34th step of this Problem, and therefore that Theorem having already been demonstrated, the Determination is consequently both true and necessary for limiting the lines given.

The

The Composition of the foregoing Probl. 8.



Suppos.

85. $B =$ the Base of a Triangle is given.
86. $P =$ the Perpendicular is given.
87. $C =$ the summ of the leggs is given.
88. $C \sqsubset \sqrt{:\square B + 4\square P:}$ (*Determination.*)

Req. to make the Triangle.

Construction.

89. By *Probl.* 4. *Chap.* 5. find a right line H, such, that its Square may be equal to $\square C - \square B$; which Effection is possible, as is evident by the Determination prescribed in 88°, therefore suppose

$$\square H = \square C - \square B.$$

90. Find likewise a right line I, such, that its Square may be equal to $\square C - \square B - 4\square P$, which Effection the Determination shews to be possible, therefore suppose

$$\square I = \square C - \square B - 4\square P.$$

91. Then by *Probl.* 8. *Chap.* 5. let it be made, as the line H to the line I, so the line B (the given Base) to a fourth Proportional, suppose it to be the line K, therefore,

$$H \,.\, I :: B \,.\, K.$$

That is, in 29°, $\sqrt{:cc - bb} \,.\, \sqrt{:cc - bb - 4pp:} :: b \,.\, a.$

In which Analogy, the first Term H is greater than the second Term I, (as is evident by *Constr.* in 90° and 91°,) therefore the third Term B shall be greater than the fourth K, (*per Schol. Prop.* 14. *Elem.* 5.) and consequently $C \sqsubset K$, for (by the Determination in 88°) $C \sqsubset B$. Thus far that hath been done which is directed by *Theor.* 2. in 34° of this *Probl.* the rest of the Construction follows.

92. Let a Triangle be made of these three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$, which is possible to be done. (*per prop.* 22. *Elem.* 1.) if $C \sqsubset K$, and that the summ of every two of those three lines be longer than the third; but that those lines are so qualified, I prove thus;

First, by what hath been said in 91°, $C \sqsubset K$, and consequently $\frac{1}{2}C - \frac{1}{2}K$ is equal to some real right line.

Secondly, the summ of B and $\frac{1}{2}C + \frac{1}{2}K$ is manifestly greater than the third line $\frac{1}{2}C - \frac{1}{2}K$.

Thirdly, the summ of the two lines $\frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$ makes C, which (by the Determination in 88°) is greater than the third line B.

Fourthly, that the summ of B and $\frac{1}{2}C - \frac{1}{2}K$ is greater than the third line $\frac{1}{2}C + \frac{1}{2}K$ may be proved thus;

It hath been shewn in 91°, that . . . . $B \sqsubset K$.
Therefore by adding $\frac{1}{2}C$ to each part, ... $B + \frac{1}{2}C \sqsubset K + \frac{1}{2}C$.
Therefore by subtracting $\frac{1}{2}K$ from each part, . . . $B + \frac{1}{2}C - \frac{1}{2}K \sqsubset \frac{1}{2}K + \frac{1}{2}C$.
Which was to be Dem.

Now since it hath been proved that $\frac{1}{2}C - \frac{1}{2}K$ is equal to some real right line, and that the summ of every two of these three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}K$ and $\frac{1}{2}C - \frac{1}{2}K$, is greater than the third, 'tis possible to make a Triangle of those three lines, (*per prop.* 22. *Elem.* 1.) Suppose then it be done, and that the Triangle so made is A R O, (in the preceding Diagram,) having its Base A R equal to the given Base B, and the greater legg A O equal to $\frac{1}{2}C + \frac{1}{2}K$, and the lesser legg R O equal to $\frac{1}{2}C - \frac{1}{2}K$. I say the Triangle A R O

A R O will ſatisfie the Problem propounded; but to render the Demonſtration thereof the more eaſie to Learners, I ſhall premiſe a few things in eight ſteps next following.

93. If the quantities of the given lines B, P and C be expreſt by numbers, it will be eaſie to diſcover the kind of the Triangle ſought, when the leggs are unequal, (as they were ſuppoſed to be in the Reſolution,) by *Theor.* 3. in 35° of this Problem; for if the fourth Proportional found out by that Theorem be leſs than the Baſe, the Perpendicular falls within the Triangle; if greater, without; if equal to the Baſe, upon the end of the Baſe.

Suppoſing then it be diſcovered, that the Perpendicular falls upon A R within the Triangle A R O, from the Center O, at the diſtance of the leſſer legg O R, ($= \frac{1}{2}C - \frac{1}{2}K$) deſcribe the Circle O R G D cutting O A in G; then produce A O to the Circumference in D, draw alſo the Semidiameter O E, and from the Center O let fall O F perpendicular to E B, therefore (*per prop.* 3. *Elem.* 3.) $FE = FR$. Then,

94. Becauſe (*per defin.* 15. *Elem.* 1.) . . . . } $OD = OR = OG$.
95. Therefore by adding A O to each part, ... } $AD = AO + OR$.
96. But by *Conſtr.* in 92°, ... } $C = AO + OR$.
97. Therefore from 95° and 96°, (*per Ax.* 1. *Chap.* 2.) . . } $AD = C$.
98. Again, by *Conſtr.* in 92°, ... } $AO = \frac{1}{2}C + \frac{1}{2}K$.
99. Alſo by *Conſtr.* in 92°, ... } $OR = \frac{1}{2}C - \frac{1}{2}K$.
100. Therefore, by ſubtracting the Equation in 99° from that in 98°, ... } $AG = K$.

101. Now I ſhall ſhew that the Triangle A R O, made as before, will ſatisfie the Problem. Firſt then by *Conſtruction* in 92°, the Baſe A R is equal to the given Baſe B; and it hath been proved in 96°, that $AO + OR = C$ the given ſumm of the leggs. So it remains only to ſhew, that the Perpendicular O F is equal to the given Perpendicular P; but that is made manifeſt by the following Demonſtration, which is form'd out of the foregoing Reſolution, by a repetition of its ſteps in a backward (not direct) order.

102. . . *Req. demonſtr.* . . . . $OF = P$.

Demonſtration.

103. By *Conſtr.* in 91°, . . H . I :: B . K.
That is, in 29°, ... $\sqrt{:cc - bb}$. $\sqrt{:cc - bb - 4pp}$ :: b . a.
104. Therefore, (*per prop.* 22. *El.* 6.) $\square H$. $\square I$ :: $\square B$. $\square K$.

Now that the Terms of the laſt Analogy may be converted into their equivalent quantities expreſſible by the lines in the Diagram, theſe ſeven Equations next following are to be well obſerved.

105. By *Conſtr.* in 89°, ... } $\square C - \square B = \square H$.
106. And from 97°, ... } $\square AD = \square C$.
107. And from 92°, ... } $\square AR = \square B$.
108. Therefore from 105°, 106°, 107°, ... } $\square AD - \square AR = \square H$.
109. Again, by *Conſtr.* in 90°, ... } $\square C - \square B - 4\square P = \square I$.
110. Therefore from 105°, 108° and 109°, . . } $\square AD - \square AR - 4\square P = \square I$.
111. And from 100°, ... } $\square AG = \square K$.

112. Therefore the Terms of the Analogy in 104° being exchanged for their equivalent quantities in 108°, 110°, 107° and 111°, that Analogy will be converted into this, *viz.*

$\square AD - \square AR$. $\square AD - \square AR - 4\square P$ :: $\square AR$. $\square AG$.

That is, in 28°,

$cc - bb$. $cc - bb - 4pp$ :: bb . aa.

113. Therefore by *altern* and *inverſe Reaſon*,

$\square AR$. $\square AD - \square AR$ :: $\square AG$. $\square AD - \square AR - 4\square P$.]

That is, in 27°,

bb . $cc - bb$ :: aa . $cc - bb - 4pp$.

Now to return backwards from the 27[th] to the 25[th] ſtep of the Reſolution, by the lines of the Diagram, ſome Analogies not expreſt in the Reſolution muſt be introduced, (which are inferr'd from the Algebraical Fraction $\frac{ccaa - bbaa}{bb}$, as before hath been hinted in 55°,) to wit, the four Analogies next following.

 114. By

114. By *Theor.* 2. in 29° of *Probl.* 9. *Chap.* 7. $\}$ AD . AR :: AE . AG.
115. Therefore (*per prop.* 22. *Elem.* 6.) . . $\}$ $\square$AD . $\square$AR :: $\square$AE . $\square$AG.
116. Therefore, by *Division of Reason*,
$\square$AD $-$ $\square$AR . $\square$AR :: $\square$AE $-$ $\square$AG . $\square$AG.
117. And inversly,
$\square$AR . $\square$AD $-$ $\square$AR :: $\square$AG . $\square$AE $-$ $\square$AG.
118. Therefore from 113° and 114°, (*per prop.* 11, & 14. *Elem.* 5.)
$\square$AE $-$ $\square$AG $=$ $\square$AD $-$ $\square$AR $-$ 4$\square$P.

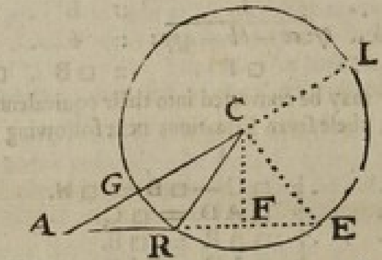
That is, in 25°, $\}$ . . $\frac{ccaa}{bb} - aa = cc - bb - 4pp.$

119. And by adding 4$\square$P, also $\square$AG to each part, and subtracting $\square$AE from each part of the Equation in 118°, this will arise,
4$\square$P $=$ $\square$AG $+$ $\square$AD $-$ $\square$AR $-$ $\square$AE.
120. But by *Theor.* 1. in 33° of this Problem,
$\square$AE $-$ $\square$AG $=$ $\square$AD $-$ $\square$AR $-$ 4$\square$OF.
121. Therefore by adding 4$\square$OF, also $\square$AG to each part, and subtracting $\square$AE from each part of the last preceding Equation, this will arise,
4$\square$OF $=$ $\square$AG $+$ $\square$AD $-$ $\square$AR $-$ $\square$AE.
122. Therefore from 119° and 121°, (*per Ax.* 1. *Chap.* 2.) $\}$ 4$\square$OF $=$ 4$\square$P.
123. Therefore, ... $\}$ $\square$OF $=$ $\square$P.
124. Therefore, ... $\}$ . . OF $=$ P.

Which was to be Demonstr. Therefore the Problem is satisfied. But for further Illustration, all the preceding Canons and Theorems raised out of the Resolution, may be exemplified by the Numbers placed near the Diagram at the beginning of this Problem.

Probl. IX.

The Base, Perpendicular and difference of the leggs of a plain Triangle being severally given, to find the Triangle. But the Base must exceed the difference of the leggs.



AR = 9	FE = 6
AC = 17	CF = 8
RC = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Prepar.

1. Suppose the Triangle ARC obtusangled at R, (the end of the Base AR,) to be that which is sought; then from the Center C, at the distance of the lesser legg CR describe the Circle CRGLE, cutting the greater legg CA in G; so shall AG be the difference of the leggs CA and CR, for CG $=$ CR.
2. Produce AC and AR to the Circumference in L and E, then is AL equal to the summ of the leggs AC and CR, for CL $=$ CR.
3. Draw the Semidiameter CE, and make CF $\perp$ RE, so shall CF cut RE into two equal parts in F, (*per prop.* 3. *Elem.* 3.) Which things being premis'd, the Resolution of the Problem propounded may be resolved in manner following.

Suppos.

4. $b =$ AR the Base of $\triangle$ ARC is given.
5. $p =$ CF the Perpendicular is given.
6. $d =$ AG the difference of the leggs AC and RC is given.

Req. to find the Triangle.

Resolution.

7. Put a for the unknown summ of the leggs, *viz.* assume
$a =$ AC $+$ RC $=$ AL.

8. Then

8. Then from *Theor.* 2. in 34° of the foregoing *Probl.* 8. of this *Chap.* this Analogy ariseth,

$\sqrt{:\overline{aa - bb}:} \;.\; \sqrt{:\overline{aa - bb - 4pp}:} \;::\; b \;.\; d.$

9. The Squares of which proportional lines shall be Proportionals also, therefore

$aa - bb \;.\; aa - bb - 4pp \;::\; bb \;.\; dd.$

10. Therefore by Conversion of Reason,

$aa - bb \;.\; 4pp \;::\; bb \;.\; bb - dd.$

11. And alternately,

$aa - bb \;.\; bb \;::\; 4pp \;.\; bb - dd.$

12. And by Composition,

$aa \;.\; bb \;::\; 4pp + bb - dd \;.\; bb - dd.$

13. And by inverse and alternate Reason,

$bb - dd \;.\; 4pp + bb - dd \;::\; bb \;.\; aa.$

14. But the sides of proportional Squares are also Proportionals, therefore from the last preceding Analogy,

$\sqrt{:\overline{bb - dd}:} \;.\; \sqrt{:\overline{4pp + bb - dd}:} \;::\; b \;.\; a.$

15. And because by *Theor.* 2. *Probl.* 10. *Chap.* 7.

$d \;.\; AE \;::\; b \;.\; a$

16. Therefore from the two last preceding Analogies, this ariseth,

$\sqrt{:\overline{bb - dd}:} \;.\; \sqrt{:\overline{4pp + bb - dd}:} \;::\; d \;.\; AE.$

The Analogy in 14° gives

THEOR. 1.

17. As the right line whose Square is equal to the excess whereby the Square of the Base of a plain Triangle exceeds the Square of the difference of the leggs, is to the right line whose Square is equal to the said excess together with the Square of the double Perpendicular; so is the Base, to the summ of the leggs.

Or thus, which is more convenient for Arithmetical practice.

As the excess of the Square of the Base above the Square of the difference of the leggs, is to the summ of the said excess and the Square of the double Perpendicular; so is the Square of the Base, to the Square of the summ of the leggs.

Therefore, the Base, Perpendicular, and difference of the leggs of a plain Triangle being severally given, the summ of the leggs shall be given also by the said *Theor.* 1. And consequently the leggs shall be given severally, by *Theor.* 9. *Chap.* 4.

The Analogy in 16° gives

THEOR. 2.

18. As the right line whose Square is equal to the excess by which the Square of the Base of a plain Triangle exceeds the Square of the difference of the leggs, is to the right line whose Square is equal to the said excess together with the Square of the double Perpendicular; so is the difference of the leggs to a fourth Proportional; which exceeds the Base, when the Perpendicular falls without the Triangle, for then 'tis the line compos'd of the Base and the double distance from the foot of the Perpendicular to the obtuse angle at the nearer end of the Base; but when the Perpendicular falls within, the said fourth Proportional is less than the Base, and is the difference of the segments of the Base made by the Perpendicular: Lastly, when the said fourth Proportional is equal to the Base, the Perpendicular falls upon the end thereof.

Therefore, if the quantities of the lines given in this *Probl.* 9. be exprest by numbers, we may discover by *Theor.* 2. above exprest, whether the Triangle sought be obtusangled, acute-angled, or right-angled at the Base.

It is also evident by the 14th and 16th steps of the Resolution, that to the end the Problem propounded may be possible, the given lines must be subject to this

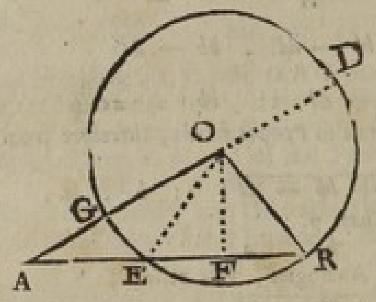
Determination.

The given Base must exceed the given difference of the leggs.

The truth of the preceding Theorems and Determination, as also the Composition of this *Probl.* 9. will be obvious to him that understands what hath been delivered in the foregoing *Probl.* 8. and therefore I shall wave the Composition.

Probl. X.

In a plain Triangle having unequal acute angles at the Base, the Perpendicular, difference of the leggs, and difference of the segments of the Base made by the Perpendicular, being severally given, to find the Triangle. But the lines given must be subject to the Determinnations hereafter exprest.



AR = 21		FR = 6
AO = 17		OF = 8
RO = 10		AF = 15
AE = 9		AG = 7
EF = 6		AD = 27

Preparat.

1. Let the Diagram belonging to the preceding *Probl.* 8. of this *Chapt.* be here repeated, and suppose the △ ARO having unequal acute angles A and R at the ends of the Base AR to be the Triangle sought; then respect being had to the Preparatory Construction in the three first steps of the said *Probl.* 8. the Resolution of this *Probl.* 10. may be formed thus;

Suppos.

2. p = OF the Perpendicular of △ ARO is given.
3. d = AG the difference of the leggs is given.
4. b = AE the difference of the segments of the Base is given.

Req. to find the Triangle.

Resolution.

5. It is manifest that if AE the given difference of the segments FA, FR be esteemed the Base of the △ AEO obtusangled at E, then AG shall be the difference of the leggs AO and EO, as well as of AO and RO, (for EO = RO = OG,) and OF a common Perpendicular to the two Triangles ARO and AEO; therefore in △ AEO, the Base AE, the Perpendicular OF, and AG the difference of the leggs AO and EO being severally given, the said △ AEO shall be given by the foregoing *Probl.* 9. of this *Chapt.* For, the summ of the leggs, to wit, AD = AO + EO = AO + OR shall be given by this following Analogy, (according to *Theor.* 1. in 17° of the said *Probl.* 9.) *viz.*

$$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} \;::\; b \;.\; AD.$$

6. Then AD and AG the summ and difference of the leggs AO and EO, or of AO and RO, being given, the said leggs shall be given severally by *Theor.* 9. *Chap.* 4.
7. Moreover, forasmuch as AR in reference to △ AEO obtusangled at E, is compos'd of the Base AE and ER, (= 2FE = 2FR,) the said AR, which is also the Base of △ ARO, shall be given by *Theor.* 2. in 18° of the preceding *Probl.* 9. For,

$$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} \;::\; d \;.\; AR.$$

From the premisses tis manifest that the Base and leggs of the Triangle sought in this *Probl.* 10. may be found out by the foregoing *Probl.* 9. But that there may be a possibility of finding out the Triangle required, the lines given must be liable to these two following Determinations, *viz.*

Determination 1.

8. The line given for the difference of the segments of the Base made by the Perpendicular falling within the Triangle, must exceed the given difference of the leggs, that is, (in the Figure belonging to this *Probl.* 10.) AE ⊏ AG, the truth whereof is manifest by *prop.* 8. *Elem.* 3.

9. Again,

9. Again, because by *Supposition* the Triangle sought hath unequal acute angles at the Base, the Perpendicular falls within, and the Base must necessarily exceed the difference of the segments of the Base made by the Perpendicular; therefore to the end the lines given may be capable of effecting the Problem propounded, the fourth Proportional (or Base) found out by the Analogy before exprest in 7° must exceed the given line A E. Hence,

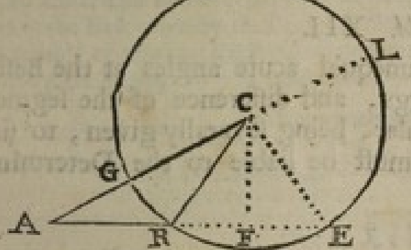
Determination 2.

10. . . . . $\dfrac{\sqrt{:4pp + bb - dd:} \times d}{\sqrt{:bb - dd:}} \sqsubset b.$

Supposing then the given Quantities to be qualified according to the tenour of the Determinations before prescribed, the industrious Learner may easily apply what hath been said in the foregoing 5th, 6th and 7th steps, as well to the Geometrical Effection, as to the Arithmetical Solution of this *Probl.* 10.

Probl. XI.

In a plain Triangle obtusangled at the Base, the Perpendicular, difference of the leggs, and the line compos'd of the Base and the double distance from the foot of the Perpendicular to the obtuse angle, being severally given, to find the Triangle. But the given lines must be subject to the Determinations hereafter exprest.



AR = 9	FE = 6
AC = 17	CF = 8
RC = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Prepar.

1. Let the Diagram belonging to the foregoing *Probl.* 9. of this *Chapt.* be here repeated, and suppose the △ A R C obtusangled at R, (the end of the Base A R,) to be the Triangle sought; then respect being had to the preparatory Construction in 1°, 2° and 3° of *Probl.* 9. the Resolution of this *Probl.* 11. may be formed thus;

Suppos.

2. p = C F the Perpendicular of △ A R C is given.
3. d = A G the difference of the leggs A C and R C is given.
4. b = A E the line compos'd of the Base A R and 2 F E, (or 2 F R,) is given.

Req. to find the Triangle.

Resolution.

5. It is manifest, that if the given line A E be esteemed the Base of the △ A E C having unequal acute angles at A and E, then A G is the difference of the leggs A C and E C, as well as of A C and R C, (for E C = R C,) and C F is a common Perpendicular to the two Triangles A E C and A R C; therefore in △ A E C, the Base A E, the Perpendicular C F, and A G the difference of the leggs A C and E C, (or R C,) being given severally, the said △ A E C shall be given by *Probl.* 9. of this *Chapt.* For A L = A C + E C, (the summ of the leggs of △ A E C,) shall be given by this following Analogy, (according to *Theor.* 2. in 17° of the said *Probl.* 9.) *viz.*

$$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} \;::\; b \;.\; \text{AL}.$$

6. Then A L and A G the summ and difference of the leggs A C and E C being severally given, the leggs themselves shall be also given severally, by *Theor.* 9. *Chap.* 4.
7. Moreover, because A R in reference to the △ A E C is the difference of the segments F A and F E, made by the Perpendicular C F, and is also the Base of the △ A R C

required,

required, the said AR shall be given by *Theor.* 2. in *Probl.* 18. of *Probl.* 9. of this *Chap.* For,

$$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} \;::\; d \;.\; AR.$$

From the premisses 'tis evident, that the three sides of the Triangle required by this *Probl* 11. are discovered by *Probl.* 9. of this *Chapt.* But the lines given must be subject to the following Determinations, that there may be a possibility of finding out a Triangle to satisfie the Problem propounded.

Determination 1.

8. The line given for the summ of the Base and double distance from the foot of the Perpendicular to the obtuse angle, must be longer than the line given for the difference of the leggs, that is, $AE \sqsubset AG$, as may be easily proved; for by *Supposition* $\triangle ARC$ is obtusangled at R, therefore $AE \sqsubset AR$, and consequently AE much greater than AG, for (*per prop.* 8. *Elem.* 3.) $AR \sqsubset AG$.

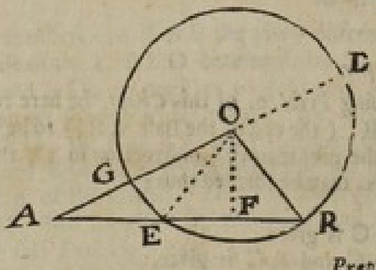
Again, because by *Supposition* the Triangle sought is obtusangled at the Base, the Perpendicular falls without, and the Base shall necessarily be less than the line compos'd of the Base and the double distance from the foot of the Perpendicular to the obtuse angle; therefore to the end the lines given may be capable of effecting the Problem propounded, the fourth Proportional (or Base) found out by the Analogy in 7°, must be less than the given line AE. Hence,

Determination 2.

$$\frac{\sqrt{:4pp + bb - dd:} \times d}{\sqrt{:bb - dd:}} \sqsupset b.$$

Probl. XII.

In a plain Triangle having unequal acute angles at the Base, the Perpendicular, summ of the leggs, and difference of the segments of the Base made by the Perpendicular, being severally given, to find the Triangle. But the lines given must be liable to the Determinations hereafter declared.



AR = 21	FR = 6
AO = 17	OF = 8
OR = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Prepar.

1. Let the Diagram belonging to the preceding *Probl.* 8, of this *Chapt.* be here repeated, and suppose the Triangle ARO having unequal acute angles A and R at the ends of the Base AR to be the Triangle sought; then respect being had to the preparatory Construction in the three first steps of the said *Probl.* 8. the Resolution of this *Probl.* 12. may be formed thus;

Suppos.

2. p = OF the Perpendicular of $\triangle$ ARO is given.
3. c = AD = AO + RO the summ of the leggs is given.
4. b = AE = FA − FR the difference of the segments of the Base is given.

Req to find the Triangle.

Resolution.

5. It is evident, that if the given line AE be esteem'd the Base of the $\triangle$ AEO obtusangled at E, then AD is the summ of the leggs AO and EO, as well as of AO and RO, for EO = RO, and OF is a common Perpendicular to the two Triangles AEO and ARO; therefore in $\triangle$ AEO, the Base AE, the Perpendicular OF, and AD the summ of the leggs AO and EO being severally given, the Triangle

A E O shall be given by the foregoing *Probl.* 8. of this *Chapt.* For A G, the difference of the leggs A O and E O, shall be given by this following Analogy, (according to *Theor.* 2. in 24° of *Probl* 8.)

$$\sqrt{:cc - bb:} \;.\; \sqrt{:cc - bb - 4pp:} \;::\; b \;.\; AG.$$

6. Then A D and A G the summ and difference of the leggs A O and E O, (or RO,) being given severally, the said leggs shall be also given severally, by *Theor.* 9 *Chap.* 4.

7. Moreover, forasmuch as A R in reference to the △ A E O obtusangled at E, is compos'd of the Base A E and E R, (= 2 F E = 2 F R,) the said A R, which is also the Base of △ A R O required, shall be given by *Theor.* 3. in 35° of *Probl.* 8. For,

$$\sqrt{:cc - bb:} \;.\; \sqrt{:cc - bb - 4pp:} \;::\; c \;.\; AR.$$

From the premisses 'tis manifest, that the Base and leggs of the Triangle sought by this *Probl.* 12. are discovered by the foregoing *Probl.* 8. But that there may be a possibility of finding out the desired Triangle, the given lines must be subject to these two following Determinations, *viz.*

Determination 1.

$$c \sqsubset \sqrt{:bb + 4pp:} \quad \text{That is,}$$

8. The line given for the summ of the leggs must exceed that right line whose Square is equal to the summ of the Square of the given Base and the Square of the double of the given Perpendicular.

This Determination doth openly shew it self in the preceding Analogy in 5°, and hath already been demonstrated in *Probl.* 8. of this Chapter.

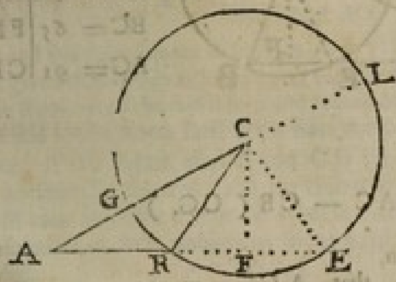
9. Again, because by *Supposition* the Triangle sought hath unequal acute angles at the Base, the Perpendicular falls within, and the Base must necessarily exceed the difference of the segments of the Base made by the Perpendicular, therefore to the end the given lines may be capable of effecting the Problem propounded, the fourth Proportional (or Base) found out by the Analogy in 7°, must exceed the given line A E. Hence,

Determination 2.

$$\frac{\sqrt{:cc - bb - 4pp:} \times c}{\sqrt{:cc - bb:}} \sqsubset b.$$

Probl. XIII.

In a plain Triangle obtusangled at the Base, the Perpendicular, summ of the leggs, and the line compos'd of the Base and double distance from the foot of the Perpendicular to the obtuse angle, being given severally, to find the Triangle. But the given lines must be subject to the Determinations hereafter declared.



AR = 9	FE = 6
AC = 17	CF = 8
CR = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Preparat.

1. Let the Diagram belonging to the foregoing *Probl.* 9. of this *Chapt.* be here repeated, and suppose the △ A R C obtusangled at R, (the end of the Base A R,) to be the Triangle sought; then respect being had to the preparatory Construction in 1°, 2° and 3° of the said *Probl.* 9. the Resolution of this *Probl.* 13. may be formed thus;

Suppos.

2. p = C F the Perpendicular of △ A R C is given.
3. c = A L = A C + R C the summ of the leggs is given.

4. b = AE

4. b = A E the line compos'd of the Base A R and 2 F R (or 2 F E) is given.
Req. to find the Triangle.

Resolution.

5. It is manifest, that if the given line A E be esteem'd the Base of the △ A E C having unequal acute angles at A and E, then A L is the summ of the leggs A C and E C, as well as of A C and R C, (for E C = R C,) and C F is a common Perpendicular to the two Triangles A R C and A E C; therefore in △ A E C, the Base A E, the Perpendicular C F, and A L the summ of the leggs A C and E C, (or R C,) being given severally, the △ A E C shall be given by the foregoing *Probl.* 8. of this Chapter. For A G the difference of the leggs shall be given by this following Analogy, (according to *Theor.* 2. in 34° of *Probl.* 8.)

$$\surd : \overline{cc - bb} : \ . \ \surd : \overline{cc - bb - 4pp} : \ :: \ b \ . \ AG.$$

6. Then A L and A G the summ and difference of the leggs A C and E C being given severally, the leggs shall also be given severally, by *Theor.* 9. *Chap.* 4.
7. Moreover because A R, in reference to the △ A E C, is the difference of the segments F A and F E made by the Perpendicular C F, and is also the Base of the △ A R C required, the said A R shall be given by *Theor.* 3. in 35° of *Probl.* 8. For,

$$\surd : \overline{cc - bb} : \ . \ \surd : \overline{cc - bb - 4pp} : \ :: \ c \ . \ AR.$$

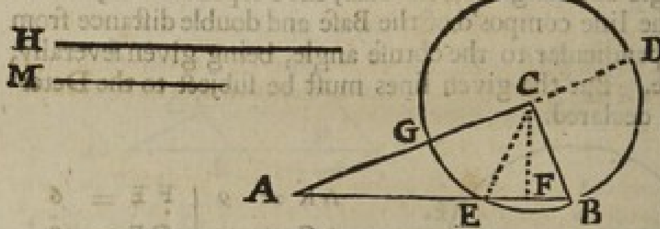
8. From the premisses 'tis evident, that the three sides of the Triangle required by this *Probl.* 13. are discovered by *Probl.* 8. of this *Chapt.* But the lines given must be subject to the following Determinations, *viz.*

Determinat. 1. . . . . . . $c \sqsubset \surd : \overline{bb + 4pp} :$

Determinat. 2. . . $\dfrac{\surd : \overline{cc - bb - 4pp} : \times c}{\surd : \overline{cc - bb} :} \sqsupset b.$

Probl. XIV.

The Hypothenusal of a right-angled Triangle being given, as also a mean Proportional between the Base and Perpendicular, to find the Triangle. But the right line arising by the Application of the Square of the given mean to the given Hypothenusal, must not be greater than half the Hypothenusal.



AB=169	AE=119
AC=156	FB= 25
BC= 65	FE= 25
AG= 91	CF= 60

Suppos.

1. A B C is a △ right-angled at C.
2. A C ⊏ C B.
3. C B G D is a ⊙, whence A G = A C − C B (C G.)
4. C F ⊥ A B.
5. b = A B the Hypothenusal is given.
6. m = M a right line given, and such, that A C . m :: m . C B.
Req. to find out △ A B C.

Resolut. 1.

7. Because by *Supposition* in 6° the right line m is a mean Proportional between A C and C B, therefore mm = ▭ A C, C B, that is, ▭ A B, C F, for (*per prop.* 41. *Elem.* 1.) each of those Rectangles is equal to 2 △ A B C; therefore $\frac{mm}{b}$ = C F the Perpendicular is given; therefore also △ A B C shall be given both Geometrically and Arithmetically,

tically, by *Probl.* 16. *Chap.* 5. But here I shall frame the Resolution and Composition of the same Problem after another manner.

Resolut. 2.

8. Put a for the difference of the leggs sought, *viz.* $a = AG = AC - CB.$

9. Then because AD is the summ, and AG the difference of the leggs AC and CB, therefore (*per Theor.* 7. *Chap* 4.) . . . . $\square M(\square AC, CB) + \frac{1}{4}\square AG = \square\frac{1}{2}AD.$

10. Therefore in the letters belonging to the Resolution, the Square of half the summ of the leggs shall be equal to . . . . . . $mm + \frac{1}{4}aa \; (= \square \frac{1}{2}AD.)$

11. Therefore the square Root of that Square of the half summ shall be the half summ of the leggs, to wit, ... $\sqrt{:}\overline{mm + \frac{1}{4}aa}: \; (= \frac{1}{2}AD.)$

12. Therefore from 8° and 11°, (by *Theor.* 9. *Chap.* 4.) the greater legg is . . . . . $\sqrt{:}\overline{mm + \frac{1}{4}aa}: + \frac{1}{2}a \; (= AC.)$

13. And the lesser legg is . . . . $\sqrt{:}\overline{mm + \frac{1}{4}aa}: - \frac{1}{2}a \; (= CB.)$

14. Therefore from 12°, (by *Theor.* 2. *Chap.* 4.) the Square of the greater legg shall be . . $mm + \frac{1}{2}aa + a \times \sqrt{:}\overline{mm + \frac{1}{4}aa}:$

15. And from 13°, (by *Theor.* 5. *Chap.* 4.) the Square of the lesser legg shall be . . . . . $mm + \frac{1}{2}aa - a \times \sqrt{:}\overline{mm + \frac{1}{4}aa}:$

16. Therefore the summ of all in 14° and 15° gives the summ of the Squares of the legs, to wit, $2mm + aa \; (= \square AC + \square CB.)$

17. And because by *Suppos.* in 1° $\angle ACB = \perp$, therefore from 5° and 16°, (*per prop.* 47. *Ele.* 1.) this Equation ariseth, ... $2mm + aa = bb \; (= \square AB.)$

18. Therefore by subtracting $2mm$ from each part of the last Equation, ... $aa = bb - 2mm.$

19. Therefore by extracting the square Root out of each part of the last Equation, the difference of the leggs is made known, *viz.* . . . $a = \sqrt{:}\overline{bb - 2mm}: \; (= AG.)$

20. Therefore from 12°, 13°, 18° and 19°, the leggs shall be given severally, *viz.*

$$\begin{cases} AC = \sqrt{:}\overline{\frac{1}{4}bb + \frac{1}{2}mm}: + \sqrt{:}\overline{\frac{1}{4}bb - \frac{1}{2}mm}: \\ CB = \sqrt{:}\overline{\frac{1}{4}bb + \frac{1}{2}mm}: - \sqrt{:}\overline{\frac{1}{4}bb - \frac{1}{2}mm}: \end{cases}$$

The Equation in 19° gives

CANON 1.

21. From the Square of the given Hypothen. subtract the double Square of the given mean Proportional, so the square root of the remainder shall be the difference of the leggs sought.

The Equations in 20° give

CANON 2.

22. To and from the Square of half the given Hypothenusal add and subtract half the Square of the given mean Proportional, and reserve the summ and remainder; then extract the square Root out of the said summ and remainder severally; lastly, the summ and difference of the said square Roots shall be the sides about the right angle of the Triangle sought.

Note. If the values of AC and CB (the sides about the right angle) before exprest in 20° be severally squared, and the Universal square Root extracted out of each Product, there will come forth the Canon delivered in *Sect.* 55. *Probl.* 16. *Chap.* 5. for the Arithmetical Resolution of such ambiguous Biquadratick Equations as fall under the Form there expounded.

But that the truth of the preceding Canons may more clearly appear, I shall propound and demonstrate them in the form of Thorems, by a repetition of the steps of the foregoing Resolution.

THEOR. 1.

23. In a right-angled Triangle having unequal leggs about the right angle, the difference of those leggs is equal to a right line whose Square is equal to the excess whereby the Square of the Hypothenusal exceeds the double Square of a mean Proportional between the said leggs.

THEOR. 2.

24. In a right-angled Triangle having unequal leggs about the right angle, the greater legg is equal to the summ of these two right lines, to wit, the right line whose Square is equal to the Square of half the Hypothenusal together with half the Square of a mean Proportional between the leggs, and the right line whose Square is equal to the excess whereby the Square of half the Hypothenusal exceeds half the Square of the said mean. But the lesser legg is equal to the difference of the said two right lines.

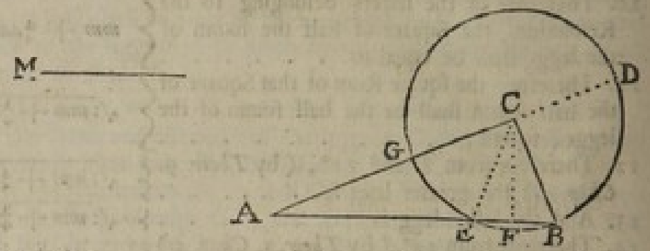
Suppos.

25. ABC is a $\triangle$ right-angled at C.

26. AC $\sqsubset$ CB.

27. CBGD is a $\odot$; and ACD is a right line, therefore

28. $AD = AC + CB$, and $AG = AC - CB$.



29. M is a right line, such, that . . . . . . $AC \,.\, M :: M \,.\, CB$.

Req. demonstr.

30. *Theor.* 1. $AG = \sqrt{:\square AB - 2\square M:}$

31. *Theor.* 2. $\begin{cases} AC = \sqrt{:\tfrac{1}{4}\square AB + \tfrac{1}{2}\square M:} + \sqrt{:\tfrac{1}{4}\square AB - \tfrac{1}{2}\square M:} \\ CB = \sqrt{:\tfrac{1}{4}\square AB + \tfrac{1}{2}\square M:} - \sqrt{:\tfrac{1}{4}\square AB - \tfrac{1}{2}\square M:} \end{cases}$

Demonstration.

32. By *Suppos.* in 28°, ... $AG = AC - CB$.

33. Therefore the Square of that Equation, (*per Theor.* 5. *Chap.* 4.) gives . . . . $\square AG = \square AC + \square CB - 2\square AC, CB$,

34. By *Suppos.* in 25° $\angle$ ACB is $\perp$, therefore (*per prop.* 47. *Elem.* 1.) . . . . $\square AB = \square AC + \square CB$.

35. Therefore from 33° and 34°, (*per Ax.* 6. *Chap.* 2.) . . . . $\square AG = \square AB - 2\square AC, CB$,

36. From 29°, (*per prop.* 17. *Elem.* 6.) . . $\square M = \square AC, CB$.

37. And consequently, ... $2\square M = 2\square AC, CB$.

38. Therefore from 35° and 37°, (*per Ax.* 6. *Chap.* 2.) . . . . $\square AG = \square AB - 2\square M$.

39. But the sides of equal Squares are also equal, therefore from 38°, ... $AG = \sqrt{:\square AB - 2\square M:}$

Which was *Theor.* 1. to be Dem.

40. Again, because by *Suppos.* in 28°, AD is the summ, and AG the difference of the leggs AC and CB, therefore (*per Theor.* 7. *Chap.* 4.) $\square \tfrac{1}{2} AD = \square AC, CB + \tfrac{1}{4}\square AG$.

41. And because from 29°, (*per prop.* 17. *Elem.* 6.) . . . . $\square M = \square AC, CB$.

42. And by taking $\tfrac{1}{4}$ of all in 38°, ... $\tfrac{1}{4}\square AB - \tfrac{1}{2}\square M = \tfrac{1}{4}\square AG$.

43. Therefore from 40°, 41° and 42°, (*per Ax.* 6. *Chap.* 2.) . . . . $\square \tfrac{1}{2} AD = \tfrac{1}{4}\square AB + \tfrac{1}{2}\square M$.

44. But the sides of equal Squares are also equal, therefore from 43°, ... $\tfrac{1}{2} AD = \sqrt{:\tfrac{1}{4}\square AB + \tfrac{1}{2}\square M:}$

45. And for the like reason, 'tis manifest from the Equation in 42°, that . . . . $\tfrac{1}{2} AG = \sqrt{:\tfrac{1}{4}\square AB - \tfrac{1}{2}\square M:}$

46. Therefore, by taking the summ and difference of the Equations in 44° and 45°, these will arise,

$$\begin{cases} \tfrac{1}{2} AD + \tfrac{1}{2} AG = \sqrt{:\tfrac{1}{4}\square AB + \tfrac{1}{2}\square M:} + \sqrt{:\tfrac{1}{4}\square AB - \tfrac{1}{2}\square M:} \\ \tfrac{1}{2} AD - \tfrac{1}{2} AG = \sqrt{:\tfrac{1}{4}\square AB + \tfrac{1}{2}\square M:} - \sqrt{:\tfrac{1}{4}\square AB - \tfrac{1}{2}\square M:} \end{cases}$$

47. And because AD is the summ, and AG the difference of AC and CB; therefore (*per Theor.* 9. *Chap.* 4.)

$$AC = \tfrac{1}{2} AD + \tfrac{1}{2} AG; \text{ and } CB = \tfrac{1}{2} AD - \tfrac{1}{2} AG.$$

48. There-

48. Therefore from 46° and 47°, (*per Ax.* 1. *Chap.* 2.)

$$\begin{cases} AC = \sqrt{:\tfrac{1}{4}\square AB + \tfrac{1}{2}\square M:} + \sqrt{:\tfrac{1}{4}\square AB - \tfrac{1}{2}\square M:} \\ CB = \sqrt{:\tfrac{1}{4}\square AB + \tfrac{1}{2}\square M:} - \sqrt{:\tfrac{1}{4}\square AB - \tfrac{1}{2}\square M:} \end{cases}$$

Which was *Theor.* 2. to be Demonstr.

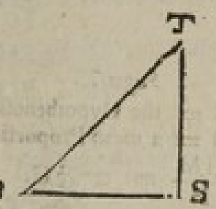
In the next place, to the end the Geometrical Effection of the foregoing *Probl.* 14. may meet with no obstruction, I shall prove the truth of the Determination annex'd to the Problem, by demonstrating this following

LEMMA.

49. In a right-angled Triangle, if the Square of a mean Proportional between the sides about the right angle, (that is, if the Rectangle of those sides) be applied to the Hypothenusal, the line thence arising shall sometimes be equal to half the Hypothenusal, and sometimes less, but never greater than the said half.

The sides about the right angle are either equal to one another, or else unequal; I shall begin with with the first Case.

Suppos. in *Case* 1.



50. RST is a $\triangle$ right-angled at S.
51. $RS = ST$.
52. M is a right line, such, that
53. $RS \,.\, M :: M \,.\, ST$.
54. ∴ . *Req. demonstr.* . . $\dfrac{\square M}{RT} = \tfrac{1}{2}RT$.

Demonstration.

55. By *Supposition* in 51°, ... } . . $RS = ST$.
56. Therefore by drawing ST as a common altitude into each part, ... } $\text{▭}RS,ST = \square ST = \square RS$.
57. Therefore, (*per Ax.* 8. *Chap.* 2.) . . } $2\text{▭}RS,ST = \square ST + \square RS$.
58. And because by *Suppos.* in 50° $\angle S$ is ⊥, therefore, (*per prop.* 47. *Elem.* 1.) . . . } . . $\square RT = \square ST + \square RS$.
59. Therefore from 57° and 58°, (*per Ax.* 1. *Chap* 2.) . . . . } $2\text{▭}RS,ST = \square RT$.
60. And consequently, ... } $\text{▭}RS,ST = \tfrac{1}{2}\square RT$.
61. But from 53°, (*per prop.* 17. *Elem.* 6.) . } $\text{▭}RS,ST = \square M$.
62. Therefore from 60° and 61°, (*per Ax.* 1. *Chap.* 2.) . . . . } . . $\square M = \tfrac{1}{2}\square RT = \text{▭}RT,\tfrac{1}{2}RT$.
63. Therefore from 62°, by Application of each part to RT, ... } . . $\dfrac{\square M}{RT} = \tfrac{1}{2}RT$.

Which was *Case* 1. to be Dem.

Suppos. in *Case* 2.

64. ABC is a $\triangle$ right-angled at C.
65. $AC \sqsubset CB$.
66. M a right line, such, that . . . $AC \,.\, M :: M \,.\, CB$.

[*See the* Diagr. *in the precedent Page.*]

67. . . *Req. demonstr.* . . . . $\dfrac{\square M}{AB} \sqsupset \tfrac{1}{2}AB$.

Demonstration.

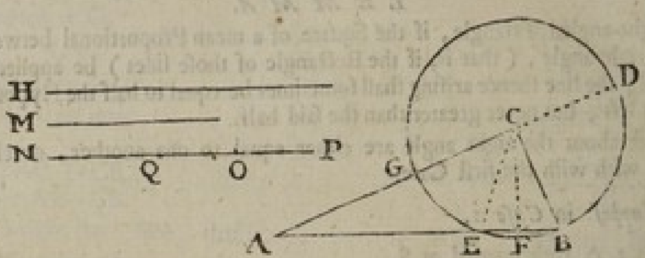
68. By the preceding *Theor.* 1. before demonstrated 'tis evident that . . . . } $2\square M \sqsupset \square AB$.
69. Therefore, by taking the half of each part, it follows that . . . . } $\square M \sqsupset \tfrac{1}{2}\square AB$.
70. Therefore from 69°, by Application of each part to AB, ... } $\dfrac{\square M}{AB} \sqsupset \tfrac{1}{2}AB$.

Which was *Case* 2. to be Dem.

71. Now because in every right-angled Triangle, the sides about the right angle are either equal or unequal between themselves, and it hath been demonstrated, that when the said sides are equal to one another, the right line arising by the Square of a mean

Proportional between the said sides to the Hypothenusal is equal to half the Hypothenusal; but when the said sides are unequal, the said right line is less than half the Hypothenusal, it is evident that the right line arising by the said Application can never be greater than half the Hypothenusal: Therefore the truth of the *Lemma* is manifest, and consequently, the lines given in *Probl.* 14. must be subject to the Determination annex'd to it, that there may be a possibility of effecting the Problem.

The Composition of the foregoing Probl. 14.



Suppos.

72. $H =$ the Hypothenusal of a right-angled Triangle is given.

73. $M =$ a mean Proportional between the sides about the right angle is given.

74. $\frac{\square M}{H}$ not $\sqsubset \frac{1}{2} H$. (*Determination.*)

Req to find the Triangle.

Construction.

75. By *Probl.* 2. *Chap.* 5. find a right line NO, such, that its Square may be equal to $\frac{1}{4}\square H + \frac{1}{2}\square M$, therefore

$$NO = \sqrt{:\tfrac{1}{4}\square H + \tfrac{1}{2}\square M:}$$

76. By the Determination in 74°, $\frac{\square M}{H}$ not $\sqsubset \frac{1}{2} H$; suppose then it be granted, or discovered by H and M given in numbers, that $\frac{\square M}{H}$ is less than $\frac{1}{2} H$, and consequently, (by multiplying each part into H,) that $\square M \sqsupset \frac{1}{2}\square H$; then it evidently follows, that 'tis possible (*per Probl.* 4. *Chap.* 5.) to find out a right line OP, such, that its Square may be equal to $\frac{1}{4}\square H - \frac{1}{2}\square M$, suppose therefore

$$OP = \sqrt{:\tfrac{1}{4}\square H - \tfrac{1}{2}\square M:}$$

77. Make $NP = NO + OP$, then from the Construction in 75° and 76°, 'tis manifest that

$$NP = \sqrt{:\tfrac{1}{4}\square H + \tfrac{1}{2}\square M:} + \sqrt{:\tfrac{1}{4}\square H - \tfrac{1}{2}\square M:}$$

78. From NO cut off $OQ = OP$, which may be done, for 'tis evident by *Constr.* in 75° and 76°, that $NO \sqsubset OP$, suppose therefore $OQ = OP$, then from 75° and 76° it follows that

$$NQ = NO - OP = \sqrt{:\tfrac{1}{4}\square H + \tfrac{1}{2}\square M:} - \sqrt{:\tfrac{1}{4}\square H - \tfrac{1}{2}\square M:}$$

79. Make $AC = NP$; also $CB = NQ$, and $CB \perp AC$; lastly, draw AB.

80. I say ABC is the right-angled Triangle required. Now we must shew that it will satisfie the Problem. First then by *Construction* in 79°, $CB \perp AC$, and consequently the angle ACB is a right angle. But that the Hypothenusal AB is equal to the given Hypothenusal H, and that the given right line M is a mean Proportional between AC and CB, (the sides about the right angle,) the following Demonstration will make manifest.

81. . . *Req. demonstr.* . . . . $\left\{ \begin{array}{l} AB = H. \text{ Also, that} \\ AC \,.\, M :: M \,.\, CB. \end{array} \right.$

Demonstration.

82. By *Constr.* in 77°, ... $NP = NO + OP$.

83. Therefore (*per Theor.* 2. *Chap.* 4.) . . $\square NP = \square NO + \square OP + 2\,\square NO,OP$.

84. By *Constr.* in 78°, ... $NQ = NO - OP$.

85. Therefore (*per Theor.* 5. *Chap.* 4.) . . $\square NQ = \square NO + \square OP - 2\,\square NO,OP$.

86. Therefore, by adding together the Equations in 83° and 85°, ... $\square NP + \square NQ = 2\square NO + 2\square OP$.

87. And

87. And from the *Constr.* in 79°, (*per prop.* 47. *Elem.* 1.)	$\square NP + \square NQ = \square AC + \square CB.$
88. Therefore from 86° and 87°, (*per Ax.* 1. *Chap.* 2.)	$2\square NO + 2\square OP = \square AC + \square CB.$
89. But by *Constr.* in 79°, $<$ ACB is ⅃, therefore (*per prop.* 47. *Elem.* 1.)	. . . . $\square AB = \square AC + \square CB.$
90. Therefore from 88° and 89°, (*per Ax* 1. *Chap.* 2.)	$2\square NO + 2\square OP = \square AB.$
91. But from the *Constr.* in 75° and 76°, (by adding together the double Squares of the Equations there exprest,) 'tis evident that . .	$2\square NO + 2\square OP = \square H.$
92. Therefore from 90° and 91°, (*per Ax.* 1. *Chap.* 2.)	$\square AB = \square H.$
93. But the sides of equal Squares are also equal, therefore from 92°, ...	$AB = H.$ Which was to be Dem.
94. Again, because by *Constr.* in 77° and 78°, N P is the summ, and NQ the difference of NO and O P, therefore (*per Theor.* 8. *Chap.* 4.)	$\sqsubset\!\sqsupset NP, NQ = \square NO - \square OP.$
95. But from the *Constr.* in 79°, ...	$\sqsubset\!\sqsupset NP, NQ = \sqsubset\!\sqsupset AC, CB.$
96. Therefore from 94° and 95°, (*per Ax.* 1. *Chap.* 2.)	$\sqsubset\!\sqsupset AC, CB = \square NO - \square OP.$
97. By *Constr.* in 75°, ...	$\frac{1}{4}\square H + \frac{1}{2}\square M = \square NO.$
98. And by *Constr.* in 76°, ...	$\frac{1}{4}\square H - \frac{1}{2}\square M = \square OP.$
99. And from 97°, by subtracting $\square$ O P from each part, ...	$\frac{1}{4}\square H + \frac{1}{2}\square M - \square OP = \square NO - \square OP.$
100. And from 98°; by adding $\frac{1}{2}\square$ M to each part, ...	$\frac{1}{2}\square M + \square OP = \frac{1}{4}\square H.$
101. And by adding the Equation in 100° to that in 99°, ...	$\frac{1}{4}\square H + \square M = \square NO - \square OP + \frac{1}{4}\square H.$
102. And from 101°, by subtracting $\frac{1}{4}\square$ H from each part, there will remain . . .	$\square M = \square NO - \square OP.$
103. But it hath been proved in 96°, that . .	$\sqsubset\!\sqsupset AC, CB = \square NO - \square OP.$
104. Therefore from 102° and 103°, (*per Ax.* 1. *Chap.* 2.)	$\sqsubset\!\sqsupset AC, CB = \square M.$
105. Wherefore from 104°, (*per prop.* 14. *Elem.* 6.)	$AC \,.\, M :: M \,.\, CB.$

Which was to be Demonstr. Therefore that is done which the Problem required.

Note. The foregoing Problem is the same in effect with this, *viz.* The summ of the Squares and the Rectangle of two right lines being given severally, to find out those lines.

Probl. XV.

The Hypothenusal and Area of a right-angled Triangle being given severally, to find out the Triangle. But the right line arising by the Application of the double Area to the Hypothenusal, must not be greater than half the Hypothenusal.

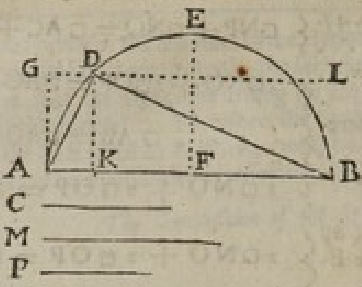
This Problem differs but little from the preceding 14th, for there, the Rectangle of the sides about the right angle, (that is, the double Area,) and Hypothenusal are given; but here, half the said Rectangle, (that is, the Area of the right-angled Triangle sought,) and the Hypothenusal are given.

Suppos.

1. A B D is a △ right-angled at D.
2. b = A B the Hypothenusal is given.
3. c = C a given right line, whose Square is equal to △ ABD, that is, $\sqsubset\!\sqsupset \frac{1}{2}$ BD,DA.

Req. to find out the Triangle.

Resolu-



AB = 169	C = $\sqrt{5070}$
BD = 156	M = $\sqrt{10140}$
AD = 65	P = 60
DK = 60	
BK = 144	
KA = 25	

Resolution.

4. Because (by *prop.* 41. *Elem.* 1.) the Rectangle of BD into DA is equal to the double Area of △ ABD, therefore from 3° the said Rectangle or double Area is . . . . $2cc.$

5. And out of 2° and 3°, (by *Theor.* 1. in 23° of the foregoing *Probl.* 14. of this *Chapt.*) the Square of the difference of the leggs about the right angle shall be . . . . $hh - 4cc.$

6. And by adding $8cc$, (to wit, four Rectangles of the leggs,) to $hh - 4cc$, (that is, the Square of the difference of the leggs,) the summ of that Addition gives (*per Theor.* 7. *Chap.* 4.) the Square of the summ of the leggs, to wit, ... $hh + 4cc.$

7. And because (by *Theor.* 3. *Chap.* 4.) a quarter of the Square of any whole right line is equal to the Square of the half; therefore from 6°, the Square of half the summ of the leggs shall be . . . . . . $\frac{1}{4}hh + cc.$

8. And consequently from 7°, half the summ of the leggs is . . . $\sqrt{:\frac{1}{4}hh + cc:}$

9. And from 5°, (by *Theor.* 3. *Chap.* 4.) the Square of half the difference of the leggs is . . . . $\frac{1}{4}hh - cc.$

10. And consequently from 9°, half the difference of the leggs is . . $\sqrt{:\frac{1}{4}hh - cc:}$

11. Therefore from 8° and 10°, (by *Theor.* 9. *Chap.* 4.) the leggs shall be given severally, *viz.*

$$BD = \sqrt{:\tfrac{1}{4}hh + cc:} + \sqrt{:\tfrac{1}{4}hh - cc:}$$
$$DA = \sqrt{:\tfrac{1}{4}hh + cc:} - \sqrt{:\tfrac{1}{4}hh - cc:}$$

From 6° and 5° ariseth

THEOR. 1.

12. In every right-angled Triangle having unequal sides about the right angle, the Square of the summ of those sides is equal to the Square of the Hypothenusal together with the quadruple of the Area: But the Square of the difference of the same sides is equal to the excess whereby the Square of the Hypothenusal exceeds the quadruple of the Area.

The Equations in 11° give

THEOR. 2.

13. In every right-angled Triangle having unequal sides about the right-angle, if to and from the Square of half the Hypothenusal, the Area be added and subtracted severally, and out of the summ and remainder severally the square Root be extracted, the summ and difference of those square Roots shall be equal to the sides about the right angle.

The truth of the Determination annex'd to this *Probl.* 15. hath already been demonstrated in the preceding *Probl.* 14. and the reason thereof will appear in the following Construction.

The Composition of the foregoing Probl. 15.

Suppos.

14. AB = the Hypothenusal of a right-angled Triangle is given.
15. C is a right line given, whose Square is equal to the Area of that Triangle.
16. $\frac{2\,\square C}{AB}$ not $\sqsubset \frac{1}{2}$ AB.

Req. to find the Triangle.

Constru-

Construction.

17. This Problem might be effected according to the direction of the foregoing Theorem in 13°, but more compendiously thus; First, by *Probl.* 2. *Chap.* 5. find a right line M, such, that its Square may be equal to 2 □ C, therefore

$$M = \sqrt{2\,\square C}.$$

18. Then by *Probl.* 7. *Chap.* 5. let it be made as A B to M, so M to a third proportional line, suppose it to be the line P, therefore

$$AB \,.\, M :: M \,.\, P.$$

19. Upon A B describe the Semicircle F A D B.

20. Make $AG \perp AB$; also $AG = P$; and $GL \parallel AB$, which Parallel G L shall necessarily either touch the Semicircle F A D B, or cut the same; for by *Suppos.* in 16°, $\frac{2\,\square C}{AB}$ not $\sqsubset \frac{1}{2} AB$, and by *Constr.* in 17° and 18°, P is equal to $\frac{2\,\square C}{AB}$; therefore P is not greater than $\frac{1}{2}$ A B, (= the Semidiameter F E.) But G L was before drawn parallel to A B at the distance of the right line P, (= A G,) and therefore the said Parallel shall either touch the Semicircle in E, or else cut the same. Supposing then the Parallel G L to cut the Semicircle in D, draw the right lines A D and D B, so shall A D B be the right-angled Triangle required. But now we must shew that it will satisfie the Problem.

21. First then by *Construction* in 19°, A B the Base of the Triangle A D B is that which in 14° was prescribed for the Hypothenusal of the right-angled Triangle sought; secondly, by *Constr.* in 19° and 20° the angle A D B is in the Semicircle F A D B, and therefore 'tis a right angle, (*per prop.* 31. *Elem.* 3.) thirdly and lastly, that the Area of the right-angled Triangle A D B is equal to the Square of the given right line C, the following Demonstration will make manifest.

Prepar.

22. From the point D in the Circumference, let fall D K perpendicular to the Diameter A B.

23. ∴ *Req. demonstr.* . . . . $\triangle ADB = \square C.$

Demonstration.

24. By *Constr.* in 18°, ... $AB \,.\, M :: M \,.\, P.$

25. And from the *Constr.* in 20° and 22°, (*per prop.* 34. *Elem.* 1.) . . . . $P = AG = DK.$

26. Therefore from 24° and 25°, by taking D K instead of P, ... $AB \,.\, M :: M \,.\, DK.$

27. Therefore from 26°, (*per prop.* 17. *Elem.* 6.) $\sqsubset AB,DK = \square M.$

28. But by *Constr.* in 17°, ... $2\,\square C = \square M.$

29. Therefore from 27° and 28°, (*per Ax.* 1. *Chap.* 2.) . . . . $\sqsubset AB,DK = 2\,\square C.$

30. And because by *Constr.* $\angle ADB$ is $\llcorner$, therefore (*per prop.* 41. *Elem.* 1.) . . . . . . $\sqsubset AB,DK = 2\,\triangle ADB.$

31. Therefore from 29° and 30°, (*per Ax.* 1. *Chap.* 2.) . . . . $2\,\triangle ADB = 2\,\square C.$

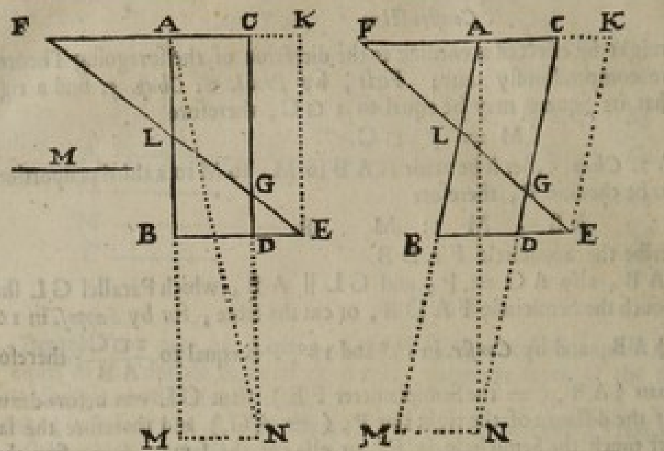
32. Therefore from 31°, (*per Ax.* 9. *Chap.* 2.) $\triangle ADB = \square C.$

Which was to be Demonstr. Therefore the Problem is satisfied.

Probl. XVI. (*Prop.* 164. *Lib.* 7. Pappi.)

A Parallelogram B A C D being given by Position, from a given point E in B D produced, to draw a right line E F to concurr with C A produced in F, so as to make the Triangle F C G equal to the given Parallelogram B A C D.

Suppos.



AB = 75
BD = 32
DE = 20
CK = 20
FA = 48
EK = 75
FK = 100
CG = 60
GD = 15
AL = 36
LB = 39
EF = 125 } *in Fig. 1.*

Suppos.

1. BACD is a Parallelogram given by Position.
2. E is a point given in BD produced.
3. b = AC = BD is given.
4. d = CD = AB is given.
5. c = DE is given.

Req. to find

6. AF a right line to be added to CA in a direct line, that EF being drawn, it may make △ FCG = BACD.

Prepar.

