WOOD, J.

1) Principles of mechanics

2) Elements of optics

THE PRINCIPLES

OF

MECHANICS:

DESIGNED FOR THE USE OF STUDENTS
IN THE UNIVERSITY.

By JAMES WOOD, D.D.

MASTER OF ST. JOHN'S COLLEGE, CAMBRIDGE.

SIXTH EDITION.

CAMBRIDGE:

Printed by J. Smith, Printer to the University;

AND SOLD BY DEIGHTON & SONS, AND NICHOLSON & SON, CAMBRIDGE;
AND J. MAWMAN, 39, LUDGATE STREET, LONDON.

1818.

THE PRINCIPLES

OF

MECHANICS:

DESIGNED FOR THE USE OF STUDENTS IN THE UNIVERSITY.

By JAMES WOOD, D.D.

MASTER OF ST. JOHN'S COLLEGE, CAMBRIDGE.

SIXTH EDITION.

CAMBRIDGE:

Printed by J. Smith, Printer to the University;

AND SOLD BY DEIGHTON & SONS, AND NICHOLSON & SON, CAMBRIDGE; AND J. MAWMAN, 39 LUDGATE STREET, LONDON.

1818

CONTENTS.

THE PRINCIPLES

OF

MECHANICS.

The term *Mechanics* has at different times, and by different writers, been applied to branches of science essentially distinct from each other. It was originally confined to the doctrine of equilibrium, or the investigation of the proportion of powers when they balance each other.

Later writers, adapting the term to their discoveries, have used it to denote that science which treats of the nature, production, and alteration of motion; giving to the former branch, by way of contra-distinction, the name of Statics.

Others, giving the term a still more comprehensive meaning, have applied it to both these sciences.

None of these definitions will exactly suit our present purpose; the first being too contracted; and the others much too extensive, for a treatise which is intended to be an introduction only, to the higher excludes every other portion from that part of space which it occupies; and thus it is capable of resistance and protrusion. "There is no idea which we receive more constantly from sensation than *solidity*. Whether we move or rest, in what posture soever we are, we always feel something under us that supports us, and hinders our farther sinking downwards; and the bodies which we daily handle make us perceive that, whilst they remain between them, they do by an insurmountable force hinder the approach of the parts of our hands that press them *."

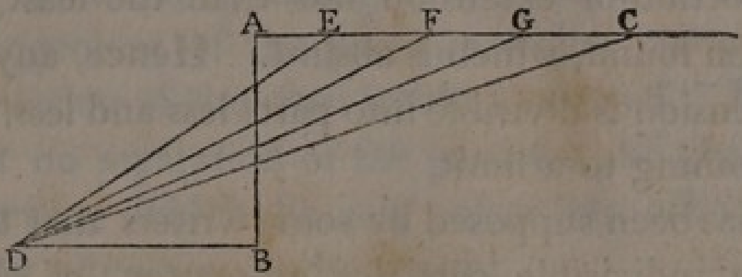
(5.) *Mobility*, or a capacity of being transferred from one place to another, is a quality found to belong to all bodies upon which we can make suitable experiments; and hence we conclude that it belongs to all matter.

(6.) *Divisibility* signifies a capacity of being separated into parts. That matter is thus divisible, our daily experience assures us. How far the division can actually be carried, is not so easily seen. We know that many bodies may be reduced to a very fine powder by trituration; by chemical solution, the parts of a body may be so far divided as not to be sensible to the sight; and by the help of the microscope we discover myriads of organized bodies, totally unknown before such instruments were invented. These and similar considerations, lead us to conclude, that the division of matter is carried to a degree of minuteness far exceeding the bounds of our faculties; and it seems not unreasonable to suppose, that this capacity of division is without limit; especially, as we can

* LOCKE's Essay, B. II. Ch. IV.

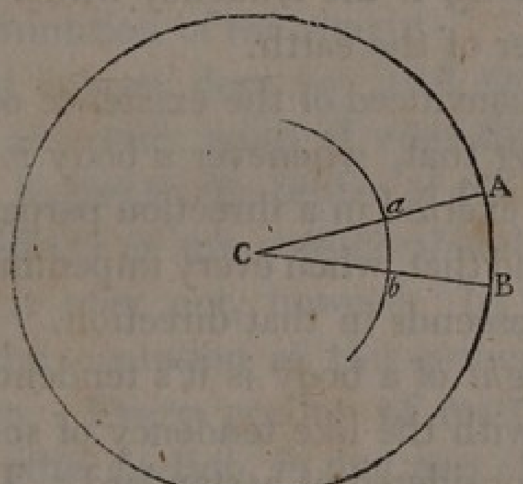
prove, theoretically, that any portion of extension is divisible into parts less and less without end*.

From the extremities of the line AB, draw AC, BD, parallel to each other, and in opposite directions; in AC take any number of points E, F, G, &c. and



join DE, DF, DG, &c. these lines will cut AB in different points; and since, in the indefinite line AC, an unlimited number of points may be taken, the number of parts into which AB is divisible, is indefinite.

This property of extension may also be proved



ex absurdo. If possible, let AB be the least portion

* Porro corporum partes divisas et sibi mutuò contiguas ab invicem separari posse ex phænominis novimus, et partes indivisas in partes minores ratione distingui posse ex mathematicâ certum est.

of a circular arc; take C the center, join CA, CB, and with the center C, and any radius Ca, less than CA, describe a circle cutting CA and CB in the points a and b; then, because AB and ab are similar arcs, they are as their radii; therefore ab is less than AB; or a portion of extension less than the least possible has been found, which is absurd. Hence, any portion of extension is divisible into parts less and less, without ever coming to a limit.

It has been supposed by some writers that there are certain indivisible particles of matter, of the same figure and dimensions, by the different modifications of which different bodies are formed. As no arguments are adduced in favour of this hypothesis, and as experiment seems to lead us to a contrary conclusion, we cannot allow it a place amongst the principles upon which our theory is to rest.

(7.) *Gravity* is the tendency which all bodies have to the center of the earth.

We are convinced of the existence of this tendency by observing that, whenever a body is sustained, it's pressure is exerted in a direction perpendicular to the horizon; and that, when every impediment is removed, it always descends in that direction.

The *weight* of a body is it's tendency to the earth, compared with the like tendency of some other body, which is considered as a standard. Thus, if a body with a certain degree of gravity be called one pound, any other body which has the same degree of gravity,

est. Utrum verò partes illæ distinctæ et nondum divisæ per vires naturæ dividi et ab invicem separari possint, incertum est. Newt. Princip. L. III. Reg. 3.

or which by it's gravity will produce the same effect, under the same circumstances, is also called a pound; and these two together, two pounds, &c.

Gravity is not an accidental property of matter arising from the figure or disposition of the parts of a body; for then, by changing it's shape, or altering the arrangement of the particles which compose it, the gravitation of the mass would be altered. But we find that no separation of the particles, no change of the structure, which human power can effect, produces any alteration in the weight.

As gravity is a property belonging to every particle of a body, independent of it's situation with respect to other particles, the gravity of the whole is the aggregate of the gravities of all it's parts. Thus, though the weight of the whole is not altered by any division, or new arrangement of the particles, yet every increase or diminution of their number produces a corresponding increase or diminution of the weight.

Our present subject does not lead us to consider gravitation in any other point of view than simply as a tendency in bodies to the center of the earth, or to attend to it's effects at any considerable distance from the surface; it may not, however, be improper to observe that the operation of this principle is much more extensive. Every portion of matter gravitates towards every other portion, in that part of the system of nature which falls under our observation. The gravitation indeed of small particles towards each other is not perceived, on account of the superior action of the earth *; yet it has been found, by the

* This attraction is not sufficient to explain the common experiment

accurate observations of Dr. Maskelyne, in Scotland, that the attraction of a mountain is sufficient to draw the plumb-line sensibly from the perpendicular.

Sir I. Newton has discovered that the moon is retained in her orbit by the agency of a cause similar to that by which a body falls to the ground, differing from it only in degree, and this in consequence of the greater distance of the moon from the earth's center. The same author has demonstrated that the planets are retained in their respective orbits by a principle of the same kind; and later writers have shewn, that the minutest irregularities in their motions may be satisfactorily deduced from the known laws of it's operation.

(8.) *Inactivity* may be considered in two lights: 1st. As an *inability* in matter to change it's state of rest or uniform rectilinear motion: 2d. As that quality by which it *resists* any such change*. In this latter sense it is usually called the *force of inactivity*, the *inertia*, or the *vis inertiæ*.

The inactivity of matter, according to the former explanation, is laid down as a law of motion; the truth of which we shall endeavour to establish in the next section.

ment of two particles of the same kind, as quicksilver, &c. when placed upon a smooth horizontal plane, running together. If the effect were not produced by some power different from gravitation, a drop of oil would run, in the same manner, towards a drop of water, which is not found to be the case.

* That is, a change from rest to uniform rectilinear motion, or a change in its uniform rectilinear motion. The term is sometimes applied to the resistance which a body makes to the production, or alteration of motion, when this resistance acts at a mechanical advantage, or disadvantage.

That a body *resists* any change in it's state of rest, or uniform rectilinear motion, is known from constant experience. We cannot move the least particle of matter without some exertion; nor can we destroy any motion without perceiving some resistance *. Thus, we see, in general, that *inertia* is a property inherent in all bodies with which we are concerned; different indeed in different cases, but existing, in a greater or less degree, in all. The quantity we are not at present considering; the existence of the property, every one, from his own observation, will readily allow. We know indeed from experience †, that the inertia of a body is not altered by altering the arrangement of it's parts; but if one portion of matter be added to another, the inertia of the whole is increased; and if any part be removed, the inertia is lessened. This clearly shews that it exists, independently, in every particle, and that the whole inertia is the aggregate of all it's parts.

Hence it follows, from our notion of quantity, that if to a body with a certain quantity of inertia, another, which has an equal quantity, be added, the whole inertia will be doubled; and that by the repeated addition of equal quantities, the whole inertia will be increased in the same proportion with the number of parts.

These properties, which are always found to exist together in the same substance, have sometimes been

* It must be observed, that this resistance is distinct from and independent of gravity; because it is perceived where gravity produces no effect: as, when a wheel is turned round it's axis, or a body moved along an horizontal plane.

† See Art. 25.

said to be essential to matter. Whether they are, or are not *necessarily* united in the same substance, it is impossible to decide, nor does it concern us to inquire. The business of natural philosophy is not to find out what might have been the constitution of nature, but to examine what it is in fact: and to account for the phænomena, which fall under our observation, from those properties of matter which we know by experience that it possesses.

(9.) By the *quantity of matter* in a body, we understand the aggregate of it's particles, each of which has a certain degree of inertia. Or, in other words, if we suppose bodies made up of particles, each of which has the same inertia, the quantity of matter in one body, is to the quantity of matter in another, as the number of such particles in the former, to the number in the latter*.

When we consider bodies as made up of parts, and compare them in this respect, it becomes necessary to give a definite and precise description of those parts; otherwise our notion of the quantity will be vague and inaccurate. Now the only properties of matter which admit of exact comparison, and which depend upon the number, and not upon the arrangement of the particles, are weight and inertia; either of which

* Quantitas materiæ est mensura ejusdem orta ex illius densitate et magnitudine conjunctim. Newt. Princip. Def. 1.

Ejusdem esse densitatis dico, quarum vires inertiæ sunt ut magnitudines. Lib. III. Prop. 6. Cor. 4.

Attendi enim oportet ad punctorum numerum, ex quibus corpus movendum est conflatum. Puncta vero ea inter se æqualia censeri debent, non quæ æque sunt parva, sed in quæ eadem potentia æquales exerit effectus. Eul. Mech. 139.

may properly be made use of as a measure of the *quantity* of matter; and since, at a given place, they are proportional to each other, as we shall shew hereafter (Art. 25), it is of little consequence which measure we adopt. The inertia has been fixed upon, because the gravity of a body, though invariable at the same place, is different at different distances from the center of the earth; whereas, the inertia is always, and under all circumstances, the same.

The *density* of a body is measured by the quantity of matter in a *given bulk;* and it is said to be *uniform* when equal quantities of matter are always contained in equal bulks.

(10.) By *motion* we understand the act of a body's changing place; and it is of two kinds, *absolute* and *relative.*

A body is said to be in *absolute motion* when it is actually transferred from one point in fixed space to another; and to be *relatively in motion,* when it's situation is changed with respect to the surrounding bodies.

These two kinds of motion evidently coincide when the bodies, to which the reference is made, happen to be fixed. In other cases, a body relatively in motion, or relatively at rest, may or may not be absolutely in motion. Thus, a spectator standing still on the shore, if his place be referred to a ship which sails by, is relatively in motion; and the several parts of the vessel are at rest, with respect to each other, though the whole is transferred from one part of space to another.

The motion of a body is *swifter* or *slower*, according as the space passed over, in a given time, is *greater* or *less*.

When a body always passes over equal parts of space in equal successive portions of time, it's motion is said to be *uniform*. When the successive portions of space, described in equal times, continually increase, the motion is said to be *accelerated*; and to be *retarded*, when those spaces continually decrease. Also the motion is said to be *uniformly* accelerated or retarded, when the increments or decrements of the spaces, described in equal successive portions of time, are always equal.

(11.) The *degree* of swiftness or slowness of a body's motion is called it's *velocity*, and it is measured by the space, *uniformly described*, in a *given* time.

The given time, taken as a standard, is usually one second; and the space described is measured in feet. Thus, when v represents a body's velocity, v is the number of feet which the body would uniformly describe in one second.

If a body's motion be accelerated or retarded, the velocity at any point is not measured by the space actually described in a given time, but by the space which would have been described in the given time, if the motion had continued uniform, from that point; or had, at that point, ceased to increase or decrease.

(12.) Cor. 1. If two bodies move uniformly on the same line, in *opposite* directions, their relative velocity is equal to the *sum* of their absolute velocities, since the space by which they uniformly approach to, or recede from, each other, in any time, is equal to the sum of the spaces which they respectively describe in that time.

When the bodies move in the *same* direction, their relative velocity is equal to the *difference* of their absolute velocities.

(13.) Cor. 2. When a body moves with an uniform velocity, the space described is proportional to the time of it's motion.

Let the body describe the space a in the time 1; then since the motion is uniform, it will describe the space ta in the time t; that is, the space described is proportional to the time.

(14.) Cor. 3. When bodies have different uniform motions, the spaces described are proportional to the times and velocities jointly*.

Let V and v be the velocities of two bodies A and B; T and t the times of their motions; S and s the spaces described. Also let S' be the space described by B in the time T:

Then $S : S' :: V : v$ (Art. 11),

$S' : s :: T : t$ (Art. 13).

Comp. $S : s :: TV : tv$;

that is, $S \propto TV$ (*Alg.* Art. 195).

Ex. Let the times be to each other as 6 : 5, and the velocities as 2 : 3; then $S : s :: 2 \times 6 : 3 \times 5 ::$ 4 : 5.

* Since the times and velocities may, in each case, be represented by numbers, there is no impropriety in speaking of their products. The truth of this observation will be evident, if the proposition be expressed in different words. When the uniform velocities of two bodies are in the ratio of the numbers V and v, and the times of their motions in the ratio of the numbers T and t, the spaces described are in the ratio of the numbers TV and tv.

(15.) Cor. 4. Since $S \propto TV$, we have $V \propto \frac{S}{T}$, and $T \propto \frac{S}{V}$, (*Alg.* Art. 205).

Ex. 1. Let A move uniformly through 5 feet in 3″, and B through 9 feet in 7″; required the ratio of the velocities.

$$V : v :: \frac{5}{3} : \frac{9}{7} :: 35 : 27.$$

Ex. 2. Let A's velocity be to B's velocity as 5 to 4; to compare the times in which they will describe 9 and 7 feet respectively.

$$T : t :: \frac{9}{5} : \frac{7}{4} :: 36 : 35.$$

(16.) Cor. 5. Since the areas of rectangles are in the ratio compounded of the ratios of their sides, if the bases of two rectangles represent the velocities of two motions, and altitudes the times, the areas will represent the spaces described.

(17.) The *quantity of motion*, or *momentum*, of a body, is measured by the velocity and quantity of matter jointly.

Thus, if the quantities of matter in two bodies be represented by 6 and 7, and their velocities by 9 and 8, the ratio of 6×9 to 7×8, or 27 to 28, is *called* the ratio of their *momenta*.

(18.) Cor. 1. If M be the momentum of a body, Q it's quantity of matter, and V it's velocity, then since $M \propto QV$, we have $Q \propto \frac{M}{V}$; and $V \propto \frac{M}{Q}$.

Ex. If the quantities of motion be as 6 to 5, and the velocities as 7 to 8, what is the ratio of the quantities of matter?

Since $Q \propto \frac{M}{V}$, we have $Q : q :: \frac{6}{7} : \frac{5}{8} :: 48 : 35$.

(19.) Cor. 2. If M be given, $Q \propto \frac{1}{V}$; and conversely if $Q \propto \frac{1}{V}$, M is invariable. (*Algebra*, Art. 206.)

(20.) Whatever changes, or tends to change, the state of rest or uniform rectilinear motion of a body, is called *force.*

Thus, impact, gravity, pressure, &c. are called forces.

When a force produces it's effect instantaneously, it is said to be *impulsive**. When it acts incessantly, it is called a *constant*, or *continued* force.

Constant forces are of two kinds, *uniform* and *variable*. A force is said to be *uniform*, when it always produces *equal* effects in equal successive portions of time; and *variable*, when the effects produced in equal times are *unequal*.

Forces, which are known to us only by their effects, must be compared by estimating those effects under the same circumstances. Thus, impulsive forces must be measured by the whole effects produced; uniform forces, by the effects produced in equal times; and variable forces, by the effects which would be produced

* Though we cannot conceive finite effects to be produced otherwise than by degrees, and consequently in successive portions of time; yet when these portions are so small as not to be distinguishable by our faculties, the effects may be said to be instantaneous.

in equal times, were the forces to become and continue uniform during those times.

The effects produced by the actions of forces are of two kinds, velocity and momentum; and thus we have two methods of comparing them, according as we conceive them to be the causes of velocity or momentum.

(21.) The *accelerating force* is measured by the *velocity* uniformly generated in a given time, no regard being had to the quantity of matter moved.

Thus, if the velocities uniformly generated, in two cases, in equal times, be as 6 to 7, the accelerating forces are *said* to be in that ratio.

The accelerating force of gravity, at the same place, is invariable; for all bodies falling freely, in an exhausted receiver, acquire equal velocities in any given time.

(22.) The *moving force* is measured by the *momentum* uniformly generated in a given time.

If the momenta thus generated, in two cases, be as 14 to 15, the moving forces are *said* to be in that ratio.

(23.) Cor. 1. Since the momentum is proportional to the velocity and quantity of matter, the moving force varies as the accelerating force and quantity of matter jointly.

The moving force of gravity varies as the quantity of matter moved, because the accelerating force is given (Art. 21).

(24.) Cor. 2. Hence it follows that the accelerating force varies as the moving force directly, and the quantity of matter inversely.

Prop. I.

(25.) *The vis inertiæ of a body is proportional to it's weight.*

The inertia, as was observed on a former occasion, is the resistance which a body makes to any change in it's state of rest or uniform rectilinear motion (Art. 8.); and this resistance is manifestly the same in two bodies, if the same force, applied in the same manner, and for the same time, communicate to each of them the same velocity.

Let two bodies, *A* and *B*, equal in *weight*, be placed in two similar and equal boxes, which are connected by a string passing over a fixed pulley; then these will exactly balance each other; and if the whole be put into motion, the gravity can neither accelerate nor retard that motion; the whole resistance therefore to the communication of motion in the system, arises from the inertia of the weights, the inertia of the string and pulley *, the friction upon the axis, and the resistance of the air †.

Now let a weight *C* be added on one side, and let the velocity generated in any given time, in the whole system, by this additional weight, be observed.

Then in the place of *A*, or *B*, substitute any other

* See Note, page 8.

† This experiment may be made with great accuracy by means of a machine, invented by Mr. Atwood, for the purpose of examining the motions of bodies when acted upon by constant forces. This machine is described in his well-known treatise on the *Rectilinear Motion and Rotation of Bodies*, (p. 299.)

mass of the same weight, and it will be found that *C* will, in the same time, generate the same velocity in this system as in the former; and, therefore, the whole resistance to the communication of motion must be the same. Also the inertia of the string and pulley, the friction of the axis, and the air's resistance, are the same in the two experiments; consequently, the resistance arising from the inertia of the weights is the same: That is, so long as the weight remains unaltered, whatever be the form or constitution of the body, the inertia is the same.

Also, since the whole quantity of inertia is the aggregate inertia of all the parts, if the *weight* be doubled, an equal quantity of *inertia* is added to the former quantity, or the whole *inertia* is doubled; and in the same manner, if the *weight* be increased in any proportion, by the repeated addition of equal *weights*, the *inertia* is increased in the same proportion.

It may be observed, that the velocity generated in a given time, is the same, whether the system begins to move from rest or not; therefore the inertia is the same, whether the system be at rest or in motion.

(26.) Cor. Since the quantity of matter is measured by the inertia (Art. 9.), it is also proportional to the weight.

SECTION II.

ON THE LAWS OF MOTION.

THE FIRST LAW.

(27.) *If a body be at rest, it will continue at rest, and if in motion, it will continue to move uniformly forward in a right line, till it is acted upon by some external force.*

That a body at rest cannot put itself in motion, we know from constant and universal experience.

That a body in motion will continue to move uniformly forward in a right line till it is acted upon by some external force, though equally certain, is not, it must be allowed, equally apparent; since all the motions which fall under our immediate observation, and rectilinear motions in particular, are soon destroyed. If however we can point out the causes which tend to destroy the motions of bodies, and shew, experimentally, that, by removing some of them and diminishing others, the motions continually become more uniform and rectilinear, we may justly conclude that any deviation from the first direction, and first velocity, must be attributed to the agency of external causes; and that there is no tendency in matter itself, either to increase or diminish any motion impressed upon it.

Now the causes which retard a body's motion, besides collision, or the evident obstruction which it meets with from sensible masses of matter, are gravity, friction, and the resistance of the air; and it will appear, by the following experiments, that when these are removed, or due allowance is made for their known effects, we are necessarily led to infer the truth of the law above laid down.

1st. If a ball be thrown along a rough pavement, it's motion, on account of the many obstacles it meets with, will be very irregular, and soon cease; but if it be bowled upon a smooth bowling-green, it's motion will continue longer, and be more rectilinear; and if it be thrown along a smooth sheet of ice, it will preserve both it's direction and it's motion for a still longer time.

In these cases, the gravity, which acts in a direction perpendicular to the plane of the horizon, neither accelerates nor retards the motion; the causes which produce the latter effect are collision, friction, and the air's resistance; and in proportion as the two former of these are lessened, the motion becomes more nearly uniform and rectilinear.

2d. When a wheel is accurately constructed, and a rotatory motion about it's axis communicated to it, if the axis, and the grooves in which it rests, be well polished, the motion will continue a considerable time; if the axis be placed upon friction wheels, the motion will continue longer; and if the apparatus be placed under the receiver of an air pump, and the air be exhausted, the motion will continue, without visible diminution, for a very long time.

In these instances, gravity, which acts aqually on opposite points of the wheel, neither accelerates nor retards the motion; and the more care we take to remove the friction, and the resistance of the air, the less is the first motion diminished in a given time.

3d. If a body be projected in any direction inclined to the horizon, it describes a curve, which is nearly the common parabola. This effect is produced by the joint action of gravity, and the motion of projection; and since the effect produced by the former is known, the effect produced by the latter may be determined. This, it is found, would carry the body uniformly forward in the line in which it was projected; as will fully appear when we come to the doctrine of projectiles. The deviation of the curve described from the parabolic form is sufficiently accounted for by the resistance of the air.

From these, and similar experiments, we are led to conclude that all bodies in motion would uniformly persevere in that motion, were they not prevented by external impediments; and that every increase or diminution of velocity, every deviation from the line of direction, is to be attributed to the agency of such causes.

(28.) It may not be improper to observe, that this law suggests two methods of distinguishing between absolute motions, and such as are only apparent; one, by considering the causes which produce the motions;

and the other, by attending to the effects with which the motions are accompanied *.

1st. We may sometimes distinguish absolute motion, or change of absolute motion, from that which is merely apparent, by considering the causes which produce them.

When two bodies are absolutely at rest, they are relatively so; and the appearance is the same, when they are moving in the same direction, at the same rate; a relative motion therefore can only arise from an absolute motion, or change of absolute motion, in one or both of the bodies. We have seen also, in the last article, that motion, or change of motion, cannot be produced but by force impressed; and therefore, if we know that such a cause exists, and acts upon one of the bodies, and not upon the other, we conclude that the relative motion arises from a change in the state of rest, or absolute motion of the former; and that with respect to the latter, the effect is merely apparent. Thus, when a person on shipboard observes the coast receding from him, he is convinced that the appearance arises from a motion, or change of motion, in the ship, upon which a cause, sufficient to produce this effect, acts, namely, the force of the wind or tide.

The precession of the equinoxes arises from a real motion in the earth, and not from any motion in the heavenly bodies; because we know that there is a force impressed upon the earth, which is sufficient to account for the appearance.

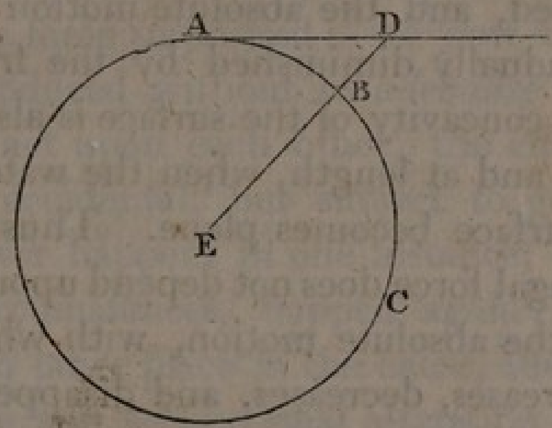
2d. Absolute motion may sometimes be disinguished from apparent motion, by the effects produced.

* Newt. Princip. Schol. ad Def.

If a body be absolutely in motion, it endeavours by it's inactivity to proceed in the line of it's direction; if the motion be only apparent, there is no such tendency.

It is in consequence of the tendency to persevere in rectilinear motion that a body revolving in a circle constantly endeavours to recede from the center. The effort thus produced is called a *centrifugal force;* and as it arises from absolute motion only, whenever it is observed, we are convinced that the motion is real.

In order to see the nature and origin of this force, suppose a body to describe the circle *ABC*; then at any point *A*, it is moving in the direction of the



tangent *AD*, and in this direction, by the first law of motion, it endeavours to proceed; also, since every point *D* in the tangent is without the circle, this tendency to move on in the direction of the tangent, is a tendency to recede from the center of motion; and the body will actually fly off, unless it is prevented by an adequate force.

The following experiment is given by Sir I. Newton to shew the effect of the centrifugal force, and to prove that it always accompanies an absolute circular motion.

Let a bucket, partly filled with water, be suspended by a string, and turned round till the string is considerably twisted; then let the string be suffered to untwist itself, and thus communicate a circular motion to the vessel. At first the water remains at rest, and it's surface is smooth and undisturbed; but as it gradually acquires the motion of the bucket, the surface grows concave towards the center, and the water ascends up the sides, thus endeavouring to recede from the axis of motion; and this effect is observed gradually to increase with the absolute velocity of the water, till at length the water and the bucket are relatively at rest. When this is the case, let the bucket be suddenly stopped, and the absolute motion of the water will be gradually diminished by the friction of the vessel; the concavity of the surface is also diminished by degrees, and at length, when the water is again at rest, the surface becomes plane. Thus we find that the centrifugal force does not depend upon the relative, but upon the absolute motion, with which it always begins, increases, decreases, and disappears.

A single instance will be sufficient to shew the great utility of this conclusion in natural philosophy.

The diurnal rotation of the heavenly bodies may, as far as the appearance is concerned, be accounted for, either by supposing the heavens to revolve from east to west, and complete a revolution in twenty-four hours; or, the earth to revolve from west to east, in the same time: but the sensible diminution of gravity as we proceed towards the equator, and the oblate figure of the earth, which are the effects of a centrifugal force, prove that the appearance is to be ascribed to a real motion in the earth.

THE SECOND LAW OF MOTION.

(29.) *Motion, or change of motion, produced in a body, is proportional to the force impressed, and takes place in the direction in which the force acts.*

It has been seen in the preceding articles, that no motion or change of motion is ever produced in a body without *some* force impressed; we now assert that it cannot be produced without an *adequate* force; that when bodies act upon each other, the effects are not variable and accidental, but subject to general laws. Thus, whatever happens in one instance, will, under the same circumstances, happen again; and when any alteration takes place in the cause, there will be a corresponding and proportional alteration in the effect produced. Were not cause and effect thus connected with, and related to, each other, we could not pretend to lay down any general rules respecting the mutual actions of bodies; experiment could only furnish us with detached and isolated facts, wholly inapplicable on other occasions; and that harmony, which we cannot but observe and admire in the material world, would be lost.

In order to understand the meaning and extent of this law of motion, it will be convenient to distinguish it into two cases; and to point out such facts, under each head, as tend to establish it's truth.

1st. The same force, acting freely for a given time, will always produce the same effect, in the direction in which it acts.

Ex. 1. If a body, in one instance, fall perpendicularly through $16\frac{1}{12}$ feet in a second, and thus acquire a velocity which would carry it, uniformly, through $32\frac{1}{6}$ feet in that time, it will always, under the same circumstances, acquire the same velocity.

The effect produced is the same, whether the body begins to move from rest or not.

Ex. 2. If a body be projected perpendicularly downwards, the velocity of projection, measured in feet (Art. 11.), will, in one second, be increased by $32\frac{1}{6}$; and if it be projected perpendicularly upwards, it will, in one second, be diminished by that quantity.

Ex. 3. If a body be projected obliquely, gravity will still produce it's effect in a direction perpendicular to the horizon; and the body, which by it's inactivity would have moved uniformly forward in the line of it's first motion, will, at the end of one second, be found $16\frac{1}{12}$ feet below that line; having thus acquired a velocity of $32\frac{1}{6}$ feet per second, in the direction of gravity.

2d. If the force impressed be increased or diminished in any proportion, the motion communicated will be increased or diminished in the same proportion.

Ex. If a body descend along an inclined plane, the length of which is twice as great as it's height, the force which accelerates it's motion is half as great as the force of gravity; and, allowing for the effect of friction, and the resistance of the air, the velocity generated in any time is half as great as it would have been, had the body fallen, for the same time, by the whole force of gravity *.

(30.) In estimating the effect of any force, two circumstances are to be attended to; first, we must consider what force is actually impressed; for this alone can produce a change in the state of motion or quiescence of a body. Thus, the effect of a stream upon the floats of a water-wheel is not produced by the whole force of the stream, but by that part of it which arises from the excess of the velocity of the water above that of the wheel; and it is nothing, if they move with equal velocities. Secondly, we must

* The experiments which most satisfactorily prove the truth of this law of motion, are made with Mr. Atwood's machine, mentioned on a former occasion.

Let two weights, each of which is represented by $9m$, balance each other on this machine; and observe what velocity is generated in one second, when a weight $2m$ is added to either of them. Again, let the weights $8m$, $8m$, be sustained, as before, and add $4m$ to one of them, then the velocity generated in one second is twice as great as in the former instance. Since, therefore, the mass to be moved is the same in both cases, viz. $20m$ together with the inertia of the machine, it is manifest that when the moving force is doubled (Art. 23.), the momentum generated is also doubled, and, by altering the ratio of the weights, it may be shewn, in any other case, that the momentum communicated is proportional to the moving force impressed.

consider in what direction the force acts; and take that part of it, only, which lies in the direction in which we are estimating the effect. Thus, the force of the wind actually impressed upon the sails of a windmill, is not wholly employed in producing the circular motion; and therefore in calculating it's effect, in this respect, we must determine what part of the whole force acts in the direction of the motion.

In the following pages, we shall see a great variety of instances in which this method of estimating the effects of forces is applied; and the conclusions thus deduced, being found, without exception, to agree with experiment, we cannot but admit the truth of the principle.

(31.) Cor. Since the effect produced upon each other by two bodies, depends upon their relative velocity, it will always be the same whilst this remains unaltered, whatever be their absolute motions.

THE THIRD LAW OF MOTION.

(32.) *Action and reaction are equal, and in opposite directions.*

Matter not only perseveres in it's state of rest or uniform rectilinear motion, but also by it's inertia resists any change. Our experience with respect to this reaction, or opposition to the force impressed, is so constant and universal, that the very supposition of it's non-existence appears to be absurd. For who can conceive a pressure without some support of that pressure? Who can suppose a weight to be raised without force or exertion? Thus far then we are assured by our senses, that whenever one body acts upon another, there is *some* reaction: The law farther asserts, that the reaction is *equal in quantity* to the action.

By *action*, we here understand moving force, which, according to the definition (Art. 22.), is measured by the momentum which is, or would be generated, in a given time; and to determine whether action and reaction, in this sense of the words, are equal or not, recourse must be had to experiment.

Take two similar and equal cylindrical pieces of wood, from one of which projects a small steel point; suspend them by equal strings, and let one of them descend through any arc and impinge upon the other at rest; then, by means of the steel point, the two bodies will move on together as one mass, and with a velocity equal to half the velocity of the impinging body. Thus the momentum, which is measured by the quantity of matter and velocity taken jointly, remains unaltered; or, as much momentum as is gained by the body struck, so much is taken from the momentum of the striking body, or communicated to it in the opposite direction.

If the striking body be loaded with lead, and thus made twice as heavy as the other, the common velocity after impact is found to be to the velocity of the impinging body :: 2 : 3; and because the joint mass after impact : quantity of matter in the striking body :: 3 : 2, the momentum after impact : momentum before :: $3 \times 2 : 2 \times 3$, or in a ratio of equality, as in the former case.

In making experiments to establish this third law of motion, allowance must be made for the air's resistance; and care must be taken to obtain a proper measure of the velocity before and after impact. See Sir. I. NEWTON's Scholium to the Laws of Motion.

(33.) The third law of motion is not confined to cases of actual impact; the effects of pressures and attractions, in opposite directions, are also equal.

When two bodies sustain each other, the pressures in opposite directions must be equal, otherwise motion would ensue; and if motion be produced by the excess of pressure on one side, the case coincides with that of impact*.

When one body attracts another, it is itself also equally attracted, and as much momentum as is thus communicated to one body, will also be communicated to the other in the opposite direction.

A loadstone and a piece of iron, equal in weight, and floating upon similar and equal pieces of cork, approach each other with equal velocities, and therefore with equal momenta; and when they meet, or are kept asunder by any obstacle, they sustain each other by equal and opposite pressures.

(34.) COR. Since the action and reaction are equal

* The effects of pressure and impact are manifestly of the same kind, and produced in the same way; excess of pressure, on one side, produces momentum, and equal and opposite momenta support each other by opposite pressures.

Thus also pressures may be compared, either by comparing the weights which they sustain, or the momenta which they would generate under the same circumstances.

at every instant of time, the whole effect of the action in a finite time, however it may vary, is equal to the effect of the reaction; because the whole effects are made up of the effects produced in every instant.

SCHOLIUM.

(35.) These laws are the simplest principles to which motion can be reduced, and upon them the whole theory depends. They are not indeed self-evident, nor do they admit of accurate proof by experiment, on account of the great nicety required in adjusting the instruments, and making the experiments; and on account of the effects of friction, and the air's resistance, which cannot entirely be removed. They are however constantly, and invariably, suggested to our senses, and they agree with experiment as far as experiment can go; and the more accurately the experiments are made, and the greater care we take to remove all those impediments which tend to render the conclusions erroneous, the more nearly do the experiments coincide with these laws.

Their truth is also established upon a different ground; from these *general* principles innumerable *particular* conclusions have been deduced; sometimes the deductions are simple and immediate, sometimes they are made by tedious and intricate operations; yet they are all, without exception, consistent with each other and with experiment: it follows therefore that the principles, upon which the calculations are founded, are true *.

* Atwood on the Motions of Bodies, p. 358.

(36.) It will be necessary to remember, that the laws of motion relate, *immediately*, to the actions of particles of matter upon each other, or to those cases in which the whole mass may be conceived to be collected in a point; not to *all* the effects that may *eventually* be produced in the several particles of a system.

A body may have a rectilinear and rotatory motion given it at the same time, and it will retain both. The action also, or reaction, may be applied at a mechanical advantage or disadvantage, and thus they may produce, upon the whole, very different momenta: these effects depend upon principles which are not here considered, but which must be attended to in computing such effects.

SECTION III.

ON THE COMPOSITION AND RESOLUTION OF MOTION.

Prop. II.

(37.) *Two lines, which represent the momenta communicated to the same or equal bodies, will represent the spaces uniformly described by them in equal times; and conversely, the lines which represent the spaces uniformly described by them in equal times, will represent their momenta.*

The momenta of bodies may be represented by numbers, as was seen Art. 17; but in many cases it will be much more convenient to represent them by lines, because lines will express not only the quantities of the momenta, but also the directions in which they are communicated.

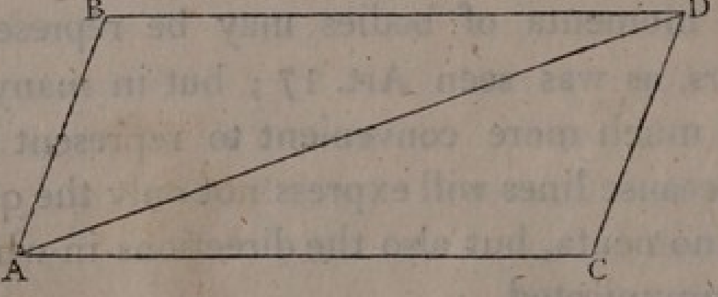
Any line drawn in the proper direction may be taken to represent one momentum; but to represent a second, a line, in the direction of the latter motion, must be taken in the same proportion to the former line, that the second momentum has to the first.

Let two lines, thus taken, represent the momenta communicated to the same, or equal bodies; then since $M \propto V \times Q$ (Art. 17.), and Q is here given, $M \propto V$; therefore the lines, which represent the momenta, will also represent the velocities, or the spaces uniformly described in equal times. Again, if the lines represent the spaces uniformly described in equal times, they represent the velocities, and since Q is given, $V \propto QV \propto M$; therefore the lines represent the momenta.

Prop. III.

(38.) *Two uniform motions, which, when communicated separately to a body, would cause it to describe the adjacent sides of a parallelogram in a given time, will, when they are communicated at the same instant, cause it to describe the diagonal in that time; and the motion in the diagonal will be uniform.*

Let a motion be communicated to a body at A, which would cause it to move uniformly from A to B

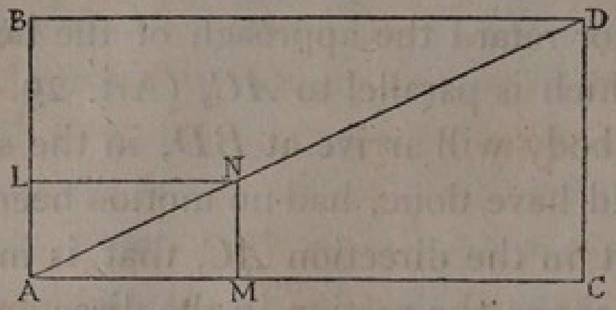


in T'', and at the same instant, another motion which alone would cause it to move uniformly from A to C in T''; complete the parallelogram BC, and draw the diagonal *AD*; then the body will arrive at the point *D*, in T'', having described *AD* with an uniform motion.

For the motion in the direction *AC* can neither accelerate nor retard the approach of the body to the line *BD* which is parallel to *AC*, (Art. 29. Ex. 3.); hence, the body will arrive at *BD*, in the same time that it would have done, had no motion been communicated to it in the direction *AC*, that is in T''. In the same manner, the motion in the direction *AB* can neither make the body approach to, nor recede from, *CD*; therefore, in consequence of the motion in the direction *AC*, it will arrive at *CD* in the same time that it would have done, had no motion been communicated in the direction *AB*, that is in T''. Hence it follows that, in consequence of the two motions, the body will be found both in *BD* and *CD* at the end of T'', and will therefore be found in *D*, the point of their intersection.

Also, since a body in motion continues to move uniformly forward in a right line, till it is acted upon by some external force (Art. 27.), the body *A* must have described the right line *AD*, with an uniform motion.

(39.) To illustrate this proposition, suppose a plane *ABDC*, as the deck of a ship, to be carried uniformly forward, and let the point *A* describe the line *AC* in T''; also, let a body move uniformly in this plane from *A* to *B*, in the same time. Complete the parallelogram *BC*, and draw the diagonal *AD*. Then at the end of T'' the body, by its own motion, will arrive at *B*; also by the motion of the plane, *AB* will be brought into the situation *CD*, and the point *B* will coincide with D; therefore the body will upon the whole, at the end of T'', be found in D. In any other time t'', let the point A be carried from A to



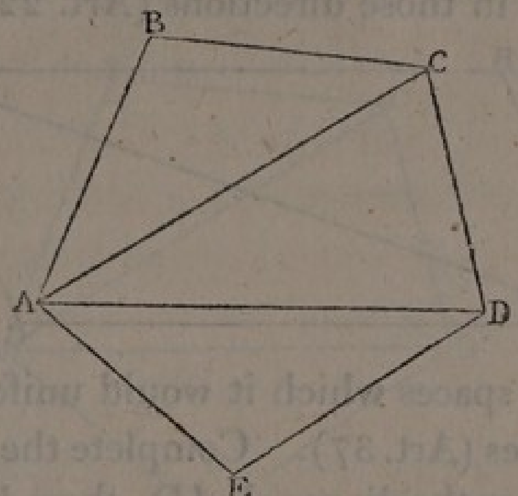
M by the motion of the plane, and the body from A to L by it's own motion; complete the parallelogram $ALNM$, and join AN; then, as in the preceding case, the body will, at the end of t'', be found in N; and since the motions in the directions AC, AB are uniform, $AC : AM :: T : t :: AB : AL$ (Art. 13.); that is, the sides of the parallelograms, about the common angle LAM, are proportional, and consequently the parallelograms are about the same diagonal AD (Euc. 26. vi.); therefore the body at the end of any time t'' will be found in the diagonal AD. It will also move uniformly in the diagonal; for, from the similar triangles AMN, ACD, we have $AD : AN :: AC : AM :: T : t$, or the spaces described are proportional to the times. (See Art. 10.)

(40.) Cor. 1. The reasoning in the last article is applicable to the motion of a point.

(41.) Cor. 2. If two sides of a triangle, AB, BD, taken in order, represent the spaces over which two uniform motions would, separately, carry a body in a given time; when these motions are communicated at the same instant to the body at A, it will describe the third side AD, uniformly, in that time.

For, if the parallelogram *BC* be completed, the same motion, which would carry a body uniformly from *B* to *D*, would, if communicated at *A*, carry it in the same manner from *A* to *C*; and in consequence of this motion, and of the motion in the direction *AB*, the body would uniformly describe the diagonal *AD*, which is the third side of the triangle *ABD*.

(42.) Cor. 3. In the same manner, if the lines *AB*, *BC*, *CD*, *DE*, taken in order, represent the spaces over which any uniform motions would, separately,



carry a body, in a given time, these motions, when communicated at the same instant, will cause the body to describe the line *AE* which completes the figure, in that time; and the motion in this line will be uniform.

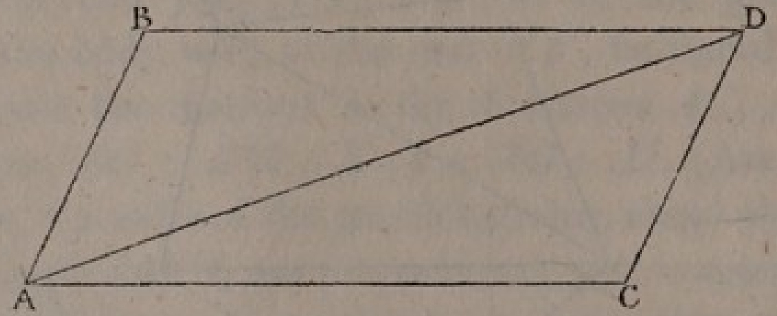
(43.) Cor. 4. If *AD* represent the uniform velocity of a body, and any parallelogram *ABDC* (Art. 38.), be described about it, the velocity *AD* may be supposed to arise from the two uniform velocities *AB*, *AC*, or *AB*, *BD*; and if one of them, *AB*, be by any means taken away, the velocity remaining will be represented by *AC* or *BD*. (See Art. 11.)

(44.) Def. A force is said to be *equivalent* to any number of forces, when it will, *singly*, produce the same effect that the others produce *jointly*, in any given time.

Prop. IV.

(45.) *If the adjacent sides of a parallelogram represent the quantities and directions of two forces, acting at the same time upon a body, the diagonal will represent one equivalent to them both.*

Let *AB*, *AC* represent two forces acting upon a body at *A*, then they represent the momenta communicated to it in those directions (Art. 22.), and conse-



quently the spaces which it would uniformly describe in equal times (Art. 37). Complete the parallelogram *CB*, and draw the diagonal *AD*; then, by the last proposition, *AD* is the space uniformly described in the same time, when the two motions are communicated to the body at the same instant; and since *AB*, *AC*, and *AD*, represent the spaces uniformly described by the same body, in equal times, they represent the momenta, and therefore the forces acting in those directions; that is, the forces *AB*, *AC**, acting at the same time, produce a force which is represented, in quantity and direction, by *AD*.

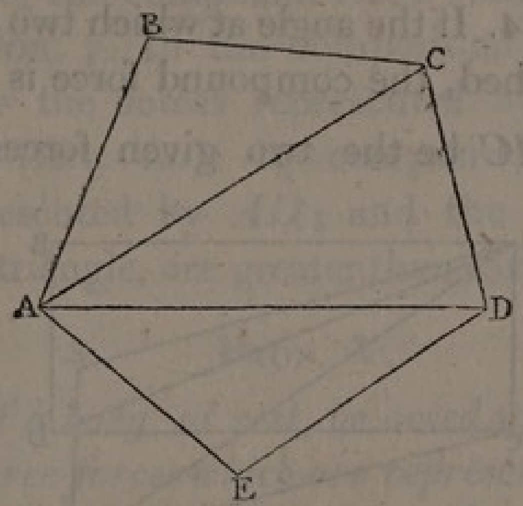
* In this, and many other cases, where forces are represented by lines, the lines are used, for the sake of conciseness, to express the forces which they represent.

Def. The force represented by AD is said to be *compounded* of the two, AB, AC.

(46.) Cor. 1. If two sides of a triangle, taken in order, represent the quantities and directions of two forces, the third side will represent a force equivalent to them both.

For a force represented by BD, acting at A, will produce the same effect that the force AC, which is equal to it and in the same direction, will produce; and AB, AC, are equivalent to AD; therefore AB, BD are also equivalent to AD.

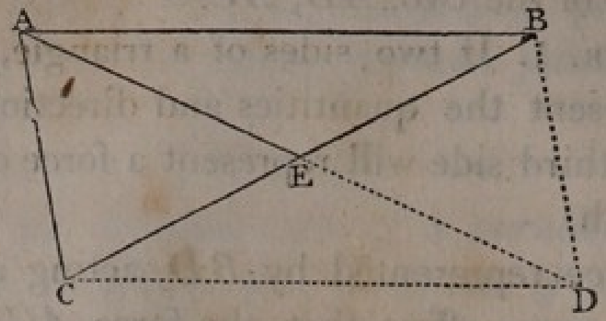
(47.) Cor. 2. If any lines AB, BC, CD, DE,



taken in order, represent the quantities and directions of forces communicated at the same time to a body at A, the line AE, which completes the figure, will represent a force equivalent to them all.

For the two AB, BC are equivalent to AC; also, AC, CD, that is, AB, BC, CD, are equivalent to AD; in the same manner AD, DE, that is, AB, BC, CD, and DE, are equivalent to AE.

(48.) Cor. 3. Let AB and AC represent the quantities and directions of two forces, join BC and draw AE bisecting it in E, then will $2AE$ represent

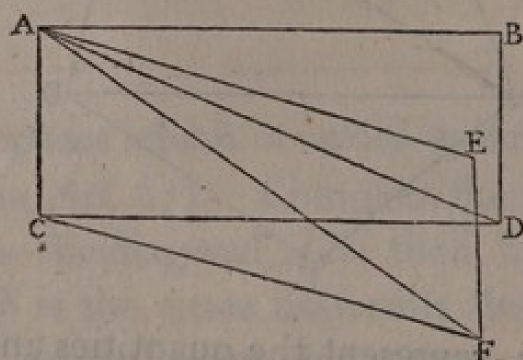


a force equivalent to them both.

For, if the parallelogram be completed, since the diagonals bisect each other, AD, which represents a force equivalent to AB and AC, is equal to $2AE$.

(49.) Cor. 4. If the angle at which two given forces act be diminished, the compound force is increased.

Let AB, AC be the two given forces; complete



the parallelogram $ABDC$ and draw the diagonal AD, this represents the compound force. In the same manner, if AE be taken equal to AB, and AE, AC, represent the two forces, then AF the diagonal of the parallelogram $AEFC$, represents the compound force; and since the angle BAC is greater than the angle EAC, ACD which is the supplement of the former, is less than ACF the supplement of the latter; also, $CF = AE = AB = CD$; therefore in the two triangles ACD, *ACF*, the sides *AC*, *CD* are respectively equal to *AC*, *CF*, and the $\angle ACD$ is less than the $\angle ACF$; consequently *AD* is less than *AF* (Euc. 24. i).

(50.) Cor. 5. Two given forces produce the greatest effect when they act in the same direction, and the least when they act in opposite directions; for, in the former case, the diagonal *AF* becomes equal to the sum of the sides *AC*, *CF*; and in the latter, to their difference.

(51.) Cor. 6. Two forces cannot keep a body at rest, unless they are equal and in opposite directions.

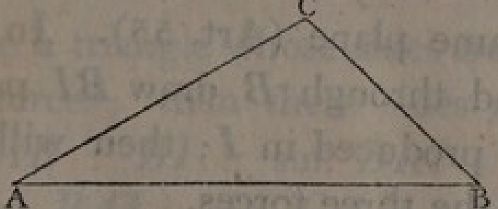
For this is the only case in which the diagonal, representing the compound force, vanishes.

(52.) Cor. 7. In the composition of forces, force is lost; for the forces represented by the two sides *AB*, *BD* (Art. 45.), by composition produce the force represented by *AD*; and the two sides *AB*, *BD*, of a triangle, are greater than the third side *AD*.

Prop. V.

(53.) *If a body, at rest, be acted upon at the same time by three forces which are represented in quantity and direction by the three sides of a triangle, taken in order, it will remain at rest.*

Let *AB*, *BC*, and *CA*, represent the quantities and



directions of three forces acting at the same time upon a body at *A*; then since *AB* and *BC* are equivalent to *AC* (Art. 46.); *AB*, *BC* and *CA* are equivalent to

AC and *CA*; but *AC* and *CA*, which are equal and in opposite directions, keep the body at rest; therefore *AB*, *BC*, and *CA*, will also keep the body at rest.

Prop. VI.

(54.) *If a body be kept at rest by three forces, and two of them be represented in quantity and direction by two sides* AB, BC*, *of a triangle, the third side, taken in order, will represent the quantity and direction of the other force.*

Since *AB*, *BC* represent the quantities and directions of two of the forces, and *AB*, *BC* are equivalent to *AC*, the third force must be sustained by *AC*; therefore *CA* must represent the quantity and direction of the third force (Art. 51).

(55.) Cor. If three forces keep a body at rest, they act in the same plane; because the three sides of a triangle are in the same plane (Euc. 2. xi).

Prop. VII.

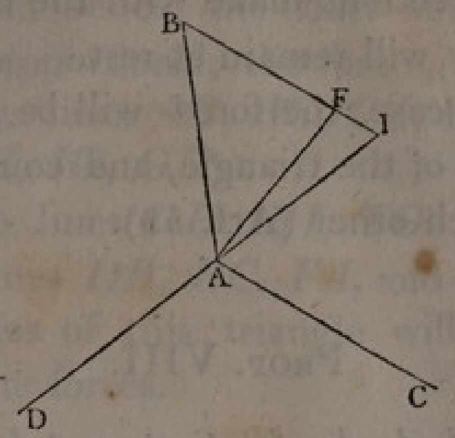
(56.) *If a body be kept at rest by three forces, acting upon it at the same time, any three lines, which are in the directions of these forces, and form a triangle, will represent them.*

Let three forces, acting in the directions *AB*, *AC*, *AD*, keep the body *A* at rest; then *AB*, *AC*, *AD* are in the same plane (Art. 55). In *AB* take any point, *B*, and through *B* draw *BI* parallel to *AC*, meeting *DA* produced in *I*; then will *AB*, *BI*, and *IA* represent the three forces.

For *AB* being taken to represent the force in that direction, if *BI* do not represent the force in the direc-

* Fig. Art. 53.

tion *AC* or *BI*, let *BF* be taken to represent it; join



AF; then since three forces keep the body at rest, and *AB*, *BF* represent the quantities and directions of two of them, *FA* will represent the third (Art. 54.), that is, *FA* is in the direction *AD*, which is impossible (Euc. 11. i. Cor.); therefore *BI* represents the force in the direction *AC*; and consequently *IA* represents the third force (Art. 54).

Any three lines, respectively parallel to *AB*, *BI*, *IA*, and forming a triangle, will be proportional to the sides of the triangle *ABI*, and therefore proportional to the three forces.

(57.) Cor. 1. If a body be kept at rest by three forces, any two of them are to each other inversely as the sines of the angles which the lines of their directions make with the direction of the third force.

Let *ABI* be a triangle whose sides are in the directions of the forces; then these sides represent the forces; and *AB* : *BI* :: sin. *BIA* : sin. *BAI* :: sin. *IAC* : sin. *BAI* :: sin. *CAD* : sin. *BAD*.

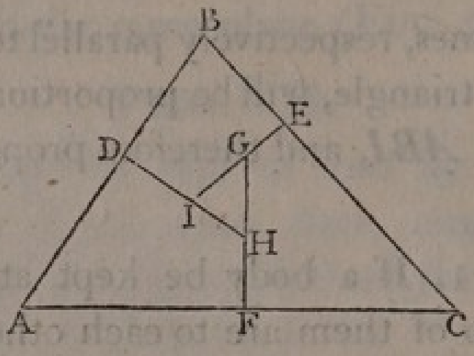
(58.) Cor. 2. If a body, at rest, be acted upon at the same time by three forces, in the directions of the sides of a triangle taken in order, and any two of them be to each other inversely as the sines of the angles which their directions make with the direction of the third, the body will remain at rest.

For, in this case, the forces will be proportional to the three sides of the triangle, and consequently they will sustain each other (Art. 53).

Prop. VIII.

(59.) *If a body be kept at rest by three forces, and lines be drawn at right angles to the directions in which they act, forming a triangle, the sides of this triangle will represent the quantities of the forces.*

Let *AB*, *BC*, *CA* be the directions in which the forces act; and let them form the triangle *ABC*; then the lines *AB, BC, CA*, will represent the forces



(Art. 56). Draw the perpendiculars *DH, EI, FG*, forming a triangle *GHI*; then since the four angles of the quadrilateral figure *ADHF* are equal to four right angles, and the angles at *D* and *F* are right angles, the remaining angles *DHF, DAF* are equal to two right angles, or to the two angles *DHF, DHG*; consequently, the angle *DAF* is equal to the angle *IHG*. In the same manner, it may be shewn, that the angles *ABC, BCA* are respectively equal to

GIH, *HGI*; therefore the triangles *ABC* and *GHI* are equiangular; hence, the sides about their equal angles being proportional, the forces, which are proportional to the lines *AB*, *BC*, *CA*, are proportional to the lines *HI*, *IG*, *GH*.

Cor. If the lines *DH*, *EI*, *FG* be equally inclined to the lines *DB*, *EC*, *FA*, and form a triangle *GHI*, the sides of this triangle will represent the quantities of the forces.

Prop. IX.

(60.) *If any number of forces, represented in quantity and direction by the sides of a polygon, taken in order, act at the same time upon a body at rest, they will keep it at rest.*

Let *AB*, *BC*, *CD*, *DE*, and *EA* (Fig. Art. 47.), represent the forces; then since *AB*, *BC*, *CD* and *DE* are equivalent to *AE* (Art. 47.); *AB*, *BC*, *CD*, *DE*, and *EA*, are equivalent to *AE* and *EA*; that is, they will keep the body at rest.

Prop. X.

(61.) *If any number of lines, taken in order, represent the quantities and directions of forces which keep a body at rest, these lines will form a polygon.*

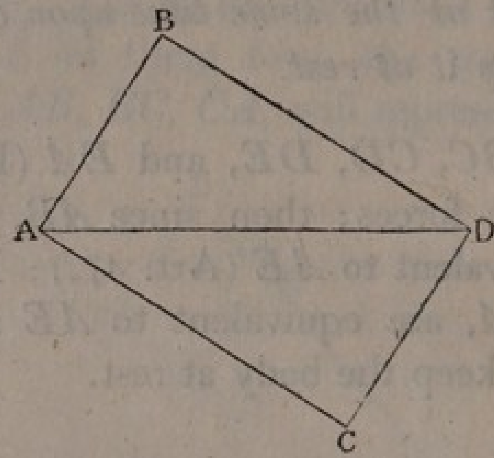
Let *AB*, *BC*, *CD* and *DE* represent forces which keep a body at rest (Fig. Art. 47.); then the point *E* coincides with *A*. If not, join *AE*, then *AB*, *BC*, *CD*, and *DE*, are equivalent to *AE*; and the body will be put in motion by a single force *AE*, which is contrary to the supposition; therefore the point E coincides with A, and the lines form a polygon.

This and the last proposition are true when the forces act in different planes.

Prop. XI.

(62.) *A single force may be resolved into any number of forces.*

Since the single force AD is equivalent to the two, AB, BD, it may be conceived to be made up of, or resolved into, the two, AB, BD. The force AD may



therefore be resolved into as many pairs of forces as there can be triangles described upon AD, or parallelograms about it. Also AB, or BD, may be resolved into two; and, by proceeding in the same manner, the original force may be resolved into any number of others.

(63.) Cor. 1. If two forces are together equivalent to AD, and AB be one of them, BD is the other.

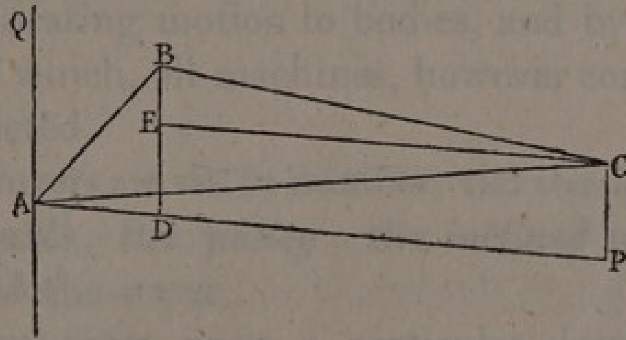
(64.) Cor. 2. If the force AD be resolved into the two, AB, BD, and AB be wholly lost, or destroyed, the effective part of *AD* is represented in quantity and direction by *BD*.

(65.) Cor. 3. In the resolution of forces, the whole quantity of force is increased. For the force represented by *AD* is resolved into the two *AB*, *BD* which are together greater than *AD* (Euc. 20. i).

Prop. XII.

(66.) *The effects of forces, when estimated in given directions, are not altered by composition or resolution.*

Let two forces *AB*, *BC*, and the force *AC* which is equivalent to them both, be estimated in the directions



AP, *AQ*. Draw *BD*, *CP* parallel to *AQ*; and *CE* parallel to *AP*. Then the force *AB* is equivalent to the two *AD*, *DB*; of which *AD* is in the direction *AP*, and *DB* in the direction *AQ*; in the same manner, *BC* is equivalent to the two *BE*, *EC*; the former of which is in the direction *BD* or *QA*, and the latter in the direction *EC* or *AP*; therefore the forces *AB*, *BC*, when estimated in the directions *AP*, *AQ*, are equivalent to *AD*, *EC*, *DB*, and *BE*; or, *AD*, *DP*,

DB and BE, because EC is equal to DP; and since DB and BE are in opposite directions, the part EB of the force DB is destroyed by BE; consequently, the forces are equivalent to AP, DE, or AP, PC. Also AC, when estimated in the proposed directions, is equivalent to AP, PC; therefore the effective forces in the directions AP, AQ are the same, whether we estimate AB and BC, in those directions, or AC, which is equivalent to them.

(67.) Cor. When AP coincides with AC, EC also coincides with it, and D coincides with E. In this case the forces DB, BE wholly destroy each other; and thus, in the composition of forces, force is lost.

SECTION IV.

ON THE MECHANICAL POWERS.

(68.) The mechanical powers are the most simple instruments used for the purpose of supporting weights, or communicating motion to bodies, and by the combination of which, all machines, however complicated, are constructed.

These powers are six in number, viz. the *lever*; the *wheel and axle*; the *pulley*; the *inclined* plane; the *wedge*; and the *screw*.

Before we enter upon a particular description of these instruments, and the calculation of their effects, it is necessary to premise, that when any forces are applied to them, they are themselves supposed to be at rest; and consequently, that they are either without weight, or that the parts are so adjusted as to sustain each other. They are also supposed to be perfectly smooth; no allowance being made for the effects of adhesion.

When two forces act upon each other by means of any machine, one of them is, for the sake of distinction, called the *power*, and the other the *weight*.

ON THE LEVER.

(69.) Def. The *Lever* is an inflexible rod, moveable, in one plane, upon a point which is called the *fulcrum*, or *center of motion*.

The power and weight are supposed to act in the plane in which the lever is moveable round the fulcrum, and tend to turn it in opposite directions.

(70.) The properties of the lever cannot be deduced immediately from the propositions laid down in the last section, because the forces acting upon the lever are not applied at a point, which is always supposed to be the case in the composition and resolution of forces; they may however be derived from the following principles, the truth of which will readily be admitted.

Ax. 1. *If two weights balance each other upon a straight lever, the pressure upon the fulcrum is equal to the sum of the weights, whatever be the length of the lever*.*

Ax. 2. *If a weight be supported upon a lever which rests on two fulcrums, the pressure upon the fulcrums is equal to the whole weight.*

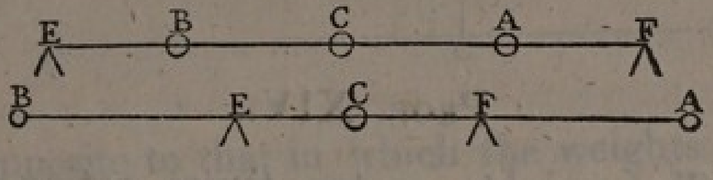
Ax. 3. *Equal forces, acting perpendicularly at the extremities of equal arms of a lever, exert the same effort to turn the lever round.*

* The effect produced by the gravity of the lever is not taken into consideration, unless it be expressly mentioned.

Prop. XIII.

(71.) *If two equal weights act perpendicularly upon a straight lever, the effort to put it in motion, round any fulcrum, will be the same as if they acted together at the middle point between them.*

Let *A* and *B* be two equal weights, acting perpendicularly upon the lever *FB*, whose fulcrum is *F*.



Bisect *AB* in *C*; make $CE = CF$; and at *E* suppose another fulcrum to be placed.

Then since the two weights *A* and *B* are supported by *E* and *F*, and these fulcrums are similarly situated with respect to the weights, each sustains an equal pressure; and therefore the weight sustained by *E* is equal to half the sum of the weights. Now let the weights *A* and *B* be placed at *C*, the middle point between *A* and *B*, and consequently the middle point between *E* and *F*; then since *E* and *F* support the whole weight *C*, and are similarly situated with respect to it, the fulcrum *E* supports half the weight; that is, the pressure upon *E* is the same, whether the weights are placed at *A* and *B*, or collected in *C*, the middle point between them; and therefore, the effort to put the lever in motion round *F*, is the same on either supposition.

(72.) Cor. If a weight be formed into a cylinder *AB* (Fig. Art. 73.) which is every where of the same density, and placed parallel to the horizon, the effort of any part *AD*, to put the whole in motion round *C*, is the same as if this part were collected at *E*, the middle point of *AD*.

For the weight *AD* may be supposed to consist of pairs of equal weights, equally distant from the middle point.

What is here affirmed of weights, is true of any forces which are proportional to the weights, and act in the same directions.

Prop. XIV.

(73.) *Two weights, or two forces, acting perpendicularly upon a straight lever, will balance each other, when they are reciprocally proportional to their distances from the fulcrum.*

Case 1. When the weights act on *contrary* sides of the fulcrum.

Let x and y be the two weights, and let them be formed into the cylinder *AB*, which is every where of the same density. Bisect *AB* in *C*; then this cylinder

will balance itself upon the fulcrum *C* (Art. 72). Divide *AB* into two parts in *D*, so that $AD : DB :: x : y$, and the weights of *AD* and *DB* will be respectively x and y; bisect *AD* in *E* and *DB* in *F*; then since *AD* and *DB* keep the lever at rest, they will keep it at rest when they are collected at E and F (Art. 72.); that is, x, when placed at E, will balance y, when placed at F; and $x : y :: AD : BD :: AB - BD : AB - AD :: 2CB - 2BF : 2AC - 2AE :: 2CF : 2CE :: CF : CE$.

Case 2. When the two forces act on the *same* side of the center of motion.

Let AB be a lever whose fulcrum is C; A and B two weights acting perpendicularly upon it; and let $A : B :: BC : AC$; then these weights will balance each other, as appears by the former Case. Now suppose a power sufficient to sustain a weight equal to the sum of the weights A and B, to be applied at C, in a direc-

tion opposite to that in which the weights act; then will this power supply the place of the fulcrum (Art. 70. Ax. 1.); also, a fulcrum placed at A, or B, and sustaining a weight A, or B, will supply the place of the body there, and the equilibrium will remain. Let B be the center of motion; then we have a straight lever whose center of motion is B, and the two forces A and $A + B$, acting perpendicularly upon it at the points A and C, sustain each other; also, $A : B :: BC : AC$; therefore $A : A+B :: BC : BA$.

