GRIFFIN'S SCIENTIFIC TEXT-BOOKS

BIOMATHEMATICS

BEING

THE PRINCIPLES OF MATHEMATICS FOR STUDENTS OF BIOLOGICAL SCIENCE

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BY

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INTRODUCTION

BY

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WITH MANY WORKED NUMERICAL EXAMPLES CHOSEN FROM THE DIFFERENT BRANCHES OF BIOLOGY, AND 125 DIAGRAMS.

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AFFECTIONATELY DEDICATED TO

MY DAUGHTER

PATRICIA

ON HER SECOND BIRTHDAY.

"He who knows mathematics and does not make use of his knowledge, to him applies the verse in Isaiah (v. 12), 'They regard not the work of the Lord, neither consider the operation of His hands.'"

The Talmud.

"The laws by which God has thought good to govern the Universe are surely subjects of lofty contemplation; and the study of that symbolical language by which alone these laws can be fully deciphered is well deserving of his [man's] noblest efforts."

Professor Sedgwick.

"The living and the dead, things animate and inanimate, we dwellers in this world and this world wherein we dwell, are bound alike by physical and mathematical law."

D'Arcy W. Thompson.

"The ultimate aim of embryology is the mathematical derivation of the adult from the distribution of growth in the germ."

Wilhelm His.

PREFACE

In a recent signed article reviewing two books on Biochemistry, the reviewer—one of the foremost of British physiologists—laments the fact that so much of what is evidently important is veiled in mathematical language. He says: "My own acquaintance with Mathematics is not deep enough to be able to criticise, and I fancy the majority of physiologists and students (even honours students) are much in the same predicament." The time is, therefore, ripe for a book which should explain to the biological student those portions of the so-called higher mathematics which are now being utilised in the study and investigation of biological problems. The present work is meant to fill this gap in biological literature.

The book is designed to fulfil a double object. It aims, in the first instance, at affording the reader sufficient mathematical knowledge to follow intelligently the records of the more modern researches in the various fields of biological science. In addition, it is hoped that a mastery of the book will enable the laboratory investigator to make use of the principles of Mathematics for the purpose of co-ordinating his experimental results. From the kindly Introduction by Professor Sir William M. Bayliss, it appears that I have succeeded in achieving both of these aims.

It is a great pleasure to acknowledge here my indebtedness to the following gentlemen for various kindnesses shown to me either in connection with the manuscript or during the progress of the book through the press: Professors Sir William M. Bayliss, D.Sc., LL.D., F.R.S., and H. E. Roaf, M.D., D.Sc., read the manuscript and made helpful suggestions. To Sir William Bayliss, I am particularly grateful for honouring me and my book with his learned Introduction. Sir Walter M. Fletcher, M.D., Sc.D., F.R.S., Secretary of the Medical Research Council, Sir Sydney Russell-Wells, M.D., F.R.C.P., M.P., ex-Vice-Chancellor of the

University of London, and Mr. Fred S. Spiers, O.B.E., B.Sc., etc., Secretary of the Faraday Society, have taken great trouble with the manuscript. Professor D'Arcy W. Thompson, C.B., D.Litt., F.R.S., has read the proofs of the earlier portions of the book, and Dr. Major Greenwood and his colleagues in the Statistical Department of the Ministry of Health have taken very great pains in connection with the proofs of the chapter on Biometrics. Dr. W. A. M. Smart, Demonstrator of Physiology at the London Hospital Medical College, supplied me with the proofs of a couple of formulæ; these are acknowledged in the text. Dr. A. G. M'Kendrick, F.R.S. (Edin.), Superintendent of the Research Laboratories of the Royal College of Physicians of Edinburgh, and especially Sir James Crichton Browne, M.D., D.Sc., LL.D., F.R.S., have shown me other very great kindnesses as the book was passing through the press. Lastly, I wish to pay a tribute of reverence and respect to the memory of my dear friend, Dr. J. W. Ballantyne, who, though not a mathematician, took a great interest in the book.

W. M. FELDMAN.

31, New Cavendish Street, W. 1.
May, 1923.

INTRODUCTION

BY

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AT the present day it is scarcely necessary to combat the old prejudice that the application of mathematical treatment to the biological sciences is a serious error. No doubt caution must be used in the process, as will be pointed out below. But in many aspects these sciences have now reached a stage in which the use of mathematics has become not only profitable, but indispensable. So far as I am aware, there is no book in the English language, or for that matter in any other language, which fills the place of this work by Dr. Feldman. Mellor's "Higher Mathematics," valuable as it is for the student of physics or chemistry, contains more than the biologist needs as yet, while it omits matter which is of importance for workers in the domain of the phenomena of life. Much more is this the case with the regular text-books of mathematics. The variety of knowledge that the physiologist, for example, has to call to his aid is so vast that he really cannot spare the time to master these text-books. The present work seems to me to have succeeded in giving just what is likely to be useful. It frequently happens that the biologist at the time in his career when the value of mathematical knowledge forces itself upon him finds that he has forgotten much of what he learned when a student, since he has had little or no occasion to use it in the meantime. Even if he has not forgotten his early studies, it is to be feared that but too often he would realise the uselessness of most of them. He has probably been taught too much dull and unedifying trigonometry and scarcely anything of the fascinating problems of the calculus of continuous changes. Dr. Feldman has wisely begun at the beginning. His book will be useful not only to research workers who wish to subject their experimental results to mathematical treatment, but also to those who merely require to be able to understand the expressions given in papers which they read. It is not to be expected that a book of this kind can be grasped completely by reading it through more or less rapidly. Each step should be mastered before the next is taken.

We may, I take it, accept the statement that the ultimate aim of all science is to express in a mathematical form the discoveries that have been made. In that part of biology with which I am most familiar, the experimental investigation of vital processes, the usual course may be described as follows. We first of all find that the presence of some particular phenomenon is always associated with that of some other, so that when we arrange that this latter shall be present, we know that the former will show itself also. It is often said that the latter is the " cause " of the former, although this way of putting it may be philosophically incorrect. Probably all that we are justified in saying is that the presence of the one always involves that of the other. It is in many cases of biological research impossible to proceed further than this first stage, at least up to the present time. But it must not be supposed that discoveries of this merely qualitative kind are of no importance. In most cases, however, a further step may be taken by *measuring* the magnitude of the " effect " in relation to that of the " cause," if we may be allowed to use these terms for convenience. Suppose that we determine the degree of abolition of response to a constant stimulus as this response is diminished by the action of successive known doses of a poison. We thus obtain two sets of numbers, one set being some mathematical function of the other set. The next step is to find out what function this is—in other words, to express the results in a mathematical formula. As would be expected, this is not always a simple matter. Help is often obtained by making a " graph," in which case we make use of the methods of co-ordinate geometry. Inspection of this curve may suggest various formulæ to be tested. In the case mentioned we may find that a curve belonging to the family of parabolas satisfies the data. But if we proceed to affirm that this shows that the phenomenon is one of adsorption, we begin to tread on dangerous ground and much caution is required if we are to make real progress. As Dr. Feldman points out on p. 89, " occasionally two or more different formulæ will give results each of which is in agreement with observation." He gives cases of this. Indeed, we have to resort to further experiment in order to test whether our interpretation is in accordance with other characteristics of the process assumed in our working hypothesis.

We see one of the uses of the application of mathematics to our experimental data. In the particular example given, if it had not been for the form of the curve, we might not have suspected adsorption as a controlling factor, and we are therefore led to make the experimental tests necessary to find out whether it is so or not.

Before we leave this example, we may note that the expression for the adsorption isothermal contains two arbitrary constants, which are given appropriate values in any actual case. This fact gives, as is clear, a large degree of flexibility to the equation, and necessitates special caution in drawing conclusions from the fact that it applies to any particular case. The same statement may be made as to the Barcroft-Hill formula for the dissociation of oxy-hæmoglobin. This can be interpreted on the basis of mass-action formulæ, but it has been stated that the data can also be fitted by an adsorption formula.

Mathematics is a powerful tool, and may do damage unless well under control. But under such control it will do what no other tool can do. It is in the application of the compound interest law to physiological phenomena, or rather in conclusions drawn from this, that it seems to the writer that unjustifiable statements are often made. As is shown in the text, there are a number of phenomena in which the rate of further change is proportional to the amount of change which has already taken place, or in other words to the amount of material actually in process of change at the moment. One of these cases is that of those simple chemical reactions in which the rate of change is proportional to the concentration of one of the reacting molecules. This is a direct result of the law of mass-action, and is known as the formula for a unimolecular reaction. Now, it is sometimes stated that when a process has been found to follow this law, we may conclude that it is a simple chemical reaction. But there are, in many cases, other facts that show that the process as a whole must be much more complex and involve physical as well as chemical changes. Thus, all that we are entitled to deduce from obedience to the unimolecular law is that the slowest of the series of processes, which controls the rate of the whole, *may* be a simple chemical reaction. It does not follow that this is the most significant or most controllable part of the phenomenon.

Equal caution is necessary in regard to conclusions drawn from temperature coefficients. Certain experiments on the rate of the heart-beat gave values directly proportional to the absolute temperature, just as if it were a simple physical process, like the expansion of a gas. Others gave the usual value for a chemical reaction. Now we know quite well that the contraction of the heart muscle is neither one of these nor the other alone. It seems as if under some conditions the physical factors were in control of the rate; under others, the chemical factors. In any case, we are led to further work.

It will be clear from what has been said that the expression of experimental data in a mathematical formula may have two desirable results. It may serve as a suggestion of a nature of a process, and thus lead to additional experimental tests of the validity of such a hypothesis, or it may for the time being merely serve to control the accuracy of the methods used. If the data fall into no sort of regular form, it may reasonably be suspected that the way in which they were obtained was not free from objection. This point of view is well put in the following quotation from the lectures by Arrhenius on "Immunochemistry" (p. 7). After showing that the process of immunisation can be expressed by a formula,

$$\text{viz. } \frac{1}{(\text{concentration})^{n-1}} = \text{const}_1 + \text{const}_2 \,.\, t.$$

Arrhenius proceeds: "The application of the formula of Madsen teaches us much more" (that is, than that the quantity of antitoxin in the blood decreases more rapidly in the early stages than later on). "It shows that the phenomenon is a regular one, and we are impelled to seek for a cause for the differences of the values of the constants n and const_2. For instance, the different values of const_2 for the three days in the experiments of Bomstein—are they really different, or do the observed differences depend only on experimental errors? This and other questions suggest themselves after the use of such an equation, and they lead to improvement in the experimental methods, and to very sharp and well-defined ideas of the natural phenomena themselves. With the help of formulæ, which may be empiric or rational, scientific progress will be much more rapid than without them; and as the experimental material increases, the empiric formulæ will probably be converted into rational ones, *i.e.*, we shall detect new laws of nature. It is, therefore, very much to be regretted that efforts have been made, especially recently, to reject the use of formulæ in the treatment of questions of serum-therapy. These efforts may be regarded as a last desperate struggle against the stringent conclusions that may be reached by means of the application of mathematical treatment —a struggle that cannot be greatly prolonged."

There appear to be some men of science who are quite satisfied to see their experimental results expressed in the form of a mathematical expression, even when this is merely an empirical one, and ask for nothing more. But most of us want to have some mental picture of what is actually happening, or at least to know the meaning of the factors of our equations in terms of other known phenomena. Thus, although the Barcroft-Hill hæmoglobin formula was at first only the equation to the curve of experimental results, it was not long before attempts were made to find the meaning of the constants—attempts which are still in progress. We see indeed how the formula has led to an immense number of valuable experimental results. The want of satisfaction above alluded to may be felt more especially in relation to conclusions drawn from thermodynamical considerations. While such conclusions are always valid in respect to the direction in which changes take place and the amount of such changes, they are unconcerned with the particular way in which the results are brought about. There is no doubt, on the other hand, as to the value of this mode of treatment, although the mechanism at work is left for future discovery.

Dr. Feldman's book concludes with an account of the most important statistical methods. This will be welcomed. In many domains of biological inquiry this is clearly the only way to arrive at reliable conclusions. At the same time, and without wishing to undervalue the method in its proper sphere, I feel bound to enter a word of protest against the uncritical use of correlation formulæ in experimental work where we have the conditions under control. It must be familiar to all who make such experiments that some of them are obviously of little or no value, while others are so good that their results far outweigh a large number of the bad ones. Indeed, it sometimes happens that one good experiment alone is sufficient to solve the problem. I assume, of course, that we know why this is so, and that we do not call an experiment a bad one merely because it gives results contrary to what we had expected or desired. Claude Bernard warns us " In physiology, more than anywhere else, on account of the complexity of the subjects of experiment, it is easier to make bad experiments than to be certain what are good experiments, that is to say, comparable with one another."

In what criticism I have made in the preceding remarks, it is far from my intention to detract from the value of mathematics in biology, which is shown to be sufficiently evident. To make exaggerated claims or an inappropriate use of the powerful tool at our disposal seems to me, however, to invite opposition from those who are unsympathetic, and to bring discredit on an extremely valuable help.

I would conclude by strongly recommending all those who wish to follow modern work in the biological field to make themselves familiar with the contents of Dr. Feldman's book.

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BIOMATHEMATICS.

CHAPTER I.

INTRODUCTORY.

Mathematics is the science and art of rapid and accurate computation, and the principal object of its application to any department of science is to attempt, as far as possible, to construct, out of the quantitative results obtained in the laboratory, a kind of algebraical picture—called a formula—by means of which these results can be co-ordinated and investigated.* **Biomathematics** is the science and art of rapid and accurate computation applied to the study and investigation of biological problems.

Chemistry, like every other branch of natural philosophy, did not become an exact science until it became quantitative, and until such time as the results obtained by the biologist are capable of being subjected to the same mathematical reasonings and manipulations as are those of astronomy, physics and chemistry, biological science—using the term in its widest sense—will be looked upon as a kind of Cinderella by the people who worship at the shrines of those sciences which are universally acknowledged as exact.

" But I assert," so wrote Kant in 1786, " that the claim of any particular branch of natural philosophy to be considered as a science, can be assessed only on the basis of the amount of mathematics employed in it." This is much more true to-day than it was when it was written some 140 years ago, and it may be said with some truth that the modern scientific investigator who does not also possess a knowledge of mathematics is like a strong man who is unarmed, for it is the ability to measure his results and

* This definition of Mathematics is, strictly speaking, incomplete, but will suffice for practical purposes. Mathematics is, of course, much more than a short road to rapid and accurate computation. It is the mode of reasoning the special kind of Logic by which we measure and compare all manner of magnitudes, all quantitative expressions involving number, size, position and time; and by which we learn to state these relations with the utmost economy of words, of symbols and of thought itself.

to co-ordinate those measurements in such a way as to enable him to foretell with confidence the quantitative results of any future experiments, that supplies the investigator with the necessary ammunition to fight his battles on more or less sure ground.

It is hardly possible to pick up any biological journal without coming upon some polemical article or correspondence. Such differences of opinion are hardly to be found in mathematical literature. This is due to the fact that whilst a biological statement not mathematically expressed is open to different interpretations, each of which can be correct under different experimental conditions—a mathematical formula can either be wholly true or wholly false ; there is no intermediate position.

For some years now a great tendency has manifested itself to introduce quantitative measurements into the investigation of biological problems. This is a very healthy sign, showing as it does that the time is fast approaching when the fairy slipper of mathematics will be fitted to the dainty foot of Cinderella, and Biology will be declared a princess amongst the exact sciences.

In the course of the following pages it will be my endeavour to give a concise, but, I hope, clear and adequate exposition of those mathematical principles and manipulations a knowledge of which is essential for an intelligent appreciation of the records of the more recent investigations in the various fields of biological science. I shall as far as possible illustrate each important mathematical process by means of examples taken from one or other branch of biological science. This, I hope, will help the reader to follow the steps of the various mathematical processes with greater interest than he would otherwise do. But at the same time it is necessary to point out that although I shall do my best to smooth away difficulties it will also be necessary for the reader who is totally new to the subject to set himself resolutely to the task of mastering the various manipulative steps. It would be wrong to pretend that the manner in which the matter is here treated will enable the reader to whom the subject is quite new to follow easily what I have to say ; but I do believe that with reasonable application and concentration, with pen in his hand and paper in front of him, any non-mathematically disposed reader of average education and ability will be able to overcome the difficulties which are bound to confront him every now and again.

The word **accurate** in our definition of mathematics requires a little further elaboration. Since all the quantities with which we have to deal in the various branches of biology are those which have been obtained as the result of weighing and measuring, it is clear that the accuracy of our calculations must necessarily depend upon the accuracy with which these quantities have been obtained, But no measurements can ever be absolutely exact, however skilful and painstaking may have been the observer who made them, and however fine may have been the apparatus with which the observations were made. For example, when we say that the height of an individual has been ascertained to be 5 ft. $6\frac{3}{4}$ in., or 5·5625 ft., we mean that if we place a measuring rod against the individual in question, a horizontal bar from the top of the person's head will meet the measuring rod at the division marked 5 ft. $6\frac{3}{4}$ in. But if we were to examine the line of contact of the horizontal bar with the measuring rod by means of a magnifying glass we would see that this point does not exactly coincide with the point marked 5 ft. $6\frac{3}{4}$ in. Or when we say that a cubic millimetre of blood is found to contain 5,000,000 red blood corpuscles, we do not of course mean that 5,000,000 is the exact number; what we mean is that the number is correct to within say 100,000 corpuscles above or below that figure. Indeed, no ordinary measurements, however skilful the observer who made them, are correct to more than 4 figures. It is true that with the aid of very delicate apparatus it is possible to obtain a far greater degree of accuracy than that. For example, A. V. Hill has described an apparatus which can detect a difference of temperature of one hundred millionth of a degree Centigrade ($10^{-8°}$ C.) and Tashiro has studied metabolism in nerves with an apparatus which can detect one ten millionth of a gramme of CO_2 (10^{-7} grm.). Similarly a delicate galvanometer can detect a current as minute as one billionth part (10^{-12}) of the initial current in a conductor; whilst Sir Jagadis Bose's crescograph, which he used in the study of growth of plants, is so delicate that it is claimed by its inventor to register $\frac{1}{100,000}$ in., which is the increment in length of the plant per half a second. But even such exceedingly delicate instruments as these do not give **absolutely exact** measurements. Hence it is clear that in all our calculations the highest accuracy we can and need so far attain is a result which is accurate to about 7 figures, and that for ordinary purposes an accuracy of 1 in 1000 *i.e.*, up to 4 figures is sufficient.

This, however, does not mean that the mathematical processes that we are about to describe are merely approximations. What it does mean is that our mathematical operations, although **with data which are absolutely exact, will yield results which are also absolutely exact,** need not be carried to a higher degree of accuracy than up to a certain number of significant figures, *i.e.*, no more than the number of figures to which the data have been originally ascertained.

Having grasped this essential fact the reader will understand that if we are confronted with such a mathematical operation as

2·5968 × 1·7324 and we are told that each of these numbers is correct only as far as the 4th significant figure (or the 3rd decimal place in this particular instance), it would be absurd to record the product to 8 decimal places (**as the product carried out by the ordinary arithmetical methods would yield**). Indeed the product, as we shall see, will only be correct to 3 figures. Hence one of the first things with which we shall have to deal in these pages is the method of approximation—it being understood that the approximation can be carried as near exactness as we like or are entitled to carry it.

Degree of Accuracy in Calculation.—One often speaks of calculations carried to so many decimal places. Such a statement gives no idea of the accuracy of the calculation. Thus supposing the result of a certain computation to be 0·000019. This goes to 6 decimal places, and yet an error of 1 in the last figure would mean an error of 1 part in 19 or about 5 per cent. If, on the contrary, the result be 3572·42, this only goes to 2 decimal places, yet an error of 1 in the last figure would mean an error of no more than 1 in 357242, or about 3 in a million, *i.e.*, 0·0003 per cent. To take concrete examples: When we say, for instance, that the width of a microscopic object such as of a bacillus is 0·00004 in., it is important that the last figure be accurate, since 0·00003 or 0·00005 in. is 25 per cent. below or above the real width. But when one speaks of the distance of the sun as 92,700,000 miles, an error of a couple of hundred thousand miles is of no consequence; and whilst an error of 10 million miles in the sun's distance would be very considerable, the same error would be negligible in the distance of Sirius, which is more than 500,000 times that of the sun. Hence we see that **the degree of accuracy depends not so much on the number of decimal places as the number of significant figures** to which the result is carried. The result ·000019 goes as far as 2 significant figures, whilst 3572·42 goes to 6 significant figures. And whenever we have to do any calculation it is never justifiable to state the result to more than the number of figures in the given data.

Note.—By the term "significant figures" is understood the figures which are definitely known, counting from the right, irrespective of the decimal point. Thus ·00001937 and 193·7 have each 4 significant figures. If it is known that the next figure is 0, *e.g.*, ·000019370 . . . and 193·70 . . each has 5 significant figures.

When we say that a measurement is correct to 4 significant figures, we mean that the first three are absolutely correct and the last one is nearest the truth. Thus both 32·174 and 32·166, if given to 4 significant figures would be 32·17, because 32·166 is nearer to 32·17 than to 32·16, and 32·174 is nearer to 32·17 than to 32·18.

Similarly, by common convention 32·165 would also be given as 32·17 to 4 significant figures. Hence we see that when we say that 32·17 is correct to 4 significant figures we mean, strictly speaking, that the number is less than 32·175 and not less than 32·165. In other words, the possible error in the number is not more than 0·005 "in excess or in defect," *i.e.*, not more than $\pm$ 0·005.

The *limits of error*, therefore, of 32·17 when described as known to 4 significant figures are $\pm$ 0·005. The actual error may, of course, have any value between these extremes.

CHAPTER II.

SIMPLIFIED METHODS IN ARITHMETIC.

LET us take the numbers 2·5968 and 1·7324 again. Supposing them to represent the sides of a rectangle and that each is known to be correct to 5 significant figures or 4 decimal places. What is the area of the rectangle and to how many places is the area correct ?

If we multiply the two numbers in the **ordinary** way we get the product = 4·49869632.

Now, to how many places is this product correct ?

In order to answer this question we must bear in mind what we have said in the last chapter (last paragraph). When we say that 2·5968 is correct to 4 decimal places we mean that the true measurement lies between 2·59675 and 2·59685. In other words, 2·5968 lies between

$$(2{\cdot}5968 - {\cdot}00005) \text{ and } (2{\cdot}5968 + {\cdot}00005).$$

Similarly 1·7324 lies between

$$(1{\cdot}7324 - {\cdot}00005) \text{ and } (1{\cdot}7324 + {\cdot}00005).$$

Hence the correct product cannot be less than

$$(2{\cdot}5968 - {\cdot}00005)\ (1{\cdot}7324 - {\cdot}00005)$$

nor more than

$$(2{\cdot}5968 + {\cdot}00005)\ (1{\cdot}7324 + {\cdot}00005).$$

Now we know that if a, b, and c represent **any** numbers then

$$(a - c)\ (b - c) = ab - c\ (a + b) + c^2$$

$$\text{and } (a + c)\ (b + c) = ab + c\ (a + b) + c^2$$

$$\therefore\ (2{\cdot}5968 - {\cdot}00005)\ (1{\cdot}7324 - {\cdot}00005)$$

$$= 2{\cdot}5968 \times 1{\cdot}7324 - {\cdot}00005\ (2{\cdot}5968 + 1{\cdot}7324) + {\cdot}0000000025 \quad . \quad (1)$$

Similarly $(2{\cdot}5968 + {\cdot}00005)\ (1{\cdot}7324 + {\cdot}00005)$

$$= 2{\cdot}5968 \times 1{\cdot}7324 + {\cdot}00005\ (2{\cdot}5968 + 1{\cdot}7324) + {\cdot}0000000025 \quad . \quad (2)$$

Therefore the true value of $2{\cdot}5968 \times 1{\cdot}7324$ lies between the values of (1) and (2). But these differ only in the middle term, viz., ·00005 (2·5968 + 1·7324),

i.e., $\cdot 00005 \times 4{\cdot}3292$ or ·00021646.

Hence the product of $2{\cdot}5968 \times 1{\cdot}7324$ (each of which is correct to **4** places) is correct to **3** places only, but the error in the 4th place is not greater than **2**. Hence we say that the **limit of error is** $\pm$ **·0002.**

Now, since our product can be correct only to *3* places, it is obvious that to multiply the two numbers in the ordinary way and get an answer to 8 places and then disregard the last 5 places is absolute waste of time and labour. Hence, whenever one has to do a multiplication of this kind (and the same applies to division) one adopts some contracted method which will give the answer to no more than the number of places required. The simplest, best and most general method is by means of logarithms, although the ordinary methods of **contracted multiplication and division** dealt with in elementary arithmetic books are frequently very useful.

LOGARITHMS.

The simplest and most generally useful method of abbreviating arithmetical work is the logarithmic method. In order to understand clearly the meaning and use of logarithms, it is necessary to refresh one's memory with regard to the meaning and laws of *Indices.*

Indices.—The expression a^m means the product of m factors each of which is equal to **a**. Thus $a^m = a \times a \times a \ldots$ to m factors. The letter **m** is called the **index** of the power to which **a** is raised, and **a** is called the **fundamental number** or **base**,

$$e.g., \ 10^4 = 10 \times 10 \times 10 \times 10 \ (i.e., \text{ to 4 factors}).$$

Laws of Indices.—The following laws are universally true whether **a**, **m** and **n** be positive or negative, integral or fractional.

(1) $a^m \times a^n = a^{m+n}$ ($e.g.$, $10^4 \times 10^3 = 10^7 = 10{,}000{,}000$).

(2) $\dfrac{a^m}{a^n} = a^{m-n}$ ($e.g.$, $\dfrac{10^5}{10^3} = 10^2 = 100$).

(3) $(a^m)^n = a^{mn}$ ($e.g.$ $(10^3)^4 = 10^{12} = 1000{,}000{,}000{,}000$).

(4) $(a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$ ($e.g.$ $(1000^2)^{\frac{1}{3}} = 1000^{\frac{2}{3}} = 100$.

The meaning of a fractional index, such as $\frac{1}{n}$ in $a^{\frac{1}{n}}$.

Since by the 1st law of indices $a^m \times a^n = a^{m+n}$

$$\therefore \quad a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

But $\quad \sqrt{a} \times \sqrt{a} = a$

$$\therefore \quad a^{\frac{1}{2}} = \sqrt{a}$$

Similarly $\quad a^{\frac{1}{3}} = \sqrt[3]{a}$

and generally $\quad a^{\frac{1}{n}} = \sqrt[n]{a}$

i.e., **a fractional index of the power of any number means some root of that number and $a^{\frac{1}{n}}$ means the nth root of a.**

$$E.g., \quad 10^{\frac{1}{2}} \text{ or } 10^{0\cdot5} = \sqrt{10} = 3\cdot1623$$
$$10^{\frac{1}{3}} \text{ or } 10^{0\cdot333} = \sqrt{10} = 2\cdot1547$$
$$10^{\frac{1}{4}} \text{ or } 10^{0\cdot25} = \sqrt{10} = \sqrt{3\cdot1623} = 1\cdot7783,$$

and so on.

Corollary.—From this it follows that $a^{\frac{m}{n}}$ means that **a** is to be raised to the mth power and is then to have the nth root of the result extracted.

$$E.g., \quad 10^{\frac{3}{4}} \text{ or } 10^{\cdot75} = \sqrt{10^3} = \sqrt{1000} = 5\cdot623$$
$$10^{\frac{4}{3}} \text{ or } 10^{1\cdot333} = \sqrt{10^4} = \sqrt{10{,}000} = 21\cdot547,$$

and so on.

The meaning of a zero index such as 0 in a^0.

Since $$\frac{a^m}{a^n} = a^{m-n}$$
$$\therefore \quad \frac{a^m}{a^m} = a^{m-m} = a^0.$$
But $$\frac{a^m}{a^m} = 1.$$
$$\therefore \quad a^0 = 1.$$

i.e., **any number raised to zero power is equal to 1.**

In order to realise the exact meaning of a^0 and why it is equal to 1, let us take the nth root of such a number as 10, where n is some large number.

$$\text{Thus, since } 10^{\frac{1}{2}} \text{ or } 10^{0\cdot5} = \sqrt{10} = 3\cdot1623$$
$$\therefore \quad 10^{\frac{1}{4}} \text{ or } 10^{0\cdot25} = \sqrt{3\cdot1623} = 1\cdot7783$$
$$\therefore \quad 10^{\frac{1}{8}} \text{ or } 10^{0\cdot125} = \sqrt{1\cdot7783} = 1\cdot3336$$
$$\therefore \quad 10^{\frac{1}{16}} \text{ or } 10^{0\cdot0625} = \sqrt{1\cdot3336} = 1\cdot1548$$
$$\therefore \quad 10^{\frac{1}{32}} \text{ or } 10^{0\cdot03175} = \sqrt{1\cdot1548} = 1\cdot0756$$
$$\therefore \quad 10^{\frac{1}{64}} \text{ or } 10^{0\cdot015875} = \sqrt{1\cdot0746} = 1\cdot0366$$
$$\therefore \quad 10^{\frac{1}{128}} \text{ or } 10^{0\cdot007938} = \sqrt{1\cdot0366} = 1\cdot0183$$
$$\therefore \quad 10^{\frac{1}{256}} \text{ or } 10^{0\cdot003969} = \sqrt{1\cdot0183} = 1\cdot009,$$

and so on.

In other words, as the fractional index $\frac{1}{n}$ becomes less and less, or as **n** becomes larger and larger, so does the value of $10^{\frac{1}{n}}$ become nearer and nearer 1, but it never becomes absolutely equal to 1, except when n becomes infinitely large, *i.e.*, when $\frac{1}{n}$ becomes infinitely small, *i.e.*, when $\frac{1}{n}$ becomes $= 0$.

Indeed $10^{\frac{1}{10,000}} = 1{\cdot}0002$

$10^{\frac{1}{100,000}} = 1{\cdot}00002$

$10^{\frac{1}{1,000,000}} = 1{\cdot}000002$,

and so on.

Hence $10^0 = 1$.

Similarly, if instead of 10 we take any other number we get the same result.

$\therefore$ **$a^0 = 1$, whatever number a stands for.**

Still another way of looking at it is this :

$a^m = 1 \times a \times a \times a$ . . . (a being repeated m times).

$\therefore$ $a^3 = 1 \times a \times a \times a$ . . . (a being repeated 3 times).

$\because$ $a^0 = 1$ not multiplied by **a** any times).

$= 1$.

The meaning of a negative index, such as $-m$ in a^{-m}.

Since $\frac{1}{a^m} = \frac{a^0}{a^m} = a^{0-m} = a^{-m}$,

$\therefore$ $a^{-m} = \frac{1}{a^m}$

e.g. $10^{-1} = \frac{1}{10}$; $10^{-2} = \frac{1}{10^2}$, etc.

Logarithms.—We know of course that all numbers which are composed of factors of 10 only, or of the reciprocals of factors of 10 only, can be represented as some integral power (positive or negative) of 10, thus :

$\frac{1}{100} = 10^{-2}$; $\frac{1}{10} = 10^{-1}$; $1 = 10^0$; $10 = 10^1$; $100 = 10^2$; $1000 = 10^3$, etc.

We have also seen that other numbers, not so composed, can be represented as some fractional power (pure or mixed) of 10, thus :

$$1{\cdot}3336 = 10^{\cdot125}; \quad 21{\cdot}547 = 10^{1\cdot333}, \text{ etc.}$$

Now it can be proved that *all conceivable numbers, whether integral or fractional, can be represented as being some power of* 10 *such as* 10^m, where m is either a positive or negative integer, or pure or mixed fraction or decimal.

Thus, to take one example, we can show in the following manner that $2 = 10^{0 \cdot 3010}$.

$$\begin{aligned} \text{For we have seen that } 10^{0\cdot 25} &= 1\cdot 7783 \\ \text{and } 10^{0\cdot 0625} &= 1\cdot 1548 \end{aligned} \Big\} \text{(p. 8)}$$

$$\therefore \quad 10^{0\cdot 25} \times 10^{0\cdot 0625} = 1\cdot 7783 \times 1\cdot 1548$$

i.e., $$10^{0\cdot 3125} = 2\cdot 0510 \quad . \quad . \quad . \quad . \quad (a)$$

Also $$\begin{aligned} 10^{0\cdot 0079} &= 1\cdot 018 \\ 10^{0\cdot 0040} &= 1\cdot 009 \end{aligned} \Big\} \text{(p. 8).}$$

$$\begin{aligned} \therefore \quad 10^{0\cdot 0119} &= 1\cdot 018 \times 1\cdot 009 \\ &= 1\cdot 027 \quad . \quad . \quad . \quad . \quad (b) \end{aligned}$$

$$\therefore \quad \frac{10^{0\cdot 3125}}{10^{0\cdot 0119}} = \frac{2\cdot 0540}{1\cdot 027}$$

i.e. $$\mathbf{10^{0\cdot 3010} = 2.}$$

(Q.E.D.)

Now, supposing we had to multiply 3·1623 by 2·154. We know from actual multiplication that the result is 6·812. . . . But supposing we were told beforehand that $3\cdot 1623 = 10^{0\cdot 5}$; that $2\cdot 154 = 10^{0\cdot 333}$ and that $10^{0\cdot 8333} = 6\cdot 812$, we could have reduced our labour very greatly because we could have performed our multiplication as follows :

$$3\cdot 162 \times 2\cdot 154 = 10^{0\cdot 5} \times 10^{0\cdot 33} = 10^{0\cdot 833} = 6\cdot 811.$$

Hence we see that by knowing the index of the power to which the number 10 has to be raised to produce any given number we can convert a tedious multiplication into a simple addition.

Similarly an operation like $\frac{3\cdot 162}{2\cdot 154}$ could be performed as follows :

$$\frac{3\cdot 162}{2\cdot 154} = \frac{10^{0\cdot 5}}{10^{0\cdot 33}} = 10^{0\cdot 167}$$

and if we knew that $10^{0\cdot 167} = 1\cdot 468$, we could say at once that

$$\frac{3\cdot 162}{2\cdot 154} = \frac{10^{0\cdot 5}}{10^{0\cdot 33}} = 10^{0\cdot 167} = 1\cdot 468.$$

So that a complicated and troublesome division could in this way be converted into a very simple subtraction.

In general, if we express any 2 numbers as 10^m and 10^n respectively, then if we also know what 10^{m+n} and 10^{m-n} are equal to, we can by simple addition of m and n or by simple subtraction of n from m perform the operations of multiplying the two numbers together or dividing one number by the other.

Now, when we express any number **a** in the form 10^m (*e.g.*, when we express 3·162 as $10^{\cdot 5}$) we say that **m is the logarithm of a to the base 10** (*e.g.*, 0·5 is the logarithm of 3·162 to the base 10; 0·33 is the logarithm of 2·154 to the case 10, and so on).

Similarly, we can represent any number as some power of 2, 3, 4, etc., since $2^2 = 4$, $2^4 = 16$, $2^{\frac{1}{2}} = \sqrt{2} = 1{\cdot}412$, etc.; $3^1 = 3$, $3^2 = 9$, $3^3 = 27$; $3^{\frac{1}{2}} = 1{\cdot}73$, etc.

Indeed, we see that any quantity **b** can be represented as some power of any other fundamental number or base **a**, *i.e.*, we can always write $b = a^m$.

Hence we arrive at the following definition:

Definition of Logarithm.—Whenever we have an expression like $a^m = b$ we say that m is the logarithm of b to the base a. In other words: *The logarithm of any number* **b** *to a given base* **a** *is the index of the power to which the base* **a** *must be raised to produce the given number* **b**.

Thus, since $31{\cdot}623 = 10^{1{\cdot}5}$, we say that the logarithm of 31·623 *to the base 10* is 1·5.

Similarly, since $1{\cdot}412 = 2^{0{\cdot}5}$

∴ the logarithm of 1·412 *to the base 2* is 0·5.

The symbolical way of writing logarithms is $m = \log_a b$, which is read: "m is the logarithm of b to the base a."

Common Logarithms.—For ordinary purposes of calculation logarithms are used to the base 10. The advantages of this base will be seen later (p. 79). Hence, when we say that the logarithm of a number **b** is **m**, we mean that $b = 10^m$, and instead of writing $m = \log_{10} b$ we simply write $m = \log b$, the base 10 being understood. Such logarithms are called *Common* logarithms.

Advantages of Logarithms.—Now, logarithms not only save much tedious labour in cases of complicated multiplications and divisions, but they become of still greater use when we have to raise a number to a certain power, and especially if we have to extract the root of any number. Thus, supposing we have the following tasks to perform:

(1) What is $(3{\cdot}162)^3$?

(2) What is $\sqrt[3]{3{\cdot}162}$?

By logarithms we do these operations quickly and simply as follows:

$$(3{\cdot}1623)^3 = (10^{0{\cdot}5})^3 = 10^{0{\cdot}5 \times 3} = 10^{1{\cdot}5} = 31{\cdot}623$$

$$\sqrt[3]{(3{\cdot}162)} = (3{\cdot}162)^{\frac{1}{3}} = 10^{\frac{0{\cdot}5}{3}} = 10^{0{\cdot}166} = 1{\cdot}468.$$

Hence we arrive at the following *rules* for working with logarithms:

(i.) **The logarithm of a product is equal to the sum of the logarithms of its separate factors,** or $\log ab = \log a + \log b$.

(ii.) **The logarithm of a quotient is equal to the difference between the logarithm of the dividend and that of the divisor,** or $\log \frac{a}{b} = \log a - \log b$.

(iii.) $\log a^n = n \log a$.

(iv.) $\log \sqrt[n]{a}$ or $\log a^{\frac{1}{n}} = \frac{1}{n} \log a$.

Example.—Find the value of $\dfrac{41{\cdot}6 \sqrt[3]{7{\cdot}67} \times (1{\cdot}4)^2}{\sqrt[3]{19{\cdot}41 \times 1{\cdot}783^2} \times 4{\cdot}62}$

$$
\begin{aligned}
\text{Let value} &= x \\
\therefore \quad \log x &= \log 41{\cdot}6 + \tfrac{1}{3} \log 7{\cdot}67 + 2 \log 1{\cdot}4 \\
&\quad - [\tfrac{1}{3} (\log 19{\cdot}41 + 2 \log 1{\cdot}783) + \log 4{\cdot}62] \\
&= 1{\cdot}6190 + \tfrac{1}{3} \times 0{\cdot}8848 + 2 \times 0{\cdot}1461 \\
&\quad - (\tfrac{1}{3} \times 1{\cdot}2880 + \tfrac{2}{3} \times 0{\cdot}2512 + {\cdot}6646) \\
&= 2{\cdot}2061 - 1{\cdot}2614 = {\cdot}9447 \\
&\qquad\qquad\qquad = \log 8{\cdot}805 \\
\therefore \quad x &= 8{\cdot}805.
\end{aligned}
$$

Explanation.—(i.) The logarithm of the fraction is equal to the logarithm of the numerator minus the logarithm of the denominator (rule (ii.)).

(ii.) The logarithm of the numerator is equal to the sum of the logarithms of its separate factors = $\log 41{\cdot}6 + \log \sqrt[3]{7{\cdot}67} + \log (1{\cdot}4)^2$ (rule (i.)).

But $\log \sqrt[3]{7{\cdot}67} = \frac{1}{3} \log 7{\cdot}67$ (rule (iv.)).

and $\log (1{\cdot}4)^2 = 2 \log (1{\cdot}4)$ (rule (iii.)).

$\therefore$ logarithm of numerator = $\log 41{\cdot}6 + \frac{1}{3} \log 7{\cdot}67 + 2 \log 1{\cdot}4$.

Similarly with the denominator.

Origin of the Term "Logarithm."—The series of numbers 10, 100, 1000, etc., is, as we shall see on p. 63, Chapter VI., a geometrical progression, *i.e.*, a series in which each term bears a constant ratio to the one immediately preceding it (*e.g.*, $\frac{100}{10} = \frac{1000}{100} = \frac{10{,}000}{1000}$, etc. = 10).

Hence	1	which	= 10^0	is the common ratio raised to the power			0
	10	,,	= 10^1	,,	,,	,,	1
	100	,,	= 10^2	,,	,,	,,	1
	3·1623	,,	= $10^{1\cdot5}$	,,	,,	,,	1·5

and so on.

So that in the expression $b = 10^m$, the index m denotes the number of times the common ratio 10 must be taken as a factor to produce b. Hence

0 is the ratio number or logarithm (λογος = ratio, αριθμος = number) of				…	…	1
1	,,	,,	,,	,,	,,	10
2	,,	,,	,,	,,	,,	100
1·5	,,	,,	,,	,,	,,	3·1623
m	,,	,,	,,	,,	,,	b

Hence the name " Logarithm."

Characteristic and Mantissa.—The integral part of any logarithm is called the **characteristic**; the decimal portion is called the **mantissa.** Thus, if $\log_{10} 15{\cdot}2 = 1{\cdot}1818$, then 1 is the characteristic and 0·1818 is the mantissa.

Since log 10 = 1 and log 100 = 2, it is clear that any number intermediate between 10 and 100 must have a logarithm lying between 1 and 2. Hence the characteristic of all numbers between 10 and 100 must be 1, although the mantissa is different in each case. Thus log 11 = **1**·0414 ; log 66 = **1**·8195 ; log 99 = **1**·9956, and so on

Similarly the characteristic of every number between 100 and 1000 is 2; that of every number between 1000 and 10,000 is 3, and so on; whilst all numbers between 1 and 10 have no "characteristic" at all, or have zero characteristic. Hence logarithm tables leave out the characteristic and only give the mantissa of the logarithm of any number. Thus the logarithm of 9458, for instance, is given as 0·975799, it being understood that since 9458 lies between 1000 and 10,000 the characteristic is 3, and that, therefore, the full logarithm is 3·975799, and so on for any number.

Now consider the number 9458 again. Its logarithm = 3·9758. This means that $10^{3\cdot9758} = 9458$. Hence $\frac{10^{3\cdot9758}}{10} = 945\cdot8$; $\frac{10^{3\cdot9758}}{10^2} = 94\cdot58$; $\frac{10^{3\cdot9758}}{10^3} = 9\cdot458$; $\frac{10^{3\cdot9758}}{10^4} = 0\cdot9458$, etc.

But $\frac{10^m}{10^n} = 10^{m-n}$ $\left(e.g., \frac{10^4}{10^2} = \frac{10000}{100} = 100 = 10^2 = 10^{4-2}\right)$

$$\therefore \quad \frac{10^{3\cdot9758}}{10} = 10^{2\cdot9758}$$
$$\frac{10^{3\cdot9758}}{10^2} = 10^{1\cdot9758}$$
$$\frac{10^{3\cdot9758}}{10^3} = 10^{0\cdot9758}$$
$$\frac{10^{3\cdot9758}}{10^4} = 10^{0\cdot9758-1}$$
$$\frac{10^{3\cdot9758}}{10^5} = 10^{0\cdot9758-2}, \text{ etc.}$$

Hence

$10^{3\cdot9758}$	= 9458	and ∴	log 9458	= 3·9758
$10^{2\cdot9758}$	= 945·8	,,	log 945·8	= 2·9758
$10^{1\cdot9758}$	= 94·58	,,	log 94·58	= 1·9758
$10^{0\cdot9758}$	= 9·458	,,	log 9·458	= 0·9758
$10^{0\cdot9758-1}$	= 0·9458	,,	log 0·9458	= 0·9758 − 1
$10^{0\cdot9758-2}$	= 0·09458	,,	log 0·09458	= 0·9758 − 2
	etc.			etc.

Hence we see that **all the numbers which contain the same significant figures in the same order, but only differ in the position of the decimal point, have the same mantissa in their logarithms, but different characteristics.** Now this is a very convenient fact, since once we know the logarithm of any number we can at once write down the logarithm of the same number multiplied or divided by any power of 10. *This fact only holds good when the base is* 10, *and hence is in itself a sufficient reason why logarithms* *for practical purposes are calculated to the base* 10. Thus, supposing, for instance, logarithms were calculated to the base 8, then since $8^{2\cdot3333} = 8^2 \times 8^{1/3} = 64 \times 2 = 128$, $\therefore$ log 128 = 2·3333, but log 12·8 is not 1·3333, because $8^{1\cdot3333} = 8 \times 8^{\frac{1}{3}} = 8 \times 2 = 16$, and not 12·8; nor would log 1·28 = 0·3333, because $8^{0\cdot3333} = 2$ and not 1·28.

Another fact that we learn from the above table of logarithms is that there is a great convenience in writing the mantissa in the form of a *positive* decimal. Thus we write log 0·9457, 0·97581 − 1 and not − 0·02409, because in the form 0·97581 − 1 the mantissa is the same as in log 9457 or log 945·7, etc. But instead of writing the logarithm of a fractional number in the form, say, 0·9758 − 1 one writes it as follows: $\bar{1}\cdot9758$. Similarly log 0·09457 = $\bar{2}\cdot9758$; the negative sign being placed *over* the characteristic instead of in front of it to show that it is only the characteristic and not the whole of the logarithm that is negative. Thus, while − 2·9758 = − 2 − 0·9758, $\bar{2}\cdot9758$ = − 2 + 0·9758.

Napierian Logarithms.—But whilst for ordinary purposes of computation we use logarithms to the base 10, or Briggsian logarithms, as they are called (having been first calculated by Briggs in 1617), in the higher mathematics, for reasons which will become clear presently (see p. 80), it is convenient to use the incommensurable number 2·71828 . . . (generally denoted by the letter ϵ or **e**) as the base. This system of logarithms is called the Napierian system, because Napier, the inventor of logarithms, calculated his logarithms to this base (in 1614) (Chapter VI, p. 80).

Antilogarithm.—By antilogarithm is meant the number corresponding to a given logarithm. Thus, if the logarithm of 2 is 0·30103, then the antilogarithm of 0·30103 is 2.

Exponential Equation.—An exponential equation is one in which the unknown quantity occurs as the index or exponent of a power. They are most easily solved by the aid of logarithms:

Example.—Find x from the equation $2^x = 9$.

Since $2^x = 9$, therefore $x \log 2 = \log 9$.

$$\therefore \quad x = \frac{\log 9}{\log 2} = \frac{\cdot9542}{\cdot3010} = 3\cdot17.$$

Example.—Solve the following equation:

$$5^{5-3x} = 2^{x+2}.$$

Taking logarithms $(5 - 3x) \log 5 = (x + 2) \log 2$,

$$i.e., \ \cdot699\,(5 - 3x) = \cdot301\,(x + 2),$$

whence $x = 1\cdot206.$

EXAMPLES.

(1) Find the logarithms of 5, 20, 25, 1·25, it being given that log 2 = ·30103.

$$\text{Log } 5 = \log \frac{10}{2} = \log 10 - \log 2 \text{ (p. 11)}.$$
$$= 1 - \cdot 30103 = \cdot 69897.$$
$$\log 20 = \log 10 + \log 2 = 1 \cdot 30103.$$
$$\log 25 = \log (5)^2 = 2 \log 5 = 2 \times \cdot 69897.$$
$$= 1 \cdot 39799.$$
$$\log 1 \cdot 25 = \log \frac{5^3}{100} = 3 \times \cdot 69897 - 2.$$
$$= 0 \cdot 09691.$$

(2) Given that log 317 = 2·5011 and log 318 = 2·5024. Find the value of log 317·2.

The *method* adopted is that of "*Proportional Parts.*"

$$\text{Thus } \log 317 = 2 \cdot 50106.$$
$$\log 318 = 2 \cdot 50243.$$

∴ A difference of 1 in the number gives a difference of ·00137 in the logarithm.

∴ A difference of 0·2 in the number will give a difference of ·00137 × 0·2 in the logarithm.

$$\therefore \quad \log 317 \cdot 2 = 2 \cdot 50106 + \cdot 000274.$$
$$= \cdot 50133.$$

The student is earnestly recommended to become familiar with the method of *proportional parts*, as it is very frequently utilised in practical work.

(3) The following formula has been found to express the influence of temperature upon the velocity of a chemical reaction, viz., $\frac{V_{t+n}}{V_t} = x^n$ (see p. 238),

where V_t = Velocity at temperature t.
V_{t+n} = „ „ $t + n$.
and x = increase in velocity per 1° C.

In the case of the tadpole Hertwig found that when the developing ovum was kept at 10° it took $3\frac{1}{3}$ times as long to reach a certain stage of development as when kept at 20°. Find the increase in rate of growth (x) per 1° C., assuming the above formula to hold good.

Putting $n = 10$ in the above formula we get

$$x^{10} = \frac{V_{20}}{V_{10}} = 3 \cdot 33.$$

$$\therefore \quad 10 \log x = \log 3 \cdot 33 = 0 \cdot 5224.$$
$$\therefore \quad \log x = \cdot 05224.$$
$$\therefore \quad x = 1 \cdot 128 \text{ (from the tables of logarithms).}$$

(4) In the case of the pea, Miss Leitch's figures for the rate of growth in rootlets at various temperatures (as modified by D'Arcy Thompson) are as follows :

At 10° the rootlet grows 0·41 mm. per hour.
„ 14° „ „ 0·61 „ „
„ 20° „ „ 1·01 „ „
„ 24° „ „ 1·43 „ „

Find the values of x for the temperature intervals 10° — 20° *and* 14° — 24°, (assuming the formula given in the last question to hold good).

$$\frac{V_{20}}{V_{10}} = \frac{1{\cdot}01}{{\cdot}41} = x_1^{10} \quad \therefore 10 \log x_1 = \log 2{\cdot}46 = {\cdot}3909 \quad \therefore x_1 = 1{\cdot}094.$$

$$\frac{V_{24}}{V_{14}} = \frac{1{\cdot}43}{{\cdot}61} = x_2^{10} \quad \therefore 10 \log x_2 = \log 2{\cdot}34 = {\cdot}3692 \quad \therefore x_2 = 1{\cdot}089.$$

(5) Robertson found that the decay of memory traces for meaningless syllables committed to memory is expressed by the equation

$$N - n = 1{\cdot}733t^{0{\cdot}056}$$

where N = number of syllables originally committed to memory, and n = number of syllables remembered after t hours.

If N = 13, find after what time all the syllables will be completely forgotten (assuming the formula to be true for all values of t).

When all syllables are forgotten $n = 0$.

$$\begin{aligned}
\therefore \quad N &= 1{\cdot}733t^{0{\cdot}056}, \\
\textit{i.e.,} \quad 13 &= 1{\cdot}733t^{0{\cdot}056}. \\
\therefore \quad \log 13 &= \log 1{\cdot}733 + 0{\cdot}056 \log t, \\
\text{or } 1{\cdot}113943 &= {\cdot}238799 + 0{\cdot}056 \log t, \\
\text{or } {\cdot}875144 &= 0{\cdot}056 \log t. \\
\therefore \quad \log t &= \frac{{\cdot}875144}{{\cdot}056} = 15{\cdot}62757. \\
\text{But } {\cdot}62757 &= \log 4{\cdot}242. \\
15{\cdot}62757 &= \log 4{\cdot}242 \times 10^{15} \text{ (p. 13)}. \\
\therefore \quad t &= 4242 \times 10^{12} \text{ hours} \\
&= \frac{4242 \times 10^{12}}{24 \times 365} \text{ years} \\
&= 484 \times 10^{9} \text{ years !}
\end{aligned}$$

In other words, memory traces are never completely wiped out. This is an interesting calculation from the point of view of hypnosis and allied conditions, since it shows that although a person may be entirely unconscious of certain memories during his waking state, still the memories exist in the subconscious mind, and when the resistance to the passage of impulses through certain cerebral areas is lowered, as in hypnosis, these memory traces reveal themselves.

(6) The viscosity coefficient (η) of water at any temperature (t) is given by the formula

$$\eta = 2{\cdot}989\,(t + 38{\cdot}5)^{-1{\cdot}4}.$$

Find η when $t = 20$.

$$\begin{aligned}
\text{Here} \quad \eta &= 2{\cdot}989 \times (58{\cdot}5)^{-1{\cdot}4} \\
\therefore \quad \log \eta &= \log 2{\cdot}989 - 1{\cdot}4 \log 58{\cdot}5. \\
&= {\cdot}475526 - 1{\cdot}4 \times 1{\cdot}767156. \\
&= {\cdot}475526 - 2{\cdot}474018. \\
&= \bar{2}{\cdot}001508. \\
\therefore \quad \eta &= {\cdot}0100.
\end{aligned}$$

(7) Dreyer found the following relationship between W (weight in grams) and λ (sitting height in centimetres) of a person :

$$W = ({\cdot}038025\lambda)^{\frac{1}{0{\cdot}319}}.$$

Find the sitting height of a person weighing 89·78 kilograms.

$$\text{Log } 89780 = \frac{1}{0 \cdot 319} (\log 0 \cdot 38025 + \log \lambda),$$

i.e.,
$$4 \cdot 953180 = \frac{1}{0 \cdot 319} (\bar{1} \cdot 580012 + \log \lambda).$$

$$\therefore \quad \log \lambda = 4 \cdot 95318 \times 0 \cdot 319 - \bar{1} \cdot 580012$$
$$= 2.$$
$$\therefore \quad \lambda = 100 \text{ cm.}$$

(8) The following equation (Dubois) represents the relationship between the surface area of the body (S) and its height (H) and weight (W) :

$$\underset{\text{Sq. cm.}}{S} = 71 \cdot 84 \; \underbrace{W^{0 \cdot 425}}_{\text{Kilograms.}} \; . \; \underbrace{H^{0 \cdot 725}}_{\text{Centimetres.}}.$$

Find the surface area when W = 60 kg. and H = 150 cm.

$$\text{Log S} = \log 71 \cdot 84 + 0 \cdot 425 \log W + 0 \cdot 725 \log H$$
$$= 1 \cdot 85637 + 0 \cdot 425 \times 1 \cdot 77815 + 0 \cdot 725 \times 2 \cdot 17609$$
$$= 4 \cdot 1917 = \log 15550.$$
$$\therefore \text{ area} = 15{,}550 \text{ sq. cm.} = 1 \cdot 555 \text{ sq. metres.}$$

(9) The population of a certain country doubles itself in 100 years. Find rate of growth per annum assuming it constant. If the population is a million at the beginning of the century, find what it will be in twenty, fifty and eighty years, respectively, from the beginning.

If P = population, then

$$P\left(1 + \frac{r}{100}\right)^{100} = \text{population after 100 years (see p. 64)}$$
$$= 2P.$$
$$\therefore \quad \left(1 + \frac{r}{100}\right)^{100} = 2.$$
$$\therefore \quad 100 \log \left(1 + \frac{r}{100}\right) = \log 2$$
$$= 0 \cdot 30103.$$
$$\therefore \quad \log \left(1 + \frac{r}{100}\right) = 0 \cdot 0030103.$$
$$= \log 1 \cdot 007.$$
$$\therefore \quad \frac{r}{100} = 0 \cdot 007. \quad \therefore \quad r = 0 \cdot 7$$

i.e., rate of growth = 7 per 1,000 per annum.

∴ at twenty years' population

$$= (1 \cdot 007)^{20} = x.$$
$$\therefore \quad 20 \log (1 \cdot 007) = \log x.$$
i.e.,
$$0 \cdot 060206 = \log x = \log 1 \cdot 1487.$$
$$\therefore \quad \text{population} = 1{,}148{,}700.$$

The student can work out for himself the population at fifty and eighty years respectively. [Answer, 1,414,200 ; 1,741, 100.]

(10) Find the value of

$$\sqrt{2\sqrt{2\sqrt{2\sqrt{2\sqrt{2\sqrt{2 \ldots}}}}}} \quad \text{to infinity.}$$

Let the value of this expression = x.

Now the expression can also be written as $2^{\frac{1}{2}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{8}} \times 2^{\frac{1}{16}} \ldots$ to infinity.

Hence $x = 2^{\frac{1}{2}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{8}} \times 2^{\frac{1}{16}} \ldots\ldots$ to infinity.

$\therefore \quad \log x = \frac{1}{2}\log 2 + \frac{1}{4}\log 2 + \frac{1}{8}\log 2 + \ldots\ldots$ to infinity

$= (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots\ldots) \log 2.$

But $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots$ to infinity $= 1$. (See p. 68.)

$\therefore \quad \log x = \log 2,$

whence $x = 2.$

We might have arrived at the answer in a somewhat different and equally pretty way :

$$x = \sqrt{2\sqrt{2\sqrt{2\sqrt{2\ldots\ldots}}}} \text{ to infinity.}$$

$$\therefore \quad x^2 = 2\sqrt{2\sqrt{2\sqrt{2\sqrt{2\ldots\ldots}}}} \text{ to infinity}$$

$$= 2x.$$

$$\therefore \quad x^2 - 2x = 0 \text{ or } x(x-2) = 0.$$

$$\therefore \quad x = 2.$$

Similarly $\sqrt{3\sqrt{3\sqrt{3\sqrt{3\ldots\ldots}}}}$ to infinity $= 3$

and generally $\sqrt{n\sqrt{n\sqrt{n\ldots\ldots}}}$ to infinity $= n$.

EXERCISES.

(1) If the number of persons born in any year is $\frac{1}{45}$ of the whole population at the beginning of the year, and the number of those who die $\frac{1}{60}$ of it, in how many years will the population be doubled, given log 181 = 2·5768 ?

[Answer, 125 years.]

(2) Evaluate $\sqrt{\dfrac{101\cdot4 \times (0\cdot2891)}{(0\cdot00854)^4 \times 7694}}$. [Answer, 4·814.]

(3) The volume of a cylinder is given by the formula $v = \pi r^2 h$ (where r = radius, and h = height). Find v if $r = 0\cdot5$, $h = 12\cdot76$ and $\pi = 3\cdot142$.

[Answer, 10.]

(4) Meeh's formula for the surface of the human body is $S = K\sqrt[3]{W^2}$. where S = surface in sq. decimetres, W = weight in kilograms and K is a constant (which, in case of children = 10·3). It has been found by Benedict and Talbot that if l = length of an infant, then the amount of heat produced by it in 24 hours is $0\cdot1265 l K \sqrt[3]{W^2}$. Calculate the theoretical amount of heat produced by an infant weighing 3·63 kilograms and measuring 52 cm. in length.

[Answer, $\log x = \log 0\cdot1265 + \log 52 + \log 10\cdot3 + \frac{2}{3}\log 3\cdot63 = 2\cdot2038$. $\therefore \; x = 160$ calories.]

(5) Using the formula $\dfrac{V_{t+n}}{V_t} = x^n$ (see p. 15), find the value of this quotient for $n = 10$ (called the temperature coefficient) for the inversion of cane sugar, given $V_{25} = 0\cdot765$, $V_{55} = 35\cdot5$).

[Answer, $\dfrac{V_{t+10}}{V_t} = 3\cdot6$.]

(6) Prove that $\sqrt[3]{2\sqrt[3]{2\sqrt[3]{2\sqrt[3]{2\ldots\ldots}}}}$ to infinity $= \sqrt{2}$. (See example 10.)

MECHANICAL AIDS TO CALCULATION.

(*a*) **The Slide Rule.**—A simple apparatus of very great service for the purposes of rapid multiplication, division, squaring and extraction of square roots in cases where no greater accuracy is required than that of three significant figures is the *slide rule*, the principle of which will be readily understood, once the idea of logarithms has been firmly grasped.

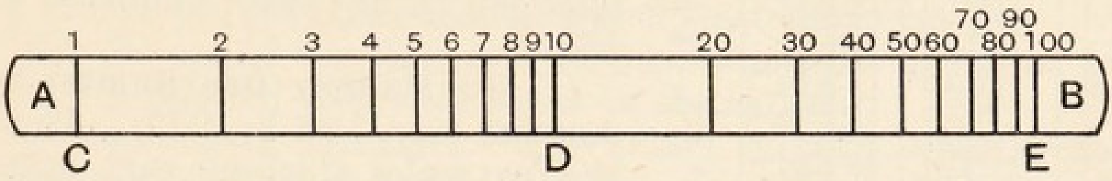

Fig. 1.—Logarithmic Scale of Slide Rule.

AB (Fig. 1) is a ruler upon which is a scale marked as follows: Suppose the portion CE to be two units long (*e.g.*, 10 in., the unit length being 5 in.). The length CE is divided into two equal parts at D, and D is marked 10. Each of the equal parts CD and DE is divided into ten parts, which are *not* equal but proportional to the logarithms of the numbers between 1 and 10. Thus the

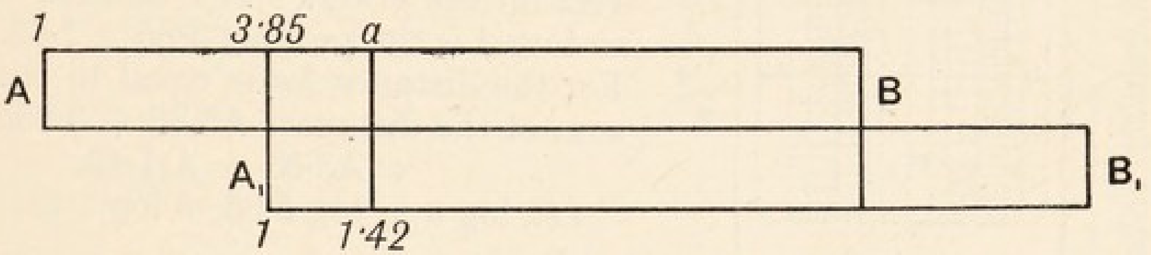

Fig. 2.—Mode of using Slide Rule for Multiplication.

point C or the zero point is marked 1;* the point 2 is put at a distance from C equal to log 2; the point 3 is placed at a distance log 3 from C, and so on. The point D, being at a distance of one unit from C is marked 10 because log 10 = 1. Also the point E, being at a distance of two units from C is marked 100, because log 100 = 2. The point 20 is placed at a distance of log 20 from

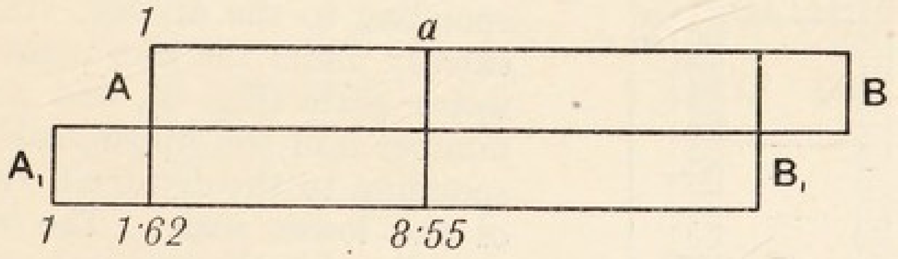

Fig. 3.—Mode of using Slide Rule for Division.

C, or of log 2 from D (since log 20 = 1 + log 2), and so with the other numbers 30, 40, etc. The intermediate spaces are further similarly subdivided. Now supposing we have another ruler,

* Because log 1 = 0.

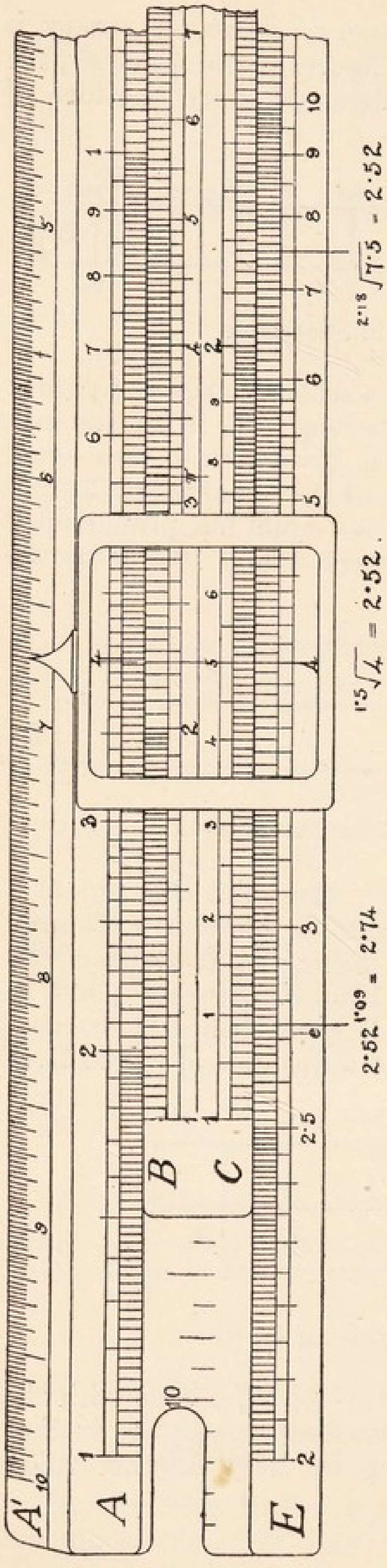

FIG. 3A.—Slide Rule.

A_1B_1, the same length as AB, graduated in exactly the same way as AB, then we can use the two rulers to find the product, or quotient of any two numbers, as follows :

(1) **To Multiply One Number by Another** (say, 3·85 by 1·42).—Place the two scales alongside each other and in such a way that the zero point of the lower scale A_1B_1 (*i.e.*, the point marked 1), is situated exactly under the point corresponding to the division 3·85 of the upper scale AB (Fig. 2), and note the number a on the upper scale which lies exactly above the number 1·42 on the lower. The number a so found is the product $3{\cdot}85 \times 1{\cdot}42$. For the distance Aa is equal to the sum of the distances $A3{\cdot}85 + 3{\cdot}85a$

$$= A3{\cdot}85 + A_1 1{\cdot}42,$$

$$i.e., \log a = \log 3{\cdot}85 + \log 1{\cdot}42.$$

$$\therefore \quad a = 3{\cdot}85 \times 1{\cdot}42.$$

Indeed, a will be found to be very nearly 5·5 on the scale (the exact product being 5·47 to two decimal places).

(2) **To Divide One Number by Another** (say, 8·55 by 1·62).—Place the point of the lower scale corresponding to the divisor, viz., 1·62, exactly under the zero point of the upper scale (Fig. 3), and note the number a of the upper scale corresponding to the dividend, viz., 8·55 of the lower scale. The number a so found is the required quotient.

For the distance

$$Aa = A_1 8{\cdot}55 - A_1 1{\cdot}62,$$

$$i.e., \log a = \log 8{\cdot}55 - \log 1{\cdot}62.$$

$$\therefore \quad a = \frac{8{\cdot}55}{1{\cdot}62}.$$

Indeed, a will be found to be as near as possible 5·28 (which is the correct quotient). The student is recommended to procure a slide rule (obtainable at about 4*s.*) and use it for the purposes indicated above. A slide rule 10 in. long gives an accuracy of up to 0·25 per cent. Longer ones give greater accuracy.

For the use of the slide rule for involution and evolution the reader is referred to special books on the subject.

(*b*) **Calculating Machines.**—The expert mathematical investigator will need an elaborate calculating machine, which is, however, too costly for the ordinary student.

(For other methods of simplifying arithmetical work see pp. 23, 38, 39 and 52.)

CHAPTER III.

A FEW POINTS IN ALGEBRA.

I ASSUME that the reader is familiar with the solutions of simple algebraical equations, such as $x + 3 = 6$, giving $x = 3$; and with the addition and subtraction of fractions such as $\frac{1}{x+1} + \frac{1}{x-1}$, when the fractions have to be reduced to a common denominator $(x + 1)(x - 1)$ and added, thus:

$$\frac{1}{x+1} + \frac{1}{x-1} = \frac{(x-1)}{(x+1)(x-1)} + \frac{(x+1)}{(x+1)(x-1)}$$
$$= \frac{2x}{(x+1)(x-1)} = \frac{2x}{(x^2-1)}.$$

Having refreshed our memories with regard to these simple matters I should like to introduce the reader to two simple points which have a very great importance in higher mathematics. These two matters are:

(1) The distinction between an *equation* and an *identity*.

(2) The meaning of the term *partial fractions*.

(1) **Equation and Identity.**—Whilst an equation like $x + 3 = 6$, or $x^2 + 4x + 4 = 9$ is a statement to the effect that two algebraical expressions are equal for certain particular values of the quantities used in the expressions—these particular values being called the **roots** of the equations—an *identity* is a statement that two algebraical expressions are equal for *all* values of the quantities used in the expressions.

Thus $x + 3 = 6$ expresses the fact that if you substitute 3 for x, the two expressions on either side of the equality sign (=) become alike, *i.e.*, $3 + 3 = 6$.

Similarly $x^2 + 2x + 4 = 9$ is only true when $x = +1$ or -5; thus if $x = 1$ we have

$$1 + 4 + 4 = 9,$$

and if $x = -5$ we have

$$25 - 20 + 4 = 9.$$

But $x + 3 = 6$ is not true if we put $x = 1$, or 2, or any other value except 3; and $x^2 + 4x + 4 = 9$ is not true if we give x any value other than $+1$ or -5.

If, however, we take such an expression as

$$x + 3 = 2 \cdot \frac{x}{2} + 3$$

or

$$x + 3 = \frac{1}{3}x + \frac{1}{3}x + \frac{1}{3}x + 3$$

or $(x - 1)(x + 1) = x^2 - 1$,

such statements are *identities*, because whatever value you give x in any of these expressions the result is *always* true.

The following identities, which the student can verify by actual multiplication, must be committed to memory, and should be familiar each way:

$$(a + b)^2 = a^2 + 2ab + b^2 \quad \ldots \quad (1)$$
$$(a - b)^2 = a^2 - 2ab + b^2 \quad \ldots \quad (2)$$
$$a^2 - b^2 = (a + b)(a - b) \quad \ldots \quad (3)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \quad \ldots \quad (4)$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \quad \ldots \quad (5)$$

Another important identity with which we shall deal later is the *binomial theorem* (see p. 64), viz.:

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \ldots + \frac{n(n-1)}{1 \cdot 2}a^2b^{n-2} + nab^{n-1} + b^n.$$

These identities will not only crop up continually in the course of our manipulations, but they are sometimes of considerable use for the purpose of simplifying arithmetical operations.

E.g., (1) $22{\cdot}9^2 - 22{\cdot}1^2 = (22{\cdot}9 + 22{\cdot}1)(22{\cdot}9 - 22{\cdot}1)$ by (3)
$= 45{\cdot}0 \times {\cdot}8 = 36{\cdot}00.$

(2) Find the square of 1·00013 correct to 6 significant figures.

$(1{\cdot}00013)^2 = (1 + 0{\cdot}00013)^2 = 1 + 2 \times 0{\cdot}00013 + (0{\cdot}00013)^2$ by (1)
$= 1{\cdot}00026.$

$(0{\cdot}00013)^2$ being equal to 0·00000169 does not affect the 6th significant figure, and may therefore be ignored.

(3) Find the value of $1{\cdot}0013 \times 0{\cdot}9987$. This can easily be put into the form $(a + b)(a - b) = a^2 - b^2$.

Thus $(1 + {\cdot}0013)(1 - {\cdot}0013) = 1^2 - {\cdot}000169$
$= 0{\cdot}009891.$

Note.—The student's attention is particularly directed to identities (1) and (2) which are of fundamental importance in the solution of quadratic equations. It will be observed that if an expression of the second degree is a complete square, then the coefficient of the third term is the square of half the coefficient of the middle term provided the coefficient of the first term is unity. Thus in the given identities, the coefficient of a^2 is 1, and, therefore, coefficient of b^2 is $\left(\frac{\pm 2}{2}\right)^2 = 1$.

EXAMPLES.

(1) Prove the following identities :

(1) $x(x+1)(x+2)(x+3)+1 = (x^2+3x+1)^2$.

(2) $x(x+2)(x+4)(x+6)+16 = (x^2+6x+4)^2$.

Hence evaluate

(a) $\sqrt{999{,}993 \times 999{,}994 \times 999{,}995 \times 999{,}996 + 1}$

(b) $\sqrt{999{,}993 \times 999{,}995 \times 999{,}997 \times 999{,}999 + 16}$.

(1) $(x+1)(x+2) = x^2+3x+2$.

$$\begin{aligned} \therefore \quad x(x+1)(x+2)(x+3) &= x(x^2+3x+2)(x+3) \\ &= (x^2+3x+2)(x^2+3x) \\ &= \{(x^2+3x+1)+1\}\{(x^2+3x+1)-1\} \\ &= (x^2+3x+1)^2 - 1. \end{aligned}$$

$\therefore \quad x(x+1)(x+2)(x+3)+1 = (x^2+3x+1)^2$.

(2) Similarly

$(x+2)(x+4) = x^2+6x+8$.

$$\begin{aligned} \therefore \quad x(x+2)(x+4)(x+6) &= (x^2+6x+8)(x^2+6x) \\ &= \{(x^2+6x+4)+4\}\{(x^2+6x+4)-4\} \\ &= (x^2+6x+4)^2 - 16. \end{aligned}$$

$\therefore \quad x(x+2)(x+4)(x+6)+16 = (x^2+6x+4)^2$.

(*a*) If we put $x = 999{,}993$ and use identity (1), we get — expression under square root sign

$$= (999{,}993^2 + 3.999{,}993 + 1)^2.$$

$$\begin{aligned} \therefore \quad \text{value of expression} &= 999{,}993^2 + 3.999{,}993 + 1 \\ &= (10^6 - 7)^2 + 3.(10^6 - 7) + 1 \\ &= 10^{12} - 14.10^6 + 49 + 3.10^6 - 21 + 1 \\ &= 10^{12} - 11.10^6 + 29 = 999{,}989{,}000{,}029. \end{aligned}$$

(*b*) By putting $x = 999{,}993$ and using identity (2) we get expression under $\sqrt{}$

$$= (999{,}993^2 + 6.999{,}993 + 4)^2.$$

$\therefore$ value of expression

$$\begin{aligned} &= (999{,}993^2 + 6.999{,}993 + 4) \\ &= (10^6 - 7)^2 + 6.(10^6 - 7) + 4 \\ &= 10^{12} - 14.10^6 + 49 + 6.10^6 - 42 + 4 \\ &= 10^{12} - 8.10^6 + 11 \\ &= 999{,}992{,}000{,}011. \end{aligned}$$

Exercise.—Evaluate $\sqrt{9991 \times 9994 \times 9997 \times 1000 + 81}$.

[Answer, 99,910,009.]

This introduces us to another important point, viz., the factorisation of a complicated algebraical expression.

Law of Factors.—If an expression like

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \ . \ . \ . \ Kx + m$$

contains $(x - a)$ as a factor, then when in the given expression x is put $= a$, the expression must become $= 0$.

This is almost obvious. For if $(x - a)$ is a factor of the expression, we must have

$$\frac{Ax^n + Bx^{n-1} + \ . \ . \ . \ + Kx + m}{(x - a)} = Q,$$

where Q is the quotient,

$\therefore \quad Q\,(x - a) = Ax^n + Bx^{n-1} + \ . \ . \ . \ + Kx + m.$

This being an identity is true for all values of x.

$\therefore$ it is true also when $x = a$.

Put $x = a$, and we get

$$Q \times 0 = Aa^n + Ba^{n-1} + \ . \ . \ . \ + Ka + m.$$

$$\therefore \quad Aa^n + Ba^{n-1} + \ . \ . \ . \ + Ka + m = 0,$$

i.e., if $Ax^n + Bx^{n-1} + \ . \ . \ . + Kx + m$ vanishes when $x = 0$, then $x = a$ contains $x - a$ as a factor.

EXAMPLES.

(1) Find the factors of $x^3 - 3x^2 - 10x + 24$.

Put $x = 1$ and the expression becomes

$$1 - 3 - 10 + 24 = 12.$$

As when x is made $= 1$ the expression does not vanish.

$\therefore \quad x - 1$ is *not* a factor.

Put $x = 2$ and the expression becomes

$$8 - 3{\cdot}4 - 10{\cdot}2 + 24 = 8 - 12 - 20 + 24 = 0,$$

$\therefore \quad x - 2$ is a factor.

Divide $x^3 - 3x^2 - 10x - 24$ by $x - 2$ and the quotient is $x^2 - x - 12$, the factors of which are $(x + 3)$ and $(x - 4)$.

Hence $x^3 - 3x^2 - 10x + 24 = (x - 2)\,(x + 3)\,(x - 4)$.

Or we might have tested for a factor $x + 3$ by putting $x = -3$, when the expression becomes

$$-27 - 27 + 30 + 24 = 0.$$

$\therefore \quad x + 3$ is a factor.

Similarly by making $x = 4$, the expression becomes

$$64 - 48 - 40 + 24 = 0.$$

$\therefore \quad x - 4$ is a factor

(2) Find the square root of $x^4 + 4x^6 + 10x^2 + 12x + 9$.

Let $\qquad \sqrt{x^4 + 4x^6 + 10x^2 + 12x + 9} = (x^2 + ax + 3)^2.$

$$\therefore \quad x^4 + 4x^3 + 10x^2 + 12x + 9 = (x^2 + ax + 3)^2.$$

$$= x^4 + 2ax^3 + (6 + a^2)x^2 + 12x + 9.$$

This being an identity the coefficients of like powers of x must be equal to one another.

$$\therefore \quad 4x^2 = 2ax^3, \text{ whence } a = 2.$$

$$\therefore \quad \text{the required root} \qquad = x^2 + 2x + 3.$$

(3) Find the factors of $a^2(b - c) + b^2(c - a) + c^2(a - b)$.

By putting $b = c$, the expression becomes

$$0 + b^2(b - a) + b^2(a - b) = 0.$$

$\therefore \quad (b - c)$ is a factor.

Similarly $(c - a)$ and $(a - b)$ are factors.

$$\therefore \quad a^2(b - c) + b^2(c - a) + c^2(a - b) = \text{K}(a - b)(b - c)(c - a) \ldots (1)$$

where K is the remaining factor.

Now (1) being an identity is true for all values of a, b, and c. Put $a = 0$, then we get

$$0 + b^2c + c^2b = \text{K}(-b)(b - c)c$$

or

$$bc(b - c) = -\text{K}bc(b - c).$$

$$\therefore \quad \text{K} = -1.$$

$\therefore \quad$ Required factors are $-(b - c)(c - a)(a - b)$.

Partial Fractions.—Whenever we have a fraction, the denominator of which consists of the product of two or more factors, then that fraction can always be expressed as the algebraical sum of a number of fractions, each of which has as its denominator only one of the factors of the original denominator. For example, we have seen that

$$\frac{2x}{x^2 - 1} = \frac{1}{x + 1} + \frac{1}{x - 1}.$$

The component fractions $\frac{1}{x + 1}$ *and* $\frac{1}{x - 1}$, *of which the original fraction* $\frac{2x}{x^2 - 1}$ *is composed, are called the partial fractions of* $\frac{2x}{x^2 - 1}$.

Now, whilst it is easy to add together several simple fractions to get a single more complicated fraction, the reverse process of splitting up a complicated fraction into its simpler constituents or its partial fractions is not always so easy, although it can always be done. In the higher mathematics such splitting of a fraction into its partial fractions is very often necessary (see pp. 148, 208 and 254), and we shall therefore take a few typical cases to show how, by making use of the properties of an identity such a splitting can be effected.

Example.—Supposing we did not know what the partial fractions of $\frac{2x}{x^2 - 1}$ were. How can we set about to find them ?

The first thing we have to do is to discover what are the factors of the denominator of the given fraction. In our case the denominator is $x^2 - 1$, and we know that $(x^2 - 1) = (x + 1)(x - 1)$, so that

$$\frac{2x}{x^2 - 1} = \frac{2x}{(x + 1)(x - 1)}.$$

Hence we know that our fraction must consist of two simpler fractions, the denominators of which are $(x + 1)$ and $(x - 1)$ respectively. The only thing, therefore, that remains to be discovered is, What are the respective numerators of these component or partial fractions ?

Call these numerators A and B, and we get

$$\frac{2x}{x^2 - 1} = \frac{2x}{(x + 1)(x - 1)} = \frac{A}{x + 1} + \frac{B}{x - 1}.$$

If now we perform the addition of these partial fractions in the ordinary way we get

$$\frac{A}{x + 1} + \frac{B}{x - 1} = \frac{A(x - 1) + B(x + 1)}{(x + 1)(x - 1)};$$

and this must be equal to $\dfrac{2x}{(x + 1)(x - 1)}$.

$$\therefore \quad \frac{2x}{(x + 1)(x - 1)} = \frac{A(x - 1) + B(x + 1)}{(x + 1)(x - 1)}.$$

$$\therefore \quad 2x = A(x - 1) + B(x + 1).$$

This being an identity is true for all values of x; let us therefore put $x = 1$, when we shall get

$$2 = A(1 - 1) + B(1 + 1) = 2B.$$

$$\therefore \quad B = 1.$$

Similarly by putting $x = -1$ we get

$$-2 = -2A$$

giving $A = 1$.

$$\therefore \quad \frac{2x}{(x + 1)(x - 1)} \text{ which } = \frac{A}{(x + 1)} + \frac{B}{x - 1}$$

$$= \frac{1}{(x + 1)} + \frac{1}{x - 1}.$$

EXAMPLES.

(1) Find the partial fractions of

$$\frac{3x + 2}{x^2 - 6x^2 + 11x - 6}.$$

We first find the factors of the denominator by using the method employed in example (1), p. 25. In that way we find that

$$x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3).$$

We therefore put $\dfrac{3x + 2}{x^3 - 6x^2 + 11x - 6} = \dfrac{A}{x - 1} + \dfrac{B}{x - 2} + \dfrac{C}{x - 3}$

$$= \frac{A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)}{x^3 - 6x^2 + 11x - 6}.$$

$$\therefore \quad 3x + 2 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2),$$

an identity which must therefore be true for all values of x.

By putting $x = 1$ we get

$$3 + 2 = A(-1)(-2) + 0$$
$$= 2A.$$
$$\therefore \quad A = \frac{5}{2}.$$

By putting $x = 2$ we get

$$B = -8$$

and by putting $x = 3$ we get

$$C = \frac{11}{3}.$$

$$\therefore \quad \frac{3x + 2}{x^3 - 6x^2 + 11x - 6} = \frac{5}{2(x-1)} - \frac{8}{(x-2)} + \frac{11}{3(x-3)}.$$

The importance of partial fractions will be realised when we come to deal with differentiation (p. 148, Chapter IX.) and integration (Chapter XVII., p. 254).

(2) Resolve into partial fractions

$$\frac{x^3 + 3x + 1}{(1 - x)^4}.$$

In cases like this where the denominator contains a power of a single factor (called a *repeating factor*), we employ a very useful artifice or dodge which simplifies our operations very considerably.

We say, let $1 - x = z$

$$\therefore \quad x = 1 - z.$$

Hence
$$\frac{x^3 + 3x + 1}{(1-x)^4} = \frac{(1-z)^3 + 3(1-z) + 1}{z^4}$$
$$= \frac{1 - 3z + 3z^2 - z^3 + 3 - 3z + 1}{z^4}$$
$$= \frac{5 - 6z + 3z^2 - z^3}{z^4}$$
$$= \frac{5}{z^4} - \frac{6}{z^3} + \frac{3}{z^2} - \frac{1}{z}$$
$$= \frac{5}{(1-x)^4} - \frac{6}{(1-x)^3} + \frac{3}{(1-x)^2} - \frac{1}{1-x}.$$

EXERCISES.

(1) Resolve into partial fractions

$$\frac{4x^2 + 2x - 14}{x^3 + 3x^2 - x - 3}$$

$$\left[\text{Answer, } \frac{3}{x+1} - \frac{1}{x-1} + \frac{2}{x+3}.\right]$$

(2) Resolve into partial fractions

$$\frac{3x^3 - 5x^2 + 4}{(x-1)^3(x^2+1)}$$

$$\left[\text{Answer, } \frac{1}{(x-1)^3} - \frac{3}{2(x-1)^2} + \frac{3}{(x-1)} + \frac{3(1-2x)}{2(x^2+1)}.\right]$$

SURDS.

Definition.—A surd is the root of an exact number which cannot be exactly determined, and which cannot therefore be expressed by an integer or by a finite fraction.

E.g., $\sqrt[2]{2}$ or $\sqrt[3]{5}$, etc.

Surds frequently occur in mathematical work, and in order to reduce the labour of calculation to a minimum, certain artifices may have to be employed.

E.g., $$\frac{\sqrt{5}-1}{\sqrt{5}+1}.$$

If we had to find the value of this expression by first extracting the square root of 5, which is 2·2361, and then performing the various operations, the work would be laborious, thus :

$$\frac{2{\cdot}2361 - 1}{2{\cdot}2361 + 1} = \frac{1{\cdot}2361}{3{\cdot}2361}$$
$$= 0{\cdot}38195.$$

But by remembering that $(a + b)\ (a - b) = a^2 - b^2$, we can simplify our work very greatly by **rationalising the denominator** as follows :

$$\frac{\sqrt{5}-1}{\sqrt{5}+1} = \frac{(\sqrt{5}-1)(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{(\sqrt{5}-1)^2}{(\sqrt{5})^2 - 1}$$
$$= \frac{5 - 2\sqrt{5} + 1}{5 - 1} = \frac{6 - 2\sqrt{5}}{4}$$
$$= \frac{3 - \sqrt{5}}{2}$$
$$= \frac{3 - 2{\cdot}2361}{2}$$
$$= \frac{{\cdot}7639}{2} = 0{\cdot}38195.$$

Leaving out the unessential steps the work would appear as follows :

$$\frac{\sqrt{5}-1}{\sqrt{5}+1} = \frac{(\sqrt{5}-1)^2}{4} = \frac{3-\sqrt{5}}{2}$$
$$= \frac{{\cdot}7639}{2} = 0{\cdot}38195.$$

EXERCISE.

Find the value of $\dfrac{1}{\sqrt{6} - \sqrt{2}}$ given $\sqrt{2} = 1{\cdot}4142$ and $\sqrt{3} = 1{\cdot}7321$.

[Answer, 0·966.]

Irrational Numbers.—All surds, as well as incommensurable numbers such as π, or e (p. 72), which cannot be expressed in the form of an integer or simple fraction, are called irrational numbers.

Imaginary Quantities.—Since the square of every number, whether positive or negative, is always positive, *e.g.*, $(+a)^2 = +a^2$, and $(-a)^2 = +a^2$, therefore there is no real quantity known whose square is negative. Hence, $\sqrt{-1}$, or $\sqrt{-2}$, or $\sqrt{-a}$ can only be imaginary, since their squares produce negative quantities, thus:

$$(\sqrt{-1})^2 = -1;\ (\sqrt{-2})^2 = -2;\ (\sqrt{-a})^2 = -a.$$

Definition.—An imaginary quantity is the square root of a negative quantity.

Such imaginary quantities frequently occur in mathematical analysis, and it is therefore necessary to say a few words about them here.

Notation.—$\sqrt{-1}$ is indicated by the letter i.

Therefore $\sqrt{-2} = i\sqrt{2}$.

$$\sqrt{-3} = i\sqrt{3}.$$

$$\sqrt{-4} = i\sqrt{-4} = 2i.$$

$$\ldots\ldots = \ldots\ldots$$

$$\sqrt{-a} = ia.$$

Properties of Imaginary Numbers.—

(1) **Powers.**—(i.) $i^1 = i$; (ii.) $i^2 = -1$;

(iii.) $i^3 = i^2 \times i = -1 \times i = -i$;

(iv.) $i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$.

After this the results recur, viz., $i^5 = i$; $i^6 = -1$, and so on.

(2) If $a + bi = c + di$, then $a = c$ and $b = d$. For $(a - c) = (d - b)i$, and, therefore, unless $a = c$ and $b = d$, when each side $= 0$, we shall have $(a - c)$ which is real, equal to $(d - b)i$ which is imaginary.

EQUATIONS AND THEIR SOLUTION.

The *degree* of an equation is determined by the index of the highest power of the unknown quantity in the equation.

Thus $ax + b$ is an equation of the first degree (or simple).

$ax^2 + bx + c$ is an equation of the second degree (or *quadratic*).

$ax^3 + bx^2 + cx + d$ is an equation of the third degree (or *cubic*).

. . . .

$ax^n + bx^{n-1} + cx^{n-2} + \ldots + lx + m$ is an equation of the nth degree.

Simple Equations.—It is assumed that the reader is familiar with the method of solving a simple equation with one unknown, such as $ax + b = 0$, when $x = -\frac{b}{a}$.

Quadratic Equations.—The solution of quadratic equations is very important. There are **three methods** of dealing with such equations, viz. :

(*a*) **Factorisation.**—*E.g.*, if $2x^2 - 5x - 3 = 0$, then by factorisation $(2x + 1)(x - 3) = 0$.

$$\therefore \text{ either } 2x + 1 = 0, \text{ giving } x = -\tfrac{1}{2},$$
$$\text{or } \quad x - 3 = 0, \text{ giving } x = 3.$$

(*b*) **Application of the Identity** $(x + \text{A})^2 = x^2 + 2\text{A}x + \text{A}^2$ (p. 23). If the factors cannot be easily detected, then we proceed to convert the equation into the form $(x + \text{A})^2$ as follows :

Let equation be $\quad ax^2 + bx + c = 0.$

$$\therefore \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

The expression is divided throughout by **a** in order to make the coefficient of x^2 unity.

Now transfer the last term to the other side, thus :

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

If now we add to each side the square of half the coefficient of x, *i.e.*, $\left(\frac{b}{2a}\right)^2$ or $\frac{b^2}{4a^2}$, then the left-hand side will become a perfect square (see note, p. 23),

thus $$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

or $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

$$\therefore \quad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}.$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Thus in the example given

$$2x^2 - 5x - 3 = 0$$
$$a = 2, b = -5, \text{C} = -3.$$

$$\therefore \quad x = \frac{5 \pm \sqrt{25 + 24}}{4} = \frac{5 + 7}{4} \text{ or } \frac{5 - 7}{4}$$
$$= 3 \text{ or } -\tfrac{1}{2}.$$

The expression $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for the roots of a quadratic equation lead to the following conclusions :

(1) If $b^2 = 4ac$, then the expression under the square root sign vanishes, and therefore $x = \frac{-b \pm 0}{2a}$, *i.e.*, the two roots are equal.

(*E.g.*, in equation $4x^2 + 12x + 9$,

$$x = \frac{-12 \pm \sqrt{144 - 36 \cdot 4}}{8} = \frac{-3}{2},$$

i.e., the two roots are each equal to $\frac{-3}{2}$.)

(2) If $b^2 > 4ac$, then the expression under the square root sign is positive, and therefore the equation has two *real* roots.

(*E.g.*, in equation $2x^2 + 5x + 3 = 0$,

$$x = \frac{-5 \pm \sqrt{25 - 24}}{4} = -\frac{5 \pm 1}{4}$$

$$= \frac{-3}{2} \text{ or } -1.)$$

(3) If $b^2 < 4ac$, then the expression under the square root sign is negative, and therefore the equation has only imaginary roots.
(*E.g.*, in equation $7x^2 + 5x + 1 = 0$,

$$x = \frac{-5 \pm \sqrt{25 - 28}}{14} = \frac{-5 \pm \sqrt{-3}}{14})$$

$$= \frac{-5 \pm i\sqrt{3}}{14}.$$

(*c*) **Graphical Methods** (see p. 117).

Higher Equations.—All equations higher than quadratic are best solved graphically (see p. 117), unless it is possible to deal with them by the method of factorisation.

Example.—In a problem on Biochemistry (see p. 223), the following equation occurs :

$$\frac{n^2}{(1 - n)(A - n)} = 4.$$

Find the value of n in terms of A.
Multiplying out we get

$$n^2 = 4A - 4An - 4n + 4n^2$$

or $$3n^2 - 4n(A + 1) + 4A = 0.$$

This is a quadratic equation in n similar in form to $ax^2 + bx + c$, except that the coefficients are different, viz. :

$$a = 3\ ;\ b = -4(A + 1)\ ;\ c = 4A.$$

$$\therefore \quad n = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ab})$$

$$= \frac{1}{6}[4(A + 1) \pm \sqrt{16(A + 1)^2 - 48A}]$$

$$= \frac{1}{6}[4(A + 1) \pm 4\sqrt{A^2 + 2A + 1 - 3A}]$$

$$= \frac{1}{3}[2(A + 1) \pm 2\sqrt{A^2 - A + 1}]$$

$$= \frac{2}{3}(A + 1 \pm \sqrt{A^2 - A + 1}).$$

Simultaneous Equations, or equations with more than one unknown quantity.

The following example will illustrate a method of dealing with such equations :

Solve $\quad 12x + 11y = 12 \quad . \ . \ . \ . \ . \ . \ . \quad$ (i.)

$\quad 42x + 22y = 40{\cdot}5 \quad . \ . \ . \ . \ . \ . \ . \quad$ (ii.)

Multiply the 1st equation throughout by 2, thus:

$$24x + 22y = 24.$$

Subtract this from (ii.) as follows :

$$\begin{array}{l} 42x + 22y = 40{\cdot}5 \\ 24x + 22y = 24 \\ \hline 18x \qquad\quad = 16{\cdot}5 \end{array}$$

$$\therefore \quad x = \frac{16{\cdot}5}{18} = \frac{11}{12}.$$

If now we substitute this value of x in either of the original equations (*e.g.*, in (i.)), we get

$$12 \times \frac{11}{12} + 11y = 12$$

or

$$11 + 11y = 12.$$

$$\therefore \quad 11y = 1.$$

$$\therefore \quad y = \frac{1}{11}.$$

Hence the solution is $x = \frac{1}{12}, y = \frac{1}{11}$

[Generally, if $ax + by + c = 0$ and $a'x + b'y + c' = 0$,

then $\quad x = \frac{b'c - bc'}{a'b - ab'}$ and $y = \frac{ac' - a'c}{a'b - ab'}$]

Simultaneous equations can also be solved graphically (see p. 117).

Simultaneous equations occur frequently in the analysis of curves.

Example.—How much bread, cheese and butter are required to supply 100 grm. of protein, 100 grm. of fat and 500 grm. of carbohydrate? The following being, in round numbers, the percentage compositions of those articles of diet.

	Protein.	Fat.	Carbohydrate.
Bread	8	1	50
Cheese	30	20	1
Butter	2	80	1

Let there be required

x grm. of bread,
y grm. of cheese,
z grm. of butter.

We then have the following equations:

$$\text{Protein} = 0{\cdot}08x + 0{\cdot}3\,y + 0{\cdot}02z = 100 \text{ grm.} \quad \text{(i.)}$$
$$\text{Fat} = 0{\cdot}01x + 0{\cdot}2\,y + 0{\cdot}80z = 100 \text{ grm.} \quad \text{(ii.)}$$
$$\text{Carbohydrate} = 0{\cdot}5x + 0{\cdot}01y + 0{\cdot}01z = 500 \text{ grm.} \quad \text{(iii.)}$$

Multiplying (ii.) by 8 we get

$$0{\cdot}08x + 1{\cdot}6y + 6{\cdot}4\,z = 800$$

Subtract (i.) $\quad 0{\cdot}08x + 0{\cdot}3y + 0{\cdot}02z = 100$

and get $\quad 1{\cdot}3y + 6{\cdot}38z = 700 \quad (a)$

Similarly (iii.) $\quad 0{\cdot}5x + 0{\cdot}01y + 0{\cdot}01z = 500$

(ii.) $\times$ 50 $\quad 0{\cdot}5x + 10\,y + 40\,z = 5000$

By subtraction $\quad 9{\cdot}99y + 39{\cdot}99z = 4500$

or in round numbers $\quad 10y + 40\,z = 4500$

i.e., $\quad y + 4z = 450 \quad (b)$

Now from (a) we have

$$1{\cdot}3y + 6{\cdot}38z = 700$$

Subtract $(b) \times 1{\cdot}3$ $\quad 1{\cdot}3y + 5{\cdot}2\,z = 585$

$$1{\cdot}18z = 115$$
$$\therefore \quad z = 97{\cdot}5 \text{ grm.}$$

Substituting this value of z in (b) we get

$$y + 390 = 450.$$
$$\therefore \quad y = 60 \text{ grm.}$$

Substituting these values of y and z in (iii.) we get

$$0{\cdot}5x + {\cdot}6 + {\cdot}98 = 500$$
$$0{\cdot}5x = 500 - 1{\cdot}6 = 498{\cdot}4.$$
$$\therefore \quad x = 996{\cdot}8 \text{ grm.}$$

$\therefore$ In round numbers there are required

1,000 grm. or 2 lb. 3 oz. of bread,
60 grm. or 2·2 oz. of cheese,
and 100 grm. or 3·5 oz. of butter.

EXERCISES.

(1) How much cane sugar ($C_{12}H_{22}O_{11}$) and dry albumen (containing 15 per cent. N and 53 per cent. C) are required in a mixed dietary to furnish 30 grm. N, and 350 grm. C. ?

[Answer. If x = amount of sugar and y = amount of albumen we have $0{\cdot}15y = 30$. $\therefore y = 200$. Also, formula for sugar shows it to contain 42 per cent. carbon. $\therefore \cdot 42x + \cdot 53y = 350$ or $\cdot 42x + 106 = 350$. $\therefore x = \frac{244}{42} = 581$. Hence 581 grm. of sugar and 200 grm. of albumen are required.]

(2) Find the amount of bread, meat, suet, butter and potatoes required by a man doing an ordinary amount of work (*i.e.*, to supply 4 oz. of protein, 3 oz. of fat, 16 oz. of carbohydrate), given the following composition :

	Protein.	Fat.	Carbohydrates.	Salts.
Beef	15	8·4	—	1·6
Bread	8	1	50	1·5
Salt butter	—	80	—	3·0
Potatoes	2	0·2	21·84	1·0

[Answer. 1 lb. beef, 5·2 oz. bread, 1·42 oz. butter, 3·8 lb. potatoes.]

CHAPTER IV.

A FEW POINTS IN ELEMENTARY TRIGONOMETRY.

Trigonometry has to do with the relations between the sides of a right-angled triangle, the other angles of which are known.

The student will notice that if one of the other two angles of a right-angled triangle is known, then the remaining angle is also known.

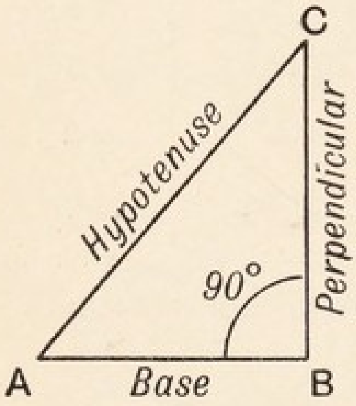

FIG. 4.—Right-angled Triangle.

For in any right-angled triangle ABC (Fig. 4), if $\angle$ B = 90°, then $\angle$s A + B also = 90°.

(Since the sum of the three angles = 180°)

$\therefore$ if, say, angle A is known, then B is also known, since $\angle$ B = 90° − A.

Thus if $\angle$ A = 30°, then $\angle$ B = 90° − 30° = 60°.

The Trigonometrical Ratios.—Supposing we fix our attention on the angle A in the right-angled triangle ABC, then

(i.) the side BC, which is opposite the angle A, is called the *perpendicular*;

(ii.) the side AC which is opposite the right angle is called the *hypotenuse*;

(iii.) the third side AB which is adjacent to the right angle and the angle A is called the *base*.

From these three sides we can form six different ratios as follows:

(i.) $\frac{BC}{AC}$ or $\frac{\text{perpendicular}}{\text{hypotenuse}}$ is called the *sine* of the angle BAC or A, and is written sin A.

(ii.) $\frac{AB}{AC}$ or $\frac{\text{base}}{\text{hypotenuse}}$ is called the *cosine* of the angle BAC or A, and is written cos A.

(iii.) $\frac{BC}{AB}$ or $\frac{\text{perpendicular}}{\text{base}}$ is called the *tangent* of the angle BAC or A, and is written tan A.

(iv.) $\frac{AC}{BC}$ or $\frac{\text{hypotenuse}}{\text{perpendicular}}$ is called the *cosecant* of the angle BAC or A, and is written cosec A.

(v.) $\frac{AC}{AB}$ or $\frac{\text{hypotenuse}}{\text{base}}$ is called the *secant* of the angle BAC or A, and is written sec A.

(vi.) $\frac{AB}{BC}$ or $\frac{\text{base}}{\text{perpendicular}}$ is called the *cotagent* of the angle BAC or A, and is written cot A.

The important ratios to remember are the first three, viz., sin A, cos A and tan A. The other three ratios are formed by inverting or taking the reciprocals of the first three. Thus,

$$\text{cosec A} = \frac{1}{\sin A}; \ \sec A = \frac{1}{\cos A}; \ \cot A = \frac{1}{\tan A}.$$

The Powers of Trigonometrical Ratios.—If we have to write the square, cube, or any other power of any of these ratios, *e.g.*, the square of sin A, we cannot write sin A^2 because this would mean the sine of the angle A^2. Hence it would be necessary to write it as $(\sin A)^2$, involving the use of brackets. A simpler way of writing it is $\sin^2 A$. We therefore write :

$\text{Sin}^2 A$ instead of $(\sin A)^2$, or $\sin^n A$ instead of $(\sin A)^n$,
$\text{Cos}^2 A$ instead of $(\cos A)^2$, or $\cos^n A$ instead of $(\cos A)^n$,
$\text{Tan}^2 A$ instead of $(\tan A)^2$, or $\tan^n A$ instead of $(\tan A)^n$,
and so on.

Note.—It is very important that the student should *distinguish* between the meanings of the terms *geometrical tangent*, which means the line touching a curve at one point only, and the *trigonometrical tangent* of an angle which, as we have just seen, means the ratio $\frac{\text{perpendicular}}{\text{base}}$ of the right-angled triangle of which the angle is the angle at the base (and opposite the perpendicular).

The following relations between the various ratios are most important :

(1) $Sin^2 A + cos^2 A = 1$, whatever the angle A may be. This is almost obvious from Fig. 4.

For $$\sin^2 A = \frac{BC^2}{AC^2} \text{ and } \cos^2 A = \frac{AB^2}{AC^2}.$$

$$\therefore \quad \sin^2 A + \cos^2 A = \frac{BC^2 + AB^2}{AC^2} = \frac{AC^2}{AC^2} = 1. \qquad \text{(Q.E.D.)}$$

Corollaries.—*Sin² A* $= 1 - \cos^2 A$; $\cos^2 A = 1 - \sin^2 A$.

(ii.) $Tan\,A = \dfrac{sin\,A}{cos\,A}$.

For $$\sin A = \frac{BC}{AC} \text{ and } \cos A = \frac{AB}{AC}.$$

$$\therefore \quad \frac{\sin A}{\cos A} = BC \div AB = \frac{BC}{AB} = \tan A. \quad \text{(Q.E.D.)}$$

(iii.) Similarly, $cot\,A = \dfrac{cos\,A}{sin\,A}$.

(iv.) $Sec^2 A = 1 + tan^2 A$.

For $$\sec^2 A = \frac{1}{\cos^2 A} = \frac{\sin^2 A + \cos^2 A}{\cos^2 A} \text{ (see (i.)),}$$

$$= \frac{\sin^2 A}{\cos^2 A} + \frac{\cos^2 A}{\cos^2 A}$$

$$= \tan^2 A + 1. \qquad \text{(Q.E.D.)}$$

(v.) Similarly $cosec^2 A = 1 + cot^2 A$.

Inverse Ratios.—When we wish to say that A is an angle whose sine, cosine or tangent, etc., is equal to m, we write as follows :

$A = \sin^{-1} m$; or $A = \cos^{-1} m$; or $A = \tan^{-1} m$, etc.,
or $A = \text{arc} \sin m$; $A = \text{arc} \cos m$; $A = \text{arc} \tan m$, etc.

Use of Trigonometrical identities for Simplification of Arithmetical Operations.—Familiarity with a couple of trigonometrical identities occasionally affords one a ready means for greatly reducing the labour involved in some complicated arithmetical operations. The identities which most readily lend themselves to such use are

$$\left.\begin{array}{l}(1)\ \sin^2\theta = 1 - \cos^2\theta \\ (2)\ \sec^2\theta = 1 + \tan^2\theta\cdot\end{array}\right\} \text{ (See also p. 52.)}$$

By means of the first of these identities a difference of two squares is reduced to one square number, and by means of the second, a sum of two squares is reduced to one square number.

E.g., find the value of $\dfrac{\sqrt{(1+x)} + \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}}$, when $x = \cdot 6202$.

Rationalising the denominator by multiplying the numerator and denominator by $\sqrt{1+x} + \sqrt{1-x}$ (see p. 29), we get

$$\frac{(\sqrt{1+x}+\sqrt{1-x})^2}{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}$$
$$=\frac{(1+x)+2\sqrt{1-x^2}+(1-x)}{(1+x)-(1-x)}$$
$$=\frac{2+2\sqrt{1-x^2}}{2x}$$
$$=\frac{1+\sqrt{1-x^2}}{x}$$
$$=\frac{1+\sqrt{1-(\cdot6202)^2}}{\cdot6202}$$

Now, bearing in mind the identity $\sin^2\theta = 1 - \cos^2\theta$, we can put $\cdot6202 = \cos\theta$,

and then $\sqrt{1-(\cdot6202)^2}$ becomes $\sqrt{1-\cos^2\theta} = \sin\theta$.

Now, by referring to a table of natural cosines we find that an angle whose cosine is $\cdot6202$ is $51^\circ\ 40'$.

$\therefore\ \sin\theta = \sin 51^\circ 40$, which is found from the table of natural sines to be $\cdot7844$.

$\therefore$ our expression becomes

$$\frac{1+\cdot7844}{\cdot6202}=\frac{1\cdot7844}{0\cdot6202}=2\cdot8771.$$

Example on the use of the identity $\sec^2\theta = 1 + \tan^2\theta$

Find the value of $(11\cdot78^2 + 5\cdot67^2)^{-\frac{1}{2}}$.

$$11\cdot78^2+5\cdot67^2 = 11\cdot78^2\left(1+\frac{5\cdot67^2}{11\cdot78^2}\right)$$
$$= 11\cdot78^2(1+\tan^2\theta)\ \left(\text{if we put } \frac{5\cdot67}{11\cdot78}=\tan\theta\right)$$
$$= 11\cdot78^2\sec^2\theta.$$

$$\therefore\quad (11\cdot78^2+5\cdot67^2)^{-\frac{1}{2}} = (11\cdot78^2\sec^2\theta)^{-\frac{1}{2}}$$
$$=\frac{1}{(11\cdot78^2\sec^2\theta)^{\frac{1}{2}}}$$
$$=\frac{1}{11\cdot78\sec\theta}$$
$$=\frac{\cos\theta}{11\cdot78}.$$

Now $\dfrac{5\cdot67}{11\cdot78} = 0\cdot4812.$

$\therefore\quad \tan\theta = 0\cdot4812.\quad \therefore\quad \theta = 25^\circ\ 42'$ (from the tables).

$\therefore\quad \cos\theta = \cos 25^\circ 42 = 0\cdot9011.$

$\therefore\quad \dfrac{\cos\theta}{11\cdot78} = \dfrac{\cdot9011}{11\cdot78} = \cdot076.$

$\therefore\quad (11\cdot78^2+5\cdot67^2)^{-\frac{1}{2}} = \cdot076.$

EXAMPLES.

(1) Prove that

$$\frac{\tan A + \cot A}{\tan A - \cot A} = \frac{1}{1 - 2\cos^2 A}$$

$$\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$= \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} = \frac{1}{\sin A \cos A}$$

$$\tan A - \cot A = \frac{\sin A}{\cos A} - \frac{\cos A}{\sin A}$$

$$= \frac{\sin^2 A - \cos^2 A}{\sin A \cos A} = \frac{\sin^2 A + \cos^2 A - 2\cos^2 A}{\sin A \cos A}$$

$$= \frac{1 - 2\cos^2 A}{\sin A \cos A}$$

$$\therefore \frac{\tan A + \cot A}{\tan A - \cot A} = \frac{1}{\sin A \cos A} \div \frac{1 - 2\cos^2 A}{\sin A \cos A}$$

$$= \frac{1}{1 - 2\cos^2 A}. \qquad \text{(Q.E.D.)}$$

(2) Prove that $\cos^4 A - \sin^4 A = 2\cos^2 A - 1$

$$(\cos^4 A - \sin^4 A) = (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$$
$$= 1 \times (\cos^2 A - \sin^2 A)$$
$$= 2\cos^2 A - \sin^2 A - \cos^2 A$$
$$= 2\cos^2 A - (\sin^2 A + \cos^2 A)$$
$$= 2\cos^2 A - 1. \qquad \text{(Q.E.D.)}$$

EXERCISES.

Prove the following identities :

(1) $\tan A \,.\, \operatorname{cosec} A = \sec A$

(2) $\sin A = \dfrac{\tan A}{\sqrt{1 + \tan^2 A}}$.

On the Measurement of Angles.—In the same way as logarithms are used to one of two bases, viz., 10 or ϵ ($= 2{\cdot}71828\ldots$), so are angles measured in one of two ways, viz. :

(i.) The rectangular measure — with the *right angle* as the unit.

(ii.) The circular measure — with the *radian* as the unit.

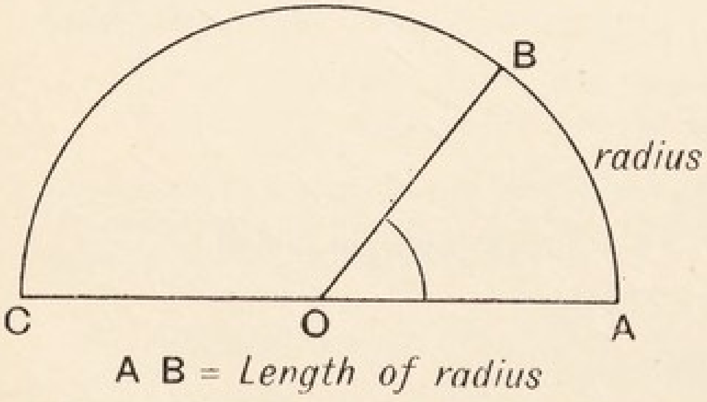

Fig. 5.—The Radian or Circular Unit of an Angle.

Definition of Radian.—If O is the centre of the semicircle ABC (Fig. 5), and the arc AB is equal in length to the radius OA or OB, then the angle AOB is called a *radian*.

Whilst the rectangular measure is used in all numerical trigonometrical calculations, the circular measure is used in all theoretical trigonometrical analysis.

The Divisions of a Right Angle.—Each right angle contains 90 degrees or 90°.

Each degree contains 60 minutes or 60′.

Each minute contains 60 seconds or 60″.

An angle 68° 15′ 36·7″ means an angle containing 68 degrees, 15 minutes and 36·7 seconds.

The reason why the circular measure is used in higher trigonometry is because the various trigonometrical ratios can be calculated by using the radian as the unit—in the same way as the various logarithms can be calculated by using ϵ as the base. Thus we shall see that

$$\sin\theta \text{ (where } \theta \text{ is expressed in radians)} = \theta - \frac{\theta^3}{1.2.3} + \frac{\theta^5}{1.2.3.4.5} - \dots$$

$$\cos\theta = 1 - \frac{\theta^2}{1.2} + \frac{\theta^4}{1.2.3.4} - \frac{\theta^6}{1.2.3.4.5.6} + \dots$$

$$\tan\theta = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \frac{\theta^7}{7} + \dots$$ (see p. 190).

To convert an Angle from one Measurement into Another.—The whole length of the circumference of a circle $= 2\pi r$. Where r = length of radius and $\pi = 3{\cdot}14159$ or $\frac{22}{7}$.

But a radian is an arc whose length = length of radius.

∴ The whole length of the circumference $= 2\pi$ radians.

Also the whole circumference contains four right angles = 360°.

$$\therefore \quad 2\pi \text{ radians} = 360°.$$

$$\therefore \quad \text{Radian} = \frac{360°}{2\pi} = \frac{180°}{\pi} = 180 \times \frac{7}{22}$$
$$= 57{\cdot}296°.$$
$$= 57°\,17'\,45{\cdot}6''$$

Conversely, $\quad$ 4 right angles $= 2\pi$ radians

$$\therefore \quad \text{A right angle} = \frac{\pi}{2} \text{ radians.}$$

and $$1° = \frac{\pi}{180} \text{ radians.}$$

Hence, by calculating a table of trigonometrical ratios by means of the formulæ given above, one can easily convert them into the rectangular measure by multiplying θ in the series by $\frac{\pi}{180}$, in the same way as one can reconvert Napierian into common logarithms by multiplying by the modulus.

Angular Velocity.—If a wheel makes 75 turns per minute, this means that it makes 1·25 turns per second. This again means that any point on the circumference moves through an angle of $360° \times 1·25 = 450°$ in one second But $360° = 6·282$ radians, therefore $450°$ or $360° \times \frac{5}{4} = 6·282 \times \frac{5}{4} = 7·8525$ radians. Hence we say that the **angular velocity,** *i.e.*, **the number of radians described by the point per unit of time** is 7·8525 radians per second.

Simple Harmonic Motion.—Let a particle P starting from A move uniformly round the circumference of a circle, of radius r, in the direction indicated by the arrow (*i.e.*, anticlockwise)

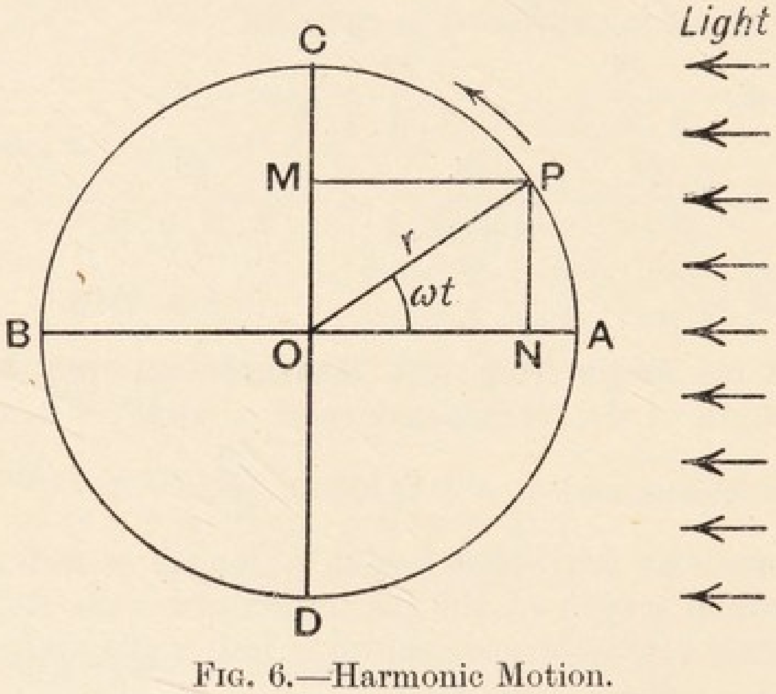

Fig. 6.—Harmonic Motion.

(Fig. 6). Imagine also that while P is moving on the right semicircumference it is being illuminated by parallel rays of light falling upon it from the right, and when moving on the left semicircumference it is illuminated by parallel light falling upon it from the left. A shadow of P will then be thrown upon the diameter CD at M, and with each change of position of P upon the circumference there will be a corresponding change of position of M upon the diameter. Similarly, if the shadow of P be projected upon the diameter AB at N, there will be a corresponding change in the position of N with each movement of P. The motion of M or N along their respective diameters is a simple harmonic motion.

Definition.—*A simple harmonic motion (S.H.M.) is the motion of the projection upon a diameter of a point moving uniformly in a circle.*

Let ω = angular velocity of P.

Then angle AOP $= \omega t$ (where t = time taken for P to move from A to P). And the distances of M and N from the centre O, will be given by

$$\text{OM} = y = r \sin \omega t$$

and

$$\text{ON} = x = r \cos \omega t.$$

The maximum distance of either projection from the centre is called the *amplitude* and is equal to r.

The *period* or *periodic time* (T) of the S.H.M. is the time taken by N to pass from A to B and back again. $\text{T} = \frac{2\pi}{\omega}$.

The *frequency* (f) of the vibration is the reciprocal of the periodic time so that $f = \frac{1}{\text{T}} = \frac{\omega}{2\pi}$.

The Trigonometrical Ratios of Certain Angles.—In the same way as logarithm tables give the logarithms of all the numbers from 1 to 100,000, so do trigonometrical tables give the values of the sines, cosines, tangents, etc., of all the angles from 1° to 45°. The method of finding these values does not concern us here. There are, however, a few angles the trigonometrical ratios of which can be easily found.

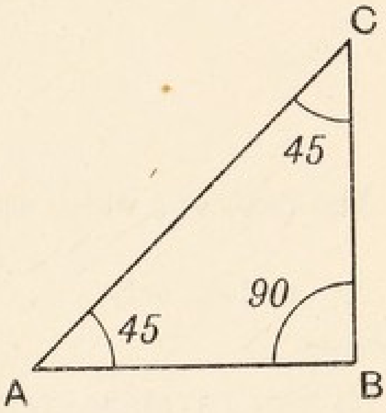

Fig. 7.—Isosceles right-angled triangle.

To find sin 45°; **cos** 45°; **tan** 45° (Fig. 7).

If $\quad \text{A} = 45°$ then C must $= 45°$.

$\therefore \quad \text{AB} = \text{BC}$.

But $\quad \text{AC}^2 = \text{AB}^2 + \text{BC}^2$

$\qquad = 2\text{AB}^2$ or 2BC^2.