7. By the given point E draw EK ∥ DC, and to concurr with AC produced in K, whence CK = DE, and EK = DC.
8. Let CD be continued to N, so, that DN = DC, whence CN = 2 DC, and therefore △ ACN = ▭ AC,CD = BACD, (*per prop.* 41. *Elem.* 1.)

Resolution.

9. Suppose that done which is required, and put } $a = \text{FA}.$
10. Then because by *Constr.* in 7° EK ∥ DC, the Triangles FCG and FKE are equiangular, (*per prop.* 29. *Elem.* 1.) therefore (*per prop.* 4. *Elem.* 6.) these are Proportionals, *viz.* . . . . }
$$\underbrace{\text{FK}}_{a+b+c} \,.\, \underbrace{\text{FC}}_{a+b} :: \underbrace{\text{KE}}_{d} \,.\, \underbrace{\text{CG}}_{\frac{da+db}{a+b+c}}.$$
11. By *Constr.* in 8°, ... } △ ACN = BACD.
12. And the Problem requires . . . . . } △ FCG = BACD.
13. Therefore from 11° and 12°, (*per Ax.* 1. *Chap.* 2.) . . . . } △ ACN = △ FCG.
14. And because those equal Triangles ACN and FCG have a common angle FCN, the sides about that angle shall be reciprocally proportional, (*per prop.* 15. *Elem.* 6.) therefore }
$$\begin{array}{cccc} \text{FC} & .\ \text{AC} :: & \text{CN} & .\ \text{CG}. \\ a+b & .\ b :: & 2d & .\ \frac{da+db}{a+b+c}. \end{array}$$
15. Therefore, by halving the Antecedents in the last Analogy, ... }
$$\tfrac{1}{2}a+\tfrac{1}{2}b \,.\, b :: d \,.\, \frac{da+db}{a+b+c}.$$
16. But it hath been shewn above in 10°, that . . . . }
$$a+b+c \,.\, a+b :: d \,.\, \frac{da+db}{a+b+c}.$$
17. Therefore from 15° and 16°, (*per prop.* 11. *Elem.* 5.) . . . . }
$$a+b+c \,.\, a+b :: \tfrac{1}{2}a+\tfrac{1}{2}b \,.\, b.$$
18. And by doubling the two latter Terms, their Reason is not alter'd, therefore . . . }
$$a+b+c \,.\, a+b :: a+b \,.\, 2b.$$
19. Therefore from the last Analogy by *Division of Reason*, ... }
$$c \,.\, a+b :: a-b \,.\, 2b.$$
20. Therefore by comparing the Rectangle of the means to the Rectangle of the extremes, }
$$aa - bb = 2bc.$$

21. There-

21. Therefore by adding *bb* to each part, ...	$aa = bb + 2bc \; (= \square FA.)$
22. Therefore, by extracting the square Root out of each part, ...	$a = \sqrt{:\overline{bb + 2bc}:} = FA.$
23. Again, from 3° and 5°, respect being had to the Diagram, ...	$BE (= BD + DE) = b + c.$
24. Therefore by squaring each part, ...	$\square BE = bb + cc + 2bc.$
25. And by subtracting $\square DE = cc$ from each part, .	$\square BE - \square DE = bb + 2bc.$
26. But from 21°, ...	. . . $\square FA = bb + 2bc.$
27. Therefore from 25° and 26°, (*per Ax.*1. *Chap.*2.)	. . . $\square FA = \square BE - \square DE.$
28. Therefore by extracting the square Root out of each part, ...	. . . . $FA = \sqrt{:\overline{\square BE - \square DE}:}$

The Equations in 21° and 27° do afford this

THEOREM.

29. If FC ∥ BE, and AB ∥ CD, and △FCG = BACD, then the Square of FA is equal to the Square of BD together with twice the Rectangle of BD into DE. Moreover, the Square of FA is equal to the excess by which the Square of BE exceeds the Square of DE.

Therefore BD and DE being given severally, FA shall be given also, and consequently EF may be drawn to solve the Problem propounded.

But to manifest the truth of the said Theorem, I shall form a Demonstration thereof by a repetition of the steps of the preceding Resolution in a direct order, to which end, let respect be had to the *Diagram*, *Suppos.* and *Prepar.* at the beginning of the Problem.

30. . . *Req. demonstr.* . . . $\square FA = \square BD + 2 \sqsubset\!\sqsupset BD, DE = \square BE - \square DE.$

Demonstration.

31. Forasmuch as △FKE and △FCG are equiangular, (for by *Constr.* in 7°, EK ∥ CD,) therefore (*per prop.*4. *Elem.* 6.)	FK . FC :: KE . CG.
32. By *Constr.* in 8°, ...	△ACN = BACD.
33. And by *Suppos.* in 29°, ...	△FCG = BACD.
34. Therefore from 32° and 33°, (*per Ax.* 1. *Chap.* 2.)	△ACN = △FCG.
35. And because ∠FCN is common to those equal Triangles ACN and FCG, therefore, (*per prop.* 15. *Elem.* 6.) .	FC . AC :: CN (or 2 CD) . CG.
36. Therefore from 35°, by halving the Antecedents, ...	$\frac{1}{2}$FC . AC :: CD (or KE) . CG.
37. But it hath been shewn in 31°, that .	FK . FC :: KE . CG.
38. Therefore from 36° and 37°, (*per prop.* 11. *Elem.* 5.)	FK . FC :: $\frac{1}{2}$FC . AC.
39. And from 38°, by doubling the two latter Terms, ...	FK . FC :: FC (or FA + AC) . 2AC.
40. And from 39°, by *Division of Reason*,	CK . FC :: FA — AC . 2AC.
41. That is, (as is evident by the Diagram,)	DE . FA + AC :: FA — AC . 2BD.
42. Therefore from 41°, (*per prop.* 16. *Elem.* 6.)	$\sqsubset\!\sqsupset \left\{ \begin{matrix} FA + AC, \\ FA - AC \end{matrix} \right\} = 2 \sqsubset\!\sqsupset BD, DE.$
43. But by *Theor.* 8. *Chap.* 4. . . .	$\sqsubset\!\sqsupset \left\{ \begin{matrix} FA + AC, \\ FA - AC \end{matrix} \right\} = \square FA - \square AC \; (\square BD.)$
44. Therefore from 42° and 43°, (*per Ax.* 1. *Chap.* 2.)	$\square FA - \square BD = 2 \sqsubset\!\sqsupset BD, DE.$
45. Therefore from 44°, by adding $\square$ BD to each part, ...	$\square FA = \square BD + 2 \sqsubset\!\sqsupset BD, DE.$

Which was to be Dem.

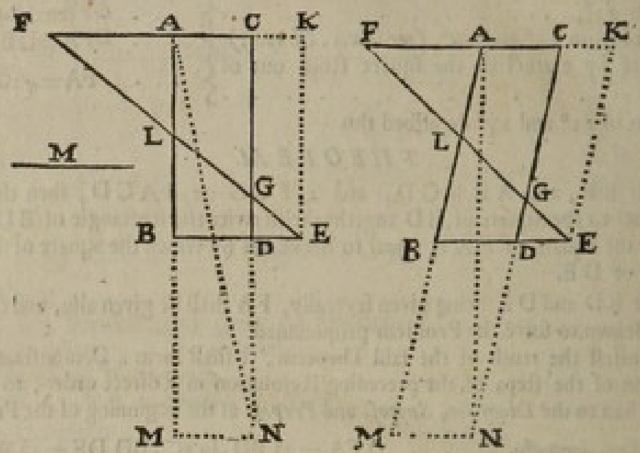
46. Again, because by *Supposition* in 2°, and by the Diagram, ...	. BE = BD + DE.
47. Therefore by squaring each part of that Equation, (*per Theor.*2. *Chap.* 4.) . .	$\square BE = \square BD + \square DE + 2 \sqsubset\!\sqsupset BD, DE.$

48. And from 47°, by ſubtracting $\square DE$ from each part, ... $\square BE - \square DE = \square BD + 2 \square BD,DE.$

49. Therefore from 45° and 48°, (*per Ax.* 1. *Chap.* 2.) . . . . . . $\square FA = \square BE - \square DE.$

Which was alſo to be Dem. Therefore the truth of the preceding Theorem is manifeſt.

The Compoſition of the foregoing Probl. 16.



Suppoſ.

50. BACD is a Prallelogram given by Poſition.

51. E is a point given in BD continued.

52. . . *Req.* to draw EF a right line, ſuch, that $\triangle FCG = BACD.$

Conſtruction.

53. By *Probl.* 9. *Chap.* 5. find a mean proportional line M between BD and BD + 2DE, therefore $BD . M :: M . BD + 2DE.$

54. Produce CA to ſuch a point F, that AF may be equal to the line M, (to wit, the mean Proportional found out in 53°,) then draw a right line from E to F, ſo ſhall the Triangle FCG be equal to the Rectangle BACD, as was required; the truth whereof will evidently appear by the following Demonſtration, form'd out of the foregoing Reſolution by a repetition of its ſteps in a backward (not direct) order. But by way of Preparation, draw EK ∥ and = DC; alſo make CN = 2CD; draw AN, and produce FC to K.

55. . . *Req. demonſtr.* . . . . $\triangle FCG = BACD.$

Demonſtration.

56. By *Conſtr.* in 53° and 54°, . . $BD . M (\text{or } FA) :: M . BD + 2DE.$

57. Therefore (*per prop.* 17. *Ele.* 6.) $\square FA = \square BD + 2 \square BD,DE.$

58. Therefore from 57°, by ſubtracting $\square BD$ from each part, $\square FA - \square BD = 2 \square BD,DE.$

59. But by *Theor.* 8. *Chap.* 4. . $\square FA - \square AC(\square BD) = \square \left\{ \begin{matrix} FA + AC, \\ FA - AC. \end{matrix} \right.$

60. Therefore from 58° and 59°, (*per Ax.* 1. *Chap.* 2.) . . . $\square \left\{ \begin{matrix} FA + AC, \\ FA - AC \end{matrix} \right\} = 2 \square BD, DE.$

61. Therefore from 60°, (*per prop.* 14. *Elem.* 6.) . . . . $DE(CK) . FA + AC :: FA - AC . 2BD(2AC.)$

62. That is, as is evident by the Diagram, ... $CK . FC :: FA - AC . 2AC.$

63. Therefore from 62°, (by *Compoſ. of Reaſon,*) . . . . . $FK . FC :: FC(FA + AC) . 2AC.$

64. And from 63°, by halving the two latter Terms, ... $FK . FC :: \frac{1}{2}FC . AC.$

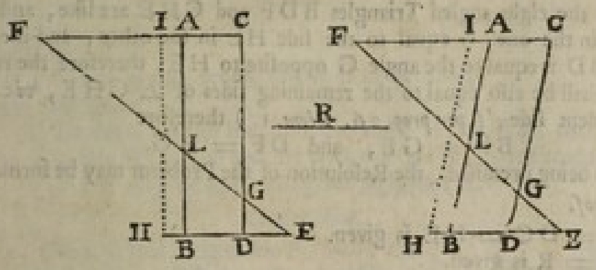
65. But

65. But because △ FKE and △ FCG are like, (for EK ∥ DC,) therefore, (*per prop. 4. Elem. 6.*)	FK . FC :: KE(CD) . CG.
66. Therefore from 64° and 65°, (*per prop. 11. El. 5.*)	½FC . AC :: CD . CG.
67. And from 66°, by doubling the Antecedents, .	FC . AC :: CN(2CD) . CG.
68. And because ∠ FCN is common to △ FCG and △ ACN, and it appears in 67°, that the sides about that common angle are reciprocally proportional, therefore (*per prop. 15. Elem. 6.*) .	△ FCG = △ ACN.
69. But by *Constr.* in 8°, ...	BACD = △ ACN.
70. Therefore from 68° and 69°, (by *Ax. 1. Ch 2.*)	△ FCG = BACD.

Which was to be Demonstr. Therefore the Problem is satisfied.

Probl. XVII.

A Parallelogram BACD being given by Position, from a given point E in BD produced, to draw a right line EF to meet with CA produced in F, that the Triangle FCG may have a given Reason to the Parallelogram BCAD, suppose as HD to BD.



Construction.

1. By the point H draw HI ∥ BA or DC (*per prop. 31. Elem. 1.*) then by the last preceding Problem draw a right line EF, so as to make the Triangle FCG equal to the Parallelogram HICD, so shall △ FCG be to BACD as HD to BD, which was required; the truth whereof will be manifest by the following Demonstration.

2. . . . *Req. demonstr.* . . . . △FCG . BACD :: HD . BD.

Demonstration.

3. Because (*per prop. 1. Elem 6.*)	HICD . BACD :: HD . BD.
4. And by *Constr.* in 1°, ...	△FCG = HICD.
5. Therefore from 3° and 4°, ...	△FCG . BACD :: HD . BD.

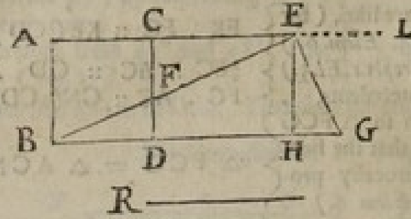
Which was to be Dem.

6. After the same manner, from the given point E a right line may be drawn so as to make the Triangle FCG equal to a given Space, suppose the Square of the right line R, by making the Parallelogram HICD equal to the Square of R, and △ FCG = HICD: For,

If by *Construction* . . . . HICD = □R,
And by *Constr.* . . . . HICD = △FCG.
Then it follows (*per Ax. 1.*) that . . △FCG = □R.

Probl. XVIII. (*Prop. 71. Lib. 7.* Pappi.)

A Square BACD (whose side is BD or DC) being given, to draw a right line from the angle B, as BE, that may so cut the side DC, and concurr with the side AC produced towards L, that FE may be equal to a given right line R.



$BD = HE = 60$
$R = FE = 91$
$BF = EG = 65$
$DG = 109$
$DF = HG = 25$
$FC = 35$
$DH = CE = 84$

Prepar.

1. Suppose that done which is required, *viz.* that BE is a right line so drawn from the angle B that it cuts DC in F, and concurrs with AC produced in E, and makes FE equal to the given right line R.
2. Make EG perpendicular to BE, and let BD be continued until it concurr with EG in G, and from E let fall EH perpendicular to BG, whence it follows (*per prop.* 8, & 2. *Elem.* 6.) that
$\triangle$ BEG, $\triangle$ BHE, $\triangle$ GHE, $\triangle$ BDF } are like (that is, equiangular) right-angled Triangles, and therefore the sides about the equal angles are Proportionals, (*per prop* 4. *Elem.*6.)
3. And because the right-angled Triangles BDF and GHE are like, and the side BD ($=$ DC) in the one, is equal to the side HE in the other, and the angle BFD opposite to BD is equal to the angle G opposite to HE, therefore the remaining sides of $\triangle$ BDF shall be also equal to the remaining sides of $\triangle$ GHE, *viz.* each side to its correspondent side, (*per prop.* 26. *Elem.* 1.) therefore
$$BF = GE, \text{ and } DF = HG.$$

These things being premised, the Resolution of the Problem may be formed thus:

Suppos.

4. $b = BD = DC = HE$ is given.
5. $d = FE = R$ is given.

Resolution.

6. For DG put a, *viz.* suppose . . . . } $a = DG$.
7. And for BF ($=$ GE) put e, *viz.* suppose . . . } $e = BF = GE$.
8. Then from 4° and 6°, ... } $b + a = BG$.
9. And from 7° and 5°, ... } $e + d = BE$.
10. The Square of the Equation in 7° gives . . . } $ee = \square BF$.
11. And the Square of the Equation in 8° gives . . . } $bb + 2ba + aa = \square BG$.
12. And the Square of the Equation in 9° gives . . . } $ee + 2de + dd = \square BE$.
13. Now because by *Constr.* in 2°, the Triangle BEG is right-angled at E, and from 3°, $\square BF = \square GE$, therefore from 10°, 11°, 12°, (*per prop.* 47. *Elem.* 1.) this Equation ariseth, *viz.*
$$\overbrace{bb + 2ba + aa}^{\square BG} = \overbrace{ee + 2ed + dd}^{\square BE} + \overbrace{ee.}^{\square GE\ (\square BF.)}$$
14. And because from 2°, ... } $\triangle$BEG and $\triangle$BHE are like.
15. Therefore (*per prop.* 4. *Elem.*6.) . . . . } BG . GE :: BE . EH.
16. That is, in the letters of the Resolution, ... } $b + a \,.\, e :: e + d \,.\, b$.
17. Which Analogy being reduced to an Equation, gives } $bb + ba = ee + ed$.
18. And by subtracting the Equation in 17° from that in 13°, this remains, ... } $ba + aa = ee + ed + dd$.
19. And if instead of $ee + ed$ in the last preceding Equation, there be taken $bb + ba$, which in 17° appears to be equal to $ee + ed$, then the Equation in 18° will be reduced to this . . . . } $ba + aa = bb + ba + dd$.
20. Whence, by subtracting ba from each part, there remains . . . . } $aa = bb + dd$.
21. Therefore by extracting the square Root out of each part of the last Equation, it gives . . . . } $a = \sqrt{:bb + dd:} = DG$.

Hence

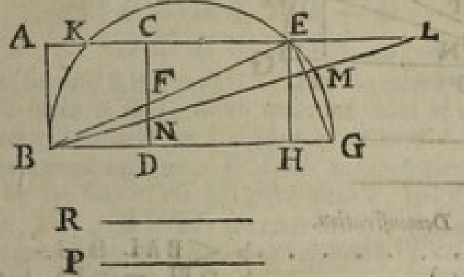
Hence this

THEOREM.

22. The right line D G is equal to that right line whose Square is equal to the summ of the Squares of BD and FE. Therefore if BD (or DC) and FE be given severally, then D G is given also, by the help whereof the Problem may be effected.

This Theorem is demonstrated in *Prop.* 71. of the 7th Book of *Pappus*'s Mathematical Collections, and the truth thereof is also manifest by the foregoing Resolution, wherein the Argumentation is clearly Geometrical as well as Algebraical, and therefore there is no need of any further Demonstration of the said Theorem.

The Composition of the foregoing Probl. 18.



BD = HE = 60
R = FE = 91
BF = EG = 65
DG = 109
DF = HG = 25
FC = 35
DH = CE = 84

Suppos.

23. B A C D is a Square given, whose side is B D or D C.
24. R is a right line given.
25. . . *Req.* To draw a right line from the angle B, as BE, that may so cut the side D C, and concurr with the side A C produced towards L, that F E may be equal to a given right line R.

Constructiion.

26. By *Probl.* 2. *Chap.* 5. find a right line P, such, that its Square may be equal to $\square$ BD + $\square$ R, therefore . } $P = \sqrt{:\square BD + \square R}:$
27. To B D add the line P, so, that B D and P may make a straight line, as B D G, therefore . . . } DG = P, and BG = BD + P.
28. Upon B G describe the Semicircle B K E G.
29. Let A C be continued towards L, so shall the line produced cut the Semicircle B K E G, I say cut it, not touch it, nor lye without it; for $\frac{1}{2}$ B G is greater than B D or B A, as may be proved thus:

Because by *Constr.* in 27°, ... DG = P.
And by *Constr.* in 26°, ... P $\sqsubset$ BD.
Therefore (*per Ax.* 4. *Chap.* 2.) . . . . DG $\sqsubset$ BD.
And by adding B D to each part, ... BD + DG $\sqsubset$ 2 BD.
But by *Constr.* in 27°, ... BD + DG = BG.
Therefore (*per Ax.* 3. *Chap.* 2.) . . . . BG $\sqsubset$ 2 BD.
And consequently, ... $\frac{1}{2}$ BG $\sqsubset$ BD or BA.

Which was to be proved. And therefore A C E, which is parallel to B G at the distance of B A shall necessarily cut the Semicircle BKEG in two points, as in K and E.

30. Lastly, draw the right line B E, so shall F E be equal to the given right line R, as was required. But that F E = R, I demonstrate thus;
31. . . *Req. demonstr.* . . . . FE = R.

Demonstration.

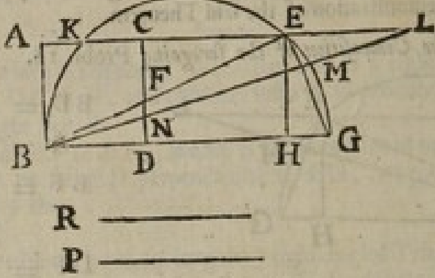
32. By *Constr.* in 26°, ... } $\square$BD + $\square$R = $\square$P.
33. And from the *Constr.* in 27°, ... } . . $\square$DG = $\square$P.
34. Therefore (*per Ax.* 1. *Chap.* 2.) . . . . . } . . $\square$DG = $\square$BD + $\square$R.
35. But by the *Theor.* in 22° of this Problem, . . } . . $\square$DG = $\square$BD + $\square$FE.
36. Therefore from 34° and 35°, (*per Ax.* 1. *Chap.* 2.) } $\square$BD + $\square$FE = $\square$BD + $\square$R.
37. And from 36°, by subtracting $\square$ BD from each part, } . . . $\square$FE = $\square$R.
38. Therefore, ... } . . . . FE = R.

Which was to be Dem.

39. But

39. But that FE is the only right line that can be found equal to the given right line R; and in such a Position as the Problem requires, I shall demonstrate in the next place; to which end, let some other line besides BE, as BL, be drawn from B to concurr with AC produced beyond E in L: Now if BL doth effect the Problem, then NL must be equal to FE; but NL is greater than FE, as will be made manifest by the following Demonstration.

40. . . . *Req. demonstr.* . . . . NL $\sqsubset$ FE.



Demonstration.

41.	Because by *Suppos.*	$>$ $\angle$BAL is $\lrcorner$.
42.	Therefore (*per prop. 47. Elem. 1.*)	$>$ $\square$BL $=$ $\square$BA $+$ $\square$AL.
43.	And in like manner, ...	$>$ $\square$BE $=$ $\square$BA $+$ $\square$AE.
44.	By *Suppos.* in 39°, AL $\sqsubset$ AE, and consequently, .	$>$ $\square$AL $\sqsubset$ $\square$AE.
45.	Therefore from 42°, 43° and 44°, (*per Ax. 4. Chap. 2.*)	$>$ $\square$BL $\sqsubset$ $\square$BE.
46.	Therefore from 45°, ...	$>$ BL $\sqsubset$ BE.
47.	Again, because by *Suppos.*	$>$ $\angle$BDC is $\lrcorner$.
48.	Therefore (*per prop. 47. Elem. 1.*)	$>$ $\square$BF $=$ $\square$BD $+$ $\square$DF.
49.	And in like manner, ...	$>$ $\square$BN $=$ $\square$BD $+$ $\square$DN.
50.	And because DF $\sqsubset$ DN, and consequently, .	$>$ $\square$DF $\sqsubset$ $\square$DN.
51.	Therefore from 48°, 49° and 50°, ...	$>$ $\square$BF $\sqsubset$ $\square$BN.
52.	Therefore from 51°, ...	$>$ BF $\sqsubset$ BN.
53.	And consequently, ...	$>$ BN $\sqsupset$ BF.
54.	And because it hath been shewn in 46°, that . . .	$>$ BL $\sqsubset$ BE.
55.	Therefore from 52°, 53°, 54°, (*per Ax. 16. Chap. 2.*)	$>$ BL $-$ BN $\sqsubset$ BE $-$ BF.
56.	That is, (as is evident by the Diagram,) . . .	$>$ NL $\sqsubset$ FE.

Which was to be Dem. And therefore BL will not effect the Problem propounded. The like Demonstration will hold good in comparing BE to any other right line that shall be drawn from B to cut DC, and to concurr with AC produced.

57. Here the Learner may observe, that in resolving a Problem by the Algebraick Art, there may often-times be found out various Equations so constituted, that every one of them may be capable of solving the Problem, but the simplest of those Equations is to be preferr'd before the rest, and chiefly to be aim'd at, though for the most part 'tis much harder to be discovered than those more compounded. As, in the foregoing *Probl.* 18. among various Equations that may be found out to solve the same, that in 21° at the end of the preceding Resolution is the simplest. But who would think, that the way to solve that Problem is to search out the quantity of the line DG, and not rather of one of these lines, to wit, BE, BF, AE, CE, DF? for by any one of these lines, from the consideration of the like right-angled Triangles BAE, FCE, BDF we may come to an Equation more easily than by the line DG, but the Geometrical Construction of such Equation will be much harder than that of the Equation whereby DG is before discovered: And because the Equation resulting upon the search of any of the said five lines, to wit, BF, BE, AE, CE, DF falls under a higher Form than any of the Equations expounded in this Book, I shall referr the more curious Reader for satisfaction concerning the same, to *Pag.* 82, 83, 84 of *Renatus des Cartes's* Geometry, set forth by *Fran. van Schooten* in 1659. yet I shall here shew how the quantities of those five lines before mentioned are also deducible from the preceding Resolution.

First then, the same things being supposed as before, draw EG; and make EH $\perp$ BG; then let the Equations in the preceding 17th and 21th steps be here repeated, *viz.*

58. It

58. It hath been shewn in 17°, that . . . . } $bb + ba = ee + ed.$
59. And in 21°, that . . . . } $a = \sqrt{:bb + dd:}$
60. Therefore if $\sqrt{:bb + dd:}$ instead of a be drawn into b, the Equation in 58° will be reduced to this, *viz.* . . . . } $bb + b\sqrt{:bb + dd:} = ee + ed.$
61. Which last Equation may be reduced into these three Proportionals, *viz.* . . . . } $e + d,\ \sqrt{:bb + b\sqrt{bb + dd}},\ e \div\!\div.$
62. Of which three Proportionals, the mean, to wit, $\sqrt{:bb + b\sqrt{bb + dd}:}$ is given, as also d the difference of the extremes $e + d$ and e, therefore the extremes shall be given severally, by the *Theor.* in 24° of *Probl.* 12. *Chap.* 5. *viz.*

$$\sqrt{:\tfrac{1}{4}dd + bb + b\sqrt{bb + dd}:} - \tfrac{1}{2}d = e = \text{BF}.$$

$$\sqrt{:\tfrac{1}{4}dd + bb + b\sqrt{bb + dd}:} + \tfrac{1}{2}d = e + d = \text{BE}.$$

63. And because EH (= DC = DB) is a mean Proportional between BH and HG, whose summ is BG; which mean and summ of the extremes are represented in the preceding Resolution by b and $b + a$, whereof b is given in 4°, and a in 21°, for 'tis there found equal to $\sqrt{:bb + dd:}$ and consequently $b + a = b + \sqrt{:bb + dd:}$ therefore by the help of the said given mean b, and the said given summ of the extremes, to wit, $b + \sqrt{:bb + dd:}$ the extremes BH (= AE) and HG (= DF) shall be given severally by the *Theor.* in 21° of *Probl.* 13. *Chap.* 5. *viz.*

$$\tfrac{1}{2}b + \sqrt{:\tfrac{1}{4}bb + \tfrac{1}{4}dd:} + \sqrt{:\tfrac{1}{4}dd - \tfrac{1}{2}bb + \tfrac{1}{2}b\sqrt{bb + dd}:} = \text{BH} = \text{AE}.$$

$$\tfrac{1}{2}b + \sqrt{:\tfrac{1}{4}bb + \tfrac{1}{4}dd:} - \sqrt{:\tfrac{1}{4}dd - \tfrac{1}{2}bb + \tfrac{1}{2}b\sqrt{bb + dd}:} = \text{HG} = \text{DF}.$$

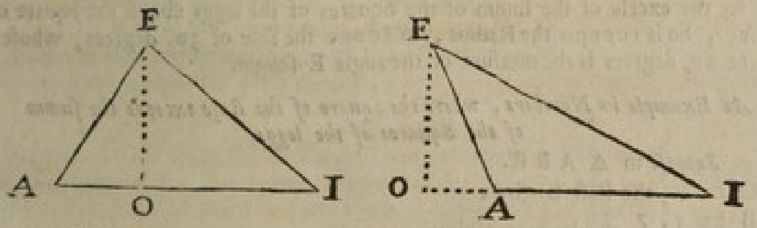
64. And because CE = DH = BH − BD, and BH and BD are given as before, therefore CE (= DH) is given also, *viz.*

$$\sqrt{:\tfrac{1}{4}bb + \tfrac{1}{4}dd:} + \sqrt{:\tfrac{1}{4}dd - \tfrac{1}{2}bb + \tfrac{1}{2}b\sqrt{bb + dd}:} - \tfrac{1}{2}b = \text{CE}.$$

Lastly, for the better illustration of the premisses, I have calculated whole Numbers (placed near the Diagram) to express the Quantities of all the Right lines given and sought in this *Probl.* 18.

A LEMMA, *leading to the following* Probl. 19.

If in any oblique-angled plain Triangle, any one of the three sides be called the Base, and the other two the leggs; Then, as the Radius, (or total Sine,) is to the Sine complement of the angle contain'd under the leggs; so is the double Rectangle of the leggs, to the difference between the summ of the Squares of the leggs, and the Square of the Base.



Suppos.

1. Let EI be the Base, } of the oblique-angled Triangle AEI.
2. AE and AI the leggs }
3. $\angle$A (that is, $\angle$EAI) is contain'd under the leggs AE, AI.
4. EO ⊥ AI.
5. R = the Radius, or total Sine.
6. Sc. $\angle$A = the Sine complement of the Angle A, (or $\angle$EAI,) that is, the Sine of the angle AEO.

Req. demonstr.

7. { If $\angle$A be acute, then, R . Sc. $\angle$A :: 2 ▭ AE,AI . □AE + □AI − □EI.
   If $\angle$A be obtuse, then, R . Sc. $\angle$A :: 2 ▭ AE,AI . □EI − □AE − □AI.

Demonstr.

Demonstration.

8. By a vulgar Axiom in the Doctrine of plain Triangles, . . } $R . Sc. \angle A :: AE . AO.$

9. Therefore by taking the common altitude AI, ... } $R . Sc. \angle A :: \square AE, AI . \square AO, AI.$

10. And by doubling the two latter Terms, ... } $R . Sc. \angle A :: 2\square AE,AI . 2\square AO,AI.$

11. And because by *Supposition* in *Case* 1. $\angle$ A is acute, therefore (*per prop.* 13. *Elem.* 2.) . . } $\square AE + \square AI - \square EI = 2 \square AO, AI.$

12. Therefore, from 10° and 11°, by exchanging equal quantities, } $R . Sc. \angle A :: 2\square AE,AI . \square AE + \square AI - \square EI.$
Which was to be Dem.

13. But when $\angle$A is obtuse, then (*per prop.* 12. *Elem.* 2.) . . . } $\square EI - \square AE - \square AI = 2 \square AO, AI.$

14. Therefore from 10° and 13°, by exchanging equal quantities, } $R . Sc. \angle A :: 2\square AE,AI . \square EI - \square AE - \square AI$

Which was also to be Dem. Therefore the truth of the *Lemma* is manifest. Hence this

COROLLARY.

15. If in an oblique-angled plain Triangle the three sides be given severally, the angles shall also be given severally, without the help of the Perpendicular; for if the side opposite to an angle sought be called the Base, and the other two sides the leggs; then as the double Rectangle of the leggs is to the difference between the summ of the Squares of the leggs and the Square of the Base; so is the Radius to the Sine complement of the angle opposite to the Base. Which angle is acute when the Square of the Base is less than the summ of the Squares of the leggs; but obtuse when greater.

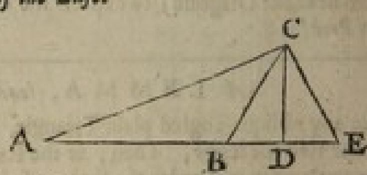
An Example in Numbers, where the summ of the Squares of the leggs exceeds the Square of the Base.

Suppos. in $\triangle$ AEC.

16. AC = 7, the Base is given.

17. AE = 8, }
18. EC = 3, } the leggs are given.

Req. to find $\angle$E.



Solution Arithmetical.

19. By the preceding Corollary; As 48 the double Rectangle of the leggs, A E, E C, is to 24 the excess of the summ of the Squares of the leggs above the Square of the Base A C; So is 100000 the Radius, to 50000 the Sine of 30. degrees, whose complement 60. degrees is the measure of the angle E sought.

An Example in Numbers, where the Square of the Base exceeds the summ of the Squares of the leggs.

Suppos. in $\triangle$ ABC.

20. AC = 7, the Base is given.

21. AB = 5, }
22. BC = 3, } the leggs are given.

Req. to find $\angle$ ABC.

Solution Arithmetical.

23. By the preceding Corollary; As 30, the double Rectangle of the leggs, A B, B C, is to 15, the excess of the Square of the Base above the summ of the Squares of the leggs; So is 100000 the Radius, to 50000 the Sine of 30. degrees, whose complement 60. degrees subtracted from 180. degrees, leaves 120. degrees for the angle ABC sought.

Note. Because in this second Example the Square of the Base exceeds the summ of the Squares of the leggs, the angle sought is obtuse; and therefore the complement of the angle relating to the Sine which is the fourth Proportional of the before-mentioned Analogy, being subtracted from 180. degrees, leaves the angle sought. But when the summ of the

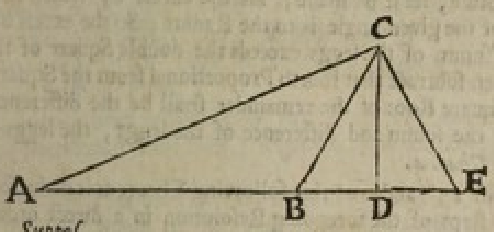
the Squares of the leggs exceeds the Square of the Base, then the complement it self of the angle relating to the said Sine (or fourth Proportional) is the angle sought; as in the first Example.

Probl. XIX.

The Base (that is, any side) of a plain Triangle being given, as also the angle opposite to the Base, and the summ of the sides (or leggs) containing that angle, to find the Triangle. But the given summ of the leggs must exceed the given Base, *(per prop. 22. Elem. 1.)*

Note. Because in this Problem the given angle is not of the same kind with the right lines given, a right line is to be found out, by the help of that angle, which may stand instead of the angle: To which end, let a Circle be described at any distance, and make an angle at the Center equal to the given angle, then from one end of that arch of the Circumference which is the measure of the said angle at the Center, let fall a Perpendicular upon a Diameter drawn to the other end of the said arch, so is the said Perpendicular the Sine of the given angle, and the segment of the Diameter between the foot of the same Perpendicular and the Center of the Circle is the Sine-complement of the given angle. Now instead of the given angle the said Sine-complement may be taken, by the help whereof, and of the Radius (or Semidiameter) of the said Circle, the Resolution of the Problem propounded may be formed in manner following.

The Problem hath three Cases, for the given angle opposite to the Base given is either right, or acute, or obtuse; the first of those Cases hath already been solved in *Probl* 4. of this *Chapt.* I shall therefore begin with the second Case, which supposeth the given angle to be acute, and the leggs containing that angle to be unequal.



AC = 7
AE = 8
EC = 3 = BC
AB = 5
$<$E = 60 degrees.
$<$ABC = 120 degrees.

Suppos.

1. b = A C the Base of $\triangle$ A C E is given.
2. c = A E + E C the summ of the leggs is given.
3. $<$E opposite to the Base A C is acute, and given.
4. r = the Radius (or total Sine) is given.
5. d = the Sine-complement of $<$ E is given.

Req. to find out the Triangle.

Resolution.

6. Suppose A C E to be the $\triangle$ sought, and put a for the difference of the leggs A E, E C; *viz.* assume . . . . a = AE − EC.
7. Therefore from 2° and 6°, (*per Theor.* 9. *Chap.* 4.) the greater legg shall be . . . . $\frac{1}{2}c + \frac{1}{2}a$ = AE.
8. And (by the same *Theor.*) the lesser legg shall be . . . . . $\frac{1}{2}c - \frac{1}{2}a$ = EC.
9. Therefore the double Product of the leggs is . . . . . . $\frac{1}{2}cc - \frac{1}{2}aa$.
10. And the summ of the Squares of the leggs is . . . . . $\frac{1}{2}cc + \frac{1}{2}aa$.
11. And because the given angle E is acute, the summ of the Squares of the leggs exceeds the Square of the Base, (*per prop.* 13. *Elem.* 2.) therefore $\frac{1}{2}cc + \frac{1}{2}aa$ exceeds bb, and the excess it self is . . $\frac{1}{2}cc + \frac{1}{2}aa - bb$.
12. And from 4°, 5°, 9° and 11°, this Analogy is manifest, (by the *Lemma* prefixt before this *Probl.*) viz.

$$r \;.\; d \;::\; \tfrac{1}{2}cc - \tfrac{1}{2}aa \;.\; \tfrac{1}{2}cc + \tfrac{1}{2}aa - bb.$$

13. Therefore from that Analogy, by Composition of Reason converse,

$$r + d \;.\; r \;::\; cc - bb \;.\; \tfrac{1}{2}cc - \tfrac{1}{2}aa.$$

14. And

14. And by doubling the two latter Terms of the last Analogy,

$$r + d \ . \ r \ :: \ 2cc - 2bb \ . \ cc - aa.$$

Hence

CANON 1.

15. When the given angle is acute, let it be made, as the summ of the Radius and the Sine-complement of the given angle is to the Radius, so the excess by which the double Square of the given summ of the leggs exceeds the double Square of the Base, to a fourth Proportional. Then subtract that fourth Proportional from the Square of the summ of the leggs, and the square Root of the remainder shall be the difference of the leggs sought. Lastly, the summ, as also the difference of the leggs being given, the leggs shall be given severally by *Theor.* 9. *Chap.* 4.

16. But when the given angle is obtuse, then the Square of the Base exceeds the summ of the Squares of the leggs, (*per prop.* 12. *Elem.* 2.) in which Case, $\frac{1}{2}cc + \frac{1}{2}aa$ in the 10th step must be subtracted from bb the Square of the Base, and the remainder will be $bb - \frac{1}{2}cc - \frac{1}{2}aa$, so instead of the Analogy in the 12th step, this ariseth,

$$r \ . \ d \ :: \ \tfrac{1}{2}cc + \tfrac{1}{2}aa \ . \ bb - \tfrac{1}{2}cc - \tfrac{1}{2}aa.$$

17. And because $r \sqsubset d$, (for the Radius or total Sine is greater than any other Sine,) therefore from the last preceding Analogy, by Conversion of Reason,

$$r \ . \ r - d \ :: \ \tfrac{1}{2}cc - \tfrac{1}{2}aa \ . \ cc - bb.$$

18. Therefore inversly,

$$r - d \ . \ r \ :: \ cc - bb \ . \ \tfrac{1}{2}cc - \tfrac{1}{2}aa.$$

19. And by doubling the two latter Terms of the last Analogy,

$$r - d \ . \ r \ :: \ 2cc - 2bb \ . \ cc - aa.$$

Hence

CANON 2.

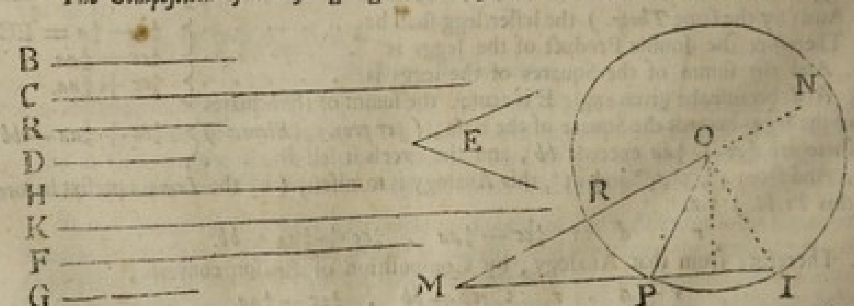
20. When the given angle is obtuse, let it be made, As the excess by which the Radius exceeds the Sine-complement of the given angle is to the Radius; So the excess by which the double Square of the given summ of the leggs exceeds the double Square of the Base, to a fourth Proportional. Then subtract that fourth Proportional from the Square of the summ of the leggs; and the square Root of the remainder shall be the difference of the leggs sought. Lastly, from the summ and difference of the leggs, the leggs shall be given severally, by *Theor.* 9. *Chap.* 4.

From the preceding Canons in 15° and 20° this following Theorem is deducible, and easie to be demonstrated by the steps of the foregoing Resolution in a direct order, *viz.* by proceeding from the beginning to the end of the Resolution.

THEOREM.

21. If any one of the three sides of an oblique-angled plain Triangle be called the Base, and the other two sides (or leggs) be unequal, then the excess of the double Square of the summ of the leggs above the double Square of the Base, shall be to the excess of the Square of the summ of the leggs above the Square of their difference, as the summ of the Radius and Sine-complement of the angle opposite to the Base is to the Radius, when the said angle is acute; but as the excess of the Radius above the said Sine-complement is to the Radius, when the said angle is obtuse.

The Composition of the foregoing Probl. 19. *when the given angle is acute.*



Suppos.

Suppos.

22. $B =$ the Base of a Triangle is given.
23. $C =$ the summ of the leggs is given.
24. $C \sqsubset B$. (*Determination.*)
25. $< E =$ to the angle opposite to the Base is acute, and given.
26. $R =$ the Radius or Semidiameter of a Circle is given.
27. $D =$ the Sine-complement of the angle E, where the Radius is equal to the line R, is given.

Req. to make the Triangle.

Construction.

28. Find a right line H that may be equal to $R + D$.
29. By *Probl.* 4. *Chap.* 5. find a right line K, such, that its Square may be equal to $2 \square C - 2 \square B$, which effection is possible, for by *Supposition* $C \sqsubset B$, therefore

$$\square K = 2 \square C - 2 \square B.$$

30. By *Probl.* 11. *Chap.* 5. let it be made as H to R, so $\square K$ to another Square, whose side suppose to be found F, therefore

$$H \,.\, R :: \square K \,.\, \square F.$$

That is, in 14°, $\quad r + d \,.\, r :: 2cc - 2bb \,.\, cc - aa.$

31. Find a right line G, such, that its Square may be equal to $\square C - \square F$, which effection is possible if $C \sqsubset F$; but that C is greater than F, I prove thus,

By the Theorem in the preceding 21th step, $\square F$ the last Term of the Analogy in 30° is equal to the excess of $\square C$ above the Square of the difference of the leggs, therefore $\square C = \square F +$ the Square of the said difference, whence 'tis manifest that $\square C \sqsubset \square F$, and consequently $C \sqsubset F$, therefore 'tis possible to find a right line G, such, that

$$\square G = \square C - \square F \; (= aa.)$$

Thus far the Construction hath been made according to the direction of *Canon* 1.

32. Now let a Triangle be made of these three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}G$, and $\frac{1}{2}C - \frac{1}{2}G$, which effection is possible (*per prop.* 22. *Elem.* 1.) if $C \sqsubset G$, and the summ of every two of those three lines be greater than the third; but these things may be made manifest thus,

First, it hath been proved in 31° that $C \sqsubset G$, and consequently $\frac{1}{2}C - \frac{1}{2}G$ (one of the above-mentioned three lines,) is greater than nothing, and therefore equal to some real right line.

Secondly, it is manifest that the summ of B and $\frac{1}{2}C + \frac{1}{2}G$ is greater than $\frac{1}{2}C - \frac{1}{2}G$.

Thirdly, the summ of $\frac{1}{2}C + \frac{1}{2}G$ and $\frac{1}{2}C - \frac{1}{2}G$ makes C, which by *Supposition* is greater than B.

Fourthly, that the summ of B and $\frac{1}{2}C - \frac{1}{2}G$ is greater than $\frac{1}{2}C + \frac{1}{2}G$, that is, $B \sqsubset G$, I prove thus;

By *Constr.* in 30°, ...	$H \,.\, R :: \square K \,.\, \square F.$
That is, as appears by *Constr.* in 28°,	$R + D \,.\, R :: 2\square C - 2\square B \,.\, \square C - \square G.$
29° and 31°, ... Therefore by Division of Reason, .	$D \,.\, R :: \square C + \square G - 2\square B \,.\, \square C - \square G.$
And inversly, ...	$R \,.\, D :: \square C - \square G \,.\, \square C + \square G - 2\square B.$
But $R \sqsubset D$, therefore from the last preceding Analogy, (*per Schol prop.* 14. *Elem* 5.)	$\square C - \square G \sqsubset \square C + \square G - 2 \square B.$
And by adding $\square G$ to each part, .	$\square C \sqsubset \square C + 2 \square G - 2 \square B.$
And by adding $2 \square B$ to each part,	$2 \square B + \square C \sqsubset \square C + 2 \square G.$
And by subtracting $\square C$ from each part,	$2 \square B \sqsubset 2 \square G.$
And by halving each part, ...	$\square B \sqsubset \square G.$
Therefore, ...	$B \sqsubset G.$ Which was to be Dem.

33. Now since it hath been shewn that the summ of every two of these three right lines, B, $\frac{1}{2}C + \frac{1}{2}G$, and $\frac{1}{2}C - \frac{1}{2}G$, is greater than the third, 'tis possible to make a Triangle of those three lines; suppose it therefore done, and that the Triangle so made is MOP, and that MP is equal to B, $MO = \frac{1}{2}C + \frac{1}{2}G$, and $OP = \frac{1}{2}C - \frac{1}{2}G$, then shall MOP be the Triangle required. Now we must shew that it will satisfie the Problem. First then by *Construction*, $MP = B$ the given Base; likewise by *Constr.*

 $MO + OP$

MO + OP = C the line prescribed for the summ of the leggs; it remains only to prove that the angle MOP is equal to the given angle E; but that will be made manifest by the following Demonstration, which is formed out of the steps of the foregoing Resolutinn, by returning backwards from the 14th step to the 11th.

Prepar.

See the last preceding Diagram. 34. From the Center O, at the distance of OP, describe the Circle OPRNS, and and produce MO to N in the Circumference; therefore $MN = MO + OP\,(ON) = C$, and $RM = OM - OP\,(OR) = G$.

35 . . *Req. demonstr.* . . . . $\angle MOP = \angle E$.

Demonstration.

36. By *Constr.* in 30°, . . . $H \,.\, R :: \square K \,.\, \square F$.

37. Therefore from 36°, 28°, 29°, 31°, by exchange of equal quantities, ... $R + D \,.\, R :: 2\square C - 2\square B \,.\, \square C - \square G$.

That is, in 14°, ... $r + d \,.\, r :: 2cc - 2bb \,.\, cc - aa$.

38. And because by *Constr.* in 33° and 34°, ... $MN = C \,.\, MP = B \,.\, MR = G$.

39. Therefore from 37° and 38°, by exchanging equal quantities, $R + D \,.\, R :: 2\square MN - 2\square MP \,.\, \square MN - \square MR$.

40. And by halving the two latter Terms, ... $R + D \,.\, R :: \square MN - \square MP \,.\, \frac{1}{2}\square MN - \frac{1}{2}\square MR$.

That is, in 13°, ... $r + d \,.\, r :: cc - bb \,.\, \frac{1}{2}cc - \frac{1}{2}aa$.

41. Therefore from 40°, by Division of Reason,

$D \,.\, R :: \frac{1}{2}\square MN + \frac{1}{2}\square MR - \square MP \,.\, \frac{1}{2}\square MN - \frac{1}{2}\square MR$.

42. Therefore inversly,

$R \,.\, D :: \frac{1}{2}\square MN - \frac{1}{2}\square MR \,.\, \frac{1}{2}\square MN + \frac{1}{2}\square MR - \square MP$.

That is, in 12°, ... $r \,.\, d :: \frac{1}{2}cc - \frac{1}{2}aa \,.\, \frac{1}{2}cc + \frac{1}{2}aa - bb$.

43. And because by *Theor.* 7. *Chap.* 4. . . . . $2\square OM, OP = \frac{1}{2}\square MN - \frac{1}{2}\square MR$.

44. And by *Theor.* 6. *Chap.* 4. $\square OM + \square OP = \frac{1}{2}\square MN + \frac{1}{2}\square MR$.

45. Therefore from 42°, 43°, 44°, by exchanging equal quantities, $R \,.\, D :: 2\square OM, OP \,.\, \square OM + \square OP - \square MP$.

46. By the last Term of the last preceding Analogy 'tis evident that . . . . $\square OM + \square OP \sqsubset \square MP$.

47. Therefore in $\triangle$ MOP, (*per prop.* 13. *Elem.* 2.) . . . . $\angle MOP$ is acute.

48. Therefore in $\triangle$MOP, (by the *Lemma* prefixt before this *Probl.* Rad. Sin. comp. $\angle MOP :: \left\{ \begin{matrix} 2\square OM, OP. \\ \square OM + \square OP - \square MP \end{matrix} \right.$

49. Therefore from 45° and 48°, (*per prop.* 11. *Elem.* 5.) . . $R \,.\, D ::$ Rad. Sin. comp. $\angle MOP$.

50. But by *Suppos.* in 26° and 27°, $R \,.\, D ::$ Rad. Sin. comp. $\angle E$.

51. Therefore from 49° and 50°, (*per prop.* 11. *Elem.* 5.) . . Rad. Sin. comp. $\angle MOP$ :: Rad. Sin. comp. $\angle E$.

52. Therefore, (*per prop.* 14. *El.* 5.) Sin. comp. $\angle MOP =$ Sin. comp. $\angle E$.

53. Therefore from 51° and 52°, (*per prop* 28. *Elem.* 3.) . . $\angle MOP = \angle E$. Which was to be Dem.

After the same manner, the third *Case* of *Probl.* 19. (*viz.* when the given angle is obtuse,) may be Geometrically effected and demonstrated.

Examples in Numbers, to illustrate the foregoing Resolution of Probl. 19.

Example 1. *where the given angle is acute.*

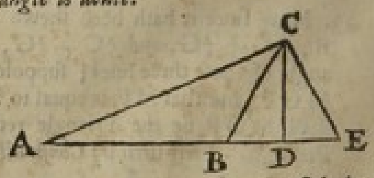
Suppos. in $\triangle$ AEC.

54. . . AC = 7, the Base is given.

55. AE + EC = 11, the summ of the leggs is given.

56. . . $\angle E = 60$ degrees is given.

Req. to find AE and EC severally.



Solution

Solution Arithmetical.

57. Suppose the Radius of a Circle to be . . . . 100000.

58. Then the Sine-complement of the given angle E, 60. degrees, that is, the Sine of 30. degrees, is . . . . 50000.

59. Therefore the summ of the Radius and that Sine-complement is . . 150000.

60. Then (by *Canon* 1. in the preceding 15th step,) As the said summ 150000 is to the Radius 100000; So is 144, (the excess whereby 242 the double Square of 11 the given summ of the leggs exceeds 98 the double Square of the given Base 7,) to a fourth Proportional 96, which subtracted from 121 the Square of the given summ of the leggs, leaves 25, whose square Root 5 is the difference of the leggs: Therefore (*per Theor.* 9. *Chap.* 4.) the leggs themselves, to wit, A E and E C shall be 8 and 3.

Example 2. *where the given Angle is obtuse.*

Suppos. in $\triangle$ A B C.

61. . . . A C = 7, the Base is given.

62. A B + B C = 8, the summ of the leggs is given.

63. . $\angle$ A B C = 120 degrees is given.

Req. to find A B and B C severally.

Solution Arithmetical.

64. Suppose the Radius of a Circle to be . . . . 100000.

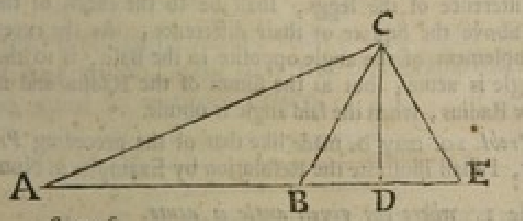
65. Then the Sine-complement of the given angle A B C 120. degrees, that is, the Sine of 30. degrees, is . . . . 50000.

66. Therefore the excess of the Radius above the Sine-complent is . . 50000.

67. Then (by *Canon* 2. in the preceding 20th step,) As the said excess 50000 is to the Radius 100000, So is 30 (the excess whereby 128 the double Square of 8 the given summ of the leggs exceeds 98 the double Square of the given Base 7,) to a fourth Proportional 60; which subtracted from 64 the Square of the given summ of the leggs, leaves 4, whose square Root 2 is the difference of the leggs sought: Therefore (*per Theor.* 9. *Chap.* 4.) the leggs themselves, to wit, A B and B C shall be 5 and 3.

Probl. XX.

The Base of a plain Triangle being given, as also the angle opposite to the Base, and the difference of the sides (or leggs) containing that angle, to find the Triangle. But the given Base must be greater than the given difference.



A C = 7
A E = 8
E C = 3 = B C.
A B = 5
$\angle$E = 60 degrees.
$\angle$ABC = 120 degrees.

Suppos.

1. b = A C the Base of $\triangle$ A C E is given.
2. c = A E — E C the difference of the leggs is given.
3. $\angle$ E opposite to the Base A C is acute, and given.
4. r = the Radius (or total Sine) is given.
5. d = the Sine-complement of $\angle$ E is given.

Req. to find A E and E C severally.

Resolution.

6. For the summ of the leggs A E and E C, put a, *viz.* suppose . . . . a = AE + E C.

7. Then out of 1°, 2°, 4°, 5° and 6°, (by the *Theor.* in 21° of the foregoing *Probl.* 19.) this following Analogy will arise, ... $r + d \,.\, r :: 2aa - 2bb \,.\, aa - cc.$

8. Therefore

8. Therefore by doubling the Consequents, (*per Schol.* Clavii *in prop.* 22. *Elem.* 5.) . . $r+d \,.\, 2r :: 2aa-2bb \,.\, 2aa-2cc.$

9. And by halving each of the two latter Terms, $r+d \,.\, 2r :: aa-bb \,.\, aa-cc.$

10. And inversly, ... $2r \,.\, r+d :: aa-cc \,.\, aa-bb.$

11. Therefore by Conversion and Inversion of Reason, ... $r-d \,.\, 2r :: bb-cc \,.\, aa-cc.$

Hence

CANON 1.

12. When the given angle is acute, let it be made, As the excess of the Radius above the Sine-complement of that angle, is to the double Radius; So the excess of the Square of the given Base above the Square of the given difference of the leggs, to a fourth Proportional: Then to that fourth Proportional add the Square of the difference of the leggs, and the square Root of the summ shall be the summ of the leggs sought. Lastly, the summ, as also the difference of the leggs being given, the leggs shall be given severally, by *Theor.* 9. *Chap.* 4.

13. But when the given angle is obtuse, then by the *Theorem* in 21° of the preceding *Probl.* 20. this Analogy will arise, ... $r-d \,.\, r :: 2aa-2bb \,.\, aa-cc.$

14. Therefore by doubling the Consequents, . $r-d \,.\, 2r :: 2aa-2bb \,.\, 2aa-2cc.$

15. And by halving each of the two latter Terms, $r-d \,.\, 2r :: aa-bb \,.\, aa-cc.$

16. Therefore inversly, ... $2r \,.\, r-d :: aa-cc \,.\, aa-bb.$

17. Therefore by Conversion and Inversion of Reason, ... $r+d \,.\, 2r :: bb-cc \,.\, aa-cc.$

Hence

CANON 2.

18. When the given angle is obtuse, let it be made, As the summ of the Radius and Sine-complement of that angle, is to the double Radius; So the excess of the Square of the given Base above the Square of the given difference of the leggs to a fourth Proportional: Then to that fourth Proportional add the Square of the difference of the leggs, and the square Root of the summ shall be the summ of the leggs sought. Lastly, the summ and difference of the leggs being given, the leggs shall be given severally, by *Theor.* 9. *Chap.* 4.

From the preceding Canons in 12° and 18° this following Theorem is deducible, and easie to be demonstrated by the steps of the Resolution in a direct order.

THEOREM.

19. If any one of the three sides of an oblique-angled plain Triangle be called the Base, and the other two sides (or leggs) be unequal; Then the excess of the Square of the Base above the Square of the difference of the leggs, shall be to the excess of the Square of the summ of the leggs above the Square of their difference; As the excess of the Radius above the Sine-complement of the angle opposite to the Base, is to the double Radius, when the said angle is acute; but as the summ of the Radius and the Sine-complement is to the double Radius, when the said angle is obtuse.

The Composition of this *Probl.* 20. may be made like that of the preceding *Probl.* 19. But waving the Composition, I shall illustrate the Resolution by Examples in Numbers.

Example 1. *where the given angle is acute.*

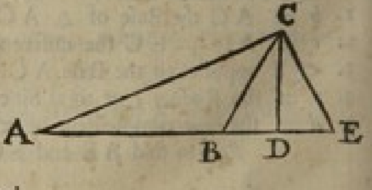
Suppos. in △ AEC.

20. . . AC = 7 the Base is given.

21. AE − EC = 5 the difference of the leggs is given.

22. . . ∠E = 60. degrees is given.

Req. to find AE and EC severally.



Solution Arithmetical.

23. Suppose the Radius of a Circle to be . . . . 100000.

24. Then the Sine-complement of the given angle E, 60. degrees, that is, the Sine of 30. degrees is . . . . 50000.

25. Therefore the excess of the Radius above the Sine-complement is . . 50000.

26. The

26. Then by *Canon* 1. (in the preceding 12th step,) As the said excess 50000 is to the double Radius 200000; So is 24, (the excess whereby 49 the Square of the Base exceeds 25 the Square of the difference of the leggs,) to a fourth Proportional 96, which increased with 25 the Square of the difference of the leggs, makes 121, whose square Root 11 is the summ of the leggs sought. Therefore (by *Theor.* 9. *Chap.* 4.) the leggs themselves, to wit, A E and E C shall be 8 and 3.

Example 2. *where the given angle is obtuse.*

Suppos. in △ ABC.

27. . . . AC = 7 the Base is given.
28. AB — BC = 2 the difference of the leggs is given.
29. . . <ABC = 120. degrees is given.
Req. to find AB and BC severally.

Solution Arithmetical.

30. Suppose the Radius of a Circle to be . . . . } 100000.
31. Then the Sine-complement of the given angle ABC, 120. degrees, that is, the Sine of 30. degrees is . . . . } 50000.
32. Therefore the summ of the Radius and that Sine-complement is . . } 150000.
33. Then (by *Canon* 2. in the preceding 18th step,) As the said summ 150000 is to the double Radius 200000; So is 45, (the excess of 49 the Square of the Base above 4 the Square of the difference of the leggs,) to a fourth Proportional 60, which increased with 4 the Square of the difference of the leggs, makes 64, whose square Root 8 is the summ of the leggs sought. Therefore (by *Theor.* 9. *Chap.* 4.) the leggs themselves, to wit, AB and BC shall be 5 and 3.

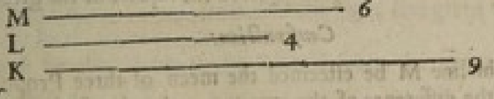
Chap. IX.

The third Classis *of Examples of the Resolution and Composition of Plane Problems.*

IN which Examples, the Resolution ends in an Analogy, wherein the Mean of three proportional right lines is given, as also the Difference or else the Summ of the Extremes, to find the Extremes severally.

Probl. I.

To find two right lines, such, that their difference may be equal to a right line given; and that the Rectangle made of the lines found out may be equal to a given Space.



Suppos.

1. d = D the difference of two right lines is given.
2. m = M a right line given, whose Square is equal to a given Space.

Req. to find

3. L and K two right lines, such, that K — L = D. Also, that
4. ▭ KL = □ M.

Resolution.

5. Put a for the lesser of the two right lines sought, *viz.* suppose . . . . } a = L.
6. Therefore from 1° and 5° the greater right line sought shall be } $a + d$ (= K.)
7. And from 5° and 6°, the Rectangle (or Product of their multiplication) shall be . . . . } $aa + da$.

8. But

8. But the said Rectangle must be equal to the given Space, supposed to be equal to the Square of m, therefore . . . $aa + da = mm$.

9. Which Equation may be resolved into these Proportionals, *viz.* . . . . $a+d \,.\, m :: m \,.\, a$.

10. Of which three Proportionals $a+d$, m and a, the mean m is given, as also d the difference of the extremes $a+d$ and a, therefore the extremes severally, which are the right lines sought by this Problem, shall be given also, *per Probl.* 12. *Chap.* 5. whence also, (respect being had to the Theorem in 24° of the same *Probl.*) there will arise this following

CANON.

11. . . . . $\left\{ \begin{array}{l} \sqrt{:\tfrac{1}{4}dd + mm:} + \tfrac{1}{2}d = K, \\ \sqrt{:\tfrac{1}{4}dd + mm:} - \tfrac{1}{2}d = L, \end{array} \right\}$ the lines sought.

That is, in words,

To the Square of half the given difference add the Square equal to the given Space, and extract the square Root of the summ: Then adding half the said difference to the said square Root, this summ shall be equal to the greater of the two right lines sought; but subtracting the said half difference from the said square Root, the remainder shall be equal to the lesser line sought.