(74.) Cor. 1. If two weights, or two forces, acting perpendicularly on the arms of a straight lever, keep each other in equilibrio, they are inversely as their distances from the center of motion.

For the weights will balance when they are in that proportion, and if the proportion be altered by increasing or diminishing one of the weights, it's effort to turn the lever round will be altered, or the equilibrium will be destroyed.

(75.) Cor. 2. Since $A : B :: BC : AC$ when there is an equilibrium upon the lever AB, whose fulcrum is C, by multiplying extremes and means, $A \times AC = B \times BC$.

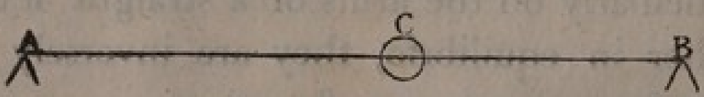
(76.) Cor. 3. When the power and weight act on the same side of the fulcrum, and keep each other in equilibrio, the weight sustained by the fulcrum is equal to the difference between the power and the weight.

(77.) Cor. 4. In the common balance, the arms of the lever are equal; consequently, the power and weight, or two weights, which sustain each other, are equal. In the false balance, one arm is longer than the other; therefore the weight, which is suspended at this arm, is proportionally less than the weight which it sustains at the other.

(78.) Cor. 5. If the same body be weighed at the two ends of a false balance, it's true weight is a mean proportional between the apparent weights.

Call the true weight x, and the apparent weights, when it is suspended at A and B, a and b respectively; then $a : x :: AC : BC$, and $x : b :: AC : BC$; therefore $a : x :: x : b$.

(79.) Cor. 6. If a weight C be placed upon a lever which is supported upon two props A and B in an

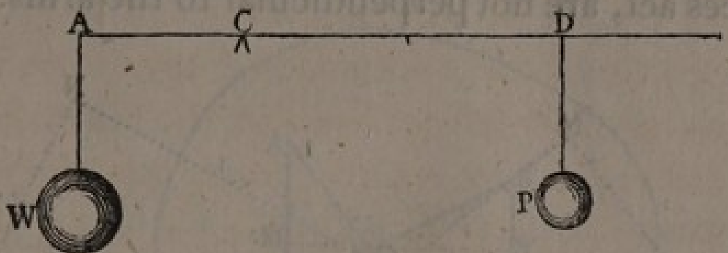


horizontal position, the pressure upon A : the pressure upon $B :: BC : AC$.

For if B be conceived to be the fulcrum, we have this proportion, the weight sustained by A : the weight

$C :: BC : AB$; in the same manner, if A be considered as the fulcrum, then the weight C : the weight sustained by $B :: AB : CA$; therefore, *ex æquo*, the weight sustained by A : the weight sustained by $B :: BC : AC$.

(80.) Cor. 7. If a given weight P be moved along the graduated arm of a straight lever, the weight W,



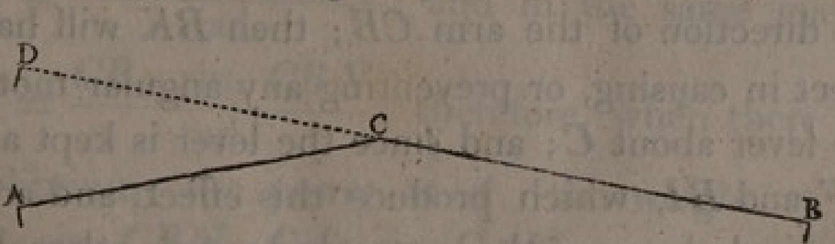
which it will balance at A, is proportional to CD, the distance at which the given weight acts.

When there is an equilibrium, $W \times AC = P \times DC$ (Art. 75.); and AC and P are invariable; therefore $W \propto DC$ (*Alg.* Art. 199.)

Prop. XV.

(81.) *If two forces, acting upon the arms of any lever, keep it at rest, they are to each other inversely as the perpendiculars drawn from the center of motion to the directions in which the forces act.*

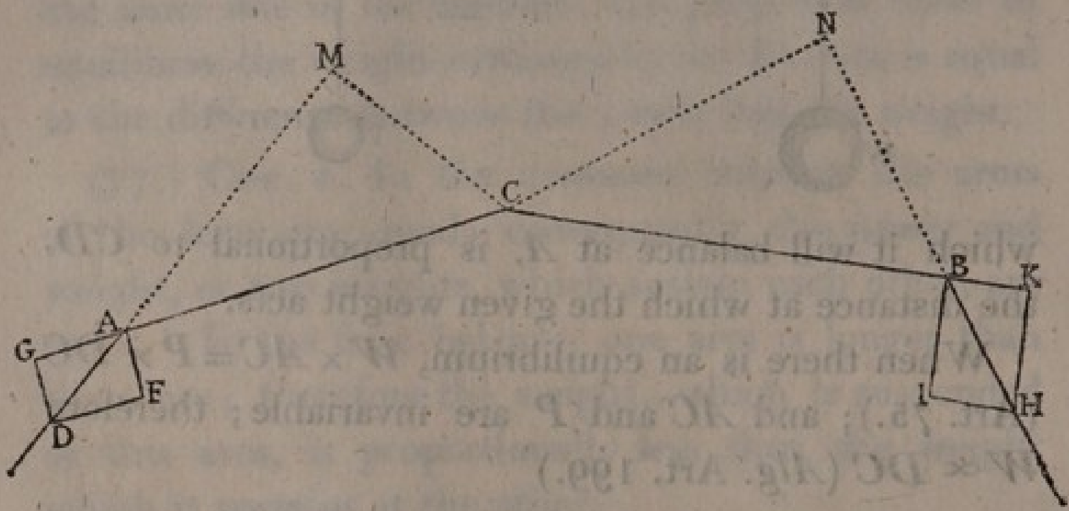
Case 1. Let two forces, A and B, act *perpendicularly* upon the arms CA, CB, of the lever ACB whose fulcrum is C, and keep each other at rest. Produce



BC to D, and make $CD = CA$; then the effort of A

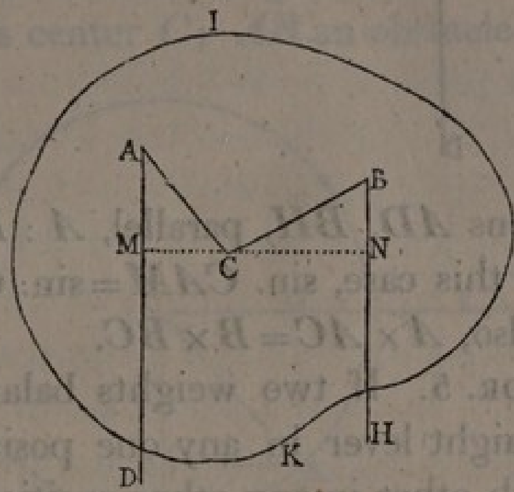
to move the lever round C, will be the same, whether it be supposed to act perpendicularly at the extremity of the arm CA, or CD (Art. 70. Ax. 3.); and on the latter supposition, since there is an equilibrium, $A : B :: CB : CD$ (Art. 74.); therefore $A : B :: CB : CA$.

CASE 2. When the directions AD, BH, in which the forces act, are not perpendicular to the arms. Take



AD and BH, to represent the forces; draw CM and CN at right angles to those directions; also draw AF perpendicular, and DF parallel to AC, and complete the parallelogram GF; then the force AD is equivalent to the two AF, AG, of which, AG acts in the direction of the arm, and therefore can have no effect in causing, or preventing any angular motion in the lever about C. Let BH be resolved, in the same manner, into the two BI, BK, of which BI is perpendicular to, and BK in the direction of the arm CB; then BK will have no effect in causing, or preventing any angular motion in the lever about C; and since the lever is kept at rest, AF and BI, which produce this effect, and act perpendicularly upon the arms, are to each other, by the 1st case, inversely as the arms; that is, $AF : BI :: CB : CA$, or $AF \times CA = BI \times CB$. Also in the similar triangles ADF, ACM, $AF : AD :: CM : CA$, and $AF \times CA = AD \times CM$; in the same manner, $BI \times CB = BH \times CN$; therefore $AD \times CM = BH \times CN$, and $AD : BH :: CN : CM$.

(82.) Cor. 1. Let a body IK be moveable about the center C, and two forces act upon it at A and B,



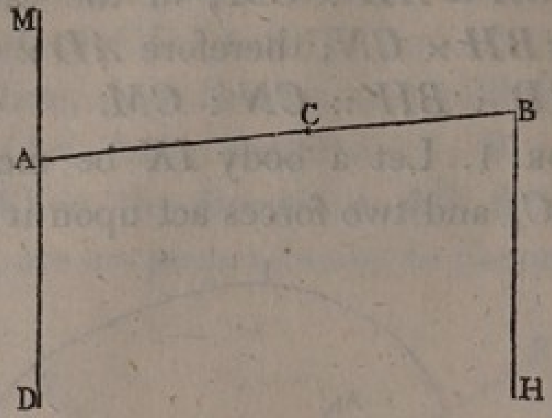
in the directions AD, BH, which coincide with the plane ACB; join AC, CB; then this body may be considered as a lever ACB, and drawing the perpendiculars CM, CN, there will be an equilibrium, when the force acting at A : the force acting at $B :: CN : CM$*.

(83.) Cor. 2. The effort of the force A, to turn the lever round, is the same, at whatever point in the direction MD it is applied; because the perpendicular CM remains the same.

(84.) Cor. 3. Since $CA : CM ::$ rad. : sin. CAM, $CM = \frac{CA \times \text{sin. } CAM}{\text{rad.}}$; and in the same manner, $CN = \frac{CB \times \text{sin. } CBN}{\text{rad.}}$; therefore, when there is an equilibrium, the power at A : the weight at B :: $\frac{CB \times \text{sin. } CBN}{\text{rad.}} : \frac{CA \times \text{sin. } CAM}{\text{rad.}} :: CB \times \text{sin. } CBN : CA \times \text{sin. } CAM$.

* See Art. 74.

(85.) Cor. 4. If the lever ACB be straight, and



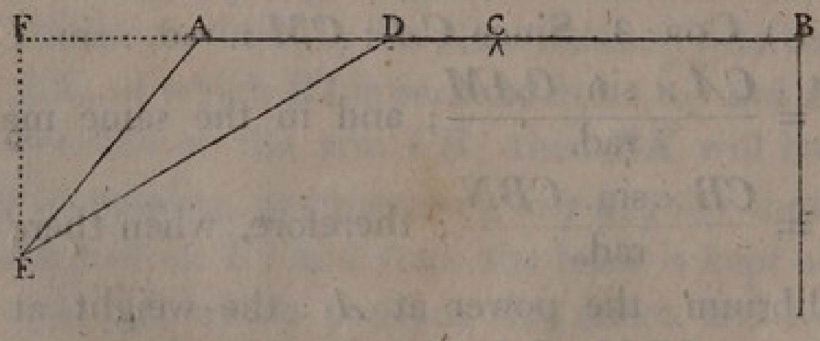
the directions AD, BH, parallel, $A : B :: BC : AC$; because, in this case, sin. $CAM =$ sin. CBH.

Hence also, $A \times AC = B \times BC$.

(86.) Cor. 5. If two weights balance each other upon a straight lever in any one position, they will balance each other in any other position of the lever; for the weights act in parallel directions, and the arms of the lever are invariable.

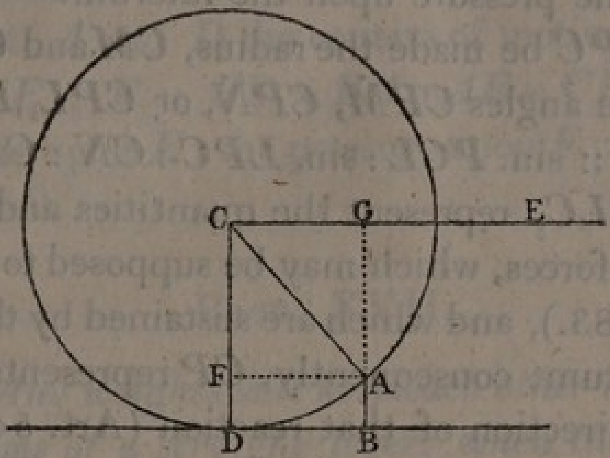
(87.) Cor. 6. If a man, balanced in a common pair of scales, press upwards by means of a rod, against any point in the beam, except that from which the scale is suspended, he will preponderate.

Let the action upwards take place at D, then the scale, by the reaction downwards, will be brought



into the situation E; and the effect will be the same as if DA, AE, DE, constituted one mass; that is, drawing EF perpendicular to CA produced, as if the scale were applied at F (Art. 83.); consequently the weight, necessary to maintain the equilibrium, is greater than if the scale were suffered to hang freely from A, in the proportion of $CF : CA$ (Art. 80.).

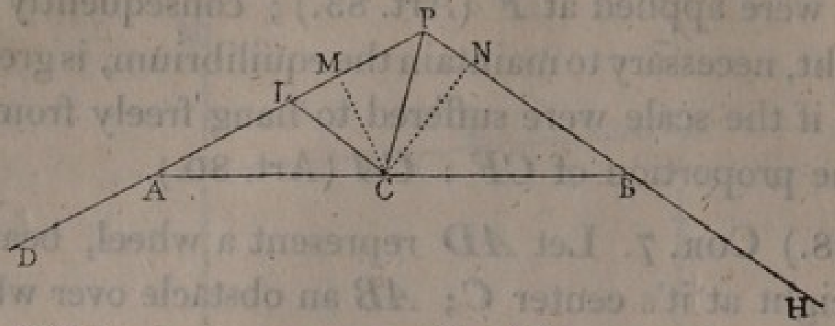
(88.) Cor. 7. Let AD represent a wheel, bearing a weight at it's center C; AB an obstacle over which



it is to be moved by a force acting in the direction CE; join CA, draw CD perpendicular to the horizon, and from A draw AG, AF, at right angles to CE, CD. Then CA may be considered as a lever whose center of motion is A, CD the direction in which the weight acts, and CE the direction in which the power is applied; and there is an equilibrium on this lever when the power : the weight :: $AF : AG$.

Supposing the wheel, the weight, and the obstacle given, the power is the least when AG is the greatest; that is, when CE is perpendicular to CA, or parallel to the tangent at A.

(89.) Cor. 8. Let two forces acting in the directions AD, BH, upon the arms of the lever ACB, keep each other in equilibrio; produce DA and HB till they meet in P; join CP, and draw CL parallel to *PB*; then will *PL*, *LC* represent the two forces,

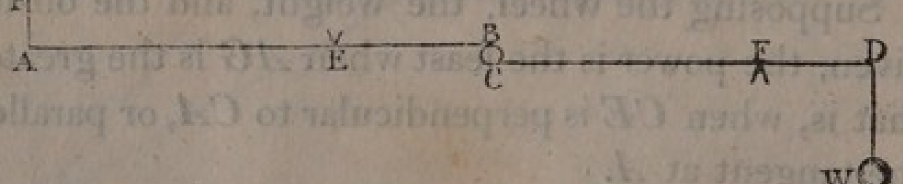


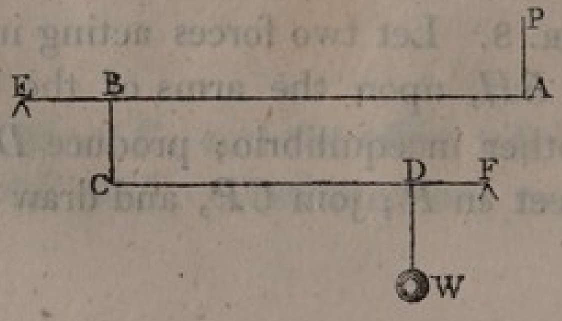
and *PC* the pressure upon the fulcrum.

For, if *PC* be made the radius, *CM* and *CN* are the sines of the angles *CPM*, *CPN*, or *CPL*, *PCL*; and *PL* : *LC* :: sin. *PCL* : sin. *LPC* : *CN* : *CM*; therefore *PL*, *LC*, represent the quantities and directions of the two forces, which may be supposed to be applied at *P* (Art. 83.), and which are sustained by the reaction of the fulcrum; consequently, *CP* represents the quantity and direction of that reaction (Art. 54.), or *PC* represents the pressure upon the fulcrum.

Prop. XVI.

(90.) *In a combination of straight levers,* AB, CD, *whose centers of motion are* E *and* F, *if they act perpendicularly upon each other, and the directions in which the power and weight are applied be also perpendicular to the arms, there is an equilibrium when* P : W :: EB × FD : EA × FC.





For, the power at A : the weight at B, or C :: EB : EA; and the weight at C : the weight at D :: FD : FC; therefore, $P : W :: EB \times FD : EA \times FC$.

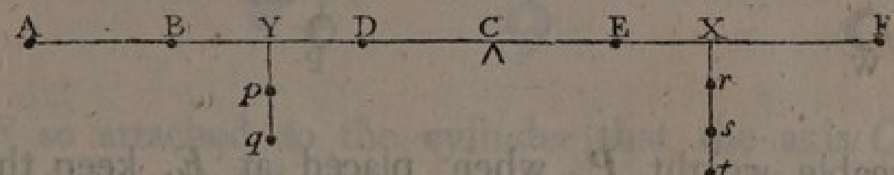
By the same method we may find the proportion between the power and the weight, when there is an equilibrium, in any other combination of levers.

(91.) Cor. If E and F be considered as the power and weight, A and D the centers of motion, we have, as before, $E : F :: FD \times BA : AE \times CD$. Hence the pressure upon E : the pressure upon F :: $FD \times BA : AE \times CD$.

Prop. XVII.

(92.) *Any weights will keep each other in equilibrio on the arms of a straight lever, when the products, which arise from multiplying each weight by it's distance from the fulcrum, are equal, on each side of the fulcrum.*

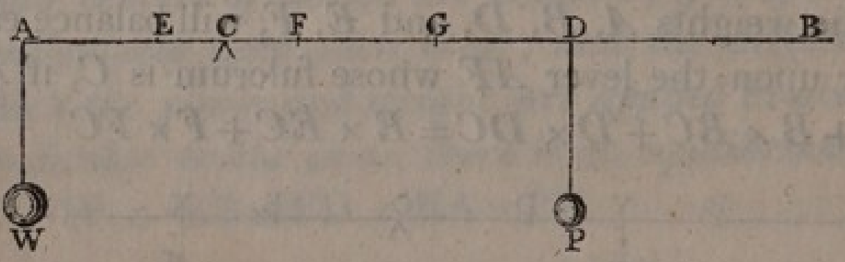
The weights A, B, D, and E, F, will balance each other upon the lever AF whose fulcrum is C, if $A \times AC + B \times BC + D \times DC = E \times EC + F \times FC$.



In CF take any point X, and let the weights r, s, t, placed at X, balance respectively, A, B, D; then $A \times AC = r \times XC$; $B \times BC = s \times XC$; $D \times DC = t \times XC$, (Art. 85.); or, $A \times AC + B \times BC + D \times DC = \overline{r+s+t} \times XC$. In the same manner, let p and q, placed at Y, balance respectively, E and F; then $\overline{p+q} \times YC = E \times EC + F \times FC$; but by the supposition $A \times AC + B \times BC + D \times DC = E \times EC + F \times FC$; therefore $\overline{r+s+t} \times XC = \overline{p+q} \times YC$, and the weights r, s, t, placed at X, balance the weights p, q, placed at Y; also A, B, D, balance the former weights, and E, F, the latter; consequently A, B, D, will balance E and F.

(93.) Cor. 1. If the weights do not act in parallel directions, instead of the distances we must substitute the perpendiculars, drawn from the center of motion, upon the directions. (See Art. 81.)

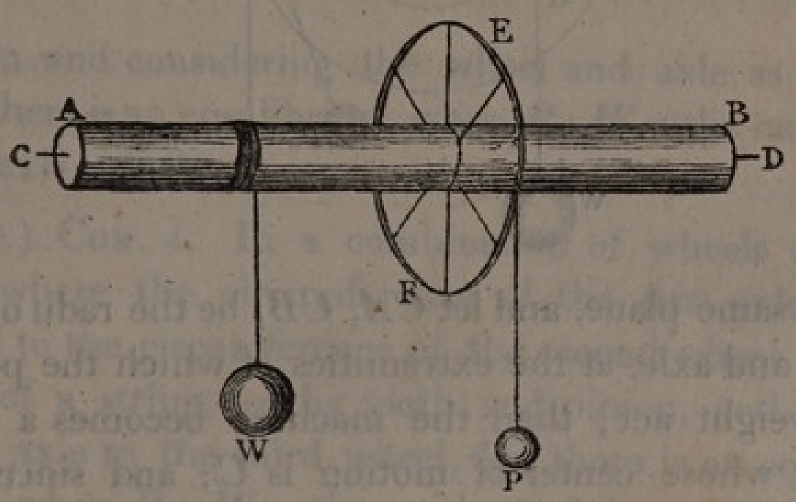
(94.) Cor. 2. In Art. 80. the lever is supposed to be without weight, or the arms AC, CD to balance each other: In the formation of the common *steel-yard* the longer arm CB is heavier than CA, and allowance must be made for this excess. Let the



moveable weight P, when placed at E, keep the lever at rest; then when W and P are suspended upon the lever, and the whole remains at rest, W sustains P, and also a weight which would support P when placed at E; therefore $W \times AC = P \times DC + P \times EC = P \times DE$; and since AC and P are invariable, $W \propto ED$; the graduation must therefore begin from E; and if P, when placed at *F*, support a weight of one pound at *A*, take *FG*, *GD*, &c. equal to each other, and to *EF*, and when *P* is placed at *G* it will support two pounds; when at *D* it will support three pounds, &c.

ON THE WHEEL AND AXLE.

(95.) The *wheel and axle* consists of two parts, a cylinder *AB* moveable about it's axis *CD*, and a circle



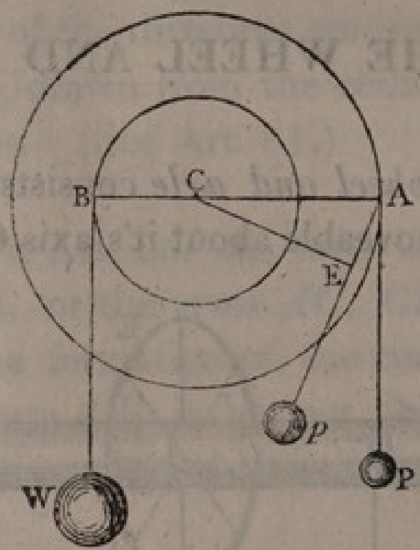
EF so attached to the cylinder that the axis *CD* passes through it's center, and is perpendicular to it's plane.

The power is applied at the circumference of the wheel, usually in the direction of a tangent to it, and the weight is raised by a rope which winds round the axle in a plane at right angles to the axis.

Prop. XVIII.

(96.) *There is an equilibrium upon the wheel and axle, when the power is to the weight, as the radius of the axle to the radius of the wheel.*

The effort of the power to turn the machine round the axis, must be the same at whatever point in the axle the wheel is fixed; suppose it to be removed, and placed in such a situation that the power and weight may act



in the same plane, and let CA, CB, be the radii of the wheel and axle, at the extremities of which the power and weight act; then the machine becomes a lever ACB, whose center of motion is C; and since the radii CA, CB, are at right angles to AP and BW, we have $P : W :: CB : CA$ (Art. 82).

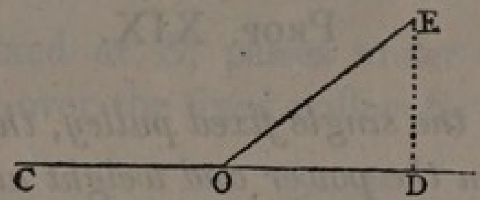
(97.) Cor. 1. If the power act in the direction Ap, draw CE perpendicular to Ap, and there will be an equilibrium when $P : W :: CB : CE$ (Art. 82).

The same conclusion may also be obtained by resolving the power into two, one perpendicular to AC, and the other parallel to it.

(98.) Cor. 2. If $2R$ be the thickness of the ropes by which the power and weight act, there will be an equilibrium when $P : W :: CB+R : CA+R$, since the power and weight must be supposed to be applied in the axes of the ropes.

The ratio of the power to the weight is greater in this case than the former; for if any quantity be added to the terms of a ratio of less inequality, that ratio is increased (*Alg.* Art. 162.).

(99.) Cor. 3. If the plane of the wheel be inclined to the axle at the angle EOD, draw ED perpendicular



to CD; and considering the wheel and axle as one mass, there is an equilibrium when $P : W ::$ the radius of the axle : ED.

(100.) Cor. 4. In a combination of wheels and axles, where the circumference of the first axle is applied to the circumference of the second wheel, by means of a string, or by tooth and pinion, and the second axle to the third wheel, &c. there is an equilibrium when $P : W ::$ the product of the radii of all the axles : the product of the radii of all the wheels. (See Art. 90.)

(101.) Cor. 5. When the power and weight act in parallel directions, and on *opposite* sides of the axis, the pressure upon the axis is equal to their *sum*; and when they act on the *same* side, to their *difference*. In other cases the pressure may be estimated by Art. 89.

ON THE PULLEY.

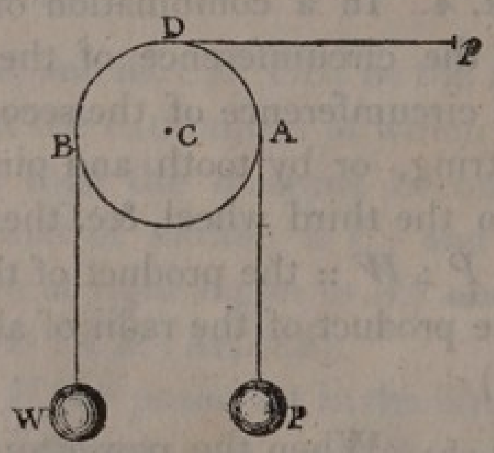
(102.) Def. A *Pulley* is a small wheel moveable about it's center, in the circumference of which a groove is formed to admit a rope or flexible chain.

The pulley is said to be *fixed*, or *moveable*, according as the center of motion is fixed or moveable.

Prop. XIX.

(103.) *In the single fixed pulley, there is an equilibrium, when the power and weight are equal.*

Let a power and weight P, W, equal to each other, act by means of a perfectly flexible rope PDW, which passes over the fixed pulley ADB; then, whatever



force is exerted at D in the direction DAP, by the power, an equal force is exerted by the weight in the direction DBW; these forces will therefore keep each other in equilibrio.

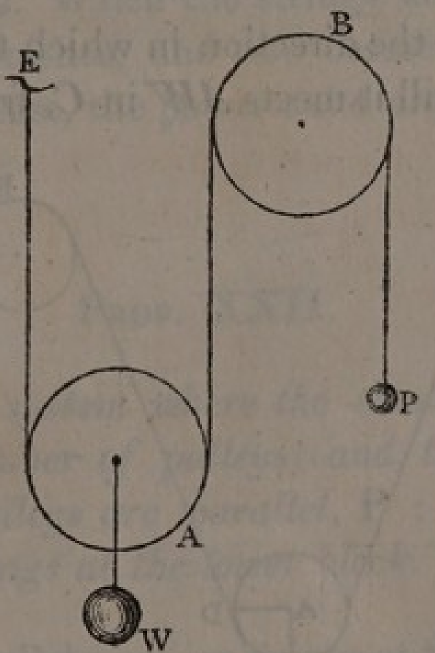
Cor. 1. Conversely, when there is an equilibrium, the power and weight are equal*.

Cor. 2. The proposition is true in whatever direction the power is applied; the only alteration made, by changing it's direction, is in the pressure upon the center of motion. (See Art. 106.)

Prop. XX.

(104.) *In the single moveable pulley, whose strings are parallel, the power is to the weight as* 1 *to* 2†.

A string fixed at *E*, passes under the moveable pulley *A*, and over the fixed pulley *B*; the weight is



annexed to the center of the pulley *A*, and the power is applied at *P*. Then since the strings *EA*, *BA* are

* See Art. 74.

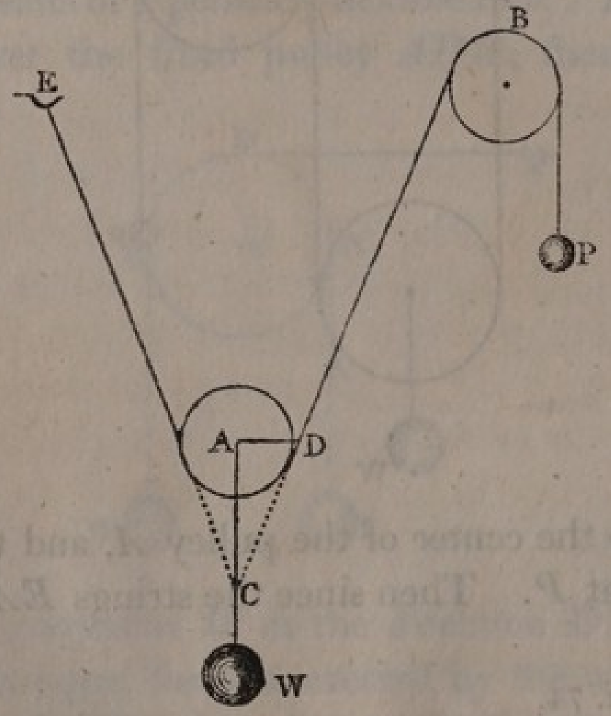
† In this and the following propositions, the power and weight are supposed to be in equilibrio.

in the direction in which the weight acts, they exactly sustain it; and they are equally stretched in every point, therefore they sustain it equally between them; or each sustains half the weight. Also, whatever weight AB sustains, P sustains (Prop. xix. Cor. 1.), therefore $P : W :: 1 : 2$.

Prop. XXI.

(105.) *In general, in the single moveable pulley, the power is to the weight, as radius to twice the cosine of the angle which either string makes with the direction in which the weight acts.*

Let AW be the direction in which the weight acts; produce BD till it meets AW in C, from A draw AD



at right angles to AC, meeting BC in D; then if CD

be taken to represent the power at P, or the power which acts in the direction DB, CA will represent that part of it which is effective in sustaining the weight, and AD will be counteracted by an equal and opposite force, arising from the tension of the string CE; also, the two strings are equally effective in sustaining the weight; therefore $2AC$ will represent the whole weight sustained; consequently, $P : W :: CD : 2AC ::$ rad. : 2 cos. DCA.

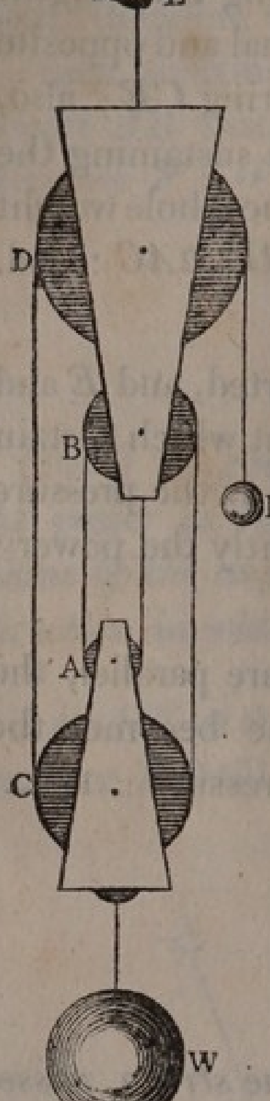
(106.) Cor. 1. If the figure be inverted, and E and B be considered as a power and weight which sustain each other upon the fixed pulley A, W is the pressure upon the center of motion; consequently the power : the pressure :: radius : 2 cos. DCA.

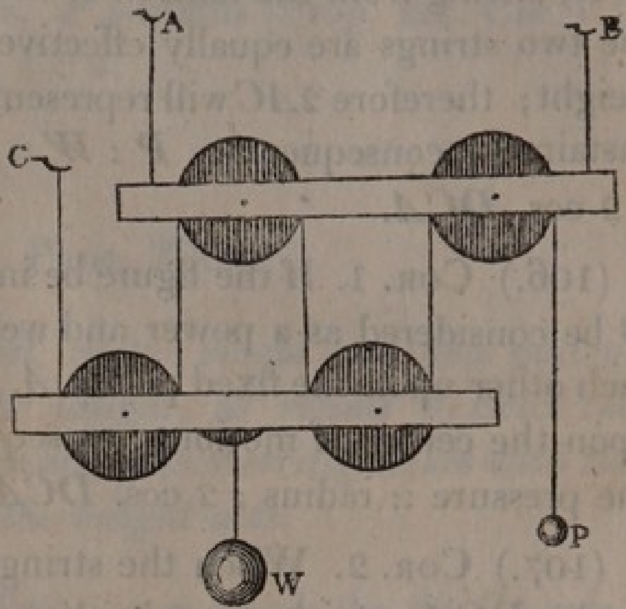
(107.) Cor. 2. When the strings are parallel, the angle DCA vanishes, and it's cosine becomes the radius; in this case, the power : the pressure :: 1 : 2.

Prop. XXII.

(108.) *In a system where the same string passes round any number of pulleys, and the parts of it between the pulleys are parallel,* P : W :: 1 : *the number of strings at the lower block.*

Since the parallel parts, or strings at the lower block, are in the direction in which the weight acts, they exactly support the whole weight; also, the tension in every point of these strings is the same, otherwise the system would not be at rest, and consequently each of them sustains an equal weight; whence it follows that, if there be n strings, each sustains





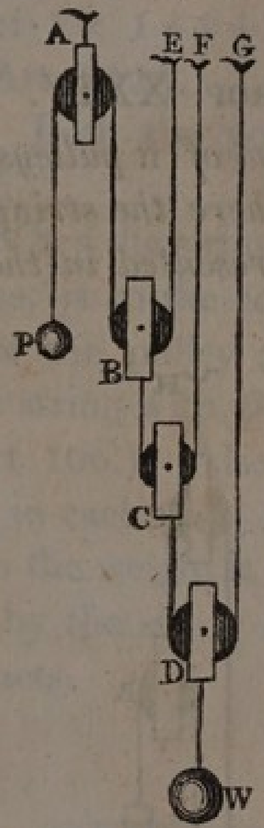
$\frac{1}{n}$*th* part of the weight; therefore, P sustains $\frac{1}{n}$*th* part of the weight, or $P : W :: \frac{1}{n} : 1 :: 1 : n$.

(109.) Cor. If two systems of this kind be combined, in which there are m and n strings, respectively, at the lower blocks, $P : W :: 1 : mn$.

Prop. XXIII.

(110.) *In a system where each pulley hangs by a separate string, and the strings are parallel,* P : W :: 1 : *that power of* 2 *whose index is the number of moveable pulleys.*

In this system, a string passes over the fixed pulley *A*, and under the moveable pulley *B*, and is fixed



at *E*; another string is fixed at *B*, passes under the moveable pulley *C*, and is fixed at *F*; &c. in such a manner that the strings are parallel.

Then, by Art. 104, when there is an equilibrium,

P : the weight at B :: 1 : 2

the weight at B : the weight at C :: 1 : 2

the weight at C : the weight at D :: 1 : 2

&c.

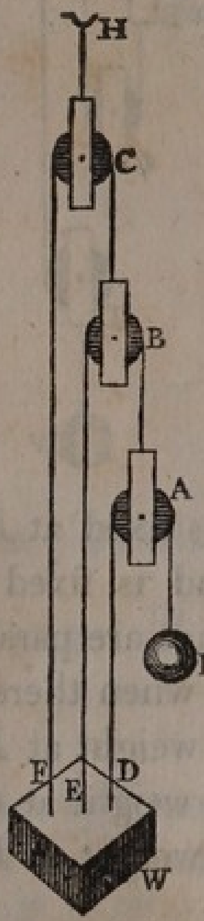
Comp. $P : W :: 1 : 2 \times 2 \times 2 \times$ &c. continued to as many factors as there are moveable pulleys; that is, when there are n such pulleys, $P : W :: 1 : 2^n$.

(111.) Cor. 1. The power and weight are wholly sustained at A, E, F, G, &c. which points sustain respectively, $2P$, P, $2P$, $4P$, &c.

(112.) Cor. 2. When the strings are not parallel, $P : W ::$ rad. : 2 cos. of the angle which the string makes with the direction in which the weight acts, in each case (Art. 105.)

Prop. XXIV.

(113.) *In a system of* n *pulleys each hanging by a separate string, where the strings are attached to the weight as is represented in the annexed figure,* $P : W :: 1 : 2^n - 1$.



A string, fixed to the weight at F, passes over the pulley C, and is again fixed to the pulley B; another string, fixed at E, passes over the pulley B, and is fixed to the pulley A; &c. in such a manner that the strings are parallel:

Then, if P be the power, the weight sustained by the string DA is P; also the pressure downwards upon A, or the weight which the string AB sustains, is $2P$ (Art. 107.); therefore the string EB sustains $2P$; &c. and the whole weight sustained is $P+2P+4P+$&c. Hence, $P : W :: 1 : 1+2+4+$&c. to n terms :: $1 : 2^n - 1$ (*Alg.* Art. 222.).

(114.) Cor. 1. Both the power and the weight are sustained at H.

(115.) Cor. 2. When the strings are not parallel, the power in each case, is to the corresponding pressure upon the center of the pulley :: rad. : 2 cos. of the angle made by the string with the direction in which the weight acts (Art. 106.). Also, by the resolution of forces, the power in each case, or pressure upon the former pulley, is to the weight it sustains :: rad. : cos. of the angle made by the string with the direction in which the weight acts.

ON THE INCLINED PLANE.

Prop. XXV.

(116.) *If a body act upon a perfectly hard and smooth plane, the effect produced upon the plane is in a direction perpendicular to it's surface.*

Case 1. When the body acts perpendicularly upon the plane, it's force is wholly effective in that direction; since there is no cause to prevent the effect, or to alter it's direction.

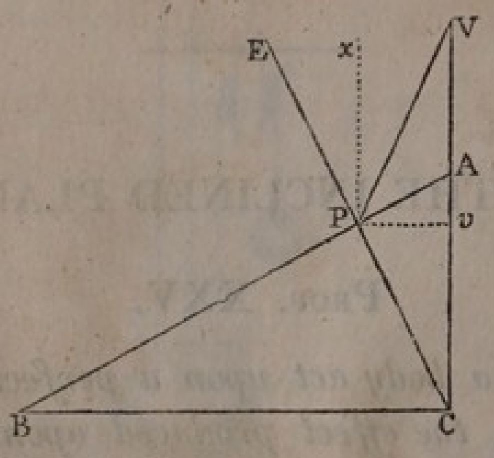
Case 2. When the direction in which the body acts is oblique to the plane, resolve it's force into two, one parallel, and the other perpendicular, to the plane; the former of these can produce no effect upon the plane, because there is nothing to oppose it in the direction in which it acts (see Art. 29.); and the latter is wholly effective (by the first case); that is, the effect produced by the force is in a direction perpendicular to the plane.

(117.) Cor. The reaction of the plane is in a direction perpendicular to it's surface (Art. 32.).

Prop. XXVI.

(118.) *When a body is sustained upon a plane which is inclined to the horizon,* P : W :: *the sine of the plane's inclination : the sine of the angle which the direction of the power makes with a perpendicular to the plane.*

Let *BC* be parallel to the horizon, *BA* a plane in-



clined to it; *P* a body, sustained at any point upon the plane by a power acting in the direction PV. From P draw PC perpendicular to BA, meeting BC in C; and from C draw CV perpendicular to BC, meeting PV in V*. Then the body P is kept at rest by three forces which act upon it at the same time; the power, in the direction PV; gravity, in the direction VC; and the reaction of the plane, in the direction CP (Art. 117.); these three forces are therefore properly represented by the three lines PV, VC, and CP (Art. 56.); or $P : W :: PV : VC ::$ sin. PCV : sin. VPC; and in the similar triangles APC, ABC (Euc. 8. vi.), the angles ACP, and CBA are equal; therefore P : W :: sin. ABC : sin. VPC.

(119.) Cor. 1. When PV coincides with PA, or the power acts parallel to the plane, $P : W :: PA : AC :: AC : AB$.

(120.) Cor. 2. When PV coincides with Pv, or the power acts parallel to the base, $P : W :: Pv : vC :: AC : CB$; because the triangles PvC, ABC are similar.

(121.) Cor. 3. When PV is parallel to CV, the power sustains the whole weight.

(122.) Cor. 4. Since $P : W ::$ sin. ABC : sin. VPC, by multiplying extremes and means, $P \times$ sin. $VPC = W \times$ sin. ABC; and if W, and the sine of the $\angle ABC$ be invariable, $P \propto \frac{1}{\text{sin. } VPC}$ (*Alg.* Art. 206.); therefore P is the least, when $\frac{1}{\text{sin. } VPC}$ is the least, or sin. VPC the greatest; that is, when sin. VPC becomes the radius, or PV coincides with PA. Also, P is indefinitely great when sin. VPC vanishes; that is, when the power acts perpendicularly to the plane.

* That PV, CV, are in the same plane, appears from Art. 55.

(123.) Cor. 5. If P and the $\angle ABC$ be given, $W \propto \sin. VPC$; therefore W will be the greatest when $\sin. VPC$ is the greatest, that is, when PV coincides with PA. Also, W vanishes when the $\sin. VPC$ vanishes, or PV coincides with PC.

(124.) Cor. 6. The power : the pressure :: PV : PC :: $\sin. PCV : \sin. PVC$ :: $\sin. ABC : \sin. PVC$.

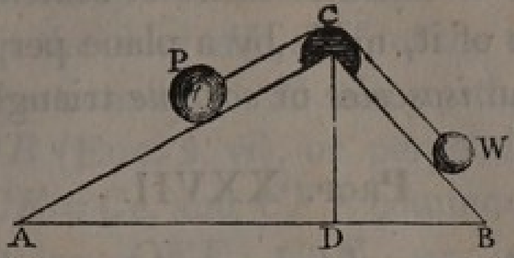
(125.) Cor. 7. When the power acts parallel to the plane, the power : the pressure :: $PA : PC$ :: $AC : BC$.

(126.) Cor. 8. When the power acts parallel to the base, the power : the pressure :: $Pv : PC$ :: $AC : AB$.

(127.) Cor. 9. $P \times \sin. PVC =$ the pressure $\times \sin. ABC$; and when P and the $\angle ABC$ are given, the pressure $\propto \sin. PVC$; therefore the pressure will be the greatest when PV is parallel to the base.

(128.) Cor. 10. When two sides of a triangle, taken in order, represent the quantities and directions of two forces which are sustained by a third, the remaining side, taken in the same order, will represent the quantity and direction of the third force (Art. 54.) Hence, if we suppose PV to revolve round P, when it falls between Px, which is parallel to VC, and PE, the direction of gravity remaining unaltered, the direction of the reaction must be changed, or the body must be supposed to be sustained against the under surface of the plane. When it falls between PE and xP produced, the direction of the power must be changed. And when it falls between xP produced, and PC, the directions of both the power and reaction must be different from what they were supposed to be in the proof of the proposition; that is, the body must be sustained against the under surface of the plane, by a force which acts in the direction VP.

(129.) Cor. 11. If the weights P, W, sustain each other upon the planes AC, CB, which have



a common altitude CD, by means of a string PCW which passes over the pulley C, and is parallel to the planes, then $P : W :: AC : BC$.

For, since the tension of the string is every where the same, the sustaining power, in each case, is the same; and calling this power x,

$$P : x :: AC : CD \text{ (Art. 119.)};$$
$$x : W :: CD : CB.$$
$$\text{comp. } P : W :: AC : CB.$$

ON THE WEDGE.

(130.) Def. A *Wedge* is a triangular prism; or a solid generated by the motion of a plane triangle parallel to itself, upon a straight line which passes through one of it's angular points*.

* See also Euc. B. XI. Def. 13.

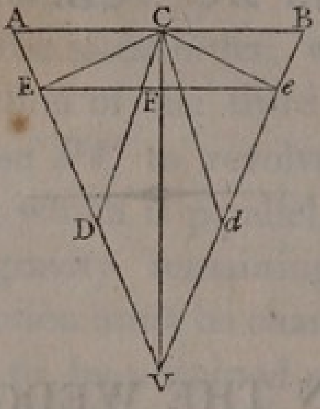
Knives, swords, coulters, nails, &c. are instruments of this kind.

The wedge is called *isosceles* or *scalene*, according as the section of it, made by a plane perpendicular to it's sides, is an *isosceles* or *scalene* triangle.

Prop. XXVII.

(131.) *If two equal forces act upon the sides of an isosceles wedge at equal angles of inclination, and a force act perpendicularly upon the back, they will keep the wedge at rest, when the force upon the back is to the sum of the forces upon the sides, as the product of the sine of half the vertical angle of the wedge and the sine of the angle at which the directions of the forces are inclined to the sides, to the square of radius.*

Let AVB represent a section of the wedge, made by a plane perpendicular to it's sides; draw VC per-



pendicular to AB; DC, dC, in the directions of the forces upon the sides; and CE, Ce, at right angles to AV, BV; join Ee, meeting CV in F.

Then, in the triangles VCA, VCB, since the angles VCA, CAV, are respectively equal to VCB, VBC, and VC is common to both, $AC = CB$, and the $\angle CVA = \angle CVB$. Again, in the triangles ACD, BCd, the angles DAC, CDA, are equal to the angles CBd, BdC, and $AC=BC$; therefore, $DC=dC$. In the same manner it may be shewn that $CE=Ce$, and $AE=Be$; hence the sides AV, BV, of the triangle AVB, are cut proportionally in E and e; therefore Ee is parallel to AB (Euc. 2. vi), or perpendicular to CV; also, since $CE=Ce$, and CF is common to the right-angled triangles CEF, CeF, we have $EF=eF$ (Euc. 47. i.)

Now since DC and dC are equal, and in the directions of the forces upon the sides, they will represent them; resolve DC into two, DE, EC, of which DE produces no effect upon the wedge, and EC, which is effective (Art. 116.), does not wholly oppose the power, or force upon the back; resolve EC therefore into two, EF, parallel to the back, and FC perpendicular to it, the latter of which is the only force which opposes the power. In the same manner it appears that eF, FC, are the only effective parts of dC, of which FC opposes the power, and eF is counteracted by the equal and opposite force EF; hence if $2CF$ represent the power, the wedge will be kept at rest*; that is, when the force upon the back : the sum of the resistances upon the sides :: $2CF$: $DC+dC$ :: $2CF$: $2DC$ :: CF : DC; and

$$CF : CE :: \sin. CEF : \text{rad.} :: \sin. CVE : \text{rad.}$$
$$CE : DC :: \sin. CDE : \text{rad.}$$

Comp. $CF : DC :: \sin. CVE \times \sin. CDE : \overline{\text{rad.}}|^2$

(132.) Cor. 1. The forces do not sustain *each other*, because the parts DE, de, are not counteracted.

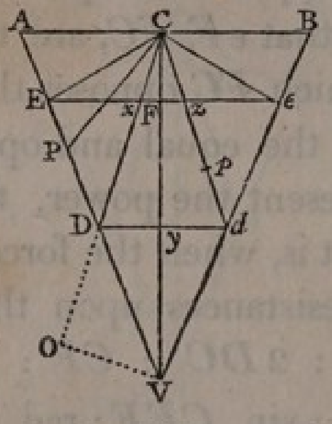
* The directions of the three forces must meet in a point, otherwise a rotatory motion will be given to the wedge.

(133.) Cor. 2. If the resistances act perpendicularly upon the sides of the wedge, the angle CDE becomes a right angle, and P : the sum of the resistances :: sin. $CVE \times$ rad. : $\overline{\text{rad.}}|^2$ :: sin. CVE : rad. :: $AC : AV$.

(134.) Cor. 3. If the directions of the resistances be perpendicular to the back, the angle $CDE = \angle CVE$, and P : the sum of the resistances : $\overline{\text{sin. } CVE}|^2 : \overline{\text{rad.}}|^2$ :: $AC^2 : AV^2$.

(135.) Cor. 4. When the resistances act parallel to the back, sin. CDA = sin. CAV, and P : the sum of the resistances :: sin $CVA \times$ sin. CAV : $\overline{\text{rad.}}|^2$ :: $CA \times CV : AV^2$ :: $CE \times AV$* : AV^2 :: $CE : AV$.

(136.) Cor. 5. In the demonstration of the proposition it has been supposed that the sides of the wedge are perfectly smooth; if on account of the friction, or by any other means, the resistances are *wholly* effective, join Dd, which will cut CV at right angles



in y, and resolve DC, dC into Dy, yC, dy, yC, of which Dy and dy destroy each other, and $2yC$ sustains the power. Hence, the power : the sum of the resistances :: $2yC : 2DC$ :: $yC : DC$ :: sin. CDy or DCA : rad.

* By similar triangles, $CE : CA :: CV : AV$; therefore $CE \times AV = CA \times CV$.

(137.) Cor. 6. If Ee cut DC and dC in x and z, the forces, xC, zC, when wholly effective, and the forces DC, dC, acting upon smooth surfaces, will sustain the same power $2CF$.

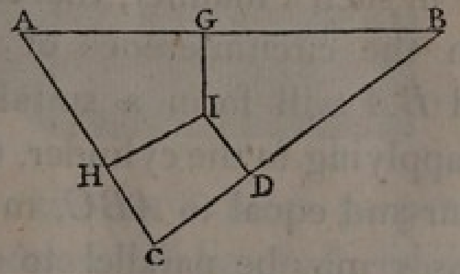
(138.) Cor. 7. If from any point P in the side AV, PC be drawn, and the resistance upon the side be represented by it, the effect upon the wedge will be the same as before; the only difference will be in the part PE which is ineffective.

(139.) Cor. 8. If DC be taken to represent the resistance on one side, and pC, greater or less than dC, represent the resistance on the other, the wedge cannot be kept at rest by a power acting upon the back; because, on this supposition, the forces which are parallel to the back are unequal.

This Proposition and it's Corollaries have been deduced from the actual resolution of the forces, for the purpose of shewing what parts are lost, or destroyed by their opposition to each other; the same conclusions may, however, be very concisely and easily obtained from Art. 142.

Prop. XXVIII.

(140.) *When three forces, acting perpendicularly upon the sides of a scalene wedge, keep each other in equilibrio, they are proportional to those sides.*



Let GI, HI, DI, the directions of the forces, meet in I; then since the forces keep each other at rest, they are proportional to the three sides of a triangle which are respectively perpendicular to those directions (Art. 59.); that is, to the three sides of the wedge.

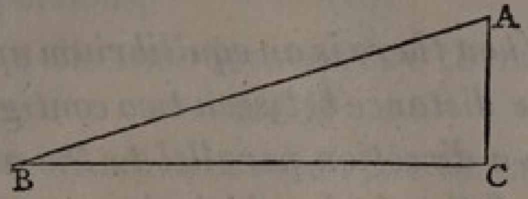
(141.) Cor. 1. If the lines of direction, passing through the points of impact, do not meet in a point, the wedge will have a rotatory motion communicated to it; and this motion will be round the center of gravity of the wedge. (See Art. 184.)

(142.) Cor. 2. When the directions of the forces are not perpendicular to the sides, the effective parts must be found, and there will be an equilibrium when those parts are to each other as the sides of the wedge.

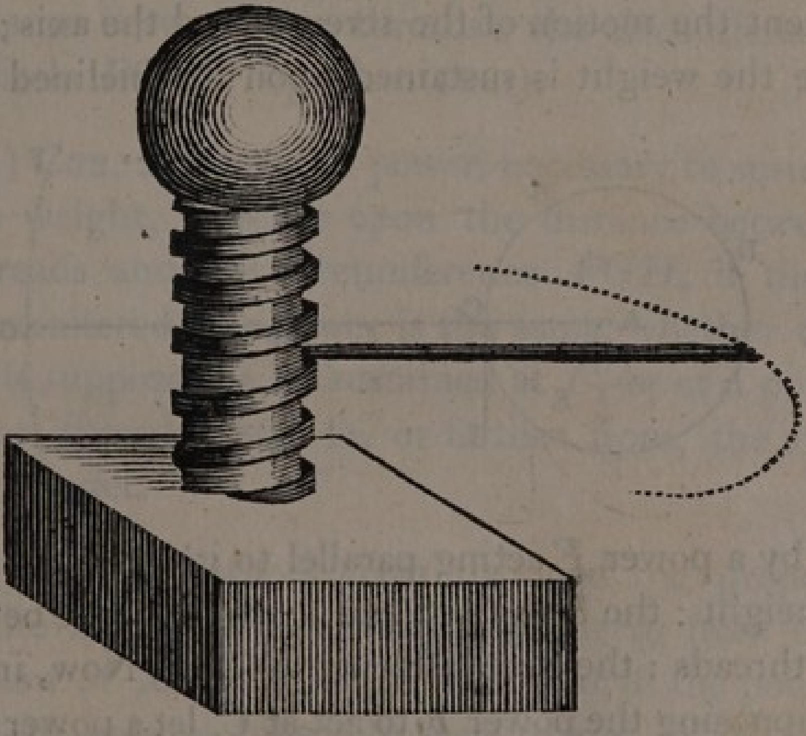
ON THE SCREW.

(143.) Def. The *Screw* is a mechanical power, which may be conceived to be generated in the following manner:

Let a solid and a hollow cylinder of equal diameters be taken, and let ABC be a right-angled plane triangle whose base BC is equal to the circumference of the solid cylinder; apply the triangle to the convex surface of this cylinder, in such a manner, that the base BC may coincide with the circumference of the base of the cylinder, and BA will form a spiral thread on it's surface. By applying to the cylinder, triangles, in succession, similar and equal to ABC, in such a manner, that their bases may be parallel to BC, the spiral thread may be continued; and supposing this thread to have thickness, or the cylinder to be protuberant where it falls, the external screw will be formed, in which the



distance between two contiguous threads, measured in a direction parallel to the axis of the cylinder, is *AC*. Again, let the triangles be applied in the same manner to the concave surface of the hollow cylinder, and where the thread falls let a groove be made, and the internal screw will be formed. The two screws being thus exactly adapted to each other, the solid or hollow cylinder, as the case requires, may be moved round

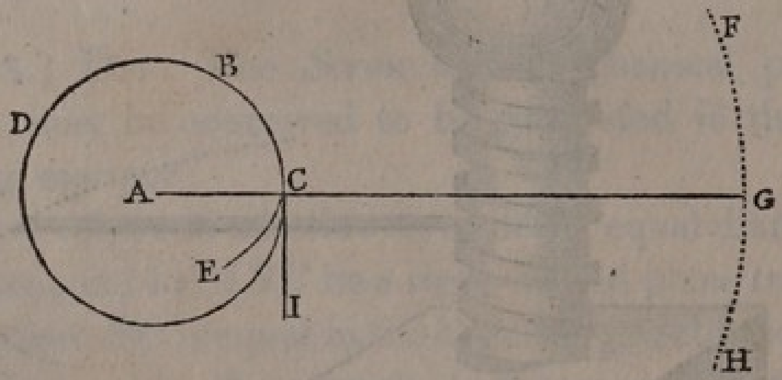


the common axis, by a lever perpendicular to that axis; and a motion will be produced in the direction of the axis, by means of the spiral thread.

Prop. XXIX.

(144.) *When there is an equilibrium upon the screw,* P : W :: *the distance between two contiguous threads, measured in a direction parallel to the axis : the circumference of the circle which the power describes.*

Let *BCD* represent a section of the screw made by a plane perpendicular to it's axis, *CE* a part of the spiral thread upon which the weight is sustained; then *CE* is a portion of an inclined plane, whose height is the distance between two threads, and base equal to the circumference *BCD*. Call *F* the power which acting at *C* in the plane *BCD*, and in the direction *CI* perpendicular to *AC*, will sustain the weight *W*, or prevent the motion of the screw round the axis; then since the weight is sustained upon the inclined plane



CE by a power *F* acting parallel to it's base, *F* : *W* :: the height : the base (Art. 120.) :: the distance between two threads : the circumference *BCD*. Now, instead of supposing the power *F* to act at *C*, let a power *P* act perpendicularly at *G*, on the straight lever *GCA*, whose center of motion is *A*, and let this power produce the same effect at *C* that *F* does; then, by the property of the lever, $P : F :: CA : GA ::$ the circumference BCD : the circumference FGH. We have therefore these two proportions,

$F : W ::$ distance between two threads : BCD
$P : F :: \quad BCD \quad : FGH$
comp. $P : W ::$ distance between two threads : FGH.

(145.) Cor. 1. In the proof of this Proposition the whole weight is supposed to be sustained at one point C of the spiral thread; if we suppose it to be dispersed over the whole thread, then, by the Proposition, the power at G necessary to sustain any part of the weight : that part :: the distance between two threads : the circumference of the circle FGH; therefore the sum of all these powers, or the whole power : the sum of all the corresponding weights, or the whole weight, :: the distance between two threads : the circumference of the circle FGH (*Alg.* Art. 183.).

(146.) Cor. 2. Since the power, necessary to sustain a given weight, depends upon the distance between two threads and the circumference FGH, if these remain unaltered, the power is the same, whether the weight is supposed to be sustained at C, or at a point upon the thread nearer to, or farther from, the axis of the cylinder.

(147.) Some Authors have deduced the properties of the mechanical powers *immediately* from the Third Law of Motion, contending that if the power and weight be such as would sustain each other, and the machine be put into motion, the momenta of the power and weight are equal; and consequently, that the power $\times$ the velocity of the power $=$ the weight

× the velocity of the weight; or *the power's velocity : the weight's velocity :: the weight : the power.*

Though this conclusion be just, the reasoning by which it is attempted to be proved is inadmissible, because the Third Law of Motion relates to the action of one body immediately upon another (Art. 36.). It may however be deduced from the foregoing Propositions; and as it is, in many cases, the simplest method of estimating the power of a machine, it may not be improper to establish it's truth.

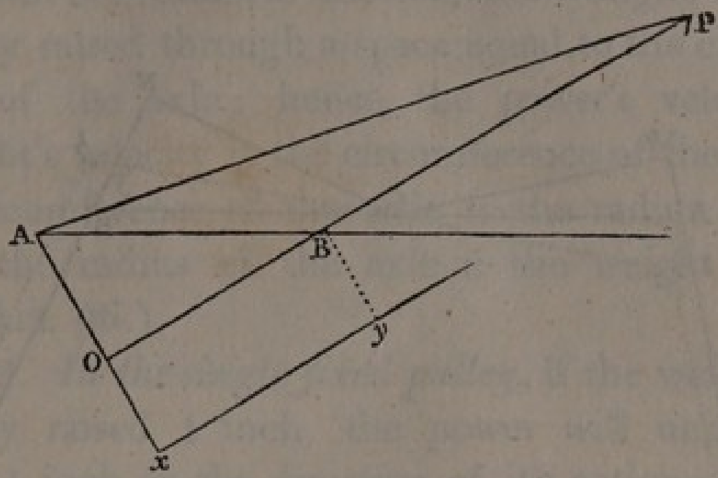
In the application of the rule, two things must be attended to: 1st, We must estimate the velocity of the power or weight in the direction in which it acts. 2dly, We must estimate that part only of the power or weight which is effective.

These two considerations are suggested by the Second Law of Motion, according to which motion is communicated in the *direction* of the force *impressed*, and is proportional to that force.

Prop. XXX.

(148.) *The velocity of a body in any one direction* AB *being given, to estimate it's velocity in any other direction* BP.

Suppose the motion of A to be produced by a force acting in the direction BP, by means of a string which passes over a pulley at P; produce PB to O, making $PO = PA$; join AO; then OB is the space which measures the approach of A to P. Now let the pulley be removed to such a distance that the angle at P may be considered as evanescent, and the power will always act in the same direction BP; also, the angles at A and O are equal; and they are right angles, because the three

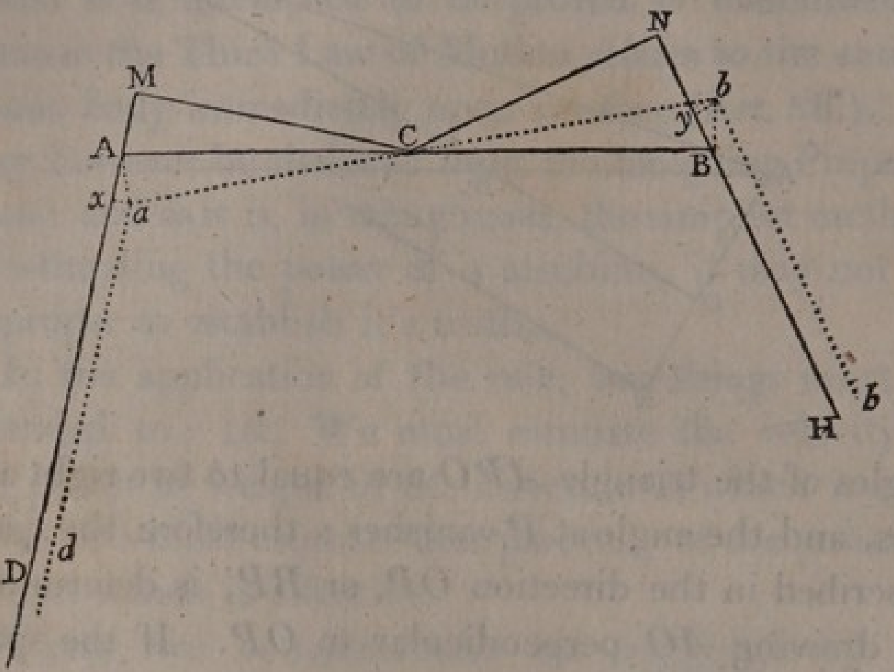


angles of the triangle APO are equal to two right angles, and the angle at P vanishes; therefore, the space described in the direction OP, or BP, is determined by drawing AO perpendicular to OP. If the space described in the direction xy, which is parallel to OP, be required, produce AO to x, and from B draw By at right angles to xy; then the figure $OByx$ is a parallelogram, and $OB = xy$ the space required. Also, if the motion in the direction AB be uniform, the motion in the direction BP, or xy, is uniform; since $AB : OB ::$ rad. : cos. ABO. Hence, the velocity in the direction AB : the velocity in the direction $BP :: AB : OB$ (Art. 11.).

Prop. XXXI.

(149.) *If a power and weight sustain each other on any machine, and the whole be put in motion, the velocity of the power : the velocity of the weight :: the weight : the power.*

Case 1. *In the lever ACB*, let a power and weight, acting in the directions *AD, BH*, sustain each other, and let the machine be moved uniformly round the



center *C*, through a very small angle ACa; Join Aa, Bb; draw *CM*, ax, at right angles to *MD*; and *CN*, by, at right angles to *NB*; then *A*'s velocity : *B*'s velocity :: $Ax : By$ (Art. 148.). Now the triangles Axa, *MCA*, are similar; because $\angle xAC = \angle AMC + \angle MCA$ (Euc. 32. i.) and $\angle aAC = \angle AMC$; therefore, $\angle xAa = \angle MCA$; and the angles at *M* and x are right angles; consequently, the remaining angles are equal; and

$$Ax : Aa :: CM : CA;$$

also, in the sim. $\triangle^s ACa, BCb, Aa : Bb :: CA : CB$; and in the sim. $\triangle^s Bby, BCN, Bb : By :: CB : CN$; by composition, $Ax : By :: CM : CN ::$ the weight : the power (Art. 81.); or the power's velocity : the weight's velocity :: the weight : the power.

Case 2. *In the wheel and axle*, if the power be made to describe a space equal to the circumference of the wheel with an uniform motion, the weight will be uniformly raised through a space equal to the circumference of the axle; hence, the power's velocity : the weight's velocity :: the circumference of the wheel : the circumference of the axle :: the radius of the wheel : the radius of the axle :: the weight : the power (Art. 96.).

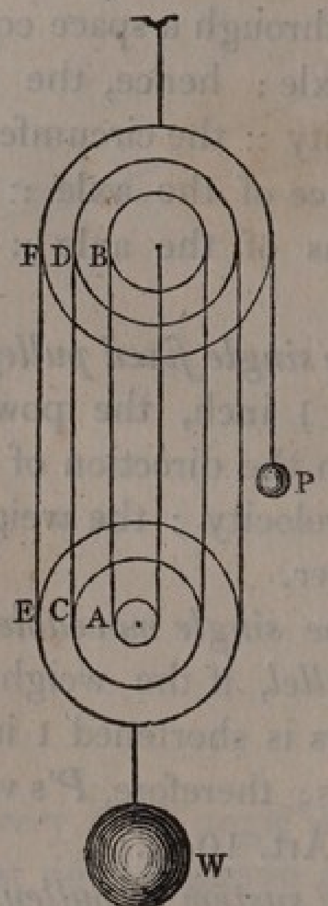
Case 3. *In the single fixed pulley*, if the weight be uniformly raised 1 inch, the power will uniformly describe 1 inch in the direction of it's action; therefore the power's velocity : the weight's velocity :: the weight : the power.

Case 4. *In the single moveable pulley where the strings are parallel*, if the weight be raised 1 inch, each of the strings is shortened 1 inch, and the power describes 2 inches; therefore, *P*'s velocity : *W*'s velocity :: *W* : *P* (Art. 104.).

Case 5. *In the system of pulleys described in Art.* 108, if the weight be raised 1 inch, each of the strings at the lower block is shortened 1 inch, and the power describes *n* inches; therefore, *P*'s velocity : *W*'s velocity :: *W* : *P*.

In this system of pulleys, whilst 1 inch of the string passes over the pulley *A*, 2 inches pass over the pulley *B*, 3 over *C*, 4 over *D*, &c.

Hence it follows, that if in the solid block *A*, the grooves *A*, *C*, *E*, &c. be cut, whose radii are 1, 3, 5, &c. and in the block *B*, the grooves *B*, *D*, *F*, &c. whose radii are 2, 4, 6, &c. and a string be passed round these grooves as in the annexed figure; the grooves will answer the purpose of so many distinct pulleys, and every point in each, moving with the velocity of the string in contact with it, the whole friction will be



removed to the two centers of motion in the blocks *A* and *B*.