$\therefore \quad \text{AC} = \text{AB}\sqrt{2}$ or $\text{BC}\sqrt{2}$.

$$\therefore \quad \sin \text{A which} = \frac{\text{BC}}{\text{AC}} = \frac{\text{BC}}{\text{BC}\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{(\sqrt{2})^2} = \frac{\sqrt{2}}{2},$$

$$\cos \text{A} \quad ,, \quad = \frac{\text{AB}}{\text{AC}} = \frac{\text{AB}}{\text{AB}\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan \text{A} \quad ,, \quad = \frac{\text{BC}}{\text{AB}} = 1.$$

$$\therefore \quad \sin 45° = \frac{\sqrt{2}}{2}; \ \cos 45° = \frac{\sqrt{2}}{2}; \ \tan 45° = 1.$$

Similarly it can be easily proved that

$$\sin 60 = \frac{\sqrt{3}}{2}; \ \cos 60 = \frac{1}{2}; \ \tan 60 = \sqrt{3};$$

$$\sin 30 = \frac{1}{2}; \ \cos 30 = \frac{\sqrt{3}}{2}; \ \tan 30 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Also, $\sin 90 = 1; \ \cos 90 = 0; \ \tan 90 = \infty,$
$\sin 0 = 0; \ \cos 0 = 1; \ \tan 0 = 0.$

EXAMPLES.

(1) Find θ from the equation $4 \sin \theta = \operatorname{cosec} \theta$.
(assuming that θ is less than a right angle.)

$$\text{Cosec } \theta = \frac{1}{\sin \theta}.$$

$$\therefore \quad 4 \sin \theta = \frac{1}{\sin \theta},$$

i.e., $4 \sin^2 \theta = 1.$

$$\therefore \quad 2 \sin \theta = 1.$$

$$\therefore \quad \sin \theta = \pm \frac{1}{2},$$

The value of θ which makes $\sin \theta = -\frac{1}{2}$ is greater than a right angle.

$\therefore$ we take only $\sin \theta = \frac{1}{2}$.

whence $\theta = 30°$.

(2) Find θ from the equation $\tan \theta = 3 \cot \theta$.
(assuming that θ is less than a right angle.)

$$\text{Cot } \theta = \frac{1}{\tan \theta}.$$

$$\therefore \quad \tan \theta = \frac{3}{\tan \theta}.$$

$$\therefore \quad \tan^2 \theta = 3.$$

$$\therefore \quad \tan \theta = \pm \sqrt{3}.$$

Ignoring $\tan \theta = -\sqrt{3}$, we get $\tan \theta = \sqrt{3}.$

$$\therefore \quad \theta = 60°.$$

(3) Find θ from the equation $\cos^2 \theta + 2 \sin^2 \theta - \frac{5}{2} \sin \theta = 0.$

$$\text{Cos}^2 \theta = 1 - \sin^2 \theta.$$

$\therefore$ equation becomes

$$1 + \sin^2 \theta - \frac{5}{2} \sin \theta = 0$$

or

$$\sin^2 \theta - \frac{5}{2} \sin \theta + 1 = 0.$$

Making the left-hand side of the equation a complete square by adding $\frac{9}{16}$ (see p. 31, we get

$$\sin^2\theta - \frac{5}{2}\sin\theta + \frac{25}{16} = \frac{9}{16}$$

$$\text{or } \left(\sin\theta - \frac{5}{4}\right)^2 = \left(\frac{3}{4}\right)^2.$$

$$\therefore \qquad \sin\theta - \frac{5}{4} = \frac{\pm 3}{4}.$$

$$\therefore \quad \sin\theta = \frac{1}{2} \text{ or } 2, \text{ whence } \theta = 30^\circ.$$

The equation $\sin\theta = 2$ is an impossible one, since there can be no angle whose sine is greater than 1.

EXERCISES.

(1) Simplify $\cos^4 A + 2\sin^2 A \cos^2 A$.

[The expression $= (\cos^2 A + \sin^2 A^2) - \sin^4 A$

$= 1 - \sin^4 A$.]

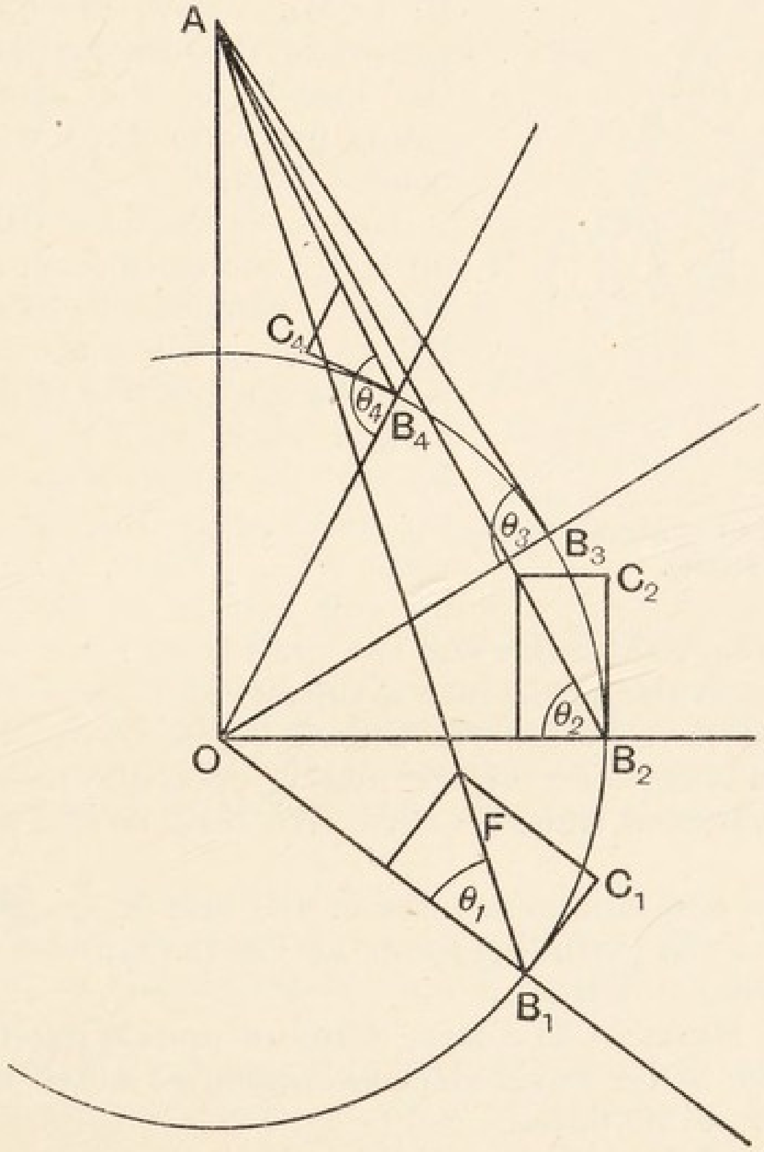

Fig. 8.—Alteration in Angle of Pull of Muscle during Contraction.

(2) Solve the equation $8 \cos^2 \theta - 8 \cos \theta + 1 = 0$.

$$\left[\text{Answer, } \cos\theta = \frac{2 \pm \sqrt{2}}{4}.\right]$$

Angle of Pull of Muscle.—The amount of work performed by a muscle depends upon a number of factors, viz. :

(1) The number of contracting fibres.
(2) Their arrangement.
(3) Their degree of contraction.
(4) The angle they make with the bone to be moved.

The angle of pull keeps on altering as the bone keeps on moving.

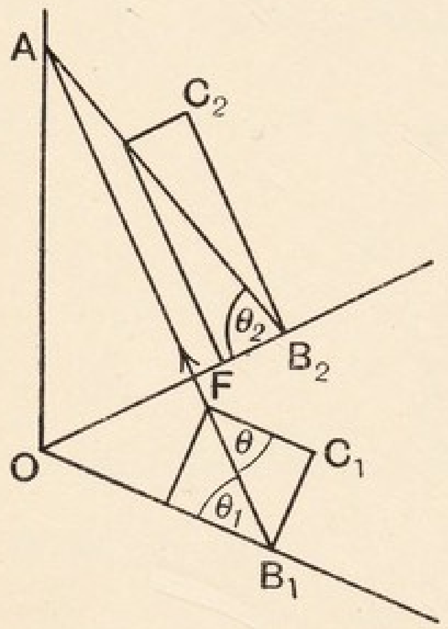

FIG. 9.—Showing increase in Value of the Effective Component BC with increase of Angle of Pull of Muscle (up to a right angle).

Thus, if A is the origin of a muscle, B is its insertion, and O the joint between the two bones (Figs. 8 and 9), it is seen that the angle of pull ABO keeps on altering as the bone OB, or the insertion B, moves upwards from B_1 to B_4. Now, the effective component of the contractile force F of the muscle is the component BC acting perpendicularly to the moving bone $= F \sin \theta$.

Hence, as the bone OB gets pulled up and θ increases from θ_1 to θ_2, the vertical component increases from BC_1 to BC_2. When the moving bone is in such a position OB_3 that θ_3 **is a right angle,** then the whole of the force of the muscle is spent in moving the bone, since B_3C_3 then coincides with AB_3. When OB gets pulled up still further to OB_4 θ_4 becomes greater than a right angle and the contractile force of the muscle becomes resolvable again into a vertical and horizontal force, so that the effective pull of the muscle begins to decrease.

The maximum force of the muscle upon the moving bone is therefore obtained when $\theta = 90$. We then have $F \sin \theta = F \sin 90 = F$.

Similarly, the minimum force of the muscle exerted upon the bone is when OB is parallel to the axis of the muscle θ is then $= 0$ and $F \sin \theta = F \sin 0 = 0$.

Force of Muscle.—The force which a muscle exerts in pulling its insertion to its origin depends, amongst other things, upon the direction of its fibres.

(*a*) **Direct Prismatic Muscle** (*e.g.*, Masseter, etc.).—In muscles of this type (Fig. 10), the fibres are all rectilinear, parallel to one another, and are attached at right angles to the lines of origin and insertion. (AB = origin, CD = insertion.)

Let f = contractile force of each fibre (of which $\frac{f}{2}$ may be considered to pull AB towards CD, and the other half, $\frac{f}{2}$, pulls CD towards AB).

Let n = number of fibres and F = total force of muscle.

Then clearly $F = nf$ (of which half, viz., $\frac{F}{2}$ or $\frac{nf}{2}$, acts in the

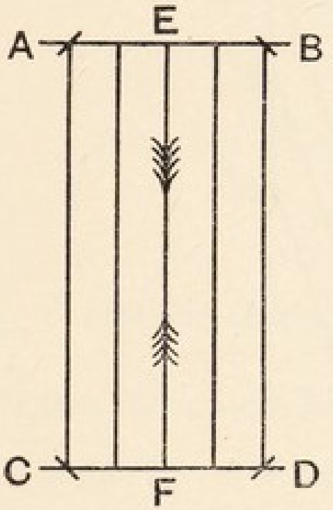

Fig. 10.—Arrangement of Fibres in Direct Prismatic Muscle.

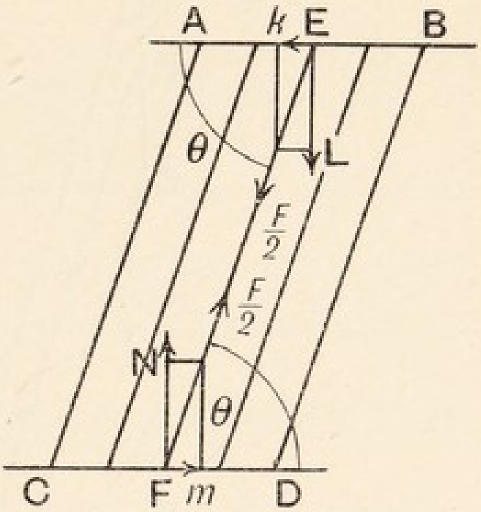

Fig. 11.—Arrangement of Fibres in Rhomboid Muscle.

direction towards CD and the other half, $\frac{F}{2}$ or $\frac{nf}{2}$, acts in the **direction** towards AB).

Also, the **line** of action of F will be where E and F are the middle points of AB and CD respectively.

(*b*) **Rhomboid Muscles** (*e.g.*, Intercostals, Rhomboids, etc.).—In this group of muscles the fibres are also rectilinear and parallel, but they are attached obliquely to the lines of origin and insertion (Fig. 11).

Here, again, $F = nf$, acting in the line EF. But as EF is oblique to AB and CD let its inclination to each of these lines be $= \theta$, and let each half of this resultant force F be split up into two components, viz.:

(1) EK and FM acting along the lines of origin and insertion but in opposite directions—tending to make the muscle prismatic.

(2) EL and FN acting perpendicularly to the lines of origin and insertion and pulling the insertion towards the origin.

$$\text{We then have EK} = \text{FM} = \frac{F}{2}\cos\theta$$

$$\text{EL} = \text{FN} = \frac{F}{2}\sin\theta.$$

Hence, total force tending to make muscle prismatic $= F\cos\theta$, and total force tending to pull the origin and insertion towards each other $= F\sin\theta$.

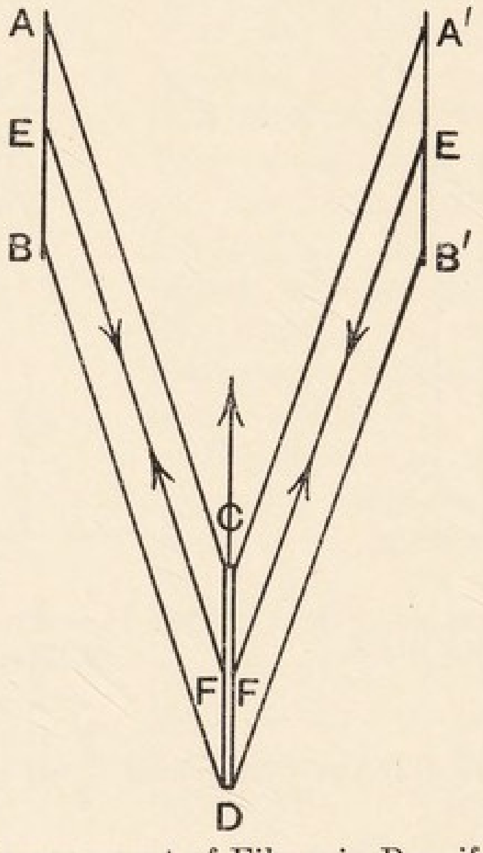

Fig. 12.—Arrangement of Fibres in Penniform Muscle.

(*c*) **Penniform Muscles** (*e.g.*, Mylohyoid, Accelerator Urinæ, etc.).—In muscles of this type the arrangement of the fibres is as shown in the diagram (Fig. 12).

AB and A′B′ are two lines of origin and CD is the line of insertion.

It will be seen, therefore, that such a muscle consists practically of two rhomboid arrangements symmetrically situated with regard to the line of insertion.

The total force pulling CD up in the direction of the arrow due to action of each half of the muscle $= F\cos\theta$ (see *b* (2), above).

$\therefore$ resultant force $R = 2F\cos\theta$.

(*d*) For **Fan-shaped muscles,** see p. 242.

EXAMPLES.

(1) The hand holds a weight of 10 lb. The forearm is inclined at 45° to the horizontal, and in this position the angle of pull of the biceps is 75°. What is the force with which the biceps must pull in order to hold the weight ?

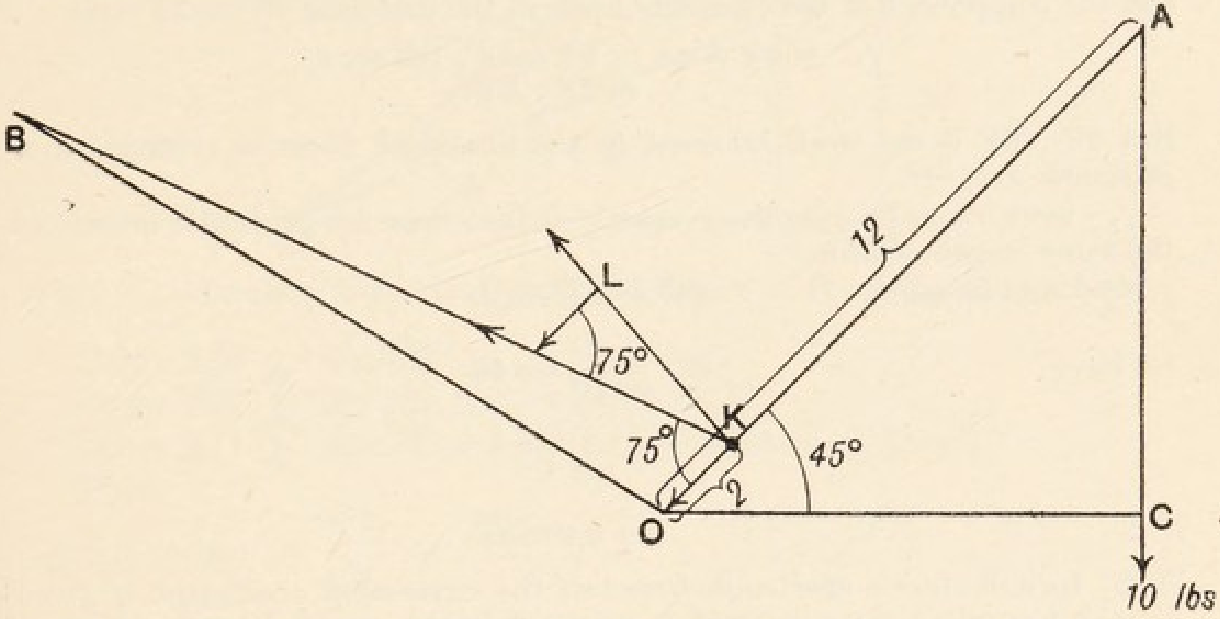

Fig. 13.—Calculation of Effective Component KL of Biceps Muscle BK.

The effective component KL of the force F of the muscle = F sin 75 (see Fig. 13).

∴ by the principle of levers

$$F \sin 75 \times OK = AC \times OC = 10 \times OA \cos 45$$
$$= 10 \times 12 \times \frac{\sqrt{2}}{2}$$
$$= 60\sqrt{2} = 84{\cdot}84 \text{ lbs.}$$
$$\therefore \quad F = \frac{84{\cdot}84}{OK \sin 75} = \frac{84{\cdot}84}{2 \sin 75} = \frac{42{\cdot}42}{0{\cdot}96593}$$
$$= 43{\cdot}9 \text{ lb.}$$

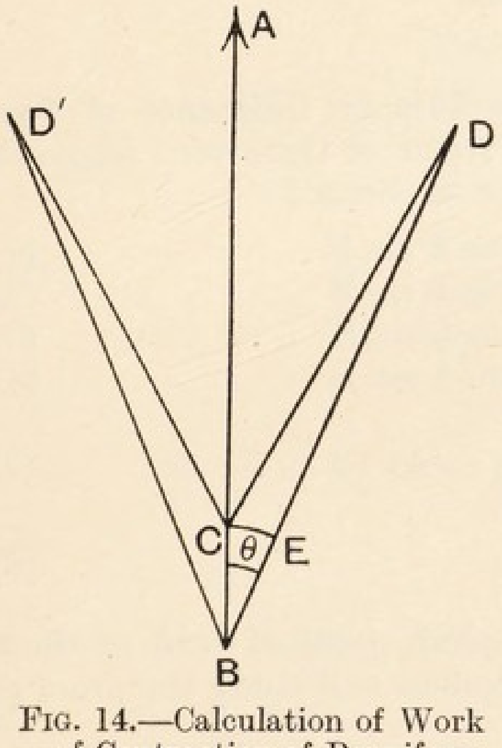

Fig. 14.—Calculation of Work of Contraction of Penniform Muscle.

(2)—(*a*) Find the work done by the contraction of a penniform muscle. (*b*) The angle made by the fibres of the mylohyoid with the central raphé is 45°. If the longest fibre contracts by $\frac{1}{10}$ inch, how far will the middle point of the hyoid bone be drawn up ?

(*a*) Work is measured by force multiplied by the distance through which it acts in its own direction.

Now, fixing our attention to the right side of the penniform muscle, let DB represent one fibre, which by contraction pulls the point B up to C. The fibre DB will therefore have contracted to DC.

Now, if CE is dropped perpendicular to DB (Fig. 14), then BE is equal to the amount of shortening of DB (since DE is very nearly = DC), and BC is the distance through which B has been moved.

But $$BC = \frac{BE}{\cos\theta} = BE \sec\theta,$$

and the component of the complete force in the direction BC = 2F cos θ.

$$\therefore \quad \text{work done} = 2F\cos\theta \,.\, BE\sec\theta$$
$$= 2F \,.\, BE.$$

But 2F . BE is the work inherent in the muscular fibres if arranged in a prismatic manner.

∴ work done by penniform muscle = that done by prismatic muscle of the same length of fibre.

(*b*) From formula $$BC = BE \sec\theta,$$

we have $$BC = \frac{1}{10}\sec 45$$
$$= \frac{1}{10}\sqrt{2}$$
$$= \cdot 14 \text{ inch.}$$

(3) In a Hüfner's spectrophotometer, the extinction coefficient is given by the formula $e = -\log\cos^2\phi$, where ϕ is the angle through which a Nichol's prism has to be rotated to restore equality of spectra. Find e, when $\phi = 61^\circ\ 52'$.

$$\text{Log}\cos^2\phi = 2\log\cos\phi = 2\log\cos 61^\circ\ 52'$$
$$= 2 \times (\bar{1}\cdot 6735) \text{ (from table)}$$
$$= 2 \times 0\cdot 6735 - 2$$
$$= 1\cdot 347 - 2$$
$$= -\cdot 653.$$
$$\therefore \quad e = -2\log\cos\phi = \cdot 653.$$

The Trigonometrical Ratios of the Sum or Difference of Two Angles.—The following identities in virtue of their very frequent occurrence in mathematical work, are important :

$$\text{Sin}(A + B) = \sin A\cos B + \cos A\sin B \quad . \quad . \quad . \quad (1)$$
$$\text{Sin}(A - B) = \sin A\cos B - \cos A\sin B \quad . \quad . \quad . \quad (2)$$
$$\text{Cos}(A + B) = \cos A\cos B - \sin A\sin B \quad . \quad . \quad . \quad (3)$$
$$\text{Cos}(A - B) = \cos A\cos B + \sin A\sin B \quad . \quad . \quad . \quad (4)$$
$$\text{Tan}(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \quad . \quad . \quad . \quad (5)$$
$$\text{Tan}(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} \quad . \quad . \quad . \quad (6)$$

It is not necessary to give a complete proof of each of these identities, but it may be well if the student will study the proof of the first of these formulæ as a type.

Let MCN = ∠A and NCK = ∠B (Fig. 15).

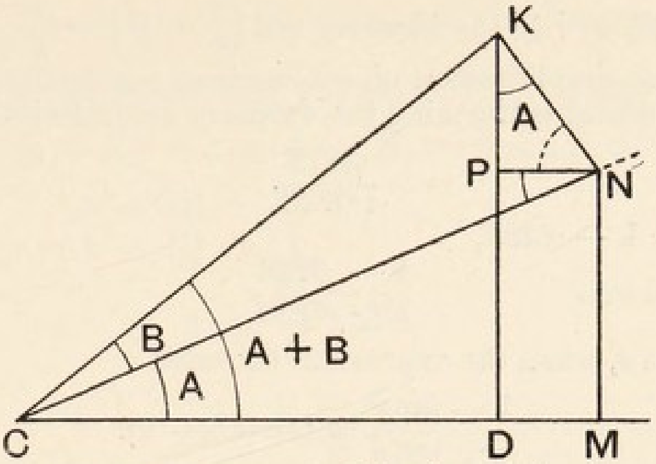

FIG. 15.—To show the Trigonometrical Ratios of the Sum of Two Angles.

Drop NM $\perp^r$ to CM.
Draw NK $\perp^r$ to CN meeting CK in K.
Drop KD $\perp^r$ to CM and NP $\parallel$ DM meeting KD in P.
Then

$$\sin(A + B) = \frac{KD}{CK} = \frac{KP + PD}{CK} = \frac{KP + NM}{CK} = \frac{KP}{CK} + \frac{NM}{CK}.$$

Now, $\frac{KP}{CK} = \frac{KP \cdot KN}{KN \cdot CK}$; and $\frac{NM}{CK} = \frac{NM \cdot CN}{CN \cdot CK}$,

But $\frac{KP}{KN} = \cos PKN$, *i.e.*, $\cos A$,

and $\frac{KN}{CK} = \sin B$.

$\therefore$ $\frac{KP}{CK} = \cos A \sin B$.

Also $\frac{NM}{CN} = \sin A$,

and $\frac{CN}{CK} = \cos B$.

$\therefore$ $\frac{NM}{CK} = \sin A \cos B$.

$\therefore$ $\sin(A + B)$ which $= \frac{KP}{CK} + \frac{NM}{CK}$

$= \cos A \sin B + \cos B \sin A$.

(Q.E.D.)

Note.—From the formula $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ it follows that if $A = 45°$ or $\frac{\pi}{4}$, then

$$\tan\left(\frac{\pi}{4} + B\right) = \frac{1 - \tan 45\ \tan B}{1 - \tan 45 \tan B} = \frac{1 + \tan B}{1 - \tan B}.$$

This identity as well as the identity $\tan\left(\frac{\pi}{4} - B\right) = \frac{1 - \tan B}{1 + \tan B}$ are also of use for simplification of arithmetical manipulations.

Example.—Without performing the ordinary arithmetical operations, find the value of

$$\frac{\cdot 3772}{1\cdot 6228}.$$

Since $\cdot 3772 = 1 - \cdot 6228$,

$\therefore$ fraction becomes $$\frac{1 - \cdot 6228}{1 + \cdot 6228}.$$

Call $0\cdot 6228$ $\tan \theta$, when the expression becomes

$$\frac{1 - \tan\theta}{1 + \tan\theta} = \tan(45 - \theta).$$

But the angle whose tangent is $\cdot 6228$ is $31°55'$ (from the table of tangents).

$$\therefore \quad \frac{\cdot 3772}{1\cdot 6228} = \tan(45 - 31\cdot 55)$$
$$= \tan 13°5'$$
$$= \cdot 2324. \quad \text{Answer.}$$

The sin (A + B) and cos (A + B), etc., formulæ may be used to calculate the numerical values of the trigonometrical ratios of 15°, 75°, etc.

Thus

$$\sin 15 = \sin(45° - 30°) = \sin 45° \cdot \cos 30° - \cos 45° \cdot \sin 30°$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{2}}{4}(\sqrt{3} - 1).$$

Similarly

$$\cos 15 = \cos(45° - 30°) = \cos 45 \cdot \cos 30 + \sin 45 \sin 30$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$
$$= \frac{\sqrt{2}}{4}(\sqrt{3} + 1).$$

Hence

$$\tan 15 = \frac{\sin 15°}{\cos 15} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{(\sqrt{3} - 1)(\sqrt{3} - 1)}{(\sqrt{3} + 1)(\sqrt{3} - 1)}$$
$$= \frac{3 - 2\sqrt{3} + 1}{(\sqrt{3})^2 - 1}$$
$$= \frac{4 - 2\sqrt{3}}{2}$$
$$= 2 - \sqrt{3}$$

Important Corollaries to the Sum and Difference Formulæ.

By adding identities (1) and (2) on p. 50, we get

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B.$$

If we put $(A + B) = P$
and $(A - B) = Q,$

we get $$A = \frac{P + Q}{2}$$

and $$B = \frac{P - Q}{2}.$$

$$\therefore \quad \sin P + \sin Q = 2 \sin \frac{(P + Q)}{2} . \cos \frac{(P - Q)}{2} \quad . \quad . \quad (a)$$

Similarly $$\sin P - \sin Q = 2 \cos \frac{(P + Q)}{2} . \sin \frac{(P - Q)}{2} \quad . \quad . \quad (b)$$

$$\cos P + \cos Q = 2 \cos \frac{(P + Q)}{2} \cos \frac{(P - Q)}{2} \quad . \quad . \quad (c)$$

$$\cos P - \cos Q = 2 \sin \frac{(P + Q)}{2} . \sin \frac{(P - Q)}{2} \quad . \quad . \quad (d)$$

These identities are of very great importance because not only are algebraic sums converted by their help into products which are amenable to logarithmic computation, but they come in very useful in various trigonometrical manipulations.

CHAPTER V.

A FEW POINTS IN ELEMENTARY MENSURATION.

THE relation between area and volume is a matter of very great importance in many physiological problems, and it is therefore necessary for the student to refresh his memory with regard to the area and volume of a few of the commoner geometrical figures.

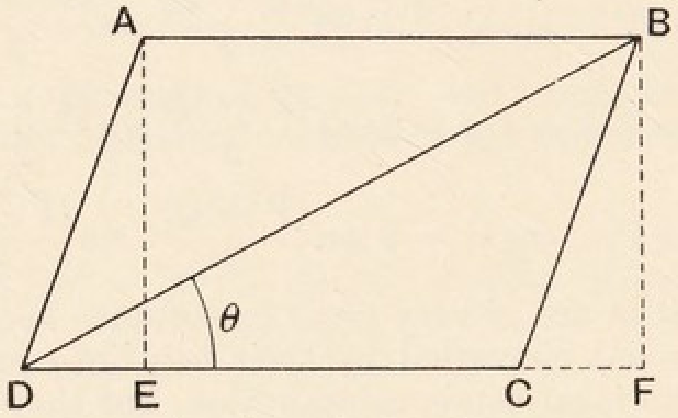

FIG. 16.—Area of a Parallelogram.

Area of a Parallelogram ABCD = DC . AE (Fig. 16) (where AE is the perpendicular dropped from A to DC).

(*a*) Hence, *area of a rectangle AEFB = AB . AE, i.e., area of rectangle = product of the two sides.*

(*b*) Further, if the four sides of the rectangle are equal the rectangle becomes a square, and its *area* = a^2 where a = length of one of the sides.

Area of a Triangle DBC = $\frac{1}{2}$ **DC . BF** (where BF is the perpendicular dropped from the apex to the base DC or to the base DC produced in Fig. 16).

(*a*) If two sides DC and DB, as well as their included angle D is known, then,

since $$\frac{BF}{DB} = \sin A, \text{ or } BF = DB \sin A,$$

$$\therefore \quad \textit{area of triangle} = \frac{1}{2}\, DC\,.\, DB \sin A,$$

i.e., area of a triangle = half the product of two sides and the sine of the included angle.

(*b*) **Area of Right-angled Triangle BDF** (Fig. 16) $= \frac{1}{2}$ **DF . BF.**

(*c*) **Area of Equilateral Triangle ABC**

$$= \frac{1}{2} BC . AB \sin B \text{ (Fig. 17)},$$

$$= \frac{1}{2} BC^2 \sin 60,$$

$$= \frac{1}{2} BC^2 . \frac{\sqrt{3}}{2} = BC^2 \frac{\sqrt{3}}{4}.$$

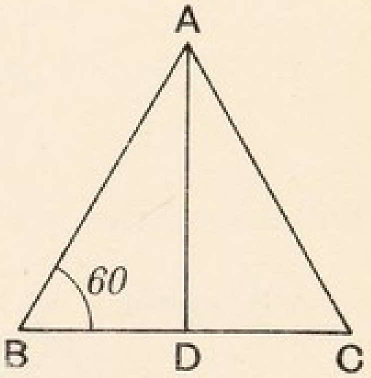

Fig. 17.—Area of Equilateral Triangle.

Area of a Circle $= \pi r^2$ (Fig. 18).

Where r = radius, and π = relation between length of circumference and diameter $= 3{\cdot}1416$ or $\frac{22}{7}$.

(*a*) **Area of Annulus, or Circular Ring** (Fig. 19).—In the diagram it is seen that the area of the circular ring is equal to the

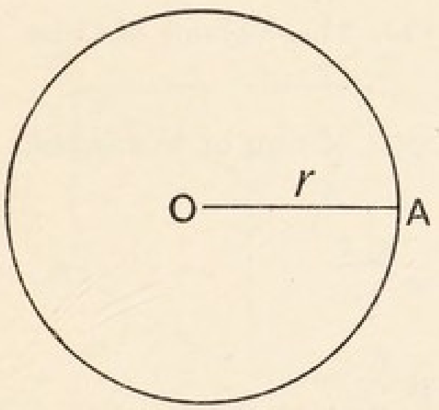

Fig. 18.—Area of Circle $= \pi r^2$.

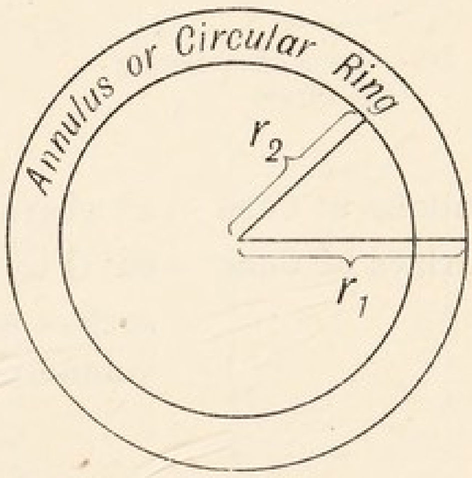

Fig. 19.—Area of Annulus.

difference between the areas of the two concentric circles or radii r_1 and r_2 respectively.

$$\therefore \text{ area of annulus} = \pi r_1^2 - \pi r_2^2.$$
$$= \pi(r_1^2 - r_2^2)$$
$$= \pi(r_1 + r_2)(r_1 - r_2).$$

E.g., if $r_1 = 10$ in., and $r_2 = 9$ in., then

$$\text{area of the annulus} = \pi(10 + 9)(10 - 9)$$
$$= 19\pi . = 59{\cdot}7 \text{ sq. in.}$$

(*b*) **Area of a Sector of a Circle AOB** (Fig. 20).—If θ be the value of the angle of the sector in radians, then

$$\frac{\text{area of sector}}{\text{total area of circle}} = \frac{\theta}{2\pi}.$$

$$\therefore \quad \text{area of sector} = \frac{\theta}{2\pi} \times \text{area of circle}$$

$$= \frac{\theta}{2\pi} \cdot \pi r^2$$

$$= \frac{\theta}{2} r^2.$$

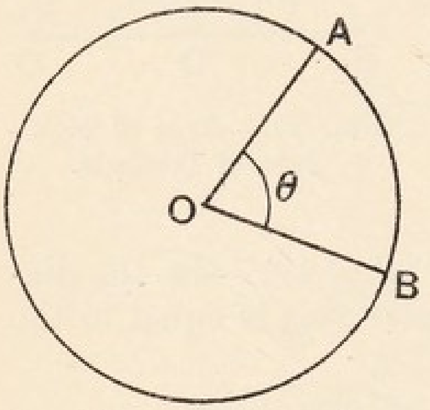

FIG. 20.—Area of a Sector of a Circle.

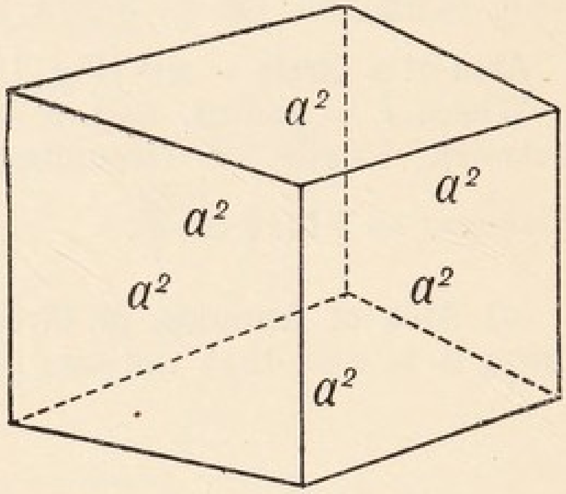

FIG. 21.—Volume of Cube.

Volume of Cube $= \mathbf{a}^3$ where $a =$ length of one of the sides.
Surface of Cube $= \mathbf{6a}^2$ (Fig. 21).

$$\therefore \quad \frac{\text{surface of cube}}{\text{volume of cube}} = \frac{6a^2}{a^3}$$

$$= \frac{6}{a}.$$

Corollary.—If side of cube is of unit length, then the extent of its surface is six times its volume $\left(\text{since}\, \frac{6}{a} = \frac{6}{1} = 6\right)$.

If $a = 2$ units, then extent of its surface is three times its volume $\left(\text{since}\, \frac{6}{a} = \frac{6}{2} = 3\right)$.

If $a = \frac{1}{2}$ unit, then extent of its surface is twelve times its volume $\left(\text{since}\, \frac{6}{a} = \frac{6}{0{\cdot}5} = 12\right)$.

Hence (a) *As a cube increases in volume, its surface relatively diminishes.*

(b) *As a cube decreases in volume its surface relatively increases.*

Sphere.— $$Volume = \frac{4}{3}\pi r^3.$$

$$Surface = 4\pi r^2.$$

$$\therefore \frac{\text{surface of sphere}}{\text{volume of sphere}} = \frac{4\pi r^2}{\frac{4}{3}\pi r^3} = \frac{1}{3r}.$$

Hence, as in the case of the cube,

(a) *As the sphere increases in volume, its surface relatively diminishes.*

(b) *As the sphere diminishes in volume its surface relatively increases.* (*Cf.* table in example 5, p. 325.)

Prism (rectangular).—If a, b, c, be the lengths of the sides of prism (Fig. 22) then,

$$Area = 2\,(ab + bc + ac).$$
$$Volume = abc.$$
$$\text{Diagonal BE} = \sqrt{\text{EG}^2 + \text{GB}^2}$$
$$= \sqrt{\text{EG}^2 + \text{GC}^2 + \text{BC}^2}$$
$$= \sqrt{c^2 + a^2 + b^2}.$$

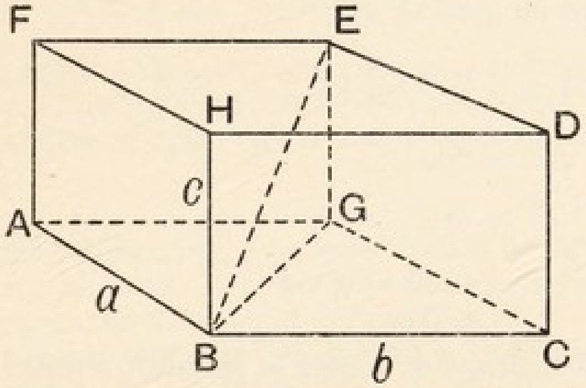

Fig. 22.—Volume and Area of Rectangular Prism.

Fig. 23.—Volume and Area of Cylinder.

Cylinder (Fig. 23).—

Area of curved surface = circumference of base × height.
$$= 2\pi rh.$$
$$Volume = \text{area of base} \times \text{height}.$$
$$= \pi r^2 h.$$
$$Total\ area = \text{area of curved surface} + \text{areas of two ends}.$$
$$= 2\pi rh + 2\pi r^2.$$
$$= 2\pi r(r + h).$$

Cone (Fig. 24).—*Volume*, $V = \frac{1}{3}\pi r^2 h$.

Curved surface, $S = \pi rl = \pi r\sqrt{h^2 + r^2}$.

Total surface = area of curved surface + area of base.
$= \pi rl + \pi r^2 = \pi r(l + r)$.

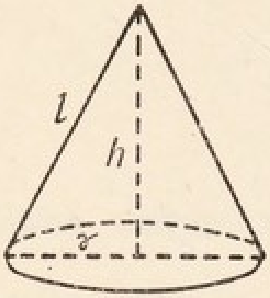

Fig. 24.—Volume of Cone.

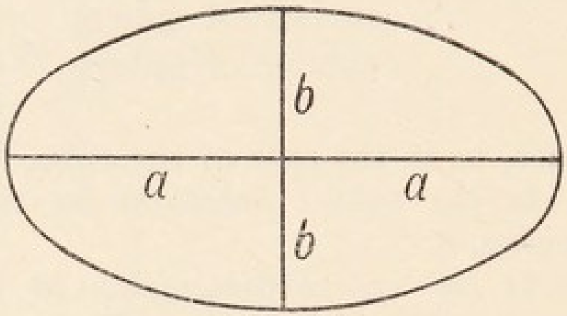

Fig. 25.—Circumference and Area of Ellipse.

Ellipse (Fig. 25).—If $2a$ and $2b$ denote the lengths of the major and minor axes, then *circumference* $= \pi(a + b)$ approximately, and *area* $= \pi ab$.

EXAMPLES.

(1) Find the radius of a circle equal in area to that of an ellipse whose axes are 10 and 8 ft.

If r = radius of the circle, then its area $= \pi r^2$.

But area of ellipse $= \pi \times \frac{10}{2} \times \frac{8}{2} = 20\pi$ sq. ft.

$$\therefore \quad \pi r^2 = 20\pi$$

$$\therefore \quad r^2 = 20$$

whence $r = \sqrt{20} = 2\sqrt{5}$
$= 4{\cdot}472$.

(2) It has been found that the average diameter of an adult's pulmonary air-cell = 0·2 mm., whilst that of an infant's air-cell (at birth) = 0·07 mm. Assuming that these air-cells are spherical, and that the total volume of the lungs = 1,617 c.c. in the adult, and 67·7 c.c. in the new-born infant, find the total number of air-cells and their total surface in the adult and in the new-born infant.

Volume of single air-cell in adult $= \frac{4}{3}\pi(0{\cdot}2)^3$ cub. mm. (p. 57)
$= 0{\cdot}004$ cub. mm.

Volume of single air-cell in new-born $= \frac{4}{3}\pi(0{\cdot}07)^3$ cub. mm.
$= 0{\cdot}00018$ cub. mm.

$\therefore$ total number of air-cells in adult $= \frac{1617 \times 10^3}{0{\cdot}004} = \frac{1617 \times 10^6}{4}$

$= 404 \times 10^6$,

And total number of air-cells in new-born $= \frac{67{\cdot}7 \times 10^3}{0{\cdot}00018}$

$= \frac{677 \times 10^7}{18}$

$= 376 \times 10^6$,

i.e., **the number is approximately the same at birth as in the full-growth adult, viz., about 4×10^8.**

Surface of single air-cell in adult $= 4\pi(0{\cdot}2)^2 = 0{\cdot}125$ sq. mm.

Surface of single air-cell in new-born $= 4\pi\left(\frac{0{\cdot}07}{2}\right)^2 = 0{\cdot}0154$ sq. mm.

$\therefore$ total surface of air-cells in adult $= 4 \times 10^8 \times 0{\cdot}125$ sq. mm.

$= 0{\cdot}5 \times 10^6 = 500{,}000$ sq. cm.

$= 50$ sq. metres.

And total surface of air-cells in new-born $= 0{\cdot}0154 \times 4 \times 10^8$ sq. mm.

$= 6$ sq. metres.

$\therefore$ total surface of air-cells in new-born is $\frac{1}{8}$ that in the adult.

Hence we see that whilst the volume of the infant's lungs is only about $\frac{1}{22}$ that in the adult, the **total surface of their alveoli are as much as $\frac{1}{8}$ that in the adult**—showing that the gaseous exchange is very active in young infants, *i.e.*, about three times as active as in the adult. (See W. M. Feldman, "Principles of Ante-Natal and Post-Natal Child Physiology," Longmans, 1920.)

(3) The following has been found to be the percentage composition of ordinary bacteria : water, 85 per cent. ; solids 15 per cent., of which 1 part in a thousand consists of sulphur.

Assuming that the weight of a molecule of any element $= M \times 8{\cdot}6 \times 10^{-22}$ mgm., where M = molecular weight of the element, how many molecules of sulphur does a micrococcus of diameter $0{\cdot}15\mu$ ($\mu = \frac{1}{1000}$ mm.) contain ?

Assuming the micrococcus to be spherical, its volume

$= \frac{4}{3}\pi(0{\cdot}075\mu)^3$

$= 18 \times 10^{-13}$ cub. mm.

Taking its sp. gr. $= 1$,

then its weight $= 18 \times 10^{-13}$ mgm.

Now, weight of sulphur molecule

$$= 32 \times 8{\cdot}6 \times 10^{-22} \text{ (since mol. wt. of S} = 32)$$
$$= 275 \times 10^{-22} \text{ mgm.}$$

But micrococcus contains $\frac{15}{100} \times \frac{1}{1000}$ part of sulphur

$$= 15 \times 10^{-5} \text{ part of sulphur.}$$

∴ total weight of sulphur in micrococcus

$$= 15 \times 10^{-5} \times 18 \times 10^{-13} = 27 \times 10^{-17} \text{ mgm.}$$

But weight of one molecule of sulphur $= 275 \times 10^{-22}$ mgm.

∴ number of molecules of sulphur in one micrococcus

$$= \frac{27 \times 10^{-17}}{275 \times 10^{-22}} = \text{about } 10{,}000.$$

(4) *Mode of Action of Renal Glomeruli.*—Brodie, by measuring the calibre of the tubules of the kidneys and the application of Poisseuille's law for the passage of fluids along narrow tubes, finds that the pressure necessary to drive the fluid along the tubules is comparable to that existing in the glomerular capillaries. Hence he believes that the pulsation of the glomerulus does not account for the flow of urine in the tubules and, accordingly, attributes a secretory action to the glomerular surface.

The following are his calculations :—

Data.	Length. Cm.	Diameter. μ. (*i.e.*, $\frac{1}{1000}$ mm.)
(1) Proximal convoluted tubule	1·2	12
Loop of Henle—		
Descending limb	0·9	10
Ascending limb	0·9	9
Distal convoluted tubule	0·2	18
Collecting tubule	2·2	16

(2) Diuresis (*i.e.*, rate at which urine was being discharged from one of the kidneys at the time of the experiment) = 1 c.c. per min.

(3) Poisseuille's law, which gives the pressure head of fluid that must have existed within each capsule in order to drive fluid out of the kidney, is

$p = \frac{8l\eta V}{\pi r^4}$ dynes per sq. cm. (see p. 212),

where

l = length of tube in cm.,

η = coefficient of viscosity of water at 35°

$= 719 \times 10^{-5}$,

V = flow in c.c. per second,

r = radius of tube in cm.

Hence

$$p = \frac{8 \times 719 \times 1^{-5}}{\pi} \cdot \frac{1}{60} \cdot \frac{1}{142000} \cdot 10^{16} \cdot \frac{l}{r^4} \text{ dynes per sq. cm. (where } r$$

is expressed in μ)

$$= \frac{8 \times 719 \times 7}{22 \times 6 \times 142 \times 1333{\cdot}2} \cdot 10^7 \cdot \frac{l}{r^4} \text{ mm. Hg.}$$

$$= 1{\cdot}611 \times 10^4 \frac{l}{r^4} \text{ mm. Hg.}$$

Consequently, for a flow of 1 c.c. per minute,

p, per centimetre of tubule, when	r is 4·5 μ	= 39·29 mm. Hg.
	r is 5 μ	= 25·78 ,,
	r is 6 μ	= 12·43 ,,
	r is 8 μ	= 3·93 ,,
	r is 9 μ	= 2·46 ,,

Hence pressure head required for :

Proximal convoluted tubule	$= 1{\cdot}2 \times 12{\cdot}43 = 14{\cdot}916$	mm. Hg.
Descending limb	$= 0{\cdot}9 \times 25{\cdot}78 = 23{\cdot}212$	,,
Ascending limb	$= 0{\cdot}9 \times 39{\cdot}29 = 35{\cdot}361$	,,
Distal convoluted tubule	$= 0{\cdot}2 \times 2{\cdot}46 = 0{\cdot}492$	,,
Collecting tubule	$= 2{\cdot}2 \times 3{\cdot}93 = 8{\cdot}646$	,,
	Total pressure head = 82·627	,,

As the mean aortic blood pressure was 120 mm. Hg., and the loss of pressure head between aorta and glomercular capillaries is about 35 mm. Hg.,

$\therefore$ B.P. within glomerular capillaries was probably about 85 mm. Hg.

In other words, practically the whole of the blood pressure is required to set up a pressure head in the fluid within the capsule sufficient to drive the secreted fluid down the tubules.

(N.B.—The bulk of opinion is against Brodie's conclusions. For criticism of the inference drawn from this calculation, see Cushny, "The Secretion of Urine," p. 57, Longmans, 1917.)

(4) The average diameter of a human capillary is $\frac{1}{100}$ mm. ; the linear velocity of blood in it is $\frac{1}{2}$ mm. per second.

Find the volume of outflow from a capillary per second.

Volume of outflow = linear velocity $\times$ cross section.

$$= \frac{1}{2} \times \pi \left(\frac{1}{200}\right)^2.$$

$$= \frac{1}{2} \times 3{\cdot}14 \times \frac{1}{40{,}000}.$$

$$= {\cdot}00004 \text{ cub. mm. per second.}$$

(5) Experiments on animals have shown that the circulation time is equal to 28 heart-beats. Assuming this to hold good for man, and also assuming the total volume of blood in the body to be 4,000 c.c., find the number of capillaries in the human body, using the data of the last example, and assuming the pulse rate to be 72 per minute.

Since circulation time = 28 heart-beats,
and since 72 beats = 1 minute = 60 seconds,

$$\therefore \quad \text{circulation time} = \frac{28}{72} \times 60 = 23{\cdot}3 \text{ seconds.}$$

But volume of capillary outflow = 0·00004 cub. mm. per second.

$\therefore$ total volume of outflow from one capillary in the circulation time = $23{\cdot}3 \times 0{\cdot}00004 = {\cdot}000932$ cub. mm.

$\therefore$ if n = number of capillaries in the body (filled with blood) we must have $0{\cdot}000932n$ = total volume of blood in body = 4,000,000 cub. mm.

$$\therefore \quad n = \frac{4 \times 10^6}{93210 \times {}^{-6}} = 4{\cdot}3 \times 10^9.$$

(See W. M. Feldman, *Proc. Physiol. Soc.*, 1912.)

CHAPTER VI.

SERIES.

A SUCCESSION of quantities, each of which is formed in accordance with some fixed rule or principle, is called a *series*. Each of the successive quantities in a series is called a *term*, and the fixed rule in accordance with which each term is formed is called the *law of the series*.

There are very many different kinds of series, with some of which we shall have to deal, *e.g.*, the series ϵ (p. 75), the exponential series (p. 78), the logarithmic series (p. 79), the various trigonometrical series (p. 189), etc. The best known types are the arithmetical and geometrical progressions.

An *arithmetical progression* (A.P.) is a series in which each term differs from its immediate predecessor by a constant quantity called the *common difference* (C.D.) Thus

$$1, 3, 5, 7, 9 \ldots\ldots\ldots$$
$$6, 3, 0, -3, -6 \ldots\ldots\ldots$$

are arithmetical progressions; the C.D. in the first being 2 ($3 - 1 = 5 - 3$, etc., $= 2$), and that in the second -3 ($3 - 6 = 0 - 3$, etc., $= -3$). The most general form of an A.P. is

$$a, a + d, a + 2d, a + 3d \ldots\ldots\ldots$$

where a is the first term, and d is the common difference (whether + ve or − ve, and whether integral or fractional).

Whenever any quantity grows in such a way that its increase (or decrease) in value during any equal intervals of time is always a constant proportion of its original value, then the successive values attained by it at the ends of these intervals of time form the terms of an arithmetical progression.

The best example of such a form of growth is money lent at a fixed rate of *simple* interest.

Thus, if £100 be invested at 10 per cent. simple interest per annum, then each year the capital increases by $\frac{1}{10}$ of its original value, viz., £10, and the amounts to which the capital has grown at the beginning of each successive year are the terms of the A.P.

$$100, 110, 120, 130 \ldots\ldots\ldots$$

Hence we can say that *the law of an arithmetical progression is analogous to the law of simple interest.*

A *geometrical progression* (G.P.) is a series in which each term bears a constant ratio—called the *common ratio* (C.R.) to its immediate predecessor. Thus

$$3, 6, 12, 24, 48 \quad \ldots\ldots\ldots\ldots$$

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81} \quad \ldots\ldots\ldots\ldots$$

are geometrical progressions; the C.R. in the first being $2\left(\frac{6}{3}=\frac{12}{6}, \text{etc.}, =2\right)$, and that in the second being $\frac{1}{3}\left(\frac{1}{3}\Big/1=\frac{1}{9}\Big/\frac{1}{3}, \text{etc.}, =\frac{1}{3}\right)$.

The most general form of a G.P. is

$$a, ar, ar^2, ar^3 \quad \ldots\ldots\ldots\ldots$$

where a is the first term, and r is the common ratio (integral or fractional, + ve or − ve).

Whenever any quantity grows in such a way that its increase (or decrease) in value during any equal intervals of time is proportional not to its original value, but to its value at the beginning of that interval, then the successive values attained by it at the ends of these intervals of time form the terms of a geometrical progression.

The best example of such a form of growth is money invested at a fixed rate of *compound* interest.

Thus, if £100 be invested at 10 per cent. compound interest per annum, then the interest during the first year $= \frac{1}{10} \times 100 = £10$. This, being added to the capital, makes the new capital at the end of the first or beginning of the second year

$$= 100 + \frac{1}{10} \cdot 100 = £100\left(1+\frac{1}{10}\right).$$

$\therefore$ The interest during the second year $= \frac{1}{10} \times 100\left(1+\frac{1}{10}\right)$,

which, when added to the increased capital, makes the new capital at the end of the second or beginning of the third year

$$= 100\left(1+\frac{1}{10}\right) + \frac{1}{10} \times 100\left(1+\frac{1}{10}\right)$$

$$= 100\left(1+\frac{1}{10}\right)\left(1+\frac{1}{10}\right) = £100\left(1+\frac{1}{10}\right)^2.$$

$\therefore$ The interest during the third year $= \frac{1}{10} \times 100\left(1+\frac{1}{10}\right)^2$

which, when added to the further increased capital, makes the new capital at the end of the third or beginning of the fourth year

$$= 100\left(1 + \frac{1}{10}\right)^2 + \frac{1}{10} \times 100\left(1 + \frac{1}{10}\right)^2$$

$$= 100\left(1 + \frac{1}{10}\right)^2\left(1 + \frac{1}{10}\right) = £100\left(1 + \frac{1}{10}\right)^3,$$

and so on.

So that the amounts to which the capital has grown at the beginning of each successive year are the terms of the G.P.

$$100,\ 100\left(1 + \frac{1}{10}\right),\ 100\left(1 + \frac{1}{10}\right)^2,\ 100\left(1 + \frac{1}{10}\right)^3 \ldots$$

or $\qquad 100,\ 100\,(1{\cdot}1),\ 100\,(1{\cdot}1)^2,\ 100\,(1{\cdot}1)^3 \ldots\ldots$

Hence we can say that *the law of a geometrical progression is analogous to the law of compound interest.*

We shall return to the compound interest law presently (see p. 84).

The Binomial Theorem.—A most important and interesting series is obtained by raising a binomial expression (*i.e.*, an expression containing two terms, like $(a + b)$, etc.) to any power n. Newton's *Binomial Theorem* states that for any value of a, b, and n,

$$(a + b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\,.\,2}a^{n-2}b^2$$
$$+ \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3$$
$$+ \ldots\ldots\ldots + \ldots\ldots$$
$$+ \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^3b^{n-3} + \frac{n(n-1)}{1\,.\,2}a^2b^{n-2}$$
$$+ \frac{n}{1}ab^{n-1} + b^n.$$

The right-hand side of this identity is called the *expansion* of $(a + b)^n$.

There is no need to give here a very rigid proof of this theorem. It will be sufficient for our purpose if we verify the theorem for several different values of n.

Thus we know from actual multiplication that

$(a + b)^2 = a^2 + 2ab + b^2$, which is the same as

$$a^2 + \frac{2}{1}a^{2-1}b + \frac{2(2-1)}{1\,.\,2}a^{2-2}b^2 + \frac{2(2-1)(2-2)}{1\,.\,2\,.\,3}a^{2-3}b^3$$
$$+ \ldots\ldots$$

[Since
$$\frac{2}{1}a^{2-1}b = 2ab$$
$$\frac{2(2-1)}{1 \,.\, 2}a^{2-2}b^2 = \frac{2 \,.\, 1}{1 \,.\, 2}a^0b^2 = b^2,$$
$$\frac{2(2-1)(2-2)}{1 \,.\, 2 \,.\, 3}a^{2-3}b^3$$
and the subsequent terms are all $= 0$, because they contain the factor $(2 - 2)$ which $= 0$].

Similarly, actual multiplication gives

$(a + b)^3 = a^3 + 3ab + 3a^2b^2 + b^3$, which is the same as
$$a^3 + \frac{3}{1}a^{3-1}b + \frac{3(3-1)}{1 \,.\, 2}a^{3-2}b^2 + \frac{3(3-1)(3-2)}{1 \,.\, 2 \,.\, 3}a^{3-3}b^3.$$

All the subsequent terms, containing as they do $(3 - 3)$ as a factor, vanish, and so on, for any value of n.

The Factorial Notation.—It is customary to denote the product of the 1st n consecutive integers
$$1 \,.\, 2 \,.\, 3 \,.\, 4 \;.\;.\;.\; (n-2)(n-1)n \text{ as } \lfloor\!\underline{n} \text{ or } n\,!$$
which is read *factorial n*.

Thus
$$\lfloor\!\underline{1} \text{ or } 1\,! = 1$$
$$\lfloor\!\underline{2} \text{ or } 2\,! = 1 \,.\, 2 = 2$$
$$\lfloor\!\underline{3} \text{ or } 3\,! = 1 \,.\, 2 \,.\, 3 = 6$$
$$\lfloor\!\underline{4} \text{ or } 4\,! = 1 \,.\, 2 \,.\, 3 \,.\, 4 = 24,$$
and so on.

Hence, the binomial theorem is written as
$$(a + b)^n = a^n + \frac{n}{1\,!}a^{n-1}b + \frac{n(n-1)}{2\,!}a^{n-2}b^2$$
$$+ \;.\;.\;.\; \frac{n(n-1)}{2\,!}a^2b^{n-2} + \frac{n}{1\,!}ab^{n-1} + b^n.$$

EXAMPLES.

Use the binomial theorem for finding the values of the following

(1) $\sqrt{10} = \sqrt{9+1} = \sqrt{9\left(1 + \frac{1}{9}\right)} = 3\sqrt{1 + \frac{1}{9}}$
$$= 3\left(1 + \frac{1}{9}\right)^{\frac{1}{2}} = 3\left(1 + \frac{1}{2}\cdot\frac{1}{9} - \frac{1}{8}\cdot\frac{1}{81} + \;.\;.\;.\;\right)$$
$$= 3{\cdot}1623.$$

(2) $\sqrt[3]{10} \quad = \sqrt{8\left(1+\frac{1}{4}\right)} = 2\left(1+\frac{1}{4}\right)^{\frac{1}{3}}$

$$= 2\left(1 + \frac{1}{3}\,\frac{1}{4} - \frac{2}{18}\cdot\frac{1}{16} + \,.\;\;.\;\;.\;\;.\;\;.\right) = 2{\cdot}1547.$$

(3) $\sqrt[4]{1{\cdot}006} = (1 + {\cdot}006)^{\frac{1}{4}} = 1 + \frac{1}{4} \times {\cdot}006 - \ldots$

$= 1{\cdot}0015$, correct to four places of decimals.

(4) $(1{\cdot}006)^4 \;= 1 + 4 \times {\cdot}006 + \frac{4\,.\,3}{1\,.\,2}\,({\cdot}006)^2 + \ldots = 1{\cdot}024$, correct to three decimal places.

From the last two examples we learn that if a is very small compared with unity, then

$$(1+a)^n \;= 1 + na.$$

$$(1+a)^{\frac{1}{n}} \;= 1 + \frac{a}{n},\; i.e.,\; \sqrt[n]{1+a} = 1 + \frac{a}{n}.$$

$$(1+a)^{-n} = 1 - na,\; i.e.,\; \frac{1}{1+a} = 1 - na.$$

$$(1+a)^{-\frac{1}{n}} = 1 - \frac{a}{n},\; i.e.,\; \sqrt[n]{\frac{1}{1+a}} = 1 - \frac{a}{n}.$$

Similarly, if a and b be very small compared with unity, then, since $\quad (1 \pm a)\,(1 \pm b) = 1 \pm a \pm b + ab,$
we may for purposes of approximation write

$$(1 \pm a\,(1 \pm b) = 1 \pm a \pm b.$$

Thus $\quad (1{\cdot}0003) \times (1{\cdot}0006) = 1 + {\cdot}0003 + {\cdot}0006$
$= 1{\cdot}0009.$

Also $\quad (1 \pm a)^m\,(1 \pm b)^n = (1 \pm ma)\,(1 \pm nb)$
$= 1 \pm ma \pm nb.$

E.g., if t is small $\sqrt{1 + \frac{t}{273}} = \left(1 + \frac{t}{273}\right)^{\frac{1}{2}} = 1 + \frac{t}{546}.$

EXAMPLES.

(1) Find $\sqrt[3]{998.}$

$$\sqrt[3]{998} = \sqrt[3]{1{,}000(1 - {\cdot}002)} = 10(1 - {\cdot}002)^{\frac{1}{3}}$$

$$= 10\left(1 - \frac{{\cdot}002}{3}\right) = 9{\cdot}993$$

(2) Find the value of $\frac{985 \times 5{\cdot}08}{1004}$.

$$\text{The expression} = \frac{1000(1 - {\cdot}015) \times 5(1 + {\cdot}016)}{1000(1 + {\cdot}004)}$$
$$= \frac{5(1 - {\cdot}015)(1 + {\cdot}016)}{1 + {\cdot}004}$$
$$= 5[1 - {\cdot}015 + {\cdot}016 - {\cdot}004]$$
$$= 4{\cdot}985.$$

(3) Similarly, $\frac{1{\cdot}0015 \times 2{\cdot}063 \times 0{\cdot}998}{({\cdot}997)^2}$

$$= (1 + {\cdot}0015) \times 2(1 + {\cdot}0315)(1 - {\cdot}002)(1 - {\cdot}003)^{-2}$$
$$= 2[1 + {\cdot}0015 + {\cdot}0315 - {\cdot}002 - {\cdot}006]$$
$$= 2{\cdot}074.$$

EXERCISES.

(1) Find $\sqrt{2}$. [Answer, 1·413.]

(2) Prove that $\sqrt[3]{26} = 2{\cdot}9625$.

There are many other types of series with which we shall have to deal in this book, *e.g.*, the exponential, the logarithmic (pp. 78 and 79), the various trigonometrical series (p. 190), and the various series that result from the expansion of binomial expressions The one which occurs with the utmost frequency in the higher mathematics is :

$$1 + \frac{1}{1} + \frac{1}{1 \,.\, 2} + \frac{1}{1 \,.\, 2 \,.\, 3} + \frac{1}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \,.\,.$$

or $$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \,.\,.\,.$$

This, as we shall see later is a series, the successive terms of which represent the successive terms in the expansion $\left(1 + \frac{1}{n}\right)^n$, when n is infinitely large and is called the series e.

Sum of a Geometrical Progression.—The sum of a geometrical series is important from our point of view, and we therefore proceed to find a formula for it.

Let $S = a + ar + ar^2 + ar^3 + \,.\,.\,.\,.\, + ar^{n-1}$
where ar^{n-1} is the nth term.

$$\therefore \quad Sr = ar + ar^2 + ar^3 + ar^4 \,.\,.\,.\, + ar^n.$$

$\therefore$ by subtraction we get
$Sr - S = ar^n - a$ (all the intermediate terms cancelling themselves out),

i.e.,
$$S(r - 1) = a(r^n - 1)$$
$$\therefore \quad S = \frac{a(r^n - 1)}{r - 1}.$$

If $r < 1$, then the sum is written as
$$S = \frac{a(1 - r^n)}{1 - r}.$$

Corollary.—Since when $r < 1$, $r^\infty = 0$ (see p. 9).

$\therefore$ when $r < 1$, the sum to infinity $= \dfrac{a}{1 - r}$.

EXAMPLES.

(1) The sum to 10 terms of $1 + 2 + 4 + 8 + \ldots$ is
$$\frac{2^{10} - 1}{2 - 1} = 1023.$$

(2) The sum to infinity of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is $\dfrac{1}{1 - \frac{1}{2}} = 2.$

The sum of a series is sometimes denoted by writing the Greek letter Σ in front of the general term. Thus $\Sigma(1 + r^2)$ stands for and is read as "the sum of such terms as $(1 + r^2)$." By placing small letters after Σ, we indicate how many terms are to be taken,

thus
$$\sum_{r=1}^{r=50} (1 + r^2)$$

denotes the sum of the terms obtained from $(1 + r^2)$ by giving r the values from 1 to 50 in succession.

$$\therefore \quad \sum_{r=1}^{r=50} = (1 + 1^2) + (1 + 2^2) + (1 + 3^2) + (1 + 4^2) + 1 + 5^2) + \ldots + (1 + 50^2)$$
$$= 2 + 5 + 10 + 17 + 26 + \ldots + 2501.$$

When the successive terms differ from one another by infinitesimally small quantities, then the symbol $\int$ (which is a long S) is substituted for Σ. We shall return to this in our section on the Integral Calculus.

Finite and Infinite Series.—If a series terminate at some assigned term, say, the nth term—where n is a finite number like 100, 200, 1,000, and so on, then it is called a finite series, *e.g.*,

1, 3, 5, 7, 9 . . . 29.

If, however, the number of terms is unlimited, it is called an infinite series, *e.g.*,

$$1, 3, 5, 7, 9 \ . \ . \ . \ 27, 29 \ . \ . \ . \text{ to infinity.}$$

Now whilst the sum of an infinite series like this will necessarily be infinitely great—increasing as it does with each term—there are some infinite series whose sum may have a **finite** value. A very good example of such a series is afforded by a recurring decimal. Thus, if we convert $\frac{1}{3}$ into a decimal fraction we obtain :

$$\frac{1}{3} = \cdot 33333 \ . \ . \ . \text{ the 3's being continued for ever,}$$

$$\textit{i.e.,} \quad \frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ . \ . \ . \text{ to infinity.}$$

The same is the case with $\frac{1}{9}$, for

$$\frac{1}{9} = \cdot 11111 \ . \ . \ . \text{ the 1's being continued for ever}$$

$$= \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ . \ . \ . \text{ to infinity,}$$

i.e., each of these infinite series has a finite value, which means that if we continually increase the number of terms in each of these series the sum will get nearer and nearer to $\frac{1}{3}$ or $\frac{1}{9}$ respectively. Thus the difference between

$$\frac{1}{9} \text{ and } 0\cdot 1111 \text{ is } \frac{1}{9} - \frac{1111}{10000} = \frac{1}{90{,}000};$$

the difference between

$$\frac{1}{9} \text{ and } 0\cdot 1111111111 \text{ is } \frac{1}{9} - \frac{1111111111}{10{,}000{,}000{,}000} = \frac{1}{9 \cdot 10^{10}},$$

and so on.

Now, $\frac{1}{9 \cdot 10^{10}}$ is such a small fraction that if we were to take this fraction of the velocity of light which is 186,000 miles per second, it would amount only to about $\frac{1}{10}$ inch ! Now, if this is the case when we take 10 terms of the series, imagine how much smaller the difference would be if we were to take 20, or 100, or 1,000, or 1,000,000 terms. Indeed, it is obvious that the greater the number of terms in this series, the nearer and nearer its sum approaches to the value $\frac{1}{9}$, and if we continue the series to an infinite number of terms its sum *ultimately* becomes the value $\frac{1}{9}$. Now, such an **ultimate** value of the series—a value which the sum of the series never actually reaches, but approaches closer and closer, and to which it may get as close as ever we please by continuing the series long enough—is called the *limit* or *limiting value* of the series.

The symbolical way of writing this is $\mathrm{Lt}_{n \longrightarrow \infty}\ 0{\cdot}111\ .\ .\ . = \frac{1}{9}$ which is read as follows: "The limit of 0·111 . . . to n terms when n is made infinitely great is equal to $\frac{1}{9}$." (The symbol ∞ stands for infinity.)

Similarly, if we take any fraction $\frac{1}{n}$, we can make it as small as we please by making n sufficiently large; thus if $n = 1000$, $\frac{1}{n} = \frac{1}{1000}$; if $n = 1{,}000{,}000$, $\frac{1}{n} = \frac{1}{1{,}000{,}000}$, and so on; so that by taking n as very very large, $\frac{1}{n}$ becomes very very small, and when n is made infinitely large, *i.e.*, $n = \infty$, then $\frac{1}{n}$ becomes $\frac{1}{\infty}$, *i.e.*, infinitely small, *i.e.*, 0. Hence, $\mathrm{Lt}_{n \longrightarrow \infty}\ \frac{1}{n} = 0$, which is read as follows: "The limit of $\frac{1}{n}$ when n is infinitely large is zero."

Similarly, since by actual division we find that

$$\frac{1}{1-x} = 1 + x + x^2 + \ .\ .\ .\ .\ .\ .\ .\ \text{to infinity},$$

we say that $\mathrm{Lt}_{n \longrightarrow \infty}\ (1 + x + x^2 + \ .\ .\ .\ .\ .) = \frac{1}{1-x}$.

Also $\frac{1}{1+x} = 1 - x + x^2 - \ .\ .\ .\ .\ .\ .$ to infinity.

$$\therefore \quad \mathrm{Lt}_{n \longrightarrow \infty}\ (1 - x + x^2 - \ .\ .\ .\ .\ .) = \frac{1}{1+x}.$$

Convergency and Divergency of Series.—When an infinite series is of such a nature that its sum to any number of terms cannot numerically exceed some finite quantity—called the limit—however large the number of terms, then such a series is said to be *convergent*.

Thus, the sum of the series

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^{n-1}} = \frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}} \quad \text{(see p. 68).}$$

Now $\frac{1}{3^n}$ becomes smaller and smaller as n is made larger and larger, and when $n = \infty$, $\frac{1}{3^n} = 0$.

$$\therefore \quad \frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}} = \frac{1 - 0}{\frac{2}{3}} = \frac{1}{2/3} = \frac{3}{2}.$$

Hence the series $1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3} \ldots$ is convergent.

When an infinite series is of such a nature that its sum to n terms can be made greater than any finite quantity by taking n large enough, then such a series is said to be *divergent*.

E.g., the sum of the series

$$1 + 3 + 3^2 + 3^3 + \ldots\ldots = \frac{1 - 3^n}{1 - 3} = \frac{3^n - 1}{3 - 1},$$

and when $n = \infty$, 3^n, and hence also $3^n - 1 = \infty$,

$$\therefore \quad \frac{3^n - 1}{3 - 1} = \infty.$$

Hence the series $1, 3, 3^2, 3^3 \ldots$ is divergent.

From what we have just said it follows, therefore, that the series

$$1 + x + x^2 + x^3 \ldots\ldots$$

is convergent or divergent, according as $x < 1$ or > 1.

Thus, the sum of the series to infinity is

$$S = 1 + x + x^2 + x^3 + \ldots \text{ to } \infty = \frac{x^n - 1}{x - 1}.$$

$$\therefore \quad \text{if } x < 1,\ x^n = 0, \text{ and sum} = \frac{1}{1 - x},$$

but if $x > 1$, $x^n = \infty$, and sum $= \infty$.

In this book we shall deal only with convergent series.

The Series ϵ or e.—The compound interest law, with which we dealt on p. 64, leads us to another series which is the most important series in the higher mathematics, as we shall see presently (p. 79).

The sum of this series, which, like π (*i.e.*, the relation between the circumference of a circle and its diameter), is an incommensurable number—although, as we shall see, it can be calculated to as many places of decimals as we wish—is called ϵ (epsilon) or **e**, after Euler, the discoverer of the series. It is, as we have already seen on p. 67,

$$1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \dots\dots$$

and its value to 5 decimal places is 2·71828 . . . . .

If money, say, £100, is invested at simple interest at the rate, say, of 10 per cent. per annum, then, since as we have seen, the amounts to which the capital has grown at the ends of the first, second, third, etc., years are £110, £120, £130, etc., therefore in ten years the capital would become £200, *i.e.*, the capital would double itself. And no matter whether the interest is collected in yearly instalments of £$\frac{100}{10}$, or in monthly instalments of £$\frac{100}{120}$, or of daily instalments of £$\frac{100}{3650}$, etc., the capital would still double itself in the same period of ten years. Indeed, if we were to divide the year into n intervals and collect £$\frac{10}{n}$ at the end of every interval, the amount collected during each year would still be £10$\left(i.e., £\frac{10}{n} \times n\right)$, and the capital would still double itself in ten years. But in the case of compound interest it is different. We have seen that if the interest, say, 10 per cent., is collected **and added to the capital every year**, then the amounts to which the capital has grown at the ends of the first, second, third, etc., years are

$$£100\left(1 + \frac{1}{10}\right), £100\left(1 + \frac{1}{10}\right)^2, £100\left(1 + \frac{1}{10}\right)^3, \text{ etc. (p. 64)},$$

so that in ten years the capital would become

$$£100\left(1 + \frac{1}{10}\right)^{10} = £100\,(1{\cdot}1)^{10}$$

$$= £100 \times 2{\cdot}59375\,;\ [10 \log 1{\cdot}1 = 0{\cdot}41393 = \log 2{\cdot}59375].$$

If, however, instead of collecting the interest yearly, we were to collect it monthly and add it to the capital each time, then the interest during the first month $= \frac{1}{120} \times 100$.

This, added to the capital, makes the new capital at the end of the first month or beginning of second month

$$= 100 + \frac{1}{120} \times 100 = £100 \left(1 + \frac{1}{120}\right).$$

$\therefore$ interest during second month $= \frac{1}{120} \times 100 \left(1 + \frac{1}{120}\right)$,

which, when added to the increased capital, makes the new capital at the end of the second or beginning of the third month

$$= 100 \left(1 + \frac{1}{120}\right) + \frac{1}{120} \times 100 \left(1 + \frac{1}{120}\right)$$

$$= 100 \left(1 + \frac{1}{120}\right) \left(1 + \frac{1}{120}\right)$$

$$= £100 \left(1 + \frac{1}{120}\right)^2.$$

Similarly, at the end of the third month, the capital has grown to

$$£100 \left(1 + \frac{1}{120}\right)^3$$

and so on.

So that at the end of ten years, or 120 months, **when at simple interest the capital would have doubled itself,** the capital would, at compound interest reckoned monthly, have grown to

$$£100 \left(1 + \frac{1}{120}\right)^{120} = £100 \left(\frac{121}{120}\right)^{120} = £100 \times 2{\cdot}707$$

[since $120 \log \frac{121}{120} = 121 (\log 121 - \log 120) = 0{\cdot}43248 = \log 2{\cdot}707$].

Supposing, however, we collected the interest and added it to the capital, not monthly, but daily.

Then, the interest during the first day $= \frac{1}{3650} \times 100$.

This, added to the capital, makes the new capital at the end of the first or beginning of the second day $100 + \frac{1}{3650} \times 100 = £100 \left(1 + \frac{1}{3650}\right)$.

$\therefore$ interest during second day $= \frac{1}{3650} \times 100 \left(1 + \frac{1}{3650}\right)$, which, when added to increased capital, makes the new capital at the end of the second day

$$= 100\left(1+\frac{1}{3650}\right)+\frac{1}{3650}\times 100\left(1+\frac{1}{3650}\right)=£100\left(1+\frac{1}{3650}\right)$$

and so on.

So that at the end of 10 years, or 3,650 days, **when at simple interest the capital would have doubled itself** the capital would, at compound interest reckoned daily, have grown to

$$£100\left(1+\frac{1}{3650}\right)^{3650}=£100\left(\frac{3651}{3650}\right)^{3650}=£100\times 2{\cdot}718$$

$$\left[3650\log\frac{3651}{3650}=3650(\log 3651-\log 3650)=0{\cdot}43435=\log 2{\cdot}718\right].$$

If we go still further and calculate the interest every hour, then in 10 years, *i.e.*, 87600 hours, the £100 would become

$$£100\left(\frac{87601}{87600}\right)^{87600}=£100\times 2{\cdot}71828.$$

Finally, it will be seen that **since the capital keeps on growing every minute, second, and, indeed, every instant,** the ultimate value of the £100 at the end of 10 years will be £100 $\left(1+\frac{1}{n}\right)^n$, where n is infinitely great.

Now, expanding $\left(1+\frac{1}{n}\right)^n$ by the binomial theorem (p. 64), we get

$$\left(1+\frac{1}{n}\right)^n=1+\frac{n}{1}\cdot\frac{1}{n}+\frac{n(n-1)}{1\,.\,2}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\cdot\frac{1}{n^3}$$

$$+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\cdot\frac{1}{n^4}+\ldots$$

$$=1+\frac{1}{1}+\frac{n^2\left(1-\frac{1}{n}\right)}{1\,.\,2}\cdot\frac{1}{n^2}+\frac{n^3\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\,.\,2\,.\,3}\cdot\frac{1}{n^3}$$

$$+\frac{n^4\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{1\,.\,2\,.\,3\,.\,4}\cdot\frac{1}{n^4}+\ldots$$

But when $n=\infty$, $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}$, etc. $=0$.

$$\therefore \ \mathrm{Lt}_{n \to \infty}\left(1+\frac{1}{n}\right)^n = 1+\frac{1}{1}+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\frac{1}{1\,.\,2\,.\,3\,.\,4}+\cdots$$

which, when worked out to the 1st six decimal places, gives the result 2·718282 . . . as follows:

	1	= 1·000000 00
divided by	1	= 1·000000 00
,,	2	= 0·500000 00
,,	3	= 0·166666 66
,,	4	= 0·041666 66
,,	5	= 0·008333 33
,,	6	= 0·001388 88
,,	7	= 0·000198 41
,,	8	= 0·000024 80
,,	9	= 0·000002 76
,,	10	= 0·000000 27
		2·718282 . . .

This series $1+\frac{1}{1}+\frac{1}{1\,.\,2}+\frac{1}{1\,.\,2\,.\,3}+\frac{1}{1\,.\,2\,.\,3\,.\,4}$ . . . . . . *is called the series* ϵ (*or* e), which, as I have said, is the most important series in the higher mathematics. We can say, therefore, that *the series* ϵ (*or* e) *is the expansion of* $\left(1+\frac{1}{n}\right)^n$ *when n is made infinitely large.*

The Meaning of e.—From the method by which we have derived e $\left[\textit{i.e.,}\ \mathrm{e} = \mathrm{Lt}_{n \to \infty}\left(1+\frac{1}{n}\right)^n\right]$ we see that **e** represents the amount which a unit quantity, growing **continuously** at a certain rate in accordance with the law of *compound* interest (*i.e.*, at a rate which is a constant proportion of its magnitude at every instant), will reach after that period of time at which the same quantity would double itself if it grew at the same rate in accordance with the law of *simple* interest (*i.e.*, at a rate which is proportional to its original value).

Thus, if two equal capitals be put out at the same rate of interest at the same time, one at simple and the other at compound interest, then at the time at which the money invested at simple interest has doubled itself the sum put out at *true* compound interest (*i.e.*, interest collected and added to the capital every instant) will become **e** times its original value.

If the money had been invested at 5 per cent. compound interest per annum, it would become £100 e in 20 years $\left(i.e., \text{ in } \frac{100}{5}\right)$. If invested at 20 per cent. it would become £100 e in five years $\left(i.e., \text{ in } \frac{100}{20}\right)$, and in general, if invested at r per cent., it would become £100 e in $\frac{100}{r}$ years.

$$\therefore \quad \text{£100 becomes £100 e in } \frac{100}{r} \text{ years.}$$

$$\therefore \quad \text{£100 becomes £100 } e^{\frac{r}{100}} \text{ in one year.}$$

$$\therefore \quad \text{£100 becomes £100 } e^{\frac{rt}{100}} \text{ in } t \text{ years.}$$

The most general way of expressing this is as follows:

If Q_o = original quantity (*e.g.*, the principal),

r = rate of growth per cent. **per unit of time** (*e.g.*, rate of interest per cent. per annum),

t = number of such units of time during which the quantity is allowed to grow **continuously** (*e.g.*, number of years during which the capital is invested),

and Q_t = the amount to which Q_o has grown in the time t,

then $Q_t = Q_o e^{\frac{rt}{100}}$.

This is one of the most fundamental formulæ with which we shall have to deal throughout this book, and the reader is most earnestly recommended to get a clear grasp of its meaning and commit it to memory.

Since we assume **r** to be constant during the period of growth,

$$\therefore \quad \frac{r}{100} \text{ is constant} = k \text{ (say).}$$

$$\therefore \quad \text{equation } Q_t = Q_o e^{\frac{r}{100}t} \text{ becomes}$$

$$\mathbf{Q_t = Q_o e^{kt}.}$$

This, then, is a somewhat modified form of the same equation for the compound interest law.

[*Note.*—**k** is called the constant of increment.]

Further, by taking logarithms on both sides, we get

$$\log_{10} Q_t = \log_{10} Q_o + kt \log_{10} e.$$

$$\therefore \quad \log_{10} Q_t - \log_{10} Q_o = kt \log_{10} e$$

or

$$kt \log_{10} e = \log \frac{Q_t}{Q_o} \cdot$$

But since

$$e = 2{\cdot}718 \ldots$$

$$\therefore \quad \log_{10} e = \log 2{\cdot}718 = 0{\cdot}4343.$$

$$\therefore \quad {\cdot}4343\ kt = \log \frac{Q_t}{Q_o}.$$

$$\therefore \quad k = \frac{1}{{\cdot}4343t} \log \frac{Q_t}{Q_o}$$

$$= 2{\cdot}302 \log \frac{Q_t}{Q_o}.$$

Now, as Q_t represents the amount to which Q_o has grown in time t, ∴ $Q_t = Q_o + x_t$ where x_t represents the increment during the time t.

∴ finally, the most general way of expressing the compound interest law is $\mathbf{k = 2{\cdot}302 \log \frac{Q_o + x_t}{Q_o}}$.

(See next Chapter, p. 86, and Chapter XIII., p. 216.)

The equation for the compound interest law may therefore be expressed in either of two forms, viz. :

either (1) exponential form $Q_t = Q_o\mathbf{e}^{kt}$.

or (2) logarithmic form $k = \frac{2{\cdot}30}{t} \log_{10} \frac{Q_o + x_t}{Q_o}$.

The second form is the one most commonly used in the literature, but sometimes it is more convenient to use the first form. *E.g.*, if one has at hand a table of logarithms to the base **e** (*i.e.*, one giving the values of $\mathbf{e}^x$ for different values of x), then the first form of the equation is very serviceable ; otherwise it is more convenient to use the second form. Whichever formula the student uses he must never forget what k stands for : it is the rate at which the magnitude increases or decreases in a unit of time.

The Distinguishing Peculiarity of the Series e.—The series $1 + \frac{1}{1} + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,2\,.\,3} + \frac{1}{1\,.\,2\,.\,3\,.\,4}$ has one primary remarkable property which renders it peculiarly adaptible to logarithmic computation. This peculiarity is that

$$\mathbf{e}^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \cdots$$

where x is any number, whether positive or negative, and whether an integer or a fraction.

That this is so can be seen at once from the following considerations :

$$\mathrm{e}^x = \left(1 + \frac{1}{n}\right)^{nx} = \left(1 \times \frac{nx}{1} \cdot \frac{1}{n}\right) + \frac{nx(nx-1)}{1 \,.\, 2} \cdot \frac{1}{n^2}$$
$$+ \frac{nx(nx-1)(nx-2)}{1 \,.\, 2 \,.\, 3} \cdot \frac{1}{n^3}$$
$$+ \; . \; . \; . \; . \; . \; .$$
$$= 1 + \frac{nx}{1} \cdot \frac{1}{n} + \frac{n^2x^2}{1 \,.\, 2}\left(1 - \frac{1}{nx}\right) \cdot \frac{1}{n^2}$$
$$+ \frac{n^3x^3\left(n - \frac{1}{nx}\right)\left(1 - \frac{2}{nx}\right)}{1 \,.\, 2 \,.\, 3} \cdot \frac{1}{n^3}$$
$$+ \; . \; . \; . \; . \; .$$
$$= 1 + \frac{x}{1} + \frac{x^2}{1 \,.\, 2} + \frac{x^3}{1 \,.\, 2 \,.\, 3} + \frac{x^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \; . \; . \; .$$

when n is infinitely large. There is no other algebraical series which has the same property.

The series $$1 + \frac{x}{1} + \frac{x^2}{1 \,.\, 2} + \frac{x^3}{1 \,.\, 2 \,.\, 3} + \; . \; . \; .$$

is called the exponential series, and the identity

$$\mathrm{e}^x = 1 + \frac{x}{1} + \frac{x^2}{1 \,.\, 2} + \frac{x^3}{1 \,.\, 2 \,.\, 3} + \; . \; . \; .$$

is called the *exponential theorem.*

The exponential theorem, then, gives an expansion in terms of the index. For another peculiarity see p. 152.

Use of Exponential Theorem for calculating Logarithms.—Supposing we wish to find the logarithm of **a**. Now, in the same way as any number can be represented as some power of 10 (see p. 9), so can also any number be represented as a power of **e**.

Let us therefore put $\mathrm{e}^m = a$ (so that $m = \log_{\mathrm{e}} a$).

$$\therefore \quad a^x = \mathrm{e}^{mx}$$
$$= 1 + \frac{mx}{1} + \frac{m^2x^2}{1 \,.\, 2} + \frac{m^3x^3}{1 \,.\, 2 \,.\, 3} + \frac{m^4x^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \; . \; . \; . \; .$$

(by the exponential theorem).

$$\therefore \quad a^x - 1 = \frac{mx}{1} + \frac{m^2x^2}{1 \,.\, 2} + \frac{m^3x^3}{1 \,.\, 2 \,.\, 3} + \; . \; . \; . \; .$$
$$= x\left\{\frac{m}{1} + \frac{m^2x}{1 \,.\, 2} + \frac{m^3x^2}{1 \,.\, 2 \,.\, 3} + \; . \; . \; . \; .\right\}$$
$$\therefore \quad \frac{a^x - 1}{x} = \frac{m}{1} + \frac{m^2x}{1 \,.\, 2} + \frac{m^3x^2}{1 \,.\, 2 \,.\, 3} + \; . \; . \; . \; .$$

This being an identity is of course true for *all* values of x, and hence it is also true for $x = 0$.

Now when $x = 0$, we have the right side of the equation $= m = \log_e a$.

$$\therefore \quad \log_e a = \frac{a^x - 1}{x} \text{ when } x = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

To find the value of $\frac{a^x - 1}{x}$, when $x = 0$, we convert a^x into a binomial expansion by putting $a = 1 + b$.

$$\text{Then } a^x = (1 + b)^x = 1 + \frac{x - b}{1} + \frac{x(x-1)b^2}{1\,.\,2} + \frac{x(x-1)(x-2)b^3}{1\,.\,2\,.\,3} + \ldots$$

$$\therefore \quad \frac{a^x - 1}{x} = \frac{b}{1} + \frac{(x+1)b^2}{1\,.\,2} + \frac{(x-1)(x-2)b^3}{1\,.\,2\,.\,3} + \frac{(x-1)(x-2)(x-3)b^4}{1\,.\,2\,.\,3\,.\,4} + \ldots$$

which, when $x = 0$, becomes

$$b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \ldots .$$

But when $a = (1 + b)$, $\log_e a$ becomes $\log_e (1 + b)$.

$$\therefore \quad \log_e (1 + b) = b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \ldots .$$

This is called the *logarithmic series.*

Hence the great reason for using **e** *as a base for logarithms is that the logarithms of any number to this base can be at once calculated by using the logarithmic series.* For this reason Napierian logarithms are also called *natural logarithms.*

As a matter of fact this particular series is in itself not of great practical value for two reasons, viz. :

(α) The series only holds good for values of **b** lying between $- 1$ and $+ 1$; since for any values of b higher than $+ 1$ or less than $- 1$, each of the terms gradually and rapidly increases in value and the series therefore becomes **divergent** instead of **convergent.**

(β) Even for values of b between $- 1$ and $+ 1$, unless b is very small, the series converges so slowly that it would be necessary to take very many terms in order to calculate the logarithm, even to a small number of decimals.

Hence for practical purposes it is necessary to convert the series into a more convergent one. This can be easily done by a little algebraical transformation, which is, however, outside the scope of the present book. Students wishing to pursue the subject further must refer to any text-book on higher algebra.

Example.—Find $\log_e 1{\cdot}5$.

In the series $\log_e (1 + b) = b - \frac{b^2}{2} + \frac{b^3}{3} \ldots$ put $b = 0{\cdot}5$, and we get

$$\log_e 1{\cdot}5 = {\cdot}5 - \frac{{\cdot}25}{2} + \frac{{\cdot}125}{3} - \frac{{\cdot}0625}{4} + \frac{{\cdot}03175}{5} - \frac{{\cdot}015875}{6}$$
$$= {\cdot}4057.$$

This can easily be converted into common logarithms by multiplying by ${\cdot}4343$ (see p 80).

$$\therefore \quad \log_{10} 1{\cdot}5 = {\cdot}4057 \times {\cdot}4343$$
$$= {\cdot}176. \; . \; . \; .$$

Napierian Logarithms.—We saw in Chapter II., p. 11, that whilst for ordinary computation purposes common logarithms (to the base 10) are universally employed, the logarithms used for the purposes of higher mathematical analysis are the Napierian ones which are calculated to the base 2·71828 . . . The thoughtful student, who has most probably wondered why such a peculiar base was chosen, will now have found the solution to the riddle. The number 2·71828 . . . is merely the sum of the series *e* which is not only so very useful for the purpose of calculating logarithms, but occurs in the consideration of all natural phenomena which take place in accordance with the compound interest law (see Chapter VII.).

The Modulus.—Having calculated the logarithm of any number to the base e it is easy to find what the logarithm of the same number is to base 10, or, indeed, to any other base. Thus it has been found that $\log_e 10 = 2{\cdot}302585$.

This means that $e^{2\cdot302585} = 10$

$$\therefore \quad e = 10^{\frac{1}{2\cdot302585}} = 10^{0\cdot434294}.$$
$$\therefore \quad e^m = 10^{0\cdot434294m} = a \text{ (say)}.$$
$$\therefore \quad m = \log_e a$$

and
$$0{\cdot}434294m = \log_{10} e^m = \log_{10} a.$$
$$\therefore \quad 0{\cdot}434294 = \frac{\log_{10} a}{\log_e a}$$
$$\therefore \quad \log_{10} a = 0{\cdot}434294 \log_e a = {\cdot}4343 \log_e a.$$

In other words, *in order to convert the logarithm of a number from the base* e *to the base* 10, *it is necessary to multiply the logarithm of the number to the base* e *by* 0·434294. *This conversion figure* 0·4343 *is called the modulus of the common system of logarithms* and is frequently denoted by the letter μ.

Conversely, if we are given any formula in which the logarithms are expressed to the base e and we want to convert it to one with logarithms to the base 10, it is necessary to multiply by $\frac{1}{\cdot4343}$, *i e.*, by 2·3025.

Thus, if
$$k = \frac{1}{t} \log_e \frac{Q_o + x_t}{Q_o}$$

then
$$k = \frac{2{\cdot}3}{t} \log_{10} \frac{Q_o + x_t}{Q_o} \text{ (see p. 77)}$$

Example.—The influence of temperature upon the velocity of protein digestion is given by the equation

$$\frac{k_2}{k_1} = e^{5285\frac{(T_2 - T_1)}{T_1T_2}},$$

where k_1 and k_2 are the velocities at the absolute temperatures T_1, T_2. At what temperature will the velocity be double that at 22·6° ?

$$\text{Log}\,\frac{k_2}{k_1} = 5285\,\frac{(T_2 - T_1)}{T_1T_2}\log e$$

$$= \cdot 4343 \times 5285\,\frac{(T_2 - T_1)}{T_1T_2}.$$

In our case $k_2 = 2k_1$ and $T_1 = 273 + 22{\cdot}6 = 295{\cdot}6.$

$$\therefore \quad \log 2 = \cdot 4343 \times 5285\,\frac{(T_2 - 295{\cdot}6)}{295{\cdot}6T_2},$$

or

$$\cdot 30103 = \frac{\cdot 4343 \times 5285\,(T_2 - 295{\cdot}6)}{295{\cdot}6T_2},$$

or

$$\frac{\cdot 30103 \times 295{\cdot}6}{\cdot 4343 \times 5285}T_2 = T_2 - 295{\cdot}6,$$

i.e., $0{\cdot}039T_2 = T_2 - 295{\cdot}6,$

or $0{\cdot}961T_2 = 295{\cdot}6.$

$$\therefore \quad T_2 = \frac{295{\cdot}6}{0{\cdot}961} = 307{\cdot}6 \text{ absolute} = 34{\cdot}6^\circ.$$

The Meaning of the Expression e^{-1}.—We have seen that e stands for the limit of the expansion $\left(1 + \frac{1}{n}\right)^n$, when $n = \infty$, and we have also seen that $e^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \ldots$

Therefore, by taking $x = -1$, we get

$$e^{-1} = 1 - \frac{1}{1} + \frac{1}{1\,.\,2} - \frac{1}{1\,.\,2\,.\,3} + \ldots.$$

But

$$\left(1 - \frac{1}{n}\right)^n = 1 + n\left(-\frac{1}{n}\right) + \frac{n(n-1)}{1\,.\,2}\left(-\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\left(-\frac{1}{n}\right)^3 + \ldots$$

$$= 1 - \frac{1}{1} + \frac{1}{1\,.\,2} - \frac{1}{1\,.\,2\,.\,3}, \text{ when } n = \infty.$$

$$\therefore \quad \text{Lt}_{n\rightarrow\infty}\left(1 - \frac{1}{n}\right)^n = e^{-1}. = \frac{1}{e}.$$

Now in the same way as the terms of the expansion $\left(1+\frac{1}{n}\right)^n$, when $n=\infty$, express the positive growth—or increase—of a certain quantity from moment to moment when the rate of increase is such that the *increment* at any moment is proportional to the value of the quantity at that particular moment, so do the terms of the expansion $\left(1-\frac{1}{n}\right)^n$ when $n=\infty$ express the negative growth—or diminution—of a certain quantity from moment to moment when the rate of diminution is such that *decrease* at any moment is proportional to the value of that quantity at that moment.

Example.—Find the value of $e^{1\cdot 3}$ to three places of decimals.

$$e^{1\cdot 3} = 1 + \frac{1\cdot 3}{1} + \frac{1\cdot 3^2}{1\,.\,2} + \frac{1\cdot 3^3}{1\,.\,2\,.\,3} + \dots$$
$$= 3\cdot 669.$$

EXERCISES.

(1) Prove that:

$$\frac{e+e^{-1}}{2} = 1 + \frac{1}{2\,!} + \frac{1}{4\,!} + \frac{1}{6\,!} + \dots$$

(2) Find, by logarithms, the value of $e^{0\cdot 32}$.

[$\cdot 32 \log e = \cdot 32 \times \cdot 4343 = \cdot 139 = \log 1\cdot 38.$
$\therefore\ e^{0\cdot 32} = 1\cdot 38$].

(3) Given $\log_{e10} 9 = \cdot 954243$, find the value of $\log_{10} 11$.

[$\text{Log}_e\, 9 = 2\cdot 3026$, $\log_{10} 9 = 2\cdot 19722$.

$$\text{But } \log_e \frac{11}{9} \text{ which } = \log_e \left(\frac{1+\frac{1}{10}}{1-\frac{1}{10}}\right) = 2\left\{\frac{1}{10} + \frac{1}{3\times 10^3} + \dots\right\}$$
$$= \cdot 2006.$$
$$\therefore\ \log_e 11 = 2\cdot 19722 + \cdot 2006 = 2\cdot 39782.$$
$$\therefore\ \log_{10} 11 = 2\cdot 39782 \times \cdot 4343 = 1\cdot 0413.]$$

CHAPTER VII.

THE SIMPLE AND COMPOUND INTEREST LAWS IN NATURE.

(A) Examples of the Simple Interest Law.

ALTHOUGH, as we shall see later (p. 84), very many of the phenomena occurring in Nature take place in accordance with the law of compound interest, there are a good few examples of natural phenomena which, at any rate, as a first approximation obey the simple interest law.

(1) **The Coefficient of Expansion.**—Supposing a body such as a thin, long metal rod to be heated, then we know from elementary physics that if l_o = length of rod at 0° C.

and a = coefficient of expansion,

then l_t, the length of the rod at t° C.

$$= l_o(1 + at).$$

Thus, putting $t = 0, 1, 2, 3, 4$, etc., we get the length of the rod at these different temperatures.

$$l_o, l_o(1 + a), l_o(1 + 2a), l_o(1 + 3a), l_o(1 + 4a) \ . \ . \ . \ ,$$

i.e., the terms of an arithmetical progression. In other words, the law of thermal increase in length is the same as the simple interest law.

(2) **Henry's Law of Solubility of Gases.**—This law states that at constant temperature the weight of gas dissolved by a unit volume of liquid is proportional to the pressure.

Thus

$$W = mp.$$

where m = amount of gas dissolved at one atmosphere and W = amount of gas dissolved at p atmospheres.

(3) **Distance covered by a Body moving with a Uniform Velocity,** when we know from elementary mechanics that

$$S = Vt,$$

where V is the velocity per unit of time and S is the distance from some fixed point covered in t units of time, measured from some fixed instant.

(B) Examples of the Compound Interest Law in Nature.

The form of growth in which the rate of increment or of decrement is at every instant proportional to the magnitude (at that instant) of that which is increasing or decreasing is of particularly frequent occurrence in Nature, and hence Lord Kelvin classified all these forms of natural growth as examples of "the compound interest law in Nature." The following are a few examples :

(1) **The Growth of a Population.**—We have already seen on p. 17, example (9), that the increase of a population—assuming the rate of growth to be constant, *i.e.*, undisturbed by undue emigration, immigration, war, epidemics, etc., takes place in accordance with the law of compound interest. Now, suppose the population of a certain country to double itself in 100 years; what is the rate of growth, per annum, assuming it to be constant? If the population is a million at the beginning of the century, what will it be in 20, 50, and 80 years respectively from the beginning?

Here the most convenient equation to use is

$$Q_t = Q_o e^{\frac{r}{100}t}.$$

$$\therefore \quad Q_{100} = Q_o e^{\frac{r}{100} \times 100} = Q_o e \;\; = 2Q_o \text{ (by hypothesis).}$$

$$\therefore \quad e^r = 2.$$

$$\therefore \quad r \log e = \log 2.$$

$$\therefore r = \frac{\log 2}{\cdot 4343} = \frac{\cdot 30103}{\cdot 4343}, \text{ whence } r = \cdot 6931,$$

i.e., the increase of the population is at the rate of 0·6931 per cent., or at the rate of 6·931 per thousand.

In 20 years' time the population is :

$$Q_{20} = Q_o e^{\frac{r}{100} \times 20} = Q_o e^{0\frac{\cdot 6931}{100} \times 20}$$
$$= Q_o e^{0 \cdot 1386}$$
$$= 1{,}000{,}000\, e^{0 \cdot 1386}$$

$$\therefore \quad \log_{10} Q_{20} = \log_{10} 1{,}000{,}000 + 0{\cdot}1386 \log_{10} e$$
$$= 6 + 0{\cdot}1386 \times \cdot 4343$$
$$= 6{\cdot}06019$$

$$\therefore \quad Q_{20} = 1{,}148{,}700.$$

In 50 years' time

$$Q_{50} = 1,000,000\ e^{\frac{\cdot 6931}{100} \times 50} = 1,000,000\ e^{\cdot 3465}$$

$$\therefore \quad \log Q_{50} = 6 + \cdot 3465 \times \cdot 4343$$
$$= 6 \cdot 1505$$
$$\therefore \quad Q_{50} = 1,414,200.$$

In 80 years' time

$$Q_{80} = 1,000,000\ e^{\frac{\cdot 6931}{100} \times 80} = 1,000,000\ e^{\cdot 5544}.$$

$$\therefore \quad \log Q_{80} = 6 + \cdot 4343 \times \cdot 5544$$
$$= 6 \cdot 2408.$$
$$\therefore \quad Q_{80} = 1,741,100.$$

The commonest way in which the compound interest law is met with in natural phenomena is when a magnitude **decreases** continually in proportion to its size at any moment.

The following are a few examples:

(2) Arseniuretted hydrogen (AsH_3), when heated, splits up into arsenic and hydrogen in accordance with the equation

$$2AsH_3 = 2As + 3H_2.$$

Now, by Guldberg and Waage's law, the rate of decomposition at any moment is proportional to the active mass of the substance undergoing decomposition, *i.e.*, to the amount of the substance present at that moment.

Therefore: If 1 grm. of AsH_3 becomes, say, 0·9 grm. after one minute,

then the (0·9) grm. of AsH_3 becomes (0·9) (0·9) grm. = $(0{\cdot}9)^2$ grm. after the second minute.

and the $(0{\cdot}9)^2$ grm. of AsH_3 becomes $(0{\cdot}9)^2(0{\cdot}9) = (0{\cdot}9)^3$ grm. after the third minute,

and the $(0{\cdot}9)^3$ grm. of AsH_3 becomes $(0{\cdot}9)^3(0{\cdot}9) = (0{\cdot}9)^4$ grm. after the fourth minute,

and so on.

Here, then, since the AsH_3 keeps on decomposing at every moment, the ultimate quantity left after ten minutes—when diminishing in accordance with the simple interest law (*i.e.*, if the amount of AsH_3 decomposed every instant would be replaced so as to make the active mass always equal to 1 grm.) the total amount of **original** AsH_3 would have entirely disappeared—is $\left(1 - \frac{1}{n}\right)^n$ when n is infinitely great. This, as we have seen on p. 81), is the same as $\left(1 + \frac{1}{n}\right)^{-n}$ when $n = \infty$, and hence we get the fundamental equation for this class of phenomena as

$$Q_t = Q_o e^{-\frac{rt}{100}} = Q_o e^{-kt} \left(\text{where } k = \frac{r}{100}\right),$$

or, taking logarithms of both sides,

$$\log_e Q_t = \log_e Q_o - kt,$$

whence

$$kt = \log_e Q_o - \log_e Q_t$$
$$= \log_e \frac{Q_o}{Q_t}.$$

But as Q_t represents the amount to which Q_o has diminished in time t,

$$\therefore \quad Q_t = Q_o - x_t$$

where x_t represents the decrement during the time t.

∴ finally the most general way of expressing the compound interest law, when the magnitude continually diminishes, is

$$k = \frac{1}{t} \log e \frac{Q_o}{Q_o - x_t}$$

or

$$k = 2{\cdot}3026 \frac{1}{t} \log_{10} \frac{Q_o}{Q_o - x_t}.$$

Hence, whenever the reader comes across either of the following equations, viz. :

$$Q_t = Q_o e^{kt}$$
$$Q_t = Q_o e^{-kt}$$
$$k = \frac{1}{t} \log \frac{Q_o + x_t}{Q_o}$$
$$k = \frac{1}{t} \log \frac{Q_o}{Q_o - x}$$

he knows at once that he is dealing with an example of growth in accordance with the compound interest law.

(3) A pane of glass obliterates 5 per cent. of the light falling upon it ; how much light gets through twenty such panes, one behind the other, assuming that they all act in the same way ?

$Q_t = Q_o e^{-kt}$. (Q_t = amount of light passing through t panes of glass.)

When $t = 1$ pane of glass, 5 per cent. of light is obliterated, *i.e.*, 95 per cent. of the light passes through.

$$\therefore \quad Q_1 = \frac{95}{100} Q_o.$$

But $\quad Q_1 = Qe^{-k.1} = Q.e^{-k}.$

$\therefore \quad e^{-k} = 0{\cdot}95.$

i.e., $\quad \frac{1}{e^k} = 0{\cdot}95.$

$\therefore \quad e^k = \frac{1}{0{\cdot}95} = 1{\cdot}053.$

$\therefore \quad k \log_{10} e = \log_{10} 1{\cdot}053$, or $\cdot 4343\, k = \log 1{\cdot}053 = \cdot 02243.$

$\therefore \quad k = \frac{\cdot 02243}{\cdot 4343} = \cdot 05.$

$\therefore \quad Q_{20} = Q_o e^{-20k}$

$\qquad = Qe^{-1}.$

$\therefore \quad \frac{Q_o}{Q_{20}} = e^1 = 2{\cdot}72.$

$\therefore \quad \frac{Q_{20}}{Q_o} = \frac{1}{2{\cdot}72} = \cdot 37.$

$\therefore$ About 37 per cent. of the light passes through.
This is the principle of the spectrophotometer.

If the thickness of one layer of glass in the last example is 10 cm., what thickness of the same glass will reduce the intensity of the light to $\frac{1}{2}$?

Here $\quad Q_t = \frac{1}{2}Q_o$

$$\therefore \quad \frac{1}{2}Q_o = Q_o e^{-kt},$$

or

$$\frac{1}{2} = e^{-kt},$$

$$\text{or } 2 = e^{kt}.$$

But $\quad k = \cdot 05.$

$$\therefore \quad 2 = e^{0{\cdot}05t} = e^{\frac{t}{20}},$$

$$2{\cdot}3 \log_{10} 2 = \frac{1}{20}t.$$

$$\therefore \quad t = 46 \log 2$$
$$= 46 \times \cdot 30103$$
$$= 13{\cdot}85 \text{ layers}$$
$$= 138{\cdot}5 \text{ cm. thick.}$$

(4) Another most interesting example is **the rate of cicatrisation of a wound,** which has been shown by Carrel, Hartmann, Lecomte

du Noüy and others, to follow the compound interest law (see *Journ. Exp. Med.*, Vol. 24, 1916, and Vol. 27, 1918).

Example.—The following results were found by Winkelman in the case of a cooling body.

t (time).	θ (temperature).
0	18·9
7·40	16·9
15·85	14·9
25·35	12·9
36·65	10·9

Show that the rate of cooling agrees with the compound interest law.
If the law of cooling is the same as the compound interest law, then

$$\theta_t = \theta_0 e^{-kt}.$$

$\therefore$ (i.) $\theta_{7\cdot 40} = \theta_0 e^{-kt},$

i.e., $16\cdot 9 = 18\cdot 9 e^{-k \times 7\cdot 4},$

i.e., $\dfrac{18\cdot 9}{16\cdot 9} = e^{7\cdot 4k},$

i.e., $1\cdot 12 = e^{7\cdot 4k}.$

$\therefore$ $7\cdot 4k = 2\cdot 3 \log 1\cdot 12 = 2\cdot 3 \times \cdot 04922 = 1\cdot 132.$

$\therefore$ $k = \dfrac{1132}{74000} = 0\cdot 0153.$

(ii.) $\theta_{15\cdot 85} = 14\cdot 9.$

$\therefore$ $14\cdot 9 = 18\cdot 9 e^{-5\cdot 85k},$

i.e., $e^{15\cdot 85k} = 1\cdot 27.$

$\therefore$ $15\cdot 85k = \cdot 2390$, whence $k = 0\cdot 0151.$

(iii.) $\theta_{25\cdot 35} = 12\cdot 9.$

$\therefore$ $12\cdot 9 = 18\cdot 9 e^{-25\cdot 35k}.$

$\therefore$ $1\cdot 47 = e^{25\cdot 35k}.$

$\therefore$ $25\cdot 35k = \cdot 3853$

whence $k = \cdot 0152.$

(iv.) As $t = 36\cdot 65$ we have

$$10\cdot 9 = 18\cdot 9 e^{-36\cdot 65k}.$$

$\therefore$ $\dfrac{18\cdot 9}{10\cdot 9} = e^{36\cdot 65k}$

i.e., $e^{36\cdot 65k} = \dfrac{18\cdot 9}{10\cdot 9} = 1\cdot 73.$

$\therefore$ $36\cdot 65k = \cdot 5481,$

whence $k = \dfrac{0\cdot 5481}{36\cdot 65} = \cdot 0149.$

Hence we have the following results :

Between the intervals, 0 and 7·40, $k = 0\cdot 0153.$
" " 0 and 15·85, $k = 0\cdot 0151.$
" " 0 and 25·35, $k = 0\cdot 0152.$
" " 0 and 36·65, $k = 0\cdot 0149.$

In other words, within the limits of experimental error, on the assumption that the law of cooling follows the compound interest law, the value of k remains constant.

Hence the assumption is most probably correct.

Note.—The student must not fall into the trap of believing that because the observed and calculated results agree, therefore the theory in question is necessarily true. For whilst it is the case that disagreement between the observed and calculated results is definite evidence against the theory in question, agreement between the two results is not absolute evidence in its favour. Occasionally two or more different formulæ will give results each of which is in agreement with observation.

The following example will illustrate this point. It is taken from Mellor's "Higher Mathematics" (Longmans, Green & Co.).

Dulong and Petit's formula for velocity of cooling is $V = b(c^{\theta} - 1)$, where $b = 2{\cdot}037$ and $c = 1{\cdot}0077$.

Stefan gives the formula :

$$V = a\{(273 + \theta)^4 - (273)^4\}$$

where $a = 10^{-9} \times 16{\cdot}72$;

and yet the results as calculated by either of these formulæ are practically identical and agree closely with the observed results.

Thus :

θ, excess of temp. of body above that of medium.	Velocity of cooling.		
	Observed.	Calculated by the formulæ of Dulong and Petit.	Stefan.
220	8·81	8·89	8·92
200	7·40	7·34	7·42
180	6·10	6·03	6·09
160	4·89	4·87	4·93
140	3·88	3·89	3·92
120	3·02	3·05	3·05
100	2·30	2·33	2·30

(See also Chapter XIX., p. 288.)

EXAMPLES.

(1) A cup of tea whose temperature five minutes ago was 100° above that of the surrounding objects is now 80° above them. What will be its temperature in another half-hour, assuming it to fall by the compound interest law ?

In five minutes the difference of temperature was reduced from 100° to 80°.

$\therefore$ From equation $Q_5 = Q_0 e^{-5k}$ we have

$$\frac{80}{100} = e^{-5k}$$

$$\therefore \quad \frac{100}{80} \text{ or } 1{\cdot}25 = e^{5k}$$

$$\therefore \quad \log 1{\cdot}25 = 5k \log e$$

$$\therefore \quad 2{\cdot}3 \log 1{\cdot}25 = 5k$$

$$\textit{i.e.,} \quad 5k = 2{\cdot}3 \times {\cdot}097 = {\cdot}2231.$$

In another half-hour, t will be 35 minutes from the beginning

$$\therefore \quad Q_{35} = Q_o e^{-35k} = Qe^{-\cdot 223 \times 7}$$
$$= Q_o e^{-1\cdot 5617}$$
$$\therefore \quad \frac{Q_o}{Q_{35}} = e^{1\cdot 5617}$$
$$\therefore \quad \log \frac{Q_o}{Q_{35}} = 1\cdot 5617 \times \cdot 4343$$
$$= \cdot 6782$$
$$\therefore \quad \frac{100}{Q_{35}} = 4\cdot 767$$
$$\therefore \quad Q_{35} = \frac{100}{4\cdot 77} = 21^\circ \text{ nearly.}$$

(2) A most interesting instance of the compound interest law is the hydrolysis of cane sugar, and of catalysis in general. The hydrolysis (in the presence of acid) may be represented as follows:

$$C_{12}H_{22}O_{11} + H_2O = \underbrace{C_6H_{12}O_6}_{\text{dextrose}} + \underbrace{C_6H_{12}O_6}_{\text{lævulose}}.$$

The acid is found not to undergo any change of concentration during the reaction. Now the amount of water in which the cane sugar is dissolved being very great compared with the amount of the sugar, the change in the concentration of the water brought about by the hydrolysis is negligible, and hence the only change of concentration to be considered is that of the cane sugar. Hence we have

$$Q_t = Q_o e^{-kt}.$$

The change of concentration can be followed by means of the polariscope, and in this way the following figures were obtained (at temperature 75° C.):

t (in mins.).	Angle of rotation.
0	25·16
56	16·95
116	10·38
176	5·46
236	1·85
371	− 3·28
∞	− 8·38

The angle of rotation at time 0 is that obtained before the beginning of the hydrolysis; the angle at time ∞ is that obtained after complete inversion.

Now, since the concentration of the cane sugar is proportional to the angle of rotation,

∴ Q_t is proportional to the *total* angle of rotation,
i.e., Q_o is proportional to (25·16 + 8·38) or 33·54.
Also $Q_o - Q_{116}$,, ,, ,, (25·16 − 10·38) or 14·78.

∴ Amount of cane sugar left at 116 mins. is proportional to (33·54 − 14·78), *i.e.*, 18·76.

Hence we can put $Q_{116} = 18{\cdot}76$ and $Q_o = 33{\cdot}54$.

$\therefore$ equation $Q_t = Q_o e^{-kt}$ becomes

$$18{\cdot}76 = 33{\cdot}54 e^{-116k}.$$

$$\therefore \quad e^{-116k} = \frac{18{\cdot}76}{33{\cdot}54}.$$

$$\therefore \quad e^{116k} = \frac{33{\cdot}54}{18{\cdot}76} = 1{\cdot}79.$$

$$\therefore \quad 116k = 2{\cdot}3 \log 1{\cdot}79 = {\cdot}5822.$$

$$\therefore \quad k = \frac{{\cdot}5822}{116} = \mathbf{0{\cdot}00502}.$$

Similarly at 176, amount of sugar formed may be put $= 25{\cdot}16 - 5{\cdot}46 = 19{\cdot}70$.

$\therefore$ Q_{176} may be put $= 33{\cdot}54 - 19{\cdot}70 = 13{\cdot}74$.

$$\therefore \quad 13{\cdot}74 = 33{\cdot}54 e^{-176k}.$$

$$\therefore \quad 176k = 2{\cdot}3 \log \frac{33{\cdot}54}{13{\cdot}74} = 2{\cdot}3 \log 2{\cdot}44 = {\cdot}8910.$$

$$\therefore \quad k = \mathbf{{\cdot}00506}.$$

Similarly for the other values of t. The value of k will be found in all these cases to be in the neighbourhood of 0·005.

(3) Cohnheim studied the affinity of albumose for HCl in the following manner:

Equal volumes (5 c.c.) of a 10 per cent. solution of cane sugar were respectively mixed with

(i.) 5 c.c. of an HCl solution containing 0·05 grm. HCl.

(ii.) 5 c.c. of a mixture of 0·025 grm. HCl and 0·25 grm. albumose.

The angle of rotation of each of these cane sugar solutions was found to be 4·422. After mixing them respectively as above they were kept for the same length of time (four hours) at the same temperature. At the end of that time the polariscope reading was 3·15 in case (i.), and 1·35 in case (ii.).

Find whether there has been any combination of albumose with HCl, and if so in what proportion.

The logarithmic form of the equation for the compound interest law gives

$$k = \frac{1}{t} \log \frac{Q_o}{Q_o - x_t}.$$

Therefore in our case we have

in case (i.) $\left.\begin{aligned} Q_o &= 4{\cdot}422 \\ x_{t_1} &= 3{\cdot}15 \end{aligned}\right\} \quad \therefore \quad Q_o - x_{t_1} = 1{\cdot}272;$

in case (ii.) $\left.\begin{aligned} Q_o &= 4{\cdot}422 \\ x_{t_2} &= 1{\cdot}35 \end{aligned}\right\} \quad \therefore \quad Q_o - x_{t_2} = 3{\cdot}072;$

and t being the same in each case, we can call t, the unit of time, 1.

$\therefore$ in (i.) $\quad k_1 = \log \dfrac{4{\cdot}422}{1{\cdot}272} = \log 3{\cdot}48 = 0{\cdot}541,$

and in (ii.) $\quad k_2 = \log \dfrac{4{\cdot}422}{3{\cdot}072} = \log 1{\cdot}44 = 0{\cdot}158.$

Now, since the inversion velocities are proportional also to the amounts of free HCl in the two cases, let us suppose that of the 0·025 grm. HCl in the second case a grms. have combined with the albumose, leaving only (0·025 — a) grm. HCl free.

∴ the concentrations in the two cases are 0·05 and (0·025 — a) respectively.

$$\therefore \quad \frac{k1}{k} = \frac{0 \cdot 05}{0 \cdot 025 - a}$$

$$i.e., \quad \frac{0 \cdot 541}{0 \cdot 158} = \frac{0 \cdot 05}{0 \cdot 025 - a}$$

$$i.e., \quad 3 \cdot 42(0 \cdot 025 - a) = \cdot 05.$$

$$\therefore \quad 3 \cdot 42 \times \cdot 025 - 3 \cdot 42a = 0 \cdot 05.$$

$$\therefore \quad a = \frac{3 \cdot 42 \times 0 \cdot 025 - 0 \cdot 05}{3 \cdot 42}$$

$$= \cdot 0104 \text{ grm.}$$

∴ 0·25 grm. albumose combined with 0·0104 grm. HCl, or in proportion of 1 to 0·0416, *i.e.*, in proportion of 100 to 4·16.

Hence the albumose combined with 4·16 per cent. of its own weight of HCl.

EXERCISES.

(1) If a hot body cools so that in 24 minutes its excess of temperature has fallen to half the initial amount, how long will it take to cool down to 1 per cent. of the original excess ?

[Since $\frac{1}{2} = e^{-24k}$, $\therefore \quad 2 = e^{24k}$, whence $k = 0 \cdot 0288$.

If t = time during which temperature has fallen to 1/100 of original excess, we have

$$\frac{1}{100} = e^{-kt} = e^{-\cdot 0288t},$$

whence $t = 160$ minutes.]

(2) The pressure P_h of the atmosphere at an altitude h kilometres is given by $P_h = P_o e^{-kh}$, P_o being the pressure at sea-level (760 mm.). The pressures at 10, 20, and 50 kilometres being 199·2, 42·2 and 0·32 respectively, find the mean value of k.