This Canon may be also Synthetically inferr'd from the given lines, by the help of *Theor.* 7, and 9. *Chap.* 4. For,

12. By considering the things given and sought in 1°, 2°, 3° and 4° of this Problem it follows by *Theor.* 7. *Chap.* 4. that . . . . $\tfrac{1}{4}dd + mm = \square : \tfrac{1}{2}K + \tfrac{1}{2}L :$

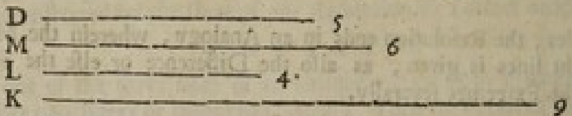
13. Therefore by extracting the square Root out of each part in 12°, ... $\sqrt{:\tfrac{1}{4}dd + mm:} = \tfrac{1}{2}K + \tfrac{1}{2}L$.

14. And from 1° and 3°, ... $\tfrac{1}{2}d = \tfrac{1}{2}K - \tfrac{1}{2}L$.

15. Therefore (by *Theor.* 9. *Chap.* 4.) the summ and difference of the two last preceding Equations gives the two lines sought by this Problem, *viz.* . . . $\sqrt{:\tfrac{1}{4}dd + mm:} + \tfrac{1}{2}d = K$. $\sqrt{:\tfrac{1}{4}dd + mm:} - \tfrac{1}{2}d = L$.

Thus you see the same Canon is discovered as before.

The Composition of Probl. 1.



Suppos.

16. D a right line equal to the difference of two right lines sought, is given.

17. M a right line given, whose Square is equal to a given Space.

Req. to find,

18. Two such right lines, that their difference may be equal to the given difference D, and that the Rectangle made of them may be equal to the Square of the given right line M.

Construction.

19. Let the given right-line M be esteemed the mean of three Proportionals, and the given right line D the difference of the extremes; then by *Probl.* 12. *Chap.* 5. find the extremes, the lesser whereof suppose to be L, therefore the greater shall be equal to L+D, and consequently these shall be Proportionals, *viz.*

$L + D \,.\, M :: M \,.\, L$.

20. Make $K = L + D$, whence $K - L = D$.

21. I say K and L are the two right lines sought, but that they will satisfie the Problem propounded, I prove thus: First by *Construction* in the 20th step, the difference of the said right lines K and L is equal to the given difference D; so it remains only to prove that the Rectangle made of the said right lines K and L is equal to the Square of the given right line M, but that is here-under demonstrated by returning backwards from the 9th step, (to wit, the last of the Resolution) to the 8th.

22. . . *Req. demonstr.* . . . . $\square K, L = \square M$.

Demonstr.

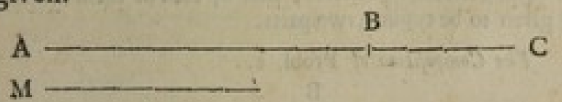
Demonstration.

23. By *Constr.* in 19° and 20°, ... $\}$ K (L + D) . M :: M . L.
That is, in 9°, ... $\}$ . . $a + d$. m :: m . a.
24. Therefore (*per* 17. *prop.* 6. *Elem.*) . . . $\}$ $\square$K,L ($\square$L + $\square$D,L) = $\square$ M.
That is, in 8°, ... $\}$ . . . $aa + da$ = mm.

Which was to be Demonstr. Therefore that is done which the Problem required.

Probl. II.

To cut a given right line into two such parts, that the Rectangle made of the parts may be equal to a Space given. But the side of a Square equal to the given Space must not be greater than half the right line given.



AB = 9
BC = 4
AC = 13
M = 6

Suppos.

1. b = A C is a right line given to be cut into two parts.
2. m = M is a right line given, whose Square is equal to a Figure or Space given.

Req. to find

3. AB and BC such parts of AC, that AB + BC = AC. Also,
4. $\square$ AB,BC = $\square$ M.

Resolution.

5. Put a for either of the parts sought, *viz.* suppose . . $\}$ a = AB.
6. Therefore from 1° and 5° the other part sought shall be $\}$ $b - a$ = BC.
7. And from 5° and 6° the Rectangle of the parts is . . $\}$ $ba - aa$.
8. But the said Rectangle must be equal to the given Space, therefore . . . . $\}$ $ba - aa = mm$.
9. Which Equation may be resolved into this Analogy, . $\}$ $b - a$. m :: m . a.
10. But that Analogy doth manifestly consist of three Proportionals, whereof the mean m is given, as also b the summ of the extremes $b - a$ and a; therefore the extremes severally, (which are the parts required by this *Probl.*) shall be given also, by *Probl.* 13. *Chap.* 5. Whence also this

CANON.

11\. . . . . . . $\left\{ \begin{array}{l} \frac{1}{2}b + \sqrt{:\overline{\tfrac{1}{4}bb - mm}:} = \text{AB}, \\ \frac{1}{2}b - \sqrt{:\overline{\tfrac{1}{4}bb - mm}:} = \text{BC}. \end{array} \right.$

That is, in words,

From the Square of half the right line given to be cut into two parts, subtract the Square equal to the given Space, and extract the square Root of the remainder; then add and subtract the said square Root to and from half the right line given to be cut; so the Summ and Remainder shall be the parts required.

This Canon may be also Synthetically inferr'd from the given lines, by the help of *Theor.* 7, and 9. *Chap.* 4. For,

12. By considering the things given and sought in 1°, 2°, 3° and 4° of this Problem, it follows by *Theor.* 7. *Chap.* 4. that . . . . $\}$ $\frac{1}{4}bb - mm = \square : \overline{\tfrac{1}{2}\text{AB} - \tfrac{1}{2}\text{BC}}:$
13. Therefore by extracting the square Root out of each part in 12°, ... $\}$ $\sqrt{:\overline{\tfrac{1}{4}bb - mm}:} = \frac{1}{2}\text{AB} - \frac{1}{2}\text{BC}.$
14. And from 1° and 3°, ... $\}$ . . . . . $\frac{1}{2}b = \frac{1}{2}\text{AB} + \frac{1}{2}\text{BC}.$
15. Therefore (by *Theor.* 9. *Chap.* 4.) the summ and difference of the two last preceding Equations gives the lines sought by this Problem, *viz.* . . . $\left\{ \begin{array}{l} \frac{1}{2}b + \sqrt{:\overline{\tfrac{1}{4}bb - mm}:} = \text{AB}. \\ \frac{1}{2}b - \sqrt{:\overline{\tfrac{1}{4}bb - mm}:} = \text{BC}. \end{array} \right.$

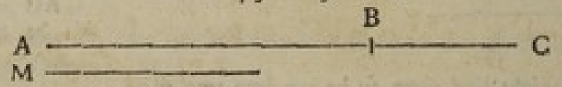
Thus you see the same Canon is discovered as before in 11°; and thereby it evidently appears, that to the end the given lines may be capable of effecting the Problem propos'd, they must be subject to this following

Determination.

16. The side of a Square equal to the given Space must not be greater than half the right line given to be cut into two parts. But that this Determination is necessary, I demonstrate thus;

Forasmuch as by the 9th step of the Resolution the right line m is found to be the mean of three Proportionals whereof the summ of the extremes is b, and in 20° of *Probl.* 13. *Chap* 5. it hath been proved that the mean of three Proportionals cannot be greater than half the summ of the extremes, it follows, that m must not be greater than $\frac{1}{2}b$. If therefore m happens to be greater than $\frac{1}{2}b$, the Problem propounded cannot possibly be solved; for then mm cannot be subtracted from $\frac{1}{4}bb$, as the Canon directs. But if m be equal to, or less than $\frac{1}{2}b$, then that which the Canon bids to be done is possible; by which also 'tis easie to perceive, that when $m = \frac{1}{2}b$, then the extreme Proportionals (which are equal to the parts required,) will be equal between themselves, that is, each of them will be equal to $\frac{1}{2}b$, to wit, half the line given to be cut into two parts.

The Composition of Probl. 2.



Suppos.

17. AC is a right line given to be cut into two parts.
18. M is a right line given, whose Square is equal to a Space given.
19. M not $\sqsubset \frac{1}{2}$ AC. (*Determination.*)

Req. to find

20. AB and BC such parts of AC, that $AB + BC = AC$. Also;
21. $\square$ AB, BC $= \square$ M.

Construction.

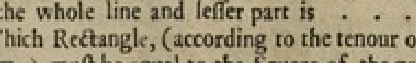
22. By *Probl.* 14. *Chap.* 5. cut the given right line AC into two such parts in B, that the line M may be a mean Proportional between the parts; which effection is possible, for by the Determination in 19°, M not $\sqsubset \frac{1}{2}$ AC; suppose therefore

$$AB \,.\, M :: M \,.\, BC.$$

23. I say AB and BC are the parts required, for their summ by *Construction* is equal to the given right line AC; and because by *Constr.* also it hath been made, As AB to M, so M to BC, therefore (*per* 17. *prop.* 6. *Elem.*) $\square$ AB, BC $= \square$ M.
Which was required to be done.

Probl. III.

To cut a given right line according to the extreme and mean Reason; that is, into two such parts, that the Rectangle made of the whole line and lesser part may be equal to the Square of the greater.



$AB = 10$
$AC = \sqrt{125} - 5$
$CB = 15 - \sqrt{125}$

Suppos.

1. $b =$ AB is a right line given.

Req. to find

2. AC and CB such parts of AB, that $\square$ AB, CB $= \square$ AC.

Resolution.

3. Put a for the greater part sought, *viz.* . . . . . $a = AC$.
4. Therefore from 1° and 3° the lesser part shall be . . $b - a = CB$.
5. Therefore from 1° and 4° the Rectangle (or Product) of the whole line and lesser part is . . . . . . $bb - ba$.
6. Which Rectangle, (according to the tenour of the Problem,) must be equal to the Square of the greater part, therefore . . . . $bb - ba = aa$.
7. Therefore, by adding ba to each part of that Equation, $bb = aa + ba$.
8. Which last Equation may be resolved into this Analogy, ... $a \,.\, b :: b \,.\, b+a$.

9. But

9. But that Analogy doth manifestly consist of three Proportionals, whereof the mean b is given, which is also the difference of the extremes a and $b + a$; therefore the extremes, whereof the lesser is equal to the greater part sought, shall be given also, by *Probl.* 12. *Chap.* 5. Whence also there will arise this following

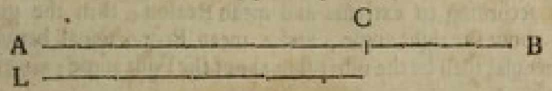
C A N O N.

10. . . . . $\sqrt{:}\,\overline{bb + \tfrac{1}{4}bb}: - \tfrac{1}{2}b = AC$, the greater segment.

That is, in words,

To the Square of the given line, add the Square of half the same line; then from the square Root of that summ subtract half the given line, and the remainder shall be the greater part sought, which subtracted from the given line leaves the lesser part.

The Composition of Probl. 3.



Suppos.

11. Let A B be a right line given to be cut into two such parts, that the Rectangle made of the whole line and the lesser part may be equal to the Square of the greater.

Construction.

12. Let the given line A B be esteemed the mean, as also the difference of the extremes of three Proportionals; then by *Probl.* 12. *Chap.* 5. find out the extremes severally, the lesser whereof we may suppose to be the right line L, and consequently the greater extreme is $AB + L$, therefore these are Proportionals, *viz.*

$$L \,.\, AB :: AB \,.\, AB + L.$$

13. From A B cut off $AC = L$, which is possible to be done, for by *Construction* in 12°, $AB \sqsubset L$, because A B is the mean of three Proportionals whose lesser extreme is L. So the given line A B is cut into two parts in C, according to extreme and mean Reason, *viz.* the Rectangle made of the whole line A B and the lesser part C B is equal to the Square of A C the greater part, as will be made manifest by the following Demonstration, formed out of the preceding Resolution by a repetition of its steps in a backward order.

14. . . *Req. demonstr.* . . . . $\Box AB,CB = \square AC.$

Demonstration.

15. By *Constr.* in 12° and 13°, ... $AC \,.\, AB :: AB \,.\, AB + AC.$
That is, in 8°, ... $a \,.\, b :: b \,.\, b + a.$
16. Therefore (*per prop.* 17. *Elem.* 6.) . . $\square AB = \square AC + \Box AB,AC.$
That is, in 7°, ... $bb = aa + ba.$
17. Therefore by subtracting $\Box$ AB, AC from each part, ... $\square AB - \Box AB,AC = \square AC.$
That is, in 6°, ... $bb - ba = aa.$
18. And because (as is evident by the Diagram,) $AB - AC = CB.$
19. Therefore, by drawing A B as a common altitude into each part of the last Equation, . $\square AB - \Box AB,AC = \Box AB,CB.$
20. Therefore from 17° and 19°, (*per Ax.* 1. *Chap.* 2.) . . . . $\Box AB,CB = \square AC.$

Which was to be Demonstr. Therefore the Problem is satisfied.

C O R O L L. 1.

21. From the preceding Resolution, the invention of a right-angled Triangle whose three sides shall be Proportionals is discovered; for if a given right line b be cut according to extreme and mean Reason, and the greater segment be a, 'tis manifest by the 7th step of the foregoing Resolution, that

$$bb = ba + aa.$$

22. And because (by *prop.* 48. *Elem.* 1.) if a Square be equal to two Squares, the sides of those three Squares will constitute a right-angled Triangle, therefore the square Roots

of the three Planes in the preceding Equation in 21°, viz. b, $\sqrt{ba}$, a shall be the three sides of a right-angled Triangle, and be Proportionals also, for the Rectangle of the extremes is manifestly equal to the Square of the mean. Hence therefore a Canon is discovered to find out a right-angled Triangle, whose three sides shall be Proportionals.

CANON.

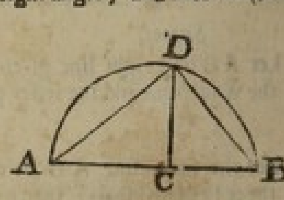
23. . . . . $\left\{ \begin{array}{l} \text{Hyp.} = b \text{ a right line or number taken at pleasure,} \\ \text{Base} = \sqrt{:bb + \tfrac{1}{4}bb:} - \tfrac{1}{2}b. \\ \text{Perp.} = \sqrt{:b \times \sqrt{bb + \tfrac{1}{4}bb} - \tfrac{1}{2}b:} \end{array} \right.$

That is, in words,

Take any right line (or number) for the Hypothenusal of a right-angled Triangle, and cut it into two parts according to extreme and mean Reason; then the greater part shall be one of the sides about the right angle, and a mean Proportional between the greater part and the Hypothenusal shall be the other side about the right angle; and those three sides shall be Proportionals.

Construction.

$\angle$ ADB is ⅃.	AB = 10
AB, AD, DB ∺.	AD = 7.8615
AC = DB.	DB = 6.1803



24. Take any right line, as AB, and (by the foregoing *Probl.* 3.) cut the same by extreme and mean Reason in C, therefore

AB . AC :: AC . CB.

25. Then upon the line AB describe the Semicircle ADB, and from C raise CD ⊥ AB. Lastly, draw AD and DB, so shall ADB be a right-angled Triangle whose three sides are Proportionals: For first, the $\angle$ ADB being in the Semicircle, is a right angle, (*per prop.* 31. *Elem.* 3.) But that the three sides AB, AD, DB are Proportionals, I prove thus;

26. . . *Req. demonstr.* . . . . AB . AD :: AD . DB.

Demonstration.

27. Because by *Constr.* in 25° ADB is a Semicircle, and CD ⊥ AB, therefore (*per Coroll. prop.* 8. *Elem.* 6.)	AB . DB :: DB . CB.
28. And because by *Constr.* in 24°, ...	AB . AC :: AC . CB.
29. Therefore out of 27° and 28°, (*per prop* 17. *El.* 6.)	▭ AB, CB = □ DB = □ AC.
30. And consequently, ...	. . . . DB = AC.
31. And because (by *Constr.* in 25°,) △ ADB is right-angled at D, and DC ⊥ AB, therefore (*per prop.* 8. *Elem* 6.)	AB . AD :: AD . AC.
32. Therefore from 30° and 31°, ...	AB . AD :: AD . DB.

Which was to be Dem.

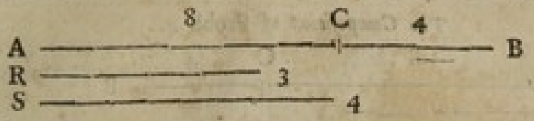
COROLL. 2.

33. From the premisses the way of solving this following Problem is also deducible; *viz.* To cut a right line given into three such Proportionals, that the Square of the greatest shall be equal to the Squares of the other two: For if it be made as the summ of the three Proportionals AB, AD, DB (being the sides of a right-angled Triangle, and also Proportionals,) is to every one of them; so the right line given to three others; the Problem will be satisfied.

Probl. IV.

To cut a given right line into two such parts, that the Rectangle made of the whole line and one of the parts, to the Square of the other part may have a given Reason.

A ——



Suppos.

1. $b = AB$ is a right line given to be cut into two parts.

2. $\left\{\begin{matrix} r = R \\ s = S \end{matrix}\right\}$ are the Terms of a given Reason.

Req. to find

3. AC and CB such parts of AB, that $AC + CB = AB$. Also, that
4. $\square$ AB, CB . $\square$ AC :: R . S.

Resolution.

5. Put a for one of the parts sought, *viz.* . . .	$a = AC.$
6. Therefore from 1° and 5° the other part shall be	$b - a = CB.$
7. Therefore the Square of the first part is . .	aa ($= \square$ AC.)
8. And the Rectangle of the given line b and latter part is	$bb - ba$ ($= \square$ AB, CB.
9. Therefore to answer what the Problem requires, these must be Proportionals, *viz.* .	$r \,.\, s :: bb - ba \,.\, aa.$
10. Now to avoid an Equation between Solids, let it be made, as r to s, so b to a fourth Proportional, which may be called d, therefore .	$r \,.\, s :: b \,.\, d.$
11. Therefore from 9° and 10°, (*per prop.* 11. *Elem.* 5.)	$b \,.\, d :: bb - ba \,.\, aa.$
12. And by drawing $b - a$ as a common altitude into b and d severally, this Analogy is manifest, (*per prop.* 1. *Elem.* 6.)	$b \,.\, d :: bb - ba \,.\, db - da.$
13. Therefore from 11° and 12°, (*per prop.* 11. *Elem.* 1.)	$bb - ba \,.\, aa :: bb - ba \,.\, db - da.$
14. And because the first Term of the last Analogy is equal to the third, the second shall be equal to the fourth, (*per prop.* 14. *Elem.* 5.) therefore, ...	$aa = db - da.$
15. Therefore by adding da to each part of the last Equation, it gives	$aa + da = db.$
16. Which last Equation may be resolved into this Analogy, ...	$a + d \,.\, \sqrt{db} :: \sqrt{db} \,.\, a.$

In which Analogy (consisting of three Proportionals,) the mean, to wit, $\sqrt{db}$ is given, as also d the difference of the extremes $a + d$ and a, therefore the extremes severally, (the lesser whereof is one of the Parts sought by this Problem,) shall be given also by *Probl.* 12. *Chap.* 5. Whence also this

CANON.

17. . . . . $\sqrt{:\tfrac{1}{4}dd + db:} - \tfrac{1}{2}d = AC.$

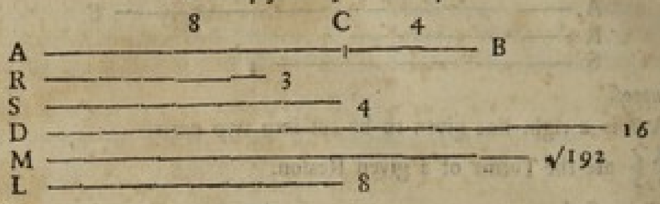
That is, in words,

Let it be made, As R the first Term of the given Reason, to the latter S; so AB the line given to be cut into two parts, to a fourth Proportional, which may be called D. Then to the Square of half that fourth Proportional, add the Rectangle made of the same Proportional and the given line AB, and from the square Root of the summ subtract half the said fourth Proportional D: The remainder shall be one of the parts sought, which may be called the first. Lastly, the said first part being subtracted from the given line AB, gives also the other part.

18. *Note.* Whether the first Term of the given Reason be greater or less than the latter, it shall always be; As the first Term is to the latter; so the Rectangle made of the given line and the latter part found out by the Canon; to the Square of the first part.

The

The Composition of Probl. 4.



Suppos.

19. A B is a right line given to be cut into two parts.
20. R and S are the Terms of a given Reason.

Req. to find

21. A C and C B such parts of A B, that $AC + CB = AB$. Also, that
22. $\square AB, CB \;.\; \square AC :: R \;.\; S.$

Construction.

23. By *Probl.* 8. *Chap.* 5. let it be made; as R to S, so A B to a fourth, which may be called D, therefore } $R \;.\; S :: AB \;.\; D.$
24. By *Probl.* 9. *Chap.* 5. find a mean proportional line, as M, between A B and D, therefore . . . . } $AB \;.\; M :: M \;.\; D.$
25. Therefore (*per prop.* 17. *Elem.* 6.) . . . . . } $\square M = \square D, AB.$
26. Then esteeming the line M to be the mean of three Proportionals, and D the difference of the extremes, find out the extremes, (*per Probl.* 12. *Chap.* 5.) the lesser whereof suppose to be L, then consequently the greater is equal to $L + D$, therefore these are Proportionals, *viz.*

$$L + D \;.\; M :: M \;.\; L.$$

27. Therefore, *per Theor.* in 24° of *Probl.* 12. *Chap.* 5. } $L = \sqrt{:\tfrac{1}{4}\square D + \square M:} - \tfrac{1}{2}D.$
28. From A B cut off $AC = L$, which effection is possible, if $AB \sqsubset L$, but that A B is greater than L, I prove thus,

First, it is manifest that . . . . } $\square AB + \tfrac{1}{4}\square D + \square D, AB \sqsubset \tfrac{1}{4}\square D + \square D, AB.$

And by extracting the square Root out of each part, ... } $AB + \tfrac{1}{2}D \sqsubset \sqrt{:\tfrac{1}{4}\square D + \square D, AB:}$

And by subtracting $\tfrac{1}{2}$D from each part, } $AB \sqsubset \sqrt{:\tfrac{1}{4}\square D + \square D, AB:} - \tfrac{1}{2}D.$

But by *Constr.* in 25° and 27°, . . } $L = \sqrt{:\tfrac{1}{4}\square D + \square D, AB:} - \tfrac{1}{2}D.$

Therefore from the two last preceding steps, (*per Ax* 3. *Chap.* 2.) . . } $AB \sqsubset L.$ Which was to be Dem.

29. I say the given line A B is cut in C into two parts, according to the tenour of the Problem propounded. Now we must shew, that the Rectangle made of the whole line A B and one of the parts, is to the Square of the other part as R to S. But that Analogy is made manifest by the following Demonstration, formed out of the steps of the preceding Resolution, by returning backwards from the 16th step to the Analogy in the 9th step.
30. . . *Req. demonstr.* . . . . $R \;.\; S :: \square AB, CB \;.\; \square AC.$

Demonstration.

31. Because by *Construction* in 26°, ... } $L + D \;.\; M :: M \;.\; L.$
32. And by *Constr.* in 28°, ... } $AC = L.$
33. Therefore from 31° and 32°, by exchange of equal quantities, ... } $AC + D \;.\; M :: M \;.\; AC.$
    That is, in 16°, ... } $a + d \;.\; \sqrt{db} :: \sqrt{db} \;.\; a.$
34. Therefore, (*per prop.* 17. *Elem.* 6.) . . . } $\square AC + \square D, AC = \square M.$
35. And because by *Constr.* in 24° and 25°, . } . . . . $\square D, AB = \square M.$
36. Therefore from 34° and 35°, (*per Ax.* 1. *Chap.* 2.) . . . . } $\square AC + \square D, AC = \square D, AB.$
    That is, in 15°, ... } $aa + da = db.$
37. Therefore by subtracting $\square$D,AC from each part of the Equation in 36°, ... } $\square AC = \square D, AB - \square D, AC.$
    That is, in 14°, ... } $aa = db - da.$

38. And

38. And from 37°, (*per prop* 7. *Elem.* 5.) this Analogy is manifest, ... $\square AB - \square AB,AC \;.\; \square AC :: \square AB - \square AB,AC \;.\; \square D,AB - \square D,AC$. that is, in 13°, $bb - ba \;.\; aa :: bb - ba \;.\; db - da$.

39. And by reason of the common altitude AB–AC, this following Analogy is manifest, (*per propos.* 1. *Elem.* 6.) . . . . . $AB \;.\; D :: \square AB - \square AB,AC \;.\; \square D,AB - \square D,AC$. that is, in 12°, $b \;.\; d :: bb - ba \;.\; db - da$.

40. Therefore from 38° and 39°, (*per prop.* 11. *Elem.* 5.) . . . . $AB \;.\; D :: \square AB - \square AB,AC \;.\; \square AC$.
That is, in 11°, ... $b \;.\; d :: bb - ba \;.\; aa$.

41. And because by *Constr.* in 23°, ... $AB \;.\; D :: R \;.\; S$.
That is, in 10°, ... $b \;.\; d :: r \;.\; s$.

42. Therefore from 40° and 41°, (*per prop.* 11. *Elem.* 5.) . . . . $R \;.\; S :: \square AB - \square AB,AC \;.\; \square AC$.
That is, in 9°, ... $r \;.\; s :: bb - ba \;.\; aa$.

43. And because (*per Diagram*,) . . . . $CB = AB - AC$.

44. And consequently, (*per prop.* 1. *Elem.* 6.) by drawing A B into each part, . . . $\square AB,CB = \square AB - \square AB,AC$.

45. Therefore from 42° and 44°, by exchanging equal quantities, ... $R \;.\; S :: \square AB,CB \;.\; \square AC$.

Which was to be Demonstr. Therefore the Problem is satisfied.

Probl. V.

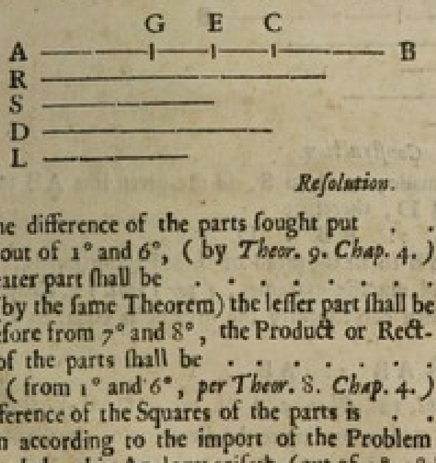
To cut a given right line into two such parts, that the difference of the Squares of the parts, to the Rectangle made of the same parts, may have a given Reason.

Suppos.

1. $b = AB$ is a right line given to be cut into two parts.
2. $r = R$
3. $s = S$ } are the Terms of a given Reason.

Req. to find

4. AC and CB, such parts of AB, that $AC + CB = AB$. Also, that
5. $\square AC - \square CB \;.\; \square AC,CB :: R \;.\; S$.



$AB = 12$
$R = 3$
$S = 2$
$D = 8$
$L = 4$
$AC = 8$
$CB = 4$

Resolution.

6. For the difference of the parts sought put . . a.
7. Then out of 1° and 6°, (by *Theor.* 9. *Chap.* 4.) the greater part shall be . . . . $\frac{1}{2}b + \frac{1}{2}a$.
8. And (by the same Theorem) the lesser part shall be $\frac{1}{2}b - \frac{1}{2}a$.
9. Therefore from 7° and 8°, the Product or Rectangle of the parts shall be . . . . $\frac{1}{4}bb - \frac{1}{4}aa$.
10. And (from 1° and 6°, *per Theor.* 8. *Chap.* 4.) the difference of the Squares of the parts is . . ba.
11. Then according to the import of the Problem propounded, this Analogy ariseth, (out of 2°, 3°, 10° and 9°, *viz.* . . . . $r \;.\; s :: ba \;.\; \frac{1}{4}bb - \frac{1}{4}aa$.
12. Now to avoid an Equation between Solids, let it be made, as r to s, so b to a fourth Proportional, which may be called d, therefore . . . . . $r \;.\; s :: b \;.\; d$.
13. Therefore out of 11° and 12°, (*per pro.* 11. *El.* 5.) $b \;.\; d :: ba \;.\; \frac{1}{4}bb - \frac{1}{4}aa$.

14. And

14. And by drawing a as a common altitude into b and d severally, this following Analogy will be manifest, (*per prop.* 1. *Elem.* 6.) *viz.* . . . $b \;.\; d \;::\; ba \;.\; da.$
15. Therefore from the 13th and 14th steps, (*per prop.* 11. *Elem.* 5.) . . . . $ba \;.\; \frac{1}{4}bb - \frac{1}{4}aa \;::\; ba \;.\; da.$
16. Therefore from the 15th step this Equation ariseth, (*per prop.* 14. *Elem.* 5.) *viz.* . . . . $\frac{1}{4}bb - \frac{1}{4}aa = da.$
17. And by multiplying each Term of the Equation in the 16th step by 4, this Equation is produced, *viz.* $bb - aa = 4da.$
18. Therefore by adding aa to each part, . . . $bb = aa + 4da.$
19. Which Equat. may be resolved into this Analogy, $a + 4d \;.\; b \;::\; b \;.\; a.$
20. But that Analogy doth manifestly consist of three Proportionals; whereof the mean b is given, as also $4d$ the difference of the extremes $a + 4d$ and a; therefore the extremes severally, the lesser whereof is the difference of the parts sought by this Problem, shall be given also, by *Probl.* 12. *Chap.* 5. whence also, (respect being had to the Theorem in 24° of the same Problem,) there will arise this

CANON.

21. . . . . . $\sqrt{:bb + 4dd}: - 2d = a$, the difference of the parts sought.

That is, in words,

Let it be made, as the first Term (whether it be the greater or lesser) of the given Reason, is to the second; so the line given to be divided, to a fourth Proportional. Then add the Square of the double of that Proportional to the Square of the line given to be divided, and from the square Root of the summ subtract the double of the said Proportional, so shall the remainder be the difference of the parts sought. Then by the summ and difference of the parts, the parts shall be given severally by *Theor.* 9. *Chap.* 4.

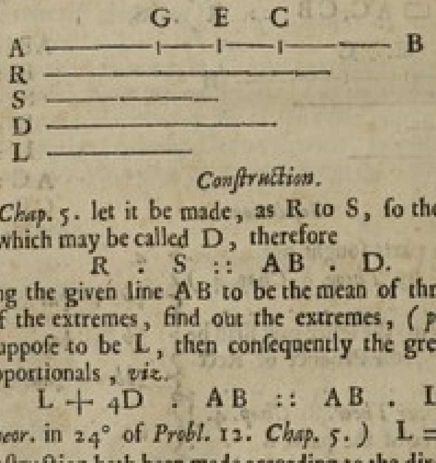
The Composition of Probl. 5.

Suppos.

22. A B is a right line given.
23. R and S are the Terms of a given Reason.

Req. to find

24. A C and C B such parts of A B, that A C + C B = A B. Also,
25. □ A C — □ CB . ▭ AC, CB :: R . S.



Construction.

26. By *Probl.* 8. *Chap.* 5. let it be made, as R to S, so the given line AB to a fourth Proportional, which may be called D, therefore

R . S :: AB . D.

27. Then esteeming the given line AB to be the mean of three Proportionals, and 4 D the difference of the extremes, find out the extremes, (*per Probl.* 12. *Chap.* 5.) the lesser whereof suppose to be L, then consequently the greater is equal to L + 4D, and these are Proportionals, *viz.*

L + 4D . AB :: AB . L.

Whence, (*per Theor.* in 24° of *Probl.* 12. *Chap.* 5.) L = $\sqrt{:}\overline{\square AB + 4\square D}: - 2D.$

Thus far the Construction hath been made according to the direction of the Canon in 21°.

28. Divide the given line AB into two equal parts in E, whence EA = EB = $\frac{1}{2}$AB.
29. From EA and EB cut off EG and EC, such parts, that each may be equal to $\frac{1}{2}$L, which is possible to be done, if $\frac{1}{2}$AB $\sqsubset$ $\frac{1}{2}$L; but that $\frac{1}{2}$AB $\sqsubset$ $\frac{1}{2}$L, I prove thus,

First, it is manifest that . . . . □AB + 4▭AB,D + 4□D $\sqsubset$ □AB + 4□D.

Therefore by extracting the square Root out of each part, ... AB + 2D $\sqsubset$ $\sqrt{:}\overline{\square AB + 4\square D}:$

And

And by subtracting 2D from each part, ...	$AB \sqsubset \sqrt{:\square AB + 4\square D:} - 2D.$
But by *Constr.* in 27°, ...	$L = \sqrt{:\square AB + 4\square D:} - 2D.$
Therefore, *(*per Ax.* 3. *Chap.* 2.)	$AB \sqsubset L$; and $\frac{1}{2}AB \sqsubset \frac{1}{2}L.$

Therefore it is possible to cut off from $\frac{1}{2}AB$, that is, from $EB = EA$, a right line equal to $\frac{1}{2}L$; suppose therefore $EC = EG = \frac{1}{2}L$.

30. I say the given right line AB is cut into two parts in C, as the Problem requires; *viz.* The difference of the Squares of the parts AC and CB, is to the Rectangle made of the same parts, as R to S; which Analogy may be demonstrated by a retrograde repetition of the steps of the preceding Resolution, in manner following.

31\. . . . *Req. demonstr.* . . . . $R . S :: \square AC - \square CB . \text{▭}AC,CB.$

Demonstration.

32. Forasmuch as by *Constr.* in 27°,	$L + 4D . AB :: AB . L.$
That is, in 19°, ...	$a + 4d . b :: b . a.$
33. Therefore from 32°, (*per prop.* 17. *Elem.* 6.)	$\square AB = \square L + 4\text{▭}DL.$
That is, in 18°, ...	$bb = aa + 4da.$
34. Therefore from 33°, by subtracting the $\square L$ from each part, . .	$\square AB - \square L = 4\text{▭}DL.$
That is, in 17°, ...	$bb - aa = 4da.$
35. And by taking a quarter of the Equation in 34°, it gives . . .	$\frac{1}{4}\square AB - \frac{1}{4}\square L = \text{▭}DL.$
That is, in 16°, ...	$\frac{1}{4}bb - \frac{1}{4}aa = da.$
36. Therefore from 35° this following Analogy will be manifest, (*per* 7. *prop.* 5. *Elem*)	$\text{▭}AB,L . \frac{1}{4}\square AB - \frac{1}{4}\square L :: \text{▭}AB,L . \text{▭}DL.$
That is, in 15°, ...	$ba . \frac{1}{4}bb - \frac{1}{4}aa :: ba . da.$
37. And by reason of the common altitude L, this following Analogy will be manifest, (*per prop.* 1. *Elem.* 6.)	$AB . D :: \text{▭}AB,L . \text{▭}DL.$
That is, in 14°, ...	$b . d :: ba . da.$
38. Therefore from 36° and 37°, (*per prop.* 11. *Elem* 5.)	$AB . D :: \text{▭}AB,L . \frac{1}{4}\square AB - \frac{1}{4}\square L.$
That is, in 13°, ...	$b . d :: ba . \frac{1}{4}bb - \frac{1}{4}aa.$
39. And because by *Constr.* in 26°, .	$AB . D :: R . S.$
That is, in 12°, ...	$b . d :: r . s.$
40. Therefore out of 38° and 39°, (*per* 11. *prop.* 5. *Elem.*)	$R . S :: \text{▭}AB,L . \frac{1}{4}\square AB - \frac{1}{4}\square L.$
That is, in 11°, ...	$r . s :: ba . \frac{1}{4}bb - \frac{1}{4}aa.$
41. And because by *Constr.* in 29°, .	$EC = \frac{1}{2}L = EG.$
42. Therefore by taking 2EC, that is, GC, instead of L in the Analogy in 40°, this will thence arise, *viz.* .	$R . S :: \text{▭}AB,GC . \frac{1}{4}\square AB - \frac{1}{4}\square GC.$
43. And because (*per Diagram,*) .	$AB = AC + CB.$
44. And by *Constr.* in 28° and 29°, .	$GC = AC - CB\ (AG.)$
45. Therefore from 43° and 44°, (*per Theor.* 8. *Chap.* 4.)	$\square AC - \square CB = \text{▭}AB,GC.$
46. And because by *Constr.* in 29°, .	$AC = \frac{1}{2}AB + \frac{1}{2}GC.$
47. And by *Constr.* also in 29°, . .	$CB = \frac{1}{2}AB - \frac{1}{2}GC.$
48. Therefore from 46° and 47°, (*per Theor.* 8. *Chap* 4. & *Prop.* 1. *Elem.* 6.)	$\text{▭}AC,CB = \frac{1}{4}\square AB - \frac{1}{4}\square GC.$
49. Therefore if instead of the two latter Terms of the Analogy in 42°, you take their equivalent quantities, to wit, the first parts of the Equations in 45° and 48°, this Analogy will arise, *viz.*	$R . S :: \square AC - \square CB . \text{▭}AC,CB.$

Which was to be demonstr. Therefore the Problem is satisfied.

Probl. VI.

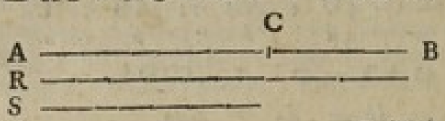
To cut a given right line into two ſuch parts, that the Rectangle made of the whole line and the difference of the parts, to the Square of the leſſer part may have a given Reaſon.

Suppoſ.

1. b = A B is a right line given to be cut into two parts.
2. $\left\{ \begin{array}{l} r = R \\ s = S \end{array} \right\}$ are the Terms of a given Reaſon.

Req. to find

3. AC and CB ſuch parts of AB, that AC + CB = AB. Alſo,
4. □ AB × $\overline{AC - CB}$. □ CB :: R . S.



AB = 16	AC = 10
R = 16	CB = 6
S = 9	

Reſolution.

5. Put a for the greater part ſought, *viz.* aſſume	a = AC.
6. Therefore out of 1° and 5°, the leſſer part ſhall be	$b - a$ = CB.
7. And from 5° and 6°, the difference of the parts is	$2a - b$.
8. And from 1° and 7°, the Rectangle of the given line into the difference of the parts is	$2ba - bb$.
9. And from 6°, the Square of the leſſer part is	$bb + aa - 2ba$.
10. Therefore from 2°, 8° and 9°, according to the import of the Problem, theſe muſt be Proportionals, *viz.*	$r \,.\, s :: 2ba - bb \,.\, bb + aa - 2ba$.
11. Therefore inverſly, ...	$s \,.\, r :: bb + aa - 2ba \,.\, 2ba - bb$.
12. And by Compoſition, ...	$s + r \,.\, r :: aa \,.\, 2ba - bb$.
13. And inverſly, ...	$r \,.\, s + r :: 2ba - bb \,.\, aa$.
14. Now to avoid an Equation between Solids, let it be made, as r to $s + r$; ſo b to a fourth Proportional, which may be called d, therefore	. . . $r \,.\, s + r :: b \,.\, d$,
15. Therefore from 13° and 14°, (*per prop.* 11. *Elem.* 5.)	. . . . $b \,.\, d :: 2ba - bb \,.\, aa$.
16. And by drawing $2a - b$ as a common altitude into b and d ſeverally, this Analogy is manifeſt, (*per prop.* 1. *Elem.* 6.) .	. . . . $b \,.\, d :: 2ba - bb \,.\, 2da - db$.
17. Therefore from 15° and 16°, (*per prop.* 11. *Elem.* 5.)	$2ba - bb \,.\, 2da - db :: 2ba - bb \,.\, aa$.
18. And from 17°, (*per prop.* 14. *Elem.* 5.)	$2da - db = aa$.
19. And by adding db to each part of that Equation, ...	$2da = aa + db$.
20. And by ſubtracting aa from each part,	$2da - aa = db$.
21. Which laſt Equation may be reſolved into this Analogy, ...	$2d - a \,.\, \sqrt{db} :: \sqrt{db} \,.\, a$.

22. But that Analogy doth manifeſtly conſiſt of three Proportionals, whereof the mean $\sqrt{db}$ is given, as alſo $2d$ the ſumm of the extremes $2d - a$ and a; therefore the extremes ſeverally, the leſſer whereof is the greater part ſought, ſhall be given alſo, by *Probl.* 13. *Ch.* 5. And from the premiſſes and the Theorem in 21° of the ſame Problem there will ariſe this following

CANON.

23. . . . . . $d - \sqrt{:dd - db:}$ = AC, the greater part ſought.

That

That is, in words,

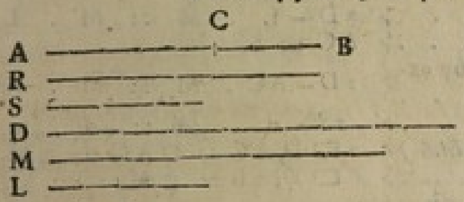
Let it be made as R the first Term of the given Reason, to R+S the summ of both Terms; so A B the line given to be cut into two parts, to a fourth Proportional, which may be called D. Then from the Square of that fourth Proportional, subtract the Rectangle made of the said fourth Proportional and the given line A B, and extract the square Root of the remainder. Lastly, subtract the said square Root from the said fourth Proportional, and this remainder shall be the greater part sought.

24. *Note.* Although the Equation found out in 20° may be expounded by either of the two extreme Proportionals mentioned in 22°, yet the lesser of them is only capable of solving the Problem propounded; for the greater extreme is greater than the line given to be cut into two parts, and therefore cannot be equal to either of the parts; which I prove thus,

If $\sqrt{db}$ be the mean of three Proportionals, and $2d$ the summ of the extremes, then (by *Probl.* 13. *Chap.* 5.) the greater extreme shall be equal to	$d + \sqrt{:\overline{dd - db}:}$
Now we must prove that	$d + \sqrt{:\overline{dd - db}:} \sqsubset b.$
First, by *Construction* in 14°, (and by *Schol. prop.* 14. *Elem* 5.) 'tis evident that	$d \sqsubset b.$
And consequently, by drawing d into each part, . . .	$dd \sqsubset db.$
Therefore, ...	$\sqrt{:\overline{dd - db}:} \sqsubset 0.$
And consequently, by adding d to each part, ...	$d + \sqrt{:\overline{dd - db}:} \sqsubset d.$
But by *Constr.* in 14°, ...	$d \sqsubset b.$
Therefore (*per Ax.* 5. *Chap.* 2.)	$d + \sqrt{:\overline{dd - db}:} \sqsubset b.$

Which was to be Dem.

The Composition of the preceding Probl. 6.



AB = 16	M = 20
R = 16	L = 10
S = 9	AC = 10
D = 25	CB = 6

Suppos.

25. A B is a right line given to be cut into two parts.
26. R and S are the Terms of a given Reason.

Req. to find

27. AC and C B such parts of A B, that A C + CB = AB. Also,
28. $\square$ AB $\times$ $\overline{AC - CB}$. $\square$ CB :: R . S.

Construction.

29. By *Probl* 8. *Chap.* 5. let it be made, as R to S+R; so AB to a fourth Proportional, which we may suppose to be the line D, therefore	R . S+R :: AB . D.
30. By *Probl.* 9. *Chap.* 5. find a mean proportional line, as M, between A B and D, therefore . . .	A B . M :: M . D.
31. Therefore it follows from the last Analogy, (*per prop.* 17. *Elem.* 6.) that	$\square$ A B, D = $\square$ M.

32. Then by *Probl* 14. *Chap.* 5. cut the double of the right line D (before found in 29°) into two such parts, that the line M may be a mean Proportional between them; which Effection is possible, if M be not greater than D; but that M is less than D, I prove thus,

By the *Construct* in 29° 'tis manifest that D $\sqsubset$ AB, and by *Constr.* in 30°, M is a mean Proportional between A B and D, therefore M $\sqsupset$ D. Therefore 'tis possible to cut 2 D into two such parts that M may be a mean Proportional between the parts. Suppose then it be done, and that the lesser part is found L, therefore these shall be Proportionals, *viz.*

2 D − L . M :: M . L.

33. And

33. And consequently, (*per Theor.* in 21° of *Probl.* 13. *Chap.* 5.)	$L = D - \sqrt{:}\,\overline{\square D - \square M}:$

34. From A B cut off A C = L, which is possible to be done if A B $\sqsubset$ L, but that A B $\sqsubset$ L, I prove thus;

First, from the *Constr.* in 29°, . . .	$D \sqsubset AB.$
Therefore, by drawing A B into each part,	$\Box D, AB \sqsubset \square AB.$
And by adding $\square$ D to each part, . .	$\square D + \Box D, AB \sqsubset \square D + \square AB.$
And by subtracting 2 $\Box$ D, A B from each part, ...	$\square D - \Box D, AB \sqsubset \square D + \square AB - 2\Box D, AB.$
And by extracting the square Root out of each part, ...	$\sqrt{:}\,\overline{\square D - \Box D, AB}: \sqsubset D - AB.$
And by adding A B to each part, . .	$AB + \sqrt{:}\,\overline{\square D - \Box D, AB}: \sqsubset D.$
And by subtracting $\sqrt{:}\,\overline{\square D - \Box D, AB}:$ from each part, ...	$AB \sqsubset D - \sqrt{:}\,\overline{\square D - \Box D, AB}:$
But it hath been shewn in 31° and 33°, that	$L = D - \sqrt{:}\,\overline{\square D - \Box D, AB}:$
Therefore from the two last preceding steps,	$AB \sqsubset L.$

Which was to be Dem. Therefore 'tis possible to cut off from A B a segment equal to L, as A C.

35. I say the given right line A B, in the point C is cut into two parts, A C and C B, which will solve the Problem; *viz.* the Rectangle made of the whole line A B, and the excess of A C above C B, is to the Square of C B; as the line R to the line S: As will be made manifest by the following Demonstration, form'd out of the foregoing Resolution, by a retrograde repetition of the steps thereof.

36\. . . . *Req. demonstr.* . . . $R . S :: \Box AB \times \overline{AC - CB} . \square CB.$

Demonstration.

37. By *Constr.* in 32°, ...	$2D - L . M :: M . L.$
38. And by *Constr.* in 34°, ...	$AC = L.$
39. Therefore from 37° and 38°, by exchanging equal quantities, ...	$2D - AC . M :: M . AC.$
That is, in 21, ...	$2d - a . \sqrt{db} :: \sqrt{db} . a.$
40. Therefore from 39°, (*per prop.* 17. *El.* 6.)	$2\Box D, AC - \square AC = \square M.$
41. But by *Constr.* in 31°, ...	$\Box D, AB = \square M.$
42. Therefore from 40° and 41°, (*per Ax.* 1. *Chap.* 2.)	$2\Box D, AC - \square AC = \Box D, AB.$
That is, in 20°, ...	$. . 2da - aa = db.$
43. And by adding $\square$ A C to each part of the Equation in 42°, ...	$2\Box D, AC = \square AC + \Box D, AB.$
That is, in 19°, ...	$. . 2da = aa + db.$
44. And by subtracting $\Box$ D, A B from each part in 43°, ...	$2\Box D, AC - \Box D, AB = \square AC.$
That is, in 18°, ...	$. . 2da - db = aa.$

45\. And from 44°, (*per prop.* 7. *Elem.* 5.) this Analogy will be manifest,

$$\left.\begin{array}{r} 2\Box AB, AC, - \square AB \;. \\ 2\Box D, AC, - \Box D, AB :: \\ 2\Box AB, AC, - \square AB \;. \\ \square AC \;. \end{array}\right\} \text{that is, in } 17°, \left\{\begin{array}{l} 2ba - bb \;. \\ 2da - db :: \\ 2ba - bb \;. \\ aa \;. \end{array}\right.$$

46\. And by reason of the common altitude 2 A C — A B, this following Analogy is manifest, (*per prop.* 1. *Elem.* 6.)

$$\left.\begin{array}{r} 2\Box AB, AC - \square AB \;. \\ 2\Box D, AC - \Box D, AB :: \\ AB \;. \\ D \;. \end{array}\right\} \text{that is, in } 16°, \left\{\begin{array}{r} 2ba - bb \;. \\ 2da - db :: \\ b \;. \\ d \;. \end{array}\right.$$

47. Therefore from 45° and 46°, (*per prop.* 11. *Elem.* 5.)	$AB . D :: 2\Box AB, AC, - \square AB . \square AC.$
That is, in 15°, ...	$b . d :: 2ba - bb . aa.$
48. But by *Construction* in 29°, . . .	$AB . D :: R . S + R.$

That

That is, in 14°, ... $b \,.\, d :: r \,.\, s + r.$

49. Therefore from 47° and 48°, (*per prop.* 11. *Elem* 5.) . . . . . $R \,.\, S + R :: 2\Box AB,AC - \square AB \,.\, \square AC.$

That is, in 13°, ... $r \,.\, s + r :: 2ba - bb \,.\, aa.$

50. Therefore inversly, ... $S + R \,.\, R :: \square AC \,.\, 2\Box AB,AC, - \square AB.$

That is, in 12°, ... $s + r \,.\, r :: aa \,.\, 2ba - bb.$

51. Therefore by Division of Reason,

$$\left.\begin{array}{r} S \,. \\ R :: \\ \square AB + \square AC - 2\Box AB,AC \,. \\ 2\Box AB,AC, - \square AB \,. \end{array}\right\} \text{that is, in } 11°, \left\{\begin{array}{l} s \,. \\ r :: \\ bb + aa - 2ba \,. \\ 2ba - bb \,. \end{array}\right.$$

52. Therefore inversly,

$$\left.\begin{array}{r} R \,. \\ S :: \\ 2\Box AB,AC, - \square AB \,. \\ \square AB + \square AC - 2\Box AB,AC \,. \end{array}\right\} \text{that is, in } 10°, \left\{\begin{array}{l} r \,. \\ s :: \\ 2ba - bb \,. \\ bb + aa - 2ba \,. \end{array}\right.$$

Now the Scope in the six steps next following is to prove, that $\Box AB \times \overline{AC - CB}$ is equal to $2\Box AB,AC - \square AB$, to wit, the third Term of the Analogy in 52°.

53. By *Constr.* in 34°, ... $AB = AC + CB.$

54. And by adding AC to each part, $AB + AC = 2AC + CB.$

55. And by subtracting CB from each part in 54°, ... $AB + AC - CB = 2AC.$

56. It is evident by the first part of the Equation in 44°, that . . . . $2AC \sqsubset AB.$

57. Therefore AB may be subtracted from each part of the Equation in 55°, and this Equation between two real right lines will remain, *viz.* . . $AC - CB = 2AC - AB.$

58. Therefore by drawing AB as a common altitude into each part of the last Equation, it will produce (by *Theor* 1. *Chap.* 4.) . . . . $\Box AB \times \overline{AC - CB} = 2\Box AB,AC, - \square AB.$

Which was to be shewn. It remains to prove that $\square CB$ is equal to $\square AB + \square AC - 2\Box AB,AC$, to wit, the fourth Term of the Analogy in 52°.

59. By *Constr.* in 34°, ... $CB = AB - AC.$

60. And because the Squares of equal right lines are also equal, therefore from 59°, (*per Theor.* 5. *Chap.* 4.) $\square CB = \square AB + \square AC - 2\Box AB,AC.$

Which was to be Demonstr.

61. Lastly, instead of the third and fourth Terms of the Analogy in 52°, their equivalent quantities, to wit, those in the first parts of the Equations in 58° and 60° being taken, this following Analogy ariseth, *viz.*

$$R \,.\, S :: \Box AB \times \overline{AC - CB} \,.\, \square CB.$$

Which was *Req. demonstr.* in 36°. Therefore the Problem is satisfied.

Probl. VII.

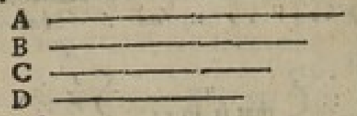
To find two right lines that their summ may be equal to a right line given, and that the difference of the Squares of those two lines, to the Square of the lesser of them, may have a given Reason.

This Problem is the same in effect with the preceding sixth; for the difference of the Squares of the two right lines sought by this Problem, is equal to the Rectangle of the summ and difference of the parts sought by the last preceding Problem.

A LEMMA,

A LEMMA, *leading to the following* Probl. 8.

If four right lines be in continual proportion, the summ of the means is a mean Proportional between the summ of the first and second, and the summ of the third and fourth Proportionals.



$A = 125$
$B = 100$
$C = 80$
$D = 64$

Suppos.

1. A, B, C, D $\div\!\div$, *viz.* A . B :: B . C :: C . D.
2. . . *Req. demonstr.* . . . . A+B . B+C :: B+C . C+D.

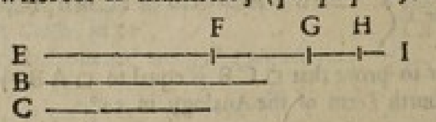
Demonstration:

3. By *Suppos.* in 1°, ... A . B :: B . C.
4. Therefore by Composition of Reason, A+B . B :: B+C . C.
5. And alternately, ... A+B . B+C :: B . C.
6. Again, by *Suppos.* . . . . B . C :: C . D.
7. Therefore by Composition, ... B+C . C :: C+D . D.
8. And alternately, ... B+C . C+D :: C . D.
9. Therefore from 6° and 8°, (*per prop.* 11. *Elem.* 5.) . . . . B+C . C+D :: B . C.
10. Likewise from 5° and 9°, (*per prop.* 11. *Elem.* 5.) . . . . A+B . B+C :: B+C . C+D.

Which was to be Demonstr.

Probl. VIII.

The summ of the extremes, and summ of the means of four right lines in Continual proportion being given severally, to find the Proportionals. But the first summ must be greater than the latter; the reason whereof is manifest, (*per prop.* 25. *Elem.* 5.)



$EF = 16$
$FG = 8$
$GH = 4$
$HI = 2$

Suppos.

1. EF, FG, GH, HI $\div\!\div$, *viz.* EF . FG :: FG . GH :: GH . HI.
2. b = EF + HI is given. Also,
3. c = FG + GH = FH is given: Therefore,
4. $d = b + c$ = EI is given.
5. $b \sqsubset c$. (*Determination.*)

Req. to find EF, FG, GH, HI.

Resolution.

6. By *Suppos.* in 1°, ... EF, FG, GH, HI $\div\!\div$.
7. Therefore by the *Lemma* prefixt before this Problem, . . . EF+FG . FG+GH :: FG+GH . GH+HI.
8. That is, as is evident by the Diagram, ... EG . FH :: FH . GI.
9. Of which three continual Proportionals the mean FH, that is, c, is given; as also EI, (= EG + GI,) that is, d, the summ of the extremes. Threfore, by the Theorem in 21° of *Probl.* 13. *Chap.* 5. the extremes shall be given severally, *viz.*

10. . . . . . $\begin{cases} \frac{1}{2}d + \sqrt{:\overline{\tfrac{1}{4}dd - cc}:} = EG = EF + FG. \\ \frac{1}{2}d - \sqrt{:\overline{\tfrac{1}{4}dd - cc}:} = GI = GH + HI. \end{cases}$

11. Therefore, from 10° and 3° 'tis manifest, that of these three Proportionals, EF, FG, GH, the summ of the first and second, to wit, EG, is given; also the summ of the second and third, to wit, FH, is given. Likewise of these three Proportionals, FG, GH,

G H, H I, the summ of the first and second, to wit, F H, is given; also G I the summ of the second and third is given; therefore, (according to the Canon in 44° of *Probl.* 5. *Chap.* 7.) F G and G H shall be given severally by these following Analogies, *viz.*

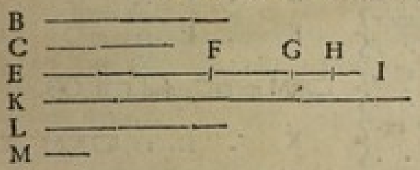
12. . . . $\begin{cases} c + \frac{1}{2}d + \sqrt{:\overline{\frac{1}{4}dd - cc}:} \ . \ c :: \frac{1}{2}d + \sqrt{:\overline{\frac{1}{4}dd - cc}:} \ . \ FG. \\ c + \frac{1}{2}d - \sqrt{:\overline{\frac{1}{4}dd - cc}:} \ . \ c :: \frac{1}{2}d - \sqrt{:\overline{\frac{1}{4}dd - cc}:} \ . \ GH. \end{cases}$

Which Analogies, respect being had to the Equations in 10°, and to the Diagram, will give this following

CANON.

13. From the Square of half the aggregate of the given summ of the extremes and the given summ of the means, subtract the Square of the summ of the means, and extract the square Root of the remainder. Then add and subtract that square Root to and from the said half aggregate, and reserve the summ and remainder. Then it shall be, as the summ reserved together with the summ of the means, is to the summ of the means; so the summ reserved to the greater mean sought. Or, as the remainder reserved together with the summ of the means, is to the summ of the means; so the remainder reserved to the lesser mean. Lastly, the summ before reserved being lessened by the greater mean, gives the greater extreme: or, the remainder reserved being lessened by the lesser mean, gives the lesser extreme.

The Composition of the foregoing Probl. 8.



B = 18	K = 36
C = 12	L = 18
EI = 30	GF = 8
EG = 24	GH = 4
GI = 6	M = 4

Suppos.

14. B = the summ of the extremes of four right lines in continual proportion is given.
15. C = the summ of the means is given.
16. B $\sqsubset$ C. (*Determination.*)

Req. to find the four Proportionals severally.

Construction.

17. Make E I = B + C.
18. Divide E I into two such parts in G, that the line C may be a mean Proportional between the parts; which effection is possible, (*per Probl.* 14. *Chap.* 5.) if C be not greater than ½ E I, but that C is less than ½ E I, I prove thus;

By the Determination in 16°, ... C $\sqsupset$ B.
Therefore by adding C to each part, ... 2 C $\sqsupset$ B + C.
But by Construction in 17°, ... E I = B + C.
Therefore, (*per Ax.* 3. *Chap.* 2.) . . . . . 2 C $\sqsupset$ E I.
And consequently, ... C $\sqsupset$ ½ E I.

Which was to be shewn.

Therefore 'tis possible to cut E I into two such parts, that C shall be a mean Proportional between them. Suppose then E I to be so cut in G, and that E G is greater than G I; therefore, ... } E G . C :: C . G I.

19. Make . . . . } K = EG + C; also, L = GI + C.
20. Let it also be made (by *Probl.* 8. *Chap.* 5.) as K to C, so E G to a fourth Proportional G F, therefore, ... } K . C :: E G . G F.
21. Again, let it be made as L to C, so G I to a fourth Proportional G H; therefore, ... } L . C :: G I . G H.
22. Then from EG cut off GF, and from G I cut off G H; (which subtractions are possible, for by Construction in 19°, K is greater than C; therefore from the Analogy in 20°, E G is greater than G F: Likewise by Construction in 19°, L is greater than C, and consequently from the Analogy in 21°, G I is greater than G H;) so the remainders E F and H I are the extreme Proportionals sought. I say E F, F G, G H and H I are four

four continual Proportionals, which will satisfie the Problem propounded. But to make the truth thereof evident, I shall prove three things, *viz.* First, that FH the summ of the means FG and GH is equal to the given summ C: Secondly, that the summ of the extremes EF and HI is equal to the given summ B: Thirdly and lastly, that the said EF, FG, GH, HI are in continual proportion in this order, *viz.*
EF . FG :: FG . GH :: GH . HI.

First, that FH the summ of the means FG and GH is equal to the given summ C, I prove thus;

Preparat.

23. To the lines K and C find a third Proportional, as M, *(per Probl. 7. Chap. 5.)* therefore, ...	K . C :: C . M.
24. . . . *Req. demonstr.*	FH = C.

Demonstration.

25. By *Constr.* in 18°, ...	EG . C :: C . GI.
26. Therefore by Composition of Reason,	EG+C . C :: C+GI . GI.
27. That is, by exchanging equal right lines according to the Construction in 19°, .	K . C :: L . GI.
28. Therefore alternly, ...	K . L :: C . GI.
29. And by drawing C as a common altitude into the two latter Terms, ...	K . L :: □C . □C,GI
30. But from the *Constr.* in 23° and 21°, *(per prop. 17, & 16. Elem. 6.)* . . .	□KM = □C; and □L,GH = □C,GI.
31. Therefore from 29° and 30°, by exchange of equal Rectangles, ...	K . L :: □K,M . □L,GH
32. But *(per prop. 1. Elem. 6.)*	K . L :: □K,M . □L,M.
33. That from 31° and 32°, *(per prop. 11. Elem. 5.)*	□K,M . □L,GH :: □K,M . □L,M.
34. Therefore from 33°, *(per prop. 14. El. 5.)*	. . . □L, GH = □L, M.
35. Therefore from 34°, *(per prop. 14. El. 6.)*	L . M :: L . GH.
36. Therefore from 35°, *(per prop. 14. El. 5.)*	. . . . . M = GH.
37. Again, by *Constr.* in 20°, ...	K . C :: EG . GF.
38. And by *Constr.* in 23°, ...	K . C :: C . M.
39. Therefore from 37° and 38°, *(per Coroll.* Herigon. *in prop. 12. Elem. 5.)* .	2K . 2C :: EG+C . GF+M.
40. And because by *Constr.* in 19°, . . .	. . . . . K = EG+C.
41. Therefore from 39° and 40°, ...	2K . 2C :: K . GF+M.
42. But *(per prop. 15. Elem. 5.)*	2K . 2C :: K . C.
43. Therefore from 41° and 42°, *(per prop. 11. Elem. 5.)*	K . GF+M :: K . C.
44. Therefore from 43°, *(per prop. 14. El. 5.)*	GF + M = C.
45. But it hath been proved in 36°, that .	GH = M.
46. Therefore from 44° and 45, *(per Ax. 6. Chap. 2.)*	GF + GH = C.
47. But 'tis evident by the Diagram, that	GF + GH = FH.
48. Therefore from 46° and 47°, *(per Ax. 1. Chap. 2.)*	FH = C. Which was to be Dem.