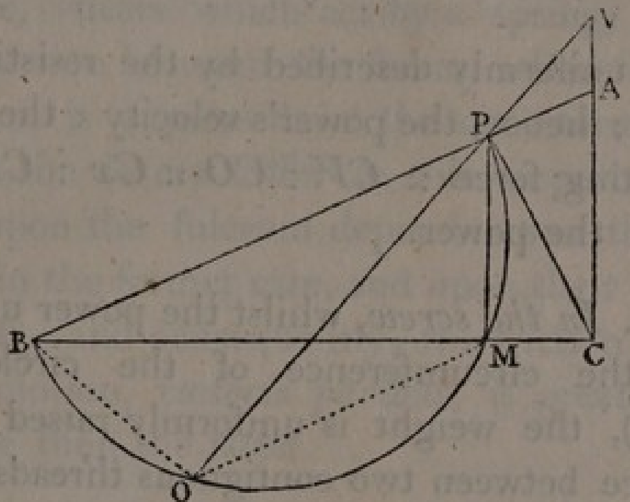
CASE 6. *In the system of pulleys described in Art.* 110, each succeeding pulley moves twice as fast as the preceding;

therefore, *W*'s velocity : *C*'s velocity :: 1 : 2
C's velocity : *B*'s velocity :: 1 : 2
B's velocity : *P*'s velocity :: 1 : 2
&c.

comp. *W*'s velocity : *P*'s velocity :: 1 : 2×2×2 ×&c. :: *P* : *W*.

CASE 7. *In the system, Art.* 113, if the weight be raised 1 inch, the pulley *B* will descend 1 inch, and the pulley *A* will descend 2+1 inches; in the same manner, the next pulley will descend $2 \times \overline{2+1}+1$ inches, or 4+2+1 inches; &c. therefore *P*'s velocity : *W*'s velocity :: 1+2+4+&c. : 1 :: *W* : *P*.

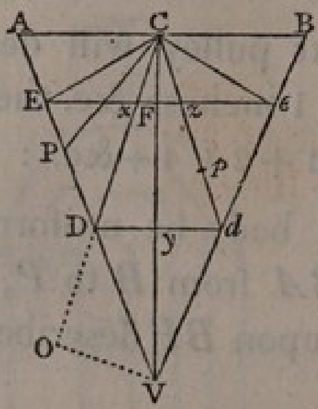
CASE 8. Let a body be uniformly raised along *the inclined plane BA* from *B* to *P*, by a power acting parallel to *PV*; upon *BP* describe a semi-circle *BOP*,



cutting *BC* in *M*; produce *VP* to *O*, join *BO, PM, MO.* Then since the angles *BOP, BMP*, in the semi-circle, are right angles, *OP* and *MP* are spaces uniformly described in the same time, by the power and weight in their respective directions (Art. 148.); also, because ∠*POM* = ∠*PBM* = ∠*PCV*, and ∠*OPM* = ∠*PVC* (Euc. 29. i.), the triangles *POM PVC* are similar, and *OP* : *MP* :: *VC* : *PV*, or the power's velocity : the weight's velocity :: the weight : the power, in the case of an equilibrium (Art. 118.)

CASE 9. *In the isosceles wedge, xC* is the only effective part of the resistance *DC* (see Art. 137.);

draw VO perpendicular to CD produced; then if the wedge be moved uniformly from C to V, CO is



the space uniformly described by the resisting force (Art. 148.); hence, the power's velocity : the velocity of the resisting force :: $CV : CO :: Cx : CF$ :: the resistance : the power.

Case 10. *In the screw,* whilst the power uniformly describes the circumference of the circle FGH (Art. 144.), the weight is uniformly raised through the distance between two contiguous threads; therefore P's velocity : W's velocity :: the circumference of the circle FGH : the distance between two threads :: $W : P$.

Case 11. *In any combination of the mechanical powers,* let $P : W$, $W : R$, $R : S$, &c. be the ratios of the power and weight in each case, when there is an equilibrium; then,

P's velocity : W's velocity :: $W : P$
W's velocity : R's velocity :: $R : W$
R's velocity : S's velocity :: $S : R$
&c.
comp. P's velocity : S's velocity :: $S : P$.

SCHOLIUM.

(150.) It has been usual to distinguish Levers into three kinds, according to the different situations of the power, weight, and center of motion; there are however only two kinds which essentially differ; those in which the forces act on *contrary* sides of the center of motion, as the common balance, steel-yard, &c. and those in which they act on the *same* side, as the stock-knife, shears which act by a spring, oars, &c. The proportion between the forces, when there is an equilibrium, is expressed in the same terms in each case; but the levers differ in this respect, that the pressure upon the fulcrum depends upon the sum of the forces in the former case, and upon their difference in the latter; and consequently, the friction upon the center of motion, *cæteris paribus,* is greater in the former case than the latter.

(151.) The pulley has, by some Writers, been referred to the lever, and they have justly deduced it's properties from the proportions which are found to obtain in that mechanical power; for, the adhesion of the pulley and the rope, which takes place at the circumference of the pulley, will overcome the friction at the center of motion; both because it acts at a mechanical advantage, and because the surface in contact is greater in the former case than in the latter; and the friction depends, not only upon the weight sustained, but also upon the quantity of surface in contact: Thus, in practice, the rope and pulley move on together, and the pulley becomes a lever.

(152.) The Wedge has hitherto chiefly been applied to the purposes of separating the parts of bodies, and it's power, notwithstanding the friction, is much greater than the theory leads us to expect; the reason is, the effect is produced by impact, and the momentum thus generated is incomparably greater than the effect of pressure, in the same time. Mr. ECKHARD, a very ingenious mechanic, by combining it with the wheel and axle, has constructed a machine, the power of which, considering it's simplicity, is much greater than that of any machine before invented.

SECTION V.

ON THE CENTER OF GRAVITY.

(153.) Def. The *Center of Gravity* of any body, or system of bodies, is that point upon which the body or system, acted upon only by the force of gravity, will balance itself in all positions*.

(154.) Hence it follows, that if a line or plane, which passes through the center of gravity, be supported, the body, or system, will be supported in all positions.

(155.) Conversely, if a body, or system, balance itself upon a line or plane, in all positions, the center of gravity is in that line or plane.

If not, let the line or plane be moved parallel to itself till it passes through the center of gravity, then we have increased both the quantity of matter on one side of the line or plane, and it's distance from the line or plane, and diminished both, on the other side; hence, if the body balanced itself in all positions in the former case, it cannot, from the nature of the

* That there is such a point in every body, or system of bodies, will be shewn hereafter.

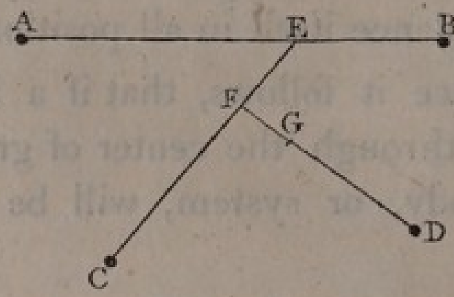
lever, balance itself in *all positions*, in the latter; consequently, the center of gravity is not in this line, or plane (Art. 154.), which is contrary to the supposition.

(156.) Cor. By reasoning in the same manner, it appears that a body, or system of bodies, cannot have more than one center of gravity.

Prop. XXXII.

(157.) *To find the center of gravity of any number of particles of matter.*

Let A, B, C, D, &c. be the particles; and suppose A, B, connected by the inflexible line AB without weight*; divide AB into two parts in E, so that



$A : B :: BE : EA$, or comp. $A+B : B :: AB : EA$; then will A and B balance each other upon E, or if E be supported, A and B will be supported in all positions (Art. 86.); let E be supported on the line CE, then are A and B supported in all positions; also the pressure upon the point E is equal to the sum of the weights A and B (Art. 70. Ax. 1.) Join EC, and take $A+B : C :: CF : FE$, or $A+B+C : C :: EC : FE$; then if F be supported, E and C will be supported, that is, A, B, and C, will be supported, in all

* The particles must be supposed to be connected, otherwise they could not act upon each other, so as to balance upon any point.

positions of the system; and the pressure upon F will be the sum of the weights, A, B, and C. In the same manner, join FD, and divide it into two parts in G, so that $A+B+C : D :: DG : FG$, or $A+B+C+D : D :: FD : FG$, and the system will balance itself in all positions upon G; that is, G is the center of gravity of the system.

(158.) Cor. 1. From this Proposition it appears that every body, or system of bodies, has a center of gravity.

(159.) Cor. 2. If the particles be supposed to be connected in any other manner, the same point G will be found to be their center of gravity (Art. 156.)

(160.) Cor. 3. The effect of any number of particles in a system, to produce or destroy an equilibrium, is the same, whether they are dispersed, or collected in their common center of gravity.

(161.) Cor. 4. If A, B, C, &c. be bodies of finite magnitudes, G, the center of gravity of the system, may be found by supposing the bodies collected in their respective centers of gravity.

(162.) Cor. 5. If the bodies A, B, C, &c. be increased or diminished in a given ratio, the same point G will be the center of gravity of the system. For the points E, F, G, depend upon the relative, and not upon the absolute weights of the bodies.

(163.) Cor. 6. If any forces, which are proportional to the weights, act in parallel directions at A, B, C, D, they will sustain each other upon the point G; and this point is still called the center of gravity, though the particles are not acted upon by the force of gravity.

(164.) Cor. 7. A force applied at the center of gravity of a body cannot produce a rotatory motion in it. For every particle resists, by it's inertia, the communication of motion, and in a direction opposite to that in which the force applied tends to communicate the motion; these resistances, therefore, of the particles, act in parallel directions, and they are proportional to the weights (Art. 25.); consequently, they will balance each other upon the center of gravity.

Prop. XXXIII.

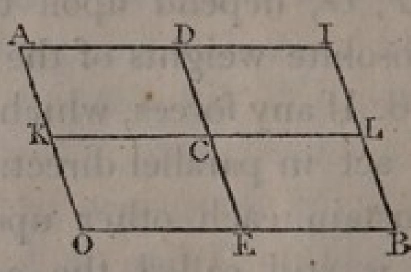
(165.) *To find the center of gravity of a right line**.

The center of gravity of a right line, composed of particles of matter which are equal to each other and uniformly dispersed, is it's middle point. For, there are equal weights on each side, equally distant from the middle point, which will sustain each other, in all positions, upon that point (Art. 86.)

Prop. XXXIV.

(166.) *To find the center of gravity of a parallelogram.*

Let *AB* be an uniform lamina of matter in the



form of a parallelogram; bisect *AO*, *AI*, in *K* and *D*;

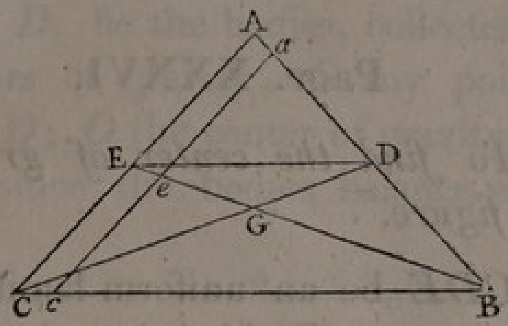
* When we speak of a line or plane as having a center of gravity, we suppose it to be made up of particles of matter, uniformly diffused over it.

draw *KL*, *DE*, respectively, parallel to *AI*, *AO*, cutting each other in *C*; this point *C* is the center of gravity of the figure. For if the parallelogram be supposed to be made up of lines parallel to *AI*, any one of these, as *KL*, is bisected by the line *DE* (since *AC*, *CI*, are parallelograms, and therefore, $KC = AD = DI = CL$); consequently, each line will balance itself upon *DE* (Art. 165.), or the whole figure will balance itself upon *DE*, in all positions; therefore, the center of gravity is in that line (Art. 155.) In the same manner it may be shewn that the center of gravity of the figure is in the line *KL*, consequently *C*, the intersection of the two lines, is the center of gravity required.

Prop. XXXV.

(167.) *To find the center of gravity of a triangle.*

Let *ABC* be an uniform lamina of matter in the form of a triangle; bisect *AB*, *AC*, in *D*, *E*; join



CD, *BE*, cutting each other in *G*, this point is the center of gravity of the triangle.

Suppose the triangle to be made up of lines parallel to *CA*, of which let *cea* be one; then since the triangles *BEC*, *Bec*, are similar,

$BE : EC :: Be : ec$; also, in the triangles BEA, Bea, $EA : BE :: ea : Be$;
by composition, $EA : EC :: ea : ec$; and $EA = EC$, therefore $ea = ec$; and consequently, the line ac will balance itself in all positions upon BE. For the same reason, every other line parallel to AC will balance itself, in all positions, upon BE, or the whole triangle will balance itself in all positions upon BE; therefore the center of gravity of the triangle is in that line. In the same manner it may be proved that the center of gravity is in the line CD; therefore it is in G, the intersection of the two lines BE, CD.

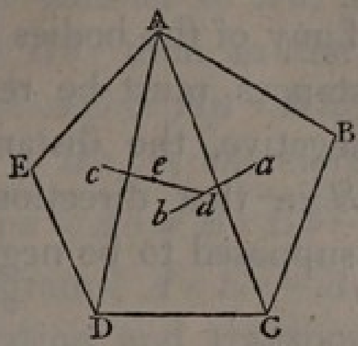
(168.) Cor. The distance of G from B is two-thirds of the line BE. Join ED; then since $AD = DB$, and $AE = EC$, ED is parallel to BC (Euc. 2. vi.); therefore, the triangles AED, ACB, are similar, and $CB : CA :: ED : EA$; alternately, $CB : ED :: CA : EA :: 2 : 1$. Also, the triangles CGB, EGD, are similar, therefore, $BG : CB :: GE : ED$; alternately, $BG : GE :: CB : ED :: 2 : 1$; hence, $BG : BE :: 2 : 3$.

Prop. XXXVI.

(169.) *To find the center of gravity of any rectilinear figure.*

Let $ABCDE$ be an uniform lamina of matter of the proposed figure. Divide it into the triangles ABC, ACD, ADE, whose centers of gravity a, b, c, may be found by the last Proposition; then if the triangles be collected in their respective centers of gravity (Art. 160.), their common center of gravity may be found as in Prop. 32.; that is, join ab and take

$db : ad :: $ the triangle ABC : the triangle ADC, and d is the center of gravity of the two triangles



ABC, ACD. Join dc, and take $ce : ed ::$ the sum of the triangles ABC, ACD : the triangle AED, and e is the center of gravity of the figure.

Prop. XXXVII.

(170.) *To find the center of gravity of any number of bodies placed in a straight line.*

Let A, B, C, D, be the bodies, collected in their respective centers of gravity; S any point in the straight line SAD; O the center of gravity of all the bodies. Then since the bodies balance each other

upon O, $A \times AO + B \times BO = C \times CO + D \times DO$ (See Art. 92.); that is, $A \times \overline{SO - SA} + B \times \overline{SO - SB}$ $= C \times \overline{SC - SO} + D \times \overline{SD - SO}$; hence, by mult. and transposition, $A \times SO + B \times SO + C \times SO + D \times$

$SO = A \times SA + B \times SB + C \times SC + D \times SD$; therefore,

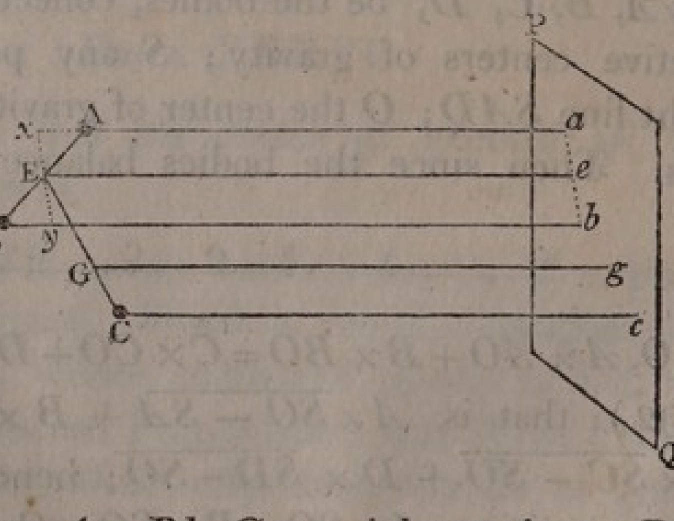
$$SO = \frac{A \times SA + B \times SB + C \times SC + D \times SD}{A + B + C + D}.$$

(171.) Cor. If any of the bodies lie the other way from S, their distances must be reckoned negative; and if SO be negative, the distance SO must be measured from S in that direction which, in the calculation, was supposed to be negative. (See *Alg.* Art. 472.)

Prop. XXXVIII.

(172.) *If perpendiculars be drawn from any number of bodies to a given plane, the sum of the products of each body, multiplied by it's perpendicular distance from the plane, is equal to the product of the sum of all the bodies multiplied by the perpendicular distance of their common center of gravity from the plane.*

Let A, B, C, &c. be the bodies, collected in their respective centers of gravity; PQ the given



plane; draw Aa, Bb, Cc, at right angles to PQ, and consequently, parallel to each other (Euc. 6. xi.); join AB, and take $AE : EB :: B : A$, then E is the center of gravity of A and B; through E draw Ee perpendicular to PQ, or parallel to Aa, and xEy perpendicular to Aa or Bb; then in the similar triangles AEx, EBy, $Ax : AE :: By : BE$, alternately, $Ax : By :: AE : BE :: B : A$; therefore $A \times Ax = B \times By$, that is, $A \times \overline{xa - Aa} = B \times \overline{Bb - yb}$, or since Ea, Eb, are parallelograms, $A \times \overline{Ee - Aa} = B \times \overline{Bb - Ee}$; and by multiplication and transposition, $A \times Ee + B \times Ee = A \times Aa + B \times Bb$, that is, $\overline{A+B} \times Ee = A \times Aa + B \times Bb$.

Again, join EC, and take $CG : GE :: A+B : C$, then G is the center of gravity of the bodies A, B, C; draw Gg perpendicular to PQ; and it may be shewn, as before, that $\overline{A+B} \times Ee + C \times Cc = \overline{A+B+C} \times Gg$, or $A \times Aa + B \times Bb + C \times Cc = \overline{A+B+C} \times Gg$. The same demonstration may be extended to any number of bodies.

(173.) Cor 1. Hence $Gg = \dfrac{A \times Aa + B \times Bb + C \times Ce}{A + B + C}$;

and if a plane be drawn parallel to PQ, and at the distance Gg from it, the center of gravity of the system lies somewhere in this plane. In the same manner two other planes may be found, in each of which the center of gravity lies, and the point where the three planes cut each other, is the center of gravity of the system.

(174.) Cor. 2. If any of the bodies lie on the other side of the plane, their distances must be reckoned negative.

(175.) Cor. 3. Wherever the bodies are situated, if their respective perpendicular distances from the plane remain the same, the distance of their common center of gravity from the plane will remain the same.

(176.) Cor. 4. Let the bodies lie in the same plane, and let perpendiculars, Aa, Bb, Cc, Gg, be drawn to any given *line* in that plane, then

$$Gg = \frac{A \times Aa + B \times Bb + C \times Cc}{A + B + C}.$$

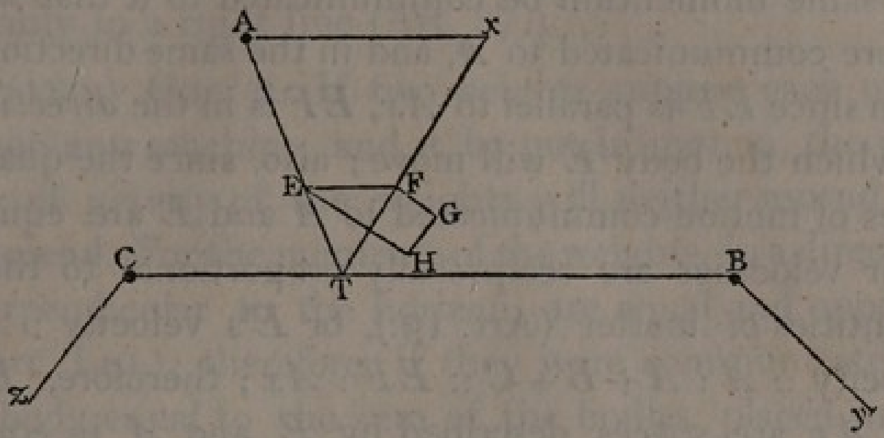
(177.) Cor. 5. If A and B be on one side of the plane, and C on the other, and the plane pass through the center of gravity, then $A \times Aa + B \times Bb = C \times Cc$. For $Gg \times \overline{A+B+C} = A \times Aa + B \times Bb - C \times Cc$, and $Gg=0$, therefore $A \times Aa + B \times Bb - C \times Cc = 0$; or $A \times Aa + B \times Bb = C \times Cc$.

Prop. XXXIX.

(178.) *If any momenta be communicated to the parts of a system, it's center of gravity will move in the same manner that a body, equal to the sum of the bodies in the system, would move, were it placed in that center, and the same momenta, in the same directions, communicated to it.*

Let A, B, C, be the bodies in the system, and the points A, B, C, their respective centers of gravity; join BC, and take $BT : TC :: C : B$; join AT, and take $TE : EA :: A : B+C$, or $TE : TA :: A : A+B+C$, then will E be the center of gravity of the system (Art. 161.)

Suppose the momentum communicated to A would cause it to move from A to x in T'', and at x let the body be stopped; join Tx, and take $TF : Tx :: A : A+B+C$, then F is the center of gravity of the bodies when they are at x, B, C; join EF, and since $TE : TA :: A : A+B+C :: TF : Tx$, EF is parallel to Ax (Euc. 2. vi.), and consequently the triangles TEF,



TAx, are similar; therefore $EF : Ax :: A : A+B+C$.

Hence if one body A in the system be moved from A to x, the center of gravity is moved from E to F; which point may be thus determined; draw EF parallel to Ax, and take $EF : Ax :: A : A+B+C$.

Next let a momentum be communicated to B, which would cause it to move from B to y in T''; at y let the body be stopped; then, according to the rule above laid down, draw FG parallel to By, and take $FG : By :: B : A+B+C$, and G will be the center of gravity of the bodies when they are at x, y, C. In the same manner, let a momentum be communicated to C, which would cause it to move from C to z in T'', and at z let the body be stopped; draw GH parallel to Cz, and take $GH : Cz :: C : A+B+C$, then H is the center of gravity of the bodies when they are at x, y, z. If now the momenta, instead of being communicated separately, be communicated *at the same instant* to the bodies, at the end of T'' they will be found in x, y, z, respectively; therefore, at the end of T'', their common center of gravity will be in H.

Now let E be a body equal to $A+B+C$, and let the same momentum be communicated to it that was before communicated to A, and in the same direction; then since EF is parallel to Ax, EF is in the *direction* in which the body E will move; also, since the quantities of motion communicated to A and E are equal, their velocities are reciprocally proportional to their quantities of matter (Art. 19.), or E's velocity : A's velocity :: $A : A+B+C$ :: $EF : Ax$; therefore, EF and Ax are spaces described by E and A in equal times (Art. 11.), or E will describe the space EF in T''. In the same manner FG is the space which the body E will describe in T'', if the momentum, before communicated to B, be communicated to it; and GH the space it will describe in T'', if the momentum before communicated to C, be communicated to it; join EH; and when the motions are communicated *at the same instant* to E, it will describe EH in T'' (Art. 42.) Hence it follows, that when the same momenta are communicated to the parts of a system, and to a body, equal to the sum of the bodies, placed in the common center of gravity, this body and the center of gravity are in the same point at the end of T''; and T may represent any time; therefore, they are always in the same point.

The same demonstration may be applied, whatever be the number of bodies in the system.

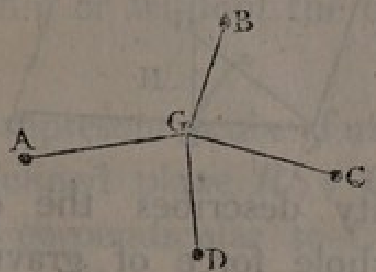
(179.) Cor. 1. If the parts of a system move uniformly in right lines, the center of gravity will either remain at rest, or move uniformly in a right line. For if the momenta communicated to the several parts of the system be communicated to a body, equal to the sum of the bodies, placed in the center of gravity of the system, it will either remain at rest or move uniformly in a right line (Art. 27.)

(180.) Cor. 2. If two weights support each other upon any machine, and it be put in motion, the center of gravity of the weights will neither ascend nor descend. For the momenta of the weights, in a direction perpendicular to the horizon, are equal and opposite (Art. 149.); therefore, if they were communicated to a body equal to the sum of the bodies, placed in the common center of gravity, they would neither cause it to ascend nor descend.

(181.) Cor. 3. The motion or quiescence of the center of gravity is not affected by the mutual action of the parts of a system upon each other. For action and reaction are equal and in opposite directions, and equal and opposite momenta communicated to a body, equal to the sum of the bodies in the system, will not disturb it's motion or quiescence.

(182.) Cor. 4. The effect of any force to communicate motion to the common center of gravity, is the same, upon whatever body in the system it acts.

(183.) Cor. 5. If G be the center of gravity of the



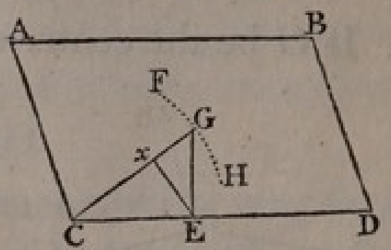
particles of matter, A, B, C, D, which are acted upon only by their mutual attractions, they will meet at *G*. For they must meet, and their common center of gravity will remain at rest (Art. 181.); therefore, they must meet at that center.

(184.) Cor. 6. If a rotatory motion be communicated to a body, and it be then left to move freely, the axis of rotation will pass through the center of gravity. For the center of gravity itself, either remaining at rest or moving uniformly forward in a right line, has no rotation.

Prop. XL.

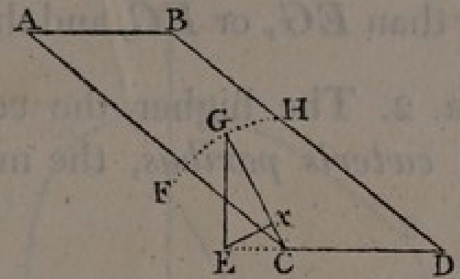
(185.) *If a body be placed upon an horizontal plane, and a line drawn from it's center of gravity perpendicular to that plane, the body will be sustained, or not, according as the perpendicular falls within or without it's base.*

Let *ABDC* represent the body, *G* it's center of gravity; draw *GE* perpendicular to the horizon; join *CG*, and with the radius *CG* describe the circular arc *HGF*; then the body cannot fall over at *C*, unless the



center of gravity describes the circular arc *GF*. Suppose the whole force of gravity applied at *G* (Art. 178.), and take *GE* to represent it; draw *Ex* perpendicular to CG; then the force GE is equivalent to the two Gx, xE, of which Gx cannot move the body either in the direction GF or GH; and when E falls within the base, xE acts at G in the direction GH; therefore the center of gravity cannot describe the arc GF, that is, the body cannot fall over at C. In the same manner it may be shewn that it cannot fall over at D.

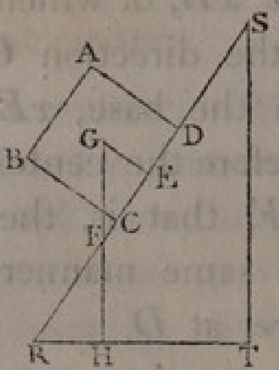
When the perpendicular GE falls without the base, xE acts in the direction GF, and since there is no



force to counteract this, the center of gravity will move in that direction, or the body will fall.

(186.) Cor. 1. In the same manner it may be shewn, that if a body be placed upon an inclined plane, and the lateral motion be prevented by friction, the body will be sustained or not, according as the perpendicular to the horizon, drawn through it's center of gravity, falls within or without the base.

Ex. Let $ABCD$ represent a cube of uniform density, placed upon the inclined plane RS; G it's center of gravity; draw GE perpendicular to CD, and GFH perpendicular to the horizon; then this body will not be sustained upon the inclined plane, if the angle of the plane's inclination SRT, exceed half a right angle. For if the ∠ *FRH* be greater than half a right angle, the



∠ *RFH*, or *GFE*, is less than half a right angle, and the ∠ *FGE* is greater than half a right angle; therefore, *EF* is greater than *EG*, or *EC*, and the body will roll.

(187.) Cor. 2. The higher the center of gravity of a body is, *cæteris paribus*, the more easily it is overturned.

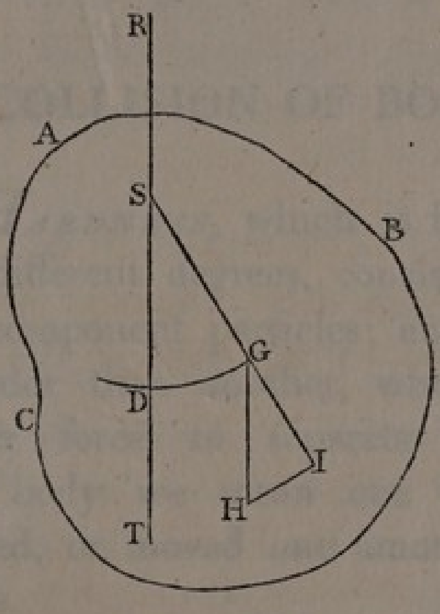
The same construction being made as in the Proposition, the whole weight of the body : that part of the weight which keeps it steady upon it's base, or opposes any power employed to overturn it :: *GE* : *xE* :: *GC* : *CE*; and when *CE* and the whole weight of the body are given, the force which keeps the body steady $\propto \frac{1}{GC}$ (*Alg.* Art. 206.); therefore as *GC* increases, that is, as *GE* increases, the force which keeps the body steady decreases, or the more easily will the body be overturned.

(188.) Cor. 3. When *CE* vanishes with respect to *GC*, the force which keeps the body steady vanishes, and the body may be overturned by a very small force. Thus it is extremely difficult to balance a body upon a *point* placed *under* the center of gravity.

Prop. XLI.

(189.) *If a body be suspended by any point, it will not remain at rest till the center of gravity is in the line which is drawn through that point, perpendicular to the horizon.*

Let S be the point of suspension of the body ABC;



G it's center of gravity; join SG and produce it; through S, and G, draw RST, and GH, perpendiculars to the horizon; then the effort of gravity, to put the body in motion, is the same that it would be, were all the particles collected at G; take GH to represent the force in that direction, and draw HI perpendicular to GI; then the force GH is equivalent to the two GI, IH, of which GI is sustained by the reaction at the point of suspension S, and IH is employed in moving the center of gravity in a direction perpendicular to SG; therefore the center of gravity cannot remain at rest till IH vanishes; that is, till the angle IGH, or GST, or GSR, vanishes, or SG coincides with RT.

(190.) Cor. Hence it follows, that if a body be suspended successively by different points, and perpendiculars to the horizon be drawn through the point of suspension, and passing through the body, the center of gravity will lie in each of these perpendiculars, and consequently, in the point of their intersection.

SECTION VI.

ON THE COLLISION OF BODIES.

(191.) DEF. *HARDNESS*, which is found in different bodies in different degrees, consists in a firm cohesion of the component particles; and that body is said to be harder than another, whose particles require a greater force to separate them. By a *perfectly hard* body we mean one whose parts cannot be separated, or moved one amongst another by any finite force.

(192.) DEF. The tendency in a body to recover it's former figure, after having been compressed, is called *elasticity*. That body is said to be more elastic than another, which recovers it's figure with the greater force, supposing the compressing force the same. By a *perfectly elastic* body we mean one which recovers it's figure with a force equal to that which was employed in compressing it.

That such a tendency exists in bodies is evident from a variety of experiments. If an ivory ball, stained with ink, be brought gently into contact with an unstained ball, the spot received by the latter will be very small, since two spheres touch each other only in a single point; but if one of the balls be made to impinge upon the other, the spot will be enlarged; and the greater the force of impact, the greater will be the surface stained; hence it is manifest, that one, or both of the balls, has been compressed, and afterwards recovered it's spherical figure. Almost all bodies with which we are acquainted are elastic in a greater or less degree; but none perfectly so. In steel balls, the force of elasticity is to the compressing force as 5 to 9; in glass, as 15 to 16; though in all cases, the force of elasticity seems to depend, in some measure, upon the diameter of the ball.

(193.) Def. The impact of two bodies is said to be *direct*, when their centers of gravity move in the right line which passes through the point of impact.

In considering the effects of collision, the bodies are usually supposed to be spheres of uniform density; and in their actions upon each other, not to be affected by gravity, or any other force but that of inertia.

Prop. XLII.

(194.) *If the impact of two perfectly hard bodies be direct, after impact they will either remain at rest, or move on, uniformly, together.*

Since there is no force to turn either body out of the line of direction, they will continue in that line after impact*. Let A and B be the two bodies, moving in the *same* direction, and let A overtake B; then will A continue to accelerate B's motion, and B will continue to retard A's, till their velocities are equal, at which time they will cease to act upon each other; and since there is no force to separate them, they will move on together, and their common velocity, by the First Law of Motion, will be uniform. When they move in *opposite* directions, if their forces be equal, they will rest after impact; if A's force be greater than B's, the whole velocity of B will be destroyed, and A's not being destroyed, A will communicate velocity to B, and B by it's reaction will retard A, till they move on together, as in the former case.

Prop. XLIII.

(195.) *If the impact of two perfectly hard bodies be direct, their common velocity may be found by dividing the whole momentum before impact, estimated in the direction of either motion, by the sum of the quantities of matter.*

Let A and B be the quantities of matter contained in the bodies, a and b their velocities; then, when they move in the same direction, $Aa+Bb$ is the whole momentum in that direction, before impact. When they move in opposite directions, $Aa-Bb$ is the whole momentum estimated in the direction in which A moves.

* The momenta of the particles in each body are proportional to their weights, since their velocities are equal; these momenta, therefore, will not turn the body to either side of the line passing through the center of gravity (Art. 163.)

In the former case, as much as Aa, the momentum of A, is diminished, so much is Bb, the momentum of B, increased by the impact (Art. 32.); therefore $Aa + Bb$ is equal to the whole momentum after impact.

In the latter case, if Aa be greater than Bb, before the bodies can begin to move together, Bb, the momentum of B, must be destroyed; and therefore A's momentum must be diminished by the quantity Bb (Art. 32.) Thus, when the bodies begin to move in the same direction, $Aa - Bb$ is their whole momentum; and as much momentum as is afterwards communicated to B, so much is lost by A; therefore $Aa - Bb$ is equal to the whole momentum after impact.

If Aa be less than Bb, the momentum after impact, in the direction of B's motion, will be $Bb - Aa$; or, in the direction of A's motion, $Aa - Bb$.

Let v be the common velocity after impact; then $\overline{A+B} \times v$ is the whole momentum; consequently, $\overline{A+B} \times v = Aa \pm Bb$, and $v = \frac{Aa \pm Bb}{A+B}$. In which expression, the positive sign is to be used when the bodies move in the same direction before impact, and the negative sign, when they move in *opposite* directions.

(196.) Cor. 1. When the bodies move in opposite directions with equal momenta, they will remain at rest after impact. In this case $Aa - Bb = 0$; therefore $v = 0$.

(197.) Cor. 2. If Bb be greater than Aa, v is negative. This shews that the bodies will move in the directions of B's motion, which was supposed, in the proposition, to be negative.

Prop. XLIV.

(198.) *In the direct impact of two perfectly hard bodies* A *and* B, *estimating the effects in the direction of* A's *motion*, A + B : A :: *the relative velocity of the two bodies : the velocity gained by* B. *And* A + B : B :: *their relative velocity : the velocity lost by* A.

The same notation being retained; when the bodies move in the same direction, $a-b$ is their relative velocity (Art. 12.); and v, their common velocity after impact, is $\frac{Aa+Bb}{A+B}$ (Art. 195.); therefore, the velocity *gained* by B, or $v-b$, is $\frac{Aa+Bb}{A+B}-b$, or $\frac{Aa-Ab}{A+B}$; hence, $A+B : A :: a-b$: the velocity gained by B. Also, $a-\frac{Aa+Bb}{A+B}$, or $\frac{Ba-Bb}{A+B}$ is the velocity *lost* by A; therefore $A+B : B :: a-b$: the velocity lost by A.

When the bodies move in opposite directions, $a+b$ is their relative velocity (Art. 12.); and $v=\frac{Aa-Bb}{A+B}$ (Art. 195.); also, the velocity communicated to B upon the whole, in the direction of A's motion, is $v+b$, or $\frac{Aa-Bb}{A+B}+b$, that is, $\frac{Aa+Ab}{A+B}$; therefore, $A+B : A :: a+b$: the velocity gained by B.

The velocity lost by A is $a-\frac{Aa-Bb}{A+B}$, or $\frac{Ba+Bb}{A+B}$; therefore, $A+B : B :: a+b$: the velocity lost by A.

Ex. Let the weights of A and B be 10 and 6*; their velocities 12 and 8, respectively; then, when they move in the same direction, $10+6 : 10 :: 12-8 : \frac{40}{16}=2\frac{1}{2}$, the velocity gained by B; and $10+6 : 6 :: 12-8 : \frac{24}{16}=1\frac{1}{2}$, the velocity lost by A.

When they move in opposite directions, $12+8$ is their relative velocity; and $10+6 : 10 :: 12+8 : \frac{200}{16}=12\frac{1}{2}$, the velocity gained by B in the direction of A's motion. Also, since it had a velocity 8 in the opposite direction before impact, it's velocity after impact is $4\frac{1}{2}$ in the direction of A's motion. Again, $10+6 : 6 :: 12+8 : \frac{120}{16}=7\frac{1}{2}$, the velocity lost by A.

(199.) Cor. 1. Whilst the relative velocity remains the same, the velocity gained by B, and the velocity lost by A, are unaltered.

(200.) Cor. 2. Hence it also follows that the velocities, gained by B, and lost by A, are the same, whether both bodies are in motion, or A impinges upon B at rest, with a velocity equal to their relative velocity in the former case.

(201.) Cor. 3. If the relative velocity be the same, the momentum communicated is the same, whether A impinges upon B, or B upon A.

Call r the relative velocity; then when A impinges upon B, $A+B : A :: r : \frac{Ar}{A+B}$, the velocity gained

* See Art. 26.

by B; therefore $\frac{ABr}{A+B}$ is the momentum gained by B. When B impinges upon A, $A+B : B :: r : \frac{Br}{A+B}$, the velocity gained by A; therefore $\frac{ABr}{A+B}$ is the momentum gained by A; which is also the momentum gained by B on the former supposition.

Prop. XLV.

(202.) *When the bodies are perfectly elastic, the velocity gained by the body struck, and the velocity lost by the striking body, will be twice as great as if the bodies were perfectly hard.*

Let A and B be the bodies; then, as in Art. 194, A will accelerate B's motion, and B will retard A's, till their velocities are equal; and if they were perfectly hard they would then cease to act upon each other, and move on together; thus, during the first part of the collision, the same effect is produced, that is, the same velocity is gained and lost, as if the bodies were perfectly hard. But, during this period, the bodies are compressed by the stroke, and since they are, by the supposition, perfectly elastic, the force with which each will recover it's former shape is equal to that with which it was compressed; therefore, each body will receive another impulse from the elasticity equal to the former, or B will gain, and A lose, upon the whole, twice as great a velocity as if both bodies had been perfectly hard.

(203.) The same demonstration may be applied to the case where one body is perfectly hard, and the other perfectly elastic.

Prop. XLVI.

(204.) *In the direct impact of two perfectly elastic bodies* A *and* B, A+B : 2A :: *their relative velocity before impact : the velocity gained by* B *in the direction of* A's *motion; and* A+B : 2B :: *their relative velocity : the velocity lost by* A, *in that direction.*

Call r the relative velocity of the bodies; x the velocity gained by B, and y the velocity lost by A, when both bodies are perfectly hard; then $2x$ is the velocity gained by B, and $2y$ the velocity lost by A, when they are perfectly elastic; and

$A+B : A :: r : x$ (Art. 198.); therefore,
$A+B : 2A :: r : 2x$ (*Alg.* Art. 185.), the velocity gained by B.

Again, $A+B : B :: r : y$ (Art. 198.); therefore,
$A+B : 2B :: r : 2y$, the velocity lost by A.

Ex. Let the weights of the bodies be 5 and 4, their velocities 7 and 5; then, when they move in the same direction, $5+4 : 10 :: 7-5 : \frac{20}{9} = 2\frac{2}{9}$, the velocity gained by B; therefore $5+2\frac{2}{9}$, or $7\frac{2}{9}$ is B's velocity after impact. Also, $5+4 : 8 :: 7-5 : \frac{16}{9} = 1\frac{7}{9}$, the velocity lost by A; therefore $7-1\frac{7}{9}$, or $5\frac{2}{9}$, is A's velocity after impact. When they move in opposite directions, $5+4:10::7+5:\frac{120}{9}=13\frac{1}{3}$, the velocity gained by *B*. Also, since it had a velocity 5 in the opposite direction, it's velocity after impact, in the direction of *A*'s motion, is $13\frac{1}{3}-5$, or $8\frac{1}{3}$. Again, $5+4:8::7+5:\frac{96}{9}=10\frac{2}{3}$ *A*'s velocity lost; and since it had a velocity 7 before impact, after impact it will move in the opposite direction with a velocity $3\frac{2}{3}$.

(205.) Cor. 1. When $A=B$, the bodies interchange velocities. For, in this case, $A+B=2A=2B$; therefore, the velocity gained by *B*, and the velocity lost by *A*, are respectively equal to their relative velocity before impact. Let a and b be their velocities before impact; then, when they move in the same direction, $a-b$ is the velocity gained by *B*, or lost by *A*; therefore $a-b+b$, or a, is *B*'s velocity after impact; and $a-\overline{a-b}$, or b, is *A*'s velocity. If b be negative, or the bodies move in opposite directions, $a+b-b$, or a, is *B*'s velocity, and $a-\overline{a+b}$, or $-b$, is *A*'s velocity after impact.

(206.) Cor. 2. If the bodies move in opposite directions with equal quantities of motion, the whole momentum of each will be destroyed during the compression, and an equal one generated by elasticity in the opposite direction; each body will therefore be reflected with a velocity equal to that which it had before impact.

(207.) Cor. 3. The relative velocity of the bodies after impact is equal to their relative velocity before impact.

Let a and b be the velocities of the bodies before impact; p and q their velocities after; then $a-b=q-p$. For, $A+B : 2A :: a-b : \frac{2A\times\overline{a-b}}{A+B}$, the velocity gained by B; therefore $q=b+\frac{2A\times\overline{a-b}}{A+B}$. Also, $A+B : 2B ::$ $a-b : \frac{2B\times\overline{a-b}}{A+B}$, the velocity lost by A; therefore $p=$ $a-\frac{2B\times\overline{a-b}}{A+B}$; and $q-p=b-a+\frac{\overline{2A+2B}\times\overline{a-b}}{A+B}$ $=b-a+2a-2b=a-b$. When the bodies A and B move in opposite directions, the sign of b is negative; in other respects the demonstration is the same.

(208.) Cor. 4. The sum of the products of each body, multiplied by the square of it's velocity, is the same before and after impact.

The notation in the last Article being retained; $Aa+Bb=Ap+Bq$ (Art. 34.); by transposition, $Aa-Ap=Bq-Bb$; or $A\times\overline{a-p}=B\times\overline{q-b}$. Also $a-b=q-p$ (Art. 207.); or $a+p=q+b$; therefore $A\times\overline{a-p}\times\overline{a+p}=B\times\overline{q-b}\times\overline{q+b}$; or $Aa^2-Ap^2=Bq^2-Bb^2$; therefore $Aa^2+Bb^2=Ap^2+Bq^2$. If any of the quantities, b, p, q, be negative, it's square will be positive, and therefore the conclusion will not be altered.

(209.) Cor. 5. If there be a row of equal elastic bodies, A, B, C, D, &c. at rest, and a motion be communicated to A, and thence to B, C, D, &c. they will all remain at rest after the impact, except the last, which will move off with a velocity equal to that with which the first moved.

For A and B will interchange velocities (Art. 205); that is, A will remain at rest, and B move on with A's velocity. In the same manner it may be shewn that all the others will remain at rest after impact, except the last, which will move off with the velocity communicated to A.

(210.) Cor. 6. If the bodies decrease in magnitude, they will all move in the direction of the first motion, and the velocity communicated to each succeeding body will be greater than that which was communicated to the preceding.

For, $A+B : 2B ::$ A's velocity before impact : the velocity lost by A; and since $2B$ is less than $A+B$, A does not lose it's whole velocity; therefore it will move on after impact in the direction of the first motion. Also, $A+B : 2A ::$ A's velocity before impact : the velocity gained by B; and since $2A$ is greater than $A+B$, the velocity gained by B is greater than A's velocity before impact. In the same manner it may be shewn that B, C, D, &c. will move on in the direction of the first motion; and that the velocity communicated to each will be greater than that which was communicated to the preceding body.

(211.) Cor. 7. If the bodies increase in magnitude, they will all be reflected back, except the last, and the velocity communicated to each succeeding body will be less than that which was communicated to the preceding.

For, in this case, $2B$ is greater than $A+B$; therefore, A loses more than it's whole velocity, or it will move in the contrary direction. Also, $2A$ is less than $A+B$; therefore, the velocity gained by B is less than A's velocity before impact. In the same manner it may be shewn that B, C, D, &c. will be reflected;

and that the velocity communicated to each will be less than that which was communicated to the preceding body.

(212.) Cor. 8. The velocity thus communicated from A through B to C, when B is greater than one of the two A, C, and less than the other, exceeds the velocity which would be communicated immediately from A to C.

Let a represent A's velocity; then

$A+B : 2A :: a : \frac{2Aa}{A+B}$, the velocity of B; and

$B+C : 2B :: \frac{2Aa}{A+B} : \frac{2Aa}{A+B} \times \frac{2B}{B+C}$, the velocity communicated from B to C.

Again, $A+C : 2A :: a : \frac{2Aa}{A+C}$, the velocity communicated immediately from A to C. Hence it follows, that the velocity communicated to C, by means of B, is greater than that which would be communicated to it immediately, if $\frac{2Aa}{A+B} \times \frac{2B}{B+C}$ be greater than $\frac{2Aa}{A+C}$; that is, if $A+C$ be greater than $\frac{\overline{A+B} \times \overline{B+C}}{2B}$, or $2A+2C$ greater than $A+C+B+\frac{AC}{B}$; or $A+C$ greater than $B+\frac{AC}{B}$. Suppose $A=B+x$, $C=B+y$; then $A+C=2B+x+y$, and $B+\frac{AC}{B} = B + \frac{B^2+Bx+By+xy}{B} = 2B+x+y+\frac{xy}{B}$; therefore, the velocity communicated to C by means of B, is greater than the velocity communicated to it without B, if $2B+x+y$ be greater than $2B+x+y+\frac{xy}{B}$, which will always be the case when xy is negative, or when x and y have different signs; that is, when B is less than one of the bodies, A, C, and greater than the other*.

(213.) Cor. 9. If the bodies be in geometrical progression, the velocities communicated to them will be in geometrical progression; and when there are n such bodies, whose common ratio is r, the velocity of the first : the velocity of the last :: $\overline{1+r}|^{n-1} : 2^{n-1}$.

Let A, Ar, Ar^2, Ar^3, &c. be the bodies; a, b, c, d, &c. the velocities successively communicated to them; then

$A + Ar : 2A :: a : b$, or

$1 + r : 2 :: a : b$; and in the same manner,

$1 + r : 2 :: b : c$

$1 + r : 2 :: c : d$, &c.

therefore $a : b :: b : c :: c : d$ &c. Also, by composition, $\overline{1+r}|^{n-1} : 2^{n-1} :: a$: the velocity of the last.

(214.) Cor. 10. If the number of mean proportionals, interposed between two given bodies A and X, be increased without limit, the ratio of A's velocity to the velocity thus communicated to X will approximate to the ratio of $\sqrt{X} : \sqrt{A}$ as it's limit.

Let A, B, C, D, ... X be the bodies; a, b, c, d, ... x the velocities communicated to them. Then since the number of bodies interposed between

* The velocity communicated from A through B to C, is a maximum when A, B, and C, are in geometrical progression. (*Flux.* Art. 21. Ex. 11.)

A and X is increased without limit, their differences will be diminished without limit; let $A+z=B$; then

$$2A+z : 2A :: a : b$$

$$\text{or } A+\frac{z}{2} : A :: a : b$$

$$\text{*and } A+\frac{z}{2} : A :: \sqrt{A+z} : \sqrt{A} :: \sqrt{B} : \sqrt{A};$$

$$\text{therefore, } \sqrt{B} : \sqrt{A} :: a : b$$

$$\text{in the same manner, } \sqrt{C} : \sqrt{B} :: b : c$$

$$\sqrt{D} : \sqrt{C} :: c : d$$

&c.

$$\text{comp. } \sqrt{X} : \sqrt{A} :: a : x.$$

Cor. The conclusion is the same when the intermediate bodies vary according to any other law, if the difference of the succeeding bodies, in every part of the series, be evanescent.

Prop. XLVII.

(215.) *In the direct impact of two perfectly elastic bodies* A *and* B, *if the compressing force be to the force of elasticity* :: 1 : m, *then* $A+B : \overline{1+m}\times A$ :: *their relative velocity before impact : the velocity gained by* B *in the direction of* A's *motion. And* $A+B : \overline{1+m}\times B$ :: *their relative velocity before impact : the velocity lost by* A, *in that direction.*

By reasoning, as in Art. 202, it appears that the

* Since $\overline{A+\frac{z}{2}}\Big|^2 : A^2 :: A^2+Az+\frac{z^2}{4} : A^2 :: A+z+\frac{z^2}{4A} : A :: B+\frac{z^2}{4A} : A$, the ratio of $\overline{A+\frac{z}{2}}\Big|^2 : A^2$, when z is continually diminished, approximates to the ratio of $B : A$, and consequently, the ratio of $A+\frac{z}{2} : A$ approximates to the ratio of $\sqrt{B} : \sqrt{A}$ as it's limit.

velocity gained by B, and the velocity lost by A during the compression, are the same as if the bodies were perfectly hard; and the velocity communicated by the elasticity is to the velocity communicated by the compression :: $m : 1$. Call r the relative velocity before impact, x the velocity gained by B, and y the velocity lost by A, during the compression; then $\overline{1+m} \times x$ is the velocity gained by B, and $\overline{1+m} \times y$ the velocity lost by A, upon the whole. Now

$$A+B : A :: r : x \text{ (Art. 198.)},$$

$$\text{and } A+B : B :: r : y;$$

therefore, $A+B : \overline{1+m} \times A :: r : \overline{1+m} \times x$, the velocity gained by B;

and $A+B : \overline{1+m} \times B :: r : \overline{1+m} \times y$, the velocity lost by A.

(216.) Cor. 1. The relative velocity before impact : the relative velocity after impact :: $1 : m$.

Let a and b be the velocities of the two bodies before impact, p and q their velocities after; then

$A+B : \overline{1+m} \times A :: a-b : \dfrac{\overline{1+m} \times A \times \overline{a-b}}{A+B}$, the velocity gained by B;

$$\text{therefore, } q = b + \frac{\overline{1+m} \times A \times \overline{a-b}}{A+B};$$

$$\text{in the same manner, } p = a - \frac{\overline{1+m} \times B \times \overline{a-b}}{A+B};$$

$$\text{hence, } q-p = b-a+\frac{\overline{1+m} \times \overline{A+B} \times \overline{a-b}}{A+B}, \text{ or}$$

$b-a+a-b+m \times \overline{a-b}$, i. e. $m \times \overline{a-b}$, is the relative velocity after impact; and $a-b : m \times \overline{a-b} :: 1 : m$.

When the bodies move in opposite directions, the sign of b is negative.

(217.) Cor. 2. Hence it appears that if the velocities of the bodies before and after impact be known, the elastic force is known.

(218.) Cor. 3. If A impinge upon B at rest, A will remain at rest after impact when $A : B :: m : 1$.

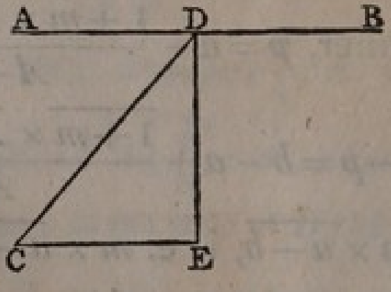
In this case A loses it's whole velocity, and $A+B : \overline{1+m} \times B :: a :$ the velocity lost by A; therefore $A+B=\overline{1+m} \times B$, and $A=mB$; consequently, $A : B :: m : 1$.

(219.) Cor. 4. The momentum communicated is the same, whether A impinges upon B, or B upon A, if the relative velocity be the same. This is the case when the bodies are perfectly hard (Art. 201.); and the effect produced in elastic bodies is in a given ratio to that which is produced when the bodies are perfectly hard.

Prop. XLVIII.

(220.) *When a perfectly hard body impinges obliquely on a perfectly hard and immoveable plane* AB, *in the direction* CD, *after impact it will move along the plane, and the velocity before impact : the velocity after :: radius : the cosine of the angle* CDA.

Take CD to represent the motion of the body before



impact; draw CE parallel, and DE perpendicular to AB.

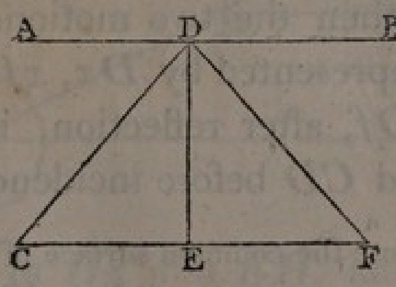
Then *CD* may be resolved into the two *CE*, *ED*, (Art. 43.), of which *ED* is wholly employed in carrying the body in a direction perpendicular to the plane; and since the plane is immoveable, this motion will be wholly destroyed, (See Art. 116.) The other motion *CE*, which is employed in carrying the body parallel to the plane, will not be affected by the impact; and consequently, there being no force to separate the body and the plane, the body will move along the plane; and it will describe $DB = CE$ in the same time that it described *CD* before impact; also, these spaces are uniformly described (Art. 27.); consequently, the velocity before impact : the velocity after :: *CD* : *CE* :: radius : sin. $\angle CDE$ :: radius : cos. $\angle CDA$.

(221.) Cor. The velocity before impact : the difference between the velocity before and the velocity after, that is, the velocity lost :: radius : rad. − cos. $\angle CDA$ :: rad. : the versed sine of the angle *CDA*.

Prop. XLIX.

(222.) *If a perfectly elastic body impinge upon a perfectly hard and immoveable plane* AB, *in the direction* CD, *it will be reflected from it in the direction* DF, *which makes, with* DB, *the angle* BDF *equal to the angle* ADC.

Let *CD* represent the motion of the impinging



body; draw *CF* parallel, and *DE* perpendicular to

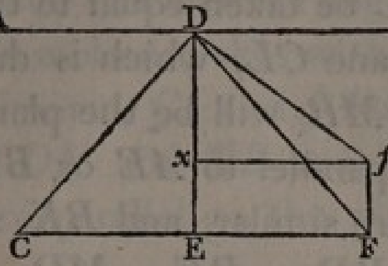
AB; make $EF=CE$, and join DF. Then the whole motion may be resolved into the two CE, ED, of which CE is employed in carrying the body parallel to the plane, and must therefore remain after the impact; and ED carries the body in the direction ED, perpendicular to the plane; and since the plane is immoveable, this motion will be destroyed during the compression, and an equal motion will be generated in the opposite direction by the force of elasticity. Hence it appears, that the body at the point D has two motions, one of which would carry it uniformly from D to E, and the other from E to F, in the same time, viz. in the time in which it described CD before the impact; it will, therefore, describe DF in that time (Art. 38.) Also, in the triangles CDE, EDF, CE is equal to EF, the side ED is common, and the $\angle CED$ is equal to the $\angle DEF$; therefore, the $\angle CDE =$ the $\angle EDF$; hence, the $\angle CDA =$ the $\angle FDB$.

(223.) Cor. 1. Since $CD=DF$, and these are spaces uniformly described in equal times, before and after the impact, the velocity of the body after reflection is equal to it's velocity before incidence.

(224.) Cor. 2. If the body and plane be imperfectly elastic, take $DE : Dx ::$ the force of compression : the force of elasticity; draw xf parallel and equal to EF, join Ff, Df; then the two motions which the body has at D are represented by Dx, xf*, and the body will describe Df, after reflection, in the same time that it described CD before incidence; therefore, the

* Here we suppose the common surface of the body and plane, during the impact, to remain parallel to AB, in which case there is no cause to accelerate or retard the motion CE (See Art. 116.)

velocity before incidence : the velocity after reflection

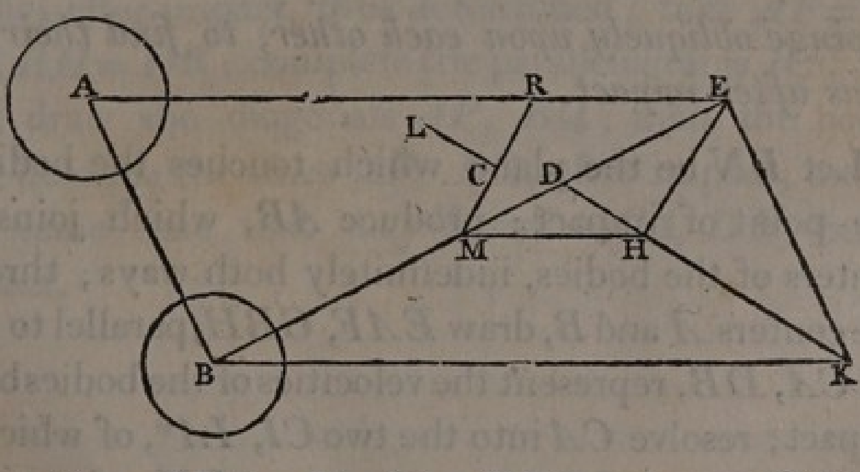


:: $CD : Df$ :: $DF : Df$ :: sin. DfF, or sin. of it's supplement EDf : sin. DFf, or sin. FDE :: sin. EDf : sin. EDC.

Prop. L.

(225.) *Having given the radii of two spherical bodies moving in the same plane, their velocities, and the directions in which they move, to find the plane which touches them both at the point of impact.*

Let AE, BE, meeting in E, be the directions in which the bodies A and B move; and let AE and BD be spaces uniformly described by them in the same time; complete the parallelogram $ABKE$; join KD, and with the center E and radius equal to the sum of the radii of the two bodies, describe a circular



arc cutting KD in H; join EH, and complete the parallelogram $EHMR$. Then R and M will be the places of the centers of the two spheres when they meet; and if RC be taken equal to the radius of the sphere A, the plane CL, which is drawn through C perpendicular to MR, will be the plane required.

Since MH is parallel to AE or BK, the triangles DMH, DBK, are similar, and $BK : BD :: MH : MD$; or $AE : BD :: RE : MD$; therefore $AE : BD :: AR : BM$ (Euc. 19. v.); and since AE and BD are spaces described in the same time by the uniform motions of A and B, AR and BM, which are proportional to them, will be described in the same time; when, therefore, the center of the body A is in R, the center of the body B is in M, and the distance $MR = HE =$ the sum of the radii of the bodies; hence, they will be in contact when they arrive at those points. Also, MR which joins their centers will pass through the point of contact; and LC will be a tangent to them both.

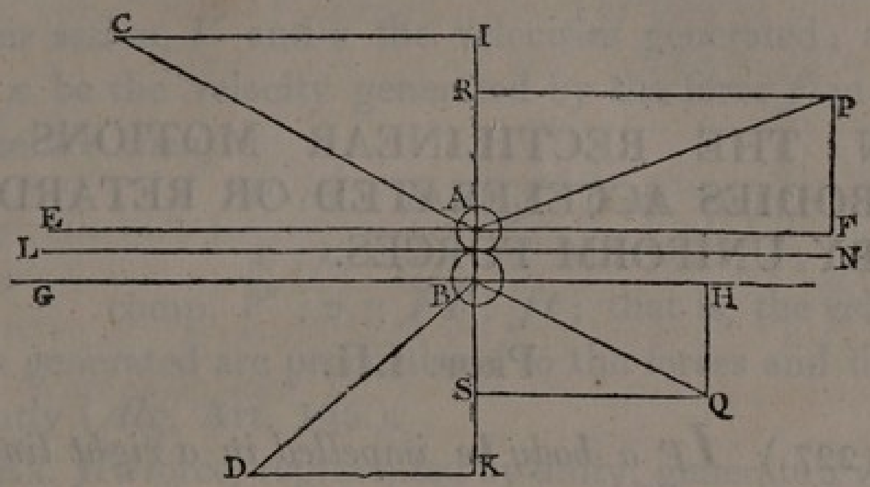
Prop. LI.

(226.) *Having given the motions, the quantities of matter, and the radii of two spherical bodies which impinge obliquely upon each other, to find their motions after impact.*

Let LN be the plane which touches the bodies at the point of impact; produce AB, which joins the centers of the bodies, indefinitely both ways; through the centers A and B, draw EAF, GBH, parallel to LN; let CA, DB, represent the velocities of the bodies before impact; resolve CA into the two CI, IA*, of which CI is parallel, and IA perpendicular to LN; also resolve

* See Art. 43.

DB into two, *DK* parallel to *LN*, and *KB* perpendicular to it. Then *CA* and the angle *CAI*, which the direction of *A*'s motion makes with *AI* perpendicular to *LN*, being known, *CI* and *IA* are known; in the same manner, *DK* and *KB* are known. Now *CI*, *DK*, which are parallel to the plane *LN*, will not



be affected by the impact; and *IA*, *KB*, which are perpendicular to it, are the velocities with which the bodies impinge directly upon each other, and their effects may be calculated by Prop. 44, when the bodies are perfectly hard; and by Prop. 47, when they are elastic. Let *AR* and *BS* be the velocities of the bodies after impact, thus determined; take $AF=CI$, and $BH=DK$; complete the parallelograms *RF*, *SH*, and draw the diagonals *AP*, *BQ*; then the bodies will describe the lines *AP*, *BQ*, after impact, and in the same time that they describe *CA*, *DB*, before impact.

SECTION VII.

ON THE RECTILINEAR MOTIONS OF BODIES ACCELERATED OR RETARDED BY UNIFORM FORCES.

Prop. LII.

(227.) *If a body be impelled in a right line by an uniform force, the velocity communicated to it is proportional to the time of it's motion*.*

The accelerating force is measured by the velocity uniformly generated in a given time (Art. 21.), and in this case, the force is invariable, by the supposition; therefore, equal increments of velocity are always generated in equal times (Art. 20.); and since a body, by the first law of motion, retains the increments of velocity thus communicated to it, if, in the time t, the velocity a be generated, in the time mt the velocity ma is generated; that is, the velocity generated is proportional to the time (*Alg.* Art. 193.).

* By *force*, in this and the following Propositions, we understand the *accelerating force*, no regard being paid to the quantity of matter moved, unless it be expressly mentioned. Also, the direction in which the force acts, to generate or destroy velocity, is supposed to coincide with the direction of the motion.

Prop. LIII.

(228.) *If bodies be impelled in right lines by different uniform forces, the velocities generated in any times are proportional to the forces and times jointly.*

Let F and f be the forces, T and t the times of their action, V and v the velocities generated; also, let x be the velocity generated by the force f in the time T; then,

$$V : x :: F \ : f \quad \text{(Art. 21.)};$$
$$x : v :: T \ : t \quad \text{(Art. 227.)};$$

comp. $V : v :: FT : ft$; that is, the velocities generated are proportional to the forces and times jointly (*Alg.* Art. 195.).

Ex. If a force, represented by unity, generate a velocity represented by $2m$, in one second of time, what velocity will the force F generate in T seconds?

Since $V \propto FT$, we have $1 \times 1 : FT :: 2m : 2mFT$, the velocity required.

(229.) Cor. Since $V \propto FT$, $T \propto \frac{V}{F}$ (*Alg.* Art. 205.)

Prop. LIV.

(230.) *If a body's motion be retarded by an uniform force, the velocity destroyed in any time is equal to that which would be generated in the same time, were the motion accelerated by the same force.*

The force impressed is the same, by the supposition, whether the body move in the direction of the force, or in the opposite direction; therefore, the velocity generated in the former case, is equal to the velocity destroyed, in the same time, in the latter (Art. 29.)

(231.) Cor. 1. Hence, the velocity destroyed by an uniform force is proportional to the time of it's action.

For, the velocity generated by the action of the force is in that ratio (Art. 227.)

(232.) Cor. 2. The velocities destroyed by different uniform forces, are proportional to the forces and times jointly (Art. 228.)

Prop. LV.

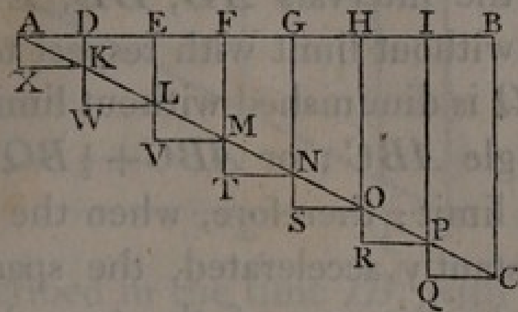
(233.) *If a body be moved through any space, from a state of rest, by the action of an uniform force, and then be projected in the opposite direction with the velocity acquired, and move till that velocity is destroyed, the whole spaces described in the two cases are equal.*

The velocity generated in any time, is equal to the velocity destroyed in the same time by the action of the same force (Art. 230.); hence, the whole times of motion, in the two cases, are equal; also, if equal times be taken, from the beginning of the motion in the former case, and from the end of the motion in the latter, the velocities at those instants are equal. Since then the whole times of motion are equal, and also the velocities at all corresponding points of time, the whole spaces described are equal.

Prop. LVI.