[Answer, $k = \frac{1 \cdot 34 + 1 \cdot 45 + 1 \cdot 45}{3} = 1 \cdot 44$.]

(3) The quantity Q of a radio-active substance which has not yet undergone transformation is known to be related to the initial quantity Q_o of the substance by the relation $Q = Q_o\ e^{-\lambda t}$, where λ is a constant and t is the time in seconds from beginning of transformation. Find the "mean life" of thorium, and of radium A, being given that for thorium $\lambda = 5$, and for radium A, $\lambda = 3 \cdot 85 \times 10^{-3}$). (By the "mean life" is meant the time required to transform half the substance.)

[Answer, For thorium, 0·14 seconds, and for radium A, 3 minutes.]

(4) An electric current left to die out in a certain circuit drops to $\frac{1}{e}$ of its value in $\frac{1}{10}$ of a second; how long will it take to drop to a millionth of its value, assuming that it decreases at a rate proportional to itself?

$$\left[\frac{1}{e} = e^{-\frac{k}{10}}. \quad \therefore \quad k = 10. \quad \text{Hence } 10^{-6} = e^{-10t}\right.$$

whence $t = 1{\cdot}4$ seconds nearly.]

(5) Cholera bacilli double themselves in number in 30 minutes. Find the number that one bacillus would give rise to in 24 hours.

$$\left[2 = e^{30k}. \quad \therefore \quad k = {\cdot}0230.\right.$$

$$\therefore \quad \text{in 24 hours, number} = e^{1440 \times 0{\cdot}0231}$$

$$= 28 \times 10^{13}.]$$

(6) In a series of experiments with anthrax spores treated with a 5 per cent. solution of phenol at a temperature of 20·2° C., Miss Chick found the following numbers of surviving bacteria present at the stated times. Examine these numbers in the light of Miss Chick's theory that the number of organisms destroyed by the disinfectant at any moment is proportional to the number of living organisms present at that moment.

t (hours) . . .	0	0·5	1·5	2·7	5·95	25·6
n (number of surviving bacteria) . . .	434	410	351	331	241	28

[The numbers of surviving bacteria agree with those found by means of the formula

$$n_t = n_o e^{-kt} \text{ or } k = \frac{2{\cdot}3}{t} \log \frac{n_o}{n_t}.$$

The mean value of k will be found to be 0·108.]

For further examples and exercises, see Chapters XIII. and XX.

CHAPTER VIII.

FUNCTIONS, VARIABLES AND CONSTANTS.

THE term **function** has a different meaning in mathematics from what it has in physiology. In mathematics we speak of one quantity being a function of another when the value of the first quantity depends upon that of the second.

Thus in all the cases we have been considering in Chapter VII., the value of Q keeps on changing from Q_o to Q_t as t changes—whatever **t** happens to represent—and hence we say that **Q** is a function of **t.**

For example, the amount to which a capital grows at simple or compound interest—**when the rate of interest is fixed**—depends upon the time t, *i.e.*, upon the number of years or the other units of time, during which the capital is allowed to grow, and hence we say that the amount to which a capital grows at a fixed rate of interest is a **function** of the time.

Again, the temperature of a cooling body is a **function** of the time during which cooling occurs because, **so long as the temperature of the surroundings is fixed and unaltered**, the temperature gradually decreases as time increases.

The hydrolysis of sugar, or, indeed, the amount of chemical transformation occurring in any chemical reaction, is a function of the time.

The amount of light passing through a given transparent substance is a function of the thickness of that substance.

Indeed, we can multiply the number of examples of functions almost indefinitely. For every problem that one investigates in the laboratory is an example of a function. When we investigate Boyle's law we are dealing with a function, viz., when the temperature is unaltered, then either the pressure of the gas is a **function** of its volume or the volume is a **function** of the pressure ; Charles' law, which says that the volume of a gas **at fixed pressure** depends upon the temperature, is an example of volume being a function of temperature, and so on.

Variables and Constants.—In the examples we have been considering, and, indeed, in the case of any mathematical function, we deal with at least two quantities which keep on changing, provided other conditions remain the same, throughout the period of the experiment. For example, with money growing at interest the quantities that keep on changing are the amount to which the capital has grown and the time during which the capital has been allowed to grow, **provided the rate of interest remains the same**. In the case of chemical transformations, the quantities that keep on changing are the amount of substance transformed and the time during which the chemical reaction has been allowed to proceed, **provided the temperature of the reacting substances is kept unaltered.**

Again, in the case of light passing through a transparent substance, the quantities which keep on changing are the amount of light passing through and the thickness of the substance through which the light is passing, provided the nature of the transparent substance remains the same.

Now, the quantities which keep on changing their value in any function are called *variables*, and the quantity or quantities which remain fixed for the duration of the experiment are called *constants*.

Hence we say that in the case of money growing at fixed interest the **variables** are the amount and time; whilst the **constant** is the rate of interest. In the case of light passing through a transparent medium, the variables are the amount of light transmitted and the thickness of the medium, the constant being the nature of the medium (*i.e.*, whether glass, water, oil, etc.).

In the case of chemical transformation, the variables are the amount of substance transformed and the length of time of the experiment, the constant being the temperature, and so on.

Dependent and Independent Variables.—Now, of the two variables, we speak of one of the variables being dependent upon the other, *e.g.*, the amount to which the capital has grown depends upon the time, and so on.

The variable whose value at any time depends upon another variable is called a **dependent** variable; whilst the other variable, the variation of which determine the value of the dependent variable is called an **independent** variable.

Thus, the amount in the case of money growing at interest is called the dependent variable; the time during which the money grows is the independent variable; whilst the rate of interest is the constant.

Similarly, in the case of chemical reactions, the amount of substance transformed is the dependent variable; the period of transformation is the independent variable; whilst the temperature of the experiment is the constant, and so on, for any of the other cases.

Other examples of functions occurring in biological inquiries are the relationships existing between the sitting height of a person and his vital capacity, or between the sitting height and the circumference of his chest, etc. Thus, Dreyer has shown that if Si = sitting height in centimetres, then

$$\text{vital capacity in c.c.} = \frac{Si^{2\cdot257}}{6\cdot1172} \text{ (in the case of males)}$$

$$\text{chest circumference in centimetres} = \frac{Si^{1\cdot1442}}{2\cdot00148} \text{ (in the case of males)}$$

$$\text{weight of person in grams} = (0\cdot38025\ Si)^{0\cdot319}.$$

(See p. 16, example (7).)

The surface (S in square decimetres) of the body is a function of the weight (W in kilograms), and $S = K\sqrt[3]{w^2}$ (where K is a constant).

Also the temperature of a person is a function of the time of the day (Fig. 26).

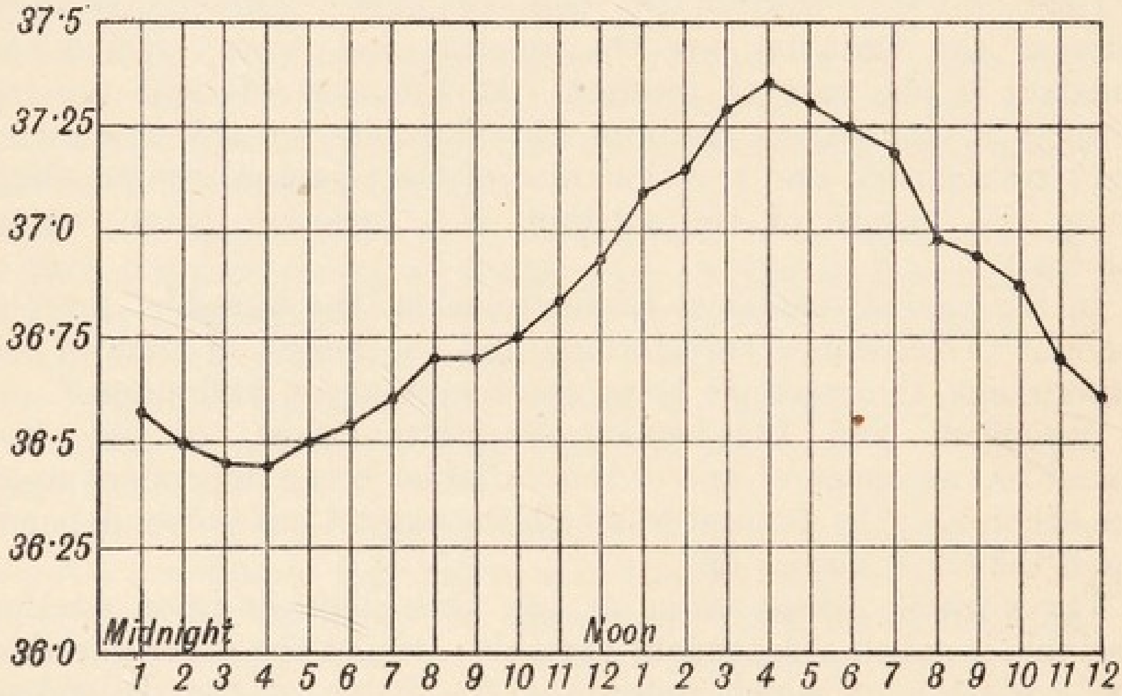

Fig. 26.—A One-Hourly Temperature Chart of a Normal Person.

The pulse of a person is a function of the body temperature (other things remaining constant). The area of a square or a cube is a function of the length of the side, $y = x^2$, and so on.

Definitions.—Hence we arrive at the following *definitions :*

(*a*) *A constant* is a quantity which during any set of operations retains the same value.

(*b*) *A variable* is a quantity which during any set of operations keeps on changing in value.

(i.) *A dependent variable* is one whose value at any moment depends upon the value of another variable at that moment.

(ii.) *An independent variable* is one whose value at any moment determines the value of the dependent variable at that moment.

(*c*) *A function* is the numerical relation between one variable and another, *i.e.*, the relation between the dependent and the independent variables, and is generally written as $y = f(x)$, which is read "y is a function of x."

One generally uses the last few letters of the alphabet to denote variables, *e.g.*, t, u, w, x, y, z, and the remaining letters, *e.g.*, a, b, c or k, l, m, etc., to denote constants, so that in any function it is not necessary to point out in any other way which are the constants and which the variables.

Classes of Functions.—There are two main classes of functions, viz. :

(1) Algebraic. (2) Transcendental.

An **algebraic** function is one which contains the algebraic sum of some powers of x—whether those powers be positive, or negative, integral or fractional ; so that if $y = f(x)$, y can be expressed in terms of x by means of an equation consisting of a *finite* number of terms :

$$e.g., \quad y = x^5 + 7x^4 - x + 2.$$

A **transcendental** function is one which cannot be expressed in terms of x by an equation containing a limited or finite number of terms :

$$e.g., \quad y = \sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \ldots \text{ to infinity.}$$

Functions are also divided into

(1) Explicit. (2) Implicit.

An explicit function is one in which the dependent variable is expressed in terms of the independent variable :

e.g., $y = x^3 + 3x + 5$,

or $y = \log x$,

etc.

An implicit function is one in which the dependent and independent variable are involved together :

e.g., $y^2x - ayx + bx^2 + C = 0.$

Whilst an explicit function is written

$$y = f(x) \text{ or } y = \phi(x), \text{ etc.,}$$

an implicit function is written

$$f(y, x) = 0 \text{ or } \phi(y, x) = 0, \text{ etc.}$$

The Graphical Representation of a Function.

The position of any point in a plane can be completely determined in one of the two following ways:

(1) We can say that the point P is situated at certain distances NP, MP from two fixed straight lines Oy, Ox in the same plane, crossing each other at right angles at the point O (Fig. 27); or

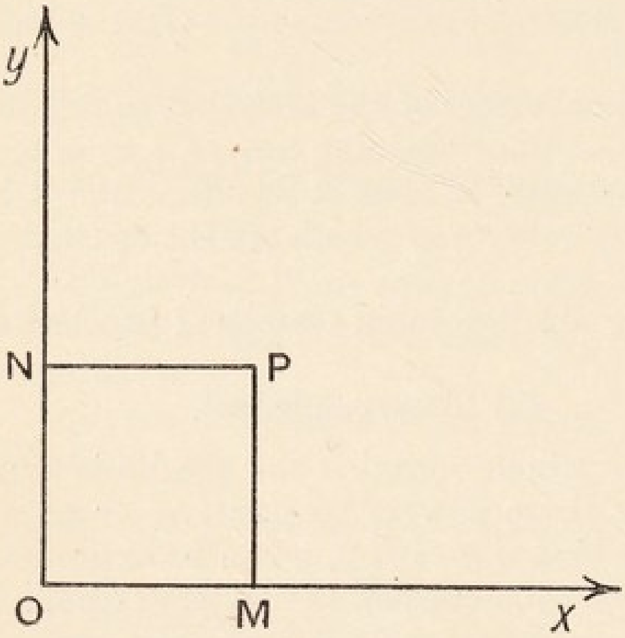

Fig. 27.—Cartesian Co-ordinates.

(2) We can say that the point P is situated at a certain distance OP from a fixed point O in the same plane, and that OP makes a certain angle θ with a fixed line Ox in the same plane (Fig. 28).

The first system of defining the position of a point is called the *Cartesian rectangular co-ordinate system* (after Réné Descartes, who invented it), whilst the second system is called the *Polar co-ordinate system.*

The lines Ox, Oy in the rectangular co-ordinate system are called the *axes*, and O is called the *origin*. The horizontal line Ox is called the *abscissa axis.* The vertical line Oy is called the *ordinate axis*. The distances NP, MP, of the point P from the two axes are called the *co-ordinates* of the point.

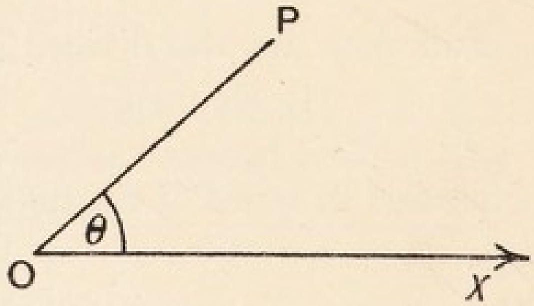

Fig. 28.—Polar Co-ordinates.

As NP = OM, which is a portion of the abscissa axis, and MP = ON, which is a portion of the ordinate axis, the co-ordinates NP, and MP of the point P are called its abscissa and ordinate respectively.

The short way of indicating the position of the point P with reference to the axis is to call it the point (x, y), where x represents the length OM or NP (*i.e.*, the abscissa), and y represents the length ON or MP (*i.e.*, the ordinate).

Thus, if NP = 3 ft. and MP = 4 ft., the period P would be designated as the point (3, 4), and so on.

In the polar system, the point P would be referred to as the point (r, θ), where $r = OP$ and $\theta = \angle POx$. Thus, if OP = 4 ft. and $\theta = 30°$, then the point P would be designated as the point (4, 30°).

In analytical geometry it is sometimes more convenient to use one system and sometimes the other.

By common convention it is agreed to call the direction along Ox, *i.e.*, east of the origin, positive or +, and the direction along Ox, *i.e.*, west of the origin, negative or −.

Also the direction along Oy, *i.e.*, north of the origin, is called positive (+), and the direction along Oy', *i.e.*, south of the origin, is called negative (−).

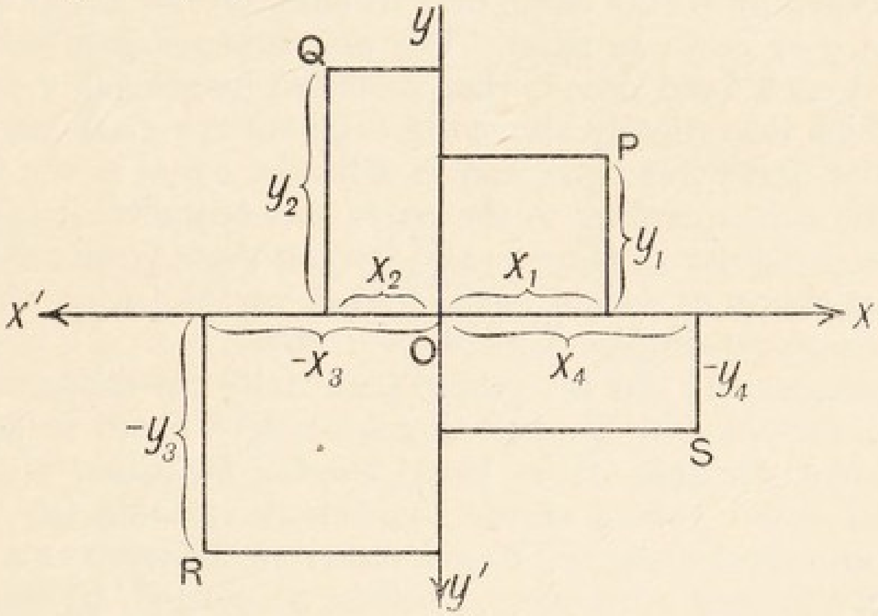

Fig. 29.—Convention with regard to Signs.

Thus the point P would be designated as the point (x_4, y_1).
„ „ Q „ „ „ „ $(-x_2\ y_2)$.
„ „ R „ „ „ „ $(-x_3, -y_3)$.
„ „ S „ „ „ „ $(x_4, -y_4)$.

Hence, if we are given the co-ordinates of any point, we can easily locate its position. Thus, to find the position of the point (4, 2)—the units being in inches, for instance — we would measure off a distance OM (along OX) = 4 in., and then measure

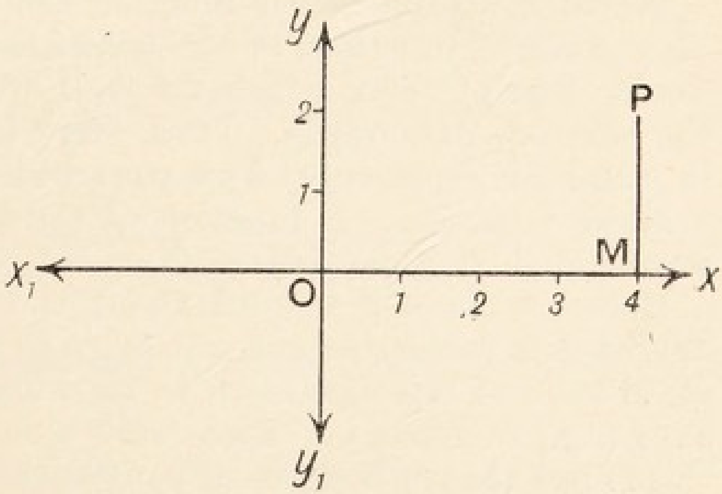

Fig. 30.—Position of a Point P in Space.

off from M, a distance along a line perpendicular to OM of 2 in. The end of this line would give the required point P.

Any one who has ever used a temperature or weight chart on which a patient's temperature or weight is recorded at fixed intervals on squared paper is already acquainted with the use of co-ordinate geometry. In each of those charts successive equal portions of time (hours, or days, as the case may be) are represented by equal lengths measured, according to a certain scale, along a horizontal line (the x or abscissa axis), and degrees of temperature (above normal) or units of weight (above a certain selected weight) are represented by equal lengths measured, either according to the same or a different scale, along a vertical line (the y or ordinate axis). The temperature or weight of the person at each fixed time is then recorded by placing a dot at the point which is vertically above the mark on the x axis corresponding to the particular time, and at a height equal to the length of the y axis corresponding to the particular temperature or weight. The line (irregular in these cases) joining these points shows how the temperature or weight of the person varied during the period under consideration (see Fig. 26 on p. 96).

In the same way one can record pictorially the relation between any two variables by representing successive equal values of the independent variable (x) by equal lengths measured along the x axis, and equal values of the dependent variable (y) by equal lengths measured along the y axis, **either on the same or a different scale.** By joining a number of points, so plotted, by means of a curve one obtains a graphic representation of the particular function.

Graph.—Such a curve, the co-ordinates of every point on which satisfy the relation between the dependent and independent variable in any function $y = f(x)$, is called a *graph* of that function.

Whilst every particular function has its own particular graph, each **group** of functions of the same kind corresponds to one particular **type** of curve which we shall investigate in the course of the next few paragraphs. Thus, every function of the first degree in x and y is represented by a straight line, and is therefore called a **linear** function. A function of the type $y = x^2$ or $(x + a)^2$ is represented by a parabola, and so on.

Here let me say a word about the **scales of representation.** Whilst it is preferable and advisable to adopt, when possible, the same scale of representation in marking the units along the two axes (*e.g.*, representing each unit along Ox and Oy by $\frac{1}{10}$ in.), this is not always possible or convenient. In such cases different scales of representation must be adopted for the two axes. Thus a unit along the x axis may be represented by $\frac{1}{10}$ in., while that along the y axis may be represented on the same diagram by $\frac{2}{10}$, or $\frac{3}{10}$, or $\frac{5}{10}$, or $\frac{10}{10}$ in., and so on. The difference in scale must be allowed for in the course of any calculation in which the graph is used. (See examples below, also Fig. 60, p. 140, etc.)

Note.—(1) The graph is also called the *locus*, which means the path traced by a point which moves in such a way as always to satisfy given conditions.

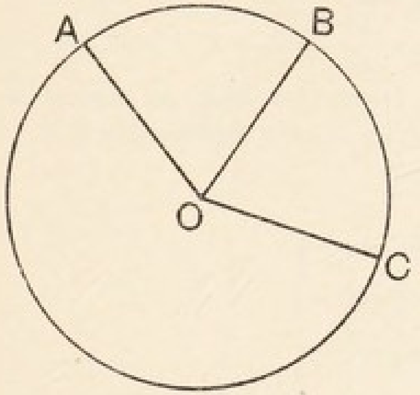

FIG. 31.

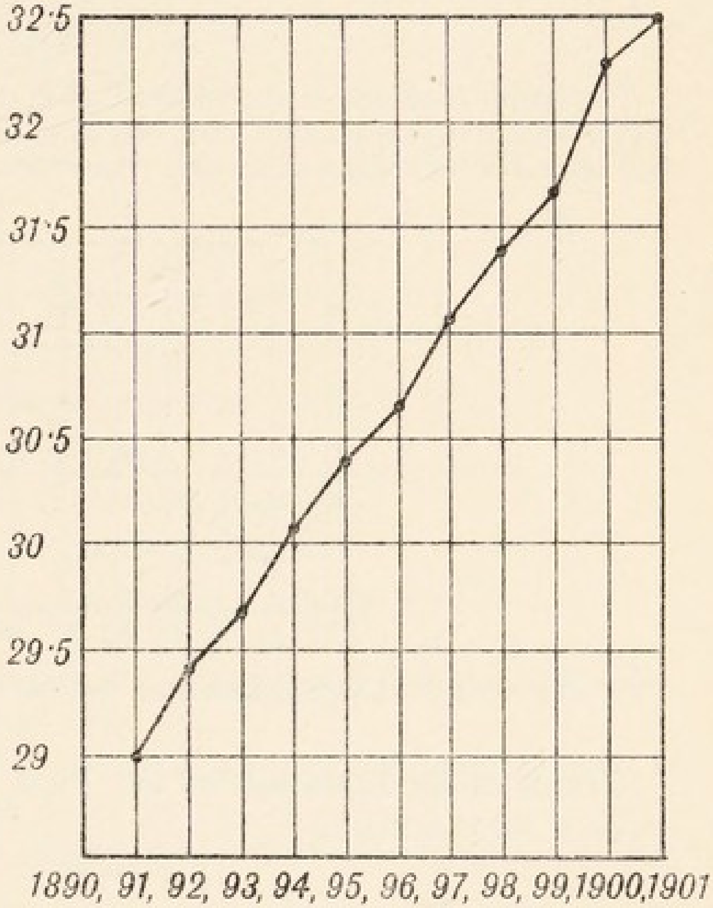

FIG. 32.—Graph of the Population in England and Wales between 1890 and 1901.

E.g., a circle is the locus of point which is always situated at a given distance from a fixed point, since any point, A, B, C, etc., in the circumference is always at the same distance from the centre (O).

Note.—(2) Whilst $y = f(x)$ signifies the fact that "y is a function of x," the expression $y = f(n)$, where n is some *fixed* number, means that "y is the value of the original function of x when the variable x assumes the value of n." Thus if we have a function like $y = 3x^2 + 2x + 1$, then $y = f(4)$ becomes $y = 3(4)^2 + 2(4) + 1 = 48 + 8 + 1 = 57$.

Similarly $y = f(o)$ would mean $y = 3(o) + 2(o) + 1 = 1$.

The expression $y = f(o)$ is one which is frequently met with.

EXAMPLES.

(1) Draw a graph for the following statistics giving the population of England and Wales in millions at the end of each year.

Year ...	1891	1892	1893	1894	1895	1896	1897	1898	1899	1900	1901
Population...	29·0	29·4	29·7	30·1	30·4	30·7	31·4	31·4	31·7	32·3	32·5

The graph is shown in the diagram (Fig. 32). (Each year interval is represented by one division along the x axis, and each half a million population by one division, on a different scale, along the y axis.)

(2) Plot a graph showing the relation between the weight and the height of women from the following statistics :

Weight in lbs. ...	100	106	113	119	130	138	144
Height in inches ...	60	61	62	63	64	65	66

The graph is shown in the diagram (Fig. 33). Here the scale of representation adopted is such that each division along the x axis represents 5 lbs., and each division along the y axis represents 1 inch.

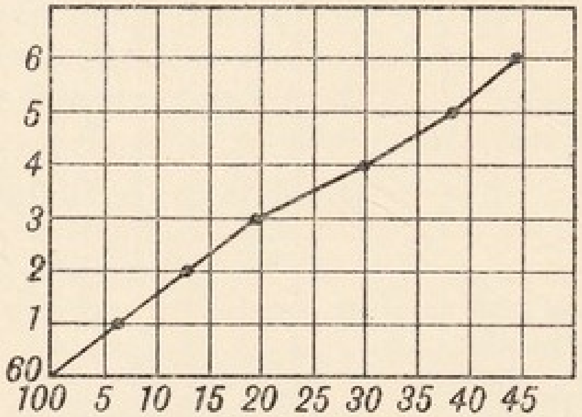

Fig. 33.—Graph showing Relation between Weight and Height of Woman.

Graph of an Equation of the First Degree in x and y.—Let us take a function like

$$y = 2x + 3.$$

In order to draw its graph we proceed as follows :—

(i.) Give to x a series of values, say, 0, 1, 2, 3 . . .

(ii.) Calculate the corresponding values of y, thus :

For $x = 0, y = 2 \times 0 + 3 = 3 \therefore$ graph contains point (0, 3)
,, $x = 1, y = 2 \times 1 + 3 = 5 \therefore$,, ,, ,, (1, 5)
,, $x = 2, y = 2 \times 2 + 3 = 7 \therefore$,, ,, ,, (2, 7)
,, $x = 3, y = 2 \times 3 + 3 = 9 \therefore$,, ,, ,, (3, 9)
,, $x = 4, y = 2 \times 4 + 3 = 11 \therefore$,, ,, ,, (9, 11)
etc., etc., etc.

Lastly, when $y = 0$, $2x + 3 = 0$ and $\therefore x = -\frac{3}{2}$.

(iii.) Plot the various points (0, 3), (1, 5), (2, 7), etc., thus found (see Fig. 34).

(iv.) Join these points by a "curve" passing through them.

It will be seen by inspection (and it is easily proved by the most

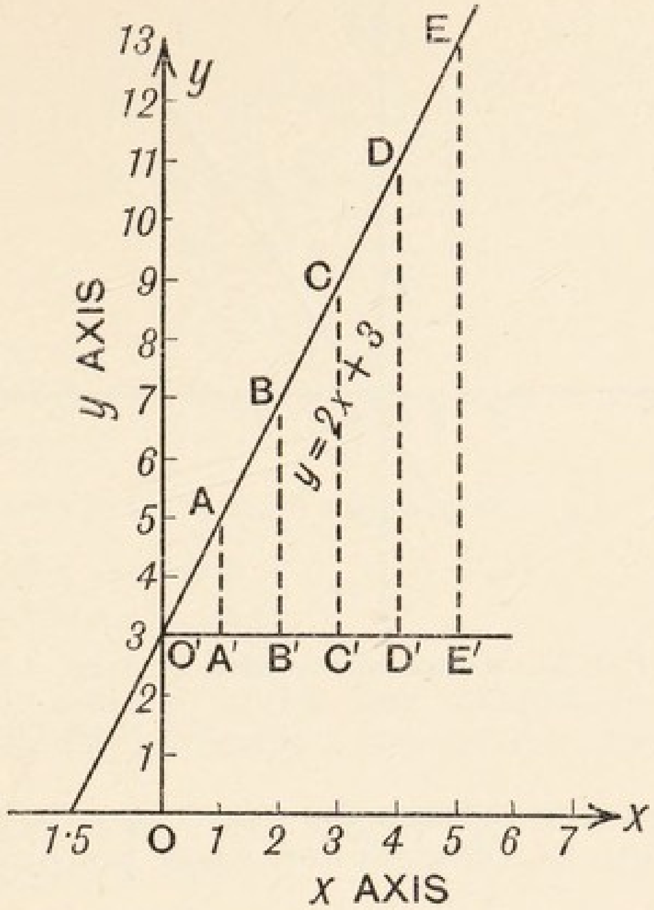

FIG. 34.—Graph of Function $y = 2x + 3$.

rudimentary geometry) that the resulting graph is a straight line, possessing the following properties, viz. :

(1) It is inclined to the axis of x at an angle whose tangent is 2 (thus, AA′/O′A′ = BB′/O′B′ = etc. = 2).

(2) It cuts the y axis at a distance of 3 from the origin.

(3) It cuts the x axis at a distance of $-\frac{3}{2}$ from the origin.

Similarly, we can draw the graph of such a function as

$$2y = 3x - 1.$$

Thus dividing by 2 we get $y = \frac{3}{2}x - \frac{1}{2}$.

For $x = 0, y = -\frac{1}{2}$. ∴ Graph contains the point $\left(0, -\frac{1}{2}\right)$

,, $x = 1, y = 1$. ∴ ,, ,, ,, ,, (1, 1).

,, $x = 2, y = 2\frac{1}{2}$. ∴ ,, ,, ,, ,, $\left(2, 2\frac{1}{2}\right)$.

etc., etc.

Lastly, when $y = 0, x = \frac{1}{3}$.

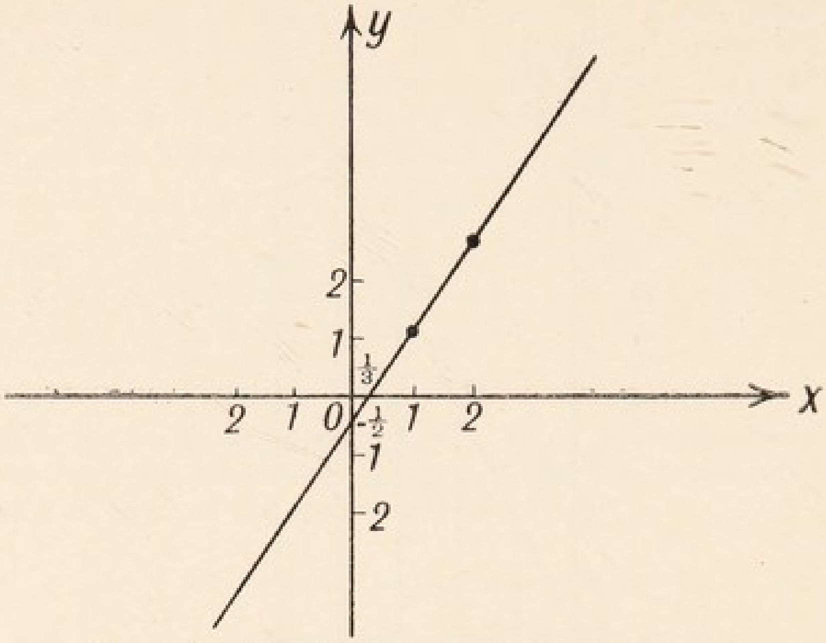

FIG. 35.—Graph of Function $2y = 3x - 1$,
or $y = \frac{3}{2}x - \frac{1}{2}$.

The curve passing through the various points thus found will again be found to be a straight line, whose properties are :

(1) It is inclined to the axis of x at angle whose tangent is $\frac{3}{2}$.

(2) It cuts the axis of y at a distance $-\frac{1}{2}$ from the origin.

(3) It cuts the axis of x at a distance $\frac{1}{3}$ from the origin.

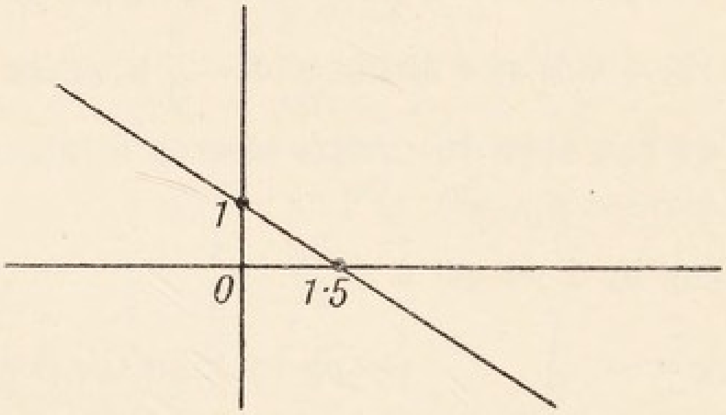

FIG. 36.—Graph of Function $y = -\frac{2}{3}x + 1$,
or $3y = -2x + 3$.

Let us take one more example :

$$y = -\frac{2}{3}x + 1.$$

When $x = 0,\ y = 1.$

,, $x = 1,\ y = \frac{1}{3}$

,, $x = 2,\ y = -\frac{1}{3}.$

,, $x = 3,\ y = -1.$

Lastly, when $y = 0,\ x = \frac{3}{2}.$

The graph will again be found to be a straight line possessing the following properties :

(1) Its inclination to the axis of x is $\tan^{-1}\left(-\frac{2}{3}\right)$.

(2) Its y intercept $= 1$ unit.

(3) Its x intercept $= 1\frac{1}{2}$ units.

In general—

The equation $y = mx + b$, (i.e., any function $y = f(x)$ of the first degree in x and y) represents a straight line whose properties are that :

(1) **Its inclination to the x axis is $\tan^{-1} m$.**

(2) **Its y intercept is b units.**

(3) **Its x intercept is $-\frac{b}{m}$ units.**

From this there result the following three *corollaries*, viz. :—

(*a*) If $b = 0$, then the line passes through the origin ; *e.g.*, the accompanying graphs represent the lines

$$y = 2x;\quad y = \frac{3}{2}x;\quad \text{and } y = -\frac{x}{2}.$$

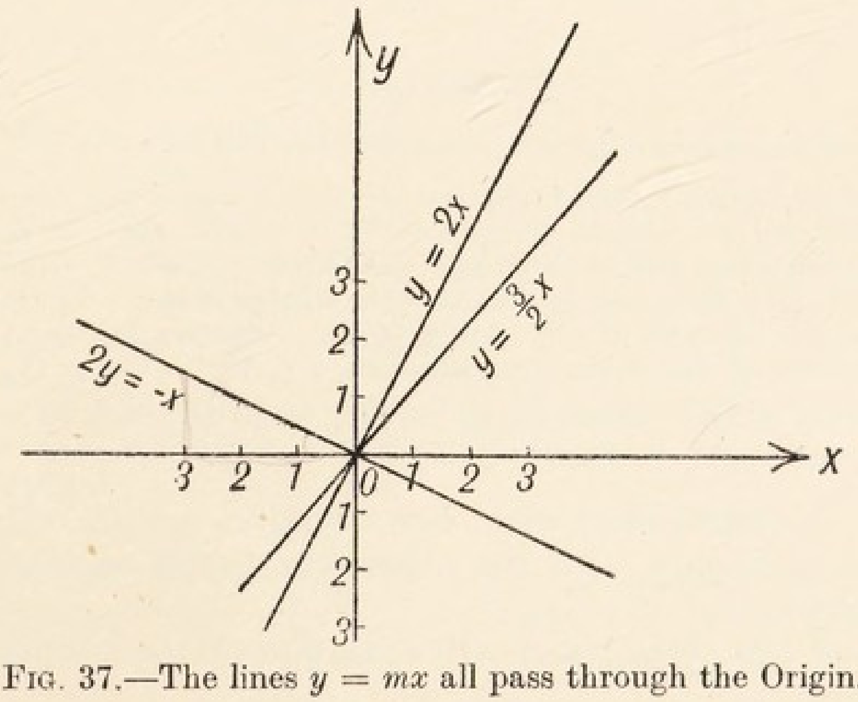

FIG. 37.—The lines $y = mx$ all pass through the Origin.

(*b*) If $x = 0$, then the graph is a line parallel to the x axis at a distance b units from it.

(*c*) If $y = 0$, then the graph is a line parallel to the y axis at a distance $-\frac{b}{m}$ from it.

(*d*) **All equations of the first degree in which the coefficients of y and the coefficients of x are respectively equal are represented by lines which are parallel to one another,** because, m being the same in all, their inclinations to the x axis are the same, viz., $\tan^{-1} m$. Similarly, **all lines in which the relationship between the coefficients of x and y is the same are parallel to one another.** Thus $y = 3x + 8$; $y = 3x + 5$; $y = 3x + \frac{3}{2}$; $y = 3x$, etc., are parallel (see **Fig.** 38).

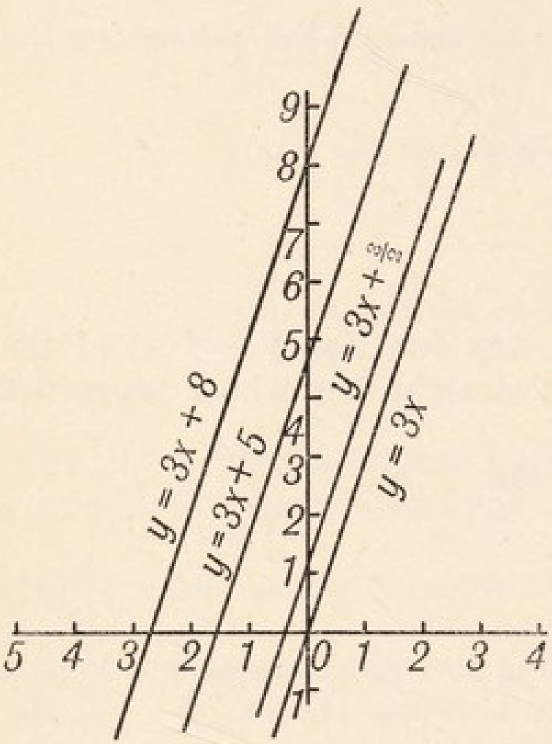

FIG. 38.—If the Value of m is constant, then the Lines are parallel.

(*e*) To draw the graph of a function $y = mx + b$, it is sufficient to plot two points only, since two points are sufficient to determine any straight line.

The most convenient points to plot are those found by making $x = 0$ and $y = 0$, viz., the points $(0, b)$ and $(-\frac{b}{m}, 0)$.

EXAMPLES.

(1) If we draw a straight line graph representing the equation $y = \frac{9}{5}x + 32$, we shall get a diagram representing the relation between the centigrade (x) and Fahrenheit (y) scales of temperature. To draw such a graph it is sufficient to plot, two points only, *e.g.*, the freezing point (0, — 32) and the boiling point (100, 212), and join them by a straight line.

(2) Sörensen's method of expressing H-ion concentration is to give the minus logarithm of the H-ion concentration, *i.e.*, $P_H = -\log C_H$. Hence by plotting C_H or [H] against P_H or log [H], we get a graph for converting one into the other (H. E. Roaf).

We see, therefore, that when two variables are so related that one varies uniformly with the other, the graph of the function is a straight line.

Hence all functions which are examples of the simple interest law may be represented as some straight line and are, therefore, as we have already said, called *linear functions*.

Examples of Linear Functions are (1) Henry's law of solubility of gases, which states that the amount of gas which can be dissolved in a liquid at constant temperature is proportional to the partial pressure of the gas.

(ii.) The law of expansion of solids, liquids and gases which states that the amount of expansion, whether linear, superficial, or cubical, is proportional to the temperature.

(iii.) The law of motion at uniform velocity which states that the distance covered is proportional to the time.

(iv.) In many cases certain other functions may by logarithmic manipulation be converted into linear functions. As an example we may take Boyle's law of gases. This, although as we shall see later, is a function whose curve constitutes a rectangular hyperbola, may yet be transformed in such a way as to be capable of being represented by a straight line. Thus the law states that

$$PV = K$$

(P = pressure ; V = volume ; and K = constant).

By taking logarithms we have

$$\log P + \log V = \log K.$$

Hence, putting $\log P = y$

$\log V = x$

and $\log K = c$

we have $y = -x + c,$

which represents a straight line.

(v.) Similarly, the relation between H-ion concentration and OH-ion concentration in water, represented by [H] . [OH] = K, can be transformed into a linear function by taking logarithms of each side: log [H] + log [OH] = log K.

Note.—[H] indicates H-ion concentration ; similarly with [OH].

In Chapter XX. we shall make frequent use of this principle.

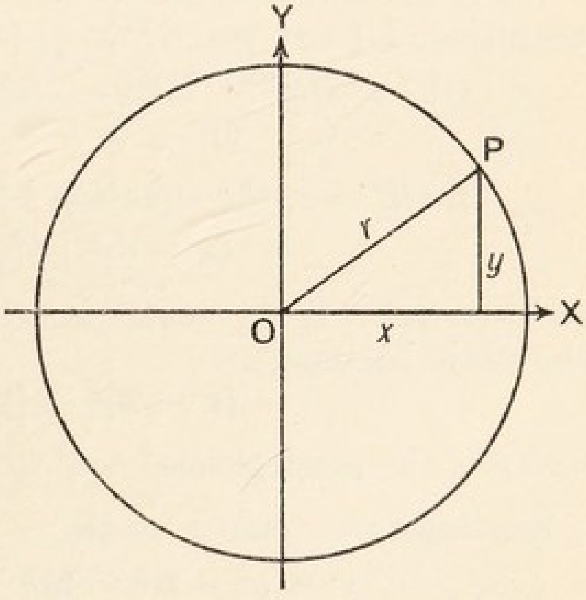

FIG. 39.—Equation of a Circle of which the Centre is the Origin is $x^2 + y^2 = r^2$.

The Equation of a Circle (see Fig. 39).—

(*a*) If we take the centre of the circle as the origin, then, since for any point $P(x, y)$ on the circumference

$$x^2 + y^2 = r^2 \text{ (where } r = \text{length of radius)},$$

$\therefore$ the equation $x^2 + y^2 = r^2$ represents a circle whose centre is at the origin.

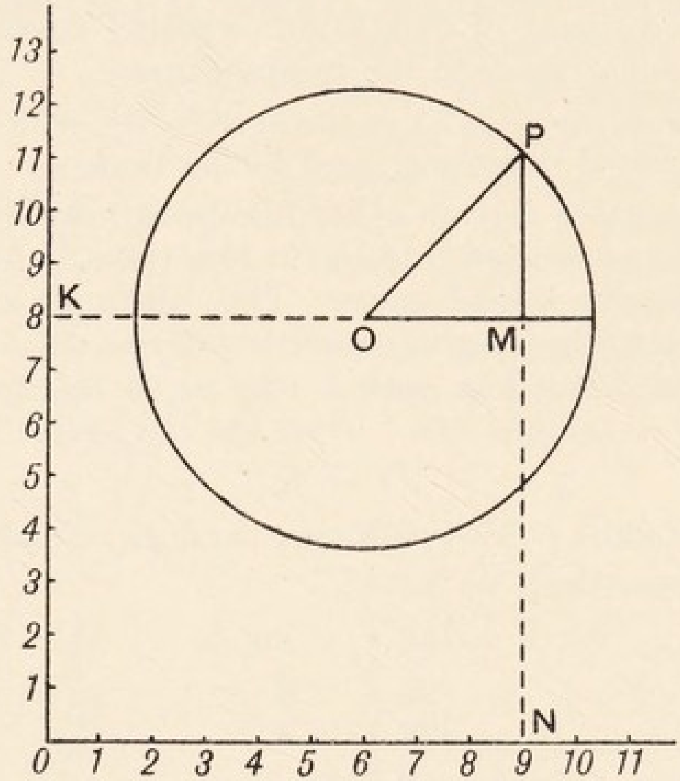

FIG. 40.—The General Equation of a Circle is $(x - h)^2 + (y - k)^2 = r^2$.

(*b*) If the centre is not at the origin, but at a point, say (6, 8), then since for any point $P(x, y)$ (see Fig. 41), we have

$$\begin{aligned} OP^2 &= OM^2 + MP^2 = (KM - KO)^2 + (NP - NM)^2 \\ &= (x - 6)^2 + (y - 8)^2. \end{aligned}$$

$\therefore$ if $OP = r$, the equation becomes

$$(x - 6)^2 + (y - 8)^2 = r^2,$$

and generally, if the centre is at the point (h, k), the equation of the circle becomes

$$(\mathbf{x} - \mathbf{h})^2 + (\mathbf{y} - \mathbf{k})^2 = \mathbf{r}^2,$$

which is the most general equation of a circle.

Expanding the equation we get

$$x^2 + y^2 - 2xh - 2yk + h^2 + k^2 - r^2 = 0,$$

and since h, K, and r are constants we can put $h^2 + k^2 - r^2 = c$.

$\therefore$ equation of a circle is

$$x^2 + y^2 - 2xh - 2yk + c = 0.$$

The Graph of the Function $y = x^2$ (or $\sqrt{y} = \pm x$).—

By putting	$x = 0$, y becomes	$= 0$, $\therefore$	curve passes through	origin
,,	$x = 1$, y ,,	$= 1$, $\therefore$	,, ,, ,,	$(1, 1)$
,,	$x = 2$, y ,,	$= 4$, $\therefore$	,, ,, ,,	$(2, 4)$
,,	$x = 3$, y ,,	$= 9$, $\therefore$	,, ,, ,,	$(3, 9)$
,,	$x = -1$, y ,,	$= 1$, $\therefore$	,, ,, ,,	$(-1, 1)$
,,	$x = -2$, y ,,	$= 4$, $\therefore$	,, ,, ,,	$(-2, 4)$
,,	$x = -3$, y ,,	$= 9$, $\therefore$	,, ,, ,,	$(-3, 9)$

and so on.

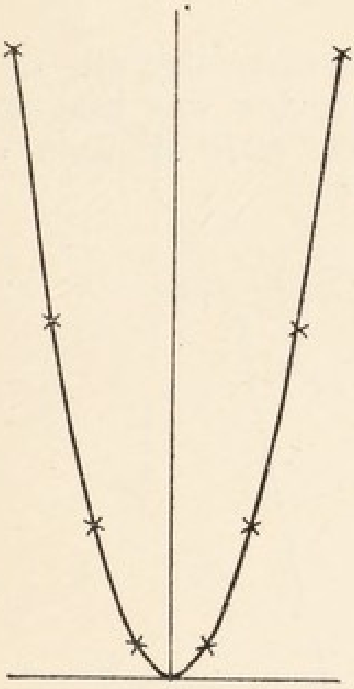

FIG. 41.—Graph of Function $y = x^2$.

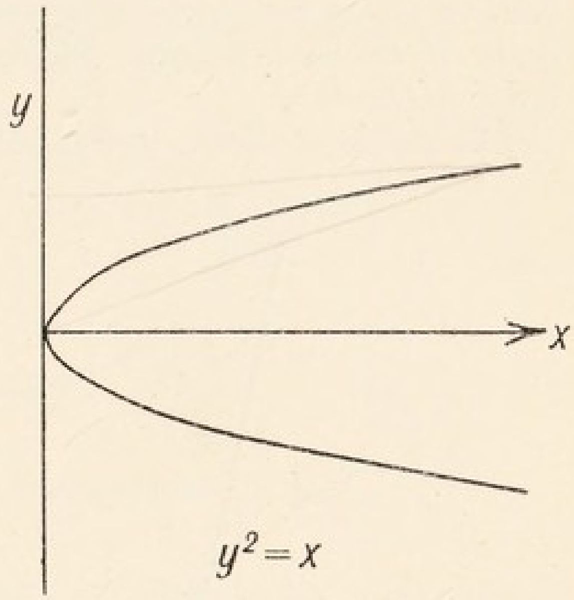

FIG. 42.—Graph of Function $y^2 = x$.

The graph is a parabola passing through the origin, and whose axis is the axis of y (see Fig. 41).

Similarly, the graph of the function $y^2 = x$ will be represented by a parabola passing through the origin, and whose axis is the axis of x (see Fig. 42).

The Property of a Parabola (Fig. 43).—The distinguishing character of a parabola is that any point on it is always the same distance from a given fixed line AB, and also from a given fixed point S.

Thus $\quad PS = PM,$
$\quad\quad\quad CS = CN$, etc.

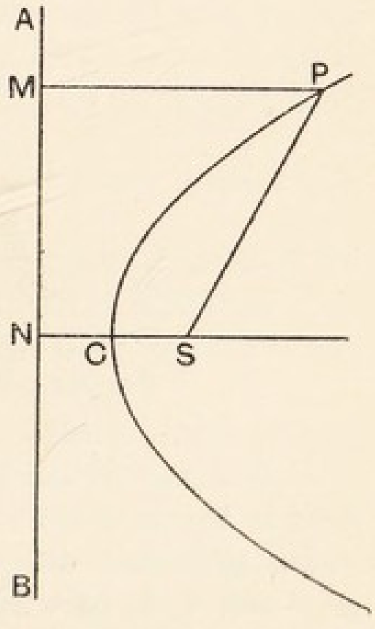

FIG. 43.—To illustrate Property of a Parabola.

The fixed point S is called the *focus*, and the fixed line AB is called the *directrix*.

Consider now such a function as $(y - 2) = 3(x + 1)^2$.

When $x =$	$0, y - 2 = 3{\cdot}1^2$	$= 3.$	$\therefore$	$y = 5$
,, $x =$	$1, y - 2 = 3{\cdot}2^2$	$= 12.$	$\therefore$	$y = 14$
,, $x =$	$2, y - 2 = 3{\cdot}3^2$	$= 27.$	$\therefore$	$y = 29$
,, $x =$	$3, y - 2 = 3{\cdot}4^2$	$= 48.$	$\therefore$	$y = 50$
,, $x =$	$-1, y - 2 = 3{\cdot}0$	$= 0.$	$\therefore$	$y = 2$
,, $x =$	$-2, y - 2 = 3(-1)^2$	$= 3.$	$\therefore$	$y = 5$

and so on.

In this case in order to make the graph of convenient size it is best to chose our scale of representation in such a way that one scale division = 5 units along Oy and one scale division = unity along Ox.

The graph is shown in Fig. 44.

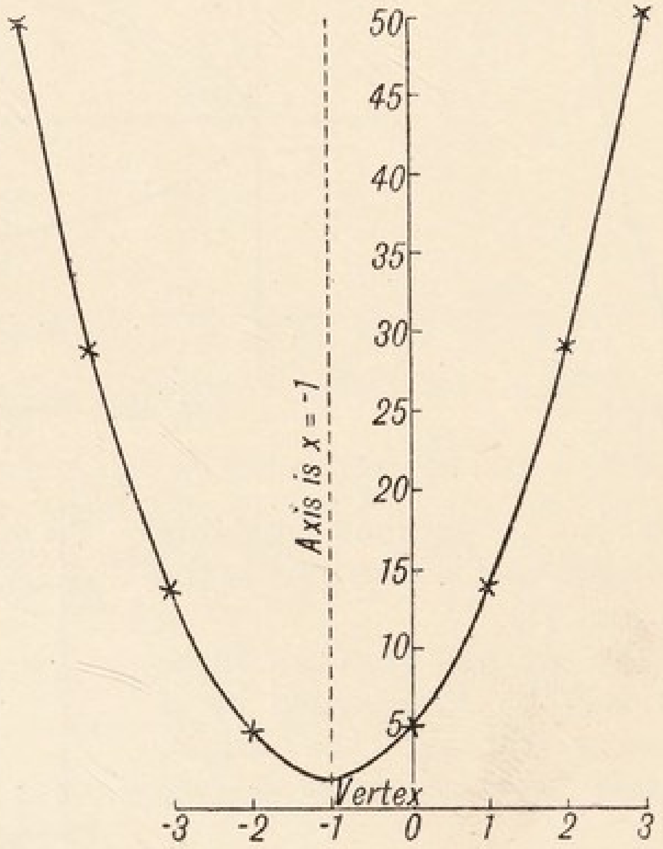

FIG. 44.—Graph of Function $(y - 2) = 3(x + 1)^2$.

It is a parabola whose vertex is at the point $(-1, 2)$ and whose axis is the line $x = -1$.

In general, an equation of the form $y = a(x + b)^2$, or $y^2 = ax + b$, in which one of the variables is of the first degree and the other of the second degree, may be represented graphically by some sort of a parabola.

Examples of Parabolic Functions are: (i.) The law of motion at uniform acceleration (a), which states that the distance covered (s) is proportional to the square of the time (t) $\left(s = \frac{t^2a}{2}\right)$.

(ii.) (α) The relation between the side of a square and its area $y = x^2$; (β) the relation between the radius of a circle and its area $y = \pi x^2$.

(iii.) Schütz Borissoff law (see p. 151).

The Ellipse (Fig. 47).—The ellipse is a curve possessing the following distinguishing properties:

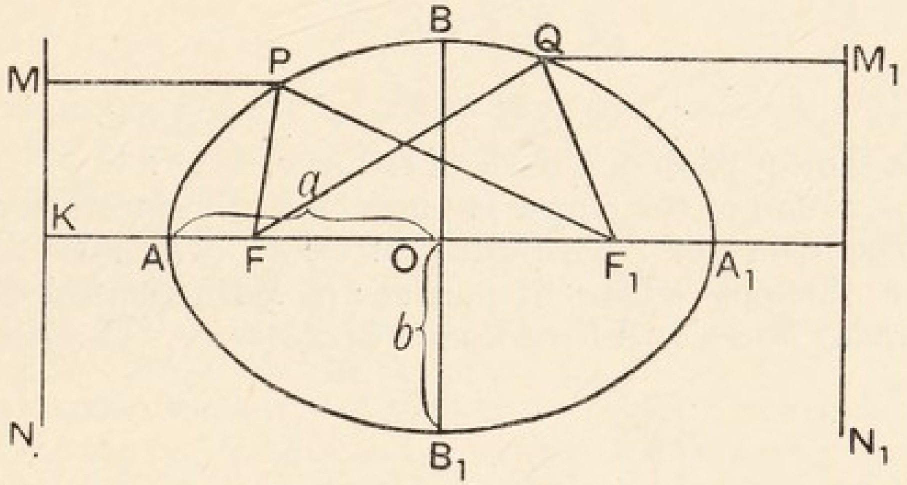

FIG. 45.—To illustrate Properties of an Ellipse.

(1) The sum of the distances of any point on the curve from two fixed points F, F_1, called the *foci*, is constant, *e.g.*, $FP + F_1P = FQ + F_1Q =$ constant.

(2) The distance of any point P on the curve from one of the foci, always bears a constant proportion (less than 1) to the distance of the same point from a corresponding fixed line MN called the *directrix*, *e.g.*, $\frac{FP}{PM} = \frac{FA}{AK} = e\,(< 1)$. This fixed ratio e is called the *eccentricity* of the ellipse, and, whilst it is constant for the same ellipse, it has different values for different ellipses.

The diameter AA_1, on which the foci are situated, is called the *major axis*, and is taken as equal to $2a$.

The diameter BB_1, which bisects the major axis at right angles, is called the *minor axis*, and is taken as equal to $2b$.

The first property of an ellipse suggests a *method of describing the curve*. Take a fine thread FPF_1 equal in length to the major axis AA_1. Fix the two ends of the thread to two pins stuck at F and F_1, and by moving a fine-pointed pencil in tight contact with the thread, the half $APBQA_1$ of the curve will be traced out, since in any position such as P, $FP + F_1P$ = length of thread $= AA_1$. By transferring the thread to the other side of the pins the lower half will be traced out.

Equation of Ellipse.—If we call the major axis $2a$ and the minor axis $2b$, and if we take these axes as the co-ordinate axes, O being the origin, then the equation of an ellipse can be shown to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \begin{pmatrix} a = \text{semi-major axis,} \\ b = \text{semi-minor axis.} \end{pmatrix}$$

Hence, if $a = b$, then the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1,$$

or

$$x^2 + y^2 = a^2,$$

which is the equation of a circle of radius a.

The equation of the ellipse is important in connection with

(i.) The theory of "Frequency Surfaces" which occurs in the study of the more advanced portions of Mathematical Statistics (see Caradog Jones's "A First Course in Statistics," Chapter XIX.).

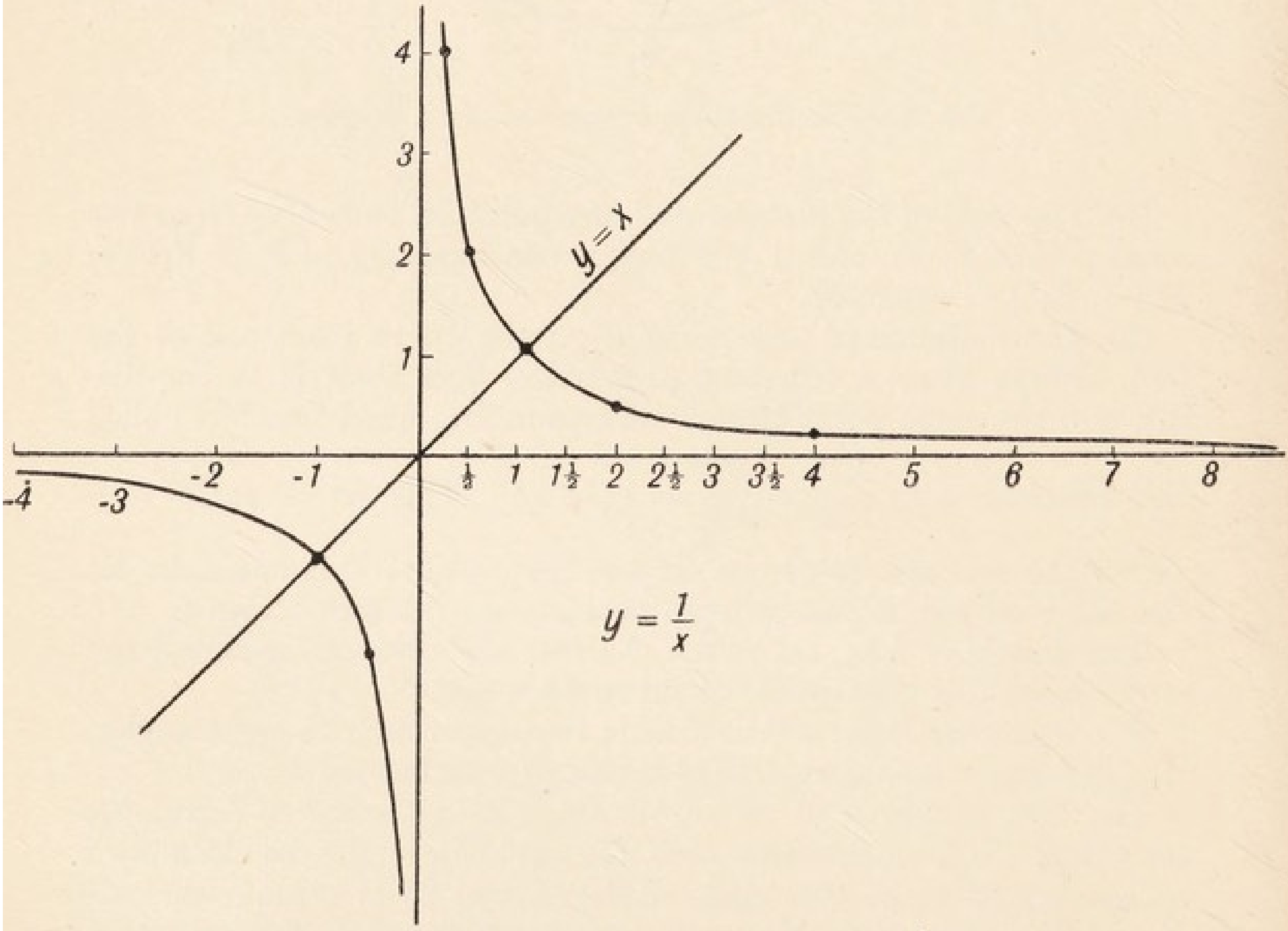

FIG. 46.—Graph of the Function, $y = \frac{1}{x}$.

(ii.) The lines of growth of certain bivalve molluscs (see D'Arcy W. Thompson's "Growth and Form," p. 583).

(iii.) Elliptical muscles (see S. Haughton's "Animal Mechanics."

The graph of the function $y = \frac{1}{x}$.

By putting	$x = 0$,	y becomes	$=$	∞
,,	$x = -4$,	y ,,	$=$	$-\frac{1}{4}$
,,	$x = -2$,	y ,,	$=$	$-\frac{1}{2}$
,,	$x = -1$,	y ,,	$=$	-1
,,	$x = 1$,	y ,,	$=$	1
,,	$x = 2$,	y ,,	$=$	$\frac{1}{2}$
,,	$x = 4$,	y ,,	$=$	$\frac{1}{4}$

The resulting graph is given in Fig. 46. It is what is called a **rectangular hyperbola.** It will be noticed that this curve possesses the following properties:

(1) It consists of two symmetrical halves situated in the first and third quadrants of the axes of co-ordinates.

(2) Each half is itself symmetrical with reference to the line $y = x$, which is the axis of the curve.

(3) Although the limbs of the curve approach nearer and nearer to the axes of y and x, they never actually touch these axes. Hence the y and x axes are what are called the *asymptotes* of the curve.

We see, therefore, that when two variables are so related that one varies as the reciprocal of the other, the graph of the function is a rectangular hyperbola.

Examples of Hyperbolic Functions are: (i.) Boyle's law of the relationship between the pressure and volume of a gas at constant temperature, which states that the volume of a gas at constant temperature is inversely proportional to the pressure to which it is subjected.

(ii.) The extent of ionic dissociation of an electrolyte is inversely proportional to the concentration. All functions of this nature can, as we have seen, be transformed into linear functions by means of logarithms (see p. 107).

(iii.) The dissociation curve of hæmoglobin (*i.e.*, the relation of oxygen pressure to oxy- and total hæmoglobin) is a rectangular hyperbola.

EXERCISE.

Plot the curve from the following data and show that it is an hyperbola.

x	0	1	2	3	4	5	6	7	8	9
y	∞	9	4·5	3	2·25	1·8	1·5	1·3	1·13	1

[xy = constant = 9. $\therefore$ curve is a rectangular hyperbola.]

The Graph of the Function $y = x^3 - 12x$.

When $x = 0, y = 0$. ∴ curve passes through origin.
,, $x = 1, y = -11$
,, $x = 2, y = -16$
,, $x = 3, y = -9$
,, $x = 4, y = 16$
,, $x = -1, y = 11$
,, $x = -2, y = 16$
,, $x = -3, y = 9$
,, $x = -4, y = -16$

The graph is shown in Fig. 47.

A better way of drawing the graph would be to take different scales of representation for x and y. Thus, if one scale unit of x is made equal to 5 scale units of y, we get a much more detailed curve as shown in Fig. 47A. (*Cf.* Scale Modulus, p. 122.)

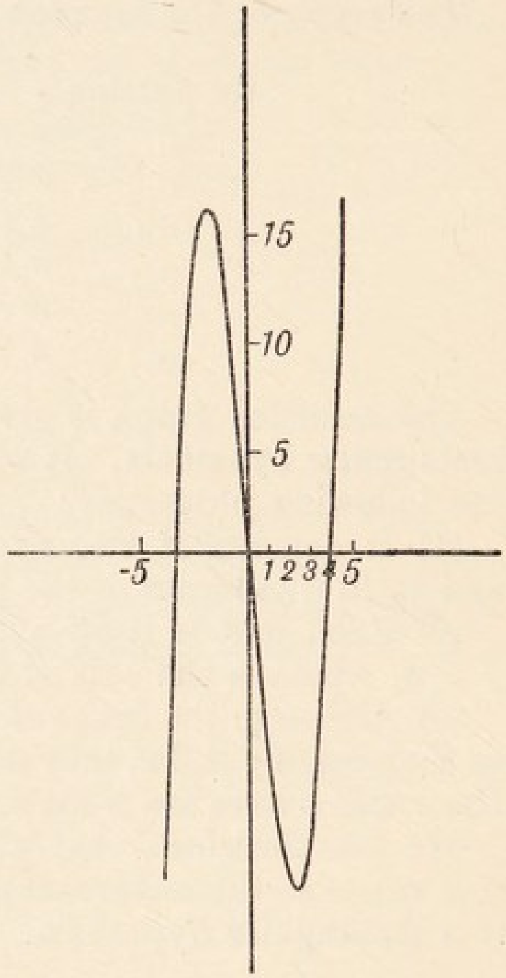

FIG. 47.—Graph of Function $y = x^3 - 12x$.

The Graph of the Function $y = e^x$ (Fig. 48).—If we write the equation in the form $2{\cdot}3 \log_{10} y = x$, we can arrange the plotting table as follows:

y.	0	1	2	3	4	5	6	7	8	9	10
$x = 2{\cdot}3 \log_{10} y$	$-\infty$	0	·69	1·1	1·38	1·61	1·8	1·95	2·08	2·2	2·3

The Graph of $y = e^{-x^2}$ (Fig. 49).—This is a most important curve in mathematical statistics (see p. 343).

If we put $2{\cdot}3 \log_{10} y = -x^2$, we get the following plotting table:

y	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1
$x = \pm\sqrt{2{\cdot}3 \log y}$	$\pm\infty$	$\pm$ 1·52	$\pm$ 1·27	$\pm$ 1·10	$\pm$ 0·96	$\pm$ 0·83	$\pm$ 0·71	$\pm$ 0·60	$\pm$ 0·47	$\pm$ 0·32	·0

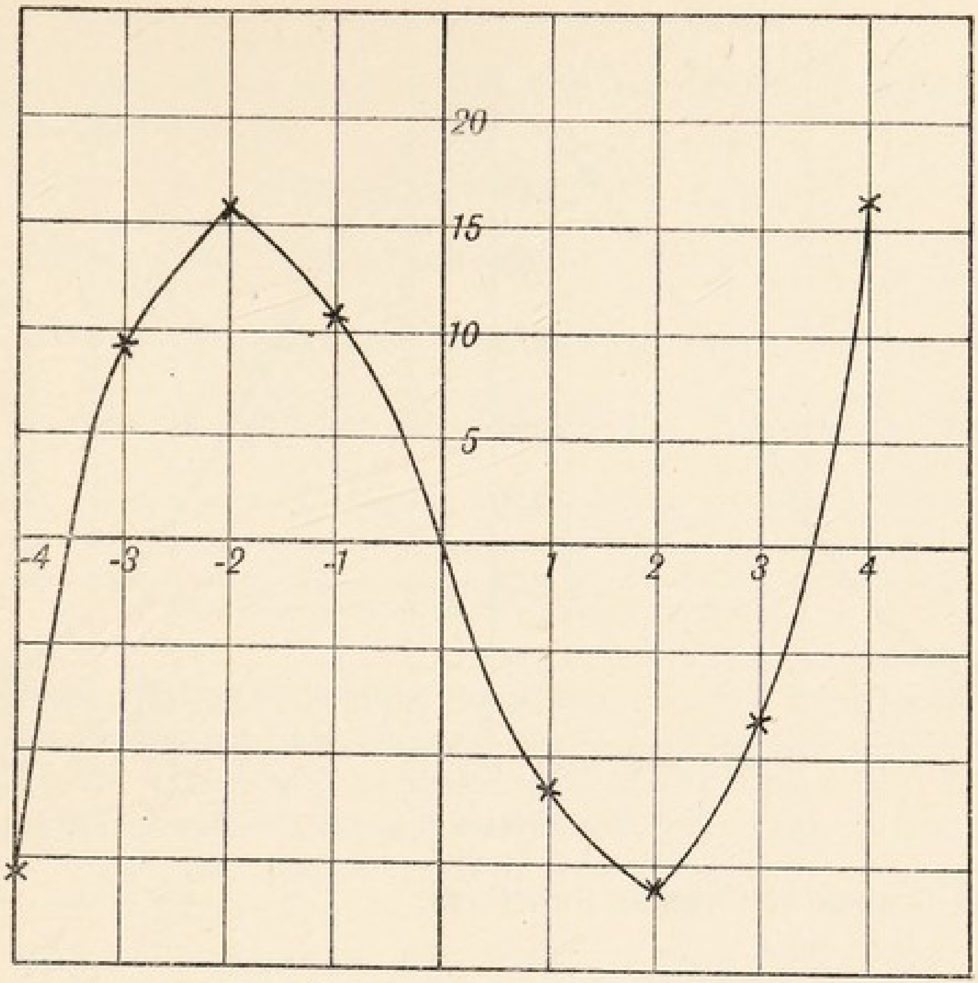

FIG. 47A.—Graph, $y = x^3 - 12x$, drawn with Different Scales of Representation for x and y.

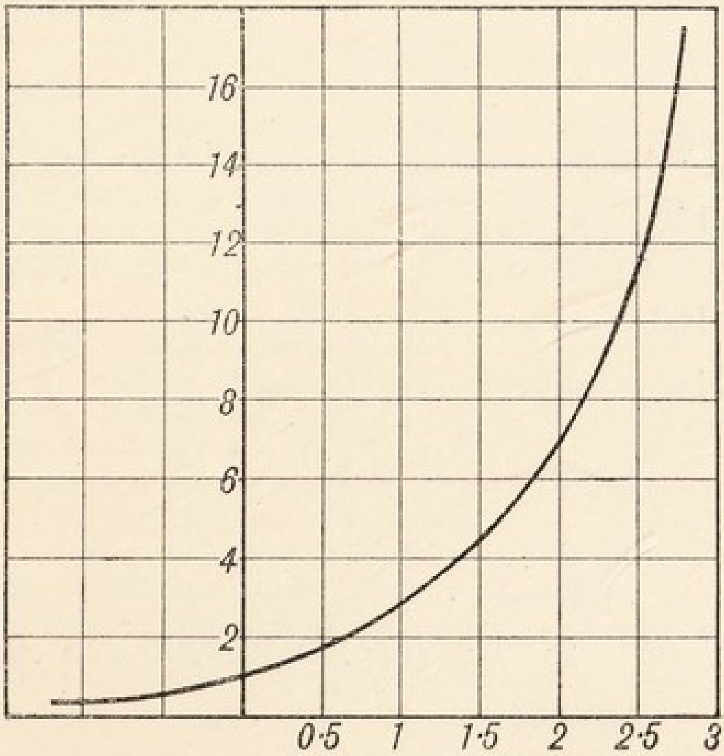

FIG. 48.—Graph of $y = e^x$

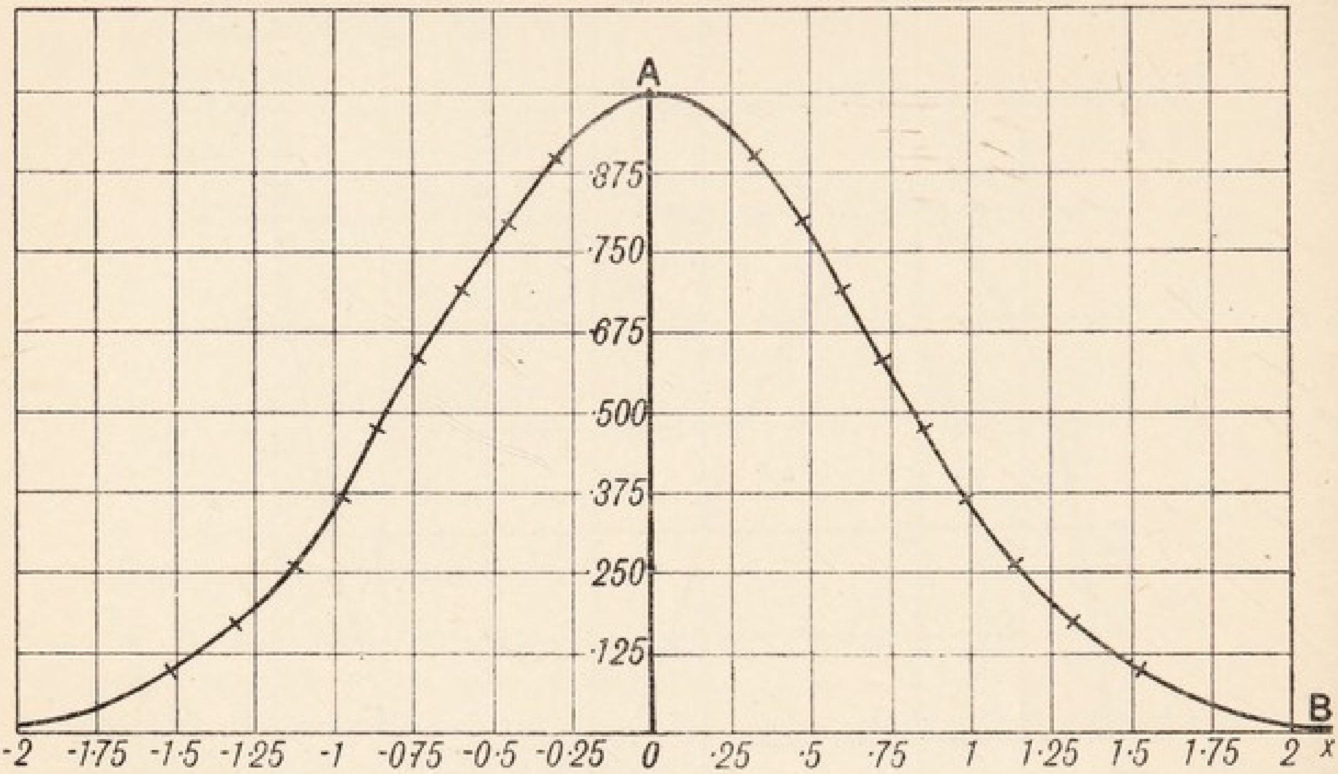

FIG. 49.—Graph of $y = e - x^2$.

The Graphs of Circular Functions.

$y = \sin x$.

When	$x = 0$,	$y = 0$	When	$x = 210$,	$y = -\frac{1}{2}$
,,	$x = 30$,	$y = \frac{1}{2}$	,,	$x = 240$,	$y = -\frac{\sqrt{3}}{2}$
,,	$x = 60$,	$y = \frac{\sqrt{3}}{2}$	,,	$x = 270$,	$y = -1$
,,	$x = 90$,	$y = 1$	,,	$x = 300$,	$y = -\frac{\sqrt{3}}{2}$
,,	$x = 120$,	$y = \frac{\sqrt{3}}{2}$	,,	$x = 330$,	$y = -\frac{1}{2}$
,,	$x = 150$,	$y = \frac{1}{2}$	,,	$x = 360$,	$y = 0$
,,	$x = 180$,	$y = 0$	etc.		

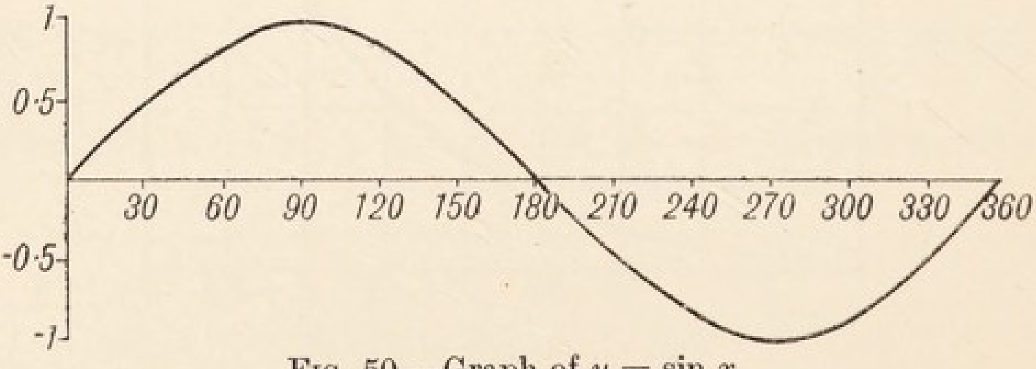

FIG. 50.—Graph of $y = \sin x$.

The graph is shown in Fig. 50.

In a similar way $y = \cos\theta$ and $y = \tan\theta$ may be represented by graphs (see Fig. 51).

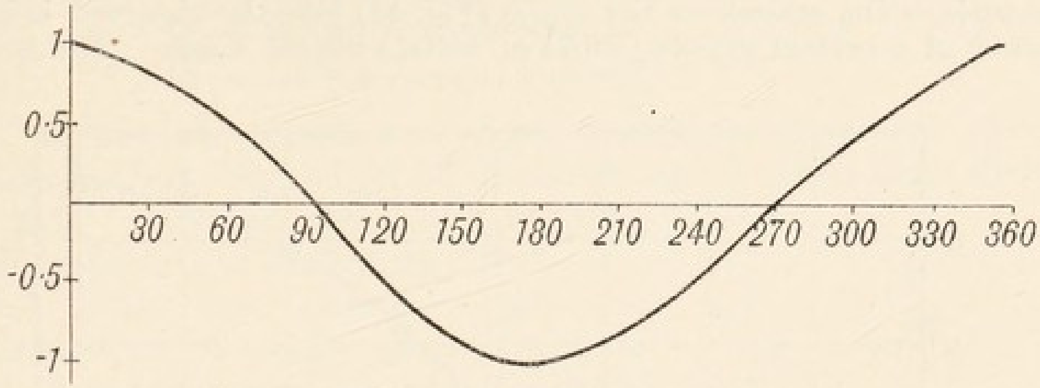

FIG. 51.—Graph of Function $y = \cos x$.

Graphical Method of Solving Equations.—It will be noticed that in all the graphs we have plotted, the points on the x axis intersected by the graph will give the root of the equation $f(x) = 0$, and hence we have a convenient method of solving equations of any degree. Thus the graph of Fig. 47A gives the roots of the equation $x^3 - 12x = 0$.

Similarly, simultaneous equations of two unknowns may sometimes be solved graphically by drawing the graph for each of the two equations. The points of intersection of the two graphs are the roots required.

EXAMPLES.

(1) Solve $x^2 - 3x + 1 = 0$.

Draw the graph in the usual way. It will be found to cut the x axis at points where $x = 0·4$ and $2·6$ very nearly. If, now, the points in the neighbourhood of $x = ·4$ and $x = 2·6$ be plotted on a large scale, it will be seen that the curve cuts the axis x very close to $x = ·38$ and $x = 2·62$ (see further, p. 261). The process can be repeated in order to get closer and closer approximations.

(2) Solve $x^3 + 3x^2 - 10x + 24 = 0$.

Proceeding as in (1), the graph will be found to cut the x axis at the points $x = -3$, $x = 2$, and $x = 4$, which are therefore the roots.

(3) Solve $y + x - 8 = 0$; $y - 2x + 6 = 0$.

The graphs of these equations will be found to intersect at the point (4·67, 3·34). $\therefore$ $x = 4·67$, $y = 3·34$.

Interpolation.—Supposing we plot a function like $y = x^2$, for values of $x = -3, -2, -1, 0, +1 +2 +3$, etc., and we wish to know what the value of y will be for some intermediate value of x, say, when $x = 1·25$. Then by referring to the point on the graph whose abscissa is 1·25, we can at once read off what the value of y is. This method is called *interpolation*, and is exceedingly useful in practical work. For example, the population during the intercensal periods can be found in this way (see also Chapter XX.).

Example.—The crosses on the graph (Fig. 52) give the observed heights in inches of a certain growing child at certain stated times. The height

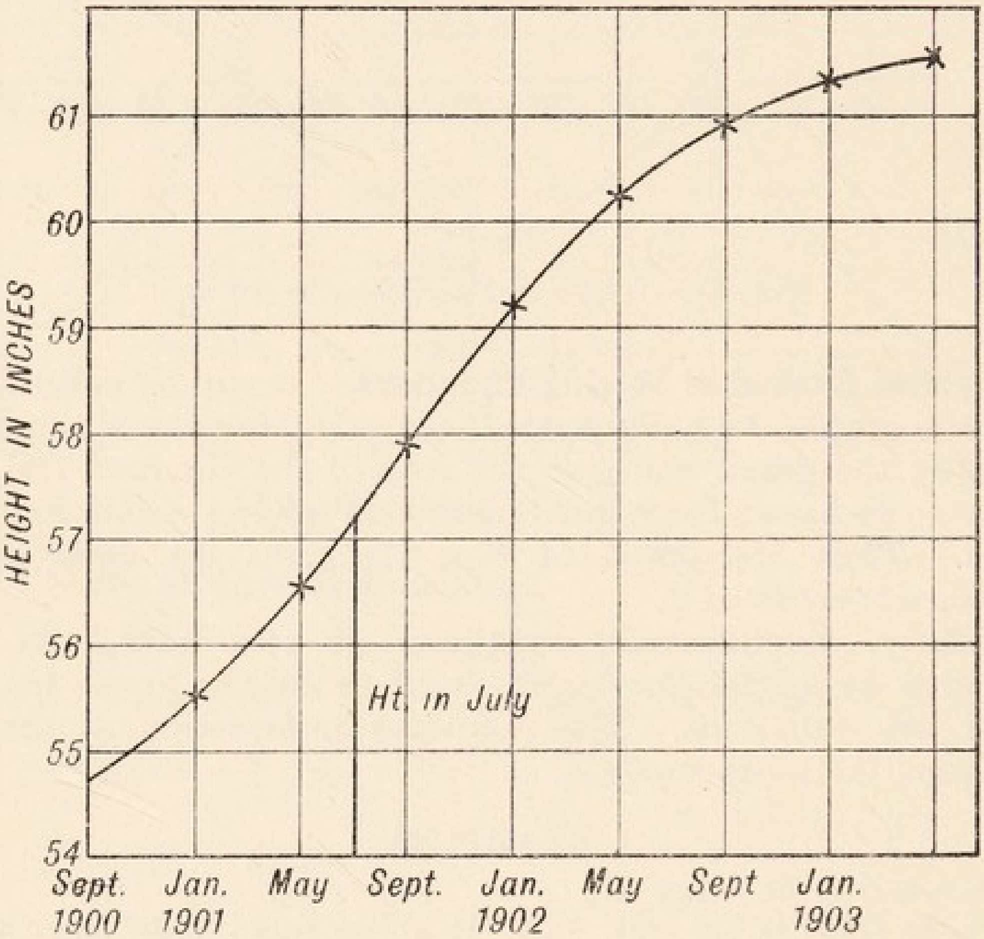

Fig. 52.—Graph to illustrate Interpolation.

of the child in July, 1901, is found by interpolation to have been 57·2 inches.

For other methods of interpolation, the reader must consult more advanced books on mathematics.

Caution.—The reader must remember that whilst it is in most cases permissible to use a formula or graph for purposes of interpolation, it is in many cases dangerous to use it for purposes of *extra* polation—outside the limits of observation (see p. 289).

Nomography.

Supposing we have some form of equation or "law" representing the relationship between *three* variables, such as the relationship between the surface area of the body and the person's height and weight, which Messrs. Dubois have expressed by the equation

$$S = 71{\cdot}84W^{0{\cdot}425} . H^{0{\cdot}725}$$

(where S = surface in square centimetres,
W = weight in kilograms,
and H = height in centimetres).

It is possible to draw a chart, called an *alignment chart* or *nomogram* (see Fig. 56, p. 124), consisting of three lines situated at fixed distances from one another and graduated in such a way that—

(i.) if the two external lines are graduated with scales representing **W** and **H,** and the middle one is graduated with a scale representing **S,** then

(ii.) any straight line ABC joining any particular graduation (A) on the **W** scale with any other graduation (B) on the **H** scale will cut the **S** scale in a point (D) corresponding in value to $71{\cdot}84W^{0{\cdot}425}H^{0{\cdot}725}$. (*Cf.* Fig. 53.)

Because such a chart enables us, by means of one diagram, to find the value of any one variable for given values of the other variables in a given "law," therefore such a chart is called a *nomogram* (nomos = law).

(1) **The Sum and Difference Nomogram.**—(*a*) In order to understand the principle of *nomography* or the construction of nomograms, we shall construct such a chart for a simple law like

$$x = y \pm z.$$

If we draw three parallel lines at equal distances apart, and call the two outside lines the y and z scales, and the middle one the x scale, and then graduate **y and z** with the **same** unit, and the scale **x** with **half** the unit (see Fig. 53), then if we join any value of y with any value of z, it will cut the x scale at a point corresponding to $y \pm z$.

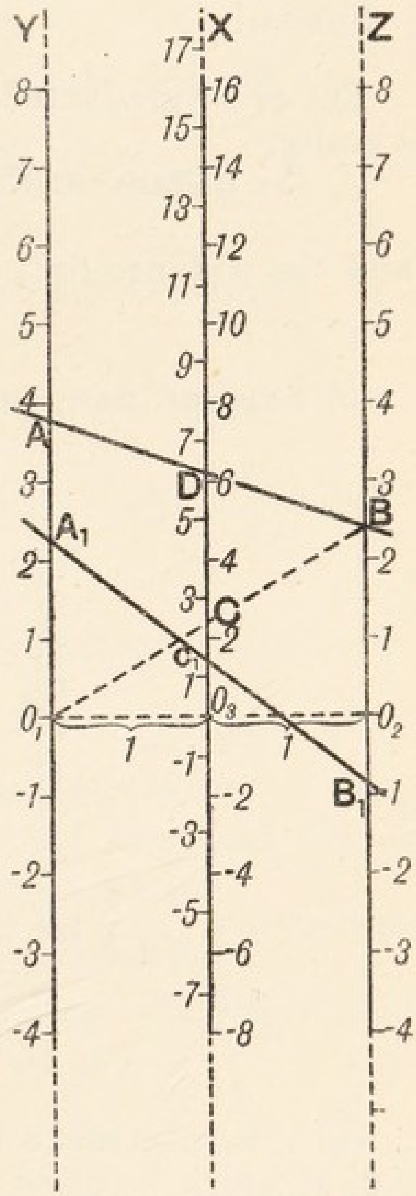

Fig. 53.—Nomogram for $x = y \pm z$.

Thus, in the diagram, the line joining the point 3·8 on the y scale with the point 2·4 on the z scale cuts the x scale at the point corresponding to 6·2 (*i.e*, 3·8 + 2·4); whilst the line joining the point 2·3 on the y scale with the point $-0·8$ on the z scale cuts the x scale at the point 1·5 (*i.e.*, $2·3 - 0·8$). This is so, because, if we look at Fig. 53, we shall see at once that by the law of similarity of triangles

$$CD = \frac{1}{2} 0_1A \text{ and } 0_3C = \frac{1}{2} 0_2B.$$

$$\therefore \quad 0_3D \text{ (which } = 0_3C_1 + CD) = \frac{1}{2}(0_2B + 0_1A).$$

Similarly $$0_3C_1 = \frac{1}{2}(0_1A_1 - 0_2B_1).$$

But as we graduated the x scale with **half** the unit of the y and z scales,

$\therefore$ the distance 0_3D as actually read off on the scale $= 0_1A + 0_2B$,

and the distance 0_3C_1 as actually read off on the scale $= 0_1A_1 - 0_2B_1$.

$$\therefore \quad x = y \pm z.$$

(*b*) Suppose, now, we have a function like

$$x = f(y) + f(z),$$

e.g., $$x = (2y) + (3z).$$

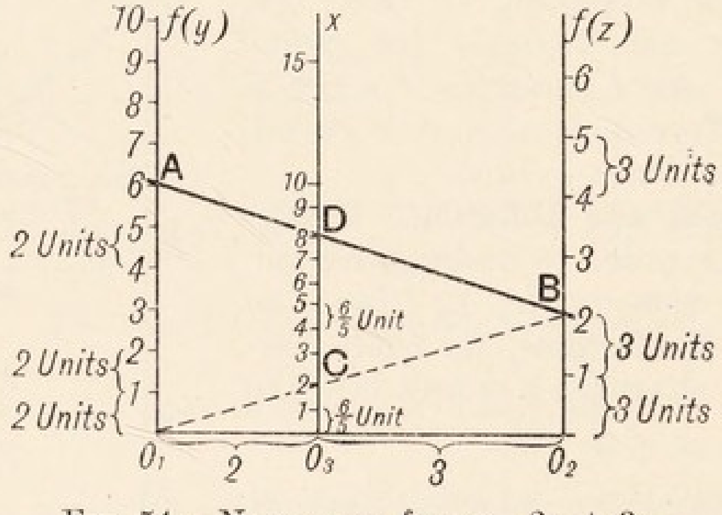

FIG. 54.—Nomogram for $x = 2y + 3z$.

Then, to construct a nomogram we take any arbitrary distance 0_10_2 at which we place the two scales, $f(y)$ and $f(z)$, and graduate them as follows :

Make each unit division along the $f(y)$ scale equal to 2 units of length (say, 2 cm. or 2 in. or anything else), and each unit division along the $f(z)$ scale 3 of the same units of length (centimetres or inches, etc). We then place the x scale at a point 0_3, such that

$$\frac{0_10_3}{0_30_2} = \frac{2}{3},$$

and graduate this x scale in such a way that each unit division along it is equal to $\frac{6}{5}\left(i.e., \frac{2 \times 3}{2+3}\right)$ of the same units. **This will complete the nomogram.**

Thus, it will be seen in the figure that a line joining the point 6 on the $f(y)$ scale with the point 2 on the $f(z)$ scale passes through the point marked 8 on the x scale (*i.e.*, $x = (2y) + (3z)$). This is so because

$$0_3C = \frac{2}{5}\ 0_2B = \frac{2}{5} \times 6 \text{ units of length}$$

(since each unit division along $f(z) = 3$ units of length).

Similarly $$CD = \frac{3}{5}\ 0_1A = \frac{3}{5} \times 12 \text{ units of length.}$$

$$\therefore \quad 0_3D = \frac{12}{5} + \frac{36}{5} = \frac{48}{5} \text{ units of length.}$$

But each division along the x scale has been made $= \frac{6}{5}$ units of length.

$\therefore$ 0_3D, which equal to $\frac{48}{5}$ units of length, must coincide with division 8.

Similarly for any other points.

$\therefore$ every point along x will solve the equation $x = (2x) + (3z)$.

Indeed, it can be easily shown that if $f(y) = m_1y$, and $f(z) = m_2y$, then the requisite nomogram is obtained as follows :

(1) Place the (m_1y) and (m_2z) scales any arbitrary convenient distance 0_10_3 apart.

(2) Place the x scale at a point 0_3 between the (m_1y) and (m_2z) scales, such that $\frac{0_10_3}{0_30_2} = \frac{m_1}{m_2}$.

(3) Graduate the (m_1y) scale in such a way that each division along it $= m_1$ units of length.

(4) Graduate the (m_2z) scale in such a way that each division along it $= m_2$ units of length.

(5) Graduate the x scale in such a way that each division along it $= \frac{m_1m_2}{m_1 + m_2} = m_3$ units of length.

(2) **Product and Quotient Nomogram.**—(*a*) $x = yz$ or $x = \frac{y}{z}$.

Place the x scale midway between the y and z scales as in the sum and difference nomogram, except that instead of graduating the y and z scales arithmetically, **we graduate them with the same unit logarithmically as in the case of the slide rule** (see p. 19) (so that the points marked 1, 2, 3 . . . − 2, − 3, etc., correspond to log 1, log 2, log 3 . . . etc.), whilst the x scale is graduated logarithmically with half the unit. We shall then get a nomogram for $x = yz$ or $x = \frac{y}{z}$.

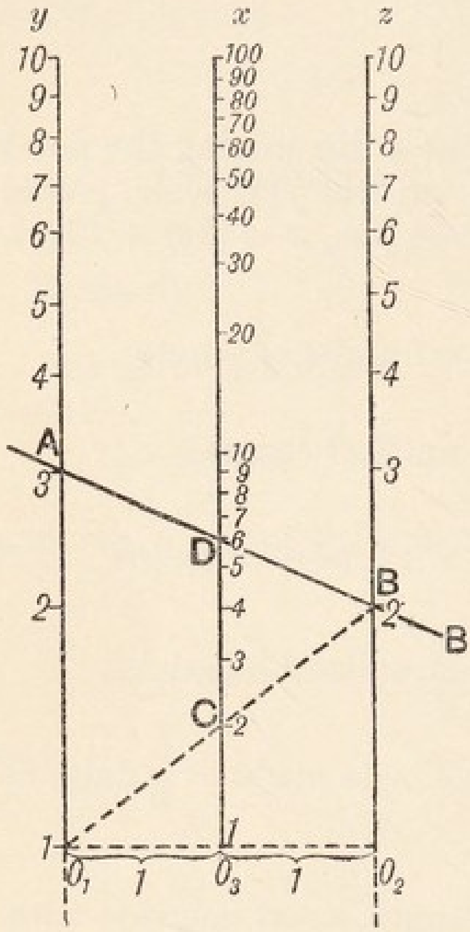

FIG. 55. — Nomogram for $x = yz$, or $x = \frac{y}{z}$.

Thus, in the diagram

$$0_3D = \frac{1}{2}(0_1A + 0_2B) \ (cf. \ (1)),$$

or

$$\log x = \frac{1}{2}(\log y + \log z) = \frac{1}{2}\log yz,$$

$\therefore$ x scale being graduated with half the unit of the y and z scales we get $\log x = \log yz$.

$\therefore \quad x = yz.$

(*b*) Similarly, because when $x = y^m z^n$, $\log x = m \log y + n \log z$,

$\therefore$ if we make $\frac{\text{the distance between the } y \text{ and } x \text{ scales}}{\text{the distance between } x \text{ and } z \text{ scales}} = \frac{m}{n}$, and graduate y, z and x, we get a nomogram for $x = y^m z^n$.

The Scale Modulus.—The numbers m_1, m_2, m_3, used for graduating the three scales, are called the scale moduli. **The scale modulus, therefore, is that number of units of length which represents the unit division of the scale.**

The scale moduli for the $f(y)$ and $f(z)$ scales may be arbitrarily chosen to suit the particular requirements, but the modulus of the x scale, as well as the position of that scale, depends, as we have seen, upon the values of the other two moduli. Conversely, if we choose the length 0_10_2, the position of the point 0_3 and the modulus of one of the scales, then the graduations of the other two scales become fixed. Thus, if the moduli for $f(y)$ and $f(z)$ scales are m_1 and m_2 respectively, then

$$\frac{0_1 0_3}{0_3 0_2} = \frac{m_1}{m_2}$$

and m_3 (modulus for x scale) $= \dfrac{m_1 m_2}{m_1 + m_2}$.

Example.—Construct a nomogram for the equation

$$S = 71 \cdot 84 W^{0 \cdot 425} \,.\, H^{0 \cdot 725} \text{ (Dubois).}$$

(S = surface in square centimetres, W = weight in kilograms, and H = height in centimetres).

It is obvious that in an equation like this both W and H must have certain limits. Let us therefore take the upper and lower limits of W as 70 and 2 kgms. respectively; and the upper and lower limits of H as 180 and 50 cm. respectively.

Taking logarithms of both sides we get log S − log 71·84 = ·425 log W + ·725 log H.

Put (0·425 log W) = y; and (0·725 log H) = z.

Since the range of W is between 2 and 70 kgm.,

∴ whole length of the y scale will be

$$\begin{aligned}\cdot 425 (\log 70 - \log 2) &= \cdot 425 (1 \cdot 8451 - \cdot 3010)\\ &= \cdot 636 \text{ of a unit of the W scale.}\end{aligned}$$

Now, a suitable length for this range will be about 10 ins.

∴ one complete unit of the scale would be represented by about $\frac{10}{\cdot 636} = 15 \cdot 7$ ins., say 16 ins.

i.e., the **modulus** of the weight scale is 16 ins.

Again, the range of H is between 50 and 180 cm.

∴ whole length of the z scale will be

$$\begin{aligned}\cdot 725 (\log 180 - \log 50) &= \cdot 725 (2 \cdot 2553 - 1 \cdot 6990)\\ &= \cdot 403 \text{ of a unit of the H scale.}\end{aligned}$$

Taking the length as suitable for this range as also about 10 ins., we shall get the **modulus** for the height scale as $\frac{10}{\cdot 403}$ = about 25 ins.

$$\therefore \quad \text{ratio between moduli} = \frac{16}{25} = \cdot 64 : 1 = 2 : 3 \text{ (approximately).}$$

$$\therefore \qquad \frac{0_1 0_3}{0_3 0_2} = \frac{2}{3}.$$

Hence, if we choose $0_1 0_2$ as 5 ins., we must make $0_1 0_3$ = 2 ins., and $0_3 0_2$ = 3 ins.

Now, since the moduli m_1 and m_2 are 16 and 25, therefore a unit length of the W scale will be 16 ins., and a unit length of the H scale will be 25 ins.

∴ ·636 of a unit length of the W scale will be

$$\cdot 636 \times 16 = \mathit{10 \cdot 2 \ ins.} = \mathit{length\ of\ W\ scale},$$

and ·403 of a unit length of the H scale will be

$$\cdot 403 \times 25 = \mathit{10 \cdot 1 \ ins.} = \mathit{length\ of\ H\ scale}.$$

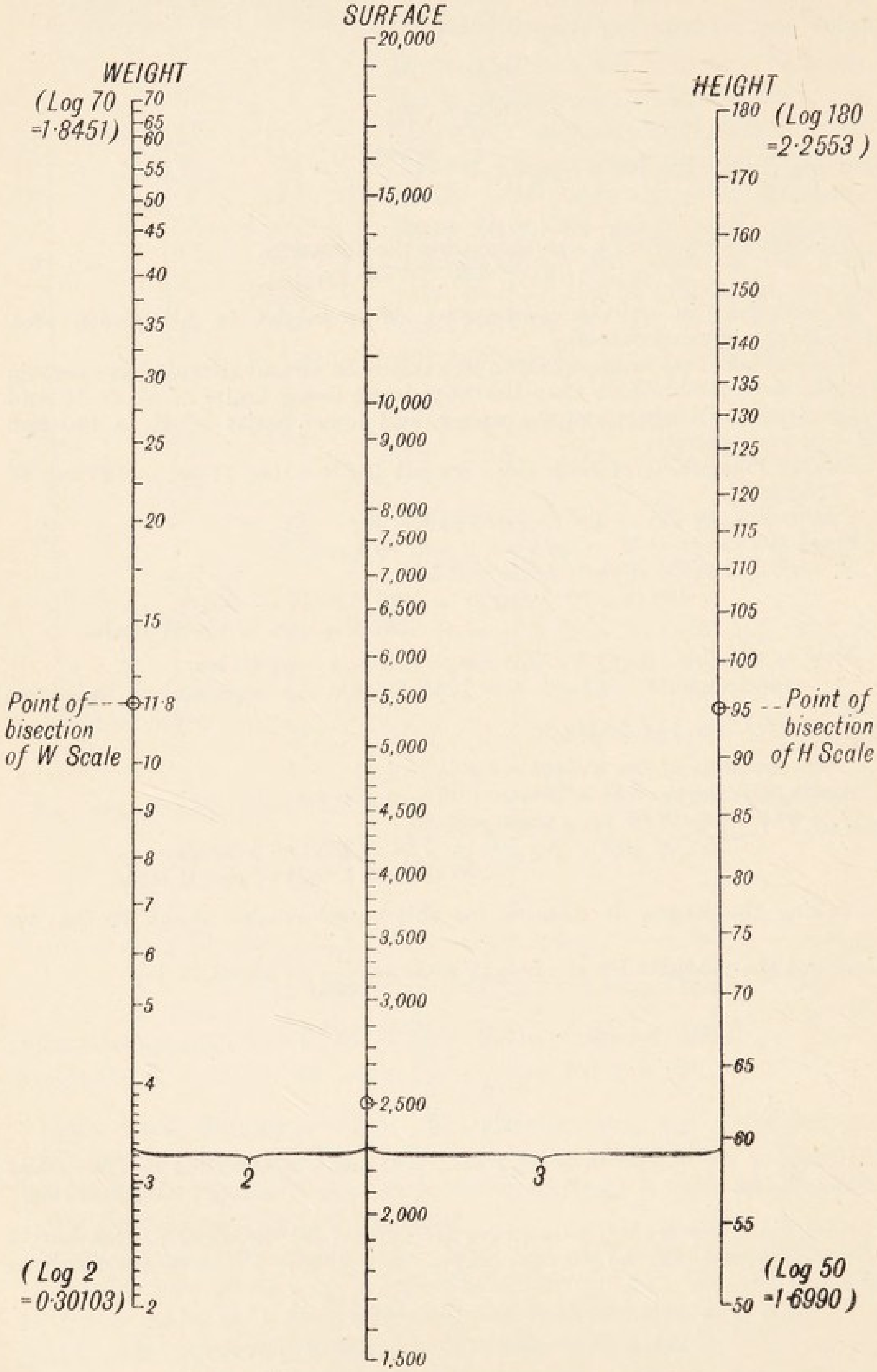

Fig. 56.—Nomogram for $S = 71{\cdot}84W^{0{\cdot}425} . H^{0{\cdot}725}$.

We must now graduate these scales.

(1) Find the middle point of the W scale by taking half the sum of the logarithm limits.

$$\text{Now } \frac{1}{2} (\log 70 + \log 2) = \frac{1}{2} (1{\cdot}8451 + {\cdot}30103)$$

$$= \frac{1}{2} \times 2{\cdot}1461 = 1{\cdot}0731 \text{ ins.}$$

But 1·073 is the logarithm of 11·83.

∴ middle point of scale = 11·83 kgm. (Put a small circle round it for reference.)

∴ the point representing 10 kgm. will be 16 × ·425(log 11·8 − log 10) ins. below the central point, *i.e.*, 0·5 in. below.

Now, calculate the position of every 5-kgm. variation in weight above and below 10 until you get to the top and bottom of the scale. This is done as follows :

5	15	20	25	30	35	40
0·699 1·000	1·1761 1	1·3010 1	1·3979 1	1·4771 1	1·5441 1	1·6021 1
− ·301	·1761	·3010	·3979	·4771	·5441	·6021

etc.

∴ the points representing these weights will be these differences multiplied by 16 × ·425 or by *6·8*, viz., 6·8 × (− ·301), 6·8 × ·1761, 6·8 × ·301, etc., *i.e.*, − 2·04, 1·2, 2·04, 2·71, 3·24, 3·7, 4·1, etc., ins., distant from the 10-kgm. point, and so on for the other points.

Now construct the H scale in the same way.

$$\text{The middle point} = \frac{1}{2} (\log 180 + \log 50)$$

$$= \frac{1}{2} (2{\cdot}2553 + 1{\cdot}6990)$$

$$= \frac{1}{2} \times 3{\cdot}9543 = 1{\cdot}9772$$

$$= \log 94{\cdot}9,$$

say, 95 cm.

(Put a small circle round it for reference.)

Calculate the position of every 5-cm. variation in height above and below 95 cm. until you get to the top and bottom of the scale. Thus :

80	85	90	100	105	110
1·9031 1·9772	1·9294 1·9772	1·9542 1·9772	2 1·9772	2·0212 1·9772	2·0413 1·9772
− ·0741	− ·0478	− ·0230	0·0228	0·0440	0·0641

∴ the points representing these heights will be 25 × ·725 times these differences, or *18·13* times these differences, viz.,

$$18{\cdot}1 \times (-\,{\cdot}0741),\ 18{\cdot}1 \times (-\ {\cdot}048),\ 18{\cdot}1 \times (-\,0230),$$
$$18{\cdot}1 \times 1{\cdot}0220,\quad 18{\cdot}1 \times 1{\cdot}044,\quad 18{\cdot}1 \times 1{\cdot}064,\quad \text{etc.,}$$

or − 1·34, − 1·05, − ·4163, ·4163, ·8, 1·19, etc., ins. from the middle point or from the 95 cm. point, and so on for the other points.

The S scale is graduated by calculating a value of S from the formula :

$$S = 71{\cdot}84W^{0{\cdot}425}.\ H^{0{\cdot}725}, \text{ for values of,}$$
$$\text{say, } W = 4, \text{ and } H = 60, \text{ which gives}$$
$$\log S = \log 71{\cdot}84 + {\cdot}425 \log 4 + {\cdot}725 \log 60$$
$$= 1{\cdot}8564 + {\cdot}2560 + \bar{1}{\cdot}2892$$
$$= 3{\cdot}4016 = \log 2520.$$

∴ a line joining the scales of W and H at the points 4 and 60 respectively will meet the scale S at a point which should be marked 2520, or it can without very great error be marked *2500.*

The other points at intervals of 500 sq. cm. may then be calculated exactly in the same way as in the case of the other two scales, except that the differences in the logarithms must be multiplied by m_3 which =

$$\frac{m_1 m_2}{m_1 + m_2} = \frac{25 \times 16}{25 + 16} = 9{\cdot}8 = \text{approximately } \mathit{10}.$$

Thus the point 2000 will be

$$10(\log 2500 - \log 2000) = 10(3{\cdot}3979 - 3{\cdot}3010)$$
$$= {\cdot}97 \text{ in. below the 2500 point,}$$

and so on for the other points.

The nomogram so constructed (see Fig. 56) gives results within a very considerable degree of accuracy. If the moduli had been made larger, the various scales would have been longer, and the degree of accuracy greater. (See Feldman and Umanski, *The Lancet*, 1922, Vol. 1 ; also Boothby and Sandiford, *Boston Med. and Surg. Journ.*, 1922, Vol. 185). The latter paper gives a comprehensive account of the use of nomographic charts for the calculation of the metabolic rate by the gasometer method.)

EXERCISES.

The student might compare the results found for S, mathematically, by the long and tedious process of solving the equation as given by the Dubois formula, and as rapidly ascertained by the nomogram, for any chosen values of W and H.

CHAPTER IX.

DIFFERENTIALS AND DIFFERENTIAL COEFFICIENTS.

In all the functions with which we have been dealing in Chapter VIII. we considered the relative increments or decrements of mutually related quantities when such increments or decrements were infinitesimally small. Such infinitesimally small increments or decrements are called **differentials**; differential being the diminutive of the word difference. The differential of any quantity x is written as **dx,** where **d** is an abbreviation of the expression "the differential of." Hence, whenever one meets with the expression **dx**, one must not think of it as "d multiplied by x" but as "the differential of x" or as "an infinitesimally small bit of x." Similarly, **dy** stands "for the differential of y," or "an infinitesimally small bit of y."

dx is read as "dee-eks,"
dy is read as "dee-wy,"

and so on.

Now, although **dx** stands for an infinitesimally small and therefore, in itself, negligible bit of x, it does not follow that such quantities as xdx, x^2dx, etc., are negligible, but $(dx)(dx)$ or $(dx)^2$ would be negligible.

Let us take an example. Supposing $x = 1{,}000{,}000$ and $dx = 0{\cdot}000001$, then $xdx = 1$; $x^2dx = 1{,}000{,}000$, and so on, *i.e., xdx* is not a negligible quantity and x^2dx is as large as the original quantity x itself; but $(dx)^2 = (10^{-6})^2 = 10^{-12}$ is an utterly negligible quantity.

Now, if we speak of any fraction $\frac{1}{n}$ as a very small proportion of the whole, then the fraction $\frac{1}{n^2}$ would be called a fraction of the second order of smallness, or, in mathematical language, $\frac{1}{n^2}$ would be called a fraction of the second order of magnitude and $\frac{1}{n^3}$ would be a fraction of the third order of smallness (or magnitude), and so on.

Hence, if we make $\frac{1}{n}$ small enough, *e.g.*, $\frac{1}{10^{10}}$ (which, as we saw on p. 69, corresponds to about 1 in. in 200,000 miles)—or smaller still—then we are justified in neglecting quantities of the second order of magnitude (or smallness) unless such quantities happen to occur as factors multiplied by some quantity which is itself very large.

In order to save repetition, let me say here a few words about the notation used in connection with very small quantities. If we want to say that a certain portion of x is so exceedingly small as to be **practically**, but **not absolutely**, negligible, we denote such a portion of x by the symbol Δx. If, however, Δx is made smaller and smaller until it becomes infinitesimally small, *i.e.*, until it becomes in itself *absolutely* negligible, or zero, then we denote such a portion of x by the symbol dx, by which we mean the differential or infinitely small bit of x.

Hence we can say that dx is the limit of Δx when Δx is made infinitely small.

Now, if in any function $y = f(x)$, the independent variable x changes in value from x to $(x + dx)$, where dx is, as we have seen, an infinitesimally small increment (or decrement) of x, then the dependent variable y will necessarily undergo a correspondingly infinitesimal change and become $y + dy$.

The ratio between the infinitesimally small change dy in the dependent variable to the infinitesimally small change dx in the independent variable, *i.e.*, the ratio $\frac{dy}{dx}$ is called the **differential coefficient** of y with respect to x. In other words, a differential coefficient represents the true rate of change (as contrasted with the average rate of change) of the dependent variable y with respect to any change of the independent variable x.

Thus if y = distance travelled by a body in time x, then $\frac{dy}{dx}$ = the true velocity of the body at any moment.

If y = length, or area, or volume, of a body that is heated to any temperature x, then $\frac{dy}{dx}$ = coefficient of expansion (linear, superficial, or cubical).

If y = concentration of any body undergoing chemical change during a period of time t, then $\frac{dy}{dx}$ = reaction, velocity, etc.

Average and Real or True Rate of Change.—The term **rate of change** has not exactly the same meaning, in its ordinary colloquial use, as it has when used in scientific terminology. Thus, when we say, for instance, that the velocity of a train during a certain period of its journey was, say, 30 miles per hour, we mean that the distance (in miles) covered by the train during that **measurable** period (in hours), divided by the length of that period, was equal to 30.

Thus the period of observation might have been 15 minutes, or 0·25 hour, during which time the train covered a distance of $7\frac{1}{2}$ miles. We, then, get as follows:

Distance in miles $= 7{\cdot}5$

Period in hours $= 0{\cdot}25$

$$\therefore \text{ speed during the 15 minutes, or } \tfrac{1}{4} \text{ hour} = \frac{7{\cdot}5}{0{\cdot}25}.$$

$$= 7{\cdot}5 \times 4 = 30 \text{ miles per hour.}$$

But it is clear that unless the speed was uniform the velocity must have kept on changing during that measured period, so that the above calculation gives us no idea of the *actual* velocity of the train **at every moment** during that $\frac{1}{4}$ hour. What it does give us is the **average** velocity during that interval of time, and if, for instance, we had taken a different period of observation, say, 5 minutes, we might have found that during that shorter period the train covered a distance of, say, 3 miles, so that the velocity during that interval was $\frac{3}{1/12} = 3 \times 12 = 36$ miles per hour.

In scientific problems, as we have said, we have to investigate the instantaneous velocity, or rate of change, of some growing or varying quantity. In other words, we have to find out what is the *actual* rate of change of the dependent variable (such as distance) at any moment, *i.e.*, during an infinitesimally small, and therefore *immeasurable*, change of the independent variable (such as time).

It might at first seem not only a contradiction, but an impossibility, to *measure* the amount of change during an *immeasurable* period, but it is the beauty of mathematics that it renders the apparently impossible not only possible, but easy. For example, we cannot reach the sun, yet we can measure its distance. In what follows it will be my object to show how the problem of finding *actual*, or *true*, or *instantaneous*, rates of change can be solved. We shall first take an example and work it out by logical reasoning from first principles, and, when that example has been thoroughly mastered and understood, the rules which we shall develop, and the methods of their development, will become quite easy.

Suppose a body to be moving with a uniform velocity of, say, 10 ft. per second, and at a certain fixed instant it has imparted to it an acceleration of 6 ft. per second per second. We know from elementary mechanics that the distance **S** covered by that body during an interval of time **t** from the fixed instant is given by the formula

$$S = 10t + 3t^2.$$

Now, with the aid of this formula we can easily ascertain what is the **average** velocity of the body during any **definite measurable** interval of time. Thus, for the average velocity during the fourth second we have

Distance covered in three seconds $= 10 \times 3 + 3 \times 9 = 57$ ft.

Distance covered in four seconds $= 10 \times 4 + 3 \times 16 = 88$ ft.

$\therefore$ distance covered during the fourth second $= 88 - 57 = 31$ ft.

$\therefore$ *average velocity during the whole of the fourth second* $= 31$ *ft. per second.*

Now let us see what is the average velocity during first half of the fourth second.

Distance covered in three seconds $= 10 \times 3 + 3 \times 9 = 57$ ft.

Distance covered in 3·5 seconds $= 10 \times 3{\cdot}5 + 3 \times (3{\cdot}5)^2 = 71{\cdot}75$ ft.

$\therefore$ distance covered during first half of the fourth second $= 71{\cdot}75 - 57 = 14{\cdot}75$ ft.

$\therefore$ *average velocity during that half second* $= \dfrac{14{\cdot}75}{0{\cdot}5} = 29$ *ft. per second.*

But this, again, does not represent the true velocity during the **whole** of the half a second, because the velocity is not constant, but is continually increasing. Let us therefore find what the average velocity is during the first $\frac{1}{10}$ of the fourth second.

We have again

Distance covered in 3 seconds $= 10 \times 3 + 3 \times 9 = 57$ ft.

Distance covered in 3·1 seconds $= 10 \times 3{\cdot}1 + 3 \times (3{\cdot}1)^2 = 59{\cdot}83$ ft.

$\therefore$ Distance covered during the first $\frac{1}{10}$ or 0·1 of the fourth second $= 59{\cdot}83 - 57 = 2{\cdot}83$ ft.

$\therefore$ average velocity during that $\frac{1}{10}$ second $= \dfrac{2{\cdot}83}{0{\cdot}1} = 28{\cdot}3$ ft. per second.

But, as we said, this again only gives the *average* velocity during the $\frac{1}{10}$ second. But by taking the distance covered in 3·01 seconds, we can find the average velocity during the first $\frac{1}{100}$ of the fourth second. Thus :

Distance covered in 3·01 seconds $= 10 \times 3{\cdot}01 + 3 \times (3{\cdot}01)^2 =$ 57·2803.

$\therefore$ distance covered in first $\frac{1}{100}$ of the fourth second

$$= 57{\cdot}2803 - 57$$
$$= {\cdot}2803 \text{ ft.}$$

$\therefore$ average velocity during that $\frac{1}{100}$ or ·01 of a second

$$= \frac{{\cdot}2803}{{\cdot}01} = 28{\cdot}03 \text{ ft. per second.}$$

Similarly, it can be shown that :

Average velocity during the first $\frac{1}{1000}$ or ·001 of the fourth second = *28*·0003 ft. per second.

Average velocity during the first $\frac{1}{10000}$ or ·0001 of the fourth second = *28*·0003 ft. per second.

Average velocity during the first $\frac{1}{100000}$ or ·00001 of the fourth second = *28*·00003 ft. per second.

Average velocity during the first $\frac{1}{1000000}$ or ·000001 of the fourth second = *28*·000003 ft. per second,

and so on, there being always as many zeros preceding the 3 in the decimal as there are in the decimal of the second considered. Hence we see that if we calculate the velocity during smaller and smaller intervals of time, we find that it approaches closer and closer to *28* ft. per second, and in the limit, when the interval of time is infinitesimally small, *i.e.*, during the time **dt**, the **actual** velocity of the body, or its instantaneous velocity at that instant, becomes 28 ft. per second. In other words, the actual or true, or instantaneous velocity at the end of the third second or at the beginning of the fourth second, of a body moving in accordance with the law

$$S = 10t + 3t^2$$

is 28 ft. per second.

Put in the notation of the calculus, we say that if

$$S = 10t + 3t^2,$$

then $$\frac{ds}{dt} \text{ (when } t = 3) = 28 \text{ ft. per second.}$$

Similarly, we can show that the actual velocity at the end of the fourth or beginning of the fifth second is 34 ft. per second, and so on.

Let us now see how we can arrive at this **actual velocity** at any moment in a simple way.

We have

$$S = 10t + 3t^2.$$

If we increase t to $(t + dt)$, **S** will correspondingly become $(S + ds)$.

$$\begin{aligned} \text{Hence} \quad S + ds &= 10(t + dt) + 3(t + dt)^2 \\ &= 10t + 10dt + 3t^2 + 6tdt + 3(dt)^2 \\ \text{But} \quad S &= 10t + 3t^2 \\ \therefore \text{ by subtraction} \quad ds &= 10dt + 6tdt + 3(dt)^2 \end{aligned}$$

But $(dt)^2$ being of the second order of smallness may in the limit be neglected (see p. 128).

$$\therefore \quad ds = 10dt + 6tdt = (10 + 6t)dt.$$

$$\therefore \quad \frac{ds}{dt} = 10 + 6t.$$

This gives the actual velocity at any moment t. Thus, if $t = 3$,

$$\frac{ds}{dt} = 10 + 6 \times 3 = 28 \text{ ft. per second.}$$

Similarly, at the end of the fourth second

$$\frac{ds}{dt} = 10 + 6 \times 4 = 34 \text{ ft. per second,}$$

and so on.

The beginner is sometimes puzzled to know why an expression like $\frac{ds}{dt}$ or $\frac{dy}{dx}$, etc., should be called a differential **coefficient.** The Germans call it a differential **quotient,** which it obviously is. But if the reader will look at the expression above, $ds = (10 + 6t)dt$, he will see at once that $(10 + 6t)$, which $= \frac{ds}{dt}$, is the **coefficient** of the differential dt. Hence the name.