Secondly, that the summ of the extremes EF and HI is equal to the given summ B, I demonstrate thus;

49. . . . *Req. demonstr.*	EF + HI = B.

Demonstration.

50. By *Constr.* in 17°, ...	EI = B + C.
51. And it hath been proved in 48°, that .	FH = C.
52. Therefore by subtracting C or FH from each part in 50°, ...	EI − FH = B.
53. But 'tis evident by the Diagram, that .	EI − FH = EF + HI.

45. There-

54. Therefore from 52° and 53°, (*per Ax.* 1. *Chap.* 2.) . . . . } EF + HI = B. Which was to be Dem.

Thirdly and lastly, that EF, FG, GH, HI are in continual proportion, I demonstrate thus;

55. . . . *Req. demonstr.* . . . . . . EF . FG :: FG . GH :: GH . HI.

Demonstration.

56. By *Constr.* in 20°, ... > . . K . C :: EG . GF.
57. And by *Constr.* in 19°, ... > EG + C = K.
58. Therefore from 56° and 57°, ... > EG + C . C :: EG . GF.
59. And because it hath been proved in 48°, that . . > FH = C.
60. Therefore from 58° and 59°, ... > EG + FH . FH :: EG . GF.
61. It is manifest by the Diagram, that of the three right lines EF, FG, GH the summ of the first and second is EG, and the summ of the second and third is FH; therefore the last preceding Analogy is qualified in every respect according to the Theorem in 45° of *Probl.* 5. *Chap.* 7. whence EF, FG, GH shall be Proportionals, *viz.*
EF . FG :: FG . GH.
62. Again, by *Constr.* in 21°, ... > . . L . C :: GI . GH.
63. And by *Constr* in 19°, ... > L = GI + C.
64. Therefore from 62° and 63°, ... > GI + C . C :: GI . GH.
65. But it hath been proved in 48°, that . . . . > FH = C.
66. Therefore from 64° and 65°, ... > GI + FH . FH :: GI . GH.
67. Therefore from 66°, in like manner as before in 61°, (*per Theor.* in 45° of *Probl.* 5. *Chap.* 7.) . } FG . GH :: GH . HI.
68. And from 61°, ... > EF . FG :: FG . GH.
69. Wherefore from 67° and 68°, (*per prop.* 11. *Elem.* 5.) . . . . } EF . FG :: FG . GH :: GH . HI.

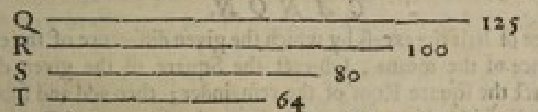
Which was to be demonstrated in the last place. Therefore the Problem is satisfied.

A LEMMA, *leading to the following* Probl. 9.

If four right lines be in continual proportion, the difference of the means is a mean Proportional between the difference of the first and second, and difference of the third and fourth Proportionals.

Suppos.

1. Q, R, S, T $\div\div$; *viz.* Q . R :: R . S :: S . T.
2. $Q \sqsubset R$. Whence $R \sqsubset S$, and $S \sqsubset T$.



3. . . . *Req. demonstr.* . . . . Q − R . R − S :: R − S . S − T.

Demonstration.

4. By *Suppos.* in 1°, ... > Q . R :: R . S.
5. Also by *Suppos.* in 2°, ... > $Q \sqsubset R$.
6. Therefore by Division of Reason, . . . > Q − R . R :: R − S . S.
7. And alternly, ... > Q − R . R − S :: R . S.
8. Again, by *Suppos.* . . . . > R . S :: S . T.
9. Therefore by Division of Reason, ... > R − S . S :: S − T . T.
10. And alternly, ... > R − S . S − T :: S . T.
11. Therefore from 8° and 10°, (*per prop.* 11. *Elem.* 5.) . . . . } R − S . S − T :: R . S.
12. Likewise from 7° and 11°, *per prop.* 11. *El.* 5. > Q − R . R − S :: R − S . S − T.

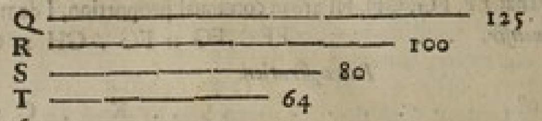
Which was to be Demonstr.

Probl. IX.

The difference of the extremes, and difference of the means of four right lines in Continual proportion being given severally, to find the

 Proportio-

Proportionals. But the given difference of the extremes muſt be greater than the triple difference of the means.



Suppoſ.

1. $Q, R, S, T \div\!\div$; *viz.* $Q . R :: R . S :: S . T$.
2. $Q \sqsubset R$, whence $R \sqsubset S$; and $S \sqsubset T$.
3. $b = Q - T$ is given.
4. $c = R - S$ is given.
5. $d = b - c$ is given.
6. $b \sqsubset 3c$. (*Determination.*)

Req. to find Q, R, S, T.

Reſolution.

7. By *Suppoſ.* in 1° and 2°, ... } $Q, R, S, T \div\!\div$; alſo $Q \sqsubset R$.
8. Therefore by the *Lemma* prefixt before this *Probl.* 9. . . . . } $Q-R . R-S :: R-S . S-T$.
9. Of which three continual Proportionals in 8° the mean $R - S$, that is, c, is given; as alſo d ($= Q + S - T - R$,) the ſumm of the extremes $Q - R$ and $S - T$; therefore by the Theorem in 21° of *Probl.* 13. *Chap.* 5. the extremes ſhall be given ſeverally, *viz.*

10. . . . . $\begin{cases} \frac{1}{2}d + \sqrt{:\overline{\frac{1}{4}dd - cc}:} = Q - R. \\ \frac{1}{2}d - \sqrt{:\overline{\frac{1}{4}dd - cc}:} = S - T. \end{cases}$

11. Therefore from 4°, 5° and 10° 'tis manifeſt, that of theſe three Proportionals Q, R, S, the difference of the firſt and ſecond, to wit, $Q - R$ is given, alſo $R - S$ the difference of the ſecond and third is given. Likewiſe of theſe three Proportionals R, S, T, the difference of the firſt and ſecond, to wit, $R - S$ is given, alſo $S - T$ the difference of the ſecond and third is given; therefore, (according to the Canon in 13. of *Probl.* 6. *Chap.* 7.) R and S ſhall be given ſeverally by theſe following Analogies, *viz.*

12. . . . $\begin{cases} \overline{\frac{1}{2}d + \sqrt{:\overline{\frac{1}{4}dd - cc}:} - c} \;.\; c :: \frac{1}{2}d + \sqrt{:\overline{\frac{1}{4}dd - cc}:} \;.\; R. \\ \overline{c - \frac{1}{2}d - \sqrt{:\overline{\frac{1}{4}dd - cc}:}} \;.\; c :: \frac{1}{2}d - \sqrt{:\overline{\frac{1}{4}dd - cc}:} \;.\; S. \end{cases}$

Which Analogies, reſpect being had to the Equations in 10°, do afford this

CANON.

13. From the Square of half the exceſs by which the given difference of the extremes exceeds the given difference of the means, ſubtract the Square of the given difference of the means, and extract the ſquare Root of the remainder; then add and ſubtract that ſquare Root to and from the ſaid half-exceſs, and reſerve the Summ and Remainder. Then it ſhall be, as the exceſs of the Summ reſerved above the given difference of the means, is to the difference of the means; ſo is the Summ reſerved to the greater mean ſought. Or, as the exceſs of the difference of the means above the Remainder reſerved, is to the difference of the means; ſo is the Remainder reſerved to the leſſer mean ſought. Laſtly, if the Summ reſerved be added to the greater mean it gives the greater extreme, and if the Remainder reſerved be ſubtracted from the leſſer mean it gives the leſſer extreme.

But to the end there may be a poſſibility of effecting the Problem propounded, the given lines muſt be liable to this

Determination.

14. The given difference of the extremes muſt be greater than the triple of the given difference of the means.

The truth of this Determination will be made manifeſt by the following Theorem, and that 'tis neceſſary, will be evident in 28° of the following Conſtruction of the Problem.

THEOREM.

15. If four right lines be in continual proportion, the difference of the extremes is greater than the triple difference of the means.

Suppoſ.

Suppos.

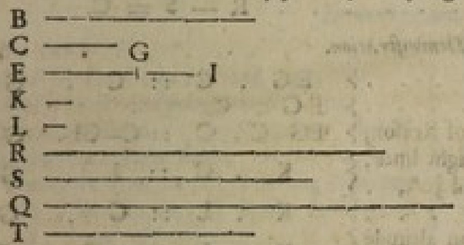
16. Q, R, S, T $\div\!\div$; *viz.* Q . R :: R . S :: S . T.
17. Q $\sqsubset$ R. Whence R $\sqsubset$ S, and S $\sqsubset$ T.
18. . . . *Req. demonstr.* . . . . Q $-$ T $\sqsubset$ 3R $-$ 3S.

Demonstration.

19. By *Suppos.* in 16° and 17°, ... } Q, R, S, T $\div\!\div$; also, Q $\sqsubset$ R.
20. Therefore by the *Lemma* prefixt before this *Probl.* 9. . . . . } Q$-$R . R$-$S :: R$-$S . S$-$T.
21. But if four quantities be Proportionals, the summ of the extremes is greater than the summ of the means, (*per prop.* 25. *Elem.* 5.) therefore from 20°, ... } Q + S $-$ R $-$ T $\sqsubset$ 2R $-$ 2S.
22. And by adding R to each part in 21°, . . } Q + S $-$ T $\sqsubset$ 3R $-$ 2S.
23. Wherefore by subtracting S from each part in 22°, ... } Q $-$ T $\sqsubset$ 3R $-$ 3S.

Which was to be Demonstr.

The Composition of the foregoing Probl. 9.



B = 61	L = 4
C = 20	R = 100
EI = 41	S = 80
EG = 25	Q = 125
GI = 16	T = 64
K = 5	

Suppos.

24. B = the difference of the extremes of four right lines in continual proportion is given.
25. C = the difference of the means is given.
26. B $\sqsubset$ 3 C. (*Determination*)

Req. to find the Proportionals severally.

Construction.

27. Make E I = B $-$ C.
28. Divide E I into two such parts in G that C may be a mean Proportional between the parts, which may be done, (*per Probl.* 14. *Chap.* 5.) if C be not greater than $\frac{1}{2}$ E I. But that C is less than $\frac{1}{2}$ E I, I prove thus;

By the Determination in 26°, ... 3 C $\sqsupset$ B.
Therefore by subtracting C from each part, . . . 2 C $\sqsupset$ B $-$ C.
But by *Constr.* in 27°, ... E I = B $-$ C.
Therefore, (*per Ax.* 3. *Chap.* 2.) . . . . 2 C $\sqsupset$ E I.
And consequently, ... C $\sqsupset$ $\frac{1}{2}$ E I.

Which was to be shewn. Therefore 'tis possible (*per Probl.* 14. *Chap.* 5.) to cut E I into two such parts that C may be a mean Proportional between them. Suppose then that E I is so cut in G, and that E G is the greater part, and G I the lesser; therefore,

E G . C :: C . G I.

29. Make K = EG $-$ C, which is possible to be done, for by *Constr.* in 28°, EG $\sqsubset$ C.
30. Make L = C $-$ GI, which is possible to be done, for by *Constr.* in 28°, C $\sqsubset$ GI.
31. By *Probl.* 8. *Chap.* 5. let it be made as K to C, so E G to a fourth Proportional, suppose it be found R, therefore

K . C :: EG . R.

32. Again, let it be made, as L to C, so G I to a fourth Proportional S, therefore

L . C :: GI . S.

33. Make Q = EG + R, whence Q $-$ R = EG.
34. Make T = S $-$ GI, whence S $-$ T = GI; but that G I is less than S, as is implied by this Effection, I prove thus;

By *Constr.* in 30°, ... $L = C - GI$.
Therefore by adding GI to each part, ... $L + GI = C$.
Therefore, ... $L \sqsupset C$.
Therefore from the Analogy in 32°, (*per Schol. prop.* 14. *Elem.* 5.) $GI \sqsupset S$.
Which was to be proved.

35. I say Q, R, S, T (found out by *Constr.* in 33°, 31°, 32° and 34°,) are the four continual Proportionals sought. But to make it manifest that they will satisfie the Problem, I shall prove three things, *viz.* First, that the difference of the means R and S is equal to the given difference C: Secondly, that the difference of the extremes Q and T is equal to the given difference B: Thirdly and lastly, that the said Q, R, S, T are in continual proportion in this order, *viz.* $Q . R :: R . S :: S . T$.

First then, that the difference of the means R and S is equal to the given difference C, I demonstrate thus:

Prepar.

36. By *Probl.* 7. *Chap.* 5. let it be made as K to C, so C to a third proportional line M, therefore . . . . $K . C :: C . M$.

Req. demonstr. . . . . $R - S = C$.

Demonstration.

37. By *Constr.* in 28°, ... $EG . C :: C . GI$.
38. And by *Constr.* in 28°, ... $EG \sqsubset C$.
39. Therefore from 37°, by Division of Reason, $EG-C . C :: C-GI . GI$.
40. That is, by exchanging equal right lines, according to the *Constr.* in 29° and 30°, . $K . C :: L . GI$.
41. Therefore alternly, ... $K . L :: C . GI$.
42. And by drawing C as a common altitude into each of the two latter Terms in 41°, . $K . L :: \square C . ▭C,GI$.
43. But from the *Constr.* in 36° and 32°, . . ▭K,M = □C; and ▭L,S = ▭C,GI.
44. Therefore from 42° and 43°, by exchange of equal Rectangles, ... $K . L ::$ ▭K,M . ▭L,S.
45. But by reason of the common altitude M, this Analogy is manifest, ... $K . L ::$ ▭K,M . ▭L,M.
46. Therefore from 44° and 45°, (*per prop.* 11. *Elem.* 5.) . . . . ▭K,M . ▭L,S :: ▭K,M . ▭L,M.
47. Therefore from 46°, (*per prop.* 14. *Elem.* 5.) ▭L,S = ▭L,M.
48. And from 47°, (*per prop.* 14. *Elem.* 6.) . $L . M :: L . S$.
49. Therefore from 48°, (*per prop.* 14. *Elem.* 5.) $M = S$.
50. Again, by *Constr.* in 31°, ... $K . C :: EG . R$.
51. And by *Constr.* in 36°, ... $K . C :: C . M$.
52. Therefore from 50° and 51°, (*per prop.* 11. *Elem.* 5.) . . . . $EG . R :: C . M$.
53. Therefore alternly, ... $EG . C :: R . M$.
54. Therefore from 38° and 53°, by Division of Reason, ... $EG-C . C :: R-M . M$.
55. And because by *Constr.* in 29°, ... $K = EG - C$.
56. Therefore from 54° and 55°, ... $K . C :: R-M . M$.
57. But by *Constr.* in 36°, ... $K . C :: C . M$.
58. Therefore from 56° and 57°, (*per prop.* 11. *Elem* 5.) . . . . $R-M . M :: C . M$.
59. Therefore from 58°, (*per prop.* 14. *Elem.* 5.) $R - M = C$.
60. But it hath been proved in 49°, that . $M = S$.
61. Therefore from 59° and 60°, (*per Ax.* 6. *Chap.* 2.) . . . . $R - S = C$. Which was to be Dem.

Secondly, that the difference of the extremes Q and T is equal to the given difference B, I demonstrate thus:

62. . . . *Req. demonstr.* . . . . $Q - T = B$.

Demonstr.

Demonftration,

63. By *Conftr.* in 33°, ...	$Q = EG + R.$
64. And by *Conftr.* in 34°, ...	$T = S - GI.$
65. And from 61°, ...	$R \sqsubset S.$
66. Therefore from 63°, 64° and 65°, . . .	$Q \sqsubset T$; and $EG + R \sqsubset S - GI.$
67. And by fubtracting the Equation in 64° from the Equation in 63°, ...	$Q - T = EG + R + GI - S.$
68. But by *Conftr.* in 28°, ...	$EI = EG + GI.$
69. Therefore from 67° and 68°, (*per Ax.* 6. *Chap.* 2.)	$Q - T = EI + R - S.$
70. Again, by *Conftr.* in 27°, ...	$B - C = EI.$
71. And it hath been proved in 61°, that . .	$C = R - S.$
72. Therefor the fumm of the Equations in 70° and 71° gives	$B = EI + R - S.$
73. Therefore from 69° and 72°, (*per Ax.* 1. *Chap.* 2.)	$Q - T = B.$ Which was to be Dem.

Thirdly and laftly, that Q, R, S, T are in continual proportion I demonftrate thus:

74. . . . *Req. demonftr.* . . . . $Q . R :: R . S :: S . T.$

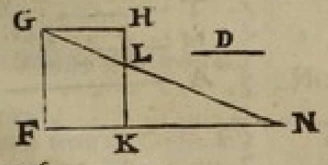
Demonftration.

75. By *Conftr.* in 31°, ...	$K . C :: EG . R.$
76. And by *Conftr.* in 29° and 33°, . . .	$EG - C = K$; and $Q - R = EG.$
77. And it hath been proved in 61°, that . .	$R - S = C.$
78. Therefore from 75°, 76° and 77°, by exchanging equal right lines, ...	$Q - R - \overline{R - S} . R - S :: Q - R . R.$
79. Therefore from the laft preceding Analogy, by the *Theor.* in 14° of *Probl.* 6. *Chap.* 7. Q, R, S are ∺, *viz.*	$Q . R :: R . S.$
80. Again, by *Conftr.* in 32°, ...	$L . GI :: C . S.$
81. And by *Conftr.* in 30° and 34°, . . .	$C - GI = L$; and $S - T = GI.$
82. And it hath been proved in 61°, that . .	$R - S = C.$
83. Therefore from 80°, 81° and 82°, by exchanging equal right lines, ...	$R - S - \overline{S - T} . S - T :: R - S . S.$
84. Therefore from the laft preceding Analogy by the *Theor.* in 14° of *Probl.* 6. *Chap.* 7. R, S, T are ∺, *viz.*	$R . S :: S . T.$
85. Wherefore from 79° and 84°, (*per prop.* 11. *Elem.* 5.)	$Q . R :: R . S :: S . T.$

Which was to be demonftrated in the laft place. Therefore the Problem is fatisfied.

Probl. X.

A Rectangle FGHK being given by Pofition, to draw a right line GN from G one of the angles oppofite to the Bafe FK, to cut the Bafe produced, fuppofe in N, fo as to make the Triangle KLN (lying without the Rectangle) equal to a given Space; fuppofe the Square of the right line D.



FK = 12	KL = 10
FG = 15	GN = 39
□D = 120	LN = 26
KN = 24	GL = 13

Suppof.

1. FGHK is a ▭ given.
2. b = FK or GH is given.
3. c = FG or KH is given.
4. d = D the fide of a Square given.

Req.

Req. to find

5. KN a right line to be added to FK, so, as FKN may be a strait line, and that GN being drawn it may make $\triangle KLN = \square D$.

Resolution.

6. Put a for the desired increase of the Base FK, *viz.* $a = KN$.
7. Then because $\triangle NKL$ and $\triangle NFG$ are equiangular, these sides are Proportionals, (*per prop.* 4. *Elem.* 6.) *viz.* $FN \,.\, FG :: KN \,.\, KL$.
8. That is, in the letters belonging to the Resolution, $a+b \,.\, c :: a \,.\, \frac{ca}{a+b}$
9. And because (*per prop.* 41. *Elem.* 1.) $\square KN, KL = 2\triangle KLN$.
10. Therefore from 8°, 9°, 4°, 5° and 6°, $a \times \frac{ca}{a+b} = 2dd = 2\triangle KLN$.
11. Now to avoid an Equation between Solids, let it be made as c to d, so $2d$ to a fourth Proportional, call it t, therefore $c \,.\, d :: 2d \,.\, t$.
12. Whence, (*per prop.* 16. *Elem.* 6.) $ct = 2dd$.
13. Therefore from 10° and 12°, (*per Ax.* 1. *Ch.* 2) $a \times \frac{ca}{a+b} = ct$.
14. Which Equation may be resolved into these Proportionals, *viz.* $a \,.\, t :: c \,.\, \frac{ca}{a+b}$
15. And by drawing $a+b$ as a common Factor into each of the two latter Terms of the last Analogy, this ariseth, $a \,.\, t :: ca+cb \,.\, ca$.
16. And by casting away the common Factor c, this Analogy ariseth, $a \,.\, t :: a+b \,.\, a$.
17. And from the last Analogy, by Division of Reason, $a-t \,.\, t :: b \,.\, a$.
18. Therefore, by comparing the Rectangle of the extremes to the Rectangle of the means, $aa - ta = tb$.
19. Which Equation may be resolved into this Analogy, $a-t \,.\, \sqrt{tb} :: \sqrt{tb} \,.\, a$.

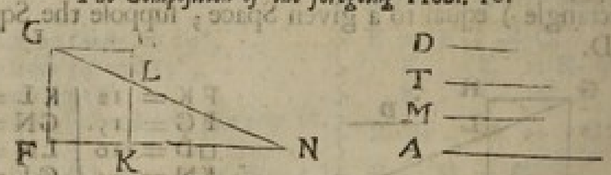
But the last Analogy doth manifestly consist of three Proportionals, whereof the mean, to wit, $\sqrt{tb}$ is given, as also t the difference of the extremes a and $a-t$; therefore the extremes severally, the greater whereof is the desired increase KN, shall be given also, *per Probl.* 12. *Chap.* 5. and the Theorem in 24° of the same *Probl.* gives this following

CANON.

20. $\frac{1}{2}t + \sqrt{:\frac{1}{4}tt + tb:} = KN$. That is, in words,

Let it be made as FG the altitude of the given Rectangle, to D the side of the given Square; so 2D the double of the same side, to a fourth Proportional, which may be called T. Then to the Square of half that fourth Proportional T, add the Rectangle of T into FK the Base of the given Rectangle, and extract the square Root of the summ. Lastly, that square Root added to half T, gives KN the desired increase of the Base.

The Composition of the foregoing Probl. 10.



Suppos.

21. FGHK is a $\square$ given
22. D is the side of a Square given.

Req. to find

23. KN a right line to be added to FK, so, as FKN may be a strait line, and that GN being drawn, it may make $\triangle KLN = \square D$.

Constr.

Construction.

24. By *Probl.* 8. *Chap.* 5. let it be made as FG to D, so 2D to a fourth proportional line, suppose it be found T; therefore, ... $FG \;.\; D :: 2D \;.\; T.$

25. By *Probl.* 9. *Chap.* 5. find a mean Proportional, as M, between T and FK, therefore . . . $T \;.\; M :: M \;.\; FK.$

26. Making M to be the mean of three Proportionals, and T the difference of the extremes, find the extremes severally, (*per Probl.* 12. *Chap.* 5.) the greater whereof suppose to be A, then the lesser shall be A — T, therefore . . . . $A - T \;.\; M :: M \;.\; A.$

That is, in 19° the last step of the Resolution, . . $a - t \;.\; \sqrt{tb} :: \sqrt{tb} \;.\; a.$

27. Produce FK to such a point N, that KN may be equal to the right line A found out in 26°, and draw GN; so shall $\triangle$ KLN be equal to the Square of the given line D, as was required. But that $\triangle KLN = \square D$, the following Demonstration, form'd out of the preceding Resolution by a repetition of the steps thereof in a backward (not direct) order will make manifest.

28. . . . *Req. demonstr.* . . . . $\triangle KLN = \square D.$

Demonstration.

29. By *Constr.* in 26° and 27°, . . $KN - T \;.\; M :: M \;.\; KN.$

That is, in 19° the last step of the Resolution, ... $a - t \;.\; \sqrt{tb} :: \sqrt{tb} \;.\; a.$

30. Therefore from 29°, (*per prop.* 17. *Elem.* 6.) . . . . $\square KN - \square T{,}KN = \square M.$

31. Likewise from the *Constr.* in 25°, ... $\square T{,}FK = \square M.$

32. Therefore from 30° and 31°, (*per Ax.* 1. *Chap.* 2.) . . . . . $\square KN - \square T{,}KN = \square T{,}FK.$

That is, in 18°, ... $aa - ta = tb.$

33. Therefore from 32°, (*per prop.* 14. *Elem.* 6.) . . . . $KN - T \;.\; T :: FK \;.\; KN.$

That is, in 17°, ... $a - t \;.\; t :: b \;.\; a.$

34. And from 33°, by Composition of Reason, ... $KN \;.\; T :: KN + FK \;.\; KN.$

That is, in 16°, ... $a \;.\; t :: a + b \;.\; a.$

35. And from 34°, by taking in the the common altitude FG, . . $KN \;.\; T :: \square FG{,}KN + \square FG{,}FK \;.\; \square FG{,}KN.$

That is, in 15°, ... $a \;.\; t :: ca + cb \;.\; ca.$

36. And because $\triangle$NKL and $\triangle$NFG are equiangular, therefore (*per prop.* 4. *Elem.* 6.) . . . . $FN \;.\; FG :: KN \;.\; KL.$

37. And consequently, (*per prop.* 16. *Elem.* 6.) . . . . $\square FN{,}KL = \square FG{,}KN.$

38. Therefore from 35° and 37°, by exchanging equal Rectangles, . $KN \;.\; T :: \square FG{,}KN + \square FG{,}FK \;.\; \square FN{,}KL.$

39. And from 38°, by rejecting the the common altitude KN + FK, that is, FN, ... $KN \;.\; T :: FG \;.\; KL.$

That is, in 14°, ... $a \;.\; t :: c \;.\; \frac{ca}{a+b}.$

40. Therefore from 39°, (*per prop.* 16. *Elem.* 6.) . . . . . . $\square KN{,}KL = \square FG{,}T.$

That is, in 13°, ... $a \times \frac{ca}{a+b} = ct.$

41. And because from the *Constr.* in 24°, (*per prop.* 16. *Elem.* 6.) . . . $2\square D = \square FG{,}T.$

42. Therefore from 40° and 41°, (*per Ax.* 1. *Chap.* 2.) . . . $\square KN{,}KL = 2\square D.$

That is, in 10°, ... $a \times \frac{ca}{a+b} = 2dd.$

43. But (*per prop.* 41. *Elem.* 1.) . $\square KN{,}KL = 2\triangle KLN.$

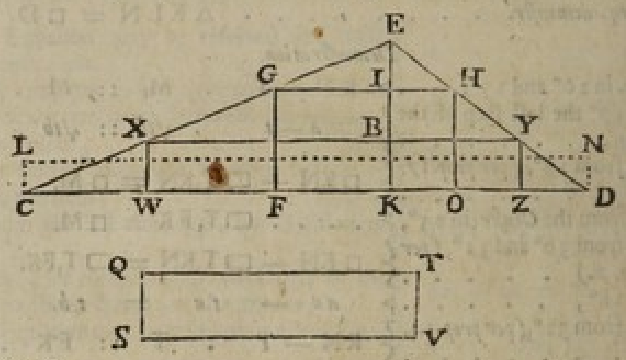
44. There-

44. Therefore from 42° and 43°, (*per Ax.* 1. *Chap.* 2.) . ⊱ 2 △ KLN = 2 □ D.
45. Therefore from 44°, (*per Ax.* 21. *Chap.* 2.) . . ⊱ △ KLN = □ D.
Which was to be demonstr. Therefore that is done which the Problem required.

Probl. XI.

In a given Triangle to inscribe a Rectangle equal to a given Rectangle. But the right line arising by the Application of the given Rectangle to the Base of the given Triangle, must not exceed a quarter of the Perpendicular falling upon that Base; and consequently the double of the Rectangle must not be greater than the Triangle.

Note. By the Base of the Triangle given in this Problem, is meant such a side as hath not an obtuse angle at either of its ends within the Triangle; for 'tis easie to conceive, that if the Triangle be obtusangled at the Base, a Rectangle cannot be inscribed within the Triangle, so, as that the Base of a Rectangle may be a segment of the Base of the Triangle, and all the angular points of the Rectangle lye in the sides of the Triangle.



Suppos.

1. CDE is a △ given.
2. b = CD the Base is given, and neither $\angle$ C nor $\angle$ D is obtuse.
3. p = EK the Perpendicular is given.
4. □ ST, and the sides thereof, to wit, SV and SQ are given.
5. $r = \dfrac{\square \mathrm{ST}}{\mathrm{CD}}$ = CL or DN is given; whence, □ CN = □ ST.

Req. to inscribe

6. FGHO a □ within the △ CDE, with condition that
7. □ FGHO may be equal to □ ST.

Resolution.

8. Put a for the altitude of the Rectangle required, *viz.* . . . . a = FG = KI = OH.
9. Which altitude subtracted from the Perpendicular EK, leaves EI, therefore from 3° and 8°, $p - a$ = EI.
10. It is manifest by the *Lemma* prefixt before *Probl.* 11. *Chap.* 7. that . . . . . . EK . EI :: CD . GH.
11. Therefore (from 3°, 9° and 2°,) in the letters belonging to the Resolution, ... $p \,.\, p-a :: b \,.\, \dfrac{bp-ba}{p}$.
12. And because according to the import of the Problem, ... □ FG,GH = □ ST, or □ CD,CL.
13. Therefore (from 8°, 11°, 2° and 5°,) in the letters of the Resolution, ... $a \times \dfrac{bp-ba}{p} = br$.
14. Which last Equation is reducible to this Analogy, (*per prop.* 14. *Elem.* 6.) . . . . $a \,.\, r :: b \,.\, \dfrac{bp-ba}{p}$.
15. Therefore from 11° and 14°, (*per prop.* 11. *Elem.* 5.) . . . . $p \,.\, p-a :: a \,.\, r$.

16. Which

16. Which last Analogy gives this Equation, (*per prop.* 16. *Elem.* 6.) . . . . } $pa - aa = pr$.

17. And that Equation may be resolved into these Proportionals, (*per prop.* 14. *Elem.* 6.) . . . . . . } $p - a \,.\, \sqrt{pr} :: \sqrt{pr} \,.\, a$.

Of which three Proportionals the mean, to wit, $\sqrt{pr}$ is given, as also p the summ of the extremes, therefore the extremes severally, (to wit, KI and EI, or EB and KB, the segments of the Perpendicular EK,) shall be given also, (by *Probl.* 13. *Chap.* 5.) and the Theorem in 21° of the same *Probl* 13. gives this following

CANON.

18. . . . . . $\left\{ \begin{array}{l} \frac{1}{2}p + \sqrt{:\overline{\tfrac{1}{4}pp - pr}:} = \mathrm{KI}\ (= \mathrm{EB},) \\ \frac{1}{2}p - \sqrt{:\overline{\tfrac{1}{4}pp - pr}:} = \mathrm{KB}\ (= \mathrm{EI},) \end{array} \right\}$ *viz.* in words,

First, let it be made as the Base of the given Triangle, to one of the sides of the given Rectangle; so the other side of the same Rectangle, to a fourth Proportional, which may be called r. Secondly, from the Square of half the Perpendicular falling upon the said Base, subtract the Rectangle made of the said Perpendicular and fourth Proportional r. Thirdly, extract the square Root of the remainder. Fourthly, add and subtract the said square Root to and from the said half Perpendicular; the summ and remainder shall be segments of the Perpendicular, either of which may be taken for the altitude of the Rectangle required to be inscribed. Lastly, the Base of the Rectangle sought is equal to the right line arising by the Application of the given Rectangle to the altitude before found.

This Canon may be propounded in the form of a Theorem, which may easily be demonstrated by a repetition of the steps of the preceding Resolution in a direct (not retrograde) order; but taking the truth of the Canon for granted, I shall proceed to the Composition of the Problem, for the effecting whereof, 'tis necessary that the given quantities be qualified according to the tenour of this

Determination.

19. The right line arising by the Application of the given Rectangle, to the Base of the given Triangle, must not exceed a quarter of the Perpendicular falling upon that Base; and consequently, the double of the Rectangle must not be greater than the Triangle.

Which Determination shews it self openly in the Canon, where it appears, that to the end there may be a possibility of subtracting pr from $\frac{1}{4}pp$, 'tis necessary that . . . } pr not $\sqsubset$ $\frac{1}{4}pp$.

And consequently, by dividing each part by p, ... } r not $\sqsubset$ $\frac{1}{4}p$.

And by doubling each part, ... } $2r$ not $\sqsubset$ $\frac{1}{2}p$.

And by drawing b into each part, ... } $2rb$ not $\sqsubset$ $\frac{1}{2}pb$.

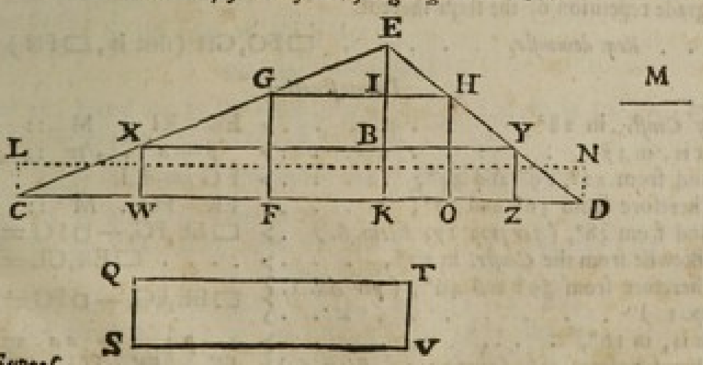
But from 2° and 5°, ... } $2rb = 2\square\mathrm{ST}$.

And by *prop.* 41. *Elem.* 1. . . . . } $\frac{1}{2}pb = \triangle\mathrm{CDE}$.

Therefore from the three last preceding steps, by exchanging equal quantities, ... } $2\square\mathrm{ST}$ not $\sqsubset$ $\triangle\mathrm{CDE}$.

Therefore from the premisses, both the rise and truth of the Determination are manifest.

The Composition of the foregoing Probl. 11.



Suppos.

20. CDE is a △ given.

21. CD the Base is given; and neither ∠C nor ∠D is obtuse.

22. EK the Perpendicular is given.

23. □ST, and the sides thereof, to wit, SV and SQ are given.

24. $CL = DN = \frac{\square ST}{CD}$ is given, (*per Probl.* 8. *Chap* 5.)

25. CL not $\sqsubset \frac{1}{4}$ EK. (*Determination.*)

Req. to inscribe

26. □FGHO in △CDE, so, that □FGHO = □ST.

Construction.

27. By *Probl.* 9. *Chap.* 5. find a mean Proportional, as M, between EK and CL, therefore,

EK . M :: M . CL.

28. By *Probl.* 14. *Chap.* 5. cut EK into two such parts in I, that M (before found) may be a mean Proportional between the parts, which effection is possible if M be not greater than $\frac{1}{2}$ EK; but that M is greater than $\frac{1}{2}$ EK, I prove thus;

By the Determination in 25°, ... } CL not $\sqsubset \frac{1}{4}$EK.

Therefore by drawing EK into each part, ... } □EK,CL not $\sqsubset \frac{1}{4}$□EK.

But from the *Constr.* in 27°, (*per prop.* 17. *Elem.* 6.) . . } □EK,CL = □M.

Therefore from the two last preceding steps, (*per Ax.* 4. *Chap.* 2.) . . . . } □M not $\sqsubset \frac{1}{4}$□EK.

Therefore by extracting the square Root out of each part, . } M not $\sqsubset \frac{1}{2}$EK.

Which was to be proved. Therefore 'tis possible to cut EK into two such parts, that M shall be a mean Proportional between them; suppose then it be done, (*per Probl.* 14. *Chap.* 5.) and that the parts are EI and KI, therefore EI + KI = EK. Also,

EI (or EK — KI) . M :: M . KI.

That is, in 17°, ... *p* — *a* . $\sqrt{pr}$ :: $\sqrt{pr}$. *a*.

29. Then set either of the said parts of EK, suppose the greater part, from K to I; and by the point I, draw GIH parallel to CD. Lastly, from the points G and H let fall GF and HO Perpendiculars to the Base CD; so shall FGHO be the incribed Rectangle required. But to make it manifest that the said Rectangle will satisfie the Problem, two things are to be proved, *viz.* First, that all the angles of the quadrilateral Figure FGHO are right angles; and then that the said Rectangle is equal to the given Rectangle SQTV.

30. . . . *Req. demonstr.* . . . . FGHO is a □.

Demonstration.

31. By *Constr.* in 29°, ... } GH ∥ FO.

32. Also by *Constr* in 29°, ... } GF and HO are ⊥ FO.

33. Therefore, (*per defin.* 10. *Elem.* 1.) . . . . . . } $\angle GFO = \lrcorner = \angle HOF$.

34. Therefore from 31°, 32°, 33°, (*per prop.* 29. *Elem.* 1.) } FGHO is □.

Which was to be Demonstr.

It remains to prove that □FG,GH (that is, □FH) = □ST; but that equality will be manifest by the following Demonstration, form'd out of the preceding Resolution by a retrograde repetition of the steps thereof.

35. . . . *Req. demonstr.* . . . . . . □FG,GH (that is, □FH) = □ST.

Demonstration.

36. By *Constr.* in 28°, ... } EK − KI . M :: M . KI.

That is, in 17°, ... } *p* — *a* . $\sqrt{pr}$ :: $\sqrt{pr}$. *a*.

37. And from 22°, 29° and 34°, ... } FG = KI.

38. Therefore from 36° and 37°, ... } EK − FG . M :: M . FG.

39. And from 38°, (*per prop.* 17. *Elem.* 6.) . } □EK,FG, — □FG = □M.

40. Likewise from the *Constr.* in 27°, . . . } . . . . □EK,CL = □M.

41. Therefore from 39° and 40°, (*per Ax.* 1. *Chap.* 2.) . . . . } □EK,FG, — □FG = □EK,CL.

That is, in 16°, ... } . *pa* — *aa* = *pr*.

42. Therefore from 41°, (*per prop.* 14. *Elem.* 6.) } EK . EK − FG :: FG . CL.

That is, in 15°, ... } *p* . *p* — *a* :: *a* . *r*.

43. By *Constr.* in 29°, ... } GH ∥ CD.

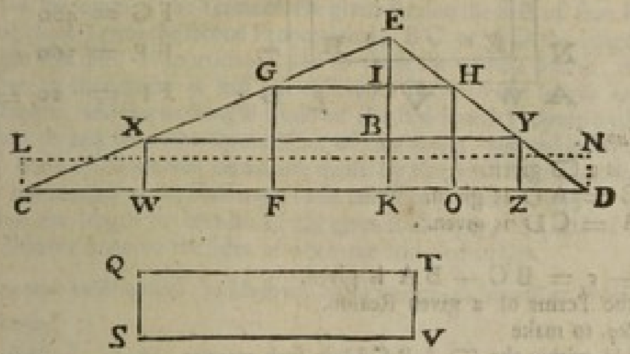
44. There-

44. Therefore by the *Lemma* prefixt before *Probl.* 11. *Chap.* 7. . . . . } EK . EK−FG :: CD . GH.
EI
That is, in 11°, ... } $p \;.\; p-a :: b \;.\; \frac{bp-ba}{p}$.
45. Therefore from 42° and 44°, (*per prop.* 11. *Elem.* 5.) . . . . } FG . CL :: CD . GH.
46. And from 45°, (*per prop.* 16. *Elem.* 6.) . } □ FG, GH = □ CD, CL.
47. But from the *Constr.* in 24°, ... } . . . □ ST = □ CD, CL.
48. Therefore from 46° and 47°, (*per Ax.* 1. *Chap.* 2.) . . . . } □ FG, GH (that is, □ FH) = □ ST.

Which was to be Demonstr. Therefore that is done which the Problem required.

49. *Note.* If KB be made equal to EI, then shall EB be equal to IK, (by reason of the common intersegment IB,) and consequently EK is cut in B as well as in I, according to the import of the preceding Construction in 28°. Therefore if by the point B a parallel be drawn to the Base CD, as XBY, and from the points X and Y, perpendiculars be let fall upon CD, as XW and YZ, the inscribed □ WY, that is, WXYZ shall be also equal to the given Rectangle ST, that is, SQTV, and the Demonstration may be formed as before, by taking KB or WX instead of KI. So two Rectangles are inscribed in the given △ CDE, each of which is equal to the given Rectangle SQTV.

Examples in Numbers to illustrate the preceding Resolution of Probl. 11.



Suppos.

50. CD = 168 the Base
51. CE = 117 } the leggs } of △ CDE are given severally.
52. DE = 75
53. SV = 84 } the sides of □ ST, therefore □ ST = 1680.
54. SQ = 20
55. □ FH = □ ST = 1680.
56. EK ⊥ CD.

Req. to find in Numbers,

57. FG or HO, } the sides of □ FH.
58. GH or FO,

Solution Arithmetical.

59. EK = 45, found out by the three sides of △ CDE given in 50°, 51°, 52°, by the help of *Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.
60. KI = 30 } found out by the Canon in 18° of this Problem.
61. IE = 15
62. FG = 30 = KI, found out in 60°.
63. GH = 56 = FO, given from 55° and 62°. For $\frac{1680}{30} = 56$.

The Proof.

64. □ FG, GH = 1680 = □ SV, SQ, (= □ ST.) Also,
65. . . { 15 (or 45 − 30) . 56 :: 45 . 168; that is,
EI (or EK − KI) . GH :: EK . CD.
66. Therefore by the converse of the *Lemma* prefixt before *Probl.* 11. *Chap.* 7. GH ∥ CD. Also G and H are in CE and DE.

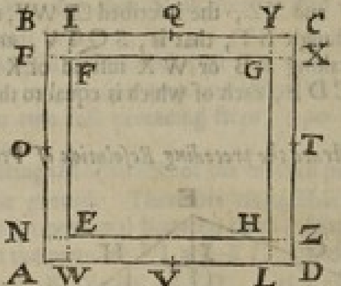
Another

Another Example.

67. Again, the same $\triangle CDE$ and its sides being given in numbers as before in 50°, 51° and 52°, you will find (by the like Operation as in *Example* 1.) $XW = 15 = YZ$, and $XY = 112 = WZ$; whence the Area of $\square XZ$ is 1680, which is the same with the Area of $\square ST$ prescribed in *Example* 1. And that the Rectangle XZ or WXYZ is inscribed in $\triangle CDE$, may be proved in like manner as before in 65° and 66°.

Probl. XII.

Within a given Rectangle to make a Rectangle, with this condition, that there may be an equal parallel distance between their sides; and that the Space lying between the sides of both the Rectangles may be to the inscribed Rectangle in a given Reason.



$BC = 440$
$BA = 400$
$R = 2$
$S = 9$
$FG = 400$
$FE = 360$
$FI = 20 = GX$

Suppos.

1. $ABCD$ is a $\square$ given.
2. $b = BC = AD$ is given.
3. $c = BA = CD$ is given.
4. $b \sqsubset c$.
5. $d = b - c = BC - BA$ is given.
6. r and s the Terms of a given Reason.

Req. to make

7. $\square EFGH$ within the $\square ABCD$ in such manner, that
8. $FI = GX = HL = EN$. Also, that
9. $\square BC,BA - \square FG,FE \,.\, \square FG,FE :: r \,.\, s$.

Prepar.

10. By viewing the Diagram, and reflecting upon what is given and required, it will be evident that } $BC = FG + 2GX\ (2FI.)$
11. Likewise, ... } $BA = FE + 2GX\ (2FI.)$
12. And by subtracting the Equation in 11° from that in 10°, this remains, *viz.* . . . . . } $BC - BA = FG - FE\ (= d.)$
13. Whence 'tis manifest that the difference between the length and breadth of the Rectangle required to be inscribed is given; for 'tis equal to the difference between the length and breadth of the given $\square ABCD$.

Resolution.

14. Put a for the shorter side of the required Rectangle EFGH, *viz.* . . . . } $a = EF = HG$.
15. Therefore from 5°, 13° and 14°, the longer side shall be . . . . } $a + d\ (= FG.)$
16. Therefore from 14° and 15°, the Area of $\square EFGH$ is equal to . . . . } $aa + da\ (= \square EFGH.)$
17. And from 2° and 3° the Area of the given Rectangle is . . . . } $bc\ (= \square ABCD.)$
18. And by subtracting the Area in 16° from that in 17°, there will remain . . . . . . } $bc - aa - da\ (= BFGCHDEA.)$
19. Therefore from 9°, 18° and 16°, according to the tenour of the Problem, this Analogy ariseth, *viz.* } $r \,.\, s :: bc - aa - da \,.\, aa + da$.

20. There-

20. Therefore from 19°, by Composition of Reason, $r + s \;.\; s :: bc \;.\; aa + da$.

21. Now to avoid an Equation between Solids, let it be made as $r + s$ to s, so b to a fourth Proportional, call it f, therefore . . . . . $r + s \;.\; s :: b \;.\; f$.

22. Therefore from 20° and 21°, (*per prop.* 11. *El.* 5.) $b \;.\; f :: bc \;.\; aa + da$.

23. And this Analogy, by reason of the common Factor c is evident, (*per prop.* 1. *Elem.* 6.) *viz.* $b \;.\; f :: bc \;.\; fc$.

24. Therefore from 22° and 23°, (*per prop.* 11. *Elem.* 5.) . . . . $bc \;.\; aa + da :: bc \;.\; fc$.

25. Therefore from 24°, (*per prop.* 14. *Elem.* 5.) $aa + da = fc$.

26. Which Equation may be resolved into these Proportionals, *viz.* . . . . $a \;.\; \sqrt{fc} :: \sqrt{fc} \;.\; a + d$.

Of which three Proportionals the mean, to wit, $\sqrt{fc}$ is given, as also d the difference of the extremes $a + d$ and a, therefore *per Probl.* 12. *Chap.* 5. the extremes shall be given severally, (which are the sides of the Rectangle required to be inscribed;) and the Theorem in 24° of the said *Probl.* 12. gives this following

CANON.

27 . . . . $\left\{ \begin{array}{l} \sqrt{:\tfrac{1}{4}dd + fc:} - \tfrac{1}{2}d = EF. \\ \sqrt{:\tfrac{1}{4}dd + fc:} + \tfrac{1}{2}d = FG. \end{array} \right\}$ *viz.* in words,

Make $r + s$ the summ of the Terms of the given Reason the first of four Proportionals, s the latter of those Terms the second Proportional, BC or AD the longer side of the given Rectangle the third Proportional, and to those three find a fourth, which may be called f. Then to the Square of half the difference between the length and breadth of the given Rectangle, add the Rectangle made of the said fourth Proportional and the said breadth. Then to and from the square Root of that summ, add and subtract the said half difference, so shall the summ and remainder made by that addition and subtraction be the desired length and breadth of the Rectangle to be inscribed; which length or breadth being subtracted from the length or breadth of the given Rectangle, the half of the remainder is the parallel distance between the sides of both the said Rectangles.

An Example in Numbers, to illustrate the preceding Resolution of Probl. 12.

Suppos.

28. BC = 440 }
29. BA = 400 } the sides of the given Rectangle ABCD.

30. R = 2 }
31. S = 9 } the Terms of the given Reason.

Req. to make

32. $\square$ EFGH within the $\square$ ABCD, in such manner, that

33. FI = GX = HL = EN. Also,

34. $\square$ BC,BA $-$ $\square$ FG,FE . $\square$ FG,FE :: R . S :: 2 . 9.

Solution Arithmetical.

35. $\square$ BC,BA = 176000, from 28° and 29°.

36. FG . . . . = . . 400, }
37. FE . . . . = . . 360, } found out by the Canon in 27°.

38. $\square$ FG,FE = 144000, from 36° and 37°.

39. $\square$ BC,BA $-$ $\square$ FG,FE = 32000, from 35° and 38°.

The Proof.

40. R . S :: $\square$ BC,BA $-$ $\square$ FG,FE . $\square$ FG,FE.

41. 2 . 9 :: 32000 . 144000.

42. FI = 20 = GX = HL = EN the parallel distance.

Another way of resolving the preceding Probl. 12.

43. The same things being given and required as before, let a be put for the side of a Square equal to the inscribed Rectangle, therefore . . . . $aa = \square$ EFGH.

44. From 2° and 3° the Area of the given Rectangle is bc.

45. Therefore the difference of those Rectangles is . $bc - aa$.

46. There-

46. Therefore according to the tenour of the Problem this Analogy ariseth, *viz.* . . . . $r \,.\, s :: bc - aa \,.\, aa.$

47. Whence, by Composition of Reason, this Analogy ariseth, which gives the Area of the Rectangle to be inscribed, ... $r + s \,.\, s :: bc \,.\, aa.$

From the last Analogy ariseth

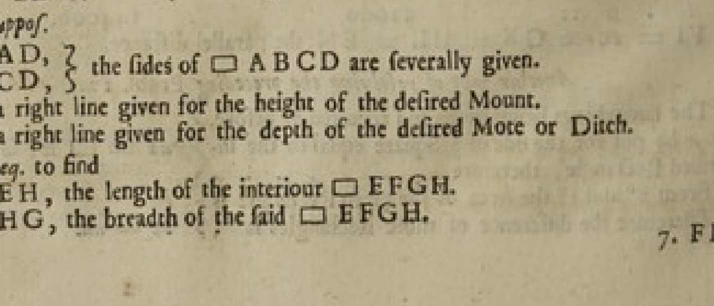
CANON 2.

As the summ of the Terms of the given Reason is to the latter Term, so is the Area of the given Rectangle to the Area of the inscribed Rectangle; therefore the Area of the inscribed Rectangle is given also. Then the Area of the inscribed Rectangle being given, as also the difference of the sides, (for this difference, as before hath been shewn in 13°, is equal to the difference of the sides of the given Rectangle,) the sides shall be given severally by *Probl.* 1. of this Chapter. And lastly, the length of the inscribed Rectangle being subtracted from the length of the given Rectangle, or the breadth from the breadth, the half of the remainder is the parallel distance between the sides of both the Rectangles.

This Canon may be exemplified by the numbers given in the preceding *Examp.* 1. And in regard the Composition of this Problem according to either of the said ways of Resolution will not be difficult to him that understands the preceding Problems of this Chapter, I shall wave the Composition, and leave it as an exercise to the industrious Learner.

Probl. XIII.

A Nobleman having made choice of a plot of ground for the making of a Garden of pleasure, gives direction to a Surveyor to trace out a Rectangle, or long-Square, whose length and breadth shall be equal to two given right lines B C and B A. Also to make another long-Square within the former, in such manner, that there may be an equal parallel distance between the sides of both the said long-Squares. Moreover, the Nobleman's design is, that the space lying between the sides of both the long-Squares shall be sunk perpendicularly, to make a Mote or Ditch whose depth shall be equal to a given right line S, and the breadth thereof such, that the earth digged out of the intended Ditch being layd upon the said interiour long-Square as a Base, may be capable of raising a rectangular Mount whose altitude shall be equal to a given right line R. The Question is, to find out the length and breadth of the interiour long-Square, as also the breadth of the Ditch, that is, the parallel distance between the sides of both the long-Squares.



Suppos.

1. BC = AD, } the sides of ▭ A B C D are severally given.
2. BA = CD, }
3. R = a right line given for the height of the desired Mount.
4. S = a right line given for the depth of the desired Mote or Ditch.

Req. to find

5. FG, or EH, the length of the interiour ▭ E F G H.
4. EF, or HG, the breadth of the said ▭ E F G H.

7. FI =

7. FI = GX = HL = EN the parallel distance.
8. R × □ EFGH = S × Plane BFGCDHEA.

Construction.

9. By the preceding *Probl.* 12. let a Rectangle or long-Square be made within the given □ ABCD, in such manner, that there may be an equal parallel distance between their sides, and that the Space lying between the sides of both Rectangles may have such proportion to the inscribed Rectangle, as the given right line R, (prescribed for the height of the Mount,) hath to the given right line S, (prescribed for the depth of the Ditch.) Now suppose that by the said 12th Problem the □ EFGH is so made within the □ABCD, that the sides of the one keep an equal parallel distance to the sides of the other, *viz.* FI = GX = HL = EN; and that as R is to S, so the interval or Plane BFGCDHEA, to the □ EFGH. Then it will be manifest (*per prop.* 34. *Elem* 11.) that R × □ EFGH (which is equal to the Solidity of the Mount,) is equal to S × Plane BFGCDHEA, (which is equal to the solidity of the Ditch;) as was required.

The quantities of the length and breadth of the inscribed Rectangle (or Base of the Mount,) as also of the parallel distance (or breadth of the Ditch) may be found out in numbers by either of the Canons of the preceding *Probl.* 12. and for the greater evidence, I shall here add

An Example in Numbers, to illustrate the preceding Construction of Probl. 13.

Suppos.

10. BC = 440 } the sides of the given □ ABCD.
11. BA = 400 }
12. R = 2 } the given altitude of the Mount to be raised perpendicularly upon □ EFGH.
13. S = 9 } the given depth of the Ditch BFGCDHEA.

Req. to find out in Numbers,

14. FG and FE the sides of □ EFGH. Also,
15. FI = GX = HL = EN the parallel distance; with condition also, that
16. R × □ FG, FE may be equal to S × Plane BFGCDHEA.

Solution Arithmetical.

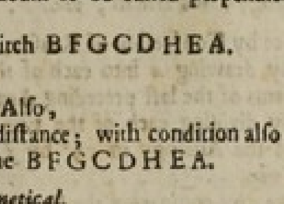
17. FG = 400 } found out by the quantities given in 10°, 11°, 12°, 13°, according
18. FE = 360 } to the preceding Construction in 9° of this *Probl.* 13.
19. FI = 20 }

The Proof.

20. □ FG, FE = 144000, the Area of □ EFGH, *viz.* the Base of the Mount.
21. R × □ FG, FE = 288000, the Solid content of the Mount.
22. □ BC, BA − □ FG, FE = 32000, the Area of BFGCDHEA.
23. S × Area of BFGCDHEA = 288000, the Solidity of the Ditch.
24. R × □ FG, FE = S × Plane BFGCDHEA = 288000; as was required.

Probl. XIV.

Within a given Rectangle AEFB to make a Rectangle CGHD, with this condition, that after the right lines EG and HF are drawn, the Spaces CGHD, EGHF, HDBF and GCAE may be equal to one another, and consequently every one of them equal to a fourth part of the given Rectangle AEFB.



Suppos.

1. AEFB is a □ given.
2. IE = LF = AC = DB.
3. EM = FK = HL = IG.
4. IL = GH = CD.
5. g = AB = EF is given.
6. f = AE = BF is given.

Req.

Req. to make

7. $\square$ CGHD = EGHF = HDBF = GCAE = $\frac{1}{4}\square$ AEFB.

Resolution.

8. Put	a = CD = GH = IL.
9. Then because EF − IL = IE + LF, 'tis manifest from 2°, 5° and 8°, that . . .	$g - a$ = IE + LF.
10. And because by *Suppos.* in 2° IE = LF, the half of the Equation in 9° gives . .	$\frac{1}{2}g - \frac{1}{2}a$ = IE = LF.
11. And the summ of the Equations in 8° and 10° gives	$\frac{1}{2}g + \frac{1}{2}a$ = IF = EL.
12. Then supposing $\square$ CGHD to be equal to $\frac{1}{4}\square$ AEFB, that is, $\frac{1}{4}fg$; let $\frac{1}{4}fg$ be divided by a, that is, GH, and the Quotient gives	$\frac{\frac{1}{4}fg}{a}$ = HD = BK.
13. And because BF − BK = KF, by subtracting $\frac{\frac{1}{4}fg}{a}$ from f, that is, BK from BF, there will remain	$f - \frac{\frac{1}{4}fg}{a}$ = KF = LH.
14. Now the Problem requires that . . .	$\square$IF,KF (that is, EGHF) = $\frac{1}{4}\square$AE,EF.
15. That is, in the letters belonging to the Resolution, (as appears by the 11th and 13th steps,)	$\overline{\frac{1}{2}g + \frac{1}{2}a} \times \overline{f - \frac{\frac{1}{4}fg}{a}} = \frac{1}{4}fg.$
16. Which last Equation may be resolved into these Proportionals, *viz.*	$\frac{1}{2}g + \frac{1}{2}a \,.\, \frac{1}{4}g :: f \,.\, f - \frac{\frac{1}{4}fg}{a}.$
17. And by doubling the two first Terms of that Analogy, this ariseth, *viz.* . . .	$g + a \,.\, \frac{1}{2}g :: f \,.\, f - \frac{\frac{1}{4}fg}{a}.$
18. Whence by Conversion of Reason, . .	$g + a \,.\, \frac{1}{2}g + a :: f \,.\, \frac{\frac{1}{4}fg}{a}.$
19. And by drawing a into each of the two latter Terms of the last preceding Analogy,	$g + a \,.\, \frac{1}{2}g + a :: fa \,.\, \frac{1}{4}fg.$
20. And by dividing each of the two latter Terms of the Analogy in 19° by f, this ariseth, *viz.*	$g + a \,.\, \frac{1}{2}g + a :: a \,.\, \frac{1}{4}g.$
21. Whence, by comparing the Rectangle of the means to the Rectangle of the extremes, this Equation ariseth, *viz.*	$aa + \frac{1}{2}ga = \frac{1}{4}gg + \frac{1}{4}ga.$
22. And by subtracting $\frac{1}{4}ga$ from each part of the Equation in 21°, this ariseth, *viz.* .	$aa + \frac{1}{4}ga = \frac{1}{4}gg.$
23. Which last Equation may be resolved into these Proportionals, *viz.*	$a + \frac{1}{4}g \,.\, \frac{1}{2}g :: \frac{1}{2}g \,.\, a.$

24. But of those three continual Proportionals, the mean, to wit, $\frac{1}{2}g$ is given, as also $\frac{1}{4}g$ the difference of the extremes $a + \frac{1}{4}g$ and a, therefore the extremes shall be given severally, (*per Probl.* 12. *Chap.* 5.) the lesser of which shall be equal to the desired line CD, (= GH = IL,) represented by a in the precedent Resolution of this Problem: And the Theorem in 24° of *Probl.* 12. *Chap.* 5. gives this following

CANON.

25. . . . . . $a = \sqrt{:\tfrac{17}{64}gg:} - \tfrac{1}{8}g$ = CD = GH = IL

That is, in words,

To the Square of one eighth part of the Base (that is, either of the sides) of the given Rectangle, add a quarter of the Square of the same Base, and from the square Root of the summ subtract one eighth part of the said Base; the remainder shall be that side of the required Rectangle which is a segment of the Base of the given Rectangle. Whence the rest of the lines in the Diagram belonging to this *Probl.* 14. shall be given also.

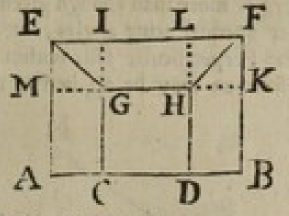
26. It is evident, that no Term of any Analogy or Equation in the foregoing Resolution exceeds the dimensions of a Square, and therefore the forming of the Composition of this Problem by a retrograde repetition of the steps of the Resolution will not be difficult to him that understands the Resolutions and Compositions of the precedent Problems of

this

this Chapter; waving therefore the Geometrical Effection and Demonstration, I shall apply the Canon before exprest to the Arithmetical Solution of the Problem propounded.

An Example in Numbers, to illustrate the preceding Resolution of Probl. 14.

Suppos.



27. AEFB is a ▭, whose Base is AB, and altitude AE.
28. AB = 10 = EF is given.
29. AE = 6 = BF is given.
30. EM = FK = HL = IG.
31. IE = LF = AC = DB.

Req. to find in numbers,

32. The quantities of the lines CD, (= IL,) IE, (= LF,) HD, (= BK,) FK, (= HL,) with this condition, that the Area of every one of these four Spaces, to wit, ▭ CGHD, EGHF, HDBF and GCAE may be equal to a quarter of the Area of the given Rectangle AEFB, *viz.*

▭ CGHD = EGHF = HDBF = GCAE = 15 = $\frac{1}{4}$ ▭ AEFB.

Solution Arithmetical.

33. From 28°, by the Canon in 25°, you will find } $\sqrt{\frac{425}{16}}-\frac{5}{4}$ = CD = GH = IL.
34. And by subtracting $\sqrt{\frac{425}{16}}-\frac{5}{4}$ from 10, that is, IL from EF, there will remain . . . } $\frac{45}{4}-\sqrt{\frac{425}{16}}$ = IE + LF.
35. And because IE = LF, the half of $\frac{45}{4}-\sqrt{\frac{425}{16}}$ gives . . . . } $\frac{45}{8}-\sqrt{\frac{425}{64}}$ = IE = LF = KH.
36. And the summ of the numbers in 33° and 35° makes . . . . } $\frac{35}{8}+\sqrt{\frac{425}{64}}$ = IF = EL.
37. Then by dividing 15 the Area of ▭ CGHD, that is, $\frac{1}{4}$ of 60 the Area of ▭ AEFB, by $\sqrt{\frac{425}{16}}-\frac{5}{4}$, that is, CD, the Quotient gives } $\frac{3}{4}+\sqrt{\frac{3825}{400}}$ = DH = BK.
38. And by subtracting $\frac{3}{4}+\sqrt{\frac{3825}{400}}$ from 6, that is, BK from BF, there will remain . . . } $\frac{21}{4}-\sqrt{\frac{3825}{400}}$ = FK = HL.