(234.) *If a body be moved from a state of rest by the action of an uniform force, the space described, reckoning from the beginning of the motion, varies as the square of the time, or as the square of the last acquired velocity.*

Take *AB* to represent the time of the body's motion; draw *BC* at right angles to *AB*, and let *BC*



represent the last acquired velocity; join *AC*; divide the time *AB* into small equal portions *AD*, *DE*, *EF*, *FG*, &c.; and from the points *D*, *E*, *F*, *G*, &c. draw *DK*, *EL*, *FM*, *GN*, &c. parallel to *BC*, meeting *AC* in the points *K*, *L*, *M*, *N*, &c.; complete the parallelograms *DX*, *EW*, *FV*, *GT*, &c.

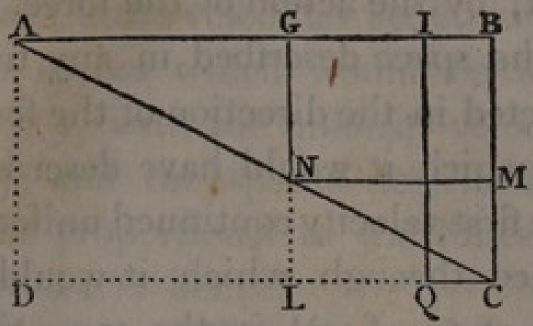
Then, in the similar triangles *ABC*, *ADK*, we have *AB* : *AD* :: *BC* : *DK*; and since *BC* represents the velocity acquired in the time *AB*, *DK* represents the velocity acquired in the time *AD* (Art. 227.); in the same manner it appears, that *EL*, *FM*, *GN*, &c. represent the velocities generated in the times *AE*, *AF*, *AG*, &c. Now, if the body move with the uniform velocity *DK*, during the time *AD*, and with the uniform velocities *EL*, *FM*, *GN*, &c. during the times *DE*, *EF*, *FG*, &c. respectively, the spaces described may properly be represented by the rectangles *DX*, *EW*, *FV*, *GT*, &c. (Art. 16.); therefore, the whole space described, on this supposition, will be represented by the sum of the rectangles, or by the triangle *ABC*, together with the sum of the triangles *AXK*, *KWL*, *LVM*, *MTN*, &c. that is, because the bases of these small triangles are respectively equal to *IB*, and the sum of their altitudes is equal to *BC*, by the triangle *ABC*, together with half the rectangle *BQ*. Let the intervals *AD*, *DE*, *EF*, *FG*, &c. be diminished without limit with respect to *AB*, and the rectangle *BQ* is diminished without limit with respect to the triangle *ABC*; or $ABC+\frac{1}{2}BQ$ approaches to *ABC* as it's limit; therefore, when the motion of the body is constantly accelerated, the space described is represented by the area of the triangle *ABC*. The space described in any other time *AG*, reckoning from the beginning of the motion, is represented, on the same scale, by the area of the triangle *AGN*; and because these triangles are similar, the space described in the time *AB* : the space described in the time $AG :: AB^2 : AG^2$.

Also, *BC*, *GN*, represent the velocities generated in the times *AB*, *AG*; and from the same similar triangles, the space described in the time *AB* : the space described in the time $AG :: BC^2 : GN^2$.

(235.) Ex. If a body be accelerated from a state of rest by an uniform force, and describe *m* feet in the first second of time, it will describe $4m, 9m, 16m, \ldots$ mT^2 feet, in the 2, 3, 4, ... *T* first seconds.

(236.) Cor. 1. The space described, reckoning from the beginning of the motion, is half that which would be described in the same time with the last acquired velocity continued uniform.

Complete the parallelogram *BD*; then, it appears from the Proposition, that the space described in the time *AB*, reckoning from the beginning of the motion : the space described in the time *IB* with the uniform velocity BC :: the triangle ABC : BQ. Also,



the space described in the time IB, with the uniform velocity BC : the space described in the time AB, with the same uniform velocity :: IB : AB (Art. 13.) :: BQ : BD; and by compounding these two proportions, we have the space described in the time AB, when the body's motion is accelerated from a state of rest : the space described in the same time with the last acquired velocity continued uniform :: the triangle ABC : the rectangle BD :: 1 : 2*.

(237.) Cor. 2. The space described in the time GB is represented by the area $GBCN$; or, if NM be drawn parallel to GB, by the rectangle GM together with the triangle NMC. Now, GM represents the space which a body would describe in the time GB, with the uniform velocity GN; and the triangle NMC, which is similar to the triangle ABC, represents the

* This proof has been misunderstood; it amounts to this:

The rectangle BQ represents the space uniformly described, with the velocity BC, in the time BI, on the same scale that the triangle ABC represents the space through which the body is drawn, by the action of the uniform force, in the time AB; and also, on the same scale that DB represents the space uniformly described in the time AB, with the velocity BC; consequently, the spaces described, when the body's motion is accelerated from rest for the time AB, and when the velocity BC remains uniform for the same time, are represented, *on the same scale,* by the triangle ABC and the rectangle BD.

space through which the body would be moved from a state of rest, by the action of the force, in the time *GB*; thus, the space described in any time, when a body is projected in the direction of the force, is equal to the space which it would have described, in that time, with the first velocity continued uniform, together with the space through which it would have been moved from a state of rest, in the same time, by the action of the force.

(238.) Cor. 3. If a body be projected in a direction opposite to that in which the uniform force acts, with the velocity *BC*, and move till that velocity is destroyed, the whole time of it's motion is represented by *BA*, (Art. 230.), and the space described by the area *ABC* (Art. 233.)

Also, the space described in the time *BG* is represented, on the same scale, by the area *BGNC*, that is, by the rectangle *BL* diminished by the triangle *CLN*, or *CNM*. Thus it appears, that the space described in the time *BG*, is equal to that which would have been described with the first velocity continued uniform during that time, diminished by the space through which the body would have been moved from a state of rest, in the same time, by the action of the uniform force.

Prop. LVII.

(239.) *When bodies are put in motion by uniform forces, the spaces described in any times, reckoning from the beginning of the motion in each case, are proportional to the times and last acquired velocities jointly.*

Let S and s be the spaces described in the times T and t; V and v the velocities acquired; then $2S$ and $2s$ are the spaces which would be described in the times T and t, with the uniform velocities V and v (Art. 236.); and the spaces described with uniform velocities are proportional to the times and velocities jointly (Art. 14.); hence,

$$2S : 2s :: TV : tv,$$

$$\text{or } S : s :: TV : tv \text{ (\textit{Alg}. Art. 184.)};$$

that is, $S \propto TV$ (*Alg*. Art. 195.)

(240.) Cor. Hence, the times vary as the spaces directly, and the last acquired velocities inversely.

Prop. LVIII.

(241.) *The spaces described, reckoning from the beginning of the motions, vary also as the forces and squares of the times; or as the squares of the velocities directly, and the forces inversely.*

In general, $S \propto TV$ (Art. 239.); and $V \propto FT$ (Art. 228.); hence, $TV \propto FT^2$ (*Alg*. Art. 203.); therefore, $S \propto FT^2$. Also, $T \propto \frac{V}{F}$; therefore, $TV \propto \frac{V^2}{F}$, and consequently, $S \propto \frac{V^2}{F}$.

(242.) Cor. 1. If T be given, $S \propto F$; that is, the spaces described in equal times, by bodies which are put in motion by uniform forces, are proportional to those forces.

(243.) Cor. 2. Since $S \propto \frac{V^2}{F}$, we have $V^2 \propto FS$ (*Alg*. Art. 203.); that is, the squares of the velocities communicated are proportional to the forces and spaces described jointly.

(244.) Cor. 3. If V be given, $S \propto \frac{1}{F}$.

(245.) Cor. 4. When bodies in motion are retarded by uniform forces, and move till their whole velocities are destroyed, the spaces described vary as the forces and squares of the times; or, as the squares of the first velocities directly and the forces inversely.

For, the time in which any velocity is destroyed, is equal to the time in which it would be generated by the same force; also, the spaces described, on supposition that the body in the latter case is moved from a state of rest, are equal (Art. 233.); therefore, the same expressions which represent the relations of the forces, spaces, times, and velocities, in accelerated motions, represent them also when the motions are retarded, and the bodies move till their whole velocities are destroyed.

Thus, when equal bodies are made to impinge upon banks of earth, sand, &c. where the retarding forces are invariable, the depths to which they sink, or the whole spaces described, are as the squares of the first velocities directly and the forces inversely; and the resisting forces are as the squares of the first velocities directly and the spaces inversely.

Prop. LIX.

(246.) *If a body be moved from a state of rest by the action of an uniform force, the spaces described in equal successive portions of time, reckoned from the beginning of the motion, are as the odd numbers* 1, 3, 5, 7, 9, &c.

If m be the space described in the first portion of time, $4m$ will be the space described in the two first portions (Art. 235.); therefore, $4m - m$, or $3m$, will be the space described in the second portion alone. Also, $9m$ will be the space described in the three first portions of time, and consequently, $9m - 4m$, or $5m$, will be the space described, in the third portion, &c. Thus the spaces described in the equal successive portions of time, are m, $3m$, $5m$, $7m$, $9m$, &c. which are as the odd numbers 1, 3, 5, 7, 9, &c.

(247.) Cor. When a body is retarded by an invariable force, the spaces described in equal portions of time, reckoning from the end of the motion, are as the odd numbers 1, 3, 5, 7, 9, &c.

For, when a body moves till it's whole motion is destroyed by an uniform force, the space described in any time is equal to that which would be described in the corresponding time, were the body moved from a state of rest by the action of the same force (See Art. 233.)

Prop. LX.

(248.) *The force of gravity, at any given place, is an uniform force, which always acts in a direction perpendicular to the horizon, and accelerates all bodies equally.*

The same body will, by it's gravity, always produce the same effect under the same circumstances; thus, it will, at the same place, bend the same spring in the same degree; it will also fall through the same space in the same time, if the resistance of the air be removed;

therefore, the force of gravity is uniform. Also, all bodies which fall freely by this force, descend in lines perpendicular to the horizon; and, in an exhausted receiver, they *all* fall through the same space in the same time; consequently, gravity acts in a direction perpendicular to the horizon (Art. 29.), and accelerates *all* bodies equally (Art. 242.)

It is found by experiments made on the descent of heavy bodies, and on the oscillations of bodies in small circular arcs (Art. 302.), that every body which falls freely *in vacuo* by the force of gravity, descends from rest through $16\frac{1}{12}$ feet in one second of time.

This fact being established, every thing relating to the descent of bodies when they are accelerated by the force of gravity, and to their ascent when they are retarded by that force, supposing the motions to be *in vacuo*, may be deduced from the foregoing Propositions.

1st. When a body falls by the force of gravity, the velocity acquired in any time, as T'', is such as would carry it uniformly over $2mT$ feet in $1''$; where $m = 16\frac{1}{12}$.

Since a body falls $16\frac{1}{12}$ feet in $1''$, it acquires a velocity which would carry it uniformly through $32\frac{1}{6}$ feet in $1''$ (Art. 236.); and when a body is accelerated by a given invariable force, the velocity generated is proportional to the time (Art. 226.); therefore, $1'' : T'' :: 32\frac{1}{6} : 32\frac{1}{6}\,T$, the velocity acquired in T''; that is, the velocity acquired is such as would carry the body uniformly over $32\frac{1}{6}\,T$ feet in $1''$. Let V be the velocity acquired, and $m = 16\frac{1}{12}$, then $V = 2mT$.

2d. The space fallen through in T'', reckoned from the beginning of the motion, is mT^2 feet.

For $S \propto T^2$ (Art. 234.); therefore, $1^2 : T^2 :: m : mT^2$, the space described in T''. That is, $S = mT^2$.

Ex. 1. In $3''$ a body falls $9m$, or $9 \times 16\frac{1}{12} = 144\frac{3}{4}$ feet.

Ex. 2. In $\frac{1}{2}''$ a body falls $\frac{m}{4}$, or $16\frac{1}{12} \times \frac{1}{4} = 4\frac{1}{48}$ feet.

3d. The space fallen through to acquire the velocity V, is $\frac{V^2}{4m}$ feet.

For, $S \propto V^2$ (Art. 234.); therefore, $\overline{2m}|^2 : V^2 :: m : S$, and $S = \frac{V^2}{4m}$ feet.

Ex. If a body fall from rest till it has acquired a velocity of 20 feet per second, the space fallen through is $\frac{20 \times 20}{64\frac{1}{3}}$, or 6.21 feet, nearly.

From the three preceding expressions, $V = 2mT$; $S = mT^2$; and $S = \frac{V^2}{4m}$; any one of the quantities S, T, V, being given, the other two may be found.

Ex. To find the time in which a body will fall 90 feet; and the velocity acquired.

Since $S = mT^2$, $T^2 = \frac{S}{m}$, and $T = \sqrt{\frac{S}{m}}$; in this case $T = \sqrt{\frac{90}{16\frac{1}{12}}} = 2.36$ seconds, nearly.

Also, $S = \frac{V^2}{4m}$; therefore, $V = 2\sqrt{mS}$; in this case, $V = 2\sqrt{16\frac{1}{12} \times 90} = 76$ feet per second, nearly.

4th. If a body fall from rest by the force of gravity, the spaces described in any equal successive portions of time, reckoning from the beginning of the motion, are as the numbers 1, 3, 5, 7, &c. Thus, the spaces fallen through in the 1st, 2d, 3d, 4th seconds, are $16\frac{1}{12}$, $3 \times 16\frac{1}{12}$, $5 \times 16\frac{1}{12}$, $7 \times 16\frac{1}{12}$ feet, respectively. Also, if a body, projected upwards, move till it's whole velocity is destroyed, the spaces described in equal successive portions of time are as the numbers 1, 3, 5, 7, &c. taken in an inverted order. Thus, if the velocity be wholly destroyed in 4″, the spaces described in the 1st, 2d, 3d, 4th seconds, are $7 \times 16\frac{1}{12}$, $5 \times 16\frac{1}{12}$, $3 \times 16\frac{1}{12}$, $16\frac{1}{12}$ feet, respectively.

5th. If a body begin to move in the direction of gravity with any velocity, the whole space described in any time is equal to the space through which the first velocity would carry the body, together with the space through which it would fall by the force of gravity in that time (Art. 237.).

Ex. If a body be projected perpendicularly downwards, with a velocity of 20 feet per second, to find the space described in 4″.

The space described in 4″, with the first velocity, is 4×20, or 80 feet; and the space fallen through in 4″, by the action of gravity, is $16\frac{1}{12} \times 16$, or $257\frac{1}{3}$ feet; therefore, the whole space described is $80 + 257\frac{1}{3}$, or $337\frac{1}{3}$ feet.

6th. If a body be projected perpendicularly upwards, the height to which it will ascend in any time is equal to the space through which it would move with the first velocity continued uniform, diminished by the space through which it would fall by the action of gravity in that time (Art. 238.).

Ex. 1. To what height will a body rise in 3″, if projected perpendicularly upwards with a velocity of 100 feet per second ?

The space which the body would describe in 3″, with the first velocity, is 300 feet; and the space through which the body would fall by the force of gravity in 3″ is $16\frac{1}{12} \times 9$, or $144\frac{3}{4}$ feet; therefore the height required is $300 - 144\frac{3}{4}$, or $155\frac{1}{4}$ feet.

Ex. 2. If a body be projected perpendicularly upwards with a velocity of 80 feet per second, to find it's place at the end of 6″.

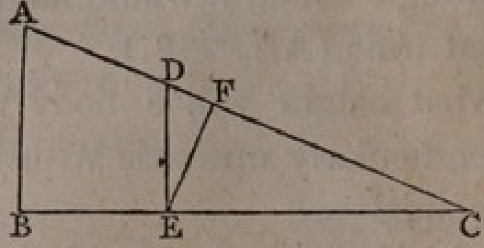
The space which would be described in 6″, with the first velocity, is 480 feet, and the space fallen through in the same time is $16\frac{1}{12} \times 36$, or 579 feet; therefore the distance of the body from the point of projection, at the end of 6″, is $480 - 579$, or -99 feet. The negative sign shews that the body will be below the point of projection (See *Alg*. Art. 472.).

Prop. LXI.

(249.) *The force which accelerates or retards a body's motion upon an inclined plane, is to the force of gravity, as the height of the plane is to it's length.*

Let AC be the plane, BC it's base, parallel to the horizon, AB it's perpendicular height, D the place of a body upon it. From the point D draw DE parallel to AB, and take DE to represent the force of gravity; from E draw EF perpendicular to AC.

Then the whole force DE is equivalent to the two



DF, FE, of which FE is perpendicular to the plane, and, consequently, is supported by the plane's reaction (Art. 116.); the other force DF, not being affected by the plane, is wholly employed in accelerating or retarding the motion of the body in the direction of the plane; therefore, the accelerating force : the force of gravity :: $DF : DE$ :: (from the similar triangles DEF, ABC) $AB : AC$.

(250.) Cor. 1. Since the accelerating force, on the same plane, is in a given ratio to the force of gravity, it is an uniform force.

(251.) Cor. 2. If H be the height of an inclined plane, L it's length, and the force of gravity be represented by unity, the accelerating force on the inclined plane is represented by $\frac{H}{L}$.

For, the accelerating force : the force of gravity (1) :: $H : L$; therefore the accelerating force $= \frac{H}{L}$.

(252.) Cor. 3. Since $H : L$ :: the sine of the plane's inclination : the radius, $\frac{H}{L}$, or the accelerating force, varies as the sine of the plane's inclination to the horizon.

(253.) Cor. 4. If a body fall down an inclined plane, the velocity V, generated in T'', is such as would carry it uniformly over $\frac{H}{L} \times 2mT$ feet in $1''$; where $m = 16\frac{1}{12}$.

In general, $V \propto FT$ (Art. 228.); therefore, the velocity acquired when a body falls by the force of gravity : the velocity acquired on the inclined plane :: the product of the numbers which represent the force and time in the former case : the product of the numbers which represent them in the latter*; also, the force of gravity being represented by unity, the accelerating force upon the plane is $\frac{H}{L}$, and the velocity generated by the force of gravity in $1''$ is $2m$; therefore, $2m$: V :: 1×1 : $\frac{H}{L} \times T$; and $V = \frac{H}{L} \times 2mT$†.

Ex. Thus, if the length of an inclined plane be twice as great as it's height, a body which falls down this plane will, in $3''$, acquire a velocity of $\frac{1}{2} \times 32\frac{1}{6} \times 3$, or $48\frac{1}{4}$ feet per second.

(254.) Cor. 5. The space fallen through in T'', from a state of rest, is $\frac{H}{L} \times mT^2$ feet.

In general, $S \propto FT^2$ (Art. 241.); therefore, the space through which a body falls by the action of gravity in $1''$: the space through which it falls down the inclined plane in T'' :: the product of the numbers which represent the force and square of the time in the

* See Note, page 12.

† In this, and the following Articles, the planes are supposed to be perfectly smooth, and the resistance of the air inconsiderable.

former case : the product of the numbers which represent them in the latter; or, if S be the space described upon the plane, $m : S :: 1 \times 1^2 : \frac{H}{L} \times T^2$, and

$S = \frac{H}{L} \times mT^2$.

Ex. 1. If $L = 2H$, the space through which a body falls in 3″ is $\frac{1}{2} \times 16\frac{1}{12} \times 9$, or $72\frac{3}{8}$ feet.

Ex. 2. To find the time in which a body will descend 12 feet down this plane.

Since $S = \frac{H}{L} \times mT^2$, $T^2 = \frac{L \times S}{H \times m} =$ (in this case) $\frac{2}{1} \times 12 \times \frac{1}{16\frac{1}{12}} = 1.49$; and $T = 1.2$, nearly.

(255.) Cor. 6. The space through which a body must fall, from a state of rest, to acquire the velocity V, is $\frac{L}{H} \times \frac{V^2}{4m}$ feet.

In general, $S \propto \frac{V^2}{F}$ (Art. 241.); therefore, the space through which the body falls by the force of gravity : the space through which it falls down the plane :: $\frac{V^2}{F}$ in the former case : $\frac{V^2}{F}$ in the latter; and if m $(16\frac{1}{12})$ be the space fallen through by the action of gravity, $2m$ is the velocity acquired; hence, $m : S :: \frac{4m^2}{1} : \frac{L}{H} \times V^2$; and $S = \frac{L}{H} \times \frac{V^2}{4m}$.

Ex. 1. If $L = 2H$, and a body fall from a state of rest till it has acquired a velocity of 30 feet per second, the space described is $\frac{2}{1} \times \frac{900}{64\frac{1}{3}} = 27.97$ feet, nearly.

Ex. 2. If a body fall 12 feet from a state of rest down this plane, to find the velocity acquired.

Since $S = \frac{L}{H} \times \frac{V^2}{4m}$, we have $V^2 = 4mS \times \frac{H}{L} =$ (in this case) $64\frac{1}{3} \times 12 \times \frac{1}{2} = 386$; hence, $V = 19.6$ feet per second, nearly.

Cor. 7. In the same manner, if a body be acted upon by any uniform force, which is to the force of gravity as $F : 1$, and V represent the velocity generated, T the time in seconds, S the space described, in feet, reckoned from the beginning of the motion, then $V = 2mFT$; $S = mFT^2$; and $V^2 = 4mFS$.

Prop. LXII.

(256.) *The velocity which a body acquires in falling down the whole length of an inclined plane, varies as the square root of the perpendicular height of the plane**.

In general, when the force is uniform, $V^2 \propto FS$ (Art. 243.); in this case, $F \propto \frac{H}{L}$, and $S = L$, by the supposition; therefore, $V^2 \propto \frac{H}{L} \times L \propto H$; and $V \propto \sqrt{H}$ (*Alg.* Art. 202†.).

(257.) Cor. 1. When the heights of two inclined planes are equal, the velocities acquired in falling down their whole lengths are equal.

* Bodies, in this, and the subsequent Propositions, are supposed to fall from a state of rest.

† See also Cor. 6. Ex. 2. of the last Proposition.

(258.) Cor. 2. The velocity which a body acquires in falling down the length of an inclined plane is equal to the velocity which it would acquire in falling down it's perpendicular height.

Prop. LXIII.

(259.) *The time of a body's descent down the whole length of an inclined plane, varies as the length directly, and as the square root of the perpendicular height inversely.*

In general, $S \propto TV$ (Art. 239.); therefore, $T \propto \frac{S}{V}$; and in this case, $V \propto \sqrt{H}$ (Art. 256.); consequently, $T \propto \frac{S}{\sqrt{H}} \propto \frac{L}{\sqrt{H}}$*.

(260.) Cor. 1. If the height, or the last acquired velocity, be given, $T \propto L$.

(261.) Cor. 2. If the inclination be given, or $H \propto L$, then $T^2 \propto \frac{L^2}{L} \propto L$, and $T \propto \sqrt{L}$. That is, the times of descent, down planes equally inclined to the horizon, vary as the square roots of their lengths.

(262.) Cor. 3. The time of descent down an inclined plane, is to the time of falling down it's perpendicular height, as the length of the plane, to it's height.

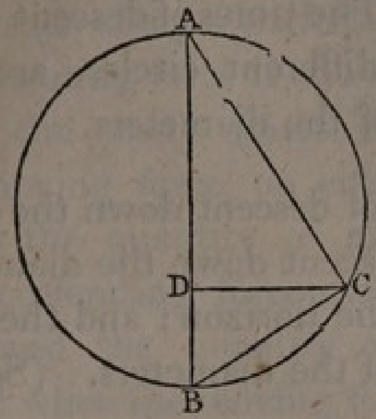
Prop. LXIV.

(263.) *If chords be drawn in a circle from the extremity of that diameter which is perpendicular to the horizon, the velocities which bodies acquire by falling down them are proportional to their lengths; and the times of descent are equal.*

Let ACB be the circle, AB it's diameter perpen-

* See also Art. 254. Ex. 2.

dicular to the horizon; BC a chord drawn from the extremity B of the diameter; join AC, and draw CD



perpendicular to AB, or parallel to the horizon. Then CB may be considered as an inclined plane whose perpendicular height is DB, and the velocity acquired in falling down it varies as $\sqrt{DB}$ (Art. 256.). Now, from the similar triangles DBC, ABC, $DB : CB :: CB : AB$; therefore, $DB = \frac{CB^2}{AB}$, and $\sqrt{DB} = \frac{CB}{\sqrt{AB}}$; consequently, $V \propto \frac{CB}{\sqrt{AB}}$, and AB is invariable; therefore $V \propto CB$.

Again, $T \propto \frac{S}{V}$ (Art. 240.), and in this case, CB, which is the space described, has been proved to be proportional to the velocity acquired; therefore $T \propto \frac{CB}{CB}$, or the time of descent is invariable.

(264.) Cor. 1. The time of descent down any chord CB, is equal to the time of descent down the diameter AB.

(265.) Cor. 2. In the same manner, the time of descent down AC is equal to the time of descent down

AB; therefore the time of descent down *AC* is equal to the time of descent down *CB*.

(266.) Cor. 3. The times of descent down the chords thus drawn, in different circles, are proportional to the square roots of the diameters.

For, the times of descent down the chords are equal to the times of descent down the diameters which are perpendicular to the horizon; and these times vary as the square roots of the diameters. (See Art. 234.)

(267.) When a body falls freely by the force of gravity, every particle in it is equally accelerated; that is, every particle descends towards the horizon with the same velocity; in this descent, therefore, no rotation will be given to the body. The same may be said when a body descends along a perfectly smooth inclined plane, if that part of the force which acts in a direction perpendicular to the plane (Art. 249.), be supported; that is, if a perpendicular to the plane, drawn from the center of gravity of the body, cut the plane in a point which is in contact with the body. If this part of the force be not sustained by the plane, the body will partly roll and partly slide, till this force is sustained; and afterwards the body will wholly slide. When the lateral motion is entirely prevented by the adhesion of the body to the plane, we have before seen on what supposition the body will roll (Art. 186.); if the adhesion be not sufficient to prevent all lateral motion, this body will partly slide and partly roll; and to estimate the space described, the time of it's motion, or the velocity acquired, we must have recourse to other principles than those above laid down.

on this subject the Reader may consult Professor VINCE's *Plan of a Course of Lectures*, p. 39.

(268.) When a body falls freely by the force of gravity, or descends along a perfectly smooth inclined plane, the accelerating force is the same, whatever be the weight of the body (Arts. 248, 249.); consequently, the moving force, on either supposition, is proportional to the quantity of matter moved. In all cases, the accelerating force varies as the moving force directly and the quantity of matter inversely (Art. 24.); and when the moving force and quantity of matter moved are invariable, the accelerating force is uniform, and it's effects may be estimated by the rules laid down in the first part of this section.

Ex. If two bodies, whose weights are P and Q, be connected by a string, and hung over a fixed pulley, to find how far the heavier P will descend in T''.

The moving force of gravity is proportional to the weight; if therefore P be taken to represent the moving force of the former body when it descends freely, Q will represent the moving force of the latter, and $P-Q$ will represent the moving force when the bodies are connected and oppose each other's motion; hence, neglecting the inertia of the string and pulley, the accelerating force of gravity : the accelerating force in this case :: $\frac{P}{P} : \frac{P-Q}{P+Q} :: 1 : \frac{P-Q}{P+Q}$; and, since $FT^2 \propto S$, $1 \times 1^2 : \frac{P-Q}{P+Q} \times T^2 :: 16\frac{1}{12} : 16\frac{1}{12} \times \frac{P-Q}{P+Q} \times T^2$, the space required.

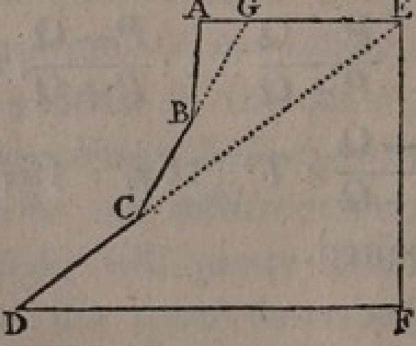
SECTION VIII.

ON THE OSCILLATIONS OF BODIES IN CYCLOIDS AND IN SMALL CIRCULAR ARCS.

Prop. LXV.

(269.) *If a body descend down a system of inclined planes, the velocity acquired, on the supposition that no motion is lost in passing from one plane to another, is equal to that which would be acquired in falling through the perpendicular height of the system.*

Let *ABCD* be the system of planes; draw *AE*,



DF, parallel to the horizon; produce *CB*, *DC*, till they meet *AE* in *G* and *E*; and draw *EF* perpendicular to *DF*. Then the velocity acquired by a body in falling from *A* to *B*, is equal to that which it would acquire in falling from *G* to *B*, because the planes *AB*, *GB*, have the same perpendicular height (Art. 257.); and since, by the supposition, no velocity is lost in passing from one plane to another, the body will begin to descend down *BC* with the same velocity, whether it fall down *AB* or *GB*; consequently, the velocity acquired at *C* will be the same on either supposition. Also, the velocity acquired at *C* is equal to that which would be acquired in falling down *EC* (Art. 257.); and no velocity being lost at *C*, the body will begin to descend down *CD* with the same velocity, whether it fall from *A* through *B* and *C* to *D*, or from *E* to *D*; and the velocity acquired in falling down *ED* is equal to the velocity acquired in falling through the perpendicular height *EF* (Art. 258.); therefore, the velocity acquired in falling down the whole system, is equal to the velocity acquired in falling through the perpendicular height of the system.

Prop. LXVI.

(270.) *If a body fall from a state of rest down a curve surface which is perfectly smooth, the velocity acquired is equal to that which would be acquired in falling from rest through the same perpendicular height.*

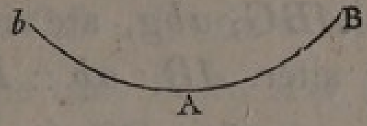
When a body passes from one plane *AB* to another *BC*, the whole velocity : the quantity by which the velocity is diminished :: radius : the versed sine of the $\angle$ *ABG* (Art. 221.); when, therefore, the angle *ABG* is diminished without limit, the velocity lost is diminished without limit; and if the lengths of the planes, as well as their angles of inclination ABG, BCE, be continually diminished, the system approximates to a curve, as it's limit, in which no velocity is lost; consequently, the whole velocity acquired is equal to that which a body would acquire in falling through the same perpendicular altitude (Art. 269*.).

(271.) Cor. 1. If a body be projected up a curve, the perpendicular height to which it will rise is equal to that through which it must fall to acquire the velocity of projection.

* When the chord of an arc is diminished without limit with respect to the diameter, the versed sine is diminished without limit with respect to the chord; because, the diameter : the chord :: the chord : the versed sine; hence, the ratio of the diameter to the versed sine, and consequently, the ratio of the radius to the versed sine, is, in this case, indefinitely greater than the ratio of the diameter to the chord. Let BC be one of the evanescent planes, V the velocity of the descending body at B, $V+v$ it's velocity at C; produce CB to G, and let GB be the space through which the body must descend to acquire the velocity V; then, $V : V+v :: \sqrt{GB} : \sqrt{GB+BC}$; and when $GB : BC$ :: the radius : an evanescent chord, $V : V+v :: GB : GB+\frac{BC}{2}$ (see Note, p. 126.); therefore, $V : v :: GB : \frac{BC}{2} :: 2GB : BC$. Also, V : the velocity lost at B :: radius : the versed sine of the angle ABG. Hence it follows, that the ratio of V to the velocity lost at B, is indefinitely greater than the ratio of V to the velocity acquired in the descent from B to C; and consequently, the velocity lost at B is indefinitely less than the velocity acquired in the descent from B to C; in the same manner, the velocity lost at any other plane is indefinitely less than the velocity acquired in the descent down that plane; therefore, the velocity lost in the whole descent is indefinitely less than the whole velocity acquired.

For the body in it's ascent will be retarded by the same degrees that it was accelerated in it's descent.

(272.) Cor. 2. If BAb be a curve in which the lowest point is A, and the parts AB, Ab, are similar and equal, a body in falling down BA will acquire a



velocity which will carry it to b; and since the velocities at all equal altitudes in the ascent and descent are equal, the whole time of the ascent will be equal to the time of descent.

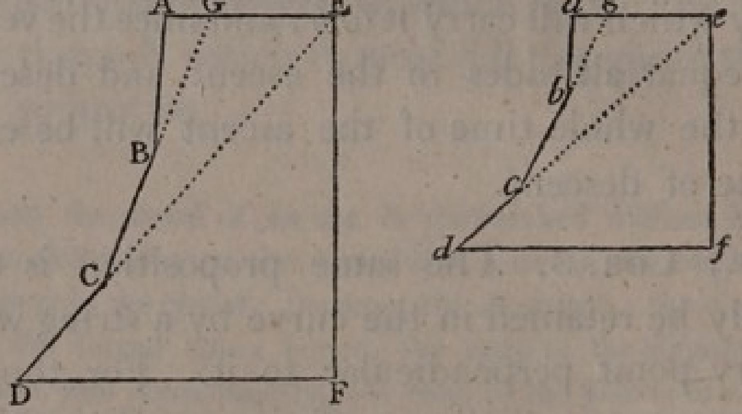
(273.) Cor. 3. The same proposition is true, if the body be retained in the curve by a string which is in every point perpendicular to it. For the string will now sustain that part of the weight which was before sustained by the curve (Art. 117.).

Prop. LXVII.

(274.) *The times of descent down similar systems of inclined planes, similarly situated, are as the square roots of their lengths, on the supposition that no velocity is lost in passing from one plane to another.*

Let $ABCD$, $abcd$, be two similar systems of inclined planes, similarly situated; that is, let $AB : ab :: BC : bc :: CD : cd$; the angles ABC, BCD, respectively equal to the angles abc, bcd; and the planes AB, ab, equally inclined to the horizon. Complete the figures as in the last Proposition; then, since $AB : ab ::$

$BC : bc :: CD : cd$, we have, $AB : ab :: AB + BC + CD : ab + bc + cd$ (*Alg.* Art. 183.); and, $\sqrt{AB} : \sqrt{ab} :: \sqrt{AB+BC+CD} : \sqrt{ab+bc+cd}$. Also, since the angles ABC, abc, are equal, their supplements, the angles ABG, abg, are equal; and the angles of inclination to the horizon BAG, bag, are equal; therefore, the triangles ABG, abg, are similar, and $AB : BG :: ab : bg$; alter. $AB : ab :: BG : bg :: BC :$

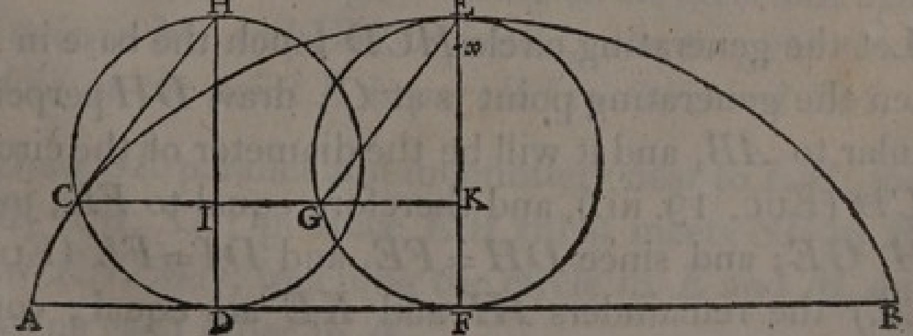


bc; consequently, $BG : bg :: BG+BC\,(GC) : bg + bc\,(gc) :: AB : ab$. In the same manner, $ED : ea :: AB : ab$. Then, because the planes AB, ab, are equally inclined to the horizon, the time of descent down AB : the time down $ab :: \sqrt{AB} : \sqrt{ab}$ (Art. (261.); and if the bodies fall down GC, gc, the time down GC : the time down $gc :: \sqrt{GC} : \sqrt{gc} :: \sqrt{AB} : \sqrt{ab}$; also, the time down GB : the time down $gb :: \sqrt{GB} : \sqrt{gb} :: \sqrt{AB} : \sqrt{ab}$; hence the whole time down GC : the whole time down gc :: the time down GB : the time down gb; therefore, the remainder, the time down BC : the remainder, the time down bc, in the same ratio, or as $\sqrt{AB} : \sqrt{ab}$ (Euc. 19. v.); and since, by the supposition, no motion is lost in passing from one plane to another, the times of descent down BC and bc are the same, whether the bodies descend from A and a, or from G and g; consequently, when the bodies descend down the systems, the time down BC : the time down bc :: $\sqrt{AB} : \sqrt{ab}$. In the same manner it may be shewn that the time down CD : the time down cd :: $\sqrt{AB}$: $\sqrt{ab}$. Hence, the time down AB : the time down ab :: the time down BC : the time down bc :: the time down CD : the time down cd; therefore, the time down $AB+BC+CD$: the time down $ab+bc+cd$ :: the time down AB : the time down ab :: $\sqrt{AB}$: $\sqrt{ab}$ (*Algebra*, Art. 183.) :: $\sqrt{AB+BC+CD}$: $\sqrt{ab+bc+cd}$.

(275.) Cor. 1. If the lengths of the planes, and their angles of inclination ABG, ACE, &c. be continually diminished, the limits, to which these systems approximate, are similar curves, similarly situated, in which no velocity is lost (Art. 270.); hence, the whole times of descent down these curves will be as the square roots of their lengths.

(276.) Cor. 2. The times of descent down similar circular arcs, similarly situated, are as the square roots of the arcs, or as the square roots of their radii.

(277.) Def. If a circle be made to *roll* in a given



L

plane upon a straight line AB, the point C in the circumference, which was in contact with AB at the beginning of the motion, will, in a revolution of the circle, describe a curve $ACEB$ called a *cycloid*.

The line AB is called the *base* of the cycloid.

The circle HCD is called the *generating circle*.

The line FE, which is drawn bisecting AB at right angles, and produced till it meets the curve in E, is called the *axis*, and the point E, the *vertex*, of the cycloid.

(278.) Cor. 1. The base AB is equal to the circumference of the generating circle; and AF to half the circumference.

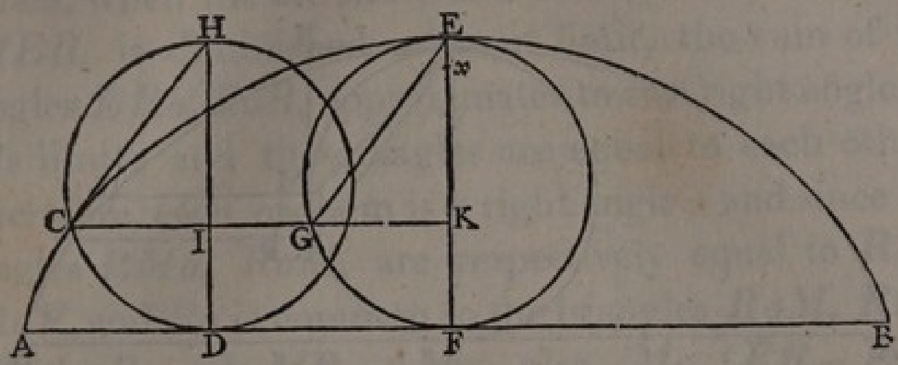
(279.) Cor. 2. The axis FE is equal to the diameter of the generating circle.

When the generating circle comes to F, draw the diameter Fx, which will be perpendicular to AB (Euc. 18. iii.); and because the circle has completed half a revolution, x is the generating point; that is, x is a point in the cycloid, or x coincides with E.

Prop. LXVIII.

(280.) *If a line* CGK, *drawn from a point* C *in the cycloid, parallel to the base* AB, *meet the generating circle, described upon the axis, in* G, *the circular arc* EG *is equal to the right line* CG.

Let the generating circle HCD touch the base in D when the generating point is at C; draw DH perpendicular to AB, and it will be the diameter of the circle HCD (Euc. 19. iii.), and therefore equal to FE; join CH, GE; and since $DH = FE$, and $DI = FK$ (Euc. 34. i.) the remainders IH and KE are equal; consequently, CI, which is a mean proportional between



HI and ID (Euc. Cor. 8. vi.), is equal to KG, which is a mean proportional between EK and KF; to each of these equals add IG, and $CG=IK$. Also, CH, which is a mean proportional between IH and HD, is equal to GE, which is a mean proportional between EK and EF; therefore, the arc $CH=$the arc GE (Euc. 28. iii.); and since every point in CD has been successively in contact with AD, $CD=AD$, and $HCD=AF$ (Art. 278.); hence, the arc $CH=DF$; therefore, the arc $EG=DF=IK=CG$.

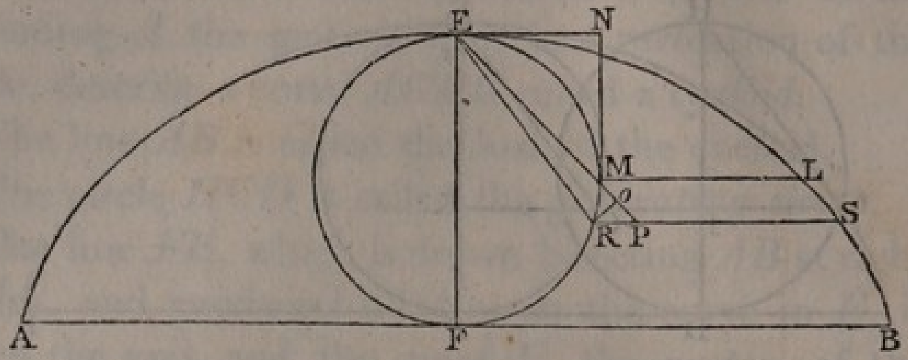
Cor. If the line KGC be always drawn perpendicular to EF the diameter of the circle EGF, and GC be taken equal to the arc EG, the locus of the point C is a cycloid, whose axis is EF.

Prop. LXIX.

(281.) *If a line* LM, *drawn from* L *parallel to the base* AB, *meet the generating circle described upon the axis in* M, *and* EM *be joined, the tangent to the cycloid at the point* L *is parallel to the chord* EM.

Draw SR parallel and indefinitely near to LM; join EM, RM, SL; produce EM till it meets SR in P; draw EN, MN, touching the circle in E and M, and meeting each other in N.

Then, since *RM* is ultimately in the direction of the



tangent *MN* (Newt. Lem. 6.), the angles *RMP*, *EMN*, are equal; and because *EN* is parallel to *RS* (Euc. 18. iii.) the angles *MPR*, *MEN*, are equal; therefore, the triangles *EMN*, *RMP*, are equi-angular, and *EN* : *MN* :: *RP* : *RM*; and since *EN*=*NM*, *RP*=*RM*=the arc *RM* (Newt. Lem. 7.). Again, since the arc *EMR*=*RS* (Art. 280.), and *RM*=*RP*, the remainders, the arc *EM* and the right line *PS* are equal; also, *ML*=the arc *EM*; therefore, *PS*=*ML*; consequently, *SL* is equal, and parallel to *PM* (Euc. 33. i*.); and since *SL* is ultimately in the direction of the tangent at *L* (Newt. Lem. 6.), *MP*, or *EM*, is parallel to the tangent at *L*.

(282.) Cor. The tangent to the cycloid at *B* or *A*, is perpendicular to *AB*.

Prop. LXX.

(283.) *The same construction being made, the cycloidal arc* EL *is double of* EM *the corresponding chord of the generating circle described upon the axis.*

* The Proposition may be justly applied, because the difference between *LM* and *SP* is evanescent with respect to *MP*, or *LS*.

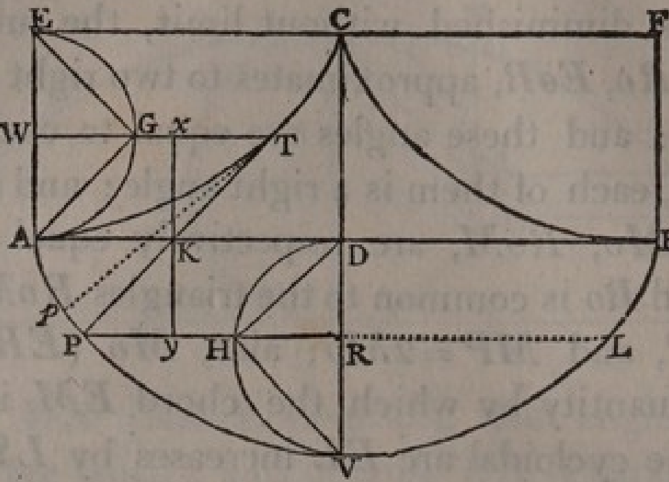
Join ER, and in EP take $Eo=ER$; join Ro. Then, when the arc MR, and consequently the angle MER, is diminished without limit, the sum of the angles ERo, EoR, approximates to two right angles as it's limit; and these angles are equal to each other; therefore, each of them is a right angle; and since the angles RMo, RoM, are respectively equal to RPo, RoP, and Ro is common to the triangles RoM, RPo, $Mo=oP$, and $MP=2Mo$; also, Mo $(ER-EM)$ is the quantity by which the chord EM increases, whilst the cycloidal arc EL increases by LS; and it appears from the demonstration of the last Proposition that $MP=LS=\text{arc}\,LS$ (NEWT. Lem. 7.); therefore, the arc $LS=2Mo$; or, the cycloidal arc EL increases twice as fast as the corresponding chord EM; and they begin together at E; consequently, the cycloidal arc EL is double of EM, the corresponding chord of the generating circle.

(284.) COR. The whole semi-cycloidal arc EB is equal to twice the axis EF.

PROP. LXXI.

(285.) *To make a body oscillate in a given cycloid.*

Let AVB be the given cycloid, placed with it's vertex downwards, and it's axis DV perpendicular to the horizon. Produce VD to C, making $DC=VD$; complete the rectangles DE, DF; upon AE describe a semi-circle AGE, and with A as the generating point, and base EC, describe a semi-cycloid ATC; this will pass through the point C, because the semi-circumference $AGE=DHV=AD=EC$; in the same manner, describe an equal semi-cycloid between C and B.



Then, if a body P be suspended from C by a string whose length is CV or CTA (Art. 284.), and made to vibrate between the cycloidal cheeks CA, CB, it will always be found in the cycloid AVB.

Let the string be brought into the situation CTP, and since it is constantly stretched by the gravity, and the centrifugal force of P, it will be a tangent to the cycloid at the point T where it leaves the curve. From T and P draw TGW, PHR, parallel to AD; join AG, GE, DH, HV; and through K draw xKy perpendicular to TG or PH. Then, since the chord AG is parallel to TP (Art 281.), and TG is parallel to AK, the figure GK is a parallelogram, and $AG=TK$, $GT=AK$; and because the length of the string is equal to CTA, and the part CT is common to the string and the cycloidal arc, $TP=AT=2AG$ (Art. 283.)$=2TK$; or $TK=KP$; hence, the triangles TKx, PKy, are similar and equal, and $Kx=Ky$; also, $Kx=AW$ and $Ky=DR$; therefore $AW=DR$, and $AE=DV$; hence, the arc $AG=$the arc DH; and the angle

GEA = the angle HVD; or, the angle GAK = the angle KDH (Euc. 32. iii.); consequently, AG is parallel to DH; and therefore, TP is parallel to DH, and the figure $KPHD$ is a parallelogram; hence, $KD = PH$. Again, since the arc $AG = GT$ (Art. 280.) $= AK$, the arc $DH = AK$; and the semi-circumference $DHV = AD$; therefore, the arc $VH = KD = HP$; that is, P is in the cycloid, whose axis is DV, and vertex V (Art. 280.).

(286.) Cor. 1. Since DH is parallel to TP, and VH to the tangent at P, the angle contained between TP and the tangent, or between TP and the curve, is equal to the angle DHV; that is, TP is always perpendicular to the curve.

(287.) Cor. 2. If Pp be an evanescent arc, the perpendiculars to the curve at P and p, ultimately meet in T; and Pp may be considered as a circular arc whose radius is TP.

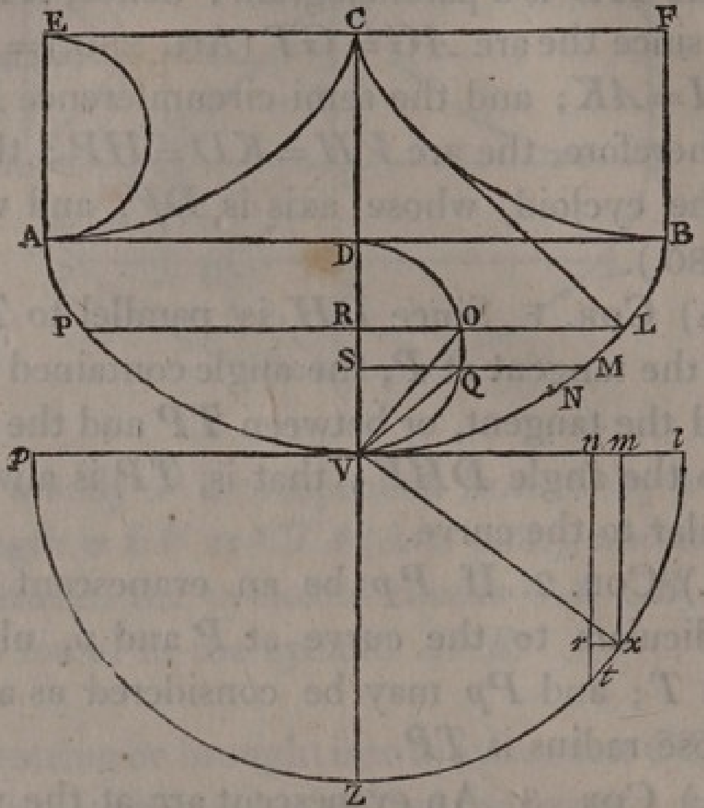
(288.) Cor. 3. An evanescent arc at the vertex of the cycloid may be considered as a circular arc whose radius is CV.

(289.) Def. If a body begin to descend in a curve, from any point, and again ascend till it's velocity is destroyed (Art. 272.), the time in which the motion is performed is called *the time of an oscillation.*

Prop. LXXII.

(290.) *If a body, vibrating in the cycloid* AVB, *begin to descend from* L, *the velocity acquired at any point* M *varies as* $\sqrt{VL^2 - VM^2}$; *or, as the right sine of a circular arc whose radius is equal to* VL, *and versed sine to* LM.

From the points L and M, draw LOR, MQS, at right angles to DV, meeting the circle DOV in O



and Q; join OV, QV; with the radius $Vl = VL$, describe the semi-circle lZp, and take $lm = LM$; draw mx, VZ, at right angles to Vl; and join Vx.

The velocity acquired in the descent from L to M, is equal to the velocity acquired in falling from R to S (Art. 270.); and therefore it varies as $\sqrt{RS}$ (Art. 241.); that is, $\propto \sqrt{RV - SV} \propto \sqrt{DV \times RV - DV \times SV}$ (because DV is invariable), $\propto \sqrt{VO^2 - VQ^2} \propto \sqrt{4VO^2 - 4VQ^2} \propto \sqrt{VL^2 - VM^2}$ (Art. 283.), $\propto \sqrt{Vl^2 - Vm^2} \propto \sqrt{Vx^2 - Vm^2} \propto \sqrt{mx^2} \propto mx$.

(291.) Cor. The velocity at M : the velocity at V :: mx : VZ :: mx : Vx.

Prop. LXXIII.

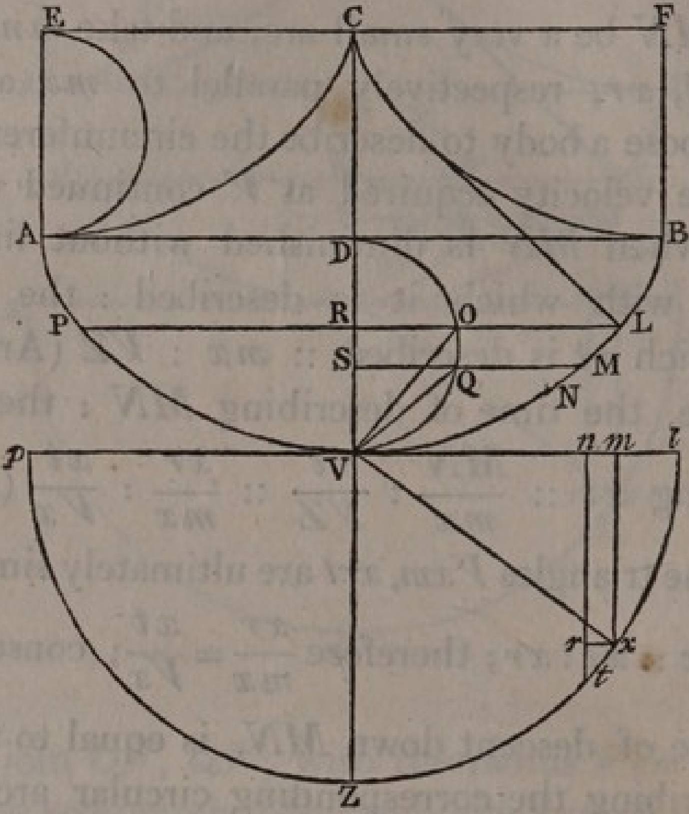
(292.) *The time of an oscillation in the arc* LVP, *is equal to the time in which a body would describe the semi-circumference* lZp, *with the velocity acquired at* V *continued uniform.*

Let MN be a very small arc, and take $mn = MN$; draw nt, xr, respectively parallel to mx and Vl; and suppose a body to describe the circumference lZp with the velocity acquired at V continued uniform. Then, when MN is diminished without limit, the velocity with which it is described : the velocity with which xt is described :: mx : VZ (Art. 291.); therefore, the time of describing MN : the time of describing xt :: $\frac{MN}{mx} : \frac{xt}{VZ} :: \frac{xr}{mx} : \frac{xt}{Vx}$ (Art. 15.) Now, the triangles Vxm, xrt are ultimately similar, and $Vx : mx :: xt : xr$; therefore $\frac{xr}{mx} = \frac{xt}{Vx}$; consequently, the time of descent down MN, is equal to the time of describing the corresponding circular arc xt with the velocity VZ; and the same may be proved of all other corresponding arcs in the cycloid and the circle; therefore the whole time of an oscillation is equal to the time of describing the semi-circumference lZp, with the velocity acquired at V continued uniform.

Prop. LXXIV.

(293.) *The time of an oscillation in a cycloid is to the time of descent down it's axis, as the circumference of a circle to it's diameter.*

If a body fall down the chord OV, the velocity acquired at V is equal to the velocity in the cycloid at V (Art. 270.); and with this velocity continued uniform, the body would describe $2OV$, or VL, or Vl, in the time of descent down OV (Art. 237.); that is, in the time of descent down DV (Art. 264.). It



appears then, that the time of an oscillation is equal to the time of describing lZp with the velocity acquired in the cycloid at V (Art. 292.); and that the time of descent down the axis DV is equal to the time of describing Vl with the same velocity; therefore, the time of an oscillation : the time of descent down the axis :: the time of describing the circumference lZp, with the velocity VZ : the time of describing Vl with the same velocity :: $lZp : Vl$ (Art. 13.) :: $2lZp : 2Vl$ :: the circumference of a circle : it's diameter.

(294.) COR. 1. The time of an oscillation in a given cycloid, at a given place, is the same, whether the body oscillate in a greater or a smaller arc.

For, the time of an oscillation bears an invariable ratio to the time of descent down the axis, which, in a given cycloid, at a given place, is given.

(295.) COR. 2. The time of an oscillation in a small circular arc whose radius is CV, is to the time of descent down $\frac{1}{2}CV$, as the circumference of a circle to it's diameter.

For, the time of an oscillation in this circular arc is equal to the time of an oscillation in an equal arc of the cycloid AVB (Art. 288.).

(296.) COR. 3. The time of an oscillation in a cycloid, or small circular arc, when the force of gravity is given, varies as the square root of the length of the string.

For, the time of an oscillation varies as the time of descent down half the length of the string; that is, as the square root of half the length of the string, or as the square root of it's whole length.

Ex. 1. To compare the times in which two pendulums vibrate, whose lengths are 4 and 9 inches.

Since $T \propto \sqrt{L}$, we have $T : t :: \sqrt{4} : \sqrt{9} :: 2 : 3$.

Ex. 2. If a pendulum, whose length is 39.2 inches, vibrate in one second, in what time will a pendulum vibrate whose length is L inches?

$\sqrt{39.2} : \sqrt{L} :: 1 : T = \sqrt{\frac{L}{39.2}}$, the time required, in seconds.

Ex. 3. To compare the lengths of two pendulums, whose times of oscillation are as 1 to 3.

Since $T \propto \sqrt{L}$, $T^2 \propto L$; therefore, $1 : 9 :: L : l$.

(297.) Cor. 4. The number of oscillations, which a pendulum makes in a given time, at a given place, varies inversely as the square root of it's length.

Let n be the number of oscillations, t the time of one oscillation; then, nt is the whole time, which, by the supposition, is given; therefore, $n \propto \frac{1}{t}$ (*Alg.* Art. 206.), and $t \propto \sqrt{L}$; consequently, $n \propto \frac{1}{\sqrt{L}}$.

Ex. 1. If a pendulum, whose lengh is 39.2 inches, vibrate seconds, or 60 times in a minute, how often will a pendulum whose length is 10 inches vibrate in the same time?

Since $n \propto \frac{1}{\sqrt{L}}$, we have $\sqrt{10} : \sqrt{39.2} :: 60 : 60 \times \sqrt{39.2} = 118.8$, nearly, the number of oscillations required.

Ex. 2. If a pendulum, whose length is 39.2 inches, vibrate seconds, to find the length of a pendulum which will vibrate double seconds, or 30 times in a minute.

Since $n \propto \frac{1}{\sqrt{L}}$, we have, $L \propto \frac{1}{n^2}$; and in this case, $\overline{30}|^2 : \overline{60}|^2 :: 39.2 : L = 4 \times 39.2 = 156.8$ inches, the length required.

Ex. 3. To find how much the pendulum of a clock, which loses one second in a minute, ought to be shortened.

Since the pendulum vibrates 59 times, whilst a pendulum of 39.2 inches vibrates 60 times, it's length may be found as in the last example; $\overline{59}|^2 : \overline{60}|^2$ :: 39.2 : 40.5, it's length; and it ought to be 39.2 inches; therefore, 40.5 − 39.2, or 1.3 inches, is the quantity by which it ought to be shortened, in order that it may vibrate seconds.

(298.) Cor. 5. If the force of gravity be not given, the time of an oscillation varies as the square root of the length of the pendulum directly, and as the square root of the force of gravity inversely.

For, the time of an oscillation varies as the time of descent down half the length of the string; and in general, the time of descent through any space $\propto \sqrt{\frac{S}{F}}$ (Art. 241.); in this case, $S = \frac{1}{2}L$; therefore $S \propto L$, and the time of descent $\propto \sqrt{\frac{L}{F}}$; hence T, the time of an oscillation, $\propto \sqrt{\frac{L}{F}}$.

(299.) Cor. 6. If the length of the pendulum be given, $T \propto \frac{1}{\sqrt{F}}$; and $F \propto \frac{1}{T^2}$.

The time in which a given pendulum vibrates, increases as it is carried from a greater latitude on the earth's surface to a less; therefore, the force of gravity decreases as the latitude decreases.

(300.) Cor. 7. The force of gravity at the equator : the force of gravity at any proposed latitude :: the length of a pendulum which vibrates seconds at the equator : the length of a pendulum which vibrates seconds at the proposed latitude.

For, $T \propto \sqrt{\frac{L}{F}}$; if therefore T be given, $\sqrt{F} \propto \sqrt{L}$, or $F \propto L$.

(301.) Cor. 8. If the chord BV* be drawn, the time of descent down the cycloidal arc BV : the time of descent down the chord :: $DB : BV$.

For, the time of descent down the arc BV is equal to half the time of an oscillation (Art. 272); therefore, the time of descent down the arc BV; the time of descent down DV :: half the circumference of a circle : it's diameter :: $DB : DV$; also, the time of descent down DV : the time of descent down the chord BV :: $DV : BV$; therefore, *ex æquo*, the time of descent down the arc BV : the time of descent down the chord :: $DB : BV$.

Prop. LXXV.

(302.) *The space through which a body falls by the force of gravity in the time of an oscillation in a cycloid, or small circular arc, is to half the length of the pendulum, as the square of the circumference of a circle to the square of it's diameter.*

The spaces through which bodies fall by the action of the same uniform force are as the squares of the times (Art. 241.); and since the time of an oscillation : the time of descent down half the length of the pendulum :: the circumference of a circle : it's diameter, the space fallen through in the time of an oscillation :

* The line BV is wanting in the figure.

half the length of the pendulum :: the square of the circumference of a circle : the square of it's diameter.

Ex. To find how far a body will fall by the force of gravity in one second, where the length of the pendulum, which vibrates seconds, is 39.2 inches.

The circumference of a circle : it's diameter :: 3.14159 : 1; consequently, the space fallen through in one second : $\frac{39.2}{2}$ :: $\overline{3.14159}|^2$: 1^2; hence, the space fallen through is $19.6 \times \overline{3.14159}|^2 = 193$ inches, or $16\frac{1}{12}$ feet, nearly.

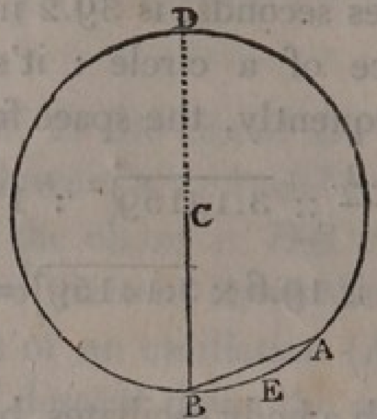
If the arc, in which a body oscillates, be diminished, the effect of the air's resistance is diminished; and when the arc is very small, this resistance does not sensibly affect the time of an oscillation. By observing, therefore, the length of a pendulum which vibrates seconds in very small arcs, we determine the space through which a body would fall *in vacuo* in one second, with sufficient accuracy for all practical purposes.

Prop. LXXVI.

(303.) *The time of descent to the lowest point in a small circular arc is to the time of descent down it's chord, as the circumference of a circle to four times the diameter.*

Let AB be the arc, C it's center, BD the diameter perpendicular to the horizon; T the time of descent down the arc AEB, t the time of descent down the chord, C the circumference of a circle, D it's diameter. Then, $2T$ is the time of an oscillation of the pendulum

CB; therefore, $2T$: the time down $\frac{CB}{2}$ :: $C : D$ (Art. 295.); and the time down $\frac{CB}{2}$: the time down



DB, or AB, :: 1 : 2 (Art. 241.); therefore, $2T$: the time down AB (t) :: $C : 2D$, and $T : t :: C : 4D$.

Prop. LXXVII.

(304.) *The force, which accelerates or retards a body's motion in a cycloid, varies as the arc intercepted between the body and the lowest point.*

Let DV represent the whole force of gravity, (See Fig. in p. 166.) from P draw PH parallel to AD meeting the circle DHV in H; join DH, HV.

Then, the whole force DV, which acts upon the body at P, may be resolved into the two DH, HV; of which DH is in the direction of the string, and therefore neither accelerates nor retards the motion of P; and HV is in the direction of the tangent at P (Art. 281.), and therefore wholly employed in accelerating or retarding the motion in the curve; consequently, the force of gravity : the accelerating force :: DV : HV; and since the force of gravity, and DV, are invariable, the accelerating force $\propto HV \propto 2HV \propto PV$ (Art. 283.).

(305.) Cor. 1. If a body move in any line, and be acted upon by a force which varies as the distance from the lowest point, the motion of this body will be similar to the motion of a body oscillating in a cycloid.

For, if an arc, measured from the vertex of a cycloid, be taken equal to the line, and the accelerating forces, in the line and the cycloid, at these equal distances from the lowest points, be equal, they will always be equal, because they vary according to the same law; and the bodies, being impelled by equal forces, will be equally accelerated, and describe equal spaces in any given time.

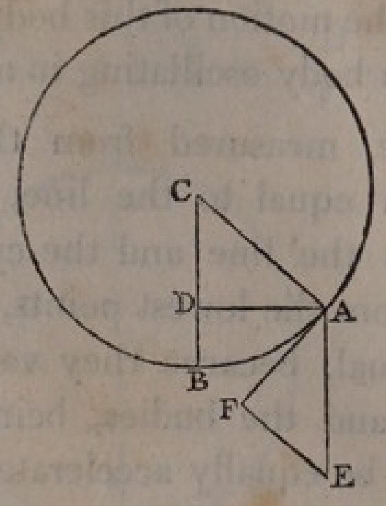
(306.) Cor. 2. The time of descent to the lowest point in the line will always be the same, from whatever place the body begins to fall.

(307.) Cor. 3. If the distance of the body from the lowest point at the beginning of the descent, be made the radius, the velocity acquired will be represented by the right sine, and the time by the arc, whose versed sine is the space fallen through.

Prop. LXXVIII.

(308.) *If a body vibrate in a circular arc, the force which accelerates or retards it's motion varies as the sine of it's distance from the lowest point.*

Let a body oscillate in a circular arc whose radius is AC; from the center C, and A the place of the body, draw CB, AE perpendicular to the horizon, and take AE to represent the force of gravity; draw AD perpendicular to CB, and EF perpendicular to *AF*, which is a tangent to the circle at *A*.



Then, the force *AE* is equivalent to the two *AF*, *FE*; of which *FE* is perpendicular to the tangent *AF*, or in the direction of the radius *CA*, and can neither accelerate nor retard the motion of the body; the other, *AF*, is in the direction of the tangent, and is wholly employed in accelerating or retarding the body's motion; therefore, the force of gravity : the accelerating force :: *AE* : *AF*, that is, from the similar triangles *AEF*, *CAD*, :: *CA* : *AD*; and consequently, the accelerating force $= \frac{\text{gravity} \times AD}{AC}$; in which expression, gravity and the radius *AC* are invariable; therefore, the accelerating force varies as *AD*.

(309.) Cor. 1. If the accelerating force were proportional to the arc, the oscillations, whether in greater or smaller arcs, would be performed in equal times (Art. 306.); but, since the sine does not increase as fast as the arc, the force in the greater arc is less than that which would be sufficient to make the time of oscillation equal to the time in the smaller arc;

therefore, the time of oscillation in the greater circular arc is greater than in the less.