As we shall constantly have to deal with biological phenomena in which quantities keep on growing or changing, it will be our business to find the value of $\frac{dy}{dx}$ (which, as we have seen, represents the rate of change of y at any moment, as x keeps on growing) for various functions of x. The method of finding the differential coefficient of a function is called *differentiation.*

Let us take as another illustration any one of the examples of the compound interest law to which we devoted so much attention in Chapter VII. The simplest case, again, is money allowed to grow at **true** compound interest. Let us take the most general equation for such a form of growth, viz. :

$Q = Q_o e^{kt}$ (see p. 76),

where Q_o = original capital,

k = constant of growth $= \frac{r}{100}$ where r = interest per cent. per annum,

t = time to which the money is allowed to grow,

$e = 2{\cdot}71828 \ldots$ (see p. 80),

and Q = amount to which Q_o has grown in time t.

To simplify matters let us put $Q_o = 1$, and let us designate Q by the letter y. Then we have here a function in which t (the time) is the independent variable and y (the amount to which the capital has grown) is the dependent variable (*i.e.*, dependent upon the value of t).

Supposing we did not know what sort of growth such a function represents, and we made it our business to find it out, we would clearly set out to ascertain the rate of change of y with each infinitesimally small increase in the value of t, *i.e.*, we would have to find the differential coefficient of y with respect to t, or $\frac{dy}{dt}$.

Let us proceed to do so, and see the result at which we arrive,

$$y = e^{kt}.$$

$\therefore$ if t becomes increased to $(t + dt)$ (where dt is infinitesimally small) then y will in consequence become $y + dy$.

Our new equation, therefore, after an infinitely small interval of time dt, will be

$$y + dy = e^{k(t + dt)} = e^{kt + kdt}.$$

But $$y = e^{kt}$$

$\therefore$ by subtraction

$$dy = e^{kt + kdt} - e^{kt} = e^{kt} \times e^{kdt} - e^{kt}$$

(see Chap. II., p. 7, Law I.).

$$= e^{kt}(e^{kdt} - 1).$$

Now, by the exponential theorem we have

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \,.\,.\,.\,.$$

(see Chap. VI., p. 78).

$\therefore$ by putting $x = kdt$ we get

$$e^{kdt} = 1 + \frac{kdt}{1} + \frac{k^2(dt)^2}{1\,.\,2} + \frac{k^3(dt)^3}{1\,.\,2\,.\,3}.$$

But $(dt)^2$, being of the second order of magnitude, may in the limit be rejected, as may also be all the other terms containing $(dt)^3$, $(dt)^4$, etc.

$$\therefore \quad e^{kdt} \text{ ultimately becomes } = 1 + kdt.$$

$$\therefore \quad e^{kdt} - 1 = 1 + kdt - 1$$
$$= kdt.$$

Hence, we get

$$dy = e^{kt}\,.\,kdt = ke^{kt}\,.\,dt.$$

$$\therefore \quad \frac{dy}{dt} = ke^{kt} = ky \text{ (since } y = e^{kt}).$$

In other words, the differential coefficient of this function which expresses the rate of change of y with respect to t, tells us that in this particular function the growing quantity y grows in such a way that its increase in growth at any moment $\left(i.e., \frac{dy}{dx}\right)$ is proportional to its value at that moment.

We shall see later (p. 152) that this function can be differentiated in a much simpler way, but the method adopted here, though somewhat complicated, will repay careful study.

Note.—It is most important to notice that, as $\frac{dy}{dx}$ represents a momentary **rate of change,** therefore if $\frac{dy}{dx} = +$ **ve** the value of y **increases** with increases of x, and if $\frac{dy}{dx} = -$ **ve** the value of y **diminishes** with every increase of x. Thus in the case of absorption of light by a transparent medium, for instance we saw that the amount of light passed through becomes less and less as the thickness of the medium increases, and that in that case the form of the function is

$$Q_t = Q_0 e^{-kt}$$

or

$$y = e^{-kt},$$

whence

$$\frac{dy}{dx} = -ke^{-kt},$$

$$i.e., \frac{dy}{dx} \text{ is negative.}$$

Hence we have the following *rules :*

(1) **In the case of any function y = f(x), so long as $\frac{dy}{dx} = +$ ve for any value of x, then y keeps on increasing with every increase of x ; but when we find $\frac{dy}{dx} = -$ ve for some value of x, then we know that with every increase of x, y keeps on decreasing.**

(2) As a corollary of (1) it follows that **if for any value of x, $\frac{dy}{dx} = 0$, then for that particular value of x the value of y is stationary.**

These useful rules are of fundamental importance in the study of maxima and minima of functions (see p. 164).

The Geometrical Meaning of a Differential Coefficient.—To understand the meaning of $\frac{dy}{dx}$ in a geometrical sense it is necessary to be perfectly clear about the two following points, viz. :

(1) The distinction between a geometrical tangent to a curve and the trigonometrical tangent of an angle. This has been explained on p. 37, Chapter IV.

(2) The meaning of the term **slope** or **gradient**.

If we take any line AB, then its slope is expressed by the tangent of the angle which the line makes with the axis of x, *i.e.*, by the ratio $\frac{MP}{AM}$ (Fig. 57).

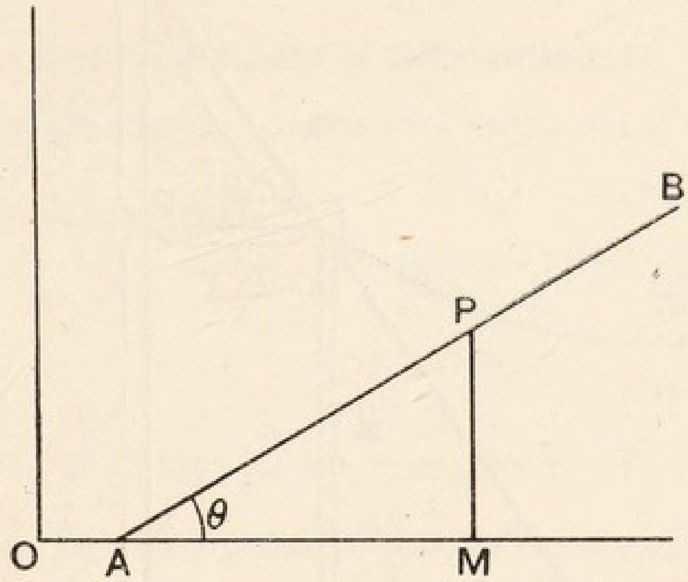

Fig. 57.—To illustrate the Meaning of Slope of a Straight Line.

Similarly, the slope of any curve at any point P is the tangent of the angle θ, which the geometrical tangent at P makes with the axis of x, *i.e.*, by the ratio $\frac{MP}{AM}$ (Fig. 58).

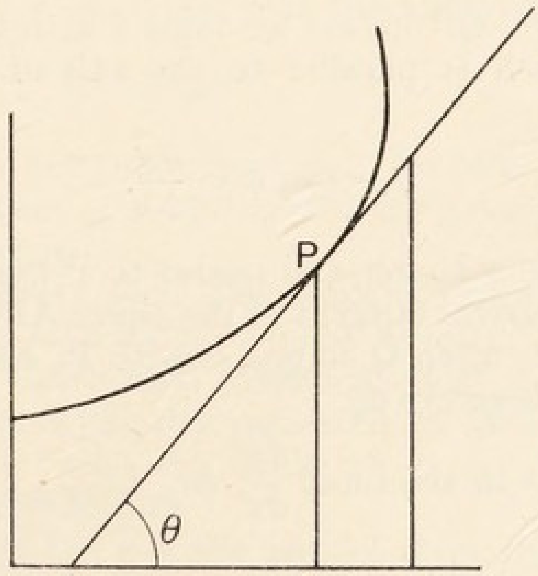

Fig. 58.—To illustrate the Meaning of the Slope of a Curve.

Now consider any function $y = f(x)$.

Let APQ be a portion of the graph of this function (Fig. 59), and let P and Q be two points very close to each other, then if the co-ordinates of P be x, y, the co-ordinates of Q will be

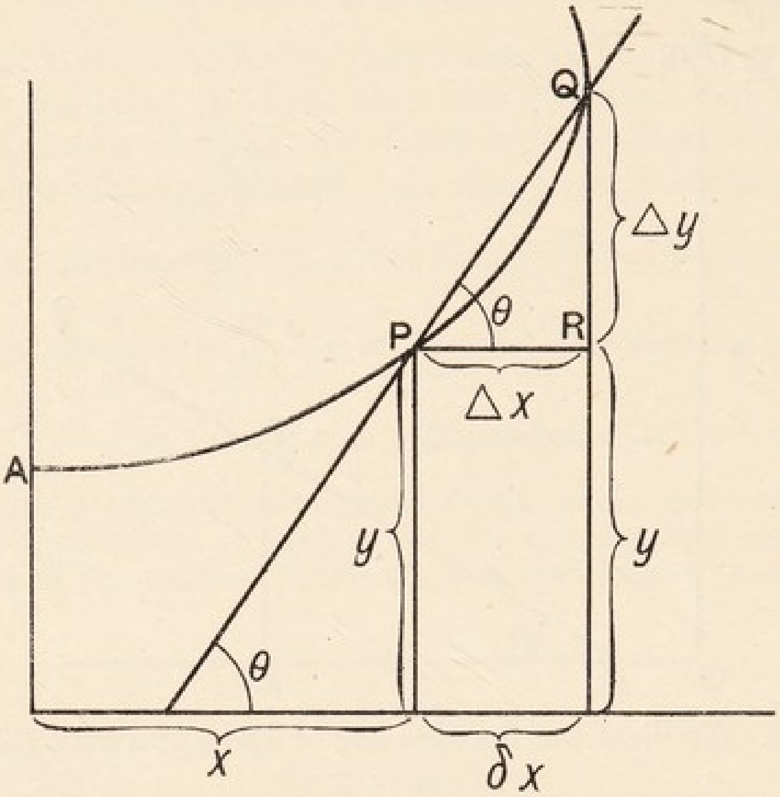

FIG. 59.—Geometrical Meaning of a Differential Coefficient.

$x + \Delta x$, $y + \Delta y$ (where Δx = a minute but finite part of x, Δy = a minute part of y).

And if the chord QP makes an angle θ with the axis of x, then the line PR, which is parallel to the axis of x, also makes an angle θ with QP.

$$\therefore \quad \tan\theta = \frac{\Delta y}{\Delta x}.$$

Now, as Q is taken nearer and nearer to P the line QP tends to become the geometrical tangent of the curve APQ at the point P.

But in the limit, when Q coincides with P, Δx becomes dx, and therefore also Δy becomes dy.

$$\therefore \quad \frac{\Delta y}{\Delta x} \text{ becomes in the limit } \frac{dy}{dx}, \textit{ i.e.}, \operatorname{Lt}_{\Delta x \to o} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

Hence $\frac{dy}{dx}$ is the trigonometrical tangent of the angle which the tangent of the curve at any point makes with the axis of x.

In other words, *the differential coefficient of a functiom* $y = f(x)$ *is the slope of the curve of that function at any definite point* (x, y).

Notation. — The differential coefficient of $f(x)$ is generally denoted by $f'(x)$.

Thus, if $$y = f(x),$$

then $$\frac{dy}{dx} = f'(x).$$

General Methods of Differentiation.

(1) **Algebraic Fractions.**—*Differential coefficient of* x^n.

Let $y = x^n$.

Then $y + dy = (x + dx)^n$.

$$= x^n + \frac{nx^{n-1}}{1} dx + \frac{n(n-1)x^{n-2}}{1 \, . \, 2} (dx)^2 + \frac{n(n-1)(n-2)x^{n-3}}{1 \, . \, 2 \, . \, 3} (dx)^3 + \ldots$$

(by the binomial theorem ; see p. 64)

But all terms to the right of $\frac{nx^{n-1}}{1} dx$ in this expansion containing quantities of the second and higher orders of magnitude may be rejected.

Hence $$(x + dx)^n = x^n + \frac{nx^{n-1}}{1} dx,$$

i.e., $$y + dy = x^n + nx^{n-1}dx.$$

But y $$= x^n,$$

$$\therefore \quad dy = nx^{n-1}dx \text{ (by subtraction).}$$

$$\therefore \quad \frac{dy}{dx} = nx^{n-1}.$$

Hence we obtain the following rule : *To obtain the differential coefficient of some power of x, multiply by the index of the power and then reduce the power by* 1.

It is most important that the student should grasp the full significance of this statement. What exactly is meant by the statement that the differential coefficient of $x^n = nx^{n-1}$?

It means that if x is slightly increased in value, then the corresponding increase in value of y is nx^{n-1}.

Let us take a few examples.

Supposing $y = x^2$, *i.e.*, y is the area of a square, each of whose sides is equal to x.

If x undergoes any very slight change in length, what will be the corresponding change in the area of the square ? Imagine, for instance, the square to be made of iron whose coefficient of expansion is 0·000012.

If the length of the side = 1 metre, what change will the area of the square undergo if it is heated 1° C. ?

Since $y = x^2 = 1$ square metre,

$\therefore \quad y + \Delta y^* = (x + \Delta x)^2 = (1 + \cdot 000012)^2.$

$\therefore \quad \Delta y$, *i.e.*, increase in area of square

$$= (1 + \cdot 000012)^2 - 1$$
$$= 2 \times \cdot 000012 + (\cdot 000012)^2.$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = 2 + (\cdot 000012) \text{ square metres,}$$

i.e., as the side increases by 0·000012 metre the area increases by 2 + ·000012 sq. cm. If the square were to be heated to $\frac{1}{100}$ C., the change in area of the square would be 2 + ·00000012. So that as the increase in the side becomes less and less the increase in area becomes nearer and nearer 2, and when the increase in the length of the side is infinitesimal and becomes dx the actual increase in the area dy equal $2xdx$ and the **rate of increase** in area $\frac{dy}{dx}$ becomes equal to $2x = 2$.

From the result of $\frac{dx^n}{dx} = nx^{n-1}$ we obtain the following differential coefficients :

$$\frac{dx^1}{dx} = 1 \cdot x^0 = 1,$$

$$\frac{dx^2}{dx} = 2 \cdot x,$$

$$\frac{dx^3}{dx} = 3x^2,$$

$$\frac{dx^4}{dx} = 4x^3,$$

and so on.

This rule holds good whether n be positive or negative, integral or fractional.

Thus, let $n = -m$,

then $y = x^{-m}$.

$$\therefore \quad y + dy = (x + dx)^{-m}$$
$$= x^{-m} - mx^{-m-1}dx + \frac{m(m+1)}{1 \cdot 2} x^{-m-2}(dx)^2$$
$$- \ldots .$$

* Whilst dx or dy represents an infinitesimally small change of x or of y, Δx and Δy are used to denote very minute but not infinitesimally small changes of the variables.

$\therefore \quad dy = -mx^{-m-1}dx$ + some higher powers of dx which may be rejected.

$$\therefore \quad \frac{dy}{dx} = -mx^{-m-1}.$$

E.g., if $y = x^{-2}$, then

$$\frac{dy}{dx} = -2x^{-3}.$$

If $y = x^{-3}$,

then $\frac{dy}{dx} = -3x^{-4}$,

and so on.

Let m be fractional $= \frac{1}{m}$,

then $y = x^{\frac{1}{m}}$

$$\therefore \quad y + dy = (x + dx)^{\frac{1}{m}}$$

$= x^{\frac{1}{m}} + \frac{1}{m} x^{\frac{1}{m}-1} dx$ + some higher power of dx which may be rejected.

$$\therefore \quad dy = \frac{1}{m} x^{\frac{1}{m}-1} dx.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{m} x^{\frac{1}{m}-1}.$$

E g, if $y = x^{\frac{1}{2}}$,

then $\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

If $y = x^{\frac{1}{3}}$,

then $\frac{dy}{dx} = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$,

and so on.

Comparison of the Geometrical with the Algebraical Method of finding $\frac{dy}{dx}$.—Plot the graph of any function, say, $y = x^2$, as in Fig. 60 (a parabola), and draw tangents at various points P, Q, R, whose co-ordinates are, say, (3, 9) (4·5, 20·25) and (7, 49), respectively. Measure the tangents of the angles made by these tan-

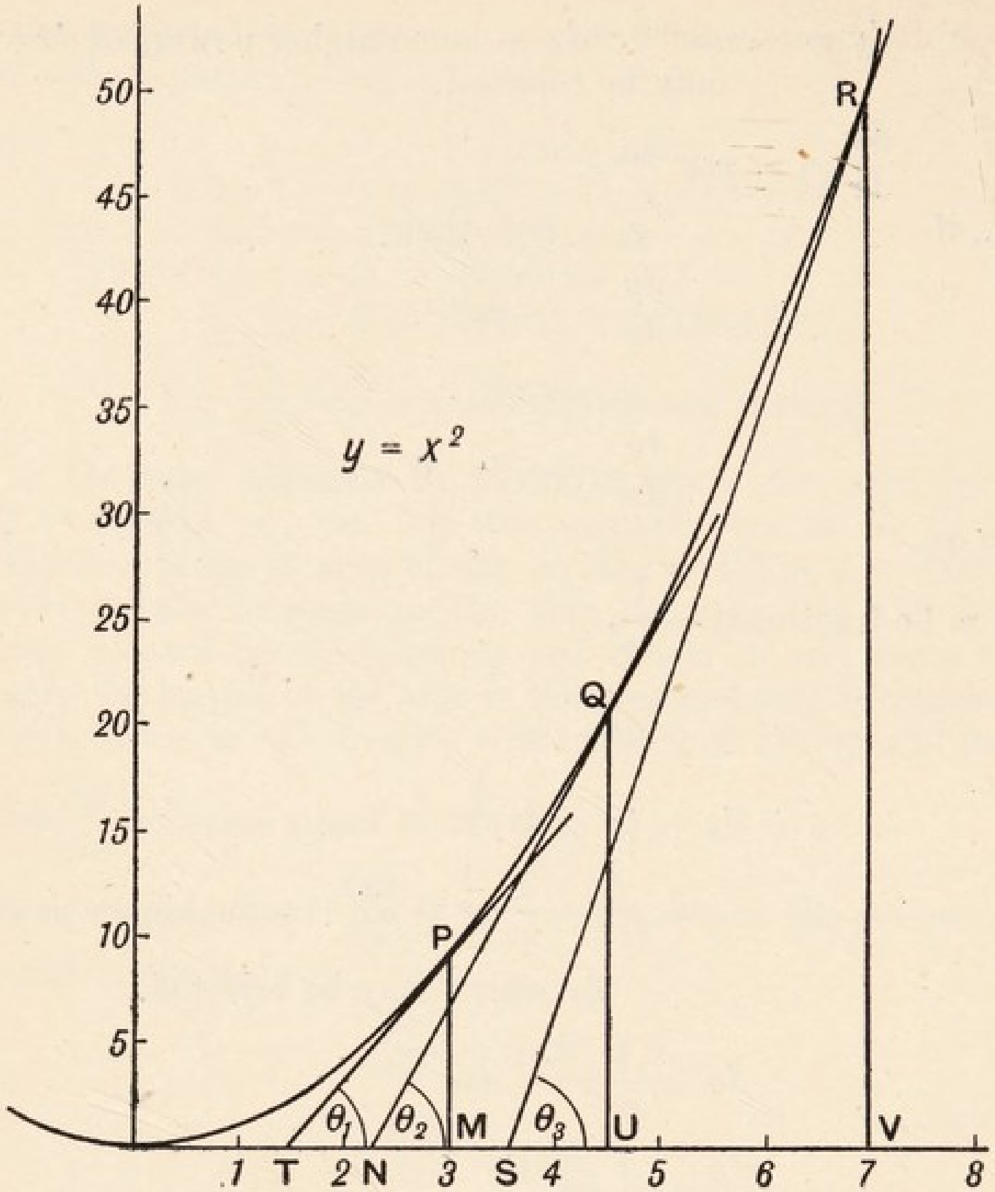

FIG. 60.—Graphical Method of showing that if $y = x^2$, then $\frac{dy}{dx} = 2x$.

gents with the x axis, *i.e.*, the ratios $\frac{\text{PM}}{\text{MT}}$, $\frac{\text{QU}}{\text{NU}}$ and $\frac{\text{RV}}{\text{SV}}$. You will find that these ratios are 6, 10, and 14 respectively.

Now we know that $$\frac{dx^2}{dx} = 2x.$$

$\therefore$ when $x = 3$, $\frac{dx^2}{dx}$ should $= 6$;

when $x = 4{\cdot}5$, $\frac{dx^2}{dx}$ should $= 9$;

when $x = 7$, $\frac{dx^2}{dx}$ should $= 14$.

Hence we see that $\frac{dy}{dx}$ at any point in the curve is truly represented by the tangent of the angle made by the geometrical tangent at that point. If we plot the values of $\frac{dy}{dx}$ or y' thus formed geometrically against the corresponding values of x, *i.e.*, the points (3, 6), (4·5, 9), (7, 14), etc., we shall get a graph which will represent the slope or first derivative of the original or primitive curve. The graph in question will be the straight line $y' = 2x$.

Example.—Find the slope of the curve, $y = \frac{1}{2}x^2$, at the points (4, 1), and (3, $4\frac{1}{2}$).

From the graph (Fig. 61) it is seen that the slope of the tangent at the point for which $x = 1$ is $\frac{\text{MP}}{\text{QM}} = 1$.

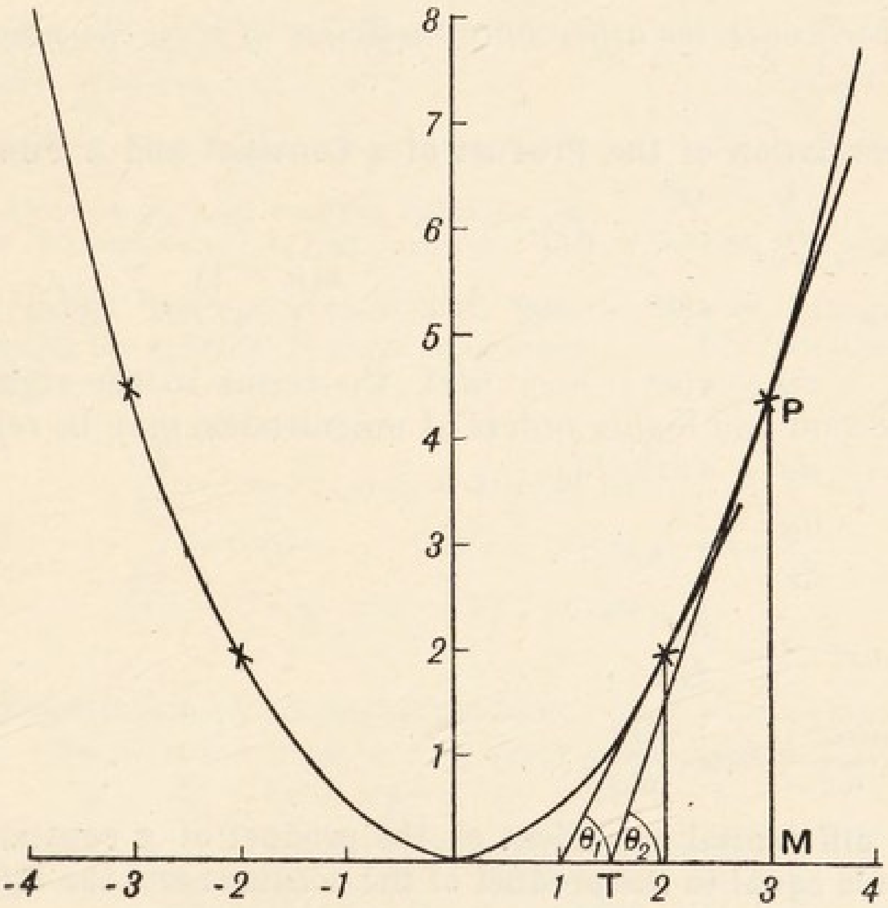

FIG. 61.—Graphical Method of Showing that if $y = \frac{x^2}{2}$, then $\frac{dy}{dx} = x$.

By the algebraical method we find $\frac{dy}{dx} = \frac{1}{2} . 2x = x$, so that when $x = 2$, $\tan \theta_1$ becomes $= 2$.

Similarly it is seen that at the point where $x = 3$, the slope, $\tan \theta_2 = 3$, and so on.

By plotting the various values of x and y' we shall get the first derivative graph which is the straight line $y = x$.

EXERCISE

Draw the graph $y = \frac{1}{2}x(x + 1)$ and find the values of $\frac{dy}{dx}$ at the points (4, 10), (8, 36).

[Answer, 4·5, 8·5.]

Differentiation of a Constant.—Since a constant means a quantity which does not vary, therefore if $y = c$, $dy = 0$.

$$\therefore \quad \frac{dy}{dx} = 0,$$

i.e., **the differential coefficient of a constant = 0.**

Hence, if $y = x^n + c$, then $\frac{dy}{dx} = nx^{n-1}$.

In other words, the differential coefficient of x^n *is the same as that of* $x^n + c_1$.

Differentiation of the Product of a Constant and a Function.—Let

$$y = ax^n,$$

$$\therefore \quad y + dy = a(x + dx)^n$$

$$= a(x^n + nx^{n-1}dx + \frac{n(n-1)}{1 \cdot 2}x^{n-2}(dx)^2 + \ldots).$$

$\therefore \quad dy = a(x^n + nx^{n-1}dx)$, the terms to the right, being of the second and higher orders of magnitudes, may be rejected.

$$\therefore \quad dy = anx^{n-1}dx,$$

$$\therefore \quad \frac{dy}{dx} = a \cdot nx^{n-1}.$$

But $nx^{n-1} = \frac{dx^n}{dx}$.

$$\therefore \quad \frac{d(ax^n)}{dx} = a \cdot \frac{dx^n}{dx};$$

i.e., **the differential coefficient of the product of a constant and a function is equal to the product of the constant and the differential coefficient of the function.**

Hence also, if $y = ax^n + b$, then $\frac{dy}{dx} = a \cdot nx^{n-1}$.

Examples.—(1) It has been found that the relation between the percentage (x) of casein dissolved in a given solution of alkali, and the time (t) of stirring, is expressed by the equation $x = Kt^m$. Find the rate at which the solution occurs (K = constant).

Differentiating we get $\frac{dx}{dt} = Kmt^{m-}$

$\therefore$ velocity of solution $= Kmt^{m-1}$

(2) At a given instant the radius of a soap bubble is increasing at the rate of 2 ins. per minute. What is the rate of the increase of volume when the radius is 3 ins. ?

$$\text{Volume of bubble, } V = \frac{4}{3}\pi r^3 \text{ (see p. 57).}$$

$$\therefore \quad \frac{dv}{dr} = \frac{4}{3}\pi \,.\, 3r^2 = 4\pi r^2.$$

$$\therefore \quad \text{when} \quad r = 3 \text{ ins.}, \frac{dv}{dr} = 4\pi \,.\, 9 = 36\pi.$$

$$\therefore \quad dv = 36\,\pi dr.$$

$$\therefore \quad \text{when} \quad dr = 2 \text{ ins.}, dv = 72\pi.$$
$$= 72 \times 3{\cdot}1416.$$
$$= 226{\cdot}2 \text{ cub. ins.}$$

(3) At what rate does the amount of light passing through the iris diaphragm of a microscope change with increase in the radius of the aperture ?

Let radius of aperture $= r$ (Fig. 62).

$\therefore$ area of aperture $A = \pi r^2$

$$\therefore \quad \frac{dA}{dr} = 2\pi r.$$

But the amount of light passing through the aperture is proportional to the area of the aperture.

$\therefore$ the rate at which the amount of light passing through the aperture changes on opening the diaphragm $= 2\pi r$.

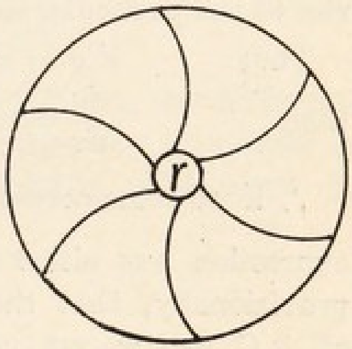

FIG. 62.—Diagram of Iris Diaphragm.

Thus when $r = 5$ mm.,

$$\frac{dA}{dr} = 10\pi \text{ sq. mm.} = 31{\cdot}4 \text{ sq. mm.}$$

When $r = 6$ mm.

$$\frac{dA}{dr} = 12\pi \text{ sq. mm.} = 37{\cdot}7 \text{ sq. mm.},$$

and so on.

The Differentiation of an Algebraic Sum.—Let $y = x^2 + x + 1$. Suppose x to increase to $(x + dx)$, then y becomes $y + dy$.

$$\therefore \quad y + dy = (x + dx)^2 + (x + dx) + 1$$
$$= x^2 + 2xdx + (dx)^2 + (x + dx) + 1.$$

$$\text{But} \quad y = x^2 \qquad\qquad + x \quad + 1$$

$$\therefore \quad dy = 2xdx + dx + (dx)^2$$
$$= 2xdx + dx$$

$$\therefore \quad \frac{dy}{dx} = 2x + 1.$$

But $2x$ is the differential coefficient of x^2,
1 ,, ,, ,, x,
and 0 ,, ,, ,, 1.

$\therefore$ **the differential coefficient of an algebraic sum is the algebraic sum of the differential coefficients of each of the terms.**

Note.—It is most important for the student to realise and remember the fact that all functions of x which only differ in respect of a constant term have the same differential coefficient (see p. 142 above).

Thus
$$y = 4x^5 + 3x^4 + 2x^3 + 7x^2 + 3x$$
$$y = 4x^5 + 3x^4 + 2x^3 + 7x^2 + 3x + 1$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$y = 4x^5 + 3x^4 + 2x^3 + 7x^2 + 3x + n$$
when differentiated give the same result,

viz.,
$$\frac{dy}{dx} = 20x^4 + 12x^3 + 6x^2 + 14x + 3.$$

The importance of this fact will be appreciated when we have to deal with integration, *i.e.*, when we are confronted with the problem of ascertaining from the differential coefficient what the original function was which gave rise to the particular differential coefficient.

Thus if $y = x^n$ or $x^n + c$ (where c = any constant),

then
$$\frac{dy}{dx} = nx^{n-1}.$$

$\therefore$ if we come across the expression $\frac{dy}{dx} = nx^{n-1}$ we know that the original expression was either simply x^n or $x^n + C$, and we therefore always say, provisionally, that the function whose differential coefficient is nx^{n-1} is $x^n + C$, and we set out to find, from the other data in the problem, the value of C. It may turn out to be zero, and then we know that the original function was $y = x^n$. (See chapter on Integral Calculus, p. 198.)

The Differentiation of a Product of Two Functions.—Suppose we have to differentiate the product $y = uv$, where u and v are functions of x.

Then
$$y + dy = (u + du)(v + dv)$$
$$= uv + vdu + udv + du \,.\, dv.$$
$$\therefore \quad dy = vdu + udv + du \,.\, dv.$$

But $du \,.\, dv$, being a quantity of the second order of magnitude, may be discarded.

$$\therefore \quad dy = vdu + udv$$
$$\therefore \quad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Hence **the differential coefficient of a product is equal to the sum of the products of each function by the differential coefficient of the other.**

E.g., let
$$y = (ax^2 + b)(cx^3 + d).$$

Now the differential coefficient of the first factor $= 2ax$ and the differential coefficient of the second factor $= 3cx^2$.

$$\therefore \frac{dy}{dx} = 2ax(cx^3 + d) + 3cx^2(ax^2 + b)$$
$$= 2acx^4 + 2axd + 3acx^4 + 3cbx^2$$
$$= 5acx^4 + 3bcx^2 + 2adx.$$

The Differentiation of a Product of more than Two Functions.—Let $y = uvw$, where u, v, and w are functions of x. Then by treating u as one factor and vw as another factor

we get $$\frac{dy}{dx} = u\frac{d(vw)}{dx} + vw\frac{du}{dx}.$$

But $$\frac{d(vw)}{dx} = v\frac{dw}{dx} + w\frac{dv}{dx}.$$

$$\therefore \quad u\frac{d(vw)}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx}.$$

$$\therefore \quad \frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}.$$

Similarly, **in the case of the product of n functions, the differential coefficient of the product is equal to the sum of n products of the differential coefficient of each of the functions multiplied by the remaining (n − 1) functions.**

Thus if $n = 5$, we have
$y = uvwst$ (say);

then $$\frac{dy}{dx} = uvws\frac{dt}{dx} + uvwt\frac{ds}{dx} + uvst\frac{dw}{dx} + uwst\frac{dv}{dx} + vwst\frac{du}{dx}.$$

Note.—The differential coefficient of a product may also in suitable cases be found by multiplying out the product and treating the result as an algebraic sum. Thus to take the same example again:

$$y = (ax^2 + b)(cx^3 + d)$$
$$= acx^5 + bcx^3 + adx^2 + bd.$$
$$\therefore \quad \frac{dy}{dx} = 5acx^4 + 3bcx^2 + 2adx,$$

the same result as before.

Differentiation of a Fraction.—If the fraction is such that its numerator and denominator contain a common factor, then one must first eliminate that common factor.

Thus, if

$$y = \frac{x^2 + 3x + 2}{x + 2}$$
$$= \frac{(x + 2)(x + 1)}{x + 2}$$
$$= x + 1.$$
$$\therefore \quad \frac{dy}{dx} = 1.$$

But supposing the fraction is given in its simplest form ; how is one to differentiate it ? We proceed as follows :

Let $$y = \frac{u}{v} \text{ (where } u \text{ and } v \text{ are functions of } x\text{).}$$

Then $$y + dy = \frac{u + du}{v + dv}.$$

Now, by performing the algebraical division of $\frac{u + du}{v + dv}$, we obtain as follows :

$$v + dv \,\Big)\, u + du \,\Big(\, \frac{u}{v} + \frac{du}{v} - \frac{u \,.\, dv}{v^2} + \ldots$$

$$\frac{u + \frac{u}{v} dv}{\quad du - \frac{u}{v} dv \quad}$$

$$\frac{du + \frac{du \,.\, dv}{v}}{\quad - \frac{u}{v} \,.\, dv - \frac{du \,.\, dv}{v} \quad}$$

$$\frac{- \frac{u}{v} \,.\, dv - \frac{u(dv)^2}{v^2}}{\quad - \frac{du \,.\, dv}{v} + \frac{u(dv)^2}{v^2}. \quad}$$

And as each of the two terms in the last remainder are of the second order of magnitude, they may be discarded.

$$\therefore \quad \frac{u + du}{v + dv} = \frac{u}{v} + \frac{du}{v} - \frac{udv}{v^2}.$$

$$\therefore \quad y + dy = \frac{u}{v} + \frac{du}{v} - \frac{udv}{v^2}.$$

But $$y = \frac{u}{v}.$$

$$\therefore \quad dy = \frac{du}{v} - \frac{udv}{v^2}.$$

$$= \frac{vdu - udv}{v^2}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Hence, the differential coefficient of a fraction is obtained as follows :

Multiply the denominator by the differential coefficient of the numerator and subtract from this the product of the numerator by the differential coefficient of the denominator, and divide the difference by the square of the denominator.

E.g., if $$y = \frac{ax^2 + b}{cx^3 + d},$$

then $$\frac{dy}{dx} = \frac{(cx^3 + d) \,.\, 2ax - (ax^2 + b) \,.\, 3cx^2}{(cx^3 + d)^2}$$

$$= \frac{2acx^4 + 2adx - 3acx^4 - 3bcx^2}{(cx^3 + d)^2}$$

$$= \frac{-acx^4 - 3bcx^2 + 2adx}{(cx^3 + d)^2}.$$

If we have to differentiate a fraction with a somewhat complicated denominator, then we have two courses open to us in order to simplify the process of differentiation, viz. :

(1) If the numerator and denominator contain one or more common factors, then the fraction should, as we have already stated on p. 145, be first simplified by cancelling these common factors out.

E.g. (*a*) $$y = \frac{x^2 - 2ax^2 + a^2}{x^3 - 3a^2x + 3ax^2 - a^3}.$$

The numerator $= (x - a)^2$.
The denominator $= (x - a)^3$.

$\therefore$ The function $y = \dfrac{(x-a)^2}{(x-a)^3} = \dfrac{1}{x-a} = (x-a)^{-1}$

$$\therefore \qquad \frac{dy}{dx} = -\frac{1}{(x-a)^2}.$$

(*b*) $$y = \frac{x^2 - a^2}{x^2 + 2ax + a^2} = \frac{(x-a)(x+a)}{(x+a)^2}$$

$$= \frac{x-a}{x+a}.$$

$$\therefore \qquad \frac{dy}{dx} = \frac{(x+a) \,.\, 1 - (x-a) \,.\, 1}{(x+a)^2}$$

$$= \frac{2a}{(x+a)^2}.$$

(2) If there is no common factor to cancel out, then split the fraction into its partial fractions and differentiate each separately.

Thus $$y = \frac{3x+1}{x^2-1} = \frac{1}{x+1} + \frac{2}{x-1}$$

(Chap. III., p. 26).

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{(x+1)^2} - \frac{2}{(x-1)^2}$$

$$= -\frac{(3x^2+2x+3)}{(x^2-1)^2}.$$

This example is given merely to illustrate the method, and not as an illustration of the type of case where splitting up into partial fractions is helpful, because in this particular instance differentiation by the ordinary rule can be done more expeditiously than by the partial fraction method. Thus

$$\frac{dy}{dx} = \frac{(x^2-1) \,.\, 3 - (3x+1) \,.\, 2x}{(x^2-1)^2}$$

$$= -\frac{3x^2+2x+3}{(x^2-1)^2}.$$

But supposing we had a function like the following :

$$y = \frac{(3x^2-2x+1)}{(x+1)^2(x-2)}.$$

Differentiation by the ordinary method would entail a considerable amount of tedious labour, but doing it by the method of partial fractions is a comparatively simple matter. The expression when split up into its component fractions becomes

$$y = \frac{2}{(x+1)} - \frac{2}{(x+1)^2} - \frac{1}{(x-2)}.$$

$$\therefore \quad \frac{dy}{dx} = -\frac{2}{(x+1)^2} + \frac{4}{(x+1)^3} - \frac{1}{(x-2)}.$$

Differentiation of a "Function of a Function."—In the function $y = (x^2+a^2)^{\frac{2}{3}}$, the expression (x^2+a^2) is a function of x, and the expression $(x^2+a^2)^{\frac{2}{3}}$ is a function of (x^2+a^2); therefore $(x^2+a^2)^{\frac{2}{3}}$ is a "function of a function" of x. To differentiate such a function we proceed as follows :

$$y = (x^2+a^2)^{\frac{2}{3}}. \qquad \therefore \quad y^{\frac{3}{2}} = x^2 + a^2 = u \text{ (say)}.$$

$$\therefore \quad \frac{du}{dy} \text{ or } \frac{d(x^2+a^2)}{dy} = \frac{3}{2}y \quad \left(= \frac{dy^{\frac{3}{2}}}{dy}\right).$$

$$\therefore \qquad d(x^2+a^2) = \frac{3}{2}y^{\frac{1}{2}}dy.$$

$$\therefore \quad \frac{d(x^2 + a^2)}{dx} = \frac{3}{2} y^{\frac{1}{2}} \frac{dy}{dx}.$$

But
$$\frac{d(x^2 + a^2)}{dx} = 2x.$$

$$\therefore \quad \frac{3}{2} y^{\frac{1}{2}} \frac{dy}{dx} = 2x$$

(*i.e.*, the differential coefficient of $y^{\frac{3}{2}}$ with respect to y (viz., $\frac{3}{2} y^{\frac{1}{2}}$), multiplied by the differential coefficient of y with respect to x, is equal to the differential coefficient of $y^{\frac{3}{2}}$ with respect to x).

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{\frac{3}{2} y^{\frac{1}{2}}} = \frac{4x}{3(x^2 + a^2)^{\frac{1}{3}}}.$$

See also example 9, p. 156.

In actual practice, the work is abbreviated thus :

$$y = (x^2 + a^2)^{\frac{2}{3}}.$$

$$\therefore \quad y^{\frac{3}{2}} = x^2 + a^2.$$

$$\therefore \quad \frac{3}{2} y^{\frac{1}{2}} \frac{dy}{dx} = 2x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{\frac{3}{2} y^{\frac{1}{2}}} = \frac{4x}{3(x^2 + a^2)^{\frac{1}{3}}}.$$

Such expressions can be dealt with somewhat differently as follows :

Put
$$x^2 + a^2 = u.$$

Then
$$y = u^{\frac{2}{3}}.$$

$$\therefore \quad \frac{dy}{du} = \frac{2}{3} u^{\frac{1}{3}}.$$

But
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

and
$$\frac{du}{dx} = 2x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{3} u^{-\frac{1}{3}} 2x$$

$$= \frac{2}{3} (x^2 + a^2)^{-\frac{1}{3}} . 2x$$

$$= \frac{4x}{3(x^2 + a^2)^{\frac{1}{3}}}.$$

EXAMPLES.

Differentiate the following expressions :

(1) $$y = 4x^{\frac{7}{4}}.$$

$$\frac{dy}{dx} = 4 \cdot \frac{7}{4} \cdot x^{\frac{7}{4}-1} = 7x^{\frac{3}{4}}.$$

(2) $$y = 3x^2 + 2x + 1.$$

$$\frac{dy}{dx} = 6x + 2.$$

(3) $$y = (x + a)\,(x + b).$$

$$\frac{dy}{dx} = (x + a)\,.\,1 + (x + b)\,.\,1$$

$$= 2x + a + b.$$

(4) $$y = \frac{x + a}{x + b} = \frac{(x + b)\,.\,1 - (x + a)\,.\,1}{(x + b)^2}$$

$$= \frac{b - a}{(x + b)^2}.$$

(5) $$y = \frac{x^m}{(x + 1)^m}.$$

$$\frac{dy}{dx} = \frac{(x + 1)^m\,.\,mx^{m-1} - x^m\,.\,m(x + 1)^{m-1}}{(x + 1)^{2m}}$$

$$= \frac{mx^{m-1}(x + 1)^{m-1}\,.\,[(x + 1) - x]}{(x + 1)^{2m}}$$

$$= \frac{mx^{m-1}}{(x + 1)^{m+1}}.$$

(6) $$y = \sqrt{x + \sqrt{1 + x^2}}.$$

$$\therefore \quad y^2 = x + \sqrt{1 + x^2}.$$

$$\therefore \quad 2y\frac{dy}{dx} = 1 + \frac{2x}{2\sqrt{1 + x^2}} = \frac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}}$$

$$= \frac{y^2}{\sqrt{1 + x^2}}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{2\sqrt{1 + x^2}}$$

$$= \frac{\sqrt{x + \sqrt{1 + x^2}}}{2\sqrt{1 + x^2}}.$$

(7) $$y = \sqrt{a + x + \sqrt{a + x + \sqrt{a + x + \text{etc.}}}},$$ to infinity.

$$\therefore \quad y^2 = a + x + y.$$

$$\therefore \quad y^2 - y = x + a.$$

By completing the square on the left-hand side the expression becomes

$$y^2 - y + \frac{1}{4} = x + a + \frac{1}{4}.$$

or

$$\left(y - \frac{1}{2}\right)^2 = \frac{4x + a + 1}{4}.$$

$$\therefore \quad y - \frac{1}{2} = \pm \frac{\sqrt{4x + a + 1}}{2}.$$

$$\therefore \quad y = \frac{1 \pm \sqrt{4x + a + 1}}{2}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{4}{4\sqrt{4x + 4a + 1}} = \frac{1}{\sqrt{4x + 4a + 1}}.$$

(8) The Schütz-Borissoff law with regard to the action of enzymes such as pepsin and rennin is expressed by the formula

$$x = \mathrm{K}\sqrt{\mathrm{F}qt},$$

where
x = amount of substance transformed,
t = time of transformation,
F = concentration of enzyme,
q = initial concentration of substrate (*e.g.*, albumen or milk), and k is a constant.

Find an expression for the velocity of hydrolysis in such a case. i.e. $\frac{dx}{dt}$.

Since $x = k\sqrt{\mathrm{F}qt}$,

$$\therefore \quad x^2 = k^2\mathrm{F}qt = \mathrm{KF}qt \text{ (where K is another constant } = k^2).$$

$$\therefore \quad t = \frac{x^2}{\mathrm{KF}q}.$$

$$\therefore \quad \frac{dt}{dx} = \frac{2x}{\mathrm{KF}q}$$

(velocity of hydrolysis).

$$\therefore \quad \frac{dx}{dt} = \frac{\mathrm{KF}q}{2x},$$

i.e., the velocity of hydrolysis in a case like this is inversely proportional to the amount of substance hydrolysed. (See, further, p. 287.)

EXERCISES.

Differentiate the following:

(1) $$y = \sqrt{\frac{a^2 + x^2}{a^2 - x^2}}.$$

$$\left[\text{Answer, } \frac{dy}{dx} = \frac{2a^2x}{\sqrt{(a^2 + x^2)(a^2 - x^2)^3}}.\right]$$

(2) $$y = \sqrt{x\sqrt{x\sqrt{x}}}, \text{ etc. . . . to infinity.}$$

$$\left[\text{Answer, } \frac{dy}{dx} = 1.\right]$$

(*Cf.* example 10, Chap. II., p. 17.)

(3) $$y = \sqrt{\frac{1-x}{1+x}}.$$

$$\left[\text{Answer,}\quad \frac{dy}{dx} = \frac{1}{(1+x)\sqrt{1-x^2}}.\right]$$

(2) **Differentiation of Exponential, Logarithmic and Circular Functions.**

(i.) **Exponential Functions.**

(*a*) *The differential coefficient of* e^x.

The exponential series has a most important *peculiarity*, viz., that **its differential coefficient is the same as itself.** Thus,

if $$y = e^x = 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \dots$$

then $$\frac{dy}{dx} = 0 + 1 + \frac{2x}{1\,.\,2} + \frac{3x^2}{1\,.\,2\,.\,3} + \frac{4x^3}{1\,.\,2\,.\,3\,.\,4} + \dots$$

$$= 1 + x + \frac{x^2}{1\,.\,2} + \frac{x^3}{1\,.\,2\,.\,3} + \dots$$

$$= e^x.$$

i.e., $$\mathbf{\frac{de^x}{dx} = e^x}.$$

This peculiarity is not shared by any other known function, and it is useful to remember it in connection with differential equations of the type $\frac{d^ny}{dx^n} = ny$. For another peculiarity see p. 77.

(*b*) *The differential coefficient of* a^x.

Let $a = e^c$.

$$\therefore \quad a^x = e^{cx} = 1 + cx + \frac{(cx)^2}{1\,.\,2} + \frac{(cx)^3}{1\,.\,2\,.\,3} + \dots$$

$\therefore$ if $a^x = y$, we have

$$\frac{dy}{dx} = 0 + c + c^2x + \frac{c^3x^2}{1\,.\,2} + \dots$$

$$= c\left\{1 + cx + \frac{(cx)^2}{1\,.\,2} + \dots\right\}$$

$$= ce^{cx} = ca^x.$$

But since $a = e^c$, $\therefore$ $c = \log_e a$.

$$\therefore \quad \frac{dy}{dx} = \log_e a\,.\,a^x = \mathrm{K}y \text{ (where } \mathrm{K} = \log_e a).$$

In other words, **the rate of growth** $\left(\frac{dy}{dx}\right)$ **of an exponential function is proportional to itself.**

If $a > 1$, then K (which $= \log_e a$) is $+$ ve, and the rate of growth of a^x is $+$ ve, *i.e.*, the function **increases** at a rate proportional to itself.

If $a < 1$, then K is $-$ ve, and the rate of growth of a^x is $-$ ve, *i.e.*, the function diminishes at a rate proportional to itself.

Corollary.—If $y = a^x$, then $\frac{1}{y}\frac{dy}{dx}$ (*i.e.*, the *proportional* rate of increase of the function is constant).

Gradient or Slope of the Curve y = e^x.—If we draw the graph $y = e^x$ (see Fig. 63) and draw tangents at various points on it, *e.g.*, at P (0, 1), Q (1, 2·72), R (2, 7·39), making the angles θ_1, θ_2, θ_3, with the x axis, we shall find that—

$$\tan \theta_1 = \frac{1}{1} = e^0,$$

$$\tan \theta_2 = \frac{2{\cdot}72}{1} = e^1,$$

$$\tan \theta_3 = \frac{7{\cdot}39}{1} = e^2.$$

and so on, so that at any point whose abscissa is x

$$\tan \theta_x = e^x.$$

Hence we have a *geometrical or pictorial proof of the truth of the statement that*

$$\frac{de^x}{dx} = e^x.$$

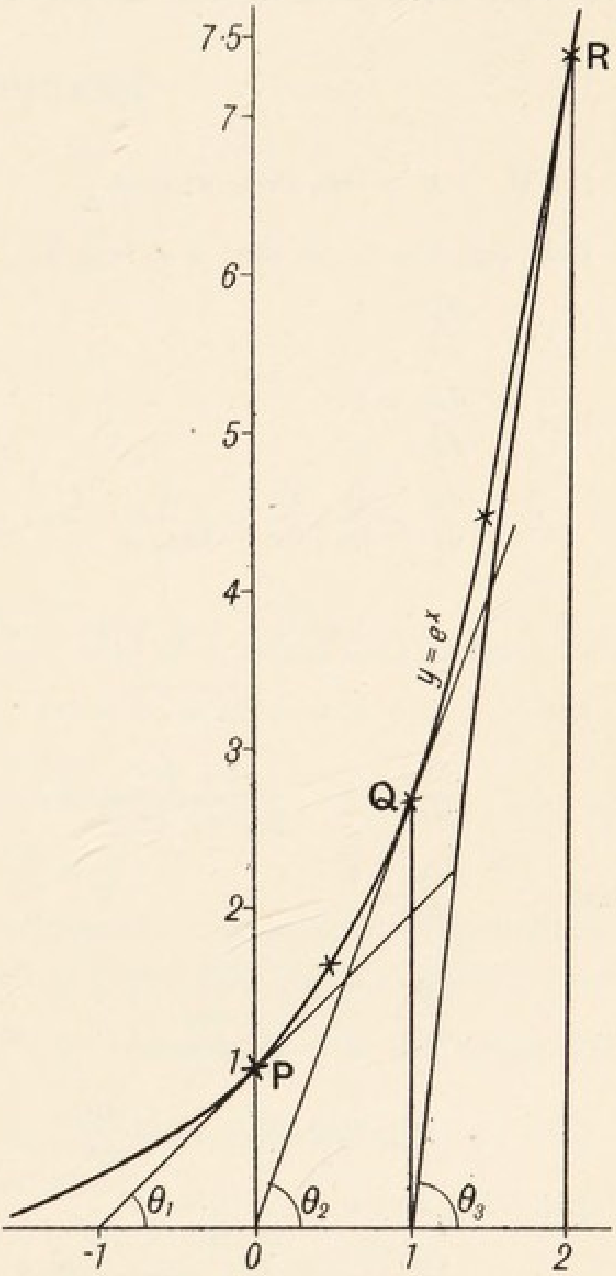

FIG. 63.—Graphical Method of showing that $\frac{de^x}{dx} = e^x$.

(ii.) **Logarithmic Functions.**

To find the differential coefficient of $\log_e x$.

Let $$y = \log_e x.$$

$$\therefore \quad e^y = x.$$

$$\therefore \quad \frac{dx}{dy} = e^y = x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}.$$

(See example (9) for the practical application of this differential coefficient as a labour-saving device.)

EXAMPLES.

(1) If $y = \log_e (\log_e x)$ find $\frac{dy}{dx}$.

Let $\log_e x = u$, so that $y = \log_e u$.

$$\therefore \quad \frac{dy}{du} = \frac{1}{u}.$$

But $$\frac{du}{dx} = \frac{1}{x}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\log_e x} \cdot \frac{1}{x} = \frac{1}{x \log_e x}.$$

(2) $y = \log (x + 1 + \sqrt{2x + x^2})$. Find $\frac{dy}{dx}$.

Let $$x + 1 + \sqrt{2x + x^2} = u.$$

$$\therefore \quad \frac{du}{dx} = 1 + \frac{2 + 2x}{2\sqrt{2x + x^2}} = 1 + \frac{1 + x}{\sqrt{2x + x^2}},$$

and $$\frac{dy}{du} = \frac{1}{u}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{\sqrt{2x + x^2} + 1 + x}{\sqrt{2x + x^2}} \cdot \frac{1}{x + 1 + \sqrt{2x + x^2}} = \frac{1}{\sqrt{2x + x^2}}.$$

(3) $y = (\log x)^n$. Find $\frac{dy}{dx}$

Let $\log x = u. \quad \therefore \quad y = u^n.$

$$\therefore \quad \frac{dy}{du} = nu^{n-1} = n (\log x)^{n-1}.$$

$$\therefore \quad \frac{dy}{dx} \left(= \frac{dy}{du} \cdot \frac{du}{dx}\right) = \frac{n(\log x)^{n-1}}{x}.$$

(4) $y = e^{e^x}$. Find $\frac{dy}{dx}$

If $e^x = u$, then $y = e^u$.

$$\therefore \quad \frac{dy}{du} = e^u = e^{e^x}.$$

$$\therefore \quad \frac{dy}{dx} = e^{e^x} . e^x$$

(5) $y = x$.

$$\therefore \quad \log y = x \log x.$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = x . \frac{1}{x} + 1 \cdot \log x = 1 + \log x.$$

$$\therefore \quad \frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x).$$

(6) $y = x^{x^{x^{\cdot^{\cdot^{\cdot \text{ to } \infty}}}}}$.

Here $y = x^y$. $\therefore$ $\log y = y \log \infty$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log x + \frac{y}{x}.$$

$$\therefore \quad \frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x} \Big/ \left(\frac{1}{y} - \log x\right)$$

$$= \frac{y^2}{x(1 - y \log x)}.$$

(7) Fechner's law states that the intensity of a sensation is proportional to the logarithm of the stimulus.

In symbolic form the law becomes

$$y = \log x,$$

where y is the perception and x is the stimulus.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}$$

i.e., the perceptibility of a sensation is inversely proportional to the stimulus.

(8) Cholera bacilli double themselves in number in 30 minutes. Find their rate of growth.

This being an example of growth in accordance with the compound interest law (see p. 93),

$$\therefore \quad 2 = e^{30k}, \text{ where } k \text{ is a constant.}$$

$$\therefore \quad 2{\cdot}302 \log_{10} 2 = 30k.$$

$$\therefore \quad k = \frac{2{\cdot}302 \times {\cdot}30103}{30} = {\cdot}0231.$$

$$\therefore \quad \text{law of growth is } y = e^{0{\cdot}0231t}.$$

$$\therefore \quad \text{rate of growth } \frac{dy}{dt} = 0{\cdot}0231e^{0{\cdot}0231t}.$$

(9) The following method of differentiation by taking logarithms first (to the base e) saves a lot of arithmetical labour in suitable cases (*i.e.*, in cases of functions consisting of a number of factors).

$$y = \frac{x\sqrt{1-x^2}}{\sqrt{1+x^2}}. \quad \therefore \quad \log y = \log x + \frac{1}{2}\log(1-x^2) - \frac{1}{2}\log(1+x^2).$$

$$\therefore \quad \frac{d}{y}\frac{dy}{dx} = \frac{1}{x} - \frac{x}{x-x^2} - \frac{x}{1+x^2} = \frac{1-2x^2-x^4}{x(1-x^2)(1+x^2)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{x\sqrt{1-x^2}}{\sqrt{1+x^2}} \frac{(1-2x^2-x^4)}{x(1-x^2)(1+x^2)} = \frac{1-2x^2-x^4}{(1+x^2)^{\frac{3}{2}}(1-x^2)^{\frac{1}{2}}}$$

Differentiation of Circular Functions.

(a) Direct Circular Functions.

(iii.) *The differential coefficient of sin x.*

Let $\quad y = \sin x.$

$$\therefore \quad y + dy = \sin(x+dx).$$

$$\therefore \quad dy = \sin(x+dx) - \sin x.$$

$$= 2\sin\frac{dx}{2} \,.\, \cos\frac{2x+dx}{2} \text{ (see Chap. IV., p. 53).}$$

$$\therefore \quad \frac{dy}{dx} = \frac{2\sin\frac{dx}{2} \,.\, \cos\frac{2x+dx}{2}}{dx}.$$

But $\mathrm{Lt}_{dx \longrightarrow 0}\dfrac{\sin dx}{dx} = 1$ (see p. 191). $\therefore \quad \dfrac{2\sin\frac{dx}{2}}{dx} = 1.$

and $$\mathrm{Lt}_{dx \longrightarrow 0} \cos\frac{2x+dx}{2} = \cos x.$$

$$\therefore \quad \frac{dy}{dx} = \cos x.$$

Similarly, $\dfrac{d\cos x}{dx} = -\sin x$

and $$\frac{d\tan x}{dx} = \frac{d\frac{\sin x}{\cos x}}{dx} = \frac{\cos x \,.\, \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$= 1 + \tan^2 x.$$

Note Regarding the Peculiarity of sin θ and cos θ.—The student will have noticed that when sin θ is differentiated with respect to θ the result is $\cos\theta$, and when $\cos\theta$ is differentiated with respect to θ it becomes $-\sin\theta$.

Similarly the differential coefficient of $\cos\theta$ is $-\sin\theta$ and the differential coefficient of $-\sin\theta$ is $\cos\theta$. Hence we get the following *two curious results*, viz.:

(i.) **Each of these functions when differentiated twice gives rise to the original function with the sign changed from + to −.**

(ii.) **Each of these functions when differentiated four times gives rise to the original function with the original sign.**

These two trigonometrical ratios are the only functions which possess these peculiarities, and we shall see that great advantage is taken of these peculiarities for the purpose of expanding $\sin\theta$ and $\cos\theta$ in powers of θ (see p. 189). They are also useful to remember when solving differential equations of the type

$$\frac{d^2y}{dx^2} = -n^2y.$$

The student must bear in mind the fact that these results are only true if the angle x is expressed in circular measure (*i.e.*, in radians). In the calculus all angles are understood to be expressed in radians and not in degrees.

(*b*) **Inverse Circular Functions.**

To find $\frac{dy}{dx}$ *if* $y = \sin^{-1}x$.

Since $y = \sin^{-1}x$. $\therefore x = \sin y$ (by definition).

$$\therefore \frac{dy}{dx} = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}.$$

$$\therefore \frac{dx}{dy} = \frac{1}{\sqrt{1-x^2}}.$$

Similarly, it can be shown by the student that—

$$\text{If } y = \cos^{-1}x, \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}.$$

$$\text{If } y = \tan^{-1}x, \frac{dy}{dx} = \frac{1}{1+x^2}.$$

$$\text{If } y = \cot^{-1}x, \frac{dy}{dx} = -\frac{1}{1+x^2}.$$

$$\text{If } y = \sec^{-1}x, \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

$$\text{If } y = \text{cosec}^{-1}x, \frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

EXAMPLES.

(1) $y = \sin nx$. Find $\frac{dy}{dx}$.

Put $nx = u$.

$$\therefore \quad \frac{dy}{du} = \cos u.$$

$$\therefore \quad \frac{dy}{dx} = \text{Cos}\, u \,.\, \frac{du}{dx} = n \cos u$$
$$= n \cos nx.$$

(2) $y = \sin^n x$.

Let $\sin x = u$. $\therefore y = u^n$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = nu^{n-1} \cos x.$$
$$= n \sin^{n-1} \cos x.$$

(3) $y = \sin x^n$.

Let $x^n = u$. $\therefore$ $y = \sin u$,

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = nx^{n-1} \cos x^n.$$

(4) $y = \tan^3 x = u^3$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3 \tan^2 x \sec^2 x.$$

(5) $y = e^{\sqrt{\sin x}} = e^u$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^{\sqrt{\sin x}} \cdot \frac{\cos x}{2\sqrt{\sin x}}.$$

(6) $y = \dfrac{\tan x}{x}$.

$$\frac{dy}{dx} = \frac{x \sec^2 x - \tan x}{x^2} = \frac{x - \frac{1}{2}\sin 2x}{x^2}.$$

(7) $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x}}}$, etc., to infinity.

$$\therefore \quad y^2 = \sin x + y.$$

$$\therefore \quad y^2 - y = \sin x.$$

$$\therefore (2y - 1)\frac{dy}{dx} = \cos x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos x}{2y - 1.}$$

(8) In example (1), p. 49, when the hand is at A, the weight moves upwards with a velocity of 10 ins. per second. What is the angular velocity of the forearm at that instant ?

Let distance AC $= x$.

Then angle AOC $(= \theta) = \sin^{-1}\dfrac{x}{12}$.

$$\therefore \quad \frac{d\theta}{dx} = \frac{d\left(\sin^{-1}\frac{x}{12}\right)}{d\left(\frac{x}{12}\right)} \cdot \frac{d\left(\frac{x}{12}\right)}{dx}$$

$$= \frac{1}{12} \cdot \frac{d\left(\sin^{-1}\frac{x}{12}\right)}{d\left(\frac{x}{12}\right)}$$

$$= \frac{1}{12} \cdot \frac{1}{\sqrt{1 - \frac{x^2}{144}}}$$

$$= \frac{1}{12} \cdot \frac{12}{\sqrt{144 - x^2}}$$

$$= \frac{1}{\sqrt{144 + x^2}}$$

$$\therefore \quad \text{angular velocity which} = \frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{1}{\sqrt{144 - x^2}} \cdot \frac{dx}{dt}$$

$$= \frac{10}{\sqrt{144 - x^2}} \text{ radians per second}$$

(since $\dfrac{dx}{dt} = 10$ ins. per second).

But since $\theta = 45°$ and AO $= 12$ ins.

$$\therefore \quad x = AC = \frac{12\sqrt{2}}{2} = 6\sqrt{2}$$

$$\therefore \quad \frac{10}{\sqrt{144 - x^2}} = \frac{10}{\sqrt{144 - 72}} = \frac{10}{\sqrt{72}} = \frac{6\sqrt{2}}{6} \text{ radians per second.}$$

$$= \frac{5\sqrt{2}}{6} \times 57{\cdot}3 \text{ degrees per second.}$$

$$= \frac{5 \times 1{\cdot}414}{6} \times 57{\cdot}3$$

$$= 67{\cdot}52° \text{ per second.}$$

(9) A galvanometer mirror M is 1 metre distant from the scale, and the spot of light is moving on the scale with a velocity of 15 cm. per second. When it is deflected 17 cm., what is the angular velocity of the beam of light at that instant, and what is the angular velocity of rotation of the mirror at that instant ?

Let AB be the scale (the student is to draw the diagram for himself), M the mirror, and MH the distance of M from AB. Let S be the position of the spot of light on the scale at any instant.

Then if we put $\quad HS = x$ and $\angle SMH = \theta$,

we have
$$\tan\theta = \frac{x}{MH} = \frac{x}{100}$$

$$\therefore \quad \theta = \tan^{-1}\frac{x}{100}$$

$$\therefore \quad \frac{d\theta}{dx} = \frac{1}{100}\frac{d\left(\tan^{-1}\frac{x}{100}\right)}{dx} = \frac{1}{100} \quad \frac{1}{1+\frac{x^2}{10000}}$$

$$= \frac{100}{10000 + x^2}.$$

$$\therefore \quad \frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt} = \frac{100}{10000 + x^2}\cdot\frac{dx}{dt}.$$

But
$$\frac{dx}{dt} = 15 \text{ cm. per second,}$$

$$\therefore \quad \frac{d\theta}{dt} = \frac{1500}{10000 + x^2} \text{ radians per second.}$$

But
$$x = 17.$$

$$\therefore \quad \frac{d\theta}{dt} = \frac{1500}{10289} \text{ radians per second.}$$

$$= \frac{1500}{10289} \times 57{\cdot}3 \text{ degrees per second.}$$

$$= 8{\cdot}35^\circ \text{ per second.}$$

But beam of light rotates twice as fast as the mirror,

$\therefore$ angular velocity of rotation of mirror $= 4{\cdot}18^\circ$ per second.

EXERCISES.

Differentiate the following :

(1) $y = \sin 2x$. [Answer, $2\cos 2x$ (see example (1)).]

(2) $y = \sin\sqrt{x}$. [Answer, $\frac{\cos\sqrt{x}}{2\sqrt{x}}$ (see example (3)).]

(3) $y = \frac{\cos 3x + \cos x}{\sin 3x - \sin x}$.

[Answer, $-\operatorname{cosec}^2 x$. Hint : use identities in Chapter IV., p. 53.
Expression becomes $= \operatorname{Cot} x$; $\frac{d \cot x}{dx} = -\operatorname{cosec}^2 x$.]

(4) For what value of θ, between 0 and 180 is $\tan\theta$ increasing four times as fast as θ ?

[Answer, $\frac{d\tan\theta}{d\theta} = \sec^2\theta = 4. \quad \therefore \cos\theta = \pm\frac{1}{2} \quad \therefore \theta = 60^\circ \text{or } 120^\circ.$]

CHAPTER X.

MAXIMA AND MINIMA.

It is often important to ascertain under what conditions some particular function of a given variable shall have a maximum or minimum value, and what that maximum or minimum value may be. For example, it is found that the reaction velocity under the influence of enzymes is very low at low temperatures and gradually rises as the temperature rises up to a certain limit, called the *optimum temperature*, but if the temperature is increased beyond that limit the velocity begins to diminish.

Again, we saw on p. 46, Chapter IV., that as the angle of pull of a muscle increases the effective force of the muscle increases until the size of the angle reaches a certain value when the force begins to diminish again.

The differential calculus affords us an easy method of ascertaining the maximum and minimum values of given functions.

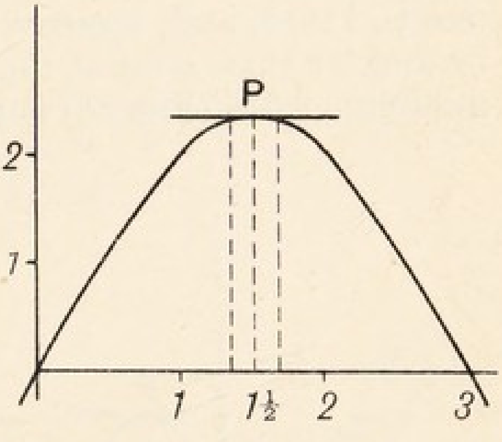

Fig. 64.—Graph of $y = 3x - x^2$.

Maximum.—If a function of x increases in value while x is increased, and then begins to diminish when x is still further increased, the value of the function when the change occurs is called a maximum. Thus, in Fig. 64, showing the graph of $y = 3x - x^2$, we see that at the point P, where $x = 1\frac{1}{2}$, y has a value equal to $2\frac{1}{4}$, and that at points on the curve close to and on either side of P, the ordinates are less than at P. At P, therefore, the value of y is a maximum.

Minimum.—If a function of x decreases in value, while x is increased, and then begins to increase when x is still further increased, the value of the function when the change occurs is called a *minimum*. Thus, Fig. 65 shows the graph of

$$y = x^2 - 3x + 3.$$

We see that at the point P, where $x = 1\frac{1}{2}$, y has a value equal to $\frac{3}{4}$, and that at the points on the curve close to and on either side of P the ordinates are greater than at P. At P, therefore, the value of y is a minimum.

It is important to remember that a **maximum** or a **minimum** ordinate is not always the greatest or smallest ordinate of the graph. For example, in Fig. 66 the ordinate at P is a maximum ordinate, although the ordinate at A is greater. Similarly, the ordinate at S_1 is a minimum, although the ordinate at B is less. All that we mean by the statement, that the ordinate y (or the function $y = f(x)$) has a maximum or minimum value at the points P and S, is that the ordinates at P or S are respectively greater or less than ordinates, close to and on either side of them. Indeed, one function may have several maxima and minima (*e.g.*, $y = \sin\theta$, $y = \cos\theta$ (see p. 116)), and, moreover, some of the minima may actually be greater than some of the maxima in the same curve, *e.g.*, the minimum at S (Fig. 66) is greater than the maximum at P.

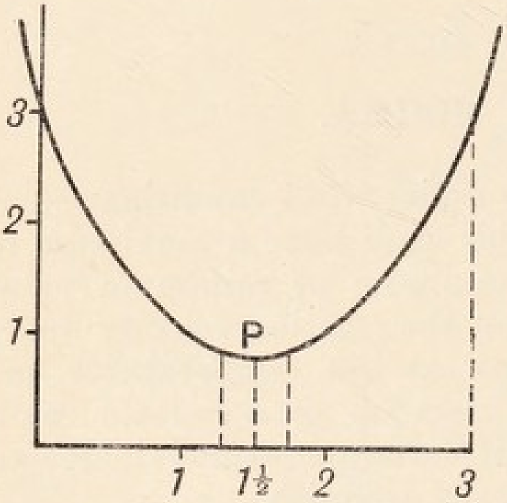

FIG. 65. — Graph of $y = x^2 - 3x + 3$.

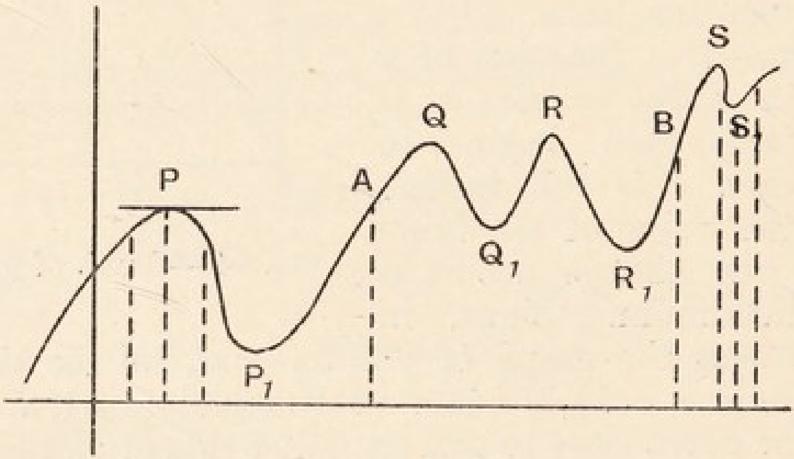

FIG. 66.—Diagram to illustrate Maxima and Minima.

Hence, the words maximum and minimum are not used mathematically with their ordinary meanings of "greatest possible" and "least possible."

Points of Inflection.—Another point to remember is that although at all points of maxima and minima the geometrical tangent is parallel to the x axis and therefore makes an angle

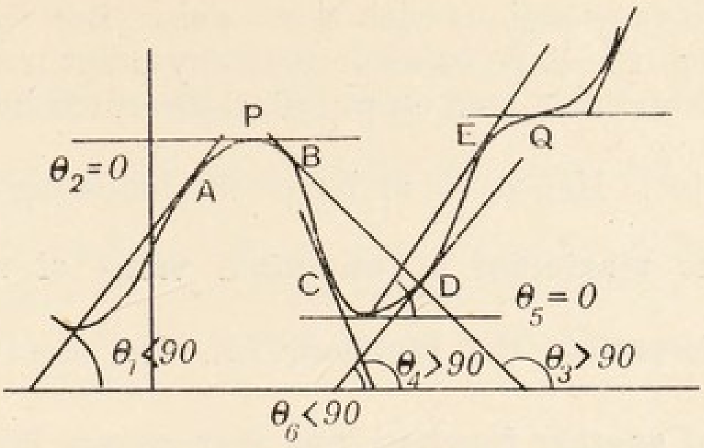

FIG. 67.—Point of Inflection at Q.

with that axis whose tangent = 0, so that $\frac{dy}{dx} = 0$, the reverse is not necessarily true, viz., we may have a point like Q (Fig. 67), where the tangent is parallel to the x axis and where, therefore, $\frac{dy}{dx} = 0$, and yet the function has neither a maximum nor a minimum value at that point. Such a point is called a *Point of inflection;* it is a point where the tangent crosses the curve, and the test for it is that $\frac{dy}{dx}$ does not change sign in passing through zero.

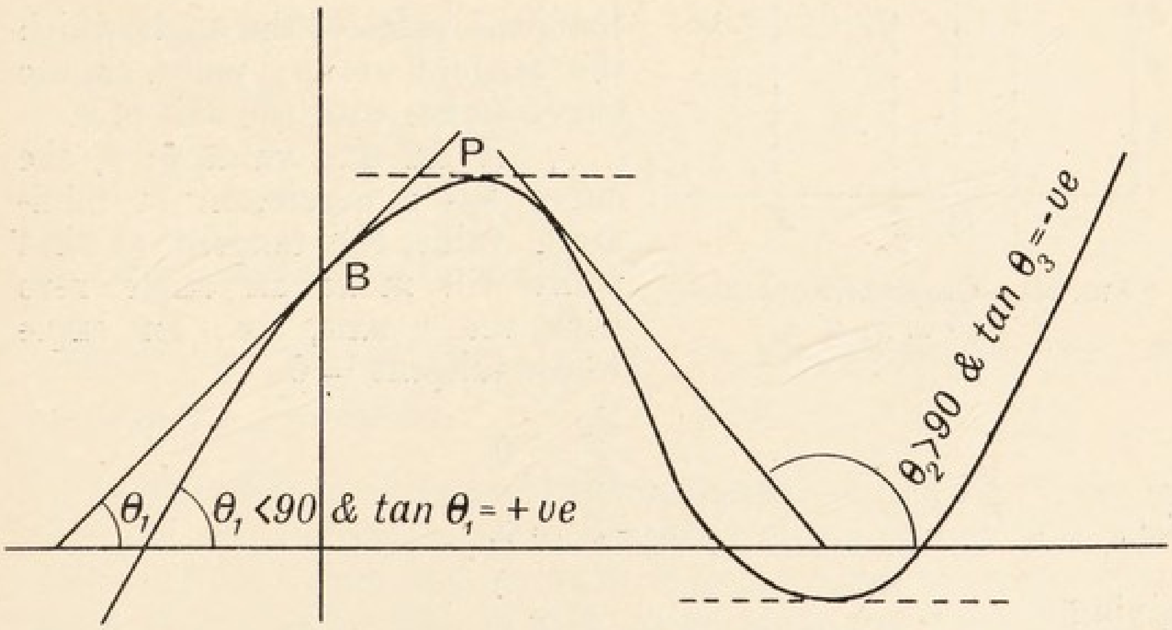

FIG. 68.

Investigation of Maximum and Minimum Values of a Function.—From Figs. 67 and 68 it is obvious that at the maximum and minimum points in a curve, the tangent is parallel to the axis of

x, *i.e.*, it makes an angle 0 with the x axis. But the tangent of the angle which the slope of a curve at any point makes with the axis of x is, as we have seen on p. 136, represented by the value of $\frac{dy}{dx}$ at that point. Hence, we arrive at the following rule :

To find the maximum or minimum value of any function $y = f(x)$, differentiate the function, *i.e.*, find the value of $\frac{dy}{dx}$, in the form of another function, and then equate this $\frac{dy}{dx}$ to zero. The value of x thus found gives the abscissa of the point whose ordinate is a maximum or a minimum.

An example will make this clear.

Let the function be $y = x^2 - 3x + 6$ (see Fig. 69). Find the maximum or minimum values of this function, *i.e.*, for which values of x will the function be a maximum or a minimum ?

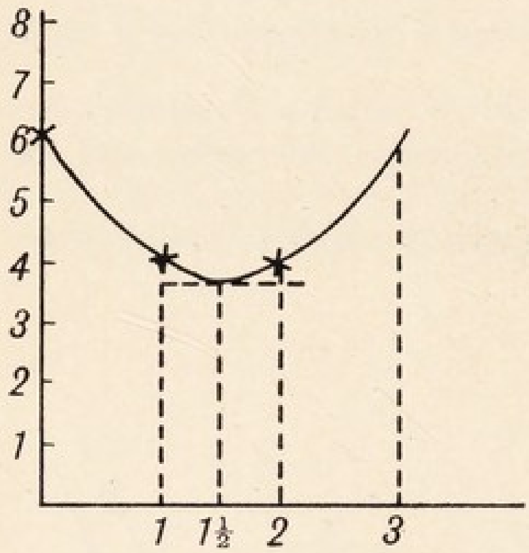

FIG. 69.—Graph of Function $y = x^2 - 3x + 6$.

Differentiating we get

$$\frac{dy}{dx} = 2x - 3.$$

This $\frac{dy}{dx}$, as we have seen, represents the value of the angle which the tangent at any point on the curve makes with the axis of x.

$\therefore$ if for any value of x, the curve has a maximum or minimum value, the tangent at that point will make an angle zero with the x axis, *i.e.*, an angle whose tangent $= 0$.

$$\therefore \quad \frac{dy}{dx} = 0,$$

i.e., $$2x - 3 = 0,$$

giving $$x = \frac{3}{2}.$$

Hence, the curve will have a maximum or a minimum value at a point whose abscissa $= \frac{3}{2}$ (see Fig. 69).

Discrimination between Maxima and Minima.—In order to ascertain whether the value so found is a maximum *or* a minimum there are several possible ways of doing so.

(1) **Trial Method.**—After finding the value of x for which $\frac{dy}{dx} = 0$, substitute this value in the function, and thus ascertain what value y assumes. Then give a slightly different value to x and see what effect such a new value of x has upon the value of y. The method is best illustrated by an example.

Take again the function

$$y = x^2 - 3x + 6.$$

We have seen that when $x = 1{\cdot}5$, $\frac{dy}{dx} = 0$.

Now, when $x = \frac{3}{2}$ we have

$$\begin{aligned} y &= (1{\cdot}5)^2 - 3 \times 1{\cdot}5 + 6 \\ &= 2{\cdot}25 - 4{\cdot}5 + 6 \\ &= 3{\cdot}75. \end{aligned}$$

Let us give x the very slightly higher value $1{\cdot}51$.

We then have

$$\begin{aligned} y &= (1{\cdot}51)^2 - 3 \times 1{\cdot}51 + 6 \\ &= 2{\cdot}2801 - 4{\cdot}53 + 6 \\ &= 3{\cdot}7501, \text{ which is greater than } 3{\cdot}75. \end{aligned}$$

Again, give x the slightly smaller value of $1{\cdot}49$.

We then have

$$\begin{aligned} y &= (1{\cdot}49)^2 - 3 \times 1{\cdot}49 + 6 \\ &= 2{\cdot}2201 - 4{\cdot}47 + 6 \\ &= 3{\cdot}7501, \end{aligned}$$

which is again higher than $3{\cdot}75$.

Hence, we see that when $\frac{dy}{dx} = 0$ in this particular case, the value of y is a minimum.

(2) **The Method of Second Differentiation.**—From Fig. 70 we see that the slope of the curve continually changes, and that when we pass from a minimum value at P to the right, the angle of the slope at A changes from 0 at P to an angle less than 90° at A. But the tangent of an angle less than 90 is + ve, *therefore the slope changes from* 0 *to* + *ve when we pass from a minimum value to the right.* But when we pass from a maximum value at Q to the right, the angle of the slope at B changes from 0 to an angle greater than 90, and whose tangent is therefore − ve. *Hence the*

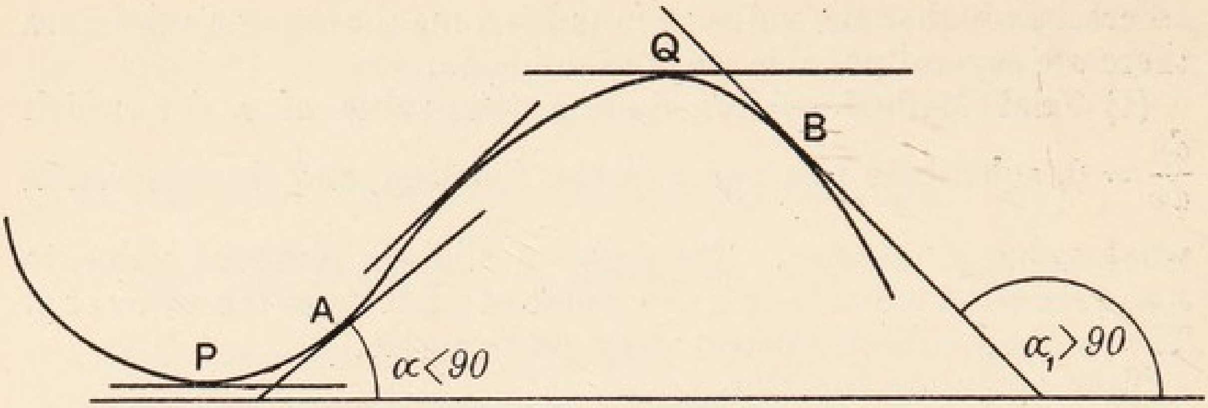

FIG. 70.—Diagram to illustrate the Method of Discriminating between a Maximum and a Minimum.

slope changes from 0 *to* − *ve when we pass from a maximum to the right.* Now if we call the slope of the curve at any point $\frac{dy}{dx}$, then $\frac{d\left(\frac{dy}{dx}\right)}{dx}$ denotes the rate of change of slope.

But $\frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d^2y}{dx^2}$, *i.e.*, the second differential coefficient of y, with respect to x. Therefore, *if* $\frac{d^2y}{dx^2}$ *is* + *ve, then the value found by equating* $\frac{dy}{dx}$ *to zero is a minimum, but if* $\frac{d^2y}{dx^2}$ *is negative, then the value found by equating* $\frac{dy}{dx}$ *is a maximum.*

Thus, in the above case,

$$\frac{dy}{dx} = 3x - 5.$$

$$\therefore \quad \frac{d^2y}{dx^2} = 3 = + \text{ve}.$$

$\therefore$ the value found is a minimum.

Let us take another example :

$$y = x^3 - 3x + 16.$$

Find the maxima and minima of this function :

$$\frac{dy}{dx} = 3x^2 - 3 = 0. \quad \therefore \quad x^2 = 1, \text{ and } x = \pm 1.$$

$$\frac{d^2y}{dx^2} = 6x.$$

When $x = +1$, $6x$ is positive,
,, $x = -1$, $6x$ is negative.
Hence, $x = 1$ corresponds to a minimum, $y = 14$,
and $x = -1$,, ,, maximum, $y = 18$.

Example.—Find the value of x which makes $\sin^3 x \cos x$ a maximum.

We have $$\frac{dy}{dx} = 3 \sin^2 x \,.\, \cos^2 x - \sin^4 x = 0.$$
$$\therefore \quad \sin^2 x\,(3 \cos^2 x - \sin^2 x) = 0.$$

When $\sin^2 x = 0$, the function $= 0$.
$\therefore$ the value $x = 0$ must be rejected.
$\therefore$ the function is a maximum when
$$\sin^2 x = 3 \cos^2 x = 3(1 - \sin^2 x),$$
i.e., $$4 \sin^2 x = 3,$$
i.e., $$\text{when } \sin x = \frac{\sqrt{3}}{2},$$
i.e., $$\text{when } x = 60°.$$
The value of the function then becomes
$$\frac{3\sqrt{3}}{8} \cdot \frac{1}{2} = \frac{3}{16}\sqrt{3}.$$

EXAMPLES.

(1) *Problem in Public Health.*—For an ellipse of given diameter, find the relation between the major and minor axes so that the area may be a maximum. Hence show that the best shape to give to a water pipe in order to prevent its bursting during frosty weather is that of an ellipse.

Whilst the perimeter of an ellipse cannot be accurately expressed in simple form, the formula $p = \pi(x + y)$ is an approximate expression for its perimeter p, if x and y are the two semi-axes, and if x is nearly equal to y (*i.e.*, if the ellipse is nearly circular).*

From the formula $p = \pi(x + y)$, we have $y = \frac{p}{\pi} - x$.

But area A of an ellipse $A = \pi xy$.
$$\therefore \quad A = \pi x\left(\frac{p}{\pi} - x\right) = px - \pi x^2.$$
$$\therefore \quad \frac{dA}{dx} = p - 2\pi x.$$

---

* A more exact formula is
$$P = \pi(x + y)\left[\frac{1}{4}\left(\frac{x-y}{x+y}\right)^2 + \frac{1}{64}\left(\frac{x-y}{x+y}\right)^4 + \ldots\ldots\right].$$
$\therefore$ if x and y are nearly equal, $P = \pi(x + a)$.

$\therefore$ for a maximum $p - 2\pi x = 0$,

whence $$p = 2\pi x$$

or $$x = \frac{p}{2\pi} = \frac{\pi}{2\pi}(x + y)$$

$$= \frac{x + y}{2}.$$

$$\therefore \quad 2x = x + y,$$

whence $$x = y.$$

$\therefore$ the given ellipse must have its semi-axes equal, *i.e.*, it must be circular.

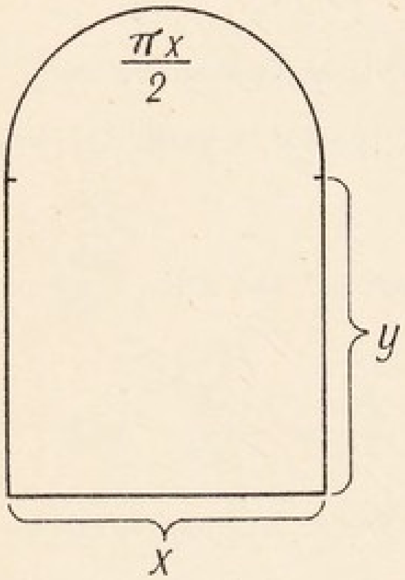

Fig. 71.

From this it follows that if a water pipe is made slightly elliptical, then when the water inside it expands as a result of freezing, the pipe will tend to become more circular, for by so doing the area of its cross-section increases without altering its perimeter—thus preventing its bursting.

(2) *Another Problem in Public Health.*—A window is in the form of a rectangle surmounted by a semicircle (Fig. 71). If the perimeter is 30 feet, find the dimensions so that the greatest possible amount of light may be admitted.

From the diagram we see that total perimeter of window

$$= 2y + x + \frac{\pi x}{2}$$

$$= 30 \text{ feet (by hypothesis).}$$

$$\therefore \quad y = \frac{60 - x(2 + \pi)}{4} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now, area (A) of window = sum of areas of semicircular and rectangular parts

$$= \frac{\pi x^2}{8} + xy$$

$$= \frac{\pi x^2}{8} + \frac{x}{4}[60 - x(2 + \pi)] \text{ (from (1)).}$$

$$\therefore \quad \frac{dA}{dx} = 15 - \frac{x}{4}(4 + \pi)$$

$$= 0 \text{ (for a maximum).}$$

$$\therefore \quad x = \frac{60}{4 + \pi} = 8{\cdot}4 \text{ ft.}$$

$$\therefore \quad y = \frac{60 - 8{\cdot}4(2 + \pi)}{4} = 4{\cdot}2.$$

$$\therefore \quad \text{height of window which} = y + \frac{x}{2} = 4{\cdot}2 + 4{\cdot}2.$$

$$= 8{\cdot}4.$$

$$\therefore \quad \text{total height} = \text{width} = 8{\cdot}4 \text{ ft.}$$

(3) *Problem in Biochemistry.*—When ethyl acetate is hydrolysed in the presence of acetic acid as a catalyser the following reaction occurs :

$$CH_3COOC_2H_5 + H_2O = CH_3COOH + C_2H_5OH.$$

The acetic acid formed thus gradually increases in amount and causes a uniform acceleration in the velocity of the reaction. But at the same time the active mass of the ester diminishes, thus causing a retardation in the reaction velocity. At what point will the velocity of reaction be a maximum ?

Let the initial concentration of acetic acid $= a$ molecules per litre.
and let the initial concentration of ester $= b$ molecules per litre.

Let x molecules be hydrolysed after a time t, producing x molecules of acetic acid.

$\therefore$ velocity due to acetic acid originally present,

$$= \frac{dx_1}{dt} = K_1 a (b - x).$$

Velocity due to acetic acid produced

$$= \frac{dx_2}{dt} = K_1 x (b - x).$$

$\therefore$ actual velocity $= \dfrac{dx_1}{dt} + \dfrac{dx_2}{dt}.$

$$= K_1(a + x)(b - x).$$

$\therefore$ for a maximum, $\dfrac{d}{dx}(a + x)(b - x) = 0,$

i.e., $-(a - b) - 2x = 0$, whence $x = \dfrac{1}{2}(a - b)$.

(4) *Problem in Physiology of Growth.*—On theoretical grounds it has been found by Robertson (see "Child Physiology," p. 249) that growth in weight of infants up to nine months old is an autocatalytic phenomenon taking place in accordance with the equation $\log \dfrac{x}{341{\cdot}5 - x} = K(t - 1{\cdot}66)$, where x is the weight of an infant in ounces at the age of t months.

At what age will the growth of the infant be most rapid ?

Since $\log \dfrac{x}{341{\cdot}5 - x} = K(t - 1{\cdot}66)$,

$$\therefore \quad \frac{1}{K}\log\frac{x}{341{\cdot}5 - x} + 1{\cdot}66 = t.$$

$\therefore$ put $\dfrac{x}{341{\cdot}5 - x} = z,$

then
$$\frac{dz}{dx} = \frac{(341{\cdot}5 - x) \times 1 - x(-1)}{(341{\cdot}5 - x)^2}$$

$$= \frac{341{\cdot}5}{(341{\cdot}5 - x)^2}$$

Now $$\frac{dt}{dz} = \frac{1}{K} \cdot \frac{1}{z} = \frac{1}{K} \cdot \frac{341{\cdot}5 - x}{x}.$$

$$\therefore \quad \frac{dt}{dx} \text{ which } = \frac{dt}{dz} \cdot \frac{dz}{dx} = \frac{1}{K} \cdot \frac{341{\cdot}5 - x}{x} \cdot \frac{341{\cdot}5}{(341{\cdot}5 - x)^2}$$

$$= \frac{1}{K} \cdot \frac{341{\cdot}5}{x(341{\cdot}5 - x)},$$

$$= \frac{C}{x(341{\cdot}5 - x)} \text{ where } C = \frac{341{\cdot}5}{K}.$$

$$\therefore \quad \text{velocity of growth } v \text{ which } = \frac{dx}{dt} = \frac{1}{\frac{dt}{dx}}$$

$$= \frac{x(341{\cdot}5 - x)}{C}$$

$$= \frac{341{\cdot}5x - x^2}{C}$$

$$\therefore \quad \frac{dv}{dx} = \frac{341{\cdot}5 - 2x}{C} = 0 \text{ (for a maximum)},$$

whence $$x = \frac{341{\cdot}5}{2} = 170{\cdot}75,$$

i.e., the growth of the infant is most rapid when the infant weighs 170·75 oz.

In order to find at what age the infant's weight is 170·75 oz. we return to the original equation:

$$\log \frac{x}{341{\cdot}5 - x} = K(t - 1{\cdot}66).$$

$\therefore$ when $x = 170{\cdot}75$ oz. we get

$$\log \frac{170{\cdot}75}{341{\cdot}5 - 170{\cdot}75} = K(t - 1{\cdot}66),$$

i.e., $$\log \frac{170{\cdot}75}{170{\cdot}75} = K(t - 1{\cdot}66),$$

i.e., $$\log 1 = K(t - 1{\cdot}66).$$

But $$\log 1 = 0. \quad \therefore \quad K(t - 1{\cdot}66) = 0,$$

whence $$t = 1{\cdot}66 \text{ months} = \text{about seven weeks}.$$

$\therefore$ the infant grows quickest at the age of seven weeks.

This result is confirmed by the table giving the weight of infants at various ages during the first nine months (see "Child Physiology," p. 251).

It will be noticed that the problem of finding when $Kx(341{\cdot}5 - x)$ is a maximum is the same as finding when $x(A - x)$ is a maximum. This is the same as finding:

(i.) How to divide a number A in such a way as to make the product a maximum. The answer is to divide it into two equal parts.

(ii.) How to divide a line into two parts so as to make the rectangle contained by the two parts a maximum. The answer is to bisect the line.

(5) *Problem in General Physiology.*—Under what H ion concentration will the sum of protein ions be a minimum in relation to the undissociated protein in solution ?

As protein is an amphoteric electrolyte (*i.e.*, it contains both H and OH ions), there must be two forms of dissociation, viz. :

(*a*) [A*p*] and [H], where A_p stands for **Anion** protein.
(*b*) [K*p*] and [OH], where K_p stands for **Kation** protein.

Let reaction velocity for first form of dissociation be K_a, then, by the law of mass action (*q.v.*, p. 215),

$$K_a = \frac{[A_p] \,.\, [H]}{[P]} \text{ or } A_p = \frac{K_a \,.\, [P]}{[H]}.$$

([P] stands for the concentration of non-dissociated protein.)

If reaction velocity for second form of dissociation be K_b,

then similarly
$$K_p = \frac{K_b \,.\, [P]}{[OH]}.$$

Hence
$$\frac{[A_p] + [K_p]}{[P]} = \frac{K_a}{[H]} + \frac{K_b}{[OH]} = u.$$

Hence we have to find for what value of [H], u will be a minimum.

We must therefore represent $\frac{K_a}{[H]} + \frac{K_b}{[OH]}$ as a function of [H].

Now if K_w = dissociation constant of water, we have

$$[OH] \,.\, [H] = K_w,$$

whence
$$[OH] = \frac{K_w}{[H]}.$$

$$\therefore \quad \frac{K_a}{[H]} + \frac{K_b}{[OH]} = \frac{K_a}{[H]} + \frac{K_b[H]}{K_w} = u.$$

Hence for a minimum we must have

$$\frac{du}{d[H]} \text{ which} = -\frac{K_a}{[H]^2} + \frac{K_b}{K_w} = 0,$$

or
$$\frac{K_a}{[H]^2} = \frac{K_b}{K_w},$$

whence
$$\frac{K_a}{K_b} = \frac{[H]^2}{K_w} = \frac{[H]^2}{[H] \cdot [OH]} = \frac{[H]}{[OH]}.$$

Hence there will be a minimum of dissociated protein ions when the H and OH ion concentrations are in the same relation as the corresponding dissociation constants or reaction velocities, *i.e.*, at the **isolectric** point.

In order to find the value of [H] under those conditions we proceed as follows :

$$[H] = \frac{K_a}{K_b}[OH].$$

But
$$[H] \,.\, [OH] = K_w = 10^{-14}.$$

(this being the dissociation constant of pure water).

$$\therefore \quad [OH] = \frac{10^{-14}}{[H]}.$$

$$\therefore \quad [H] = \frac{K_a}{K_b}\,\frac{10^{-14}}{[H]}.$$

$$\therefore \quad 10^{-14}\,\frac{K_a}{K} = [H]^2$$

$$\therefore \quad [H] = 10^{-7}\sqrt{\frac{K_a}{K_b}}.$$

That this value of H gives a minimum and not a maximum is seen from the fact that $\dfrac{d^2u}{d[\mathrm{H}]^2} = \dfrac{2\mathrm{K}a}{[\mathrm{H}]^3} = +$ ve.

(6) *Problem in Morphology of Blood Vessels.*—John Hunter wrote as follows: "To keep up a circulation **sufficient for the part and no more,** Nature has varied the angle of origin of the arteries accordingly." Prove the truth of this statement, on the assumption that the loss of pressure between any point A on the main trunk and a point D on the branch artery BD is mainly due to friction of the blood-stream against the arterial walls, and is therefore in accordance with Hess's law proportional to

$$\frac{\mathrm{AB}}{\mathrm{R}} + \frac{\mathrm{BD}}{r}.$$

where R and r are the radii of the vessels.

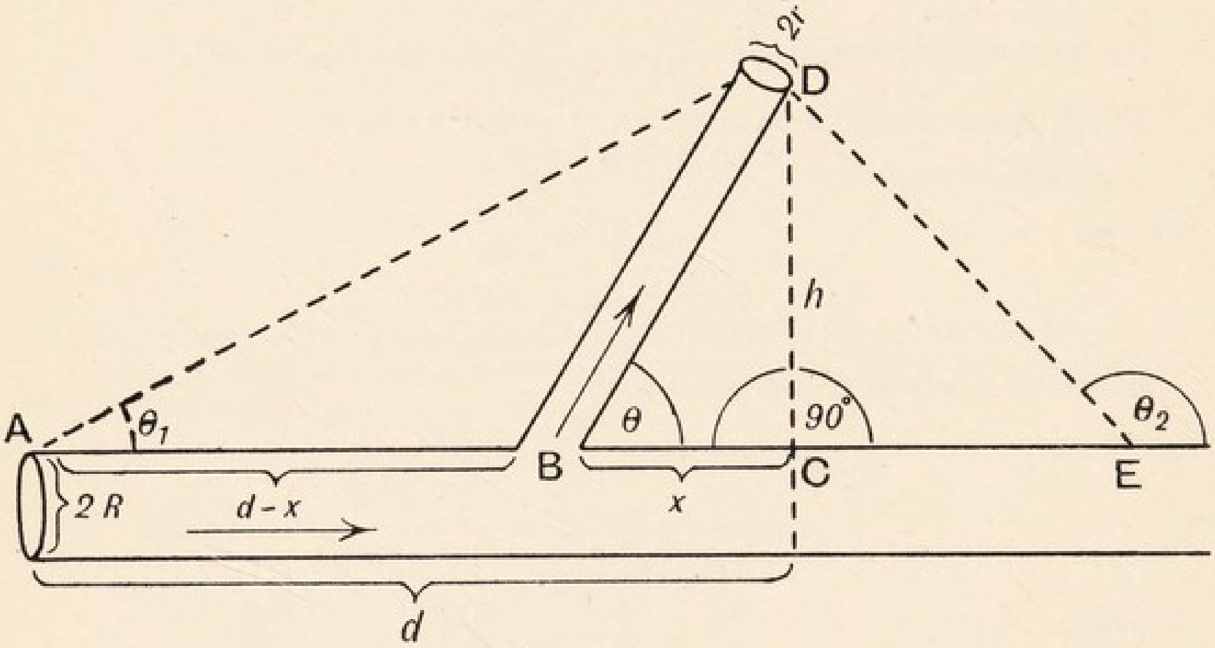

FIG. 72.

The route by which the blood could be conveyed from A to D can be any one of the following (Fig. 72). Either—

(1) By a branch from A to D inclined to the main trunk at an angle θ; or
(2) By a branch from C to D at right angles to the main trunk; or
(3) By a branch from any point B (between A and C) to D, making an acute angle θ, with the main trunk; or
(4) By a branch from any point E (beyond C) to D, making an obtuse angle θ_2 with the main trunk.

The problem which remains to be solved, therefore, is, along which route will the loss of pressure between A and D be a minimum?

Let us take any point B on the main trunk. Then loss of pressure between A and D is according to Hess's law proportional to

$$\frac{\mathrm{AB}}{\mathrm{R}} + \frac{\mathrm{BD}}{r}.$$

Now, D being a fixed point on the branch vessel, let its perpendicular distance DC from the main trunk be $= h$.

Also let distance of B from C $= x$ (*i.e.*, the unknown point at which the branch vessel comes off is x units distant from C).

Also let distance AC $= d$.

$\therefore$ we have the following :

$$AB = d - x.$$

$$BD = \sqrt{h^2 + x^2}.$$

(being the hypotenuse of a right angled triangle).

$\therefore$ fall of pressure between A and D is proportional to

$$\frac{d - x}{R} + \frac{\sqrt{h^2 + x^2}}{r}.$$

$\therefore$ the route must be such that

$$\frac{d - x}{R} + \frac{\sqrt{h^2 + x^2}}{r}$$

is a minimum.

Calling this function y, we must have

$$\frac{dy}{dx} = 0,$$

i.e.,
$$-\frac{1}{R} + \frac{2x}{2r\sqrt{x^2 + h^2}} = 0,$$

whence
$$\frac{r}{R} = \frac{x}{\sqrt{x^2 + h^2}} = \cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$\therefore$ the angle at which the vessel comes off is such that its cosine is proportional to $\frac{r}{R}$.

To find the distance x, we have

$$\frac{r}{R} = \frac{x}{\sqrt{x^2 + h^2}}.$$

$$\therefore \quad \frac{x^2}{x^2 + h^2} = \frac{r^2}{R^2},$$

i.e.,
$$x^2(R^2 - r^2) = h^2r^2,$$

whence
$$x = \frac{hr}{\sqrt{R^2 - r^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) we learn that

(i.) all branch vessels of the same radius coming off from the same trunk will make the same angle with the main trunk ;

(ii.) If $\frac{r}{R}$ is very small, then $\cos\theta$ is very small, *i.e.*, a branch of very small calibre comes off practically at a right angle (viz. at C) ;

(iii.) If $\frac{r}{R}$ is nearly unity, then $\cos\theta = 0$, *i.e.*, a branch of very large calibre comes off practically parallel to the main trunk (*e.g.*, the external and internal carotids). (See D'Arcy W. Thompson's "Growth and Form," and Burns' "Biophysics.")

(7) *The Economy of the Bee.*—A bee cell may be considered as a regular hexagonal prism modified in the following manner (J. Salpeter) :

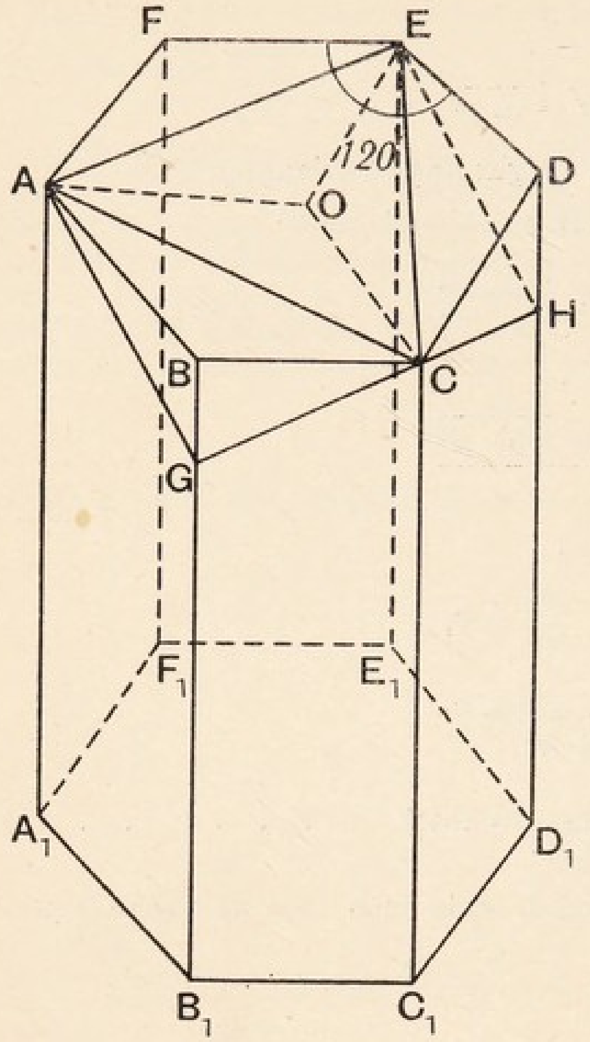

FIG. 73.

Let ABCDEF be a base of the prism (Fig. 73). Join AC and let a plane ACG inclined at an angle α to the base ABCDEF, cut the border BB_1 in the point G.

Rotate the pyramid ABCG round AC until the triangle ABC comes to lie in the position AOC. The apex G will then project upwards. Pass similar planes through CE and AE, also making angles α with the base ABCDEF, and then rotate those pyramids until the triangles CDE and AFE coincide with the triangles COE and AOE respectively.

The three little pyramids will then meet in a common apex S (Fig. 74). **The resulting figure is then similar to a bee cell.**

It is clear that by turning the pyramids ABCG and the other two pyramids round the lines AC, etc., the *volume* of the cell has not been diminished, but the *area* has been diminished, since although the three rhombi (AGCS and the other two analogous rhombi) have been added to the area, yet there have been removed the whole of the surface ABCDEF, and the surfaces of the six triangles (ABG, CBG, and the other four analogous triangles). Let us therefore consider **what must be the angle α between each of the planes and the base in order to make the area of the resulting cell a**

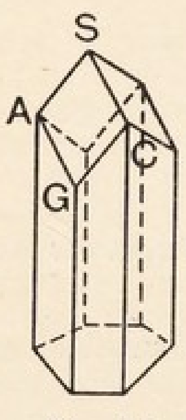

FIG. 74.

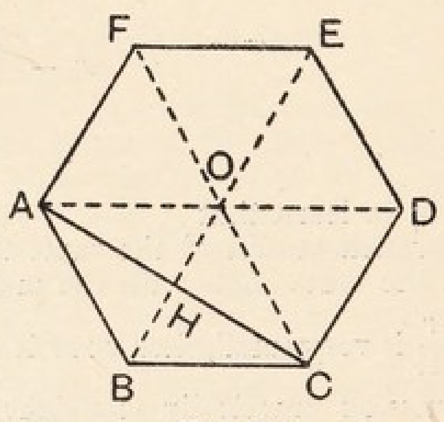

FIG. 75.

minimum. When this occurs the bees will obviously have to use less wax in order to build a cell containing a given amount of honey.

In order to solve this problem it will be better to consider what must be the length BG (Fig. 73), since this length also determines the shape of the cell.

Let this length BG $= x$.

Now if we have a hexagon (Fig. 75), and we divide it by means of the three diagonals into six equilateral triangles, OAB, OBC, etc., then the height h (*e.g.*, AH) of each triangle is given by

$$h = \sqrt{AB^2 - BH^2} = \sqrt{a^2 - \frac{a^2}{4}} = \frac{a\sqrt{3}}{2}$$

(where a = length of side of hexagon).

$$\therefore \quad AC = 2h = a\sqrt{3}.$$

$\therefore$ surface of hexagon which $= 3$ times surface ABCO
$= 6$ times surface ABC

$$= 6\frac{AC}{2} \cdot BH = 6 \cdot \frac{a\sqrt{3}}{2} \cdot \frac{a}{2} = \frac{3a^2\sqrt{3}}{2}.$$

Again (in Fig. 73) $\quad GC^2 = BC^2 + x^2 = a^2 + x^2.$

$$\therefore \quad GC = \sqrt{a^2 + x^2}.$$

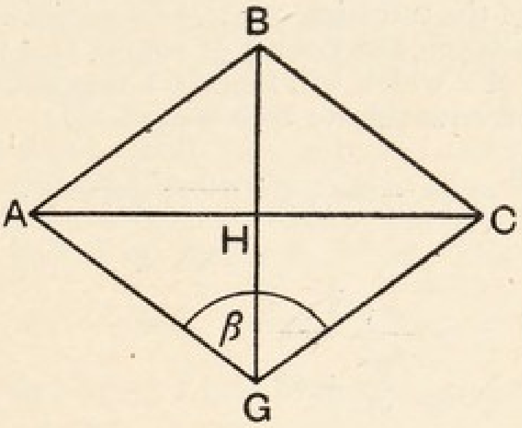

FIG. 76.

Now, surface of rhombus AGCS (Fig. 74) or AGCB (Fig. 76)
= twice the surface of the triangle AGC
= AC . GH.

But $\quad GH = \sqrt{GC^2 - HC^2},$

$$= \sqrt{(a^2 + x^2) - \frac{3a^2}{4}}$$

$$\left(\text{since } HC = \frac{AC}{2} \text{ and } AC = a\sqrt{3}\right)$$

$$= \sqrt{\frac{a^2}{4} + x^2}.$$

$\therefore$ surface of rhombus (which $= AC \cdot GH$),

$$= a\sqrt{3} \cdot \sqrt{\frac{a^2}{4} + x^2}.$$

$\therefore$ to form the bee's cell, one has to take away from the surface of the prism—

(i.) The surface of the base,

i.e., $$\frac{3a^2}{2}\sqrt{3}.$$

(ii.) The surfaces of six triangles like ABG,

i.e., $$6 \cdot \frac{ax}{2} = 3ax.$$

On the other hand one has to add—
(i.) The three rhombi like AGCS (Fig. 74),

$$= 3a\sqrt{3}\sqrt{\frac{a^2}{4} + x^2}.$$

$\therefore$ the total diminution in surface is

$$y = \frac{3a^2\sqrt{3}}{2} + 3ax - 3a\sqrt{3}\sqrt{\frac{a^2}{4} + x^2}.$$

Now for the area of the resulting cell to be a minimum (for a given volume) the portion y that is taken away must of course be a maximum. Hence we must find the value of x which will make y a maximum.

Differentiating and equating to zero we get

$$\frac{dy}{dx} = 3a - 3a\sqrt{3} \cdot \frac{2x}{2\sqrt{\frac{a^2}{4} + x^2}} = 0,$$

i.e., $$1 = \frac{x\sqrt{3}}{\sqrt{\frac{a^2}{4} + x^2}}$$

or $$\frac{a^2}{4} + x^2 = 3x^2,$$

whence $$2x^2 = \frac{a^2}{4},$$

giving $$x = \frac{a\sqrt{2}}{4}.$$

To show that this value of x makes the surface of the *portion to be removed* a maximum and not a minimum, differentiate again and get

$$\frac{d^2y}{dx^2} = -\text{ ve}.$$

$\therefore\ x = \frac{a\sqrt{2}}{4}$ makes the surface of the *resulting* cell a minimum.

To find the angle β (*i.e.*, angle AGC) of the rhombus AGCS (Fig. 74) or AGCB (Fig. 76) under such conditions, we have

$$\sin\frac{\beta}{2} = \frac{\text{AH}}{\text{AG}} = \frac{a\sqrt{3}}{2} \cdot \sqrt{\left(\frac{a}{4}\sqrt{2}\right)^2 + a^2}$$
$$= \frac{1}{3}\sqrt{6},$$

whence $$\beta = 109^\circ\ 28'\ 14''.$$

This value agrees very closely with the value of the angle as found by actual measurement.

There is an interesting story associated with this angle β. It is narrated that Maraldi measured the angle and found it to be $= 109° 28'$, and later, at the instigation of Reaumur, König calculated the value of β as $109° 26'$. Maclaurin, dissatisfied with the discrepancy of $2'$ between the calculated and observed results, repeated Maraldi's measurements and found them correct. He then discovered that König's logarithm tables were not absolutely correct, so that the bee cell has served to discover a mistake in logarithmic tables.

This story is very interesting and pretty, but unfortunately can only be regarded as an anecdote and nothing more, since recent measurements undertaken by Vogt in the case of 4,000 bee cells gave the average value of β as $107°$!

What the cause of such a discrepancy between the calculated and expected result is, it is impossible to say. Possibly there may be factors operating other than those of surface tension—to which latter is to be ascribed the tendency for production of minimal surfaces (see "Child Physiology," p. 94). Perhaps this discrepancy may help to bring about further discoveries in the same way as the observed perturbation in the calculated orbit of Uranus helped to bring about the discovery of Neptune.

(8) *Problem in Neurophysiology.*—The ratio between the radius of the axon and that of the myelin sheath of a nerve has been found to be 1 : 1·6. Assuming that the myelin coat has an insulating function like the covering in a submarine telegraph cable, is this value of the ratio such as to make the velocity of a nerve impulse a maximum ? it being known that the speed of signalling along a submarine cable varies as $x^2 \log_e \frac{1}{x}$, where x is the ratio between the radius of the core and that of the covering.

If $\quad y =$ velocity of impulse,

then $$y = Kx^2 \log_e \frac{1}{x} = -Kx^2 \log x.$$

$$\therefore \quad \frac{dy}{dx} = -K\left(2x \log x + x^2 \cdot \frac{1}{x}\right)$$
$$= -Kx(2 \log x + 1).$$

$\therefore$ for a maximum, $\log x = -\frac{1}{2}$.

$$\therefore \quad x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} = 1 : 1{\cdot}65.$$

Hence, on the given assumption, the observed and calculated results are in perfect agreement.

That $x = 1 : 1{\cdot}65$ gives a maximum and not a minimum speed is seen from a second differentiation :

$$\frac{d^2y}{dx^2} = -K \cdot 2 \log x - 2Kx \cdot \frac{1}{x} - K$$
$$= -K(2 \log x + 3) = -\text{ve}.$$

$$\therefore \quad x = 1 : 1{\cdot}65 \text{ gives a maximum.}$$

(See W. M. Feldman, *Proc. Physiol. Soc.*, 1923.)

EXERCISES.

(1) An open tank is to be constructed with a square base and vertical sides, so as to contain a given quantity of water. What must be the relation between depth and width so as to make the expense of lining it with lead a minimum?

[Answer. Depth $= \frac{1}{2}$ width.]

(2) If $y = x^3 - 6x^2 + 11x - 6$, find for which values of x y will be a maximum or a minimum.

$$\left[\text{Answer,}\quad \text{For maximum } x = 2 - \frac{\sqrt{3}}{3},\right.$$

$$\left.\text{For minimum } x = 2 + \frac{\sqrt{3}}{3}.\right]$$

(3) If $y = \frac{\log x}{x}$, find for which value of x y will be a maximum.

[For a maximum $\log x = 1$. $\therefore$ $x = e$ and $y = e^{-1}$.]

(4) If $y = x^{\frac{1}{x}}$, prove that the minimum value of y is when $x = e$.

$$\left[\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x^2}(\log x - 1). \quad \therefore \quad \text{for minimum, } \log x = 1.\right]$$

Curvature.—By the curvature of a circle is meant the rate at which the circumference curves round. Thus, if we look at the various circles in the diagram (Fig. 77), we see that in the largest

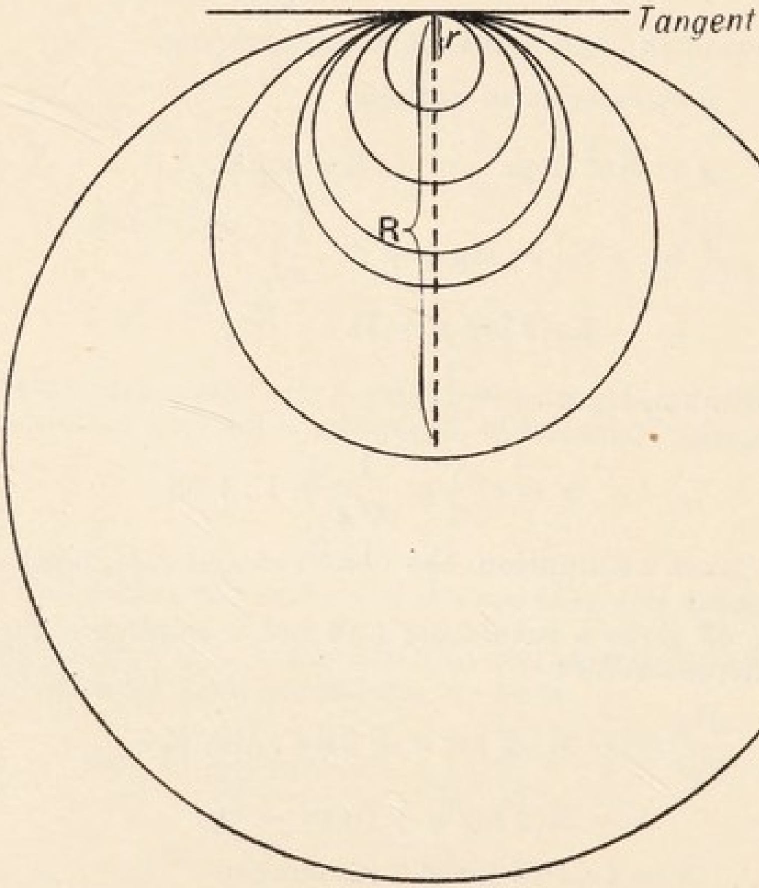

FIG. 77.—Diagram to illustrate the Meaning of the Term Curvature.

circle the rate of curving is least and in the smallest circle it is most. Hence we see that the curvature of a circle is measured by the reciprocal of its radius. Thus, if the radii of the smallest and largest circles be r and R respectively, then the curvature of the smallest circle $= \frac{1}{r}$ and the curvature of the largest circle $= \frac{1}{R}$.

Radius of Curvature.—Let DE be any curve and A, B, C, three points on it very close together (Fig. 78). Then a circle ABCF can be drawn through these three points, and it will be seen that as the distance between the three points becomes less and less until A and C ultimately coincide with B, the circle passing through these points will have the same curvature as the curve at the point B.

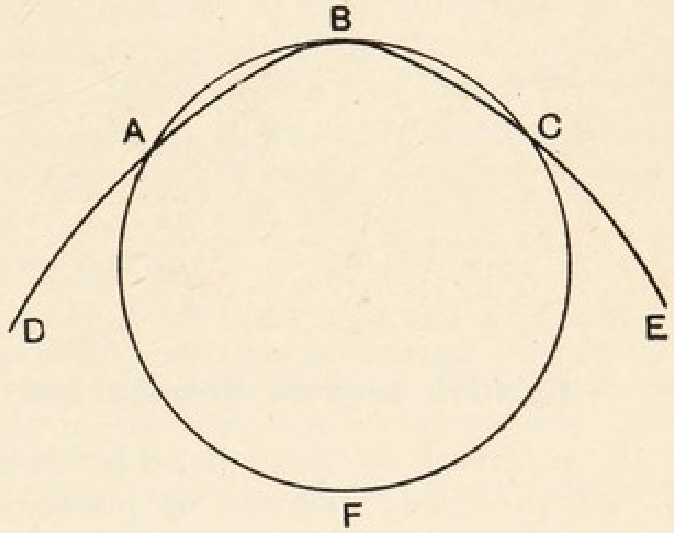

Fig. 78.—Diagram to illustrate the meaning of the Term "Radius of Curvature."