So in the six last preceding steps the quantities of all the lines sought by *Probl.* 14. are found out in numbers; but that they will satisfie the condition prescribed in 32°, will be evident by

The Proof.

39. The Product of . . . . $\overline{-\frac{5}{4}+\sqrt{\frac{425}{16}}}$ into $\overline{\frac{3}{4}+\sqrt{\frac{3825}{400}}}$ = 15.
That is, ... CD into DH = CGHD.

40. The Product of . . . . . $\overline{\frac{35}{8}+\sqrt{\frac{425}{64}}}$ into $\overline{\frac{21}{4}-\sqrt{\frac{3825}{400}}}$ = 15.
That is, ... IF into FK = EGHF.

41. The Product of . . . . . $\overline{\frac{3}{4}+\sqrt{\frac{3825}{400}}}$ into $\overline{\frac{45}{8}-\sqrt{\frac{425}{64}}}$]
That is, ... BK into KH] = ▭ HDBK.

42. The Product of . . . . . $\overline{\frac{21}{8}-\sqrt{\frac{3825}{1600}}}$ into $\overline{\frac{45}{8}-\sqrt{\frac{425}{64}}}$]
That is, ... $\frac{1}{2}$KF into KH] = △ HKF.

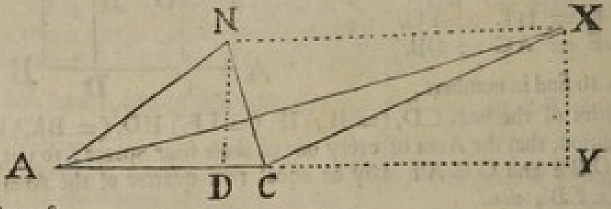
43. The summ of the Products in 41° and 42° makes . . . . } 15 = ▭ HDBK + △ HKF = HDBF.
44. And from 27°, 30°, 31° and 43°, ... } GCAE = HDBF = 15.
45. Therefore from 39°, 40°, 43° and 44°, 'tis evident that
▭ CGHD = EGHF = HDBF = GCAE = 15 = $\frac{1}{4}$ ▭ AE, FB.

Which was to be done. All which Calculations will be evident to him that understands the Arithmetick of Surd-numbers, handled at large in *Chap.* 9. *Book* II. of this Treatise.

Probl. XV.

The Base, Perpendicular and Proportion of the leggs of a plain Triangle being severally given, to find out the Triangle. But the given lines must be subject to the Determination hereafter exprest.

Note. There is more than enough given in this Problem, unless it requires a Triangle that hath either unequal acute angles, or else an obtuse angle at the Base; in the first of those Cases the Perpendicular falls within the Triangle, in the latter without; but the following Resolution may be applied to each Case.



Suppos.

1. △ ACN is acute-angled at the ends of the Base AC.
2. △ ACX is obtuse-angled at C, the end of the Base AC.
3. $b =$ AC the Base is given.
4. $p =$ ND $=$ XY the Perpendicular is given.
5. r and s are the given Terms of the Proportion of the leggs, *viz.*

$$r \,.\, s :: AN \,.\, NC :: AX \,.\, XC.$$

6. $r \sqsubset s$.

Req. to find the Triangle.

Resolution.

7. Put a for the distance from the foot of the Perpendicular to the remoter end of the Base, *viz.* suppose	$a =$ DA or YA.
8. Therefore from 3° and 7°, the distance from the foot of the Perpendicular to the nearer end of the Base is . . .	$b - a$, or $a - b$; viz. DC or YC.
9. The Square of which distance is . .	$aa - 2ba + bb$ ($= \square$ DC or $\square$ YC.)
10. The Square of the distance in 7° is	aa ($= \square$ DA or $\square$ YA.)
11. The Square of the given Perpendicular in 4° is	pp ($= \square$ ND $= \square$ XY.)
12. By *prop.* 47. *Elem.* 1.	$\square$ DC $+ \square$ ND $= \square$ CN.
13. Likewise, ...	$\square$ YC $+ \square$ YX ($\square$ND) $= \square$ CX.
14. Therefore from 9°, 11°, 12° and 13°, the Square of the lesser legg, in the letters of the Resolution, is . . .	$aa - 2ba + bb + pp$ ($= \square$CN or $\square$CX.)
15. Again, by *prop.* 47. *Elem.* 1. . .	$\square$ DA $+ \square$ ND $= \square$ AN.
16. Likewise, ...	$\square$ YA $+ \square$ YX ($\square$ND) $= \square$ AX.
17. Therefore from 10°, 11°, 15° and 16°, the Square of the greater legg is	$aa + pp$ ($= \square$ AN or $\square$ AX.)
18. And consequently the greater legg is	$\sqrt{:\overline{aa + pp}:}$ (= AN or AX.)
19. And from 14°, the lesser legg is .	$\sqrt{:\overline{aa - 2ba + bb + pp}:}$ (=CN or CX.)
20. Therefore from 5°, 18° and 19°, according to the tenour of the Problem,	$r \,.\, s :: \sqrt{:\overline{aa+pp}:} \,.\, \sqrt{:\overline{aa-2ba+bb+pp}:}$
21. Therefore from 20°, (*per prop.* 22. *Elem.* 6.)	$rr \,.\, ss :: aa + pp \,.\, aa - 2ba + bb + pp.$
22. Now in order to find out an Equation wherein the highest Power of the line a sought may not exceed a Square; to r and s find a third Proportional, which may be called t, therefore, . . .	$r \,.\, s :: s \,.\, t.$

23. There-

23. Therefore from 22°, (*per Coroll. prop.* 20. *Elem.* 6.)	$rr \,.\, ss :: r \,.\, t.$
24. Therefore from 21° and 23°, (*per prop.* 11. *Elem.* 5.)	$r \,.\, t :: aa + pp \,.\, aa - 2ba + bb + pp.$
25. Therefore from 24°, by Conversion of Reason, ...	$r \,.\, r - t :: aa + pp \,.\, 2ba - bb.$
26. Therefore inversly, ...	$r - t \,.\, r :: 2ba - bb \,.\, aa + pp.$
27. Let it be made as $r - t$ to r, so b to a fourth Proportional, which may be called f, therefore, ...	$r - t \,.\, r :: b \,.\, f.$
28. Therefore from 26° and 27°, (*per prop.* 11. *Elem.* 5.)	$b \,.\, f :: 2ba - bb \,.\, aa + pp.$
29. And by drawing $2a - b$ as a common Factor into b and f severally, this Analogy is manifest, (*per prop.* 1. *El.* 6.)	$b \,.\, f :: 2ba - bb \,.\, 2fa - fb.$
30. Therefore from 28° and 29°, (*per prop.* 11. *Elem.* 5.)	$2ba - bb \,.\, aa + pp :: 2ba - bb \,.\, 2fa - fb.$
31. Therefore from 30°, (*per prop.* 14. *Elem* 5.) this Equation ariseth, . .	$2fa - fb = aa + pp.$
32. Whence, by adding fb to each part,	$2fa = aa + pp + fb.$
33. And by subtracting aa from each part of the last Equation, this ariseth, . .	$2fa - aa = pp + fb.$
34. Which last preceding Equation may be converted into this Analogy, *viz.*	$2f - a \,.\, \sqrt{:pp + fb:} :: \sqrt{:pp + fb:} \,.\, a.$

35. But that Analogy doth manifestly consist of three continual Proportionals, whereof the mean, to wit, $\sqrt{:pp + fb:}$ is given, as also $2f$ the summ of the extremes $2f - a$ and a; therefore the extremes shall be given severally, (by *Probl.* 13. *Chap.* 5.) either of which may be taken for the line a sought, *viz.*

$$a = f + \sqrt{:ff - pp - fb:} = \text{YA};$$

$$\text{Or,} \quad a = f - \sqrt{:ff - pp - fb:} = \text{DA}.$$

36. From 34° and 35° 'tis easie to perceive that $\sqrt{:pp + fb:}$ cannot be greater than f; for the mean of three Proportionals never exceeds half the summ of the extremes, (as hath been shewn in 20° of *Probl.* 13. *Chap.* 5.) But the said $\sqrt{:pp + fb:}$ may sometimes be equal to, and sometimes less than f; to the end therefore there may be a possibility of finding out a Triangle to satisfie the Problem propounded, the given lines must be subject to this following

Determination, ... $\sqrt{:pp + fb:}$ not $\sqsubset f$.

That is, in words,

First, if it be made as r to s, so s to a third Proportional t. Secondly, as the excess of r above t, to r; so the given Base b to a fourth Proportional f. Then the side of a Square equal to the summ of the Square of the given Perpendicular p and the Rectangle of f into b, must not be greater than f, for when the said side happens to be greater than f, 'tis impossible to find a Triangle qualified as the Problem requires, by the help of the given lines r, s, b and p.

This Determination is discovered by the three Proportionals in 34°, which are rightly inferr'd from the preceding Resolution; and since the Resolution is clearly Geometrical as well as Arithmetical, I shall take the truth of the Determination for granted.

37. It hath before been declared in 35°, that the distance sought, which is represented by a in the Resolution, may be either of the two right lines or extreme Proportionals found out in the said 35th step; which two right lines will be equal to one another when $\sqrt{:pp + fb:} = f$, for then each of those lines will be equal to f; (as is evident by the Equations in 35°,) in which Case, there can but one Triangle be found out to solve the Problem, and that Triangle will always be obtuse-angled at the Base. But when it happens that $\sqrt{:pp + fb:} \sqsupset f$, then the said extreme Proportionals, (to wit, the values of a in 35°,) will be unequal between themselves, and in this Case the Problem propounded may be solved by either of those two right lines, or extreme Proportionals,

viz. two different Triangles may be found out wherein these three things will be common, to wit, the Base, the Perpendicular, and the Proportion of the leggs; of which Triangles, that which is formed by the help of the greater of the said two right lines, (or extreme Proportionals,) will always be obtuse-angled at the Base; but the other Triangle form'd by the help of the lesser of the said two right lines will sometimes be obtuse-angled at the Base, sometimes acute-angled, and sometimes right angled. Now to discover which of those three kinds of Triangles will happen, I shall give three Rules, which presuppose the quantities of the given lines to be exprest by Numbers.

Rule I.

38. If $\frac{pp}{b} + b \sqsubset f$; but $\frac{pp}{f} + b$ not $\sqsubset f$; then the lesser value of a in 35°, (that is, $f - \sqrt{:}\overline{ff - pp - fb}:$) is greater than the Base b, and consequently the Triangle form'd by the help of the said lesser value shall be obtuse-angled at the Base.

Rule II.

39. If $\frac{pp}{b} + b \sqsupset f$; then the lesser value of a in 35° is less than the Base, and consequently the Triangle form'd by the help of the said lesser value shall be acute-angled at the Base.

Rule III.

40. If $\frac{pp}{b} + b = f$; then the lesser value of a in 35° is equal to the Base, and consequently, the Triangle form'd by the help of the said lesser value shall be right-angled at the Base.

The truth of *Rule* 1. may be demonstrated thus;

41. . . . *Suppos.* in *Rule* 1. . . $\frac{pp}{b} + b \sqsubset f$; but $\frac{pp}{f} + b$ not $\sqsubset f$.

42. . . . *Req. demonstr.* . . . $f - \sqrt{:}\overline{ff - pp - fb}: \sqsubset b$; as is affirmed in *Rule* 1.

Demonstration.

43. By *Suppos.* in 41°, ... $\frac{pp}{b} + b \sqsubset f$.

44. Therefore by multiplying each part by b, . $pp + bb \sqsubset fb$.

45. And by adding ff to each part in 44°, . . . $ff + pp + bb \sqsubset ff + fb$.

46. And by subtracting pp from each part in 45°, (which the *Supposition* in 41°, or the *Determination* in 36° shews to be possible,) it follows that $ff + bb \sqsubset ff + fb - pp$.

47. Likewise by subtracting $2fb$ from each part in 46°, ... $ff + bb - 2fb \sqsubset ff - pp - fb$.

48. And by extracting the square Root out of each part in 47°, ... $f - b \sqsubset \sqrt{:}\overline{ff - pp - fb}:$

49. And by adding b to each part in 48°, . . . $f \sqsubset \sqrt{:}\overline{ff - pp - fb}: + b$.

50. Wherefore by subtracting $\sqrt{:}\overline{ff - pp - fb}:$ from each part in 49°, ... $f - \sqrt{:}\overline{ff - pp - fb}: \sqsubset b$.

Which was to be Demonstr.

After the same manner the truth of the preceding second and third Rules may be demonstrated; and from the premisses the following Canon is deducible, for the Arithmetical Solution of the Problem propounded.

CANON.

51. Let it be made as r the greater Term of the given Reason, (or Proportion,) to s the lesser; so the same s to a third Proportional, which may be called t. Let it also be made as the excess of r above t to r; so the given Base A C to a fourth Proportional, which may be called f. Then from the Square of f subtract the Square of the given Perpendicular N D (= X Y) together with the Rectangle made of f into the Base A C, and out of the remainder, if any happen, extract the square Root. That done, add the the said square Root to f before found, and the summ shall be the distance from the foot of the Perpendicular falling without the Triangle upon the Base continued to the remoter end of the Base; which distance we may suppose to be AY in the following *Fig.* 1, 2, and 3.

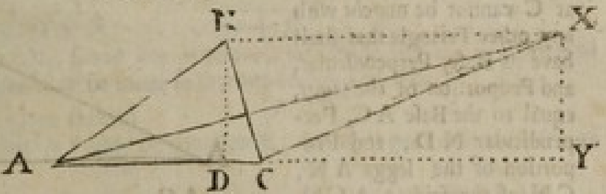
and 3. Whence $CY = AY - AC$ is given; and consequently, (*per prop.47.Elem.1.*) $CX = \sqrt{:\square CY + \square XY}:$ is given, and $AX = \sqrt{:\square AY + \square YX}:$ is given also. Therefore $\triangle ACX$, whose angle ACX is obtuse, is given, which I shall call the first of the two Triangles that will solve the Problem propounned.

Again, subtract the square Root before found, from the before mentioned fourth Proportional f, and reserve the remainder. Then observe whether the said remainder be less, greater, or equal to the given Base AC; if less, then the said remainder shall be equal to AD, to wit, the greater segment of the Base AC made by the falling of the Perpendicular ND within the Triangle ANC in *Fig.*1. Whence $AN = \sqrt{:\square AD + \square ND}:$ is given. Likewise $CN = \sqrt{:\square CD + \square DN}:$ is given, and therefore $\triangle ACN$ acute-angled at A and C is given; which I call the latter of the two Triangles that will solve the Problem. But if the remainder before reserved happens to be greater than the given Base AC, then the said remainder shall be the distance from the foot of the Perpendicular falling without the Triangle to the remoter end of the Base, which distance we may suppose to be AD in *Fig.* 2. whence $CD = AD - AC$ is given, and consequently, (*per prop* 47. *Elem.* 1.) $CN = \sqrt{:\square CD + \square DN}:$ is given. Likewise $AN = \sqrt{:\square AD + \square DN}:$ is given; and therefore in *Fig.* 2. $\triangle ACN$ obtuse-angled at C is given, which shall be the latter of two Triangles that will solve the Problem. But if the remainder before reserved happens to be equal to the given Base AC, then the latter of two Triangles that will solve the Problem shall be right-angled at the Base, as the $\triangle ACN$ right-angled at C, in *Fig.* 3. and consequently, $AN = \sqrt{:\square AC + \square CN}:$ (*per prop.* 47. *Elem.* 1.) is given. Therefore in *Fig.* 3. $\triangle ACN$ is given also.

Lastly, when it happens that nothing remains after subtraction is made of the summ of the Square of the given Perpendicular ND and the Rectangle of the given Base AC into the fourth Proportional f, from the Square of the same f; then f it self shall be the distance from the foot of the Perpendicular falling upon the Base continued to the remoter end of the said Base; which distance we may suppose to be AD in $\triangle ADN$ in *Fig.* 4. Whence $CD = AD - AC$ is given, and consequently, (*per prop.*47.*Elem.*1.) $CN = \sqrt{:\square CD + \square DN}:$ is given: Likewise $AN = \sqrt{:\square AD + \square DN}:$ is given. Therefore in *Fig* 4. $\triangle ACN$ obtuse-angled at C is given, which is the only Triangle in this Case that will solve the Problem.

TRIANGLES in Numbers, to illustrate the preceding Canon *of* Probl. 15.

FIG. 1.



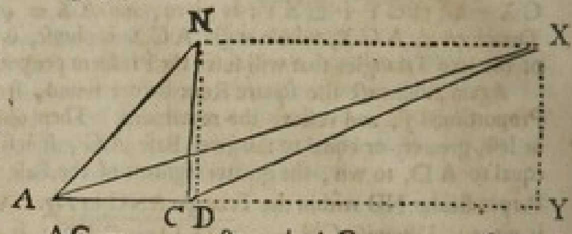
52. In this *Fig.* 1. the Triangles ACX and ACN, the first of which is obtuse-angled at the Base AC, and the latter acute-angled, have one common Base AC, also equal Perpendiculars ND and XY; and the leggs AX, CX of $\triangle ACX$ have the same Proportion one to the other, as the leggs AN, CN of $\triangle ACN$.

$AC = 39$	$AC = 39$
$AN = 40$	$AX = 8\sqrt{153}$
$CN = 25$	$CX = 5\sqrt{153}$
$ND = 24$	$XY = 24$
$DA = 32$	$YA = 96$
$DC = 7$	$YC = 57$

$AN . CN :: AX . CX :: 8 . 5.$

FIG. 2.

FIG. 2.

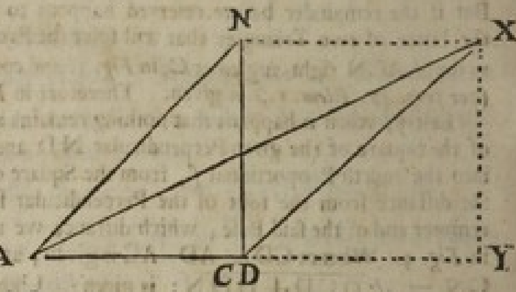


53. In this *Fig.* 2. the Triangles ACX and ACN, each of which is obtuse-angled at the Base AC, have one common Base AC, also equal Perpendiculars ND and XY, and the leggs AX, CX of △ ACX have the same Proportion one to the other, as the leggs AN, CN of △ ACN.

AC =	812	AC =	812
AN =	$4\sqrt{102544}$	AX =	$4\sqrt{570256}$
CN =	$3\sqrt{102544}$	CX =	$3\sqrt{570256}$
ND =	960	XY =	960
DC =	36	YC =	2052
DA =	848	YA =	2864

AN . CN :: AX . CX :: 4 . 3.

FIG. 3.

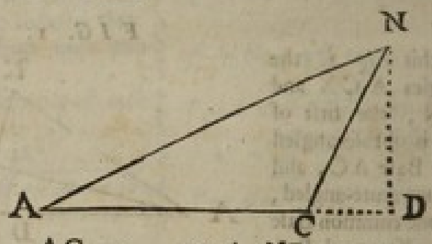


54. In this *Fig.* 3. the Triangles ACX and ACN, the first of which is obtuse-angled at the Base AC, and the latter right-angled, have one common Base AC, also equal Perpendiculars ND and XY; and the leggs AX, CX of △ ACX have the same Proportion one to the other, as the leggs AN, CN of △ ACN.

AC =	8	AC =	8	XY =	6
AN =	10	AX =	$5\sqrt{13}$	YC =	9
CN =	6	CX =	$3\sqrt{13}$	YA =	17

AN . CN :: AX . CX :: 5 . 3.

FIG. 4.



55. In this *Fig.* 4. the Triangle ACN obtuse-angled at C cannot be matcht with any other Triangle that shall have its Base, Perpendicular, and Proportion of the leggs, equal to the Base AC, Perpendicular ND, and Proportion of the leggs AN, CN of the said △ ACN.

AC =	30	ND =	20
AN =	$2\sqrt{500}$	DC =	10
CN =	$1\sqrt{500}$	AD =	40

AN . CN :: 2 . 1.

56. It is prescribed by the preceding Determination in 36°, that $\sqrt{:\overline{pp+fb}:}$ must not be greater than f, I shall therefore divide the Composition of this *Probl.* 15. into two Cases, *viz.*

Case 1. when $\sqrt{:\overline{pp+fb}:} \sqsupset f$. *Case* 2. when $\sqrt{:\overline{pp+fb}:} = f$.

The Composition of Case 1. Probl. 15.

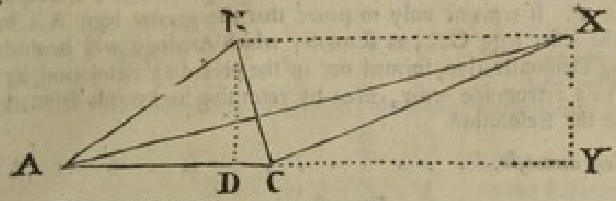
Suppos.

57. B a right line equal to the Base of a plain Triangle is given.
58. P a right line equal to the Perpendicular is given.

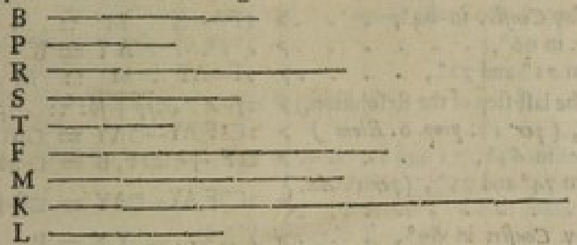
59. R and S

59. R and S two right lines expressing the Reason (or Proportion) of the leggs of the same Triangle, are given.

60. $R \sqsubset S$.



Req. to find out the Triangle.



Construction.

61. To the given lines R and S find a third Proportional, (by *Probl.* 7. *Chap.* 5.) suppose the line T, therefore . . . $R \,.\, S :: S \,.\, T.$

62. Also (by *Probl.* 8. *Chap.* 5.) let it be made as $R - T$ to R; so the given Base B to a fourth Proportional line F, therefore . . . . $R - T \,.\, R :: B \,.\, F.$

63. By *Probl.* 2. *Chap.* 5. find a right line M, such that its Square may be equal to $\square P + \square B,F$, therefore . . $\square M = \square P + \square F,B.$

64. By *Probl.* 14. *Chap.* 5. divide the double of F into two such parts, that the line M may be a mean between them; which Effection is possible, for by *Suppos.* in *Case* 1. (before exprest in 56°,) the line M (that is, $\sqrt{:pp + fb:}$) is less than F; suppose then that 2 F is cut into two parts, whereof the greater is equal to the line K, and the lesser equal to the line L; and that the line M is a mean Proportional between K and L, therefore these are Proportionals, *viz.*

$$\underset{L}{2F - K} \,.\, M :: M \,.\, K, \quad \Big\| \quad \underset{K}{2F - L} \,.\, M :: M \,.\, L.$$

Each of which Analogies is correspondent to that in the 34[th] step of the preceding Resolution, *viz.*

$$2f - a \,.\, \sqrt{:pp + fb:} :: \sqrt{:pp + fb:} \,.\, a.$$

Now by the help of the line K, found out as above, an obtuse-angled plain Triangle to solve the Problem propounded may be made in manner following, *viz.*

65. Make $AC = B$, (the given Base)

66. Produce AC to Y, so that AY may be equal to K, which is greater than AC, as may be proved thus;

It is manifest that . . . . $R \sqsubset R - T.$

Therefore from the Analogy in 62°, (*per Coroll.* of 14. *prop* 5. *Elem.*) . . . . $F \sqsubset B.$

But by *Constr.* in 65°, ... $AC = B.$

Therefore (*per Ax.* 3. *Chap.* 2.) . . . . $F \sqsubset AC$ (or B.)

And because the greatest of three Proportionals is greater than half the summ of the extremes, therefore from 64°, . . $K \sqsubset F.$

Therefore (*per Ax.* 5. *Chap.* 2.) . . . . AY or $K \sqsubset AC.$

Which was to be demonstr.

67. Make $YX \perp AY$, also make $YX = P$, the given Perpendicular.

68. Lastly, from A and C (the ends of the Base AC) draw the right lines AX and CX to meet with the top of the Perpendicular YX in X, so the Triangle ACX obtusangled at C, (for as before hath been proved in 66°, $AY \sqsubset AC$) shall be one

one of the two Triangles which in *Case* 1. will satisfie the Problem; which I prove thus,

69. First, by *Construction* in 65° the Base AC is equal to the given Base B; secondly, by *Constr.* in 67° the line YX is perpendicular to AC continued, and equal to the given Perpendicular P. It remains only to prove that the greater legg AX hath such proportion to the lesser legg CX, as R to S; which Analogy will be made manifest by the following Demonstration, formed out of the preceding Resolution by a repetition of its steps in a retrograde order, *viz.* by returning backwards from the end to the beginning of the Resolution.

70. . . . *Req. demonstr.* . . . . R . S :: AX . CX.

Demonstration.

71. Forasmuch as by *Constr.* in 64°, . . . > $2F - K \,.\, M :: M \,.\, K.$

72. And by *Constr.* in 66°, ... > . . . . . $AY = K.$

73. Therefore from 71° and 72°, ... > $2F - AY \,.\, M :: M \,.\, AY.$

That is, in 34°, the last step of the Resolution,) > $2f-a \,.\, \sqrt{:pp+fb:} :: \sqrt{:pp+fb:} \,.\, a.$

74. But from 73°, (*per* 17. *prop.* 6. *Elem*) > $2\Box F,AY - \square AY = \square M.$

75. And by *Constr.* in 63°, ... > $\square P + \Box F,B = \square M.$

76. Therefore from 74° and 75°, (*per* 1. *Ax. Chap.* 2.) . . . . } $2\Box F,AY - \square AY = \square P + \Box F,B.$

77. And because by *Constr.* in 67°, . . . > . . . . . $YX = P.$

78. And consequently, ... > . . . . $\square YX = \square P.$

79. And by *Constr.* in 65°, ... > . . . . . $AC = B.$

80. Therefore out of 76°, 78° and 79°, . > $2\Box F,AY - \square AY = \square YX + \Box F,AC$

That is, in 33°, ... > . . . $2fa - aa = pp + fb.$

81. And from 80°, by adding $\square AY$ to each part, > $2\Box F,AY = \square AY + \square YX + \Box F,AC.$

That is, in 32°, ... > . . $2fa = aa + pp + fb.$

82. And by subtracting $\Box F,AC$ from each part of the Equation in 81°, ... } $2\Box F,AY - \Box F,AC = \square AY + \square YX.$

That is, in 31°, ... > . . $2fa - fb = aa + pp.$

83. And this following Analogy is manifest, (*per prop.* 7. *Elem.* 5.) for the first and third Proportionals are one and the same, and the second and fourth equal one to the other, (as hath before been proved in 82°;)

$$\left.\begin{array}{r} 2\Box AC,AY - \square AC \,. \\ \square AY + \square YX :: \\ 2\Box AC,AY - \square AC \,. \\ 2\Box F,AY - \Box F,AC \,. \end{array}\right\} \text{that is, in } 30°, \left\{\begin{array}{l} 2ba - bb \,. \\ aa + pp :: \\ 2ba - bb \,. \\ 2fa - fb \,. \end{array}\right.$$

84. And by reason of the common altitude $2AY - AC$ in the two latter Terms of the subsequent Analogy, it will be manifest (*per* 1. *prop.* 6. *Elem.*) that

$$\left.\begin{array}{r} AC \,. \\ F :: \\ 2\Box AC,AY - \square AC \,. \\ 2\Box F,AY - \Box F,AC \,. \end{array}\right\} \text{that is, in } 29°, \left\{\begin{array}{l} b \,. \\ f :: \\ 2ba - bb \,. \\ 2fa - fb \,. \end{array}\right.$$

85. And because the two latter Terms of the Analogy in 84°, are the same, and in the same order with the two latter Terms of the Analogy in 83°, therefore from 83° and 84° (*per* 11. *prop.* 5. *Elem.*) these shall be Proportionals, *viz.*

$$\left.\begin{array}{r} AC \,. \\ F :: \\ 2\Box AC,AY - \square AC \,. \\ \square AY + \square YX \,. \end{array}\right\} \text{that is, in } 28°, \left\{\begin{array}{l} b \,. \\ f :: \\ 2ba - bb \,. \\ aa + pp \,. \end{array}\right.$$

86. But from the *Constr.* in 62° and 65°, . . . $AC \,.\, F :: R-T \,.\, R.$

That is, in 27°, ... $b \,.\, f :: r-t \,.\, r.$

87. Therefore from 85° and 86°, (*per* 11. *prop.* 5. *Elem.*) these shall be Proportionals, *viz.*

$$\left.\begin{array}{r} R-T \,. \\ R :: \\ 2\Box AC,AY - \square AC \,. \\ \square AY + \square YX \,. \end{array}\right\} \text{that is, in } 26°, \left\{\begin{array}{l} r-t \,. \\ r :: \\ 2ba - bb \,. \\ aa + pp \,. \end{array}\right.$$

88. And

88. And from 87°, by Reason Inverse, these are Proportionals, *viz.*

$$\left.\begin{array}{r} R\ . \\ R - T\ :: \\ \square AY + \square YX\ . \\ 2\square AC,AY - \square AC\ . \end{array}\right\} \text{that is, in } 25°, \left\{\begin{array}{r} r\ . \\ r - t\ :: \\ aa + pp\ . \\ 2ba - bb\ . \end{array}\right.$$

89. And from 88°, by Conversion of Reason, these are Proportionals, *viz.*

$$\left.\begin{array}{r} R\ . \\ T\ :: \\ \square AY + \square YX\ . \\ \square AY - 2\square AC,AY + \square AC + \square YX\ . \end{array}\right\} \begin{array}{l}\text{which answer to those in } 24° \\ \text{in the Resolution, } viz.\end{array}$$

$$r\ .\ t\ ::\ aa + pp\ .\ aa - 2ba + bb + pp.$$

90. And because by *Constr.* in 61°, ... } $R\ .\ S\ ::\ S\ .\ T.$
That is, in 22°, ... } $r\ .\ s\ ::\ s\ .\ t.$
91. Therefore (*per Coroll.* of 20. *prop.* 6. *Elem.*) . . . } $R\ .\ T\ ::\ \square R\ .\ \square S.$
That is, in 23°, ... } $r\ .\ t\ ::\ rr\ .\ ss.$
92. Therefore from 89° and 91°, (*per* 11. *prop.* 5. *Elem.*) these shall be Proportionals, *viz.*

$$\left.\begin{array}{r} \square R\ . \\ \square S\ :: \\ \square AY + \square YX\ . \\ \square AY - 2\square AC,AY + \square AC + \square YX\ . \end{array}\right\} \begin{array}{l}\text{which answer to those in } 21°, \\ viz.\end{array}$$

$$rr\ .\ ss\ ::\ aa + pp\ .\ aa - 2ba + bb + pp.$$

93. And because by *Constr.* in 67°, ... } $YX \perp AY.$
94. Therefore (*per* 47. *prop.* 1. *Elem.*) . . . } $\square AX = \square AY + \square YX.$
95. Likewise . . . . } $\square CX = \square CY + \square YX.$
96. Moreover by *Constr.* in 66°, and by the *Diagr.* } $CY = AY - AC.$
97. And consequently, (*per* 5. *Theor.* 4. *Chap.*) } $\square CY = \square AY - 2\square AC,AY + \square AC.$
That is, in 9°, ... } $\square\overline{a-b} = aa - 2ba + bb.$
98. Therefore if instead of $\square CY$ in 95°, we take that which in 97° is found equal to $\square CY$, the Equation in 95° will be reduced to this, *viz.*

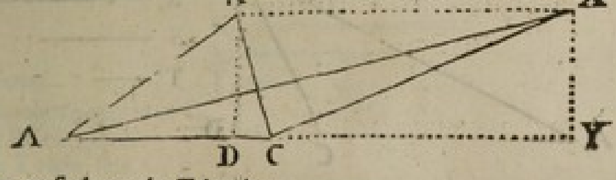
$$\square CX = \square AY - 2\square AC,AY + \square AC + \square YX.$$

99. Likewise if instead of the third and fourth Proportionals in 92°, we take those Squares which are found equal to them respectively in 94° and 98°, the Analogy in 92° will be reduced to this, *viz.* $\square R\ .\ \square S\ ::\ \square AX\ .\ \square CX.$

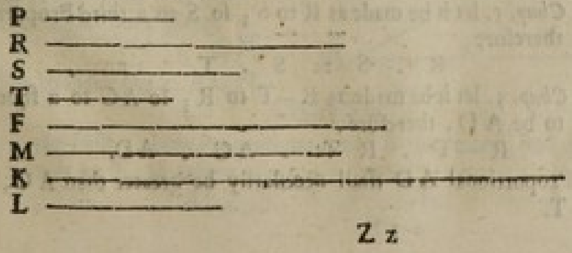
100. Wherefore (*per prop.* 22. *Elem.* 6.)

$$R\ .\ S\ ::\ AX\ .\ CX. \qquad \text{Which was to be Dem.}$$

101. Another Triangle to solve the Problem in *Case* 1. before exprest in 56°, may by the help of the lesser Root L before found in 64°, be formed thus, *viz.* Let the lines before given and found out in 57°, 58°, 59°, 60°, 61°, 62°, 63°, 64°, together with the Diagram standing between 60° and 61° be here repeated, then will the Construction be as followeth.



Req. to find out the Triangle.



Construction.

102. Make A C = B (the given Base.)

103. Upon A C, continued if need be, make A D = L, which lesser Root L, (as before hath been shewn,) will sometimes be greater than the Base; but supposing it be discovered (by *Rule* 2. in 39° of this *Probl.*) that L is lesser than B, or A C, cut off from A C a segment equal to L, as A D.

104. Make D N ⊥ A C in the point D, also make D N = P the given Perpendicular.

105. Lastly, from the ends of the Base A C draw the right lines A N and C N to meet with the top of the Perpendicular D N, in N; so the Triangle A C N acute-angled at A and C, (for by *Supposition* A D is lesser than A C,) will satisfie the Problem, as well as the △ A C X before found. For first, by *Construction* in 102° the Base A C is equal to the given Base B: Secondly, the Perpendicular D N (by *Constr.* in 104°) is equal to the given Perpendicular P; and by a repetition of the steps of the Resolution in a backward order, in like manner as before in the preceding Demonstration, saving that L must be used here instead of K, and N D instead of X Y; it may easily be proved that the leggs A N and C N are in the given Reason of R to S.

Moreover, when the lesser Root L is greater than the Base, the Triangle formed by the help of such lesser Root shall be obtuse-angled at the Base, and the Construction and Demonstration in every respect like to that by the greater Root.

But it must be remembred, that when the Perpendicular falls within the Triangle, then the Square of D C is equal to the Square of A C — A D; but when it falls without, then the Square of Y C is equal to the Square of Y A — A C: So that before the Construction and Demonstration by the lesser Root be entred upon, it will be requisite to find out the kind of the Triangle, by the help of the three preceding Rules in 38°, 39°, 40°; and when it happens that $r . s :: \sqrt{:}\overline{bb + pp}: . p$, then 'tis evident (by 47. *prop.* 1. *Elem.*) that the Triangle formed by the lesser Root will be right-angled at the Base, and in such Case there is no need of further proof.

The Composition of Case 2. Probl. 15.

106. Which Case, in the letters belonging to the preceding Resolution presupposeth . . . . } . $f = \sqrt{:}\overline{pp + fb}: = a.$

107. And consequently, by squaring each part, . . . } . $ff = pp + fb = aa.$

108. That is, in the lines of the ensuing *Constr.* and *Diagr.* } □AD = □DN + □AD,AC.

Suppos.

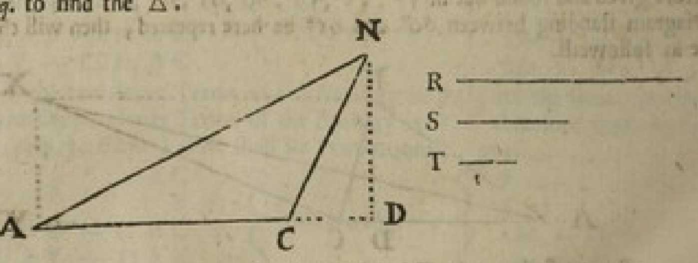
109. A C = the Base of a Triangle is given.

110. DN = the Perpendicular is given.

111. R and S two right lines expressing the Reason of the leggs are given.

112. R ⊏ S.

Req. to find the △.



Construction.

113. By *Probl.* 7. *Chap.* 5. let it be made as R to S; so S to a third Proportional, suppose it be found T, therefore,

R . S :: S . T.

114. By *Probl.* 8. *Chap.* 5. let it be made as R—T to R; so A C to a fourth Proportional, suppose it to be A D, therefore,

R—T . R :: . AC . AD.

Which fourth Proportional A D shall necessarily be greater than A C, because R is greater than R— T.

115. Make

115. Make $DN \perp AD$ in the point D; then from A and C, the ends of the given Base AC, draw the right lines AN and CN to meet with the top of the Perpendicular DN in N; so shall ACN be the Triangle required. For first, the Base AC is equal to the given Base; also the Perpendicular ND is equal to the given Perpendicular. But that the leggs AN and CN are in the given Reason of R to S, it may easily be demonstrated by a backward repetition of the steps of the foregoing Resolution, in like manner as before in the Composition of *Case* 1; with this Caution, That as often as a is found in the Resolution, f must be taken instead of a, because in this second Case f is equal to a; for since by *Supposition* in 106°, $f = \sqrt{:pp + fb}$: it will be evident from 35°, that $f = a$. But in regard the Demonstration of this second Case differs not from that of the following *Probl.* 16. I shall wave it here.

COROLLARY.

116. From the premisses it follows, that the Perpendicular DN of the Triangle ACN formed in *Case* 2. (before exprest in 106°, 107° and 108°,) is a mean Proportional between AD and DC the distances from D the foot of the Perpendicular falling without the Triangle to the ends of the Base; and consequently, (*per prop. 6. Elem. 6.*) the Triangles ADN and CDN are equiangular. See the last preceding Diagram, and compare it with this following Demonstration.

117. . . . *Req. demonstr.* . . . . . . { AD . DN :: DN . DC. Also, △ADN and △CDN are equiangular.

Demonstration.

118. By *Suppos.* in 108°, ... $\square AD = \square DN + \square AD, AC.$

119. Therefore by subtracting $\square$ AD, AC from each part, ... $\square AD - \square AD, AC = \square DN.$

120. And from 119°, (*per prop.* 14. *Elem.* 6.) these are Proportionals, *viz.* . . . . AD . DN :: DN . AD − AC.

121. But 'tis evident by the last preceding Diagram, that . . . . $DC = AD - AC.$

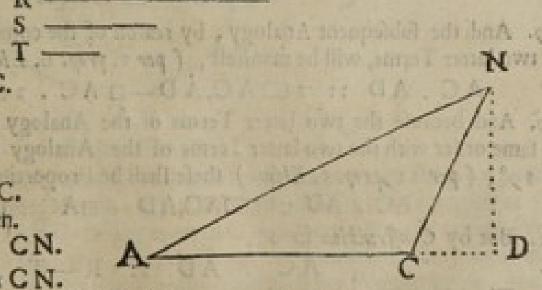
122. Therefore from 120° and 121°, . . AD . DN :: DN . DC.

123. Therefore from 122°, (*per prop.6.Elem.6.*) △ADN and △CDN are equiangular. Which was to be Demonstr.

From the preceding *Corollary* and *Construction* of *Case* 2. the following *Probl.* 16. is deducible.

Probl. XVI.

To find a plain Triangle obtuse-angled at the Base, and that the Base may be equal to a right line given. Also, that the Perpendicular falling upon the Base continued may be a mean Proportional between the distances from the foot of the Perpendicular to the ends of the Base: And that the leggs of the Triangle may be in a given Reason, suppose as R to S.



Suppos.

1. △ACN is obtuse-angled at C.
2. AC the Base is given.
3. ACD is a right line.
4. $DN \perp AD$.
5. AD . DN :: DN . DC.
6. R and S are right lines given.
7. R . S :: AN . CN.

Req. to make the △ ACN.

Construction.

8. Making R the first of three Proportionals, and S the second, find a third, (*per 7.Probl. 5.Chap.*) let it be T, therefore

R . S :: S . T.

 9. Also

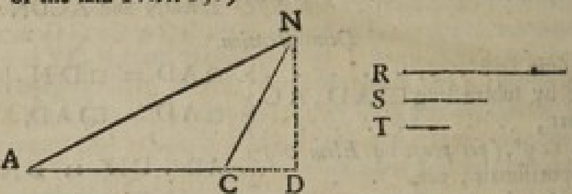
9. Also making R — T the first of four Proportionals, R the second, and the given Base AC the third, find a fourth, (*per* 8. *Probl.* 5. *Chap.*) suppose it be AD, therefore

R — T . R :: AC . AD.

Which fourth Proportional AD shall necessarily be greater than AC, because the second Proportional R is greater than the first R — T.

10. Find a mean Proportional, as DN, between AD and DC, (*per* 9. *Probl.* 5. *Chap.*) therefore, AD . DN :: DN . DC (= AD — AC.)

11. Make DN ⊥ AD in the point D; then draw the right lines AN and CN, so shall ACN be the Triangle required. Now we must shew that it will satisfie the Problem: First then, AC the Base is equal to the right line prescribed for the Base, and from the 9th step it is less than AD; therefore the angle ACN is obtuse: Secondly, the Perpendicular ND, (by *Construction* in 10°,) is a mean Proportional between AD and DC, (to wit, the two distances from D the foot of the Perpendicular ND, to A and C the ends of the Base AC.) It remains only to prove, That the leggs AN and CN of the Triangle ACN, are in such proportion one to the other as R to S; which Analogy I shall make manifest by the following Demonstration, formed out of the Resolution of the preceding *Probl.* 15. by a backward repetition of the steps of the said Resolution, in *Case* 2. (but respect must be had to the Caution given in 115° of the said *Probl.* 15.)



12. . . . *Req. demonstr.* . . . . R . S :: AN . CN.

Demonstration.

13. Forasmuch as by *Constr.* in 10°, . . . } AD . DN :: DN . DC (AD — AC.)
14. Therefore (*per* 17. *prop.* 6. *Elem.*) . . } □AD — ▭AD,AC = □DN.
15. And from 14°, by equal addition of ▭AD,AC, } □AD = □DN + ▭AD,AC.
16. And from 15°, by equal addition of □AD, } 2□AD = □AD + □DN + ▭AD,AC.
17. And from 16°, by equal subtraction of ▭AD,AC, ... } 2□AD — ▭AD,AC = □AD + □DN.

18. And because in the following Analogy the first and third Terms are one and the same, and the second and fourth equal one to the other, (as hath been proved in 17°,) therefore (*per* 7. *prop.* 5. *Elem.*)

2▭AC,AD — □AC .
□AD + □DN ::
2▭AC,AD — □AC .
2□AD — ▭AD,AC .
} Proportionals.

19. And the subsequent Analogy, by reason of the common altitude 2AD — AC in the two latter Terms, will be manifest, (*per* 1. *prop.* 6. *Elem.*) *viz.*

AC . AD :: 2▭AC,AD — □AC . 2□AD — ▭AD,AC.

20. And because the two latter Terms of the Analogy in 19° are the same and in the same order with the two latter Terms of the Analogy in 18°, therefore from 18° and 19°; (*per* 11. *prop.* 5. *Elem.*) these shall be Proportionals, *viz.*

AC . AD :: 2▭AC,AD — □AC . □AD + □DN.

21. But by *Construction* in 9°,

AC . AD :: R — T . R.

22. Therefore from 20° and 21°, (*per* 11. *prop.* 5. *Elem.*)

R — T . R :: 2▭AC,AD — □AC . □AD + □DN.

23. And from 22°, by Reason inverse,

R . R — T :: □AD + □DN . 2▭AC,AD — □AC.

24. And

24. And from 23°, by Conversion of Reason, these shall be Proportionals, *viz.*

$$\left.\begin{array}{r} R\ . \\ T\ :: \\ \square AD + \square DN\ . \\ \square AD + \square DN + \square AC - 2\square AC,AD\ . \end{array}\right\}\ \text{Proportionals.}$$

25. And because by *Constr.* in 8°, ... } $R\ .\ S\ ::\ S\ .\ T.$
26. And consequently, (*per Coroll.* of 20. *prop.* 6. *Elem*) . . . . } $R\ .\ T\ ::\ \square R\ .\ \square S.$
27. Therefore from 24° and 26°, (*per* 11.*prop.* 5.*Elem.*) these shall be Proportionals, *viz.*

$$\left.\begin{array}{r} \square R\ . \\ \square S\ :: \\ \square AD + \square DN\ . \\ \square AD + \square DN + \square AC - 2\square AC,AD\ . \end{array}\right\}\ \text{Proportionals.}$$

28. And because by *Constr.* in 11°, . . . } $DN \perp AD.$
29. Therefore, *per* 47. *prop.* 1. *Elem.* (respect being had to the Diagram,) . . . . . } $\square AN = \square AD + \square DN.$
30. Likewise, ... } $\square CN = \square DC + \square DN.$
31. Again, by *Constr.* in 9°, and by view of the Diagram, ... } $DC = AD - AC.$
32. And consequently, (*per Theor.* 5. *Chap.* 4.) } $\square DC = \square AD + \square AC - 2\square AD,AC.$
33. Therefore if instead of $\square DC$ in 30°, we set that which in 32° is found equal to $\square DC$, the Equation in 30° will be reduced to this, *viz.*

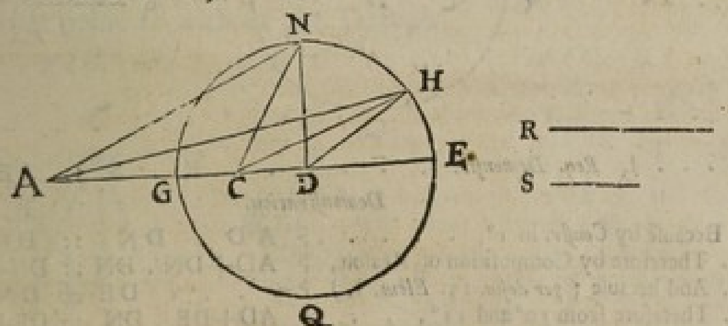
$$\square CN = \square AD + \square AC - 2\square AD,AC + \square DN.$$

34. Likewise, if instead of the third and fourth Proportionals in 27°, we take those Squares which are found equal to them respectively in 29° and 33°, the Analogy in 27° will be reduced to this, $\square R\ .\ \square S\ ::\ \square AN\ .\ \square CN.$
35. Wherefore . . . $R\ .\ S\ ::\ AN\ .\ CN.$

Which was to be demonstrated. Therefore that is done which the Problem required.

Probl. XVII. (Probl. *Apollon. Pergæi.*)

Two points (A and C) being given in a Plane, to describe a Circle in the same Plane, that two right lines drawn from those points to concurr in any point of the Circumference may have a given Reason; suppose the greater line to the less, as R to S.



Construction.

1. Upon the given line AC as a Base, to wit, the shortest distance between the given points A and C, make (by the preceding *Probl.* 16.) a Triangle ACN obtuse-angled at C, and such, that the Perpendicular ND falling upon AC produced, may be a mean Proportional between AD and DC; also, that the leggs AN and CN may be in the given Reason of R to S. Therefore by that Construction these are Proportionals, *viz.*

$$AD\ .\ DN\ ::\ DN\ .\ DC.$$
$$R\ .\ S\ ::\ AN\ .\ CN.$$

2. Then from the Center D, at the distance of the Perpendicular DN, describe the Circle DGNEQ, which shall necessarily cut AC; for by *Construction* DN is a mean Propor-

Proportional between D A and D C, which D C being but part of D A is less than D A, therefore the mean, or Semidiameter D N or D G is less than D A, but greater than D C. Now I say the Circle D G N E Q is that which is required by the Problem, and therefore we must shew that if two right lines be drawn from the given points A and C to meet in any point of the Circumference of that Circle, those right lines shall have such proportion one to the other as the given lines R and S; the demonstration whereof I shall divide into three Cases, in regard there may be a threefold position of the point taken in the Circumference; for the point may be either E, or else G, to wit, the ends of that Diameter which lyes in the same straight line with the given line A C; or lastly, the point may be taken in any other part of the Circumference, as H; which Cases I shall demonstrate in their order.

Preparat.

3. Forasmuch as in the Triangles A D N and C D N the angle at D is common, and the sides about that angle are Proportionals, for by *Construction* in 1° it hath been made, as AD . DN :: DN . DC, therefore (*per prop. 6. Elem. 6.*)
$\triangle$ ADN and $\triangle$ CDN are equiangular.

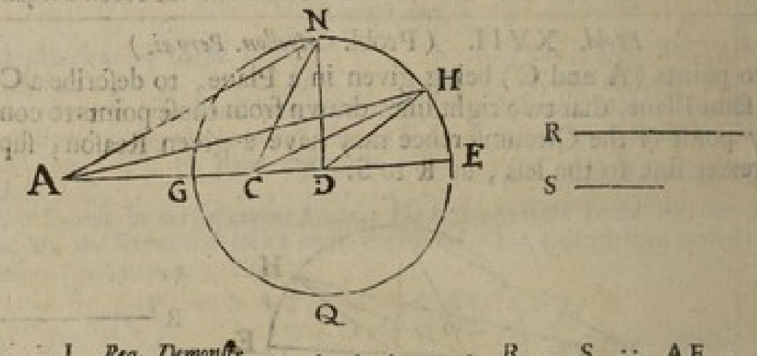
4. But we must enquire which angles in those like Triangles are equal one to the other. First then, because the angle at D is common, the angle C N D in $\triangle$ C D N must be equal either to the angle A N D, or to the angle N A D in $\triangle$ A D N; but the angle C N D being but part of the angle A N D cannot be equal to it, therefore
$\angle$CND = $\angle$NAD. Also, $\angle$NCD = $\angle$AND.

5. In like manner, because by *Constr.* in 1° and 2°,
AD . DN (or DH) :: DN (or DH) . DC.

6. Therefore, (*per prop. 6. Elem. 6.*)
$\triangle$ ADH and $\triangle$ CDH are equiangular.

7. And for the like reason as before in 4°,
$\angle$CHD = $\angle$HAD. Also, $\angle$HCD = $\angle$AHD.

These things premised, I shall proceed to the Demonstration of the three *CASES* before mentioned.



8. ∴ I. *Req. Demonstr.* . . . . R . S :: AE . CE.

Demonstration.

9. Because by *Constr.* in 1°, ... AD . DN :: DN . DC.
10. Therefore by Composition of Reason, AD+DN . DN :: DN+DC . DC.
11. And because (*per defin.* 15. *Elem.* 1.) . . . . DE = DN.
12. Therefore from 10° and 11°, . . . AD+DE . DN :: DE+DC . DC.
13. That is, as is evident by the *Diagram*, AE . DN :: CE . DC.
14. Therefore alternly, ... AE . CE :: DN . DC.
15. Again, it hath been proved in 3°, that $\triangle$ ADN and $\triangle$ CDN are equiangular.
16. And in 4°, that . . . . $\angle$NCD = $\angle$AND.
17. Therefore from 15° and 16°, (*per prop.* 4. *Elem.* 6.) . . . . AN . DN :: CN . DC.
18. Therefore alternly, ... AN . CN :: DN . DC.
19. But by *Constr.* in 1°, ... AN . CN :: R . S.
20. Therefore from 18° and 19°, (*per* 11. *prop.* 5. *Elem.*) . . . . R . S :: DN . DC.

21. But

21. But it hath been proved in 14°, that . > AE . CE :: DN . DC.
22. Therefore from 20° and 21°, (*per* 11. *prop.* 5. *Elem.*) . . . . > R . S :: AE . CE.
Which was to be demonstr.
23. . . . II. *Req. Demonstr.* . . . . . . R . S :: AG . CG.

Demonstration.

24. Forasmuch as by *Constr.* in 1°, . . . > AD . DN :: DN . DC.
25. Therefore by Division of Reason, . . > AD−DN . DN :: DN−DC . DC.
26. And because (*per defin.* 15. *Elem.* 1.) > . . . . . DG = DN.
27. Therefore from 25° and 26°, . . . > AD−DG . DN :: DG−DC . DC.
28. That is, (as is evident by the *Diagram*,) > AG . DN :: CG . DC.
29. Therefore alternly, ... > AG . CG :: DN . DC.
30. But before in 20°, it hath been proved that > R . S :: DN . DC.
31. Therefore from 29° and 30°, (*per* 11. *prop.* 5. *Elem.*) . . . . > R . S :: AG . CG.
Which was to be demonstrated in the second place.
32. . . . III. *Req. Demonstr.* . . . . . . R . S :: AH . CH.

Demonstration.

33. It hath before been proved in 6°, that . > △ ADH and △ CDH are equiangular.
34. And in 7°, that . . . . > < HCD = < AHD.
35. Therefore from 33° and 34°, (*per* 4. *prop.* 6. *Elem.*) . . . . > CH . CD :: AH . HD.
36. And alternly, ... > CH . AH :: CD . HD(or DN.)
37. Therefore inversly, ... > AH . CH :: DN . CD.
38. But it hath been proved in 20°, that . > R . S :: DN . CD.
39. Therefore from 37° and 38°, (*per* 11. *prop.* 5. *Elem.*) . . . . > R . S :: AH . CH.

Which was to be demonstrated in the last place. Therefore that is done which the Problem required.

Probl. XVIII.

To divide a given Triangle ABC into two parts which shall be in a given Reason, suppose as AH to HB, by a right line DK drawn from a given point D without the Triangle.

FIG. 1.

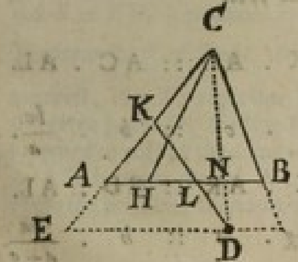


FIG. 2.

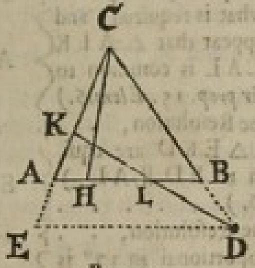
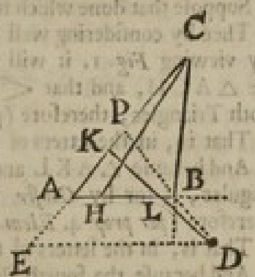


FIG. 3.



Prepar.

1. By the given point D draw DE parallel to the Base AB, and continue the leggs CA, CB beneath the Base; then the point D will either lye between the Increases of the leggs, as in *Fig.* 1. or else in one of the said Increases, as in *Fig.* 2. or lastly, between the Increases of the Base and one of the leggs, as in *Fig.* 3.
2. Divide the Base AB in H in the given Reason, and draw CH, therefore (*per prop.* 1. *Elem.* 6.) △ ACH . △ HCB :: AH . HB.
3. In *Fig.* 1. let a line be drawn from the given point D to C the angle opposite to the Base AB, as the line DC, which will either cut the Base AB in the point H, in which the

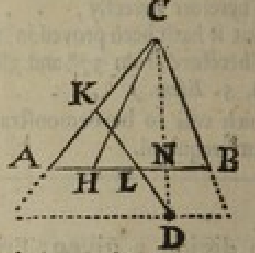
the Base is divided in the given Reason, in which Case the Problem is evidently satisfied; or else in some other point N, and then the point H will either lye between N and A, or between N and B; if H lye between N and A, then the desired line of partition to be drawn from D will cut AB and AC; but if H lye between N and B, then the said line of partition will cut AB and BC.

4. In *Fig.* 2. where the given point D lyes in CB increased, 'tis evident that the line of partition to be drawn from D shall necessarily cut AB and AC.

5. In *Fig.* 3. the line of partition to be drawn from D will sometimes cut AB and AC, sometimes it may pass by the angular point B and cut AC only, and sometimes it will cut BC and AC; but which of these lines will happen to be cut when the given point D is posited according to the Definition in 1°, relating to *Fig.* 3. may be discovered by the Rule hereafter given in 34° of this Problem.

6. In every one of those three Cases before defined in 1°, which may happen by the various position of the given point D, the Resolution of the Problem propos'd will be one and the same. Supposing then it be discovered, that the line of partition to be drawn from D must cut AB and AC in each of the three preceding Figures, the Scope of the Resolution is to find a point in AC, as K, to which a right line being drawn from D, as DK, this line DK may cut the Base AB between H and N in *Fig.* 1. or between H and B in *Fig.* 2. likewise in *Fig.* 3. between H and B, (or else pass by the angular point B,) so as to make the Triangle AKL equal to the Triangle ACH, whence it evidently follows that LKCB = △HCB, and △AKL . LKCB :: AH . HB. These things premised, the Resolution of the Problem propounded may be formed in manner following.

FIG. 1.



Suppos.

7. ABC is a △ given in *Fig.* 1.
8. D is a point given without the △ ABC.
9. AH and HB are in a given Reason.
10. b = AC is given.
11. c = AH is given.
12. g = AE is given.
13. h = ED (∥ AB) is given.

Req. to find

14. AK such a segment of AC, that DK being drawn, it may make
15. △ALK . LKCB :: AH . HB :: △ACH . △HCB.

Resolution.

16. Suppose that done which is required, and put . . . a = AK.

17. Then by considering well what is required, and by viewing *Fig.* 1. it will appear that △ALK = △ACH, and that ∠CAL is common to both Triangles; therefore (*per prop.* 15. *Elem.* 6.) . . . AK . AH :: AC . AL.

18. That is, in the letters of the Resolution, . . . a . c :: b . $\frac{bc}{a}$.

19. And because △AKL and △EKD are equiangular, (for by *Constr.* in 1° ED ∥ AL,) therefore (*per prop.* 4. *Elem.* 6.) . . . . . EK . AK :: ED . AL.

20. That is, in the letters of the Resolution, . . . $a+g$. a :: h . $\frac{ha}{a+g}$.

21. And because the fourth Proportional in 17° is the same with the fourth in 19°, therefore the fourth Proportionals in 18° and 20° shall also be equal to one another, *viz.* . . . . . $\frac{bc}{a} = \frac{ha}{a+g}$.

22. Now to avoid an Equation between Solids, let it be made as h to b, so c to a fourth Proportional, which may be called m; therefore, ... h . b :: c . m.

23. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, *viz.* . . . . . . $hm = bc$.

24. There-

24. Therefore from 21° and 23°, by exchanging equal quantities, ... $\frac{hm}{a} = \frac{ha}{a+g} = \frac{bc}{a}$.

25. Whence 'tis easie to inferr that these are Proportionals, *viz.* . . . . $a \,.\, m :: h \,.\, \frac{ha}{a+g} \left(= \frac{bc}{a}\right)$

26. But it hath been shewn in 20°, that . . . . $a+g \,.\, a :: h \,.\, \frac{ha}{a+g}$.

27. Therefore from 25° and 26°, (*per prop.* 11. *Elem.* 5.) . . . . $a \,.\, m :: a+g \,.\, a$.

28. And from 27°, by comparing the Rectangle of the extremes to the Rectangle of the means, . . $aa = ma + mg$.

29. And from 28°, by subtracting ma from each part, ... $aa - ma = mg$.

30. Which last Equation may be resolved into these Proportionals, *viz.* . . . . $a - m \,.\, \sqrt{mg} :: \sqrt{mg} \,.\, a$

31. Of which three Proportionals, the mean, to wit, $\sqrt{mg}$ is given, as also m the difference of the extremes a and $a - m$; therefore the extremes shall be given severally, (*per Probl.*12. *Chap.*5.) the greater whereof is equal to the desired line AK, which (by the Theorem in 24° of the said *Probl.* 12. *Chap.* 5) will be found equal to this right line, (or number,) *viz.*

$$\tfrac{1}{2}m + \sqrt{:\tfrac{1}{4}mm + mg:} = AK = a.$$

From which Equation and premisses we may deduce this following

CANON.

32. Let it be made as ED to AC, so AH to a fourth Proportional, which may be called M; then to the half of M add the square Root of the summ of the Square of half M and the Rectangle of M into AE, so shall the summ of that addition be the value of AK sought.