(310.) Cor. 2. Call F the force in the direction AE; then, since AC is invariable, the accelerating force in the curve $\propto F \times AD$; and if $F \times AD \propto AB$, or $F \propto \frac{AB}{AD}$, the accelerating force varies as AB, and the times of oscillation in different arcs are equal (Art. 306.).

SCHOLIUM.

(311.) In this Section we have considered the vibrations of a simple pendulum only, or of a single particle of matter, suspended by a string, the gravity of which is neglected. The Propositions are indeed applicable in practice, when the diameter of the body is small with respect to the length of the string by which it is suspended, and the weight of the string inconsiderable when compared with the weight of the body. That the conclusions are not strictly true in this case, is evident from the consideration that two particles of matter, at different distances from the axis of suspension, do not vibrate in the same time (Art. 296.); and consequently, that when they are connected together, they affect each other's motion; thus, the time of vibration of the two particles when united, is different from the time in which either would vibrate alone.

The method of determining the time of vibration of a compound pendulum, the Reader will find in the *Principles of Fluxions*, Art. 63; to which place he is also referred for the investigation of the rules for determining the centers of *Gyration* and *Percussion*; questions properly belonging to Mechanics, but inserted in that part of the Work, because the rules cannot easily be applied to the determination of those points, even in the most simple cases, without the assistance of the fluxional calculus.

(312.) To avoid the introduction of analytical demonstrations in subjects professedly geometrical, Sir I. Newton and other Writers, have had recourse to indefinitely small or evanescent increments, which continually approximate to the true increments of the quantities whose finite values are required. This method may be applied with success in all cases where the *difference* between the *assumed* and the *true* increments continually decreases, and at length vanishes, with respect to the increments themselves; or, which amounts to the same thing, when the ratio which the sum of the differences bears to one of the increments, does not exceed a finite ratio: for, by observing the *limit* to which the sum of the assumed increments approaches, when their number is increased and their magnitudes are diminished *in infinitum*, it is evident that the sum of the real increments is obtained. In the same manner, when there are two ranks of quantities, in which the assumed increments continually approximate to the real increments, as in the former instance, and the limiting ratio of the sums of the assumed increments in these cases, when their numbers are increased and their magnitudes diminished without limit, is obtained, the exact ratio of the quantities themselves is obtained. These propositions are laid down by Sir. I. Newton in the first Section of the Principia, Lem. 3d and 4th, and the same mode of reasoning has been applied in Art. 292, to compare the time of an oscillation in the cycloid BVA, with the time of describing the arc lZp with the velocity acquired at V continued uniform. In this Art. it is supposed that the time of describing MN, with the *uniform* velocity mx, is the increment of the former time, and that the time of describing xt, the side of a triangle *similar* to Vxm, with the velocity VZ, is the increment of the latter; these *assumed* increments, it is manifest, differ from the true increments of the times under consideration; but when they are diminished without limit, they differ from them by quantities which are evanescent with respect to the whole increments, and therefore by determining the limiting ratio of the sums of the assumed increments, we obtain the ratio of the actual times of describing the corresponding arcs.

SECTION IX.

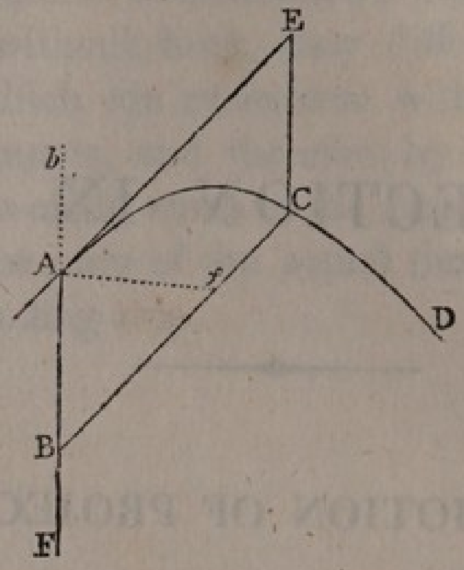
ON THE MOTION OF PROJECTILES.

Prop. LXXIX.

(313.) *A body projected in any direction, not perpendicular to the horizon, will describe a parabola, on supposition that the force of gravity is uniform, and acts in parallel lines, and that the motion is not affected by the resistance of the air.*

Let a body be projected from A in the direction AE, from which point draw ABF perpendicular to the horizon; also, let AE be the space over which the velocity of projection would carry the body in any time, T, and AB the space through which the force of gravity would cause it to descend in the same time; complete the parallelogram AC; then, in consequence of the two motions, the body will be found in C at the end of that time. For, the motion in that direction

AE can neither accelerate nor retard the approach of the body to the line *BC* (Art. 29.); therefore, at the end of the time *T*, the body will be in the line *BC*;



and, by the same mode of reasoning, it appears that it will be in the line *EC* at the same time; consequently, it will be at *C*, the point of their intersection, at the end of the time *T*. Now, since *AE* is the space which would be described in the time *T*, with the velocity of projection continued uniform, $AE \propto T$ (Art. 13.); and $BC = AE$; therefore $BC \propto T$, and $BC^2 \propto T^2$. Also, since *AB* is the space through which the body would fall by the force of gravity in the time *T*, $AB \propto T^2$ (Art. 241.); hence, $AB \propto BC^2$; and this is the property of a parabola, in which *AF* is a diameter, and *BC* an ordinate to the abscissa *AB*.

(314.) Cor. 1. The axis, and all the diameters of the parabola described, being parallel to *AF*, are perpendicular to the horizon.

(315.) Cor. 2. The direction of projection AE is parallel to the ordinate BC, and therefore it is a tangent to the curve at A.

(316.) Cor. 3. The time in which a body describes the arc AC is equal to the time in which it would fall from A to B by the force of gravity, or describe AE with the first velocity continued uniform.

(317.) Cor. 4. If the $\angle EAf$ be made equal to the $\angle EAb$, the line Af will pass through the focus of the parabola described.

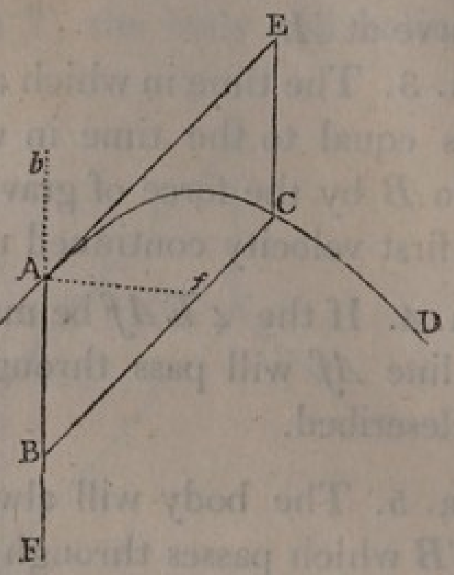
(318.) Cor. 5. The body will always be found in the plane ACB which passes through the direction of projection and the perpendicular to the horizon.

(319.) Cor. 6. If the motion in the direction AE be produced by the action of an uniform force, $AE \propto T^2 \propto AB$; or $AB \propto BC$; therefore, the locus of the point C is a right line.

Prop. LXXX.

(320.) *The velocity of the projectile, at any point in the parabola, is such as would be acquired in falling through one fourth part of the parameter belonging to that point.*

Let AB be the space through which a body must fall by the force of gravity to acquire the velocity of the projectile at A; and AE the space described with that velocity continued uniform, in the time of falling through AB; then $2AB = AE$ (Art. 236.); and, completing the parallelogram AC, $2AB = BC$; hence, $4AB^2 = BC^2$. Also, since C is a point in the parabola, and BC an ordinate to the abscissa AB (Art. 313.),



if P be the parameter belonging to the point A, $P \times AB = BC^2 = 4AB^2$; therefore $AB = \frac{1}{4}P$.

(321.) Cor. 1. If the velocity at A be given, the parameter at that point is the same, whatever be the direction of the body's motion.

(322.) Cor. 2. If AB be the space through which a body must fall to acquire the velocity at A, and a circle be described from the center A with the radius AB, the focus of the parabola described will lie in the circumference of this circle, whatever be the direction of projection; since the distance of any point in the parabola from the focus, is one fourth part of the parameter belonging to that point.

(323.) Cor. 3. The velocity in a parabola varies as the square root of the parameter.

Since the velocity is such as would be acquired in falling through $\frac{1}{4}P$, it varies as $\sqrt{\frac{1}{4}P}$ (Art. 241.), or as $\sqrt{P}$.

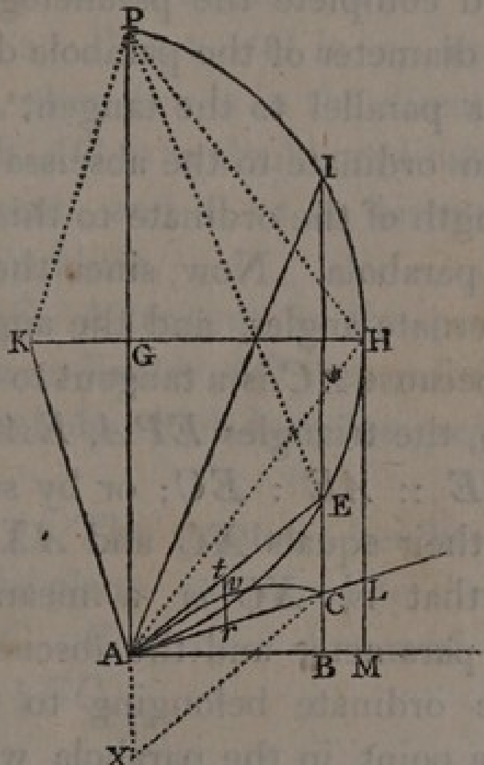
(324.) Cor. 4. The velocity is the least in the vertex of a parabola; and at equal distances from the vertex the velocities are equal.

For, the parameter belonging to the vertex is the least; and the parameters at equal distances from the vertex are equal.

Prop. LXXXI.

(325.) *To find the direction in which a body must be projected from a given point, with a given velocity, to hit a given mark.*

Let A be the point from which the body is to be



projected, C the given mark; join AC, and from A

draw AB parallel, and AP perpendicular to the horizon; take AP equal to four times the space through which a body must fall to acquire the given velocity of projection, (determined by Art. 248, Case 3.); then will AP be the parameter belonging to the point A of the parabola described (Art. 320.). Draw AK perpendicular to AC; bisect PA in G, and draw KGH perpendicular to AP, meeting AK in K; join KP; then the triangles KGP, KGA, being similar and equal, $KP = KA$. From K as a center, with the radius KA, or KP, describe a circle AHP, cutting KGH in H; through C draw CEI parallel to AP, and cutting the circle in E and I; join AE, AI; and if a body be projected, with the given velocity, in the direction AE or AI, it will hit the mark C.

Let the body be projected in the direction AE; join PE, and complete the parallelogram $AECX$; then AX is a diameter of the parabola described; and XC, which is parallel to the tangent AE, is in the direction of an ordinate to the abscissa AX; if then XC be the length of the ordinate to this abscissa, C is a point in the parabola. Now, since the angles AEC, EAP, are alternate angles, and the angle EAC = the angle EPA, because AC is a tangent to the circle at A (Euc. 16. iii.), the triangles EPA, EAC, are similar; and $AP : AE :: AE : EC$; or by substituting for AE and EC their equals XC and AX, $AP : XC :: XC : AX$; that is, XC is a mean proportional between the parameter and the abscissa; and therefore it is the ordinate belonging to that abscissa; hence, C is a point in the parabola which the body describes.

In the same manner it may be shewn that, if the body be projected with the same velocity in the direction *AI*, it will hit the mark *C*.

(326.) Cor. 1. Join *AH*, *HP*; then the angle *HAP* = the angle *HPA* = the angle *HAC*.

(327.) Cor. 2. Because *KH* is drawn through the center of the circle perpendicular to the chord *EI*, it bisects it, and consequently it bisects the arc *IHE* (Euc. 30. iii.); therefore, the angle *IAH* = the angle *HAE*. That is, the two directions *AE*, *AI*, make equal angles with *AH*, which bisects the angle *PAC*.

(328.) Cor. 3. Draw *HLM* touching the circle in *H*; then, when the point *C* coincides with *L*, the two directions *AE*, *AI*, coincide with *AH*.

(329.) Cor. 4. If the point *C* be taken in the plane *AL*, beyond *L*, the line *CEI* will not meet the circle. In this case, the velocity of projection is not sufficient to carry the body to the distance *AC*.

(330.) Cor. 5. Bisect *AC* in *r*, and draw *tvr* parallel to *AP*; then *tvr* is in the direction of a diameter, to which *AC* is a double ordinate. Also, *rt* is the sub-tangent; and if *rt* be bisected in *v*, this point is in the parabola.

(331.) Cor. 6. A tangent to the parabola at *v* is parallel to the ordinate *AC*; therefore, *v* is the point in the parabola which is at the greatest distance from *AC**.

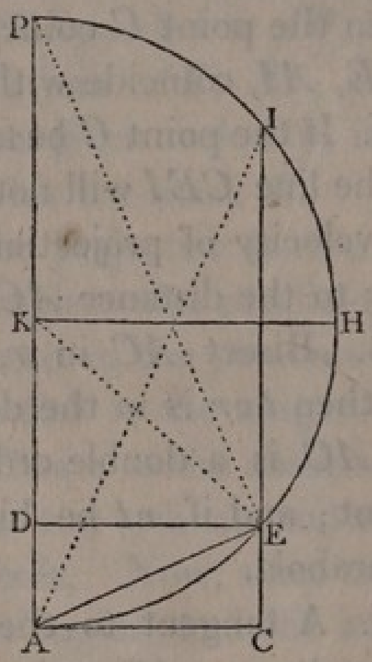
(332.) Cor. 7. The greatest height of the projectile above the plane, measured in the direction of gravity, is $\frac{1}{4}EC$. For, $rv = \frac{1}{2}rt$, and $rt = \frac{1}{2}EC$; therefore, $rv = \frac{1}{4}EC$.

* The properties of the parabola here referred to, may be found in any Treatise of Conic Sections.

Prop. LXXXII.

(333.) *Having given the velocity and direction of projection, to find where the body will strike the horizontal plane which passes through the point of projection.*

Let AC (Art. 325.) coincide with the horizontal line AB; then AK coincides with AG; and PHA is



a semi-circle; also, HAC is an angle of 45°, and AE, AI, are equally inclined to AH.

From the last Proposition it appears, that if the velocity of projection be such as would be acquired in falling through $\frac{1}{4}PA$, and AE, or AI, be the direction of projection, the range is AC. From E draw ED parallel to AC, or perpendicular to AP; join EK; then ED, or it's equal AC, is the sine of the $\angle EKA$ to the radius KA; and the $\angle EKA = 2 \angle EPA = 2 \angle EAC$;

therefore, AC is the sine of $2\angle EAC$ to the radius KA; and the sine of a given angle is proportional to the radius; consequently, rad. : KA :: sin. $2\angle EAC$: AC; hence, $AC = \frac{\text{sin. } 2\angle EAC \times KA}{\text{rad.}}$. If V be taken to represent the velocity of projection, P the parameter AP, and $m = 16\frac{1}{12}$, then $AC = \frac{\text{sin. } 2\angle EAC \times P}{2 \text{ rad.}} = \frac{\text{sin. } 2\angle EAC \times V^2}{2 \text{ rad.} \times m}$ (Art. 248.)

In the same manner, if AI be the direction of projection, $AC = \frac{\text{sin. } 2\angle IAC \times P}{2 \text{ rad.}} = \frac{\text{sin. } 2\angle IAC \times V^2}{2 \text{ rad.} \times m}$.

(334.) Cor. 1. Hence, $AC \propto \text{sin. } 2\angle EAC \times V^2$.

(335.) Cor. 2. If the velocity of projection be invariable, the horizontal range varies as sin. $2\angle EAC$.

(336.) Cor. 3. The range is the greatest when sin. $2\angle EAC$ is the greatest; that is, when the $\angle EAC$ is 45°. In this case, $AC = \frac{\text{rad.} \times P}{2 \text{ rad.}} = \frac{1}{2} P$.

(337.) Cor. 4. If the angle EAC be 15°, or 75°, sin. $2\angle EAC = \frac{1}{2}$ rad. and $AC = \frac{1}{4} P$.

(338.) Cor. 5. If the range and the parameter be given, the angle of elevation may be found.

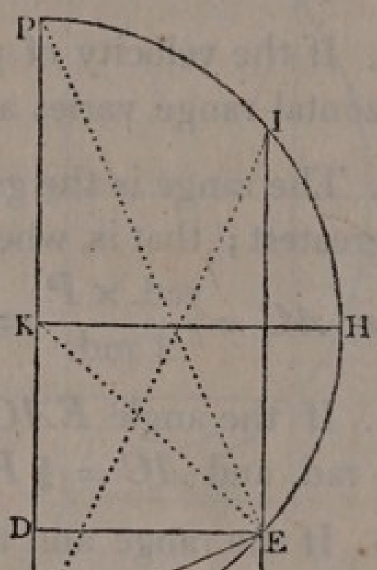
For, $AC = \frac{\text{sin. } 2\angle EAC \times P}{2 \text{ rad.}}$; therefore, sin. $2\angle EAC = \frac{2AC \times \text{rad.}}{P}$.

(339.) Cor. 6. If AC and the angle EAC be known, $P = \frac{2AC \times \text{rad.}}{\text{sin. } 2\angle EAC}$; and $V^2 = \frac{2 \text{ rad.} \times mAC}{\text{sin. } 2\angle EAC}$.

Prop. LXXXIII.

(340.) *The same things being given, to find the time of flight.*

The time in which the velocity of projection, V, would be acquired, or the time of descent down $\frac{1}{4}PA$, is $\frac{V}{2m}''$ (Art. 248.); hence, the double of this time, or the time of descent down PA, is $\frac{V}{m}''$ (Art. 241.); which is also the time of descent down EA (Art. 264.) Let T be the time of descent down PA, or EA, t the time of descent down EC, or the time of flight (Art. 316.);



then $T : t :: EA : EC$ (Art. 262.); that is, $T : t ::$ rad. : sin. $\angle EAC$, and $t = \frac{\text{sin. } \angle EAC \times T}{\text{rad.}} = \frac{\text{sin. } \angle EAC \times V}{\text{rad.} \times m}$.

(341.) Cor. 1. If the velocity of projection be invariable, the time of flight $\propto$ sin. $\angle EAC$.

(342.) Cor. 2. Hence, the time of flight is the greatest, when sin. $\angle EAC$ is the greatest. In this case, the time becomes $\frac{\text{rad.} \times T}{\text{rad.}}$, or T; that is, the greatest time of flight is equal to the time of descent down the parameter.

Prop. LXXXIV.

(343.) *The same things being given, to find the greatest height to which the projectile rises above the horizontal plane.*

The greatest height is $\frac{1}{4}EC$, or $\frac{1}{4}AD$; and AD is the versed sine of the $\angle AKE$, or $2 \angle EAC$, to the radius AK; consequently, since the versed sine of a given angle varies as the radius, rad. : AK ($\frac{1}{2}P$) :: the versed sine of $2 \angle EAC$: AD. Hence, $AD = \frac{\text{ver. sin. } 2 \angle EAC \times P}{2 \text{ rad.}}$; and $\frac{1}{4}AD$, the greatest height,

$$= \frac{\text{ver. sin. } 2 \angle EAC \times P}{8 \text{ rad.}} = \frac{\text{ver. sin. } 2 \angle EAC \times V^2}{8 \text{ rad.} \times m}$$

(344.) Cor. 1. The greatest height $\propto$ ver. sin. $2 \angle EAC \times V^2$; and when V is given, the greatest height $\propto$ ver. sin. $2 \angle EAC$.

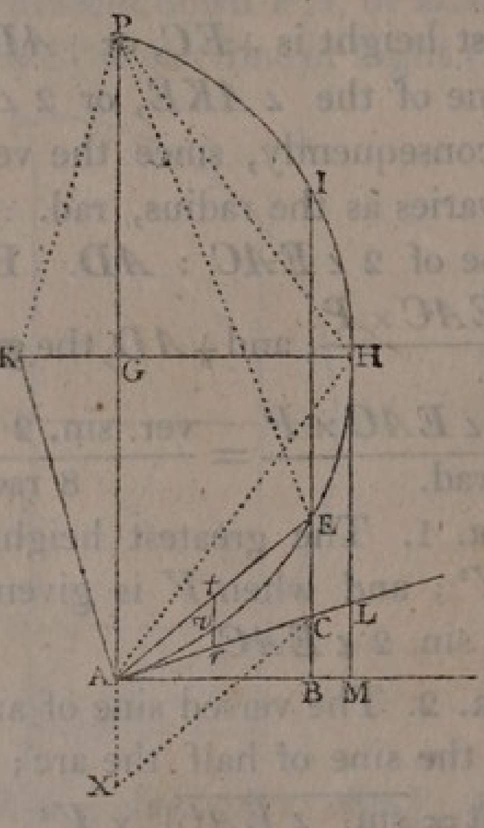
(345.) Cor. 2. The versed sine of an arc varies as the square of the sine of half the arc; therefore the greatest height $\propto \overline{\text{sin. } \angle EAC}|^2 \times V^2$.

(346.) Cor. 3. A body, projected with a given velocity, will rise to the greatest height above the horizontal plane, when the angle of elevation EAC is a right angle. In this case, ver. sin. $2 \angle EAC = 2$ rad. and the greatest altitude is $\frac{2 \text{ rad.} \times P}{8 \text{ rad.}} = \frac{P}{4}$.

Prop. LXXXV.

(347.) *The velocity and direction of projection being given, to find where the body will strike a given inclined plane which passes through the point of projection.*

It appears from Art. 325, that if a body be projected from A, in the direction AE, with the velocity acquired in falling down $\frac{1}{4}PA$, it will strike the



plane AC in the point C. Let I be the angle of inclination CAB; E the angle of elevation EAC; Z the angle EAP. Then, in the triangle EAP, $AE : AP :: \sin. \angle EPA : \sin. \angle AEP$; and the $\angle EPA =$ the $\angle EAC = E$; also the $\angle AEP =$ the $\angle ECA =$ the supplement of the $\angle ACB$; hence, AE

$: AP :: \sin. E : \cos. I$; therefore, $AE = \dfrac{\sin. E \times AP}{\cos. I}$.

Again, in the triangle EAC, $AC : AE :: \sin. \angle AEC$ ($\sin. Z$) : $\sin. \angle ACE$ ($\cos. I$); therefore, $AC = \dfrac{\sin. Z \times AE}{\cos. I}$; and by substituting for AE it's value $\dfrac{\sin. E \times AP}{\cos. I}$, $AC = \dfrac{\sin. E \times \sin. Z \times AP}{\overline{\cos. I}|^2} = \dfrac{\sin. E \times \sin. Z \times V^2}{\overline{\cos. I}|^2 \times m}$.

(348.) Cor. Hence, $AC \propto \dfrac{\sin. E \times \sin. Z \times V^2}{\overline{\cos. I}|^2}$.

Prop. LXXXVI.

(349.) *The same things being given, to find the time of flight.*

Let T be the time of descent down PA, t the time of descent down EC or the time of flight; then, $T^2 : t^2 :: PA : EC$; and since, in the similar triangles PAE, AEC, $PA : AE :: AE : EC$, we have $PA^2 : AE^2 :: PA : EC :: T^2 : t^2$; and $PA : AE :: T : t$; but $PA : AE :: \sin. \angle PEA : \sin. \angle EPA :: \sin. \angle ECA : \sin. \angle EAC :: \cos. I : \sin. E$; therefore, $T : t :: \cos. I : \sin. E$, and $t = \dfrac{\sin. E \times T}{\cos. I} = \dfrac{\sin. E \times V}{\cos. I \times m}$.

(350.) Cor. Hence, $t \propto \dfrac{\sin. E \times V}{\cos. I}$; and if V be invariable, $t \propto \dfrac{\sin. E}{\cos. I}$.

Prop. LXXXVII.

(351.) *The same things being given, to find the greatest height of the projectile above the plane* AC, *measured in the direction of gravity.*

The greatest height is $\frac{1}{4}EC$ (Art. 332.); and in the triangle AEC, $EC : AE$ :: sin. E : cos. I; therefore, $EC = \frac{\text{sin. } E \times AE}{\text{cos. } I}$; and, by substituting for AE it's value $\frac{\text{sin. } E \times AP}{\text{cos. } I}$ (Art. 347.), we have $CE = \frac{\overline{\text{sin. } E}|^2 \times AP}{\overline{\text{cos. } I}|^2}$; and $\frac{1}{4}EC = \frac{\overline{\text{sin. } E}|^2 \times AP}{4\ \overline{\text{cos. } I}|^2} = \frac{\overline{\text{sin. } E}|^2 \times V^2}{\overline{\text{cos. } I}|^2 \times 4m}$, the greatest height required.

(352.) Cor. The greatest height varies as $\frac{\text{sin. } E^2 \times V^2}{\overline{\text{cos. } I}|^2}$.

SCHOLIUM.

(353.) The theory of the motion of projectiles, given in this section, depends upon three suppositions, which are all inaccurate; 1st. that the force of gravity, in every point of the curve described, is the same; 2d. that it acts in parallel lines; 3d. that the motion is performed in a non-resisting medium. The two former of these, indeed, differ insensibly from the truth. The force of gravity, without the Earth's surface, varies inversely as the square of the distance from the center; and the altitude to which we can project a body from the surface is so small, that the variation of the force, arising from the alteration of the distance from the center of the Earth, may safely be neglected. The direction of the force is every where perpendicular to the horizon; and if perpendiculars be thus drawn, from any two points in the curve which we can cause a body to describe, they may be considered as parallel, since they only meet at, or nearly at, the center of the Earth. Even the resistance of the air does not materially affect the motions of heavy bodies, when they are projected with small velocities. In other cases, however, this resistance is so great as to render the conclusions, drawn from the theory, almost entirely inapplicable in practice. From experiments made to determine the motions of cannonballs, it appears that when the initial velocity is considerable, the air's resistance is 20 or 30 times as great as the weight of the ball; and that the horizontal range is often not $\frac{1}{20}$ part of that which the preceding theory leads us to expect. It appears also, that when the angle of elevation is given, the horizontal range varies nearly as the square root of the velocity of projection; and the time of flight as the range; whereas, according to the theory, the time varies as the velocity, and the range as the square of the velocity of projection (Arts. 340. 334.) These experiments, made with great care, and by men of eminent abilities, shew how little the parabolic theory is to be depended upon in determining the motions of military projectiles. See ROBIN'S *New Theory of Gunnery*, and HUTTON'S *Mathematical Dictionary*, article *Gunnery*.

Besides diminishing the velocity of the projectile, the air's resistance will also change it's direction, whenever the body has a rotatory motion about an axis which does not coincide with the direction in which it is moving. For the velocity with which that side of the body, strikes the air, on which the rotatory and progressive motions conspire, is greater than the velocity with which the other side strikes it, where they are contrary to each other; and therefore the resistance of the air, which increases with the velocity, will be greater in the former case than in the latter, and cause the body to deviate from the line of it's motion; this deviation will also be from the *plane* of the first motion, unless the axis of rotation be perpendicular to that plane.

Upon this principle Sir I. Newton explains the irregular motion of a tennis-ball*, and the same cause has been assigned by Mr. Robins for the deviation of a bullet from the vertical plane†. Mr. Euler, indeed, in his remarks on the *New Theory of Gunnery*, contends that the resistance of the air can neither be increased nor diminished by the rotation of the ball; because such a motion can produce no effect but in the direction of a tangent to the surface of the revolving body; and the tangential force, he affirms, is almost entirely lost. In this instance, the learned writer seems to have been misled by the common theory of resistances, according to which the tangential force produces no effect; whereas, from experiments lately made, with a view to ascertain the quantity and laws of the air's resistance, it appears that every theory which neglects the tangential force must be erroneous.

* Phil. Trans. Vol. VI. p. 3078. Maclaurin's *Newton*, p. 120.

† Tracts, Vol. I. pp. 151. 198. 214.

APPENDIX.

ON THE EFFECTS PRODUCED BY WEIGHTS ACTING UPON MACHINES IN MOTION, AND ON THE ROTATION OF BODIES.

The investigation of the effects produced by bodies when the machines on which they act are in motion, has not usually been introduced into elementary Treatises; but as the theory depends upon the principles already laid down, and may, by the help of the simplest analytical operations, be easily deduced from them, it may not improperly be added, by way of Appendix, here.

Prop. LXXXVIII.

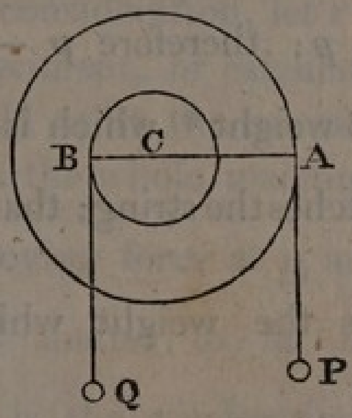
(354.) *To find what weight* x, *placed at* A *upon a machine in motion, resists the rotation as much as* y *placed at* B.

Let a and b be the velocities of the weights; then xa and yb are their momenta; and since these momenta produce equal effects on the machine, or, are sufficient to balance each other, $xa : yb :: b : a$ (Art. 149.); therefore $xa^2 = yb^2$, and $x = \frac{yb^2}{a^2}$.

Prop. LXXXIX.

(355.) *If two weights acting upon a wheel and axle put the machine in motion, to determine the velocity acquired by the descending body, and the tension of the string by which it acts.*

Let C be the center of motion; CA, CB the radii of the wheel and axle; p and q the two weights, of which p descends; $CA = a$, $CB = b$, then a and b are proportional to the velocities of p and q. And



let $z =$ the weight which q would sustain at p; and $x =$ the weight which placed at p would resist the communication of rotation as much as q resists it; $v =$ the velocity generated in the time t; $m = 16\frac{1}{12}$ feet.

Then $a : b :: q : z = \frac{qb}{a}$, and $x = \frac{qb^2}{a^2}$ (Art. 354.); hence, $p - \frac{qb}{a} =$ the force at p to move the machine, and $p + \frac{qb^2}{a^2} =$ the inertia to be moved, neglecting the inertia of the machine; consequently,

$$\frac{p - \frac{qb}{a}}{p + \frac{qb^2}{a^2}} = \text{the accelerating force, that of gravity}$$

being represented by unity (Art. 268.); and since $v = 2mft$ (Prop. LXI. Cor. 7.), we have, in this case, $v = \dfrac{p - \frac{qb}{a}}{p + \frac{qb^2}{a^2}} \times 2mt = \dfrac{pa^2 - qab}{pa^2 + qb^2} \times 2mt.$

Again, since $\dfrac{pa^2 - qab}{pa^2 + qb^2}$ is the accelerating force at p, the moving force, which generates p's velocity, is $\dfrac{pa^2 - qab}{pa^2 + qb^2} \times p$; therefore $p - \dfrac{pa^2 - qab}{pa^2 + qb^2} \times p$ is that part of p's weight* which is sustained, or the weight which stretches the string; that is, $\dfrac{pqb^2 + pqab}{pa^2 + qb^2}$, or $\dfrac{\overline{a + b}.bpq}{pa^2 + qb^2}$ is the weight which stretches the string AP.

(356.) Cor. 1. The tension of the string AP is just sufficient to sustain the tension of the string BQ; therefore $b : a :: \dfrac{\overline{a + b}.bpq}{pa^2 + qb^2} : \dfrac{\overline{a + b}.apq}{pa^2 + qb^2}$ = the tension of the string BQ.

(357.) Cor. 2. The pressure on the center of motion is the sum of the tensions of the strings AP

* In this operation, the moving force and the quantity of matter are, respectively, represented by the weight.

and BQ (Art. 101.), or, $\frac{\overline{a+b}.b}{pa^2+qb^2} \times pq + \frac{\overline{a+b}.a}{pa^2+pb^2} \times pq = \frac{\overline{a+b}|^2.pq}{pa^2+qb^2}$.

(358.) Cor. 3. When a and b are equal, the pressure on the center is $\frac{4pq}{p+q}$.

(359.) Cor. 4 Since s, the space which p descends from rest in t seconds $= mft^2$ (Prop. LXI. Cor. 7.), $s = \frac{pa^2 - qab}{pa^2 + qb^2} \times mt^2$.

(360.) Cor. 5. The same reasoning may be applied when the bodies act upon any other machine.

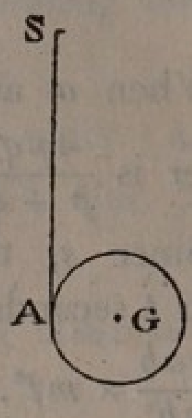
(361.) Cor. 6. If the inertia of the machine is to be taken into consideration, let r be the weight determined by experiment, or calculation, which when placed at p, would resist the communication of rotation as much as the whole machine resists it; then $p - \frac{qb}{a}$ is the moving force at p, and $r + p + \frac{qb^2}{a^2}$ is the quantity of matter to be moved; therefore, $\frac{pa^2 - qab}{ra^2 + pa^2 + qb^2}$ is the accelerating force at p, the accelerating force of gravity being represented by unity.

Prop. XC.

(362.) *If a string be wrapped round a hollow cylinder* G, *and one end fixed at* S, *to find the tension of the string when the cylinder is suffered to descend.*

Let a = the weight of the cylinder, collected in the circumference; x = the tension of the string.

Then, since the motion of the center of gravity of the cylinder is the same at whatever point of the



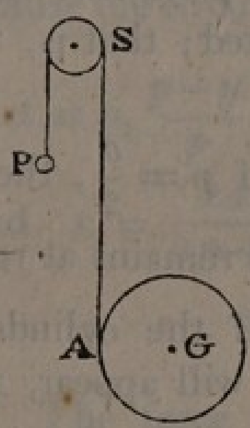
body the force is applied (Art. 182.), $a-x$ is the moving force by which the center of gravity of the cylinder descends, and $\frac{a-x}{a}$ is the accelerating force. Again, x is the weight, or moving force, which applied at the circumference of the cylinder, generates the rotation, and $\frac{x}{a}$ is the accelerating force; and since accelerating forces are proportional to the velocities generated in the same time, and, from the nature of the case, the center of gravity of the cylinder descends as fast as the string is unfolded, that is, the velocities of the center of gravity and rotation are always equal, we have $\frac{a-x}{a}=\frac{x}{a}$; hence, $x=\frac{a}{2}$; or the tension of the string is half the weight of the cylinder.

(363.) Cor. The accelerating force is $\frac{a-x}{a}=\frac{1}{2}$, the accelerating force of gravity being represented by unity.

Prop. XCI.

To find the tension when the string passes over a fixed pulley and a weight is attached to it.

(364.) Let p be the weight of the body attached to the string; x the tension; a the weight of the



cylinder. Then $p - x$ is the moving force on p; $\frac{p-x}{p}$, the accelerating force; $\frac{a-x}{a}$, the force which accelerates the cylinder; $\frac{x}{a}$, the accelerating force which produces the rotation; which quantities are proportional to the velocities generated in the same time. Also, the spaces descended by p and A are, together, always equal to the length of the string disengaged, therefore, $\frac{p-x}{p}+\frac{a-x}{a}=\frac{x}{a}$; hence, $x = \frac{2ap}{2p+a}$.

(365.) Cor. 1. The pressure on the center of the pulley is $2x$, or $\frac{4ap}{2p+a}$.

(366.) Cor. 2. The accelerating force on $p = \frac{p-x}{p} = \frac{2p-a}{2p+a}$.

(367.) Cor. 3. The accelerating force on the cylinder $= \frac{a-x}{a} = \frac{a}{2p+a}$.

(368.) Cor. 4. If $p=a$, the body and the cylinder are equally accelerated; that is, they descend at the same rate.

(369.) Cor. 5. If $p = \frac{a}{2}$, the accelerating force on p vanishes, and p remains at rest.

(370.) Cor. 6. If the cylinder be solid and of uniform density, it will appear, nearly in the same manner, that the tension of the string is $\frac{2ap}{3p+a}$; the force which accelerates the cylinder, $\frac{p+a}{3p+a}$; and the force which accelerates p, $\frac{3p-a}{3p+a}$.

Prop. XCII.

(371.) *To find the tension of the string, when the weight of the pulley is taken into the account.*

Let c be the weight which, placed at the circumference of the pulley, would resist the communication of motion as much as the pulley; and let $y =$ the tension of the string SP, $x =$ the tension of SA. Then $\frac{p-y}{p} =$ the accelerating force on p, $\frac{a-x}{a} =$ the accelerating force on A; $\frac{y-x}{c}$ = the accelerating force on the circumference of the pulley; and $\frac{x}{a}$ = the accelerating force which produces the rotation of the cylinder. Then, as in the last Proposition, $\frac{p-y}{p} + \frac{a-x}{a} = \frac{x}{a}$; also, because the circumference of the pulley always moves as fast as p, $\frac{p-y}{p} = \frac{y-x}{c}$; from which equations we find $x = \frac{2ap + ac}{2p + a + 2c}$; and $y = \frac{\overline{a+c}.2p}{2p+a+2c}$.

(372.) Cor. 1. The force which accelerates the cylinder is $\frac{a-x}{a} = \frac{a+c}{2p+a+2c}$; and the force which accelerates p, $= \frac{p-y}{p} = \frac{2p-a}{2p+a+2c}$.

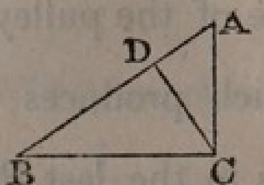
(373.) Cor. 2. The accelerating force being known, the space, time, and velocity, may be found in terms of each other (Prop. LXI. Cor. 7).

Prop. XCIII.

(374.) *If any weights* A, B, C, *act upon a machine and put it in motion, and* x, y, z, *be the spaces described in the direction of gravity,* a, b, c, *the actual velocities of the weights,* m $= 16\frac{1}{12}$ *feet, then*

$4m \times \overline{Ax + By + Cz} = Aa^2 + Bb^2 + Cc^2$.

Let AB be the direction of A's motion, AC perpendicular to the horizon; take $AB = s$, the space described by A in a very small time; draw BC



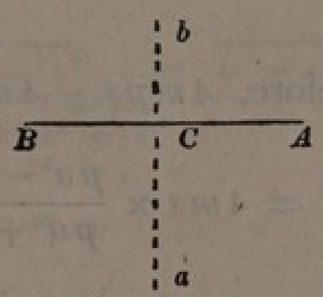
parallel to the horizon, and CD perpendicular to AB; let f be the force which accelerates A's motion; F the accelerating force in the direction of gravity.

Then $2mf\dot{s} = a\dot{a}$ (VINCE's *Flux.* Art. 82.); and $F : f :: AC : AD :: AB : AC :: \dot{s} : \dot{x}$; therefore, $f\dot{s} = F\dot{x}$, and $2mF\dot{x} = 2mf\dot{s} = a\dot{a}$, consequently $F = \frac{a\dot{a}}{2m\dot{x}}$; hence, the effective moving force on A, in the direction of gravity, $= \frac{Aa\dot{a}}{2m\dot{x}}$, and $A - \frac{Aa\dot{a}}{2m\dot{x}}$ is that part of A's whole weight, or moving force, which is sustained by the action of the other bodies in the system, that is, with which A urges the machine in the direction of gravity. In the same manner, $B - \frac{Bb\dot{b}}{2m\dot{y}}$ is the part of B's moving force sustained, and the weight at A which would balance this : $B - \frac{Bb\dot{b}}{2m\dot{y}} :: \dot{y} : \dot{x}$ (Art. 149.), therefore $\frac{B\dot{y}}{\dot{x}} - \frac{Bb\dot{b}}{2m\dot{x}}$ is the weight at A which would balance B's pressure upon the machine, and $\frac{Bb\dot{b}}{2m\dot{x}} - \frac{B\dot{y}}{\dot{x}}$ is that part of A's moving force which is sustained by B. In the same manner $\frac{Cc\dot{c}}{2m\dot{x}} - \frac{C\dot{z}}{\dot{x}}$ is that part of A's moving force which is sustained by C; consequently $A - \frac{Aa\dot{a}}{2m\dot{x}} = \frac{Bb\dot{b}}{2m\dot{x}} - \frac{B\dot{y}}{\dot{x}} + \frac{Cc\dot{c}}{2m\dot{x}} - \frac{C\dot{z}}{\dot{x}}$, and $2m \times \overline{A\dot{x} + B\dot{y} + C\dot{z}} = Aa\dot{a} + Bb\dot{b} + Cc\dot{c}$, and taking the fluents, which require no correction, $2m \times \overline{Ax + By + Cz} = \frac{Aa^2}{2} + \frac{Bb^2}{2} + \frac{Cc^2}{2}$, or $4m \times \overline{Ax + By + Cz} = Aa^2 + Bb^2 + Cc^2$*.

(375.) Cor. If any of the bodies move in a direction opposite to that which is here supposed to be positive, the space described must be reckoned negative.

Ex. 1. If the weights A and B be attached to the lever AB, to find the velocity acquired by A during the motion of the lever, round the pivot C, from an horizontal to a vertical position.

Let $CA = a$, $CB = b$, $v =$ the velocity acquired by A; then $a : b :: v : \frac{bv}{a} =$ the velocity acquired



by B; therefore, by the Proposition, $4mAa - 4mBb = Av^2 + \frac{Bb^2v^2}{a^2}$, and $v^2 = 4ma^2 \times \frac{Aa - Bb}{Aa^2 + Bb^2}$;

* For this very concise demonstration, the Author is indebted to the suggestions of the Rev. D. M. Peacock.

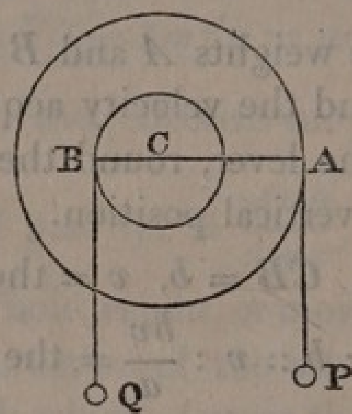
and by extracting the square root of this quantity, v is obtained.

Ex. 2. If the weight p raise q by the wheel and axle, and descend through s feet, to find the velocity acquired by p.

Let $CA=a$, $CB=b$, and v = the velocity required.

Then $a : b :: v : \frac{bv}{a} = q$'s velocity,

and $a : b :: s : \frac{bs}{a}$ = the space through which

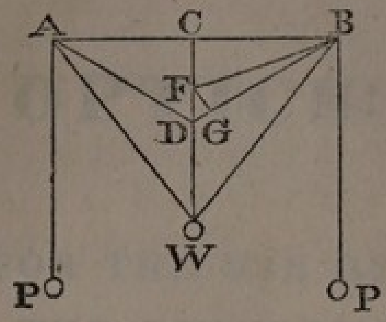


q is raised; therefore, $4mps - 4m \times \frac{bqs}{a} = pv^2 + q \times \frac{b^2v^2}{a^2}$, and $v^2 = 4ms \times \frac{pa^2 - qab}{pa^2 + qb^2}$; whence v is known.

Ex. 3. If two equal weights p, p, attached to a string which passes over the pullies A and B, raise the weight w through the space WD, to find the velocity communicated to W.

Suppose AB to be parallel to the horizon, and WDC perpendicular to it. Take $BC = CA = a$;

$WC = b$; $WB = c$; $DC = x$; $v =$ the velocity of



w; $y =$ the velocity of p; DF the small increment of WD; draw FG at right angles to DB.

Then $v : y :: DF : DG :: DB : DC$, and $v^2 : y^2 :: DB^2 : DC^2 :: a^2 + x^2 : x^2$, therefore $y^2 = \frac{x^2 v^2}{a^2 + x^2}$. Also, WD, or $b - x$ is the space which w has described in a direction perpendicular to the horizon, and $WB - DB$, or $c - \sqrt{a^2 + x^2}$, is the space through which p has descended; therefore, by the Proposition, $4m.2p.\overline{c - \sqrt{a^2 + x^2}} - 4mw.\overline{b - x} = \frac{2px^2v^2}{a^2 + x^2} + wv^2$, and $v^2 = 4m.\overline{a^2 + x^2} \times \frac{2pc - 2p.\sqrt{a^2 + x^2} - w.\overline{b - x}}{2px^2 + w.\overline{a^2 + x^2}}$; whence v is known.

THE END.

THE ELEMENTS

OF

OPTICS:

DESIGNED FOR THE USE OF STUDENTS
IN THE UNIVERSITY.

By JAMES WOOD, D.D.

MASTER OF ST. JOHN'S COLLEGE, CAMBRIDGE.

FOURTH EDITION.

CAMBRIDGE:

Printed by J. Smith, Printer to the University;

AND SOLD BY DEIGHTON & SONS, AND NICHOLSON & SON, CAMBRIDGE;
AND J. MAWMAN, 39 LUDGATE STREET, LONDON.

1818.

CONTENTS.

THE ELEMENTS

OF

OPTICS.

Sect. I.

ON THE NATURE OF LIGHT, AND THE LAWS OF REFLECTION AND REFRACTION.

(Art. 1.) By Optics we understand that branch of Natural Philosophy which treats of the nature and properties of Light, and the Theory of Vision.

(2.) Modern Philosophers have made two hypotheses to explain the manner in which vision is produced by luminous objects. Des Cartes, Huygens and Euler, suppose that there is a subtile, elastic medium which penetrates all bodies, and fills all space; and that vibrations, excited in this fluid by the luminous body, are propagated thence to the eye, and produce the sensation of vision, in the same manner that the vibrations of the air, striking against the ear, produce the sensation of sound.

It has been objected to this hypothesis, and the objection has never been answered, that the vibrations of an elastic fluid are propagated in every direction, and into every corner to which the fluid extends; on the supposition therefore that light is nothing more than the effect of the vibrations of such a fluid, there could be no shadow, or darkness.

If it be said that the fluid, by means of which vision is excited, is different from all other elastic fluids, the effect is ascribed to a cause, the nature of which is unknown; and the hypothesis amounts to nothing more than a confession, that we are ignorant in what manner vision is produced.

The other hypothesis, adopted by Sir I. Newton and his followers, is, that light consists of very small particles of matter, which are constantly thrown off from luminous bodies, and which produce the sensation of vision by actual impact upon the proper organ.

In favour of this hypothesis, it is observed that the motion of light is conformable to the laws which regulate the motions of small bodies, under the same circumstances: Thus, where it meets with no impediment, it moves uniformly forward in right lines*; and in it's passage into, and reflection from different mediums, the direction of it's motion is changed as it would be, did it consist of small particles of matter, attracted towards, or repelled from the surfaces upon which they are incident†.

Whether light has other properties of matter or not, is a question which does not appear to have been fairly decided; we may however be allowed to consider it as material, and to speak of it as consisting of particles of *matter*, till a more satisfactory hypothesis can be framed; especially, as we deduce no conclusions from the supposition, nor build any theory upon it. Those properties of light from which our theory of vision is

* See *Mechanics*, Art. 27. † Newt. *Principia*, Prop. 94, 96.

derived, are discovered by experiment, and they are wholly independent of any hypothesis respecting the manner in which the sensation is produced *.

(3.) Mr. Roemer, a Danish Astronomer, first discovered that light is propagated in time, and not communicated instantaneously from the luminous body to the eye. The discovery was made by observing that the eclipses of Jupiter's Satellites happen sooner, when he is in opposition, and later when he is in conjunction, than they ought to do according to calculations made on supposition that he is at his mean distance from the earth. To reconcile this difference between the observations and calculations, it is necessary to allow about 8′ for the time in which light passes over a radius of the earth's orbit; and the truth of the supposition is fully confirmed by Dr. Bradley's discovery of an apparent change of place in the fixed stars, which arises from the progressive motion of light, combined with the motion of the earth in it's orbit †.

(4.) Though we are, in many respects, ignorant of the nature of light, we know that it consists of distinct and independent parts.

For, it may be stopped one moment, and the next suffered to proceed; or a portion may be stopped, whilst the rest of the light is suffered to go on.

(5.) DEF. The least portion of light, which may be stopped alone, or propagated alone, or do or suffer any thing which the rest of the light does not or suffers not, is called a *Ray of Light*.

---

* See *Horsley's Newt.* vol. IV. p. 305.

† The velocity of light, determined by these different observations, is nearly the same, and about 195,000 miles per second. Hence we conclude that the velocity of light is uniform; and that direct and reflected rays move at the same rate.

Rays of light are represented by lines, drawn in the directions in which the particles move.

(6.) Def. Whatever affords a passage to the rays of light is called a *Medium*; as glass, water, air, &c. and in this sense, a vacuum is called a medium.

(7.) Def. The *density* of light is measured by the number of parts, or particles uniformly diffused over a *given* surface.

Cor. If the surface be not given, the density varies as the number of particles directly, and inversely as the area over which they are uniformly diffused.

(8.) Rays of light are not lines of *contiguous* particles.

For, rays proceed from every visible point in the universe to every other point; and, in their progress, pass freely through torrents of light issuing in all directions from different suns, and different systems; but were the particles in each ray contiguous, one ray could not cross another without producing some confusion and irregularity in each; and thus vision would be rendered indistinct and precarious. Neither is such contiguity of the particles of light necessary to produce constant vision; for, if a burning coal be made to describe a circle, with a sufficient velocity, the whole circumference appears luminous; which shews that the impression made by the light upon the sensorium, when the coal is in any one point of the circumference, remains till the coal returns again to the same point*.

* It is observed that if the revolution of the coal be performed in 7‴, the whole circle appears luminous; that is, if the particles succeed each other at an interval which does not exceed that time, constant vision is produced: and since light passes over rather more than 22,000 miles in 7‴, if the distance of the particles in a ray be not

(9.) There is something extremely subtile in the nature of light; and it's properties can with difficulty be explained, either on the supposition of it's materiality, or on that of it's being only a quality of an elastic medium. The facility and regularity with which it is transmitted through bodies of considerable density, cannot be accounted for on either hypothesis. If it consist of particles of matter, which is much the more probable supposition, their minuteness greatly exceeds the limits of our faculties, even the power of human imagination. Notwithstanding the astonishing velocity of these particles (Art. 3), their momentum is not so great as to discompose the delicate texture of the eye; and when they are collected in the focus of a powerful burning glass, it seems doubtful, whether they are capable of communicating motion to the thinnest lamina of metal that can be exposed to their impact.

Prop. I.

(10.) *A ray of light, whilst it continues in the same uniform medium*, proceeds in a straight line.*

For, objects cannot be seen through bent tubes; and the shadows of bodies are terminated by straight lines. Also, the conclusions, drawn from calculations made on this supposition, are found by experience to be true.

(11.) Cor. Hence it follows, that the density of light varies inversely as the square of the distance from a luminous point; supposing no particles to be stopped in their progress.

---

greater than 22,000 miles, they are sufficiently near to answer the purposes of constant vision. Sir Isaac Newton supposes the impression to continue about one second of time. See *Optics*, Qu. 16.

* In speaking of a medium, we always suppose it to be uniform, unless the contrary be expressed.

For, if the point from which the light proceeds be considered as the common center of two spherical surfaces, the same particles, which are uniformly diffused over the first, will afterwards be diffused, in the same manner, over the latter; and since the density of light varies, in general, as the number of particles directly, and inversely as the space over which they are uniformly diffused (Art. 7), in this case, it varies inversely as the space over which they are diffused, because the number of particles is the same; therefore, the density at the first surface : the density at the latter :: the area of the latter surface : the area of the former, that is, :: the square of the distance in the latter case : the square of the distance in the former *.

(12.) Def. When a ray of light, incident upon any surface, is turned back into the medium in which it was moving, it is said to be *reflected*.

(13.) Def. When a ray of light passes out of one medium into another, and has it's direction changed at the common surface of the two mediums, it is said to be *refracted*.

(14.) Def. The angle contained between the incident ray and the perpendicular to the reflecting, or refracting surface at the point of incidence, is called the *angle of incidence*.

(15.) Def. The angle contained between the reflected ray and the perpendicular to the reflecting surface at the point of incidence, is called *the angle of reflection*.

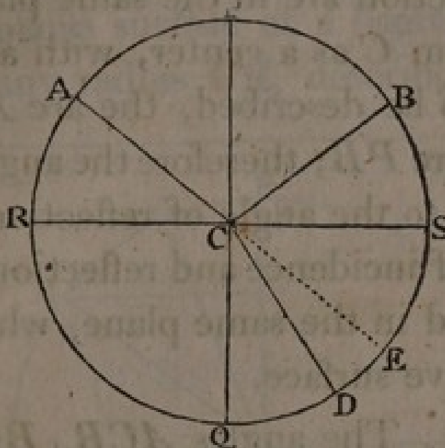
(16.) Def. The angle contained between the refracted ray and the perpendicular to the refracting surface, at the point of incidence, is called *the angle of refraction*.

---

* *Fluxions*, page 91.

(17.) Def. The angle contained between the incident ray produced and the reflected or refracted ray, is called *the angle of deviation.*

If *RS* represent the reflecting surface, *AC* a ray incident upon it, *CB* the reflected ray, and *PCQ* be



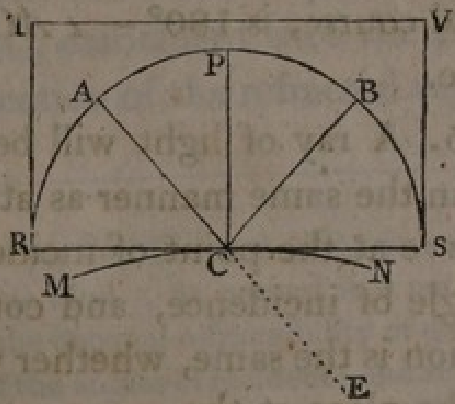
drawn, through *C*, perpendicular to *RS*, and *AC* be produced to *E*; then *ACP* is the angle of incidence, *PCB* the angle of reflection, and *BCE* the angle of deviation.

If *RS* be a refracting surface, and *CD* the refracted ray; then *QCD* is the angle of refraction, and *ECD* the angle of deviation.

Prop. II.

(18.) *The angles of incidence and reflection are in the same plane, and they are equal to each other.*

Let a ray of light *AC*, admitted through a small hole



into a dark chamber, be incident upon the reflecting surface *RS* at the point *C*; and let *CB* be the reflected ray; draw *CP* perpendicular to the reflector. Then, if the plane surface of a board *TS* be made to coincide with *CA* and *CP*, the reflected ray *CB* is found also to coincide with the plane *TS*; or the angles of incidence and reflection are in the same plane.

Again, if from *C* as a center, with any radius *CA*, the circle *RPS* be described, the arc *AP* is found to be equal to the arc *PB*; therefore the angle of incidence, *ACP*, is equal to the angle of reflection, *BCP*.

The angles of incidence and reflection are also found to be equal, and in the same plane, when rays are reflected at a curve surface.

(19.) Cor. 1. The angles *ACR*, *BCS*, which are the complements of the angles of incidence and reflection, are also equal.

(20.) Cor. 2. If *BC* be the incident ray, *CA* will be the reflected ray. For, the angle *PCA* is equal to the angle *PCB*, and in the same plane; therefore *CA* is the reflected ray.

(21.) Cor. 3. If the ray *PC* be incident perpendicularly upon the reflecting surface, it will be reflected in the perpendicular *CP*.

(22.) Cor. 4. If *AC* be produced to *E*, the angle *BCE*, which measures the deviation of the ray *AC* from it's original course, is $180° - \angle ACB$; or $180° - 2 \angle$ of incidence.

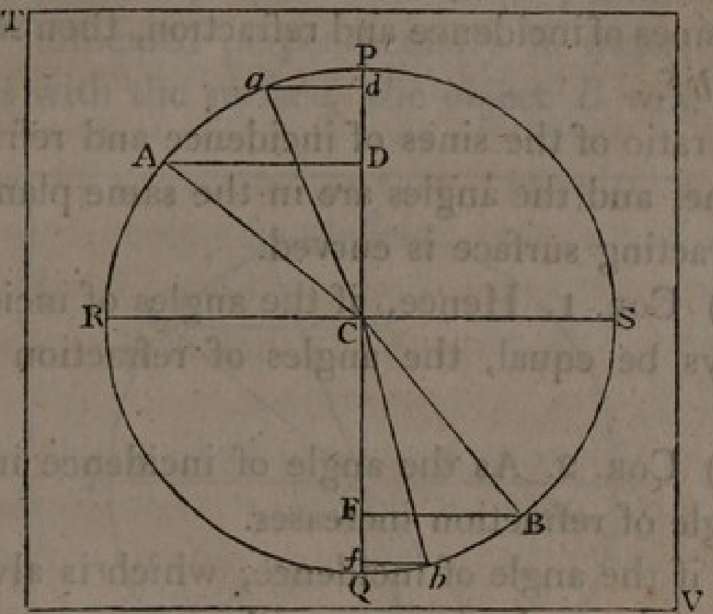
(23.) Cor. 5. A ray of light will be reflected at a curve surface, in the same manner as at a plane which touches the curve at the point of incidence.

For, the angle of incidence, and consequently the angle of reflection is the same, whether we suppose the reflection to take place at the curve, or at the plane.

PROP. III.

(24.) *The angles of incidence and refraction are in the same plane; and, whilst the mediums are the same, the sine of the angle of incidence is to the sine of the angle of refraction, in a given ratio*.*

Upon the plane surface of a board *TV*, with the center *C* and any radius *CA*, describe a circle *PRQ*,



draw the diameters *RS*, *PQ* at right angles to each other, and immerse the board into a vessel of water, in such a manner that *PQ* may be perpendicular to, and *RS* coincide with the surface of the water. Then, if a ray of light, admitted through a small hole into a dark chamber, be incident upon the surface *RS* in the direction *AC*, coincident with the plane of the board, *CB*, the direction of the refracted ray, is found to coin-

* The latter part of this proposition is only to be understood of rays of the same kind. At present it is not necessary to take into consideration the unequal refrangibility of differently coloured rays.

The sines of the angles of incidence and refraction are usually, for the sake of conciseness, called *the sines of incidence and refraction*.

cide with that plane; that is, the angles of incidence and refraction are in the same plane.

Also, if AD and BF be drawn at right angles to PQ, they are the sines of incidence and refraction, to the radius CA; and it is found that AD has to BF the same ratio, whatever be the inclination of the incident ray to the refracting surface. That is, if aC be any other incident ray, Cb the refracted ray, ad and bf the sines of incidence and refraction, then $AD : BF :: ad : bf$.

The ratio of the sines of incidence and refraction is the same, and the angles are in the same plane, when the refracting surface is curved.

(25.) Cor. 1. Hence, if the angles of incidence of two rays be equal, the angles of refraction are also equal.

(26.) Cor. 2. As the angle of incidence increases, the angle of refraction increases.

For, if the angle of incidence, which is always less than a right angle, increase, it's sine increases; and therefore the sine of refraction, which bears an invariable ratio to the sine of incidence, increases; and consequently the angle of refraction increases.

(27.) Cor. 3. When the angle of incidence vanishes, the angle of refraction vanishes also. In this case the ray suffers no refraction.

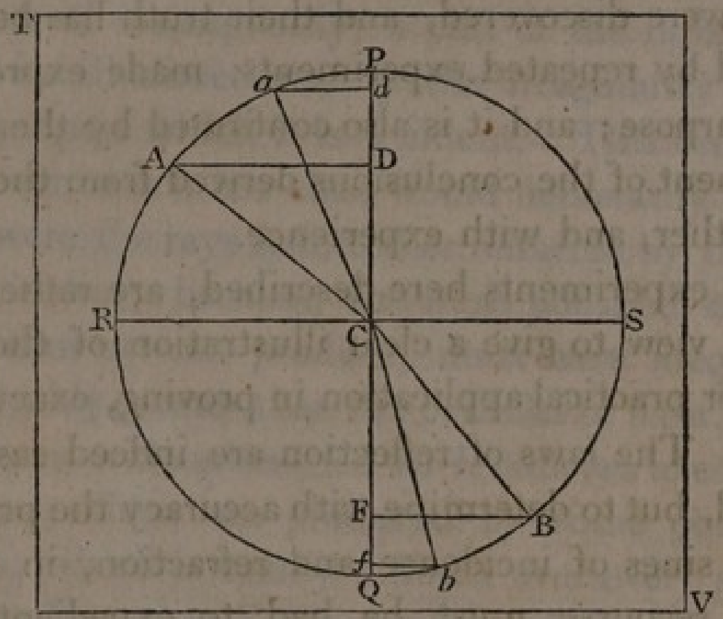
(28.) Cor. 4. A ray of light is refracted at a curve surface, in the same manner as at a plane which touches the curve at the point of incidence, if the refracting power of the mediums be the same.

For, the angle of incidence, and consequently the angle of refraction, is the same, whether we suppose the refraction to take place at the curve, or at the plane, supposing them to be mediums of the same kind.

Prop. IV.

(29.) *If a ray* AC *be refracted at the surface* RS *in the direction* CB, *then a ray* BC, *coming the contrary way, will be refracted in the direction* CA.

The construction being made as before, let a small object be placed upon the board at *B*; and when the board is immersed perpendicularly in water, till *RS* coincides with the surface, the object *B* will be seen



from *A*, in the direction *AC*; and since the motion of light, in the same medium, is rectilinear (Art. 10), the ray, by which the object is seen, is incident at *C*, and refracted in the direction *CA*.

(30.) Cor. 1. The angle of deviation of the ray *AC*, is equal to the angle of deviation of the ray *BC*, which is incident in the contrary direction.

(31.) Cor. 2. When a ray of light passes out of air into water, the sine of incidence : the sine of refraction :: 4 : 3; consequently, when a ray passes out of water into air, the sine of incidence : the sine of refraction :: 3 : 4*.

In the same manner, out of air into glass, the sine of incidence : the sine of refraction :: 3 : 2; therefore out of glass into air, the sine of incidence : the sine of refraction :: 2 : 3 *.

SCHOLIUM.

(32.) The preceding propositions, which are usually called the Laws of Reflection and Refraction, are the principles upon which the theory of vision is founded. They were discovered, and their truth has been established by repeated experiments, made expressly for this purpose; and it is also confirmed by the constant agreement of the conclusions derived from them, with each other, and with experience.

The experiments here described, are rather chosen with a view to give a clear illustration of these laws, than for practical application in proving, exactly, their truth. The laws of reflection are indeed easily established, but to determine with accuracy the proportion of the sines of incidence and refraction, in different cases, recourse must be had to expedients which cannot, in this place, be explained. The learner, when a little farther advanced in the subject, may consult on this head, Sir I. Newton's *Optics*, Sect. 2. and the *Encyclopædia Britannica*, Art. *Telescopes*, p. 356.

(33.) When a ray of light passes out of a rarer medium into a denser, that is, out of one which is

---

* These numbers do not express the exact proportions, as will be seen hereafter; but they are sufficiently accurate for our present purpose.

specifically lighter, into one which is specifically heavier, it is, in general, turned towards the perpendicular; and the contrary.

Though this is not universally the case*, yet in the subsequent part of the work, when we have occasion to speak of a denser medium, we shall always suppose it to have a greater refracting power.

(34.) When light is reflected or refracted at a polished surface, the motion of the general body of the rays is conformable to the laws above laid down; some are indeed thrown to the eye in whatever situation it is placed; and consequently, a part of the light is dispersed, in all directions, by the irregularity of the medium upon which it is incident. This dispersion is, however, much less than would necessarily be produced, were the rays reflected or refracted by the solid parts of bodies; because, the most polished surfaces, that human art can produce, must have inequalities incomparably greater than the particles of light. This, and other considerations, led Sir I. Newton to conclude that these effects are produced by some power, or medium, which is evenly diffused over every surface, and extends to a small, though finite distance from it†.

That bodies do act upon light before it comes into contact with them, is manifest from the shadows of hairs, small needles, &c. which are much larger than they ought to be, on supposition that rays pass by them in straight lines. In order to examine this phenomenon more minutely, Sir I. Newton admitted a small beam of light into a darkened chamber, and causing it to pass near the edge of a sharp knife, he

---

* *Newt. Optics*, Book II. Part 3. Prop. 10.

† Ibid. Book II. Prop. 8.

found that the rays were turned considerably from their rectilinear course, and that those rays were more inflected, or bent, which passed at a less distance from the edge, than those which were more remote. He also observed, that some of the rays were turned towards the edge, and others from it; so that rays of light, at different distances from the surfaces of bodies, are apparently acted upon by two different powers, one of which attracts, and the other repels them *.

The laws, according to which these powers vary, have not yet been discovered; but supposing the effects produced by them, at the same distance from a given surface, to be always the same, Sir I. Newton has shewn, that if small bodies were reflected and refracted by them, the angles of incidence and reflection would be equal; and the sines of the angles of incidence and refraction, in a given ratio to each other †. These conclusions leave us little room to doubt but that reflection and refraction are produced by such powers; and they afford some ground for presuming that the particles of light are material.

---

* *Newt. Optics,* Book III.

† *Principia,* Prop. 94, 96. *Optics,* Book I. Part I. Prop. 6.

Sect. II.

ON THE REFLECTION OF RAYS AT PLANE AND SPHERICAL SURFACES.

DEFINITIONS.

Art. (35.) By a *pencil of rays*, we understand a number of rays taken collectively, and distinct from the rest.

These pencils consist either of *parallel*, *converging*, or *diverging* rays.

Converging rays are such as approach to each other in their progress, and, if not intercepted, at length meet.

Diverging rays are such as recede from each other, and whose directions meet if produced backwards.

(36.) The *focus* of a pencil of rays is that point towards which they converge, or from which they diverge.

If the rays in a pencil, after reflection, or refraction, do not meet exactly in the same point, the pencil must be diminished; and the focus is the limit of the intersections of the extreme rays, when they approach nearer and nearer to each other, and at length coincide. In this case, the focus is usually called the *Geometrical Focus*.

The focus is *real*, when the rays actually meet in that point; and *imaginary*, or *virtual*, when their directions must be produced to meet.