Hence, *the radius of curvature* at any point B of a given curve is the radius of the circle which has the same curvature as the curve at that point.

Formula for Curvature.—It can be shown that,

if ρ = radius of curvature of the curve, $y = f(x)$,

then curvature which $= \frac{1}{\rho}$ is given by the formula

$$\frac{1}{\rho} = \frac{y_2}{(1 + y_1^2)^{\frac{3}{2}}},$$

where $$y_1 = \frac{dy}{dx},$$

and $$y_2 = \frac{d^2y}{dx^2},$$

and that $$\therefore \quad \rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2}.$$

Example.—Find the radius of curvature of the parabola $y = 2x^2$ at the points where $x = 0$ and $x = \frac{1}{3}$.

From the formula

$$\rho = \frac{(1 + y_1{}^2)^{\frac{3}{2}}}{y_2}$$

we have, since $y_1 = 4x$ and $y_2 = 4$,

$$\rho = \frac{(1 + 16x^2)^{\frac{3}{2}}}{4}.$$

$\therefore$ where $x = 0, \quad \rho = \frac{1}{4}$

and where

$$x = \frac{1}{3}, \quad \rho = \frac{\left(1 + \frac{16}{9}\right)^{\frac{3}{2}}}{4} = \left(\frac{25}{9}\right)^{\frac{3}{2}} \Big/ 4$$

$$= \left(\frac{5}{3}\right)^3 \Big/ 4$$

$$= \frac{125}{108}.$$

Relation between Tension and Radius of Curvature.

If we stretch an elastic band between two points on a flat surface it will obviously exercise no pressure at all on any part of the surface below. But if the band is stretched over a curved surface (*e.g.*, a cylinder), then the downward pressure of the band will be proportional to the curvature (or inversely proportional to the radius of curvature) of the surface. Hence,

if p = downward pressure of band,
T = tension with which it is stretched,
R = radius of curvature of surface,

then $p = \frac{\mathrm{T}}{\mathrm{R}}$ (per unit of surface).

If, instead of a cylinder which is curved only in one direction, the band be stretched over a surface which has curvatures in two directions, then

$$p = \mathrm{T}\left(\frac{1}{\mathrm{R}_1} + \frac{1}{\mathrm{R}_2}\right),$$

where R_1 and R_2 are the two radii of curvature.

Further, if $\mathrm{R}_1 = \mathrm{R}_2 = \mathrm{R}$ (*e.g.*, in the case of a sphere),

then

$$p = \frac{2\mathrm{T}}{\mathrm{R}}.$$

$$\therefore \quad \mathrm{T} = p\frac{\mathrm{R}}{2}.$$

But the thickness of a wall of a hollow vessel must be proportional to the tension to which it is subjected.

Therefore, when P is constant, the thickness is proportional to the radius of curvature. Hence, *in the case of the uterus, the thickness of muscle is greater at the fundus*, where the radius of curvature is greater, *than at the cervix*, where the radius of curvature is less. Also the hemispherical aortic valves need have only half the thickness of the cylindrical aorta.

Similarly, *the cardiac apex*, which has the greatest curvature (or least radius of curvature), *is the thinnest part of the heart.* The same is the case with blood vessels. Other illustrations are the constipation associated with intestinal distension (*e g.* megalocolon) and the acceleration of labour after the rupture of the membranes, because the diminution in radius of curvature enables the same tension in the uterine wall to exert a higher pressure on the contents.

EXAMPLES.

(1) The lumen of the sheep's carotid is 3 mm.; that of the ox's carotid is 6 mm. The blood pressure in these vessels has been found to be 40 mm. in the case of the sheep, and 60 mm. in the case of the ox. If the thickness of the walls of the sheep's carotid is ·616, what would you expect to be the thickness of the coats of the carotid in the ox ?

The radii of curvature of the sheep's and ox's carotids are in the proportion of 3 : 6 or 1 : 2.

The pressures in the vessels in these cases are in the proportion of 40 : 60 or 2 : 3.

Now, thickness (which is proportional to the tension) is proportional to the pressure and radius of curvature.

$$\therefore \quad t_{(\text{sheep})} = KR_sP_s \text{ (where } R_s = \text{radius of sheep's carotid,}$$
$$P_s = \text{pressure in sheep's carotid,}$$
$$\text{and } K = \text{constant),}$$

and $$t_{(\text{ox})} = KR_oP_o \quad (o \text{ standing for ox}).$$

$$\therefore \quad \frac{t_{(\text{sheep})}}{t_{(\text{ox})}} = \frac{R_sP_s}{R_oP_o} = \frac{1\,.\,2}{2\,.\,3}$$
$$= \frac{1}{3}.$$

$\therefore$ Thickness of ox's carotid should be three times that of sheep's carotid $= 3 \times \cdot 616 = 1 \cdot 848.$

(Actual observation shows $t_{(\text{ox})}$ to be 1·744.)

(2) If the radius of a capillary is 0·000005 cm., find the amount of intracapillary tension that will maintain a difference of pressure of 50 mm. Hg between the inside and the outside of the capillary.

A pressure of 50 mm. Hg $= 5 \times 13 \cdot 6 = 68$ grms. per sq. cm.

$$\therefore \quad 68 = \frac{T}{0 \cdot 000005}, \text{ giving } T = 0 \cdot 34 \text{ mgm. per cm. length.}$$

[See Cranston Walker, *Br. Med. Journ.*, Feb. 18, 1922; and Correspondence by Gillispie, McQueen, Leonard Hill and others, *Ibid.* 1921.]

CHAPTER XI.

SUCCESSIVE DIFFERENTIATIONS.

In all the examples of differentiation that we have been considering in Chapter IX. we saw that the differential coefficient also formed a function of x.

Thus, in the case of linear functions,

$$y = mx + b,$$

we get
$$\frac{dy}{dx} = m, \text{ i.e., } \frac{dy}{dx} = mx^0.$$

In the case of such functions as

$$y = x^2 + bx + c,$$

we get
$$\frac{dy}{dx} = 2x + b.$$

When
$$y = x^3 + ax^2 + bx + c,$$

we get
$$\frac{dy}{dx} = 3x^2 + 2ax + b,$$

and so on.

Hence the differential coefficient of every function can itself be differentiated a number of times. Thus, let us take as an example

$$y = x^6 + 4x^3 + 3x + 4.$$

1st differential coefficient $= 6x^5 + 12x^2 + 3.$
2nd ,, ,, $= 6 \times 5x^4 + 12 \times 2x = 30x^4 + 24x.$
3rd ,, ,, $= 6 \times 5 \times 4x^3 + 12 \times 2 \times 1$
$= 120x^3 + 24.$
4th ,, ,, $= 6 \times 5 \times 4 \times 3x^2 = 360x^2.$
5th ,, ,, $= 6 \times 5 \times 4 \times 3 \times 2x = 720x.$
6th ,, ,, $= 720 \times 1 = 720.$
7th ,, ,, $= \quad = 0.$

The notations employed for denoting the successive differential coefficients of the function $y = f(x)$ are as follows :

1st differential coefficient $= f'(x)$ or $\frac{dy}{dx}$.

2nd differential coefficient $= f''(x)$ or $\dfrac{d\left(\dfrac{dy}{dx}\right)}{dx}$, *i.e.*, $\dfrac{d^2y}{dx^2}$.

3rd ,, ,, $= f'''(x)$ or $\dfrac{d\left(\dfrac{d^2y}{dx^2}\right)}{dx}$, *i.e.*, $\dfrac{d^3y}{dx^3}$.

. . . .

nth ,, ,, $= f^n(x)$ or $\dfrac{d^ny}{dx^n}$.

From the above example it will be seen that for any rational algebraic function of the nth degree all the differential coefficients beyond the nth are zero. Thus in $y = x^6$, etc., the 7th and higher differential coefficients are zero. This is, however, not so in the case of irrational and transcendental functions. (Compare the various differential coefficients of e^x, $\sin x$, $\cos x$, etc., also of $\log x$.)

The Physical Meaning of $\frac{d^2y}{dx^2}$.—We have seen that $\frac{dy}{dx}$ represents the rate of change of a function, and if y represents space and x represents time, then $\frac{dy}{dx}$ represents *velocity* (*i.e.*, rate of change of space with time). The second differential coefficient $\frac{d^2y}{dx^2}$ represents rate of change of velocity, *i.e.*, *acceleration.*

If we represent a function by a curve, then $\frac{dy}{dx}$ means the slope of the curve and $\frac{d^2y}{dx^2}$ means the rate of change of slope.

Thus, in the case of a straight line, $y = mx + b$ (Fig. 79), the slope $\frac{dy}{dx}$ is constant and $= m$.

$$\therefore \quad \frac{d^2y}{dx^2} = 0.$$

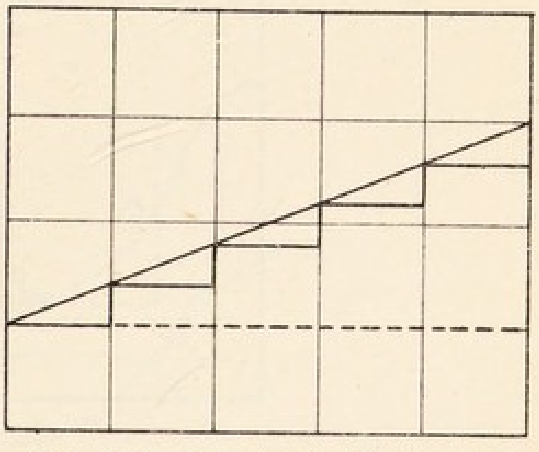

FIG. 79.—Constant Slope of a Straight Line.

But in the case of a curve representing a function of higher degree than the first, the slope, *i.e.*, $\frac{dy}{dx}$ keeps on changing, *e.g.*, in Fig. 80, the slope is gradually increasing, whilst in Fig. 81 it is gradually diminishing, and the rate at which the slope is changing is represented by $\frac{d^2y}{dx^2}$.

Concavity and Convexity.—Inspection of Figs. 80 and 81 reveals an interesting fact, viz., that if $\frac{dy}{dx}$ gradually increases as you go to

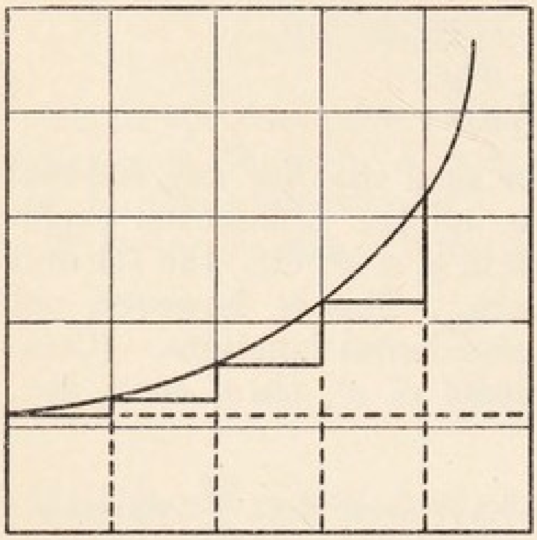

FIG. 80.—Increase of Slope in Case of Convex Curve.

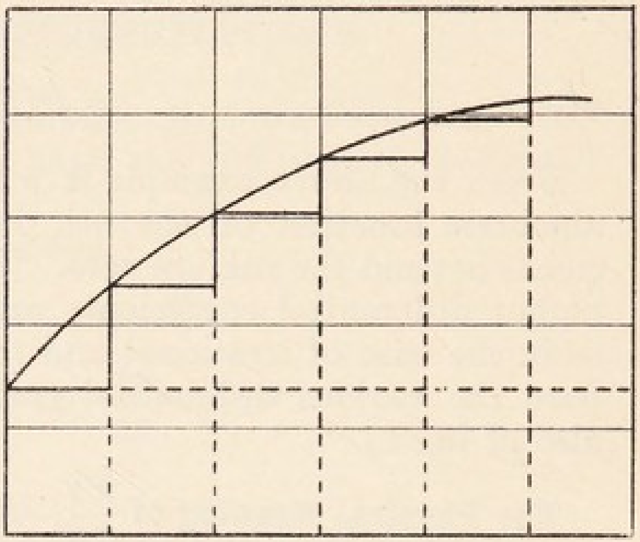

FIG. 81.—Decrease of Slope in Case of Concave Curve.

the right, *i.e.*, **when $\frac{d^2y}{dx^2}$ is positive, then the curve is convex downwards, and when $\frac{d^2y}{dx^2}$ is negative, then the curve is convex upwards.**

Point of Inflection.—The point A, or A′ (Fig. 82), where the

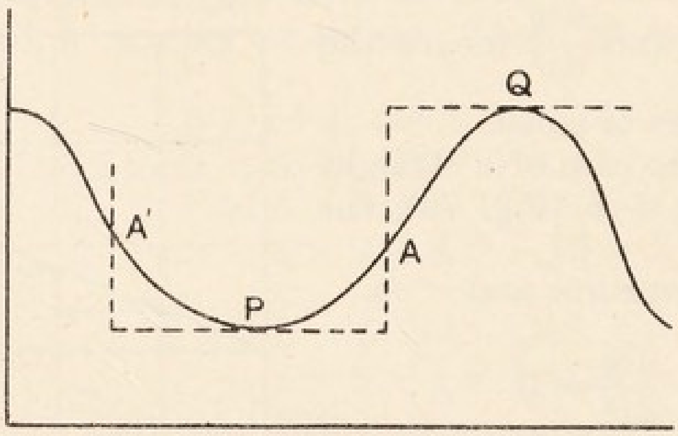

FIG. 82.

curve changes its direction from convexity to concavity, or *vice versâ*, is called *a point of inflection*, and *at this point* $\frac{d^2y}{dx^2} = 0$.

We may now summarise what we said on p. 164 about maxima and minima and what we have just said about a point of inflection :

(i.) At a *maximum or minimum* point on a curve, $\frac{dy}{dx} = 0$.

(ii.) If from a point where $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2}$ *becomes positive* as we go to the right, then we know that that point was a *minimum ;* whilst if $\frac{d^2y}{dx^2}$ *becomes negative* as we go to the right, then we know that that point was a *maximum.* This, as we have seen, is a matter of great importance in the consideration of problems on maxim and minima.

(iii.) The point where $\frac{d^2y}{dx^2} = 0$ is a *point of inflection.*

EXAMPLES.

(1) Investigate the points of inflection of the curve $y = \sin x$.

Here $$\frac{dy}{dx} = \cos x,$$

and $$\frac{d^2y}{dx^2} = -\sin x.$$

$\therefore$ for a point of inflection we must have $-\sin x = 0$.

This condition is fulfilled at the points where $x = 0$; $x = \pm \pi$; $x = \pm 2\pi$ . . . . $x = \pm n\pi$.

At these points $y = 0$.

$\therefore$ the points of inflection of this sine curve lie on the x axis at distances

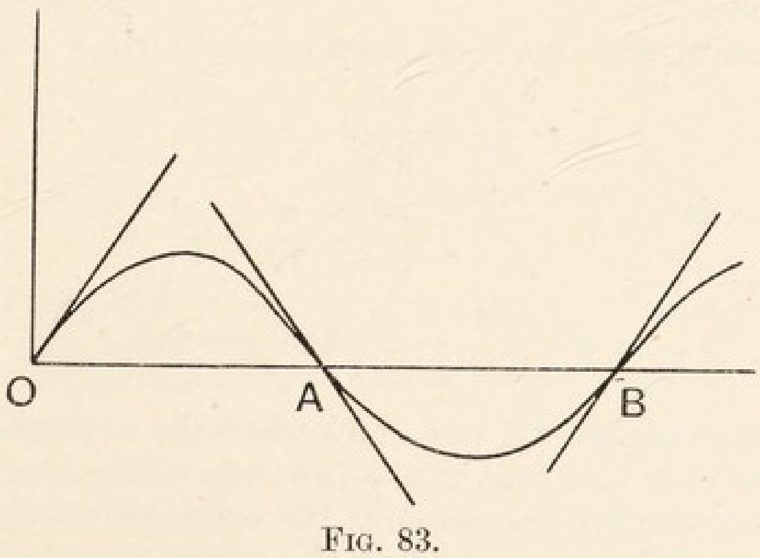

Fig. 83.

of π from one another. The points O, A, B, etc., are points of inflection (Fig. 83).

(2) Find the point of inflection of the normal curve of error whose equation is

$$y = \mathrm{A}e^{-\frac{x^2}{2\sigma^2}} \text{ (see p. 343)},$$

$$\frac{dy}{dx} = -\frac{\mathrm{A}x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}.$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{\mathrm{A}}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} + \frac{\mathrm{A}x}{\sigma^2}\cdot\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}} = 0.$$

$$\therefore \quad \frac{\mathrm{A}}{\sigma^2}e^{-\frac{x}{2\tau}}\left(\frac{x^2}{\sigma^2} - 1\right) = 0.$$

whence $\qquad x = \pm\,\sigma.$

Higher Differential Coefficients.

The differential coefficients higher than the first or second, viz., $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4} \ldots \frac{d^ny}{dx^n}$, cannot be interpreted geometrically and have no physical meaning. They are, however, of great importance in mathematical analysis, and we shall therefore devote a few paragraphs to them.

Importance of Successive Differentiations.—Apart from the importance of the second differential coefficient as a symbol for acceleration, and from the point of view of the investigation of maximum and minimum values of functions, the successive differentiations of a function lead to very important mathematical series, because by this means we can expand any function in powers of x.

Expansion of $\mathbf{e}^x$.—We have seen on p. 78 that

$$\mathrm{e}^x = 1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \cdot \quad \cdot$$

and that $\qquad \frac{d\mathrm{e}^x}{dx} = 1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \frac{x^4}{4\,!} + \ldots = e^x$

$$\therefore \quad \frac{d^2e^x}{dx^2} = e^x; \ \frac{d^3e^x}{dx^3} = e^x, \text{ and generally } \frac{d^ne^x}{dx^n} = e^x.$$

Now, supposing we did not know what the expansion of e^x was, but that we were told that e^x is a function whose successive differential coefficients are the same as the original function, we would then have no difficulty in finding the expansion of e^x. We would proceed as follows :

Let $\mathrm{e}^x = \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \mathrm{E}x^4 + \mathrm{F}x^5 + \ldots$, etc., in which the coefficients A, B, C, D, etc., are for the present unknown and have to be determined.

By differentiation

$$\frac{d\mathrm{e}^x}{dx} = \mathrm{B} + 2\mathrm{C}x + 3\mathrm{D}x^2 + 4\mathrm{E}x^3 + 5x^4 + \ldots$$

But since by hypothesis $\frac{d\mathrm{e}^x}{dx} = \mathrm{e}^x$,

therefore
$$\mathrm{B} + 2\mathrm{C}x + 3\mathrm{D}x^2 + 4\mathrm{E}x^3 + \ldots$$
$$\equiv \mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \ldots \quad \ldots \ldots \quad \text{(i.)}$$

($\equiv$ stands for identically equal to).

Now, this being an identity must be true for all values of x.

By putting $x = 0$, we get
$$\mathrm{B} = \mathrm{A} \quad \ldots \ldots \ldots \quad (a)$$

Differentiating e^x a second time, *i.e.*, by differentiating each side of (i.) we get
$$2\mathrm{C} + 3 \cdot 2\mathrm{D}x + 4 \cdot 3\mathrm{E}x^2 + \ldots$$
$$\equiv \mathrm{B} + 2\mathrm{C}x + 3\mathrm{D}x^2 + \ldots \quad \ldots \ldots \quad \text{(ii.)}$$

Hence, by putting $x = 0$, we get
$$\mathrm{B} = 2\mathrm{C} \quad \ldots \ldots \ldots \quad (b)$$

By differentiating each side of (ii.), we get
$$3 \cdot 2 \cdot \mathrm{D} + 4 \cdot 3 \cdot 2 \cdot \mathrm{E}x + \ldots$$
$$= 2\mathrm{C} + 3 \cdot 2 \cdot \mathrm{D}x + \ldots$$

Hence, by putting $x = 0$, we get
$$\mathrm{C} = 3\mathrm{D} \quad \ldots \ldots \ldots \quad (c)$$

and so on.

Collecting our results, we get

$$\mathrm{B} = \mathrm{A};$$

$$\mathrm{C} = \frac{\mathrm{B}}{2} = \frac{\mathrm{A}}{2};$$

$$\mathrm{D} = \frac{\mathrm{C}}{3} = \frac{\mathrm{B}}{3 \cdot 2} = \frac{\mathrm{A}}{3 \cdot 2}, \text{ and so on.}$$

$$\therefore \quad \mathrm{e}^x \equiv \mathrm{A}\left(1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^2}{1 \cdot 2 \cdot 3} + \ldots\right).$$

Now, this again must be true for all values of x and is therefore true when $x = 0$.

$\therefore \quad \mathrm{e}^0 = \mathrm{A}.$

But $\quad \mathrm{e}^0 = 1.$

$\therefore \quad \mathrm{A} = 1.$

$$\therefore \text{ finally,} \quad \mathrm{e}^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots$$

In this way we can find the expansion of any function in powers of x.

The reader will do well to try and find the expansion of any function the result of which is already known to him, *e.g.*,

$$y = (a + b)^x.$$

Maclaurin's or Stirling's Theorem.—This is a theorem which gives the coefficients of the various powers of x in the expansion of any function in powers of x.

Thus, if $y = f(x)$ can be expanded into a series of the form

$$y = A + Bx + Cx^2 + Dx^3 + Ex + Fx^5 + \dots$$

where A, B, C, etc., are constants, Maclaurin's theorem enables us to find these coefficients.

Thus let $y = A + Bx + Cx^2 + Dx^3 + Ex^4 + \dots$

$$\therefore \quad \frac{dy}{dx} = B + 2Cx + 3Dx^2 + 4Ex^3 + \dots$$

$$\therefore \quad \frac{d^2y}{dx^2} = 2C + 3 \,.\, 2 \,.\, Dx + 4 \,.\, 3 \,.\, Ex^2 + \dots$$

$$\therefore \quad \frac{d^3y}{dx^3} = 3 \,.\, 2 \,.\, D + 4 \,.\, 3 \,.\, 2 \,.\, Ex + \dots$$

$$\therefore \quad \frac{d^4y}{dx^4} = 4 \,.\, 3 \,.\, 2 \,.\, E + \dots$$

etc. = etc.

By making $x = 0$ in each of these identities, we get :

$$y = A.$$

$$\frac{dy}{dx} = B.$$

$$\frac{d^2y}{dy^2} = 2C \text{ or } C = \frac{1}{2\,!}\frac{dy^2}{dx^2}.$$

$$\frac{d^3y}{dx^3} = 3 \,.\, 2 \,.\, D. \quad \therefore \; D = \frac{1}{3\,!}\frac{d^3y}{dx^3}.$$

$$\frac{d^4y}{dx^4} = 4 \,.\, 3 \,.\, 2 \,.\, E. \; \therefore \; E = \frac{1}{4\,!}\frac{d^4y}{dx^4}.$$

Maclaurin's theorem may, therefore, be expressed symbolically as follows :

$$f(x) = f(0) + \frac{x}{1\,!}f'(0) + \frac{x^2}{2\,!}f''(0) + \frac{x^3}{3\,!}f'''(0) + \dots$$

Taylor's Theorem.—Taylor's theorem may be regarded as a more general expression than Maclaurin's, and is applicable where the latter is not. Thus let $f(x) = \log x$.

Expanding by Maclaurin's theorem we get

$$\log x = \log 0 + \dots$$

But since $\log 0 = -\infty$, it is obvious that the expansion is impossible.

Taylor's theorem gives the expansion of $f(x+b)$:

Let $\quad f(x+h) = A + Bx + Cx^2 + Dx^3 + \ldots \quad \ldots \quad (1)$

By putting $\quad x = 0$ we get

$$f(h) = A + B \,.\, 0 + \ldots = A.$$

By differentiating (1) we get

$$f'(x+h) = B + 2Cx + \ldots \qquad \ldots \qquad (2)$$

$\therefore$ by putting $\quad x = 0$ we get

$$f'(h) = B.$$

Similarly $\quad f''(h) = 2\,!\,C$ or $C = \dfrac{f''(h)}{2\,!}$,

$$f'''(h) = 3\,!\,D \text{ or } D = \frac{f'''(h)}{3\,!},$$

$$''''(h) = 4\,!\,E \text{ or } E = \frac{f''''(h)}{4\,!},$$

etc.

$$\therefore \quad f(x+h) = f(h) + \frac{xf'(h)}{1\,!} + \frac{x^2f''(h)}{2\,!} + \frac{x^3f'''(h)}{3\,!} + \ldots$$

Trigonometric Series.—Let us now proceed to make use of successive differentiation for determining the expansion of any function the expansion of which we have so far not yet dealt with.

Expansion of $\sin x$ in powers of x. We proceed as before.

Let $\quad \sin x = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ldots \quad . \quad (1)$

$$\therefore \quad \frac{d \sin x}{dx}, \text{ i.e., } \cos x = B + 2Cx + 3Dx^2 + 4Ex^3 + \ldots \quad (2)$$

$$\therefore \quad \frac{d^2 \sin x}{dx^2}, \text{ i.e., } \frac{d \cos x}{dx} \text{ or } -\sin x = 2C + 3\,.\,2\,.\,Dx$$
$$+ 4\,.\,3\,.\,Ex^2 + \ldots \quad (3)$$

$$\therefore \quad \frac{d^3 \sin x}{dx^3}, \text{ i.e., } \frac{d(-\sin x)}{dx} \text{ or } -\cos x = 3\,.\,2\,.\,D$$
$$+ 4\,.\,3\,.\,2\,.\,Ex + \ldots \quad (4)$$

$$\therefore \quad \frac{d^4 \sin x}{dx^2}, \text{ i.e., } \frac{d(-\cos x)}{dx} \text{ or } + \sin x = 4\,.\,3\,.\,2\,.\,E + \ldots \quad (5)$$

From (1) and (3) we have

$$\sin x \equiv A + Bx + Cx^2 + Dx^3 + \ldots$$
$$-\sin x \equiv 2C + 3\,.\,2\,.\,Dx + 4\,.\,3\,.\,Ex^2 + \ldots$$

By putting $\quad x = 0$, we get

$$\sin 0 = A, \text{ whence } A = 0$$
$$-\sin 0 = 2C, \text{ whence } C = 0.$$

From (2) and (4) we get

$$\cos x \equiv B + 2Cx + 3Dx^2 + \ldots$$
$$-\cos x \equiv 3\,.\,2\,.\,D\,. + 4\,.\,3\,.\,2.\,Ex + \ldots$$

By putting $\quad x = 0$, we obtain

$$\cos 0 = 1 = B, \text{ whence } B = 1$$
$$-\cos 0 = -1 = 3\,.\,2\,.\,D, \text{ whence } D = -\frac{1}{3\,.}$$

From (4) we have $\sin x \equiv 4 \,.\, 3 \,.\, 2 \,.\, 1 \,.\, \mathrm{E} +$ powers of x.

$\therefore$ when $x = 0$, $\mathrm{E} = 0$.

$\therefore$ we have, $\mathrm{A} = 0$, $\mathrm{B} = 1$, $\mathrm{C} = 0$, $\mathrm{D} = -\frac{1}{3 \,.\, 2}$, $\mathrm{E} = 0$, etc.

$$\therefore \quad \sin x = x - \frac{x^3}{3 \,.\, 2 \,.\, 1} + \frac{x^5}{5 \,.\, 4 \,.\, 3 \,.\, 2 \,.\, 1} - \ldots + \ldots \textit{ ad inf.}$$

Similarly we can prove that

$$\cos x = 1 - \frac{x^2}{2 \,.\, 1} + \frac{x^4}{4 \,.\, 3 \,.\, 2 \,.\, 1} - \ldots + \ldots \textit{ ad inf.}$$

Or, as an alternative method, $\cos x$ may be derived by differentiating $\sin x$ as follows :

$$\sin x = x - \frac{x^3}{3 \,.\, 2 \,.\, 1} + \frac{x^5}{5 \,.\, 4 \,.\, 3 \,.\, 2 \,.\, 1} - \ldots$$

$$\therefore \quad \frac{d \sin x}{dx} = 1 - \frac{x^2}{2 \,.\, 1} + \frac{x^4}{4 \,.\, 3 \,.\, 2 \,.\, 1} - \ldots$$

In a similar manner it can be shown that

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{3 \,.\, 5} + \ldots$$

$$\sin^{-1} x = x + \frac{x^3}{3 \,.\, 2 \,.\, 1} + \frac{3x^5}{5 \,.\, 4 \,.\, 3 \,.\, 2 \,.\, 1} + \ldots$$

and

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

(Gregory's series.)

Exponential Values of sin x and cos x.—

Since

$$e^{kx} = 1 + \frac{kx}{1} + \frac{k^2x^2}{2\,!} + \frac{k^3x^3}{3\,!} + \frac{k^4x^4}{4\,!} + \frac{k^5x^5}{5\,!} + \ldots$$

$$\therefore \quad e^{ix} = 1 + \frac{ix}{1} + \frac{i^2x^2}{2\,!} + \frac{i^3x^3}{3\,!} + \frac{i^4x^4}{4\,!} + \frac{i^5x^5}{5\,!} + \ldots \quad (i = \sqrt{-1}).$$

$$= 1 + ix - \frac{x^2}{2\,!} - \frac{ix^3}{3\,!} + \frac{x^4}{4\,!} + \frac{ix^5}{5\,!} + \ldots$$

$$= \left(1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} + \ldots\right) + i\left(x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \ldots\right)$$ (see p. 30),

i.e., $e^{ix} = \cos x + i \sin x$.

Similarly, $e^{-ix} = \cos x - i \sin x$,

$$\therefore \quad \frac{e^{ix} + e^{-ix}}{2} = \cos x \text{ (by addition), and } \frac{e^{ix} - e^{-ix}}{2i} = \sin x$$

(by subtraction).

These results are important in the solution of certain types of differential equations (see p. 284).

Calculation of the values of $\sin x$, $\cos x$, *etc.*—Supposing we wish to find the value of $\sin 30°$, then since $30° = 0{\cdot}5236$ radians, therefore $x = 0{\cdot}5236$.

$\therefore$ putting this value of x in the series $\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \ldots$

we get
$$\sin 30° = (0{\cdot}5236) - \frac{(0{\cdot}5236)^3}{3\,!} + \frac{(0{\cdot}5236)^5}{5\,!}$$
$$= 0{\cdot}5236 - 0{\cdot}0239 + 0{\cdot}0003 - \ldots$$
$$= \mathbf{0{\cdot}5000}$$

Similarly
$$\sin 60° = (1{\cdot}0472) \quad \frac{(1{\cdot}0472)^3}{3\,!} + \ldots\ldots = \mathbf{0{\cdot}8660},$$
$$\cos 30 = 1 - \frac{(0{\cdot}5236)^2}{2\,!} + \frac{(0{\cdot}5236)^4}{4\,!} - \ldots = \mathbf{0{\cdot}8660},$$

and so on, for other trigonometrical ratios of any other angle.

To prove that $\mathrm{Lt}_{x \to 0} \frac{\sin x}{x} = 1$, and $\mathrm{Lt}_{x \to 0} \cos x = 1$.

Since
$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \ldots$$
$$\therefore \quad \frac{\sin x}{x} = 1 - \frac{x^2}{3\,!} + \frac{x^4}{5\,!} - \ldots$$

$\therefore$ when $x \to 0$ the right-hand side of the equation becomes 1. (Q.E.D.)
Similarly for $\mathrm{L}t_{2 \to 0} \cos x$.

EXAMPLES.

(1) If
$$y = (\mathrm{A} + \mathrm{B}x)e^{-ax}, \text{ prove that}$$
$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + a^2y = 0.$$

$$\frac{dy}{dx} = -ae^{-ax}(\mathrm{A} + \mathrm{B}x) + \mathrm{B}e^{-ax} \text{ (p. 144).}$$
$$\therefore \quad \frac{d^2y}{dx^2} = \left\{a^2e^{-ax}(\mathrm{A} + \mathrm{B}x) - \mathrm{B}ae^{\ \ ax}\right\} - \mathrm{B}ae^{\ \ ax}$$
$$= a^2e^{-ax}(\mathrm{A} + \mathrm{B}x) - 2\mathrm{B}ae^{-ax}$$

and
$$2a\frac{dy}{dx} = -2a^2e^{-ax}(\mathrm{A} + \mathrm{B}x) + 2\mathrm{B}ae^{-ax}.$$
$$\therefore \quad \frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + a^2y = \left\{a^2e^{-ax}(\mathrm{A} + \mathrm{B}x) - 2\mathrm{B}ae^{-ax}\right\}$$
$$+ \left\{-2a^2e^{-ax}(\mathrm{A} + \mathrm{B}x) + 2\mathrm{B}ae^{-ax}\right\}$$
$$+ \left\{a^2e^{-ax}(\mathrm{A} + \mathrm{B}x)\right\}$$
$$= 0 \text{ identically.} \qquad \text{(Q.E.D.)}$$

This is a most important result, because it teaches us that if

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + a^2y = 0,$$

then $y = (A + Bx)e^{-ax}$, a fact of fundamental importance in the solution of differential equations of the second order (see p. 283). A and B are constants.

(2) Find the nth differential coefficient of $\log x$.

Since $\frac{dy}{dx} = \frac{1}{x}$, $\therefore \frac{d^2y}{dx^2} = -\frac{1}{x^2}$, $\frac{d^3y}{dx^3} = +\frac{2}{x^3}$,

$$\frac{d^4y}{dx^4} = -\frac{2 \cdot 3}{x^4}; \frac{d^5y}{dx^5} = +\frac{2 \cdot 3 \cdot 4}{x^5}; \frac{d^6y}{dx^6} = -\frac{2 \cdot 3 \cdot 4 \cdot 5}{x^6}.$$

. . . .

$$\therefore \quad \frac{d^ny}{dx^n} = (-1)^{n+1} \cdot \frac{(n-1)!}{x^n}.$$

Leibnitz's Theorem.—Leibnitz's theorem is a theorem by means of which the *nth differential coefficient of a product of two functions of x* can be written down, provided we know the successive differential coefficients of the separate factors.

We have seen that if $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{vdu}{dx}.$$

$$\therefore \quad \frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + \frac{du}{dx} \cdot \frac{dv}{dx} + v\frac{d^2u}{dx^2} + \frac{dv}{dx}\frac{du}{dx}$$

$$= u\frac{d^2v}{dx^2} + 2\frac{du}{dx} \cdot \frac{dv}{dx} + v\frac{d^2u}{dx^2}.$$

$$\therefore \quad \frac{d^3y}{dx^3} = u\frac{d^3v}{dx^3} + \frac{d^2v}{dx^2} \cdot \frac{du}{dx} + 2\frac{du}{dx} \cdot \frac{d^2v}{dx^2} + 2\frac{dv}{dx}\frac{d^2u}{dx^2}$$

$$+ \frac{dv}{dx} \cdot \frac{d^2u}{dx^2} + v\frac{d^3u}{dx^3}$$

$$= u\frac{d^3v}{dx^3} + 3\frac{du}{dx} \cdot \frac{d^2v}{dx^2} + 3\frac{d^2u}{dx^2} \cdot \frac{dv}{dx} + v\frac{d^3u}{dx^3}.$$

It will be noticed that the numerical coefficients in the above formulæ are the same as those occurring in the expansions of successive powers of $(a + b)$ by the binomial theorem.

Hence $\frac{d^ny}{dx^n} = \frac{d^n(uv)}{dx^n}$

$$= u\frac{d^nv}{dx} + n\frac{du}{dx} \cdot \frac{d^{n-1}v}{dx^{n-1}} + \frac{n(n-1)}{1 \cdot 2}\frac{d^2u}{dx^2} \cdot \frac{d^{n-2}v}{dx^{n-2}}$$

$$+ \ldots + n\frac{d^{n-1}u}{dx^{n-1}} \cdot \frac{dv}{dx} + v \cdot \frac{d_nu}{dx^n}$$

This is *Leibnitz's Theorem.*

The analogy between this theorem and Newton's binomial theorem is thus seen to be a very close one. It is remarkable that the two discoverers of the differential calculus should have discovered two such closely similar theorems.

Partial Differentiation.—We have so far dealt with the differentiation of functions of only *one* independent variable, and the differential coefficients we have obtained were the *complete* or *total* differential coefficients of those functions. In the study of biochemistry, however, one frequently has to deal with functions of more than one independent variable. Thus we know that the velocity of a chemical reaction depends not only upon the concentration of the reacting substances, but also upon the temperature; or that the volume of a gas depends not only upon the temperature, but also upon the pressure. Whenever we have to deal with such functions of more than one variable, we may differentiate the function with respect to one independent variable at a time, treating the other independent variables as if they were constants for the time being. Each of these differential coefficients is called a *partial* differential coefficient, and the total differential coefficient in such cases is a combination of the various partial coefficients.

Thus, in the case of a gas, we know that

$$PV = RT \text{ (where } P = \text{pressure, } V = \text{volume, } T = \text{absolute temperature, and } R = \text{constant).}$$

$$\therefore \quad V = \frac{RT}{P}$$

$\therefore$ when T is constant

$$\frac{dV_T}{dP} = -\frac{RT}{P^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(The little T put as a subscript shows that T has been taken as a constant.)

and when P is constant

$$\frac{dV_P}{dT} = \frac{R}{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

A more convenient way of writing a partial differential coefficient is to use the Greek delta (δ) instead of (d), when the subscript may be omitted.

Thus $$\frac{\delta V}{\delta P} = -\frac{RT}{P^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$\left(\frac{\delta V}{\delta P}\right.$ is called the coefficient of compressibility.$\left.\right)$

and $$\frac{\delta V}{\delta T} = \frac{R}{P} \quad \ldots \ldots \ldots \quad (2a)$$

$\left(\frac{\delta V}{\delta T}\right.$ is called the coefficient of expansion.$\left.\right)$

Now, from (1) we have

$$dV_T = -\frac{RT}{P^2} \cdot dP$$

and from (1a) $$-\frac{RT}{P^2} = \frac{\delta V}{\delta P}.$$

$$\therefore \quad dV_T = \frac{\delta V}{\delta P} \cdot dP \quad \ldots \ldots \ldots \quad (A)$$

Similarly $$dV_P = \frac{\delta V}{\delta T} \cdot dT \quad \ldots \ldots \ldots \quad (B)$$

A and B are *partial differentials* of V.

$\therefore$ the total variation of V, when both P and T vary together, is given by

$$\begin{aligned} dV &= dV_T + dV_P \\ &= \frac{\delta V}{\delta P} \cdot dP + \frac{\delta V}{\delta T} \cdot dT. \\ &= -\frac{RT}{P^2} \cdot dP + \frac{R}{P} \cdot dT. \end{aligned}$$

This is called the *total* or *complete differential* of V, and we thus see that

The total or complete differential of two or more independent variables is equal to the sum of their partial differentials.

EXAMPLES.

(1) Find the partial differential coefficients of

$$z = \frac{x^3}{3} - 2x^3y - 2y^2x + \frac{y}{3},$$

and find the value of dz (the complete differential).

$$\frac{\delta z}{\delta x} = x^2 - 6yx^2 - 2y^2.$$

$$\frac{\delta z}{\delta y} = -2x^3 - 4yx + \frac{1}{3}.$$

$$\begin{aligned} \therefore \quad dz &= \frac{\delta z}{\delta x} \cdot dx + \frac{\delta z}{\delta y} dy \\ &= (x^2 - 6yx^2 - 2y^2)dx - \left(2x^3 + 4yx + \frac{1}{3}\right)dy. \end{aligned}$$

(2) Find the total differential of $y = u^3 \sin v$.

$$\frac{\delta y}{\delta u} = 3u^2 \sin v\,; \; \frac{\delta y}{\delta v} = u^3 \cos v.$$

$$\therefore \quad dy = 3u^2 \sin v du + u^3 \cos v dv.$$

(3) If $y = \sin\theta + \cos\phi$, then $\frac{\delta y}{\delta\theta} = \cos\theta\,; \; \frac{\delta y}{\delta\phi} = -\sin\phi.$

$$\therefore \quad dy = \cos\theta d\theta - \sin\phi d\phi.$$

Maxima and Minima of Functions of more than One Independent Variable.—The conditions for maximum and minimum are that each partial differential coefficient should vanish (*i.e.* $= 0$).

EXAMPLES.

(1) Find maximum and minimum of

$$z = y + 2x - 2\log_e y - \log_e x.$$

$$\frac{\delta z}{\delta x} = 2 - \frac{1}{x}$$

$$\frac{\delta z}{\delta y} = 1 - \frac{2}{y}.$$

$\therefore$ for maximum or minimum

$$2 - \frac{1}{x} = 0, \text{giving } x = \frac{1}{2},$$

and
$$\frac{2}{y} - 1 = 0 \text{ giving } y = 2.$$

(2) Find maximum and minimum of

$$z = \frac{e^{x+y}}{xy}.$$

Since $e^{x+y} = e^x \,.\, e^y$.

$$\therefore \quad \frac{\delta z}{\delta x} = \frac{1}{y} \cdot \frac{xe^y e^x - e_x \,.\, e^y}{x^2} = 0, \text{ giving } e^{x+y}(x-1) = 0, \text{ or } x = 1,$$

and
$$\frac{\delta z}{\delta y} = \frac{1}{x} \cdot \frac{ye^y \,.\, e^x - e_x \,.\, e^y}{y^2} = 0, \text{ giving } y - 1 = 0, \text{ or } y = 1.$$

(3) Find the condition which must subsist between the initial concentrations a and b when $(a + b)$ is constant, so that the velocity of reaction shall be a maximum in a bimolecular reaction.

$$\text{Velocity V} = \text{K}(a - x)\,(b - x).$$

$$\therefore \quad \frac{\delta \text{V}}{\delta a} = -\text{K}(b - x) = 0, \text{ giving } b = x.$$

$$\frac{\delta \text{V}}{\delta b} = -\text{K}(a - x) = 0, \text{ giving } a = x.$$

$\therefore$ condition is that $a = b$.

Note.—We have seen that if

$$z = f(x, y),$$

then

$$dz = \frac{\delta z}{\delta x} dx + \frac{\delta z}{\delta y} dy.$$

Hence if we are given an equation like

$$\mathrm{M}dx + \mathrm{N}dy = \mathrm{A},$$

such an equation can only be a complete differential if

$$\mathrm{M} = \frac{\delta z}{\delta x}, \text{ and } \mathrm{N} = \frac{\delta z}{\delta y}.$$

This is an important result in connection with the solutions of differential equations.

CHAPTER XII.

INTEGRAL CALCULUS.

THE integral calculus may be considered as the inverse of the differential calculus. Thus, in the section on the differential calculus we concerned ourselves with the methods of differentiating any given function in order to ascertain the rate at which the dependent variable y changed with every momentary or infinitesimal change of the independent variable x in that function. The object of the integral calculus is the exact opposite of this, viz., to discover the original function from which the given differential coefficient or expression has been obtained.

Integration is the name given to this process of finding in terms of x the value of y from the given value of $\frac{dy}{dx}$, and is indicated by the symbol $\int$, which being merely a long S stands for "the sum of such quantities as." Thus, since dy stands for an infinitesimally small bit of y, therefore $\int dy$ (which is read "the integral of dy," and means the sum of all the infinite number of little bits "dy" of which y is made up), is equal to y, *i.e.*, $\int dy = y$.

Similarly $\qquad \int dx = x.$

Hence, given $\qquad \frac{dy}{dx} = 1,$

we have $dy = dx$ and $\int dy = \int dx$ or $y = x$. (But see p. 198, regarding addition of a constant.)

In many easy cases our knowledge of the differential calculus is sufficient to enable us to write down by mere inspection what is the original function whose differential coefficient is presented to us. Thus, since when $y = x^2$, we get $\frac{dy}{dx} = 2x$, therefore we can say that when $\frac{dy}{dx} = 2x$, $y = x^2$, or if $dy = 2xdx$, then $y = x^2$.

But

$$\int dy = y.$$

$$\therefore \int 2x dx = x^2.$$

Similarly, since when $y = x^4$, $\frac{dy}{dx} = 4x^3$,

$$\therefore \int 4x^3 dx = x^4.$$

Again, since when $y = \sin x$, $\frac{dy}{dx} = \cos x$,

$$\therefore \int \cos x dx = \sin x.$$

If, therefore, we are familiar with the differential coefficient of any function, we can at once write down the original function of which the given expression is the differential coefficient.

The function to be integrated is called *the integrand.*

Addition of Constant.—There is, however, one point (to which we have already alluded on p. 144), of very great importance in connection with the integration of known differential expressions. Since an infinite number of functions which differ only in respect of the constant term have the same differential coefficient (see p. 144), it will be clear that if we work back from the differential coefficient to the original function, it will be necessary to add some symbolical constant, C, called the **"integration constant."**

Thus

$$y = x^3$$

$$y = x^3 + \frac{1}{2},$$

$$y = x^3 - \frac{1}{3},$$

$$y = x^3 + 71,$$

and $y = x^3 + \text{C}$ (where C stands for any constant),

have all the same differential coefficient, viz., $\frac{dy}{dx} = 3x^2$, and therefore it is not quite correct to say that $\int 3x^2 dx = x^3$, because $x^3 + \frac{1}{2}$, $x^3 - \frac{1}{3}$, etc., would also have the same differential coefficient $\frac{dy}{dx} = 3x^2$, and therefore $\int 3x^2 dx$ might also be $= x^3 + \frac{1}{2}$ or $x^3 - \frac{1}{3}$, or $x^3 + 71$, or $x^3 + \text{C}$, etc., where C stands for any constant.

Hence we say that $\int 3x^2dx = x^3 + C.$

Similarly $\int 4x^3dx = x^4 + C,$

$\int \cos x dx = \sin x + C,$

and so on.

Evaluation of the Integration Constant.—If no further data are given us, we cannot by mere inspection of the differential coefficient find out the value of C. In actual practice, however, some further information is given which makes it easy to evaluate this constant.

Thus, supposing we are given

$$\frac{dy}{dx} = 5x^4,$$

and we are told that when $x = 0$, $y = 7$, then we can at once write down the whole original function (including the constant term) which gave rise to $\frac{dy}{dx} = 5x^4$.

For since $\int 5x^4dx = x^5 + C,$

$\therefore$ $y = x^5 + C.$

But when $x = 0, y = 7$ (by condition of the problem),

$\therefore$ when $x = 0, y = C = 7.$

$\therefore$ if $\frac{dy}{dx} = 5x^4$ and when $x = 0, y = 7,$

then $\int 5x^4dx = x^5 + 7.$

In all the above cases, if we indicate $\frac{dy}{dx}$ by y', then $dy = y'dx$, and $y = \int y'dx$.

The symbol $\int y'dx$ is read as "*the integral of y' with respect to x.*"

Geometrical Interpretation of C.—Consider three straight lines (Fig. 84).

$$y = mx.$$
$$y = mx + a,$$
$$y = mx - b.$$

All of them are inclined to the x axis at the angle $\tan^{-1} m$.

$\therefore$ m is the differential coefficient (*i.e.*, the slope) of all of them.

Hence, when we are asked to draw the graph whose differential coefficient is m, all we can do is to draw a line whose inclination

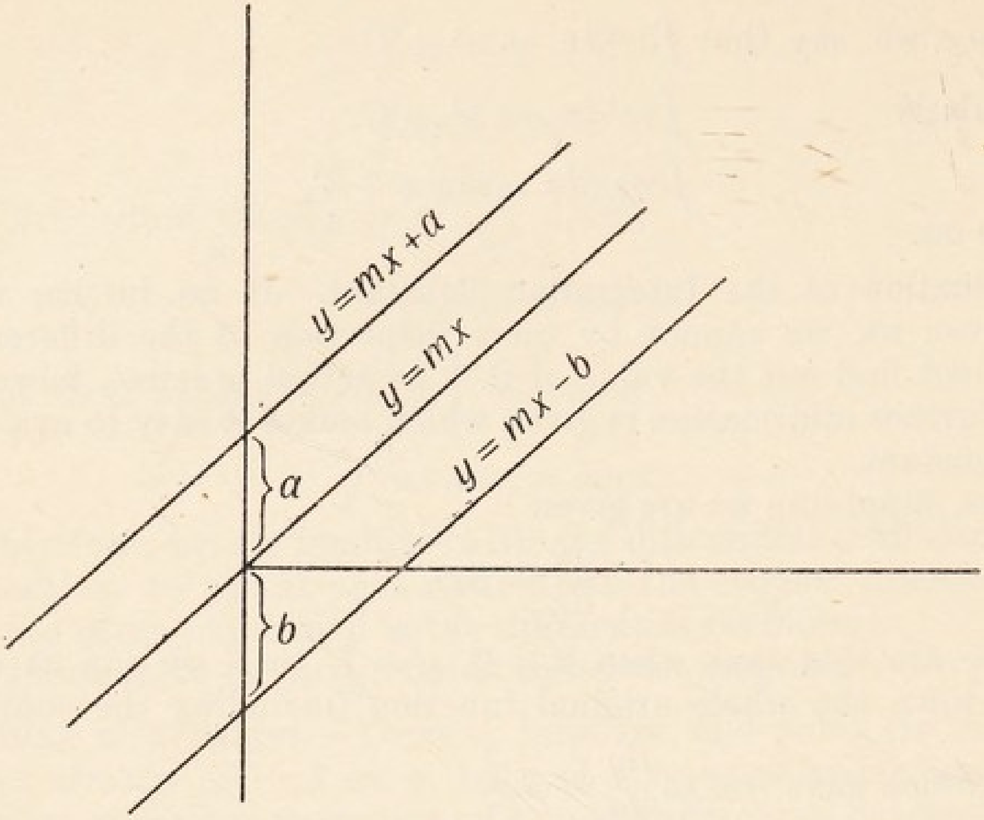

FIG. 84.

to the x axis is $\tan^{-1} m$. Such a line might either pass through the origin ($y = mx$) or cut off an intercept a from the y axis ($y = mx + a$), or cut off any other intercept such as $-b$ from the same axis ($y = mx - b$). Hence, if we are told that $\frac{dy}{dx} = m$, we provisionally write down the equation of the original function as $y = mx + C$, where C may be any constant from $-\infty$ through 0 to $+\infty$. If, however, we are also told that the graph cuts the y axis at a point (0, 3), then by writing down $y = mx + C$, and putting $x = 0$, we get

$$y = m \cdot 0 + C = C.$$

But when $x = 0, y = 3$.

$\therefore$ original function is $y = mx + 3$.

Technique of Integration.

Algebraic Functions.

To find the integral of a power of x, such as x^m.

Since, when
$$y = \frac{x^{m+1}}{m+1}$$
$$\frac{dy}{dx} = (m+1) \cdot \frac{x^m}{m+1} \text{ (see p. 137)}$$
$$= x^m.$$

$$\therefore \quad dy = x^m \,.\, dx.$$

$$\therefore \quad \int dy = \int x^m \,.\, dx,$$

i e., $$y = \int x^m \,.\, dx,$$

or $$\frac{x^{m+1}}{m+1} = \int x^m \,.\, dx.$$

Hence, the integral of $x^m = \dfrac{x^{m+1}}{m+1} + \text{C}.$

We therefore have the following *rule :*

To find the integral of any power of x, add unity to the index (converting, for instance, x^m into x^{m+1}) ; divide the result by the index thus increased (i.e., by $m + 1$) and add the integration constant C.

EXAMPLES.

$$\int x^2 dx = \frac{x^3}{3} + \text{C}.$$

$$\int x^7 dx = \frac{x^8}{8} + \text{C}.$$

$$\int x^{\frac{2}{3}} dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + \text{C} = \frac{x^{\frac{5}{3}}}{5/3} + \text{C} = \frac{3}{5} x^{\frac{5}{3}} + \text{C}.$$

$$\int x^{-4} dx = \frac{x^{-4+1}}{-4+1} + \text{C} = -\frac{x^{-3}}{3} + \text{C}.$$

$$\int x^{-\frac{7}{8}} dx = \frac{x^{-\frac{7}{8}+1}}{-\frac{7}{8}+1} + \text{C} = \frac{x^{\frac{1}{8}}}{1/8} + \text{C} = 8x^{\frac{1}{8}} + \text{C}.$$

$$\int \frac{7}{9} x^{-\frac{2}{5}} dx = \frac{7}{9} \cdot \frac{x^{-\frac{2}{5}+1}}{-2/5+1} + \text{C} = \frac{7}{9} \frac{x^{\frac{3}{5}}}{3/5} + \text{C} = \frac{35}{27} x^{\frac{3}{5}} + \text{C}.$$

To verify any of these results differentiate them and you get the expressions upon which the integration has been performed. Thus, to take a couple of the above examples.

$$y = \frac{3}{5} x^{\frac{5}{3}} + \text{C}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{5}{3} \times \frac{3}{5} x^{\frac{5}{3}-1} = x^{\frac{2}{3}},$$

$$y = 8x^{\frac{1}{8}} + \text{C}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{8} \cdot 8x^{\frac{1}{8}-1} = x^{-\frac{7}{8}}.$$

$$y = \frac{35}{27} x^{\frac{3}{5}} + \text{C}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{35}{27} \cdot \frac{3}{5} x^{\frac{3}{5}-1} = \frac{7}{9} x^{-\frac{2}{5}}.$$

Exception to the above Rule in the Case of x^{-1}.—Whilst the above rule is generally true, there is one exception—and **one only**—to which this rule does not apply. This important exception is $\frac{1}{x}$ or x^{-1}. If we apply the ordinary rule we get

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \frac{x^{-1+1}}{-1+1} + C$$

$$= \frac{x^0}{0} + C$$

$$= \infty + C.$$

This is not an infinite but an *indefinite* expression, since the value of C may be anything between $-\infty$ and $+\infty$, and thus $\int \frac{1}{x}\,dx$ may be anything between 0 and ∞. But if we refer to p. 154, we find that $\frac{1}{x}$ is the differential coefficient of $\log_e x$.

Thus if $$y = \log_e x,$$

then $$\frac{dy}{dx} = \frac{1}{x}.$$

$$\therefore \quad dy = \frac{1}{x}\,dx.$$

$$\therefore \quad \int dy = \int \frac{1}{x}\,dx,$$

i.e., $$y \text{ or } \log_e x = \int \frac{1}{x}\,dx.$$

$$\therefore \quad \boldsymbol{\int \frac{1}{x}\,dx = \log_e x + C.}$$

This integral is one of the most important ones in Biomathematics (see p. 216).

There follows from it that $\int \frac{1}{a \pm x}\,dx = \pm \log_e (a \pm x) + C$.

Exponential Functions.—

$\int e^x dx$. Since when $$y = e^x$$

$$\frac{dy}{dx} = e^x.$$

$$\therefore \quad \int e^x dx = e^x + C.$$

$\int a^x dx$. Let $$a = e^k.$$

$$\therefore \quad a^x = e^{kx}.$$

$$\therefore \quad \int a^x dx = \int e^{kx} dx$$

Put $kx = z.$

$$\therefore \quad \frac{dz}{dx} = k \text{ or } dx = \frac{dz}{k}.$$

$$\therefore \quad \int e^{kx}dx = \int \frac{e^z dz}{k}$$

$$= \frac{1}{k}\int e^z dz$$

$$= \frac{1}{k} e^z + C.$$

But $a = e^k,$

$$\therefore \quad \log_e a = k.$$

$$\therefore \quad \frac{1}{k} e^z = \frac{e^z}{\log_e a}$$

$$= \frac{e^{kx}}{\log_e a}$$

$$= \frac{ax}{\log_3 a}.$$

$\therefore$ finally $$\int a^x dx = \frac{a^x}{\log_e a} + C.$$

Trigonometrical Functions. (Circular and Inverse Circular.)—Since, when $y = \sin x$

$$\frac{dy}{dx} = \cos x.$$

$$\therefore \quad \int \cos x dx = \sin x + C.$$

Also, when $y = \cos x,$

$$\frac{dy}{dx} = -\sin x.$$

$$\therefore \quad \int \sin x dx = -\cos x + C.$$

Similarly for all the other trigonometrical functions.

Fundamental Formulæ.—Collecting all the results so far obtained we get the following table of formulæ :

$y = \dfrac{x^{m+1}}{m+1}, \dfrac{dy}{dx} = x^m$ (p. 201) $\qquad \displaystyle\int x^m dx = \frac{x^{m+1}}{m+1} + C.$

except when $m = -1.$

$y = \log_e x, \quad \dfrac{dy}{dx} = \dfrac{1}{x}$ (p. 154) $\qquad \displaystyle\int \frac{dx}{x} = \frac{1}{x} + C.$

$$y = e^x, \quad \frac{dy}{dx} = e^x \text{ (p. 152)} \qquad \int e^x dx = e^x + C.$$

$$y = \sin x, \quad \frac{dy}{dx} = \cos x \text{ (p. 156)} \qquad \int \cos x dx = \sin x + C.$$

$$y = -\cos x, \quad \frac{dy}{dx} = \sin x \text{ (p. 156)} \qquad \int \sin x dx = -\cos x + C.$$

$$y = \tan x, \quad \frac{dy}{dx} = \sec^2 x \text{ (p. 156)} \qquad \int \sec^2 dx = \tan x + C.$$

$$y = -\cot x, \quad \frac{dy}{dx} = \operatorname{cosec}^2 x \qquad \int \operatorname{cosec}^2 x dx = -\cot x + C.$$

$$y = \sin^{-1} x, \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} \text{ (p. 157)} \qquad \int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C.$$

$$y = \tan^{-1} x, \quad \frac{dy}{dx} = \frac{1}{1 + x^2} \text{ (p. 157)} \qquad \int \frac{dx}{1 + x^2} = \tan^{-1} x + C.$$

$$y = \sec^{-1} x, \quad \frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}} \text{ (p. 157)} \qquad \int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + C.$$

Integration of a Function containing a Constant as a Factor.—

If $\frac{dy}{dx} = a f(x)$, where a is a constant

then $\int a f(x)\, dx = a \int f(x)\, dx.$

In other words, **if there is a constant factor in the integrand then the constant can always be placed outside the integration sign.**

Thus
$$\int 5x^4 dx = 5\int x^4 dx = 5\frac{x^5}{5} + C$$
$$= x^5 + C.$$

Integration of a Constant by Itself.

Since when $y = ax$

$$\frac{dy}{dx} = a.$$

$$\therefore \int a dx = a.$$

Integration of the Algebraic Sum of Several Functions.—The algebraical sum of several functions is integrated by integrating each function separately, adding them together (algebraically) and then adding the constant of integration.

Thus

$$\int\left(x^4 - 3x^3 + 7x^2 + 5x + \frac{a}{x} + \frac{b}{2\sqrt{x}} + 3\right)dx$$

$$= \int x^4 dx - 3\int x^3 dx + 7\int x^2 dx + 5\int x dx + a\int \frac{1}{x}dx$$

$$+ \frac{b}{2}\int \frac{1}{\sqrt{x}}dx + 3\int dx$$

$$= \frac{x^5}{5} - \frac{3}{4}x^4 + \frac{7}{3}x^3 + \frac{5}{2}x^2 + a\log_e x + \frac{b}{2}\cdot 2x^{\frac{1}{2}} + 3x + C$$

$$= \frac{x^5}{5} - \frac{3}{4}x^4 + \frac{7}{3}x^3 + \frac{5}{2}x^2 + a\log_e x + b\sqrt{x} + 3x + C.$$

Note.—Although when each of the separate functions $\int x^4 dx$, $3\int x^3 dx$, etc., is evaluated a constant must be added to each, thus: $\int x^4 dx = \frac{x^5}{5} + C_1$; $3\int x^3 dx = \frac{3}{4}x^4 + C_2$, etc., yet when integrating their algebraic sum it is sufficient to add only one constant C; since the algebraic sum of all the separate constants $C_1 + C_2 + \ldots$ is in itself a constant, and C can be considered to represent this sum.

Similarly,

$$\int\left(\frac{7x^6 + 3x^4 + 2x - 3}{x^2}\right)dx = \frac{7x^5}{5} + x^3 + 2\log_e x + \frac{3}{x} + C.$$

EXAMPLES.

(1) Find the integral of $\qquad y' = \dfrac{x^4 + x^2 + 1}{1 + x^2}.$

$$\int\left(\frac{x^4 + x^2 + 1}{1 + x^2}\right)dx = \int\left[\frac{x^2(1 + x^2) + 1}{1 + x^2}\right]dx$$

$$= \int\left(x^2 + \frac{1}{1 + x^2}\right)dx$$

$$= \int x^2 dx + \int \frac{1}{1 + x^2}dx$$

$$= \frac{1}{3}x^3 + \tan^{-1}x + C \text{ (see p. 204).}$$

(2) Find the integral of $y' = 3x^4 - 4\cos x$.

$$\int(3x^4 - 4\cos x)dx = \int 3x^4 dx - 4\int \cos x\, dx = \frac{3}{5}x^5 + 4\sin x + C.$$

(3) Find the integral of $\quad y' = \dfrac{3}{\sin^2 x - \sin^4 x}$.

$$\int \frac{3}{\sin^2 x - \sin^4 x}dx = 3\int \frac{1}{\sin^2 x(1 - \sin^2 x)}dx$$

$$= 3\int \frac{1}{\sin^2 x \cos^2 x}dx$$

$$= 3\int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}dx$$

$$= 3\int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x}\right)dx$$

$$= 3\int \frac{1}{\cos^2 x}dx + 3\int \frac{1}{\sin^2 x}dx$$

$$= 3\int \sec^2 x dx + 3\int \operatorname{cosec}^2 x dx$$

$= \tan x + \cot x + C$ (see p. 204.)

(4) Integrate the following expression :

$$y' = 4x^3 + 3\tan^2 x.$$

$$\int(4x^3 + 3\tan^2 x)dx = \int\left(4x^3 + 3\frac{\sin^2 x}{\cos^2 x}\right)dx$$

$$= \int 4x^3 dx + 3\int \frac{\sin^2 x}{\cos^2 x}dx$$

$$= x^4 + 3\int \frac{(1 - \cos^2 x)}{\cos^2 x}dx$$

$$= x^4 + 3\int(\sec^2 x - 1)\,dx$$

$$= x^4 + 3\int \sec^2 x dx - 3\int dx$$

$$= x^4 + 3\tan x - 3x + C.$$

(5) Find the value of $\displaystyle\int \frac{1 + \sqrt{1 - x^2}}{\sqrt{1 - x^2}}dx$

$$\int \frac{1 + \sqrt{1 - x^2}}{\sqrt{1 - x^2}}dx = \int\left(\frac{1}{\sqrt{1 - x^2}} + 1\right)dx$$

$= \sin^{-1} x + x + C$ (see p. 204).

(6) Integrate $\sin 2x dx$.

Put $2x = z$.

$$\therefore \quad \frac{dz}{dx} = 2.$$

$$\therefore \quad dz = 2dx \text{ or } dx = \frac{dz}{2}.$$

$$\therefore \quad \int \sin 2x dx = \int \frac{\sin z}{2} . dz$$

$$= \frac{1}{2}\int \sin z dz$$

$$= -\frac{1}{2}\cos z + C$$

$$= -\frac{1}{2}\cos 2x + C.$$

Similarly $\int \sin ax = -\frac{1}{a}\cos ax + C.$

(7) Integrate $\int \frac{1}{(a+bx)^n} dx.$

Put $a + bx = z,$

then $\frac{dz}{dx} = b.$

$$\therefore \quad dz = bdx \text{ or } dx = \frac{dz}{b}.$$

$$\therefore \quad \int \frac{1}{(a+bx)^n} dx = \int \frac{1}{bz^n} . dz$$

$$= \frac{1}{b}\int z^{-n} dz$$

$$= \frac{1}{b(1-n)} \cdot z^{(1-n)} + C$$

$$= \frac{1}{b(1-n)} (a+bx)^{1-n} + C$$

$$= \frac{1}{b(1-n)} \cdot \frac{1}{(a+bx)^{n-1}} + C.$$

(8) Find the value of $\int e^{ax} dx$.

Put $ax = z$.

$$\therefore \quad \frac{dz}{dx} = a \text{ or } dx = \frac{dz}{a}.$$

$$\therefore \quad \int e^{ax} dx = \frac{1}{a}\int e^z dz$$

$$= \frac{1}{a}e^z + C$$

$$= \frac{1}{a}e^{ax} + C.$$

(9) Integrate $\dfrac{x^3dx}{x^2 - 3x + 2}$.

Divide the numerator by the denominator until the numerator contains a lower power of x than the denominator.

Thus $$\frac{x^3}{x^2 - 3x + 2} = x + 3 + \frac{7x - 6}{x^2 - 3x + 2}.$$

$$\therefore \int \frac{x^3dx}{x^2 - 3x + 2} dx = \int (x + 3)dx + \int \frac{7x - 6}{x^2 - 3x + 2} dx$$

$$= \frac{x^2}{2} + 3x + \int \frac{7x - 6}{x^2 - 3x + 2} dx.$$

By resolving $\dfrac{7x - 6}{x^2 - 3x + 2}$ into partial fractions (p. 26),

we get $$\frac{7x - 6}{x^2 - 3x = 2} = \frac{8}{x - 2} - \frac{1}{x - 1}.$$

$$\therefore \int \frac{7x - 6}{x^2 - 3x + 2} dx = \int \frac{8}{x - 2} dx - \int \frac{1}{x - 1} dx$$

$$= 8 \log_e (x - 2) - \log_2 (x - 1).$$

$$\therefore \int \frac{x^3}{x^2 - 3x + 2} dx = \frac{x^2}{2} + 3x + 8 \log (x - 2) - \log (x - 1) + C.$$

(10) Integrate $\dfrac{3x^2dx}{1 + x^3}$.

Here we notice that $3x^2$ is the differential coefficient of $1 + x^3$.

Hence by putting $1 + x^3 = z$ we obtain

$$\frac{dz}{dx} = 3x^2 \text{ or } dx = \frac{dz}{3x^2}.$$

$$\therefore \int \frac{3x^2dx}{1 + x^3} = \int \frac{3x^2}{z} \cdot \frac{dz}{3x^2}$$

$$= \int \frac{dz}{z}$$

$$= \log z + C$$

$$= \log (1 + x^3) + C.$$

Hence we obtain the following *most important rule:*

If the numerator of a fraction is equal to the differential coefficient of the denominator, then the integral of the fraction is equal to the natural logarithm of the denominator + C.

Thus $$\int \frac{4x^3}{1 + x^4} dx = \log (1 + x^4) + C.$$

$$\int \frac{2x}{1+x^2}\,dx = \log(1+x^2) + \text{C}.$$

$$\int \frac{nx^{n-1}\,dx}{1+x^n} = \log(1+x^n) + \text{C}.$$

$$\int \frac{4x^3+7x^2+3x+5}{x^4+\frac{7}{3}x^3+\frac{3}{2}x^2+5x+7} = \log\left(x^4+\frac{7}{3}x^3+\frac{3}{2}x^2+5x+7\right) + \text{C}.$$

(11) Integrate $\frac{5x^6}{1+x^7}\,dx.$

$$\int \frac{5x^6}{1+x^7}\,dx = \int \frac{\frac{5}{7}\cdot 7x^6}{1+x^7}\,dx$$

$$= \frac{5}{7}\int \frac{7x^6}{1+x^7}\,dx$$

$$= \frac{5}{7}\log(1+x^7) + \text{C}$$

$$= \log(1+x^7)^{\frac{5}{7}} + \text{C}.$$

(12) $$\int \frac{xdx}{1+x^2} = \frac{1}{2}\log(1+x^2) + \text{C}$$

$$= \log\sqrt{1+x^2} + \text{C}.$$

(13) Find the value of $\int \tan x dx$ and $\int \cot x dx$.

$$\int \tan x dx = \int \frac{\sin x}{\cos x}\,dx.$$

But $$\sin x = -\frac{d\cos x}{dx}.$$

$$\therefore \quad \int \tan x dx = -\log\cos x + \text{C}.$$

$$\int \cot x dx = \int \frac{\cos x}{\sin x}\,dx.$$

But $$\cos x = \frac{d\sin x}{dx},$$

$$\therefore \quad \int \frac{\cos x}{\sin x}\,dx = \log_e \sin x + \text{C}.$$

The following is an instructive example of how an expression can be manipulated so as to make the numerator the differential coefficient of the denominator :

(14) Find the value of $\int \frac{d\theta}{\sin(\beta - \theta)}$.

$$\frac{1}{\sin(\beta - \theta)} = \frac{1}{2\sin\frac{1}{2}(\beta - \theta) \, . \cos\frac{1}{2}(\beta - \theta)}$$

$$= \frac{\sin\frac{1}{2}(\beta - \theta)}{2\sin^2\frac{1}{2}(\beta - \theta) \, . \, \cos\frac{1}{2}(\beta - \theta)}$$

$$= \frac{1}{2}\operatorname{cosec}^2\frac{1}{2}(\beta - \theta) \, . \, \tan\frac{1}{2}(\beta - \theta)$$

$$= \frac{\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}(\beta - \theta)}{\cot\frac{1}{2}(\beta - \theta)}.$$

But $\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}(\beta - \theta)$ is the differential coefficient of $\cot\frac{1}{2}(\beta - \theta)$,

$$\therefore \int \frac{d}{\sin(\beta - \theta)} \text{ which} = \int \frac{\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}(\beta - \theta)\,d\theta}{\cot\frac{1}{2}(\beta - \theta)}$$

$$= \log\cot\frac{1}{2}(\beta - \theta) + \mathrm{C}.$$

This is a very important integral (see p. 246).

Definite Integral.—All the integrals we have considered up to now, viz., those of the form $\int f(x) \, . \, dx$, are called *indefinite*, or *general*, integrals, because when the integration has been performed, an expression is obtained which is another function of x of the form F (x), and whose value is undetermined so long as the value of x is undetermined. If, however, we write the integral in the form $\int_b^a f(x)dx$, we get a *definite integral*, because the symbol $\int_b^a$ tells us that, having found the expression $\mathrm{F}(x)$, of which the function $f(x)$ is the differential coefficient, we are first to substitute a for x, then substitute b for x, and finally subtract the latter from the former value.

A definite integral of the form $\int_b^a f(x)dx$ is read as follows: "The integral of $f(x)dx$ between the limits a and b." The upper value (a) is called the *superior limit,* and the lower value (b) is called the *inferior limit.*

A couple of examples will make this clear.

(a) Find the value of $\int_2^4 x^3dx.$

The general or indefinite integral is, of course, $\frac{1}{4}x^4 + \text{C}.$

Putting $x = 4$ we get $\frac{1}{4}.4^4 + \text{C} = \frac{1}{4}.256 + \text{C} = 64 + \text{C}.$

Putting $x = 2$ we get $\frac{1}{4}.2^4 + \text{C} = \frac{1}{4}.16 + \text{C} = 4 + \text{C}.$

Subtracting $4 + \text{C}$ from $64 + \text{C}$ we get 60.

$$\therefore \quad \int_2^4 x^3dx = 60.$$

It will be noticed that the **integration constant,** which is always added in the general integral, **disappears by subtraction in the definite integral.**

(b)
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin x\,dx.$$

Proceeding in the same way we get

$$\int \sin x dx = -\cos x + \text{C}.$$

$$\therefore \quad \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin x dx = -\cos\frac{\pi}{4} - \left(-\cos\frac{\pi}{6}\right)$$
$$= -\cos 45^\circ + \cos 60^\circ$$
$$= -\frac{\sqrt{2}}{2} + \frac{1}{2} = -\frac{1}{2}(\sqrt{2} - 1)$$
$$= -0{\cdot}205.$$

(c) Generally $\int_b^a f(x)\,dx = \text{F}(a) - \text{F}(b).$

Note.—Since
$$\int_b^a f(x)dx = F(a) - F(b),$$
and
$$\int_a^b f(x)dx = F(b) - F(a),$$
$$\therefore \quad \int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Poisseuille's Law for the Flow of a Viscous Liquid through a Tube.

If v = velocity of a layer of the liquid parallel to the axis and at a distance r from it, then the tangential stress due to viscosity (per unit area) $= \eta \dfrac{dv}{dr}$ (η = coefficient of viscosity).

$\therefore$ total retardation over length l is $F = -2\pi r l \eta \dfrac{dv}{dr}$ (retardation being opposed to the velocity must have a minus sign).

But F = difference of thrusts at ends of the tube $= \pi r^2 p$ (where p = pressure of fluid inside the tube).

$$\therefore \quad \pi r^2 p = -2\pi r l \eta \frac{dv}{dr},$$
or
$$-l\eta dv = \frac{1}{2} prdr,$$
i.e.,
$$-l\eta \int dv = \frac{1}{2} p \int rdr,$$
or
$$-l\eta v = \frac{1}{4} pr^2 + C.$$

But when the layer is at the periphery, $r = R$ (radius of tube) and its velocity $v = 0$.

$$\therefore \quad C = -\frac{1}{4} pR^2.$$
$$\therefore \quad l\eta v = \frac{1}{4} p(R^2 - r^2).$$
$$\therefore \quad v = \frac{p(R^2 - r^2)}{4l\eta}.$$

Now, cross-section of annulus of thickness dr is $2\pi r dr$.

$\therefore$ volume of flow (per unit of time) due to this annulus is given by the equation :
$$dV = \frac{p(R^2 - r^2) \cdot 2\pi r dr}{4l\eta}$$
$$= \frac{p\pi r(R^2 - r^2)dr}{2l\eta}.$$

∴ total volume of flow from the tube per unit of time is

$$V = \int_0^R \frac{p\pi r(R^2 - r^2)dr}{2l\eta}$$

$$= \frac{p\pi}{2l\eta}\int_0^R r(R^2 - r^2)dr$$

$$= \frac{p\pi}{2l\eta}\left(\frac{R^4}{2} - \frac{R^4}{4}\right)$$

$$= \frac{p\pi R^4}{8l\eta}$$

$$\therefore \quad p = \frac{Vl\eta}{8\pi R^4} \text{ or } \eta = \frac{\pi p R^4}{8Vl},$$

$V = \frac{\pi r^4}{8\eta} p$

which is Poisseuille's law.

I am indebted for the above proof to Dr. W. A. M. Smart.

Viscosity is the force per unit area required to produce a difference of flow equal to unity in two layers of liquid unit distance apart.

CHAPTER XIII.

BIOCHEMICAL APPLICATIONS OF INTEGRATION.

Value of Integration.—Integration serves many useful and valuable purposes. In scientific work one frequently forms an hypothesis regarding the process of a certain phenomenon. Such an hypothesis is expressed in the form of a differential equation. In order to test the validity of such an hypothesis, however, the differential equation is in itself of no use, because a differential coefficient expresses an **instantaneous** rate of change which, of course, it is impossible to measure experimentally. If, however, by means of integration, we can convert the differential expression into some relation between y and x, in which no differentials are found, we can at once subject this relationship to the test of experiment and compare the observed with the calculated results.

As an example let us consider the case of two chemical interacting substances A and B giving rise to the substances A′ and B′. Such transformation does not occur instantaneously.

If we plot a graph (Fig. 85) showing the amount of transforma-

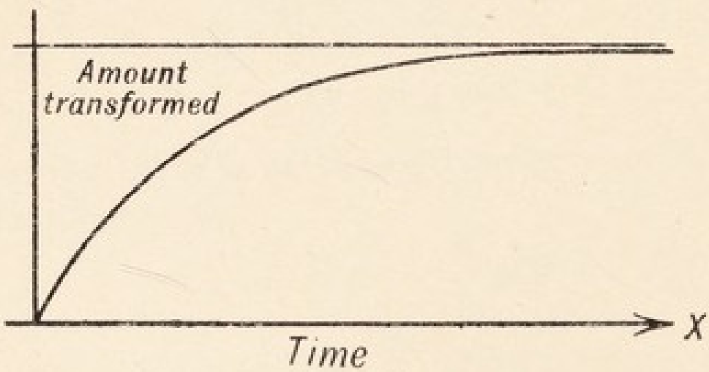

FIG. 85.

tion in time t we get a curve like the one in the diagram. From this graph we learn the following facts, viz. :

(i.) At the beginning of the reaction, when $t = 0$, the amount of substance transformed $= 0$.

(ii.) The reaction starts rapidly and then gradually slows off. This is shown by the fact that the curve, which is steep at first, gradually becomes flatter and flatter.

Now imagine the reaction to occur between the molecules of A and B in such a way that single molecules of A and B are respectively transformed into single molecules of A′ and B′. Then, since reaction occurs between molecules in contact, it is clear that the rapidity of the reaction or transformation will depend upon the frequency with which A and B will meet. In other words, **the reaction velocity, or the rate of change of concentration, must be proportional to the concentration of each of the reacting substances.** But when one quantity varies as, or is proportional to, two or more other quantities, then it varies as the product of these other quantities multiplied together.

$$\therefore \quad \textbf{reaction velocity} = \mathbf{KC_A \cdot C_B},$$

where C_A = concentration of A (*i.e.*, number of molecules per litre).

C_B = ,, B (*i.e.*, number of molecules per litre).

and K = constant.

This is *Guldberg and Waage's law of mass action.*

If the reaction takes place between two molecules of A and one molecule of B, according to the equation

$$2A + B = A' + 2B',$$

then, since the left hand side of the equation may be written as $A + A + B$,

$$\therefore \quad \text{reaction velocity} = KC_A \cdot C_A \cdot C_B$$
$$= KC_A^2 \cdot C_B.$$

Similarly, if the equation for the reaction is

$$mA + nB + kC + lD + \ldots = m'A' + n'B' + k'C' + l'D' + \ldots$$

the reaction velocity $= KC_A^m \cdot C_B^n \cdot C_C^k \cdot C_D^l \ldots$

Unimolecular Reactions.—Supposing now we have a reaction like the inversion of cane sugar,

$$C_{12}H_{22}O_{11} + H_2O = \underset{\text{Dextrose}}{C_6H_{12}O_6} + \underset{\text{Lævulose.}}{C_6H_{12}O_6}$$

The reaction velocity $= KC_s \cdot C_w$,

where C_s = concentration of cane sugar,

C_w = concentration of water.

If we start with a dilute solution, then the change in the concentration of the water is negligible (see p. 90), and therefore KC_w remains constant during the reaction.

Let $KC_w = k$, $\therefore$ reaction velocity $= k \cdot C_s$.

But reaction velocity $= \dfrac{dx}{dt}$,

where x is the amount of cane sugar transformed during a time t.

Hence, if a = the original amount of cane sugar, we have C_s, *i.e.*, the concentration of cane sugar at any instant = $(a - x)$.

$$\therefore \quad \frac{dx}{dt} = k(a - x).$$

This, then, is the theoretical expression for the reaction between cane sugar and water. Now it is clear that such an expression, or **differential equation,** does not lend itself to experimental verification, since the interval of time would have to be taken fairly large, say a good few minutes, to give any appreciable change of concentration, and during that time x would have gone on increasing, and $a - x$ would have been correspondingly diminishing. But if we can integrate this expression and get a relation between x and t without any differentials, then it would be easy to subject the equation to the test of experiment. Let us therefore proceed to integrate it.

Since $$\frac{dx}{dt} = k(a - x).$$

$\therefore$ By **separating the variables,** *i.e.*, by grouping all the x's with the (dx)'s, and all the t's with the (dt)'s, we get

$$\frac{dx}{a - x} = kdt.$$

$$\therefore \quad \int kdt = \int \frac{dx}{a - x},$$

i.e., $$kt = -\log(a - x) + C \text{ (see p. 202).}$$

It remains therefore to find the value of the constant C. This can be done as follows: When $t = 0$, *i.e.*, before the reaction has started, $x = 0$ (*i.e.*, no transformation has taken place). Substituting these values of t and x in the equation $kt = -\log(a - x) + C$, we get

$$k \,.\, 0 = -\log(a - 0) + C,$$

i.e., $$0 = -\log a + C.$$

$$\therefore \quad C = \log a.$$

Hence we get finally

$$kt = \log a - \log(a - x)$$

$$= \log \frac{a}{a - x},$$

whence $$k = \frac{1}{t} \log \frac{a}{a - x} \text{ (see pp. 77 and 86).}$$

Hence, by measuring at different intervals $t_1, t_2, t_3 \ldots t_n$, the corresponding values of $x_1, x_2, x_3 \ldots x_n$, we ought to find (if the original supposition of the law of mass action holds good, viz., $\frac{dx}{dt} = k(a - x))$, that

$$\frac{1}{t_1}\log\frac{a}{a - x_1} = \frac{1}{t_2}\log\frac{a}{a - x_2} = \frac{1}{t_3}\log\frac{a}{a - x_3} = \ldots = k.$$

The following table shows the value of k at different times t. From this it will be seen that within the limits of experimental error k is constant, and therefore there is *prima facie* evidence that the law of mass action holds good in this particular case.

t (in minutes from start).	$\frac{1}{t}\log_{10}\frac{a}{a - x} = k.$
1435	0·2348
4315	0·2359
7070	0·2343
11360	0·2310
14170	0·2301
16935	0·2316
19815	0·2291
29925	0·2330
Mean	0·2328

EXAMPLES.

(1) Madsden and Famulener investigated the loss of activity of vibriolysin at 28° C. by measuring the hæmolytic power of a vibriolysin solution kept in a thermostat for different periods of time, $t_1, t_2, t_3 \ldots$.

Now assuming that the loss of activity takes place in accordance with Guldberg and Waage's law for a monomolecular reaction, viz., that

$$K = \frac{1}{t}\log\frac{a}{a - x},$$

where a = amount of original lysine, and x = amount of lysine destroyed in time t, one can easily calculate k, and therefore, by a transformation of the equation, we get $a - x = ae^{-kt}$ (see p. 86).

The following are the observed and calculated results of $a - x$ at different times :

Time in minutes.	$(a - x)$ observed.	$(a - x)$ calculated.
0	100	100
10	78·3	83·2
20	67·6	69·5
30	59·3	57·9
40	49·8	48·3
50	40·8	40·8
60	34·4	33·6

from which it is seen that the agreement is very close, suggesting that the hypothesis is correct.

(2) After what interval of time will the initial concentration of a substance, undergoing chemical transformation in accordance with the law of a unimolecular reaction, be halved ?

The equation for a unimolecular reaction is

$$kt = \log \frac{a}{a - x}.$$

When $$x = \frac{1}{2} a, \text{ we get}$$

$$kt = \log \frac{a}{a/2} = \log_e 2.$$

$$\therefore \quad t = \frac{1}{k} \log_e 2$$

$$= \frac{2.3}{k} \log_{10} 2$$

$$= \frac{\cdot 69}{k}.$$

In other words, **the time is independent of the initial concentration in all cases of unimolecular reactions. It only depends upon the reaction constant k, with which it varies inversely.** This is a fact of great practical value when it is necessary to determine the order of any particular reaction.

For example, when diphtheria antitoxin was injected into the bodies of animals in various strengths and amounts, Bomstein found that the quantity of antitoxin present in the blood four days after injection was in every case the same fraction of the original amount. Hence, he infers that the rate of disappearance of the antitoxin takes place according to the law of a unimolecular reaction. He confirmed this by plotting log n ($n =$ content of antitoxin) against time, when a straight line was obtained (see Chapter XX., pp. 303 *et seq.*).

Bimolecular Reaction.—Let us now consider such a reaction as the following :

$$\underbrace{CH_3COOC_2H_5}_{\text{Ethyl acetate.}} + NaOH = CH_3COONa + C_2H_5OH.$$

Let a and b represent the original concentrations of $CH_3COOC_2H_5$ and NaOH respectively, and x the amount of each (*i.e.*, in molecules) transformed during the time t. Then by the law of mass action

$$\frac{dx}{dt} = k(a - x)(b - x),$$

$$\therefore \quad \frac{dx}{(a - x)(b - x)} = kdt.$$

In order to integrate this equation it is necessary to split up $\frac{dx}{(a-x)(b-x)}$ into partial fractions (see p. 26).

If we do that we find that

$$\frac{dx}{(a-x)(b-x)} = \frac{dx}{(a-b)}\left\{\frac{1}{b-x} - \frac{1}{a-x}\right\}$$

$$\therefore \quad \int\frac{dx}{(a-x)(b-x)} = \frac{1}{a-b}\left[\int\frac{dx}{b-x} - \int\frac{dx}{a-x}\right]$$

$$= \frac{1}{a-b}\left\{\log\frac{b}{b-x} - \log\frac{a}{a-x}\right\} + \text{C}$$

$$= \frac{1}{a-b}\left\{\begin{array}{l}\log b - \log(b-x) - \log a \\ \qquad + \log(a-x)\end{array}\right\} + \text{C}$$

$$= \frac{1}{a-b}\left\{\log\frac{b}{a} + \log\frac{(a-x)}{b-x}\right\} + \text{C}$$

$$= \frac{1}{a-b}\log\frac{b}{a}\frac{(a-x)}{(b-x)} + \text{C}.$$

$\therefore$ we have

$$\int kdt = \frac{1}{a-b}\log\frac{b}{a}\frac{(a-x)}{(b-x)} + \text{C},$$

i.e.,

$$kt = \frac{1}{a-b}\log\frac{b}{a}\frac{(a-x)}{(b-x)} + \text{C},$$

whence

$$k = \frac{1}{t(a-b)}\log\frac{b}{a}\frac{(a-x)}{(b-x)} + \text{C}.$$

Supposing we start with equal number of molecules of the reacting substances, we then have $a = b$, and our differential equation becomes

$$\frac{dx}{dt} = k(a-x)^2.$$

This yields on integration

$$k = \frac{1}{at}\frac{x}{(a-x)}.$$

EXAMPLES.

(1) The following figures have been obtained for the values of $a - x$ at different times t in the case of the hydrolysis of ethyl acetate by means

of NaOH. Prove that the reaction proceeds as a bimolecular one. The reaction was started with equivalent quantities of the reacting substances.

t (mins.)		$a-x$
0	.. ..	8·04
4	.. ..	5·30
6	.. ..	4·58
8	.. ..	3·91
10	.. ..	3·51
12	.. ..	3·12

This gives $a = 8{\cdot}04$, since at $t = 0$, $x = 0$. $\therefore x$ at any time $= 8{\cdot}04$ less the corresponding value of $a - x$. Thus at $t = 4$, $x = 8{\cdot}04 - 5{\cdot}30 = 2{\cdot}74$, and so on.

In order to prove that the reaction is a bimolecular one it will be sufficient if we prove that by putting $k = \frac{1}{at}\frac{x}{(a-x)}$, and substituting the various corresponding values of x and t in this equation, we shall get uniform results for the value of k.

By doing so we find the following results :

$$k_1 = \frac{1}{8{\cdot}04 \times 4} \times \frac{2{\cdot}74}{5{\cdot}30} = 0{\cdot}0160.$$

$$k_2 = \frac{1}{8{\cdot}04 \times 6} \times \frac{3{\cdot}46}{4{\cdot}58} = 0{\cdot}0156.$$

$$k_3 = \frac{1}{8{\cdot}04 \times 8} \times \frac{4{\cdot}13}{3{\cdot}91} = 0{\cdot}0164.$$

$$k_4 = \frac{1}{8{\cdot}04 \times 10} \times \frac{4{\cdot}53}{3{\cdot}57} = 0{\cdot}0160.$$

$$k_5 = \frac{1}{8{\cdot}04 \times 12} \times \frac{4{\cdot}82}{3{\cdot}12} = 0{\cdot}0162.$$

Hence, within the limits of experimental error the value of k is uniform, and therefore the reaction proceeds as a bimolecular reaction (see further pp. 310 *et seq.*).

(2) The following figures have been obtained by Senter in the study of the decomposition of H_2O_2 by hæmase (blood-catalase). The reaction was followed by withdrawing a portion of the solution at fixed intervals (t), stopping the catalysis by means of excess of H_2SO_4, and titrating with $KMnO_4$ solution.

t (mins.).	$a - x$ (c.c. $KMnO_4$).	x (c.c. $KMnO_4$).	k.
0	46·1	0	—
5	37·1	9·0	0·0435
10	29·8	16·3	0·0438
20	19·6	26·5	0·0429
30	12·3	33·8	0·0440
50	5	41·1	0·0444

Prove that the reaction proceeds as a unimolecular one.

Proceeding as in the last example, we find that putting $k = \frac{1}{t} \log \frac{a}{a - x}$, the values of k are those given in the last column, *i.e.*, are uniform. Therefore reaction is unimolecular.

(3) Madsen and Walbum studied the destruction of coli-agglutinin by means of trypsin, at 35·6° C., in the following manner : They observed the quantity of agglutinin (q) which must be added to a suspension of *Bacillus coli* to obtain a given agglutination in a given time. The strength (S) of agglutinin is, of course, inversely proportional to q.

The following values of S were found for the corresponding values of t. Show that these values agree with those one would expect to find in the case of a bimolecular reaction.

Time (hours).	S (observed).	S (calculated on the supposition that reaction is bimolecular).
0	1000	1000
0·5	775	763
1	610	600
2·25	389	395
3	280	329
4·17	259	261
5	233	227
6	189	197
8	149	155
10	140	128
12	108	109
55	59	56

$$k = 68.10^{-5}.$$

It is seen that the agreement between the observed and calculated results is very close.

Corollary.—Since
$$k = \frac{x}{at(a - x)},$$
$$\therefore \quad t = \frac{x}{ka(a - x)}.$$

Hence, when the reaction is half complete, *i.e.*, when

$$x = \frac{a}{2}, \text{ we get}$$

$$t = \frac{\frac{a}{2}}{ka \times \frac{a}{2}} = \frac{1}{ka}.$$

In other words, the time taken to change $\frac{1}{2}$ the original quantity is inversely proportional to the concentration a, *i.e.*, $t_{\frac{1}{2}} \propto \frac{1}{a}$

Similarly, in the case of *an n-molecular reaction*, in which the original concentration of the reacting substances is the same, it can be easily shown by integrating the equation $\frac{dx}{dt} = k(a - x)^n$, that the time taken to change $\frac{1}{2}$ the original quantity is proportional to $\frac{1}{a^{n-1}}$, *i.e.*, inversely proportional to a^{n-1} (see Chapter XXI., p. 313).

Chemical Equilibrium.—If in the *reversible reaction*

$$CH_3COOH + C_2H_5OH \rightleftarrows CH_3COOC_2H_5 + H_2O,$$

k_1 represents the velocity constant of esterification,
and k_2 represents the velocity constant of the opposite reaction, then we have

$$v_1 = k_1 C_{acid} \times C_{alcohol},$$

and

$$v_2 = k_2 C_{ester} \times C_{water},$$

where v_1 and v_2 are the reaction velocities in the two directions.

$\therefore$ when equilibrium exists

$$v_1 = v_2,$$

i.e.,

$$k_1 C_{acid} \times C_{alcohol} = k_2 C_{ester} \times C_{water},$$

wherefore

$$\frac{k_1}{k_2} = \frac{C_{ester} \times C_{water}}{C_{acid} \times C_{alcohol}}.$$

Now, experiment shows that if one starts with equal numbers of molecules of acid and alcohol, then equilibrium is reached when $\frac{2}{3}$ of the original amount of the substance present is decomposed. But when equal numbers of molecules are present, then $C_{acid} = C_{alcohol} = C$.

$$\therefore \quad \frac{k_1}{k_2}, \text{ or } k = \frac{\frac{2}{3}C \times \frac{2}{3}C}{\frac{1}{3}C \times \frac{1}{3}C} = 4.$$

From this value of k one can always predict when equilibrium will be established if we start with any known amounts of acid and alcohol.

EXAMPLES.

(1) One molecule of acetic acid is mixed with **a** molecules of alcohol. Find the number of molecules, n, of acid decomposed when equilibrium has been established.

Since original $C_{\textbf{Acid}} = 1$,

$$\therefore \quad C_{\textbf{Acid}} \text{ when equilibrium exists} = 1 - n,$$

and

$$C_{\text{alcohol}} \quad ,, \quad ,, \quad ,, \quad = a - n,$$

$$C_{\text{ester}} \quad ,, \quad ,, \quad ,, \quad = n,$$

$$C_{\text{water}} \quad ,, \quad ,, \quad ,, \quad = n.$$

$$\therefore \quad K = 4 = \frac{n \times n}{(1-n)(a-n)} = \frac{n^2}{(1-n)(a-n)}.$$

$$\therefore \quad n^2 = 4(a - an - n + n^2).$$

$$\therefore \quad 3n^2 - 4n(a+1) + 4a = 0,$$

whence $n = \frac{2}{3}(a + 1 - \sqrt{a^2 - a + 1})$ (see p. 33).

(2) Berthelot and Péan de St. Giles tested the above formula by mixing one molecule of acetic acid with the following number of molecules (**a**) of alcohol, and after the establishment of equilibrium found the following values of n.

—	a	n
(1)	0·05	0·05
(2)	0·08	0·078
(3)	0·18	0·171
(4)	0·28	0·226
(5)	1	0·665
(6)	8	0·966

Find whether the calculated agree with the observed results.
From formula

$$n = \frac{2}{3}(a + 1 - \sqrt{a^2 - a + 1}) \text{ we get}$$

(1)
$$n_1 = \frac{2}{3}(1{\cdot}05 - \sqrt{{\cdot}9525})$$
$$= \frac{2}{3}(1{\cdot}05 - {\cdot}975) = \frac{2}{3} \times {\cdot}075,$$
$$= {\cdot}05.$$

(2)
$$n_2 = \frac{2}{3}(1{\cdot}08 - \sqrt{{\cdot}9264}),$$
$$= \frac{2}{3}(1{\cdot}08 - {\cdot}96) = \frac{2}{3} \times {\cdot}12,$$
$$= 0{\cdot}08.$$

(3)
$$n_3 = \frac{2}{3}(1{\cdot}18 - \sqrt{{\cdot}8524}) = \frac{2}{3}(1{\cdot}18 - {\cdot}923),$$
$$= \frac{2}{3} \times {\cdot}257 = 0{\cdot}172.$$

Similarly $n_4 = {\cdot}26$; $n_5 = {\cdot}67$; and $n_6 = {\cdot}96$.

Application of the Law of Mass Action to the Dissociation of Oxyhæmoglobin.

The equation for this reversible reaction is

$$Hb + O_2 \rightleftarrows HbO_2.$$

Put $\quad$ Hb concentration $= C_{(Hb)}$

$\quad\quad O_2 \quad ,, \quad = C_{(O_2)}$

and $\quad HbO_2 \quad ,, \quad = C_{(HbO_2)}$.

We then have, when equilibrium is established,

$$K_1C_{(Hb)} \cdot C_{(O_2)} = K_2C_{(HbO_2)}$$

$$\therefore \quad \frac{K_1}{K_2} = K = \frac{C_{(HbO_2)}}{C_{(Hb)} \cdot C_{(O_2)}}.$$

Now, percentage saturation of Hb with oxygen is obviously

$$\frac{100C_{(HbO_2)}}{C_{(Hb)} + C_{(HbO_2)}}.$$

$\therefore$ if we designate the oxygen concentration by x,
and the percentage saturation of Hb by y,

we get from $\quad K = \frac{C_{(HbO_2)}}{C_{Hb} \cdot C_{O_2}}, \ Kx = \frac{C_{(HbO_2)}}{C_{(Hb)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

Also from $\quad y = \frac{100\ C_{(HbO_2)}}{C_{(Hb)} + C_{(HbO_2)}}$

we get $\quad \frac{100}{y} = \frac{C_{(Hb)} + C_{(HbO_2)}}{C_{(HbO_2)}}$

$$\therefore \quad \frac{100 - y}{y}\left(i.e., \frac{100}{y} - 1\right) = \frac{C_{(Hb)}}{C_{(HbO_2)}}\left(i.e., \frac{C_{Hb} + C_{(HbO_2)}}{C_{(HbO_2)}} - 1\right)$$

$$\therefore \quad \frac{y}{100 - y} = \frac{C_{(HbO_2)}}{C_{(Hb)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But from (1), $\quad \frac{C_{(HbO_2)}}{C_{Hb}} = Kx.$

$$\therefore \quad \frac{y}{100 - y} = Kx.$$

$$\therefore \quad \frac{100 - y}{y} = \frac{1}{Kx}.$$

$$\therefore \quad \frac{100}{y} = \frac{1 + Kx}{Kx}$$

(*i.e.*, by adding 1 to each side of the equation).

$$\therefore \quad y = \frac{100Kx}{1 + Kx}.$$

I am indebted to Dr. W. A. M. Smart for the method of deriving this formula.

The general equation of a reversible reaction may be written in the form :

$$aA + bB + cC + \ldots . \rightleftarrows a_1A_1 + b_1B_1 + c_1C_1 + \ldots .$$

(where a molecules of A react with b molecules of B, c molecules of C, etc., to form a_1 molecules of A_1, b_1 molecules of B_1, c_1 molecules of C_1, etc.).

$$\therefore \quad K_1C_A^a \,.\, C_B^b \,.\, C_C^c \ldots . = K_2C_{A_1}^{a_1} \,.\, C_{B_1}^{b_1} \,.\, C_{C_1}^{c_1} \ldots .$$

where C_A, C_B, etc., have the same meaning as similar symbols on p. 215.

$\therefore$ when equilibrium is established we have

$$K = \frac{K_1}{K_2} = \frac{C_{A_1}^{a_1} \,.\, C_{B_1}^{b_1} \,.\, C_{C_1}^{c_1} \ldots .}{C_A^a \,.\, C_B^b \,.\, C_C^c \ldots .}$$

From this it follows that, if instead of taking one molecule of Hb and combining it with one molecule of O_2, we take n molecules of each, we get our reversible equation

$$K = \frac{C_{(HbO_2)}^n}{C_{(Hb)}^n \,.\, C_{(O_2)}^n} \quad \therefore \quad Kx^n = \frac{C_{(HbO_2)}^n}{C_{(Hb)}^n}.$$

Also $$y = \frac{100\ C_{(HbO_2)}^n}{C_{(Hb)}^n + C_{(HbO_2)}^n}$$

$$\therefore \quad y = \frac{100\ Kx^n}{1 + Kx^n}.$$

which is A. V. Hills' equation for dissociation of oxyhæmoglobin.

CHAPTER XIV.

THERMODYNAMIC CONSIDERATIONS AND THEIR BIOLOGICAL APPLICATIONS.

Thermodynamic Equations.—This is a convenient place to take up the mathematical consideration of a few points in connection with the general gas equation PV = RT, where P = pressure, V = volume, T = absolute temperature, and R = a constant, called the *gas constant*. This equation, as we shall see, is of great importance in the study of numerous problems of biological interest.

(1) **To Find the Numerical Value of R.**—

Since $PV = RT$,

$$\therefore \quad R = \frac{PV}{T} \text{ for \textbf{any} value of P, V and T.}$$

If we take a gram-molecule of any gas at 0° C. and 76 cm. pressure (*i.e.*, atmospheric pressure), we have

$P = 76 \times 13{\cdot}6 = 1033$ grams per sq. cm. (since the sp. gr. of mercury = 13·6).

$V = 22{\cdot}4$ litres = 22,400 c.c. (since by Avogadro's law a gram-molecule of any gas at N.T.P. occupies 22·4 litres)

and $T = 273$.

$$\therefore \quad R = \frac{PV}{T} = \frac{1033 \times 2240}{273} = 84{,}760 \text{ grm.-cm.}$$

$$= 848 \text{ kilogram-metres.}$$

Also, since 42,640 grm.-cm. is equivalent to one calorie,

$$\therefore \quad \mathbf{R} = \frac{84{,}760}{42{,}640} = \textbf{2 calories.}$$

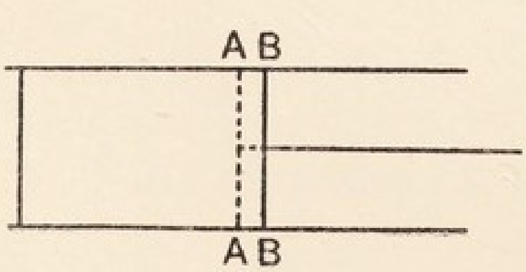

Fig. 86.

(2) **Work done by a Gas during Isothermal Expansion.**—By isothermal expansion is meant the expansion which takes place at constant temperature, as when the gas is allowed to expand inside a cylinder (with a movable piston) (Fig. 86), which is

enclosed in a thermostat supplying heat to the gas to keep its temperature constant. Let the original volume of the gas be V, when the piston is at AA. Let the gas expand so as to shift the piston through the infinitesimally small distance AB. The volume of the gas will then become $V + dV$ (where dV = the infinitesimally small increase in V). Let dW represent the infinitesimally small amount of work done by the gas during this expansion from V to $V + dV$.

Then, since the pressure of the gas may be considered to remain constant during this infinitesimally small increase in volume; and since also work is measured by force or pressure multiplied by the space through which it acts, we have

$$dW = PdV.$$

We now have to find the amount of work done **by** the gas when it expands by a **finite** amount, *i.e.*, from volume V_1 to a volume V_2. This we do by integrating, between limits, as follows:

$$\int dW = \int_{V_1}^{V_2} PdV.$$

Now whilst $$\int dW = W,$$

$$\int_{V_1}^{V_2} PdV \text{ is } \textbf{not} \text{ equal to } P\int_{V_1}^{V_2} dV, \text{ or } P(V_2 - V_1),$$

because, although P was practically constant during the minute increase of volume dV, it keeps on varying during the finite increase in volume from V_1 to V_2. But remembering the gas equation

$$PV = RT$$

(where T is the absolute temperature, which under isothermal conditions remains constant and R is the gas constant), we can put

$$P = \frac{RT}{V}.$$

Hence our differential equation becomes

$$dW = RT\frac{dV}{V}.$$

$$\therefore \quad \int dW = \int_{V_1}^{V_2} RT\frac{dV}{V},$$

i.e., $$W = RT\int_{V_1}^{V_2} \frac{dV}{V}$$

or $$\mathbf{W = RT\log_e \frac{V_2}{V} = 2{\cdot}3\ RT\log_{10}\frac{V_2}{V_1}}.$$

Application of the above Equation to Solutions.—Since it has been shown experimentally that electrolytes in solution behave like gases and since the concentration of a substance is inversely proportional to the volume of the solution, we have

$$\frac{c_1}{c_2} = \frac{V_2}{V_1}.$$

∴ the amount of work that must be done **on** a solution to increase its concentration from c_1 to c_2 is

$$\mathbf{W = 2{\cdot}3\ RT\ log_{10}} \frac{c_2}{c_1}.$$

Further, since the depression in freezing point (Δ) of a solution is proportional to the concentration—

∴ the amount of work that must be done on a solution to change the freezing point from Δ_1 to Δ_2 is

$$\mathbf{W = 2{\cdot}3\ RT\ log_{10}} \frac{\Delta_2}{\Delta_1}.$$

This is an equation of fundamental importance in biochemistry.

Note.—The gas constant has the following values, viz. :

$$\left.\begin{aligned} R &= 2 \text{ gram-calories} \\ &= 0{\cdot}85 \text{ kilogram-metre} \\ &= 0{\cdot}82 \text{ litre-atmosphere} \end{aligned}\right\} \text{(see p. 226).}$$

Example.—The freezing point of blood is $-0{\cdot}56°$ C. ; that of urine is $-1{\cdot}85°$ C. Assuming a healthy adult to pass 1·5 litres of urine a day, calculate the work done by the kidneys in a day.

The work done by the kidneys is to concentrate **one litre** of a glomerular filtrate which has a $\Delta_1 = -0{\cdot}56$ to a urine whose $\Delta_2 = -1{\cdot}85$, at body temperature 37° C.

$$\therefore \quad W = 2{\cdot}3\ R\ (273 + 37) \log_{10} \frac{1{\cdot}85}{0{\cdot}56}.$$

Taking R = 0·85 kilogram-metre we get

$$\begin{aligned} W &= 2{\cdot}3 \times 0{\cdot}85 \times 310 \ (\log 1{\cdot}85 - \log 0{\cdot}56) \\ &= 606\ ({\cdot}2672 - \bar{1}{\cdot}7482) \\ &= 606 \times {\cdot}519 \\ &= 315 \text{ kilogram-metres.} \end{aligned}$$

∴ to concentrate **1·5 litres** of the glomerular filtrate, the amount of work necessary = 1·5 × 315
= 472·5 kilogram-metres.

Note.—This calculation of the work done by the kidneys takes into account only the total concentration of the blood and urine, and is meant merely as an illustration of the application of the isothermal expansion formula to physiology. In order to calculate the actual work of the kidneys, account must be taken of the change of concentration of each of the urinary constituents. (See Cushing, "The Secretion of Urine" ; and Bayliss, "The Principles of General Physiology," 1920, p. 340.)

Application of the Isothermal Expansion Equation to Concentration Cells.—If we put two electrodes of the same metal, such as copper, into two solutions of the same salt, but of different concentrations, c_1 and c_2, then physico-chemical considerations, similar to those in the preceding paragraph, lead to the equation

$$E = 2{\cdot}3 \text{ RT} \log \frac{c_2}{c_1},$$

where E is the E.M.F. (electromotive force) in volts,
R is the gas constant expressed in coulomb × volts
= 8·36 (since 1 calorie = 4·18 coulomb × volt).
∴ at room temperature, *i.e.*, at 17° C.

$$E = 2{\cdot}3 \times 8{\cdot}36 \times 290 \log \frac{c_2}{c_1} \text{ coulomb} \times \text{volts}$$

$$= \frac{2{\cdot}3 \times 8{\cdot}4 \times 290}{96{,}530} \log \frac{c_2}{c_1} \text{ faradays}$$

(1 faraday being the amount of electricity that liberates one gram of hydrogen = 96,530 coulombs)

$$= 0{\cdot}058 \log \frac{c_2}{c_1} \text{ faradays.}$$

This formula must also be multiplied by $\frac{u-v}{u+v}$, where u and v are the velocities of the ions in question (because it is the ions that carry the electrical charge).

∴ finally, $$E = 0{\cdot}058 \frac{u-v}{u+v} \log \frac{c_2}{c_1}.$$

Hence, if the electrode be of hydrogen (*e.g.*, platinum saturated with hydrogen), this equation enables one to calculate the H-ion concentration of a given solution. (See further, books on Physical Chemistry.)

(3) **The Relation between the Two Specific Heats of Gases.**

(i.) **The Value of the Arithmetical Difference $S_p - S_v$.**

(*a*) If we raise 1 grm. of gas from 0° to 1° C., **keeping the volume constant,** then the quantity of heat required is called the **specific heat at constant volume** and is designated S_v.

(*b*) If the same quantity of the same gas is raised from 0° to 1°, but is allowed to expand **at constant pressure,** then the quantity of heat required for the purpose is called the **specific heat at constant pressure,** and is designated S_p.

It is clear that $S_p > S_v$, since if the gas is allowed to expand from V_1 to V_2 at the constant pressure P, it does an amount of work which is measured by $P(V_2 - V_1)$, and hence the quantity of heat supplied must be sufficient not only to raise the temperature of the gas by 1° C. but also to supply heat to make up for the cooling entailed by the work of expansion.

$$\therefore \quad S_p - S_v = P(V_2 - V_1).$$

But $$PV_1 = RT^1,$$

and $$PV_2 = RT\ .$$

$$\therefore \quad P(V_2 - V_1) = R(T_2 - T_1).$$

$\therefore$ when $$T_2 - T_1 = 1^\circ,$$

$$P\,(V_2 - V_1) = R.$$

$$\therefore \quad S_p - S_v = R.$$

If, instead of heating 1 grm. of the gas to 1° C., we heat 1 gram-molecule, the amounts of heat required in each case are called the molecular heats at constant volume or pressure. In that case, the *difference between the* 2 *molecular heats* = 2 *calories* (which is the value of R for a gram-molecule of gas).

(ii.) **The Ratio between the Two Specific Heats**

$$\left(i.e., \textbf{ the value of } \gamma \textbf{ which } = \frac{S_p}{S_v}\right).$$

(*a*) Let dQ be the minute amount of heat imparted to a gram-molecule of gas **at constant volume.** This will go entirely to raise the temperature of the gas by dT.

$$\therefore \quad dQ = S\,dT.$$

(*b*) If dQ be the minute quantity of heat imparted to a gram-molecule of gas **at constant pressure**, then dQ is distributed among two forms of energy, viz. (*a*) a portion goes to raise the temperature by dT; (β) another portion goes to perform work of expansion.

The portion under (*a*) is represented by $S_v dT$ and the portion under (β) is represented by PdV.

$$\therefore \quad dQ = S_v dT + PdV.$$

But $$PV = RT.$$

$\therefore$ when P_1V_1 and T are variable we get by differentiation

$$\frac{d\,(PV)}{dT} = R,$$

i.e., $$\frac{PdV}{dT} + \frac{VdP}{dT} = R \text{ (p. 144)}.$$

$$\therefore \quad PdV + VdP = RdT$$

and $$\frac{PdV}{R} + \frac{VdP}{R} = dT.$$

But $$dQ = S_v dT + PdV.$$

$$\therefore \quad dQ = S_v\left(\frac{PdV}{R} + \frac{VdP}{R}\right) + PdV$$

$$= PdV\left(\frac{S_v + R}{R}\right) + \frac{S_v VdP}{R}$$

$$= \frac{S_p PdV}{R} + \frac{S_v VdP}{R} \quad \text{(see p. 230).}$$

If the cylinder containing the gas be now surrounded by a non-conducting envelope so as to prevent heat from entering or leaving the cylinder, and P_1, V_1 and T be allowed to change **adiabatically** (*i.e.*, without transfer of heat to or from it), then $dQ = 0$.

$$\therefore \quad S_p PdV + S_v VdP = 0,$$

or
$$\frac{S_p}{S_v} PdV + VdP = 0,$$

i.e.,
$$\gamma PdV + VdP = 0,$$

or, dividing by PV,

$$\gamma \frac{dV}{V} + \frac{dP}{P} = 0.$$

$$\therefore \quad \gamma \int_{V_1}^{V_2} \frac{dV}{V} + \int_{P_1}^{P_2} \frac{dP}{P} = 0,$$

i.e.,
$$\gamma \log \frac{V_2}{V_1} + \log \frac{P_2}{P_1} = 0,$$

or
$$\gamma \log \frac{V_2}{V_1} = \log \frac{P_1}{P_2}.$$

$$\therefore \quad \left(\frac{V_2}{V_1}\right)^{\gamma} = \frac{P_1}{P_2}.$$

(4) **Law of Adiabatic Expansion** (*i.e.*, expansion without transfer of heat to or from the gas).

Since $dQ = S_v dT + PdV$ (see equation above).
and also $PdV = RdT - VdP$ (p. 230),

$$\therefore \quad dQ = S_v dT + RdT - VdP$$

$$= (S_v + R)\, dT - VdP$$

$$= S_p dT - VdP \quad \text{(see p. 230).}$$

$$= S_p dT - RT\frac{dP}{P} \quad \text{(since } PV = RT\text{).}$$

$\therefore$ For an adiabatic expression when $dQ = 0$,

$$S_p dT = RT\frac{dP}{P}.$$

But $$R = S_p - S_v \text{ (see p. 230).}$$

$$\therefore \quad S_p dT = (S_p - S_v)T\frac{dP}{P},$$

or $$\frac{S_p}{S_p - S_v}\frac{dT}{T} = \frac{dP}{P}.$$

$$\therefore \quad \frac{S_p}{S_p - S_v}\int_{T_1}^{T_2}\frac{dT}{T} = \int_{P_1}^{P_2}\frac{dP}{P},$$

i.e., $$\frac{S_p}{S_p - S_v}\log\frac{T_2}{T_1} = \log\frac{P_2}{P_1},$$

or $$\frac{S_p/S_v}{\frac{S_p}{S_v} - 1}\log\frac{T_2}{T_1} = \log\frac{P_2}{P_1},$$

i.e., $$\frac{\gamma}{\gamma - 1}\log\frac{T_2}{T_1} = \log\frac{P_2}{P_1}.$$

$$\therefore \quad \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = \frac{P_2}{P_1},$$

or $$\left(\frac{T_2}{T_1}\right)^{\gamma} = \left(\frac{P_2}{P_1}\right)^{\gamma-1}.$$

But we have seen (p. 231) that

$$\frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^{\gamma},$$

$$\therefore \quad \left(\frac{T_2}{T_1}\right)^{\gamma} = \left\{\left(\frac{V_1}{V_2}\right)^{\gamma}\right\}^{\gamma-1},$$

or $$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1}.$$

(5) **Efficiency of an Engine.**—By the efficiency of an engine is meant the proportion of the heat developed by the machine or engine which is transformed into mechanical work. Considerations based upon the second law of thermodynamics lead to the conclusion that the *maximum efficiency* is obtainable only from a *reversible engine*, *i.e.*, an engine which, " after converting a certain fraction of the heat into work, will return to its original state in

every respect if made to act backwards step by step." In order therefore to estimate the maximum efficiency of an engine, we must consider *Carnot's reversible cycle.*

Carnot's Cycle (Fig. 87).—(i.) Let a gram-molecule of gas of volume V_1, pressure P_1 and Temperature T_1 (represented in the diagram by the point A) be allowed to *expand* **isothermally** to the point B (V_2, P_2).

Then, if during the process, Q_1 units of heat have been **absorbed from** the thermostat, we have

$$Q_1 = \int_{V_1}^{V_2} PdV = RT_1 \log \frac{V_2}{V_1}.$$

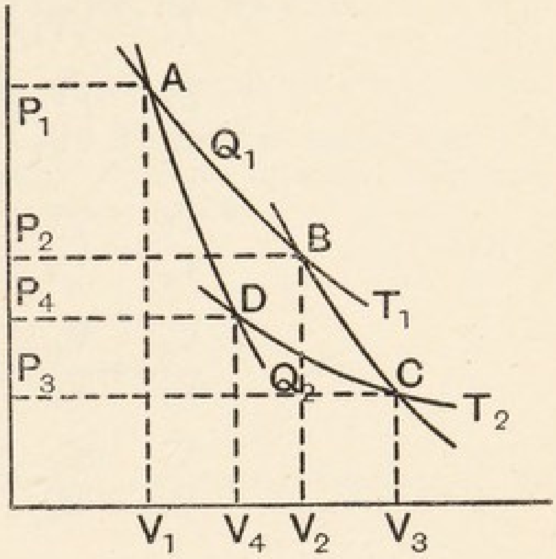

FIG. 87.—Carnot's Cycle.

(ii.) Now let the gas *expand* **adiabatically,** *i.e.*, without allowing the gas to gain or lose heat, to C (V_3, P_3). The temperature of the gas will fall from T_1 to T_2, and we have

$$\frac{T_1}{T_2} = \left(\frac{V_3}{V_2}\right)^{\gamma-1} \text{(see p. 232)}.$$

(iii.) Now let the gas be *compressed* isothermally at temperature T_2 to D (V_4, P_4). Then if Q_2 is the amount of heat *evolved* from the gas **into** the thermostat, we have

$$-Q_2 = \int_{V_3}^{V_4} PDV = RT_2 \log \frac{V_4}{V_3}$$

(heat evolved being of opposite sign to that absorbed),

$$\text{or } Q_2 = RT_2 \log \frac{V_3}{V_4}$$

$$\left(\text{since} \qquad \log \frac{V_4}{V_3} = -\log \frac{V_3}{V_4}\right).$$

$$\therefore \frac{Q_1}{Q_2} = \frac{T_1 \log \frac{V_2}{V_1}}{T_2 \log \frac{V_3}{V_4}}.$$

(iv.) Finally, let the gas be *compressed* **adiabatically** from D (V_4, P_4) to A (V_1, P_1), *i.e.*, back again to its original volume, pressure and temperature. The temperature will rise from T_2 to T_1, and we have

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_4}\right)^{\gamma-1} \text{ (p. 232),}$$

or

$$\frac{T_1}{T_2} = \left(\frac{V_4}{V_1}\right)^{\gamma-1}.$$

But

$$\frac{T_1}{T_2} = \left(\frac{V_3}{V_2}\right)^{\gamma-1} \text{ (by (ii.), on p. 233).}$$

$$\therefore \frac{V_3}{V_2} = \frac{V_4}{V_1},$$

or

$$\frac{V_2}{V_1} = \frac{V_3}{V_4}.$$

$\therefore$ equation

$$\frac{Q_1}{Q_2} = \frac{T_1 \log \frac{V_2}{V_1}}{T_2 \log \frac{V_3}{V_4}}$$

becomes

$$\frac{Q_1}{Q_2} = \frac{T_1}{T_2} \text{ or } \frac{Q_2}{Q_1} = \frac{T_2}{T_1},$$

or

$$\frac{Q_2}{Q_1} - 1 = \frac{T_2}{T_1} - 1,$$

or

$$\frac{Q_2 - Q_1}{Q_1} = \frac{T_2 - T_1}{T_1},$$

or

$$\frac{Q_1 - Q_2}{Q_1} = \frac{T_1 - T_2}{T_1}.$$

But since Q_1 is the amount of heat absorbed by the gas and Q_2 is the amount evolved,

$\therefore \dfrac{Q_1 - Q_2}{Q_1}$ is the proportion of absorbed heat which is converted into work, and represents the *efficiency* of the system, which being a reversible cycle must possess the maximum possible efficiency.

$\therefore$ *maximum efficiency* of a reversible engine (or any reversible process) $= \dfrac{T_1 - T_2}{T_1}$, where T_1 and T_2 are the absolute temperatures of the first and second isothermal operations.