33. This Canon serves to find out the value of the line AK in every one of the three preceding Figures, and when the given point D is posited according to the Definition of the first and second Cases in 1°, as in *Fig.* 1, and 2, it is easie to discover from what hath been said in 3° and 4°, which of the sides of the given Triangle ABC will be cut by the line of partition to be drawn from D. But when the point D is posited according to the Definition of the third Case in 1°, as in *Fig.* 3. then it may be doubtfull which of the sides are to be cut; to remove therefore this ambiguity, observe the following Directions, *viz.* First, draw a right line from the given point D (in *Fig.* 3.) to pass by the angular point B, as DBP; then is EABD a Trapezium, having (by *Construction*) two parallel sides AB and ED, and the other two sides EA and DB which are not Parallels, being continued will meet in some point in AC, as in P, for (by *Construction*) EAC is a right line. Now if AB, ED and EA be severally given in numbers, the line AP shall be also given in number; for putting $g = AE$, and $h = ED$, (as before in the Resolution,) also $k = AB$; the line AP (by the Theorem in 9° of *Probl.* 18. *Chap.* 7.) will be found equal to $\frac{gk}{h-k}$. It is also manifest, that if a right line be drawn from any point in AC, between P and C, to the given point D, the line so drawn must necessarily cut BC, for the line PBD is supposed to pass by the angular point B; but if a right line be drawn from any point in AC between P and A to the point D, the line so drawn will evidently cut AB. From the premisses therefore we may inferr this following

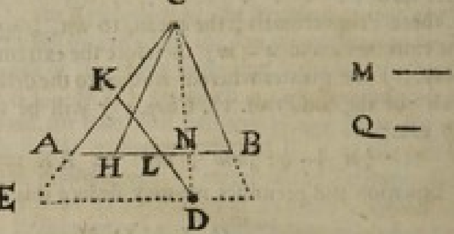
RULE.

34. If $\tfrac{1}{2}m + \sqrt{:\tfrac{1}{4}mm + gm:}$ the value of AK, be not greater than $\frac{gk}{h-k}$ the value of AP in *Fig.* 3. then the line of partition to be drawn from the given point D, shall either pass by the angular point B, as the line DBP, or else cut AB in some point between B and A, and AC in some point between P and A. But if the said value of AK be greater than the said value of AP, then the line of partition will cut BC and AC, and in this latter Case a Parallel is to be drawn by the point D to BC, which is to be esteemed the Base, and then the given lines being

respectively changed, the line found out by the Canon must be set from C towards A.

35. The Problem propounded needs not any Determination to be annexed to it, either to limit the quantities of the given lines, or the position of the given point without the given Triangle. But because from what hath been said in 1°, 2°, 3°, 4° and 5°, it evidently appears, that in every one of the three preceding Figures the given side A C must be greater than the quantity of the sought line A K, except only in one Case in *Fig.* 1. when H lyes in the line D C, and is the same with the point N, for then A K is equal to A C; it will be requisite to prove, that in all other cases the side A C is greater than that right line which the Canon finds out for the value of A K, *viz.* in the letters of the Resolution, that $b \sqsubset -\frac{1}{2}m + \sqrt{:}\,\overline{\tfrac{1}{4}mm + gm}:$ The truth hereof I shall first demonstrate in this following

FIG. 1.



Prepar.

36. Let it be made as E D to A H, so A C to a fourth proportional line, which may be called M, therefore,	ED . AH :: AC . M.
37. Let it also be made as E D to A H, so A E to a fourth proportional line, Q, therefore, . . .	ED . AH :: AE . Q.
38. Then from those Analogies it follows (*per prop.* 11. *Elem.* 5.) that	AE . Q :: AC . M.
39. . . . *Req. demonstr.*	$AC \sqsubset -\frac{1}{2}M + \sqrt{:}\,\overline{\square AE,M + \tfrac{1}{4}\square M}:$
That is, in the letters of the Resolution,	$b \sqsubset -\frac{1}{2}m + \sqrt{:}\,\overline{gm + \tfrac{1}{4}mm}:$

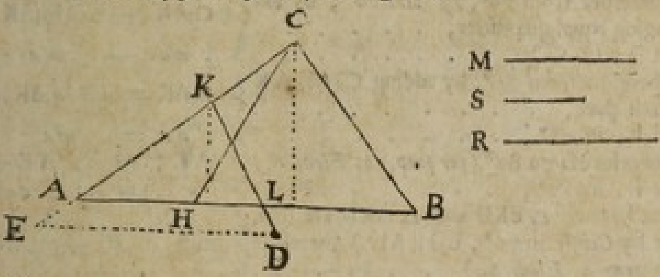
Demonstration.

40. By *Constr.* in 1°, AN ∥ ED, therefore in *Fig.* 1. △ ACN and △ ECD are equiangular, and consequently, (*per prop.* 4. *Elem.* 6.)	EC . ED :: AC . AN.
41. Therefore from 40°, (*per prop.* 16. *Elem* 6.)	$\square$ EC, AN = $\square$ ED, AC.
42. By *Suppos.* in *Fig.* 1.	AN $\sqsubset$ AH.
43. Therefore by drawing E C into each part, ...	$\square$ EC, AN $\sqsubset$ $\square$ EC, AH.
44. Therefore from 41° and 43° (*per Ax.* 4. *Chap.* 2.)	$\square$ ED, AC $\sqsubset$ $\square$ EC, AH.
45. And because, (as is evident by *Fig.* 1.)	AC + AE = EC.
46. Therefore from 45°, by drawing A H into each part, ...	$\square$ AC, AH + $\square$ AE, AH = $\square$ EC, AH.
47. Therefore from 44° and 46°, (*per Ax.* 4. *Chap.* 2.)	$\square$ ED, AC $\sqsubset$ $\square$ AC, AH + $\square$ AE, AH.
48. From 36° it follows (*per prop.* 16. *Elem.* 6.) that	$\square$ ED, M = $\square$ AC, AH.
49. Likewise from 37°, that . . .	$\square$ ED, Q = $\square$ AE, AH.
50. And by adding the Equation in 49° to that in 48°, it makes	$\square$ED,M+$\square$ED,Q=$\square$AC,AH+$\square$AE,AH
51. Therefore from 47° and 50°, (*per Ax.* 4. *Chap.* 2.)	$\square$ ED,AC $\sqsubset$ $\square$ ED,M + $\square$ ED,Q.
52. Therefore from 51°, by casting away the common altitude E D, . .	AC $\sqsubset$ M + Q.

53. And from 52°, by taking in the common altitude AC, ...	$\square AC \sqsubset \square AC,M + \square AC,Q.$
54. But from 38° it follows (*per prop.16. Elem. 6.*) that	$\square AE,M = \square AC,Q.$
55. Therefore from 53° and 54°, (*per Ax. 6. Chap. 2.*)	$\square AC \sqsubset \square AC,M + \square AE,M.$
56. And from 55°, by subtracting □ AC, M from each part, . . .	$\square AC - \square AC,M \sqsubset \square AE,M.$
57. And by adding ¼□M to each part in 56°, ...	$\square AC + \frac{1}{4}\square M - \square AC,M \sqsubset \square AE,M + \frac{1}{4}\square M.$
58. From 52° 'tis evident, that . . .	$AC \sqsubset \frac{1}{2}M.$
59. And by *Theor.* 5. *Chap.* 4. . . .	$\square:\overline{AC-\frac{1}{2}M}: = \square AC + \frac{1}{4}\square M - \square AC,M.$
60. Therefore from 57° and 59°, (*per Ax.* 4. *Chap.* 2.)	$\square:\overline{AC-\frac{1}{2}M}: \sqsubset \square AE,M + \frac{1}{4}\square M.$
61. And because if one Plane exceeds another, the side of a Square equal to the former shall exceed the side of a Square equal to the latter, therefore from 60°, ...	$AC - \frac{1}{2}M \sqsubset \surd:\overline{\square AE,M + \frac{1}{4}\square M}:$
62. Therefore from 61°, by adding ½M to each part, ...	$AC \sqsubset \frac{1}{2}M + \surd:\overline{\square AE,M + \frac{1}{4}\square M}:$

Which was to be Dem. The like Demonstration may be applied to *Fig.* 2, and 3. after N is set in the place of B in *Fig.* 2. and C in the place of P in *Fig.* 3.

The Composition of the preceding Probl. 18.



Suppos.

63. △ ABC is given.
64. AH and HB are in a given Reason.
65. D is a point given without the △ ABC.
66. DE is ∥ AB, and given.
67. AE the Increase of CA continued until it cut DE is given.

Req. to find

68. AK such a segment of AC, that DK being drawn, it may make
69. △ ALK . LKCB :: AH . HB.

Construction.

70. Supposing (by what hath been said in 1°, 2°, 3°, 4°, 5°, 33° and 34°,) it be discovered, that the line of partition to be drawn from D must cut the sides AB and AC; draw DE parallel to AB, and cutting CA continued in E.

71. Then by *Probl.* 8. *Chap.* 5. let it be made, as ED to AC, so AH to a fourth proportional line M, therefore	ED . AC :: AH . M.
72. Find a mean proportional line, as S, between M and AE, therefore	M . S :: S . AE.
73. Then esteeming the line S to be the mean of three Proportionals, and the line M the difference of the extremes, find out the extremes, (*per Probl.* 12. *Chap.* 5.) the greater whereof suppose to be the line R, whence the lesser shall be equal to R — M, therefore these are Proportionals, *viz.*	R−M . S :: S . R.
That is, in 30°, the last step of the Resolution, . .	$a-m . \sqrt{mg} :: \sqrt{mg} . a.$

74. From

74. From A C cut off AK = R, which may be done, for that A C is greater than R, I prove thus;

By *Constr.* in 73°, ...	$R - M \,.\, S :: S \,.\, R.$
Therefore by the Theorem in 24° of *Prob.* 12. *Chap.* 5.	$R = \frac{1}{2}M + \sqrt{:\overline{\square S + \frac{1}{4}\square M}}:$
And because from 72°, (*per prop.* 17. *Elem.* 6.)	$\square AE, M = \square S.$
Therefore from the two last steps, (*per Ax.* 6. *Chap.* 2.)	$R = \frac{1}{2}M + \sqrt{:\overline{\square AE, M + \frac{1}{4}\square M}}:$
But it hath been shewn in 62°, that	$AC \sqsubset \frac{1}{2}M + \sqrt{:\overline{\square AE, M + \frac{1}{4}\square M}}:$
Therefore from the two last preceding steps, (*per Ax.* 3. *Chap.* 2.)	$AC \sqsubset R.$ Which was to be Dem.

75. Lastly, draw the line DK, cutting AB in L; then shall the Triangle ALK be to the Trapezium LKCB, as A H to H B; which was required. The truth whereof will be made manifest by the following Demonstration, formed out of the foregoing Resolution by a repetition of its steps in a backward order.

76. . . . *Req. demonstr.* . . . . . . $\triangle ALK \,.\, LKCB :: AH \,.\, HB.$

Demonstration.

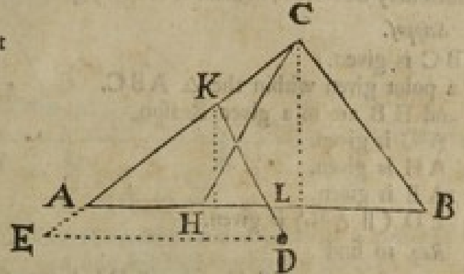
77. By *Constr.* in 73°, ...	$R - M \,.\, S :: S \,.\, R.$
That is, in 30°, (the last step of the Resolution,)	$a - m \,.\, \sqrt{mg} :: \sqrt{mg} \,.\, a.$
78. Therefore from 77°, (*per prop.* 17. *Elem.* 6.)	$\square R - \square MR = \square S.$
79. And because by *Constr.* in 74°, ...	$AK = R.$
80. And from the *Constr.* in 72°, (*per prop.* 17. *Elem.* 6.)	$\square M, AE = \square S.$
81. Therefore from 78°, 79° and 80°, by exchanging equal quantities, ...	$\square AK - \square M, AK = \square M, AE.$
That is, in 29°, ...	$aa - ma = mg.$
82. Therefore from 81°, by adding $\square M, AK$ to each part, ...	$\square AK = \square M, AK + \square M, AE.$
That is, in 28°, ...	$aa = ma + mg.$
83. Therefore from 82°, (*per prop.* 14. *Elem.* 6.)	$AK \,.\, M :: AK + AE \,.\, AK.$
That is, in 27°, ...	$a \,.\, m :: a + g \,.\, a.$
84. But because $\triangle$ EKD and $\triangle$ AKL are like, (for by *Constr.* in 70°, ED ∥ AL,) therefore (*per prop.* 4. *Elem.* 6.)	$ED \,.\, AL :: AK + AE \,.\, AK.$
That is, in 26°, ...	$b \,.\, \frac{ba}{a+g} :: a + g \,.\, a.$
85. Therefore from 83° and 84°, (*per prop.* 11. *Elem.* 5.)	$AK \,.\, M :: ED \,.\, AL.$
That is, in 25°, ...	$a \,.\, m :: b \,.\, \frac{ba}{a+g} (= \frac{bc}{a}.)$
86. But by *Constr.* in 71°;	$M \,.\, AC :: AH \,.\, ED.$
87. Therefore from 85° and 86°, (*per prop.* 23. *Elem.* 5.) agreeable to *Defin.* 8. *Chap.* 3. concerning Inordinate proportion, ...	$AK \,.\, AC :: AH \,.\, AL.$
That is, in 18°, ...	$a \,.\, b :: c \,.\, \frac{bc}{a}.$
88. And because $\triangle$ ALK and $\triangle$ ACH have a common angle, to wit, $<$ KAL, and (as appears by the Analogy in 87°,) the sides about that angle are reciprocally proportional, therefore (*per prop* 15. *Elem.* 6.)	$\triangle ALK = \triangle ACH.$
89. Therefore by subtracting $\triangle$ ALK and $\triangle$ ACH severally from the $\triangle$ ABC, the remaining spaces shall be equal one to another, *viz.*	Trapez. $LKCB = \triangle HCB.$
90. But (*per prop.* 1. *Elem.* 6.)	$\triangle ACH \,.\, \triangle HCB :: AH \,.\, HB.$
91. Therefore from 88°, 89° and 90°, by exchanging equal spaces, ...	$\triangle ALK \,.\, LKCB :: AH \,.\, AB.$

Which was to be demonstrated. Therefore that is done which the Problem required.

An

An Example in Numbers, to illustrate the foregoing Resolution *of* Probl. 18.

Suppos.



92. $\triangle$ ABC is given.
93. D is a point given without the $\triangle$ ABC.
94. DE $\parallel$ AB.
95. CAE is a right line.
96. AB = 200.
97. AC = 160.
98. BC = 120.
99. DE = 140.
100. AE = 25.
101. AH = 70. } in the given Reason of 7 to 13.
102. HB = 130. }

Req. to find

103. The quantity of AK in number, such, that DK being drawn, it may make $\triangle$ ALK . LKCB :: AH . HB :: 7 . 13.

Solution Arithmetical.

104. AK = 100, found out by the Canon in 32°, by the help of the lines given in numbers in 99°, 97°, 101° and 100°.

105. AL = 112, for $\left\{ \begin{array}{l} \text{AK} + \text{AE . ED :: AK . AL.} \\ \quad 125 \quad . \ 140 \ :: \ 100 \ . \ 112. \end{array} \right.$

106. CL ($\perp$ AB) = 96, found out by the three sides of $\triangle$ ABC, by the help of *Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.

107. KH ($\perp$ AL) = 60, for $\left\{ \begin{array}{l} \text{AC . CL :: AK . KH.} \\ 160 \ . \ 96 \ :: \ 100 \ . \ 60. \end{array} \right.$

108. $\frac{1}{2}$AB $\times$ CL = 9600, the Area of $\triangle$ ABC.

109. $\frac{1}{2}$AL $\times$ KH = 3360, the Area of $\triangle$ ALK.

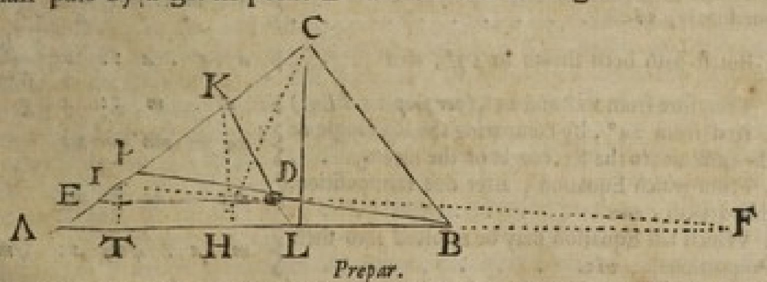
110. $\triangle$ ABC — $\triangle$ ALK = 6240, the Area of LKCB.

The Proof.

111. . . . $\left\{ \begin{array}{l} \triangle \text{ALK . LKCB :: AH . HB :: 7 . 13.} \\ \quad 3360 \ . \ 6240 \ :: \ 70 \ . \ 130 \ :: \ 7 \ . \ 13. \end{array} \right.$

Probl. XIX.

To divide a given Triangle ABC into two parts which shall be in any possible Reason given, suppose as AH to HB, by a right line KL that shall pass by a given point D within the Triangle.



Prepar.

1. By the given point D draw a Parallel to one of the sides of the given Triangle, as DE parallel to the Base AB, and cutting AC in E.
2. Divide the Base AB into two parts in the given Reason, as in the point H, and draw CH; therefore, (*per prop.* 1. *Elem.* 6.) $\triangle$ ACH . $\triangle$ HCB :: AH . HB.
3. Then supposing it be discovered (by the Rule hereafter given in 31° of this Problem,) that the line of partition required to pass by the given point D must cut AB and AC, the Scope of the Resolution is to find a point in AC, as K, from which a right line being drawn to pass by D, as KDL, the Triangle AKL shall be equal to the Triangle AHC,

AHC, whence it evidently follows that LKCB = △ HCB, and △ AKL . LKCB :: AH . HB. These things being premised, the Resolution of the Problem propounded may be formed in manner following.

Suppos.

4. △ ABC is given.
5. D is a point given within the △ ABC.
6. AH and HB are in a given Reason.
7. b = AC is given.
8. c = AH is given.
9. g = AE is given.
10. h = ED (∥ AB) is given.

Req. to find

11. AK such a segment of AC, that KDL being drawn, it may make
12. △ AKL . LKCB :: AH . HB :: △ ACH*. △ HCB.

Resolution.

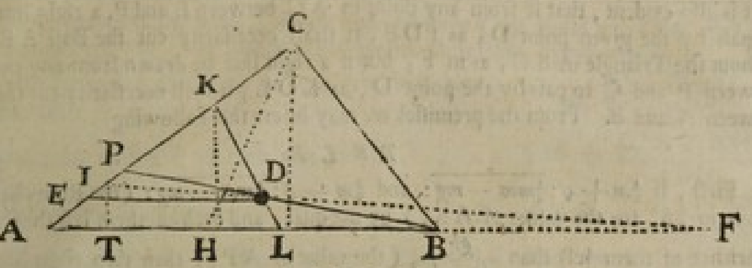
13. Suppose that done which is required, and put	a = AK.
14. Then by considering well what is required, and viewing the Diagram, it will appear that △ AKL = △ ACH, and that the ∠ CAL is common to both those Triangles, therefore (*per prop.* 15. *El.* 6.)	AK . AH :: AC . AL.
15. That is, in the letters of the Resolution, .	$a \,.\, c :: b \,.\, \frac{bc}{a}$.
16. And because △ AKL and △ EKD are like, (for by *Constr.* in 1°, DE ∥ AL,) therefore (*per prop.* 4. *Elem.* 6.)	EK . AK :: ED . AL.
17. That is, in the letters of the Resolution, . .	$a-g \,.\, a :: h \,.\, \frac{ha}{a-g}$.
18. And because the fourth Proportional in 14° is the same with the fourth in 16°, therefore the fourth Proportionals in 15° and 17°, shall be equal to one another, *viz.*	$\frac{bc}{a} = \frac{ha}{a-g}$.
19. Now to avoid an Equation between Solids, let it be made as h to b, so c to a fourth Proportional, which may be called m, therefore . . .	$h \,.\, b :: c \,.\, m$.
20. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, *viz.*	$hm = bc$.
21. Therefore from 18° and 20°, by exchanging equal Rectangles, 'tis evident that	$\frac{hm}{a} = \frac{ha}{a-g} = \frac{bc}{a}$.
22. Whence 'tis easie to inferr that these are Proportionals, *viz.*	$a \,.\, m :: h \,.\, \frac{ha}{a-g} \left(= \frac{bc}{a}.\right)$
23. But it hath been shewn in 17°, that . . .	$a-g \,.\, a :: h \quad \frac{ha}{a-g}$.
24. Therefore from 22° and 23°, (*per prop.* 11. *El.* 5.)	$a \,.\, m :: a-g \,.\, a$.
25. And from 24°, by comparing the Rectangle of the extremes to the Rectangle of the means, . .	$aa = ma - mg$.
26. From which Equation, after due transposition, this ariseth, *viz.*	$ma - aa = mg$.
27. Which last Equation may be resolved into these Proportionals, *viz.*	$m-a \,.\, \sqrt{mg} :: \sqrt{mg} \,.\, a$.

28. Of which three Proportionals, the mean, to wit, $\sqrt{mg}$ is given, as also m the summ of the extremes, therefore the extremes shall be given severally (*per Probl.* 13. *Chap.* 5.) the values whereof, by the Theorem in 21° of the said *Probl.* 13. *Chap.* 5. will be found equal to these right lines, (or numbers,) *viz.*

$$\left.\begin{cases} \frac{1}{2}m + \sqrt{:\tfrac{1}{4}mm - mg:} \\ \frac{1}{2}m - \sqrt{:\tfrac{1}{4}mm - mg:} \end{cases}\right\} \text{the two Roots of the Equation in 26°.}$$

Hence this

CANON.



CANON.

Let it be made as E D to A C, fo A H to a fourth Proportional, which may be called M. Then to and from the half of M, add and fubtract the fquare Root of the excefs of the Square of half M above the Rectangle of M into A E, fo fhall the fumm and remainder made by that Addition and Subtraction be the two values of A K fought, (reprefented by a in the foregoing Refolution.)

29. But to the end the faid values of A K may be Real, that is, greater than nothing, the given lines muft be fubject to this

Determination. g not $\sqsubset$ $\frac{1}{4}m$, or $\frac{bc}{4b}$; that is, in words,

The line A E muft not exceed the right line arifing by the Application of the Rectangle made of the lines A C and A H to the quadruple of the line E D.

The truth of this Determination, which is difcovered both by the Analogy in 27°, and by the values of a in 28°, may be proved thus,

It hath been difcovered in 27°, that thefe are Proportionals, *viz.*	$m - a \cdot \sqrt{mg} :: \sqrt{mg} \cdot a.$
Of which Proportionals the fumm of the extremes is evidently m, and the mean is $\sqrt{mg}$; therefore (as hath been fhewn in 20° of *Probl.* 13. *Chap.* 5.) . . .	$\sqrt{mg}$ not $\sqsubset$ $\frac{1}{2}m$.
Whence, by fquaring each part it follows, that . .	mg not $\sqsubset$ $\frac{1}{4}mm$.
And by Application of each part to m, ...	g not $\sqsubset$ $\frac{1}{4}m$.
But by *Conftr.* in 19°, ...	$\frac{bc}{b} = m$.
And confequently, by taking $\frac{1}{4}$ of each part, . .	$\frac{bc}{4b} = \frac{1}{4}m$.
Therefore from the three laft fteps, ...	g not $\sqsubset$ $\frac{1}{4}m$, or $\frac{bc}{4b}$.

Which was to be Demonftr.

30. It is alfo eafie to perceive by the two Roots or extreme Proportionals found out in 28°, that if $g = \frac{1}{4}m$, and confequently $mg = \frac{1}{4}mm$, then thofe Roots will be equal to one another, *viz.* each Root equal to $\frac{1}{2}m$, which, if it fall within the limits hereafter declared in the Rule in 31° of this Problem, fhall be equal to the line A K fought; But if $g \sqsupset \frac{1}{4}m$, then the faid Roots will be unequal, and the Equation in 26° may be expounded by each of thofe Roots; and fometimes either of them may be taken for the value of the line A K fought, but fometimes only one of the faid Roots, and fometimes neither of them. To dicover therefore whether there be a poffibility of effecting the Problem or not, by the lines given in fuch manner as before is expreft, the value of the line A P muft be enquired; to which end, firft, draw a right line that may pafs by the angular point B and the given point D, (in the precedent Diagram,) then is E A B D a Trapezium having (by *Conftr.* in 1°,) two parallel fides A B and E D, and the other two fides AE and BD, which are not Parallels, being continued will meet in fome point in AC, as in P. Now if AB, ED and AE be feverally given in numbers, the line EP fhall be alfo given in number, for putting $g =$ A E, and $b =$ E D, (as before in the Refolution,) alfo $k =$ A B; the line E P (by the Theorem in 9° of *Probl.* 18. *Chap.* 7.) will be found equal to $\frac{gb}{k-b}$, to which adding g, that is, A E, it makes $\frac{gk}{k-b}$ for the value of A P.

It is

It is also evident, that if from any point in A C between E and P, a right line be drawn to pass by the given point D, as I D F, it shall necessarily cut the Base A B continued without the Triangle A B C, as in F; but if a right line be drawn from any point in A C between P and C to pass by the point D, as K D L, it shall necessarily cut the Base A B between A and B. From the premisses we may inferr this following

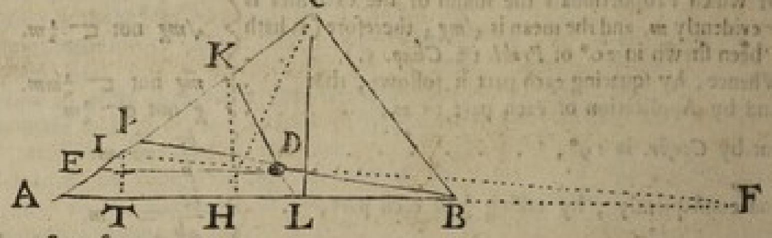
RULE.

31. First, if $\frac{1}{2}m + \sqrt{:\frac{1}{4}mm - mg:}$ and $\frac{1}{2}m - \sqrt{:\frac{1}{4}mm - mg:}$ (the Roots before found out in 28° for the values of A K,) be unequal, and each of them less than A C, but neither of them less than $\frac{gk}{k-b}$, (the value of A P,) then two right lines equal to those Roots or values of A K being set from A towards C will end in two points, from which if two right lines be drawn to pass by the given point D, each of them shall divide the given Triangle A B C into two parts in the given Reason.

Secondly, if only one of those Roots or values of A K happens to be less than A C, and not less than the said value of A P, then the given Triangle A B C can be divided only by one right line passing by the given point D, so as to cut both A B and A C, or A C only, to divide the said Triangle into two parts in the given Reason.

Thirdly and lastly, if neither of the said Roots or values of A K happens to be within the limits above exprest, then 'tis impossible to draw a right line that shall pass by the given point D, and cut both A B and A C, or A C only, and pass by the angular point B, to divide the given Triangle into two parts in the given Reason: And if the like impossibility happen after a Parallel is drawn by the given point D to each of the other two sides A C, B C, and tryal made as before, it shall be impossible to perform what the Problem requires; but when there is a possibility, then oftentimes which two sides you please may be cut by the line of partition.

The Composition of the foregoing Probl. 19.



Suppos.

32. △ A B C is given.
33. A H and H B are in a given Reason.
34. D is a point given within the △ A B C.
35. D E (∥ A B) is given.
36. A E is given, and not greater than $\frac{\square AC, AH}{4 ED}$, agreeable to the preceding *Determination* in 29°.

Req. to find

37. A K such a segment of A C, that K D L being drawn, it may make
38. △ ALK . LKCB :: AH . HB :: △ ACH . △ HCB.

Construction.

39. By the given point D draw . . . . } DE ∥ AB.
40. Then supposing it be discovered by the preceding Rule in 31°, that the line of partition which is to pass by the given point D may cut A B and A C, let it be made (by *Probl.* 8. *Chap.* 5.) as E D to A C, so A H to a fourth proportional line, which may be called M, therefore, ... } ED . AC :: AH . M.
41. Find a mean proportional line, as S, between M and A E, therefore . . . . } M . S :: S . AE.
42. Divide the line M (before found) into two such parts, that the line S may be a mean Propor-

Proportional between the parts, which may be done (by *Probl.* 14. *Chap.* 5.) if S be not greater than $\frac{1}{2}$M; but that S is not greater than $\frac{1}{2}$M, I prove thus:

From the foregoing *Constr.* in 40°, it is easie to perceive that	$M = \frac{\square AC, AH}{ED}$.
Whence, by taking $\frac{1}{4}$ of each part, ...	$\frac{1}{4}M = \frac{\square AC, AH}{4ED}$.
By the Determination in 36°, ...	AE not $\sqsubset \frac{\square AC, AH}{4ED}$.
Therefore from the two last steps, (*per Ax.* 3. *Chap.* 2.)	AE not $\sqsubset \frac{1}{4}$M.
Therefore by drawing M into each part, . . .	$\square$M, AE not $\sqsubset \frac{1}{4}\square$M.
And because from 41°, (*per prop.* 17. *Elem.* 6.) .	$\square S = \square$ M, AE.
Therefore from the two last preceding steps, (*per Ax.* 4. *Chap.* 2.)	$\square$S not $\sqsubset \frac{1}{4}\square$M.
But if one Square exceeds another, the side of the former exceeds also the side of the latter, therefore from the last step, ...	S not $\sqsubset \frac{1}{2}$M.

Which was to be Demonstr. Therefore 'tis possible to cut the line M into two such parts, that the line S may be a mean Proportional between them; suppose then the right line R be found the greater part, and T the lesser, therefore these are Proportionals, *viz.*

$$\begin{cases} M - R \;.\; S :: S \;.\; R. \\ M - T \;.\; S :: S \;.\; T. \end{cases}$$

That is, in 27°, . . . $m - a \;.\; \sqrt{mg} :: \sqrt{mg} \;.\; a.$

Which two lines R and T do answer to the two Roots or values of AK before exprest in the Canon in 28°, *viz.*

$$\left.\begin{array}{l} R = \frac{1}{2}m + \sqrt{:\tfrac{1}{4}mm - mg}: \\ T = \frac{1}{2}m - \sqrt{:\tfrac{1}{4}mm - mg}: \end{array}\right\} \text{ the values of AK.}$$

43. Now supposing that by the limits in 31° it be discovered that the greater Root or line R is less than AC, and not less than AP, let the line R be set from A towards C, as to K, *viz.* make AK = R, and draw the right line KDL; then shall △AKL be equal to △ACH, and consequently △AKL . LKCB :: AH . HB :: △ACH . △HCB, as was required.

And if the lesser Root or line T happens to be less than AC, but not less than AP, let the line T be set also from A towards C, (as to I in the following Diagram belonging to the latter Example of this Problem,) and then a right line being drawn from the point in AC where T doth end, to pass by the given point D, it shall likewise divide the given Triangle ABC into two parts in the given Reason.

But if either of the said lines R and T, which are to be set (as before) from A towards C, happens to fall between E and P, then the line of partition to pass by D will cut the Base AB continued without the Triangle ABC; as, if AI be supposed equal to T, and the line ID be drawn and continued till it concurr with the Base AB continued, as in F, then although △AIF be equal to △ACH, yet it solves not the Problem, in regard part of △AIF lyes without the given △ABC.

It remains to prove that △ALK . LKCB :: AH . AB, but this will be made manifest by the following Demonstration, formed out of the preceding Resolution by a repetition of its steps in a backward (not direct) order.

44. . . . *Req. demonstr.* . . . . . . △ALK . LKCB :: AH . HB.

Demonstration.

45. By *Constr.* in 42°, ...	M — R . S :: S . R.
That is, in 27°, (the last step of the Resolution,)	$m - a \;.\; \sqrt{mg} :: \sqrt{mg} \;.\; a.$
46. Therefore from 45°, (*per prop.* 17. *Elem.* 6.)	$\square$MR — $\square$R = $\square$S.
47. And because by *Constr.* in 43°, . . .	AK = R.
48. And from the *Constr.* in 41°, (*per prop.* 17. *Elem.* 6.)	$\square$M, AE = $\square$S.
49. Therefore from 46°, 47° and 48°, by exchanging equal quantities, ...	$\square$M, AK — $\square$AK = $\square$M, AE.
That is, in 26°, ...	$ma - aa = mg.$

50. There-

50. Therefore from 49°, by adding □ AK, and fubtracting ▭ M, AE from each part, ... □AK = ▭ M,AK − ▭ M,AE.

That is, in 25°, ... $aa = ma - mg$.

51. Therefore by refolving the Equation in 50° into Proportionals, ... AK . M :: AK − AE . AK.

That is, in 24°, ... $a \,.\, m :: a-g \,.\, a$.

52. But becaufe △ EKD and △ AKL are like, (for by *Conftr.* in 39°, ED ∥ AL,) therefore (*per prop.* 4. *Elem.* 6.) . . . . ED . AL :: AK − AE . AK.

That is, in 23°, ... $b \,.\, \frac{ba}{a-g} :: a-g \,.\, a$.

53. Therefore from 51° and 52°, (*per prop.* 11. *Elem.* 5.) . . . . AK . M :: ED . AL.

That is, in 22°, ... $a \,.\, m :: b \,.\, \frac{ba}{a-g} (= \frac{bc}{a}.)$

54. But by *Conftr.* in 40°, ... ED . AC :: AH . M.

55. Therefore from 53° and 54°, (*per prop.* 23. *Elem.* 5.) agreeable to *Defin.* 8. *Chap.* 3. . AK . AC :: AH . AL.

That is, in 15°, ... $a \,.\, b :: c \,.\, \frac{bc}{a}$.

56. And becaufe △ AKL and △ ACH have a common angle, to wit, ∠ KAL, and (as appears by the Analogy in 55°,) the fides about that angle are reciprocally proportional, therefore (*per prop.* 15. *Elem.* 6.) . . . . . △ AKL = △ ACH.

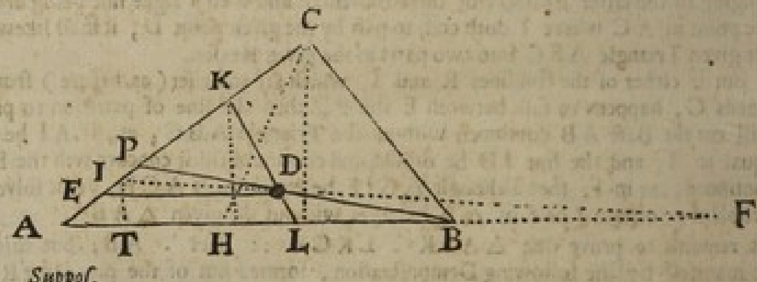
57. Therefore by fubtracting △ AKL and △ ACH feverally from △ ABC, the remaining Spaces fhall be equal one to another, *viz.* . . . . Trapez. LKCB = △ HCB.

58. But (*per prop.* 1. *Elem.* 6.) . . . . . △ ACH . △ HCB :: AH . HB.

59. Therefore from 56°, 57° and 58°, by exchanging equal Spaces, ... △ AKL . LKCB :: AH . HB.

Which was to be demonftrated. Therefore that is done which the Problem required.

An Example in Numbers, to illuftrate the precedent Refolution *of* Probl. 19. *in which Example the greater of the two Roots or values of* AK *before expreft in* 28°, *is only capable of folving the* Problem.



Suppof.

60. AB = $231\frac{1}{4}$ the Bafe } of △ ABC are given.
61. AC = 185 } the leggs
62. BC = $138\frac{1}{4}$
63. D is a point given within the △ ABC.
64. DE ∥ AB; and DE = 112 is given.
65. EA = 25 is given.
66. AH = $94\frac{88}{148}$ } in a given Reafon, *viz.* as 560 to 809.
67. HB = $136\frac{22}{148}$
68. CL, KH, IT are each ⊥ AB.

Req. to find in number,

69. AK, fuch, that the right line KDL being drawn, it may make
70. △ AKL . LKCB :: AH . HB :: 560 . 809.

Solution

Solution Arithmetical.

71. AK = 125 = the greater of the two Roots in 28°, found out by the Canon there exprest.

The Proof.

72. AL = 140; for $\left\{ \begin{array}{l} AK - AE \;.\; ED :: AK \;.\; AL. \\ \;\;100 \;\;\;.\; 112 :: 125 \;.\; 140. \end{array} \right.$

73. CL = 111, found out by the three sides of △ ABC given in 60°, 61° and 62°, with the help of *Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.

74. KH = 75; for $\left\{ \begin{array}{l} AC \;.\; CL :: AK \;.\; KH. \\ 185 \;.\; 111 :: 125 \;.\; 75. \end{array} \right.$

75. $\frac{1}{2}$AB × CL = $12834\frac{3}{8}$ the Area of △ ABC.

76. $\frac{1}{2}$AL × KH = 5250 the Area of △ AKL.

77. △ ABC — △ ALK = $7584\frac{3}{8}$ the Area of LKCB.

78. $\left\{ \begin{array}{l} \triangle AKL \;.\; LKCB :: AH \;.\; HB :: 560 \;.\; 809. \\ \;\;5250 \;\;.\; 7584\frac{3}{8} :: 94\frac{88}{148} \;.\; 136\frac{23}{148} :: 560 \;.\; 809. \end{array} \right.$

Which was to be done.

79. *Note.* In this Example, AI the lesser of the two Roots or values of AK in 28° falls between E and P, and therefore it cannot solve the Problem; for if ID be drawn and continued, it will cut AB continued without the △ ABC, as in F. But 'tis worth observing, that the △ AIF is equal to the △ AKL, as the following Proof will make manifest.

80. AI = $31\frac{1}{4}$ = the lesser Root, found out by the Canon in 28°.

81. EA = 25, given in 65°.

82. EI = $6\frac{1}{4}$ = AI — AE.

83. DE = 112, given in 64°.

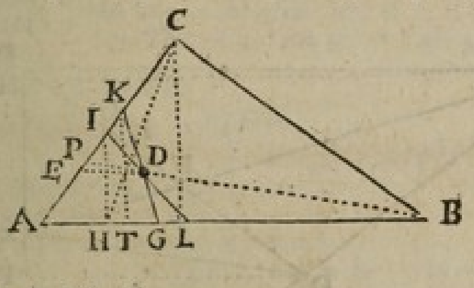
84. AF = 560; for $\left\{ \begin{array}{l} EI \;.\; ED :: AI \;.\; AF. \\ 6\frac{1}{4} \;.\; 112 :: 31\frac{1}{4} \;.\; 560. \end{array} \right.$

85. IT = $18\frac{3}{4}$; for $\left\{ \begin{array}{l} AK \;.\; KH :: AI \;.\; IT. \\ 125 \;.\; 75 :: 31\frac{1}{4} \;.\; 18\frac{3}{4}. \end{array} \right.$

86. $\frac{1}{2}$AF × IT = 5250 = $\frac{1}{2}$AL × KH, therefore △ AIF = △ AKL.

Which was to be proved.

Another Example in Numbers, referring to the subsequent Figure, where each of the two Roots or values of AK *before exprest in* 28° *is capable of solving the foregoing* Probl. 19.



Suppos.

87. AB = $231\frac{1}{4}$ the Base
88. AC = $138\frac{3}{4}$ } the leggs } of △ ABC are given.
89. BC = 185

90. D is a point given within the △ ABC.

91. DE ∥ AB, and DE = $36\frac{42}{111}$ is given.

92. EA = 38 is given.

93. AH = 40
94. HB = $191\frac{1}{4}$ } in a given Reason, *viz.* as 32 to 153.

95. CL, KT, IH are each ⊥ AB.

Req. to find in numbers,

96. AK and AI, such, that the right lines KDG and IDL being drawn, these Analogies shall ensue, *viz.*

97. $\left\{ \begin{array}{l} \triangle AKG \;.\; GKCB :: AH \;.\; HB :: 32 \;.\; 153. \text{ Also,} \\ \triangle AIL \;.\; LICB :: AH \;.\; HB :: 32 \;.\; 153. \end{array} \right.$

Solution

Solution Arithmetical.

98. $AK = 84\frac{6}{251}$, } the two Roots of the Equation in 26°, found out by the *Canon* in 28°.
99. $AI = 69\frac{1}{8}$, }

The Proof by the greater Root AK.

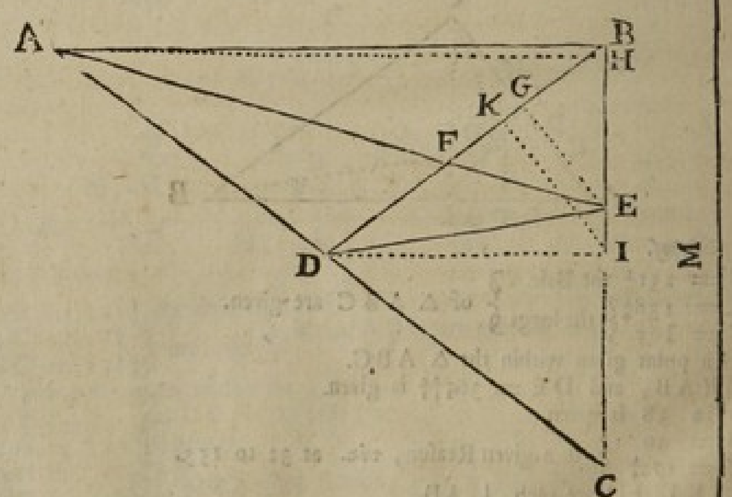
100. $AK = 84\frac{6}{251}$ before found out in 98°.
101. $AE = 38$ given in 92°.
102. $EK = 46\frac{6}{251} = AK - AE$.
103. $ED = 36\frac{20}{111}$ given in 91°.
104. $AG = 66\frac{1}{19}$; for $\left\{\begin{array}{ccccccc} EK & . & ED & :: & AK & . & AG. \\ 46\frac{6}{251} & . & 36\frac{20}{111} & :: & 84\frac{6}{251} & . & 66\frac{1}{19}. \end{array}\right.$
105. $CL = 111$ found out by the three given fides of $\triangle$ ABC.
106. $KT = 67\frac{55}{251}$; for $\left\{\begin{array}{ccccccc} AC & . & CL & :: & AK & . & KT. \\ 138\frac{3}{4} & . & 111 & :: & 84\frac{6}{251} & . & 67\frac{55}{251}. \end{array}\right.$
107. $\frac{1}{2}AB \times CL = 12834\frac{3}{8}$ = the Area of $\triangle$ ABC.
108. $\frac{1}{2}AG \times KT = 2220$ = the Area of $\triangle$ AKG.
109. $\triangle ABC - \triangle AKG = 10614\frac{3}{8}$ = the Area of GKCB.
110. $\left\{\begin{array}{ccccccccccc} \triangle AKG & . & GKCB & :: & AH & . & HB & :: & 32 & . & 153. \\ 2220 & . & 10614\frac{3}{8} & :: & 40 & . & 191\frac{1}{4} & :: & 32 & . & 153. \end{array}\right.$

The Proof by the leffer Root AI, *found out in* 99°.

111. $AL = 80$; for $\left\{\begin{array}{ccccccc} AI - AE & . & ED & :: & AI & . & AL. \\ 31\frac{1}{8} & . & 36\frac{20}{111} & :: & 69\frac{1}{8} & . & 80. \end{array}\right.$
112. $IH = 55\frac{3}{10}$; for $\left\{\begin{array}{ccccccc} AC & . & CL & :: & AI & . & IH. \\ 138\frac{3}{4} & . & 111 & :: & 69\frac{1}{8} & . & 55\frac{3}{10}. \end{array}\right.$
113. $\frac{1}{2}AB \times CL = 12834\frac{3}{8} = \triangle ABC$.
114. $\frac{1}{2}AL \times IH = 2220 = \triangle AIL$.
115. $\triangle ABC - \triangle AIL = 10614\frac{3}{8} = LICB$.
116. $\left\{\begin{array}{ccccccccccc} \triangle AIL & . & LICB & :: & AH & . & HB & :: & 32 & . & 153. \\ 2220 & . & 10614\frac{3}{8} & :: & 40 & . & 191\frac{1}{4} & :: & 32 & . & 153. \end{array}\right.$

Probl. XX.

To find the length of a right line AB, when we cannot come to either of its ends A or B.



Prepar.

First, fuppofing C to be a Station where A and B may be feen, and that there is liberty to meafure from C towards AB without impediment, meafure CD a diftance at pleafure, (in Feet, or what equal parts you pleafe,) yet fo, as CDA may be a ftraight line;

line; meafure likewife C E, a diftance at pleafure, yet fo, as C E B may be a ftraight line. Again, meafure E D; alfo D F, that is, a diftance directly towards B, until you come to the point F, where D B and E A cut one another; meafure alfo F E.

Secondly, in folving this Problem Arithmetically according to the following Refolution, by the help of thofe five lines meafured as above is directed, thefe are two principal Cafes, *viz.* firft, when the angle A C B is acute; fecondly, when 'tis obtufe; and in regard the three fides of the Triangle C D E are fuppofed (as above) to be feverally given in numbers, we may (by the Corollary in 45° of *Probl.* 10. *Chap* 7.) difcover the kind of every one of the angles of the faid Triangle. Firft then, fuppofe it be found that the angles D C E and C E D in the Diagram before expreft are each of them acute, then if a Perpendicular be let fall from the angle C D E, as D I, upon the Bafe C E, it will neceffarily fall within the faid Triangle C D E; likewife if the angle D C E be acute, and the angle C E D obtufe, then a Perpendicular from E will fall within the faid Triangle upon the oppofite fide C D; but if the angle D C E be obtufe, as in the following Diagram belonging to the latter Example of this Problem 'tis fuppofed to be,) then a Perpendicular from D or E will fall without the Triangle D C E.

Thirdly, let a Perpendicular be fuppofed to fall from E upon D B, as E G; this Perpendicular fhall be lefs than the Perpendicular D I, for E G is manifeftly lefs than the Perpendicular I K, which is lefs than I D the Hypothenufal of the right-angled Triangle I K D.

Fourthly, fuppofing a Perpendicular to fall from A upon C B produced infinitely, it will either fall upon the point B, or elfe within or without the Triangle C B A; but in this Example I fhall fuppofe that Perpendicular to be A H, falling within the Triangle C B A. Now by the help of thofe five lines C D, C E, E D, D F, F E meafured in Feet, or what equal parts you pleafe, the length of the inacceffible diftance AB may be found out in the fame kind of parts, in manner following.

Suppof.

1. b = CE = 152 }
2. c = CD = 210 }
3. d = DF = 90 } Lines given.
4. f = FE = 100 }
5. g = DE = 170 }
6. p = DI = 168 }
7. k = IE = 26 } These are confequently given, for they may be fonnd out by
8. l = IC = 126 } the help of the given fides of △ C E D and △ D F E,
9. n = EG = 80 } (*per Theor.* in 29° and 30° of *Probl.* 9. *Chap.* 7.)

I. *Req.* to find E B and F B.

Refolution.

10. Put . . . . } a = EB.
11. Then it follows from 7° and 10°, that . . } $a + k$ = IB.
12. The Square of the laft Equation gives . . . } $aa + 2ka + kk = \square$ IB.
13. To which Square adding the Square of p, that is, } . . . . . . $pp = \square$ DI.
14. The fumm makes . . . . } $aa + 2ka + kk + pp = \square$DB.
15. And becaufe in △ E I D right-angled at I, . } $gg = kk + pp = \square$ DE.
16. Therefore from 14° and 15°, (*per Ax.6. Ch.2.*) } . $aa + 2ka + gg = \square$DB.
17. And by extracting the fquare Root of the laft Equation, this arifeth, ... } $\sqrt{:aa + 2ka + gg:}$ = DB.
18. By *Suppofition* the angle C E D is acute, and confequently (*per Coroll prop.* 13. *Elem.* 1.) the angle D E B is obtufe, therefore (*per prop.* 41. *Elem.* 1.) } $\square$DI,EB=$\square$EG,DB(=2△DEB)
19. That is, in the letters of the Refolution, . . } $pa = n \times \sqrt{:aa + 2ka + gg:}$
20. Which laft preceding Equation may be refolved into thefe Proportionals, *viz.* . . . . . . } $p \,.\, n :: \sqrt{:aa + 2ka + gg:} \,.\, a$.
21. The Squares of which Proportionals are alfo Proportionals, *viz.* . . . . } $pp \,.\, nn :: aa + 2ka + gg \,.\, aa$.
22. Now to avoid an Equation between Solids; to p and n find a third Proportional, call it q, therefore } $p \,.\, n :: n \,.\, q$.

23. There-

23. Therefore from the last Analogy this will arise, (*per Coroll.* of 20. *prop.* 6. *Elem.*) . . } $pp \,.\, nn :: p \,.\, q.$

24. Therefore from the Analogies in 21° and 23° this is manifest, (*per* 11. *prop.* 5. *Elem*) } $p \,.\, q :: aa + 2ka + gg \,.\, aa.$

25. And from 24°, by Division of Reason, . } $p - q \,.\, q :: 2ka + gg \,.\, aa.$

26. Then unto $2k$ and g find a third Proportional, which may be called s, therefore . . } $2k \,.\, g :: g \,.\, s.$

27. Therefore by resolving that Analogy into an Equation, it gives . . . . } $2ks = gg.$

28. Suppose . . . . } $r = p - q.$

29. Then from 25°, 27° and 28°, by exchange of equal quantities, ... } $r \,.\, q :: 2ka + 2ks \,.\, aa.$

30. Let it be made . . . . } $r \,.\, q :: 2k \,.$ (to a fourth) $t.$

31. Then from 29° and 30°, it's manifest (*per* 11. *prop.* 5. *Elem.*) that . . . . . . } $2k \,.\, t :: 2ka + 2ks \,.\, aa.$

32. And by drawing $a + s$ into $2k$ and t severally, this Analogy is manifest, ... } $2k \,.\, t :: 2ka + 2ks \,.\, ta + ts.$

33. Therefore from 31° and 32°, (*per* 11. *prop.* 5. *Elem.*) . . . . } $2ka + 2ks \,.\, aa :: 2ka + 2ks \,.\, ta + ts.$

34. In which last Analogy, the first Term is equal to the third, therefore the second shall be equal to the fourth, *viz.* . . . . . . } $aa = ta + ts.$

35. Therefore by subtracting ta from each part of the last Equation, ... } $aa - ta = ts.$

36. Therefore (by the Canon in 43° of *Probl.* 12. *Chap.* 5.) . . . . } $a = \frac{1}{2}t + \sqrt{:\frac{1}{4}tt + ts:} = EB.$

From 22°, 26°, 28°, 30° and 36°, 'tis easie to deduce this following

CANON 1.

37. First, to p and n, (that is, DI and EG,) find a third Proportional, which may be called q; therefore, ... } $p \,.\, n :: n \,.\, q \; (= \frac{228}{21}.)$

Secondly, unto $2k$ and g, (that is, 2IE and DE,) find a third Proportional, which may be called s; therefore, ... } $2k \,.\, g :: g \,.\, s \; (= \frac{2223}{13})$

Thirdly, make . . . . } . . . $r = p - q \; (= \frac{2238}{21}.)$

Fourthly, let it be made as r to q, so $2k$ to a fourth Proportional, which may be called t; therefore, } $r \,.\, q :: 2k \,.\, t \; (= \frac{1808}{341}.)$

Fifthly and lastly, by the Canon in 43° of *Probl.* 12. *Ch.* 5. the quantity of EB (represented by a in the Resolution) will be made known, *viz.* } $\frac{1}{2}t + \sqrt{:\frac{1}{4}tt + ts:} = 100 = EB = a.$

The Demonstration of the said Canon is manifest by the preceding Resolution, which is argued Geometrically as well as Arithmetically, and no quantity in any step thereof exceeds the dimensions of a Plane. Moreover, by converting the Analogy in 21° into an Equation according to the vulgar way, the following Canon will arise, which is the same in substance with the former, but not so apt for Demonstration.

CANON 2.

38. Multiply k, and the Square of g, severally by the Square of n, and divide each Product by the excess of the Square of p, above the Square of n; then add the latter Quotient to the Square of the first, and extract the square Root of the summ; lastly, the said square Root added to the first Quotient will give the same value of a, (that is, $100 = EB$) as before.

39. Now in order to find FB, first, . . . } $IE + EB = IB = 126.$

40. Then . . . . } $\sqrt{:\square IB + \square ID:} = DB = 210.$

41. And . . . . } $DB - DF = FB = 120.$

II. To find DA.

42. The lines CD, DB, BC, DF, CE being given or found out, (as before,) the length of DA may be discovered by the Canon in 17° of *Probl.* 19. *Chap.* 7. *viz.*

First,

First, . . . DF . DB :: CE . (to a fourth) M . ($= 354\frac{4}{5}$.)
Then, . . . M — CB . EB :: CD . DA ($= 204\frac{6}{11}$.)

III. To find AH and CH.

43. By *Supposition* AH is perpendicular to CH, and each of them may be found out thus,

$$CD \,.\, CA :: \begin{cases} DI \,.\, AH\ (= 331\frac{2}{11}.) \\ CI \,.\, CH\ (= 248\frac{8}{11}.) \end{cases}$$

IV. To find AB the distance required.

44. Because in this Example CH happens to be less than CB, (that is, CE + EB,) subtract CH from CB, and the remainder is HB, *viz.*

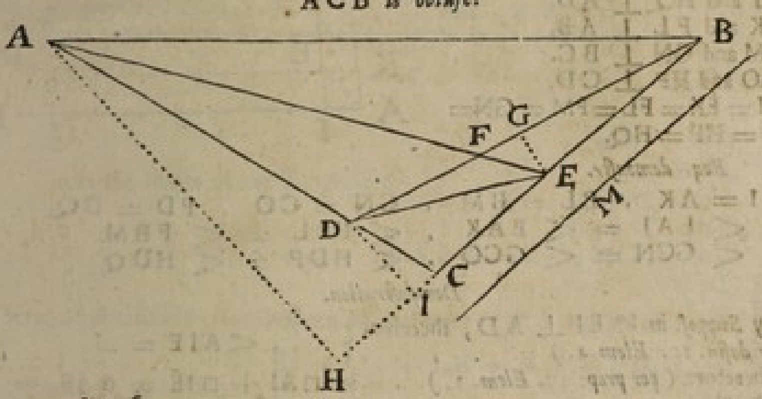
$$CB - CH = HB = 3\frac{2}{11}.$$

Then in $\triangle$ AHB right-angled at H,

$$\sqrt{:\square AH + \square HB:} = AB = \frac{\sqrt{13309200}}{11}, \text{ that is, } 331,652, \&c.$$

45. *Note.* When CH happens to exceed CB, then the Perpendicular AH will fall without the $\triangle$ CAB, upon CB produced; in which Case CB is to be subtracted from CH, and then the Square of the remainder being added to the Square of the Perpendicular AH, the square Root of the summ shall be the desired distance AB. And lastly, when CH happens to be found equal to CB, then the Perpendicular AH is the same with the distance AB.

The Arithmetical Solution of Case 2. Probl. 20. viz. *when the angle* ACB *is obtuse.*



Suppos.

46. $\left\{ \begin{array}{l} b = CE = 63\frac{3}{5}. \\ c = CD = 41\frac{3}{5}. \\ d = DF = 60\frac{132}{169}. \\ f = FE = 29\frac{111}{169}. \\ g = DE = 85. \end{array} \right\}$ Lines given.

$\left\{ \begin{array}{l} p = DI = 40. \\ k = IE = 75. \\ l = CI = 11\frac{3}{5}. \\ n = EG = 20\frac{8}{13}. \end{array} \right\}$ These are consequently given, for they may be found out by the help of the given sides of $\triangle$CED and $\triangle$DFE, (*per Theor.* 2. in 31° and 32° of *Probl.* 10. *Chap.* 7.)

Req. to find AB.

Operation Arithmetical.

First, by *Canon* 2. in 38° of the preceding Resolution you will find by the help of the respective numbers given in 46°, ... } EB $= 82\frac{1}{2}$.
Secondly, ... } IE + EB = IB $= 157\frac{1}{2}$.
Thirdly, in $\triangle$ DIB right-angled at I, . . . } $\sqrt{:\square IB + \square ID:} = DB = 162\frac{1}{2}$.
Fourthly, (by the Canon in 17° of *Probl.* 19. *Chap.* 7.) let it be made,

$$\begin{cases} DF \,.\, DB :: CE \,.\, (\text{to a fourth,})\ M = 169\frac{11424}{30807};\ \text{then} \\ M - CB \,.\, EB :: CD \,.\, (\text{to a fourth,})\ DA = 146\frac{243}{51603}. \end{cases}$$

Fifthly, ... } CD + DA = CA $= 187\frac{44842}{34809}$.

Sixthly,

Sixthly, because △ CDI and △ CAH are like, these are Proportionals, *viz.*

$$\begin{cases} CD \,.\, CI :: CA \,.\, CH \; (= 52\tfrac{26628}{174045}.) \\ CD \,.\, DI :: CA \,.\, AH \; (= 180\tfrac{10248}{38011}.) \end{cases}$$

Seventhly, ... $\succ$ $CH + CB = HB = 198\tfrac{41261}{116030}$.

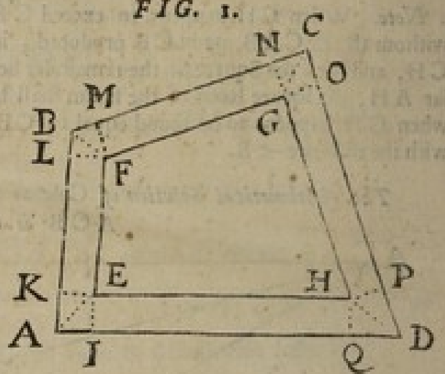
Lastly, in the △ AHB right-angled at H, $\succ$ $\sqrt{:}\, \square HB + \square HA := AB = 268\tfrac{3}{10000}$, &c.

Note. When the angle ACB happens to be a right angle, then after EB and DA are found out in like manner as before in 38° and 42° of this Problem, there will be given CA and CB the sides about the right angle ACB of △ ACB; therefore (*per prop.* 47. *Elem.* 1.) the square Root of the summ of the Squares of those sides shall be the quantity of the Hypothenusal AB, to wit, the distance sought.

LEMMA 1. *leading to* Probl. 21.

Suppos.

FIG. 1.



1. ABCD is a Trapezium.
2. EFGH is a Trapezium.
3. EF ∥ AB.
4. FG ∥ BC.
5. GH ∥ CD.
6. EH ∥ AD.
7. AE, BF, CG, DH are right lines.
8. EI and HQ ⊥ AD.
9. EK and FL ⊥ AB.
10. FM and GN ⊥ BC.
11. GO and HP ⊥ CD.
12. EI = EK = FL = FM = GN = GO = HP = HQ.

Req. demonstr.

13. AI = AK . BL = BM . CN = CO . PD = DQ.

14. $\begin{cases} \angle EAI = \angle EAK \;.\; \angle FBL = \angle FBM. \\ \angle GCN = \angle GCO \;.\; \angle HDP = \angle HDQ. \end{cases}$

Demonstration.

15. By *Suppos.* in 8° EI ⊥ AD, therefore (*per defin.* 10. *Elem.* 1.) . . . . . $\}$ . . . $\angle AIE = \perp$.

16. Therefore (*per prop.* 47. *Elem.* 1.) . . $\succ$ $\square AI + \square IE = \square AE$.

17. Likewise, ... $\succ$ $\square AK + \square KE = \square AE$.

18. Therefore from 16° and 17°, (*per Ax.* 1. *Chap.* 2.) . . . . $\}$ $\square AI + \square IE = \square AK + \square KE$.

19. By *Suppos.* in 12°, IE = KE, and consequently $\succ$ . . . . . $\square IE = \square KE$.

20. Therefore from 18° and 19°, (*per Ax.* 9. *Chap.* 2.) . . . . $\}$ . . . . . $\square AI = \square AK$.

21. Therefore from 20°, (*per Schol. prop.* 48. *Elem.* 1.) . . . . $\}$ . . . . . AI = AK.

Which was to be Demonstr.

22. Again, because AE is common to the Triangles AIE and AKE, and the other sides of those Triangles are correspondently equal to one another, (as appears in 12° and 21°,) therefore (*per prop.* 8. *Elem.* 1.) . . . $\}$ $\angle EAI = \angle EAK$.

Which was also to be Demonstr.

And by the like argumentation the truth of the rest of the Equations in 13° and 14° may be demonstrated.

LEMMA 2.

Let the same things be supposed here as before in *Lemma* 1. then, (respect being had to *Fig.* 1.)

Req.

1\. . . . *Req. demonstr.* As Radius is to the summ of the Tangents of $\angle$ AEI, $\angle$ BFL, $\angle$ CGN, $\angle$ DHP;
So EI (or EK) to AI + LB + NC + PD.

Demonstration.

2\. By *Supposition* in 12° of *Lemma* 1. . . . EI = FL = GN = HP.