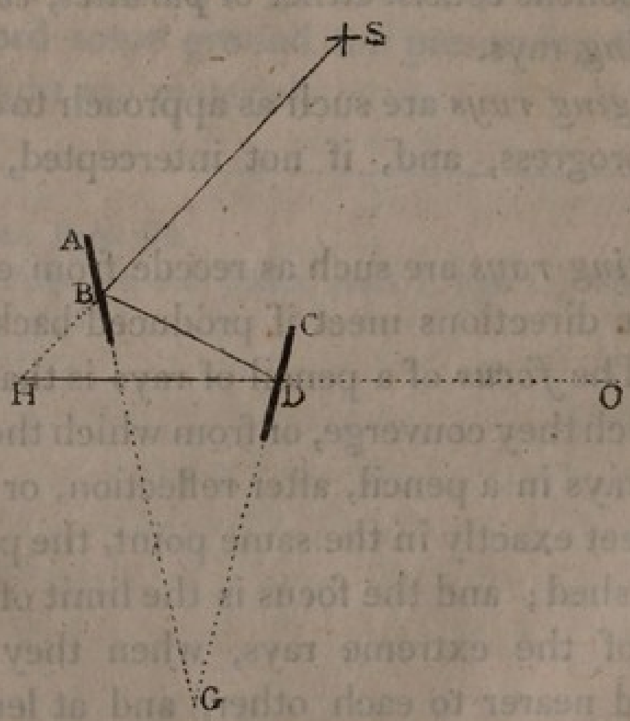
(37.) The *axis* of a pencil is that ray which is incident perpendicularly upon the reflecting or refracting surface.

(38.) The *principal focus* of a reflector, or refractor, is the geometrical focus of parallel rays incident nearly perpendicularly upon it.

Prop. V.

(39.) *If a ray of light be reflected once by each of two plane surfaces, and in a plane which is perpendicular to their common intersection, the angle contained between the first and last directions of the ray, is equal to twice the angle at which the reflectors are inclined to each other.*

Let *AB*, *CD* be two plane reflectors, inclined at the angle *AGD*; *SB*, *BD*, *DH*, the course of a ray



reflected by them. Produce *HD* to *O*, and *SB* till it meets *DH* in *H*. Then, because the ∠ *HBG* = the ∠

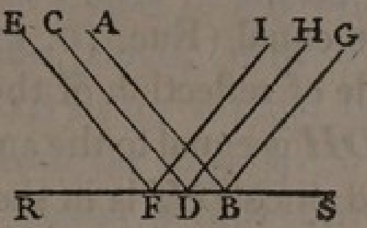
ABS = the ∠ *DBG* (Art. 19), the whole angle *DBH* = 2 ∠ *DBG*. In the same manner, the ∠ *BDO* = 2 ∠ *BDC*. And since the ∠ *BGD* = the ∠ *BDC* − the ∠ *DBG**, we have 2 ∠ *BGD* = 2 ∠ *BDC* − 2 ∠ *DBG* = the ∠ *BDO* − the ∠ *DBH* = the ∠ *BHD**.

Prop. VI.

(40.) *Parallel rays, reflected at a plane surface, continue parallel.*

Case 1. When the angles of incidence are in the *same* plane.

Let *RS* be the reflecting surface; *AB*, *CD* the

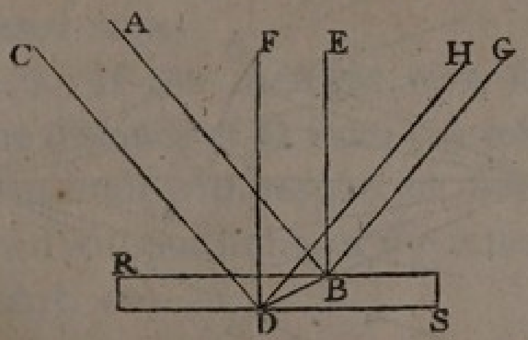


incident, *BG*, *DH* the reflected rays.

Then the ∠ *ABR* = the ∠ *GBS*, and the ∠ *CDR* = the ∠ *HDS* (Art. 19); but, since *AB* and *CD* are parallel, the ∠ *ABR* = the ∠ *CDR*; therefore the ∠ *GBS* = the ∠ *HDS*, and *BG*, *DH* are parallel (Euc. 28. 1).

Case 2. When the angles of incidence are in *different* planes.

Let *AB*, *CD* be the incident rays; *BE*, *DF* per-



pendiculars to the reflecting surface at the points of

* *Euclid*, 32. 1.

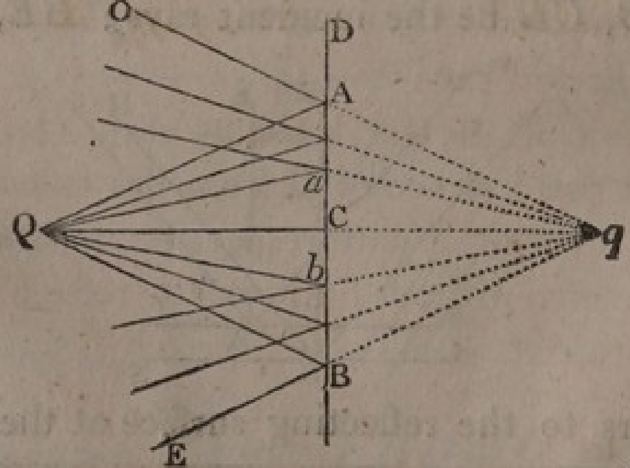
incidence; join BD; and let AB be reflected in the direction BG; also, let DH be the intersection of the planes CDF, GBD.

Then, since BE, DF, which are perpendicular to the same plane, are parallel (Euc. 6. 11), and AB, CD are parallel, by the supposition, the angles of incidence ABE, CDF are equal (Euc. 10. 11); therefore the angles of reflection are equal. Again, since EB and FD are parallel, as also AB and CD, the planes ABG, CDH are parallel (Euc. 15. 11), and they are intersected by the plane $GBDH$; consequently, DH is parallel to BG (Euc. 16. 11); therefore the angles EBG, FDH are equal (Euc. 10. 11); but the angle EBG is the angle of reflection of the ray AB; therefore the angle FDH is equal to the angle of reflection of the ray CD; and since DH is in the plane CDF, CD is reflected in the direction DH (Art. 18), which has before been shewn to be parallel to BG.

PROP. VII.

(41.) *If diverging or converging rays be reflected at a plane surface, the foci of incident and reflected rays are on contrary sides of the reflector, and equally distant from it.*

Let QAB be a pencil of rays diverging from Q,



and incident upon the plane reflector ACB; draw QC

perpendicular to the surface; then will QC be reflected in the direction CQ (Art. 21). Let QA be any other ray; and since a perpendicular to the surface at A, is in the same plane with QC and QA (Euc. 6. and 7. 11), QA will be reflected in this plane (Art. 18). Produce CA to D, and make the angle DAO equal to the angle QAC, then will AO be the reflected ray (Art. 19). Produce OA, QC till they meet in q. Then, since the $\angle qAC =$ the $\angle OAD =$ the $\angle QAC$, and also the $\angle qCA =$ the $\angle QCA$, and the side CA is common to the two triangles QCA, CAq, the side QC is equal to Cq. In the same manner it may be shewn, that every other reflected ray in the pencil, will, if produced backwards, meet the axis in q; that is, the rays, after reflection, diverge from the focus q.

If $OABE$ be a pencil of rays converging to q, they will, after reflection at the surface ACB, converge to Q (Art. 20); therefore, in this case also, the foci of incident and reflected rays are on contrary sides of the reflector, and equally distant from it.

(42.) Cor. 1. The divergency, or convergency of rays, is not altered by reflection at a plane surface.

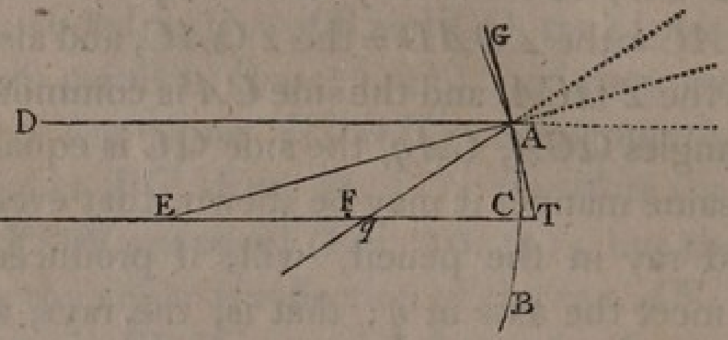
(43.) Cor. 2. In the triangles QAC, CAq, Aq is equal to QA; if therefore any reflected ray AO be produced backwards to q, making $Aq = AQ$, q is the focus of reflected rays.

(44.) Cor. 3. If the incident rays QA, Qa, be parallel, or the distance of Q from the reflector be increased without limit with respect to Aa, the distance of q is increased without limit, or the reflected rays are parallel (See Art. 40).

Prop. VIII.

(45.) *If parallel rays be incident nearly perpendicularly upon a spherical reflector, the geometrical focus of reflected rays is the middle point in the axis between the surface and center.*

Let ACB be a spherical reflector whose center is E; DA, EC, two rays of a parallel pencil incident upon it,



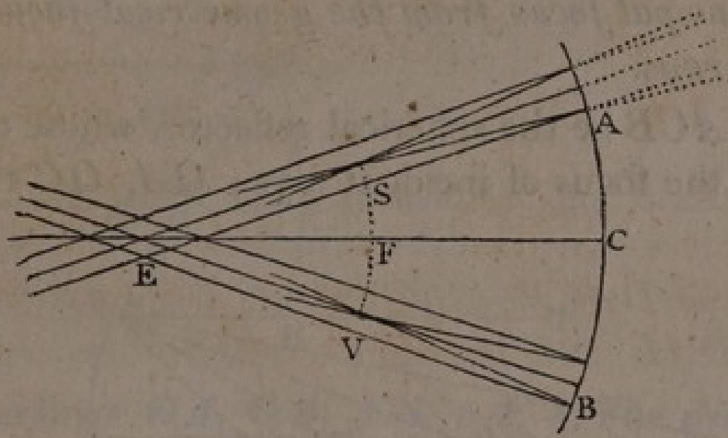
of which, EC passes through the center, and is therefore reflected in the direction CE; join EA, and in the plane $DACE$, make the angle EAq equal to the angle DAE, and DA will be reflected in the direction Aq (Art. 18); draw GAT, in the same plane, touching the reflector in A, and let it meet EC produced in T. Then, since the $\angle EAq =$ the $\angle DAE =$ the $\angle AEq$ (Euc. 29. 1), $Eq = Aq$; also, the $\angle qAT =$ the $\angle DAG$ (Art. 19) $=$ the $\angle ATq$ (Euc. 29. 1); therefore $Aq = qT$; consequently $Eq = qT$; that is, q bisects ET the secant of the arc AC. Now let DA approach to EC, and the arc AC will decrease, and it's secant, at length, become equal to the radius; consequently the limit of the intersections of Aq and CE is F, the middle point between E and C*.

* It is manifest that the rays, incident nearly perpendicularly, do not meet accurately in F; but when the arc is small in comparison of the radius, in all calculations, made for the construction of

If the rays be incident upon the convex side of the reflector, the reflected rays must be produced backwards to meet the axis; and in this case, F, the middle point between E and C, may be shewn to be the limit of the intersections of CE and Aq, as before.

(46.) Cor. 1. As the arc AC decreases, Eq, or Fq, decreases. Thus, when AC is 60°, $Eq = EC$; and when AC is 45°, Aq is perpendicular to EC, and $Eq : EC :: 1 : \sqrt{2}$.

(47.) Cor. 2. If different pencils of parallel rays be respectively incident, nearly perpendicularly, upon the



reflector, the foci of reflected rays will lie in the spherical surface SFV, whose center is E and radius EF.

(48.) Cor. 3. If the axes, EA, EC, EB, of these pencils, lie in the same plane, the foci will lie in the circular arc SFV.

(49.) Cor. 4. If any point S, in the arc SFV, whose radius EF is one half of EC, be the focus of a

---

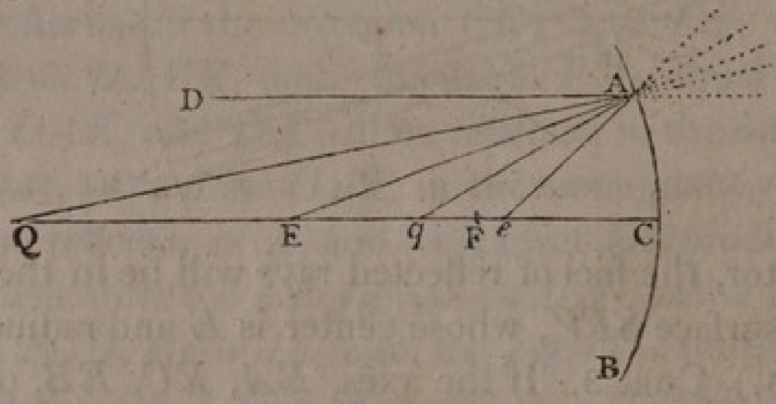
of optical instruments, F and q may be considered as coincident. Thus, if the arc AC be 20′, and the radius be divided into 100,000 equal parts, Fq is less than one of those parts; and all the rays which are incident upon the surface generated by the revolution of the arc AC about the axis EC, after reflection, cut the axis between F and q.

pencil of rays incident nearly perpendicularly upon the reflector, these rays will be reflected parallel to each other, and to EA the axis of that pencil (Art. 20).

Prop. IX.

(50.) *When diverging or converging rays are incident nearly perpendicularly upon a spherical reflector, the distance of the focus of incident rays from the principal focus, measured along the axis of the pencil, is to the distance of the principal focus from the center, as this distance is to the distance of the principal focus from the geometrical focus of reflected rays.*

Let ACB be the spherical reflector, whose center is E; Q the focus of incident rays; QA, QC two rays

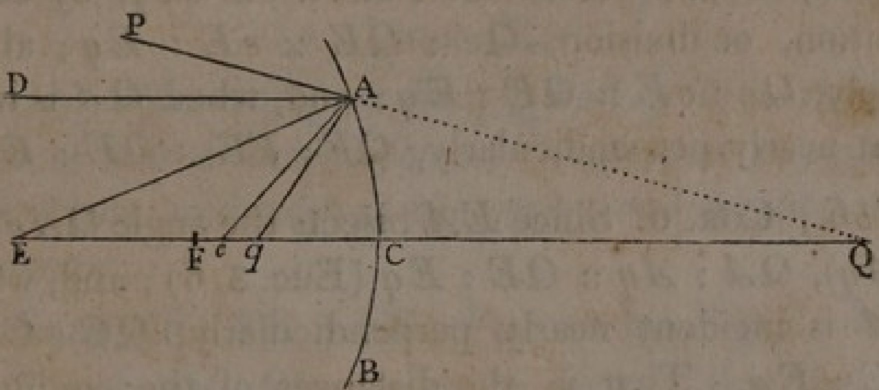


of the pencil, of which QC passes through the center E, and is therefore reflected in the direction CQ; join EA; and, in the plane $QACE$, make the angle EAq equal to the angle EAQ; then the ray QA will be reflected in the direction Aq.

Draw DA parallel to QC, and make the angle EAe equal to the angle EAD; bisect EC in F. Then, since the $\angle DAE =$ the $\angle EAe$, and the $\angle QAE =$ the $\angle EAq$, the $\angle DAQ$, or it's equal AQe, is equal to the $\angle eAq$; also, the $\angle qeA$ is common to the two triangles AQe, Aqe; therefore they are similar, and

$Qe : eA :: eA : eq$; or, since $eA = eE$ (Art. 45), $Qe : eE :: eE : eq$. Now let the arc AC be diminished without limit, or the ray QA be incident nearly perpendicularly, then e coincides with F (Art. 45); and the limit of the intersections, of CQ and Aq, is determined by the proportion $QF : FE :: FE : Fq$*.

The preceding figure is constructed for the case in which diverging rays are incident upon a concave spherical surface, and the same demonstration is applicable when the incident rays converge, as is represented in the annexed figure.



If the lines DA, QA, EA, eA, qA be produced, the figures serve for those cases in which the rays are incident upon the convex surface.

(51.) Cor. 1. If q be the focus of incident, Q will be the focus of reflected rays (Art. 20); and Q and q are called *conjugate* foci.

(52.) Cor. 2. If the distance QF be very great when compared with FE, Fq is very small when compared

---

* When QA is incident *nearly* perpendicularly upon the reflector, q may be considered as coincident with that point which is determined by the proportion $QF : FE :: FE : Fq$ (See note page 20); and all other rays, incident nearly perpendicularly, will, after reflection, cut the axis in the same point; therefore that point is the focus of reflected rays.

with it. Thus, if the rays diverge from a point in the sun's disc, and fall upon a reflector whose radius does not exceed a few feet, F and q may, for all practical purposes, be considered as coincident.

(53.) Cor. 3. When Q coincides with E, all the rays are incident perpendicularly upon the reflector, and therefore they are reflected perpendicularly (Art. 21), or q coincides with E.

(54.) Cor. 4. The point e bisects the secant of the arc AC (Art. 45).

(55.) Cor. 5. Since $Qe : eE :: eE : eq$, by composition, or division, $Qe : QE :: eE : Eq$; alternately, $Qe : eE :: QE : Eq$; and, when QA is incident nearly perpendicularly, $QF : FE :: QE : Eq$.

(56.) Cor. 6. Since EA bisects the angle QAq (or PAq), $QA : Aq :: QE : Eq$ (Euc. 3. 6); and, when QA is incident nearly perpendicularly, $QC : Cq :: QE : Eq$. That is, the distances of the conjugate foci from the center, are proportional to their distances from the surface.

(57.) Cor. 7. Since $QE : Eq :: QF : FE$ (Art. 55), and $QE : Eq :: QC : Cq$ (Art. 56), we have, ultimately, $QF : FE :: QC : Cq$.

(58.) Cor. 8. As the arc AC decreases, Eq, the distance of the intersection of the reflected ray and the axis from the center, decreases; unless Q coincide with E, or lie between E and e.

For, $Qe : eE :: QE : Eq$ (Art. 55), and as AC decreases Ee decreases (Art. 54); therefore, when Q is in eE produced, the terms of the ratio of greater inequality, $Qe : Ee$, are equally diminished, and that ratio, or it's equal $QE : Eq$, increases (Alg. Art. 163); and, since QE is invariable, Eq decreases.

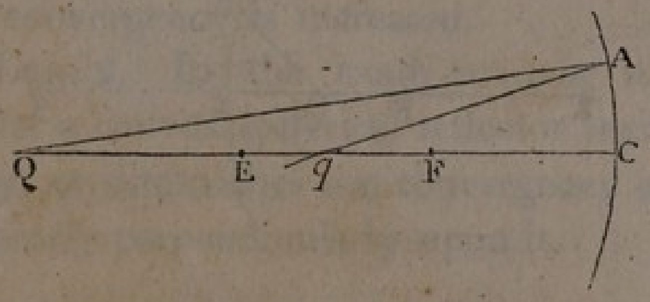
When Q is in Ee produced, as AC decreases Qe increases, and Ee decreases; therefore the ratio of $Qe : Ee$, or of $QE : Eq$ increases; and consequently, as before, Eq decreases.

But when Q lies between E and e, as AC decreases the terms of a ratio of less inequality $Qe : Ee$ are equally diminished; therefore that ratio, or it's equal $QE : Eq$, decreases (Alg. Art. 163); and since QE is invariable, Eq increases. When Q coincides with E, q also coincides with it, whatever be the magnitude of the arc AC.

Prop. X.

(59.) *The conjugate foci,* Q *and* q, *lie on the same side of the principal focus; they move in opposite directions, and meet at the center and surface of the reflector.*

Since $QF : FE :: FE : Fq$, we have $QF \times Fq = \overline{FE}|^2$; that is, Q and q are so situated that the rectangle under QF and Fq is invariable. Also, when Q coincides with E, q coincides with it (Art. 53); in



this case then, QF and Fq are measured in the same direction from F; and, since their rectangle is invariable, they must always be measured in the same direction (Alg. Art. 471).

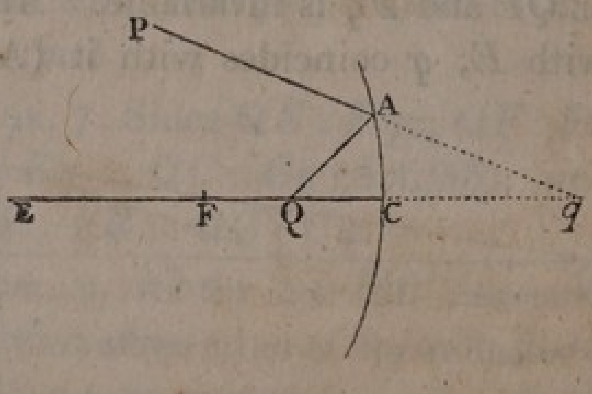
That Q and q move in opposite directions may thus be proved: the rectangle $QF \times Fq$ is invariable; and therefore as one of these quantities increases, the other decreases; also, Q and q lie the same way from the fixed point F; they must therefore move in opposite directions.

Having given, the place of Q, and FE the focal length of the reflector, to determine the place of the conjugate focus q, we must take $QF : FE :: FE : Fq$, and measure FQ and Fq in the same direction from F.

Thus, when Q, the focus of incident rays, is farther from the reflector than E, and on the same side of it, FQ is greater than FE, therefore FE is greater than Fq; or q, the focus of reflected rays, lies between F and E.

When Q is between E and F, q lies the other way from E; and whilst Q moves from E to F, q moves in the opposite direction from E to an infinite distance.

When Q is between F and C, QF is less than FE or FC; therefore FC is less than Fq; and, since FQ



and Fq are measured in the same direction from F, q is on the convex side of the reflector.

When Q coincides with C, QF is equal to FC; therefore FC is equal to Fq; or q coincides with C.

When converging rays are incident upon the concave surface of the reflector, QF is greater than FC; therefore FC is greater than Fq; or q lies between F and C.

(60.) Cor. 1. A concave spherical reflector lessens the divergency, or increases the convergency of all pencils of rays incident nearly perpendicularly upon it.

For, if the rays diverge from a point farther from the reflector than the principal focus, they are made to converge.

If they diverge from F, they are reflected parallel to CE.

If the focus of incidence lie between F and C, q is on the other side of the surface; or the rays diverge after reflection; and because $QF : FE :: QC : Cq$ (Art. 57), and QF is less than FE, QC is less than Cq; also, the subtense AC is common; therefore the angle contained between the incident rays QA, QC, is greater than the angle contained between the reflected rays AP, CQ; or the reflected rays diverge less than the incident rays.

If converging rays fall upon the reflector, QF (Fig. page 23) is greater than FE, therefore QC is greater than Cq; or the reflected rays converge to a focus nearer to the reflector than the focus of incident rays; and their convergency is increased.

(61.) Cor. 2. In the same manner it may be shewn, that a convex spherical reflector increases the divergency, or diminishes the convergency of all rays incident nearly perpendicularly upon it.

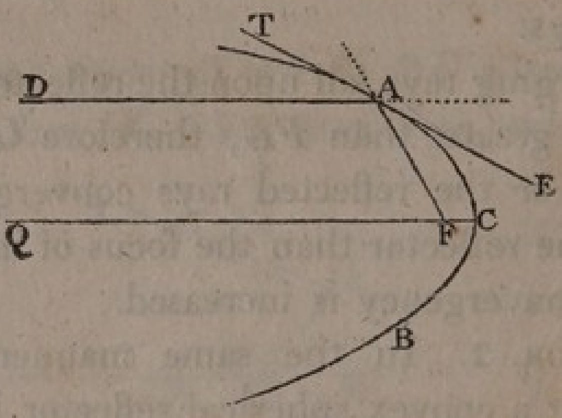
SCHOLIUM.

(62.) It appears from Art. 58, that unless the focus coincide with the center, a spherical reflector does not cause all the rays in a pencil either to converge or diverge accurately. This circumstance produces some confusion in vision, when these reflectors are made use of; and, by increasing the breadth of each pencil, or, which is the same thing, by enlarging the aperture of the reflecting surface, in order to increase the quantity of light, the indistinctness thus produced is increased, as we shall have occasion to observe hereafter.

To remedy this inconvenience, it has been proposed to make use of reflecting surfaces formed by the revolution of conic sections about their axes; and it may be proper to shew that such surfaces will, in particular cases, cause rays to converge or diverge accurately.

(63.) *Parallel rays may be made to converge, or diverge accurately, by means of a parabolic reflector.*

Let ACB be a parabola, by the revolution of which about it's axis QC, a parabolic reflector is generated;



take F the focus; let DA, which is parallel to QC, be a ray of light incident upon the concave side of this reflector; and join AF. Draw TAE in the plane DAF, and touching the paraboloid in A. Then since the angle TAD is equal to the angle EAF, from the nature of the parabola, the ray DA will be reflected in the direction AF (Art. 19). In the same manner it may be shewn, that any other ray, parallel to QC, will be reflected to F; and therefore the reflected rays converge accurately to this point.

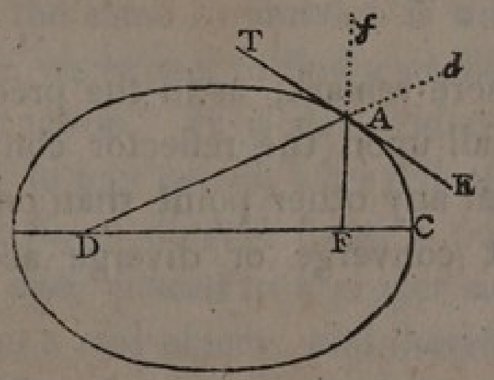
If *DA*, *FA* be produced, it is manifest that rays, incident upon the convex surface of the paraboloid, parallel to the axis, will, after reflection, diverge accurately from *F*.

The advantage, however, of a parabolic reflector is not so great as might, at first, be expected; for, if the pencil be inclined to the axis of the parabola, the rays will not be made to converge or diverge accurately; and the greater this inclination is, the greater will the error become.

Cor. If *F* be the focus of incidence, the rays will be reflected parallel to the axis.

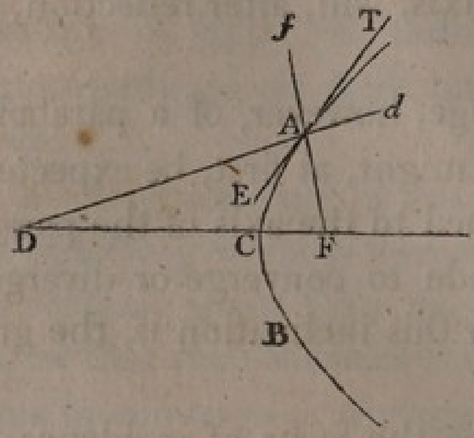
(64.) *Diverging or converging rays may be made to converge or diverge accurately, by a reflector in the form of a spheroid; and to diverge or converge accurately, by one in the form of an hyperboloid.*

Let *F* and *D* be the foci of the conic section, by the



revolution of which, about it's axis, the reflecting surface is formed; *F* the focus of incident rays; then will *D* be the focus of reflected rays.

For, let *FA* be an incident ray; join *DA*, and produce it to *d*; draw *TAE* in the plane *DAF*, and



touching the reflector in *A*; then the angle *EAF* is equal to the angle *DAT*, in the ellipse, and to *dAT* in the hyperbola; therefore *AD* is the reflected ray in the former case, and *Ad* in the latter; thus *D* is the focus of reflected rays.

If *FA* be produced to *f*, the figures serve for the cases in which rays are incident upon the convex surfaces.

We may here remark, as in the preceding article, that if rays fall upon the reflector converging to, or diverging from any other point than one of the foci, they will not converge or diverge accurately after reflection.

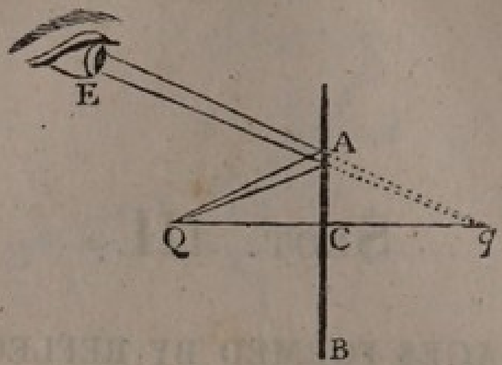
Sect. III.

ON IMAGES FORMED BY REFLECTION.

Art. (65.) The rays of light which diverge from any point in an object, and fall upon the eye, excite a certain sensation in the mind, corresponding to which, as we know by experience, there exists an external substance in the place from which the rays proceed; and whenever the same impression is made upon the organ of vision, we expect to find a similar object, and in a similar situation. It is also evident, that if the rays belonging to any pencil, after reflection or refraction, converge to, or diverge from, a point, they will fall upon the eye, placed in a proper situation, as if they came from a real object; and therefore the mind, insensible of the change which the rays may have undergone in their passage, will conclude that there is a real object corresponding to that impression.

In some cases indeed, chiefly in reflections, the judgment is corrected by particular circumstances which have no place in naked vision, as the diminution of light, or the presence of the reflecting surface, and we are sensible of the illusion; but still the impression is made, and a representation, or *image* of the object, from which the rays originally proceeded, is formed.

Thus, the rays which diverge from Q, after reflection at the plane surface ACB, enter an eye, placed



at E, as if they came from q; or q is the image of Q.

If then the rays, which diverge from any visible point in an object, fall upon a reflecting or refracting surface, the focus of the reflected or refracted rays is the *image* of that point.

(66.) The image is said to be *real*, or *imaginary*, according as the foci of the rays by which it is formed are *real*, or *imaginary*.

(67.) The image of a physical line is determined by finding the images of all the points in the line; and of a surface, by finding the images of all the lines in the surface, or into which we may suppose the surface to be divided.

PROP. XI.

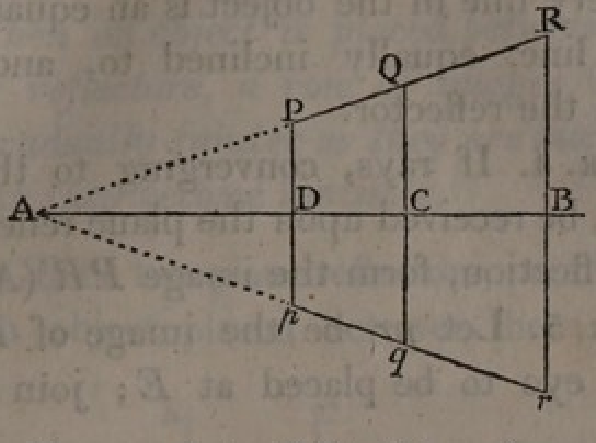
(68.) *The image of a straight line, formed by a plane reflector, is a straight line, on the other side of the reflector; the image and object are equally distant from, and equally inclined to, the reflecting plane; and they are equal to each other.*

Let PR be a straight line*, placed before the plane

---

* It is almost unnecessary to remind the reader, that the lines, which are considered as objects, must be physical lines, of sufficient thickness to reflect as many rays as are necessary for the purposes of vision.

reflector AB; produce RP, if necessary, till it meets the surface in A; draw RBr at right angles to AB,



and make Br equal to RB; join Ar; and from P draw PDp perpendicular to AB, meeting Ar in p; then will pr be the image of PR.

Since RBr is perpendicular to AB, and Br is equal to BR, r is the image of R (Art. 41).

Also, from the similar triangles ABR, ADP, $RB : AB :: PD : AD$; and from the similar triangles ABr, ADp, $AB : Br :: AD : Dp$; ex æquo, $RB : Br :: PD : Dp$; and since RB is equal to Br, PD is equal to Dp; or p is the image of P. In the same manner it may be shewn, that the image of every other point in PQR is the corresponding point in pqr; that is, pr is the whole image of PR.

Again, since BR is equal to Br, and AB common to the two triangles ABR, ABr, and also the angles at B are right angles, the angles of inclination RAB, BAr are equal, and AR is equal to Ar. In the same manner AP is equal to Ap; therefore PR is equal to pr.

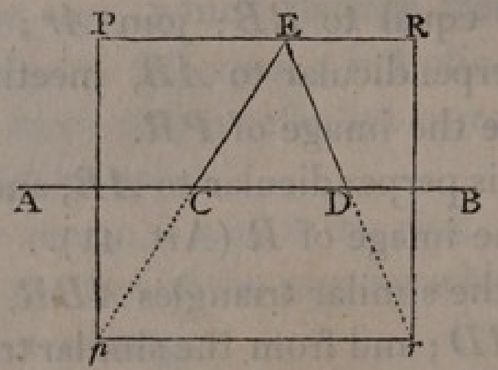
(69.) Cor. 1. If the object PR be parallel to the reflector, the image, pr will also be parallel to it.

(70.) Cor. 2. If PR be a curve, pr will be a curve, similar and equal to PR, and similarly situated on the other side of the reflector.

(71.) Cor. 3. Whatever be the form of the object, the image will be similar and equal to it. For, the image of every line in the object is an equal and corresponding line, equally inclined to, and equally distant from the reflector.

(72.) Cor. 4. If rays, converging to the several points in *pr*, be received upon the plane reflector, they will, after reflection, form the image *PR* (Art. 20).

(73.) Cor. 5. Let *pr* be the image of *PR*; and suppose an eye to be placed at *E*; join *pE*, *rE*,



cutting the reflector in *C* and *D*; then, considering the pupil as a point, the image will be seen in the part *CD* of the reflector; and it will subtend the angle *CED* at the eye; because all the rays enter the eye as if they came from a real object *pr* (Art. 65).

(74.) Cor. 6. When *PR* is parallel to *AB*, and *E* is situated in *PR*, *CD* is the half of *pr*, or *PR*.

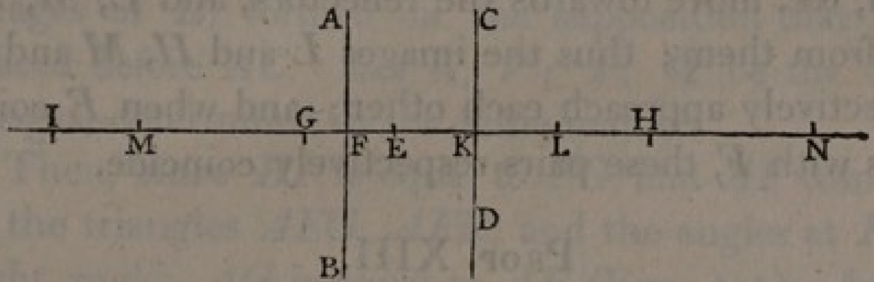
For, in this case, *pr* is parallel to *AB* (Art. 69), and therefore $CD : pr :: ED : Er :: 1 : 2$.

No rays enter the eye from any other part of the reflector.

Prop. XII.

(75.) *When an object is placed between two parallel plane reflectors, a row of images is formed which are gradually fainter as they are more remote, and at length they become invisible.*

Let *AB*, *CD* be two plane reflectors, parallel to each other; *E* an object placed between them; through



E draw the indefinite right line *NEI* perpendicular to *AB*, or *CD*. Take *FG*=*FE*; *KH*=*KG*; *FI*=*FH*, &c. Also, take *KL*=*KE*; *FM*=*FL*; *KN*=*KM*, &c.

Then, the rays which diverge from *E* and fall upon *AB*, will, after reflection, diverge from *G* (Art. 41); or *G* will be an image of *E*. Also, these rays, after reflection at *AB*, will fall upon *CD* as if they proceeded from a real object at *G*, and after reflection at *CD* they will diverge from *H*; that is, *H* will be an image of *G*; or a second image of *E*, &c. In the same manner, the rays which diverge from *E*, and fall upon *CD*, will form the images *L*, *M*, *N*, &c.

It is found by experiment, that all the light incident upon any surface, however well polished, is not regularly reflected from it. A part is dispersed in all directions (Art. 34); and a considerable portion enters the surface, and seems to be absorbed by the body.

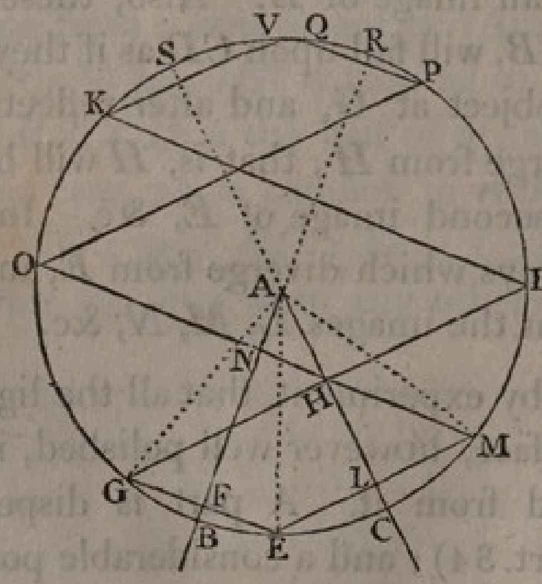
In the passage also of light through any uniform medium, some rays are continually dispersed, or absorbed ; and thus, as it is thrown backward and forward through the plate of air contained between the two reflectors *AB, CD*, it's quantity is diminished. On all these accounts, therefore, the succeeding images become gradually fainter, and, at length, wholly invisible.

(76.) Cor. If *E* move towards *F*, the images *G*, *H*, *I*, &c. move towards the reflectors, and *L*, *M*, *N*, &c. from them; thus the images *L* and *H*, *M* and *I*, respectively approach each other; and when *E* coincides with *F*, these pairs respectively coincide.

Prop. XIII.

(77.) *If an object be placed between two plane reflectors inclined to each other, the images formed will lie in the circumference of a circle, whose center is the intersection of the two planes, and radius the distance of the object from that intersection.*

Let *AB, AC* be two plane reflectors inclined at the



angle *BAC*; *E* an object placed between them. Draw

EF perpendicular to AB, and produce it to G, making $FG = EF$; then the rays which diverge from E and fall upon AB, will, after reflection, diverge from G; or G will be an image of E. From G, draw GH perpendicular to AC, and produce it to I, making $HI = GH$, and I will be a second image of E, &c. Again, draw ELM perpendicular to AC, and make $LM = EL$; also, draw MNO perpendicular to AB, and make $NO = MN$, &c. and M, O, &c. will be images of E, formed on the supposition that it is placed before AC. Let K, V; P, Q be the other images, determined in the same manner.

Then, since EF is equal to FG, and AF common to the triangles AFG, AFE, and the angles at F are right angles, AG is equal to AE (Euc. 4. 1). In the same manner it appears, that AI, AK, &c. AM, AO, AP, &c. are equal to each other, and to AE; that is, all the images lie in the circumference of the circle $EMIK$ whose center is A, and radius AE.

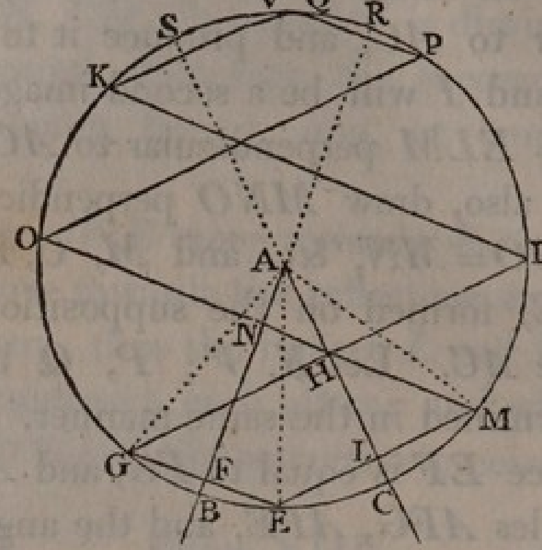
(78.) Cor. If the angle BAC be finite, the number of images is limited. For, BA and CA being produced to R and S, the images Q, V, will at length be formed between those points, and the rays which are reflected by either surface, diverging from any point Q between S and R, will not meet the other reflector; that is, no image of Q will be formed.

Prop. XIV.

(79.) *Having given the inclination of two plane reflectors, and the situation of an object between them, to find the number of images.*

It appears from the construction in the last proposition, that the lines EG, MO, IK, PQ, &c. are parallel, as also EM, GI, OP, KV, &c. Hence it follows, that the arcs EG, MI, OK, PV, &c. are equal;

as also, the arcs EM, GO, IP, KQ, &c. Let $BC=a$, $EB=b$, $EC=c$; then, the arc $EG=2b$; $EM=2c$;



$EO=EG+GO=EG+EM=2b+2c=2a$; $EK=EO+OK=EO+EG=2a+2b$; $EOQ=EK+KQ=EK+EM=2a+2b+2c=4a$, &c. Thus there is one series of images, formed by the reflections at AB, whose distances from E, measured along the circular arc EOR, are $2b$, $2a+2b$, $4a+2b$, ... $2na-2a+2b$ $(2na-2c)$, where n is the number of images; this series will be continued as long as $2na-2a+2b$, or $2na-2c$ is less than the arc EOR, or $180°+b$; and consequently n, the number of images in this series, is that whole number which is next inferior to $\frac{180+b+2c}{2a}$, or to $\frac{180+a+c}{2a}$. There is also a second series of images, formed by reflections at the same surface, whose distances from E are $2a$, $4a$, $6a$, ... $2ma$, continued as long as $2ma$ is less than $180+b$, and therefore m, the number of these images, is that whole number which is next inferior to $\frac{180+b}{2a}$.

In the same manner, the number of images formed by reflections at the surface AC, is found by taking the whole numbers next inferior to $\frac{180+a+b}{2a}$, and $\frac{180+c}{2a}$.

(80.) Cor. 1. If a be a measure of 180, the number of images formed will be $\frac{360}{a}$.

For, if a be contained an even number of times in 180, or $2a$ be a measure of 180, the number of images in each series is $\frac{180}{2a}$*; and the number upon the whole is $4 \times \frac{180}{2a} = \frac{360}{a}$. If a be contained an odd number of times in 180, $2a$ is a measure of $180+a$, or $180-a$; and the number of images is $\frac{180+a}{2a} + \frac{180-a}{2a}$† $+ \frac{180+a}{2a} + \frac{180-a}{2a} = \frac{360}{a}$.

(81.) Cor. 2. When a is a measure of 180, two images coincide.

For, if a be contained an even number of times in 180, then the number of images in the second series, formed by reflections at the surface AB, is $\frac{180}{2a}$; and the distance EOQ, $(2ma)$, of the last image from E, is 180°. In the same manner, the distance EIV, of the last image in the second series formed by reflections

* $\frac{180+a+c}{2a} = \frac{180}{2a} + \frac{a+c}{2a}$; the latter part, $\frac{a+c}{2a}$, being less than unity, is neglected.

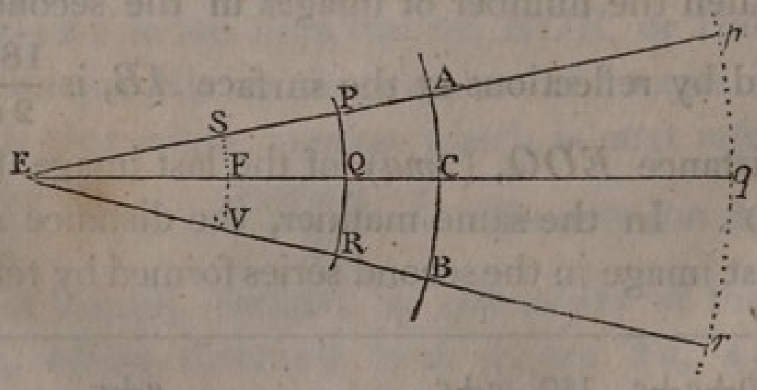
† $\frac{180+b}{2a} = \frac{180-a+a+b}{2a} = \frac{180-a}{2a} + \frac{a+b}{2a}$; the latter part, $\frac{a+b}{2a}$, being less than unity, is neglected.

at AC, is $180°$; therefore the two images, Q and V, coincide in EA produced. If a be contained an odd number of times in 180, then the number of images, in the first series, formed by reflections at AB, is $\frac{180+a}{2a}$; and the distance EOK, of the last of these images from E, is $\frac{180+a}{2a} \times 2a - 2c$; or $180° + a - 2c$. Also, the distance EMP, of the last image in the first series formed by reflections at AC, is $180° + a - 2b$; therefore $EOK + EMP = 360° + 2a - 2c - 2b = 360°$; that is, K and P coincide.

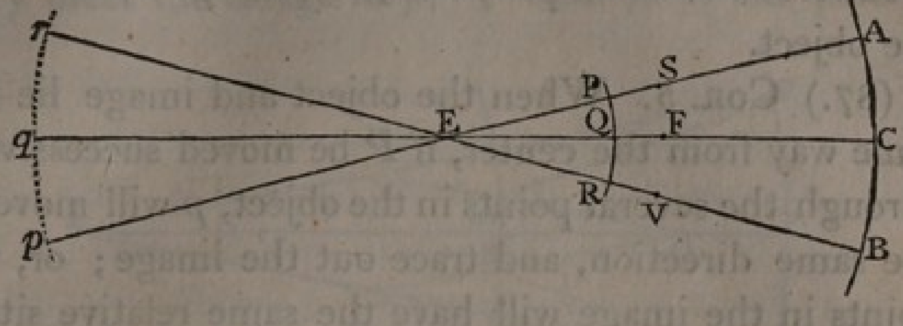
Prop. XV.

(82.) *If the object placed before a spherical reflector be a circular arc concentric with it, the image will also be a circular arc concentric with, and similar to the object.*

Let ACB be the spherical reflector, PQR the circular arc; E their common center. Take any points P, Q, R, in the object; join EP, EQ, ER, and



produce these lines, if necessary; bisect EC in F, and take $FQ : FE :: FE : Fq$, measuring Fq and FQ in the same direction from F; with the center E and radii Eq, EF, describe the circular arcs pqr, SFV, cutting EP, ER, or those lines produced, in p, r, and



S, V; then will pqr be the image of PQR.

For, since $FQ : FE :: FE : Fq$, and FQ and Fq are measured in the same direction from F, q is the image of Q (Art. 50.) Also, since EP is equal to EQ, ES to EF, and Ep to Eq, SP is equal to FQ, and Sp to Fq; therefore $SP : SE :: SE : Sp$, and since S is the principal focus of rays incident parallel to EA, and SP and Sp are measured in the same direction from S, p is the image of P. In the same manner it may be proved, that the image of every other point in PQR, is the corresponding point in pqr; that is, pqr is the image of PQR.

Again, the image and object, since their extremities are determined by straight lines which pass through the center E, subtend the same, or equal angles at that center; therefore they are similar arcs.

(83.) Cor. 1. In the same manner it appears that, if pqr be the object, PQR will be it's image; and if rays are incident the contrary way, converging to the several points in PQR, pqr will be the image of PQR.

(84.) Cor. 2. Because similar arcs are proportional to their radii, $PR : pr :: QE : Eq$.

(85.) Cor. 3. Since $QF : FE :: QE : Eq$ (Art. 55), $PR : pr :: QF : FE$.

(86.) Cor. 4. If the object be placed any where between E and C, QF is less than FE; therefore the object is less than the image. If it be placed in EC produced, or in CE produced, the image is less than the object.

(87.) Cor. 5. When the object and image lie the same way from the center, if P be moved successively through the several points in the object, p will move in the same direction, and trace out the image; or, the points in the image will have the same relative situation that the corresponding points in the object have; that is, the image will be erect. But if P and p be on contrary sides of E, they will move in opposite directions; or the image will be inverted.

(88.) Cor. 6. When the object is in FC, or in FC produced, the image is erect (Art. 87). When it is in FE, or FE produced, the image is inverted (See Art. 59).

(89.) Cor. 7. Since $QE : Eq :: QC : Cq$, the object and image are in the ratio of their distances from the reflector.

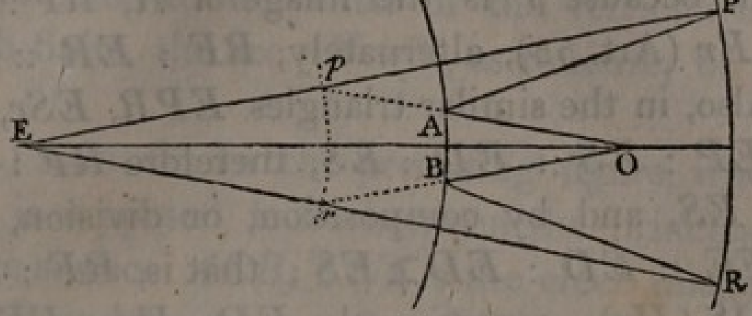
(90.) Cor. 8. If QC and qC be known, the radius of the reflector may be found. For, $QF : FE :: QC : qC$; therefore $QF \mp FE : FE :: QC \mp qC : qC$, or $QC : FE : QC \mp Cq : qC$; hence FE is known, and $2\,FE$ is the radius sought.

The upper sign is to be used when Q and q are on different sides of the reflector, and the lower, when they are on the same side.

(91.) Cor. 9. If the object be a spherical *surface*, generated by the revolution of PQR about the axis EC, the image will be a similar surface, and the magnitude of the object : the magnitude of the image :: $\overline{EQ}|^2 : \overline{Eq}|^2$.

(92.) Cor. 10. If O, the place of the eye, be given, the part of the object seen in a given portion AB, of the reflector, may be thus determined:

Join OA, OB, and produce them, if necessary, till they meet the image in p, r; then pr is the part of



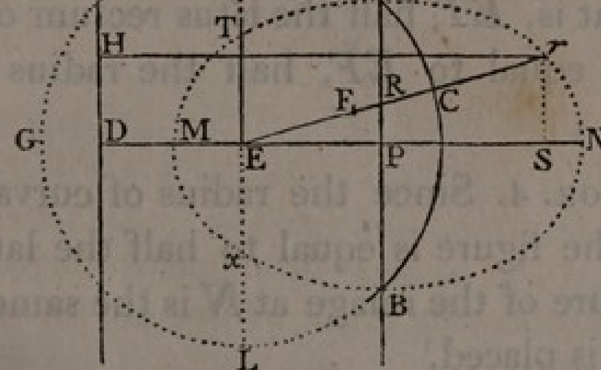
the image seen (Art. 65); join Ep, Er, and let these lines cut the object in P, R, and PR is the corresponding part of the object.

In this corollary, the pupil is supposed to be a point; and to receive the rays which are incident nearly perpendicularly upon the reflector.

Prop. XVI.

(93.) *If the object placed before a spherical reflector be a straight line, the image is a conic section.*

Let ACB be the reflector, E it's center; PR the object placed before it; through E draw $DEPN$ at



right angles to RP; take any point R in the object, join ER and produce it; bisect EC in F, and let r be the image of R; in the line PED take ED equal to EP, and draw DH at right angles to PD; from r draw rH, rS, respectively parallel to DN, and RP.

Then, because r is the image of R, $RF : FE :: ER : Er$ (Art. 55), alternately, $RF : ER :: FE : Er$. Also, in the similar triangles EPR, ESr, $ER : Er :: EP : ES :: ED : ES$, therefore $RF : FE :: ED : ES$, and by composition, or division, $RF : RF \mp FE :: ED : ED \mp ES$; that is, $RF : ER :: ED : DS$ (Hr); consequently, $ED : Hr :: FE : Er$; alternately, $ED : FE :: Hr : Er$; that is, Hr bears an invariable ratio to Er, and therefore r is a point in the conic section whose focus is E and directrix DH.

(94.) Cor. 1. When EP is equal to EF, Hr is equal to Er, and the conic section is a *Parabola*. It is an *Ellipse* or *Hyperbola*, according as EP is *greater* or *less* than EF.

(95.) Cor. 2. When the distance of the object is so great that the rays which come from any point in it may be considered as parallel, the image is a circle whose radius is EF.

(96.) Cor. 3. If ET be drawn perpendicular to EN, when r coincides with T, Hr becomes equal to ED; and since $ED : EF :: Hr : Er$, Er becomes equal to EF. That is, ET, half the latus rectum of the conic section, is equal to EF, half the radius of the reflector.

(97.) Cor. 4. Since the radius of curvature at the vertex of the figure is equal to half the latus rectum, the curvature of the image at N is the same, wherever the object is placed.

(98.) Cor. 5. If the radius of the reflector be finite, the evanescent arc rN is equal to the ordinate rS

(*Newton*, Lem. 7); and therefore $RP : rN :: EP : ES :: EP : EN$.

Also, whilst the angle REP, which the straight line subtends at the center of the reflector is small, though finite, the arc rN will, as to sense, be a right line.

(99.) Cor. 6. In the preceding figure, where the object is supposed to lie between the principal focus and the surface ACB, TNx is the erect image of the line RP indefinitely produced both ways, and TMx it's inverted image. The part ANB is formed by reflection from the concave surface ACB; AT and Bx, by reflection from the convex surfaces AK, BL; and TMx by reflection from the concave surface KGL.

Sect. IV.

ON THE REFRACTION OF RAYS AT PLANE AND SPHERICAL SURFACES.

Prop. XVII.

Art. (100.) *When a ray of light passes out of one medium into another, as the angle of incidence increases, the angle of deviation also increases.*

Let A and B be the angles of incidence and refraction; and let sin. A : sin. B :: m : n. Then by composition and division, sin. A + sin. B : sin. $A \sim$ sin. B :: $m+n : m\sim n$; but sin. A+sin. B : sin. $A\sim$sin. B :: tang $\frac{A+B}{2}$: tang. $\frac{A\sim B}{2}$*; therefore tang. $\frac{A+B}{2}$: tang. $\frac{A\sim B}{2}$ :: $m+n : m\sim n$. Now let A increase, then

---

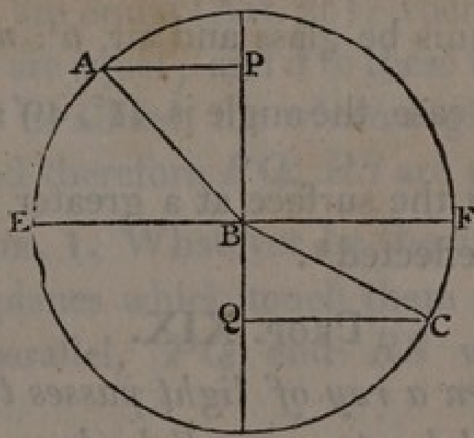
* This is deducible from the common principles of trigonometry. The sides of a triangle are proportional to the sines of the opposite angles; therefore the sum of two sides : their difference :: the sum of the sines of the opposite angles : the difference of their sines; and the sum of the sides : their difference :: the tangent of half the sum of the opposite angles : the tangent of half their difference; consequently, the sum of the sines of the angles : the difference of their sines :: the tangent of half their sum : the tangent of half their difference.

B also increases (Art. 26); therefore $\frac{A+B}{2}$ increases; and $\frac{A+B}{2}$ is less than a quadrant; therefore tang. $\frac{A+B}{2}$ increases; hence, tang. $\frac{A\sim B}{2}$, and consequently $\frac{A\sim B}{2}$, and also, $A\sim B$, the angle of deviation, increases.

Prop. XVIII.

(101.) *A ray of light cannot pass out of a denser into a rarer medium if the angle of incidence exceed a certain limit.*

Let a ray of light AB be incident upon the surface EBF of a rarer medium, and refracted in the direction BC. Through B, draw PBQ perpendicular to the surface; take BC equal to AB, and draw AP, CQ



at right angles to PQ. Then, since the ray passes out of a denser medium into a rarer, the sine of refraction is greater than the sine of incidence (Art. 33); and these sines are always in a given ratio (Art. 24); therefore the sine of refraction will become equal to the radius sooner than the sine of incidence; let the angle of incidence be increased till the sine of refraction is equal to the radius, and let the angle of incidence, and consequently the sine of incidence, be farther increased; then, if the ray be refracted, the sine of refraction must also be increased, which is impossible; therefore the ray cannot, consistently with the general laws of refraction, pass out of the denser into the rarer medium, when the angle of incidence exceeds this limit.

To determine the limit, let the sine of incidence be to the sine of refraction, out of the denser medium into the rarer, $:: n : m$; and let x be the sine of incidence when the corresponding sine of refraction is r, the radius; then $n : m :: x : r$, and $x = \frac{nr}{m}$; and the sine of incidence being known, the corresponding angle may be found from a table of sines.

Thus, if the two mediums be water and air, $n : m :: 3 : 4$; therefore $x = \frac{3r}{4}$; and the angle, whose sine is $\frac{3}{4}$ of the radius, is $48°. 36'$, nearly.

If the mediums be glass and air, $n : m :: 2 : 3$, and $x = \frac{2r}{3}$; in this case, the angle is $41°. 49'$; and the rays which fall upon the surface at a greater angle of incidence will be reflected*.

Prop. XIX.

(102.) *When a ray of light passes through a medium contained by two parallel plane surfaces, the directions in which it is incident and emergent are parallel to each other*†.

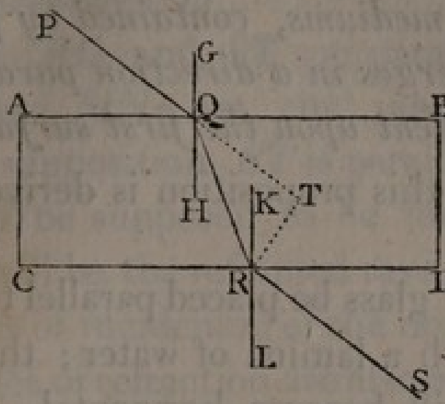
Let *ABDC* be the medium, *PQ*, *QR*, *RS*, the

---

* The reflection thus made, is much stronger than can be produced by any polished metallic surface, or glass speculum.

† In this, and similar cases, the ambient medium is supposed to be uniform.

course of a ray refracted through it; *GQH*, *KRL* perpendiculars to *AB* and *CD* at the points *Q* and *R*. Now,



the effect of the refraction is the same, whether *PQRS* be the course of the ray, or *QR* pass both ways out of the medium (Art. 30); let the latter supposition be made; then, since *AB* is parallel to *CD*, the alternate angles *BQR*, *QRC* are equal; and therefore their complements *RQH*, *QRK*, which are the angles of incidence, are equal; hence, the angles of refraction *PQG*, *SRL*, are equal (Art. 25); therefore the angles *PQA*, *SRD* are equal; and if to these the equal angles *AQR*, *QRD* be added, the whole angles *PQR*, *QRS* are equal, and therefore *PQ*, *RS* are parallel.

(103.) Cor. 1. Whatever be the form of the surfaces, if the planes which touch them at the points *Q* and *R* be parallel, *PQ* and *RS* will be parallel (Art. 28).

(104.) Cor. 2. If *RT* be drawn perpendicular to *PQ* produced, *QR* : *RT* :: rad. : sin. *RQT*; wherefore, when the angle of incidence *PQG*, and consequently the angle of deviation *RQT*, is small, and the thickness of the medium also small, *PQRS* may, without sensible error, be considered as a straight line.

Prop. XX.

(105.) *When a ray of light is refracted through two contiguous mediums, contained by parallel plane surfaces, it emerges in a direction parallel to that in which it is incident upon the first surface.*

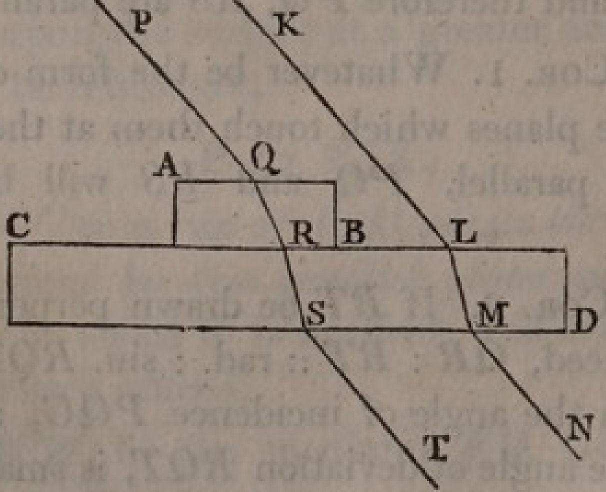
The truth of this proposition is derived from experiment.

Let a plate of glass be placed parallel to the horizon, and covered with a lamina of water; then the surface of the water will become horizontal, and therefore parallel to the surfaces of the glass plate; and if the sun's light be refracted through the two mediums, the incident and emergent rays are found to be parallel*.

Prop. XXI.

(106.) *A ray of light is as much refracted in passing through one medium into another, when they are terminated by parallel plane surfaces, as it is in passing immediately into the latter medium.*

Let *AB*, *CD* be the mediums; *PQRST* the course



of a ray refracted through them in the plane of the

* Newt. *Lectiones Opticæ*, Par. I. Sect. II.

paper; *KLMN* the course of a ray which is incident parallel to *PQ*, and refracted, in the same plane, through the medium *CD*.

Then, since *PQ* and *ST* are parallel (Art. 105), and also *KL* and *MN* (Art. 102), and *PQ* is parallel to *KL* by the supposition, *ST* is parallel to *MN*; and if *TS* and *NM* be supposed to be the incident rays, *SR* and *ML* will be the refracted rays (Art. 29); also, since the angles of incidence, of the rays *TS*, *NM*, are equal, the angles of refraction are also equal (Art. 25); hence, the complements of these angles are equal, and *SR*, *ML* are parallel; that is, the sum, or difference, of the deviations at *Q* and *R*, is equal to the deviation at *L*.

(107.) Cor. Hence it follows, that if a ray pass through any number of mediums, contained by parallel plane surfaces, it will be as much bent from it's original course as if it passed immediately out of the first medium into the last*; and the ratio of the sine of incidence to the sine of refraction, out of the first medium into the last, is the same, whether the ray pass immediately out of the first into the last, or through any number of mediums.

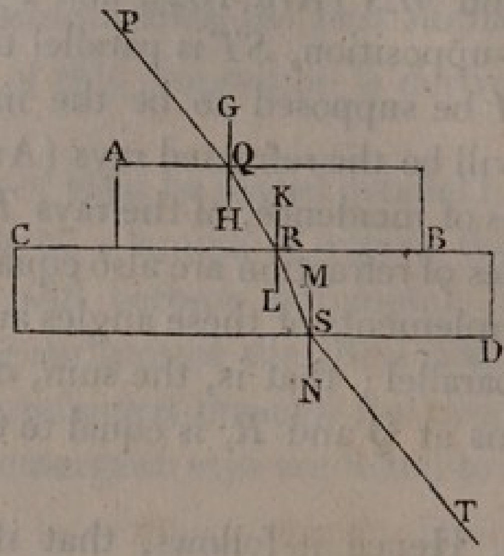
Prop. XXII.

(108.) *Having given the ratio of the sines of incidence and refraction, when a ray passes out of one medium into each of two others, to find the ratio of the sine of incidence to the sine of refraction out of one of the latter mediums into the other.*

Let *AB*, *CD* be two contiguous mediums contained by parallel plane surfaces; *PQRST* the course of a

* Newton's *Optics*, Book II. Part III. Prop. X.

ray refracted through them; through the points *Q*, *R*, *S*, draw *GQH*, *KRL*, *MSN* at right angles to the surfaces; and let $a : b ::$ sin. incidence : sin. refraction out of the surrounding medium into *AB*; $c : d ::$ sin.



incidence : sin. refraction out of the surrounding medium into *CD*.

Then, since *PQ* is parallel to *ST* (Art. 105), and *GH* to *MN*, the angles *PQG*, *NST* are equal; also, the angles *HQR*, *LRS*, are respectively equal to the angles *QRK*, *RSM*. Now, from the hypothesis, sin. *PQG* : sin. *HQR* $:: a : b$; inversely, sin. *HQR* : sin. *PQG* $:: b : a$; or

sin. *QRK* : sin. *NST* $:: b : a$; also,
sin. *NST* : sin. *RSM*(*SRL*) $:: c : d$; by composition, sin. *QRK* : sin. *SRL* $:: bc : ad$.

Ex. When a ray passes out of air into water, sin. incidence : sin. refraction $:: 4 : 3 :: a : b$; out of air into glass, sin. incidence : sin. refraction $:: 3 : 2 :: c : d$; therefore, out of water into glass, sin. incidence : sin. refraction $:: bc : ad :: 9 : 8$.

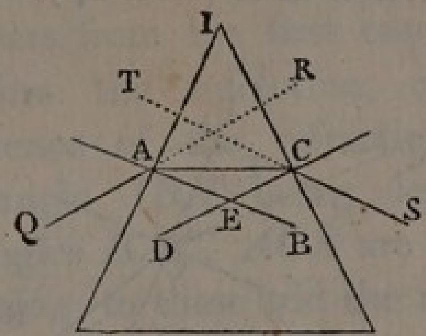
(109.) Def. A *Prism*, in optics, is a solid terminated by three rectangular parallelograms, and two similar, equal and parallel triangles.

(110.) A line drawn through the center of gravity of the prism, parallel to the intersections of the parallelograms, is called the *axis* of the prism.

Prop. XXIII.

(111.) *A ray of light which passes through a prism, in a plane perpendicular to it's axis, is turned* towards *the thicker part, or* from *it, according as the prism is* denser, *or* rarer, *than the surrounding medium.*

Let *AIC* represent a section of the prism, made by a plane which is perpendicular to it's axis, and therefore



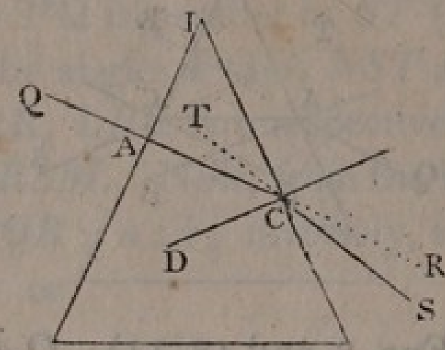
to it's surfaces (Euc. 8. and 18. 11); *QA* a ray incident in the plane *AIC*; *AC*, *CS* the course of the refracted ray, in that plane (Art. 24). Then, the effect of the refraction is the same, whether we suppose the ray to pass thus through the prism, or *AC* to pass both ways out of the prism (Art. 30); let this latter supposition be made, and the proposition resolves itself into the three following cases:

Case 1. When *AC* makes two acute angles with the sides of the prism.