E.g., if a steam engine receives steam at 140° and exhausts into the air at 105°, then the efficiency is

$$\frac{140 - 100}{273 + 140} = \frac{40}{413} = 0{\cdot}097 = 9{\cdot}7 \text{ per cent.}$$

The ordinary efficiency of a steam engine is about 20 per cent.

Example.—Calorimetric experiments have shown that the efficiency of human muscle (*i.e.*, the proportion of energy converted by muscle into work) is 20 per cent. Find whether it acts as a steam engine.

If it acts as a steam engine, then we have

$$\text{Efficiency} = \frac{T_1 - T_2}{T_1}.$$

If we take T_2 as the absolute body temperature

$$= 273 + 37 = 310,$$

then

$$\frac{1}{5} = \frac{T_1 - 310}{T_1} \text{ (since } \frac{Q_2 - Q_1}{Q_1} = \frac{1}{5} \text{ by hypothesis).}$$

$$\therefore \quad T_1 = 5T_1 - 1550.$$

$$\therefore \quad 4T_1 = 1550,$$

or

$$T_1 = 387{\cdot}5° \text{ absolute}$$

$$= 114{\cdot}5 \text{ C.}$$

Hence the body must show a fall of temperature during work from an internal temperature of, presumably, 114·5 to 37° C., which is absurd, as the body never attains such a temperature as 114·5.

If, on the other hand, we call the absolute body temperature T_1, then we have

$$\frac{1}{5} = \frac{310 - T_2}{310}$$

or

$$62 = 310 - T_2.$$

$$\therefore \quad T_2 = 310 - 62 = 248 \text{ absolute} = -25° \text{ C.}$$

In other words, during contraction the temperature of the body must fall to − 25° C., which is equally absurd. Hence muscle does **not** act like a steam engine.

Deductions from the Efficiency Equation.

(1) **Connection between the Latent Heat of Vaporisation and Change of Vapour Pressure with Temperature.**—When any fluid is heated the pressure of the saturated vapour varies with the temperature, *e.g.*, in the case of water the vapour pressure at 10° = 9·165 mm. Hg, and that at 100° = 760 mm. Hg. This means that if water be heated to 10° at 9·165 mm. pressure or to 100° at 760 mm., it will turn into vapour. In other words, the boiling point of water at 9·165 mm. is 10°, and at atmospheric pressure is 100°. Now, suppose water at 99·5° (whose vapour pressure is 746·5) has the pressure reduced from atmospheric pressure 760 mm. to 746·5 mm., then it will boil at 99·5° instead of at 100°.

If we indicate the minute change of pressure by dP and the minute change of temperature by dT, then the work done by a gram-molecule of water in such a process is

$$\frac{(V_2 - V_1)dP}{Q} = \frac{dT}{T},$$

(where V_2 = volume of a gram-molecule of the vapour
V_1 = ,, ,, ,, liquid
Q = amount of heat required to change a gram-molecule of the liquid **to the same volume** of vapour). But as V_1 is negligible in comparison with V_2 the equation may be written

$$\frac{V_2 dP}{Q} = \frac{dT}{T}.$$

But since $PV_2 = RT, \quad \therefore \quad V_2 = \frac{RT}{P},$

$$\therefore \quad \frac{RT}{PQ}dP = \frac{dT}{T},$$

$$\therefore \quad \frac{dP}{P} = \frac{QdT}{RT^2}$$

$$= \frac{Q}{2}\frac{dT}{T^2} \text{ [since R = 2 (see p. 226)].}$$

Now if we assume that Q remains constant between the limits of temperature T_1 and T_2, then

$$\int_{P_1}^{P_2} \frac{dP}{P} = \frac{Q}{2}\int_{T_1}^{T_2} \frac{dT}{T^2},$$

i.e.,
$$\log_e \frac{P_2}{P_1} = \frac{Q}{2}\left(\frac{1}{T_1} - \frac{1}{T_2}\right),$$

or
$$2{\cdot}3 \log_{10} \frac{P_2}{P_1} = \frac{Q}{2}\frac{(T_2 - T_1)}{T_1 T_2}.$$

Hence, if we know the values of P_2 and P_1 for the known temperatures T_1 and T_2, Q may easily be calculated. If, now, we add to the value of Q thus found the amount of work done by the expansion of the vapour from V_1 to V_2, which is PV_2 (since V_1 is negligible) $= 2T$, the quantity so obtained is the mean latent heat of vaporisation of a gram-molecule of liquid between T_1 and T_2.

(2) **Influence of Temperature upon the Velocity of a Chemical Reaction.**—Since substances in solution behave as if they were gases occupying the same volume as the solution, therefore P

in the case of a solution (*i.e.*, the osmotic pressure) is proportional to the concentration of the solution (*i.e.*, inversely to the volume). But since by the law of mass action the reaction velocity is proportional to the concentration, therefore P is proportional to the reaction velocity at that temperature.

Hence, if reaction velocity at $T_1 = K_1$,
and „ „ $T_2 = K_2$.

We have $$\log \frac{K_2}{K_1} = \frac{Q}{2} \frac{(T_2 - T_1)}{T_1 T_2},$$

or $$\frac{K_2}{K_1} = e^{\frac{Q}{2} \frac{(T_2 - T_1)}{T_1 T_2}}$$ (Van't Hoff-Arrhenius law).

Here Q is the amount of heat evolved in the reaction.
(For numerical examples, see pp. 238 *et seq.*)

Note (1).—Since $\frac{1}{P}$ is the same $\frac{d \log_e P}{dP}$, therefore the equation $\frac{dP}{P} = \frac{QdT}{RT^2}$ (on p. 236) may be written as $\frac{d \log_e P}{dT} = \frac{Q}{RT^2}$.

And as this equation enables us to calculate the influence of temperature upon equilibrium at **constant volume**, it is called by Van't Hoff the **isochore** equation in contradistinction to the mass action equation, which deals with the influence of change of concentration at **constant temperature**, and is therefore called the reaction **isotherm**.

Note (2).—It follows from the equation $\frac{K_2}{K_1} = e^{\frac{Q}{2} \frac{(T_1 - T_2)}{(T_1 T_2)}}$ that when $Q = 0$ then $K_2 = K_1$. This means that when the reaction is "thermo-neutral," (*i.e.*, is not accompanied by any thermal change), then temperature has no influence upon its velocity. Thus, in the reaction

$$\underbrace{CH_3(CH_2)_2COOC_2H_5}_{\text{Ethyl Butyrate.}} \rightleftarrows \underbrace{CH_3(CH_2)_2COOH}_{\text{Butyric Acid.}} + C_2H_5OH,$$

the calorific value of $CH_3(CH_2)_2COOH = 851{\cdot}3$ calories, and those of $CH_3(CH_2)_2COOH$ and C_2H_5OH are 325·7 and 524·4 respectively, giving a total of 850·2 calories. Therefore there is practically no evolution or absorption of heat in this reaction, and, according to theory, temperature should have no influence upon its velocity. Experiment has confirmed theoretical expectation, for it has been found that in similar reactions the velocity at 10° is practically the same as that at 220°.

Note (3).—If the temperature interval is sufficiently small to make the product T_1T_2 fairly constant, then $\frac{Q}{2} \frac{(T_2 - T_1)}{T_1 T_2}$ may be written as $C(T_2 - T_1)$, where $C = \frac{Q}{2T_1T_2}$.

$$\therefore \frac{K_2}{K_1} = e^{C(T_2 - T_1)} \text{ or } \log_{10} \frac{K_2}{K_1} = {\cdot}434\, C(T_2 - T_1)$$
$$= A(T_2 - T_1).$$

$$\therefore \frac{K_2}{K_1} = 10^{A(T_2 - T_1)}.$$

If $T_2 - T_1 = 10$, this equation becomes

$$\frac{K_{t+10}}{K_t} = 10^{10A}.$$

$\frac{K_{t+10}}{K_t}$ is called the **temperature coefficient** (see p. 15), and is generally found to lie between 2 and 3.

The efficiency equation has also been applied to calculate

(3) **The Heat of Solution of a Substance.**—In this case

$$2{\cdot}3 \log_{10} \frac{C_2}{C_1} = \frac{Q}{2} \frac{(T_2 - T_1)}{T_1 T_2},$$

where C_1 and C_2 are the concentrations of the saturated solutions at T_1 and T_2, and Q = heat of solution.

(4) **Heat of Dissociation of Gases,** where a similar equation applies.

EXAMPLES.

(1) Madsen and his co-workers found the influence of temperature upon the velocity of hæmolysis, agglutination and precipitation to be the same as for other chemical reactions, viz.,

$$\frac{K_2}{K_1} = e^{\frac{Q}{2}\frac{(T_2 - T_1)}{T_1 T_2}}.$$

If the addition of 0·085 c.c. of an ammonia solution to 8 c.c. of a 1 per cent. solution of horse's erythrocytes produces hæmolysis in ten minutes at a temperature of 34·8° C., how many c.c. of the same ammonia solution added to the same quantity of erythrocytes will produce the same amount of hæmolysis in the same time at a temperature of 29·7° C. ? [Q in this case = 26,760.]

As at 29·7° the velocity will be less than at 34 8°. ∴ A greater amount of NH_3 solution will be required at 29·7°, to produce the same result in the same time, than at 34·8, and if x be the quantity required, then

$$\frac{K_{34\cdot8}}{K_{29\cdot7}} = \frac{x}{0{\cdot}085}.$$

$$\therefore \quad \frac{x}{0{\cdot}085} = e^{\frac{Q}{2}\frac{(T_{34\cdot8} - T_{29\cdot7})}{T_{34\cdot8} \cdot T_{29\cdot7}}} = e^{13380 \times \frac{5\cdot1}{307\cdot8 \times 302\cdot7}}$$

$$= e^{0\cdot67}.$$

$$\therefore \quad \log x - \log 0{\cdot}085 = {\cdot}67 \times {\cdot}4343 = {\cdot}291.$$

$$\therefore \quad \log x - \bar{2}{\cdot}9294 = {\cdot}291, \text{ or } \log x = \bar{1}{\cdot}2204,$$

whence $\underline{x = 0{\cdot}166 \text{ c.c.}}$

(The observed result was found to be 0·17 c.c.)

Note.—It is easy to evaluate Q in any particular case, for T_1 and T_2 being known, and K_1 and K_2 being ascertainable for any reaction, Q is readily calculated. This is seen from the next example.

(2) Spohr found the velocity of reaction in the case of inversion of cane sugar to be 9·67 at 25° and 139 at 45°. Find what should be the theoretical velocity at 40° C.

Here we first have to find the value of Q from the two data at 25° and 45° and then substitute it in the formula for K at 40°.

Thus

$$\frac{K_{45}}{K_{25}} = e^{\frac{Q}{2} \frac{(20)}{298 \times 318}},$$

i.e.,

$$\frac{139}{9 \cdot 67} = e^{\frac{10Q}{94764}}$$

$$\therefore \quad 2 \cdot 3 \log \frac{139}{9 \cdot 67} = \frac{10Q}{94764}$$

i.e.,

$$2 \cdot 3(2 \cdot 1430 - \cdot 9854) \times \frac{94,764}{10} = Q,$$

i.e.,

$$Q = 2 \cdot 3 \times 1 \cdot 1576 \times 9476 \cdot 4$$
$$= 25,230.$$

$\therefore$ K at 40 is given by the equation

$$\frac{K_{40}}{K_{25}} = e^{\frac{25230}{2} \frac{(T_{40} - T_{25})}{313 \times 298}}$$

or

$$2 \cdot 3 \log K_{40} - 2 \cdot 3 \log 9 \cdot 67 = 12665 \times \frac{15}{93274}$$

or

$$2 \cdot 3 \log K_{40} - 2 \cdot 266 = 2 \cdot 036.$$

$$\therefore \quad \log K_{40} = \frac{4307}{2 \cdot 3} = 1 \cdot 87.$$

$$\therefore \quad K_{40} = 74 \cdot 2.$$

(The observed value of K_{40} was 73·4.)

(3) Barcroft found the influence of temperature upon the dissociation velocity of oxyhæmoglobin to be of the usual character, viz. :

$$\frac{K_{t_2}}{K_{t_1}} = e^{\frac{Q}{2} \frac{(T_2 - T_1)}{T_2 \cdot T_1}}.$$

Calculate the molecular weight of hæmoglobin from the following data :

(1) Percentage of hæmoglobin converted into oxyhæmoglobin at various temperatures when the pressure of oxygen was kept constant at 10 mm. Hg were found to be as follows :

Temperature.	Percentage converted.
16°	92
24°	71
32°	37
38°	18
49°	6

(2) Calorific value of 1 grm. hæmoglobin = 1·85 calories.

(3) The dissociation is a reversible reaction as follows :

$$\text{Oxyhæmoglobin} \rightleftarrows \text{hæmoglobin} + \text{oxygen}.$$

If we call the concentration of oxyhæmoglobin $C_{(HbO)}$, the concentration of hæmoglobin $C_{(Hb)}$, and the concentration of oxygen $C_{(O)}$, then, by the law of mass action (see p. 215),

$$k_1 C_{(HbO)} = k_2 C_{(Hb)} \times C_{(O)}.$$

Now, by Henry's law, the oxygen concentration $C_{(O)}$ is proportional to the pressure of the oxygen, p.

Therefore we can put the equation into the form

$$k_1 \, C_{(HbO)} = k_2 C_{(Hb)} \, . \, p.$$

$$\therefore \quad \frac{k_1}{k_2} = \frac{C_{(Hb)} \cdot p}{C_{(HbO)}},$$

or

$$k = \frac{C_{(Hb)} \cdot p}{C_{(HbO)}}$$

(where k is the dissociation constant).

Hence if the pressure of oxygen (p) is kept constant at any temperature, then k is proportional to $\frac{C_{(Hb)}}{C_{(HbO)}}$.

$\therefore$ k_{16} is proportional to and may be put as equal to $\frac{C_{(Hb)}}{C_{(HbO)}}$ at 16°, viz. $\frac{8}{92}$.

k_{24} ,, ,, ,, ,, 24°, ,, $\frac{29}{71}$.

k_{32} ,, ,, ,, ,, 32°, ,, $\frac{63}{37}$

k_{38} ,, ,, ,, ,, 38°, ,, $\frac{82}{18}$.

k_{49} ,, ,, ,, ,, 49°, ,, $\frac{94}{6}$.

Now,

$$\frac{k_{24}}{k_{16}} = e^{\frac{Q}{2}\left(\frac{T_{24} - T_{16}}{T_{24} \, . \, T_{16}}\right)}.$$

$$\therefore \quad \frac{29}{71} \times \frac{92}{8} = e^{\frac{4Q}{297 \, . \, 289}} = e^{\frac{Q}{21458}}$$

or

$$4{\cdot}7 = e^{\frac{Q}{21458}}.$$

$$\therefore \quad \log 4{\cdot}7 = \frac{Q}{21458} \times {\cdot}4343.$$

$$\therefore \quad 2{\cdot}3 \times {\cdot}6718 \times 21458 = Q = 32{,}450.$$

Similarly $\frac{k_{32}}{k_{24}}$ or $\frac{63}{37} \times \frac{71}{29} = e^{\frac{Q}{22646}}$,

giving

$$Q = 32{,}700.$$

Also $\frac{k_{38}}{k_{32}}$ or $2{\cdot}675 = e^{\frac{Q}{31618}}$,

giving

$$Q = 31{,}120.$$

Lastly $\frac{k_{49}}{k_{38}}$ or $3{\cdot}439 = e^{\frac{Q}{18207}}$,

giving

$$Q = 22{,}500.$$

The mean of these four values of Q

$$= \frac{(32{,}450 + 32{,}700 + 31{,}120 + 22{,}500)}{4} = 29{,}700.$$

Hence amount of heat given off by the union of 1 grm.-molecule of hæmoglobin with oxygen = 29,700 calories. But from (2) the union of 1 grm. of hæmoglobin with oxygen liberates 1·85 calories.

$$\therefore \quad \text{molecular weight of hæmoglobin} = \frac{29{,}700}{1{\cdot}85} = 16{,}500,$$

which is practically identical with the known weight of hæmoglobin.

Note.—Amongst the vital processes which are accelerated by a rise of temperature in accordance with the above law may be mentioned the following: the development of sea-urchin and other eggs (see p. 15); the conduction of impulse along a nerve; the action of drugs upon muscle; the rate of the heart-beat; phagocytosis; the reaction velocity of disinfection; the rhythm of the small intestine; respiration in plants, etc. Thus, in the case of nerve, experiment shows that

$$\frac{\text{Velocity of nerve impulse at } t_n + 10}{\text{Velocity of nerve impulse at } t_n} = 2.$$

Hence it follows that metabolism occurs more rapidly in cases of fever than normally. Similarly in the case of phagocytosis. But in this case it has been shown by Madsen and his collaborators that the temperature of the body at the time the corpuscles have been obtained is the optimum.

CHAPTER XV.

USE OF INTEGRAL CALCULUS IN ANIMAL MECHANICS.

The Relation between the Actual and Potential or Inherent Work of Fan-shaped Muscles.—Fan-shaped muscles are muscles whose one attachment (origin or insertion) is a point, and whose other attachment (insertion or origin) is a line. These muscles are divisible into three groups :

(1) **Circular Muscles,** *i.e.*, those in which the line of attachment is the arc of a circle (*e.g.*, Pectoralis Major, Latissimus Dorsi, etc.).

(2) **Elliptical Muscles,** *i.e.*, those in which the line of attachment is a portion of the circumference of an ellipse (*e.g.*, Temporalis, etc.).

(3) **Triangular Muscles,** *i.e.*, those in which the line of attachment is a straight line (*e.g.*, Trapezius).

The circular muscles are the easiest ones to deal with mathematically, whilst the elliptical ones are the most complicated of all. Here we shall deal with the circular and triangular varieties only. But first, let me explain what is meant by the *actual and potential work* of a muscle. We have seen (p. 47) that in the case of a prismatic muscle, both the total force of the muscular fibres, as well as the direction in which the insertion moves towards the origin, are entirely in the line of the fibres themselves, whilst in the case of muscles like Rhomboid muscles, in which the fibres are not attached to their origin and insertion perpendicularly, the **effective component** of the contractile force of each fibre is less than the actual force with which the fibre contracts, and also the **effective movement** of the insertion is not the same as the actual amount by which the fibres contract. Hence we say that *the actual work* of a muscle is the product of its effective force by the effective shortening. *The potential or inherent work* of a muscle is the amount of work that the muscle is capable of doing under the most favourable arrangement of its fibres (*i.e.*, if its fibres were arranged in a prismatic manner), when both the force of its contraction as well as the direction of shortening is entirely in the direction of its fibres.

(*a*) **Circular Muscle** (Fig. 88).—Let AB = circular arc representing the origin of the muscle, and O = centre of the circle representing the insertion. Let $\angle$ AOB $= 2\theta$, and let OC be the middle fibre bisecting the angle AOB, so that $\angle$ AOC $=$ $\angle$ BOC $= \theta$. Also let L $=$ length of each fibre. Then, since each fibre must contract with the same force,

$\therefore$ resultant of any pair of fibres like OA and OB, or OD, and OE, etc., making equal angles at opposite sides of the bisector OC, will act along OC.

$\therefore$ OC will represent the direction of the resultant of the whole muscle.

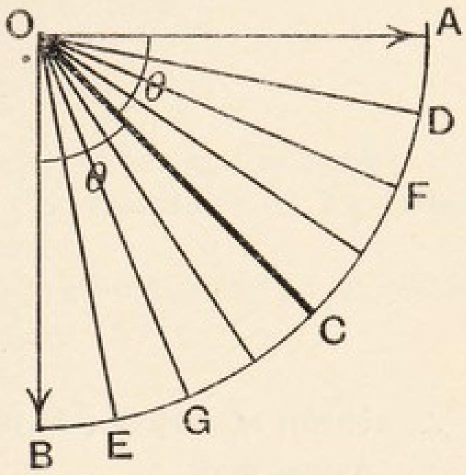

FIG. 88. — Arrangement of Fibres in a Circular Muscle.

Now since the component of the force f of, say, the fibre OA along OC $= f \cos \theta$.

$$\therefore \quad \frac{d\mathrm{R}}{d\theta} = f \cos \theta \text{ (R being the resultant of } all \text{ the fibres).}$$

$$\therefore \quad d\mathrm{R} = f \cos \theta d\theta = \text{force of a single fibre.}$$

$$\therefore \quad \mathrm{R} = \int_{-\theta}^{+\theta} f \cos \theta d\theta = 2f \sin \theta,$$

i.e., the resultant R of the forces of all the fibres, estimated in the direction OC is equal to $2f \sin \theta$ . . . . (i.)

Also, since the length of each fibre is the same $=$ L.

$\therefore$ the amount of shortening of each fibre during contraction is also the same $= l$, (say)

$\therefore$ amount of shortening in direction of resultant $= l$.

$\therefore$ total amount of work done by muscle $= 2fl \sin \theta$.

$\therefore$ *actual work of muscle* $= 2fl \sin \theta$ . . . . (ii.)

But Potential or Inherent work of the muscle is the amount of work that all fibres are capable of performing, and as each fibre acts with a force f, and contracts by an amount l,

$\therefore$ amount of inherent work of each fibre $= fl$.

$\therefore$ amount of *inherent work of whole muscle* $= 2fl\theta$.

$$\therefore \quad \frac{\text{actual work}}{\text{inherent work}} \text{ of a circular muscle } = \frac{2fl \sin \theta}{2fl\theta} = \frac{\sin \theta}{\theta}.$$

Example.—Find the amount of work lost by (1) the *Pectoralis Major*, (2) the *Latissimus Dorsi*, (3) the Iris, as the result of the circular fan-shaped arrangement of their fibres; given that the angle between the extreme fibres of these muscles is:

(1) In the case of the *Pectoralis Major* $= 90°$

(2) In the case of the *Latissimus Dorsi* $= 35°$.

(3) In the case of the Iris = 180° (since the fibres radiate through a semicircle). (See S. Haughton, "Animal Mechanics.")

(1) $\frac{\text{Actual work}}{\text{Inherent work}}$ of *Pectoralis Major*

$$= \frac{\sin 45}{\frac{\pi}{4}} = \frac{\frac{\sqrt{2}}{2}}{\frac{\pi}{4}}$$

$$= \frac{2\sqrt{2}}{\pi}$$

$$= \frac{2{\cdot}82}{3{\cdot}14}$$

$$= 0{\cdot}90.$$

∴ amount of work lost = 10 per cent.

(2) $\frac{\text{Actual work}}{\text{Inherent work}}$ of *Latissimus Dorsi*

$$= \frac{\sin 17^\circ{\cdot}30'}{{\cdot}3054} = \frac{{\cdot}3007}{{\cdot}3054} = 0{\cdot}98.$$

∴ amount of work lost = 2 per cent.

(3) In the case of the Iris, $\theta = 90$.

$$\therefore \quad \frac{\sin\theta}{\theta} = \frac{1}{\pi/2} = \frac{2}{\pi} = {\cdot}64.$$

∴ amount of work lost = 36 per cent.

(*b*) **Triangular Muscle** (Fig. 89).—

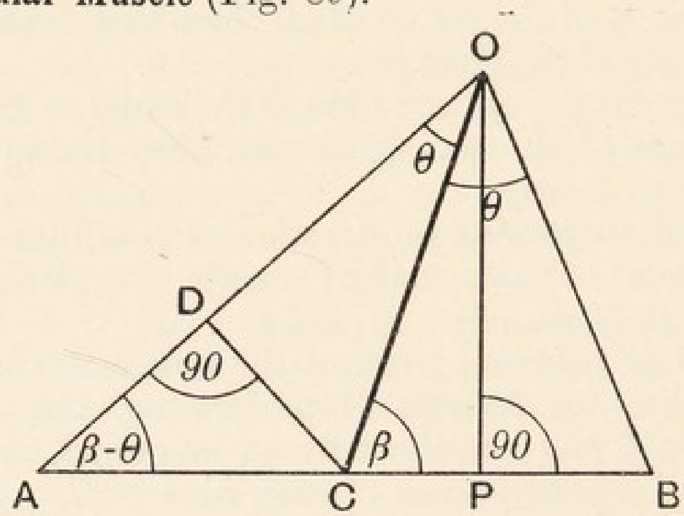

FIG. 89.—Diagram of a Triangular Muscle.

O = Insertion.
AB = Origin.
∠AOB = 2θ.
OC = bisector of angle AOB making an angle = β with the base.
OP = perpendicular to base.
CD = perpendicular from C to side OA.

Each fibre contracts with the same force f independent of the length of the fibre (which, of course, is variable).

Let L = the variable length of a fibre

and l = the amount of shortening of a fibre

$$\left(\frac{l}{L} \text{ being constant} = K\right).$$

Now, since the contractile force of each fibre is the same,

∴ OC represents the direction of the resultant of the whole muscle (as in the case of a circular muscle).

∴ as in the case of a circular muscle, the resultant of the forces of all the fibres estimated in the direction OC is

$$R = \int_{-\theta}^{+\theta} f \cos\theta = 2f\sin\theta.$$

Now let b = amount of shortening of the bisector OC.

∴ actual work done by triangular muscle $= 2fb \sin\theta$.

But $$\frac{b}{OC} = K$$

$$\therefore \quad b = K \,.\, OC.$$

$$\therefore \quad \text{actual work } W = 2fK \,.\, OC \sin\theta.$$

But $$\frac{DC}{OC} = \sin\theta,$$

$$\therefore \quad DC = OC\sin\theta.$$

$$\therefore \quad \textit{actual work } W = 2fK \,.\, DC \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now let us find the *inherent work* of the muscle.

$$\frac{OP}{OC} = \sin\beta.$$

$$\therefore \quad OP = OC\sin\beta.$$

Also, if we take any fibre such as OA, whose length = L, we have

$$\frac{OP}{OA} \text{ or } \frac{OP}{L} = \sin(\beta - \theta).$$

$$\therefore \quad OP = L\sin(\beta - \theta).$$

$$\therefore \quad L\sin(\beta - \theta) = OC\sin\beta.$$

$$\therefore \quad L = \frac{OC\sin\beta}{\sin(\beta - \theta)}.$$

∴ l (*i.e.*, amount of shortening of any fibre) which = KL,

$$= \frac{K \,.\, OC\sin\beta}{\sin(\beta - \theta)}.$$

We therefore have as follows :

Amount of shortening of a fibre $= \dfrac{\text{K . OC} \sin \beta}{\sin (\beta - \theta)}$.

Force of contraction of a fibre $= f$.

$\therefore$ amount of work inherent in a fibre

$$= \frac{f \,.\, \text{K . OC} \sin \beta}{\sin (\beta - \theta)}.$$

$\therefore$ *inherent work* of whole triangular muscle is

$$\text{W}' = \int_{-\theta}^{+\theta} \frac{f \,.\, \text{K . OC} \,.\, \sin \beta}{\sin (\beta - \theta)} \,.\, d\theta$$

$$= f \,.\, \text{K . OC} \,.\, \sin \beta \int_{-\theta}^{+\theta} \frac{d\theta}{\sin (\beta - \theta)}$$

$$= f \,.\, \text{K . OC} \,.\, \sin \beta \log \frac{\cot \frac{1}{2} (\beta - \theta)}{\cot \frac{1}{2} (\beta + \theta)} \text{ (see p. 210).}$$

$$= f \,.\, \text{K . OP} \log \frac{\cot \frac{1}{2} (\beta - \theta)}{\cot \frac{1}{2} (\beta + \theta)}.$$

$\therefore \dfrac{\text{actual work}}{\text{inherent work}}$ of a triangular muscle

$$= \frac{2 f \text{K . DC}}{f \text{K . OP} \log \dfrac{\cot \frac{1}{2} (\beta - \theta)}{\cot \frac{1}{2} (\beta + \theta)}} = \frac{2 \text{ DC}}{\text{OP} \log \dfrac{\cot \frac{1}{2} (\beta - \theta)}{\cot \frac{1}{2} (\beta + \theta)}}.$$

Example.—Find the amount of work lost during the contraction of the two Trapezii muscles, the following measurements being given (Fig. 90).

$$\beta = 83^\circ.$$
$$\theta = 47^\circ \ 30'.$$
$$\text{OP} = 7 \text{ ins.}$$
$$\text{DC} = 5{\cdot}16 \text{ ins.}$$

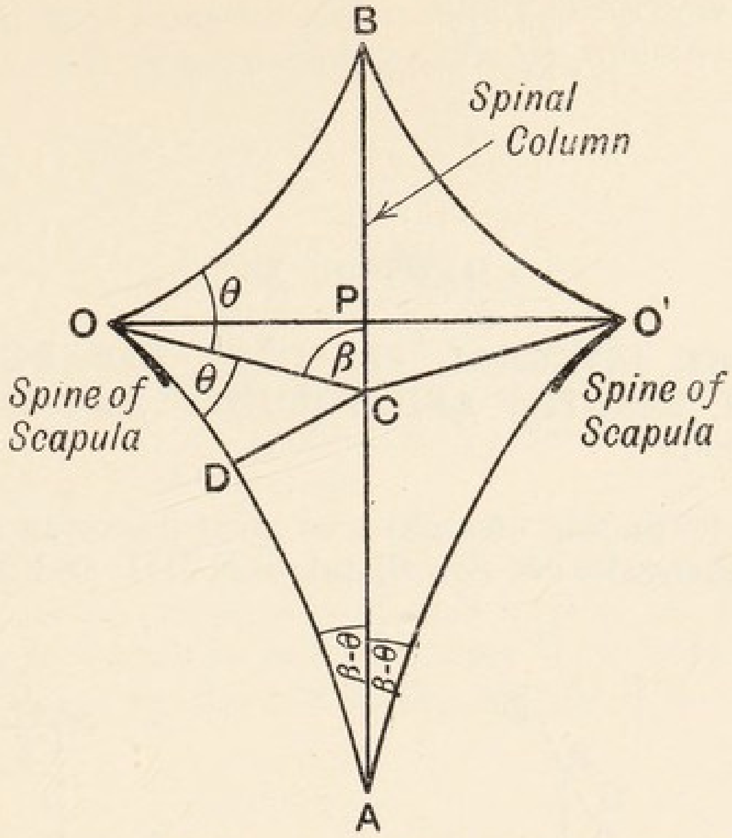

FIG. 90.—Diagram for Calculating the Work done by the Contraction of the two Trapezii Muscles.

$$\frac{W}{W'} = \frac{2\ DC}{OP \log \dfrac{\cot \frac{1}{2}(\beta - \theta)}{\cot \frac{1}{2}(\beta + \theta)}}$$

$$= \frac{2 \times 5{\cdot}16}{7 \log \dfrac{\cot \frac{1}{2}(83 - 47^\circ\ 30')}{\cot \frac{1}{2}(83 + 47^\circ\ 30')}}$$

$$= \frac{10{\cdot}32}{7 \log \left(\dfrac{\cot 17^\circ\ 45'}{\cot 65^\circ\ 15'}\right)} = 0{\cdot}77.$$

∴ each Trapezius muscle loses 23 per cent. of its inherent work, in virtue of the fan-shaped arrangement of its fibres.

CHAPTER XVI.

USE OF THE INTEGRAL CALCULUS FOR DETERMINING AREAS, LENGTHS AND VOLUMES AND MOMENTS OF INERTIA.

Areas.—The finding of areas is of great importance in all practical mathematical work (see Chapters XVIII. and XXI.).

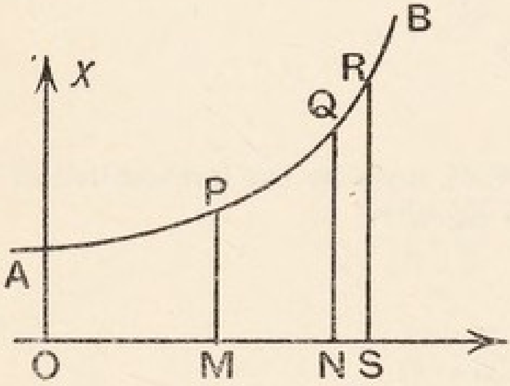

Fig. 91.

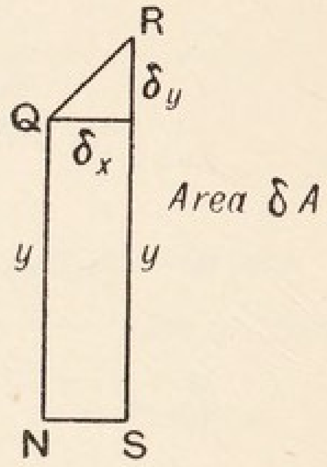

Fig. 92.

Let AB (Fig. 91) be any curve whose equation is $y = f(x)$, and let it be required to find the area (A) of the portion PMNQ.

If we call ON $= x$, then NQ $= y$.

Now take NS $= \delta x$, and draw the ordinate SR,

then SR $= y + \delta y$

and area PMSR $= \text{A} + \delta\text{A}$.

$$\therefore \text{ area QNSR, which } = \text{PMSR} - \text{PMNQ}$$
$$= \text{A} + \delta\text{A} - \text{A}$$
$$= \delta\text{A}.$$

Now if the short distance QR were straight, then, as in the diagram (Fig. 92), we would have

Area $$\delta\text{A} = y \,.\, \delta x + \frac{1}{2}\,\delta x \,.\, \delta y.$$

$$\therefore \qquad \frac{\delta\text{A}}{\delta x} = y + \frac{1}{2}\,\delta y.$$

$\therefore$ as δx and δy get smaller and smaller and become dx and dy, δA becomes dA and we ultimately get (in the limit when dx and $dy = 0$)

$$\frac{dA}{dx} = y.$$

$$\therefore \quad dA = ydx = f(x)dx.$$

$$\therefore \quad \int dA = \int ydx = \int f(x)dx.$$

$\therefore$ $A = F(x)$ where $F(x)$ is the integral of $f(x)dx$. Hence, if we have any curve whose integral is known or can be found, then its area A can be determined.

Whenever we have to find the area of any curve we have always to integrate between limits. Thus, in the above case, the limits are OM, ON, because we are concerned with finding the area, not of the surface under the whole curve, but only that under PQ, which is limited on the left by PM and on the right by QN.

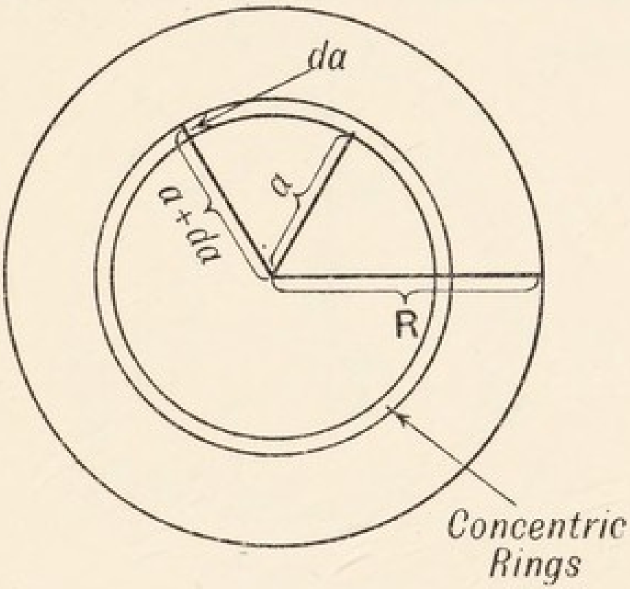

FIG. 93.

Example.—Find the area of a circle of radius r. The whole area may be considered as being made up of a series of concentric rings. Consider one such ring. It consists of two concentric circles whose radii are a and $a + da$. As the circumference of the inner of these two circles $= 2\pi a$, and the circumference of the outer of these two circles $= 2\pi(a + da)$,

$\therefore$ area of ring $= 2\pi(a + da) \cdot da = 2\pi ada$ (since $(da)^2$ is of the second order of magnitude).

$$\therefore \quad \text{area of circle} = 2\pi\int_0^R ada = \pi R^2.$$

Lengths of a Curve.—Supposing it to be required to find the length of the curve APQR in Fig. 91 :

If we refer to Fig. 92 we see that $QR^2 = (\delta x)^2 + (\delta y)^2$.

$$\therefore \quad QR = \sqrt{(\delta x)^2 + (\delta y)^2},$$

i.e., $$\delta l = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \, . \, \delta x \text{ (where } l = \text{length of curve).}$$

But the limit of the chord QR (Fig. 91) is equal to the arc QR (Fig. 90).

$$\therefore \quad dl = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, . \, dx$$

$$\therefore \quad l = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, . \, dx.$$

Hence, if the equation of the curve is known, $\frac{dy}{dx}$ is known and l can be calculated.

Example.—What is the length of the circumference of a circle of radius r ?

Equation of circle is $$x^2 + y^2 = r^2$$

$$\therefore \quad y^2 = r^2 - x^2$$

$$\therefore \quad 2y\frac{dy}{dx} = -2x$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y}.$$

$$\therefore \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{x^2}{y^2}}$$

$$= \frac{1}{y}\sqrt{x^2 + y^2}$$

$$= \frac{r}{y}$$

$$= \frac{r}{\sqrt{r^2 - x^2}}.$$

$$\therefore \quad l = 4\int_0^r \frac{r}{\sqrt{r^2 - x^2}} \, . \, dx = 4r\left[\sin^{-1}\frac{x}{r}\right]_0^r \text{ (see p. 204)}$$

$$= 2\pi r.$$

Determination of Moments of Inertia.—In the study of locomotion, as well as of the physics of bone and in the construction of various physiological recording apparatus, the moment of inertia is a constant of great importance. The integral calculus affords a ready method of evaluating this constant.

Definitions.—(1) The moment of inertia of a **particle** about a given point or line is the product of the mass of that particle by the square of its distance from that point or line, and represents the kinetic energy of the rotating particle.

(2) The moment of inertia of a **body** about a given point or line is the sum of the products of the mass of every particle in that body by the square of its own distance from that point or line, and represents the kinetic energy of rotation of the whole body.

If m be the mass of a unit volume of the substance,
dx be the size of the particle,
x be the distance of the particle from the point or line,
then $m\int x^2 dx = \mathrm{I}$ (where I stands for the moment of inertia).

EXAMPLES.

(1) To find the moment of inertia of a circle about its centre (*i.e.*, about a line passing as an axis through its centre).

Let ABC represent the circle (of radius r) and A′B′C a thin annulus of thickness dx at a distance x from the centre.

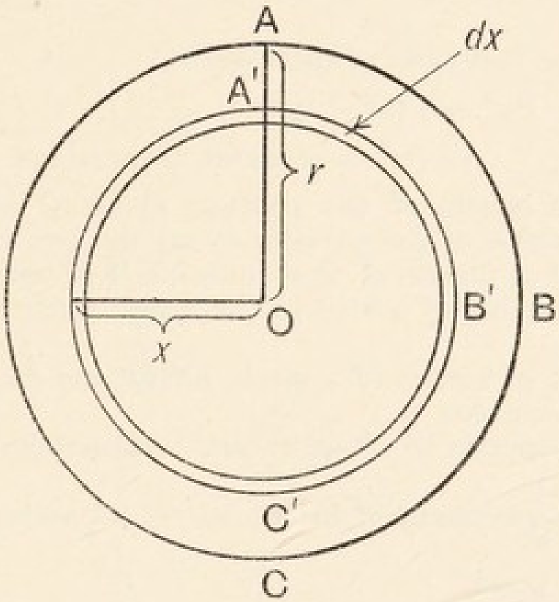

Fig. 94.

Then the moment of inertia of every particle in that annulus about O is mx^2dx ; (m = mass of unit volume of substance).

∴ moment of inertia of entire annulus about O is

$$2\pi x \cdot mx^2dx = 2\pi mx^3dx.$$

∴ moment of inertia of whole circle is given by

$$\mathrm{I} = 2\pi m\int_o^r x^3dx = \frac{\pi mr^4}{2}.$$

(2) Find the moment of inertia of a circle about one of its diameters (*e.g.* of a transverse section of a circular bone).

The moment of inertia of any particle P about the diameter AB is Px^2 (where P, represents its mass).

The moment of inertia of P about the diameter CD is Py^2.

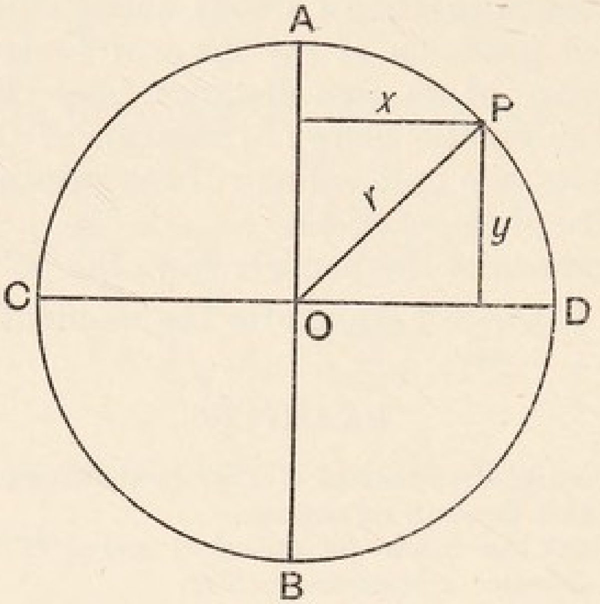

FIG. 95.

$\therefore$ sum of these moments of inertia

$$= Px^2 + Py^2 = P(x^2 + y^2)$$
$$= Pr^2 = \text{moment of inertia of P about O},$$

i.e., the moment of inertia of the particle about O is equal to the sum of the moments of inertia of the particle about the two diameters.

$\therefore$ the moment of inertia of the whole circle about the centre is equal to the sum of the moments of inertia of the whole circle about its diameters at right angles.

But the moment of inertia of a circle about one diameter is equal to that about the other diameter.

$\therefore$ moment of inertia of circle about a diameter

$$= \frac{1}{2} \text{ moment of inertia about its centre}$$

$$= \frac{\pi m r^4}{4}.$$

For an approximate method of finding the moment of inertia of a section of bone by means of squared paper (similar to that mentioned under 2 (*a*) on p. 263 in connection with approximate integration), see J. C. Koch, *Am. J. Anatomy*, 1917.

CHAPTER XVII.

SPECIAL METHODS OF INTEGRATION.

THE integrals so far considered are so called standard or fundamental integrals with which the student is expected to be familiar. When he meets with other expressions which he has to integrate he must by some means bring them into one or other of these standard forms to enable him to perform the integrations. The following are the methods most frequently employed for the purpose.

(1) **The Method of Substitution,** by which a new variable is introduced which enables the integrand to be put in a simple form.

EXAMPLES.

(1) Find the integral of $\cos^3 x dx$.

Put $$\sin x = z.$$

$$\therefore \quad \frac{dz}{dx} = \cos x.$$

$$\therefore \quad dx = \frac{dz}{\cos x}.$$

$$\therefore \quad \int \cos^3 x dx = \int \cos^3 x \frac{dz}{\cos x}$$

$$= \int \cos^2 x dz.$$

But $$\cos^2 x = 1 - \sin^2 x = 1 - z^2.$$

$$\therefore \quad \int \cos^2 x dz = \int (1 - z^2) dz$$

$$= \int dz - \int z^2 dz$$

$$= z - \frac{1}{3} z^3 = \sin x \left(1 - \frac{1}{3} \sin^2 x \right) + C.$$

(2) Find value of $\int \sec^4 x dx$.

Put $\tan x = z.$

$$\therefore \quad \frac{dz}{dx} = \sec^2 x.$$

$$\therefore \quad dx = \frac{dz}{\sec^2 x}.$$

$$\therefore \quad \int \sec^4 x dx = \int \sec^4 x \ \frac{dz}{\sec^2 x}$$

$$= \int \sec^2 x dz.$$

But

$$\sec^2 x = 1 + \tan^2 x$$

$$= 1 + z^2.$$

$$\therefore \quad \int \sec^4 x dx = \int (1 + z^2) dz$$

$$= \int dz + \int z^2 dz$$

$$= z + \frac{1}{3} z^3 + C$$

$$= \tan x + \frac{1}{3} \tan^3 x + C.$$

(2) **The Method of Partial Fractions.**—By this method a complicated algebraic fraction is broken up into a sum of a number of simpler fractions which render themselves very suitable for integration.

Example.—Find the value of $\int \frac{x^3 - 7x + 1}{x^3 - 6x^2 + 11x - 6} dx.$

Splitting the expression up into its partial fractions in the manner described on p. 26, one finds that

$$\frac{x^3 - 7x + 1}{x^3 - 6x^2 + 11x - 6} = \frac{9}{x - 2} - \frac{11}{2(x - 3)} - \frac{5}{2(x - 1)}.$$

$$\therefore \int \frac{x^3 - 7x + 1}{x^3 - 6x^2 + 11x - 6} dx = 9 \int \frac{1}{x - 2} dx - \frac{11}{2} \int \frac{1}{x - 3} dx - \frac{5}{2} \int \frac{1}{x - 1} dx$$

$$= 9 \log (x - 2) - \frac{11}{2} \log (x - 3) - \frac{5}{2} \log (x - 1) + C$$

$$= \log \frac{(x - 2)^9}{\sqrt{(x - 3)^{11} (x - 1)^5}} + C.$$

EXERCISE.

Find the value of $\int \frac{dx}{(x + 1)(x + 2)^2 (x^2 + 1)}.$

[Answer.

$$\frac{1}{2}\log(x+1)+\frac{1}{5}\frac{1}{(x+2)}-\frac{9}{25}\log(x+1)-\frac{7}{5}\log\sqrt{x^2+1}-\frac{1}{50}\tan^{-1}x.\Big]$$

(3) **The Method of Trigonometrical Transformation,** whereby trigonometrical differential expressions are reduced to easily integrable forms.

EXAMPLES.

(1) Find the value of $\int\cos^2\theta d\theta$.

Since $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$.

$$\therefore \quad \cos^2\theta = \frac{\cos 2\theta + 1}{2}.$$

$$\therefore \quad \int\cos^2\theta d\theta = \frac{1}{2}\int(\cos 2\theta + 1)d\theta$$

$$= \frac{1}{2}\int\cos 2\theta d\theta + \frac{1}{2}\int d\theta$$

$$= \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C.$$

(2) An important integral is $\int\sqrt{a^2 - x^2}dx$.

It can easily be integrated by transforming it into the form $\int\cos^2\theta d\theta$, thus causing the square root to disappear, as follows:

Put $x = a\sin\theta$, then $\sqrt{a^2 - x^2}$ becomes $a\cos\theta$ and dx becomes $a\cos\theta d\theta$.

$$\therefore \quad \int\sqrt{a^2 - x^2}dx = a^2\int\cos^2\theta d\theta$$

$$= a^2\left(\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right) + C$$

$$= \frac{a^2}{2}(\sin\theta\cos\theta + \theta) + C$$

$$= \frac{a^2}{2}\left(\frac{x}{a}.\frac{\sqrt{a^2 - x^2}}{a} + \sin^{-1}\frac{x}{a}\right) + C$$

$$= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$$

(4) **Integration by Parts.**—This method is based on the rule for differentiating a product of two functions.

Thus $d(uv) = vdu + udv$.

$$\therefore \quad vdu = d(uv) - udv.$$

$$\therefore \quad \int vdu = \int d(uv) - \int udv.$$

$$\therefore \quad \int vdu = uv - \int udv.$$

Hence, if the integral of vdu is not known, but that of udv is known, then by means of this formula, $\int vdu$ can be found.

As an example, let us take again the one we considered under the last heading, viz., $\int \cos^2\theta dv$.

Put $\quad \cos\theta d\theta = dv$ and $\cos\theta = u$,

then
$$\int \cos^2\theta d\theta = \int udv$$
$$= uv - \int vdu$$
$$= \cos\theta \,.\, \sin\theta + \int \sin^2\theta d\theta \text{ (since } v = \int dv = \int \cos\theta d\theta = \sin\theta \text{ and } du = \sin\theta d\theta)$$
$$= \frac{1}{2}\sin 2\theta + \int (1 - \cos^2\theta)d\theta$$

i.e.,
$$\int \cos^2\theta d\theta = \frac{1}{2}\sin 2\theta + \theta - \int \cos^2\theta d\theta.$$
$$\therefore \quad 2\int \cos^2\theta d\theta = \frac{1}{2}\sin 2\theta + \theta.$$
$$\therefore \quad \int \cos^2\theta d\theta = \frac{1}{4}\sin 2\theta + \frac{\theta}{2} + \text{C}.$$

Example.—Find $\int \log x dx$

Let $\quad u = \log x$, and $dv = dx$.

$$\therefore \quad \int \log x dx = \int udv = uv - \int vdu$$
$$= x\log x - \int x \,.\, \frac{dx}{x}$$
$$= x\log x - x + \text{C}$$
$$= x(\log x - 1) + \text{C}.$$

Multiple Integration.—Just as it is possible to differentiate a function successively, so it is possible to perform successive integration. Thus, if

$$\frac{d^3y}{dx^3} = a, \text{ then } \frac{d^2y}{dx^2} = ax + \text{C}_1.$$

A second integration gives $\quad \dfrac{dy}{dx} = \dfrac{1}{2}ax^2 + \text{C}_1x + \text{C}_2.$

A third and final integration yields $y = \dfrac{1}{6}ax^3 + \dfrac{1}{2}\text{C}_1x^2 + \text{C}_2x + \text{C}_3.$

These operations might have been written as follows :

$$\int \frac{d^3y}{dx^3} . dx = ax + C_1.$$

$$\iint \frac{d^3y}{dx^3} . dx . dx = \frac{1}{2} ax^2 + C_1x + C_2.$$

$$\iiint \frac{d^3y}{dx^3} . dx . dx . dx = \frac{1}{6} ax^3 + \frac{1}{2} C_1x^2 + C_2x + C_3.$$

In this case the successive integrations have been performed with respect to the same independent variable. It is sometimes necessary, however, to perform the task with respect to a different independent variable each time. Thus we might have a multiple integration as follows :

$$\iiint U \, dx \, dy \, dz,$$

where U stands for some function of x, y, z. The expression would then mean that we have to integrate first with respect to one of the variables (say x), then integrate the result with respect to any other of the variables (say y), and finally integrate the second result with respect to the third variable (z).

Thus if $\int U \, dx = A$, then $\iiint U \, dx \, dy \, dz = \iint A \, dy \, dz$,

and if $\int A \, dy = B$, then $\iint A \, dy \, dz = \int B \, dz$.

Each integration can be performed between given limits,

e.g., $\int_o^a \int_o^b \int_o^c U \, dx \, dy \, dz.$

Examples.—(1) Find the value of $\int_{-\infty}^{+\infty} e^{-x^2} dx.$

Let $$I = \int_{-\infty}^{+\infty} e^{-x^2} dx.$$

Then since the limits of x are between $+\infty$ and $-\infty$, it is obvious that if we put y instead of x, the value of the integral between $+\infty$ and $-\infty$ would still be the same.

$$\therefore \quad I = \int_{-\infty}^{+\infty} e^{-y^2} dy.$$

$$\therefore \quad I^2 = \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy = \int_{-\infty}^{+\infty} e^{-x^2} dx . e^{-y^2} dy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy.$$

By changing to polar co-ordinates we get

$$I^2 = \int_{r=o}^{r=\infty}\int_{\theta=o}^{\theta=2\pi} e^{-r^2} r dr d\theta$$

$$= \int_o^\infty e^{-r^2} r dr \int_o^{2\pi} d\theta = \frac{1}{2}\int_o^\infty e^{-r^2} dr^2 \int_o^{2\pi} d\theta$$

$$= \frac{1}{2}\Big[-e^{-r^2}\Big]_o^\infty \Big[\theta\Big]_o^{2\pi}$$

$$= \left(\frac{1}{2}\right)(2\pi) = \pi.$$

$$\therefore \quad I = \sqrt{\pi},$$

i.e., $$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}.$$

This is a **very important integral** in the theory of statistics (see p. 343).

(2) The gravid uterus may be considered as a prolate spheroid, *i.e.*, the solid of revolution generated by the rotation of an ellipse about its major axis. Find its volume and its surface, if its major and minor axes are 12 and 8 ins. respectively. Also find the radius of a sphere having the same volume as the uterus.

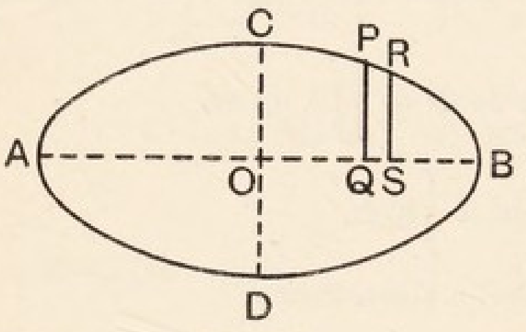

Fig. 96.

If AB and CD are the two axes of the ellipse as well as the co-ordinate axes, and PR, two points on the circumference, then, as the ellipse revolves round AB, P and R will describe circles of radii PQ and RS. Now, if P and R are infinitesimally close to each other, QS will be equal to dx, and PQ will become ultimately equal to RS. Therefore the volume traced out by the revolution of the slice PQSR will be $\pi \,.\, (PQ)^2 \,.\, dx = y^2 dx$.

But $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ [where a and b are the semi-axes (major and minor).]

$$\therefore \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

$$\therefore \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

$$\therefore \quad \text{volume of ellipse which} = 2\int_o^a \pi y^2 dx$$

$$= 2\int_o^a \pi \,.\, \frac{b^2}{a^2}(a^2 - x^2) dx$$

$$= 2\pi \frac{b^2}{a^2} \int_o^a (a^2 - x^2)\, dx$$

$$= 2\pi \frac{b^2}{a^2} \int_o^a a^2 dx - 2\pi \frac{b^2}{a^2} \int_o^a x^2 dx$$

$$= 2\pi b^2 \,.\, a - 2\pi \frac{b^2}{a^2} \cdot \frac{1}{3} a^3$$

$$= 2\pi b^2 a \left(1 - \frac{1}{3}\right)$$

$$= \frac{4}{3} \pi b^2 a.$$

$\therefore$ volume of uterus $= \frac{4}{3} \pi \cdot \left(\frac{8}{2}\right)^2 \cdot \frac{12}{6}$.

$= 402{\cdot}2$ cub. ins.

The surface of the prolate spheroid

$$= 2 \int_o^a 2\pi y dx.$$

$$= 4\pi \int_o^a \frac{b}{a} \sqrt{a^2 - x^2} dx$$

$$\left[\text{since } y^2 = \frac{b^2}{a^2} (a^2 - x^2)\right]$$

$$= 4\pi \frac{b}{a} \int_o^a \sqrt{a^2 - x^2} dx$$

$$= 2\pi ab \left(\frac{\sin^{-1} e}{e} + \sqrt{1 - e^2}\right),$$

$\left(\text{where } e \text{ is the } \textit{eccentricity} \text{ (p. 111)} = \sqrt{1 - \frac{b^2}{a^2}}\right)$

$$= 2\pi \,.\, 6 \,.\, 4 \left(\frac{\sin^{-1} \sqrt{\frac{20}{36}}}{\sqrt{\frac{20}{36}}} + \frac{4}{6}\right)$$

$$= 48\pi \left(\frac{\sin^{-1} \frac{\sqrt{5}}{3}}{\frac{1}{3}\sqrt{5}} + \frac{2}{3}\right)$$

$$= 48\pi \left(\frac{0{\cdot}84 \text{ radians}}{{\cdot}475} + {\cdot}67\right)$$

$= 270{\cdot}7$ sq. ins.

If a and b are equal, then the spheroid becomes a sphere, and its volume

$$\frac{4}{3}\pi ab^2 \text{ becomes } \frac{4}{3}\pi a^3 = \frac{4}{3}\pi r^3 \text{ (where } r = \text{radius).}$$

$$\therefore \quad \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \,.\, 16 \,.\, 6.$$

$$\therefore \quad r^3 = 96.$$

$$\therefore \quad 3 \log r = \log 96 = 1{\cdot}982271.$$

$$\therefore \quad r = 4{\cdot}579.$$

$\therefore$ mean curvature of uterus is that of a sphere whose radius = 4·579 ins.

Approximate Integration.—There are some functions the integral of which cannot be expressed in finite terms. For example, $\int e^{x^2}dx$ cannot be expressed in finite terms because we do not know any function whose differential coefficient is e^{x^2}. Another integral of the same type is $\int e^{-x^2}dx$. Such integrals, however, are frequently met with in practical work. Thus $\int e^{-x^2}dx$ is of great importance in statistical work because the equation of the probability curve—whose area between given limits of x has to be found—is $y = \text{A}e^{-\frac{x^2}{2\sigma^2}}dx$ (see p. 342). In physics also this particular integral is frequently encountered. There are several methods by which such integrals may be calculated.

(1) Method of Converting into an Infinite Series.—

EXAMPLES.

(1) Find the value of $\int e^{-x^2}dx.$

Expanding e^{-x^2} into an infinite series we get

$$e^{-x^2} = 1 - \frac{x^2}{1} + \frac{x^4}{1\,.\,2} - \frac{x^6}{1\,.\,2\,.\,3} + \frac{x^8}{1\,.\,2\,.\,3\,.\,4} - \ldots + \ldots$$

$$\therefore \int e^{-x^2}dx = \int dx - \int x^2dx + \frac{1}{2}\int x^4dx - \frac{1}{6}\int x^6dx + \frac{1}{24}\int x^8dx - \ldots + \ldots$$

$$= x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \frac{1}{216}x^9 - \ldots + \ldots + \text{C}.$$

(2) $$\int_0^{\cdot\frac{1}{2}} e^{-x^2}dx = \frac{1}{2} - \frac{1}{24} + \frac{1}{320} - \frac{1}{5376} + \ldots = {\cdot}545.$$

For the value of the integral when x is equal to ∞ see p. 257.

(3) Find the value of x which makes

$$\frac{1}{\pi}\int e^{-x^2}\,dx = \frac{1}{4}.$$

Here we have to find x from the equation

$$\int e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4} = \cdot 4431.$$

Expanding and integrating as in the first example we get

$$x - \frac{x^3}{3} + \frac{x^5}{10} - \ldots + \ldots = \cdot 4431.$$

Ignoring in this case terms of x higher than x^5, we may put

$$y = x - \frac{x^3}{3} + \frac{x^5}{10} - \cdot 4431.$$

We therefore have to find a value of x which will make $y = 0$ (see p. 117).
Tabulating the values of y corresponding to various values of x, we get the following table:

x	x	$-\frac{x^3}{3}$	$+\frac{x^5}{10}$	$-\cdot 4431$	y
0	0	0	0	− ·4431	− ·4431
·1	·1	− ·0003	+ ·000001	− ·4431	− ·3434
·2	·2	− ·0026	+ ·000032	− ·4431	− ·2457
·3	·3	− ·0090	+ ·00024	− ·4431	− ·1519
·4	·4	− ·0213	+ ·00102	− ·4431	− ·0234
·5	·5	− ·0417	+ ·00313	− ·4431	+ ·0183

Plot the curve which will be found to cut the x axis at the point where x is between 0·4 and 0·5, as can be seen from the table, since when $x = \cdot 4$, y is + ve and when $x = \cdot 5$ y is − ve. Hence x must lie between 0·4 and 0·5.

Now tabulate the corresponding values of y and x for $x = \cdot 41, \cdot 42, \cdot 43, \ldots \cdot 49$, when it will be found that x lies between ·47 and ·48.

We may get a still closer approximation by tabulating for values of $x = \cdot 471, \cdot 472, \ldots$ etc., when x is found to lie between ·476 and ·477.

Tabulating still further we get $x = \cdot 4769$ to four places of decimals.

For exercises on approximate solutions of complicated equations, see p. 262, exercises 2—4.

(4) Find the value of π.

We know that $$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

But $$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}\cdot\frac{x^2}{1} + \frac{3}{4}\cdot\frac{x^4}{2!} + \frac{15}{8}\cdot\frac{x^6}{3!} + \cdots$$

$$\therefore \quad \int \frac{dx}{\sqrt{1-x^2}} = x + \frac{1}{6}x^3 + \frac{3}{20}x^5 + \frac{15}{336}x^7 + \ \ldots + C \ . \ . \ . \quad (B)$$

Equating A and B we get

$$\sin^{-1}x = x + \frac{x^3}{6} + \frac{3x^5}{20} + \frac{5}{112}x^7 + \ldots$$

Now, if $x = \frac{1}{2}$ then $\sin^{-1}x = \frac{\pi}{6}$.

$$\therefore \quad \frac{\pi}{6} = \frac{1}{2} + \frac{1}{6}\cdot\frac{1}{8} + \frac{3}{20}\cdot\frac{1}{32} + \frac{5}{112}\cdot\frac{1}{128} + \ldots + C,$$
$$= \cdot 52342 + C.$$

But since $\sin^{-1}0 = 0$. $\therefore$ C = 0.

$$\therefore \quad \frac{\pi}{6} = \cdot 52342, \text{ whence } \pi = 3\cdot 141 \ldots.$$

EXERCISES.

(1) Prove that $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$$\left[\int \frac{dx}{1+x} = \log(1+x) \text{ and also } = \int (1 - x + x^2 - x^3 + \ldots)\,dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \right]$$

(2) Find a value of x which satisfies the equation

$$x^2 - 5\log_{10} x - 2\cdot 531 = 0.$$

Proceeding as in example 3 above, we get as follows :

x	1·99	2·00	2·01	2·02
y	− 0·065	− 0·036	− 0·07	+ 0·023

Hence x lies between 2·01 and 2·02. By tabulating further (and plotting if desired), x is found to be 2·012.

(3) Solve the equation $x + \log x = 2$. [Answer. $x = 1\cdot 755$.]

(4) In the theory of developmental mechanics the following equation occurs :

$$\frac{\pi}{4} - \theta + \theta\cot^2\theta - \cot\theta = 0. \quad \text{(See “Child Physiology,” p. 101.)}$$

Find the value of θ which will satisfy this equation (notice that in the equation θ is in radians).

Tabulating we get :

θ (in degree)	$\frac{\pi}{4}$	$-\theta$	$-\cot\theta$	$+\theta\cot^2\theta$	$= y$
34° 37′	·7854	− ·6042	− 1·4487	+ 1·2680	= ·0005
34° 38′	·7854	− ·6045	− 1·4478	+ 1·2671	= ·0002
34° 39′	·7854	− ·6048	− 1·4469	+ 1·2661	= ·0002

$\therefore$ θ must lie between 34° 38′ and 34° 39′, *i.e.*, θ = 34° 37′ 30″.

(2) **Graphically.**—Plot the curve, whose integral is to be determined, and find its area (enclosed between two given values of x and the x axis) in one of the following ways:

(*a*) Count the squares on the squared paper on which the area has been traced (Fig. 97).

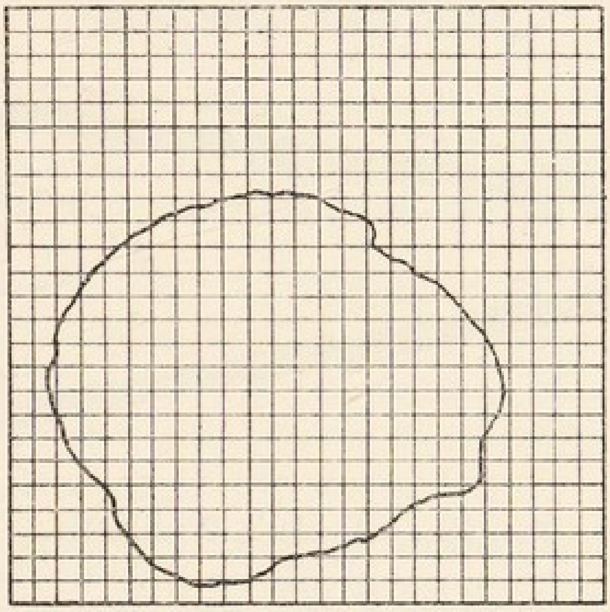

FIG. 97.—Squared Paper Method of Approximate Integration.

(*b*) Cut out the area, weigh it, and compare its weight with that of a known area of the same paper. This is a method adopted for finding the surface area of the human body.

(*c*) By the *Planimeter*, which is a special instrument that rapidly and accurately measures the area of any irregular curve.

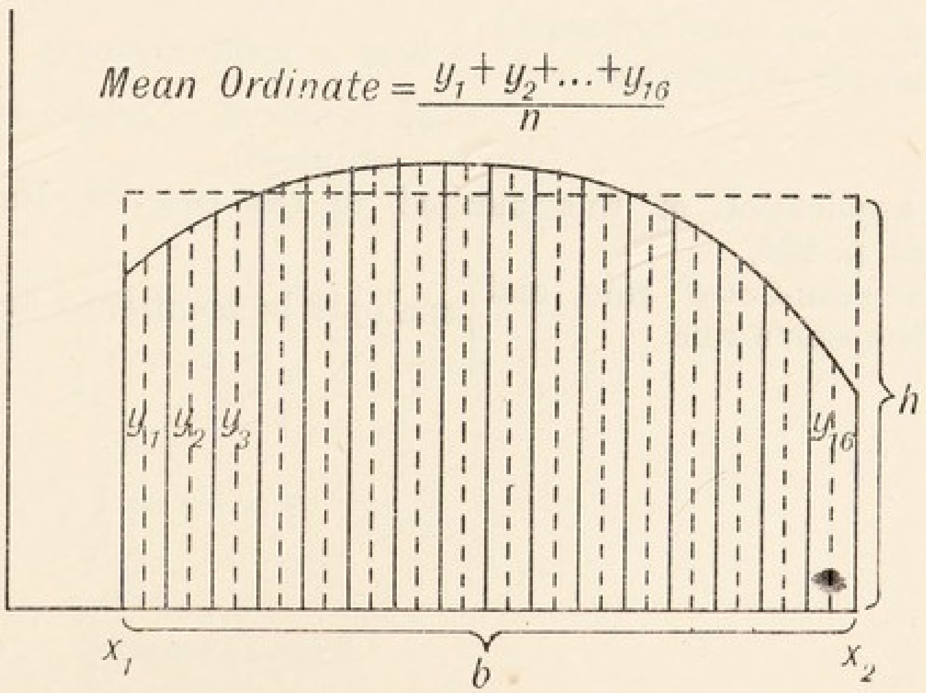

FIG. 98.

The one commonly used is *Amsler's planimeter*. It consists of two arms pivoted together. The end of one arm is fixed down to any point either inside or outside the area, whilst the point fixed to the end of the other arm is made to move round the outline of the curve. A graduated wheel records the area traced out.

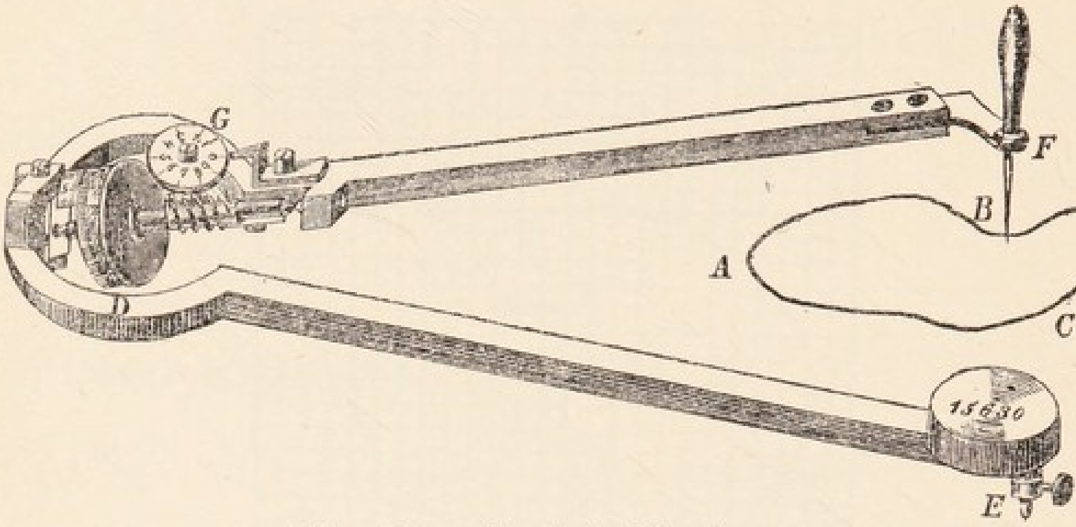

FIG. 98A.—Amsler's Planimeter.

(3) **Mean-Ordinate Method** (Fig. 98).—Since any area can be split up into a number of rectangles, as in the figure, it is obvious that the whole area is equal to the product of the base and the **average height** of all these rectangles. Hence, by dividing the base into a number of equal parts and drawing perpendiculars at the mid-point of each (dotted lines in the figure), the area of the figure is obtained by multiplying the base by

$$\frac{\text{(the sum of all these ordinates)}}{\text{number of ordinates}} = \text{base} \times \text{mean ordinate}.$$

$$= bh \text{ (where } h = \text{height of mean ordinate).}$$

See application of this method in the case of Harmonic Analysis, p. 266.

(4) Newton-Cotes' rule, and
(5) Simpson's rule } can only be mentioned here.

CHAPTER XVIII.

FOURIER'S THEOREM.

We have seen that Maclaurin's theorem enables us to expand any function of x into a series of ascending powers of x. If we have to deal with a *periodic* function, then Fourier's theorem enables us to expand any such function into a series of sines and cosines of multiples of the independent variable.

Thus, while Maclaurin's theorem states that

$$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \dots$$

Fourier's theorem in the case of periodic functions states that

$$f(x) = A_0 + (A_1 \sin x + B_1 \cos x) + (A_2 \sin 2x + B_2 \cos 2x) + (A_3 \sin 3x + B_3 \cos 3x) + \dots .$$

This theorem, apart from its enormous importance in all branches of physics, is of interest to the student of the physiology of hearing. It expresses the fact that every musical sound, such as a violin note, may always be resolved into a number of simple tones corresponding to the fundamental and its partials, and Helmholtz's theory of hearing assumes that the various radial fibres of the basilar membrane and the corresponding arches of Corti will be excited only by those partials of the compound sound to which the fibres are tuned.

To evaluate the constants $A_0, A_1 \dots A_n, B_1, B_2, \dots B_n$. Since, during a complete period, x changes from 0 to 2π, therefore we integrate both sides of the equation between the limits of 0 and 2π.

Thus $$\int_0^{2\pi} f(x)dx = A_0 \int_0^{2\pi} dx + A_1 \int_0^{2\pi} \sin x dx + A_2 \int_0^{2\pi} \sin 2x dx + \dots$$

$$+ B_1 \int_0^{2\pi} \cos x dx + B_2 \int_0^{2\pi} \cos 2x dx + \dots$$

Now $$\int_0^{2\pi} \sin nx dx = \left[-\frac{1}{n} \cos nx \right]_0^{2\pi} = 0.$$

$$\therefore \quad \int_0^{2\pi} \sin x dx = 0$$

$$\int_0^{2\pi} \sin 2x dx = 0;$$

etc.

Similarly $\int_0^{2\pi} \cos nx dx = \left[\frac{1}{n}\sin nx\right]_0^{2\pi} = 0.$

$$\therefore \quad \int_0^{2\pi} \cos x dx = 0.$$

$$\int_0^{2\pi} \cos 2x dx = 0;$$

etc.

$$\therefore \quad \int_0^{2\pi} f(x)dx = A_0 \int_0^{2\pi} dx = 2\pi A_0.$$

$$\therefore \quad A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\int_0^{2\pi} y dx.$$

$\therefore$ when $\int_0^{2\pi} f(x)dx$ is known, A_0 is known.

To find the values of A_n and B_n, multiply throughout by $\sin nx$ and $\cos nx$ respectively and integrate.

$$\text{Thus} \int_0^{2\pi} f(x)\sin nx dx = A_0\int_0^{2\pi} \sin nx dx + A_1\int_0^{2\pi} \sin nx \sin x dx$$

$$+ \ldots + A_n\int_0^{2\pi} \sin^2 nx dx + \ldots$$

$$+ B_1 \sin nx \cos x dx + B_2\int_0^{2\pi} \sin nx \cos 2x dx$$

$$+ \ldots + B_n\int_0^{2\pi} \sin nx \cos nx dx + \ldots$$

But $\int_0^{2\pi} \sin nx = 0$ (see p. 265).

$$\int_0^{2\pi} \sin nx \sin x dx \equiv \frac{1}{2}\int_0^{2\pi} \cos\frac{(n-1)}{2} x dx - \frac{1}{2}\int_0^{2\pi} \cos\frac{(n+1)}{2} x dx = 0$$

(see p. 53).

Similarly for all the other terms except $\int_0^{2\pi} \sin{}^2 nx dx,$

which $= \frac{1}{2}\int_0^{2\pi}(1 - \cos 2nx)dx = \frac{1}{2}\int_0^{2\pi}dx - \frac{1}{2}\int_0^{2\pi}\cos 2nxdx = \pi - 0 = \pi.$

$$\therefore \quad \int_0^{2\pi} f(x) \sin nxdx = A_n\pi.$$

$$\therefore \quad A_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nxdx.$$

Similarly $$B_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nxdx.$$

Hence $$A_1 = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin xdx\text{ ; } A_2 = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin 2xdx\text{ ; etc.}$$

$$B_1 = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos xdx\text{ ; } B_2 = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos 2xdx\text{ ; etc.}$$

That $\int_0^{2\pi}$ sin $nxdx$ and $\int_0^{2\pi}$ cos $nxdx$ is each equal to zero is easily seen by looking at the sine and cosine curves (Figs. 50, 51). For each period the area of the portion below the abscissæ is equal and negative to that above.

We shall now make use of these formulæ for the purpose of **Harmonic Analysis, i.e., the analysis of a periodic curve into its component curves.**

Suppose we are given a curve (Fig. 99) representing some harmonic vibration—say, a sound wave—and we wish to find its equation or its component waves, viz., the fundamental and the partials. (Example after W. N. Rose, "Mathematics for Engineers"—D.N. series.) To simplify matters, suppose we are told that the wave consists only of the fundamental tone and its first harmonic.

If we call the fundamental x and the 1st harmonic $2x$ then the equation of the curve will be

$$y = f(x) = A_0 + A_1 \sin x + A_2 \sin 2x + B_1 \cos x + B_2 \cos x.$$

The problem is to evaluate the constants A_0, A_1, A_2, and B_1, B_2.

The base of the whole curve is the period $= 2\pi = 360°$. Divide the base into ten equal parts representing the angles 36°, 72°, 108°, etc., as shown in the figure, and erect the mid-ordinates y_1, y_2, y_3 . . . y_{10} at the points corresponding to the angles 18°, 54°, 90°, etc. (Compare p. 264.) Measure these mid-ordinates and tabulate their values as follows:

$y_1 = 1{\cdot}56$	$y_6 = -1{\cdot}13$
$y_2 = 3{\cdot}75$	$y_7 = -2{\cdot}91$
$y_3 = 4$	$y_8 = -4$
$y_4 = 2{\cdot}91$	$y_9 = -3{\cdot}75$
$y_5 = 1{\cdot}13$	$y_{10} = -1{\cdot}56$

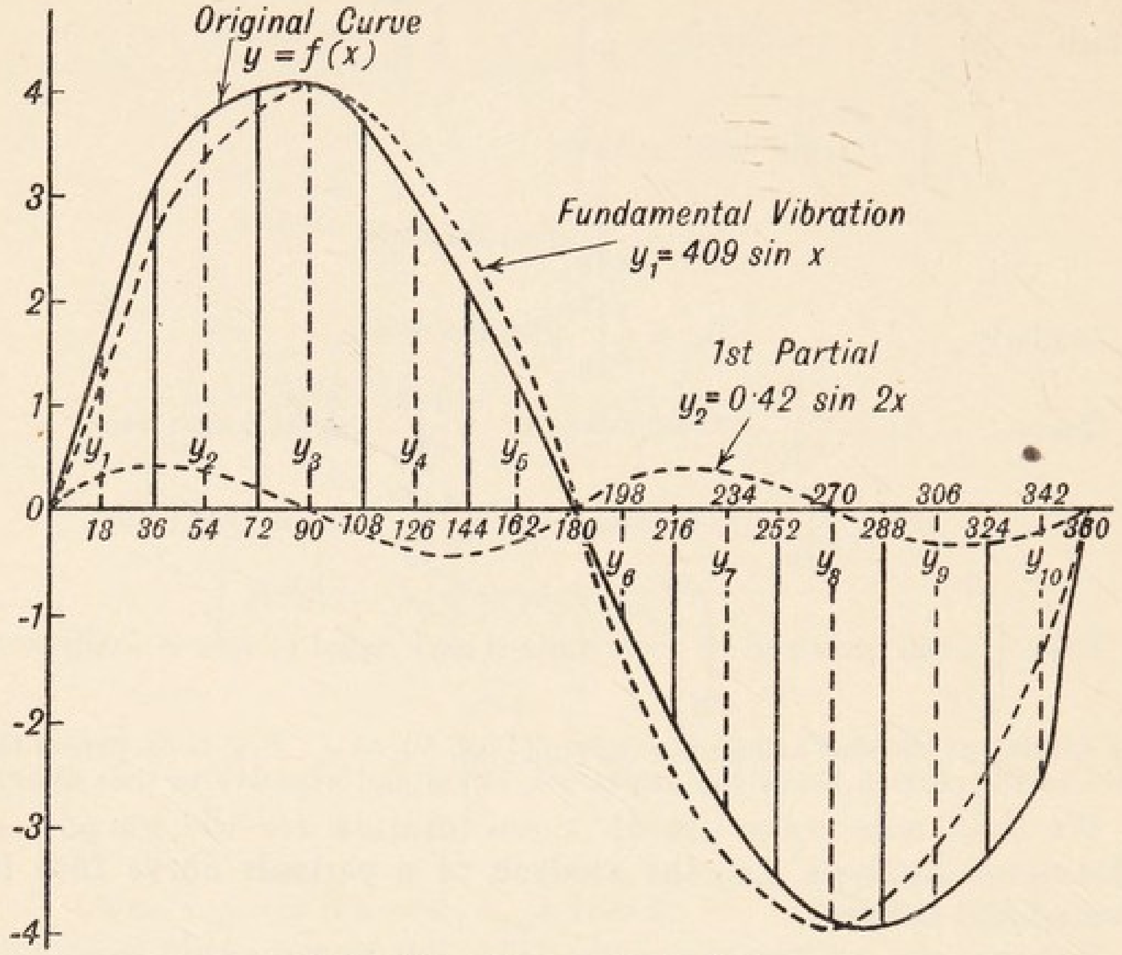

FIG. 99.—Harmonic Analysis.

Then since the sum of these ordinates $= 0$,

$\therefore \quad \int_0^{2\pi} y dx$ (which, of course, is equivalent to $(y_1 + y_2 + \ldots + y_{10}) = 0$. (See mean ordinate method of approximate integration, p. 264.)

$$\therefore \quad A_0 \text{ which} = \frac{1}{2\pi}\int_0^{2\pi} y dx = 0.$$

$\therefore$ the equation of the curve is

$$y = A_1 \sin x + A_2 \sin 2x + B_1 \cos x + B_2 \cos 2x.$$

$$\text{Now, } A_1 = \frac{1}{\pi}\int_0^{2\pi} y \sin x dx$$

which is approximately the same as

$$\frac{1}{\pi}\left[y_1 \sin 18° + y_2 \sin 54° + y_3 \sin 90° + \ . \ . \ . + y_{10} \sin 342°\right]$$

$$= \frac{1}{\pi}\Big[1{\cdot}56 \times {\cdot}309 + 3{\cdot}75 \times {\cdot}809 + 4 \times 1 + 2{\cdot}91 \times {\cdot}809 + 1{\cdot}13 \times {\cdot}309$$
$$+ (-1{\cdot}13)(-{\cdot}309) + -2{\cdot}91(-{\cdot}809) + (-4)(-1)$$
$$+ (-3{\cdot}75)(-{\cdot}809) + (-1{\cdot}56)(-{\cdot}309).\Big]$$

(See table below.)

$$= \frac{1}{\pi}[{\cdot}309\,(1{\cdot}56 + 1{\cdot}13 + 1{\cdot}13 + 1{\cdot}56) + 4\,(1 + 1) + {\cdot}809\,(3{\cdot}75 + 2{\cdot}91 + 2{\cdot}91 + 3{\cdot}75)]$$

$$= \frac{1}{\pi}[{\cdot}309\,(3{\cdot}12 + 2{\cdot}26) + 8 + {\cdot}809\,(7{\cdot}5 + 5{\cdot}82)]$$

$$= \frac{1}{\pi}({\cdot}309 \times 5{\cdot}38 + 8 + {\cdot}809 \times 13{\cdot}32) = \frac{1}{\pi}(1{\cdot}662 + 8 + 10{\cdot}776)$$

$$= \frac{1}{\pi} \times 20{\cdot}44.$$

But since 10 parts of the base $= 2\pi$.

$\therefore \quad \pi = 5$ parts.

$$\therefore \quad \mathbf{A_1 = \frac{1}{5} \times 20{\cdot}44 = 4{\cdot}09.}$$

Similarly, $A_2 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} y \sin 2x dx.$

$$= \frac{1}{5}[y_1 \sin 36° + y_2 \sin 72° + y_3 \sin 108° + \ . \ . \ . + \ . \ . \ . \ y_{10} \sin 684°]$$

$$= \frac{1}{5}[1{\cdot}56 \times {\cdot}588 + 3{\cdot}75 \times {\cdot}951 + 4 \times 0$$
$$+ 2{\cdot}91(-{\cdot}951) + 1{\cdot}13(-{\cdot}588)$$
$$+ (-1{\cdot}13) \times {\cdot}588 + (-2{\cdot}91) \times {\cdot}951$$
$$+ (-4) \times 0 + (-3{\cdot}75)(-{\cdot}951)$$
$$+ (-1{\cdot}56)(-{\cdot}588)]$$

$$= \frac{1}{5}[{\cdot}588\,(1{\cdot}56 - 1{\cdot}13 - 1{\cdot}13 + 1{\cdot}56) + {\cdot}951\,(3{\cdot}75 + 2{\cdot}91 - 2{\cdot}91 + 3{\cdot}75)]$$

$$= \frac{1}{5}({\cdot}588 \times {\cdot}86 + {\cdot}951 \times 1{\cdot}68).$$

$$\therefore \quad \mathbf{A_2 = \frac{1}{5} \times 2{\cdot}104 = {\cdot}421.}$$

Similarly, $B_1 = \frac{1}{\pi}\int_0^{2\pi} y \cos x dx$

$= \frac{1}{5}[\cdot 951\,(1\cdot 56 - 1\cdot 13 + 1\cdot 13 - 1\cdot 56)$
$\quad + \cdot 588\,(3\cdot 75 - 2\cdot 91 + 2\cdot 91 - 3\cdot 75)]$

$= 0$

and $\quad B_2 = \frac{1}{5}[\cdot 809\,(1\cdot 56 + 1\cdot 13 - 1\cdot 13 - 1\cdot 56)$
$\quad + 1\,(-4 + 4) + \cdot 309\,(-3\cdot 75 - 2\cdot 91$
$\quad + 2\cdot 91 - 3)75)]$

$= 0.$

∴ the equation contains no *cos* terms.

Hence we get the final equation

$$y = \mathbf{4{\cdot}09 \sin x + {\cdot}42 \sin 2x.}$$

In other words, the components of the function

are $\quad y_1 = 4\cdot 09 \sin x$
and $\quad y_2 = \cdot 42 \sin 2x.$

The original curve, together with its two components, is shown in Fig. 99. To facilitate the work the values are tabulated as follows:

y	x	Sin x	Cos x	Sin $2x$	Cos $2x$
$y_1 = 1\cdot 56$	18°	·309	·951	·588	·809
$y_2 = 3\cdot 75$	54°	·809	·588	·951	− ·309
$y_3 = 4$	90°	1	0	0	− 1
$y_4 = 2\cdot 91$	126°	·809	·588	− ·951	− ·309
$y_5 = 1\cdot 13$	162°	·309	− ·951	− ·588	·809
$y_6 = -1\cdot 13$	198°	− ·309	− ·951	·588	·809
$y_7 = -2\cdot 91$	234°	− ·809	− ·588	·951	− ·309
$y_8 = -4$	270°	− 1	0	0	− 1
$y_9 = -3\cdot 75$	305°	− ·809	·588	− ·951	− ·309
$y_{10} = -1\cdot 56$	342°	− ·309	·951	− ·588	·809

EXAMPLES.

Analyse the curve resulting from the following plotting table :

$x°$	0	45°	90°	135°	180°	225°	270°	315°	360°
y	0	21·5	31·25	11·25	0	9	30	26·5	0

assuming it to consist of the fundamental and the first two harmonics.

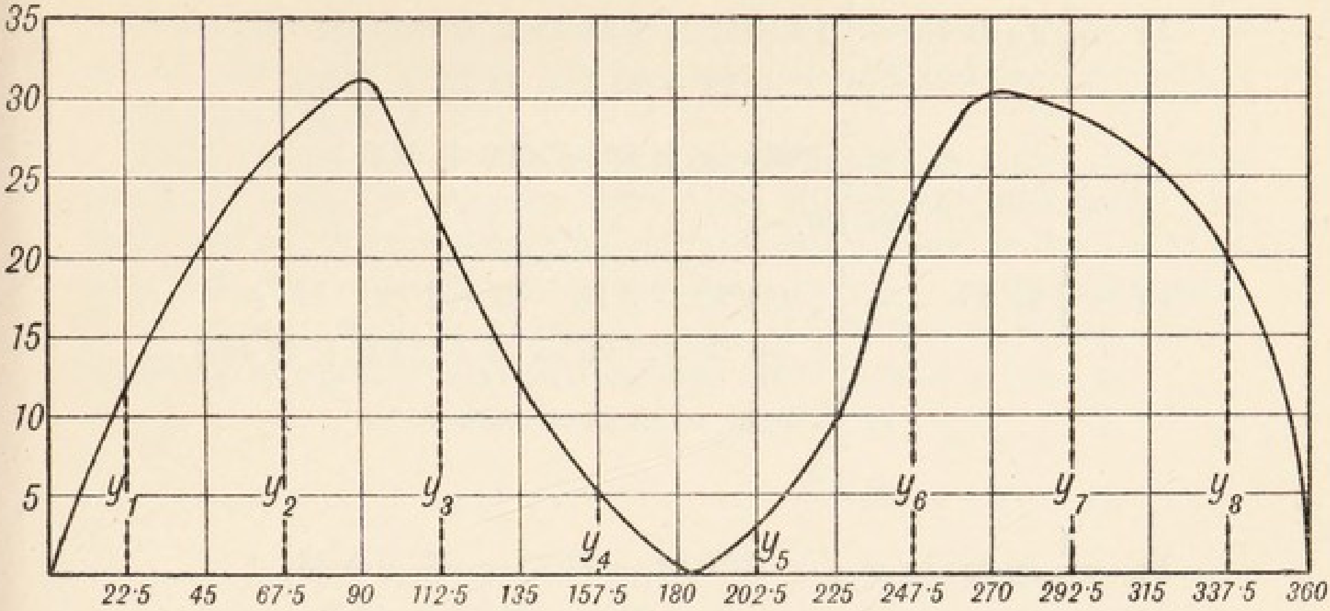

FIG. 100.—Another Example of Harmonic Analysis.

The curve is shown in Fig. 100. If we draw the mid-ordinates and measure them we can tabulate the values as follows:

y	x	Sin x	Cos x	Sin $2x$	Cos $2x$	Sin $3x$	Cos $3x$
$y_1=12$	22·5	·3827	·9239	·7071	·7071	·9239	—·3827
$y_2=27·75$	67·5	·9239	·3827	·7071	—·7071	—·3827	—·9239
$y_3=23$	112·5	·9239	—·3827	—·7071	—·7071	—·3827	·9239
$y_4=5·25$	157·5	·3827	—·9239	—·7071	·7071	·9239	—·3827
$y_5=3$	202·5	—·3827	—·9239	·7071	·7071	—·9239	—·3827
$y_6=23·5$	247·5	—·9239	— 3827	·7071	—·7071	·3827	·9239
$y_7=28·75$	292·5	—·9239	·38827	—·7071	—·7071	·3827	—·9239
$y_8=19·5$	337·5	—·3827	·9239	—·7071	·7071	—·3827	·9239

$$\therefore \quad \text{value of mean ordinate} = \frac{\Sigma y}{8} = 17·4,$$

i.e.,

$$A_0 = 17·4$$

$$A_1 = \frac{1}{4}\Sigma y \sin x = \frac{1}{4}[·3827(12 + 5·25 - 3 - 19·5) + ·9239(27·75 + 23 - 23·5 - 28·75)]$$

$$= \frac{1}{4}[·383(-5·25) + ·924(-1·50)]$$

$$= -·85.$$

$$A_2 = \frac{1}{4}\Sigma y \sin 2x = \frac{1}{4}[·71(12 + 27·75 + 3 + 23·5 - 23 - 5·25 - 28·75 - 19·5)]$$

$$= \frac{·71}{4}(66·25 - 76·5) = -1·8.$$

$$A_3 = \frac{1}{4}\Sigma y \sin 3x = \frac{1}{4}[\cdot 924(12 + 5\cdot 25 - 3) - \cdot 383\,(27\cdot 75 + 23 - 23\cdot 5 - 28\cdot 75 + 19\cdot 5)]$$

$$= \frac{1}{4}[\cdot 924 \times 14\cdot 25 - \cdot 383 \times 18]$$

$$= 1\cdot 6.$$

$$B_1 = \frac{1}{4}\Sigma y \cos x = \frac{1}{4}[\cdot 924(12 - 5\cdot 25 - 3 + 19\cdot 5) + \cdot 383(27\cdot 75 - 23 - 23\cdot 5 + 28\cdot 75)]$$

$$= \frac{1}{4}(\cdot 1924 \times 23\cdot 25 + \cdot 383 \times 10)$$

$$= 6\cdot 3.$$

$$B_2 = \frac{1}{4}\Sigma y \cos 2x = \frac{1}{4} \times \cdot 707(12 - 27\cdot 75 - 23 + 5\cdot 25 + 3 - 23\cdot 5 - 28\cdot 75 + 19\cdot 5)$$

$$= -\,11\cdot 2.$$

$$B_3 = \frac{1}{4}\Sigma y \cos 3x = \frac{1}{4}[\cdot 383(12 - 5\cdot 25 - 3) - \cdot 924\,(27\cdot 75 - 23 - 23\cdot 5 + 28\cdot 75 - 19\cdot 5)]$$

$$= \frac{1}{4}[\cdot 383 \times 3\cdot 75 = 9\cdot 24 \times (-\,9\cdot 5)]$$

$$= 2\cdot 5.$$

Hence, the equation of the curve is

$$y = 17\cdot 4 - \cdot 85 \sin x - 1\cdot 8 \sin 2x + 1\cdot 6 \sin 3x + 6\cdot 3 \cos x - 11\cdot 2 \cos 2x + 2\cdot 5 \cos 3x.$$

Note.—The number of ordinates given is not sufficient to ensure any accurate results.

Composition of Harmonic Motions.—The reverse process of finding the resultant curve when its components are known is very easy. Thus, supposing we were given that the two components are

$$y_1 = 4\cdot 09 \sin x,\ y_2 = \cdot 425 \sin 2x,$$

we tabulate as follows :

x	0	18°	54°	90°	126°	162°	198°	234°	270°	306°	342°
Sin x	0	·309	·809	1	·809	−·309	·309	−·809	−1	−·809	−·309
Sin $2x$	0	·588	·951	0	−·951	−·588	·588	·951	0	−·951	−·588
y_1	0	1·264	3·309	4·09	3·309	1·264	−1·264	−3·309	−4·09	−3·309	−1·264
y_2	0	·247	·399	0	−·399	−·247	·247	·399	0	−·399	−·247
$y = y_1 + y_2$	0	1·511	3·708	4·09	2·910	1·017	−1·017	−2·910	−4·09	−3·708	−1·511

If we plot the three curves from the values of y_1, y_2 and y, we get on the same diagram the two components and their resultant curve (see Fig. 99).

If the actual graphs of the two components are given, then the resultant curve is quickly drawn by adding (**algebraically**) the ordinates at various points on the abscissa. Thus, at the abscissal point 54°, the ordinate of the resultant curve = **arithmetical sum** of y_1 and y_2 and at abscissal point 126°, the ordinate of the resultant is equal to the **arithmetical difference** of y_1 and y_2.

CHAPTER XIX.

DIFFERENTIAL EQUATIONS.

Definitions.—(1) A differential equation is an equation which connects the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, etc., with the variables x and y themselves.

E.g.,

$$\frac{dy}{dx} = \text{K}$$

$$\frac{dy}{dx} + ay - b = 0$$

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 9y = 0$$

$$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0.$$

Differential equations are of great importance in all scientific work, since not only can many of the so-called "laws" be expressed in their most general form by such equations, but they continually occur as the result of the mathematical analysis of various phenomena.

(2) By the *solution* of a differential equation is meant the obtaining from such an equation another called the primitive, which contains the variables alone without the differentials. Thus, as we shall see presently, the solution of

$$b\frac{dy}{dx}, + ay = 0,$$

is $$y = \text{A}e^{-\frac{a}{b}x}, \text{ where A is a constant.}$$

(3) **Order and Degree of a Differential Equation.**—The *order* of a differential equation is determined by the order of the highest differential coefficient occurring in it. Thus, an equation containing $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$ or $\frac{d^ny}{dx^n}$, is called an equation of the 1st, 2nd, 3rd . . . or nth order. The *degree* of a differential equation is

determined by the degree or highest power of the differential coefficient occurring in it. Thus, an equation containing $\frac{dy}{dx}$, $\left(\frac{dy}{dx}\right)^2$, $\left(\frac{dy}{dx}\right)^3$ . . . or $\left(\frac{dy}{dx}\right)^n$ is called an equation of the 1st, 2nd, 3rd . . . or nth degree.

Examples.

$$3\frac{d^2y}{dx^2} + 25\frac{dy}{dx} - 18y = 0$$

is an equation of the second order and first degree, *i.e.*, a linear equation of the second order; but

$$7\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$

is an equation of the first order and second degree.

(4) **Homogeneous and Non-homogeneous Equations.**—A differential equation is homogeneous or non-homogeneous, according as the sum of the exponents of the variables is or is not the same in each term of the equation.

Thus
$$(ax + by)dx + (a'x + b'y)dy = 0$$
$$x^2 + 2xy\frac{dy}{dx} - y^2 = 0$$
are homogeneous.

But
$$(ax + by + c)dx + (a'x + b'y + c')dy = 0$$
$$(x + y^2)dx + 2xydy = 0$$
are non-homogeneous.

(5) **Exact and Non-exact Equations.**—A differential equation is exact or inexact according as it is presented, exactly as derived by the differentiation of a function of x and y, or it has subsequently been modified by cancelling out some common factor consisting of some function of x and y.

Thus $(x + y^2)dx + 2xydy = 0$ is an exact differential equation because it has been derived directly from the differentiation of $\frac{1}{2}x^2 + xy^2 = c$.

On the other hand, $ydx - xdy = 0$ is a non-exact equation because after differentiation of the primitive $\left(\text{viz.}, \frac{x}{y} = c\right)$ yielding $\frac{1}{y^2}(ydx - xdy) = 0$, the factor $\frac{1}{y^2}$ has been cancelled out. When this factor is restored the equation $\frac{1}{y^2}(ydx - xdy) = 0$ becomes exact.

Integrating Factor.—While every exact equation can be solved directly, a non-exact equation can only be solved after it has been made exact by restoring the factor which has been cancelled out. Such a factor is, therefore, called an *integrating factor* and is generally denoted by the letter μ.

The great difficulty of solving non-exact equations consists in finding the appropriate integrating factor,—although it is to be noted that not only has every non-exact equation **some** integrating factor, but it has an infinite number of such factors.

Thus, let $ydx - xdy = 0$.

We have seen that $\frac{1}{y^2}$ is an integrating factor yielding the primitive $\frac{x}{y} = c$.

Similarly $\frac{1}{x^2}$ is an integrating factor yielding the primitive $\frac{y}{x} = c_1$.

Again, $\frac{1}{xy}$ is also an integrating factor because it converts the equation into $\frac{dx}{x} - \frac{dy}{y} = 0$, whose solution is $\log \frac{x}{y} = c_2$.

Euler's Criterion of Integrability.—If the equation be expressed in the most general form $\mathrm{M}dx + \mathrm{N}dy = 0$, where M and N are functions of x and y, then the equation is exact if $\frac{\delta \mathrm{M}}{\delta y} = \frac{\delta \mathrm{N}}{\delta x}$.

Thus $(x + y^2)dx + 2xydy = 0$ is exact, because

$$\frac{\delta(x + y^2)}{\delta y} = 2y, \text{ and } \frac{\delta(2xy)}{\delta x} = 2y,$$

so that $$\frac{\delta \mathrm{M}}{\delta y} = \frac{\delta \mathrm{N}}{\delta x}.$$

On the other hand, $ydx - xdy = 0$ is not exact, because

$$\frac{\delta y}{\delta y} = 1, \text{ and } \frac{\delta(- x)}{\delta x} = -1,$$

so that $$\frac{\delta \mathrm{M}}{\delta y} \text{ is not equal to } \frac{\delta \mathrm{N}}{\delta x}.$$

By restoring any one of the integrating factors, however, the equation becomes exact.

Thus, putting $\mu = \frac{1}{y^2}$, equation becomes $\frac{1}{y} dx - \frac{x}{y^2} dy = 0$, and

$$\frac{\delta M}{\delta y} = -\frac{1}{y^2}; \frac{\delta\left(-\frac{x}{y^2}\right)}{\delta x} = -\frac{1}{y^2}, \therefore \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}.$$

Putting $\mu = \frac{1}{x^2}$, equation becomes $\frac{ydx}{x^2} - \frac{1}{x} dy = 0$, and both

$$\frac{\delta M}{y} \text{ and } \frac{\delta N}{\delta x} = -\frac{1}{x^2}.$$

Similarly, $\mu = \frac{1}{xy}$ makes the equation $\frac{dx}{x} - \frac{dy}{y}$, and both $\frac{\delta M}{\delta y}$ and $\frac{\delta N}{\delta x}$ are equal to 0.

For the methods of finding integrating factors the student is referred to the standard text-books. In this chapter we can only deal with a few of the simpler types of differential equations.

Solution of Differential Equations.

(i.) **Separation of Variables.**—The first and most essential point is to "separate the variables" so as to group all the x's with the (dx)'s and all the y's with the (dy)'s. When the resulting equation is integrated the solution is obtained giving an equation containing the variables alone without the differentials.

EXAMPLES.

(1) $ydx + xdy = 0$.

We separate the variables as follows :

Divide throughout by xy and get

$$\frac{dx}{x} + \frac{dy}{y} = 0.$$

$$\therefore \quad \int \frac{dx}{x} + \int \frac{dy}{y} = C,$$

i.e., $\log x + \log y = \log A$ (where $\log A = C$; the constant is put in the logarithmic form to make it uniform with $\log x$ and $\log y$),

or $$\log xy = \log A.$$

$$\therefore \quad xy = A \text{ is the solution.}$$

(2) Similarly the solution of $ydx - xdy = 0$

is $$\frac{x}{y} = A.$$

(3) Solve the equation

$$\frac{dy}{dx} = \sqrt{1 - y^2}.$$

Separating the variables we get :

$$\frac{dy}{\sqrt{1-y^2}} = dx.$$

$$\therefore \quad \int \frac{dy}{\sqrt{1-y^2}} = \int dx + C,$$

or $$\sin^{-1} y = x + C \text{ (see p. 204).}$$

$$\therefore \quad y = \sin(x + C).$$

(4) Solve $$ay + b\frac{dy}{dx} = 0.$$

Separation of variables gives

$$\frac{b}{a}\frac{dy}{y} = -\,dx.$$

$$\therefore \quad \frac{b}{a}\int\frac{dy}{y} = -\int dx + C,$$

or $$\frac{b}{a}\log y = -\,x + C; \quad (C = \log C_1).$$

$$\therefore \quad y = e^{-\frac{a}{b}x + \frac{Ca}{b}} = e^{-\frac{a}{b}x} \cdot e^{\frac{Ca}{b}},$$

$$\therefore \quad y = Ae^{-\frac{a}{b}x}, \text{ where } A = e^{\frac{aC}{b}}.$$

(5) Solve $$\frac{dy}{dx} = y + 3$$

$$\int \frac{dy}{y+3} = \int dx + C.$$

$$\therefore \quad \log(y + 3) = x + \log A_1,$$

$$\therefore \quad y = Aex - 3.$$

(6) If $$\frac{dy}{dx} = n\frac{y}{x}, \text{ then } \frac{dy}{y} = n\frac{dx}{x}.$$

$$\therefore \quad \log y = n\log x + \log A,$$

$$\therefore \quad y = Ax^n.$$

(ii.) **Cases where Variables are not Directly Separable.**—If it is impossible to separate the variables directly it may be possible to effect the separation by introducing a new variable, called an auxiliary variable.

(*a*) *Homogeneous Equations.*—In these equations the introduction of a new variable z, such that $z = \frac{y}{x}$ (or $y = xz$) enables the separation to be effected.

Example.—Solve $$x^2 + 2xy\frac{dy}{dx} - y^2 = 0.$$

This is the same as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{\left(\frac{y}{x}\right)^2 - 1.}{2\frac{y}{x}}.$$

Putting $$z = \frac{y}{x}, \text{ or } y = zx \text{ we get}$$

$$\frac{d(zx)}{dx} = \frac{z^2 - 1}{2z},$$

i.e., $$z + \frac{xdz}{dx} = \frac{z^2 - 1}{2z},$$

or $$\frac{z^2 - 1}{2z} - z = \frac{xdz}{dx},$$

i.e., $$-\frac{(z^2 + 1)}{2z} = \frac{xdz}{dx}.$$

It is now possible to bring the z's and dz's on one side and the x's and dx's on the other. We then get

$$\int\frac{2zdz}{1 + z^2} = -\int\frac{dx}{x},$$

or $$\log(1 + z^2) = -\log x + C = -\log x + \log A \text{ (where } \log A = C),$$

i.e., $$\log\frac{(x^2 + y^2)}{x^2} = -\log x + \log A,$$

or $$\log(x^2 + y^2) = \log x + \log A = \log Ax.$$

$$\therefore \quad x^2 + y^2 = Ax. \quad \text{Whence } y = \sqrt{Ax - x^2}.$$

(*b*) *Non-homogeneous Equations.*—Such equations can generally be rendered homogeneous by the following artifice :

In the equation $(ax + by + c)\,dx + (a'x + b'y + c)\,dy$ put $x = (v + h)$, $y = (w + k)$, when it becomes $\{(av + bw) + (ah + bk + c)\}\,dv + \{(a'v + b'w) + (a'h + b'k + c')\}\,dw = 0$.

If now h and k are so chosen that

$$ah + bk + c = 0$$
$$\text{and } a'h + b'k + c' = 0,$$

then the equation reduces itself to

$$(av + bw)\,dv + (a'v + b'w)\,dw = 0,$$

which is homogeneous in v and w and may be solved in the usual manner.

The following example will illustrate the process :

Solve $$(2x - 3y + 4)\,dx + (3x - 2y + 1)\,dy = 0.$$

Putting $x = (v + h)$ and $y = (w + k)$, it becomes

$$\{(2v - 3w) + (2h - 3k + 4)\}\, dv + \{(3v - 2w) + (3h - 2k + 1)\}\, dw = 0.$$

Now if we choose h and k so that

$$2h - 3k + 4 = 0$$
$$\text{and } 3h - 2k + 1 = 0,$$

i.e., if we make $h = 1$ and $k = 2$ (as found by solving the two simultaneous equations either algebraically or graphically), we get $(2v - 3w)\, dv + (3v - 2w)\, dw = 0$.

This equation being now homogeneous in v and w,

put $$z = \frac{w}{v} \text{ or } w = vz \text{ and get}$$

$$(2 - 3z)\, dv + (3 - 2z)\, (zdv + vdz) = 0$$

or $$2\,(1 - z^2)\, dv - v\,(2z - 3)\, dz = 0,$$

i.e., $$2\frac{dv}{v} = \frac{(2z - 3)}{1 - z^2}\, dz = -\frac{5\, dz}{2\,(1 + z)} - \frac{dz}{2(1 - z)}$$

(see p. 26).

$$\therefore\quad 4\int\frac{dv}{v} = -5\int\frac{dz}{1 + z} - \int\frac{dz}{1 - z},$$

or $$4 \log v = -5 \log (1 + z) + \log (1 - z) + C.$$

But $$v = (x - 1), w = (y - 2) \text{ and } z = \frac{w}{v} = \frac{y - 2}{x - 1}$$

$$\therefore\quad 4 \log (x - 1) = -5 \log \frac{(x + y - 3)}{(x - 1)} + \log \frac{(x - y + 1)}{(x - 1)} + C$$

or $$\log (x - y + 1) - 5 \log (x + y - 3) + \log A \quad (\text{where } \log A = C) = 0.$$

$\therefore$ final solution is

$$A\,(x - y + 1) = (x + y - 3)^5.$$

The method just described fails in the case where $\frac{a}{a'} = \frac{b}{b'}$. Thus if $(2x - 3y + 4)\, dx + (4x - 6y + 1)\, dy = 0$, it is impossible to choose h and k, so that

$$2h - 3k + 4 = 0$$
and $$4h - 6k + 1 = 0.$$

For the proper substitution in such a case the reader is referred to the regular text-books on differential equations.

General Solution of Linear Equations of the First Order.—The most general type of such an equation is

$$\frac{dy}{dx} + Py + Q = 0.$$

Put $y = z\phi(x)$, where z is a new variable and $\phi(x)$ is some arbitrary function of x.

$\therefore$ equation becomes

$$\frac{d(z\phi(x))}{dx} + Pz\phi(x) + Q = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But
$$\frac{dz\phi(x)}{dx} = \frac{zd\phi(x)}{dx} + \frac{\phi(x)dz}{dx}.$$

$$\therefore \quad \frac{zd\phi(x)}{dx} + \frac{\phi(x) \, . \, dz}{dx} + Pz\phi(x) + Q = 0,$$

or
$$\phi(x)\frac{dz}{dx} + z\left\{\frac{d\phi(x)}{dx} + P\phi(x)\right\} + Q = 0 \; . \quad . \quad . \quad . \quad (2)$$

Now as $\phi(x)$ is an arbitrary function of x we can choose it so as to make the bracketed expression, viz., $\frac{d\phi(x)}{dx} + P\phi(x)$, vanish (when we get rid of the term containing z).

Putting therefore
$$\frac{d\phi(x)}{dx} = -P\phi(x),$$

we get
$$\frac{d\phi(x)}{\phi x} = -Pdx.$$

$$\therefore \quad \int\frac{d\phi x}{\phi x} = -\int Pdx,$$

giving
$$\log \phi(x) = -\int Pdx,$$

whence
$$\phi(x) = e^{-\int Pdx}.$$

Now, since this value of $\phi(x)$ makes the bracketed expression in (2) vanish, therefore equation (2) now becomes

$$\phi(x)\frac{dz}{dx} + Q = 0$$

or
$$dz = -\frac{Qdx}{\phi x} = -\frac{Qdx}{e^{-\int Pdx}}$$

$$= -e^{\int Pdx}Qdx.$$

$$\therefore \quad z = -\int e^{\int Pdx}Qdx + C.$$

But $$y = z\phi(x) = -ze^{-\int \text{P}dx}$$

$\therefore$ finally $$\mathbf{y} = \mathbf{e}^{-\int \mathbf{P}d\mathbf{x}}\left[\mathbf{C} - \int \mathbf{e}^{\int \mathbf{P}d\mathbf{x}}\,\mathbf{Q}d\mathbf{x}\right].$$

EXAMPLES.

(1) Solve $$\frac{dy}{dx} + \frac{y}{x} - x^2 = 0.$$

Here $$\text{P} = \frac{1}{x};\ \text{Q} = -x^2.$$

$$\therefore \quad \int \text{P}dx = \int \frac{dx}{x} = \log x.$$

$$\therefore \quad e^{\int \text{P}dx} = e^{\log x} = x,$$

and $$e^{-\int \text{P}dx} = e^{-\log x} = \frac{1}{x}.$$

$$\therefore \quad y = \frac{1}{x}\left[\text{C} - \int . x . (-x^2)dx\right]$$

$$= \frac{1}{x}\left[\text{C} + \int x^3 dx\right]$$

$$= \frac{\text{C}}{x} + \frac{1}{4}x^4.$$

(2) If L is the coefficient of self-induction, R the resistance, E the external electromotive force in a circuit, and i the current,

then $$\text{L}\frac{di}{dt} + \text{R}i = \text{E}.$$

Find i in terms of R, L and t.

Using the formula

$$y = e^{-\int \text{P}dx}\left[\text{C} - \int e^{\int \text{P}dx}\text{Q}dx\right]$$

we have $$i = e^{-\int \frac{\text{R}}{\text{L}}dt}\left[\text{C} + \frac{\text{E}}{\text{L}}\int e^{\frac{\text{R}}{\text{L}}}dt\right]$$

$$= \text{C}e^{-\frac{\text{R}}{\text{L}}t} + \frac{\text{E}}{\text{R}}.$$

Solution of Linear Equations of the Second Order.—As a type we shall take $\frac{d^2y}{dx^2} + \frac{\text{P}dy}{dx} + \text{Q}y = 0$, where P and Q are constants.

Since $$\frac{d(Ae^{Kx})}{dx} = AKe^{Kx}$$

$$\therefore \quad \frac{d^2x}{dx^2} = AK^2e^{Kx}.$$

$\therefore$ if we **put y = Ae^Kx**, our equation becomes

$$AK^2e^{Kx} + PAKe^{Kx} + Qe^{Kx} = 0$$

or $$Ae^{Kx}(K^2 + PK + Q) = 0.$$

$\therefore$ $\mathbf{K^2 + PK + Q = 0}$. (This equation is called the **auxiliary** equation.)

$$\therefore \quad K = \frac{-P \pm \sqrt{P^2 - 4Q}}{2} \text{ (p. 31).}$$

The following three possibilities arise, viz.:

(1) $P^2 = 4Q$, when the two roots K_1 and K_2 are each equal to $-\frac{P}{2}$.

$$\therefore \quad y = Ae^{-\frac{Px}{2}}, \text{ will satisfy the equation.}$$

Also $$y = Bxe^{-\frac{Px}{2}} \text{ will satisfy the equation.}$$

$\therefore$ complete solution is $y = (A + Bx)e^{-\frac{Px}{2}}$ (see p. 192).

Example.

$$\frac{d^2y}{dx^2} + \frac{2dy}{dx} + y - 0.$$

Here $K = -1$.

$$\therefore \quad y = (A + Bx)e^{-x}.$$

(2) $P^2 > 4Q$, when there are two roots which are real but unequal, say, K_1 and K_2.

$\therefore$ $y = Ae^{K_1x}$ will satisfy the equation.

Also $y = Be^{K_2x}$ will satisfy the equation.

$\therefore$ complete solution is $y = Ae^{K_1x} + Be^{K_2x}$.

Example $$\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 3y = 0.$$

Here $K_1 = 1$ and $K_2 = -3$,

$$\therefore \quad y = Ae^{x} + Be^{-3x}.$$

(3) $P^2 < 4Q_1$ when the roots are imaginary, viz :

$$K_1 = -\frac{P + i\sqrt{4Q - P^2}}{2}$$

$$K_2 = -\frac{P - i\sqrt{4Q - P^2}}{2}, \text{ (p. 32).}$$

Hence, putting $4Q - P^2 = a$,

$$K_1 = -\frac{P + ia}{2},\ K_2 = -\frac{P - ia}{2}.$$

∴ complete solution is

$$y = Ae^{-\frac{(P-ia)x}{2}} + Be^{-\frac{(P+ia)x}{2}}$$

$$= e^{-\frac{Px}{2}}\left(Ae^{-\frac{iax}{2}} + Be^{-\frac{iax}{2}}\right).$$

But $e^{\frac{iax}{}} = \cos\frac{ax}{2} + i\sin\frac{ax}{2}$

and $e^{-\frac{iax}{2}} = \cos\frac{ax}{2} - i\sin\frac{ax}{2}$ (see p. 190).

∴ final solution is

$$y = e^{-\frac{Px}{2}}\left[A\left((\cos\frac{ax}{2} + i\sin\frac{ax}{2}\right) + B\left(\cos\frac{ax}{2} - i\sin\frac{ax}{2}\right)\right]$$

$$= e^{-\frac{Px}{2}}\left[(A + B)\cos\frac{ax}{2} + (A - B)\, i\sin\frac{ax}{2}\right]$$

or $$y = e^{-\frac{Px}{2}}\left(C_1\cos\frac{ax}{2} + iC_2\sin\frac{ax}{2}\right)$$

(where $C_1 = (A + B)$ and $C_2 = (A - B)$).

Example.

$$\frac{d^2y}{dx^2} + \frac{2dy}{dx} - 3y = 0.$$

Here $K_1 = (-1 + i\sqrt{2})$, and $K_2 = (-1 - i\sqrt{2})$.

$$\therefore\ y = e^{-x}(C_1\cos\sqrt{2}\,.\,x + iC_2\sin\sqrt{2}\,.\,x).$$

Linear equations of the second order are of very great importance in connection with the study of *damped oscillation* (*e.g.*, that of the mercury in a sphygmomanometer).

Linear Equations of the *n*th Order.—Equations of the type

$$\frac{d^n y}{dx^n} + \frac{Pd^{n-1}y}{dx^{n-1}} + \ldots \frac{Rdy}{dx} + Sy = 0$$

can be solved in the same way as equations of the second order by putting $y = Ae^{Kx}$. When we get the auxiliary equation

$$K^n + PK^{n-1} + \ldots + RK + S = 0.$$

Graphical Solution of Differential Equations.—In the same way as it is possible in the case of ordinary equations to find approximate solutions by graphic methods when ordinary algebraic methods fail us (see p. 117), so in the case of differential equations, it is often possible to obtain approximate solutions graphically. It would, however, be beyond the scope of this book to enter into a consideration of this subject here.

EXAMPLES.

(1) Solve the differential equation :

$$\frac{d^3y}{dx^3} - \frac{6d^2y}{dx^2} + \frac{11dy}{dx} - 6y = 0.$$

The auxiliary equation here is :

$$K^3 - 6K^2 + 11K - 6 = 0,$$

i.e.,

$$(K - 1)(K - 2)(K - 3) = 0.$$

$\therefore$ complete solution is $y = Ae^x + Be^{2x} + Ce^{3x}$.

(2) Solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{2dy}{dx} = 0.$

The auxiliary equation here is $K^3 + K^2 - 2K = 0$,

or

$$K(K - 1)(K + 2) = 0.$$

$\therefore$ complete solution is $y = A + Be^x + Ce^{-2x}$.

(3) Solve $\frac{d^2y}{dx^2} + \frac{7dy}{dx} + 10y = 5.$

Writing the equation as $\frac{d^2y}{dx^2} + \frac{7dy}{dx} + 10(y - 0{\cdot}5) = 0$,

we can put $(y - 0{\cdot}5) = e^{Kx}$.

Auxiliary equation is $K^2 + 7K + 10 = 0$, *i.e.*, $K_1 = -2$; $K_2 = -5$.

$\therefore$ complete solution is $y - 0{\cdot}5 = Ae^{-2x} + Be^{-5x}$.

(4) Prove that in the case of ferments—like emulsin—which gradually get destroyed in the course of the reaction which they catalyse, a limit must be reached to the amount of substance decomposed by the ferment, so that even if the original amount of ferment is large and the reaction is allowed to proceed indefinitely a certain amount of substance undergoing decomposition under the influence of the ferment must remain unaltered. (See Ernst Cohen's "Physical Chemistry.")

Let A = original amount of ferment.
B = original amount of substance which is catalysed.
x = amount of ferment which becomes inactive.
t = time in which this amount x has been destroyed.
y = amount of substance decomposed in time t.

Then, by Guldberg and Waage's law :

$$\frac{dy}{dt} = \mathrm{K}(\mathrm{A} - x)\,(\mathrm{B} - y).$$

Now it has been shown experimentally that the decomposition of a ferment progresses as a monomolecular reaction.

Hence, if c = velocity constant of the decomposition of the ferment,

$$c = \frac{1}{t}\log_e \frac{\mathrm{A}}{\mathrm{A} - x},$$

i.e.,
$$e^{ct} = \frac{\mathrm{A}}{\mathrm{A} - x},$$

or
$$\mathrm{A} - x = \frac{\mathrm{A}}{e^{ct}} = \mathrm{A}e^{-ct}.$$

Hence our equation becomes

$$\frac{dx}{dt} = \mathrm{KA}e^{-ct}\,(\mathrm{B} - y),$$

or
$$\int\frac{dy}{\mathrm{B} - y} = \mathrm{KA}\int e^{-ct}dt.$$

i.e.,
$$-\log\frac{(\mathrm{B} - y)}{\mathrm{B}} = -\frac{\mathrm{KA}}{c}e^{-ct} + \mathrm{C}.$$

To find the value of C put $t = 0$ (*i.e.*, when $y = 0$), and we then have

$$-\log 1,\ i.e.,\ 0 = -\frac{\mathrm{KA}}{c} + \mathrm{C}.$$

$$\therefore\quad \mathrm{C} = \frac{\mathrm{KA}}{c}.$$

$\therefore$ substituting this value of C we have

$$\log\frac{(\mathrm{B} - y)}{\mathrm{B}} = \frac{\mathrm{KA}}{c}e^{-ct} - \frac{\mathrm{KA}}{c}$$

$$= \frac{\mathrm{KA}}{c}\,(e^{-ct} - 1).$$

Hence if we put $t = \infty$, we get

$$\log\frac{(\mathrm{B} - y)}{\mathrm{B}} = -\frac{\mathrm{KA}}{c},$$

Whence
$$\frac{\mathrm{B} - y}{\mathrm{B}} \text{ or } 1 - \frac{\mathrm{B}}{y} = e^{-\frac{\mathrm{KA}}{c}} = \frac{1}{e^{\frac{\mathrm{KA}}{c}}}.$$

$$\therefore\quad y = \mathrm{B}\left(1 - \frac{1}{e^{\frac{\mathrm{KA}}{c}}}\right),$$

i.e., the amount of substance (y) transformed after any time is always less by $\dfrac{\text{B}}{e^{\frac{\text{KA}}{c}}}$ than the amount originally present, and will only be completely decomposed when $t = \infty$.