3\. Therefore these four following Analogies will be manifest by a vulgar Axiom in the Doctrine of plain Triangles, *viz.*

Radius .	EI	::	Tangent $\angle$ AEI .	AI.
Radius .	FL, or EI	::	Tangent $\angle$ BFL .	LB.
Radius .	GN, or EI	::	Tangent $\angle$ CGN .	NC.
Radius .	HP, or EI	::	Tangent $\angle$ DHP .	PD.

4\. And from those four Analogies this that follows is deducible, (*per Schol. prop.* 12. *Elem.* 5.) *viz.*

As . . . . 4 Rad.
To . . . . 4 EI;
So is the summ of the Tangents of these angles, to wit, $\angle$ AEI, $\angle$ BFL, $\angle$ CGN, $\angle$ DHP;
To . . . . AI + LB + NC + PD.

5\. But (by *prop.* 15. *Elem.* 5.) . . . 4 Rad. 4 EI :: Rad. EI.

6\. Therefore from the two last preceding Analogies in 4° and 5° this ariseth, (*per prop.* 11. *Elem.* 5.) *viz.*

As . . . . Rad.
To . . . . EI;
So is the summ of the Tangents of . . . . . $\angle$ AEI, $\angle$ BFL, $\angle$ CGN, $\angle$ DHP;
To . . . . AI + LB + NC + PD.

7\. Therefore alternately, As Radius is to the summ of the Tangents of $\angle$ AEI, $\angle$ BFL, $\angle$ CGN, $\angle$ DHP;
So EI to AI + LB + NC + PD.
Which was to be Demonstr.

LEMMA 3.

Let the same things be supposed here as before in *Lemma* 1. then (respect being had to *Fig.* 1.)

Req. demonstr.

1\. $$\left.\begin{array}{l} \square EI \times \overline{AD - AI} + \square EI \times \overline{AB - LB} + \\ \square EI \times \overline{BC - NC} + \square EI \times \overline{CD - PD} \end{array}\right\} = \text{Space AEFBCGHDA.}$$

That is, in words,

The Rectangle made of the parallel distance EI, (= EK,) and the excess by which the summ of the four sides AD, AB, BC, CD of the Trapezium ABCD, exceeds the summ of the four segments AI, LB, NC, PD, is equal to the Interval or Space AEFBCGHDA.

Demonstration.

2\. By *Suppos.* in 8° and 12° of *Lemma* 1. > EI and HQ $\perp$ IQ. Also, EI = HQ.

3\. Therefore (*per prop.* 27, 33, 34. *Elem.* 1.) > EH = IQ.

4\. It is evident by *Fig.* and *Lemma* 1. respect being had to the last Equation, that } IQ = EH = AD − AI − QD (PD.)

5\. And by adding AD to each part of the last Equation, ... } AD + EH = 2AD − AI − QD (PD.)

6\. And by arguing as in 4° and 5°, this Equation will be manifest, *viz.* . . . . } AB + EF = 2AB − AK (AI) − LB.

 7. Likewise,

7. Likewise, ... $>$ $BC + FG = 2BC - BM(LB) - NC.$
8. Likewise, ... $>$ $CD + GH = 2CD - CO(NC) - PD(QD.)$
9. And by comparing the half summ of the first parts of the four last preceding Equations to the half summ of the latter parts, this Equation ariseth; *viz.*

$$\left.\begin{array}{l}\frac{1}{2}AD + \frac{1}{2}EH + \\ \frac{1}{2}AB + \frac{1}{2}EF + \\ \frac{1}{2}BC + \frac{1}{2}FG + \\ \frac{1}{2}CD + \frac{1}{2}GH\end{array}\right\} = \left\{\begin{array}{l}AD - AI + \\ AB - LB + \\ BC - NC + \\ CD - PD.\end{array}\right.$$

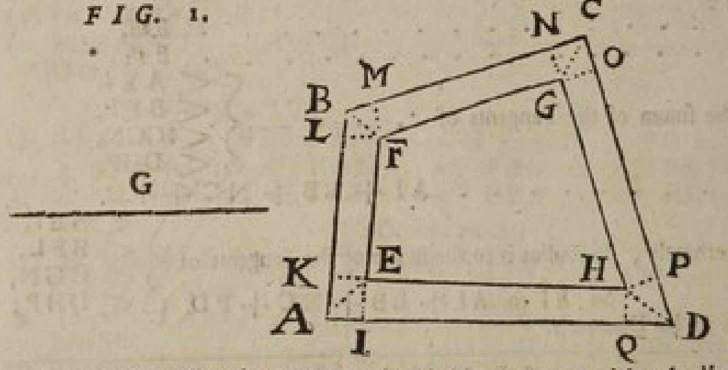
10. By *Suppos.* the Trapezium AEHD hath two parallel sides AD and EH, and EI $\perp$ AD, therefore (*per Theor.* 2. in 13° of *Probl.* 18. *Chap.* 7.) . . $\square EI \times \overline{\tfrac{1}{2}AD + \tfrac{1}{2}EH} =$ Trapez. AEHD.
11. In like manner, (respect being had to the *Suppos.* in 12° of *Lemma* 1.) . . $\square EI \times \overline{\tfrac{1}{2}AB + \tfrac{1}{2}EF} =$ Trapez. AEFB.
12. Likewise, ... $>$ $\square EI \times \overline{\tfrac{1}{2}BC + \tfrac{1}{2}FG} =$ Trapez. BFGC.
13. Likewise, ... $>$ $\square EI \times \overline{\tfrac{1}{2}CD + \tfrac{1}{2}GH} =$ Trapez. DHGC.
14. By viewing *Fig.* 1. it will be evident, that the summ of the four Trapezia exprest in the four last preceding Equations is equal to the Interval or Space AEFBCGHDA, therefore from 9°, 10°, 11°, 12° and 13°, by exchanging equal quantities this Equation ariseth, *viz.*

$$\left.\begin{array}{l}\square EI \times \overline{AD - AI} + \square EI \times \overline{AB - LB} + \\ \square EI \times \overline{BC - NC} + \square EI \times \overline{CD - PD}\end{array}\right\} = \text{Space AEFBCGHDA.}$$

Which was to be Demonstr.

LEMMA 4.

FIG. 1.



Let the same things be suppos'd here as in *Lemma* 1. and let the line G be found out by this Analogy, *viz.*

1. . . As the summ of the Tangents of . . . . $\angle$AEI, $\angle$BFL, $\angle$CGN, $\angle$DHP,
To . . . . Radius;
So . . . . $AB + BC + CD + DA$,
To a fourth Proportional . . . . G. Then

2. . . . *Req. demonstr.* . . . . . . $\frac{1}{2}G \sqsubset EI$, (the parallel distance.)

Demonstration.

3. By inverting the Terms of the Analogy demonstrated in 7° of the foregoing *Lemma* 2. these are Proportionals, *viz.*

4. . . As the summ of the Tangents of . . . . $\angle$AEI, $\angle$BFL, $\angle$CGN, $\angle$DHP,
To . . . . Radius;
So is . . . . $AI + LB + NC + PD$,
To the parallel distance . . . . EI.

5. Therefore

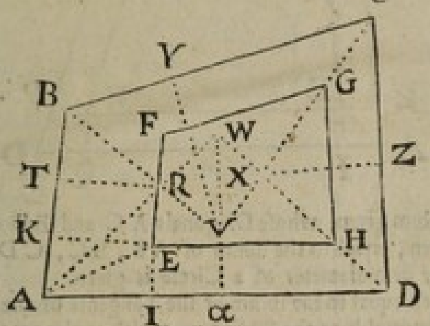
5. Therefore from 1° and 4°, (*per prop.* 11. *Elem.* 5.)
   $AB + BC + CD + DA \;.\; G :: AI + LB + NC + PD \;.\; EI.$
6. But by *prop.* 15. *Elem.* 5.
   $AB + BC + CD + DA \;.\; G :: \frac{1}{2}AB + \frac{1}{2}BC + \frac{1}{2}CD + \frac{1}{2}DA \;.\; \frac{1}{2}G.$
7. Therefore from 5° and 6°, (*per prop.* 11. *Elem.* 5.)
   $\frac{1}{2}AB + \frac{1}{2}BC + \frac{1}{2}CD + \frac{1}{2}DA \;.\; \frac{1}{2}G :: AI + LB + NC + PD \;.\; EI.$
8. In which last Analogy the first Term is greater than the third; for 'tis evident by *Fig.* 1. and by what hath been demonstrated in *Lemma* 1. that the double of the first Term is greater than the double of the third, in regard the double of the third is but part of the double of the first: Therefore (*per prop.* 14. *Elem.* 5.) the second Term is greater than the fourth, *viz.*

   $\frac{1}{2}G \sqsubset EI$ (the parallel distance.)

   Which was to be Demonstr.

LEMMA 5.

FIG. 2.



If the angles A, B, C, D of the Trapezium ABCD be severally divided into two equal parts by the right lines AW, DW, BV, CV; and if from the points W, R, V and X where every two of the said lines that lye next to one another do intersect, four right lines $W\alpha$, RT, VY and XZ be drawn perpendicularly upon AD, AB, BC and CD; and if the Perpendicular RT be shorter, or, if not shorter, yet not longer than any one of the other three Perpendiculars VY, XZ and $W\alpha$: And lastly, if a Trapezium be made within the before-mentioned Trapezium ABCD, so, as that the sides of the one are parallel to the sides of the other, and every where separated by an equal parallel distance: Then I say that the said parallel distance shall be less than the said Perpendicular RT.

For if it be said that the parallel distance is equal to the Perpendicular RT, then by the point R let ERF be drawn parallel to AB, and finish the Trapezium EFGH so as FG may be parallel to BC, likewise $GH \parallel CD$ and $EH \parallel AD$; draw also AE: Now if the sides of the interiour Trapezium EFGH be every where distant from the sides of the exteriour Trapezium by an equal parallel distance, then by *Lemma* 1. the angle EAI is equal to the angle EAK; but by *Supposition* in this *Lemma* 5. $\angle RAI = \angle RAK$, and how short soever the parallel line ERF be drawn, the angle EAI will be but part of, and consequently less than $\angle RAI$, ($= \angle RAK$,) as is evident by *Fig.* 2. therefore the angle EAI cannot be equal to the angle EAK, and consequently it contradicts *Lemma* 1. before demonstrated. The like contradiction will ensue if a Parallel be drawn to AB at a distance greater than RT. Wherefore I conclude, that if a Trapezium be made within a Trapezium in such manner as is above supposed, the parallel distance shall be less than RT, which is suppos'd to be shorter, or, if not shorter, yet not longer than any one of the three Perpendiculars VY, XZ and $W\alpha$ before mentioned.

Probl. XXI.

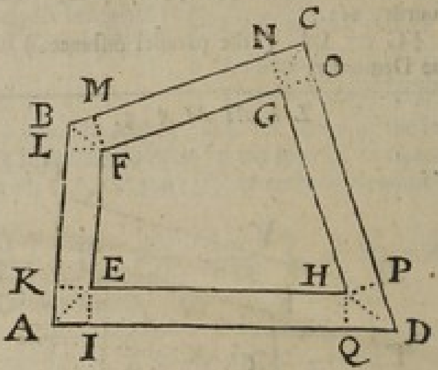
A Trapezium being given by Position, as also the quantities of all its sides and angles severally, to make a Trapezium within the former, in such manner that the sides of the one may be every where separated

 from

from the fides of the other by an equal parallel diftance; and that the Space lying between the fides of both the Trapezia may be equal to any poffible Space (or Figure) given.

Note. This Problem hath two Cafes, *viz.* Firft, when each of the Diagonals of the given Trapezium lyes within the fame; Secondly, when one of the Diagonals lyes without. I fhall handle the firft Cafe only, for this well underftood, will be a fufficient light to fhew the induftrious Analyft how to folve the latter Cafe, which is briefly done by the Learned *Fran. à Schooten*, in his *Tractat. de concinnandis Demonftrationibus ex Calculo Algebraico.*

FIG. 1.



Suppof.

1. ABCD is a Trapezium given, whofe Diagonals AC and BD do lye within the fame.
2. c = a right line given, equal to the fumm of AB, BC, CD, DA.
3. r = the Radius, or Semidiameter of a Circle is given.
4. s = a right line given equal to the fumm of the Tangents of the angles AEI, BFL, CGN and DHP, agreeable to the Radius r; which angles are the complements of the halves of the given angles of the Trapezium ABCD.
5. b = the fide of a given Square equal to the Interval or Space lying between the fides of the given Trapezium ABCD, and the fides of the Trapezium required to be made within the former.

Req. to make

6. EFGH a Trapezium, fuch, that EF ∥ AB; FG ∥ BC; GH ∥ CD, and EH ∥ AD. Alfo,
7. Space AEFBCGHDAE = □b, (or bb.) Alfo,
8. The Perpendiculars EI, FL, GN and HP to be equal between themfelves, that there may be an equal parallel diftance between the fides of the Trapezium given, and of that required.

Refolution of CASE 1.

9. Suppofe that done which is required, and put a for the parallel diftance, *viz.* . . . . $\}$ $a = EI = EK = FM = GO.$

10. Then by the foregoing *Lemma* 2. this Analogy is manifeft, *viz.*

As Radius, to the fumm of the Tangents of . . . . $\left\{\begin{array}{l} \angle AEI, \\ \angle BFL, \\ \angle CGN, \\ \angle DHP; \end{array}\right.$

So is the parallel diftance EI, to . . AI + LB + NC + PD.

11. Therefore in the letters of the Refolution,

$$r \,.\, s :: a \,.\, \frac{sa}{r} \quad (= AI + LB + NC + PD.)$$

12. By the preceding *Lemma* 3. this Equation is manifeft, *viz.*

$$\left.\begin{array}{l} \square EI \times \overline{AD - AI} + \square EI \times \overline{AB - LB} + \\ \square EI \times \overline{BC - NC} + \square EI \times CD - PD \end{array}\right\} = \text{Space } AEFBCGHDA.$$

13. Therefore in the letters of the Refolution,

$$a \times c - \frac{sa}{r} = bb \; (= \text{Space } AEFBCGHDA.$$

14. Which

14. Which Equation in 13° may be resolved into these Proportionals, *viz.* . . . . $b \,.\, c - \frac{sa}{r} :: a \,.\, b.$

15. And by drawing r as a common Factor into each of the two first Terms of the last preceding Analogy, this ariseth, *viz.* . . . . $br \,.\, cr - sa :: a \,.\, b.$

16. Now to avoid an Equation between Solids, which would arise by comparing the Rectangle of the extremes to the Rectangle of the means of the last Analogy, let it be made, as s to r, so b to a fourth Proportional, which may be called d, therefore . . . . $s \,.\, r :: b \,.\, d.$

17. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, *viz.* . . . . $sd = br.$

18. Again, let it be made, as s to r, so c to a fourth Proportional, call it g, therefore . . . . . . $s \,.\, r :: c \,.\, g.$

19. Therefore from the last Analogy, by comparing the Rectangle of the extremes to the Rectangle of the means, ... $sg = cr.$

20. Therefore from 15°, 17° and 19°, by exchanging equal Rectangles, this Analogy ariseth, *viz.* . . $sd \,.\, sg - sa :: a \,.\, b.$

21. Therefore from 20°, by casting the common Factor s out of the two first Terms, ... $d \,.\, g - a :: a \,.\, b.$

22. From which last Analogy, by comparing the Rectangle of the means to the Rectangle of the extremes, this Equation is produced, *viz.* . . . . . . $ga - aa = bd.$

23. Which Equation may be resolved into these Proportionals, *viz.* . . . . $g - a \,.\, \sqrt{bd} :: \sqrt{bd} \,.\, a.$

24. But of those three continual Proportionals, the mean, to wit, $\sqrt{bd}$ is given, as also g the summ of the extremes $g - a$ and a, therefore (*per Probl.* 13. *Chap.* 5.) the extremes shall be given severally, which by the Theorem in 21° of the said *Probl.* 13. are equal to these right lines, (or numbers,) *viz.*

$$\left.\begin{array}{l} \frac{1}{2}g + \sqrt{:\frac{1}{4}gg - bd:} \\ \frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:} \end{array}\right\} = \text{the extreme Proportionals in } 23^{\circ}.$$

Which extreme Proportionals, (or Roots of the Equation in 22°,) are equal one to the other when $\frac{1}{2}g = \sqrt{bd}$, in which Case each of the said extremes is evidently equal to $\frac{1}{2}g$; but if $\frac{1}{2}g \sqsubset \sqrt{bd}$, then the said extremes are unequal, as they happen to be in the Resolution of this Problem, but the lesser of them only (for the reason hereafter given in 26°) shall be the parallel distance sought. Hence this

CANON.

25. $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:} =$ EI $=$ EK the parallel distance. That is, in words,

Let it be made, As (s) the summ of the Tangents of the complements of the halves of the angles of the given Trapezium ABCD, is to the Radius (r), So (b) the given side of a Square equal to the prescribed Interval or Space between the sides of both the Trapezia, to a fourth Proportional (d); and so (c) the summ of the four sides of the given Trapezium to a fourth Proportional (g). Then subtract the square Root of the excess whereby the Square of half g exceeds the Rectangle made of b and d, from half g; the remainder shall be EI ($=$ EK) the parallel distance sought.

Which Canon may be propounded in the form of a Theorem, the Demonstration whereof may be easily framed by a repetition of the steps of the foregoing Resolution, by proceeding in a direct order from the beginning to the end thereof; for the Argumentation therein used is clearly Geometrical, as well as Arithmetical: But waving the Demonstration of the Canon, I shall in the next place shew what Determinations are necessary for limiting the given lines, that there may be a possibility of effecting the Problem propounded.

Determinat. 1.

26. . . . $\frac{1}{2}g \sqsubset \sqrt{bd}$; that is, $\frac{cr}{2s} \sqsubset \sqrt{:b \times \frac{br}{s}:}$

Although

Although there be a possibility to find out the extreme Proportionals in 24°, (or the two Roots of the Equation in 22°,) if $\frac{1}{2}g$ be not less than $\sqrt{bd}$, yet the parallel distance is not possible unless $\frac{1}{2}g$ be greater than $\sqrt{bd}$; for if $\frac{1}{2}g = \sqrt{bd}$, then each of the said extreme Proportionals is equal to $\frac{1}{2}g$; but by the preceding *Lemma* 4. $\frac{1}{2}g$ is greater than E I or E K (the parallel distance,) therefore neither of those equal extremes can be the parallel distance: But if $\frac{1}{2}g \sqsubset \sqrt{bd}$, as *Determinat.* 1. requires, then the two extreme Proportionals before mentioned are unequal; for if $\frac{1}{2}g$, that is, half the summ of the extremes of three Proportionals be greater than the mean, to wit, $\sqrt{bd}$, then the extremes are unequal, and consequently in such Case $\frac{1}{2}g$ is greater than the lesser extreme; therefore agreeable to *Lemma* 4. the lesser extreme shall be the parallel distance required; but the greater extreme is to be neglected; because 'tis greater than $\frac{1}{2}g$, and consequently doth contradict the said *Lemma* 4.

Determinat. 2.

27. . . . . $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd}: \sqsupset$ RT.

The perpendicular line R T in *Fig.* 2. prefix'd before the precedent *Lemma* 5. is supposed to be shorter, or if not shorter, yet not longer than any one of the other three Perpendiculars V Y, X Z and W α. But the quantities of the said four Perpendiculars may be found out in numbers by the Doctrine of Plain Triangles; for in each of the Triangles A R B, B V C, C X D and A W D, the Base is given, as also the angles at the Base, to find the Perpendicular. Which Bases are the given sides of the Trapezium A B C D, and the angles at the Bases are the halves of the given angles of the said Trapezium, therefore the said Perpendiculars are given also, whence it may be known which of them is the shortest. Now because by the foregoing Canon in 25°, and by what hath been said in 26°, the parallel distance is found equal to $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd}$: and because by the preceding *Lemma* 5. the said parallel distance must be less than R T, which is suppos'd to be the shortest of the said four Perpendiculars, or if not the shortest, yet not longer than any one of the other three, therefore $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd}$: must be less than R T; otherwise that cannot be done which the Problem requires.

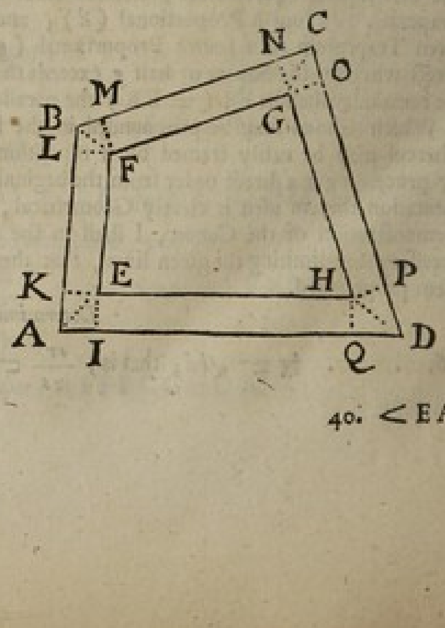
But if the given lines be qualified according to the import of the precedent Determinations, then the parallel distance E I or E K shall be given by the foregoing Canon; and then, after the angles A, B, C, D of the given Trapezium A B C D in *Fig.* 1. are severally divided into two equal parts by the right lines A E, B F, C G and D H; and after right lines are drawn parallel to the sides A B, B C, C D, D A, at the distance of the said E I or E K, the points where those Parallels do concurr in the lines bisecting the said angles will form the desired Trapezium E F G H: And it will not be difficult to demonstrate, by a repetition of the steps of the foregoing Resolution in a backward order, (in like manner as in divers preceding Problems of this Chapter,) that the Area of the Interval or Space lying between the sides of both the Trapezia is equal to the Square of the given side b. But leaving the Composition of this Problem to the Learners practice, I shall prove the truth of its Solution by an Example in Numbers.

An Example in Numbers, to illustrate the preceding Resolution of Probl. 21.

Suppos.

28. ABCD is a Trapezium.
29. EFGH is a Trapezium.
30. $\left\{\begin{array}{l}\text{EH} \parallel \text{AD} . \text{EF} \parallel \text{AB}.\\ \text{FG} \parallel \text{BC} . \text{GH} \parallel \text{CD}.\end{array}\right.$
31. AE, BF, CG, DH are right lines.
32. EI and HQ $\perp$ AD.
33. EK and FL $\perp$ AB.
34. FM and GN $\perp$ BC.
35. GO and HP $\perp$ CD.

$$\left.\begin{array}{l}36.\ AD = 40\\ 37.\ AB = 24\\ 38.\ BC = 29\\ 39.\ CD = 36\end{array}\right\}\text{ are given.}$$



40. $\angle$EAI

		Gr.	*Min.*	
40. $\angle$ EAI	$= \angle$ EAK $=$	44 :	6.	Are given.
41. $\angle$ FBL	$= \angle$ FBM $=$	55 :	56.	
42. $\angle$ GCN	$= \angle$ GCO $=$	44 :	43.	
43. $\angle$ HDP	$= \angle$ HDQ $=$	35 :	15.	
44. $\angle$ AEI	$= \angle$ AEK $=$	45 :	54.	These are consequently given, because the angles at I, K, L, M, N, O, P, Q are right angles.
45. $\angle$ BFL	$= \angle$ BFM $=$	34 :	4.	
46. $\angle$ CGN	$= \angle$ CGO $=$	45 :	17.	
47. $\angle$ DHP	$= \angle$ DHQ $=$	54 :	44.	

48. $c = 129 = AD + AB + BC + CD$ is given; (from 36°, 37°, 38°, 39°.)
49. $bb = 450\frac{22}{1000}$ is given for the Area of the Space AEFBCGHDA.
50. $r = 100000$ the Radius of a Circle is given.

Req. to find

51. $EI = FL = GN = HP$, (the parallel distance.)

Solution Arithmetical.

		Gr.	*Min.*		
52. 103192	$=$ Tangent of	45 :	54	$= \angle$ AEI	$= \angle$ AEK.
53. 67620	$=$ Tangent of	34 :	04	$= \angle$ BFL	$= \angle$ BFM.
54. 100994	$=$ Tangent of	45 :	17	$= \angle$ CGN	$= \angle$ CGO.
55. 141409	$=$ Tangent of	54 :	44	$= \angle$ DHP	$= \angle$ DHQ.

56. $413215 = s =$ the summ of those four Tangents.

Then according to the foregoing Canon in 25°, let it be made,

$$57.\ \text{As} \begin{cases} s & r & & b & & d. \\ 413215 \ . & 100000 & :: & \sqrt{:}\,450\frac{22}{1000} & . & 5\frac{134}{1000},\ \&c. \end{cases}$$

$$58.\ \text{As} \begin{cases} s & r & & c & & g. \\ 413215 \ . & 100000 & :: & 129 & . & 31\frac{216}{1000},\ \&c. \end{cases}$$

Lastly, by the precedent Canon in 25°,

59. $\frac{1}{2}g - \sqrt{:}\,\overline{\tfrac{1}{4}gg - bd}: = 4$ (very near) $= EI = EK$.

The Proof.

To r, s and EI find a fourth Proportional, (agreeable to the foregoing *Lemma* 2.) *viz.*

$$60. \ldots \begin{cases} r & & s & & EI & & AI + LB + NC + PD. \\ 100000 & . & 413215 & :: & 4 & . & 16\frac{5286}{10000}. \end{cases}$$

Then (by *Lemma* 3.)

$$EI \times \begin{Bmatrix} AD - AI + AB - LB + \\ BC - NC + CD - PD \end{Bmatrix} = \text{Space AEFBCGHDA.}$$

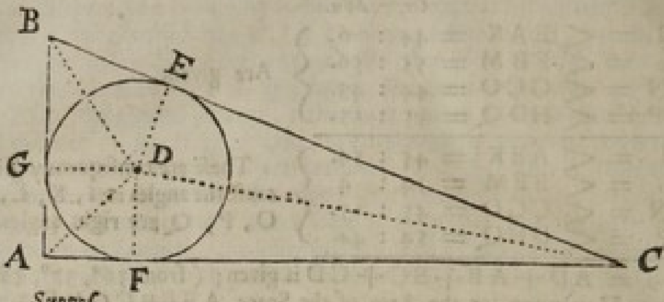
That is, 4 into $\overline{129 - 16\frac{5286}{10000}} = 449\frac{886}{1000}$, &c.

Which Area doth not want $\frac{1}{20}$ of $450\frac{22}{1000}$ the prescribed Area of the Interval AEFBCGHDA, the defect arising from this, that such numbers as were found out near the truth, were assumed to be exactly true, to avoid tediousness of Calculation. Therefore the truth of the Solution of the Problem propounded is evident.

Probl. XXII.

In a right-angled plain Triangle, the Area being given, as also the Perimeter, (that is, the summ of all the three sides,) to find out the Triangle. But the quadruple Area must be less than the Square of the Perimeter.

Suppos.



$AC = 156$
$AB = 65$
$BC = 169$
$DF = 26$
$DG = 26$
$DE = 26$

Suppos.

1. $\triangle ABC$ is right-angled at A.
2. $\square \frac{1}{2} AC, AB =$ the Area of $\triangle ABC$ is given.
3. $AC + AB + BC =$ the Perimeter is given.

Req. to find out AC, AB, BC severally.

Prepar.

4. Supposing ABC to be the Triangle sought, bisect (by *Probl. 9. Elem. 1.*) the angles BAC and ABC by the lines AD and BD meeting in D; and make $DF \perp AC$. $DG \perp AB$. $DE \perp BC$; and draw DC: Then is DF ($= DG = DE$) the Semidiameter of the inscribed Circle DFGE, which toucheth the Triangle in the points F, G and E, (*per prop. 4. Elem. 4.*)

Resolut. I.

5. From the premisses 'tis easie to perceive, that in any plain Triangle, the Semiperimeter multiplied by the Semidiameter of the inscribed Circle produceth the Area of the Triangle; and consequently the Area divided by the Semiperimeter gives the Semidiameter of the inscribed Circle: As in $\triangle ABC$ before expos'd, the Rectangle (or Product) of $\frac{1}{2}$ AC into DF is equal to the $\triangle ADC$, (*per prop. 41. Elem. 1.*) Likewise, $\square \frac{1}{2} AB, DG\ (DF) = \triangle ABD$: Also, $\square \frac{1}{2} BC, DE\ (DF) = \triangle BCD$; and those three Triangles are evidently equal to $\triangle ABC$.
6. Moreover, in every right-angled plain Triangle, if the Diameter of the inscribed Circle be subtracted from the Perimeter, the remainder is equal to the double of the Hypothenusal: As in $\triangle ABC$ before expos'd, if $AF + AG$, ($= DF + DG =$ the Diameter of the inscribed Circle DFGE,) be subtracted from the Perimeter $AC + AB + BC$, there evidently remains $FC + GB + BC = 2BE + 2EC = 2BC$; for $FC = EC$, and $GB = BE$. Therefore,

From 5° and 6° we may deduce

CANON 1.

7. Divide the given Area by the Semiperimeter, and subtract the double of the Quotient from the whole Perimeter; the half of the remainder shall be the Hypothenusal; which subtracted from the Perimeter leaves the summ of the sides about the right angle. Then the Hypothenusal being given, as also the summ of the sides about the right angle, the sides shall be given severally, both Geometrically and Arithmetically, by *Probl. 4. Chap. 8.*

Another way to find out the Hypothenusal.

Suppos.

8. $\triangle ABC$ is right-angled at A.
9. $cc = \square \frac{1}{2} AC, AB = \triangle ABC$ is given.
10. $b = AC + AB + BC$ is given.

Req. to find BC the Hypothenusal.

Resolut. II.

11. Put a for the Hypothenusal, *viz.*	$a = BC$.
12. Therefore from 10° and 11°, the summ of the sides about the right angle is	$b - a$ ($= AC + AB$.)
13. And from 11° the Square of the Hypothenusal is .	aa.
14. And from 12°, the Square of the summ of the sides about the right angle is	$aa - 2ba + bb$.

15. Therefore

15. Therefore from 9°, 13° and 14°, (by *Theor.* 1. in 12° of *Probl.* 15. *Chap.* 8.) this Equation ariseth, *viz.* $aa - 2ba + bb = aa + 4cc$.
16. From which Equation reduced, the Hypothenusal will be made known, *viz.* $a = \frac{1}{2}b - \frac{2cc}{b}$.
Hence

CANON 2.

17. From the Semiperimeter subtract the Quotient of the double Area divided by the Perimeter, and the remainder shall be the Hypothenusal. Then the sides about the right angle shall be given as before in *Canon* 1.
18. But to the end there may be a possibility of finding out a right-angled Triangle to solve the Problem propounded, it is evident by *Canon* 2, that the given quantities b and cc must be such, that $\frac{1}{2}b \sqsubset \frac{2cc}{b}$.
19. Whence, by doubling each part, $b \sqsubset \frac{4cc}{b}$.
20. Therefore from 19°, by multiplying each part into b, $bb \sqsubset 4cc$.
21. And consequently, $4cc \sqsupset bb$.

Therefore the reason of the Determination added to *Probl.* 22. is manifest; and the Canons may be exemplified by any right-angled Triangle in Rational numbers.

CHAP. X.

The fourth Classis *of Examples of the Resolution and Composition of Plane Problems.*

IN which Examples, the Resolution ends in an Analogy consisting of three Squares in continual Proportion, whereof the Mean is given; as also a Square equal either to the Difference, or else to the Summ of the Extremes; and therefore the Extremes shall be given severally, by *Probl.* 15, or 16. of *Chap.* 5.

Probl. I.

The difference of the Squares and the Rectangle of two right lines being given severally, to find out those lines.

Suppos.

1. $d =$ the given side of a Square equal to the difference of the Squares of two right lines.
2. $m =$ the given side of a Square equal to the Rectangle of the same lines.

Req. to find out the lines.

Resolution.

3. For the lesser of the two right lines sought put a.
4. Therefore the Square of the lesser line is aa.
5. And from 1° and 4°, the Square of the greater line is $aa + dd$.
6. Therefore from 5° the greater line is $\sqrt{:\overline{aa + dd}:}$
7. And from 3° and 6° the Rectangle made of the two right lines sought is $a \times \sqrt{:\overline{aa + dd}:}$
8. Which Rectangle (according to the import of the Problem) must be equal to the Square of the given right line m; therefore, $a \times \sqrt{:\overline{aa + dd}:} = mm$.
9. And that Equation may be resolved into these Proportionals, *viz.* $\sqrt{:\overline{aa + dd}:} \,.\, m :: m \,.\, a$.

Which Analogy is qualified in every respect like that in 60° of *Probl.* 15. *Chap.* 5. therefore the lines sought, which are represented by a and $\sqrt{:\overline{aa + dd}:}$ shall be given by the Geometrical Construction of the said *Probl.* 15. and their quantities in Numbers shall be given also by the Canon in 77° of the same Problem.

Probl. II.

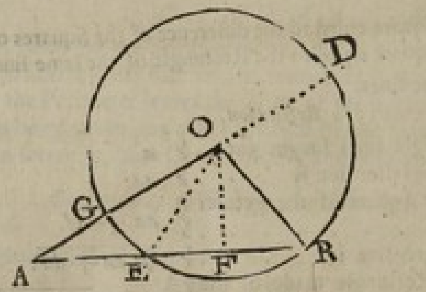
In a plain Triangle, any two sides and the Area being severally given, to find out the Triangle. But eight times the given Area must not exceed the excess by which the Square of the summ of the two given sides is greater than the Square of their difference.

Note. The two given sides of the Triangle sought are either equal one to the other, or unequal; if equal, then the Triangle is either equicrural, or equilateral; each of which is compos'd of two equal right-angled Triangles, having a common Perpendicular falling from the angle contain'd under two equal sides upon the middle of the third side, (or Base;) in which Case, the Problem propounded may be stated thus; *viz.* The Hypothenusal of a right angled Triangle is given, as also the Area, (that is, half the given Area,) to find the sides about the right angle; which sides may be found out both Geometrically and Arithmetically, by *Probl.* 15. *Chap* 8. and then either of those sides about the right angle being doubled, gives the Base of a Triangle, either equicrural or equilateral, to solve the Problem. But if the two sides given be unequal, then the Triangle sought is either right-angled, or else obtuse-angled, or hath unequal acute angles at the Base; if right-angled, then the third side or Base shall be given by *prop.* 47. *Elem.* 1. by which *Prop.* it may be discovered whether the Triangle sought, when the two given sides are unequal, and exprest by numbers, be right-angled at the Base or not; for if the Square of the lesser side given be subtracted from the Square of the greater, and the square Root of the remainder be multiplied by half the said lesser side, and if that Product be equal to the given Area, then the Triangle sought is right-angled at the Base. Now when the Triangle sought is neither right-angled at the Base, nor hath equal acute angles at the Base, the Resolution of the Problem propos'd may be formed in manner following.

Suppos.

1. ARO is a $\triangle$ whose angles A and R are acute and unequal, also $< R \sqsubset < A$, and consequently $AO \sqsubset RO$, (*per prop.* 19. *Elem.* 1.)
2. $b = AO$ is given.
3. $k = RO$ is given.
4. $c = b + k = AD$ is given.
5. $d = b - k = AG$ is given.
6. $m =$ a given right line, whose Square is equal to the Triangle ARO.

Req. to find out the Triangle.



AR = 21
AO = 17
RO = 10
OF = 8
AE = 9
EF = 6
FR = 6
AF = 15
AG = 7
$\square M = 84 = \triangle ARO$.

Resolution.

7. Put a for the Base (or third side) sought, *viz.* . . . . . $a = AR$.
8. Then by *Theor.* 5. in 69° of *Probl.* 8. *Chap.* 8. this Equation is manifest, *viz.*

$$\sqrt{:\tfrac{1}{4}aa - \tfrac{1}{4}dd:} \text{ into } \sqrt{:\tfrac{1}{4}cc - \tfrac{1}{4}aa:} = mm.$$

9. Which Equation may be resolved into these Proportionals, *viz.*

$$\sqrt{:\tfrac{1}{4}aa - \tfrac{1}{4}dd:} \ . \ m \ :: \ m \ . \ \sqrt{:\tfrac{1}{4}cc - \tfrac{1}{4}aa:}$$

10. And their Squares are also Proportionals, *viz.*

$$\tfrac{1}{4}aa - \tfrac{1}{4}dd \ . \ mm \ :: \ mm \ . \ \tfrac{1}{4}cc - \tfrac{1}{4}aa.$$

11. Which Proportionals being severally quadrupled give these, *viz.*

$$aa - dd \ . \ 4mm \ :: \ 4mm \ . \ cc - aa.$$

12. And

12. And by extracting the square Root out of every one of the four last Proportionals, these also shall be Proportionals, *viz.*

$$\sqrt{:aa - dd}: \quad . \quad 2m \quad :: \quad 2m \quad . \quad \sqrt{:cc - aa}:$$

13. Now forasmuch as the two last Analogies in 11° and 12° are in every respect like to those in 57° of *Probl.* 16. *Chap.* 5. therefore by the Geometrical Construction of that Problem, respect being had to what hath been deliver'd in the said *Sect.* 57. two right lines may be found out, either of which may be taken for the value of a, that is, the Base of the Triangle sought; which right lines are equal to one another when $8mm = cc - dd$, but unequal when $8mm$ is less than $cc - dd$; in which latter Case, two Triangles may be found out having unequal Bases, but each Triangle shall have the same Area and leggs, as will hereafter appear in the Composition of this Problem. But to the end there may be a possibility of effecting the same by the help of the lines given, they must be subject to the following Determination, which is discovered by the Analogy in 11°.

Determination.

14. The octuple of the given Area must not be greater than the excess by which the Square of the summ of the two given sides exceeds the Square of their difference.

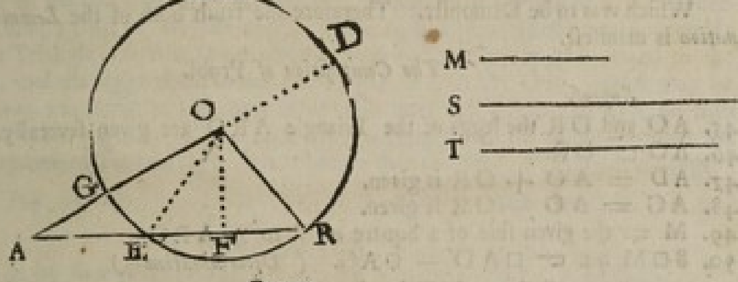
The truth of this Determination will be evident by the following

LEMMA.

15. The octuple of the Area of a plain Triangle having unequal leggs, is never greater than the excess by which the Square of the summ of the leggs exceeds the Square of their difference.

Suppos.

16. AR the Base . . . . } of △ ARO.
17. AO and OR the leggs }
18. $AO \sqsubset OR$.
19. M is a right line, such, that □M = △ ARO.
20. $AD = AO + OR$.
21. $AG = AO - OR$.
22. . . . *Req. demonstr.* . . . 8□M, or 8△ARO, not $\sqsubset$ □AD—□AG.



Prepar.

23. By *Probl.* 4. *Chap.* 5. find a right line S, such, that its Square may be equal to □AR − □AG; therefore,

$$S = \sqrt{:\square AR - \square AG}:$$

24. Likewise by the same *Probl.* find a right line T, such, that its Square may be equal to □AD − □AR; therefore,

$$T = \sqrt{:\square AD - \square AR}:$$

Demonstration.

25. Forasmuch as by *Theor.* 5. in 69° of *Probl.* 8. *Chap* 8. (respect being had to the Diagram here in view,)

$$\square \text{ of } \sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG}: \times \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR}: = \triangle ARO.$$

26. And by *Supposition* in 19°, ... □M = △ARO.

27. Therefore from 25° and 26°, (*per Axiom.* 1. *Chap.* 2.)

$$\square \text{ of } \sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG}: \times \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR}: = \square M.$$

 28. And

28. And by resolving the last Equation into Proportionals, this Analogy is manifest, *viz.*
$$\sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG} : . \ M :: M . \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR}.$$
29. The Squares of which Proportionals are also Proportionals, *(per prop.22.Elem.6.) viz.*
$$\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG . \square M :: \square M . \tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR.$$
30. And by quadrupling all in 29°,
$$\square AR - \square AG . 4\square M :: 4\square M . \square AD - \square AR.$$
31. But the sides of proportional Squares are also Proportionals, therefore from 30°,
$$\sqrt{:\square AR - \square AG} . 2M :: 2M . \sqrt{:\square AD - \square AR}.$$
32. And because by *Constr.* in 23°, ... $S = \sqrt{:\square AR - \square AG}$.
33. Also by *Constr.* in 24°, ... $T = \sqrt{:\square AD - \square AR}$.
34. Therefore from 31°, 32° and 33°, by exchanging equal right lines, this Analogy is manifest, *viz.*
$$S . 2M :: 2M . T.$$
35. And because the last Analogy consists of three Proportionals, therefore by what hath been said in 20° of *Probl.* 13. *Chap.* 5.
$$4M \text{ not } \sqsubset S + T.$$
36. And by squaring each part in 35°,
$$16\square M \text{ not } \sqsubset \square S + \square T + 2\,\square S,T.$$
37. And because by *Constr.* in 23°, . . . $\square AR - \square AG = \square S$.
38. And by *Constr.* in 24°, ... $\square AD - \square AR = \square T$.
39. Therefore by adding the two last Equations together,
$$\square AD - \square AG = \square S + \square T.$$
40. Therefore from 36° and 38°, by exchanging equal Squares,
$$16\square M \text{ not } \sqsubset \square AD - \square AG + 2\,\square S,T.$$
41. But from 34°, *(per 17. prop. Elem. 1.)* . . . $4\square M = \square S,T$.
42. And consequently, ... $8\square M = 2\,\square S,T$.
43. Therefore from 40° and 42°,
$$16\square M \text{ not } \sqsubset \square AD - \square AG + 8\square M.$$
44. Wherefore from 43°, by subtracting $8\square M$ from each part,
$$8\square M \text{ not } \sqsubset \square AD - \square AG.$$

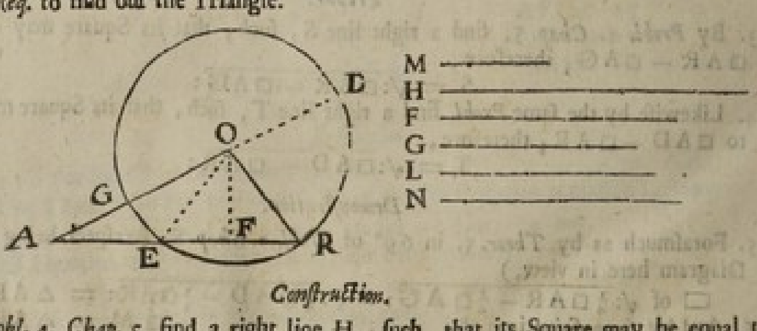
Which was to be Demonstr. Therefore the truth both of the *Lemma* and *Determination* is manifest.

The Composition of Probl. 2.

Suppos.

45. AO and OR the leggs of the Triangle ARO are given severally.
46. $AO \sqsubset OR$.
47. $AD = AO + OR$ is given.
48. $AG = AO - OR$ is given.
49. $M =$ the given side of a Square equal to $\triangle$ ARO.
50. $8\square M$ not $\sqsubset \square AD - \square AG$. *(Determination.)*

Req. to find out the Triangle.



Constrution.

51. By *Probl.* 4. *Chap.* 5. find a right line H, such, that its Square may be equal to $\square AD - \square AG$; therefore,
$$\square H = \square AD - \square AG.$$

52. Then

52. Then making H to be the Hypothenusal of a right-angled Triangle, and 2M a mean Proportional between the Base and Perpendicular, find out (*per Probl.* 16. *Chap.* 5.) the Base and Perpendicular; but such a Triangle cannot be found out unless it be proved that $\frac{4\,\square M}{H}$ is not greater than $\frac{1}{2}$H; for, (agreeable to the Determination annexed to the said *Probl.* 16.) the right line arising by the Application of the Square of the given mean Proportional to the given Hypothenusal, must not be greater than half the Hypothenusal; I shall therefore first prove that $\frac{4\,\square M}{H}$ is not greater than $\frac{1}{2}$H.

Req. demonstr. . . . . $\frac{4\,\square M}{H}$ not $\sqsubset$ $\frac{1}{2}$H.

Demonstration.

By the Determination in 50°, ... } $8\,\square M$ not $\sqsubset$ $\square AD - \square AG$.
And by *Constr.* in 51°, ... } . . . $\square H = \square AD - \square AG$.
Therefore, (*per Ax.* 3. *Chap.* 2.) . . . . } $8\,\square M$ not $\sqsubset$ $\square H$.
And consequently, ... } $4\,\square M$ not $\sqsubset$ $\frac{1}{2}\square H$.
Wherefore by Application of each part to H, . } $\frac{4\,\square M}{H}$ not $\sqsubset$ $\frac{1}{2}$H.

Which was to be Demonstr.

53. Having proved that 'tis possible to find out a right-angled Triangle which shall have the right line H for a Hypothenusal, and the double of the right line M for a mean Proportional between the Base and Perpendicular, suppose that (by *Probl.* 16. *Chap.* 5.) the right lines F and G are found equal to the said Base and Perpendicular; therefore by such Construction,

F . 2M :: 2M . G.

Also, ... $\square F + \square G = \square H$.

54. By *Probl.* 2. *Chap.* 5. find out a right line L, such, that its Square may be equal to $\square AG + \square G$; therefore, $\square L = \square AG + \square G$.

55. Likewise, find out a right line N, such, that its Square may be equal to $\square AG + \square F$; therefore, $\square N = \square AG + \square F$.

56. Now either of the said right lines N and L, (being the two values of *a* in the foregoing Resolution,) shall be the Base of a Triangle, to satisfie the Problem propounded; therefore let a Triangle, as ARO be made, so, that its Base AR may be equal to the right line N, and the leggs equal to the given right lines AO, OR; which may be done, (by *prop.* 22. *Elem.* 1.) if those three right lines AR, AO and OR be such that every two taken together as one right line, is longer than the third; but that the summ of every two of the said right lines is longer than the third, I demonstrate thus:

57. . . . I. *Req. demonstr.* . . . . AR + AO $\sqsubset$ OR.

Demonstration.

58. Forasmuch as by *Supposition* in 46°, . . . } . . . . AO $\sqsubset$ OR.
59. Therefore much more . . . . } AR + AO $\sqsubset$ OR.
60. . . . II. *Req. demonstr.* . . . . AR + OR $\sqsubset$ AO.

Demonstration.

61. By *Constr.* in 55°, ... } $\square AG + \square F = \square N$.
62. Also by *Constr.* in 56°, AR = N, and consequently, ... } . . . . $\square AR = \square N$.
63. Therefore from 61° and 62°, (*per Ax.* 1. *Chap.* 2.) . . . . } $\square AR = \square AG + \square F$.
64. Therefore from 63°, ... } $\square AR \sqsubset \square AG$.
65. And consequently, ... } . AR $\sqsubset$ AG.
66. But by *Suppos.* in 48°, ... } . AG = AO − OR.
67. Therefore from 65° and 66°, (*per Ax* 3. *Ch.* 2.) } . AR $\sqsubset$ AO − OR.
68. Wherefore by adding OR to each part in 67°, } AR + OR $\sqsubset$ AO.

Which was to Demonstr.

69. III. *Req.*

69. . . . III. *Req. demonstr.* . . . . . . $AO + OR \sqsubset AR$.

Demonstration.

70. By *Construction* in 51°, ... $\square AD - \square AG = \square H$.
71. Also by *Constr.* in 53°, ... $\square F + \square G = \square H$.
72. Therefore from 70° and 71°, (*per Ax.* 1. *Chap.* 2.) . . . . $\square AD - \square AG = \square F + \square G$.
73. And from 72°, by adding $\square AG$ to each part, ... $\square AD = \square F + \square G + \square AG$.
74. And from 73°, by equal subtraction of $\square G$, $\square AD - \square G = \square F + \square AG$.
75. But by *Constr.* in 55°, ... $\square N = \square F + \square AG$.
76. Therefore from 74° and 75°, (*per Ax.* 1. *Chap.* 2.) . . . . $\square AD - \square G = \square N$.
77. And because by *Constr.* in 56°, $AR = N$, and consequently, ... $\square AR = \square N$.
78. Therefore from 76° and 77°, (*per Ax.* 1. *Chap.* 2.) . . . . $\square AD - \square G = \square AR$.
79. And by adding the $\square G$ to each part in 78°, $\square AD = \square AR + \square G$.
80. From 79° 'tis evident, (*per Ax.* 25. *Chap.* 2.) that . . . . $\square AD \sqsubset \square AR$.
81. And consequently, ... $AD \sqsubset AR$.
82. And by *Suppos.* in 47°, ... $AD = AO + OR$.
83. Wherefore from 81° and 82°, (*per Ax.* 4. *Chap.* 2.) . . . . $AO + OR \sqsubset AR$.

Which was to be Demonstr.

84. Having proved that of the said three right lines AR, AO and OR every two taken together as one right line is longer than the third, it is possible to make a Triangle of them, (by *prop.* 22. *Elem.* 1.) suppose it then to be $\triangle ARO$; I say the Triangle so made is that which was required. Now we must shew that it will satisfie the Problem propounded.

85. First then, by *Construction* in 84°, the leggs AO and OR are equal to the two right lines given in 45° to be the leggs of the Triangle sought; and that the $\triangle ARO$ is equal to the Square of the given right line M, the following Demonstration, formed by a retrograde repetition of the steps of the Resolution aforegoing, will make manifest.

86. . . . *Req. demonstr.* . . . . $\triangle ARO = \square M$.

Demonstration.

87. By *Construction* in 53°, ... $F \,.\, 2M :: 2M \,.\, G$.
88. Therefore, (*per* 22. *prop.* 6. *Elem.*) . . . $\square F \,.\, 4\square M :: 4\square M \,.\, \square G$.
89. By *Constr.* in 55°, ... $\square N = \square AG + \square F$.
90. Also by *Constr.* in 56°, $N = AR$, and consequently, ... $\square N = \square AR$.
91. Therefore from 89° and 90°, (*per Ax.* 1.) $\square AR = \square AG + \square F$.
92. And from 91°, by equal subtraction of $\square AG$, $\square AR - \square AG = \square F$.
93. It hath been proved in 79°, that . . . . $\square AD = \square AR + \square G$.
94. And consequently, by subtracting $\square AR$ from each part, ... $\square AD - \square AR = \square G$.
95. Therefore from 88°, 92° and 94°, by exchanging equal Planes, this Analogy will be manifest, *viz.* $\square AR - \square AG \,.\, 4\square M :: 4\square M \,.\, \square AD - \square AR$.

That is, in 11°, $aa - dd \,.\, 4mm :: 4mm \,.\, cc - aa$.

96. And by taking a quarter of all in 95°,

$$\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG \,.\, \square M :: \square M \,.\, \tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR.$$

That is, in 10°, $\tfrac{1}{4}aa - \tfrac{1}{4}dd \,.\, mm :: mm \,.\, \tfrac{1}{4}cc - \tfrac{1}{4}aa$.

97. And because (*per prop.* 22. *Elem.* 6.) the sides of proportional Squares are also Proportionals, therefore from 96°,

$$\sqrt{:\overline{\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG}:} \,.\, M :: M \,.\, \sqrt{:\overline{\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR}:}$$

That is, in 9°, $\sqrt{:\overline{\tfrac{1}{4}aa - \tfrac{1}{4}dd}:} \,.\, m :: m \,.\, \sqrt{:\overline{\tfrac{1}{4}cc - \tfrac{1}{4}aa}:}$

98. And

98. And from 97°. (*per* 17. *prop.* 6. *Elem.*)

$$\square \text{ of } \sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG:} \times \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR:} = \square M.$$

That is, in 8°,

$$\sqrt{:\tfrac{1}{4}aa - \tfrac{1}{4}dd:} \text{ into } \sqrt{:\tfrac{1}{4}cc - \tfrac{1}{4}aa:} = mm.$$

99. But by *Theor.* 5. in 69° of *Probl.* 8. *Chap.* 8. respect being had to the last preceding Diagram,

$$\square \text{ of } \sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG:} \times \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR:} = \triangle ARO.$$

100. Therefore from 98° and 99°, (*per Axiom.* 1. *Chap.* 2.) . . . . $\triangle ARO = \square M.$

Which was to be Demonstr.

Therefore one Triangle is found out to solve the Problem; and by the like Construction and Argumentation another Triangle may be made upon the right line L as a Base, which latter Triangle shall have its Area and leggs correspondently equal to the Area and leggs of the first Triangle.

101. But in order to raise an Arithmetical Canon for the finding out of the third side (or Base) of the Triangle sought, first let us suppose the given quantities to be exprest by numbers, and then let the Analogy in 11° be here repeated, *viz.*

$$aa - dd \;.\; 4mm \;::\; 4mm \;.\; cc - aa.$$

102. Which Analogy is reducible to this following Equation, *viz.*

$$ccaa + ddaa - aaaa = 16mmmm + ccdd.$$

103. Now if that Equation be resolved according to the Canon in 55° of *Probl.* 16. *Chap.* 5. this following Canon will thence arise, whereby the Base of the Triangle sought may be found out Arithmetically.

CANON.

$$\sqrt{:\tfrac{1}{2}cc + \tfrac{1}{2}dd + \sqrt{:\tfrac{1}{4}cccc + \tfrac{1}{4}dddd - \tfrac{1}{2}ccdd - 16mmmm:}} = a.$$

Also,

$$\sqrt{:\tfrac{1}{2}cc + \tfrac{1}{2}dd - \sqrt{:\tfrac{1}{4}cccc + \tfrac{1}{4}dadd - \tfrac{1}{2}ccdd - 16mmmm:}} = a.$$

An Example in Numbers, to illustrate the foregoing Resolution *of* Probl. 2.

Suppos.

104. $b = 17$ } the leggs of a plain Triangle are given; whence,
105. $k = 10$ }
106. $c = 27$ the summ of the leggs is given, and
107. $d = 7$ the difference of the leggs is given.
108. $mm = 84$ the Area of the same Triangle is given.

Req. to find out the Base or third side.

Solution Arithmetical.

109. By the help of the numbers given in 106°, 107°, 108°, and of the Canon in 103°, the numbers 21 and $\sqrt{337}$ will be found out, either of which may be taken for the value of a in the foregoing Resolution, that is, the Base of a Triangle which shall have 17 and 10 for its leggs, and 84 for its Area: For as well from these three sides 21, 17 and 10, as from these, $\sqrt{337}$, 17 and 10, the Area will be found 84, (by *Canon* 1. in 77° of *Probl.* 8. *Chap.* 8.) And therefore the Problem propounded is solved both Arithmetically and Geometrically.

Probl. III.

The Base, Perpendicular and Rectangle of the leggs of a plain Triangle being given severally, to find out the Triangle. But the given quantities must be subject to the Determinations hereafter exprest.

Suppos.

1. $b =$ the Base of a Triangle is given.
2. $p =$ the Perpendicular is given.
3. $r =$ the side of a Square equal to the Rectangle of the leggs is given.

Req. to find the Triangle.

4. *Vieta*,

4. *Vieta*, in *Probl.* 1. of his Appendix to his *Appollonius Gallus*, shews the Construction of this Problem with great facility, by the help of this Theorem, *viz.* The Rectangle of any two sides containing an angle of a plain Triangle is equal to the Rectangle of the Perpendicular falling from that angle upon the opposite side, and the Diameter of the Circle circumscribing the same Triangle. Afterwards *Marinus Ghetaldus* in his Treatise of Mathematical Resolution and Composition supplies *Vieta*'s Construction with Determinations: But it will be difficult for Learners to deduce the Arithmetical Solution from that Geometrical Effection; I shall therefore form Resolutions and Compositions to solve this Problem both Arithmetically and Geometrically in all Cases, which are these five, *viz.*

Case 1.	When	$rr = \frac{1}{4}bb + pp.$
Case 2.	When . . . . And	$rr \sqsubset \frac{1}{4}bb + pp.$ $rr \sqsubset \frac{1}{2}bb.$
Case 3.	When . . . . And	$rr \sqsubset \frac{1}{4}bb + pp.$ $rr \sqsupset \frac{1}{2}bb.$
Case 4.	When . . . . And	$rr \sqsubset \frac{1}{4}bb + pp.$ $rr = \frac{1}{2}bb.$
Case 5.	When	$rr \sqsupset \frac{1}{4}bb + pp.$

Now to solve the proposed Problem in every one of those Cases, I shall give a peculiar Resolution, with a Canon and an Example in numbers; and if the given Quantities be exprest by numbers, it will be easie to discern under which of those Cases they fall.

The Resolution of CASE 1.

5. One Triangle may be always easily found out to solve the Problem in *Case* 1. *viz.* when $rr = \frac{1}{4}bb + pp$; for if upon the middle of the given Base, the given Perpendicular be erected, and from the ends of the Base two right lines be drawn to meet at the top of the Perpendicular, then the equicrural Triangle so formed will satisfie the Problem. For in every Isosceles, or equicrural Triangle, the Square of half the Base, (or a quarter of the Square of the whole Base,) together with the Square of the Perpendicular is (*per prop.* 47. *Elem.* 1.) equal to the Square of either of the equal leggs, that is, the Rectangle of the leggs.

Moreover, when in *Case* 1. it happens that $b \sqsubset 2p$, then besides that equicrural Triangle, another having unequal leggs may be found out to solve the Problem, by this following Resolution, *viz.*

6. For the unknown difference of the leggs put . . . . . . $a.$
7. Then by *Theor.* 7. *Ch.* 4. the Square of the summ of the leggs is $aa + 4rr.$
8. Therefore from 6° and 7°, (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.)

$$aa + 4rr - bb \ . \ aa + 4rr - bb - 4pp \ :: \ bb \ . \ aa.$$

9. And because by *Suppos.* in *Case* 1. . . . . $rr = \frac{1}{4}bb + pp.$
10. And consequently, ... $4rr = bb + 4pp.$
11. Therefore from 8° and 10°, by exchanging equal quantities,

$$aa + bb + 4pp - bb \ . \ aa + bb + 4pp - bb - 4pp \ :: \ bb \ . \ aa \ . \text{ Proportionals.}$$

12. Whence, by casting away such equal quantities as destroy one another by reason of contrary signs + and —, this Analogy ariseth,

$$aa + 4pp \ . \ aa \ :: \ bb \ . \ aa.$$

13. Therefore from 12°, (*per prop.* 14. *Elem.* 5.) this Equation ensues, *viz.* . . . . $aa + 4pp = bb.$
14. And by subtracting $4pp$ from each part of that Equation, . $aa = bb - 4pp.$
15. Therefore by extracting the square Root out of each part of the last Equation, ... $a = \sqrt{:\overline{bb - 4pp}:}$
16. And out of 7° and 14°, the Square of the summ of the leggs will be found equal to . . . . $bb - 4pp + 4rr.$
17. And from 10° and 16°, by exchanging $4rr$ for its equivalent quantity $bb + 4pp$, the Square of the summ of the leggs is also equal to . . . . $bb - 4pp + bb + 4pp.$

18. Which

18. Which last quantity being first contracted, and then the square Root extracted, the summ of the leggs will be made known, to wit, . . . $\sqrt{2bb}$.

The preceding 15^{th} and 18^{th} steps do afford this

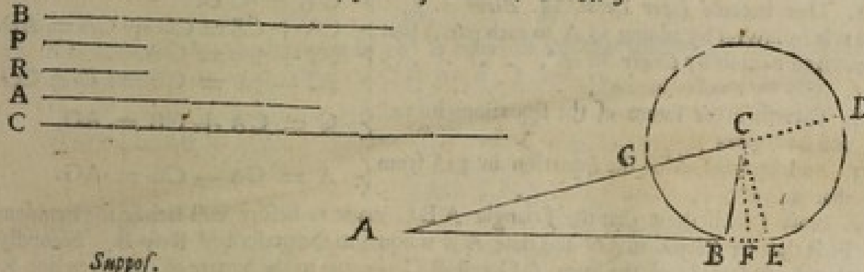
CANON.

19. . . $\left\{ \begin{array}{l} \sqrt{: bb - 4pp :} = \text{the Difference} \\ \sqrt{2bb} = \text{the Summ} \end{array} \right\}$ of the Leggs sought.

Then the summ and difference of the leggs sought being given, the leggs shall be given severally, (*per Theor.* 9. *Chap.* 4.) and consequently the Triangle.

By which Canon it appears, that the Triangle sought in *Case* 1. cannot have unequal leggs unless the given Base exceeds the double of the given Perpendicular: For by the Canon, the difference of the leggs is equal to $\sqrt{: bb - 4pp :}$ which cannot be a real Quantity, (to wit, greater than nothing,) unless $b \sqsubset 2p$.

The Composition of Case 1. Probl. 3.



Suppos.

20. B = the Base of a Triangle is given.
21. P = the Perpendicular is given.
22. R = the side of a Square equal to the Rectangle of the leggs is given.
23. $\square R = \frac{1}{4}\square B + \square P$. Also, $B \sqsubset 2P$.

Req. to find the Triangle.

Construction.