Draw *AB*, *CD* at right angles to *IA*, *IC*, and let them meet in *E*. Then, since the ∠ *CAI* is less than the ∠ *EAI*, *CA* is nearer to the vertex *I* than *EA*; and as they cross each other at *A*, *CA* produced is farther from the vertex than *EA* produced; also, the ray *CA*, when the prism is denser than the surrounding medium, is turned from the perpendicular; that is, from *EA* produced, or towards the thicker part of the prism. In the same manner it may be proved, that the ray *AC* is refracted at *C* towards the thicker part of the prism; consequently, the bending upon the whole is in that direction.

Case 2. When *AC* makes a right angle with one side of the prism.

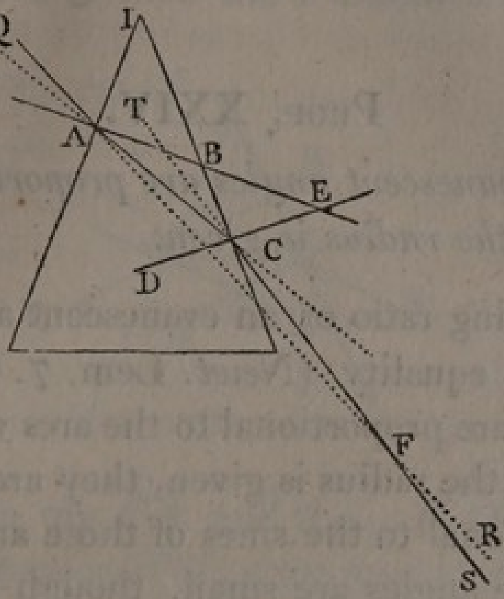
Let the angle at *A* be a right angle; then, since there is no refraction at *A* (Art. 27), the whole bend-



ing is at *C*, which may be shewn, as before, to be towards the thicker part of the prism.

Case 3. When *AC* makes an obtuse angle with one side of the prism.

Let the angle *IAC* be obtuse, and the construction being made as before, *CA* lies nearer to the base of the prism than *EA*; and *CA* produced lies farther from the base than *EA* produced; also, the ray *CA*, in passing from a denser medium into a rarer, is turned from the perpendicular; that is, from *EA* produced,



or from the base. Thus then, the refraction at *A* is from the thicker part of the prism, and the refraction at *C*, as appears from the first case, is the contrary way; therefore the refraction, upon the whole, is the difference of the refractions at *A* and *C*. Now, the angle *IBA* being less than a right angle, the angles *BAC*, *ACB* are together less than one right angle; to these add the right angle *BCE*, and the angles *BAC*, *ACB*, *BCE*, or *BAC*, *ACE*, are together less than two right angles; therefore *AB* and *CE* will meet, if produced, above *AC* (Euc. Ax. 12); let them meet in *E*. Then, since the exterior angle *DCA*, of the triangle *CAE*, is greater than the interior and opposite angle *CAE*, the angle of incidence of the ray *AC* is greater than the angle of incidence of the ray *CA*, and therefore the deviation at *C* is greater than the deviation at *A* (Art. 100); or the excess of the deviation is towards the thicker part of the prism.

In the same manner it may be proved, that a ray of light will be bent *from* the thicker part of a prism which is *rarer* than the surrounding medium.

Prop. XXIV.

(112.) *Evanescent angles are proportional to their sines, when the radius is given.*

The limiting ratio of an evanescent arc to it's sine is a ratio of equality (*Newt.* Lem. 7. Cor. 1); and since angles are proportional to the arcs which subtend them, when the radius is given, they are, in this case, also proportional to the sines of those arcs.

When the angles are small, though of finite magnitude, the same proposition is nearly true; and sufficiently accurate, if the conclusions drawn from it be considered in a practical light, and applied to the construction of optical instruments, or the explanation of the phænomena of refraction.

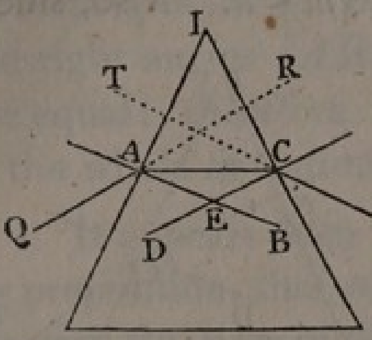
Prop. XXV.

(113.) *When a ray of light passes through a prism, in a plane which is perpendicular to it's axis, and the angles of incidence are small, the vertical angle of the prism is to the angle of deviation, as the sine of incidence, out of the prism into the ambient medium, is to the difference of the sines of incidence and refraction.*

The same construction and supposition being made as in Art. 111, the proposition will, in like manner, resolve itself into three cases.

Case 1. When *AC* makes two acute angles with the sides of the prism.

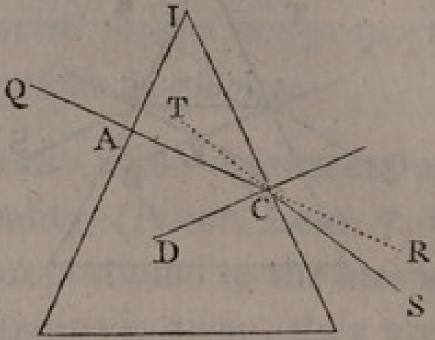
Let $m : n$ :: sin. incidence : sin. refraction, out of the prism into the ambient medium; produce QA, SC, to R and T; then the $\angle\, CAE$ is the angle of



incidence of the ray CA, and the $\angle\, EAR$ is equal to the angle of refraction of the same ray; also the $\angle\, ECA$ is the angle of incidence, and the $\angle\, ECT$ equal to the angle of refraction of the ray AC; and since the angles of incidence, and consequently the angles of refraction are small, they are proportional to their sines (Art. 112); therefore $EAC : EAR :: m : n$; and $EAC : EAC \sim EAR\ (CAR) :: m : m \sim n$. In the same manner, $ECA : ACT :: m : m \sim n$; hence $EAC : CAR :: ECA : ACT$, and $EAC + ECA : CAR + ACT :: ECA : ACT :: m : m \sim n$ (Euc. 12. 5). Again, since the four angles of the quadrilateral figure $IAEC$ are equal to four right angles, and the angles IAE, ICE are right angles, the two angles AEC, AIC are together equal to two right angles, or to the two angles AEC, AED; consequently, the angle AIC is equal to the angle AED; and AED is equal to the sum of the angles EAC, ECA; therefore the angle AIC is equal to the sum of those angles; also, the sum of the angles CAR, ACT is, in this case, the whole deviation (Prop. 23. Case 1.); therefore, from the last proportion, AIC : the whole deviation :: $m : m \sim n$.

Case 2. When AC is at right angles to one side of the prism.

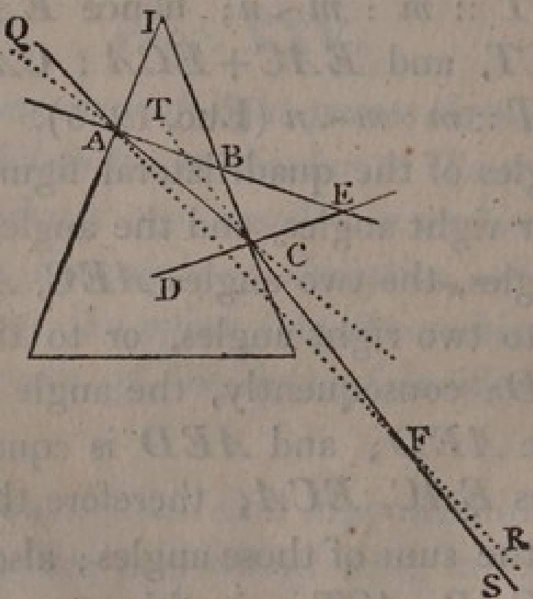
Let the angle CAI be a right angle; then the whole refraction is at C; and in this case, as before, $DCA : ACT :: m : m \sim n$. Also, since the right angle



DCI is equal to the sum of the two ACI, AIC, take away the common angle ACI, and the remaining angles DCA, AIC are equal; consequently $AIC : ACT :: m : m \sim n$.

Case 3. When AC makes an obtuse angle with one side of the prism.

It may be shewn in the same manner as in Case 1,



that $ACD : ACT :: m : m \sim n$; and $CAE : CAF :: m : m \sim n$. Hence, $ACD - CAE : ACT - CAF ::$

$m : m \sim n$ (Euc. 19. 5.); and since ACD is the exterior angle of the triangle ACE (Prop. 23. Case 3), $ACD - CAE = CEA$; also, $ACT - CAF$ is the whole deviation; therefore CEA : the deviation :: $m : m \sim n$. Again, since the triangles AIB, BCE, have vertical angles at B, and right angles IAB, BCE, the angles AIB, BEC are equal; therefore, from the last proportion, AIB : the whole deviation :: $m : m \sim n$.

(114.) Cor. 1. It appears from the demonstration of the foregoing proposition, that when the ray makes two acute angles with the sides of the prism, the angle at the vertex is equal to the *sum* of the angles of incidence; and when it makes an obtuse angle with one side, the angle at the vertex is equal to the *difference* of the angles of incidence.

(115.) Cor. 2. Hence it follows, that the angles of incidence cannot be small, unless the angle at the vertex of the prism be also small.

Ex. 1. Let the angle at the vertex of a glass prism, placed in air, be $1°$; then $m : n :: 2 : 3$, and $m : m \sim n :: 2 : 1 :: 1° : \frac{1}{2}°$, the angle of deviation when the angles of incidence are small.

Ex. 2. Let the same prism be placed in water, then $m : n :: 8 : 9$; and $m : m \sim n :: 8 : 1 :: 1° : \frac{1}{8}°$, the angle of deviation in this case.

(116.) Cor. 3. When the angle at the vertex of the prism vanishes, or the sides become parallel, the angle of deviation also vanishes (See Art. 102).

(117.) Cor. 4. If the quantity and direction of the deviation, and the angle at the vertex of the prism be known, the ratio of the sine of incidence to the sine of refraction may be determined.

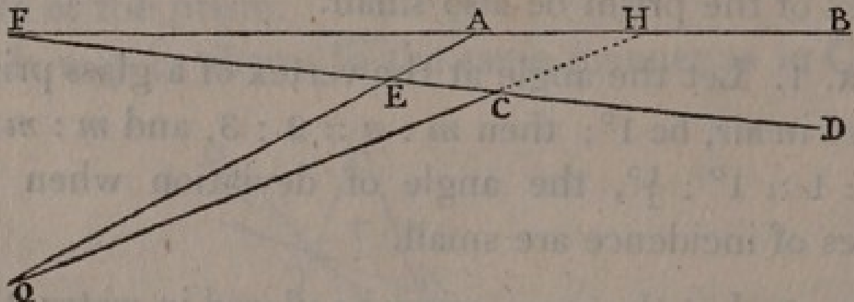
Thus, if the deviation be $\frac{1}{3}$ of the angle at the vertex, and towards the thicker part of the prism, $m : n - m :: 3 : 1$; and by composition, $m : n :: 3 : 4$.

If the deviation be towards the thinner part of the prism, $m : m - n :: 3 : 1$; therefore $m : n :: 3 : 2$. In these, as in the former cases, the angles of incidence and refraction are supposed to be proportional to their sines.

Prop. XXVI.

(118.) *When two rays are refracted through a prism, in the same plane, and the angles of incidence on each surface are small, the emergent rays are inclined to each other at an angle equal to that which is contained between the incident rays.*

Let QA, QC be the incident rays; AB, CD the directions of the refracted rays; and, if possible, let



AB, CD be parallel. Produce QC till it meets FAB in H; then since QH falls upon the parallel lines AB, ED, the angles ECQ and AHC are equal; and the exterior angle FAE, of the triangle AHQ, is greater than the interior and opposite angle AHQ; therefore it is also greater than the angle ECQ; but, because the angle at the vertex of the prism is invariable, the angles FAE, ECQ, which are equal to the angles of deviation of the rays QA, QC, are equal to each other (Art. 113), which is absurd; therefore AB, and CD

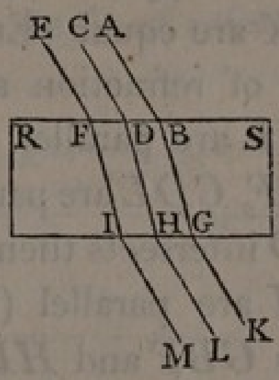
are not parallel*. Let them meet in *F*; then since the vertical angles *FEA*, *QEC* are equal, and also the angles *FAE*, *ECQ*, the remaining angles *AFE*, *EQC* are equal.

Prop. XXVII.

(119.) *Parallel rays, refracted at a plane surface, continue parallel*†.

Case 1. When the angles of incidence are in the same plane.

Let *RS* be the plane refracting surface; *AB*, *CD* the incident, *BG*, *DH* the refracted rays. Then, since *AB*, *CD* are parallel, the angles *ABR*, *CDR* are equal; and therefore the complements of these, or the angles of incidence are equal; hence, the angles of refraction are equal, and consequently the comple-



ments of the angles of refraction, that is, the angles *SBG*, *SDH* are equal; and therefore *BG* and *DH* are parallel.

Case 2. When the angles of incidence are not in the same plane.

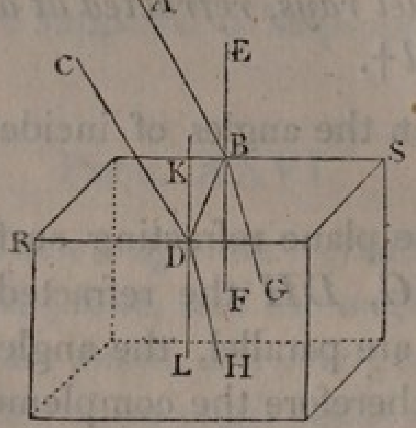
Let *AB*, *CD* be the incident rays; *EBF*, *KDL*

---

* See also Prop. 30.

† In this, and the following propositions, the rays are supposed to be equally refrangible.

perpendiculars to the refracting surface *RS*, at the points of incidence *B* and *D*; join *BD*; and let *AB* be refracted in the direction *BG*, which lies in the plane *ABF* (Art. 24); also, let *DH* be the intersection of the planes *GBD*, *CDL*.

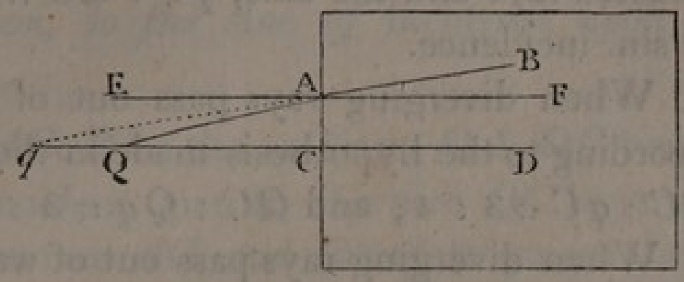


Then, since *EF*, *KL* are parallel (Euc. 6. 11), as also *AB*, *CD*, by the supposition, the angles of incidence *ABE*, *CDK* are equal (Euc. 10. 11); consequently the angles of refraction are equal. Again, because *EF* and *KL* are parallel, and also *AB* and *CD*, the planes *ABF*, *CDL* are parallel (Euc. 15. 11); and the plane *GBD* intersects them; hence it follows, that *BG* and *DH* are parallel (Euc. 16. 11); and therefore the angles *GBF* and *HDL* are equal (Euc. 10. 11); but the angle *GBF* is the angle of refraction of the ray *AB*; therefore *HDL* is equal to the angle of refraction of the ray *CD*; and since *DH* is in the plane *CDL*, *CD* is refracted in the direction *DH* (Art. 24), which has before been shewn to be parallel to *BG*.

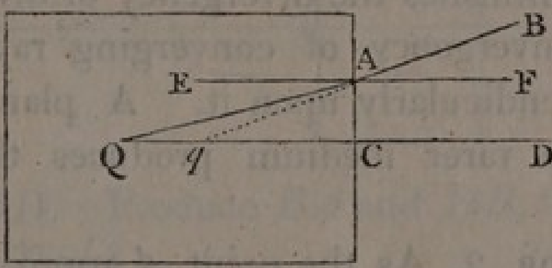
Prop. XXVIII.

(120.) *When diverging or converging rays are incident nearly perpendicularly upon a plane refracting surface, the distance of the focus of incident rays* *from the surface, is to the distance of the geometrical focus of refracted rays from the surface, as the sine of refraction to the sine of incidence.*

Let AC be the plane refracting surface; QA, QC two of a pencil of rays diverging from Q, of which QC is perpendicular to the surface, and therefore suffers no



refraction. Through A, draw EAF parallel to QC; and let QA be refracted in the direction AB; produce BA till it meets CQ, or CQ produced, in q* Then, the ∠ EAQ is the angle of incidence of the ray QA, and the ∠ BAF the angle of refraction; also, the ∠



EAQ is equal to the alternate angle AQC, and the ∠ BAF is equal to the interior and opposite angle AqC; therefore, sin. EAQ, or sin. incidence, $=$ sin. $AQC=$ sin. AQq; and sin. BAF, or sin. refraction, $=$ sin. $AqC=$ sin. AqQ; hence, $QA : qA$ :: (sin. AqQ : sin. AQq ::) sin. refraction : sin. incidence. Now, let A approximate to C, and QA is ultimately equal to QC,

* If they do not meet, AB coincides with AF; that is, the angle of refraction vanishes with respect to the angle of incidence; or the refracting power is infinite.

and qA to qC; therefore, the proportion becomes, $QC : qC ::$ sin. refraction : sin. incidence*.

Let q be the focus of rays incident the contrary way; then BAF is the angle of incidence, and EAQ the angle of refraction (Art. 29); and it may be proved as before, that, when Q is the limit of the intersections of the refracted rays and the axis, $qC : QC ::$ sin. refraction : sin. incidence.

Ex. 1. When diverging rays pass out of air into water, according to the hypothesis made in the proposition, $QC : qC :: 3 : 4$; and $QC : Qq :: 3 : 1$.

Ex. 2. When diverging rays pass out of water into air, $QC : qC :: 4 : 3$; and $QC : Qq :: 4 : 1$.

Ex. 3. When converging rays pass, in the same manner, out of air into glass, $QC : qC :: 2 : 3$; when they pass out of glass into air, $QC : qC :: 3 : 2$.

(121.) Cor. 1. A plane refracting surface of a denser medium, diminishes the divergency of diverging rays, and the convergency of converging rays, incident nearly perpendicularly upon it. A plane refracting surface of a rarer medium produces the contrary effect.

(122.) Cor. 2. As the point A approaches to C, q approaches to Q.

Let sin. incidence : sin. refraction $:: m : n$; then $QA : qA :: n : m$; and $QA^2 : qA^2 :: n^2 : m^2$; or $QC^2 + CA^2 : qC^2 + CA^2 :: n^2 : m^2$; hence $QC^2 + CA^2 : QC^2 \sim qC^2 :: n^2 : n^2 \sim m^2$; and since the point Q is fixed, and QC invariable, as also the ratio of $n^2 : n^2 \sim m^2$, when CA decreases, $QC^2 + CA^2$ decreases; and therefore $QC^2 \sim qC^2$ decreases; consequently Qq decreases.

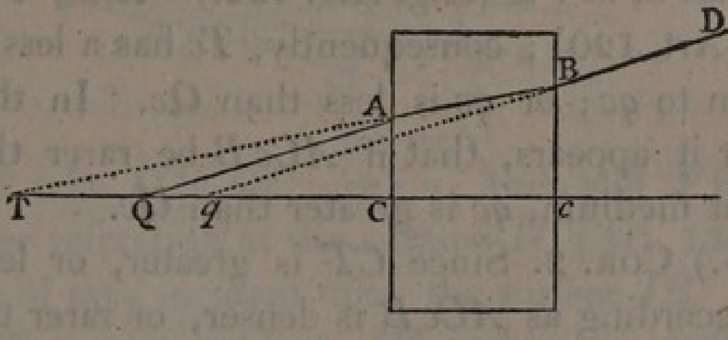
---

* See Note, page 20.

Prop. XXIX.

(123.) *When diverging or converging rays pass, nearly perpendicularly, through a medium contained by parallel plane surfaces, the distance of the foci of incident and emergent rays, is to the thickness of the medium, as the difference of the sines of incidence and refraction, to the sine of incidence upon the first surface.*

Let $ACcB$ be the medium, QA, QC two rays of a pencil incident upon it, of which QC is perpendicular to the surface AC, and therefore passes through the medium without suffering any refraction; let QA be refracted in the direction AB, and emergent in the



direction BD. Produce BA and DB, till they meet the axis in T and q.

Then, because QA and qB are parallel (Art. 102), $TA : AB :: TQ : Qq$ (Euc. 2. 6); and because AC is parallel to Bc, $TA : AB :: TC : Cc$; therefore $TQ : Qq :: TC : Cc$; and alternately, $TQ : TC :: Qq : Cc$. Now let A approximate to C, and T is, ultimately, the geometrical focus of the rays after the first refraction; therefore $QC : TC ::$ sin. refraction : sin. incidence (Art. 120); and by division, $QT : TC ::$ sin. incidence$\sim$sin. refraction : sin incidence; consequently, $Qq : Cc ::$ sin. incidence$\sim$sin. refraction : sin. incidence.

If rays, incident the contrary way, converge to q, they will, after both refractions, converge to Q (Art. 29); therefore, as before, $Qq : Cc ::$ sin. incidence $\sim$ sin. refraction : sin. incidence on the first surface.

Ex. If the medium be glass, placed in air, $Qq : Cc :: 1 : 3$; if water, placed in air, $Qq : Cc :: 1 : 4$.

(124.) Cor. 1. When the incident rays diverge, the geometrical focus of emergent rays is nearer to c, or farther from it than the focus of incident rays, according as $ACcB$ is denser, or rarer than the ambient medium.

Let $ACcB$ be denser than the ambient medium; and let sin. incidence : sin. refraction $:: m : n$. Then, $TC : QC :: m : n$; therefore Tc has to Qc a less ratio than that of $m : n$ (Alg. Art. 162.) Also, $Tc : qc :: m : n$ (Art. 120); consequently, Tc has a less ratio to Qc than to qc; or qc is less than Qc. In the same manner it appears, that if $ACcB$ be rarer than the ambient medium, qc is greater than Qc.

(125.) Cor. 2. Since CT is greater, or less than CQ, according as $ACcB$ is denser, or rarer than the ambient medium (Art. 120), it is manifest that T and q lie on opposite sides of Q.

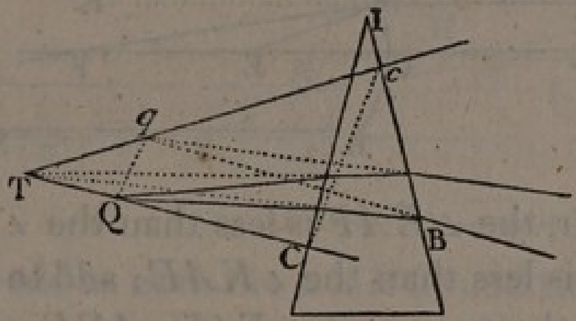
(126.) Cor. 3. When $ACcB$ is denser than the surrounding medium, as A approaches to C, T approaches to Q (Art. 122); and in consequence of this motion of T, q approaches to c. Again, as A approaches to C, B approaches to c; and on this account q approaches to T (Art. 122), or recedes from c. Thus the two motions of q are in opposite directions, and the aberration of oblique rays, from the geometrical focus, is less than when the rays are refracted at a single surface.

The same may be shewn when $ACcB$ is rarer than the ambient medium.

Prop. XXX.

(127.) *Having given the focus of incidence of a pencil of rays which passes nearly perpendicularly through the sides of a prism, and also the ratio of the sine of incidence to the sine of refraction, out of the ambient medium into the prism, to find the focus of emergent rays.*

Let CIB be the prism; Q the focus of incident rays; take $m : n ::$ sin. incidence : sin. refraction out of the ambient medium into the prism. From Q, draw QC perpendicular to IC; and in CQ, or CQ



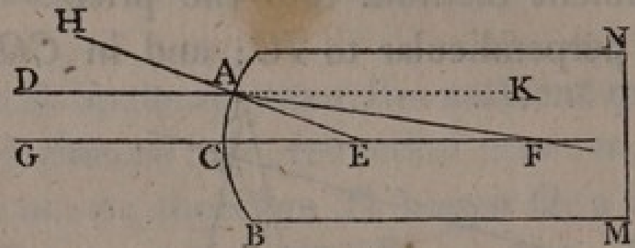
produced, take $TC : QC :: m : n$; then will T be the focus after refraction at the surface IC (Art. 120), or the focus of rays incident upon the surface IB. From T, draw Tc perpendicular to IB, and in cT, or cT produced, take $qc : Tc :: n : m$; and q will be the focus of emergent rays (Art. 120).

(128.) Cor. Since $QC : TC :: n : m :: qc : Tc$, if Cc and Qq be joined, these lines are parallel (Euc. 2. 6); and therefore $Qq : Cc :: TQ : TC :: m \sim n : m$.

Prop. XXXI.

(129.) *Parallel rays, refracted at a convex spherical surface of a denser, or a concave of a rarer medium, into which they pass, are made to converge; and refracted at a concave spherical surface of a denser, or convex of a rarer medium, they are made to diverge.*

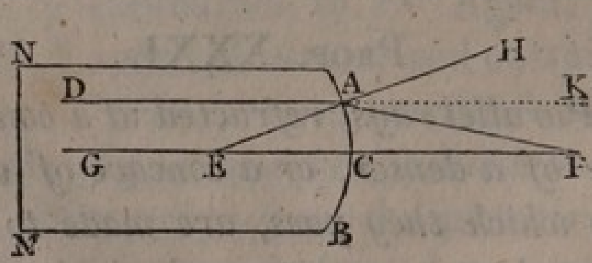
1. Let *DA*, *GC* be two rays of a parallel pencil, passing out of a rarer medium into a denser, and incident upon the convex spherical surface *ACB*, whose center is *E*. Let *GCE* pass through the center of the surface, and it suffers no refraction. Join *EA*, and produce it to *H*; also, produce *DA* to *K*; and let *DA* be refracted in the direction *AF*; then, *DAH* is the angle of incidence, and *EAF* the angle of refraction of this ray; and since it passes out of a rarer medium



into a denser, the ∠ *EAF* is less than the ∠ *HAD*, and therefore it is less than the ∠ *KAE*; add to each the ∠ *AEF*, and the two angles *FAE*, *AEF* are together less than the two *KAE*, *AEF*; and therefore they are less than two right angles (Euc. 29. 1); consequently, *AF*, and *GE*, if produced, will meet.

2. When the rays pass out of a denser medium into a rarer, and the surface of the medium into which they are refracted is spherically concave.

The construction being made as before, since the ray *DA* passes out of a denser medium into a rarer,

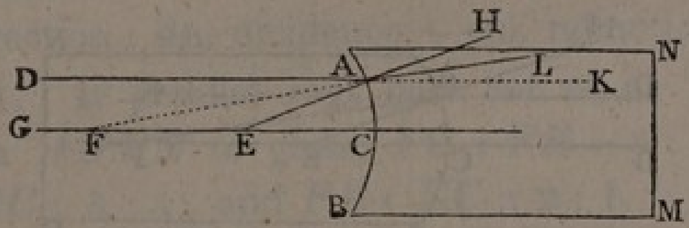


the angle of incidence *DAE*, or it's equal *AEC*, is less than the angle of refraction *HAF*; add to each the angle *EAF*, and the two *EAF*, *AEF*, are together less than the two *EAF*, *HAF*; that is, they are together less than two right angles; therefore *AF* and *EC*, if produced, will meet.

3. When the rays pass out of a rarer medium into a denser, and the surface of the medium into which they are refracted is spherically concave.

The same construction being made, let *DA* be refracted in the direction *AL*, and produce *LA* to *F*.

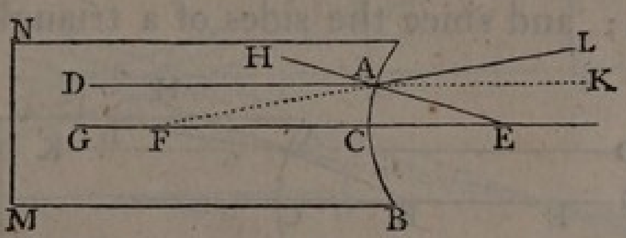
Then, since the ray *DA* passes out of a rarer medium into a denser, the ∠ *DAE* is greater than the ∠ *HAL*,



or *FAE*; add to each the ∠ *AEG*, and the two *FAE*, *AEG*, are together less than the two *DAE*, *AEG*; that is, they are less than two right angles; therefore *AF* and *EG* will meet.

4. When the rays pass out of a denser medium into a rarer, and the surface of the medium into which they are refracted is spherically convex.

In this case, as before, the ∠ *DAH*, or it's equal *AEC*, is less than the ∠ *EAL*; add to each the ∠



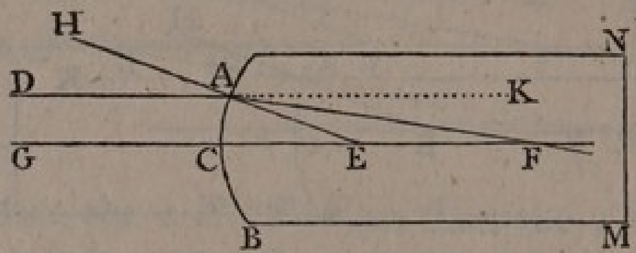
EAF, and the two *EAF*, *AEC*, are together less than the two *EAF*, *EAL*; that is, they are less than two right angles; therefore *AF* and *EC*, if produced, will meet.

In the two last cases, the refracted rays meet if produced backwards; that is, they diverge.

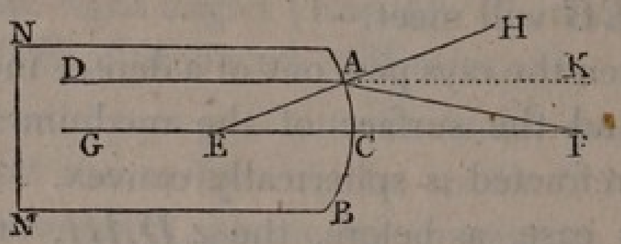
Prop. XXXII.

(130.) *When parallel rays are incident nearly perpendicularly upon a spherical refracting surface, the distance of the geometrical focus of refracted rays from the surface, is to it's distance from the center, as the sine of incidence to the sine of refraction.*

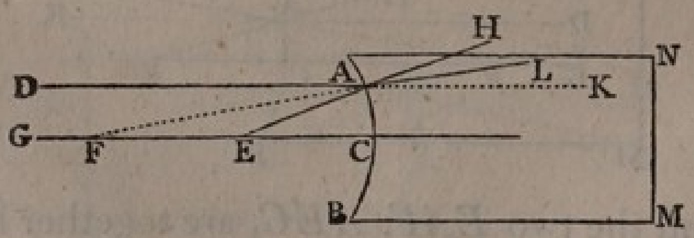
The construction being made as in the last proposition, the angle AEF, in each of the four cases, is



either equal to the angle of incidence of the ray DA, or to it's supplement; therefore sin. $\angle AEF =$ sin.

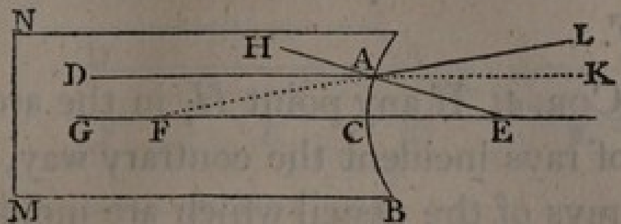


incidence. In the same manner, sin. $\angle EAF =$ sin. refraction; and since the sides of a triangle are proportional to the sines of the opposite angles,



portional to the sines of the opposite angles, $FA : FE ::$ sin. $\angle AEF$: sin. $\angle EAF$ :: sin. incidence : sin. refraction. Now let the point A approximate to C, and

FA is, ultimately, equal to *FC*; therefore, the pro-



portion becomes *FC* : *FE* :: sin. incidence : sin refraction.

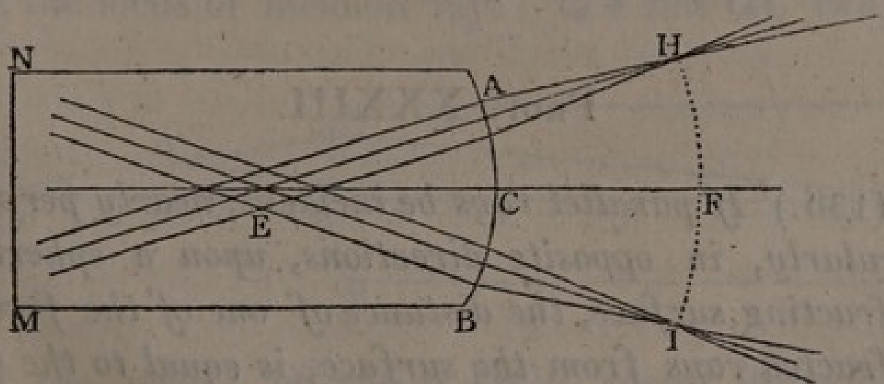
(131.) Cor. 1. Since *FC* : *FE* :: sin. incidence : sin. refraction, by division, *FC* : *CE* :: sin. incidence : sin. incidence ∼ sin. refraction. Also, *FE* : *CE* :: sin. refraction : sin. incidence ∼ sin. refraction.

Ex. 1. If parallel rays pass out of air into the medium *ABMN* of glass, *FC* : *FE* :: 3 : 2; also, *FC* : *EC* :: 3 : 1; and *FE* : *EC* :: 2 : 1.

Ex. 2. When the rays pass out of glass into air, *FC* : *FE* :: 2 : 3; also, *FC* : *EC* :: 2 : 1; and *FE* : *EC* :: 3 : 1.

(132.) Cor. 2. If *EC* be diminished whilst the refracting power remains unaltered, *FC* is also diminished.

(133.) Cor. 3. If the axes of several pencils of



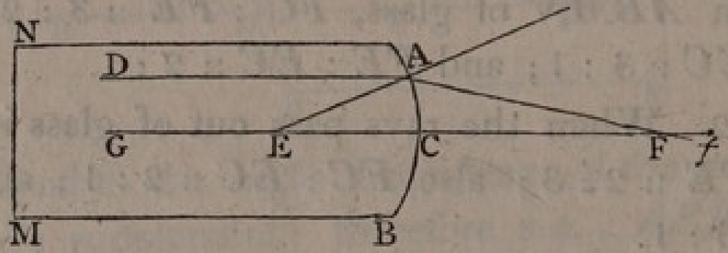
parallel rays be inclined to each other, in the same plane, the foci of refracted rays will lie in the circumference of a circle, *HFI*, whose center is *E*, and radius

EF. If the axes be in different planes, the foci will lie in the surface of a sphere, whose center is *E*, and radius *EF*.

(134.) Cor. 4. If any point *H*, in the arc *HFI*, be the focus of rays incident the contrary way, join *HE*, and those rays of the pencil which are incident nearly perpendicularly, will be refracted parallel to each other, and to *HE* (Art. 29).

(135.) Cor. 5. The distance *EF*, of the intersection of the refracted ray and the axis, from the center, is the greatest, when the arc *AC* is evanescent.

Let f be the geometrical focus; $m : n$ the ratio of the sine of incidence to the sine of refraction. Then,

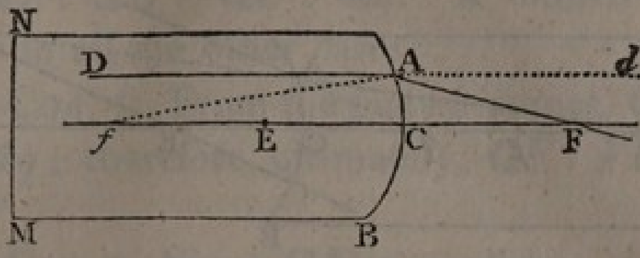


$Ef : EC :: n : m \sim n$; also, $EF : AF :: n : m$; therefore $EF : EF \sim AF :: n : m \sim n$; hence $Ef : EC :: EF : EF \sim AF$; but $EF \sim AF$ is less than EA, or EC (Euc. 20. 1); consequently, EF is less than Ef.

Prop. XXXIII.

(136.) *If parallel rays be incident, nearly perpendicularly, in opposite directions, upon a spherical refracting surface, the distance of one of the foci of refracted rays from the surface, is equal to the distance of the other from the center of the refractor.*

Let *F* be the focus when the rays pass out of the denser medium into the rarer, and f the focus when they pass out of the rarer into the denser; then FC :

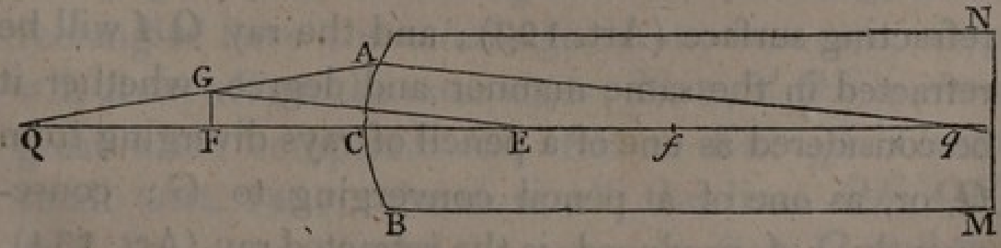


$CE :: n : m-n$ (Art. 131); also, $fE : CE :: n : m-n$; therefore $FC : CE :: fE : CE$; or $FC = fE$. By adding EC to, or subtracting it from each of these equal quantities, $FE = fC$.

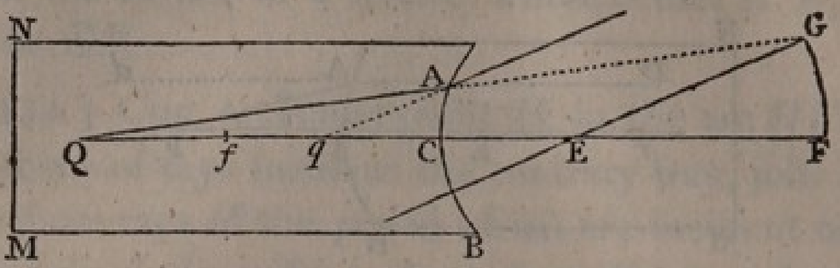
Prop. XXXIV.

(137.) *When diverging or converging rays are incident nearly perpendicularly upon a spherical refracting surface, the distance of the focus of incident rays from the principal focus of rays coming in the contrary direction, is to its distance from the center of the refractor, as it's distance from the surface, to it's distance from the geometrical focus of refracted rays.*

Let ACB be the refracting surface; E it's center; Q the focus of incident rays; QA and QC two rays



of the pencil, of which QCE passes through the center, and therefore suffers no refraction. Take F the principal focus of rays incident in the contrary direction, parallel to EC; and from the center E, with the radius EF, describe the arc FG, cutting QA,



or QA produced, in G; join EG; draw Aq parallel to EG, and let it meet the axis in q.

1. When diverging rays are incident upon a convex spherical refractor of a denser medium.

The ray QA will be refracted at A, in the same manner and degree, whether it be considered as one of a pencil of rays diverging from Q, or as one of a pencil diverging from G; and, on the latter supposition, it will be refracted parallel to GE (Art. 134); therefore Aq is the refracted ray*. And since the triangles QGE, QAq are similar, $QG : QE :: QA : Qq$; let the point A approximate to C, that the ray QA may be incident nearly perpendicularly upon the refracting surface, and the point G approximates to F; therefore ultimately, $QF : QE :: QC : Qq$.

2. When diverging rays are incident upon a convex spherical surface of a rarer medium.

In this case, Q and F are on contrary sides of the refracting surface (Art. 129); and the ray QA will be refracted in the same manner and degree, whether it be considered as one of a pencil of rays diverging from Q, or, as one of a pencil converging to G; consequently, qA, produced, is the refracted ray (Art. 134).

---

* This is only true when GA is incident nearly perpendicularly upon AC; and therefore the intersection of the refracted ray and the axis is only determined in that case.

Hence, as before, $QG : QE :: QA : Qq$; and ultimately, $QF : QE :: QC : Qq$. A similar proof is applicable in all the other cases*.

(138.) Cor. 1. From the same triangles, $QG : GE :: QA : Aq$; therefore, ultimately, $QF : FE :: QC : Cq$.

(139.) Cor. 2. Since GE is parallel to Aq one side of the triangle QAq, the other sides QA, Qq, or those sides produced, are cut proportionally (Euc. 2. 6); therefore $QG : GA :: QE : Eq$; and ultimately, $QF : FC :: QE : Eq$.

(140.) Cor. 3. If f be the other principal focus, and q the focus of incident rays, Q is the focus of refracted rays (Art. 29); therefore, $qf : fE :: qC : QC$ (Art. 138); invertendo, $fE : qf :: QC : Cq :: QF : FE$; hence, $QF : FE :: Ef : fq$.

Prop. XXXV.

(141.) *The distances* QF *and* Qq *must be measured in the same, or opposite directions from* Q, *according as* QC *and* QE *are measured in the same, or opposite directions from that point.*

Since $QF : QE :: QC : Qq$, we have $QF \times Qq = QE \times QC$; and, measuring these lines from Q, if the rectangles have the same sign in any one case, they will always have the same sign. Now, if QF be very great when compared with FE or EC, qf is very small (Art. 140); therefore all the lines QF, QC, QE, Qq, are measured the same way from Q, and the

---

* Since the incident rays may either *converge* or *diverge*, and fall upon a *convex* or *concave* surface of a *rarer* or *denser* medium, the proposition admits of eight cases.

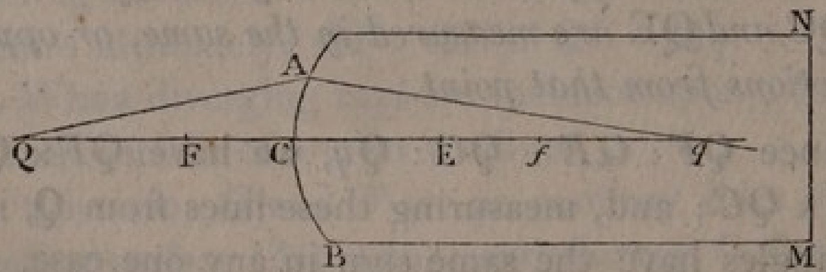
rectangles $QF \times Qq$, and $QE \times QC$, in this case, have the same sign; consequently, they will always be either both positive, or both negative; and according as QC and QE have the *same*, or *different* signs, QF and Qq must have the *same*, or *different* signs; that is, QF and Qq must be measured, from Q, in the *same*, or *different* directions, according as QC and QE are measured in the *same*, or *different* directions (Alg. Art. 471).

(142.) Nearly in the same manner, it may be shewn that QF and fq must always be measured in opposite directions from F and f.

Prop. XXXVI.

(143.) *The conjugate foci,* Q *and* q, *move in the same direction upon the indefinite line* QCq, *and they coincide at the surface and center of the refractor.*

Let the rays be incident nearly perpendicularly on ACB, a convex spherical refracting surface of a denser



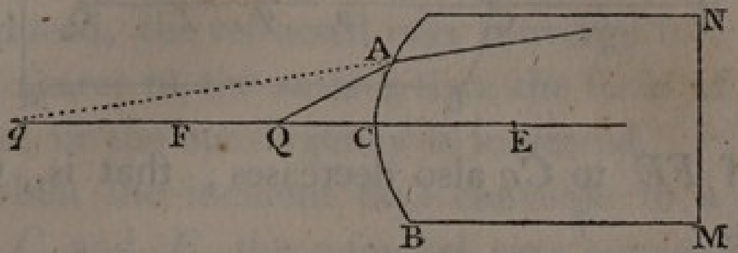
medium; take f the principal focus of rays, thus incident, and F the other principal focus.

When the incident rays are parallel, the refracted rays converge to f.

As Q approaches towards F, since $QF : FE :: QC : Cq$ (Art. 137), and the ratio of $QF : QC$ decreases (Alg. Art. 163), the ratio of $FE : Cq$ decreases; therefore Cq increases.

When Q coincides with F, the distance Cq is indefinitely great.

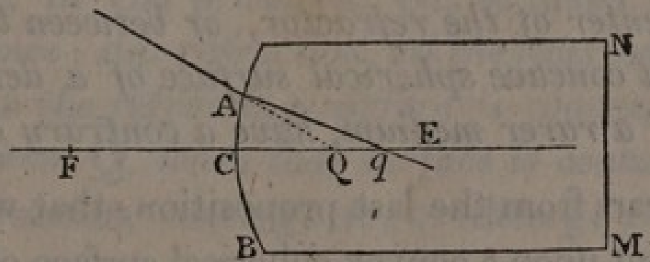
When Q is between F and C, QF and Qq are measured in the same direction from Q (Art. 141); and



as the ratio of QF to QC increases, the ratio of FE to Cq increases, or Cq decreases.

When Q coincides with C, the ratio of $QF : FE$ is finite; therefore the ratio of $QC : Cq$ is finite, and since QC vanishes, qC also vanishes; that is, q coincides with C.

When Q is between C and E, Qq must be measured from Q towards E (Art. 141); and since $QF : QE ::$

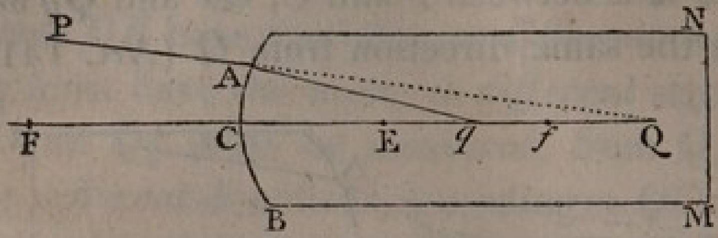


$QC : Qq$ (Art. 137), and QF is greater than QC, QE is greater than Qq; consequently, q lies between Q and E.

When Q coincides with E, since QF is equal to FE, QC is equal to Cq; or, q coincides with E.

When Q is in CE produced, Qq must be measured from Q towards C; and since $QF : QE :: QC : Qq$, and QF is greater than QC, QE is greater than Qq; that is, q lies between Q and E. Also, since QF :

$FE :: QC : Cq$ (Art. 138), and as QC or QF increases, the ratio of QF to QC decreases (Alg. Art. 162), the



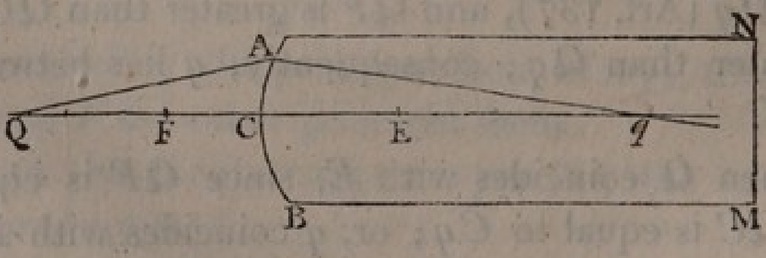
ratio of FE to Cq also decreases; that is, Cq increases.

The same demonstration, mutatis mutandis, may be applied to all the other cases.

Prop. XXXVII.

(144.) *A convex spherical refracting surface of a denser, and a concave of a rarer medium, diminish the divergency, or increase the convergency of all pencils of rays incident nearly perpendicularly upon them, unless the focus of incident rays be in the surface or center of the refractor, or between those two points; a concave spherical surface of a denser, and convex of a rarer medium, have a contrary effect.*

It appears from the last proposition, that when rays are incident upon a convex spherical surface of a denser



medium diverging from Q, a point farther from the surface than F, they are made to converge.

When the incident rays diverge from F, the refracted rays are parallel.

When *Q* is between *F* and *C*, the refracted rays diverge from a point which is farther from the surface than *Q*; therefore the divergency of the rays is diminished *.

When the incident rays converge to any point in *CE* produced, the refracted rays converge to a point which is nearer to the surface than the focus of incident rays; or the convergency is increased.

But when the incident rays converge to a point between *C* and *E*, the refracted rays converge to a point farther from the surface than the focus of incident rays; or the convergency is diminished.

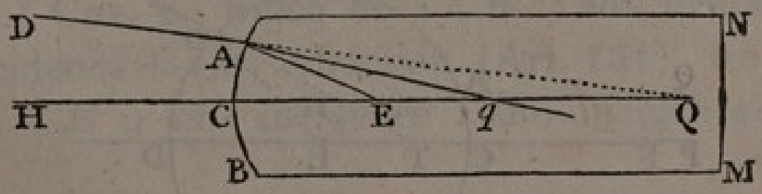
When *Q* coincides with *C* or *E*, the convergency, or divergency, is not altered.

In the same manner, the proposition may be proved in the other cases.

Prop. XXXVIII.

(145.) *If* E *be the center of a spherical refractor* ACB, *and, in* CE *produced,* QE *be taken to* EC :: *sin. incidence : sin. refraction, all the rays converging to* Q, *when the refracting surface is convex, and diverging from* Q, *when that surface is concave, will, after refraction, converge to, or diverge from, one point.*

When the refracting surface is convex, let *DA* and



HC be two rays of the pencil, of which *HC* passes

* See Art. 60.

through the center, and therefore suffers no refraction; and let DA be refracted in the direction Aq; join EA.

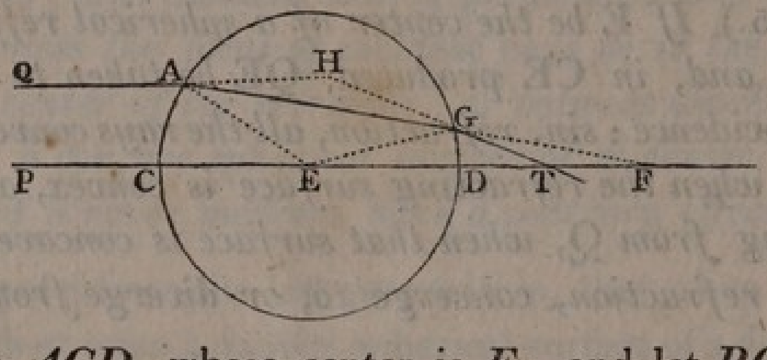
Then, since sin. incidence : sin. refraction :: QE : EA :: sin. $\angle QAE$: sin. $\angle AQE$, and the $\angle QAE$ is equal to the angle of incidence, the angle AQE is equal to the angle of refraction, that is, to the $\angle EAq$; also, the $\angle AEQ$ is common to the two triangles QAE, AqE; therefore these triangles are similar, and $QE : EA :: EA : Eq$, the three first terms of which proportion being invariable, the fourth, Eq, is also invariable; that is, all the refracted rays meet the axis in the same point.

The proposition may be proved in the same manner when the refracting surface is concave.

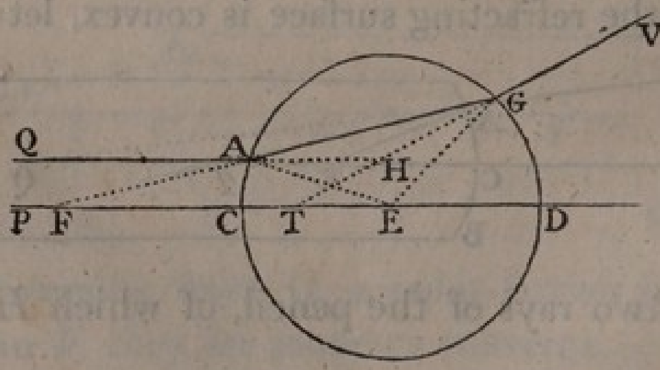
Prop. XXXIX.

(146.) *To find the principal focus of a sphere.*

Let a pencil of parallel rays be incident upon the



sphere ACD, whose center is E; and let PCE be



that ray which passes through the center, and therefore suffers no refraction at either surface (Art. 27); also, let QA, any other ray of the pencil, be refracted in the direction AG, and emergent in the direction GT, or GV, which, produced backwards, or forwards, as the case may require, cuts the axis in T. Produce QA, and TG, or VG, till they meet in H; join EA, EG.

Then, if two rays GA, AG pass out of the sphere at A and G, the angles of incidence EAG, EGA are equal, and therefore the angles of deviation are equal; but the deviation at A is the same, whether GA or QA be the incident ray (Art. 30); consequently, when the ray QA is refracted through the sphere, the deviation at A is equal to the deviation at G; or, the $\angle HAG =$ the $\angle FGT$. Also, the $\angle HAG =$ the $\angle GFT$; therefore the $\angle GFT =$ the $\angle FGT$, and $FT = TG$. Now, let the point A approximate to C, and F is, ultimately, the principal focus of rays after the first refraction; also, the point G approximates to D*; consequently, TG is ultimately equal to TD, and therefore $FT = TD$; that is, the principal focus bisects the distance between the focus after the first refraction, and the farther extremity of the diameter in the direction of which the rays are incident.

(147.) Cor. 1. Since $2\,TD = FD$, and $2\,DE = CD$, we have $2\,TD \mp 2\,DE = FD \mp CD$; that is, $2\,TE = FC$.

(148.) Cor. 2. Since $FC : CE$ :: sin. incidence : sin. incidence $\sim$ sin. refraction (Art. 131), we have $2\,TE : CE$ :: sin. incidence : sin. incidence $\sim$ sin.

* Otherwise, F would coincide with C, which cannot be the case so long as the refracting power is finite (Art. 130).

refraction; or, $TE : CE ::$ sin. incidence : 2 sin. incidence $\sim$ 2 sin. refraction.

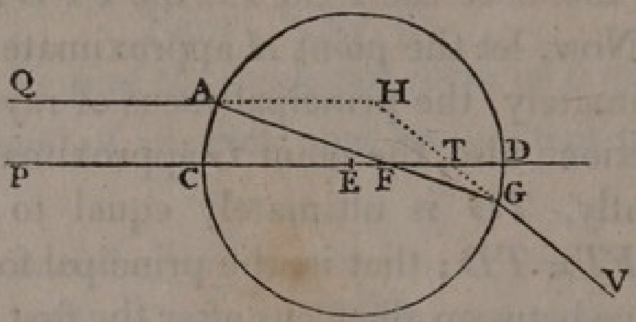
The distance TE is called the *focal length* of the sphere.

Ex. 1. If the sphere be glass, placed in air, sin. incidence : sin. refraction :: 3 : 2; therefore $TE : CE ::$ 3 : 2.

Ex. 2. If the sphere be water, placed in air, sin. incidence : sin. refraction :: 4 : 3; and $TE : CE ::$ 4 : 2 :: 2 : 1.

Ex. 3. If sin. incidence : sin. refraction :: 2 : 1, $TE : CE ::$ 2 : 2; or T coincides with D.

Ex. 4. If the sine of incidence be to the sine of refraction in a greater ratio than that of 2 : 1, T falls



within the sphere; and the rays, after the second refraction, diverge from T.

(149.) Cor. 3. If the axes of different pencils of parallel rays, incident upon the sphere, lie in the same plane, the foci will lie in the circumference of a circle whose center is E, and radius ET.

(150.) Cor. 4. If T be the focus of rays incident nearly perpendicularly upon the sphere, in the contrary direction, these rays will be refracted parallel to each other, and to TE.

(151.) Cor. 5. If the radius of the sphere, and it's focal length be known, the ratio of the sine of incidence to the sine of refraction may be determined.

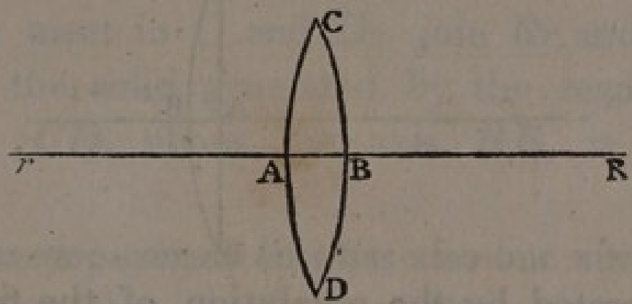
Let $m : n ::$ sin. incidence : sin. refraction; then, $2TE : EC :: m : m \sim n$; therefore $2TE : 2TE \mp EC :: m : n$; where the negative sign is to be used when the rays converge after the first refraction, and the positive sign, when they diverge.

(152.) Def. A *Lens* is a thin piece of glass, or other transparent substance, whose surfaces are either both spherical, or one plane and the other spherical.

This definition comprises the six following sorts of lenses: the *double convex*, the *double concave*, the *plano convex*, the *plano concave*, the *meniscus*, and the *concavo-convex* lens.

1. A *double convex lens* is bounded by two convex spherical surfaces.

Let R and r be the centers of two circular arcs CAD, CBD which are concave towards each other,



and which meet in C and D; join Rr, and suppose the figure CD to revolve about Rr as an axis; the solid, thus generated, is called a *double convex lens*.

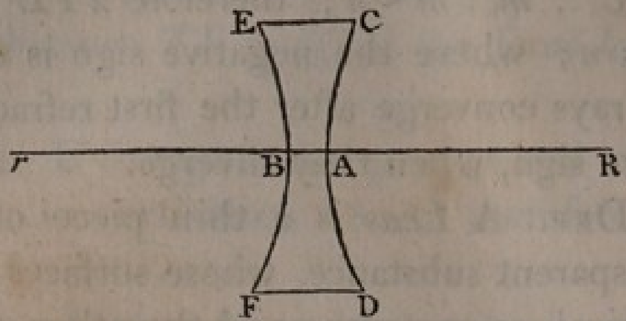
The line Rr is called the *axis* of the lens.

The figure $CADB$ represents a section of the lens, made by a plane which passes through the axis.

If CD be joined, this line is called the *diameter*, or *linear aperture* of the lens.

2. A *double concave lens* has both it's surfaces concave.

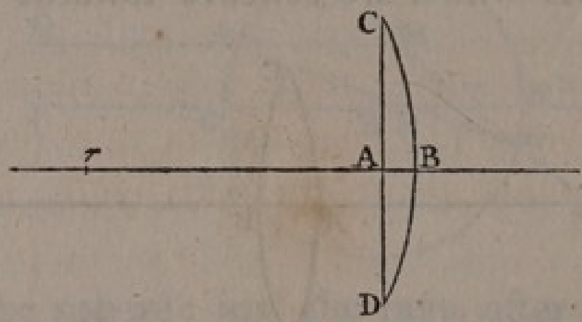
Let R and r be the centers of two circular arcs which are convex towards each other; join *Rr*; draw



EC, *FD* parallel to *Rr*, and equally distant from it; then the solid generated by the revolution of the figure *ECDF*, about the axis *Rr*, is called *a double concave lens*.

3. A *plano-convex lens* is bounded by a plane, and a convex spherical surface.

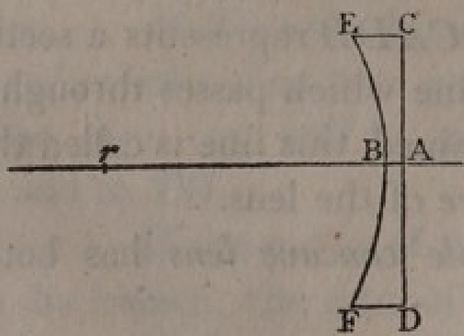
Let *CBD* be a circular arc whose center is *r*, and chord *CD*; draw *rAB* at right angles to *CD*; and the



solid generated by the revolution of the figure *CD*, about the axis *rB*, is called a *plano-convex lens*.

4. A *plano-concave lens* is bounded on one side by a plane, and on the other by a concave spherical surface.

Let *EBF* be a circular arc whose center is *r*; *CAD*

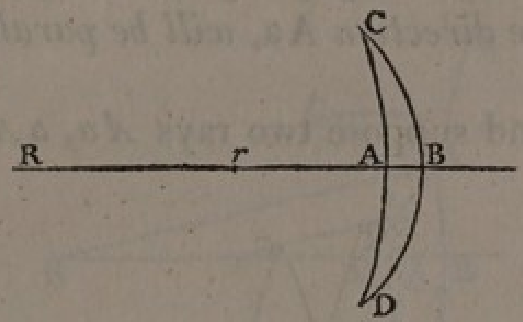


a line, perpendicular to the radius *rB* produced;

draw *EC*, *FD* parallel to *rA*, and equally distant from it; and the solid generated by the revolution of the figure *ECDF*, about the axis *rA*, is called a *plano-concave lens*.

5. A *meniscus* is bounded by a concave and a convex spherical surface which meet, if continued.

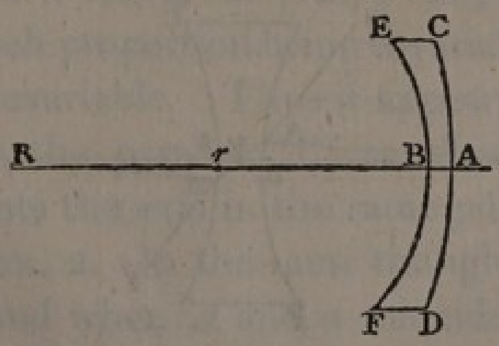
Let *R* and *r* be the centers of two arcs *CAD*, *CBD*, which have their convexities the same way,



and which meet in *C* and *D*; join *Rr* and produce it to *B*; the solid generated by the revolution of the figure *CD*, about the axis *RB*, is called a *meniscus*.

6. A *concavo-convex lens* has also one surface concave and the other convex, but the convex surface, which has the less curvature, does not, if continued, meet the concave surface.

The manner in which the lens is generated is suffi-



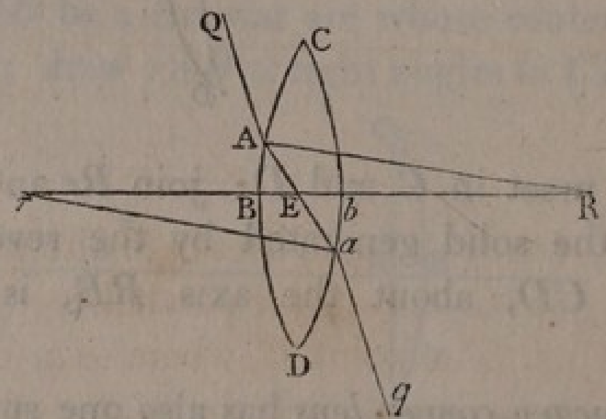
ciently evident from the preceding descriptions.

The thickness of these lenses is, in general, supposed to be inconsiderable.

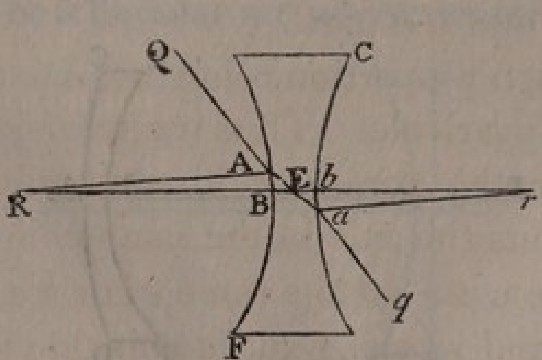
Prop. XL.

(153.) *If the radii* RA, ra, *of the surface of a lens, be drawn parallel to each other, the incident and emergent parts of a ray of light which passes through the lens in the direction* Aa, *will be parallel.*

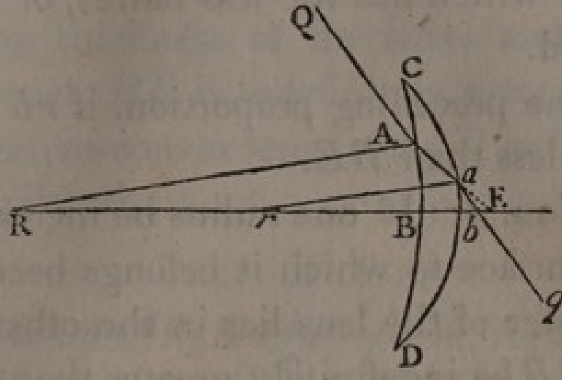
Join *Aa*, and suppose two rays *Aa*, *aA* to pass out



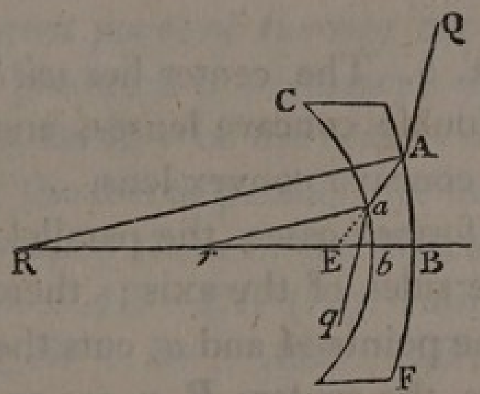
of the lens, in the directions *aq*, *AQ* (Art. 29), which are on opposite sides of *Aa* produced (Art. 33), and in the plane which passes through *RA*, *ra* (Art. 24).



Then, the angles of incidence at *A* and *a* being equal, the angles of deviation are equal; therefore the angles



QAa, *Aaq*, which are the supplements of the angles



of deviation, are equal; and these are alternate angles; consequently, *AQ* and *aq* are parallel.

(154.) Def. The point *E*, where *Aa*, or *Aa* produced, cuts the axis, is called the *center* of the lens.

(155.) Cor. 1. The center *E*, of the same lens, is a fixed point.

In the similar triangles *RAE*, *raE*, $RA : ra :: RE : rE$; therefore, by composition or division, $RA \mp ra : ra :: RE \mp rE\ (Rr) : rE$; the three first terms in which proportion being invariable, the fourth, *rE*, is also invariable. Thus it appears, that in whatever manner the parallel radii are drawn, *Aa*, or *Aa* produced, cuts the axis in the same point.

(156.) Cor. 2. In the same triangles, $AE : aE :: RA : ra$; and when *A* and *a* coincide with the axis in *B* and *b*, $BE : bE :: RB : rb$.

(157.) Cor. 3. The center of the lens is nearer to that surface which has the less radius, or which is the more curved.

For, in the preceding proportion, if rb be less than RB, bE is less than BE.

(158.) Cor. 4. If one radius be increased without limit, the surface to which it belongs becomes plane; and the center of the lens lies in the other surface.