(5) The Schütz-Borissoff law states that in the case of gastric digestion (pepsin and rennin), in the early stages,

$x = \text{K}\sqrt{\text{F}qt}$ (see p. 151).

Where x = amount of substrate transformed (*i.e.*, milk clotted, or protein hydrolysed).

F = concentration of enzyme (rennin or pepsin).

q = initial concentration of substrate (milk or egg albumen).

t = time in hours.

K = constant.

This equation leads on differentiation to the differential equation :

$$\frac{dx}{dt} = \frac{\text{K}^2\text{F}q}{2x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.})$$

i.e., **the velocity of transformation is inversely proportional to the amount of substance transformed.** Arrhenius explains this peculiar fact as being due to a combination of the acid of the gastric juice with a product of the hydrolysis to form an inactive compound. Then, as the concentration of the enzyme is very small compared with that of the substrate, the amount of inactive compound formed at the beginning of digestion is practically equal to the whole of the initial concentration (q) of the enzyme. Therefore the trace of active free enzyme which remains is given by the following equation :

$$\text{Concentration of free enzyme} = \frac{\text{KF}}{x} \quad . \quad . \quad . \quad . \quad . \quad (\text{ii.})$$

where F is the initial concentration of the enzyme; which is of the same form as (i.) above.

To find the actual velocity of hydrolysis we must apply the law of mass action, by which the velocity is proportional to both the concentration of the enzyme as well as that of the substrate at any moment.

Now, Concentration of enzyme at the time $t = \dfrac{\text{KF}}{x}$,

Concentration of substrate at the same moment $= (a - x)$.

$$\therefore \qquad \frac{dx}{dt} = \frac{\text{KF}}{x}(a - x).$$

$$\therefore \qquad \frac{xdx}{a - x} = \text{KF}dt.$$

$$\therefore \qquad \int \frac{x}{a - x}\,dx = \text{KF}t.$$

But $$\frac{x}{a - x} = \frac{a}{a - x} - 1.$$

$$\therefore \quad \int \frac{x}{a-x}\,dx = \int \left(\frac{a}{a-x} - 1\right) dx,$$

$$= \int \frac{a}{a-x}\,dx - \int dx,$$

$$= \log \frac{a}{a-x} - x.$$

Hence the theoretical formula for peptic digestion is :

$$\mathrm{KF}t = \log \frac{a}{a-x} - x.$$

Bayliss has shown that the Schütz-Borissoff empirical formula from which this theoretical formula is derived is only true in the early stages of digestion. This is so because in the early stages x is very small compared with a, and therefore $a - x$ is practically the same as a. Hence the differential equation,

$$\frac{dx}{dt} = \frac{\mathrm{KF}}{x}(a - x)$$

is the same as

$$\frac{dx}{dt} = \frac{\mathrm{KF}a}{x}.$$

$$\therefore \quad \int x\,dx = \mathrm{KF}a \int dt,$$

i.e.,

$$\frac{x^2}{2} = \mathrm{KF}at,$$

whence

$$x = \sqrt{2\mathrm{KF}at},$$

which is of the same form as the Schütz-Borissoff law.

The above example is very instructive for the following reasons :

(1) It is an illustration of the fact that two totally different formulæ may—within a certain range of values of the independent variable—equally express the relationship between the two variables. Thus, for small values of x (*i.e.*, up to $x = 33{\cdot}7$ per cent.) either of the two formulæ will give equally satisfactory results, as the following table shows (see "Principles of Biochemistry," by T. B. Robertson) :

Time (in Hours).	Protein Digested.	$\mathrm{KF} = \frac{1}{t}\left\{\log \frac{a}{a-x} - x\right\}$	$\sqrt{2\mathrm{KF}a} = \frac{x}{\sqrt{t}}$
	Per cent.		
2	10·5	3·0	7·5
4	16·4	3·8	8 2
6	19·9	3·8	8·1
8	22·7	3·8	8·0
12	27·0	3·7	7·7
16	30·4	3·6	7·6
20	33·7	3·7	7·5
32	40·0	3·4	7·1
48	45·1	3·2	6·5
64	50·8	3·1	6·3
96	57·4	2·8	5·9

But the complete logarithmic formula gives practically uniform results for all values of x, whilst with the Schütz-Borissoff formula the results are not uniform after $x = 33{\cdot}7$ per cent.

(2) It shows how risky it is to use an empirical formula for purposes of *extrapolation*, *i.e.*, for calculating values of the dependent variable outside the range of observation for values of the independent variable. Thus, although the Schütz-Borissoff formula may be safely employed for purposes of interpolation, *i.e.*, for calculating values of the dependent variable for values *within* the range of observation for values of x, yet if used for extrapolation would give widely discordant results (see, further, next Chapter, p. 300, and exercise (2) on p. 325).

(6) The rate of multiplication of micro-organisms at any moment in the presence of a limited supply of nutriment, such as obtains in test-tube experiments, is proportional both to the number of organisms as well as to the concentration of the foodstuff at that moment. Assuming that the organisms multiply by a simple conversion of available food material into other organisms, find the number of organisms y present after an interval of time t, taking the original number of organisms (*i.e.*, those present at time $t = 0$) to be y_0. (A. G. M'Kendrick and M. Kesava Pai, *Proc. R. Soc. Edinb.* xxxi., 1911.)

Let a = original concentration of foodstuff.

Then, by hypothesis,

$$a - y = \text{concentration of foodstuff at any moment } t,$$

and $$\frac{dy}{dt} = by\,(a - y), \text{ where } b \text{ is a constant.}$$

$$\therefore \quad \frac{dy}{y(a-y)} = bdt.$$

Splitting into partial fractions we get

$$\frac{dy}{y(a-y)} = \frac{1}{a}\left\{\frac{dy}{y} + \frac{dy}{a-y}\right\}$$

$$\therefore \quad \frac{1}{a}\int\frac{dy}{y} + \frac{1}{a}\int\frac{dy}{a-y} = b\int dt,$$

or $$\frac{1}{a}\log y - \frac{1}{a}\log(a-y) = bt + \text{C}. \quad (\text{C} = \text{integration constant.})$$

$\therefore \quad \log\dfrac{y}{a-y} = abt + a\text{C}$ (compare similar equation in connection with human growth, see p. 294).

To evaluate **C** put $t = 0$, when y becomes y_0 and a becomes $a - y_0$.

$$\therefore \quad \log\frac{y_0}{a-y_0} = a\text{C}.$$

$$\therefore \quad \text{C} = \frac{1}{a}\log\frac{y_0}{a-y_0}.$$

Hence the complete equation becomes

$$\log \frac{y}{a-y} = abt + \log \frac{y_0}{a-y_0}$$

or

$$\log \frac{y(a-y_0)}{y_0(a-y)} = abt.$$

$$\therefore \quad \frac{y(a-y_0)}{y_0(a-y)} = e^{abt}.$$

i.e.,

$$ya - yy_0 = y_0 ae^{abt} - yy_0 e^{abt}.$$

$$\therefore \quad y\left[a + y_0 e^{abt} - y_0\right] = y_0 ae^{abt}.$$

$$\therefore \quad y = \frac{y_0 ae^{abt}}{a - y_0 + y_0 e^{abt}}.$$

$$= \frac{a}{\frac{a-y_0}{y_0} e^{-abt} + 1}.$$

Note.—When the organism stops growing $\frac{dy}{dt} = 0$.

$$\therefore \quad \log(a-y) = 0,$$

i.e.,

$$y = a.$$

Hence, a = number of organisms when growth ceases.

For further examples in differential equations with separable variables the reader is referred to the various problems in connection with chemical kinetics as well as the physiology of growth that have been considered on previous pages.

(3) **Hæmodynamics of Mitral Incompetence.**—The case of a heart with incompetent mitral or tricuspid valves is analogous to that of an open vessel (which, for the sake of simplicity, we shall take as cylindrical or prismatic instead of conical, as in the case with the heart) which receives water at a uniform rate at the top, and which is losing water at the same time at a uniform rate through an aperture at the bottom. In this way Emil Schwartz has calculated the time it would take for a mitral regurgitant heart to acquire sufficient intraventricular pressure to overcome the diastolic pressure in the aorta and thus open the aortic valves.

Suppose the bottom of our cylindrical vessel to have its orifice plugged, then if the pressure of the water increases at the rate of p per unit of time, it will increase during the interval of time dt by the amount of pdt, whilst the level of the water x will at the same time rise by an amount dx. We shall then have $dx = pdt$. Now suppose the bottom orifice to be opened, then if its sectional area = **a**, and the sectional area of the cylinder = **A**, then an amount of water equal to dQ will flow out of the aperture during the time dt, and the fall in level will be $-\frac{dQ}{A}$.

$$\therefore \quad dx \text{ will} = pdt - \frac{dQ}{A}.$$

But by Torricelli's theorem $dQ = \mu a \sqrt{\dfrac{2gx}{1 - \dfrac{a^2}{A^2}}}\, dt$ (where μ = coefficient of discharge and g = acceleration due to gravity).

$$\therefore \quad dx = pdt - \frac{\mu a}{A}\sqrt{x}\sqrt{\frac{2gA^2}{A^2 - a^2}} \,.\, dt$$

$$= \left(p - \mu a\sqrt{x} \,.\, \sqrt{\frac{2g}{A^2 - a^2}}\right) dt.$$

Here the variables are easily separable, for the right side of the equation is the same as

$$\left(\frac{p}{\mu a}\sqrt{\frac{A^2 - a^2}{2g}} - \sqrt{x}\right) . \mu a \sqrt{\frac{2g}{A^2 - a^2}} \,.\, dt.$$

$$\therefore \quad \frac{dx}{\left(\dfrac{p}{\mu a}\sqrt{\dfrac{A^2 - a^2}{2g}} - \sqrt{x}\right)} = \mu a \sqrt{\frac{2g}{A^2 - a^2}}\, dt.$$

To simplify, put $\qquad \dfrac{\sqrt{A^2 - a^2}}{\mu a \sqrt{2g}} = R,$

and our differential equation becomes

$$\frac{R dx}{pR - \sqrt{x}} = dt.$$

$$\therefore \quad \int dt = R \int \frac{dx}{pR - \sqrt{x}}.$$

By performing the integration we get the value of t, *i.e.*, the time required for the ventricle to acquire sufficient pressure to open the aortic valves.

Put $\qquad x = z^2$ and $\displaystyle\int \frac{dx}{pR - \sqrt{x}}$ becomes $\displaystyle 2\int \frac{zdz}{pR - z}.$

Now $\frac{z}{pR - z} = \frac{pR}{pR - z} - 1.$

$$\therefore \quad 2\int \frac{zdz}{pR - z} = 2\int \frac{pR}{pR - z}\,dz - 2\int dz,$$

$$= -\,2pR \log \frac{pR}{pR - z} - 2z,$$

$$= -\,2pR \log \frac{pR}{pR - \sqrt{x}} - 2\sqrt{x},$$

$$\therefore \quad t = -\,2R\left[pR \log \frac{pR}{pR - \sqrt{x}} - \sqrt{x}\right].$$

Now when the ventricle is in diastole the intraventricular pressure $x_1 = 0$, and when the ventricle is in complete systole the pressure within it must be at least equal to the diastolic pressure in the aorta x_2 (assumed by Schwartz to be 150 mm. Hg, or 2 metres H_2O—now known to be erroneous, the diastolic pressure being only about 75 mm. Hg, or 1 metre H_2O).

Integrating between the limits x_1 and x_2, we get

$$t = R\int_{x_1}^{x_2} \frac{dx}{pR - \sqrt{x}} = 2R\left[2{\cdot}3pR \log_{10} \frac{pR - \sqrt{x_1}}{pR - \sqrt{x_2}} + \sqrt{x_1} - \sqrt{x_2}\right]$$

$$= 2R\left[2{\cdot}3pR \log_{10} \frac{pR}{pR - \sqrt{2}} - \sqrt{2}\right].$$

Now p represents change of intraventricular pressure per unit of time, *i.e.*, per second. But in the normal heart the change of pressure is from 0 to 2 metres in 0·1 second, therefore $p = 20$ (since the change of pressure is uniform). Hence, assuming A to be constant (which is, of course, not the case in the conical heart, in which it is a function of x) and $= 8$ sq. cm., and taking $\mu = 0{\cdot}62$, and $g = 9{\cdot}8$ metres Hg, we can calculate the values of t for the various values of **a** (*i.e.*, area of leak), thus :

a.	$R = \frac{\sqrt{64 - a^2}}{\mu a \sqrt{19{\cdot}6}}$	t.	Increase in per cent. of t.
1	2·89	0·1021	2·1
2	1·41	0·1031	3·1
3	0·90	0·1060	6·5
4	0·63	0·1080	8
7·8	0·08	0·441	3·41
7·9	0·06	∞	∞

CHAPTER XX.

MATHEMATICAL ANALYSIS APPLIED TO THE CO-ORDINATION OF EXPERIMENTAL RESULTS.

In order to be able to form an opinion of the nature of the process or processes responsible for any particular phenomenon under investigation, it is necessary to ascertain if there is any mathematical relationship existing between the dependent and independent variables—such as between the height and weight; weight and age, surface area and weight of a person; amount of chemical transformation and time; reaction-velocity and temperature, etc. Such a relationship expressed in the form of a mathematical formula, embodying the results found in the laboratory in a concise form, and which enables one to foretell with considerable certainty the quantitative results of any future observation of a similar nature constitutes the *law* of the phenomenon in question, and forms a powerful weapon, not only for the detection of deviations between calculated and observed results, but also enables one to form some idea of the causes of such deviations. It tells one, for instance, whether these deviations are due to experimental errors only or whether the discrepancies between the expected and observed values are greater than can be accounted for by errors due to faulty laboratory technique. In the former case such a formula leads the investigator to improve his laboratory methods with the object of minimising the magnitude of his errors; whilst in the latter the mathematical formula affords one an opportunity to ascertain what are the **inherent** factors responsible for such discrepancies, thereby helping to bring about further scientific discoveries.

The most classical, as well as the most dramatic, example of an important scientific discovery made in this way is the prediction by Adams and Leverrier, in 1846, of the mass, position and orbit of Neptune—as the result of the measured deviations of Uranus at different points in its calculated orbit—before its discovery and identification, a few months later, by Galle, who directed his telescope to the place in the heavens indicated by the theoretical calculations.

Theoretical and Empirical Formulæ.—(*a*) If there is any theoretical consideration to lead one to believe that the relationship between the dependent and independent variables is such as to follow some well-known law, then the problem is fairly easy. All one has to do is to compare the observed results with those that one would expect to find by means of the formula, and to see whether, allowing for errors of observation, the observed and calculated results agree. If they do, then there is a great probability (though by no means an absolute certainty) that the assumed formula is the correct one. A formula so established constitutes a *theoretical formula.*

The following examples will make this clear:

(1) **Problem in Physiology of Growth.**—Robertson, as the result of certain theoretical considerations (see *Child Physiology*, Chapter XVIII.), concluded that the growth in weight of a fœtus, as well as of a young infant, is an autocatalytic phenomenon for which he derived the following equation:

$$K = \frac{1}{t - 1{\cdot}66} \log_{10} \frac{x}{341{\cdot}5 - x},$$

where x = weight in ozs. of infant (or fœtus) at t months from birth (in the case of the infant t is + ve, and of the fœtus t is − ve), and K = growth constant.

The following table gives the corresponding values of t and x of fœtuses and infants at various ages. Find whether the formula is true. Also calculate the theoretical weight of an infant at the age of eight months, using the mean value of K thus found.

t	− 0·75	− 0·42	− 0·08	0	+ 0·25	+ 0·58	+ 0·92
x	111	117	127	127	137	145	146

If the formula represents the true relationship between t and x, then by substituting in it any corresponding pair of values of t and x, the value of K thus found should be practically identical in each case (but for errors of observation). If we do so we get the results given in the table on p. 295.

The agreement between the various values of K is therefore very good and the formula probably, therefore, represents the true relationship between t and x. (But see note on p. 89, Chapter VII.)

To find the weight of an infant eight months old put $t = 8$, we then have

$$\text{mean } K = {\cdot}136 = \frac{1}{8 - 1{\cdot}66} \log \frac{x}{341{\cdot}5 - x}$$

$$= \frac{1}{6{\cdot}34} \log \frac{x}{341{\cdot}5 - x}.$$

t	x	$\frac{1}{t - 1{\cdot}66} \log_{10} \frac{x}{341{\cdot}5 - x} = K.$
$-0{\cdot}75$	111	$-\frac{1}{2{\cdot}41} \log \frac{111}{230{\cdot}5} = {\cdot}132$
$-0{\cdot}42$	117	$-\frac{1}{2{\cdot}08} \log \frac{117}{224{\cdot}5} = {\cdot}136$
$-0{\cdot}08$	127	$-\frac{1}{1{\cdot}74} \log \frac{127}{214{\cdot}5} = {\cdot}131$
0	127	$-\frac{1}{1{\cdot}66} \log \frac{127}{214{\cdot}5} = {\cdot}136$
$+0{\cdot}25$	137	$-\frac{1}{1{\cdot}41} \log \frac{137}{204{\cdot}5} = {\cdot}123$
$+0{\cdot}58$	145	$-\frac{1}{1{\cdot}08} \log \frac{145}{196{\cdot}5} = {\cdot}122$
$+0{\cdot}92$	146	$-\frac{1}{0{\cdot}74} \log \frac{146}{195{\cdot}5} = {\cdot}171$
		Mean $= {\cdot}136$

$$\therefore \quad {\cdot}136 \times 6{\cdot}34, \textit{ i.e.}, {\cdot}862 = \log \frac{x}{341{\cdot}5 - x}.$$

But ·862 is the logarithm of 7·278.

$$\therefore \quad 7{\cdot}278 = \frac{x}{341{\cdot}5 - x}.$$

This gives $x = 300$ ozs., *i.e.*, the theoretical weight of a normal infant eight months old should be 300 ozs. (This also is the observed weight.)

(2) **Problem in Psychology.**— There is some experimental evidence of a chemical nature to show that mental processes are of the nature of an autocatalytic chemical reaction. Investigate the following figures obtained by Ebbinghaus in a certain memory test to see whether they agree with expectation, assuming the particular autocatalytic equation in this case to be

$$\log \frac{x}{43{\cdot}6 - x} = 0{\cdot}001468t - 0{\cdot}526 \text{ (Robertson)}$$

x	7 (1)	16 (30)	24 (44)	36 (55)
t	2·8	192	422·4	792

x = number of meaningless syllables repeated a number of times (shown in brackets) at the rate of 0·4 second per syllable.

t = number of seconds required to fix that number of syllables upon the memory.

If we substitute the various values of t in the equation

$$\log \frac{x}{43{\cdot}6 - x} = 0{\cdot}001468t - 0{\cdot}526$$

we get as follows :

When $t = 2{\cdot}8$, $\log \frac{x}{43{\cdot}6 - x} = 0{\cdot}001468 \times 2{\cdot}8 - 0{\cdot}526$

$$= - {\cdot}5218896,$$

or $$\log \frac{43{\cdot}6 - x}{x} = {\cdot}5218896 = \log 3{\cdot}326,$$

or $$\frac{43{\cdot}6 - x}{x} = 3{\cdot}326,$$

whence $$x = 10{\cdot}7.$$

Similarly, when $t = 192$, $\log \frac{x}{43{\cdot}6 - x} = - {\cdot}244$,

$$\therefore \quad \frac{43{\cdot}6 - x}{x} = 1{\cdot}754.$$

$$\therefore \quad x = \frac{43{\cdot}6}{2{\cdot}754} = 15{\cdot}8.$$

Working in the same way with the other values of t, we get

when $t = 422{\cdot}4$, $x = 24{\cdot}2$

,, $t = 792{\cdot}0$, $x = 35{\cdot}4$.

Hence, except for the first value of t, the agreement between the observed and calculated values of x is excellent.

(b) If, however, there is no theoretical basis to guide one in the selection of the formula, then the procedure is as follows :

A number of pairs of values of the two variables is taken and these are plotted in the form of a graph. If the result seems to be a straight line (within the limits of errors of observation) then one knows that the required formula is one of the first degree in x and y and is of the form $y = mx + b$ (which is the general equation of a straight line (p. 105)). What remains, therefore, is to find the values of the constants m and b. This can be done very easily either by simple algebra or by inspection, since m is the slope, and b is the y intercept of the line.

EXAMPLES.

(1) The following table records the solubility of $NaNO_3$ in water at various temperatures (S = weight in grams of $NaNO_3$ dissolved in 100 grams of water; t = temperature of the water in degrees Centigrade).

S	68·4	72·9	87·5	102
t	− 6	0	20	40

By plotting the graph we find it to be a straight line (Fig. 101). We therefore write down the empirical formula as

$$S = mt + b.$$

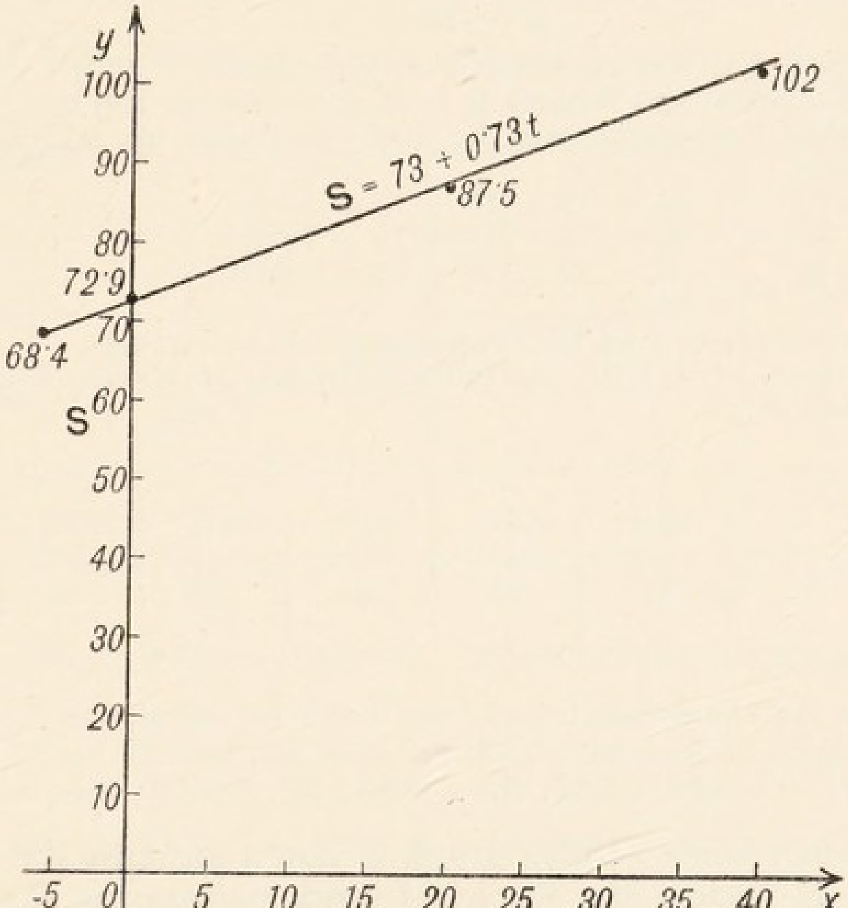

FIG. 101.—Graph of Solubility of $NaNO_3$ in Water at Various Temperatures.

The value of b is found at once, because when $t = 0$, S becomes $= m \times 0 + b = b$.

But when $t = 0$, $S = 72·9$.

$$\therefore \quad b = 72·9.$$

Hence the equation becomes

$$S = mt + 72·9.$$

To find the value of m take any other pair of values of S and t from the table and put them into the equation. For example, take $S = 68{\cdot}4$ and $t = -6$. We then get

$$68{\cdot}4 = -6m + 72{\cdot}9,$$
$$\text{giving } m = {\cdot}75.$$

Hence the equation becomes

$$S = 75t + 72{\cdot}9.$$

This equation, however, is only approximately correct, because if we were to take any other pairs of values of S and t, we would get slightly different values of m.

Thus when $S = 87{\cdot}5$ and $t = 20$, the equation becomes $87{\cdot}5 = 20m + 72{\cdot}9$, whence $m = {\cdot}78$.

Similarly when $S = 102$ and $t = 40$, the value of m becomes ·7275.

It is possible by utilising the *method of least squares* (see next Chapter, p. 345) to obtain a more accurate equation. When this is done the final result becomes :

$$S = {\cdot}73t + 73.$$

(Direct measurement of the slope of the line gives $m = {\cdot}73$.)

Having obtained this formula it is obvious that the solubility of $NaNO_3$ at any other temperature can be easily predicted by calculation or by interpolation.

Thus at 10° $S = {\cdot}73 \times 10 + 73 = 80{\cdot}3,$

" 35° $S = {\cdot}73 \times 35 + 73 = 98{\cdot}6,$

" 50° $S = {\cdot}73 \times 50 + 73 = 109{\cdot}5,$

and so on.

The student might work the next example as an exercise.

(2) In the case of KBr, the following values of S were found for the stated values of t:

S	53·4	64·6	74·6	84·7	93·5
t	0	20	40	60	80

Find the law connecting S and t and use it for calculating S at 10°, 55°, and 100° C.

[Answer, $S = 54{\cdot}2 + 0{\cdot}5t$; $S_{10} = 59{\cdot}2$; $S_{55} = 82{\cdot}0$; $S_{100} = 99{\cdot}5$.]

It is when the resulting graph is not a straight line that the difficulty of finding the empirical formula from the shape of the curve becomes very considerable, since it is, as a rule, impossible to tell by mere inspection whether the portion of the curve plotted belongs to a parabola, a hyperbola, a logarithmic curve, etc. The empirical formula must in such cases be obtained by trial. Generally one has some sort of an idea as to what form of curve one may expect, and in order to test the correctness of one's expectations one can by means of certain artifices attempt to convert the curve into a straight-line graph. If the attempt is successful then the presumption is that the formula used is the correct one.

The method is best illustrated by means of a few worked-out examples.

EXAMPLES.

(1) Sjöquist studied the course of pepsin digestion by measuring the electrical conductivity of the protein solution. He found the following values of x at the various times t (in hours) :

x	0	10·5	16·41	19·93	22·68	24·00	27·04	30·36	33·68
t	0	2	4	6	8	9	12	16	20

Find the law connecting x and t, and state to what family of curves the graph belongs.

The graph (Fig. 102) shown in the diagram looks like a portion of a

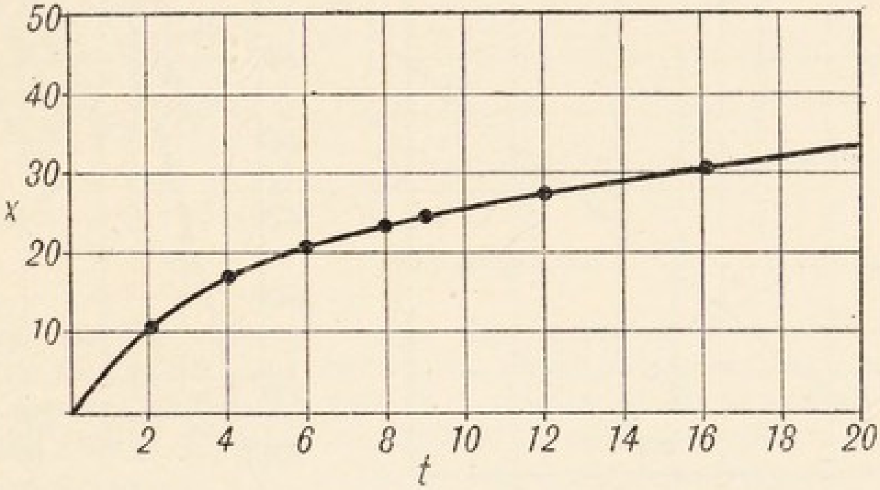

FIG. 102.—Graph of Peptic Digestion. (Result of plotting x against t.)

parabola. To see whether this is so, let the law connecting x and t be put into the form

$$t = kx^2, \text{ or } \sqrt{t} = Kx \text{ (where K is another constant } = \sqrt{k}).$$

If the supposition is correct then x plotted against $\sqrt{t}$ should give a straight line.

The plotting table will be :

x	0	10·5	16·41	19·93	22·68	24·00	27·04	30·36	33·68
$\sqrt{t}$	0	1·41	2	2·46	2·82	3	3·47	4	4·46

As will be seen from Fig. 103, the graph is a straight line passing through the origin.

Therefore the law is $\sqrt{t} = Kx$, and the graph is a parabola. In other words, the course of pepsin digestion follows the Schütz-Borissoff law.

The value of K is seen from the graph to be

$$= \frac{1}{8}\left(e.g., \frac{PM}{OM} = \frac{19{\cdot}93}{2{\cdot}46} = 8\right).$$

$$\therefore \quad \text{the law is} \qquad \sqrt{t} = \frac{1}{8}x \text{ or } x = 8\sqrt{t}.$$

This example furnishes a very good illustration of the danger of using a formula for extrapolation purposes. Thus Bayliss has shown that the

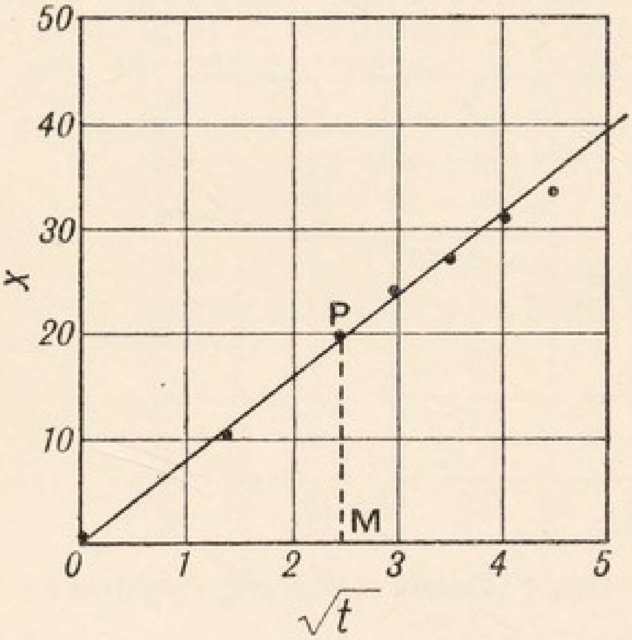

Fig. 103.—Graph of Peptic Digestion. (Result of plotting x against $\sqrt{t}$.)

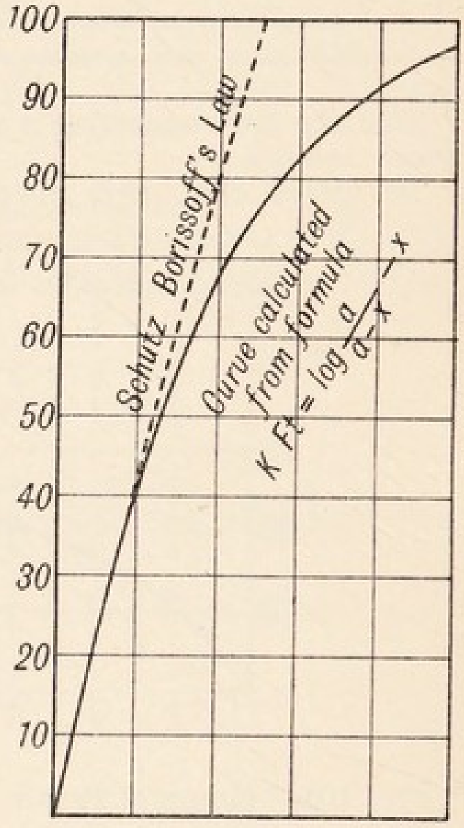

Fig. 104.—Linear and Logarithmic Graphs of the Schütz-Borissoff Law.

Schütz-Borissoff law as expressed by the relation $x = K\sqrt{t}$ only holds good for a certain stage of digestion, and therefore whilst it is safe to use this formula for purposes of interpolation within certain values of x or t, its adoption for the purpose of extrapolation outside those limits would give totally erroneous results.

Thus we have seen on p. 288 that the true equation for peptic digestion is $KFt = \log \frac{a}{a-x} - x$, but that for small values of x the relation $x = K\sqrt{aFt}$ holds good. Fig. 104 shows the course of the reaction according to the logarithmic equation as well as according to the Schütz-Borissoff law. It is seen that up to a certain point the two graphs are practically coincident. But that if extrapolation were tried from the moment where the straight line is shown dotted the results would be totally misleading.

(2) The following values of x and y are believed to be connected by an equation of the form $y = a + bx^2$:

x	0	1	2	3	4	5	6	7	8
y	2	2·05	2·2	2·45	2·8	3·25	3·8	4·45	5·2

Test the correctness of this assumption and find the values of the constants a and b.

Plotting y and x gives the curve shown in Fig. 105; but it is obvious that mere inspection altogether fails to identify the type of curve to which

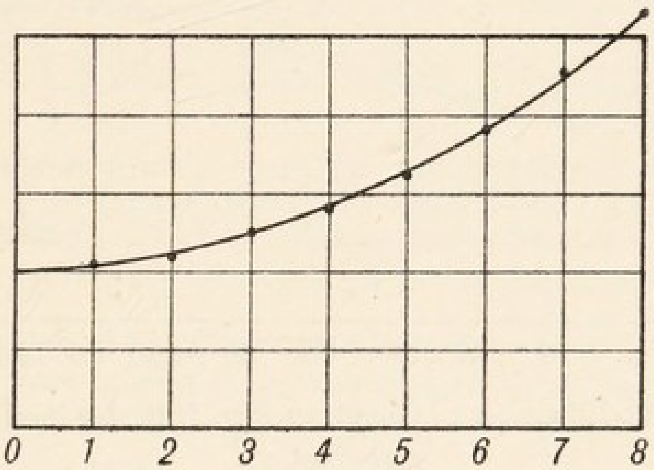

Fig. 105.

the plotted portion belongs. From the formula $y = a + bx^2$ we would expect it to be a portion of a parabola. This it might well be, but it might from its appearance equally well be a portion of an hyperbola or of an exponential curve. If, however, instead of plotting y and x, as we have done, we plot y and x^2, then, if the resulting graph is a straight line we know that $y = a + bx^2$ is the correct formula connecting the given values of x and y.

For, putting $x^2 = X$, the equation becomes $y = a + bX$, which is a straight line.

If we look at the graph (Fig. 106) resulting from plotting y against x^2 we see that it is a straight line and therefore the assumption is correct.

To find the values of a and b, we proceed exactly as in example (1) on p. 297.

When $x = 0, y = a.$

$$\therefore \quad a = 2.$$

To find b, take any pair of values of X (or x^2) and y, and put them into the equation $y = a + bx^2.$

Thus when $y = 2{\cdot}2,\ x^2 = 4.$

$$\therefore \quad 2{\cdot}2 = 2 + 4b.$$

$$\therefore \quad b = \frac{2}{4} = {\cdot}05.$$

$\therefore$ equation is $y = 2 + {\cdot}05x^2.$

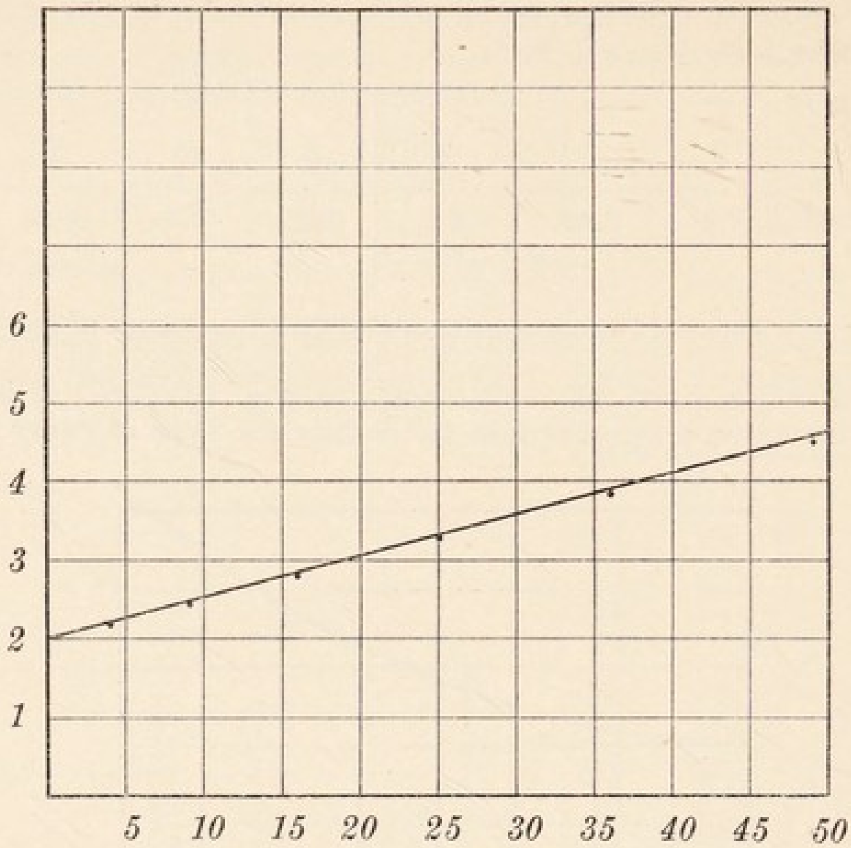

Fig. 106.—Modification of Graph in Fig. 105 obtained by plotting y against x^2.

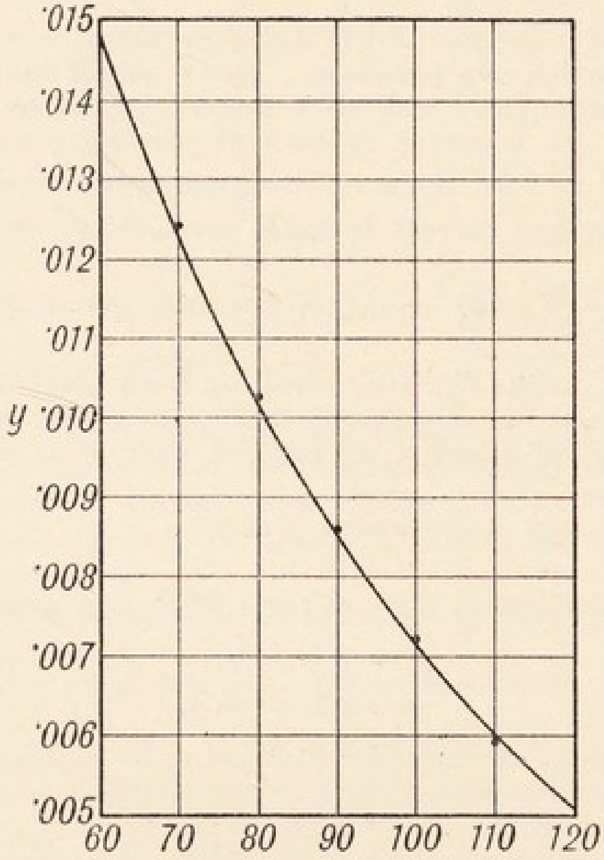

Fig. 107.

The value of the slope b can also be read off directly from the straight line graph, when it is seen to be equal to 0·05.

Note.—Whenever we have reason to expect that $\frac{dy}{dx}$ is proportional to x, the formula $y = a + bx^2$ must be tried (since $\frac{dy}{dx} = 2bx$).

(3) The following values of x and y are believed to be related by an equation of the form $y = Ae^{bx}$ (*i.e.*, the phenomenon in question is supposed to be an example of the compound interest law). Examine if this is so and then evaluate A and b.

x	120	110	100	90	80	70	60
y	·0051	·0059	·0071	·0085	·0102	·0124	·0148

If we were to plot x and y we would get a portion of a curve (Fig. 107) which might be exponential, but might equally be a portion of a parabola or of some other curve. But by taking logarithms of both sides we get:

$$\log_{10} y = \log_{10} A + bx \log e = \log_{10} A + \cdot 4343\, bx,$$

which is an equation of the first degree in x and y. If, therefore, the given values of x and y are related by an equation of the form $y = Ae^{bx}$, then, by plotting x and $\log y$, we should get a straight line whose y intercept is $\log_{10} A$ and whose slope is ·4343b.

The plotting table will be:

x	120	110	100	90	80	70	60
$\log y$	$\bar{3}$·7076	$\bar{3}$·7709	$\bar{3}$·8573	$\bar{3}$·9294	$\bar{2}$·0086	$\bar{2}$·0934	$\bar{2}$·1703

and the graph so plotted is seen to be a straight line (Fig. 108).

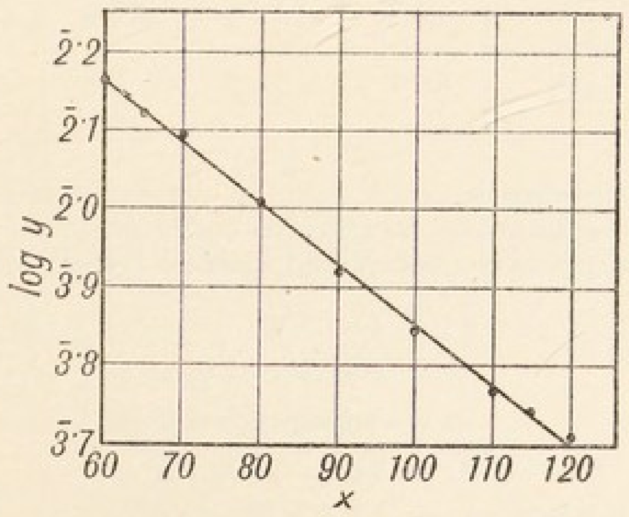

FIG. 108.—Modification of Graph in Fig. 107, obtained by plotting $\log y$ against x.

Hence the assumption that $y = Ae^{bx}$ is true. In order to find the values of A and b, we take any two pairs of values on the line so plotted,

e.g., when $x = 80$, $\log y = \bar{2}{\cdot}0086$.

when $x = 100$, $\log y = \bar{3}{\cdot}8573$.

Hence

$$\bar{2}{\cdot}0086 = \log A + {\cdot}4343 \times 80b \quad . \quad . \quad . \quad (1)$$

$$\bar{3}{\cdot}8573 = \log A + {\cdot}4343 \times 100b \quad . \quad . \quad . \quad (2)$$

By subtraction $\bar{1}{\cdot}8487 = {\cdot}4343 \times 20b$.

i.e., $-{\cdot}1513 = 8{\cdot}686b$.

$$\therefore \quad b = -\frac{1513}{86860} = -{\cdot}018.$$

Substituting for b in equation (2) we get

$$\bar{3}{\cdot}8573 = \log A - {\cdot}4343 \times 1{\cdot}8$$

$$= \log A - {\cdot}7817.$$

$$\therefore \quad \log A = \bar{3}{\cdot}8573 + {\cdot}7817$$

$$= \bar{2}{\cdot}6390.$$

$$\therefore \quad A = {\cdot}0435.$$

$\therefore$ final equation is

$$y = {\cdot}0435e^{-{\cdot}018x}.$$

The value of b could be ascertained by mere inspection.

$\therefore$ since ${\cdot}4343b = -\tan\theta$, where $-\theta$ is the angle made by the line with the x axis.

Here $\tan\theta$ is obviously $-\dfrac{{\cdot}47}{120} = -{\cdot}0078$.

$$\therefore \quad b = -\frac{{\cdot}0078}{4343} = -0{\cdot}018.$$

(4) The activity (in arbitrary units) of a certain volume of radium emanation was as follows :

Time (hours)	0	20·8	187·6	354·9	521·9	786·9
Activity .	100	85·7	24	6·9	1·5	·19

Find if the relationship between t and A corresponds to the compound interest law, and calculate the radioactive constant.

If this is an example of the compound interest law, then

$$A_t = A_0\epsilon^{-\lambda t},$$

where $A =$ activity at time t

and $\lambda =$ radioactive constant,

$$\therefore \quad \log A_t = \log A_0 = \lambda t \log \epsilon,$$

or $$2{\cdot}3 \log_{10} A_t = 2{\cdot}3 \log A_0 - \lambda t.$$

Hence by plotting $\log A_t$ against t we ought to get a straight line.

The plotting table now becomes :

t	0	20·8	187·6	3521·9	521·9	786·9
log A_t	2	1·933	1·380	·839	1·76	$\bar{1}$·279

The resulting graph is a straight line (Fig. 109). Hence the formula $A_t = A_0\epsilon^{-\lambda t}$ is correct. In the graph λ represents the tangent of the angle made by the line with the axis of x. This is obviously $-\frac{2}{580} = -3{\cdot}4 \times 10^{-3}$.

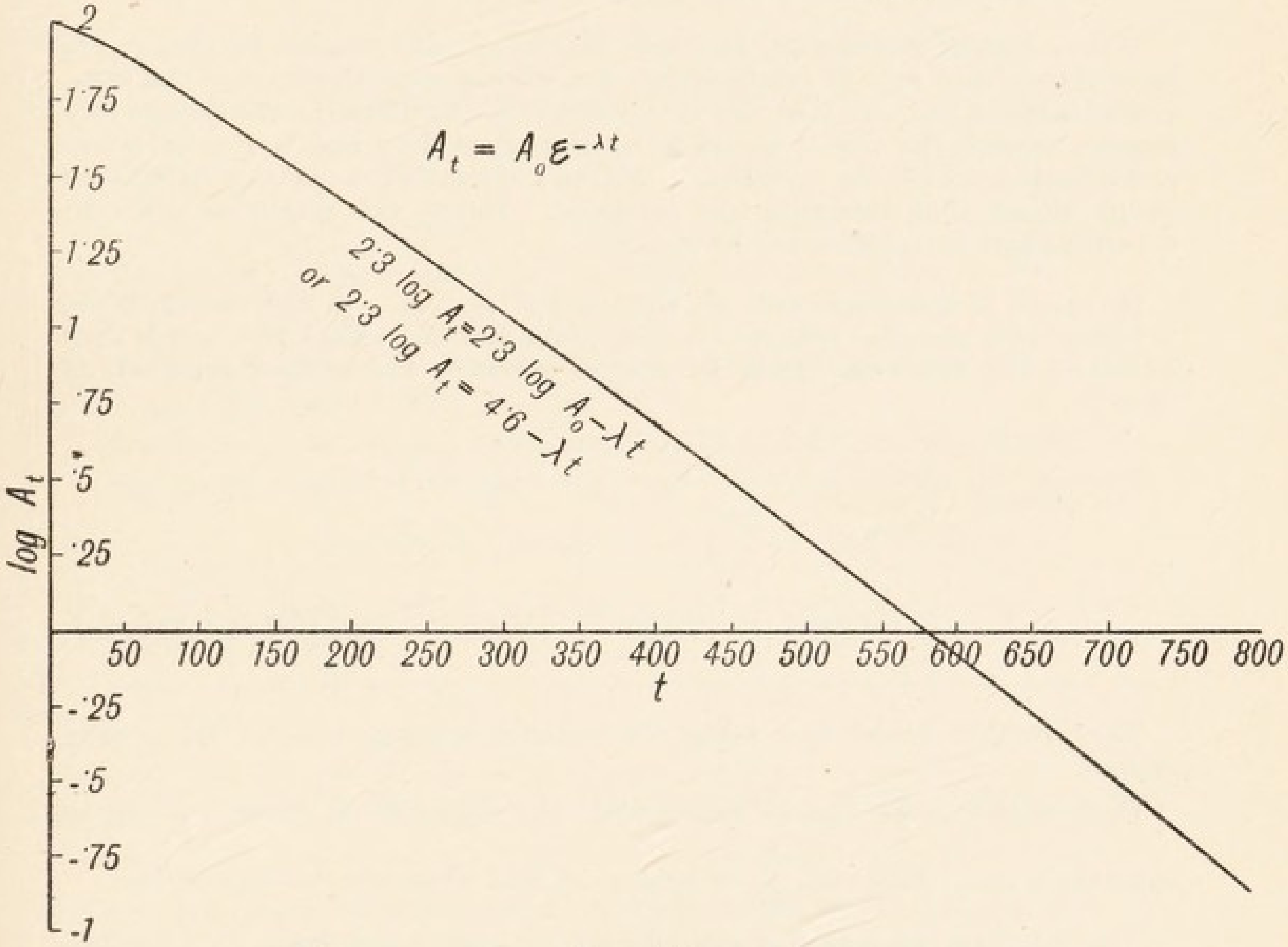

Fig. 109.

(5) The area of a wound was determined every four days by applying a sheet of transparent cellophane and making drawings on it of the edges of the wound. The areas of these drawings were then measured by the planimeter, and the following are the corrected results obtained :

No. of days . . .	0	4	8	12	16	20	24	28	32	36
Area in sq. cm. (A.) .	107	88	74·2	61·8	51	41·6	33·6	26·9	21·3	16·8

Show that cicatrisation follows the compound interest law.

By plotting log area against time we get the following table (to two places of decimals):

t (days) .	0	4	8	12	16	20	24	28	32	36
log A .	2·03	1·94	1·87	1·79	1·71	1·62	1·53	1·43	1·33	1·23

The resulting graph is a straight line.

Note.—Carrel, Hartmann, Lecomte du Noüy, and others, *loc. cit.*, p. 88, have shown that rate of cicatrisation of a wound generally follows the compound interest law. In this way it is possible to calculate (in the case of any aseptic wound) the size it would be at any given date and to foretell when cicatrisation would be complete. Marked deviation from the calculated graph shows that infection has occurred. Hence the action of different dressings and antiseptics can be studied.

(6) A. G. M'Kendrick and M. Kesava Pai (*loc. cit.*, p. 289, example 6), working with *Bacillus coli*, found that when starting with 2,850 bacilli they obtained the following numbers y after growing for various intervals of time t:

t (in hours)	0	1	2	3	4
y	2,850	17,500	105,000	625,000	2,250,000

They further found that when the number reached 100,000,000 growth ceased.

Find whether the figures agree with the equation of growth given in example 6 on p. 289, viz., $\frac{dy}{dt} = by(a - y)$, and evaluate the constants.

Also calculate the period of a generation, *i.e.*, the time it takes for a bacillus to double itself.

From
$$\frac{dy}{dt} = by(a - y)$$
we get
$$\frac{1}{y}\frac{dy}{dt} = b(a - y).$$
But
$$\frac{1}{y} = \frac{d \log_e y}{dy}$$
$$\therefore \quad \frac{d \log_e y}{dy} \cdot \frac{dy}{dt} = b(a - y)$$
i.e.,
$$\frac{d \log_e y}{dt} = b(a - y).$$

In other words, the slope of the curve obtained by plotting log y against t is $b(a - y)$.

Now, when $t = 0$, $y_0 = 2850$, and when growth has ceased $y = a = 100{,}000{,}000$ (see example 6, p. 290).

$\therefore$ slope at commencement of curve, which is $\frac{d \log y_0}{dt} = b(a - y_0) = ba$ (since y_0 is small compared with a).

Hence, by plotting log y against t the slope at the commencement of the curve $= ba$. But a is known, and therefore b can be found.

The new plotting table is:

t	0	1	2	3	4
$\log_{10} y$	3·455	4·243	5·021	5·796	6·352

The graph will be found to be a straight line whose slope = ·80.

$$\therefore \quad ab = 0·8.$$

To find the period of a generation, we notice that ab is the rate of change of $\log_{10} y$ per unit of time (*i.e.*, per hour).

$\therefore$ in one hour $\log_{10} y$ has changed from 3·455 to 4·243, *i.e.*, by 0·80.

But when y has doubled itself $\log_{10} y$ has changed by $\log_{10} 2$, *i.e.*, by ·301.

$$\therefore \quad \text{period of a generation} = \frac{·301}{·80} \text{ hr.} = 22·5 \text{ mins.}$$

(7) The following values of x and y have been found. Find a formula connecting them.

x	0	1	2	3	4	5	6	7
y	0	0·7485	0·5988	0·5614	0·5444	0·5347	0·5284	0·5241

Here y decreases with increase of x; we therefore try the formula $y = \frac{ax}{x-b}$, or $x = \frac{ax}{y} + b$ and plot x against $\frac{x}{y}$ as follows:

x	0	1	2	3	4	5	6	7
$\frac{x}{y}$	0	1·336	3·339	5·343	7·348	9·357	11·30	13·36

The resulting graph is a straight line.

$\therefore$ The assumption that the observed values are connected by the equation $x = a\frac{x}{y} + b$ is true.

To find a and b we proceed exactly as in the other cases of straight line laws (pp. 297, 298), and we get $a = 0{\cdot}5$ and $b = 0{\cdot}33$.

$\therefore$ equation is $$x = 0{\cdot}5\frac{x}{y} + 0{\cdot}33,$$

or $$xy = 0{\cdot}5x + 0{\cdot}33y.$$

When other formulæ fail then one uses the equation

$$y = a + bx + cx^2 + dx^3 + ex^4 + \ldots$$

because we know that such an equation will satisfy any values of x and y provided we take a sufficient number of terms. Thus, if we stop at the first power of x we get $y = a + bx$, which, of course, represents a straight line; $y = a + bx + cx^2$ will correspond to a parabola; whilst, as we have seen, any function, whether logarithmic, exponential, trigonometric, etc., can be expanded into a series of ascending powers of x like the above. In order to evaluate the coefficients $a, b, c, d \ldots$ we make use of Maclaurin's theorem (p. 188), according to which

$$a = f(o)\,;\ b = f'(o)\,;\ c = \frac{f''(o)}{2!}\,;\ d = \frac{f'''(o)}{3!}\ \ldots$$

Hence **a** is simply the numerical value of the intercept of the y axis cut off by the original, or primitive, curve. To determine the numerical values of $b, c, d \ldots$ we must plot the curves $y' = f'(x)$, $y'' = f''(x)$, etc. $\mathbf{y}' = \mathbf{f}'(\mathbf{x})$, *i.e.*, the first slope curve or first derivative curve can be plotted, by drawing tangents to the original curve at various points and calculating the slopes at those points in the manner explained on p. 141 and in example 6 on p. 319. The values of y' so obtained are plotted against corresponding values of x, and we thus get the first derivative curve. By repeating this process we get the 2nd derivative curve $y'' = f''(x)$. The process is repeated until we get a graph which does not differ perceptibly from a straight line.

If, now, we measure the intercept of the y axis cut by each of these derivative curves we obtain the values $f'(o), f''(o), f'''(o)$, etc., and, consequently, we get the numerical values of b, c, d, etc.

Note.—In practice one does not often employ the method just described for obtaining the various derived curves since, although theoretically perfect, it is practically very inaccurate (see p. 313). For the most exact methods of plotting these curves the reader is referred to more advanced books on practical mathematics. By choosing appropriate scales for x and y, however, the curve can be drawn in such a way as to make the above methods applicable with reasonable accuracy.

EXAMPLES.

(1) The following have been found to be the respective values of x and y. Find the law connecting them:

x	0	1	2	3	4	5	6	7	8
y	2	1·85	1·8	1·85	2	2·25	2·6	3·04	3·06

Let the equation of the plotted curve be $y = a + bx + cx^2 + \ldots$

By drawing tangents at the points where $x = 1, 2, 3 \ldots$ we obtain the numerical values of the slopes as follows :

x	1	2	3	4	5	6
y'	$-0·1$	0	0·1	0·2	0·3	0·4

When these points are plotted they are found to be on a straight line (which is, of course, $y' = f'x = b + cx$), whose slope is 0·1 and whose intercept on the y axis is $-$ ·2.

$\therefore$ equation of line is $y' = 0·1x - 0·2$.

$\therefore$ $b = -0·2$ and $c = 0·05 \left(= \frac{f''(o)}{2\,!} \right)$

But $a = f(o) = 2$.

$\therefore$ the law connecting the two variables, or the equation of the primitive curve is $y = 2 - 0·2x + 0·05x^2$.

(2) The following values have been found for x and y. Find the law connecting them :

x	-3	-2	-1	0	1	2	3	4	5
y	0	24	30	24	12	0	-6	0	24

Plot the curve by making one unit length on the x axis equal to 10 units on the y axis. It will then be found that the slopes can be determined with fair accuracy at various points. The following will be found to be the values of y' for the stated values of x :

x	-3	-2	-1	0	2	3	4
y'	35	14	1	-10	-10	-1	14

This will plot into what looks like a parabola, crossing the y axis at $y = -10$, and calculating the slopes at various points on this first derivative curve, one gets the following values of y'' and x :

x	-1	0	1	2
y	-12	-6	0	6

The graph of this is $y'' = 6x - 6$ ($= 6dx - 6$).

$$\therefore \quad a = f(o) = 24 \; ; \; b = f''(o) = -10 \; ;$$

$$c = \frac{f''(o)}{2\,!} = -\frac{6}{2} = -3 \; ; \; d = \frac{f'''(o)}{3\,!} = 1.$$

$\therefore$ law connecting x and y is

$$y = x^3 - 3x^2 - 10x + 24.$$

Summary.—If there is no theoretical basis to guide us and the smooth graph plotted from the pairs of values of x and y is not a straight line, then one tries any of the following methods :

(1) Plot y against x^2 or against $\sqrt{x}$. If the result is a straight line then the equation is of the form $y = a + bx^2$ or $y = a + b\sqrt{x}$.

(2) Plot y against $\log x$ or $\log y$ against x. If the result is a straight line then the equation is of the form $y = \mathrm{A}e^{bx}$.

(3) Plot $\log y$ against $\log x$. If the result is a straight line the equation is of the form $y = ax^n$.

(4) If $y = \mathrm{K}\frac{y}{x}$, try the formula $y = \frac{ax}{x - b}$.

(5) If the functions seem to be periodic, try Fourier's series :
$y = a_0 + a_1 \sin x + b_1 \cos x + a_2 \sin 2x + b_2 \cos 2x + \; . \; . \; .$
(see pp. 264 *et seq.*).

(6) If all the above formulæ fail, use the general equation $y = a + bx + cx^2 + dx^3 + \; . \; . \; .$ and evaluate the constants in the manner described above. It may, for instance, be found that $\frac{dy}{dx} = \mathrm{K}x$, when we know that the equation is of the form $y = a + bx^2$ (Schütz-Borissoff law), or it may be found that $\frac{dy}{dx} = \mathrm{K}y$, when it is an example of the compound interest law, $y = \mathrm{A}e^{bx}$, etc.

Determination of the Order of a Chemical Reaction.—Another most useful application of the method of graphical analysis is for the purpose of determining the **order** of a chemical reaction—by which is meant the number of molecules taking part in the reaction. For example, the splitting of hydrogen peroxide into water and oxygen in the presence of hæmase (blood catalase) is unimolecular, or a reaction of the first order, thus $H_2O_2 = H_2O + O$.

The saponification of an ester is bimolecular, or of the second order, thus :

$$C_2H_5OOC_2H_3 + NaOH = NaOOC_2H_3 + C_2H_5OOH.$$

In many cases in biochemistry it is not easy to represent the reaction by means of an exact equation, and hence it is difficult or impossible to determine the order of the reaction by mere inspection. Moreover, even when it is possible to represent the reaction by means of an equation, it is not always possible to say with certainty whether the reaction is uni-, bi- or multi-molecular. For instance, in the case of decomposition of H_2O_2 by hæmase, although we know that one molecule of H_2O_2 gives rise to one molecule of H_2O, it is not possible, without the aid of mathematics, to say whether the order is first or second, *i.e.*, whether the equation is:

$$H_2O_2 = H_2O + O,$$

or

$$2H_2O_2 = 2H_2O + O_2.$$

(1) One method of determining the order of a reaction we have already dealt with in Chapter XIII. It consists in calculating the velocity constant K by means of the formulæ for unimolecular, bimolecular, etc., reactions. The reaction is of that order for which the formula gives reasonably constant values of K.

The following are *graphical methods* of determining the order of a reaction.

(2) We have seen that $\frac{dx}{dt} = K(a - x)^n$, where n = order of the reaction.

If we put $(a - x) = C$ (*i.e.*, concentration of the original substance at any instant t),

then
$$\frac{dx}{dt} = -\frac{dC}{dt}$$

and
$$(a - x)^n = C^n.$$

∴ equation becomes

$$-\frac{dC}{dt} = KC^n.$$

$$\therefore \quad \frac{dC}{C^n} = -Kdt.$$

$$\therefore \quad \int\frac{dC}{C^n} = -Kt + A \text{ (A = integration constant).}$$

Hence,

(i.) **In the case of unimolecular reactions** ($n = 1$)

$$\int\frac{dC}{C} = -Kt + A,$$

i.e.,

$$\mathbf{\log C = -Kt + A.}$$

(ii.) **In the case of bimolecular reactions** ($n = 2$)

$$\int \frac{dC}{C^2} = -Kt + A,$$

i.e.,

$$-\frac{1}{C} = -Kt + A,$$

or

$$\mathbf{\frac{1}{C} = Kt - A.}$$

(iii.) **In the case of termolecular reactions** ($n = 3$)

$$\int C^{-3} dC = -Kt + A,$$

or

$$-\frac{1}{2C^2} = -Kt + A,$$

or

$$\mathbf{\frac{1}{2C^2} = Kt - A.}$$

(iv.) **Generally in the case of n-molecular reactions**

$$\int c^{-n} dC = -Kt + A,$$

or

$$\mathbf{\frac{1}{(1-n)\,C^{n-1}} = Kt - A.}$$

Hence, we get the following *rule* :

To find the order of a reaction we have to ascertain by trial which of the following expressions gives a straight line when plotted against time as abscissa.

(i.) **log C – unimolecular.**

(ii.) $\mathbf{\frac{1}{C}}$ **– bimolecular.**

(iii.) $\mathbf{\frac{1}{C^2}}$ **– termolecular.**

(iv.) $\mathbf{\frac{1}{C^{n-1}}}$ **– n-molecular.**

This method, therefore, like the last, involves a certain number of trials in order to arrive at the solution. There is, however, another graphical method, which by drawing the graph one can, by means of a mathematical formula, arrive at the solution at once.

(3) **The Differential Method** (see examples, pp. 319 *et seq.*).—Draw on a large scale a graph representing the change of concentration (x) with time (t). Then the velocity of any moment can be determined by measuring the angle which the tangent at that point

makes with the t axis, *i.e.*, by the slope of the curve at that point. Thus, if when the change in concentration is x_1, the angle made by the tangent with the t axis is ϕ_1, and when the change in concentration is x_2, the angle made by the tangent is ϕ_2, then the velocities at these two points are given by $\tan\phi_1$ and $\tan\phi_2$ respectively $\left(\text{since } \tan\phi = \frac{dx}{dt}\right)$, and can be read off directly from the graph,

$$\therefore \quad \left.\begin{array}{l} \tan\phi_1 = \mathrm{K}(c - x_1)^n \\ \text{and} \quad \tan\phi_2 = \mathrm{K}(c - x_2)^n \end{array}\right\} \text{where } n = \text{order of reaction.}$$

$$\therefore \quad \frac{\tan\phi_1}{\tan\phi_2} = \frac{(c - x_1)^n}{(c - x_2)^n}.$$

$$\therefore \quad \log\frac{\tan\phi_1}{\tan\phi_2} = n\log\frac{(c - x_1)}{(c - x_2)},$$

whence
$$n = \frac{\log\dfrac{\tan\phi_1}{\tan\phi_2}}{\log\dfrac{(c - x_1)}{(c - x_2)}} = \frac{\log\dfrac{u_1}{u_2}}{\log\dfrac{(c - x_1)}{(c - x_2)}},$$

(where $u_1 = \tan\phi_1$ and $u_2 = \tan\phi_2$).

As n must be an integer, it must be taken as the integer nearest to the value given by the right-hand side of the equation.

The two great objections to this method are the following:

(1) Errors may occur in drawing the curve.

(2) There is a great difficulty of reading off the value of the slope at any point with sufficient accuracy.

(4) **Still another method** is to "start the reaction with equivalent quantities of the reacting substances and determining in two experiments (which differ in concentration) the time required to consume half of the substance" (Nernst).

In the case of a *unimolecular* reaction the *time* is *independent* of the original concentration (Chapter XIII., p. 218).

In the case of a *multimolecular* reaction the *time* is *inversely proportional to* a^{n-1}, where n is the order of the reaction (Chapter XIII., p. 222).

Thus, if $n = 2$; $t \propto \frac{1}{a}$.

if $n = 3$; $t \propto \frac{1}{a^2}$.

if $n = 4$; $t \propto \frac{1}{a^3}$,

etc.

EXAMPLES.

(1) Madsen and Famulener in an investigation of the attenuation of vibriolysin at 28° C. found the following results :

Time in Minutes.	Concentration.
0	100
10	78·3
20	67·6
30	59·3
40	49·8
50	40·8
60	34·4

Find the **order** of this reaction.

If we take the logarithms of the concentrations and plot them against time, we obtain the following plotting table :

t . .	0	10	20	30	40	50	60
log C .	2	1·894	1·83	1·775	1·699	1·61	1·537

The graph is a straight line (Fig. 110), hence the reaction is unimolecular.

(2) Miss Chick and Professor Martin studied the coagulation of

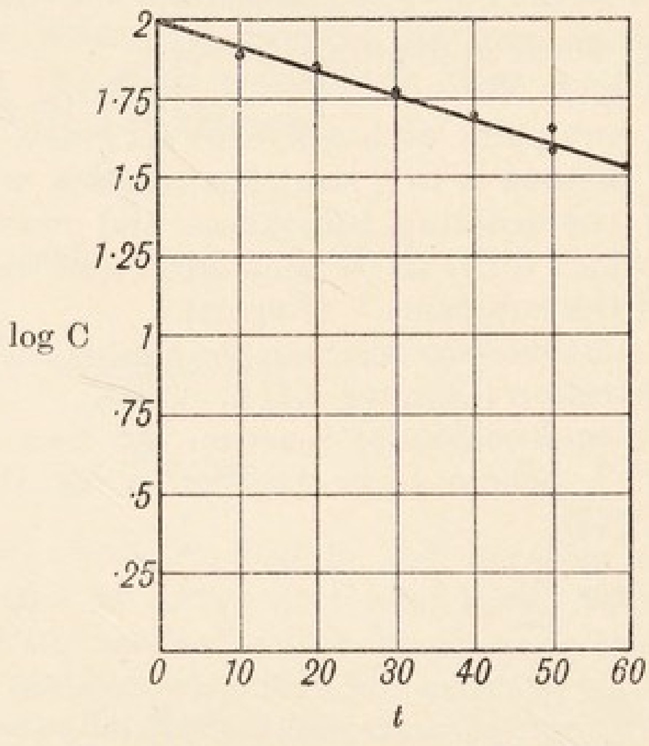

FIG. 110.—Attenuation of Vibriolysin.

hæmoglobin at various temperatures and obtained the following results at 70·4° C. :

t (in mins.)	0	2	4	6	7·5
C (concentration of Hb)	100	52·5	25·3	14·1	7·6

To which order does this reaction belong ?
What is the value of K ?
By taking the logarithms of C we get the following table :

t . .	0	2	4	6	7·5
log C . .	2	1·72	1·41	1·15	0·89

When log C is plotted against t, the resulting graph is a straight line (Fig. 111).

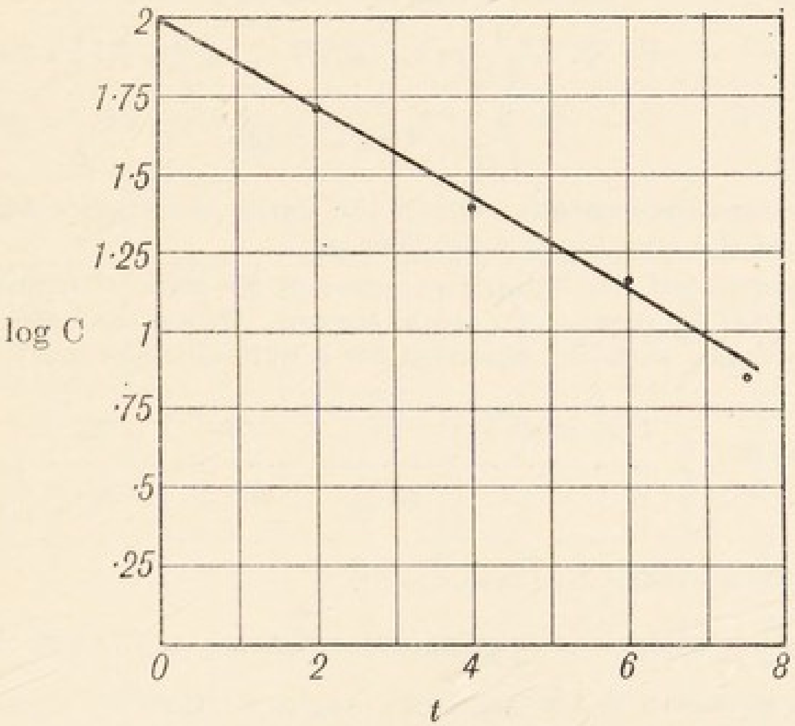

FIG. 111.—Coagulation of Hæmoglobin.

Therefore the coagulation proceeds as a monomolecular reaction.
Now the equation of a monomolecular reaction is

$$K = \frac{1}{t} \log \frac{a}{a - x} = \frac{1}{t} [\log a - \log (a - x)],$$

where a = initial concentration = C_o (= 100 per cent.),
and $(a - x)$ = concentration at any time $t = C_t$,

$$\therefore \quad K = \frac{1}{t} (2 - \log C_t).$$

By taking any pair of corresponding values of t and $\log C_t$ we can therefore find K.

Thus, when $t = 2, \log C_t = 1{\cdot}72.$

$$\therefore \quad K = \frac{1}{2}(2 - 1{\cdot}72)$$

$$= \frac{1}{2}(0{\cdot}28) = 0{\cdot}140.$$

When $t = 4, \log C_t = 1{\cdot}41.$

$$\therefore \quad K = \frac{1}{4}(2 - 1{\cdot}41)$$

$$= \frac{1}{4} \times 0{\cdot}59 = 0{\cdot}148.$$

When $t = 6, \log C_t = 1{\cdot}15.$

$$\therefore \quad K = \frac{1}{6}(2 - 1{\cdot}15)$$

$$= \frac{1}{6} \times {\cdot}85 = 0{\cdot}142.$$

When $t = 7{\cdot}5, \log C_t = {\cdot}89.$

$$\therefore \quad K = \frac{1}{7{\cdot}5}(2 - {\cdot}89)$$

$$= \frac{1}{7{\cdot}5} \times 1{\cdot}11 = 0{\cdot}148.$$

Hence K is practically constant (within the limits of experimental error) again proving that the reaction is unimolecular.

(3) Victor Henri found the following figures in the case of hæmolysis of chicken erythrocytes by means of normal serum. Prove that the reaction proceeds in accordance with the equation for a unimolecular reaction.

Quantity of serum = 0·3 c.c.	t (in mins.)	24	63	94	190
	x . .	33%	56%	78%	96%

The equation for a unimolecular reaction is

$$K = \frac{1}{t}\log\frac{a}{a - x} \text{ (where } a = \text{original concentration} = 100 \text{ per cent.).}$$

Hence if this equation holds good, we ought to find by substituting various corresponding values of t and x that K is the same in each case.

	$\frac{1}{t}\log\frac{100}{100 - x} = K.$	
$t = 24$, $x = 33$	$\frac{1}{24}\log\frac{100}{67}$	$\frac{1}{24}(2-1{\cdot}826) = {\cdot}0072$
$t = 63$, $x = 56$	$\frac{1}{63}\log\frac{100}{44}$	$\frac{1}{63}(2-1{\cdot}643) = {\cdot}0057$
$t = 94$, $x = 78$	$\frac{1}{94}\log\frac{100}{22}$	$\frac{1}{94}(2-1{\cdot}342) = {\cdot}0070$
$t = 190$, $x = 96$	$\frac{1}{190}\log\frac{100}{4}$	$\frac{1}{190}(2-{\cdot}602) = {\cdot}0073$

So that K is practically constant within the limits of experimental error, and therefore the reaction proceeds in accordance with the equation for a unimolecular reaction.

We could have shown the same thing graphically by plotting t against $\log(a - x)$ thus:

t	24	63	94	190
$\log(a - x)$	·174	·357	·658	1·398

The result is a straight line.

(4) Madsen and Walbum studied the progress of tryptic digestion by subjecting 10 grm. of casein powder to 100 c.c. of a 1 per cent. solution of trypsin at constant temperature, and testing the amount of remaining casein $(a - x)$ at various times t by means of nitrogen determinations (by Kjeldahl's method). The following figures represent the results found.

Find whether these figures agree with the supposition that the process is a bimolecular reaction.

Time (hrs.)	0	0·5	2·5	6	11	24	33	48	72
Nitrogen	·11	0·108	0·102	0·1	0·096	0·076	0·07	0·06	0·049

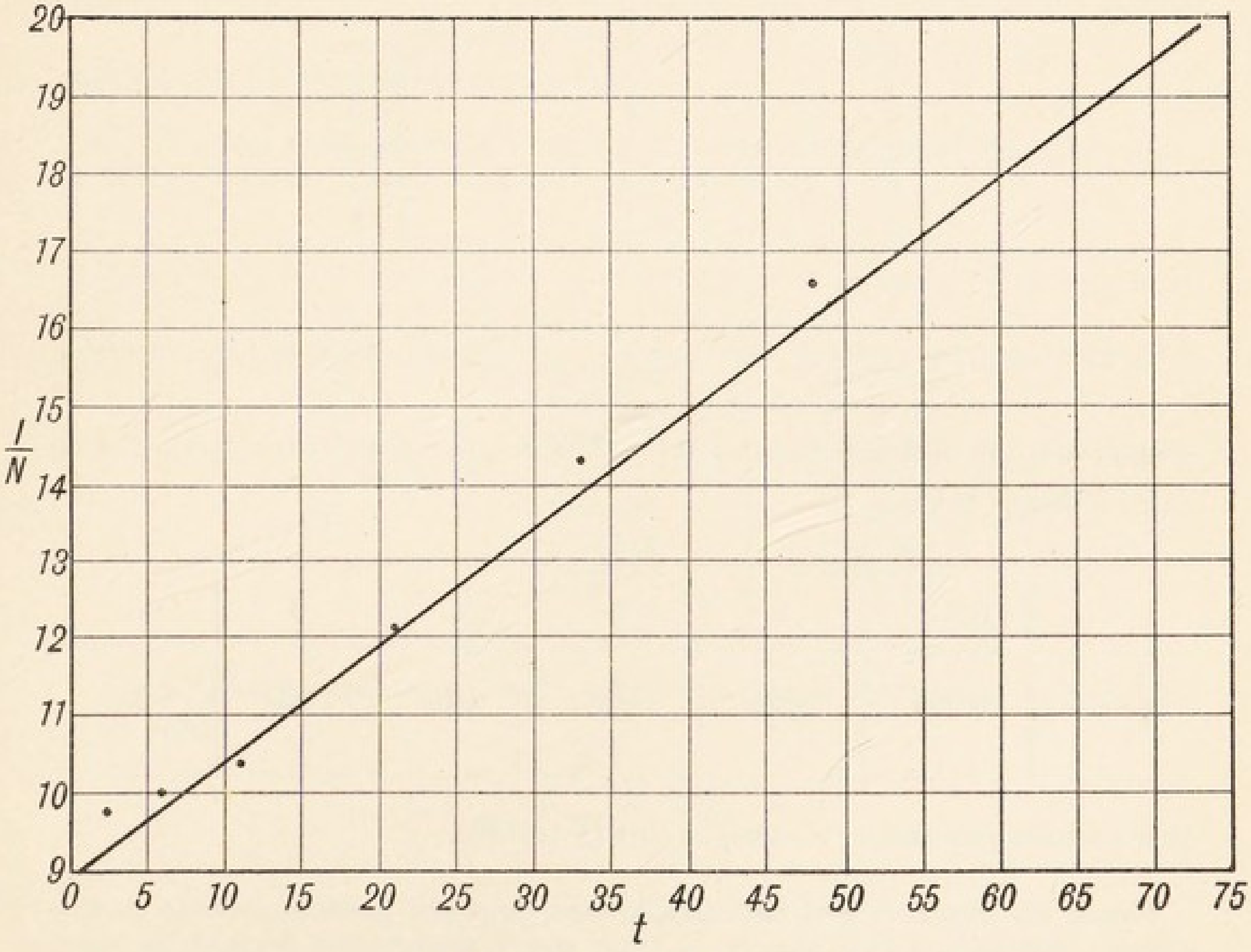

Fig. 112.—Tryptic Digestion.

If the process is a bimolecular reaction, then t when plotted against the reciprocal of nitrogen concentration should be a straight line. The plotting table would be :

t .	0	0·5	2·5	6	11	24	33	48	72
$\frac{1}{N}$.	9	9·2	9·8	10·0	10·4	13·0	14·3	16·6	20·0

The graph (see Fig. 112) is practically a straight line, therefore the process goes on as a bimolecular reaction.

(5) Madsen and Walbum studied the decomposition of tetanolysin by means of peptone. Quantitative examinations would suggest the following scheme under which the reaction occurs, viz. :

Two molecules of tetanolysin + 3 molecules of peptone
= lysinpeptone.

Investigate if this scheme is correct. The following results being given in the case of a certain experiment :

Time in Hours.	Concentration of Tetanolysin.
0·5	47·7
1	39·7
2	30·3
4	22·3
6	18·1
8	17·0

In this case by plotting log C or $\frac{1}{C^2}$, $\frac{1}{C^3}$, $\frac{1}{C^4}$, etc., against t the resulting graphs are not straight lines, but by plotting $\frac{1}{C}$ against time according to the following table :

t .	0·5	1	2	4	6	8
$\frac{1}{C}$.	0·021	·025	·033	·045	·055	·059

there results practically a straight line (Fig. 113).

Hence the reaction is a bimolecular one, and not pentamolecular as the scheme suggests. The reason is that the lysinpeptone formed is very unstable and decomposes as soon as it is formed with reformation of peptone,

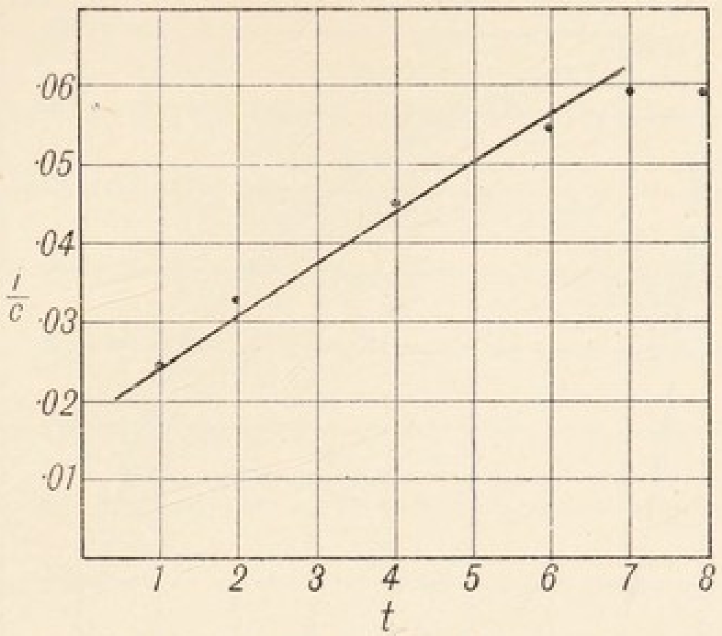

Fig. 113.—Decomposition of Tetanolysin.

so that the concentration of peptone remains constant. Hence in the equation :

$$\text{Reaction velocity} = K(C_T)^2 \,.\, (C_p)^3$$

the C_p being constant, we get

$$\text{Reaction velocity} = K(C_T)^2,$$

making the reaction a bimolecular one.

(6) The following values of x and $(c - x)$ were observed at the given times t. Find the order of the reaction by method (3).

t	x	$c - x$
1	0·01434	0·04816
1·75	0·01998	0·04252
3	0·02586	0·03664
4·5	0·03076	0·03174
7	0·03612	0·02638
11	0 04102	0·02148
17	0·04502	0·01748

The graph (t against x) is shown in Fig. 114. u_1 is seen to be $=$ ·0023 and $u_2 =$ ·0053.

$$\therefore \qquad \frac{u_1}{u_2} = \frac{23}{53} = \cdot 434.$$

$$\therefore \quad \log \frac{u_1}{u_2} = \log 0{\cdot}434 = \bar{1}{\cdot}6375 = -\ \cdot 3625.$$

$$(c - x_2) = 0{\cdot}04252, \text{ and } (c - x_1) = 0{\cdot}03174.$$

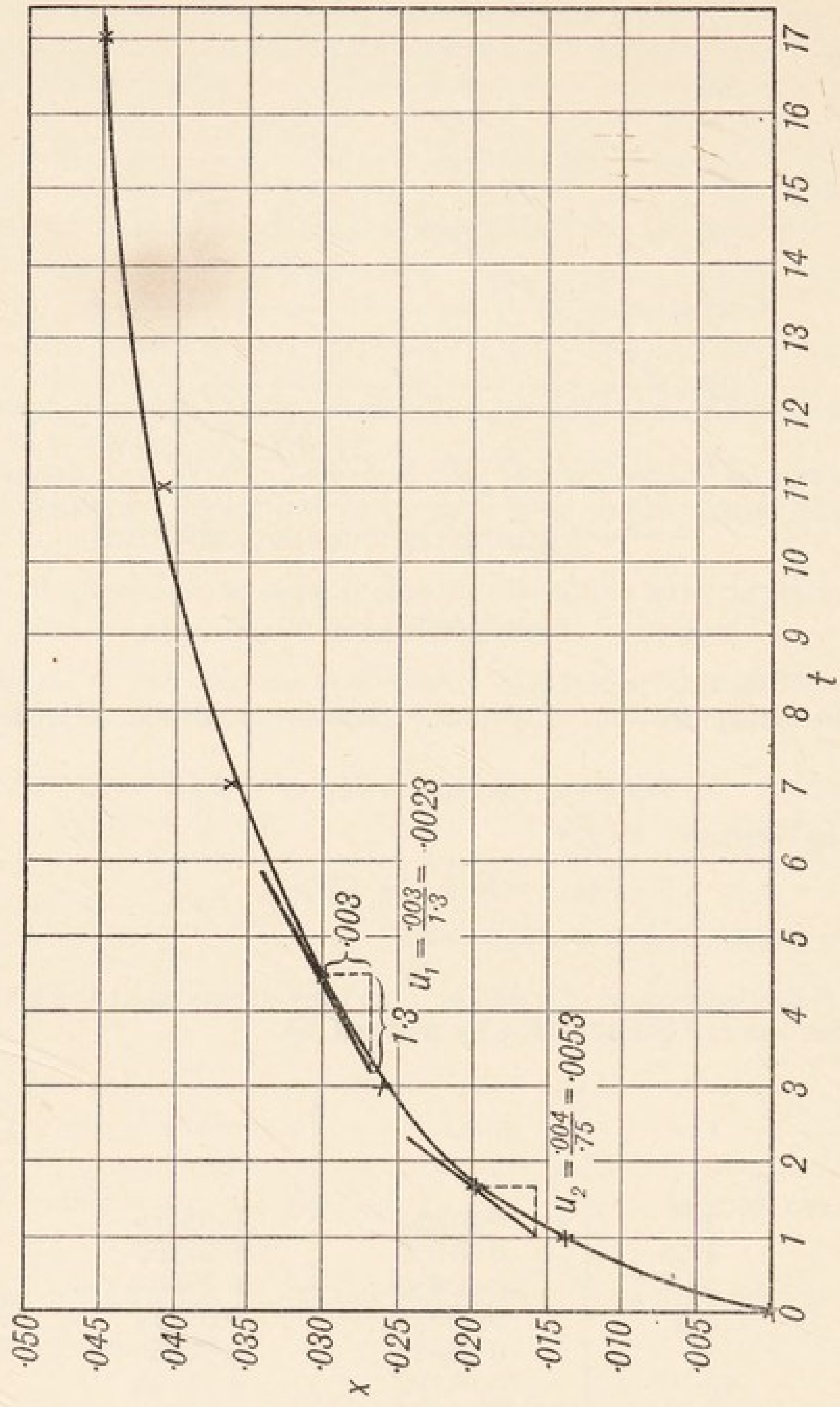

Fig. 114.

$$\therefore \qquad \frac{c - x_1}{c - x_2} = \frac{3{,}174}{4{,}252} = \cdot 723.$$

$$\therefore \quad \log\left(\frac{c - x_1}{c - x_2}\right) = \log 0\cdot 723 = \bar{1}\cdot 8591 = -\ \cdot\mathit{1409}.$$

$$\therefore \qquad n = \frac{3625}{1409} = 2\cdot 6 = \mathbf{3} \text{ (to nearest integer)}.$$

$\therefore$ reaction is trimolecular.

(7) The following values of x and $(c - x)$ were found in the case of destruction of vibriolysin by peptone. Find the order of the reaction.

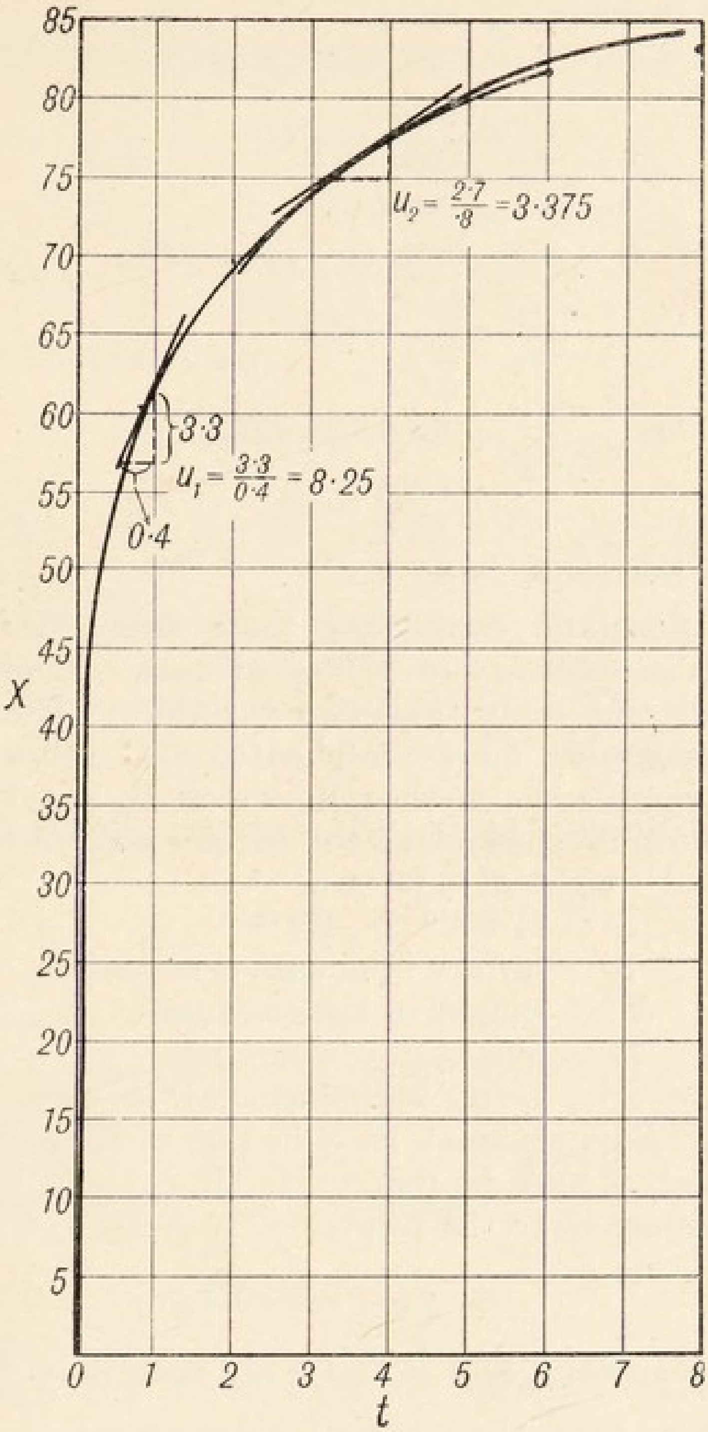

Fig. 115.

t	x	$c - x$
0·5	52·3	47·7
1	60·3	39·7
2	69·7	30·3
4	77·7	22·3
6	81·9	18·1
8	83	17

The graph is shown in Fig. 115.

$$u_1 = 8{\cdot}25, \text{ and } u_2 = 3{\cdot}375.$$

$$\therefore \quad \frac{u_1}{u_2} = \frac{8{\cdot}25}{3{\cdot}375} = 2{\cdot}44.$$

$$\log 2{\cdot}44 = {\cdot}\mathit{3874}.$$

$$(c - x_1) = 39{\cdot}7, \ (c - x_2) = 22{\cdot}3.$$

$$\therefore \quad \frac{c - x_1}{c - x_2} = \frac{39{\cdot}7}{22{\cdot}3} = 1{\cdot}8.$$

$$\therefore \quad \log \frac{(c - x_1)}{(c - x_2)} = \log 1{\cdot}8 = {\cdot}\mathit{2553}.$$

$$\therefore \quad n = \frac{3{,}874}{2{,}553} = \mathbf{1{\cdot}5}.$$

$\therefore$ reaction is bimolecular.

Empirical Formulæ connecting more than Two Variables.— Sometimes it is necessary in biological work to find an empirical formula connecting more than two variables. The process then becomes considerably more complicated and difficult. We shall take as an example the method by which D. and E. F. Du Bois found their formula connecting the surface area of the body with the height and weight of a person,

viz., $S = 71{\cdot}8 W^{0{\cdot}425} . H^{0{\cdot}725}$,

where S = surface in square centimetres.
W = weight in kilogrammes.
H = height in centimetres.

The surface area being bidimensional, it may be put $= A^2$. Weight being tridimensional, let it be put $= M^3$, and height being unidimensional, it may be put $= H$. It is clear, therefore, that any formula must be of the form

$$S = K . W^{\frac{1}{m}} . H^{\frac{1}{n}},$$

where K is a constant, and m and n are such that $\frac{3}{m} + \frac{1}{n} = 2$.

For by taking logarithms of both sides we get

$$\log S = \log K + \frac{1}{m} \log W + \frac{1}{n} \log H,$$

or

$$\log A^2 = K + \frac{1}{m} \log M^3 + \frac{1}{n} \log H$$

(where K stands for the constant $\log$ K),

i.e.,

$$2 \log A = \frac{3}{m} \log M + \frac{1}{n} \log H + K,$$

$$\therefore \quad 2 = \frac{3}{m} + \frac{1}{n}.$$

Possible values of $\frac{1}{m}$ and $\frac{1}{n}$ are $\frac{1}{3}$ and 1, or $\frac{1}{2}$ and $\frac{1}{2}$

$$\left(\text{thus } \frac{3}{3} + 1 = 2, \text{ and } \frac{3}{2} + \frac{1}{2} = 2\right).$$

Let us take $$\frac{1}{m} = \frac{1}{3} \text{ and } \frac{1}{n} = 1,$$

The formula then becomes

$$S = KW^{\frac{1}{3}} . H \text{ or } K = \frac{S}{W^{\frac{1}{3}}H}.$$

The following are a few of the values found by actual measurement :

	(1)	(2)	(3)	(4)	(5)
S	8473	16720	12320	20760	14907
W	24·2	64·0	36·5	87·1	45·2
H	110·3	164·3	146·0	182·8	171·8

From this table we get (by using formula $S = KW^{\frac{1}{3}}H$) and $K = \frac{S}{W^{\frac{1}{3}}H}$.

(1) $K = 26·7$
(2) $K = 25·5$
(3) $K = 25·4$ } The corrected average was found to be $K = 25·6$.
(4) $K = 25·8$
(5) $K = 24·8$

By using the formula $S = KW^{\frac{1}{2}}H^{\frac{1}{2}}$

$$\text{and } K = \frac{S}{W^{\frac{1}{2}}H^{\frac{1}{2}}},$$

the values of K were found by using the above table of values of S, W and H.

(1) $K = 164·0$
(2) $K = 163·0$
(3) $K = 168·6$ } The corrected average was found to be 167·2.
(4) $K = 164·5$
(5) $K = 169·2$

Now, it is seen that in case (5), for instance, the percentage variation of K from the corrected mean value is positive when the first formula is used, but negative when the second formula was used. Hence, the indication is that $\frac{1}{m}$ must be greater than

$\frac{1}{3}$ and less than $\frac{1}{2}$, whilst $\frac{1}{n}$ must be less than 1 and greater than $\frac{1}{2}$ (always providing $\frac{3}{m} + \frac{1}{n} = 2$).

Supposing one takes $\qquad \frac{1}{m} = \frac{1}{2{\cdot}5} = \frac{2}{5}.$

This would make $\qquad \frac{1}{n} = 2 - \frac{3{\cdot}2}{5} = \frac{4}{5}.$

The resulting value of K would then be

$$K = \frac{S}{W^{\cdot 4}H^{\cdot 8}}.$$

By using this formula, better results were obtained than by means of either of the other two previous formulæ.

The best value was found to be

$$\frac{1}{m} = \frac{1}{2{\cdot}35} = {\cdot}425$$

$$\frac{1}{n} = \frac{1}{1{\cdot}38} = {\cdot}725$$

$$\left(\frac{3}{m} + \frac{1}{n} = 3 \times {\cdot}425 + {\cdot}725 = 1{\cdot}275 + {\cdot}725 = 2\right).$$

∴ the formula is

$$S = KW^{0{\cdot}425}H^{0{\cdot}725},$$

and

$$K = \frac{S}{W^{0{\cdot}425}H^{0{\cdot}725}}.$$

(1) $K = 71{\cdot}30$
(2) $K = 70{\cdot}65$
(3) $K = 72{\cdot}01$
(4) $K = 71{\cdot}22$
(5) $K = 70{\cdot}36$

The corrected average was found to be 71·84.

∴ formula is $\qquad S = 71{\cdot}84W^{0{\cdot}425}H^{0{\cdot}725}.$

(See p. 123 *et seq.*, for a nomogram for this equation.)

EXERCISES.

(1) Find the law connecting x and y from the following observation (allowing for errors of observation).

x	2·5	3·5	4·4	5·8	7·5	9·6	12·0
y	13·5	17·6	22·2	28·0	35·5	47·4	56·1

[Answer, $y = 1{\cdot}42 + 4{\cdot}66x$.]

(2) The following values have been found for x and y :

x	4	5	6	7	8	9	10	11
y	6·29	5·72	5·22	4·78	4·35	4·06	3·75	3·48

It is found that the following two empirical formulæ seem to be nearly equally good :

$$y = \frac{a}{b + x}, \text{ and } y = \alpha e^{-\beta x}.$$

Find the best values of a and b, α and β.

[Answer, $a = 54·53$, $b = 4·67$; $\alpha = 8·706$, $\beta = 0·084$].

(3) The following values of x and y are believed to be related by an equation of the form $y = \mathrm{A}e^{bx}$. Examine if this is so and calculate A and b.

x	0·1	0·2	0·3	0·4	0·5	0·6
y	0·4254	0·4093	0·3704	0·3352	0·3032	0·2744

[Answer, $y = 0·5e^{-x}$.]

(4) Prove that the following values of x and t, found in the case of plant protease, are in agreement with the Schütz-Borissoff law (x = milligrammes ; t = hours).

x	5·34	8·42	9·82	11·92	12·98	13·70	17·22
t	1	2	3	4	5	6	9

[Answer, $\frac{x}{\sqrt{t}}$ is practically constant (5·34 – 5·74).]

(5) The following numbers represent the relation between the weight in kilogrammes (W) and the surface in square decimetres (S) of a number of children. If the law connecting S and W is $\mathrm{S} = \mathrm{AW}^m$, find the values of A and m. How much milk would an infant weighing 2·50 kilogrammes require per day if the amount of heat lost by the body is 1,700 calories per square metre per day, and the calorific value of milk is 736 calories per litre ?

W	2	3	4	5	6	7	8	9	10
S	16·3	21·4	26·0	30·1	34·0	37·7	41·2	44·6	47·9

[Answer. Plot log W against log S, the resulting graph is the straight line log S = log A + m log W. m will be seen at once to be $= \frac{2}{3}$, and A can be found by substitution to be = 10·3.

$$\therefore \quad \text{law is } S = 10{\cdot}3 \sqrt[3]{W^2}.$$

When W = 2·50, S becomes = 19 (either by calculation or by interpolation), *i.e.*, = ·19 square metre,

$$\therefore \quad \text{amount of milk required is } \frac{{\cdot}19 \times 1{,}700}{536} \text{ litres} = 440 \text{ c.c.}]$$

(6) The following are the pulse-rates P of people of different height H :

H (in cms.)	50	69·8	79·6	86·7	98·6	167·5 (=height of adult)
P . .	134	111·0	108	104	98	73 (rate in adult)

Find whether an equation of the form P = AH^m (where A and m are constants) will express the relationship between P and H, and evaluate the constants.

[Plotting log P against log H there results a straight line whose $m = -\frac{1}{2}$.

$$\therefore \quad P = \frac{A}{\sqrt{H}}. \quad \text{But when } P = 73,\ H = 167{\cdot}5. \quad \therefore \quad 73 = \frac{A}{\sqrt{167{\cdot}5}}.$$

$$\therefore \quad A = 73\sqrt{167{\cdot}5}. \qquad \therefore \quad \text{equation is } P = 73\sqrt{\frac{167{\cdot}5}{H}}.]$$

(7) The following have been found to be the rates of the rabbit's heart (r) at the given temperatures (t). Allowing for errors of observation, find by plotting whether these figures are in agreement with Van't Hoff-Arrhenius law. [Assume T_1T_2 to be constant (see p. 237).]

$t°$ C.	0·4	5·6	6·4	12·8	13·6	14	16
r .	5·9	11·7	12·3	25·9	28·4	29·7	37·8

[Answer. When T_1T_2 is constant, the Van't Hoff-Arrhenius law becomes

$$\frac{r_2}{r_1} = 10^{A(T_2 - T_1)} \text{(see p. 237)}$$

or

$$\log r_2 - \log r_1 = A(T_2 - T_1).$$

Hence if the above figures obey the law, then $\log\left(\frac{r_2}{r_1}\right)$ plotted against $T_2 - T_1$ should give a straight line. This will be found to be the case.]

(8) Show from the figures in exercise (6) on p. 93, that disinfection is a unimolecular reaction.

CHAPTER XXI.*

BIOMETRICS.

As explained in the opening chapter none of the quantitative results obtained in the laboratory can ever be absolutely exact. Even in the domain of the so-called exact sciences, such as astronomy, physics and chemistry, where methods of measurement are as near perfection as human ingenuity can achieve, and where the object measured is of constant size, it is found that several apparently equally reliable measurements taken of the same quantity even by the same observer are never exactly the same. Thus, notwithstanding the exceedingly skilful and minute precaution taken by Crookes to ensure accuracy in his atomic weight determinations of Thallium, in a series of twelve results the values ranged between 203·628 and 203·666, with an average of 203·642. Similarly, in the astronomical observations of the position of a star, the several results never absolutely agree. These discrepancies are due to **errors of observation** inherent in the observer. When we come to deal with the measurements of living things, we are, in addition to the errors of observation, faced with the difficulty that the thing we are measuring is not constant, but variable. No two human beings, for instance, of exactly the same age are ever of exactly the same height or weight. In other words, the biological investigator, in addition to the actual errors of observation, is confronted with the discrepancies resulting from the inherent variability of the objects he measures. One of the objects of modern statistical methods is to discover what is called **"the most probable result"** out of a series of discordant results obtained in the laboratory.

Biometrics is the application of modern statistical methods to the measurements of biological (variable) objects.

Laws of Probability.—As the whole of modern theory of statistics is based upon the theory of probability, it is necessary to discuss and elucidate a few of the more elementary laws of probability before we can consider the subject of biometrics, with which we are concerned in this chapter.

* In the writing of this chapter the author made great use of Caradog Jones's "A First Course in Statistics," G. Bell & Sons, Ltd., 1921. He wishes to acknowledge his great indebtedness to that book.

LAW I.—**If an event may happen in a ways and fail in b ways, then the probability of its happening is $\frac{a}{a+b}$, and that of its failing is $\frac{b}{a+b}$.**

Suppose a bag contains a dozen perfectly round balls made of the same material, size and weight and of the same degree of smoothness, but that 6 of the balls are white and 6 are black. It is clear that the probability of a person drawing a white ball (without looking) is $\frac{6}{12}$, and the probability of his drawing a black ball (**or of his failing to draw a white one**) is also $\frac{6}{12}$. Similarly, if 7 of the balls are white and 5 are black, the probability of drawing a white ball would be $\frac{7}{12}$ and that of failing to draw a white one $\frac{5}{12}$. We do not, of course, mean that out of 12 draws the proportion of white and black balls drawn would be 6 : 6 (or 1 : 1) in the first case, and 7 : 5 in the second case. What we do mean is that out of a very large number, say, 12,000 draws, the proportion of white to black balls would be very nearly 1 : 1 in the first case and 7 : 5 in the second case, and the greater the number of draws, the more nearly will whites and blacks approach these proportions. In general terms, therefore, we may say that if an event may happen in **a** ways (*e.g.*, 7) and fail in **b** ways (*e.g.*, 5), then the probability of its happening is $\frac{a}{a+b}\left(e.g., \frac{7}{7+5}=\frac{7}{12}\right)$, and that of its failing is $\frac{b}{a+b}$ $\left(e.g., \frac{5}{7+5}=\frac{5}{12}\right)$.

Note.—Since an event must either happen or fail, *e.g.*, a draw will be certain to bring forth either a white or a black ball,

$$\therefore \quad certainty = \frac{a}{a+b} + \frac{b}{a+b} = \frac{a+b}{a+b} = 1.$$

Hence if we indicate the probability of an event happening by p, and that of its failing by q, then it follows that

$$\text{(a) } p + q = 1,$$
$$\text{and (b) } q = 1 - p.$$

LAW II.—**The probability of the occurrence of any one of several**

exclusive events (*i.e.*, **of events which cannot occur together**) **is equal to the sum of the separate probabilities.**

Suppose a bag contains 25 balls, 3 of which are white, 4 black and 18 red, what is the probability that the first ball drawn will be a white or a black one ?

The probability of the first ball being white is $\frac{3}{25}$; that of the first ball being black is $\frac{4}{25}$, therefore the probability of the first ball being either white or black is $\frac{3+4}{25}=\frac{7}{25}$. Similarly, the probability of the first ball being either white or red is $\frac{3+18}{25}=\frac{21}{25}$.

LAW III.—**The probability of the occurrence together of two independent events is the product of their separate probabilities.**

Suppose we have 2 separate bags, one of which contains, say, a dozen balls, of which 5 are white and 7 black, and the other contains 25 balls, out of which 3 are white and 22 are black. What is the probability that in two draws, one from each of the 2 bags, a white ball will appear each time ? It is obvious that the probability in this case of the two events happening together must be smaller than the probability of the happening of one event alone. In fact, the probability of a white ball coming out at the **first** draw in each case must be the product of the two separate probabilities, viz., $\frac{5}{12}\times\frac{3}{25}=\frac{1}{20}$.

Possibility and Probability of certain Events occurring.—Let us take again the case of a bag containing balls of different colours. Suppose, for instance, it contains 10 balls, of which 5 are white and 5 are black, and that we make a number of successive draws (always returning the ball drawn before making another draw), what are the probabilities of the appearance of a certain number of whites and of another number of blacks in a given number of draws.

(1) In *one draw* there are 2 (*i.e.*, 2^1) *possible events*, viz., white or black—each of which is equally probable. These possibilities may be denoted by

$$(W, B)$$

(where W and B are not numerical quantities, but are only symbolical expressions denoting events).

(2) In *two draws* there are 4 (*i.e.*, 2^2) *possible events*, which may be denoted by

$$(W, B)\,(W, B) = (WW, WB, BW, BB),$$

which means that the W or B of the first draw may combine with the W or B of the second draw to give either white followed by white (WW), or white followed by black (WB), or black followed by white (BW), or black followed by black (BB). In other words, the different possibilities may be represented by the terms (together with their respective coefficients) of the algebraical expansion $(W + B)^2 = W^2 + 2WB + B^2$.

(3) In *three successive draws* there are 8 (*i.e.*, 2^3) *possible events*, because the 4 possible events of the first 2 draws may be combined with the 2 possible events of the third draw, thus

$$(W + B)(W^2 + 2WB + B^2) = W^3 + 3W^2B + 3WB^2 + B^3,$$

i.e., the 8 different possibilities will correspond to the terms (together with their respective coefficients) of the expansion $(W + B)^3$. It is to be noted that such terms as W^3, W^2B, etc., are short ways of expressing the fact that 3 white balls, or 2 white and 1 black ball have been drawn, and that the numerical coefficient of each term denotes the number of different possible ways in which this event may happen.

(4) Similarly, in *four successive draws* there are 16 (*i.e.*, 2^4) possibilities, corresponding to the terms of the expansion

$$(W + B)^4 = W^4 + 4W^3B + 6W^2B^2 + 4WB^3 + B^4.$$

(5) *In general*, in n successive draws there are 2^n possible events corresponding to the terms of the expansion

$$(W + B)^n = W^n + nW^{n-1}B + \frac{n(n-1)}{2!}W^{n-2}B^2 + \frac{n(n-1)(n-2)}{3!}W^{n-3}B^3 + \ldots + B^n.$$

As an illustration of the working of the laws of probability in biology we might take **the case of Mendelian expectations** when both parents are impure dominants with respect to one unit character. We then have

$$(D + R)(D + R) = D^2 + 2DR + R^2,$$

so that the offspring will consist of pure dominants (D^2), impure dominants (DR) and recessives (R^2) in the proportion of 1 : 2 : 1. For the calculation of Mendelian expectations when the parents differ in more than one unit character the reader is referred to "Child Physiology," pp. 58—60.

So much for the *possibilities*.

Now what are the *probabilities* of the different events happening ? Let us take, for example, the case of 4 draws. There are 16 equally probable events, viz. :

One of "4 successive whites."
Four of "3 whites and 1 black."
Six of "2 whites and 2 blacks."
Four of "one white and 3 blacks."
One of "4 blacks."

Hence, the respective probabilities or frequencies of the above combinations are $\frac{1}{16}$, $\frac{4}{16}$, $\frac{6}{16}$, $\frac{4}{16}$, and $\frac{1}{16}$ respectively. In other words, the probability of any particular combination appearing (or its *frequency*) corresponds to the respective term of the binomial expansion $\left(\frac{1}{2}+\frac{1}{2}\right)^4$. Similarly, the frequency of each of the combinations in 5 draws would correspond to the respective term of the expansion $\left(\frac{1}{2}+\frac{1}{2}\right)^5$, and, in general, the frequency of each combination happening in n draws would be represented by the terms of the binomial expansion $\left(\frac{1}{2}+\frac{1}{2}\right)^n$, viz.,

$$\left(\frac{1}{2}\right)^n, \frac{n}{2^n}, \frac{n(n-1)}{2^n \cdot 2!}, \frac{n(n-1)(n-2)}{2^n \cdot 3!} \cdots \frac{n}{2^n}, \frac{1}{2^n}.$$

Normal Frequency Curve.—If we now plot these frequencies as ordinates at equal intervals along a horizontal axis as abscissæ, we shall get a diagram or polygon which, as the number of

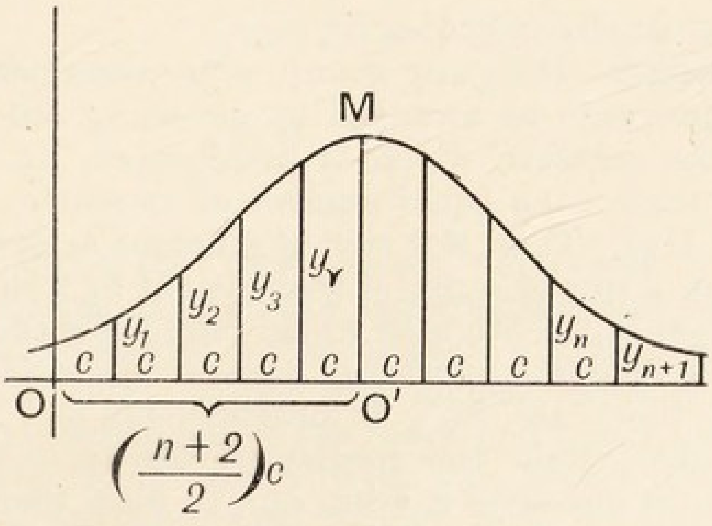

Fig. 116.—Normal Frequency Curve.

ordinates is increased, comes to resemble more and more a curve like the one in the diagram (Fig. 116). Such a curve is often called a *normal frequency curve* and its importance to the biologist lies in the fact that **not only do errors of observation in experimental work usually correspond in frequency with the ordinates of such a curve** (so that small errors of observation occur with greater frequency than large ones, and positive and negative errors of the same magnitude occur with the same frequency), **but certain biological and anthropological statistics are found to be adequately represented by curves of this type.** From the fact that the curve fits errors of observation, it is also called the *normal curve of error*, or the *Gauss-Laplace curve.*

We shall return to the analysis of this curve on p. 339 *et seq.*

Statistical Constants.—Averages and Dispersion.—In every statistical inquiry two kinds of numerical summaries must be used to give the main facts of the set of measurements under consideration, viz. :

(1) A number which represents the *type* of the group of measures or observations.

(2) A number which represents the *measure of the dispersion* or degree of variability of the measures from the type.

Averages.—The measure which represents the type of the group is called an *average*. Statistically, one speaks of three kinds of average, viz., (*a*) the *common average* or *arithmetic mean*, (*b*) the *median*, and (*c*) the *mode*.

(*a*) The **Arithmetic mean** (A.M.) or **Common average** is obtained by dividing the sum of the individual measures by their number. Thus, if the group consists of n measures, $x_1, x_2, x_3 \ldots x_n$, then

$$\text{Arithmetic mean} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}.$$

The A.M. is usually indicated by $\bar{x}$.

(*b*) **The Median.**—If in any group of measurements the individual measurements be arranged in ascending order of magnitude, then the measure which is middlemost, *i.e.*, above and below which there is an equal number of measures, is called the median (Fig. 117). Thus, if 9 men of different heights or weights be arranged in a row in order of their height or weight, then the height or weight of the 5th man would form the median—because there would be an equal number of men on either side of the 5th. Generally, if there are $2n + 1$ measures ($2n + 1$ representing any odd number), then the median is the $(n + 1)^{th}$ measure. If the number of measures is even, *e.g.*, 10 men, then the median is a height or weight intermediate between that of the 5th and the 6th man.

(*c*) **The Mode,** as its name implies, is the most frequent or **fashionable** (*la mode*) measure. Thus, if we have a table giving the heights of 43 boys—ranging from 53 to 68·75 inches, with intervals of ·25 inch, and the largest number of boys were to be 58·5 inches tall, then 58·5 would be the *mode*. In what follows we shall have to deal only with the arithmetic mean, but it is necessary that the student should be familiar with the meanings of the other two averages.

Dispersion or Degree of Variability.—There are 4 measures of variability, viz., (*a*) *average or mean deviation*, (*b*) *standard deviation*, (*c*) *quartile deviation*, and (*d*) *coefficient of variation*.

(*a*) **The average or mean deviation** is the arithmetical mean of the various individual deviations from the arithmetical mean of the measurements, regardless of algebraical sign. By a deviation is meant the difference between any individual measure and the average. For example, if the average height of a group of 50 boys is 59 inches, and that of a particular boy is 54·25, then the deviation of the height of that particular boy from the mean is $54{\cdot}25 - 59 = -4{\cdot}75$ inches. On the other hand, a boy 60 inches tall would deviate by $+1$ from the mean. If, for instance, the sum of the various deviations of the 50 boys (**independent of their algebraical sign,** *i.e.*, treating them all as positive) is 133·75 inches then the average deviation is $\frac{133{\cdot}75}{50} = 2{\cdot}675$ inches.

(*b*) **The standard deviation (S.D. or σ)** is the square root of the arithmetical average of the squares of the individual deviations from the mean, or, **the root mean square deviation from the mean.** Thus, if the deviations are $d_1, d_2, d_3 \ldots d_n$, and the number of boys $= n$, then

$$\text{S.D. or } \sigma = \sqrt{\frac{d_1^2 + d_2^2 + d_3^2 + \ldots + d_n^2}{n}},$$

$$= \sqrt{\frac{\Sigma d^2}{n}}.$$

(Where Σd^2 means the "sum of such terms as d^2.")

(*c*) **The quartile deviation or semi-interquartile range.** In the same way as the median is that measure which occupies a position midway between the lowest and highest measure, so there are two other measures termed the lower and upper quartiles, which occupy positions midway between the lowest measure and the median, and between the median and the highest measure respectively. In any distribution of 31 individuals shown in Fig. 117, the magnitude of the 16th individual (viz., 31·52) represents the median; that of the 8th (viz., 23·23) the lower quartile; whilst that of the 24th (viz., 39·27) is the upper quartile.

The deviation of the lower quartile from the median is $23{\cdot}23 - 31{\cdot}52 = -8{\cdot}29$.

The deviation of the upper quartile from the median is $39{\cdot}27 - 31{\cdot}52 = 7{\cdot}75$.

$\therefore$ average deviation irrespective of sign of the quartiles from the median, *i.e.*, the semi-interquartile range

$$= \frac{8{\cdot}29 + 7{\cdot}75}{2} = 8{\cdot}02.$$

The *lower quartile*, therefore, is the measure which occupies the position one-quarter of the way along the series of measures (arranged in ascending order of magnitude) and the *upper quartile* is the measure which occupies a position three-quarters of the way along the same series (see Fig. 117).

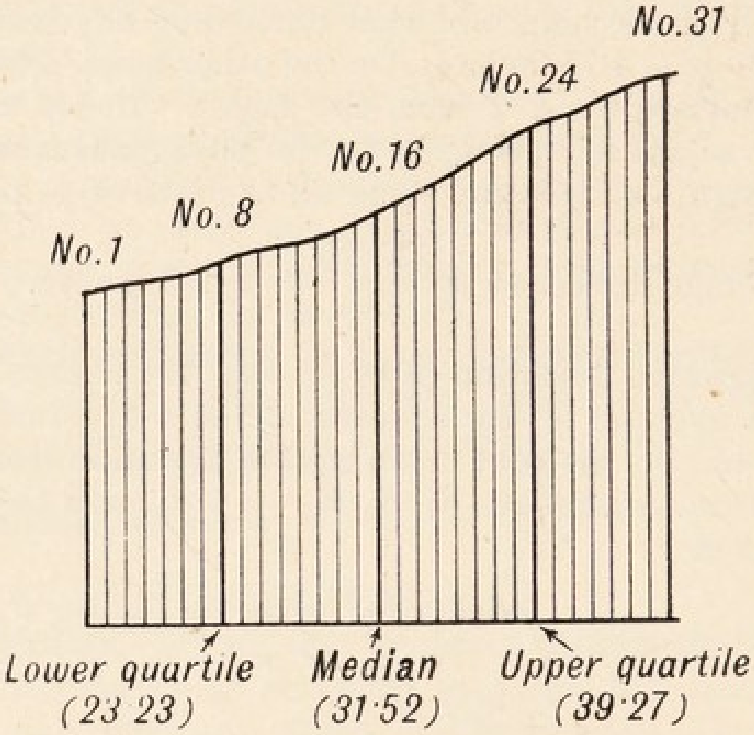

FIG. 117.

The semi-interquartile range is the same measure as the *probable error*, with which we shall deal later (p. 352 *et seq.*), when we shall show that

$$\textit{the quartile deviation} = \frac{2}{3}\,\sigma \text{ approximately (see p. 355).}$$

In statistical analysis one does not deal much with the mean deviation, but the standard deviation and probable error are continually used, and it is therefore necessary that the student should familiarise himself thoroughly with the meanings of these measures of variability.

(*d*) **The coefficient of variation (C. of V.)** is the standard deviation σ expressed as a percentage of the mean M,

$$i.e., \text{C. of V.} = \frac{100\,\sigma}{\text{M}}.$$

This coefficient of variation, therefore, measures the amount of variability σ in terms of the mean.

Weighted Averages.—We stated on p. 332 that the A.M. is found by dividing the sum of the individual observations by their number. This is only true if all observations are of equal importance or reliability, or, as we say, of equal weight. If, however, the various observations are not of equal value, we must first "weight" each observation according to its importance or reliability—although it is not always easy to decide what weight to give to any particular measurement. The following examples will elucidate the meaning of "weighting."