24. By *Probl.* 4. *Chap.* 5. find a right line A, such, that its Square may be equal to $\square B - 4\square P$: Which Effection is possible, for by *Suppos.* in 23°, $B \sqsubset 2P$; therefore, $A = \sqrt{: \square B - 4\square P :}$ (that is, $a = \sqrt{: bb - 4pp :}$)

25. Again, by *Probl.* 2. *Chap.* 5. find a right line C, such, that its Square may be equal unto $2\square B$, therefore, $C = \sqrt{2\square B}$ ($= \sqrt{2bb}$.)

Thus far the Construction hath been made according to the direction of the Canon in 19°.

26. Now let a Triangle be made of these three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}A$, and $\frac{1}{2}C - \frac{1}{2}A$; which Effection is possible, (*per prop.* 22. *Elem.* 1.) if $C \sqsubset A$, and the summ of every two of those lines be greater than the third: But that those lines are so qualified, I prove thus;

First, from the Construction in 24° and 25° it is easie to inferr that $C \sqsubset A$, and consequently $\frac{1}{2}C - \frac{1}{2}A$ is equal to some real right line.

Secondly, it is manifest that the summ of the right lines B and $\frac{1}{2}C + \frac{1}{2}A$ is greater than the third line $\frac{1}{2}C - \frac{1}{2}A$.

Thirdly, that the summ of the two lines $\frac{1}{2}C + \frac{1}{2}A$ and $\frac{1}{2}C - \frac{1}{2}A$ is greater than the third line B, *viz.* that $C \sqsubset B$, is evident by *Constr.* in 25°.

Fourthly, that the summ of the two lines B and $\frac{1}{2}C - \frac{1}{2}A$ is greater than the third line $\frac{1}{2}C + \frac{1}{2}A$, *viz.* that $B \sqsubset A$, I prove thus;

By *Constr.* in 24°, ... $\square B - 4\square P = \square A$.
Therefore by adding $4\square P$ to each part, . . . $\square B = \square A + 4\square P$.
Therefore, (*per Ax.* 25. *Chap.* 2.) . . . . $\square B \sqsubset \square A$.
Therefore, ... $B \sqsubset A$.

Which was to be demonstrated in the fourth and last place.

Having proved that $\frac{1}{2}C - \frac{1}{2}A$ is a real right line, and that the summ of every two of these three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}A$ and $\frac{1}{2}C - \frac{1}{2}A$, is greater than the third,

'tis possible to make a Triangle of those three lines; (*per prop.* 22. *Elem* 1.) Suppose then it be made, and that the Triangle found out is A B C, having its Base $AB = B$, (the Base prescribed in the Problem,) and the greater legg $AC = \frac{1}{2}C + \frac{1}{2}A$, and the lesser legg $BC = \frac{1}{2}C - \frac{1}{2}A$: I say the Triangle A B C will satisfie the Problem propounded. But to render the Demonstration thereof the more clear and easie, I shall premise a few things in seven steps next following.

27. If the given Quantities be exprest by numbers, the kind of the Triangle sought in *Case* 1. when the leggs are unequal, as they were supposed in the foregoing Resolution, may be discovered by the help of the Canon in 19° of this Problem, and of *Theor.* 3. in 35° of *Probl.* 8. *Chap.* 8. Supposing then it be discovered that the Perpendicular falls without the Triangle A B C upon A B increased, from C as a Center at the distance of C B ($= \frac{1}{2}C - \frac{1}{2}A$) describe the Circle C B G D cutting C A in G; then produce A C and A B to the Circumference in D and E; draw also the Semidiameter C E, and from the Center C make $CF \perp BE$, whence $FE = FB$, (*per prop.* 3. *Elem.* 3.)

28. Then because (*per Defin.* 15. *Elem* 1.) . . . $CB = CD.$

29. It follows, (by adding C A to each part,) that . . . $CA + CB = CD + CA = AD.$

30. And because by *Constr.* in 26°, ... $\frac{1}{2}C + \frac{1}{2}A = CA.$

31. Also by *Constr.* in 26°, ... $\frac{1}{2}C - \frac{1}{2}A = CB = CD = CG.$

32. Therefore the summ of the Equations in 30° and 31° gives . . . . $C = CA + CB = AD.$

33. And by subtracting the Equation in 31° from that in 30°, ... $A = CA - CB = AG.$

34. Now I shall shew that the Triangle A B C made as before will satisfie the Problem. First then by *Constr.* in 26° the Base A B is equal to the prescribed Base B. Secondly, that the Rectangle of the leggs A C and B C is equal to the Square of the given line R, (to wit, the prescribed Rectangle,) I shall here demonstrate.

35. . . . *Req. demonstr.* . . . . $\square AC,BC = \square R.$

Demonstration.

36. By *Constr.* in 24° and 26°, ... $\square A = \square AB - 4\square P.$

37. And from 33°, ... $\square A = \square AG.$

38. Therefore from 36° and 37°, (*per Axiom.* 1. *Chap.* 2.) . . . . $\square AG = \square AB - 4\square P.$

39. And by adding $4\square P$ to each part in 38°, . . $4\square P + \square AG = \square AB.$

40. And by adding $\square$ AB to each part of the last Equation, ... $\square AB + 4\square P + \square AG = 2\square AB.$

41. But from 32°, 25° and 26°, ... $\square AD = 2\square AB.$

42. Therefore from 40° and 41°, (*per Axiom.* 1. *Chap.* 2.) . . . . $\square AB + 4\square P + \square AG = \square AD.$

43. And by subtracting $\square$ AG from each part in 42°, ... $\square AB + 4\square P = \square AD - \square AG.$

44. From 32° and 33°, AD is the summ, and AG the difference of CA and CB, therefore, (*per Theor.* 7. *Chap.* 4. . . . . $4\square AC,BC = \square AD - \square AG.$

45. Therefore from 43° and 44°, (*per Axiom.* 1. *Chap.* 2.) . . . . $\square AB + 4\square P = 4\square AC,BC.$

46. But from 23° and 26°, ... $\square AB + 4\square P = 4\square R.$

47. Therefore from 45° and 46°, (*per Ax.* 1.) $4\square AC,BC = 4\square R.$

48. Therefore, (*per Ax.* 21. *Chap.* 2.) . . . . $\square AC,BC = \square R.$

Which was to be Demonstr.

Thirdly and lastly, that the Perpendicular C F is equal to the given Perpendicular P, I shall here demonstrate by a retrograde repetition of the steps of the preceding Resolution.

49. . . . *Req. demonstr.* . . . . $CF = P.$

Demonstration.

50. By *Constr.* in 24°, ... $A = \sqrt{:}\square B - 4\square P:$

51. And it hath been shewn in 33°, that . . . . $A = AG.$

52. There-

52. Therefore from 50° and 51°, (*per Axiom.* 1. *Chap.* 2.) . . . . $AG = \sqrt{:\square B - 4\square P}:$

53. And because by *Constr.* in 26° AB = B, therefore from 52°, ... $AG = \sqrt{:\square AB - 4\square P}:$

That is, in 15°, ... $a = \sqrt{:bb - 4pp}:$

54. And from 53°, by comparing the Squares of each part one to another, ... $\square AG = \square AB - 4\square P.$

That is, in 14°, ... $aa = bb - 4pp.$

55. And by adding $4\square P$ to each part of the Equation in 54°, ... $\square AG + 4\square P = \square AB.$

That is, in 13, ... $aa + 4pp = bb.$

56. And from 55°, (*per prop.* 7. *Elem.* 5.) this Analogy ariseth, *viz.*

$$\square AG + 4\square P \;.\; \square AG :: \square AB \;.\; \square AG.$$

That is, in 12°,

$$aa + 4pp \;.\; aa :: bb \;.\; aa.$$

57. It is manifest that the Analogy in 56° is equal to this that follows, (for $\square AB - \square AB = 0$. Also, $4\square P - 4\square P = 0$,) *viz.*

$$\left.\begin{array}{r} \square AG + \square AB + 4\square P - \square AB \;. \\ \square AG + \square AB + 4\square P - \square AB - 4\square P \;:: \\ \square AB \;. \\ \square AG \;. \end{array}\right\} \text{Proportionals.}$$

That is, in 11°,

$$\left.\begin{array}{r} aa + bb + 4pp - bb \;. \\ aa + bb + 4pp - bb - 4pp \;:: \\ bb \;. \\ aa \;. \end{array}\right\} \text{Proportionals.}$$

58. From 23° and 26°, ... $4\square R = \square AB + 4\square P.$

That is, in 10°, ... $4rr = bb + 4pp.$

59. Therefore from 57° and 58°, by exchanging equal quantities, this Analogy ariseth,

$$\left.\begin{array}{r} \square AG + 4\square R - \square AB \;. \\ \square AG + 4\square R - \square AB - 4\square P \;:: \\ \square AB \;. \\ \square AG \;. \end{array}\right\} \text{That is, in 8°, } \left\{\begin{array}{r} aa + 4rr - bb \;. \\ aa + 4rr - bb - 4pp \;:: \\ bb \;. \\ aa \;. \end{array}\right.$$

60. But it hath been shewn in 47°, that . . . $4\square AC, BC = 4\square R.$

61. Therefore by setting $4\square AC, BC$ in the place of $4\square R$ in the Analogy in 59°, this ariseth, *viz.*

$$\left.\begin{array}{r} \square AG + 4\square AC, BC - \square AB \;. \\ \square AG + 4\square AC, BC - \square AB - 4\square P \;:: \\ \square AB \;. \\ \square AG \;. \end{array}\right\} \text{are Proportionals.}$$

62. And because from 32° and 33°, AD is the summ, and AG the difference of AC and BC, therefore by *Theor.* 7. *Chap.* 4.

$$\square AD = \square AG + 4\square AC, BC.$$

63. Therefore from 61° and 62°, by exchanging equal quantities, this Analogy ariseth, *viz.*

$$\square AD - \square AB \;.\; \square AD - \square AB - 4\square P :: \square AB \;.\; \square AG.$$

64. And from 63°, by Conversion of Reason,

$$\square AD - \square AB \;.\; 4\square P :: \square AB \;.\; \square AB - \square AG.$$

65. And from 65°, by altern and inverse Reason,

$$\square AB \;.\; \square AB - \square AG :: \square AD - \square AB \;.\; 4\square P.$$

66. But by *Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.

$$\square AB \;.\; \square AB - \square AG :: \square AD - \square AB \;.\; 4\square CF.$$

 67. There-

67. Therefore from 65° and 66°, (*per prop.* 11. and 14. *Elem.* 5.)

$$4\square CF = 4\square P.\ \text{And consequently,}\ CF = P.$$

Which was to be Demonstrated in the last place; and therefore *Case* 1. *Probl.* 4. is solved.

An Example in Numbers, to illustrate the foregoing Resolution *of* Case 1. Probl. 3.

Suppos.

68. $b = 20$ the Base of a Triangle is given.
69. $p = 6$ the Perpendicular is given.
70. $rr = 136$ the Rectangle of the leggs is given.
71. $rr = \frac{1}{4}bb + pp$; also, } agreeable to *Case* 1.
72. $b \sqsubset 2p$, ... }

Req. to find the Triangle.

Solution Arithmetical.

73. $16 =$ the difference of the leggs, is found out of 68° and 69°, by the Canon in 19° of this Problem.
74. $\sqrt{800} =$ the summ of the leggs, is found out of 68°, by the same Canon.
75. $\left\{\begin{matrix}\sqrt{200} + 8 = \\ \sqrt{200} - 8 =\end{matrix}\right\}$ the leggs are found out of 73° and 74°, (*per Theor.* 9. *Chap* 4.)

The Proof.

76. $\square$ of $\left\{\begin{matrix}\sqrt{200}, + 8, \\ \sqrt{200} - 8\end{matrix}\right\} = 136$ the given Rectangle. Also,
77. If . . $\left\{\begin{matrix}20 = \text{the Base} \ldots \\ \left.\begin{matrix}\sqrt{200} + 8, \\ \sqrt{200} - 8\end{matrix}\right\}\ \text{the leggs}\end{matrix}\right\}$ of a Triangle.
78. Then (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) the Perpendicular will be found 6, which is the same with the given Perpendicular in 69°.

Moreover, $\sqrt{136}$ and $\sqrt{136}$ shall be the leggs of an equicrural Triangle having the same Base, Perpendicular and Rectangle of the leggs as are before given in 68°, 69°, and 70°.

The Resolution of CASE 2. Probl. 3.

Suppos.

1. $b =$ the Base of a Triangle is given.
2. $p =$ the Perpendicular is given.
3. $r =$ a right line given, whose Square is equal to the Rectangle of the leggs of the said Triangle.
4. $rr \sqsubset \frac{1}{4}bb + pp$, } by *Suppos.* in Case 2.
5. $rr \sqsubset \frac{1}{2}bb$, ... }

Req. to find the Triangle.

6. The Triangle sought in this second Case, as also in the third, fourth and fifth Cases of the Problem propounded, cannot be equicrural, for in every equicrural Triangle $rr = \frac{1}{4}bb + pp$, which agrees only with the *Suppos.* in *Case* 1. and consequently the leggs of the Triangle sought in all the other Cases are unequal; therefore,

For the difference of the leggs sought put . . . } a.

7. Then from 4°, 5° and 6°, (*per Theor.* 7. *Chap.* 4.) the Square of the summ of the leggs shall be . . . } $aa + 4rr$.
8. Therefore from the premisses (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.) this Analogy ariseth, *viz.*

$$aa + 4rr - bb \;.\; aa + 4rr - bb - 4pp :: bb \;.\; aa.$$

9. Therefore alternately,

$$aa + 4rr - bb \;.\; bb :: aa + 4rr - bb - 4pp \;.\; aa.$$

10. From 4° 'tis evident that . . . . } . . $4rr - bb - 4pp \sqsubset 0$.
11. And from 10°, by adding aa to each part, . . . } $aa + 4rr - bb - 4pp \sqsubset aa$.

12. There-

12. Therefore from the Analogy in 9°, by Division of Reason, (which is evidently possible from the Equation in 11°,) these are Proportionals, *viz.*

$$aa + 4rr - 2bb \;.\; bb :: 4rr - bb - 4pp \;.\; aa.$$

13. From 5° 'tis easie to perceive that $4rr - 2bb$ is greater than nothing, suppose therefore . . $\Big\}$ $cc = 4rr - 2bb.$

14. It appears also in 10°, that $4rr$ exceeds $bb + 4pp$, we may therefore suppose . . . . $\Big\}$ $dd = 4rr - bb - 4pp.$

15. Then from 12°, 13° and 14°, by exchanging equal quantities, this Analogy ariseth, *viz.* . . $\Big\}$ $aa + cc \;.\; bb :: dd \;.\; aa.$

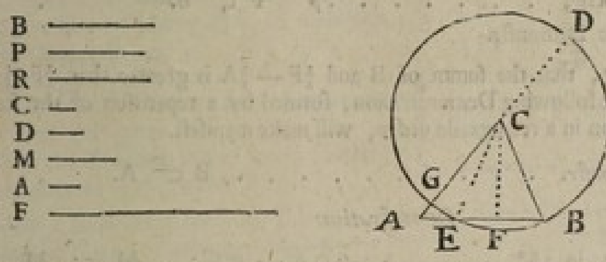
16. And from 15°, (*per prop.* 22. *Elem.* 6.) . . $\}$ $\sqrt{:}\,\overline{aa + cc}: \;.\; b :: d \;.\; a.$

17. Then between b and d find a mean Proportional, suppose it to be m, therefore, ... $\Big\}$ $b \;.\; m :: m \;.\; d.$

18. Therefore from 16° and 17°, by exchanging the mean Proportionals, (according to *Defin.* 8. *Chap* 3. concerning Inordinate proportion,) this Analogy ariseth, *viz.* . . . . $\Big\}$ $\sqrt{:}\,\overline{aa + cc}: \;.\; m :: m \;.\; a.$

19. Which last Analogy doth evidently consist of three Proportionals, whereof the greater extreme, to wit, $\sqrt{:}\,\overline{aa + cc}:$ is equal to the Hypothenusal of a right-angled Triangle, having c for its Base, and a for its Perpendicular: Now in that right-angled Triangle the Base c is given, also m, a mean Proportional between the Hypothenusal $\sqrt{:}\,\overline{aa + cc}:$ and the Perpendicular a, is given; therefore the Perpendicular a shall be given also, (*per Probl.* 15. *Chap.* 5.) which Perpendicular is equal to the difference of the leggs of the Triangle sought by this *Probl.* 3. in *Case* 2. Then the Rectangle and the difference of the leggs being given, the leggs shall be given severally, by *Probl.* 1. *Chap.* 9. Therefore the Triangle sought is given also.

The Composition of CASE 2. Probl. 3.



Suppos.

20. B = the Base of a Triangle is given.
21. P = the Perpendicular is given.
22. R = the side of a Square equal to the Rectangle of the leggs, is given.
23. $\square R \sqsubset \frac{1}{4}\square B + \square P,$ $\Big\}$ by *Suppos.* in *Case* 2.
24. $\square R \sqsubset \frac{1}{2}\square B,$ . . . . $\Big\}$

Req. to make the Triangle.

Construction.

25. By *Probl.* 4. *Chap* 5. find a right line C, such, that its Square may be equal to $4\square R - 2\square B$, (which Effection is evidently possible from the *Suppos.* in 24°;) therefore,

$$\square C = 4\square R - 2\square B.$$

26. Find likewise a right line D, such, that its Square may be equal to $4\square R - \square B - 4\square P$, (which may be done, for 'tis easie to inferr from the *Suppos.* in 23°, that $4\square R \sqsubset \square B + 4\square P$,) therefore,

$$\square D = 4\square R - \square B - 4\square P.$$

27. By

27. By *Probl.* 9. *Chap.* 5 find a mean proportional line M between the given Base B and the line D, (found out in 26°;) therefore,

$$B \ . \ M \ :: \ M \ . \ D.$$

28. Then esteeming the line C to be the Base of a right-angled Triangle, and the line M to be a mean Proportional between the Hypothenusal and Perpendicular; find out (by *Probl.* 15. *Chap.* 5.) the Perpendicular it self, suppose it be the right line A: Then if A be the Perpendicular, and C the Base, the Hypothenusal shall be equal to $\sqrt{:\square A + \square C:}$ and from that Effection this Analogy ariseth, *viz.*

$$\sqrt{:\square A + \square C:} \ . \ M \ :: \ M \ . \ A.$$

Which line A shall be equal to the difference of the leggs of the Triangle sought.

29. By *Probl.* 2. *Chap* 5. find a right line F, such, that its Square may be equal to $4\square R + \square A$; therefore,

$$\square F = 4\square R + \square A.$$

Which line F shall be equal to the summ of the leggs of the Triangle sought.

30. Then let a Triangle be made of these three right lines, *viz.* B, $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$; which may be done (by *Prop.* 22. *Elem.* 1.) if F be greater than A, and the summ of every two of those three lines be greater than the third: First, that $F \sqsubset A$, is evident from the *Constr.* in 29°. Secondly, it is also evident that the summ of B and $\frac{1}{2}F + \frac{1}{2}A$ exceeds $\frac{1}{2}F - \frac{1}{2}A$. Thirdly, that the summ of $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$ exceeds the Base B, *viz.* that $F \sqsubset B$, I demonstrate thus;

By *Constr.* in 29°, ... } $4\square R + \square A = \square F.$

And from the *Constr.* in 25°, ... } $2\square R - \frac{1}{2}\square C = \square B.$

But 'tis evident (*per Axiom.* 25. *Chap.* 2.) that . . . . } $4\square R + \square A \sqsubset 2\square R - \frac{1}{2}\square C.$

Therefore from the three last preceding steps, (*per Ax.* 3. *Chap.* 2.) . . . } $\square F \sqsubset \square B.$

And consequently, ... } $F \sqsubset B.$

Which was to be Demonstr.

Fourthly and lastly, that the summ of B and $\frac{1}{2}F - \frac{1}{2}A$ is greater than $\frac{1}{2}F + \frac{1}{2}A$; *viz.* that $B \sqsubset A$, the following Demonstration, formed by a repetition of the steps of the foregoing Resolution in a retrograde order, will make manifest.

31. . . . *Req. demonstr.* . . . . $B \sqsubset A.$

Demonstration.

32. Because by *Constr.* in 28°, . . . } $\sqrt{:\square A + \square C:} \ . \ M \ :: \ M \ . \ A.$

That is, in 18°, (the last step of the Resolution,) . . . . } $\sqrt{:aa + cc:} \ . \ m \ :: \ m \ . \ a.$

33. And by *Constr.* in 27°, ... } $M \ . \ B \ :: \ D \ . \ M.$

34. Therefore from 32° and 33°, by exchanging the mean Proportionals, (according to *Defin.* 8. *Chap.* 3.) this Analogy ariseth, *viz.* . . . . . . } $\sqrt{:\square A + \square C:} \ . \ B \ :: \ D \ . \ A.$

That is, in 16°, ... } $\sqrt{:aa + cc:} \ . \ b \ :: \ d \ . \ a.$

35. But the Squares of proportional lines are also Proportionals, therefore from 34°, (*per prop.* 22. *Elem.* 6.) . . } $\square A + \square C \ . \ \square B \ :: \ \square D \ . \ \square A.$

That is, in 15°, ... } $aa + cc \ . \ bb \ :: \ dd \ . \ aa.$

36. And because by *Constr.* in 25°, . . } $4\square R - 2\square B = \square C.$

37. Also by *Constr.* in 26°, ... } $4\square R - \square B - 4\square P = \square D.$

38. Therefore from 35°, 36° and 37°, by exchanging equal quantities, this Analogy ariseth, *viz.*

$$\square A + 4\square R - 2\square B \ . \ \square B \ :: \ 4\square R - \square B - 4\square P \ . \ \square A.$$

That is, in 12°,

$$aa + 4rr - 2bb \ . \ bb \ :: \ 4rr - bb - 4pp \ . \ aa.$$

39. There-

39. Therefore from 38°, by Composition of Reason,

$$\square A + 4\square R - \square B \;.\; \square B \;::\; \square A + 4\square R - \square B - 4\square P \;.\; \square A.$$

That is, in 9°,

$$aa + 4rr - bb \;.\; bb \;::\; aa + 4rr - bb - 4pp \;.\; aa.$$

40. But the first Proportional in 39° is evidently greater than the third, therefore (*per prop.* 14. *Elem.* 5.) the second shall be greater than the fourth, *viz.*

$$\square B \sqsubset \square A; \text{ therefore } B \sqsubset A.$$

Which was to be Demonstr.

41. Now because it hath been demonstrated that $F \sqsubset A$, and consequently $\frac{1}{2}F - \frac{1}{2}A$ is equal to a real right line, and that the summ of every two of these three right lines, to wit, B, $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$ is longer that the third, 'tis possible to make a Triangle of them (*per prop.* 22. *Elem.* 1.) Suppose then it be done, and that the Triangle so made is A B C, having its Base A B equal to the right line B, (the Base prescribed in the Problem;) also $AC = \frac{1}{2}F + \frac{1}{2}A$, and $BC = \frac{1}{2}F - \frac{1}{2}A$; I say the Triangle A B C will satisfie *Case* 2. *Probl.* 3. before propounded. But to render the Demonstration thereof the more easie, I shall premise a few things in seven steps next following.

42. First, if the quantities given in the Problem be exprest by numbers, the kind of the Triangle sought in *Case* 2. shall be known: For the Base was first given, and by what hath been said in 19°, the leggs are given also; therefore by the Corollary in 45° of *Probl.* 10. *Chap.* 7. it may be discovered whether the Triangle sought be obtuse-angled, acute-angled, or right-angled at the Base. Supposing then it be found that the Perpendicular falls upon the Base A B within the Triangle; from the Center C, at the distance of C B, $(= \frac{1}{2}F - \frac{1}{2}A,)$ the lesser legg of the Triangle A B C made as before, describe the Circle C B G D cutting C A in G; then produce A C to the Circumference in D, and draw the Semidiameter C E; and from the Center C let fall $CF \perp EB$, therefore $FE = FB$, (*per prop.* 3. *Elem* 3.) Then,

43. Because (*per defin.* 15. *Elem.* 1.) . . > $CD = CB = CG.$
44. Therefore by adding A C to each part, > $AD = AC + CB.$
45. By *Constr.* in 41°; . . . . . . > $AC = \frac{1}{2}F + \frac{1}{2}A.$
46. Also by *Constr.* in 41°, ... > $CB = \frac{1}{2}F - \frac{1}{2}A.$
47. Therefore the summ of the Equations in 45° and 46° gives . . . . } $AD = F = AC + CB.$
48. And the Equation in 46° being subtracted from that in 45°, leaves . . . . . } $AG = A = AC - CB.$

Now I shall shew that the Triangle A B C, formed as before is exprest in 41°, will satisfie *Case* 2. *Probl.* 3. First then, by *Constr.* in 41° the Base A B is equal to the prescribed Base B. Secondly, that the Rectangle made of the leggs A C and B C is equal to the given Rectangle, that is, the Square of the right line R, I shall here-under demonstrate.

49. . . . *Req. demonstr.* . . . . $\square AC, BC = \square R.$

Demonstration.

50. By *Constr.* in 29°, ... > $\square F = 4\square R + \square A.$
51. And because from 47°, ... > $\square AD = \square F.$
52. And from 48°, ... > $\square AG = \square A.$
53. Therefore from 50°, 51° and 52°, by exchanging equal quantities, ... } $\square AD = 4\square R + \square AG.$
54. And from 53°, by subtracting $\square AG$ from each part, ... } $\square AD - \square AG = 4\square R.$
55. It hath been shewn in 47° and 48°, that A D is the summ, and A G the difference of A C and B C, therefore (*per Theor.* 7. *Chap.* 4.) . . . . } $\square AD = 4\square AC, BC + \square AG.$
56. And from 55°, by subtracting $\square AG$ from each part, ... } $\square AD - \square AG = 4\square AC, BC.$

57. There-

57. Therefore from 54° and 56°, (*per Axiom.* 1. *Chap.* 2.) . . . . $\} \; 4\square AC, BC = 4\square R.$
58. Therefore from 57°, (*per Ax.* 21. *Chap.* 2.) . . $\rangle \; \square AC, BC = \square R.$
Which was to be Demonftr.

Thirdly and laftly, that the Perpendicular CF is equal to the given Perpendicular P, the following Demonftration, form'd out of the fteps of the preceding Refolution by a repetition of its fteps in a backward order, will make manifeft.

59. . . . *Req. demonftr.* . . . . $CF = P.$

Demonftration.

It hath been proved in 39°, that

60. . $\square A + 4\square R - \square B \;.\; \square B :: \square A + 4\square R - \square B - 4\square P \;.\; \square A.$
That is, in 9°,
$aa + 4rr - bb \;.\; bb :: aa + 4rr - bb - 4pp \;.\; aa.$
61. And becaufe from 41° and 48°, . . . $\square AB = \square B$; and $\square AG = \square A.$
62. Therefore from 60° and 61°, by exchanging equal quantities,
$\square AG + 4\square R - \square AB \;.\; \square AB :: \square AG + 4\square R - \square AB - 4\square P \;.\; \square AG.$
63. Therefore from 62°, by alternate Reafon,
$\square AG + 4\square R - \square AB \;.\; \square AG + 4\square R - \square AB - 4\square P :: \square AB \;.\; \square AG.$
That is, in 8°,
$aa + 4rr - bb \;.\; aa + 4rr - bb - 4pp :: bb \;.\; aa.$
64. It hath been proved in 53°, that . . . . . . $\square AD = \square AG + 4\square R.$
65. Therefore from 63° and 64°, by exchanging equal quantities,
$\square AD - \square AB \;.\; \square AD - \square AB - 4\square P :: \square AB \;.\; \square AG.$
66. Therefore from 65°, by Converfion of Reafon,
$\square AD - \square AB \;.\; 4\square P :: \square AB \;.\; \square AB - \square AG.$
67. And from 66°, by Altern and Inverfe Reafon,
$\square AB \;.\; \square AB - \square AG :: \square AD - \square AB \;.\; 4\square P.$
68. But by *Theor.* 4. in 68° of *Probl.* 8.
$\square AB \;.\; \square AB - \square AG :: \square AD - \square AB \;.\; 4\square CF.$
69. Therefore from 67° and 68°, (*per prop.* 11. *Elem.* 5.)
$\square AD - \square AB \;.\; 4\square CF :: \square AD - \square AB \;.\; 4\square P.$
70. Therefore from 69°, (*per prop.* 14. *Elem.* 5.) . . . . $\rangle \; 4\square CF = 4\square P.$
71. Therefore from 70°, (*per Ax.* 21. *Chap.* 2.) . . . . $\rangle \; \square CF = \square P.$
72. But the fides of equal Squares are alfo equal, therefore from 71°, $\rangle$. $CF = P.$
Which was to be Demonftr.
73. An Arithmetical Canon to find out the difference of the legs of the Triangle fought in *Cafe* 2. may be deduced from the Analogy in 12°, for by comparing the Rectangle of the extremes to the Rectangle of the means of that Analogy, this arifeth, *viz.*
$$aaaa + 4rraa - 2bbaa = 4rrbb - bbbb - 4ppbb.$$
Which Equation being refolved according to the Canon in 77° of *Probl.* 15. *Chap.* 5. will give this following

CANON.

74. . . . . $\sqrt{(2)} : \sqrt{4rrrr - 4bbpp} + bb - 2rr : = a.$

An Example in Numbers, to illuftrate the foregoing Refolution of Cafe 2. Probl. 3.

Suppof.

75. $b = 14$ the Bafe of a Triangle is given.
76. $p = 12$ the Perpendicular is given.
77. $rr = 195$ the Rectangle of the leggs is given.
78. $rr \sqsubset \frac{1}{4}bb + pp$, $\}$ (*per Suppof.* in *Cafe* 2.)
79. $rr \sqsubset \frac{1}{2}bb$,

Req. to find the Triangle.

Solution Arithmetical.

80. $2 =$ the difference of the leggs, is found out of 74°, 75°, 76° and 77°.
81. $28 =$ the fumm of the leggs, is found out of 7°, 77° and 80°.
82. 15 and $13 =$ the leggs are found out of 80° and 81°, by *Theor.* 9. *Chap.* 4.

The

The Proof.

83. $15 \times 13 = 195$ the given Rectangle; and if

84. $\left\{\begin{array}{l} 14 = \text{the Base} \\ \left.\begin{array}{l}13\\15\end{array}\right\} = \text{the leggs} \end{array}\right\}$ of a Triangle, then

85. $12 =$ the Perpendicular will be found out, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the same with the given Perpendicular in 76°.

The Resolution of CASE 3. Probl. 3.

Suppos.

1. $b =$ the Base of a Triangle is given.
2. $p =$ the Perpendicular is given.
3. $r =$ the side of a Square equal to the Rectangle of the leggs is given.
4. $rr \sqsubset \frac{1}{4}bb + pp$, $\}$ (*per Suppos.* in *Case* 3.)
5. $rr \sqsupset \frac{1}{2}bb$, $\}$

Req. to find the Triangle.

The leggs of the Triangle sought in this *Case* 3. are unequal, for the reason before given in 6° of the Resolution of *Case* 2. therefore,

6. For the difference of the leggs put . . . . $\}$ a.
7. Therefore from 3° and 6°, the Square of the summ of the leggs (*per Ther.* 7. *Chap.* 4.) shall be . . . . $\}$ $aa + 4rr$.
8. And from 1°, 2°, 4°, 6° and 7°, (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.) this Analogy ariseth, *viz.* $aa + 4rr - bb \,.\, aa + 4rr - bb - 4pp :: bb \,.\, aa$.
9. Therefore alternately,
$$aa + 4rr - bb \,.\, bb :: aa + 4rr - bb - 4pp \,.\, aa.$$
10. From 4° 'tis evident that . . . . $\}$ $4rr - bb - 4pp \sqsubset 0$.
11. Whence it follows, by adding aa to each part, . . $\}$ $aa + 4rr - bb - 4pp \sqsubset aa$.
12. Therefore from the Analogy in 9°, by Division of Reason, (which is evidently possible from the Equation in 11°,) these are Proportionals, *viz.*
$$aa + 4rr - 2bb \,.\, bb :: 4rr - bb - 4pp \,.\, aa.$$
13. It is evident by the first Term of the last Analogy, that $\}$ $aa + 4rr \sqsubset 2bb$.
14. And 'tis easie to inferr from the *Suppos.* in 5°, that . . $\}$ $2bb \sqsubset 4rr$.
15. Therefore from 13°, by subtracting $4rr$ from each part, $\}$ $aa \sqsubset 2bb - 4rr$.
16. It is evident, that if $2bb - 4rr$ be subtracted from aa, the remainder $aa + 4rr - 2bb$ is the same with the first Term of the Analogy in 12°; therefore if cc be put equal to $2bb - 4rr$, and $aa - cc$ be taken instead of the said first Term $aa + 4rr - 2bb$ that Analogy will be converted into this, *viz.*
$$aa - cc \,.\, bb :: 4rr - bb - 4pp \,.\, aa.$$
17. And by putting $dd = 4rr - bb - 4pp$, the last preceding Analogy will be converted into this that follows, *viz.* $aa - cc \,.\, bb :: dd \,.\, aa$.
18. But the sides of proportional Squares are Proportionals also, (*per prop.* 22. *Elem.* 6.) therefore,
$$\sqrt{:\overline{aa - cc}:} \,.\, b :: d \,.\, a.$$
19. Between b and d find a mean Proportional, which may be called m; therefore,
$$b \,.\, m :: m \,.\, d.$$
20. Therefore from 18° and 19°, by exchanging the mean Proportionals, according to *Defin.* 8. *Chap.* 3. concerning Inordinate Proportion, this Analogy ariseth, *viz.*
$$\sqrt{:\overline{aa - cc}:} \,.\, m :: m \,.\, a.$$
21. Which three continual Proportionals last exprest being well examined, it will appear that the greater extreme a may be esteem'd the Hypothenusal of a right-angled Triangle whose Base is c, and the Perpendicular is $\sqrt{:\overline{aa - cc}:}$ Now in that right-angled Triangle the Base c is given, as also m, a mean Proportional between the Hypothenusal a and the Perpendicular $\sqrt{:\overline{aa - cc}:}$ therefore the Hypothenusal, which is equal to the difference of the leggs of the Triangle sought in *Case* 3. shall be given also, (*per Probl.* 15. *Chap.* 5.) Then the Rectangle and difference of the leggs being given severally, the summ of the leggs shall be given also, (by *Probl.* 1. *Chap.* 9.) And lastly, the summ and difference of the leggs being given, the leggs shall be given severally, (by *Theor.* 9. *Chap.* 4.)

 This

This third Case needs not any other Determination than what is implied in the Suppositions; and the Composition may be formed (by the help of *Probl.* 15. *Chap.* 5.) in like manner as before in *Case* 2.

22. But for the Learner's fuller satisfaction, an Arithmetical Canon to find out the value of a, to wit, the difference of the leggs of the Triangle sought in *Case* 3. is deducible from the premisses: For the Rectangle of the extremes of the Analogy in 12° being compared to the Rectangle of the means, and it being observed (as is easie to inferr from the *Suppos.* in 5°,) that $2bb \sqsubset 4rr$, this following Biquadratick Equation ariseth, *viz.*

$$aaaa - \overline{2bb - 4rr} \times aa = 4rrbb - bbbb - 4ppbb.$$

Which Equation, if it be resolved according to the Canon in 58° of *Probl.* 15. *Chap.* 5. gives this

CANON.

23. . . . $a = \sqrt{:\, bb - 2rr + \sqrt{4rrrr - 4ppbb}\,:} =$ the difference of the leggs.

An Example in Numbers, to illustrate the precedent Resolution *of* Case 3. Probl. 3.

Suppos.

24. $b =$ 10185 the Base of a Triangle is given.
25. $p =$ 4752 the Perpendicular is given.
26. $rr =$ 50307696 the Rectangle of the leggs is given.
27. $rr \sqsubset \frac{1}{4}bb + pp$, } agreeable to the *Suppos.* in *Case* 3.
28. $rr \sqsupset \frac{1}{2}bb$, }

Req. to find the Triangle.

Solution Arithmetical.

29. 5529 = the difference of the leggs is found out of 23°, 24°, 25° and 26°.
30. 15225 = the summ of the leggs is found out of 7°, 26° and 29°.
31. $\left\{\begin{matrix} 10377 \\ 4848 \end{matrix}\right\}$ = the leggs are found out of 29° and 30°, (by *Theor.* 9. *Chap.* 4.)

The Proof.

32. $10377 \times 4848 = 50307696$ the given Rectangle: And, if
33. $\left\{\begin{matrix} 10185 = \text{the Base} \\ \left.\begin{matrix} 10377 \\ 4848 \end{matrix}\right\} = \text{the leggs} \end{matrix}\right\}$ of a Triangle; then, by those three sides,
34. 4752 = the Perpendicular will be found out, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the same with the Perpendicular given in 25°.

The Resolution of CASE 4. Probl. 3.

Suppos.

1. $b =$ the Base of a Triangle is given.
2. $p =$ the Perpendicular is given.
3. $r =$ the side of a given Square equal to the Rectangle of the leggs.
4. $rr \sqsubset \frac{1}{4}bb + pp$, } (*per Suppos.* in *Case* 4.)
5. $rr = \frac{1}{2}bb$, }

Req. to find the Triangle.

6. For the difference of the leggs sought put . . . . . . $> a$.
7. Then proceed as before in the Resolution of *Case* 3. from the 6th step to the 12th, and let the Analogy in that 12th step be here repeated, *viz.*

$$aa + 4rr - 2bb \;.\; bb \;::\; 4rr - bb - 4pp \;.\; aa.$$

8. Now because by *Suppos.* in 5°, ... $> 4rr = 2bb$.
7. And consequently from 8°, by subtracting bb from each part, $> 4rr - bb = bb$.
10. Therefore 'tis evident from 8° and 9°, that by exchanging equal quantities the Proportionals in 7° are reducible to these, *viz.*

$$aa \;.\; bb \;::\; bb - 4pp \;.\; aa.$$

11. But the sides of proportional Squares are also Proportionals, therefore from 10°,

$$a \;.\; b \;::\; \sqrt{:\, bb - 4pp\,:} \;.\; a.$$

12. There-

12. Therefore from 11°, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation ariseth, *viz.*

$$aa = b\sqrt{:bb - 4pp:}$$

13. Therefore from 12°, by extracting the square Root out of each part, the difference of the leggs sought is made known, *viz.*

$$a = \sqrt{:b \times \sqrt{bb - 4pp}:}$$

14. And because by *Theor.* 7. *Chap.* 5. the Square of the summ of any two right lines (or numbers) is equal to four times the Rectangle together with the Square of their difference; therefore from 5°, 6°, 12° and 13°, the summ of the leggs sought is also made known, *viz.*

$$\sqrt{:aa + 4rr:} = \sqrt{:b\sqrt{bb - 4pp} + 4rr:}$$

The two last preceding Equations give this following

CANON.

15. . . . . $\begin{cases} \sqrt{:b\sqrt{bb - 4pp}:} = \text{the difference of the leggs.} \\ \sqrt{:b\sqrt{bb - 4pp} + 4rr:} = \text{the summ of the leggs.} \end{cases}$

Then the summ and difference of the leggs being given, the leggs shall be given severally by *Theor.* 9. *Chap.* 4.

By which Canon and Resolution aforegoing, the Geometrical Effection and Demonstration of *Case* 4. is very easie to be made, and therefore I shall leave the same to the Learner's practice.

An Example in Numbers, to illustrate the foregoing Resolution *of* Case 4. Probl. 3.

Suppos.

16. $b = 5$ the Base of a Triangle is given.
17. $p = 2$ the Perpendicular is given.
18. $rr = 12\tfrac{1}{2}$ the Rectangle of the leggs is given.
19. $rr \sqsubset \tfrac{1}{4}bb + pp,$ }
20. $rr = \tfrac{1}{2}bb,$ } agreeable to the *Suppos.* in *Case* 4.

Req. to find the Triangle.

Solution Arithmetical.

21. $\sqrt{15}$ = the difference of the leggs is found out of 15°, 16° and 17°.
22. $\sqrt{65}$ = the summ of the leggs is found out of 15°, 16°, 17° and 18°.
23. $\left\{\begin{matrix} \sqrt{\tfrac{65}{4}} + \sqrt{\tfrac{15}{4}} \\ \sqrt{\tfrac{65}{4}} - \sqrt{\tfrac{15}{4}} \end{matrix}\right\}$ = the leggs, found out of 21° and 22°, (*per Theor.* 9. *Chap.* 4.)

The Proof.

24. . . . . $\sqrt{\tfrac{15}{4}} + \sqrt{\tfrac{15}{4}}$ into $\sqrt{\tfrac{65}{4}} - \sqrt{\tfrac{15}{4}} = 12\tfrac{1}{2}$ the given Rectangle.

25. And, if $\begin{cases} 5 = \text{the Base of a Triangle.} \\ \left.\begin{matrix} \sqrt{\tfrac{65}{4}} + \sqrt{\tfrac{15}{4}} \\ \sqrt{\tfrac{65}{4}} - \sqrt{\tfrac{15}{4}} \end{matrix}\right\} \text{the Leggs,} \end{cases}$

26. Then 2 = the Perpendicular will be found out, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the same with the given Perpendicular in 17°.

The Resolution of CASE 5. Probl. 3.

Suppos.

1. b = the Base of a Triangle is given.
2. p = the Perpendicular is given.
3. r = the side of a Square equal to the Rectangle of the leggs is given.
4. $rr \sqsupset \tfrac{1}{4}bb + pp$, agreeable to the preceding *Suppos.* in *Case* 5.

Req. to find the Triangle.

5. For

5. For the difference of the leggs ſought put . . . a.

6. Then from 3° and 5°, (*per Theor* 7. *Chap.* 4.) the Square of the ſumm of the leggs is . . . . $aa + 4rr$.

7. And from 1°, 2°, 5° and 6°, (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.) this Analogy ariſeth, *viz.*

$$aa + 4rr - bb \ . \ aa + 4rr - bb - 4pp \ :: \ bb \ . \ aa.$$

8. Therefore alternly,

$$aa + 4rr - bb \ . \ bb \ :: \ aa + 4rr - bb - 4pp \ . \ aa.$$

9. Therefore inverſly,

$$aa \ . \ aa + 4rr - bb - 4pp \ :: \ bb \ . \ aa + 4rr - bb.$$

10. By quadrupling all in 4°, and ſubtracting $\frac{1}{4}bb + pp$ from each part, it will be evident that . . . . $4rr - bb - 4pp \sqsupset 0$.

11. And by adding aa to each part in 10°, ... $aa + 4rr - bb - 4pp \sqsupset aa$.

12. Therefore from 9°, by Converſion of Reaſon, (which the 11th ſtep ſhews is poſſible,) theſe are Proportionals, *viz.*

$$aa \ . \ bb + 4pp - 4rr \ :: \ bb \ . \ 2bb - 4rr - aa.$$

13. It is evident from the *Suppoſ.* in 4°, that . . . $bb + 4pp \sqsubset 4rr$.

14. Therefore for the exceſs whereby $bb + 4pp$ exceeds $4rr$ we may put dd, whence, ... $dd = bb + 4pp - 4rr$.

15. By viewing the 11th ſtep it will appear that the third Proportional in 8° is leſs than the fourth, therefore the firſt is leſs than the ſecond, *viz.* . . . . $aa + 4rr - bb \sqsupset bb$.

16. Whence, by adding bb to each part, ... $aa + 4rr \sqsupset 2bb$.

17. And from 16°, by comparing the latter part to the firſt, 'tis eaſie to inferr that . . . . $2bb \sqsubset 4rr$.

18. Therefore for the exceſs whereby $2bb$ exceeds $4rr$, we may put cc, whence, ... $cc = 2bb - 4rr$.

19. Then from the Analogy in 12°, by exchanging equal quantities according to the Poſitions in 14° and 18°, this Analogy ariſeth, *viz.* . . . . . . $aa \ . \ dd \ :: \ bb \ . \ cc - aa$.

20. And becauſe (*per prop.* 22. *Elem.* 6.) the ſides of proportional Squares are alſo Proportionals, therefore from 19°, ... $a \ . \ d \ :: \ b \ . \ \sqrt{:cc - aa:}$

21. Between b and d find a mean Proportional, which may be called m, therefore . . . . $b \ . \ m \ :: \ m \ . \ d$.

22. Therefore from 20° and 21°, by exchanging the mean Proportionals according to *Defin.* 8. *Chap.* 3. this Analogy ariſeth, *viz.* . . . . $a \ . \ m \ :: \ m \ . \ \sqrt{:cc - aa:}$

23. Which laſt Analogy doth evidently conſiſt of three continual Proportionals, whereof the extremes a and $\sqrt{:cc - aa:}$ may be eſteem'd the Baſe and Perpendicular of a right-angled Triangle whoſe Hypothenuſal c is given, as alſo m a mean Proportional between the Baſe and Perpendicular; therefore (*per Probl.* 16. *Chap.* 5.) the Baſe and Perpendicular ſhall be given ſeverally, either of which may be taken for the difference of the leggs of a Triangle to ſatisfie the Problem propounded in *Caſe* 5. For here, two different Triangles may be always found out that ſhall have theſe three things common, to wit, the Baſe, the Perpendicular, and Rectangle of the leggs, except when it happens that $\frac{bp}{r} = r$, for then the two values of a will be equal to one another, in which Caſe there can only one Triangle be found out to agree with the preceding *Suppoſ.* in 1°, 2°, 3° and 4°.

But that there may be a poſſibility of finding out a Triangle to ſolve *Probl.* 3. in *Caſe* 5. the right line repreſented by $\frac{bp}{r}$ muſt not be longer than r: Now to make it evident that this Determination is neceſſary in *Caſe* 5,

24. . . . *Req. demonſtr.* . . . . $\frac{bp}{r}$ not $\sqsubset r$.

Demonſtr.

Demonstration.

25. First, if c be given for the Hypothenusal, and m for a mean Proportional between the Base and Perpendicular of a right-angled Triangle, (as before is supposed in 23°,) then to find out that Triangle by *Probl.* 16. *Chap.* 5. this Determination is necessary, *viz.* . . . . . $\frac{mm}{c}$ not $\sqsubset \frac{1}{2}c$.

26. Therefore from 25°, by multiplying each part into c, ... mm not $\sqsubset \frac{1}{2}cc$.

27. But from 21°, (*per prop.* 17. *Elem.* 6.) $bd = mm$.

28. Therefore from 26° and 27°, (*per Ax.* 4. *Chap.* 2.) . . . . bd not $\sqsubset \frac{1}{2}cc$.

29. And because (as is evident in 14°,) . . $\sqrt{:bb + 4pp - 4rr:} = d$.

30. Therefore from 28° and 29°, by exchanging equal Factors, ... $b\sqrt{:bb + 4pp - 4rr:}$ not $\sqsubset \frac{1}{2}cc$.

31. And because from 18°, ... $bb - 2rr = \frac{1}{2}cc$.

32. Therefore from 30° and 31°, (*per Ax.* 3. *Chap.* 2.) . . . . $b\sqrt{:bb+4pp-4rr:}$ not $\sqsubset bb - 2rr$.

33. And from 32°, (by dividing each part by b,) $\sqrt{:bb+4pp-4rr:}$ not $\sqsubset b - \frac{2rr}{b}$.

34. And from 33°, by squaring each part,

$$bb + 4pp - 4rr \text{ not } \sqsubset bb - 4rr + \frac{4rrrr}{bb}.$$

35. And from 34°, by adding $4rr$ unto, and substracting bb from each part, ... $4pp$ not $\sqsubset \frac{4rrrr}{bb}$.

36. And from 35°, by extracting the square Root out of each part, ... $2p$ not $\sqsubset \frac{2rr}{b}$.

37. And from 36°, by multiplying each part by b, $2bp$ not $\sqsubset 2rr$.

38. And by halving each part in 37°, . . . bp not $\sqsubset rr$.

39. Therefore from 38°, by dividing each part by r, ... $\frac{bp}{r}$ not $\sqsubset r$. Which was to be Dem.

40. Supposing then that $\frac{bp}{r}$ is not greater than r, the Composition of *Case* 5. *Probl.* 3. may be easily formed out of the last preceding Resolution, by the help of *Probl.* 16. *Chap.* 5. in like manner as before in *Case* 2. and an Arithmetical Canon to find the difference of the leggs of the Triangle sought in the said 5th Case may be deduced from the Analogy in the foregoing 12th step: For the Rectangle of the extremes of that Analogy being compared to the Rectangle of the means, this following Biquadratick Equation ariseth, *viz.*

$$\overline{2bb - 4rr} \times aa, - aaaa = bbbb + 4ppbb - 4rrbb.$$

Which Equation being resolved according to the Canon in 55° of *Probl.* 16. *Chap.* 5. gives this

CANON.

41. $a = \sqrt{:bb-2rr+\sqrt{4rrrr-4bbpp}:}$ Also, $a = \sqrt{:bb-2rr-\sqrt{4rrrr-4bbpp}:}$

42. Now either of those Roots or values of a may be taken for the difference of the leggs of a Triangle to solve *Case* 5. *Probl.* 3. before propounded; and if it happens that $\frac{bp}{r} \sqsupset r$, (and consequently, $bp \sqsupset rr$,) then those values of a are unequal, and there may be two different Triangles found out to satisfie the said *Case* 5. But when $\frac{bp}{r} = r$, and consequently, $bp = rr$,) then the said values of a are equal to one another, each being equal to $\sqrt{:bb - 2rr:}$ which shall be the difference of the leggs of a Triangle having a right angle opposite to the Base, (as appears by Canon 1. in 21° of *Probl.* 14. *Chap.* 8.) Which Triangle, when $\frac{bp}{r} = r$, is the only Triangle that can be found to solve *Case* 5. *Probl.* 3.

Examples

Examples in Numbers, to illustrate the preceding Resolution of Case 5. Probl. 3.

Examp. 1. *Where two Triangles are found out to solve* Case 5.

Suppos.

43. $b = 51$ the Base of a Triangle is given.
44. $p = 12$ the Perpendicular is given.
45. $rr = 740$ the Rectangle of the leggs is given.
46. $rr \sqsupset \frac{1}{4}bb + pp$, agreeable to the *Suppos.* in *Case* 5.

Req. to find the Triangle.

Solution Arithmetical.

47. $17 =$ the difference of the leggs is found out of 43°, 44° and 45°, by the lesser Root in 41°.
48. $57 =$ the summ of the leggs is found out of 6°, 45° and 47°.
49. 37 and $20 =$ the leggs are found out of 47° and 48°, (*per Theor.* 9. *Chap.* 4.)

The Proof.

50. $37 \times 20 = 740$ the given Rectangle.
51. And if $\left\{ \begin{array}{l} 51 = \text{the Base} \\ \left.\begin{array}{l} 37 \\ 20 \end{array}\right\} = \text{the leggs} \end{array} \right\}$ of a Triangle; then,
52. $12 =$ the Perpendicular will be found out of 51°, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the same with the given Perpendicular in 44°.

Again, from the same things given as before in 43°, 44°, 45°, another Triangle may be found out by the greater Root in 41°, to solve *Case* 5. *Probl.* 3. the sides of which latter Triangle are here-under exprest, *viz.*

53. $\sqrt{1953} =$ the difference of the leggs.
54. $\sqrt{4913} =$ the summ of the leggs.
55. $\sqrt{\frac{4913}{4}} + \sqrt{\frac{1953}{4}}$ } the leggs.
56. $\sqrt{\frac{4913}{4}} - \sqrt{\frac{1953}{4}}$ }

The Proof is easie to be made, in like manner as in *Example* 1.

Examp. 2. *Where only one Triangle can be found out to solve* Case 5. Probl. 3.

57. $b = 169$ the Base of a Triangle is given.
58. $p = 60$ the Perpendicular is given.
59. $rr = 10140$ the Rectangle of the leggs is given.
60. $rr \sqsupset \frac{1}{4}bb + pp$, agreeable to the *Suppos.* in *Case* 5.

Req. to find the Triangle.

Solution Arithmetical.

61. $91 =$ the difference of the leggs is found out of 57°, 58° and 59°, by either of the Roots in 41°.
62. $221 =$ the summ of the leggs is found out of 6°, 59° and 61.
63. 156 and $65 =$ the leggs are found out of 61° and 62°, (*per Theor.* 9. *Chap.* 4.)

The Proof.

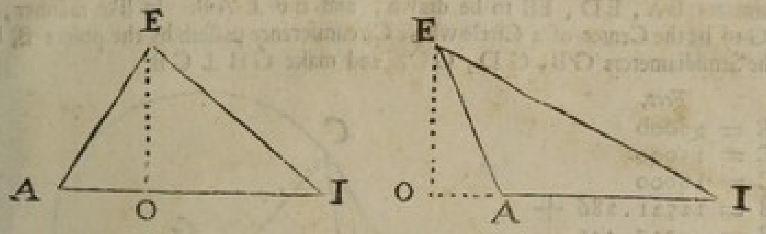
64. $156 \times 65 = 10140$ the Rectangle given in 59°.
65. And if $\left\{ \begin{array}{l} 169 = \text{the Base} \\ \left.\begin{array}{l} 156 \\ 65 \end{array}\right\} = \text{the leggs} \end{array} \right\}$ of a Triangle, then
66. $60 =$ the Perpendicular will be found out of 65°, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is equal to the given Perpendicular in 58°.

Note. This last Triangle hath a right angle opposite to the Base, agreeable to what was before hinted in 42°.

A LEMMA, *leading to the following* Probl. 4.

In any plain Triangle, As the Sine of an angle is to the Radius, (or total Sine;) So is the double Area of the Triangle, (that is, the Rectangle made of the Perpendicular and Base,) to the Rectangle of the leggs containing the said angle.

Suppos.



Suppos.

1. E I the Base ⎫
2. $\left\{\begin{matrix}AE\\AI\end{matrix}\right\}$ the leggs ⎭ of the oblique-angled △ A I E.
3. $\angle$ A, (that is, $\angle$ E A I) is contain'd under the leggs A E, A I.
4. E O ⊥ A I.
5. R = the Radius, (or total Sine.)
6. S $\angle$ A = the Sine of the angle A.
7. . . . *Req. demonstr.* . . . . . S$\angle$A . R :: ▭E O, AI . ▭A E, AI.

Demonstration.

8. By a vulgar *Axiom* in the Doctrine of plain Triangles this Analogy is manifest, *viz.* S$\angle$A . R :: EO . AE.
9. Therefore by drawing AI into each of the two latter Terms of that Analogy, this ariseth, *viz.* S$\angle$A . R :: ▭EO,AI . ▭AE,AI.

Which was to be Demonstr.

Probl. IV.

The Base of a plain Triangle being given, as also the Perpendicular, and angle opposite to the Base, to find the Triangle.

Construction.

Let a Circle be described by a Radius (or Semidiameter) taken at pleasure, and according to the *Note* at the beginning of *Probl.* 19. *Chap.* 8. find out a right line that shall be the Sine of an angle equal to the given angle. Then (by *Probl.* 9. *Chap.* 5.) find out a Square equal to a Rectangle made of the given Base and Perpendicular. That done, let it be made, (by *Probl.* 11. *Chap.* 5.) As the Sine, (found out as above,) to the Radius first assumed; So the said Square to another Square, which Square (or fourth Proportional) found out, shall be equal to the Rectangle of the leggs containing the given angle, (as is evident by the foregoing *Lemma*) Now there is given the Base and Perpendicular, as also a Square equal to the given Rectangle of the leggs, to find out the Triangle; and therefore if those given quantities be exprest by numbers, this *Probl.* 4. may be solved both Geometrically and Arithmetically in all Cases, by the help of the preceding *Probl.* 3.

Note. Some subtil Geometrical Problems, wherein the measures of angles as well as of right lines are given in numbers, may be solved Arithmetically by the Doctrine of Plain Triangles, without the help of *Algebra*; of such kind is the following Problem, with which I shall conclude this Treatise.

Probl. V.

The distances A B, A C, B C between three Towers A, B and C not standing in a straight line, being given severally in Feet. Also a fourth Tower being suppos'd to stand within the Triangle A B C, as at D; and the measures of the angles A D B, B D C and C D A being given severally in Degrees; to find the distance between the fourth Tower D and each of the other three, *viz.* the measures of the three right lines D B, D A and D C in Feet.

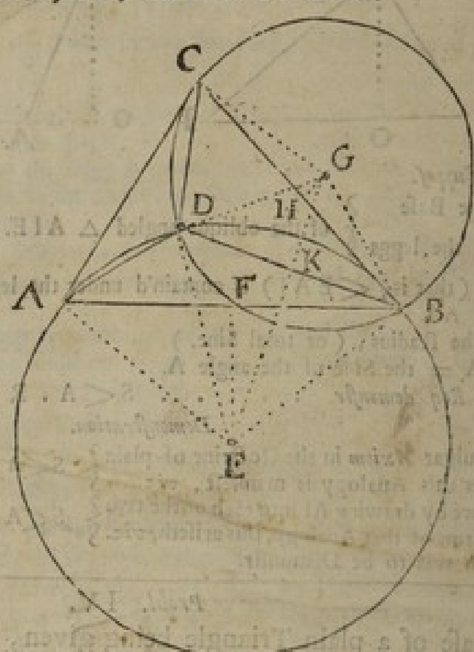
Prepar.

Forasmuch as by *Suppos.* the point D lyes within the Triangle A B C, and consequently the three points A, D, B do not lye in a straight line; let the Circumference of a Circle be supposed to pass by those points, as A D B, whose Center is E; suppose also the Semi-

Semidiameters E A, E D, EB to be drawn, and $EF \perp AB$. In like manner, supposing G to be the Center of a Circle whose Circumference passeth by the points B, D, C, draw the Semidiameters G B, G D, G C, and make $GH \perp CB$.

	Feet.
AB =	20000
AC =	15000
BC =	18000
EB =	12521.286 +
GB =	9317.445 +
DB =	13888.610 +
DA =	8283.832 +
DC =	8406.944 +

	Gr. Min.
$\angle$ ADB =	127: 0
$\angle$ BDC =	105: 0
$\angle$ CDA =	128: 0
$\angle$ FEB =	53: 0
$\angle$ EBA =	37: 0
$\angle$ HGB =	75: 0
$\angle$ GBC =	15: 0
$\angle$ CBA =	46: 8
$\angle$ GBE =	98: 8
$\angle$ GEB =	33:41
$\angle$ DBA =	19:19
$\angle$ DBC =	26:49



Solution Arithmetical, by the Doctrine of plain Triangles.

First, subtract the given $\angle$ADB from two right angles, (*viz.* from 180.Degrees,) the remainder shall be the summ of the unknown angles DAB and DBA, (*per prop.* 32. *Elem.*1.)

Secondly, forasmuch as (by *prop.*20. *Elem.* 3.) $\angle DEB = 2\angle DAB$, and $\angle DEA = 2\angle DBA$; it follows that $\angle AEB = 2\angle DAB + 2\angle DBA$; therefore in $\triangle$FEB right-angled at F, the $\angle$FEB (that is, $\frac{1}{2}\angle AEB) = \angle DAB + \angle DBA$ is given; and by *Suppos.* $FB = \frac{1}{2}AB$ is given, therefore the Semidiameter $EB = ED = EA$ shall be given also.

Thirdly, by arguing as above in the first and second steps $\angle HGB = \angle HGC$ is given; also $GD = GC = GB$ the Semidiameter of the Circle G B D C is given.

Fourthly, because $EF \perp AB$ and $\angle$FEB is given as before, therefore $\angle$EBA the Complement of the $\angle$FEB to a right angle is given; likewise the $\angle$GBC the Complement of $\angle$HGB to a right angle is given; and the $\angle$CBA is given, for it may be found out by the three given sides A B, A C, B C; therefore $\angle$GBE the summ of those three angles, E B A, C B A, G B C is given.

Fifthly, in $\triangle$ G B E, the sides G B and E B, (to wit, the Semidiameters of the two Circles G B D C and E A D B,) are given severally, as also the angle G B E comprehended by those sides, therefore the angle G E B is given also.

Sixthly, because the two Triangles E G B and E G D have two sides GB, E B equal to the two sides G D, E D, *viz.* $GB = GD$, and $EB = ED$, also the Base G E common to both those Triangles; the angles contain'd under equal right lines shall be equal, *viz.* $\angle GEB = \angle GED = \frac{1}{2}\angle DEB$: But $\angle GEB$ ($= \frac{1}{2}\angle DEB$) is given in the fifth step, and (*per prop.* 20. *Elem.* 3.) $\angle$DAB is equal to $\frac{1}{2}\angle DEB$ ($= \angle GEB$,) therefore $\angle$DAB is given. Now in $\triangle$ADB there is given $\angle$DAB, as also $\angle$ADB and the side A B, therefore the sides D B and D A, (to wit, two of the Distances sought,) are given also.

Seventhly and lastly, in $\triangle$ D A C there is given D A, as also A C, and $\angle$ADC, therefore D C (the third Distance sought,) is given.

The End of the Fourth and last BOOK.

SOLI DEO GLORIA.