(1) Out of a total of 251 infants born the duration of pregnancy was, 190 days in one case, 210 days in one case, 240 days in one case, 250 days in 2 cases, 260 days in 22 cases, 270 days in 38 cases, 280 days in 79 cases, 290 days in 78 cases, 300 days in 16 cases, 310 days in 9 cases, 320 days in 3 cases, and 330 days in one case. What is the average duration of pregnancy ?

It is obvious that the various numbers, 190, 250, 260, etc., days are not of equal importance, because whilst some are derived from no more than one observation, others are based upon many, such as 22, 79, etc., observations. Hence, each observation in this case must, in order to receive its just value, be **weighted** by multiplying it by its frequency. The A.M. therefore

$$\begin{aligned} &= (190 \times 1 + 210 \times 1 + 240 \times 1 + 250 \times 2 + 260 \times 22 \\ &\quad + 270 \times 38 + 280 \times 79 + 290 \times 78 + 300 \times 16 \\ &\quad + 310 \times 9 + 320 \times 3 + 330 \times 1)/251. \\ &= 281{\cdot}8 \text{ days.} \end{aligned}$$

Hence, A.M. $= \frac{\Sigma fx}{n}$, where f is the frequency or "weight" of each observation x, and n the total number of observations. As $n = \Sigma f$

$$\therefore \quad \text{A.M.} = \frac{\Sigma fx}{\Sigma f}.$$

(2) A number of persons have been weighed and it was found that 3 weighed 67·5 lb., 9 weighed 72·5 lb., 142 weighed 77·5 lb., and so on. Here, again, the observations are not of equal value, their relative importance being in the ratio 3, 9, 142, etc., so that, in calculating the mean, each observation has to be suitably

weighted as follows, $67 \cdot 5 \times 3 + 72 \cdot 5 \times 9 + 77 \cdot 5 \times 142$, etc., and their sum divided by the total number of persons weighed.

Similarly, the *standard deviation* $= \sqrt{\frac{\Sigma f d^2}{n}}$, where f is the frequency of each deviation.

Example.

Out of a total of 251 infants born the duration of pregnancy was as given on p. 335.

Find the standard deviation.

Average period of gestation $= \frac{\Sigma f x}{n}$ (where f = frequency),

$$= \frac{190 \times 1 + 210 \times 1 + 240 \times 1 + 250 \times 2 + 260 \times 22 + \ldots}{251}$$

$= 281 \cdot 8$ days.

$\therefore$ deviations of each from mean are

$$190 - 281 \cdot 8 = -91 \cdot 8.$$
$$210 - 281 \cdot 8 = -71 \cdot 8.$$
$$240 - 281 \cdot 8 = -41 \cdot 8.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$330 - 281 \cdot 8 = 48 \cdot 2.$$

$\therefore$ standard deviation $= \sqrt{\frac{\Sigma f d^2}{n}}$

$$= \sqrt{\{91 \cdot 8^2 \times 1 + 71 \cdot 8^2 \times 1 + 41 \cdot 8^2 \times 1 + 31 \cdot 8^2 \times 2 + 21 \cdot 8^2 \times 22 + 11 \cdot 8^2 \times 38 + 1 \cdot 8^2 \times 79 + 8 \cdot 2^2 \times 78 + 18 \cdot 2^2 \times 16 + 28 \cdot 2^2 \times 9 + 38 \cdot 2^2 \times 3 + 48 \cdot 2^2 \times 1\}/251}$$

$$= \sqrt{\frac{57757 \cdot 24}{251}}$$

$$= 15 \cdot 7.$$

Simplified Method of Finding the A.M.—The ordinary method, described in the last paragraph, of finding the arithmetic mean involves a great deal of arithmetical labour. The latter can, however, be greatly reduced by adopting two simple devices, which will be readily understood from a simple example. Supposing we have a group of 50 men whose weights (to the nearest 5 lb.) are distributed as follows :

7	men of	160	lb.
12	,,	155	,,
15	,,	150	,,
9	,,	145	,,
5	,,	140	,,
2	,,	135	,,

To find the arithmetic mean of the weights, we can see that the average is somewhere in the neighbourhood of 150; we therefore take 150 as an **arbitrary** mean. The various weights are therefore (150 + 10), (150 + 5), (150 + 0), (150 − 5), (150 −10), (150 − 15), and the weights multiplied by their frequencies, *i.e.*, Σfx, become $(150 + 10) \times 7 + (150 + 5) \times 12 + 150 \times 15 + (150 - 5) \times 9 + (150 - 10) \times 5 + (150 - 15) \times 2$, *i.e.*,

$$150\,(7 + 12 + 15 + 9 + 5 + 2)$$
$$+ 10 \times 7 + 5 \times 12 - 5 \times 9 - 10 \times 5 - 15 \times 2$$
$$= 150\Sigma f + (10 \times 7 + 5 \times 12 - 5 \times 9 - 10 \times 5 - 15 \times 2)$$
$$= 150\Sigma f + 5(2 \times 7 + 1 \times 12 - 1 \times 9 - 2 \times 5 - 3 \times 2).$$

∴ A.M. which

$$= \frac{\Sigma fx}{\Sigma f}$$
$$= \frac{150\Sigma f + 5(2 \times 7 + 1 \times 12 - 1 \times 9 - 2 \times 5 - 3 \times 2)}{\Sigma f}$$
$$= 150 + \frac{5 \times 1}{50} = 150{\cdot}1 \text{ lb.}$$

i.e., the arithmetic mean of the distribution is found by the following process:

(i.) *Choose a convenient number as origin or zero* (150 in our case).

(ii.) *Choose a convenient unit as the unit deviation* of each weight from the origin (5 is the unit in our case).

(iii.) Multiply each deviation by its frequency (2 × 7, 1 × 12, − 1 × 9, etc.).

(iv.) Take the algebraic sum of (iii.) and multiply it by the unit of deviation (*i.e.*, 5).

(v.) Divide (iv.) by the number of observations (50).

(vi.) Add (v.) to the origin (*i.e.*, to 150).

Note.—The algebraic sum of the products of the deviations from the origin by their respective frequencies is called the *first moment of the distribution with reference to the origin*, and *the first moment divided by Σf is called the first moment coefficient.* In general we may say that

$$\bar{x} = x + p.$$

Where $\bar{x}$ = true mean,

x = arbitrary mean,

and p = first moment coefficient of the distribution about the arbitrary mean as origin.

Corollary.—Hence, it follows that if $x = \bar{x}$, *i.e.*, when the arbitrary mean coincides with the true mean $p = 0$. In other words, *the first moment of the distribution about the true mean as origin is equal to zero.*

Thus, in our present case the true mean being 150·1,

$$p = 9{\cdot}9 \times 7 + 4{\cdot}9 \times 12 - 0{\cdot}1 \times 15 - 5{\cdot}1 \times 9 - 10{\cdot}1 \times 5 - 15{\cdot}1 \times 2$$
$$= 128{\cdot}1 - 128{\cdot}1 = 0.$$

Definition.—In the same way as Σfx is called the first moment of the distribution about the arbitrary mean (x being the deviation of any observation from that mean in terms of any convenient unit), so Σfx^2 *is called the second moment,* Σfx^3 *the third moment, and in general* Σfx^n *the nth moment* of the distribution about the arbitrary mean. *The various moments divided by* Σf (*or by* n) *are called the first, second . . . nth moment coefficients.*

Example.

Calculate from the data given on p. 335 the average duration of pregnancy.

Take 280 days as the origin, and arrange the work as in the following table :

(1) Duration of Pregnancy in days.	(2) Deviation (x) from arbitrary mean (280) in terms of 10 as unit.	(3) Frequency (f).	(4) fx.	(5) fx^2.
190	− 9	1	− 9	81
200	− 7	1	− 7	49
240	− 4	1	− 4	16
250	− 3	2	− 6	18
260	− 2	22	− 44	88
270	− 1	38	− 38	38
280	0	79	0	0
290	+ 1	78	+ 78	78
300	+ 2	16	+ 32	64
310	+ 3	9	+ 27	81
320	+ 4	3	+ 12	48
330	+ 5	1	+ 5	25
		251	154 − 109 = 45	586

$$\therefore \quad \text{A.M.} = 280 + \frac{45 \times 10}{251} = 281{\cdot}8 \text{ days.}$$

Simplified Method of finding the Standard Deviation.—In the same way as the A.M. can be found with less labour by choosing some arbitrary number as the origin, so can a great deal of labour be saved in computing σ by choosing some arbitrary number as origin. Thus, it can be shown that if S = root mean square deviation **from any arbitrary origin** 0, and σ = standard deviation **from the A.M.** as origin, then $\sigma^2 = S^2 - \bar{x}^2$, where $\bar{x}$ = deviation of arbitrary origin from the A.M. Thus, in the diagram (Fig. 118)

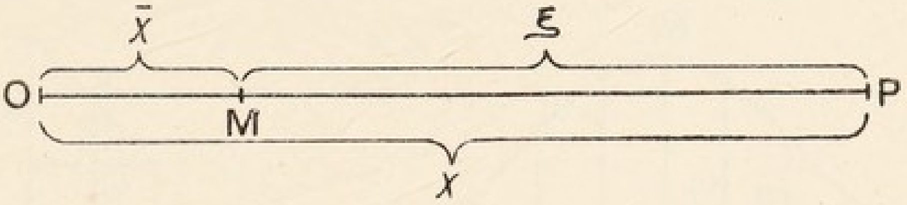

FIG. 118.

let P represent the position of any variable x, and M the position of the mean.

Then $x = \bar{x} + \xi$, where ξ = deviation of P from the mean.

$$\text{But } S^2 = \frac{f_1x_1^2 + f_2x_2^2 + f_3x_3^2 + \ldots + f_nx_n^2}{f_1 + f_2 + f_3 + \ldots + f_n} \text{ (by definition)}$$

$$= \frac{[f_1(\bar{x} + \xi_1)^2 + (\bar{x} + \xi_2)^2 + \ldots + (\bar{x} + \xi_n)^2]}{f_1 + f_2 + \ldots + f_n}$$

$$= \frac{[\bar{x}^2(f_1 + f_2 + f_3 + \ldots + f_n) + 2\,\bar{x}\,(f_1\xi_1 + f_2\xi_2 + \ldots + f_n\xi_n) + (f_1\xi_1^2 + f_2\xi_1^2 + \ldots + f_n\xi_n^2)]}{f_1 + f_2 + f_3 + \ldots + f_n}$$

Also $f_1\xi_1 + f_2\xi_2 + \ldots + f_n\xi = 0$ (by corollary on p. 337),

and $$\frac{f_1\xi_1^2 + f_2\xi^2 + \ldots + f_n\xi_n^2}{f_1 + f_2 + \ldots + f_n} = \sigma^2,$$

$$\therefore \quad S^2 = \bar{x}^2 + \sigma^2. \qquad \therefore \quad \sigma^2 = S^2 - \bar{x}^2.$$

Example.—In the pregnancy data in the example on p. 338, we have

$S^2 = \frac{58600}{251}$, since $\Sigma fx^2 = 586$ (see column 5 in above example),

and the unit deviation $= 10$.

Also $$\bar{x}^2 = 1{\cdot}8^2 = 3{\cdot}24.$$

$$\therefore \quad \sigma^2 = \frac{58600}{251} - 3{\cdot}24.$$

$$= 230{\cdot}22.$$

$$\therefore \quad \sigma = \sqrt{230{\cdot}22}$$

$$= 15{\cdot}17$$

Equation of the Normal Probability Curve.—Let the ordinates be $y_1, y_2 \ldots y_{n+1}$ and their respective abscissæ $x_1, x_2 \ldots x_{n+1}$.

Let the distance between each two consecutive ordinates $= c$, so that $x_1 = c$, $x_2 = 2c$, $x_3 = 3c$, $\ldots x_{n+1} = (n+1)c$.

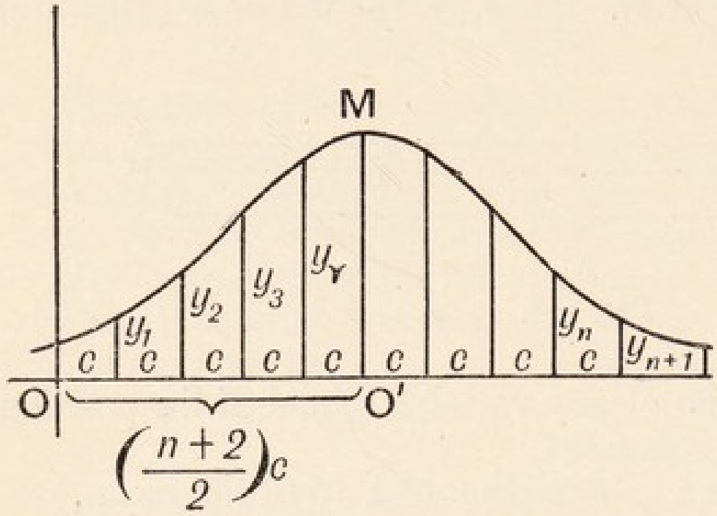

FIG. 119.—Normal Frequency Curve.

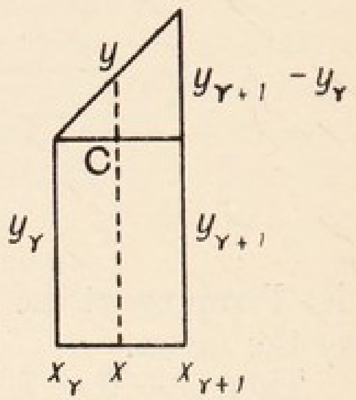

FIG. 120.

We then have any abscissa $x = rc$,
and its corresponding ordinate

$$y_r = \frac{n(n-1)(n-2)\ldots(n-r+2)}{1.2.3\ldots(r-1)} \cdot \frac{1}{2^n}.$$

(since the ordinates represent the frequencies or probabilities see p. 331.)

Similarly $\quad x_{r+1} = (r+1)c,$

and $$y_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r} \cdot \frac{1}{2^n}.$$

Hence, the *slope of the curve between the ordinates y_r and y_{r+1}* is $\frac{y_{r+1} - y_r}{c}$ (see p. 136),

$$= \frac{1}{2^n c}\left[\left\{\frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{1.2.3\ldots(r-1).r}\right\} - \left\{\frac{n(n-1)(n-2)\ldots(n-r+2)}{1.2.3\ldots(r-1)}\right\}\right]$$

$$= \frac{1}{c}\cdot\frac{1}{2^n}\cdot\frac{n(n-1)(n-2)\ldots(n-r+2)}{1.2.3\ldots(r-1)}\left(\frac{n-r+1}{r} - 1\right)$$

$$= \frac{1}{c}\cdot y_r \cdot\left(\frac{n-r+1}{r} - 1\right)$$

$$= \frac{1}{cr}\cdot y\,(n - 2r + 1).$$

Now, if n is very large and c correspondingly small, the slope of the curve at the point (x, y) midway between (x_r, y_r) and (x_{r+1}, y_{r+1}) will be the same as that between the ordinates y_r, and y_{r+1}.

But $x = \frac{1}{2}(x_r + x_{r+1}) = \frac{1}{2}[rc + (r+1)c)$.

$$= \frac{c}{2}(2r+1).$$

and $\quad y = \frac{1}{2}(y_r + y_{r+1})$.

$$= \frac{1}{2} \cdot \frac{1}{2^n}\left[\frac{n(n-1)\ldots(n-r+2)}{1.2.3\ldots(r-1)} + \frac{n(n-1)\ldots(n-r+2)(n-r+1)}{1.2.3\ldots(r-1)r}\right]$$

$$= \frac{1}{2} \cdot \frac{1}{2^n} \frac{n(n-1)\ldots(r-r+2)}{1.2.3\ldots(r-1)}\left[1 + \frac{(n-r+1)}{r}\right]$$

$$= \frac{y_r}{2}\frac{(r+n-r+1)}{r}$$

$$= \frac{y_r}{2r}(n+1).$$

$$\therefore \quad y_r = \frac{2ry}{n+1}.$$

$\therefore$ slope at point (x, y), which is

$$\frac{1}{cr} \cdot y_r(n - 2r + 1) \text{ (see above).}$$

$$= \frac{1}{cr} \cdot \frac{2ry}{n+1}(n - 2r + 1)$$

$$= \frac{2ry}{(n+1)} \frac{(n-2r+1)}{cr}$$

$$= \frac{2y}{(n+1)c}[(n+2) - (2r+1)].$$

But from $x = \frac{c}{2}(2r+1)$ we have $2r + 1 = \frac{2x}{c}$.

$\therefore$ slope at any point $(x, y) = \frac{2y}{(n+1)c}\left(n + 2 - \frac{2x}{c}\right)$.

Now, when n is infinitely large, the lines joining the tops of the ordinates y_1, y_2 . . . will form a smooth curve whose slope at (x,y) will be

$$\frac{dy}{dx} = \frac{2y}{(n+1)c}\left(n + 2 - \frac{2x}{c}\right).$$

By transferring the origin to the point $(n+2)\frac{c}{2}$, *i.e.*, to the point O′ (the foot of the middle ordinate O′M), x will become $x + \frac{(n+2)c}{2}$, and hence the equation will become

$$\frac{dy}{dx} = \frac{2y}{(n+1)c}\left[n + 2 - \frac{2}{c}\left\{x + \left(\frac{n+2}{2}\right)c\right]\right.$$

$$= \frac{2y}{(n+1)c}\left[(n+2) - \frac{2x}{c} - (n+2)\right]$$

$$= \frac{2y}{(n+1)c}\left(-\frac{2x}{c}\right)$$

$$= -\frac{4xy}{(n+1)c^2}$$

$$\therefore \quad \frac{dy}{y} = -\frac{4xdx}{(n+1)c^2},$$

$$\therefore \quad \int\frac{dy}{y} = -\frac{4}{(n+1)c^2}\int xdx,$$

$$i.e., \log y = -\frac{2x^2}{(n+1)c^2} + \log A$$

(log A = constant of integration. See p. 277).

$$\therefore \quad y = Ae^{-\frac{2x^2}{c^2(n+1)}} = Ae^{-\frac{x^2}{2\sigma^2}}$$

where $\sigma^2 = (n+1)\frac{c^2}{4}$ (σ will be found to be the S.D. See p. 351).

We now have to find the value of A.

The total frequency is the total probability, which = 1 (see p. 328). But the total probability or total frequency is the area of the curve between the limits $x = \pm\infty$.

Therefore to find total frequency we must integrate between the limits $\pm\infty$.

Thus,
$$\int_{-\infty}^{+\infty} y\,dx$$
$$= A\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2\sigma^2}} \, . \, dx$$
$$= A\sigma\sqrt{2\pi} \text{ (see p. 258).}$$
$$\therefore \quad A = \frac{1}{\sigma\sqrt{2\pi}},$$

$\therefore$ **final equation of the Probability Curve is**

$$\mathbf{y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}}.$$

(see "*A First Course in Statistics*" *by C. Caradog Jones*).

The equation may also be written

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}.$$

where
$$h = \frac{1}{\sigma\sqrt{2}}.$$

Properties of the Normal Probability Curve.—(1) The curve is symmetrical about the y axis because for any value of y there are two equal and opposite values of x. Thus, whether $x = \pm n$, the value of y is the same.

(2) It is entirely situated above the x axis since $y = Ae^{-\frac{x^2}{2\sigma^2}}$ can never be < 0.

(3) The axis of x is an asymptote, *i.e.*, the curve tails off to the right and left of the y axis but never touches the x axis, for as x tends to $\pm \infty$ y tends to become 0 but never less.

(4) The value of the maximum ordinate O′M is $\frac{1}{\sigma\sqrt{2\pi}}$, since this is the value of y when $x = 0$, ($e^{-\frac{x^2}{2\sigma^2}}$ being $= e^0 = 1$).

(5) The point P_1 on the curve, whose abscissæ or $x = \sigma$ (Fig. 123, p. 356) is a point of inflexion, *i.e.*, where the curve changes from concave to convex. This has been proved on p. 186.

(6) The height of the ordinate N_1P_1 (Fig. 123) is given by the equation

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}} \text{ (since } x = \pm\sigma)$$

$$= \frac{0{\cdot}607}{\sigma\sqrt{2\pi}} = 0{\cdot}607 \text{ the maximum ordinate (see (4))}.$$

(7) It further follows from (5) and (6) that the position of the point of inflexion P_1 depends upon the value of σ as well as upon the value of the maximum ordinate. All probability curves which have the same maximum ordinate will have the points of inflexion at the same level, but their distances from the maximum ordinate will be equal to the respective standard deviations.

Theorem of Least Squares.—If a number of observations are made upon a quantity and the errors of each of these noted, *i.e.*, as nearly as can be estimated, then from a knowledge of these errors it is possible to find the most probable value of the quantity on the assumption that the errors follow the normal distribution.

Let n observations be taken, and let the errors be $x_1, x_2 \ldots x_n$.

Also suppose all measurements to be equally good, *i.e.*, the fineness of reading (measured by $\sigma\sqrt{2}$, or h, in the normal probability formula) being the same throughout.

The probability of an error lying between x_1 and $x_1 + \delta x$ will be

$$P_1 = \delta x \times y_1$$

where $$y_1 = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x_1^2}{2\sigma 2}}$$

i.e., $$P_1 = \delta x \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x_1^2}{2\sigma^2}}$$

Similarly $$P_2 = \delta x \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x_2^2}{2\sigma 2}}$$

Now, since x_1 and x_2 are quite independent,

$\therefore$ probability of two errors falling within the range δx from x_1 and x_2 respectively will be the product of the separate probabilities (Law III.),

i.e., $$P_1 \times P_2$$

or $$\frac{\delta x \,.\, e^{-\frac{x1^2}{\sigma}}}{\sigma\sqrt{2\pi}} \times \frac{\delta x \,.\, e^{-\frac{x2^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$= \frac{(\delta x)^2}{2\sigma^2\pi} e^{-\frac{1}{2\sigma^2}(x1^2 + x2^2)}.$$

Similarly, the probability of n errors $x_1, x_2 \ldots x_n$ falling within the range δx from $x_1, x_2, x_3 \ldots x_n$ respectively is given by the equation

$$P = \frac{(\delta x)^n}{\sigma^n(2\pi)^{\frac{n}{2}}} \cdot e^{-\frac{1}{2\sigma^2}(x_1^2 + x_2^2 + \ldots + x_n^2)}$$

$$= Ke^{-\frac{1}{2\sigma^2}(x_1^2 + x_2^2 + \ldots + x_n^2)}$$

$$= \frac{K}{e^{\frac{1}{2}\sigma^2 \Sigma x^2}} \left(\text{where K stands for } \frac{(\delta x)^n}{\sigma^n(2\pi)^{\frac{n}{2}}}\right).$$

$\therefore$ P will be a maximum when $e^{\frac{1}{2\sigma^2}\Sigma x^2}$ is a minimum, *i.e.*, when Σx^2 is a minimum. In other words, *the most probable value of the quantity to be determined is that which makes the sum of the squares of the errors least.*

Application of Method of Least Squares.—Let the observed magnitude y depend upon x in such a way that

$$y = mx + b,$$

It is required to find the most probable value of m and b.—For perfect accuracy we must have for the varying observations

$$\begin{aligned} y_1 &= mx_1 + b \text{ so that } & y_1 - mx_1 - b &= 0 \\ y_2 &= mx_2 + b & y_2 - mx_2 - b &= 0 \\ &\cdots\cdots & &\cdots\cdots \\ y_n &= mx_n + b & y_n - mx_n - b &= 0. \end{aligned}$$

Let the actual observation equations found be

$$\left.\begin{aligned} y_1 - mx_1 - b &= A_1 \\ y_2 - mx_2 - b &= A_2 \\ &\cdots\cdots \\ y_n - mx_n - b &= A_n \end{aligned}\right\} \text{ where } A_1, A_2, \ldots A_n \text{ are therefore the actual deviations.}$$

$\therefore$ by theorem of least squares we ought to have that

$$A_1^2 + A_2^2 + \ldots + A_n^2 \text{ is a minimum,}$$

i.e., $$\Sigma(y - mx - b)^2 \text{ is a minimum,}$$

$$\therefore \frac{\delta}{\delta b}[\Sigma(y - mx - b)^2] \text{ must } = 0,$$

i.e., $$\Sigma(y - mx - b) = 0 \quad \ldots \quad (1)$$

and $$\frac{\delta}{\delta m}[\Sigma(y - mx - b)^2] \text{ must } = 0,$$

i.e., $$\Sigma x(y - mx - b) = 0 \quad \ldots \quad (2)$$

If there are n equations

(1) becomes $\Sigma y - m\Sigma x - nb = 0$

and (2) becomes $\Sigma xy - m\Sigma x^2 - b\Sigma x = 0.$

By solving these two simultaneous equations for m and b we get

$$m = \frac{\Sigma(x) \,.\, \Sigma(y) - n\Sigma(xy)}{\Sigma[(x)]^2 - n\Sigma(x^2)},$$

and $$b = \frac{\Sigma(x) \,.\, \Sigma(xy) - \Sigma(x^2) \,.\, \Sigma(y)}{[\Sigma(x)]^2 - n\Sigma(x^2)}.$$

Corollary.—From this result it follows that if

$$\Sigma x = 0 \text{ and } \Sigma y = 0,$$

then (1) $$m = \frac{-n\Sigma(xy)}{-n\Sigma(x^2)} = \frac{\Sigma(xy)}{\Sigma(x^2)},$$

and (2) $$b = 0.$$

These are two important results in statistical theory because they teach us that if we take the means of the observations $x_1, x_2, x_3 \ldots x_n$ and $y_1, y_2, y_3 \ldots y_n$ as the origin, that is, when $b = 0$ (see p. 105), then

$$m = \frac{\Sigma(xy)}{\Sigma(x^2)}.$$

Example.

Find the most probable values of m and b in the equation

$$s = mt + b,$$

representing the solubility of $NaNO_3$, using the data given in the example on p. 297.

t	s	t^2	st
−6	68·4	36	−410·4
0	72·9	0	0
20	87·5	400	1750
40	102	1600	4080
$\Sigma t = 54$	$\Sigma s = 330{\cdot}8$	$\Sigma t^2 = 2036$	$(st) = 5419{\cdot}6$

$$\therefore \quad m = \frac{\Sigma t \,.\, \Sigma s - n\Sigma(st)}{(\Sigma t)^2 - n\Sigma t^2} = \frac{54 \times 330{\cdot}8 - 4 \times 5419{\cdot}6}{54^2 - 4 \times 2036}$$

$$= \frac{3815{\cdot}2}{5228} = {\cdot}73.$$

$$b = \frac{\Sigma t\,.\,\Sigma(st) - \Sigma t^2\,.\,\Sigma s}{(\Sigma t)^2 - n\Sigma t^2} = \frac{54 \times 5419{\cdot}6 - 2036 \times 330{\cdot}8}{54^2 - 4 \times 2036}$$

$$= \frac{380850{\cdot}4}{5228} = 72{\cdot}9.$$

$\therefore$ the equation is $s = {\cdot}73t + 72{\cdot}9.$

Theorem.—The Arithmetical mean of a series of observed values is the most probable value of the quantity measured.

Proof.—Let $x_1, x_2, x_3, x_4, \ldots x_n$, be the respective observations (n in number). Let their arithmetic mean $= \bar{x}$,

so that $$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}.$$

Let $x_p =$ the most probable value of the magnitude.

Then $(x_1 - x_p)\,(x_2 - x_p), (x_3 - x_p), \ldots =$ residual errors.

Now the probability of making this system of errors is by Law III., p. 329.

$$\mathrm{P} = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{h^2}\left\{\Sigma(x_1 - x_p)^2\right\}}$$

But, if x_p is the most probable value, it is obvious that the probability of making the series of errors $(x_1 - x_p)$, $(x_2 - x_p)$, etc., must be a maximum.

Hence $\Sigma\,(x_1 - x_p)^2$ must be a minimum.

$\therefore \quad \Sigma x_p - \Sigma x_1 = 0,$

or $\quad \Sigma x_p = \Sigma x_1.$

But $\quad \Sigma x_p = nx_p.$

$$\therefore \quad x_p = \frac{\Sigma x_1}{n} = \bar{x} \text{ (the arithmetical mean).}$$

Example.

The following four observations of a certain quantity were made with equal care :

10, 9·4, 10·3, 10·1.

Find the most probable value.

$$\text{The A.M.} = \frac{10 + 9{\cdot}4 + 10{\cdot}3 + 10{\cdot}1}{4} = \frac{39{\cdot}8}{4} = 9{\cdot}95.$$

The most probable value is therefore 9·95. This can be verified by actual calculation.

The residual errors are

$(10 - 9{\cdot}95), (9{\cdot}4 - 9{\cdot}95), (10{\cdot}3 - 9{\cdot}95), (10{\cdot}1 - 9{\cdot}95),$

i.e., $\quad 0{\cdot}05, - {\cdot}55, {\cdot}35, {\cdot}15.$

$\therefore$ sum of squares of residual errors

$$= ({\cdot}05)^2 + ({\cdot}55)^2 + ({\cdot}35)^2 + ({\cdot}15)^2$$
$$= {\cdot}0025 + {\cdot}3025 + {\cdot}1225 + {\cdot}0225$$
$$= {\cdot}45.$$

Now let us assume the most probable value to be less than 9·95, *e.g.*, 9·6, then sum of squares of residual errors

$$= \cdot(4)^2 + (\cdot 2)^2 + (\cdot 7)^2 + (\cdot 5)^2$$
$$= \cdot 16 + \cdot 04 + \cdot 49 + \cdot 25$$
$$= \cdot 94.$$

Similarly if most probable value be taken as greater than 9·95, *e.g.*, 10·2, then sum of squares of residual errors

$$= (\cdot 2)^2 + (\cdot 8)^2 + (\cdot 1)^2 + (\cdot 1)^2$$
$$= \cdot 04 + \cdot 64 + \cdot 01 + \cdot 01$$
$$= \cdot 7.$$

But both ·94 and ·7 are greater than ·45. Similarly for any other figure higher or lower than 9·95. Hence the arithmetical mean is the most probable value.

The Standard Deviation about the True Value.—We have seen that the standard deviation is given by the formula $\sigma = \sqrt{\frac{\Sigma d^2}{n}}$. This value of σ gives the standard deviation about the mean of the various observations. It is clear, however, that this standard deviation does not measure the dispersion of the observations about the *true* value, since, although the mean of the several observations is the most probable value of those observations, it is not necessarily the *true* value unless the number of observations is infinitely large. Thus out of one series of ten observations of the height of boys of a given age we might find a mean value of 5 ft., whilst out of another series of ten observations of the height of boys of the same age the mean might be 5 ft. $\frac{1}{2}$ in. or 4 ft. $11\frac{1}{2}$ in. Unless, therefore, the true value happens to coincide exactly with the mean of the observations, the standard deviation about the true value will be a little larger than σ, since the theorem of least squares tells us that Σd^2 is a minimum at the mean. It can be shown that:

(1) The standard deviation about the true value is

$$\sigma' = \sqrt{\frac{\Sigma d^2}{(n-1)}} = \sigma\sqrt{\frac{n}{n-1}}.$$

(2) Standard deviation of arithmetical mean is

$$\sigma_m = \sqrt{\frac{\Sigma d^2}{n(n-1)}} = \frac{\sigma}{\sqrt{n-1}}.$$

Example.

In a certain blood count, four vertical rows of the hæmocytometer were counted in fifteen fields with the following results:

First row	218
Second row	224
Third row	245
Fourth row	209

Find the standard deviation from the *true* value and the standard deviation of the mean.

The arithmetical mean $= \dfrac{218 + 224 + 245 + 209}{4}$

$= 224.$

$\therefore$ deviations from mean are

$(218 - 224), (224 - 224), (245 - 224)$ and $(209 - 224)$,

i.e., $- 6, 0, + 21$ and $- 15$ respectively.

$$\therefore \quad \sigma' = \sqrt{\frac{\Sigma d^2}{(n-1)}} = \sqrt{\frac{36 + 0 + 441 + 225}{4 - 1}}$$

$$= \sqrt{\frac{702}{3}}$$

$$= \pm\ 15{\cdot}3 \text{ cells,}$$

and $$\sigma m = \sqrt{\frac{702}{4{\cdot}3}} = \sqrt{\frac{702}{12}} = \pm\ 7{\cdot}6 \text{ cells.}$$

Note.—In practice the distinction between σ and σ' is of very small importance; if n is large the value of $\sqrt{\frac{n}{n-1}}$ is practically unity. On the other hand, when n is small, the distribution of errors of sampling around the "True" value is not adequately described by a normal curve of error, and therefore a more intricate method of treatment is required (see "Biometrika," x., 1914, 522).

Unsymmetrical Distribution.—Let us now consider a case where the distribution is not symmetrical, *e.g.*, instead of the white and black balls (p. 328) being equally distributed, let the bag contain, say, one white ball and five black ones. The probability **p** of drawing white is $\frac{1}{6}$ and that (**q**) of drawing black is $\frac{5}{6}$ (and $p + q = 1$).

Now, since by Law III. the probability of two independent events occurring together is the product of their separate probabilities, therefore the possibilities and the respective probabilities for any number of draws 1, 2, 3, . . . n are those shown in the following table :

Number of Draws.	Different Possibilities.	Different Probabilities.
1	W, B	p, q
2	W^2, 2WB, B^2	$p^2, 2pq, q^2$
3	W^3, $3W^2B$, $3WB^2$, B^3	$p^3, 3p^2q, 3pq^2, q^3$
. . . .	. . . .	. . . .
n	$W^n, nW^{n-1}B, \frac{n(n-1)}{1\,.\,2}W^{n-2}B^2 \ldots B^n$	$p^n, np^{n-1}q, \frac{n(n-1)}{1\,.\,2}p^{n-1}, q \ldots q^n$

In other words, the different possibilities and probabilities are respectively given by the successive terms of the expansions $(W + B)^n$ and $(p + q)^n$.

We can now construct a frequency table, showing the probabilities of getting 0, 1, 2, 3 . . . r . . . n, whites (or other similar successes) in n draws (or other similar events), the probability in each case being denoted by the proportional frequencies of these different successes.

(1) Number of successes.	(2) Frequency.	(3) Product of numbers in columns (1) and (2).	(4) Product of numbers in columns (1) and (3).
(x)	(f)	(fx)	(fx^2)
0	q^n	0	0
1	$nq^{n-1}p$	$nq^{n-1}p$	$nq^{n-1}p$
2	$\frac{n(n-1)}{1 . 2}q^{n-2}p^2$	$n(n-1)\,q^{n-2}p^2$	$2n(n-1)q^{n-2}p^2$
. . . .	. . . .	. . . .	. . . .
r	$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1 . 2 . 3 \ldots r}q^{n-r}p^r$	$\frac{n(n-1)\ldots(n-r+1)}{1 . 2 . 3 \ldots (r-1)}q^{n-r}p^r$	$\frac{rn(n-1)\ldots(n-r+1)}{1 . 2 . 3 \ldots (r-1)}q^{n-r}p$
n	p^n	np^n	n^2p^n
Sum.	$(q+p)^n = 1$	np	$np[1 + (n-1)p]$

The curve would not be symmetrical, the median, mean and mode not being coincident (see p. 351).

The sum of the frequencies (see column 2), *i.e.*, Σf, is obviously

$$q^n + nq^{n-1}p + n\frac{(n-1)}{1 . 2}q^{n-2}p^2 + \ldots + p^n = (q+p)^n = 1$$
$$(\text{since } (p+q) = 1).$$

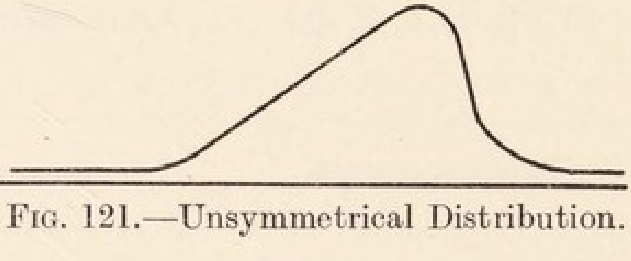

FIG. 121.—Unsymmetrical Distribution.

The sum of the terms in column 3 is

$$nq^{n-1}p + n(n-1)\,q^{n-2}\,p^2 + \ldots + np^n$$
$$= np[q^{n-1} + (n-1)\,q^{n-2}p + \ldots + p^{n-1}]$$
$$= np(q+p)^{n-1} = np = 1st\ moment\ about\ the\ origin,$$

taking the origin at the first term of the binominal.

The sum of the terms in column 4 is

$$nq^{n-1}p + 2n(n-1)q^{n-2}p^2 + \frac{3n(n-1)(n-2)}{1.2}q^{n-3}p^3 + \ldots + n^2p^n$$

$$= np\left[\left\{q^{n-1} + (n-1)q^{n-2}p + \frac{(n-1)(n-2)}{1.2}q^{n-3}p^2 + \ldots + np^{n-1}\right\} + \left\{(n-1)q^{n-2}p + \frac{2(n-1)(n-2)}{1.2}q^{n-3}p^3 + \ldots + (n-1)p^{n-1}\right\}\right]$$

$$= np[(q+p)^{n-1} + (n-1)p(q+p)^{n-2}]$$

$$= np[1 + (n-1)p] = \textit{2nd moment about the origin.}$$

The *arithmetic mean* of the distribution is equal to the sum of the terms in column 3 divided by the sum of the terms in column 2.

$$= \frac{\Sigma(fx)}{\Sigma(f)} = \frac{np}{1} = np.$$

The *mean square deviation* referred to zero, or "no success" as origin

$$= \frac{\Sigma(fx^2)}{\Sigma(f)} = np\,[1 + (n-1)p], \text{ (since } \Sigma f = 1).$$

Hence the *standard deviation* σ (see p. 333), which is the mean square deviation from the arithmetic mean np, is given by

$$\sigma^2 = np\,[1 + (n-1)\,p] - n^2p^2 \text{ (see p. 339).}$$
$$= np\,[1 - p + np] - n^2p^2$$
$$= np\,(1-p) + n^2p^2 - n^2p^2$$
$$= np\,q \text{ (since } 1 - p = q).$$

$$\therefore \quad \sigma = \sqrt{npq} \quad \left(\text{If } p = q, \sigma = \frac{\sqrt{n}}{2}\right)$$

Skewness.—Since a curve of frequency can be evolved from the binomial expansion $(p+q)^n$ it must vary in shape according as p and q are equal or unequal. If $p = q$, then we get the normal frequency curve in which there is a perfect balance between the frequency of observation on either side of the maximum ordinate. In such a curve, therefore, the mean, median and mode coincide. If, however, p and q are unequal, then the curve is unsymmetrical about its maximum ordinate. This asymetry, or lack of symmetry, of a curve is called its *skewness* and is indicated by the fact that the mean, median and mode do not coincide. It is also shown when the quartiles are not equidistant from the median. The degree of skewness can therefore be measured by either of these facts.

Measurement of Skewness.—There are several methods by which skewness can be measured.

(1) *Pearson's Method.*—Pearson has defined skewness as

$$= \frac{(\text{mean} - \text{mode})}{\sigma}$$

$$\text{or} = \frac{3\ (\text{mean} - \text{median})}{\sigma},$$

where σ = standard deviation.

(2) *Quartile Ratio.*—Let q_1 = excess of median over lower quartile, and q_2 = excess of upper quartile over median, then

$$\text{skewness} = \frac{q_2 - q_1}{q_2 + q_1}.$$

Pearson's Generalised Probability Curve.—If p and q are not equal and the distribution cannot be covered by the binominal form, then it can be shown that the differential equation,

$$\frac{dy}{dx} = \frac{y\,(x + b)}{Ax^2 + Bx + C},$$

where b, A, B and C are constants involving p, q and n only, gives on integration a set of curves known as Pearson's Generalised Probability Curves which cover most of the homogeneous statistics ordinarily met with. The types of curves resulting from the integration of the above equation are twelve in number. It is beyond the province of this book to enter into a consideration of all the different types,* but it may be stated that if $\frac{B^2}{4AC}$, which is generally denoted by the letter κ, is zero, and both B and A = 0, then the resulting curve is the normal frequency curve. κ is called the **criterion** because its value determines the type of the curve. There are simple relationships between the various constants and the moments of the distribution about the mean which makes it possible to determine what type of curve covers the given statistical data (see p. 366).

Error and Probable Error.—The deviation of any particular observation (out of a series of observations) from the mean is termed the **error** of the observation. If these deviations, or errors, are symmetrically distributed on either side of the mean then they can be usually fitted by a normal probability curve.

* See Prof. K. Pearson's Memoir, *Phil. Trans. A.* ccxvi., 1916, p. 429.

Such a curve is often called the *normal curve of error*, and has as its equation

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}},$$

where N = number of observations
and σ = standard deviation of the errors.

The *probable error* (*p.e.*) is that value of the abscissal unit on either side of the mean (X_r and X_{r_1}) such that the ordinates X_rN, $X_{r_1}N_1$ enclose half the areas of the curve so that there is an even chance that the true value of the quantity to be determined lies between these limits. It is usually denoted by $\pm r$. Thus the period of gestation has been found to be 282·5 ± 0·55 days (± 0·55 being the probable error). This means that the chances are even that the true value of the period of pregnancy lies between (282·5 + ·55) and (282·5 − ·55), *i.e.*, between 283·05 and 281·95 days.

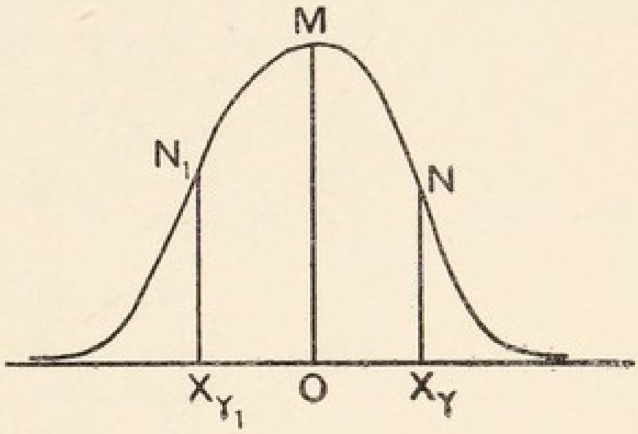

FIG. 122.

If, in the probability curve (Fig. 122), $OX_r = OX_{r_1}$ = the probable error, then by definition, the ordinates X_rN, $X_{r_1}N_1$ bisect each half of the curve on either side of the mean OM, so that area of portion $X_{r_1}N_1MNX_r = \frac{1}{2}$ total area of the probability curve. But the quartiles also bisect each half of the curve, therefore the ordinates $X_{r_1}N_1$ and X_rN coincide with the lower and upper quartiles respectively. Hence we see that **the range of the probable error $\pm r$ is exactly the same as the interquartile range** (see p. 334).

Note.—The term "probable error" is somewhat confusing, since the error in question is not by any means the most probable, and as Whipple suggests, "median deviation" is much more descriptive of its character.

Unit of Variability.—In all statistical calculations the standard deviation is taken as the *unit* of measurement.

Formula for Magnitude of Probable Error in Terms of Standard Deviation.—Since, by definition, the chance of an error falling within the limits $\pm r$ is exactly equal to the chance of an error falling outside these limits; and since the chance of the occurrence of an error of any undefined magnitude (*i.e.*, within the limits

$\pm\infty$) is geometrically expressed by the total area of the curve of error, *i.e.*, by 1,

$\therefore$ area covered by the portion of the curve representing the frequency of errors between the limits of $\pm r$ (*i.e.*, of the probable error) must $= \frac{1}{2}$ total area of probability curve $= \frac{1}{2}$.

But area of curve between the limits $x = \pm r$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-r}^{+r} e^{-\frac{x^2}{2\sigma^2}}\,dx.$$

which is the same as $$\frac{2}{\sigma\sqrt{2\pi}}\int_{0}^{+r} e^{-\frac{x^2}{2\sigma^2}}\,dx.$$

$$\therefore\quad \frac{1}{2} = \frac{2}{\sigma\sqrt{2\pi}}\int_{0}^{+r} e^{-\frac{x^2}{2\sigma^2}}\,dx.$$

i.e., $$\frac{\sqrt{2\pi}}{4} = \frac{1}{\sigma}\int_{0}^{+r} e^{-\frac{x^2}{2\sigma^2}}\,dx.$$

or $$\cdot 6266 = \frac{1}{\sigma}\int_{0}^{+r} e^{-\frac{x^2}{2\sigma^2}}\,dx.$$

Evaluating the integral by expanding $e^{-\frac{x^2}{2\sigma^2}}$ into a series we have

since $$e^{-\frac{x^2}{2\sigma^2}} = 1 - \frac{x^2}{2\sigma^2} + \frac{x^4}{2\,!\,4\sigma^4} - \frac{x^6}{3\,!\,8\sigma^6} + \ldots$$

(p. 260).

$$\therefore\quad \frac{1}{\sigma}\int_{0}^{r} e^{-\frac{x_2}{2\sigma^2}}\,dx = \frac{1}{\sigma}\int_{0}^{r}\left[1 - \frac{x^2}{2\sigma^2} + \frac{x^4}{8\sigma^4} - \frac{x^6}{48\sigma^6} + \ldots\right]dx$$

$$= \frac{1}{\sigma}\left[x - \frac{x^3}{6\sigma^2} + \frac{x^5}{40\sigma^4} - \frac{x^7}{336\sigma^6} + \ldots\right]_o^r$$

$$= \frac{1}{\sigma}\left(r - \frac{r^3}{6\sigma^2} + \frac{r^5}{40\sigma^4} - \ldots\right)$$

$$\therefore\quad \cdot 6266 = \frac{1}{\sigma}\left(r - \frac{r^3}{6\sigma^2} + \frac{r^5}{40\sigma^4} - \ldots\right)$$

or $$\frac{r}{\sigma} - \frac{r^3}{6\sigma^3} + \frac{r^5}{40\sigma^5} - \cdot 6266 = 0$$

(the higher terms of r being negligible).

This equation is satisfied by

$$\frac{r}{\sigma} = \cdot 6745 \text{ (\textit{cf.} p. 261).}$$

$$\therefore \quad r = \cdot 6745\sigma.$$

In other words, *the probable error is approximately* $\frac{2}{3}$ *of the standard deviation.*

Probability Integral Tables.—In actual work this tedious arithmetical labour is saved by means of tables giving the values of y or x for given values of x or y in the equation $y = \frac{1}{\sigma\sqrt{2\pi}}\int_0^x e^{-\frac{x^2}{2\sigma^2}}dx$, except that x (the error) is given in terms of the standard deviation σ which, as we have seen, is the unit of variability (see p. 353), when the integral becomes

$$P = \frac{1}{\sqrt{2\pi}}\int^{\xi} e^{-\frac{\xi^2}{2}}d\xi \left(\text{where } \xi = \frac{x}{\sigma}\right).$$

(P stands for the probability.)

EXAMPLES.

(1) In the example on p. 349 we saw that $\sigma' = 15 \cdot 3$ cells. Therefore the probable error $= \pm\, 0 \cdot 6745 \times 15 \cdot 3$
$= \pm\, 10 \cdot 3.$

Hence in the average counting of 224 cells there is a probable error of $\pm\, 10 \cdot 3$ cells, or of $\pm\, 4 \cdot 6$ per cent.

Similarly $\sigma m = \pm\, 7 \cdot 6$ cells.

$\therefore$ probable error of mean $= \pm\, \cdot 6745 \times 7 \cdot 6$
$= +\, 5 \cdot 2$ cells.

(2) Find the probability of an error lying between the following limits in a normal symmetrical distribution:

(i.) $\pm\,\sigma$; (ii.) $\pm\, 2\sigma$; (iii.) $\pm\, 3\sigma$;

and explain the significance of the results.

(i.) The probability of an error lying between $\pm\,\sigma$ is

$$P = \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\sigma}^{+\sigma} e^{-\frac{x^2}{2\sigma^2}}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{+1} e^{-\frac{\xi^2}{2}}d\xi \left(\text{where } \xi = \frac{x}{\sigma}\right)$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{+1} e^{-\frac{\xi^2}{2}}d\xi$$

$$= 0 \cdot 6827 \text{ (from the tables).}$$

This means that the probability that the error shall exceed $\pm \sigma$ is $1 - \cdot 683 = \cdot 317$.

In other words, the odds against an error exceeding this amount are ·683 to ·317 or approximately 2 : 1.

(ii.) The probability of the error lying between $\pm 2\sigma$ is

$$P = \frac{1}{\sqrt{2\pi}} \int_{0}^{+2} e^{-\frac{\xi^2}{2}} d\xi$$

$$= \cdot 955,$$

i.e., the probability of an error exceeding $\pm 2\sigma$ is $1 - \cdot 955 = \cdot 045$, so that the odds against an error exceeding this amount are 955 to 45 or 21 : 1.

(iii.) The probability of an error lying between $\pm 3\sigma$ is

$$P = \frac{2}{\sqrt{2\pi}} \int_{0}^{+3} e^{-\frac{\xi^2}{2}} d\xi$$

$$= \cdot 9973,$$

which means that the odds against an error exceeding this amount are ·9973 to ·0027, or 370 : 1.

From this we learn that a range six times the S.D. includes, in the case of normal distributions, the vast majority of the observations.

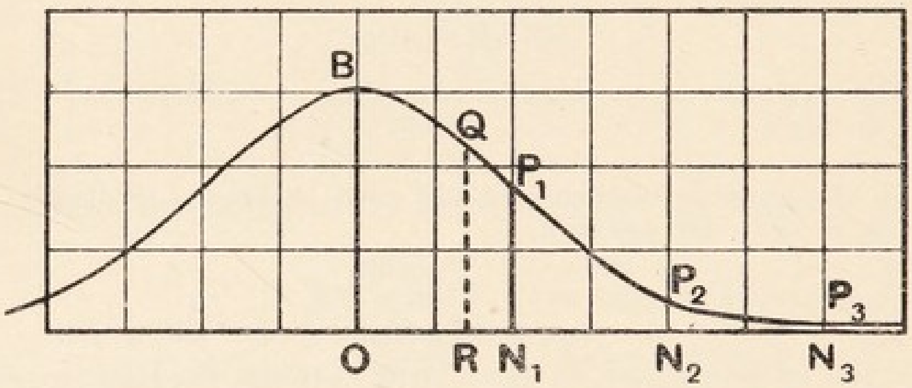

FIG. 123.—Curve of $y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$ (when $\sigma = 5$ and $N = 100$).

$$= \frac{100}{5\sqrt{2\pi}} e^{-\frac{x^2}{50}}$$

$$= 7\cdot 98 e^{-\frac{x^2}{50}}.$$

The graphic verification of these results is shown in the diagram (where $\sigma = 5$, and $N = 100$). The area ON_1P_1B represents the frequencies of errors 0 to σ.

The area ON_2P_2B represents the frequencies of errors 0 to 2σ, and the area ON_3P_3B represents the frequencies of errors 0 to 3σ.

From this it is seen that the deviation from OB exceeding 3σ, being represented by the portion of the curve beyond N_3P_3 is negligible. The point N_1 corresponds to $x = \sigma$ and P_1 is the point of inflection of the curve. The point R corresponds to $x = 0{\cdot}6745\sigma = r$ (the probable error).

EXAMPLES.

(1) Taking the mean human stature for the British Isles as 67·46 inches, the mean for Cambridge students as 68·85 inches, and the common S.D. as 2·56 inches, what percentage of Cambridge students exceed the British mean in stature, assuming the distribution to be normal? (Udny Yule.)

If ON = ordinate at mean, *i.e.*, at 68·85 inches, and PQ = ordinate at 67·46 inches (*i.e.*, the mean for the British Isles), then the area of the curve between PQ and the extreme right limit, *i.e.*, area PQNx represents the frequency of students whose height exceeds 67·46 inches.

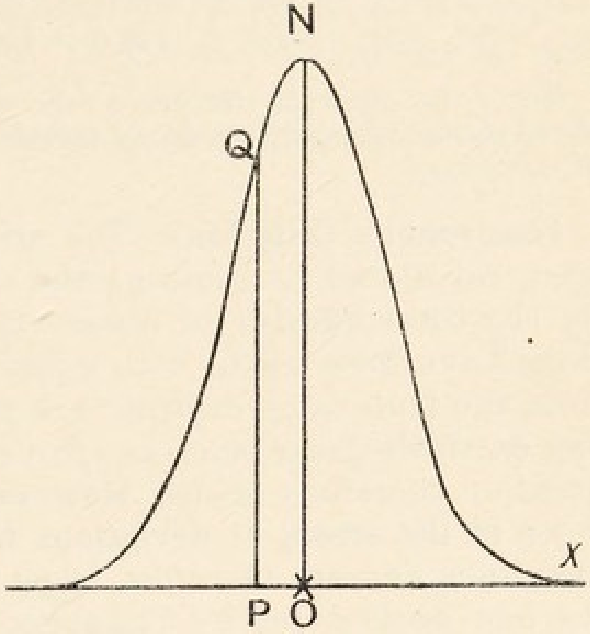

FIG. 124.

Now, total area = 1.

$\therefore$ area ONx = 0·5.

Hence we have to calculate only the area of the portion PQNO. If this area = A, then A + 0·5 = frequency required.

$$\text{Now, abscissa} \qquad OP = 67{\cdot}46 - 68{\cdot}85 = -1{\cdot}39.$$

$$\therefore \quad \xi \text{ which} = \frac{OP}{\sigma} = -\frac{1{\cdot}39}{2{\cdot}56} = -{\cdot}54.$$

$$\therefore \quad \text{area PQNO} = \frac{1}{\sqrt{2\pi}}\int_{-{\cdot}54}^{0} e^{-\frac{\xi^2}{2}}\,d\xi.$$

$$\therefore \quad A = \frac{1}{\sqrt{2\pi}}\int_{0}^{+{\cdot}54} e^{-\frac{\xi^2}{2}}\,d\xi.$$

Now, the probability integral table gives the following areas:

When $\xi = {\cdot}50$, A = ·19146.

When $\xi = {\cdot}55$, A = ·20884.

$\therefore$ a difference in ξ of 0·05 corresponds to a difference in A of ·01738.

$\therefore$ a difference in ξ of ·04 corresponds to a difference in A of $\frac{4}{5} \times {\cdot}01738$, *i.e.*, of 013904.

$\therefore$ for $\xi = {\cdot}54$, A = ·19146 + ·013904

= ·20536 (compare example (2), Chapter II., p. 15).

$\therefore$ area of PQMX = ·20536 + 0·5

= ·7054.

$\therefore$ percentage required = 70·5.

(2) In breeding certain stocks, 408 hairy and 126 glabrous plants were obtained. If the expectation is 25 per cent. glabrous, is the divergence significant, or might it have occurred as a fluctuation of sampling? (Udny Yule.)

$$\sigma = \sqrt{\cdot 25 \times \cdot 75 \times 534} \text{ (see p. 351).}$$
$$= 10.$$

As the expectation is 25 per cent. $= \frac{1}{4} \times 534 = 133 \cdot 5$, and the observed result is 126.

∴ difference between observed and theoretical result is

$$133 \cdot 5 - 126 = 7 \cdot 5.$$

Hence the observed difference falls well within the standard deviation of the expected number and might therefore occur as the result of fluctuations of sampling.

Chauvenet's Criterion.—The arithmetical mean is, as we have seen, calculated by dividing the sum of the various observations by the total number of observations—provided all the observations have been made with equal care and that their deviations from the true value or mean are purely fortuitous and not due to any extrinsic cause (such as error in arithmetical work, etc.). The question therefore arises, **How can we tell whether any one or more of the errors or deviations from the mean are probably not fortuitous and are therefore to be rejected ?** The simplest test is the one worked out by Chauvenet and is therefore called *Chauvenet's Criterion* for the rejection of observations whose deviations from the mean are too great to be merely fortuitous.

Let x_1 be the magnitude of a given error a, expressed in terms of the standard deviation σ, so that $a = x_1\sigma$.

Then the probability of the error $\pm x_1\sigma$ or any smaller error occurring is expressed by the equation

$$\mathrm{P} = \frac{2}{\sigma\sqrt{2\pi}}\int_0^{x_1\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx.$$

In other words, the *proportion* of observations, the deviations of which from the mean are less than a or $x_1\sigma$, is

$$\frac{2}{\sigma\sqrt{2\pi}}\int_0^{x_1\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx.$$

Hence, if N = total number of observations,

then $$\mathrm{NP}\left(=\frac{2\,\mathrm{N}}{\sigma\sqrt{2\pi}}\int_0^{x_1\sigma} e^{-\frac{x^2}{2\sigma^2}}\,dx\right)$$

is the actual number of observations whose deviations from the mean are *less* than a or $x_1\sigma$.

$\therefore$ number of observations whose deviations from the mean are *greater* than a or $x\sigma_1$

$$= N - NP = N(1 - P).$$

Hence, if $N(1 - P) < \frac{1}{2}$, then the probability of a deviation greater than a or $x_1\sigma$ occurring is less than the probability that no such deviation will occur.

We may therefore, in a limited number of observations, reject any deviation equal to or greater than a as being probably not fortuitous.

Chauvenet's criterion for rejection is therefore obtained from the equation:

$$N - NP = \frac{1}{2}$$

i.e.,

$$N - \frac{1}{2} = NP$$

or

$$\frac{2N - 1}{2N} = P = \frac{2}{\sigma\sqrt{2\pi}}\int_0^{x_1\sigma} e^{-\frac{x}{2\sigma_2}}dx,$$

i.e.,

$$\frac{2}{\sigma\sqrt{2\pi}}\int_0^{x_1\sigma} e^{-\frac{x}{2\sigma^2}}dx = \frac{2N - 1}{2N}.$$

EXAMPLES.

(1) In a series of 251 observations on the duration of pregnancy, the standard deviation from the mean (282) has been found to be 15·2 days. The observations gave values ranging from 190 to 340 days. Find what values may be excluded as improbably fortuitous (*e.g.*, due to pathological causes, errors in dating the beginning of pregnancy, etc.).

Applying Chauvenet's criterion we have

$$\frac{2}{\sigma\sqrt{2\pi}}\int_0^{x_1\sigma} e^{-\frac{x^2}{2\sigma^2}}dx = \frac{2 \times 251 - 1}{2 \times 251} = 0{\cdot}99801.$$

$$\therefore \int_0^{x_1\sigma} e^{-\frac{x^2}{2\sigma^2}}dx = {\cdot}99801 \times \frac{\sqrt{2\pi}}{2}$$

$$= 99801 \times \frac{2{\cdot}506}{2}$$

$$= 1{\cdot}250.$$

Using the same method as in example (3), p. 261, or by referring to a special integral table, we find $x_1 = 3{\cdot}09$.

$$\therefore\ a = \sigma x_1 = 15{\cdot}2 \times 3{\cdot}09,$$

$$= 47.$$

∴ all observations which give a deviation from 282 greater than 47 must be rejected, so that any period of gestation greater than 329 (= 282 + 47) days or less than 235 (= 282 − 47) days must be rejected as probably not fortuitous.

Hence a first application of Chauvenet's criterion gives the probable range of duration of pregnancy as 235 to 329 days.

[*Note.*—By applying Chauvenet's criterion to the observations remaining after excluding those whose duration is less than 235 or more than 329 days, Robertson found that the probable duration of pregnancy ranged between 242 and 322 days. A third application of the criterion narrows down the limits to 243 to 321 days. After this, no further rejections are allowable.]

(2) A. J. Clark found the following to be the rates of the rabbit's excised heart at the corresponding temperatures. Find whether these figures are in agreement with the Van't Hoff-Arrhenius formula, viz.,

$$\frac{K_2}{K_1} = e^{\frac{Q}{2}\frac{(T_2-T_1)}{T_1T_2}} \quad \text{(p. 237).}$$

15° C. — 25	30° C. — 82	38° C. — 170.
25° C. — 64	34° C. — 120	

Applying the formula we obtain $\frac{K_{25}}{K_{15}} = \frac{64}{25} = 2{\cdot}56 = e^{\frac{Q}{2}\cdot\frac{10}{298 \times 288}}$, giving $Q = 16130$. Similarly $\frac{K_{30}}{K_{15}}$ gives $Q = 13730$; $\frac{K_{34}}{K_{15}}$ gives $Q = 14600$, and $\frac{K_{38}}{K_{15}}$ gives $Q = 14880$. Therefore mean $Q = 14835$.

$$\therefore \text{ standard deviation} = \sqrt{\frac{(14835 - 16130)^2 + (14835 - 13730)^2 + \ldots}{4}}$$

i.e., $\sigma = 860.$

Applying Chauvenet's criterion, we get maximum deviation probable is x_1,

where
$$\frac{2}{\sigma\sqrt{2\pi}}\int_0^{\sigma x_1} e^{-\frac{x^2}{2\sigma_2}}\,dx = \frac{2 \times 4 - 1}{2 \times 4} = {\cdot}875.$$

Probability integral table gives $x_1 = 1{\cdot}55$.

Hence a deviation of $1{\cdot}55 \times 860$ or 1333 may still be considered fortuitous.

As the maximum deviation in the given cases is only 1295, it is quite a probable one. Hence the figures are in agreement with the Van't Hoff-Arrhenius law.

The following are the observed and calculated results :

Observed	$K_{25} = 64{\cdot}0$	$K_{30} = 82{\cdot}0$	$K_{34} = 120$	$K_{38} = 170$
Calculated	$K_{25} = 59{\cdot}5$	$K_{30} = 89{\cdot}6$	$K_{34} = 123{\cdot}4$	$K_{38} = 174{\cdot}3$

EXERCISE.

In a series of twenty-eight observations in a certain psychological experiment, the mean was found to be 55 and $\sigma = 6{\cdot}98$. Find the limit (x_1) of allowable deviation from the mean, it being given that when

$$\frac{2}{\sigma\sqrt{2\pi}}\int_0^{\sigma x_1} e^{-\frac{x^2}{2\sigma^2}}dx = {\cdot}49107,$$

then $$x_1 = 2{\cdot}37.$$

[Here $$\frac{2 \times 28 - 1}{2 \times 28} = {\cdot}49107. \quad \therefore \quad x_1 = 2{\cdot}37.$$

$$\therefore \quad a = 6{\cdot}98 \times 2{\cdot}37 \quad = 17.$$

$\therefore$ all observations greater than 72 and less than 38 must be rejected.]

[*Note.*—The propriety of rejecting observations on theoretical grounds has not been admitted by a consensus of trained workers. See Brunt's "The Combination of Observations," Cambridge, 1917, pp. 130—132.]

Curve Fitting.—We have seen in Chapter XX, that having obtained a number of laboratory results in a particular investigation, we plot a graph from the corresponding pairs of values of the two variables and then endeavour not only to determine the class of curves (straight line, parabola, hyperbola, exponential, etc.) to which this particular graph belongs, but also by evaluating the constants in the general equation of the curve, establish the particular equation connecting the two variables in the curve so plotted, and thus determine the "law" of the phenomenon under investigation. When we have to deal with masses of statistical data we are confronted with similar tasks, viz. :

(1) We have to decide whether the particular distribution agrees with that of a normal frequency group, and, if not, which of the other types of frequency curves best agrees with the distribution under consideration.

(2) Having found the nature of the curve which best fits in with our data, we set out to determine the constants of the curve from the observed statistics. The process of finding which particular class of frequency curves best agrees with the particular distribution and then of evaluating the constants so as to determine the equation of the particular curve, constitutes what is called *curve-fitting*.

The process of curve-fitting will be best understood by working out an actual numerical example (after Caradog Jones, "A First Course in Statistics").

The table on p. 362 illustrates the distribution of marks obtained by 514 candidates in a certain examination.

Find (*a*) what type of curve will best agree with this distribution.

(*b*) What is the exact equation of the curve so found.

(*c*) How far the given frequency data correspond with the frequencies calculated from the equation, *i.e.*, what is the "goodness of the fit"?

(*a*) To find the type of curve suitable for any distribution it is necessary to evaluate from the given data the magnitude of A, and B and C in the denominator $Ax^2 + Bx + C$ of the **general**

Marks obtained. (x)	Number of Candidates. (f)
1—5	5
6—10	9
11—15	28
16—20	49
21—25	58
26—30	82
31—35	87
36—40	79
41—45	50
46—50	37
51—55	21
56—60	6
61—65	3
Total . .	514

differential equation for a frequency curve (p. 352). When the numerical values of these coefficients are known, then the type of curve is, as we have seen, established. Now, we have seen that these coefficients can be expressed in terms of the various moments (about the mean) of the observed distribution, hence by finding these moments we can at once establish the type of curve which will best suit the given distribution.

The first thing, therefore, which we must proceed to do is to evaluate the various moments of the distribution, and to do this we must find the mean.

(i.) **To find the Mean of the Given Distribution.**—Since the x's vary discretely or discontinuously, *i.e.*, at definite and fixed intervals of five marks (which we may call the **unit**) and not continuously (*i.e.*, by infinitesimal increments), we must take the mid-value of each interval to represent the average values of the particular interval (*e.g.*, 3 in the " 1—5 " interval ; 8 in the " 6—10 " interval, and so on), and treat each of the candidates of any particular group as though he received the number of marks corresponding to the mid-value of the class-interval of his group. Thus each of the 5 candidates of the class-interval " 1—5 " is considered as having received 3 marks ; each of the 9 candidates with the class-interval of " 6—10 " is considered as having received 8 marks, and so on. This assumption is, of course, not quite correct, and certain adjustments will be necessary to eliminate the concomitant errors. With these we shall deal later (see p. 364). If now we take some arbitrary, but convenient, number such as *33 marks* (which is the mid-value of the class-interval " 31—35," *i.e.*, the interval lying midway between the 2 extreme class-intervals) as the *origin*, and the number of marks in each class-interval (*i.e.*, *5*) as the *unit*, we get the following table :

Mean No. of Marks.	Deviation from 33.	Frequency.	First Moment.	Second Moment.	Third Moment.	Fourth Moment.
	x (= 5 marks as unit)	f	(fx)	(fx^2)	(fx^3)	(fx^4)
3	− 6	5	− 30	180	− 1080	6480
8	− 5	9	− 45	225	− 1125	5625
13	− 4	28	− 112	448	− 1792	7168
18	− 3	49	− 147	441	− 1323	3969
23	− 2	58	− 116	232	− 464	928
28	− 1	82	− 82	82	− 82	82
33	0	87	..	..	..	..
38	+ 1	79	+ 79	+ 79	+ 79	+79
43	+ 2	50	100	200	400	800
48	+ 3	37	111	333	999	2997
53	+ 4	21	84	336	1344	5376
58	+ 5	6	30	150	750	3750
63	+ 6	3	18	108	648	3888
—	—	514	− 110	2814	− 1646	41142
		$N=\Sigma f$	$N_1{}^1=\Sigma xf$	$N_2{}^1=\Sigma x^2f$	$N_3{}^1=\Sigma x^3f$	$N_4{}^1=\Sigma x^4f$

The *arithmetic mean* of the distribution is (see p. 337) :

$$33 + 5\bar{x} = 33 + 5\left(-\frac{110}{514}\right) = 31{\cdot}92996 \quad . \quad . \quad . \quad \text{(A)}$$

(ii.) The *next step* is to calculate directly from the statistics the values of :

(*a*) N (total frequency) $= f_1 + f_2 + f_3 + \ . \ . \ . \ + f_n = \Sigma f.$
$= 514.$

(*b*) $N_1{}^1$ (1*st moment* about the *fixed value*, 33, as origin)
$= x_1 f_1 + x_2 f_2 + \ . \ . \ . \ + x_n f_n = \Sigma fx = -110.$

(*c*) $N_2{}^1$ (2*nd moment* about *fixed value* as origin)
$= x_1{}^2 f_1 + x_2{}^2 f_2 + \ . \ . \ . \ + x_n{}^2 f_n = \Sigma fx^2 = 2{,}814.$

(*d*) $N_3{}^1$ (3*rd moment* about *fixed value* as origin)
$= x_1{}^3 f_1 + x_2{}^3 f_2 + \ . \ . \ . \ + x_n{}^3 f_n = \Sigma fx^3 = -1{,}646.$

(*e*) $N_4{}^1$ (4*th moment* about *fixed value* as origin)
$= x_1{}^4 f_1 + x_2{}^4 f_2 + \ . \ . \ . \ + x_n{}^4 f_n = \Sigma fx^4 = 41{,}142.$

(iii.) Having found N, $N_1{}^1$, $N_2{}^1$, $N_3{}^1$, $N_4{}^1$, we evaluate the following ratios (*i.e.*, the moment coefficients), viz. :

$$\nu_1{}^1 = \frac{N_1{}^1}{N} = -\frac{110}{514}$$

$$\nu_2{}^1 = \frac{N_2{}^1}{N} = \frac{2814}{514} = 5{\cdot}4747$$

$$\nu_3{}^1 = \frac{N_3{}^1}{N} = -\frac{1646}{514} = -3{\cdot}2023$$

$$\nu_4{}^1 = \frac{N_4{}^1}{N} = \frac{41142}{514} = 8{\cdot}004.$$

(iv.) Now transfer the origin from the fixed arbitrary value 33 to the mean 31·92996 ; the ν's then become ν_1, ν_2, ν_3 and ν_4.

But we have seen that when the deviations are measured from the mean as origin, the first moment vanishes (see p. 338).

$$\therefore \quad \nu_1 = 0.$$

We have also seen that

$$\nu_2 = \nu_2{}^1 - \bar{x}^2 \text{ (see p. 339)}$$

$$= \frac{2814}{514} - \left(\frac{110}{514}\right)^2.$$

$$= 5{\cdot}4747 - 0{\cdot}0458 = 5{\cdot}4289.$$

Similarly, it can be shown that

$$\nu_3 = \nu_3{}^1 - 3\nu_2\bar{x} - \bar{x}^3$$

$$= -\frac{1646}{514} - 3 \times 5{\cdot}4289 \times \left(-\frac{110}{514}\right) - \left(\frac{110}{514}\right)^3$$

$$= 0{\cdot}29296,$$

and $\nu_4 = \nu_4{}^1 - 4\nu_3\bar{x} - 6\nu_2\bar{x}^2 - \bar{x}^4$

$$= \frac{41142}{514} - 4 \times {\cdot}29296 \times \left(-\frac{110}{514}\right) - 6 \times 5{\cdot}4289\left(-\frac{110}{514}\right)^2 - \left(-\frac{110}{514}\right)^4$$

$$= 78{\cdot}79964.$$

so that **we have arrived at the numerical values of the various moments measured from the arithmetic mean as origin.**

(v.) **Sheppard's Adjustments.**—But all the above moments have been calculated on the assumption, only approximately correct, that each of the candidates in any particular group has received a number of marks corresponding to the mid-value of the class-interval of that group. We must therefore try and make certain corrections for the fact that the number of marks actually received was not constant for each candidate in the group. These corrections have been worked out by Dr. W. F. Sheppard and are known as *Sheppard's adjustments.* It is not necessary to enter into the detailed working out of these adjustments, for, although the requisite mathematics is not at all difficult, it is rather complicated and tedious ; but it will be sufficient if we only give the results. They are as follows :

1st adjusted moment coefficient μ_1 = 1st unadjusted moment $\nu_1 = 0$.

2nd adjusted moment coefficient μ_2 = 2nd unadjusted moment $\nu_2 - \frac{1}{12}$

$$= 5{\cdot}42891 - {\cdot}0833$$
$$= 5{\cdot}34558$$

3rd adjusted moment coefficient μ_3 = 3rd unadjusted moment ν_3

$$= 0{\cdot}29296.$$

$$\text{4th adjusted moment coefficient } \mu_4 = \nu_4 - \frac{1}{2}\nu_2 + \frac{7}{240}$$

$$= 78{\cdot}79964 - \frac{1}{2} \times 5{\cdot}42891 + {\cdot}02917$$
$$= 76{\cdot}11436.$$

[*Note.*—For cases in which these corrections are not adequate see "Biometrika," xii., 1909, 231—258.]

(vi.) If we call the ratio $\frac{\mu_3^2}{\mu_2^3} = \beta_1$

and the ratio $\frac{\mu_4}{\mu_2^2} = \beta_2$

then we get

$$\beta_1 = \frac{({\cdot}29296)^2}{(5{\cdot}34558)^3} = 0{\cdot}00056$$

and
$$\beta_2 = \frac{76{\cdot}11436}{(5{\cdot}34558)^2} = 2{\cdot}66365.$$

But the various coefficients A, B, C, in the denominator $Ax^2 + Bx + C$ of the general differential equation

$$\frac{dy}{dx} = \frac{y\,(x+b)}{Ax^2 + Bx + C}$$

for a frequency curve can be expressed in terms of β_1 and β_2 as follows:

$$A = -\frac{(2\beta_2 - 3\beta_1 - 6)}{2(5\beta_2 - 6\beta_1 - 9)} = -\frac{(2 \times 2{\cdot}66365 - 3 \times 0{\cdot}00056 - 6)}{2(5 \times 2{\cdot}66365 - 6 \times 0{\cdot}00056 - 9)}$$

$$= \frac{0{\cdot}6744}{8{\cdot}63} = {\cdot}08$$

$$B = -\frac{\sqrt{\mu_2\beta_1} \,.\, (\beta_2 + 3)}{2(5\beta_2 - 6\beta_1 - 9)} = -\,{\cdot}03$$

and

$$C = -\frac{\mu_2(4\beta_2 - 3\beta_1)}{2(5\beta_2 - 6\beta_1 - 9)}$$

Hence κ, which $= \frac{B^2}{4AC} = \frac{\beta_1(\beta_2 + 3)^2}{4(4\beta_2 - 3\beta_1)\,(2\beta_2 - 3\beta_1 - 6)}$

$$= \frac{0{\cdot}00056 \times (5{\cdot}66365)^2}{4(10{\cdot}65292)\,(-\,0{\cdot}67438)}$$

$$= -\,0{\cdot}00063.$$

As κ is very small and A and B are also small,

$\therefore$ the curve **best agreeing** with this distribution is the *normal frequency curve* (see p. 353),

viz.,
$$y = \frac{N}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}}.$$

Note.—From the above formulæ for A it is seen that if β_1 is very small, and β_2 does not differ much from 3, then A is very small. Also formula for B shows that when β_1 is very small, then B is very small. Therefore if in any distribution β_1 is found to be very small and β_2 approximately equal to 3, and if also κ is small, then a normal curve may be fitted to it.

(*b*) If in the equation $y = \frac{N}{\sigma\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma^2}}$

we put $N = 514$

and $\sigma = \sqrt{\mu_2} = \sqrt{5{\cdot}3456} = 2{\cdot}312,$

the equation for the curve representing the distribution in question becomes

$$y = \frac{514}{2{\cdot}312\sqrt{2\pi}}\, e^{-\frac{x^2}{2 \times 5{\cdot}35}} = \frac{514}{2{\cdot}312 \times 2{\cdot}506}\, e^{-\frac{x^2}{2 \times 5{\cdot}35}}$$

$$= 88{\cdot}69\, e^{-\frac{x^2}{10{\cdot}7}}$$

(88·69 being the value of y at the mean 31·93.)

From this equation we are able to draw the curve by calculating the value of y for the various given values of x. *E.g.*, to find the theoretical value of y_{28} (*i.e.*, at the point where $x = 28$), we subtract 31·92996 (the mean) from 28, getting − 3·92996 as the distance (in number of marks) from the mean as origin.

Dividing − 3·92996 by 5 (which is the unit) we get

$$x_{28} = - \cdot 786.$$

$$\therefore \quad y_{28} = 88\cdot 69\, e^{-\frac{(-\cdot 786)^2}{10\cdot 7}}$$

$$= 88\cdot 69\, e^{-\cdot 058}$$

$$\therefore \quad \log_{10} y_{28} = \log_{10} 88\cdot 69 - \cdot 058 \log_{10} e$$

$$= \log_{10} 88\cdot 69 - \cdot 4343 \times \cdot 058$$

$$= 1\cdot 94788 - \cdot 02519$$

$$= 1\cdot 92269$$

$$\therefore \quad y_{28} = 83\cdot 7.$$

Similarly, for any other ordinate (see table, p. 368). The formula for the curve now enables us to calculate the area of a portion of curve corresponding to any given class-interval. As area in a frequency curve represents frequency we can in this way evaluate the **theoretical** frequencies for the various given class-intervals, as well as for any class-interval not given in the data.

An example will make this clear.

Find the theoretical frequency for the class-interval 11—15.

The area of the portion of the curve lying between $x_{-\infty} = -\infty$ and $x_{15\cdot 5} = 16\cdot 42996$ (see below) gives the frequency of candidates with marks lying between 0 and 15·5. Call this A.

[*Note.*—X_n does not represent the number of marks, but the deviation from the mean number of marks.]

Similarly, the area of the portion of the curve lying between $x_{-\infty} = -\infty$ and $x_{10\cdot 5} = 21\cdot 42996$ gives the frequency of candidates with marks lying between ·0 and 10·5. Call this B.

Therefore frequency of candidates with marks lying between 10 and 15 inclusive = A − B.

Now $\quad x_{15\cdot 5} = 31\cdot 92996 - 15\cdot 5 = 16\cdot 42996$

$$\sigma = 5\sqrt{5\cdot 34558} = 11\cdot 56025$$

$$\therefore \quad \xi_{15\cdot 5}\left(= \frac{x_{15\cdot 5}}{\sigma}\right) = \frac{16\cdot 42996}{11\cdot 56025} = 1\cdot 4212$$

$$\therefore \quad \mathrm{A} = \frac{\mathrm{N}}{\sqrt{2\pi}} \int_{-\infty}^{\xi_{15\cdot 5}} e^{-\frac{\xi^2}{2}}\, d\xi$$

$$= \frac{514}{\sqrt{2\pi}} \int_{-\infty}^{1\cdot 4212} e^{-\frac{\xi^2}{2}}\, d\xi$$

$$= 514 \times \cdot 177632.$$

$$\text{Similarly, } x_{10\cdot5} = 31\cdot92996 - 10\cdot5 = 21\cdot42996$$

$$\therefore \quad \xi_{10\cdot5} = \frac{21\cdot42996}{11\cdot36025} = 1\cdot8538$$

$$\therefore \quad \mathrm{B} = \frac{514}{\sqrt{2\pi}}\int_{-\infty}^{1\cdot8538} e^{-\frac{\xi^2}{2}}\,d\xi$$

$$= 514 \times \cdot131896$$

$$\therefore \quad \mathrm{A} - \mathrm{B} = 514\,(\cdot177632 - \cdot131896)$$

$$= 23\cdot5,$$

i.e., the theoretical frequency should be 23·5 (as compared with the observed frequency of 28).

This leads us to a consideration of the third portion of our question, viz., the test for "goodness of fit."

(*c*) **Goodness of Fit.**—Having calculated the theoretical frequencies, construct a table showing both the observed and calculated frequencies for each class-interval as follows :

Mean Number of Marks. (1)	Frequency.		Deviation (d) $= (fc - fo)$. (4)	Square of Deviation (d^2). (5)	$\frac{d^2}{fc}$ (6)
	Observed fo. (2)	Calculated. fc. (3)			
3	5	5·7	+ 0·7	0·49	0·09
8	9	10·7	+ 1·7	2·89	0·27
13	28	23·5	− 4·5	20·25	0·86
18	49	43·1	− 5·9	34·81	0·81
23	58	65·6	+ 7·6	57·76	0·88
28	82	83·1	+ 1·1	1·21	0·01
33	87	87·6	+ 0·6	0·36	0·00
38	79	76·8	− 2·2	4·84	0·06
43	50	56·0	+ 6·0	36·00	0·69
48	37	34·0	− 3·0	9·00	0·26
53	21	17·1	− 3·9	15·21	0·89
58	6	7·2	+ 1·2	1·44	0·20
63	3	3·9	+ 0·5	0·25	0·70
	514	513·9		184·51	$\chi^2 = 5\cdot04$

In order to test whether the correspondence between the observed and calculated values is as good as can be expected in any random sample, and is not due to the non-obeyance by the distribution of the normal frequency law, we take the square of each deviation as in column 5, *i.e.*, $(fc - fo)^2$, and divide it by the calculated frequency fc. We then get an expression $\frac{(fc-fo)^2}{fc}$, relating each deviation to the calculated frequency of its own group. The sum of these expressions, viz., $\Sigma \frac{(fc-fo)^2}{fo}$, is called χ^2, and is given at the bottom of column 6. From the value of χ^2 so found it is possible to estimate the probability (P) that the distribution in question is a true example of the type of curve applied to it. A table has been constructed (see Palin Elderton, "Biometrika," Vol. I.), giving the values of P corresponding to different values of χ^2 and to values of n', the total number of frequency groups (13 in this case). From such table we learn that when $n' = 13$ and $\chi^2 = 5{\cdot}04$ then $P = 0{\cdot}956$, so that the chances are nearly 0·96 to 0·04, or 24 to 1, that the differences between the calculated and observed frequencies are no greater than can be expected from random sampling. Hence the fit is excellent.

Example.

3,404 boys between fourteen and fifteen years old were weighed, and it was found that the following were the frequencies for the various weight intervals :

Weight.	Number of Boys.
65— 70	3
70— 75	9
75— 80	142
80— 85	301
85— 90	289
90— 95	380
95—100	416
100—105	404
105—110	315
110—115	320
115—120	262
120—125	221
125—130	131
130—135	76
135—140	52
140—145	20
145—150	29
150—155	14
155—160	10
160—165	2
165—170	2
170—175	5
175—180	1

Calculate the mean weight and standard deviation and ascertain whether a normal curve will fit the given distribution ?

Taking 102·5 as the origin, then

$$\text{1st moment coefficient about origin} = \frac{874}{3404} = \cdot 2568.$$

$$\therefore \quad \text{A.M.} = 102{\cdot}5 + {\cdot}2568 \times 5 = 103{\cdot}784 \text{ lb.}$$

$$\text{2nd moment coefficient about origin} = \frac{37492}{3404} = 11{\cdot}014.$$

$$\therefore \quad \text{2nd moment coefficient about mean} = 11{\cdot}014 - ({\cdot}2568)^2 = 10{\cdot}948.$$

Applying Sheppard's adjustment, we get

$$\sigma^2 = 10{\cdot}948 - \frac{1}{12} = 10{\cdot}865.$$

$$\therefore \quad \sigma = \sqrt{10{\cdot}865} = 3{\cdot}296, \textit{ i.e.}, 3{\cdot}296 \times 5 = \underline{16{\cdot}48 \text{ lb.}}$$

$$\text{3rd moment coefficient about origin} = \frac{107110}{3404} = 31{\cdot}466.$$

$$\therefore \quad \text{3rd moment coefficient about mean} = 31{\cdot}466 - 3 \times ({\cdot}2568) \times 11{\cdot}014 + 2 \times ({\cdot}2568)^2 = \underline{23{\cdot}01.}$$

$$\therefore \quad \text{Adjusted 3rd moment coefficient} = 23{\cdot}01.$$

$$\text{4th moment coefficient about origin} = \frac{1502932}{3404} = 441{\cdot}519.$$

$$\therefore \quad \text{4th moment coefficient about mean} = 441{\cdot}519 - 4 \times ({\cdot}2568) \times 31{\cdot}466 + 6 \times ({\cdot}2568)^2 \times 11{\cdot}014 - 3 \times ({\cdot}2568)^4 = 413{\cdot}542.$$

$$\therefore \quad \text{Adjusted 4th moment coefficient, } 413{\cdot}542 - \frac{1}{2} \times 10{\cdot}865 + \frac{7}{240} = 408{\cdot}10.$$

$$\therefore \quad \beta_1 = \frac{(23{\cdot}01)^2}{(10{\cdot}865)^3} = {\cdot}4134 \,; \quad \beta 2 = \frac{413{\cdot}542}{(10{\cdot}865)^2} = 3{\cdot}457.$$

Since β_1 is not small, therefore a normal curve will not fit the distribution.

Correlation and Regression.

The reader is asked to consider the following cases :

(1) The cubical **expansion** of a metal with heat.

(2) The cubical **contraction** of ice with heat (between − 4° and 0° C.).

(3) The relation between the volume of a gas and the pressure to which it is subjected (at constant temperature).

(4) The relation between the height of a person and his weight.

(5) The relationship between the height of a person and his susceptibility to, say, heart disease.

(6) The relationship between a history of rheumatic fever and the presence of heart disease.

(7) The relation between overcrowding and infantile mortality.

(8) The relation between the vaccinal condition of a person and his susceptibility to small-pox.

In each of the first three cases there is a constant and perfectly definite relationship between the two measurements—although the nature of the relationship is different in each case, as we shall see presently—so that when the temperature of the metal is increased or diminished by a fixed amount there is a concomitant perfectly fixed increase or diminution in the volume of the metal; when the temperature of the ice is increased or diminished by a fixed amount (between $-4°$ and $0°$ C.) there is a corresponding fixed diminution or increase in the volume of the ice; and when the pressure of the gas is increased or diminished by a fixed amount there is a corresponding fixed diminution or increase in the volume of the gas. In each of these cases, therefore, in virtue of the perfect correspondence between the measurements, we can predict with absolute certainty the size of one variable when that of the other is known. **When there is such perfect correspondence between two sets of measurements we say that there is complete or perfect correlation between them.**

Now take the relationship between the height of a person and his susceptibility to, say, heart disease. As far as we know there is absolutely no relationship at all between the two, so that we cannot in any way guess the condition of the heart from the person's height and *vice versâ*. In such a case, therefore, there is no concomitant variation between the two conditions, and we say that there is no correlation or there is zero (0) correlation between height and the condition of the heart.

Again, let us take the relationship between height and weight, or the relationship between rheumatic fever and heart disease, or that between overcrowding and infantile mortality. We know from statistics and general experience that, although we cannot predict with certainty the weight of a person from his height, the condition of his heart from a past positive or negative history of rheumatic fever, or the extent of the infantile mortality in a given district from a knowledge of the density of population in that district, yet there is a certain greater or lesser tendency to concomitant variation between the corresponding pairs of measurements in each of the three cases, so that **on the average** a tall person will be heavier than a short one; a person with a history of rheumatic fever will be more likely to have heart disease than one in whom there has been no such history; and a district in which there is considerable overcrowding is likely to have a higher infantile mortality than one in which the density of the population is very low. In these cases, therefore, where there is only **a tendency to concomitant variation**, but there is no absolute correspondence between the pairs of measurements, we say there is *imperfect correlation* between them. And as the correspondence is such that an **increase** in one measurement leads us to expect an **increase** in the other measurement and *vice versâ*, we say the correlation between them is of a *positive* kind.

Lastly, consider the relationship between vaccination and small-pox. It has, of course, been abundantly proved that vaccinated persons have a smaller susceptibility to small-pox than non-vaccinated, so that communities with a *higher* percentage of vaccination will, on the average, have *smaller* small-pox mortality and *vice versâ*. Here, then, we have a case of imperfect correlation again, but of a *negative* kind, or we can say there is imperfect positive correlation between vaccination and immunity against small-pox. In the case of ice we can say that there is *perfect negative correlation* between the expansion of the ice and its temperature.

Measurement of Correlation.—Perfect positive correlation, as in the case of thermal **expansion** of metals, is indicated by $+1$. Absence of correlation, as in the case of height and heart disease, is indicated by zero (0), whilst perfect negative correlation, as in the case of thermal **contraction** of ice, or the relation between pressure and volume of a gas, is indicated by -1. Hence we see that when the correlation is imperfect, as in the cases of the other examples which we considered in the preceding paragraph, it must be represented by some decimal fraction ranging between 0 and $+1$ in the case of positive correlation and between -1 and 0 in the case of negative correlation. **This number lying between the limits ± 1, which measures the degree of correlation between two sets of measurements, is called a correlation coefficient** or **correlation-ratio,** according as the nature of the graph representing the correlation is rectilinear (as in the ideal case of, say, thermal expansion, or any other example of simple interest law) or curvilinear, as in the ideal case of pressure-volume law of a gas that we considered above.

In what follows, we shall only consider cases in which the relationship can be expressed by a straight line graph, and where, therefore, the *correlation coefficient* is the constant, which we shall endeavour to find. Those cases of imperfect correlation, where the graph is not a straight line, are beyond the scope of an elementary book like this.

Method of Finding a Correlation Coefficient.—In the domain of physics it is perfectly easy to determine the correlation coefficient, because whether we plot y against x or x against y, the result will always, in the case of a simple interest law, be the same straight line (but, of course, for errors of observation). Hence, when these two lines coincide, the correlation is always perfect and the correlation coefficient is unity. Similarly, in cases of zero correlation, if we plot y against x we shall get a line parallel to the x axis, and if we plot x against y our graph will be a line parallel to the y axis. In other words, we shall get two lines at right angles to each other in cases of zero correlation and two coincident lines in cases of perfect correlation. In cases of imperfect correlation we shall get two lines inclined to each other at an acute angle, and hence we see that the angle between the two lines is in some way a measure of the degree of relationship between the two sets of measurements.

Example on Correlation.

A close study of the following example will illustrate the method of finding the correlation coefficient between two sets of measurements.

The Census Report of 1911 gives the percentage of overcrowding (*i.e.*, the percentage of the population which has less than one room per two persons) in each of twenty-nine metropolitan boroughs, together with the infantile mortality in each borough. It is required to find the correlation coefficient between overcrowding and infantile mortality.

The procedure is as follows:

(1) Draw up a table of double entry showing the frequencies of given mortality intervals for the various percentage intervals of overcrowding (see table, p. 374). The **middle unbracketed figure** in each square denotes the **frequency** or number of boroughs having a percentage x^n of overcrowding, and y_n mortality. Thus four boroughs have 10 to 15 per cent. overcrowding and 120 to 130 infant mortality.

(2) Treat each x interval as if its value were located at its mid-value (compare p. 362). Thus the x_1 interval, 0—5, is to be taken as 2·5; the x_2 interval, 5—10, is to be taken as 7·5, etc. The error involved in this assumption will be later corrected by means of Sheppard's adjustments (compare p. 364).

(3) Deal with the y intervals in the same way. Thus the interval y_1 (70—80) is to be taken as 75; the y_2 interval (80—90) is to be taken as 85, etc., and the necessary corrections will be made later.

(4) Find what mean value of y (infant mortality) is associated with any x (percentage of overcrowding). Thus the mean value of y associated with $x = 5—10$ is

$$\frac{(75 \times 1 + 95 \times 1 + 105 \times 1 + 125 \times 1)}{4} = \frac{400}{4} = 100;$$

that associated with $x = 10—15$ is

$$\frac{(105 \times 3 + 115 \times 1 + 125 \times 4 + 135 \times 1 + 145 \times 2)}{11} = 123{\cdot}17,$$

and so on. Enter these values under the appropriate x columns as shown in the table.

(5) Similarly, find what mean value of x is associated with any y. Thus the mean value of x (percentage of overcrowding) associated with infantile mortality $y = 100—110$, is

$$\frac{2{\cdot}5 \times 1 + 7{\cdot}5 \times 1 + 12{\cdot}5 \times 3 + 22{\cdot}5 \times 1 + 27{\cdot}5 \times 1}{7} = \frac{97{\cdot}5}{7} = 13{\cdot}93,$$

and so on for the others. Enter these at the end of the appropriate rows, as shown in the table.

(6) Plot (*a*) the various *actual* values of x, and corresponding average y's, viz., (2·5, 105), (7·5, 100), (12·5, 123·17), etc. (PQ in Fig. 125).

—	Percentage of Population Overcrowded.									
Infant Mortality.	x_1 0—5 (−3)	x_2 5—10 (−2)	x_3 10—15 (−1)	x_4 15—20 (0)	x_5 20—25 (+1)	x_6 25—30 (+2)	x_7 30—35 (+3)	x_8 35—40 (+4)	Total.	Average x.
y_1 70—80 (−5)	—	(10) 1 (10)	—	—	—	—	—	—	1	7·5
y_2 80—90 (−4)	—	—	—	—	—	—	—	—	0	0
y_3 90—100 (−3)	—	(6) 1 (6)	—	—	—	—	—	—	1	7·5
y_4 100—110 (−2)	(6) 1 (6)	(4) 1 (4)	(2) 3 (6)	—	(−2) 1 (−2)	(−4) 1 (−4)	—	—	7	13·93
y_5 110—120 (−1)	—	—	(1) 1 (1)	—	—	(−2) 1 (−2)	—	—	2	20
y_6 120—130 (0)	—	(0) 1 (0)	(0) 4 (0)	(0) 1 (0)	(0) 1 (0)	—	—	—	7	13·93
y_7 130—140 (+1)	—	—	(−1) 1 (−1)	(0) 1 (0)	—	—	—	—	2	15
y_8 140—150 (+2)	—	—	(−2) 2 (− 4)	0 — 0	—	(+4) 1 (+4)	(+6) 1 (+6)	—	4	21·25
y_9 150—160 (+3)	—	—	—	—	(+3) 2 (+6)	—	(·9) 1 (9)	(12) 1 (12)	4	28·75
y_{10} 160—170 (+4)	—	—	—	—	—	—	—	(16) 1 (16)	1	37·5
Totals .	1	4	11	2	4	3	2	2	29	—
Average y	105	100	123·17	130	135	121·67	150	160	—	—
$f\xi\eta$ . .	6	20	2	0	4	−2	15	28	—	—

(*b*) The various actual values of y and corresponding average values of x, viz. (75, 7·5), (85, 0), (95, 7·5), (105, 13·93), etc. (RS in Fig. 125). It will be found that the graphs best fitting these values is a straight line in each case (Fig. 125, RS). These lines intersect at a point whose x can be proved to be the mean of *all* the x's, and whose y is the mean of *all* the y's.

(7) To find the mean of all the x's and that of all the y's, we proceed as on p. 362, viz., we take some arbitrary but convenient mid-value of x such as 12·5 (*i.e.*, mid-value of 15—20 interval) as the origin, and the value of the interval, viz., 5, as the limit; also some arbitrary but convenient mid-value of y, such as 125 (*i.e.*, mid-value of 120—130 interval) as the origin,

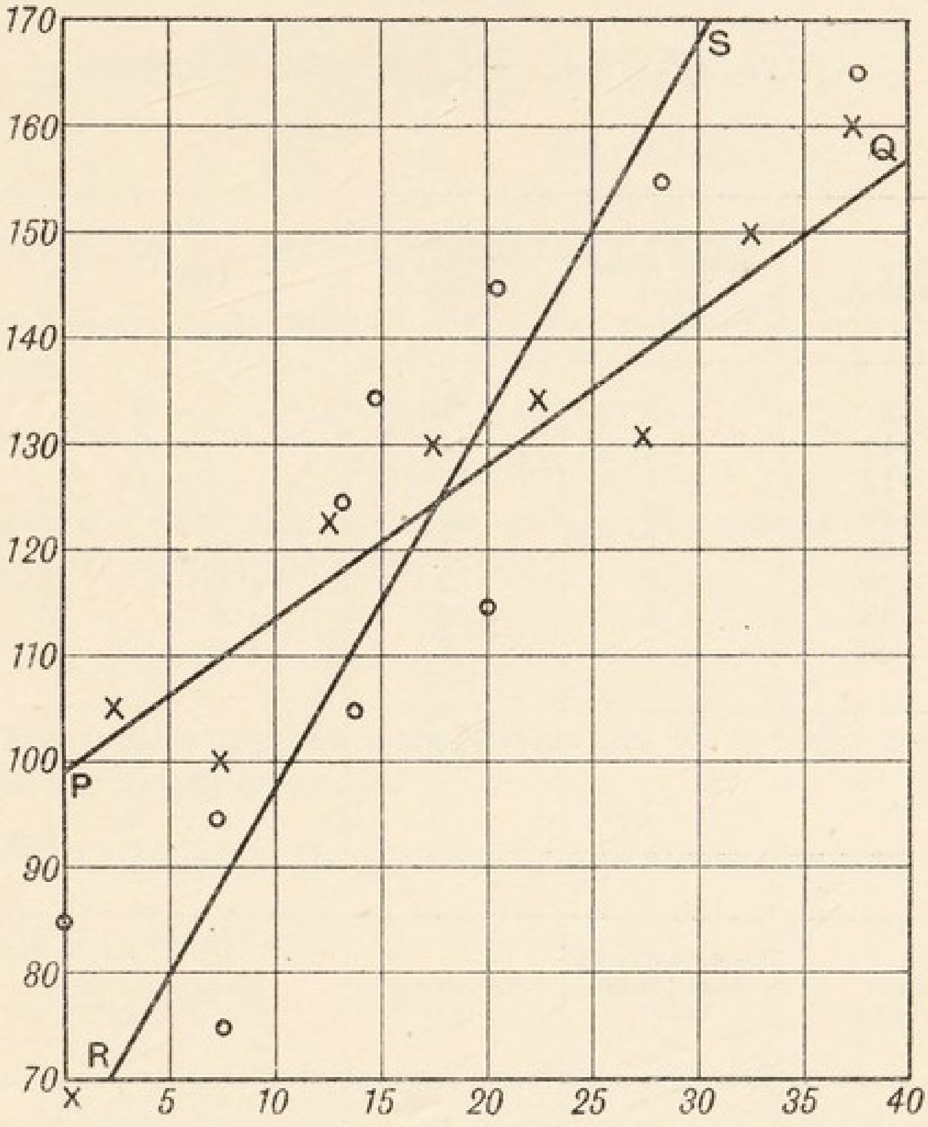

Fig. 125.

and the value of the interval, viz., 10, as the unit, we then get the tables given on p. 376.

$$\therefore \quad \textit{Arithmetic mean of all } x\text{'s} = 17{\cdot}5 + \frac{\Sigma f\xi}{\Sigma f} \times 5 \text{ (see p. 337)}$$

$$= 17{\cdot}5 + \frac{2}{29} \times 5$$

i.e., $\bar{x} = \underline{17{\cdot}85}$,

and *arithmetic mean of all y's* $= 125 + \frac{2}{29} \times 10$,

i.e., $\bar{y} = 125{\cdot}69$.

Hence the point of intersection of the two lines is the point (17·85, 125·7).

(8) We shall want the standard deviations of the x's and of the y's. These can easily be evaluated from the tables.

$$\nu_2{}^{1*} \text{ (in the case of } x) = \frac{\Sigma f\xi^2}{\Sigma f} = \frac{102}{29},$$

$$= 3{\cdot}517.$$

$$\therefore \quad \nu_2{}^* = 3{\cdot}517 - \left(\frac{2}{29}\right)^2 = 3{\cdot}5170 - {\cdot}0048$$

$$= 3{\cdot}5122.$$

Over-crowding.	Deviation from chosen Mean (17·5) (ξ)	Frequency (Number of Boroughs) (f)	($f\xi$)	($f\xi^2$)
0—5	− 3	1	− 3	9
5—10	− 2	4	− 8	16
10—15	− 1	11	− 11	11
15—20	0	2	0	0
20—25	+ 1	4	+ 4	4
25—30	+ 2	3	+ 6	12
30—35	+ 3	2	+ 6	18
35—40	+ 4	2	+ 8	32
		29	24 − 22 = 2	102

Infant Mortality.	Deviation from chosen Mean (125) η	Frequency (f)	$f\eta$	$f\eta^2$
70—80	− 5	1	− 5	25
80—90	− 4	0	0	0
90—100	− 3	1	− 3	9
100—110	− 2	7	− 14	28
110—120	− 1	2	− 2	2
120—130	0	7	0	0
130—140	+ 1	2	+ 2	2
140—150	+ 2	4	+ 8	16
150—160	+ 3	4	+ 12	36
160—170	+ 4	1	+ 4	16
		29	26 − 24 = 2	134

* See p. 364 for the meaning of these letters.

$$\therefore \quad \mu_2{}^* = 3{\cdot}512 - \frac{1}{12} = 3{\cdot}512 - {\cdot}083$$
$$= 3{\cdot}429.$$
$$\therefore \quad \sigma_x{}^* = 5\sqrt{3{\cdot}429} = 5 \times 1{\cdot}85$$
$$= 9{\cdot}25.$$

Similarly, $\sigma y = 10\sqrt{\frac{134}{29} - \left(\frac{2}{29}\right)^2 - \frac{1}{12}} = 10\ \sqrt{4{\cdot}6209 - {\cdot}0048 - {\cdot}083}$
$$= 10\sqrt{4{\cdot}5328.}$$
$$= 21{\cdot}29.$$

(9) *Consider the line* PQ. Let its equation be $y = mx + b$.

Now we have seen (p. 346) that the best value of m is given by the equation

$$m = \frac{\Sigma(x)\Sigma(y) - n\Sigma(xy)}{[\Sigma(x)]^2 - n\Sigma(x^2)}$$

and that when $b = 0$, that is, when we take the means of all the x's and of all the y's as the origin, then

$$m = \frac{\Sigma(xy)}{\Sigma(x^2)},$$

where **x and y now represent the deviations of x and y respectively from the means as origin.**

But $\frac{\Sigma(xy)}{n}$ = mean of all the products of corresponding pairs of deviation $= p$ (say)

$$\therefore \quad \Sigma(xy) = np.$$

Also $\frac{\Sigma(x^2)}{n} = \sigma^2{}_x$ (where σ_x = S.D. of all x's).

$$\therefore \quad \Sigma(x^2) = n\sigma^2{}_x.$$
$$\therefore \quad m = \frac{np}{n\sigma^2{}_x} = \frac{p}{\sigma^2{}_x}.$$

$\therefore$ hence equation of the line $y = mx$ becomes

$$y = \frac{p}{\sigma^2{}_x}x,$$

which tells us that for each unit deviation of x from the mean of all the x's, there is an average deviation of $\frac{p}{\sigma^2{}_x}$ in y from the mean of all the y's.

* See p. 364 for the meaning of these letters

But, in order to make the two deviations comparable, we must express them in terms of their respective standard deviations as units (see p. 353); for the standards of variability of x and y not being the same (except when their standard deviations happen to be the same), the two deviations can only be made comparable when they are each expressed in terms of their standard deviations as unit. Now y measured in terms of σ_y as unit becomes $\frac{y}{\sigma_y}$; and x measured in terms of σ_x as unit becomes $\frac{x}{\sigma_x}$.

$$\therefore \quad \text{equation } y = \frac{p}{\sigma_x^2} x \text{ becomes}$$

$$\frac{y}{\sigma_y} = \frac{p}{\sigma_x^2 \sigma_y} x$$

$$= \frac{p}{\sigma_x \sigma_y} \cdot \frac{x}{\sigma_x}.$$

This equation tells us that for each unit deviation of x from the mean (in terms of its own standard deviation) there is an average deviation of y from the mean of $\frac{p}{\sigma_x \sigma_y}$ units (in terms of its own standard deviation as unit). The coefficient $\frac{p}{\sigma_x \sigma_y}$, which is written r, is called the **coefficient of correlation,** since it measures in terms of their respective standard deviations as units, the average change in one variable corresponding to a unit change in the other variable. Hence, in the problem under consideration,

we have $$r = \frac{p}{\sigma_x \sigma_y},$$

i.e., $$r = \frac{\Sigma xy}{29 \sigma_x \sigma_y} = \frac{\Sigma xy}{29 \times 9{\cdot}25 \times 21{\cdot}29}.$$

In order to find Σxy (where x and y are the respective deviations from the true means as origin) we must first find the value of $\Sigma(\xi\eta)$, where ξ and η are the deviations of each x and y from the arbitrarily chosen means 17·5 and 125 as origins. This is easily done by entering just above the frequency number in each square the particular product $\xi\eta$, and just below the frequency number, the particular product $f(\xi\eta)$, where f is the frequency number. *E.g.*, row (100—110) is two class intervals less than the row containing the origin (120—130), and column (10—15) is one class interval less than the column containing the origin (15—20). Therefore the square common to row 100—110 and to column

10—15 has a product deviation $\xi\eta$ of $-2 \times (-1) = 2$ (shown in brackets above the frequency number 3) and a product $f\xi\eta$ of $3 \times 2 = 6$ (shown in brackets below the frequency number 3), and so on for the others. Therefore $\Sigma f(\xi\eta)$ is obtained by adding all the $f\xi\eta$ products in each column (or row) and then summing these results together. Thus,

$$f\xi\eta \text{ in column } 1 = 6 \qquad\qquad = 6$$
$$\text{,, } \qquad \text{,, } 2 = 10 + 6 + 4 = 20,$$

and so on, as shown in the lowest row of the correlation table.

$$\therefore \quad \Sigma\xi\eta = \text{sum of all the numbers in the bottom row}$$
$$= 6 + 20 + 2 + 0 + 4 - 2 + 15 + 28$$
$$= 73.$$

$\therefore \quad \Sigma xy$ (referred to the means as origin)

$$= 73 - \mathrm{N}d_x d_y = 73 - 29 \cdot \frac{2}{29} \cdot \frac{2}{29}$$
$$= 72{\cdot}862.$$

But since each x interval is 5 and each y interval is 10, therefore the value of each xy must be multiplied by 5×10, *i.e.*, by 50.

$$\therefore \quad \Sigma xy = 72{\cdot}862 \times 50 = 36431.$$

$$\therefore \quad \mathrm{r} = \frac{36431}{29 \times 9{\cdot}25 \times 21{\cdot}29} = {\cdot}65,$$

i.e., there is a very marked correlation between infant mortality and overcrowding.

Note.—Now that we know that $\frac{p}{\sigma x \sigma y} = r$ (the correlation coefficient), we can write the equation $\frac{y}{\sigma y} = \frac{p}{\sigma x \sigma y} \cdot \frac{x}{\sigma x}$ in the form $\frac{y}{\sigma y} = r\frac{x}{\sigma x}$,

or
$$y = r\frac{\sigma y}{\sigma x} . x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, if we had plotted the various values of y and the corresponding mean values of x (line RS) we would have arrived at the equation

$$x = r\frac{\sigma x}{\sigma y} . y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Each of these two equations expresses the fact that the deviation from the mean in the case of one of two correlated quantities is never the same as that in the case of the other of the two quantities.

The two lines represented by equations 1 and 2 are called **regres-**

sion lines, and the coefficients $r\frac{\sigma_y}{\sigma_x}$ and $r\frac{\sigma_x}{\sigma_y}$ are called **regression coefficients.**

Example on Regression.

The mean height of 1,000 fathers is 67·68 in., with S.D. = 2·70 ins. The mean height of all the sons of these fathers is 68·65 in., with S.D. = 2·71 in. If r for stature between fathers and sons = 0·514, find the average height of sons whose fathers are 70 ins. tall.

From equation $y = r\frac{\sigma_y}{\sigma_x} . x$ we get

$$y = \cdot 514 \times \frac{2\cdot 71}{2\cdot 70} x = \cdot 516x.$$

But x in case of fathers 70 ins. high

$$= 70 - 67\cdot 68 = 2\cdot 32.$$

$\therefore$ $y = \cdot 516 \times 2\cdot 32 = 1\cdot 197.$

$\therefore$ height of sons $= 68\cdot 65 + 1\cdot 2 = 69\cdot 85$ ins.,

i.e., fathers whose height is 2·32 ins. above the average have sons whose height is on the average only 1·2 ins. above the general average.

Similarly, fathers whose height is 60 ins., or 7·68 ins. below the average height of all fathers, have sons whose height is on the average $- \cdot 516 \times 7\cdot 68$, or only 3·96 ins. below the general average height of all sons (*i.e.*, 68·65 − 3·96 = 64·69 ins.). Hence we see that in the case of hereditary transmission of height there is a tendency on the part of the offspring to **go back** or **regress** from the condition of the fathers towards the general average of the population, so that children of very tall or very short parents are on the average neither so tall nor so short as their respective parents, but approximate more to the average height of the race. This is Galton's law of **filial regression,** and explains the origin of the term regression line, since the law of regression holds good for any pair of correlated variables.

Properties of Regression Lines.—The lines

$$y = r\frac{\sigma_y}{\sigma_x} . x \text{ and } x = r\frac{\sigma_x}{\sigma_y} y$$

teach us that :

(1) When $r = 0$, *i.e.*, when correlation is zero, these lines are coincident with the axes.

(2) When $r = \pm 1$, the two regression lines coincide.

(3) When $r = \pm 1$ and $\sigma_x = \sigma_y$ (*i.e.*, the two characters are also equally variable), the regression lines not only coincide, but also bisect the angle between the axes.

(4) Since $r\frac{\sigma_y}{\sigma_x} = \tan \theta^1$, where θ_1 = angle made by the line

$y = r\frac{\sigma_x}{\sigma_y}x$ with the x axis, and $r\frac{\sigma_x}{\sigma_y} = \tan\ \theta_2$, where θ_2 = angle made by line $x = r\frac{\sigma_x}{\sigma_y}y$ with the y axis.

$$\therefore \quad r\frac{\sigma_y}{\sigma_x} \times r\frac{\sigma_x}{\sigma_y} = \tan \theta_1 \tan \theta_2.$$

i.e.,
$$r^2 = \tan \theta_1 \tan \theta_2.$$

Hence if we can plot the two regression lines from a few given data we can find r roughly by measuring the angles θ_1 and θ_2.

Thus, in Fig. 125, $\tan \theta_1$ and $\tan \theta_2$ are seen by inspection to be $\frac{1}{1{\cdot}4}$ and $\frac{1}{1{\cdot}8}$ respectively, so that $r^2 = \frac{1}{1{\cdot}4} \times \frac{1}{1{\cdot}8} = 0{\cdot}4$.

$\therefore \quad r = 0{\cdot}63$ approximately.

An interesting application of the method of correlation has been made by C. Powel White, who, from a study of the percentage composition of urine in 53 specimens taken from cancer patients, found that—

(i.) The correlation coefficient between Na, K and Cl, and the amount of urine (H_2O) ranged between 0·598 and 0·669, showing that NaCl and KCl probably pass through the glomeruli.

(ii.) The correlation coefficient between Ca, Mg, and P_2O_5 and H_2O ranged between 0·161 and 0·317, suggesting that earthy phosphates are not so closely associated.

(iii.) The correlation coefficient between H_2O and SO_3 is negative (– 0·0878), and that between nitrogen, urea, and uric acid and water, is negligibly small, suggesting that they are excreted by the tubules. Hence, statistical evidence seems to favour the secretory rather than the filtration theory.

(iv.) Correlation of Na with Cl, and of P_2O_5 with Mg, is very marked, but is negligible with uric acid, showing that the salts are excreted as NaCl and $MgHPO_4$, rather than as $MgCl_2$ and NaH_2PO_4.

In the examples we have chosen, and, indeed, in most of the examples met with in biometrical work, the regression curves are rectilinear (*i.e.*, are straight lines). In these cases r measures the correlation coefficient. Sometimes, however, the regression curve is not rectilinear, but forms some more or less complicated curve. In such cases of skew correlation, correlation is measured by what is called the **correlation ratio,** which is, however, beyond the scope of this book.

Note.—The probable error of the correlation coefficient can be shown to be
$$0{\cdot}6745\,\frac{1 - r^2}{\sqrt{n}}$$
provided $n > 30$.

EXAMPLES.

(1) From a table giving the heights and weights of a number of boys the following results were found :

$\Sigma xy = 1630{\cdot}49$

$\Sigma x^2 = 644{\cdot}77$ (x = deviation in height).

$\Sigma y^2 = 14642{\cdot}06$ (y = deviation in weight).

Find the correlation coefficient between height and weight.

$$r = \frac{\Sigma(xy)}{n\sigma x\sigma y} = \frac{\Sigma(xy)}{n\sqrt{\frac{\Sigma x^2}{n} \cdot \frac{\Sigma y^2}{n}}}$$

$$= \frac{\Sigma(xy)}{\sqrt{\Sigma x^2 \cdot \Sigma y^2}}$$

$$= \frac{1630{\cdot}49}{\sqrt{644{\cdot}77 \times 14642{\cdot}00}}$$

$$= \frac{1630{\cdot}49}{3073}$$

$$= {\cdot}53.$$

(2) The correlation coefficient for cephalic index between either parent and child should be 0·33, but in the case of American Indians (in whom the family relations are somewhat loose) the coefficient was found to be 0·33 between mother and child and only 0·14 between the mother's husband and child. Find the proportion of children not due to the reputed father. (Udny Yule.)

Let the total number of pairs of observations between reputed father and child $= n$; with $r = 0{\cdot}14$. Let n_1 of these pairs be in the true relationship of father and child, with $r_1 = 0{\cdot}33$, and n_2 pairs be those between whom there is no blood relationship, and $r_2 = 0$.

∴ assuming that the mean and standard deviation of the cephalic index are the same in both sets of observations, we have (since $n = n_1 + n_2$)

$$r = \frac{\Sigma xy}{(n_1 + n_2)\,\sigma x\sigma y} = 0{\cdot}14,$$

and

$$r_1 = \frac{\Sigma xy}{n_1\sigma x\sigma y} = 0{\cdot}33.$$

$$\therefore \quad \frac{n_1}{n_1 + n_2} = \frac{r}{r_1} = \frac{0{\cdot}14}{0{\cdot}33}.$$

$$\therefore \quad 1 - \frac{n_1}{n_1 + n_2} = 1 - \frac{{\cdot}14}{{\cdot}33},$$

i.e.,

$$\frac{n_2}{n_1 + n_2} = \frac{{\cdot}33 - {\cdot}14}{{\cdot}33}.$$

$$= \frac{19}{33} = 0{\cdot}58.$$

∴ proportion of children not due to reputed father = 58 per cent.

Table I.—Table of Logarithms of Numbers from 1 to 1000.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10

TABLE I.—TABLE OF LOGARITHMS OF NUMBERS FROM 1 TO 1000.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	3	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5

Table I.—Table of Logarithms of Numbers from 1 to 1000.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
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98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Table II.—Natural Sines and Cosines.

Angle.	Sine.	D 1°	Angle	Cosine.	D 1°
0	0·0000	—	90	1·0000	—
1	0·0175	175	89	0·9998	02
2	0·0349	174	88	0·9994	04
3	0·0523	174	87	0·9986	08
4	0·0698	175	86	0·9976	10
5	0·0872	174	85	0·9962	14
6	0·1045	173	84	0·9945	17
7	0·1219	174	83	0·9925	20
8	0·1392	173	82	0·9903	22
9	0·1564	172	81	0·9877	26
10	0·1736	172	80	0·9848	29
11	0·1908	172	79	0·9816	32
12	0·2079	171	78	0·9781	35
13	0·2250	171	77	0·9744	37
14	0·2419	169	76	0·9703	41
15	0·2588	169	75	0·9659	44
16	0·2756	168	74	0·9613	46
17	0·2924	168	73	0·9563	50
18	0·3090	166	72	0·9511	52
19	0·3256	166	71	0·9455	56
20	0·3420	164	70	0·9397	58
21	0·3584	164	69	0·9336	61
22	0·3746	162	68	0·9272	64
23	0·3907	161	67	0·9205	67
24	0·4067	160	66	0·9135	70
25	0·4226	159	65	0·9063	72
26	0·4384	158	64	0·8988	75
27	0·4540	156	63	0·8910	78
28	0·4695	155	62	0·8829	81
29	0·4848	153	61	0·8746	83
30	0·5000	152	60	0·8660	86
31	0·5150	150	59	0·8572	88
32	0·5299	149	58	0·8480	92
33	0·5446	147	57	0·8387	93
34	0·5592	146	56	0·8290	97
35	0·5736	144	55	0·8192	98
36	0·5878	142	54	0·8090	102
37	0·6018	140	53	0·7986	104
38	0·6157	139	52	0·7880	106
39	0·6293	136	51	0·7771	109
40	0·6428	135	50	0·7660	111
41	0·6561	133	49	0·7547	113
42	0·6691	130	48	0·7431	116
43	0·6820	129	47	0·7314	117
44	0·6947	127	46	0·7193	121
45	0·7071	124	45	0·7071	122
Angle.	Cosine.	D 1°	Angle.	Sine.	D 1°

Table III.—Probability Integral Table Giving Value of $\frac{1}{\sqrt{2\pi}}\int_{-\xi}^{\xi} e^{-\frac{1}{2}\xi^2}\,d\xi$ in Terms of Corresponding Abscissa (after Sheppard).

ξ	$\frac{1}{2}(1+a)$	a	ξ	$\frac{1}{2}(1+a)$	a
·00	·50000	·00000	·76	·77637	·55274
·10	·53983	·07966	·77	·77935	·55870
·20	·57926	·15852	·78	·78230	·56460
·30	·61791	·23582	·79	·78524	·57040
·40	·65542	·31084	·80	·78814	·57628
·45	·67364	·34728	·85	·80234	·60468
·50	·69146	·38292	·90	·81594	·63188
·55	·70884	·41768	·95	·82894	·65788
·60	·72575	·45150	1·00	·84134	·68268
·65	·74215	·48430	1·05	·85314	·70628
·70	·75804	·51608	1·10	·86433	·72866
·71	·76115	·52230	1·50	·93319	·86638
·72	·76424	·52848	2·00	·97725	·95450
·73	·76730	·53460	2·50	·99379	·98758
·74	·77035	·54070	3·00	·99865	·99730
·75	·77337	·54674	3·50	·99977	·99954

$= \frac{x}{\sigma}$. (See p. 355.)

a represents *twice* the area of the portion of the probability curve enclosed between any ordinate and the ordinate at the mean. *E.g.*, in Fig. 124 on p. 357, a represents twice the area of PQNO. $\therefore \quad \frac{1}{\sqrt{2\pi}}\int_{-\xi}^{\xi} e^{-\frac{1}{2}\xi^2}\,d\xi = \frac{a}{2}$.

$\frac{1}{2}(1+a)$ represents the area of the portion PQNX in the same figure.

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