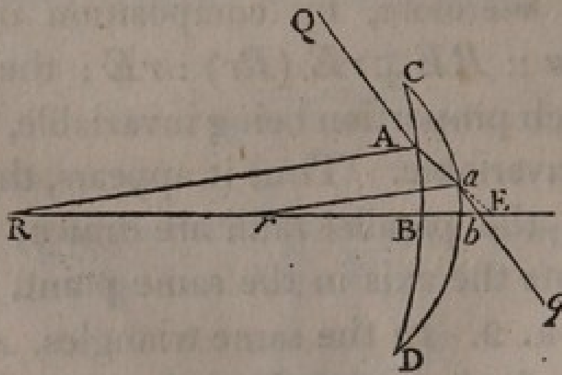
For, if RB be indefinitely greater than rb, BE becomes indefinitely greater than bE, or E coincides with b.

(159.) Cor. 5. The center lies *within* the double convex and double concave lenses, and *without* the meniscus and concavo convex lens.

In the two former cases, the parallel radii RA, ra, lie on opposite sides of the axis; therefore the line which joins the points A and a, cuts the axis. In the two latter cases, the centers R, r, are on the same side of the lens; therefore the parallel radii RA, ra lie on the same side of the axis; consequently Aa must be produced to meet the axis.

(160.) Cor. 6. The center of a meniscus may be at any distance from it's surface.

In the similar triangles RAE, raE, $RA : ra :: AE : aE$; and by division, $RA : RA - ra :: AE :$



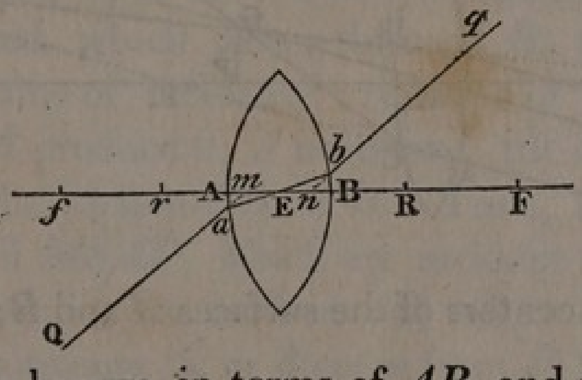
$AE - aE$ (Aa); consequently, when A and a coincide with B and b, $RB : RB - rb :: BE : Bb$. If, then, the difference of the radii decrease with respect to one of them RB, the distance BE increases with respect to the thickness of the lens; and when RB and rb are equal, BE is indefinitely great.

In the concavo-convex lens, when R and r coincide, E coincides with them.

Prop. XLI.

(161.) *If a ray of light* Qabq *be refracted through a lens,* AB, *in the direction* ab *which passes through it's center* E, *to find where the directions of the incident and emergent parts of the ray cut the axis.*

Let Qa, qb, produced if necessary, cut the axis in m and n. Also, let R, r be the centers of the surfaces A, B; F and f the foci of parallel rays incident nearly perpendicularly upon them in the directions fA, FB. Then, $AR : Br :: AE : EB$ (Art. 156.) and consequently, $AR \pm Br : Br :: AB : EB$; hence EB



and EA are known, in terms of AB, and the radii of the surfaces. Also in the similar triangles Ebn, Eam, $Eb : En :: Ea : Em$. Now considering E as the focus of rays incident on the surface B, $Ef : Er :: Eb : En$ (Art. 137), and in the same manner $EF : ER :: Ea : Em :: Eb : En$; therefore $Ef \pm EF\ (Ff) : Er \pm ER\ (Rr) :: Eb : En$; and ultimately when ab is nearly coincident with the axis, $Ff : Rr :: EB : En$. In the same manner $Ff : Rr :: EA : Em$.

(162.) Cor. 1. Hence it appears that $EB \pm EA$ $(AB) : En \pm Em\ (mn) :: Ff : Rr$.

(163.) Cor. 2. If AB, the thickness of the lens, be evanescent, $QaEbq$ may be considered as a straight line, unless Ff also vanishes when compared with Rr.

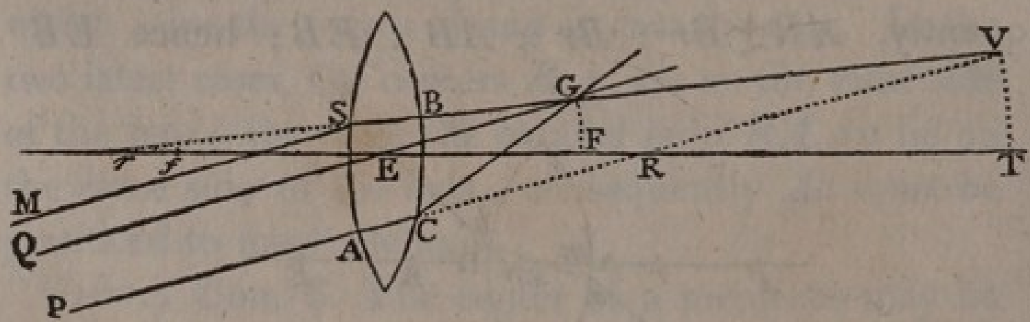
(164.) Cor. 3. If AB be a sphere, E is it's center, and m, and n, coincide with E.

(165.) Def. These points are sometimes called the *focal centers* of the lens.

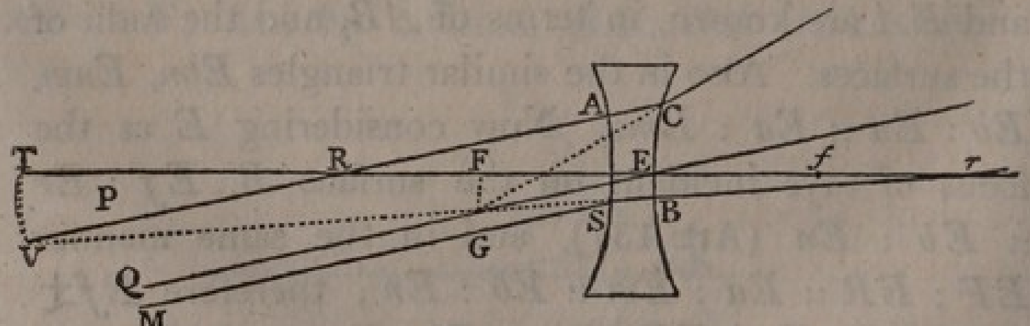
Prop. XLII.

(166.) *To find the principal focus of a lens, whose thickness is inconsiderable.*

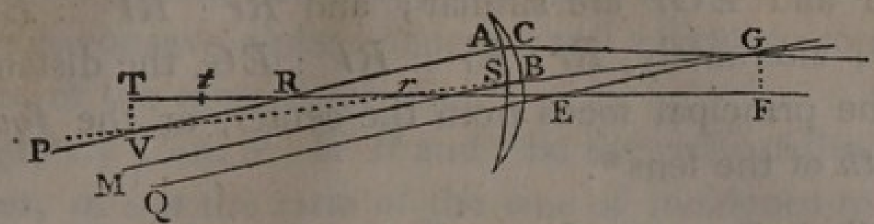
Let AB be a lens, whose axis is rT, and center E;



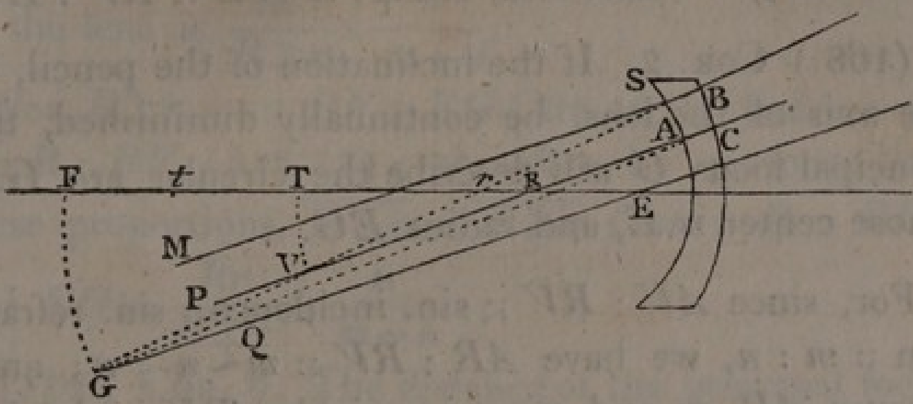
R and r the centers of the surfaces A and B; PA, QE,



MS a pencil of parallel rays incident upon it; of which QE passes through the center, and may therefore be considered as proceeding in that direction after the second refraction (Art. 162); consequently, the



focus of emergent rays will be in QE, or QE produced.



Let PA be that ray of the pencil which is incident perpendicularly upon the surface A; and in PA, or PA produced, which passes through R, take AV : RV :: the sine of incidence : the sine of refraction; join Vr, and produce it, if necessary, till it cuts QE in G, and the surface CB in B. Then, all the rays in the pencil $MSAP$, which are incident nearly perpendicularly upon the surface AS, will, after the first refraction, converge to, or diverge from V (Art. 130), and in this state they will fall upon the surface B; of this pencil, that ray which is incident at B coincides with the direction of the radius rB, and is therefore incident perpendicularly upon the surface B; consequently, it will proceed in the direction SB (Art. 27); and the focus of emergent rays will be somewhere in the line BSV, or BSV produced. The focus will also, as was before observed, be somewhere in QEG; therefore G, the intersection of the two lines, QEG and BSV, is the focus of emergent rays.

Now, since RV is parallel to EG, the triangles RVr and EGr are similar; and $Rr : RV :: Er : EG$; alternately, $Rr : Er :: RV : EG$, the distance of the principal focus from the center, or the *focal length* of the lens*.

(167.) Cor. 1. Since $RA \mp rB : rB :: Rr : Er$ (Art. 155), we have also, $RA \mp rB : rB :: RV : EG$.

(168.) Cor. 2. If the inclination of the pencil, to the axis of the lens, be continually diminished, the principal focus G will describe the circular arc GF, whose center is E, and radius EG.

For, since $AV : RV ::$ sin. incidence : sin. refraction $:: m : n$, we have $AR : RV :: m \sim n : n$; and, because AR, m and n are invariable, RV is also invariable. Again, $Rr : Er :: RV : EG$; and, since the three first terms in this proportion are invariable, the fourth, EG, is also invariable.

(169.) Cor. 3. If any point G, in the arc FG, be the focus of rays incident in the contrary direction, these rays will emerge parallel to each other and to GE (Art. 29).

(170.) Cor. 4. It appears from the construction of the figures, that parallel rays are made to converge, by a double convex lens, a plano convex lens, and a

---

* It is necessary to observe, that we only determine the ultimate intersection of the rays, when they are incident nearly perpendicularly on each surface; that is, when their inclination to the axis of the lens is diminished without limit. The conclusion is, however, nearly true, when they are inclined at a *small*, though *finite* angle to that axis.

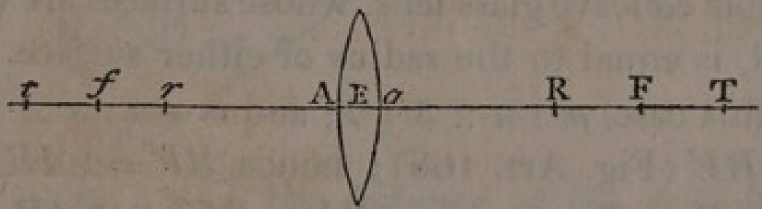
meniscus, of greater density than the surrounding medium. And that they are made to diverge, by a double concave, a plano concave, and a concavo convex lens, of the same description*.

(171.) Cor. 5. If R and r be the radii of the surfaces, $m : n$ the ratio of the sine of incidence to the sine of refraction at the first surface, the focal length of the lens is $\frac{Rr}{R \mp r} \times \frac{n}{m-n}$.

For, $R \mp r : r :: RV : EG$ (Art. 167); and $m-n : n :: R : RV$ (Art. 131; therefore, by compounding these proportions, $\overline{m-n} \times \overline{R \mp r} : nr :: R : EG$; and $EG = \frac{Rr}{R \mp r} \times \frac{n}{m-n}$.

(172.) Cor. 6. The distance of the principal focus from the center is the same on each side of the lens.

Let F be the principal focus when the rays are inci-



dent in the direction tA; f the principal focus when they are incident in the contrary direction.

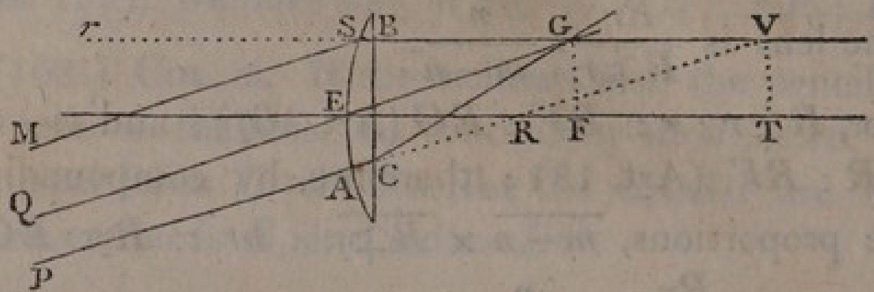
Then, since m and n are the same in both cases, and R and r are alike concerned in the expression $\frac{Rr}{R \mp r}$, the value of $\frac{Rr}{R \mp r} \times \frac{n}{m-n}$ is the same, whether the rays are first incident on the surface A, or on a; that is, $EF = Ef$.

* Lenses are always supposed to be denser than the surrounding medium, unless the contrary be specified.

(173.) Cor. 7. If the radii of the surfaces be given, the focal length varies as $\frac{n}{m-n}$.

(174.) Cor. 8. If one surface be plane, the focal length of the lens is equal to the focal length of the other surface.

When the distance of r is indefinitely increased,



Vr is parallel to ER, and the figure $GERV$ becomes a parallelogram; therefore $EG = RV = R \times \frac{n}{m-n}$.

(175.) Ex. 1. The focal length of a double convex, or double concave glass lens, whose surfaces are equally curved, is equal to the radius of either surface.

In this case, $m : n :: 3 : 2$; and $m - n : n :: 1 : 2 :: AR : RV$ (Fig. Art. 166); hence $RV = 2AR$. Also, $RA + rB : rB :: 2 : 1 :: (RV : EG ::)\ 2AR : EG$; consequently, $EG = AR$.

Ex. 2. If one surface of a glass lens be plane, the focal length is equal to the diameter of the other surface.

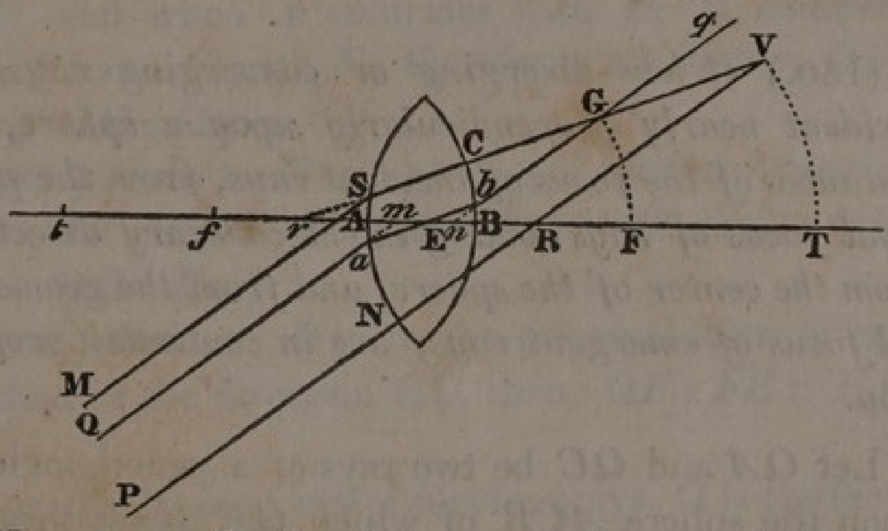
Here, $EG = RV$ (Art. 174); that is, $EG = 2AR$.

Prop. XLIII.

(176.) *To determine the focal length of a lens whose thickness is not inconsiderable.*

Let AB be the lens whose axis is rR, and center E; R and r the centers of it's surfaces; MS, Qa, *PN* a pencil of parallel rays incident upon it, of which *Qabq* passes through the center; *m*, *n* the focal centers; also, let *PN* be that ray which is incident perpendicularly on the surface *A*, and therefore *PN*, or *PN* produced, passes through *R*; take $NV : RV ::$ sin. *I* : sin. *R*, and *V* is the focus after the first refraction. Join *Vr* meeting the surface *A* in *S*, and the line *bq* in *G*, then *G* is the focus of emergent rays, as appears from the reasoning in Art. 166.

Then, because *nG* is parallel to *Qa*, and therefore to *RV*, the triangles *RrV*, *nrG* are similar, and $Rr :$



$RV :: nr : nG$, the distance of the principal focus from the focal center *n*.

(177.) Cor. 1. If the inclination of the pencil to the axis of the lens be diminished continually, the foci *V* and *G* will describe the circular arcs *VT*, *GF*, whose centers are *R* and *n* *; and $Rr : RT :: nr : nF$.

(178.) Cor. 2. If *f* be the other principal focus, *mf* is equal to *nF*.

Since $Rr : RT :: nr : nF$, we have $Rr \times nF = nr \times RT$; in the same manner, when the rays are incident

* See Art. 168.

in the opposite direction, $Rr \times mf = mR \times rt$. Also,

$$En : Em :: rB : RA$$

and, $Er : ER :: rB : RA$; consequently,

$$En \pm Er : Em \pm ER :: rB : RA :: rt : RT^*$$

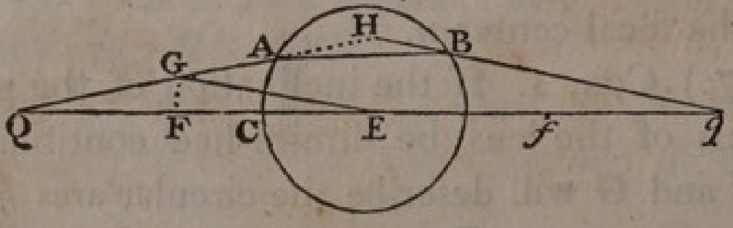
or, $nr : mR :: rt : RT$; hence, $nr \times RT = mR \times rt$; therefore, $Rr \times nF = Rr \times mf$; and $nF = mf$.

(179.) Cor. 3. If any point G, in the arc GF, be the focus of rays incident in the contrary direction, they will emerge parallel to each other, and to Gn†.

Prop. XLIV.

(180.) *When diverging or converging rays are incident nearly perpendicularly upon a sphere, the distances of the focus of incident rays, from the principal focus of rays coming in the contrary direction, from the center of the sphere, and from the geometrical focus of emergent rays, are in continual proportion.*

Let QA and QC be two rays of a pencil incident upon the sphere ACB, of which QE passes through



the center E, and therefore suffers no refraction. Let QA be refracted in the direction AB, and emergent

---

* See Art. 131.

† The focal length may also be determined by finding the focus after the first refraction Prop. 32; and this point being the focus of rays incident upon the second surface, the focus of emergent rays may be found by Prop. 28 or 34.

in the direction Bq; produce QA and qB till they meet in H; take F the principal focus of rays incident in the contrary direction; and from the center E with the radius EF, describe the circular arc FG, cutting QA, or QA produced, in G; join GE. Then, since the ray QA will be refracted in the same manner and degree, whether it be considered as belonging to the focus Q, or to the focus G, Bq is parallel to GE (Art. 150); therefore the triangles QGE, QHq are similar; whence, $QG : QE :: QH : Qq$. Also, since the angles HAB, HBA are equal (Art. 146), $HA = HB$; and when A coincides with C, H coincides with E, and G with F; therefore, ultimately, $QF : QE :: QE : Qq$.

(181.) Cor. 1. From the same similar triangles, $QG : GE :: QH : Hq$; therefore, ultimately, $QF : FE :: QE : Eq$.

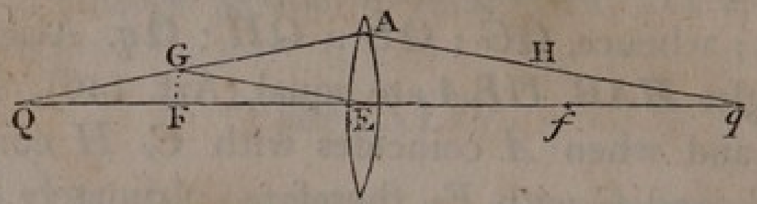
(182.) Cor. 2. If f be the principal focus of rays incident in the direction QC, then, $QF : FE :: Ef : fq$.

For, if q be the focus of incident rays, Q is the focus of refracted rays (Art. 29); therefore $qf : fE :: qE : EQ$ (Art. 181); invertendo, $Ef : fq :: QE : Eq :: QF : FE$; that is, $QF : FE :: Ef : fq$.

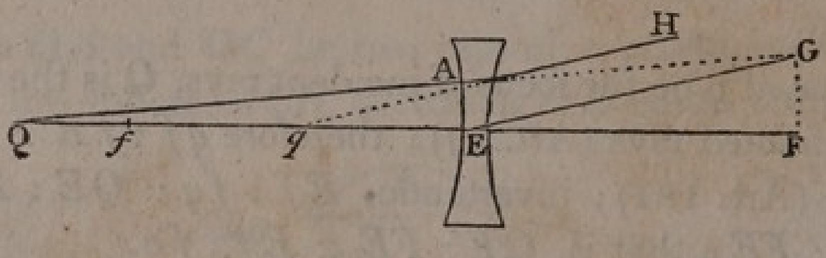
Prop. XLV.

(183.) *When diverging or converging rays are incident nearly perpendicularly upon a lens, whose thickness is inconsiderable, the distances of the focus of incident rays from the principal focus of rays coming in the contrary direction, from the center of the lens, and from the geometrical focus of emergent rays, are in continual proportion.*

Let AE be the lens; E its center; Q the focus of incident rays; QA, QE two rays of the pencil, of which QE is coincident, or nearly coincident with the axis of the lens, and therefore suffers no refraction (Art. 163); let QA be emergent in the direction AH; produce AH backwards or forwards as the case requires, till it meets the axis in q; take F the principal focus of



rays incident the contrary way; and from the center E, with the radius EF, describe the circular arc FG, meeting QA, or QA produced, in G; join GE. Then, since the ray QA will be refracted in the same manner,



and degree, whether it be considered as belonging to the focus Q, or to the focus G, AH is parallel to GE (Art. 169). Hence, the triangles QGE, QAq are similar, and $QG : QE :: QA : Qq$; therefore, ultimately, $QF : QE :: QE : Qq$.

(184.) Cor. 1. In the same triangles, $QG : GE :: QA : Aq$; and ultimately, $QF : FE :: QE : Eq$.

(185.) Cor. 2. Since $QF = QE \pm FE$, we have $QE \pm FE : FE :: QE : Eq$; therefore $Eq = \frac{QE \times FE}{QE \pm FE}$;

and $\frac{1}{Eq}=\frac{1}{FE}\pm\frac{1}{QE}$. From this equation, if the nature of the lens be known, any two of the three quantities FE, QE, and Eq being given, the third may be found.

(186.) Cor. 3. If f be the other principal focus, $QF : FE :: Ef : fq$.

For, if q be the focus of incident rays, Q is the focus of refracted rays (Art. 29); therefore $qf : fE :: qE : EQ$ (Art. 184); invertendo, $Ef : fq :: QE : Eq :: QF : FE$; that is, $QF : FE :: Ef : fq$.

Prop. XLVI.

(187.) *The distances* QF *and* Qq *must always be measured in the same direction from* Q.

Since $QF : QE :: QE : Qq$ (Art. 183), we have $QF\times Qq=QE^2$; therefore the sign of the rectangle $QF\times Qq$ is invariable; and, when the distance of Q from F is very great, the distance of q from f is very small (See Art. 186); measuring, therefore, the lines QF and Qq from Q, their rectangle, in this case, is positive, consequently it is always positive, or QF and Qq must always be measured the same way from Q (*Alg.* Art. 471).

(188.) Nearly in the same manner, it may be proved that EQ and Eq must be measured in the same, or opposite directions from E, according as FQ and FE are measured in the same, or opposite directions from F. As also, that QF and fq must be measured in opposite directions from F and f.

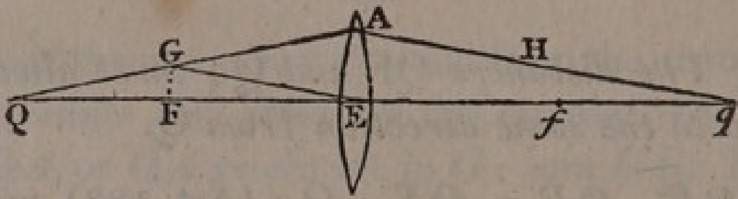
(189.) Cor. Because $QF\times fq = FE\times Ef$ (Art. 186), QF varies inversely as fq; and since these distances are measured in opposite directions from F and *f*, it is manifest that the conjugate foci, Q and q, move in the same direction upon the indefinite line QEq.

Prop. XLVII.

(190.) *A convex lens increases the convergency, or diminishes the divergency of rays incident nearly perpendicularly upon it, unless they converge to, or diverge from the center.*

Parallel rays are refracted converging to the principal focus.

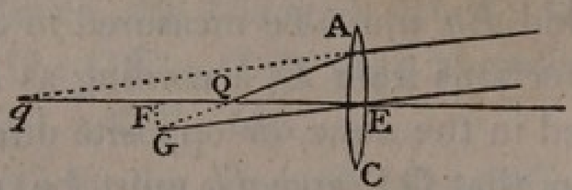
When the incident rays diverge from a point farther



from the lens than it's principal focus, since $QF : QE :: QE : Qq$, and QF is less than QE, QE is less than Qq; also, QF and Qq are always measured the same way from Q; therefore q is beyond the lens; or the refracted rays converge.

When Q coincides with F, the refracted rays are parallel.

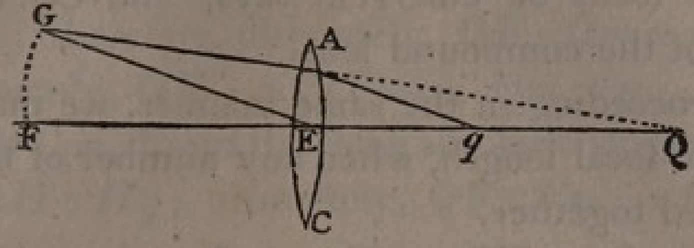
When Q is between F and E, q is on the same side of the lens, and farther from E than Q is (Art. 187);



therefore, the refracted rays diverge less than the incident rays.

When Q coincides with E, q also coincides with it, and the convergency, or divergency, is not altered.

When converging rays are incident upon the lens, QF is greater than QE; therefore QE is greater than



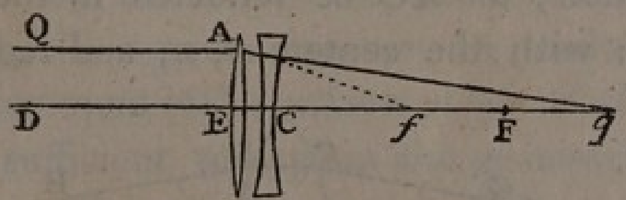
Qq; and q lies between Q and E; consequently, the refracted rays converge more than the incident rays.

(191.) In the same manner it may also be proved, that a concave lens increases the divergency, or diminishes the convergency of rays incident nearly perpendicularly upon it; except when the focus of incident rays coincides with the center of the lens.

Prop. XLVIII.

(192.) *To find the focal length of a compound lens.*

Let the two lenses A and C be placed close together, in such a manner that their axes may coincide; and



let QA and DE be two rays of a parallel pencil incident upon them, of which DE is coincident with their common axis. Take f the principal focus of rays incident upon the lens A, in the direction DE; and F the principal focus of rays incident, the contrary way, upon the lens C. Then, after refraction at the lens A, the rays converge to f, and are thus incident upon the lens C; if, therefore, we take $fF : FC :: Cf : Cq$, and measure Cq and Cf in the same, or opposite directions from C, according as Ff and FC are measured in the same, or opposite directions from F (Art. 188), q is the focus of emergent rays, and Cq the focal length of the compound lens.

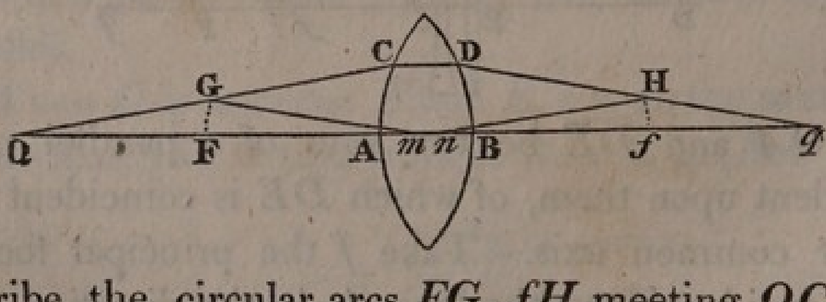
By proceeding in the same manner, we may determine the focal length, when any number of lenses are combined together.

(193.) Cor. When F and f are coincident, the emergent rays are parallel.

Prop. XLIX.

(194.) *When diverging or converging rays are incident upon a lens, whose thickness is not inconsiderable, to find the geometrical focus of emergent rays.*

Let AB be the lens; Qaq it's axis; m, n it's focal centers; F and f, the principal foci of rays incident in the directions qB, QA; QA, QC two rays of the pencil diverging from Q, of which QA is coincident, or nearly coincident with the axis, and therefore suffers no refraction; let QC be refracted in the directions CD, Dq; with the centers m, n, and radii mF, nf,



describe the circular arcs FG, fH meeting QC and Dq in G and H; join Gm, nH.

Then the ray QC will be refracted at C and D, in the same manner and degree, whether we consider it as proceeding from Q or from G, and on the latter supposition Dq is parallel to Gm (Art. 179). Again, if qD be the incident ray, CQ is the emergent ray, and, as before, CQ is parallel to Hn. Hence it follows, that the triangles QGm, nHq are similar, and $QG : Qm :: nH : nq$; therefore, ultimately, $QF : Qm :: nf : nq$; or, alternately, $QF : nf\ (Fm) :: Qm : nq$.

(195.) Cor. From the same similar triangles, $QG : Gm :: nH : Hq$; ultimately, $QF : Fm :: nf : fq$.

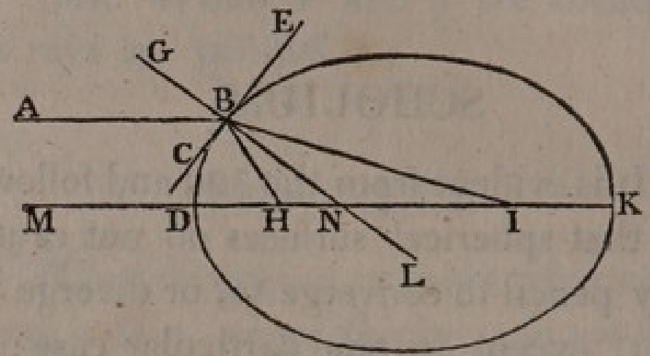
SCHOLIUM.

(196.) It is evident from the 32d and following propositions, that spherical surfaces do not cause all the rays in any pencil to converge to, or diverge from the same point, except in one particular case; and this will be shewn more distinctly in the 7th section.

To remedy the imperfection of optical instruments arising from this cause, it has been proposed to adopt such refractors as are generated by the revolution of the ellipse or hyperbola. But as these are never resorted to in practice, on account of the great difficulty of giving them the exact form, and because the same effect may, in a great measure, be produced by the proper adjustment of the surfaces of spherical refractors, it will be sufficient to explain the geometrical principles upon which the properties of the proposed refractors depend.

(197.) *If a prolate spheroid be generated by an ellipse whose major axis is to the distance between it's foci, as the sine of incidence to the sine of refraction out of the ambient medium into the solid, a pencil of parallel rays, incident in the direction of it's axis, will be refracted, converging accurately, to the farther focus.*

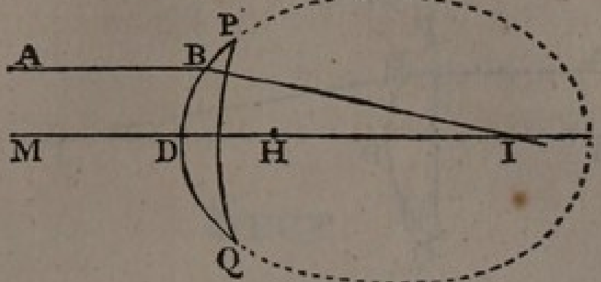
Let *BDK* be the ellipse, by the revolution of which, about it's major axis *DK*, the spheroid is generated; *H* and *I* it's foci; then, by the supposition, *DK* : *HI* :: sin. incidence : sin. refraction. Let *AB*, which is parallel to *DK*, be a ray of light incident upon the spheroid; join *HB*, *IB*; draw *EBC* touching the



generating ellipse in *B*; through *B* draw *GBL* at right angles to *EBC*, meeting *DK* in *N*.

Then, since the ∠ *HBC* is equal to the ∠ *IBE*, by the nature of the ellipse, and the ∠ *NBC* to the ∠ *NBE*, the angles *HBN*, *NBI* are equal; therefore, *IB* : *BH* :: *IN* : *NH* (Euc. 3. vi.) comp. *IB* : *IB* + *BH* (*DK*) :: *IN* : *IH*, alt. *IB* : *IN* :: *DK* : *IH* :: sin. incidence : sin. refraction; also, *IB* : *IN* :: sin. *INB* : sin. *IBN* :: sin. *BNH*, or sin. *ABG* : sin. *IBL*; therefore, sin. *ABG* : sin. *IBL* :: sin. incidence : sin. refraction; and, since sin. *ABG* is the sine of incidence, sin. *IBL* is the sine of refraction; and because the angle *LBI* is less than a right angle, *BI* is the refracted ray. In the same manner it may be shewn, that every other ray in the pencil will be refracted to *I*.

(198.) Cor. 1. If from the center *I*, with any radius less than *ID*, a circular arc *PQ* be described, the solid generated by the revolution of PDQ about the axis



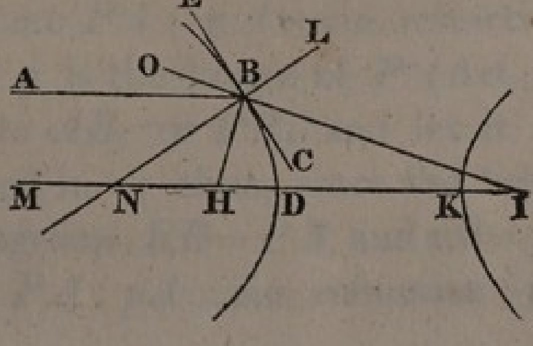
DI, will refract all the rays, incident parallel to DI, accurately to I.

For, after refraction at the surface PDQ the rays converge to I; and they suffer no refraction at the surface PQ, because they are incident perpendicularly upon it.

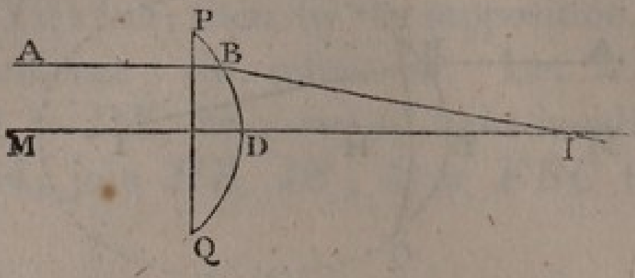
(199.) Cor. 2. Rays diverging from I will be refracted parallel to ID.

(200.) *If an hyperboloid, whose major axis is to the distance between the foci as the sine of incidence to the sine of refraction out of the solid into the ambient medium, be generated in a similar manner, parallel rays, incident in the direction of the axis, and refracted out of the hyperboloid, will converge to the farther focus.*

The proof is nearly the same as in the former case.



(201.) Cor. 1. If PQ be drawn perpendicular to the axis of the hyperbola, and meet the curve in P and



Q, the solid generated by the revolution of PDQ, about the axis MDI, will refract all the rays, incident parallel to MI, accurately to I.

For, the rays will suffer no refraction at the plane surface PQ.

(202.) Cor. 2. Rays diverging from I, and incident upon the surface PDQ, will be refracted parallel to ID.

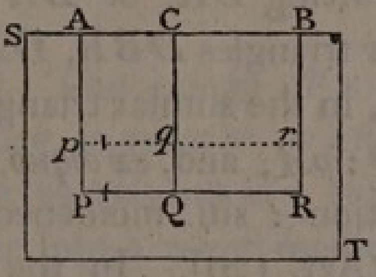
Sect. V.

ON THE IMAGES FORMED BY PLANE AND SPHERICAL REFRACTORS.

Prop. L.

(Art. 203.) *THE image of a straight line, formed by a plane refracting surface, is a straight line.*

Case 1. Let PQR be a straight line, parallel to the plane refracting surface ACB; from P and R, draw

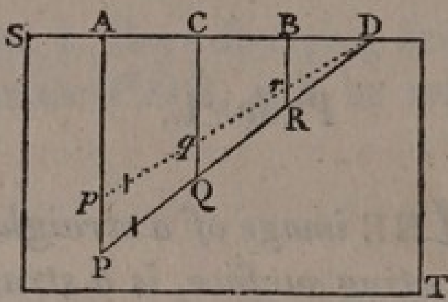


PA, RB, at right angles to AB; and in AP, or AP produced, take $PA : pA ::$ sin. refraction : sin. incidence; and p is the image of P (Art. 120). Draw pr parallel to AB, or PR, and let it meet BR, or BR produced in r; then, since the figures AR, Ar, are parallelograms, $RB = PA$, and $rB = pA$; therefore $RB : rB :: PA : pA ::$ sin. refraction : sin. incidence, consequently r is the image of R. In the same manner it may be shewn, that the image of any other point Q, is q, the corresponding point in pr, determined by drawing QC perpendicular to AB, and producing it, if necessary, till it meets pr; consequently, pr is the whole image of PR.

In this case, since pR is a parallelogram, the image is equal and parallel to the object.

CASE 2. When PQR is inclined to the refracting surface.

Produce PR, if necessary, till it meets the surface in D; from P and R, draw PA, RB, at right angles



to AD; and in AP, or AP produced, take $PA : pA$ :: sin. refraction : sin. incidence; then is p the image of P. Join Dp, cutting BR, or BR produced in r; and in the similar triangles DBR, DAP, $RB : BD$ :: $PA : AD$; also, in the similar triangles DBr, DAp, $BD : rB$ :: $AD : pA$; and, *ex æquo*, $RB : rB$ :: $PA : pA$ :: sin. refraction : sin. incidence; therefore r is the image of R (Art. 120). In the same manner it may be shewn, that the image of any other point Q, in PR, is q, the corresponding point in pr, found by drawing QC perpendicular to AD, and producing CQ if necessary; that is, pr is the whole image of PR.

In this case, $PQ : pq$ :: $QD : qD$ :: $QR : qr$ (Euc. 2. vi.); that is, the corresponding parts of the image and object are proportional.

(204.) Cor. 1. The image and object are on the same side of the refracting surface; and the image is *nearer* to, or *farther* from the surface than the object, according as the rays pass out of a *denser* into a *rarer*, or out of a *rarer* into a *denser* medium.

Ex. If the medium ST be water, contiguous to air, $PA : pA :: 4 : 3$; and $PA : Pp :: 4 : 1$. Thus, the image of the bed of a river is nearer to the surface than the bed itself, by one fourth part of the whole depth.

(205.) Cor. 2. Any two points p, r, in the image, have the same relative situation that the corresponding points P, R, of the object have; therefore the image is erect.

(206.) Cor. 3. If PR, the $\angle PDA$, and the ratio of the sines of incidence and refraction, be known, the $\angle pDA$, and pr may be found.

For, DA being made the radius, tang. PDA : tang. pDA :: $PA : pA$ :: sin. refraction : sin. incidence; therefore the $\angle pDA$ may be found from the tables. Again, $PR : pr :: PD : pD$ :: sec. PDA : sec. pDA.

(207.) Cor. 4. The image of a straight line inclined to the surface, is *greater*, or *less* than the object, according as the rays pass out of a *rarer* into a *denser*, or out of a *denser* into a *rarer* medium.

(208.) Cor. 5. If the figure ST move parallel to itself, on a line which is perpendicular to it's plane, PQR, and pqr, will generate planes, the latter of which is the image of the former.

(209.) Cor. 6. When the object is a plane, parallel to the refracting surface, the image is equal and parallel to the object.

(210.) Cor. 7. If the object be a plane, inclined to the refracting surface, the breadths of the object and image, measured by corresponding lines which are parallel to their common intersection, are equal; but their breadths PR, pr, measured by corresponding lines perpendicular to that intersection, are unequal. In this case, the image and object are not similar.

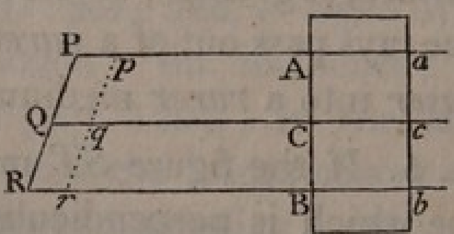
(211.) Cor. 8. If pr be the image of PR, and the eye be so placed as to receive the rays which are incident nearly perpendicularly upon the surface AB, they will enter the eye as if they came from a real object in the situation pr.

Cor. 9. If the rays be refracted at a second surface, pr may be considered as an object placed before that surface, and its image determined in the same manner.

Prop. LI.

(212.) *The image of a straight line, formed by a medium contained by parallel plane surfaces, is a straight line, equal and parallel to the object.*

Let $ABba$ be the medium, PQR the object placed before it. From P, and R, draw PAa, RBb at right



angles to AB; and in AP, or AP produced, according as Ab is denser, or rarer than the surrounding medium, take $Pp : Aa ::$ sin. incidence $\sim$ sin. refraction : sin. incidence; and p is the image of P (Art. 123). Draw pr parallel to PR, and let it meet BR, or BR produced in r. Then, since Pr and Ab are parallelograms, $Rr = Pp$, and $Bb = Aa$; therefore $Rr : Bb ::$ $Pp : Aa ::$ sin. incidence $\sim$ sin. refraction : sin. incidence, or r is the image of R. In the same manner it may be shewn, that the image of any other point Q in the object, is q, the corresponding point in pr, determined by drawing QC perpendicular to AB, and producing it, if necessary, till it meets pr. It appears, from the construction, that pr is equal and parallel to PR.

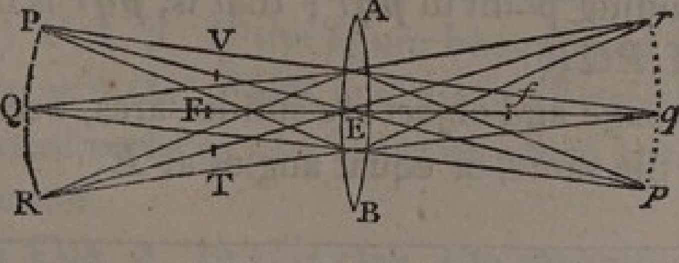
Ex. If the medium Ab be glass, surrounded by air, $Pp : Aa :: 1 : 3$.

(213.) Cor. Whatever be the form of the object, the image will be similar and equal to it (See Art. 71).

Prop. LII.

(214.) *If the object placed before a sphere, or lens whose thickness is inconsiderable, be a circular arc concentric with it, the image will also be a circular arc* concentric with, and similar to the object.*

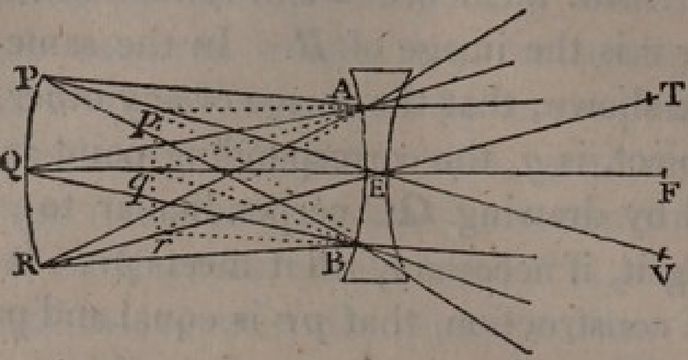
Let AB be the refractor, E it's center; PQR a circular arc whose center is E; in PQR take any



point Q, and join QE; let F be the principal focus of rays incident in the opposite direction to QE. In

* In the case of the lens, the proposition is not accurately true, as appears by the observation contained in the next Note.

QE, or QE produced, take $QF : FE :: QE \cdot Eq$, EQ and Eq being measured in the same, or opposite



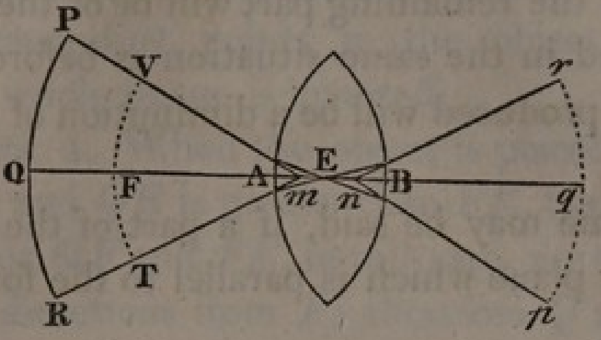
directions from E, according as Q and E lie in the same, or opposite directions from F; and q will be the image of Q (Art. 188). From the center E, with the radii EF, Eq, describe the circular arcs VFT, pqr; and from the points P and R, in the object, draw PE, and RE, producing them, if necessary, till they meet pqr in p and r; then will pr be the image of PR. For, since $EP = EQ$, and $EV = EF$, the sum, or difference of EP and EV, is equal to the sum, or difference of QE and EF; that is, $PV = QF$; also, $Ep = Eq$, by the construction; and $QF : FE :: QE : Eq$; therefore $PV : VE :: PE : Ep$, or p is the image of P*. In the same manner it may be shewn, that the image of every other point in PQR, is the corresponding point in pqr; that is, pqr is the whole image of PQR.

The image and object are similar arcs, because they subtend the same, or equal angles at E.

---

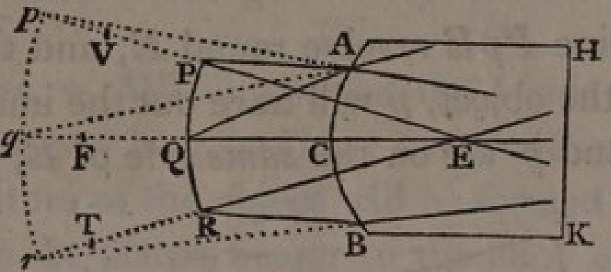
* Here it is supposed that the foci q, p, of direct, and oblique pencils, are equally distant from E. This is not accurately true in the case of the lens, and consequently, the image is not a circular arc; it does not, however, sensibly differ from that form, when the angle which PQR subtends at E is small.

(215.) In the same manner it may be proved, from Art. 194, when the thickness of the lens is not in-



considerable, that if the object be a circular arc whose center is m, the image is a similar arc whose center is n*.

Also, if the refractor be a spherical surface with which the object is concentric, it may be shewn, from



the 139th article, that the image is similar to, and concentric with the object.

(216.) Cor. 1. Since PR and pr are similar arcs, $PR : pr :: EQ : Eq$; hence, in the lens, or sphere, $PR : pr :: QF : FE$ (Arts. 184. 181).

(217.) Cor. 2. If the figure be supposed to revolve about the axis Qq, PQR and pqr will generate similar spherical surfaces, the latter of which is the image of the former.

(218.) Cor. 3. In this case, the magnitude of the object : the magnitude of the image :: $\overline{EQ}|^2 : \overline{Eq}|^2$.

---

* In the subsequent Propositions, the thickness of the lens is supposed to be inconsiderable, unless the contrary be expressed.

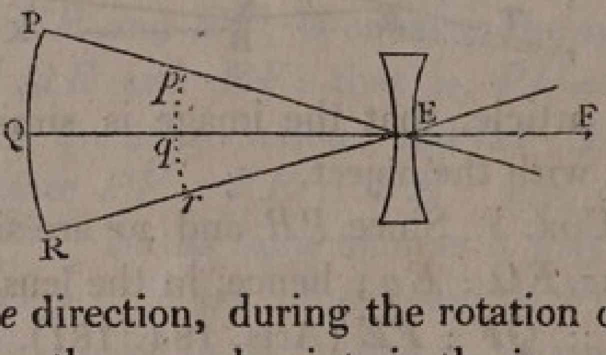
(219.) Cor. 4. If the half of a lens be covered, or cut off by a plane passing through the axis, the image formed by the remaining part will be of the same magnitude, and in the same situation as before; the only alteration produced will be a diminution of the brightness.

The same may be said, if a part of the lens be cut off by any plane which is parallel to the former.

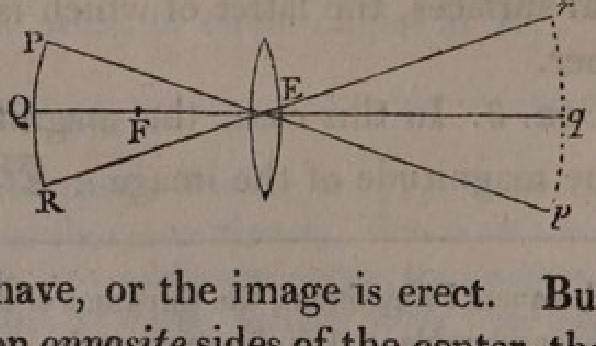
Prop. LIII.

(220.) *When the image and object are on the same side of the center of the refractor, the image is erect with respect to the object; when they are on opposite sides of the center, it is inverted.*

If the line PpE revolve round E, and the point P trace out the object, p will trace out the image. Also, when P and p are on the *same* side of E, they move



in the *same* direction, during the rotation of the line PpE; thus, the several points in the image have the same relative position that the corresponding points in



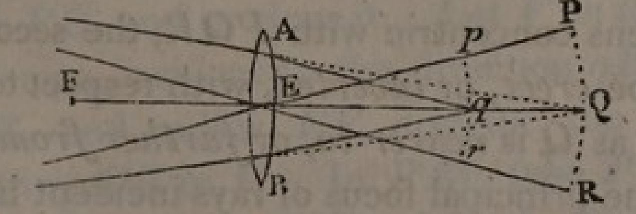
the object have, or the image is erect. But, when P and p are on *opposite* sides of the center, they move in *opposite* directions, and the situation of any two points in the image is inverted, if compared with the situation of the corresponding points in the object; consequently the whole image is inverted.

(221.) Cor. 1. When the object is placed before a convex lens, and QE is greater than FE, the image is inverted. For, QF and FE, in this case, are measured in opposite directions from F; therefore Q and q are on opposite sides of E (Art. 188). When the object is between F and E, the image is erect.

In the former case, QF may be greater than, equal to, or less than FE; therefore the image may be less than, equal to, or greater than the object (Art. 216). In the latter case, the image is always greater than the object.

(222.) Cor. 2. When the refracted rays actually meet, if a screen be placed at their concourse, an image or picture of the object will be formed upon it. If the screen be placed nearer to the lens, or farther from it, than the focus of refracted rays, the image will be indistinct; because the rays, which proceed from a single point, will be diffused over some space upon the screen, and mixed with the rays which diverge from other points in the object; and this indistinctness will increase as the distance of the screen, from the focus of refracted rays, increases.

(223.) Cor. 3. When converging rays, which tend to form an image PQR, are received by a convex lens



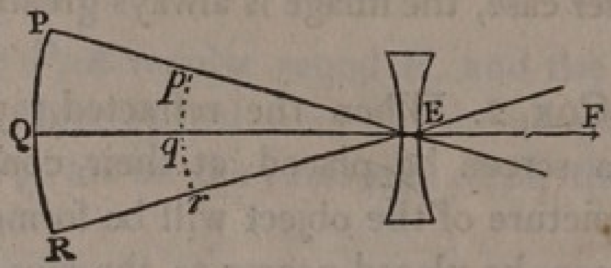
which is concentric with PQR, another image pqr

will be formed, nearer to the lens, and erect with respect to the first image.

Let rays, converging to the several points in *PQR*, be intercepted by the convex lens *AB*; take *F* the principal focus of rays incident in the contrary direction.

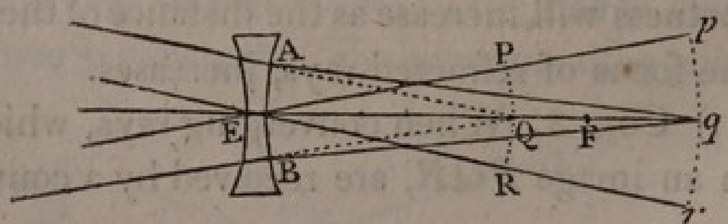
Then, since *QF* is greater than *FE*, *QE* is greater than *Eq*; consequently, *PQR* is greater than *pqr*. Also, since *Q* and *E* are on the same side of *F*, *Q* and *q* are on the same side of the center *E*; and therefore, *pqr* is erect with respect to *PQR*.

(224.) Cor. 4. When the object is placed before a double concave lens, since *QF* and *FE* are measured



the same way from *F*, *Q* and *q* are on the same side of *E*; that is, the image is erect. Also, since *QF* is greater than *FE*, the object is greater than the image.

(225.) Cor. 5. If converging rays, which tend to form the image *PQR*, be intercepted by a double



concave lens concentric with *PQR*, the second image *pqr* will be *erect*, or *inverted*, with respect to the first, according as *Q* is *nearer to*, or *farther from*, the lens than *F*, the principal focus of rays incident in the contrary direction.

When Q is between F and E, EQ and Eq are measured in the same direction from E (Art. 188); consequently, pqr is erect with respect to PQR. When F is between Q and E, PQR and pqr are on opposite sides of the center; and therefore the image pqr is inverted.

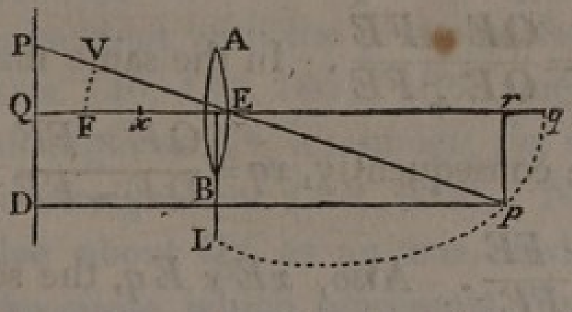
(226.) Cor. 6. When Q lies between F and E, or pqr is erect, QF is less than FE, and consequently EQ is less than Eq, or PQR is less than pqr. In other cases, PQR may be greater than, equal to, or less than pqr.

(227.) Cor. 7. When Q coincides with F, the emergent rays, in each pencil, are parallel.

Prop. LIV.

(228.) *The image of a straight line, formed by a lens or sphere, is the arc of a conic section.*

Let AB be a lens, or sphere, whose center is E; PD a straight line placed before it; through E, draw QEq at right angles to PQ; in PD take any point



P; join PE, and produce it. Let F be the principal focus of rays incident in the direction qE; with the center E, and radius EF, describe the circular arc FV, cutting PE in V. In PEp, take $PV : PE :: PE : Pp$, measuring PV and Pp in the same direction from P; then p is the image of P (Art. 187*). Draw pD parallel to qQ. Then, since the triangles PEQ, PpD are similar, $PE : Pp :: QE : Dp$; consequently, $PV : PE :: QE : Dp$. Also, $PV : VE :: PE : Ep$ (Arts. 184. 181); alternately, $PV : PE :: VE\ (FE) : Ep$; therefore $QE : Dp :: FE : Ep$; and alternately, $QE : FE :: Dp : Ep$; consequently the locus of the point p, is a conic section, whose focus is E, and directrix PD†.

(229.) Cor. 1. The curve is an *ellipse, parabola,* or *hyperbola,* according as QE is *greater* than, *equal* to, or *less* than FE.

(230.) Cor. 2. When Ep coincides with EL, that ordinate to the axis which passes through the focus, Dp becomes equal to QE, and therefore $EL=EF$; that is, half the latus rectum of the conic section is equal to the focal length of the glass.

(231.) Cor. 3. The curvature of the image, at it's vertex, is the same, wherever the object is placed.

(232.) Cor. 4. If xq be the major axis of the conic section, $Qq : Eq :: QE : FE$; and by division, or composition, $QE : Eq :: QF : FE$; therefore $Eq = \frac{QE \times FE}{QF} = \frac{QE \times FE}{QE \mp FE}$. In the same manner, $Ex = \frac{QE \times FE}{QE \pm FE}$; consequently, $xq = \frac{QE \times FE}{QE \sim FE} \pm \frac{QE \times FE}{QE + FE} = \frac{2QE^2 \times FE}{EQ^2 \sim FE^2}$. Also, $xE \times Eq$, the square of the semi-axis minor, $= \frac{QE^2 \times FE^2}{QE^2 \sim FE^2}$.

---

* See Note, p. 93. † See Art. 93.

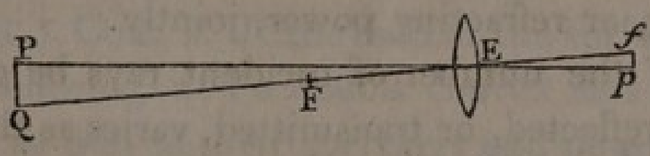
(233.) Cor. 5. If the focal length of the refractor be finite, and pr be drawn perpendicular to the axis, the evanescent arc pq is equal to pr; and $QP : qp :: EQ : Eq$.

Also, whilst the angle QEP, which QP subtends at the center of the glass, is small, though finite, the image pq, when formed at a finite distance from the refractor, will, as to sense, be a right line, and $QP : qp :: EQ : Eq$.

Prop. LV.

(234.) *The sun's image, formed by a spherical refracting surface, lens or sphere, is a circle, and nearly in the principal focus of the refractor.*

Let E be the center of the refractor; F and f it's principal foci; PQ a radius of the sun's disc. Then



since FE is inconsiderable with respect to QF, the image of Q, may, for all practical purposes, be considered as coincident with the principal focus f of the refractor (Arts. 140. 182. 186); also, since QP subtends a small angle at E*, it's image, fp, may be considered as a straight line (Art. 233). Now, let the figure revolve about Qf as an axis, and whilst QP generates the circle which represents the sun's disc, fp will generate it's image, which is, therefore, a circle.

* About 16′.

In the same manner it may be shewn, that the sun's image, formed by a spherical reflector, is a circle, and in the principal focus of the reflector.

(235.) Cor. 1. Since the angle fEp is given, fp the radius of the image, is proportional to Ef, the focal length of the glass.

(236.) Cor. 2. The area of the image varies as the square of it's radius; and therefore as the square of the focal length of the reflector, or refractor.

(237.) Def. By the reflecting, or refracting *powers* of different substances, we understand the ratio of the number of rays reflected, or transmitted by them, if the number of incident rays be the same.

Thus, if one surface reflect two thirds, and another one third of the incident rays, the reflecting *powers* are said to be as 2 : 1.

(238.) Cor. The number of rays reflected, or transmitted, varies as the number incident, and the reflecting, or refracting power, jointly.

For, if the number of incident rays be given, the number reflected, or transmitted, varies as the power; if the power be the same, the number of rays reflected, or transmitted, varies as the number incident; therefore, when both vary, the number of rays reflected, or transmitted, varies as the number incident, and the reflecting, or refracting power, jointly.

Prop. LIV.

(239.) *The density of rays in the sun's image varies directly as the area of the aperture of the reflector, or refractor by which it is formed, and the reflecting, or refracting power, jointly; and inversely as the square of the focal length of the reflector, or refractor.*

The density of rays in the image varies directly as their number, and inversely as the space over which they are diffused*; that is, directly as the number, and inversely as the square of the focal length of the reflector, or refractor (Art. 236). Also, the number of rays reflected, or transmitted, varies as the number incident, and the reflecting, or refracting power, jointly; that is, as the area of the aperture through which the incident rays pass, and the power, jointly; consequently, the density of rays in the image, varies, directly, in the compound ratio of the aperture and power, and inversely, as the square of the focal length of the reflector, or refractor †.

(240.) Cor. 1. When the apertures are circular, the density varies, directly, in the compound ratio of the square of the linear aperture and power; and inversely as the square of the focal length of the reflector, or refractor.

(241.) Cor. 2. If the radii of the surfaces of a concave reflector, and a double convex lens of glass, be equal, as well as their apertures and powers, since the focal length of the reflector : the focal length of the lens :: 1 : 2 (Arts. 45. 175), the density of rays in the image formed by the reflector : the density in the image formed by the lens :: 4 : 1.

(242.) Cor. 3. The focal length of a glass sphere, is three times as great as the focal length of a reflector of the same radius (Art. 148); therefore, on the

* Here we suppose the rays to be *uniformly* diffused over the image, which is not the case; it is, however, true at points similarly situated in images formed by rays which are diffused according to the same law.

† The rays lost in passing through the air, are not taken into the account.

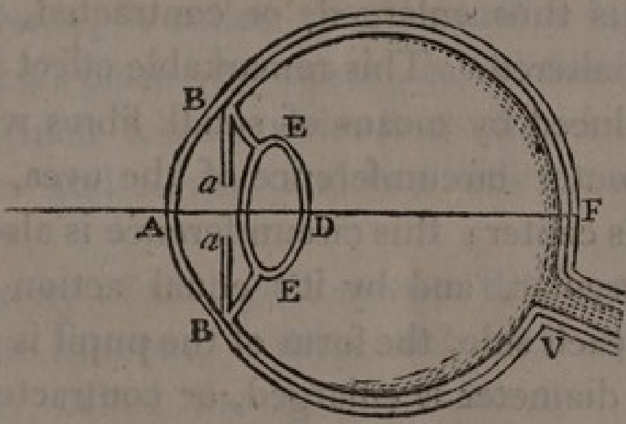
former supposition, the density of rays in the image formed by the reflector : the density in the image formed by the sphere :: 9 : 1.

(243.) Cor. 4. If the rays which tend to form the sun's image, be received by a double convex lens, another image, nearer to the lens, and consequently less than the former, will be produced (Art. 223). Hence it appears, that independent of the rays lost in their passage through the lens thus employed, the burning power of a reflector, or refractor may be increased.

Sect. VI.

ON THE EYE AND THEORY OF VISION.

Art. (244.) The annexed figure represents a section of the human eye, made by a plane which is perpendicular to the surfaces of the coats which contain it's several humours, and also to the nose. It's form is



nearly spherical, and would be exactly so, were not the forepart a little more convex than the remainder; the parts *BFB, BAB,* are, in reality, segments of a greater and a less sphere.

The humours of the eye are contained in a firm coat *BFBA,* called the *sclerotica;* the more convex, or protuberant part of which, *BAB,* is transparent, and from it's consistency, and horny appearance, it is called the *cornea*. This coat is represented by the space contained between the two exterior circles *BFBA*.

Contiguous to the sclerotica is a second coat of a softer substance, called the *choroeides*. This coat is represented by the next white space, and extends, along the back part of the sclerotica, to the cornea.

From the junction of the choroeides and cornea arises the *uvea*, *Ba*, *Ba*, a flat, opaque membrane, in the forepart of which, and nearly in it's center*, is a circular aperture called the *pupil*.

The pupil is capable of being enlarged, or contracted with great readiness †; by which means, a greater or less number of rays may be admitted into the eye, as the circumstances of vision require. In weak light, too few rays might render objects indistinct; and in strong light, too many might injure the organ. Whilst the pupil is thus enlarged, or contracted, it's figure remains unaltered. This remarkable effect is thought to be produced by means of small fibres which arise from the outer circumference of the uvea, and tend towards it's center; this circumference is also *supposed* to be muscular, and by it's equal action upon the fibres, on each side, the form of the pupil is preserved, whilst it's diameter is enlarged, or contracted.

At the back part of the eye, a little nearer to the nose than the point which is opposite to the pupil, enters the *optic nerve V*, which spreads itself over the whole of the choroeides like a fine net; and from this

---

* In some eyes, the pupil is a little nearer to the nose than the center of the uvea.

† The limit of it's aperture, in the eyes of adult persons, appears to be from about $\frac{1}{4}$ to $\frac{1}{10}$ or $\frac{1}{12}$ of an inch. Harris's *Optics*, p. 94.

circumstance is called the *retina*. It is immersed in a dark mucus which adheres to the choroeides.

These three coats, the *sclerotica*, the *choroeides*, and the *retina*, enter the socket of the eye at the same place. The sclerotica is a continuation of the *dura mater*, a thick membrane which lies immediately under the scull. The choroeides is a continuation of the *pia mater*, a fine thin membrane which adheres closely to the brain. The retina proceeds from the brain.

Within the eye, a little behind the pupil, is a soft transparent substance *EDE*, nearly of the form of a double convex lens, the anterior surface of which is less curved than the posterior, and rounded off at the edges, *E*, *E*, as the figure represents. This humour, which is nearly of the consistency of hard jelly, decreasing gradually in density from the center to the circumference, is called the *crystalline* humour. It is kept in it's place by a muscle, called the *ligamentum ciliare*, which takes it's rise from the junction of the choroeides and cornea, and is a little convex towards the uvea*.

The cavity of the eye, between the cornea and the crystalline humour, is filled with a transparent fluid like water, called the *aqueous humour*. The cavity between the crystalline humour and the back part of the eye, is also filled with a transparent fluid, rather

---

* The anterior surface of this muscle, and the posterior surface of the uvea, are covered with a black mucus, evidently designed to absorb any of the extreme rays which may happen to reach so far, and which might be reflected to the retina, and produce confusion in the vision.

more viscous than the former, called the *vitreous humour.*

(245.) It is not easy to ascertain, with great accuracy, the refracting powers of the several humours; the refracting powers of the aqueous and vitreous humours, are nearly equal to that of water; the refracting power of the crystalline humour is somewhat greater *.

(246.) The surfaces of the several humours of the eye are so situated as to have one line perpendicular to them all. This line *ADF* is called the axis of the eye, or the *optic axis* †.

(247.) The point in the axis at which the object, and the image upon the retina, subtend equal angles, is not far distant from the posterior surface of the crystalline lens ‡, though it's situation is subject to a small change, as the figure of the eye, or the distance of the object is changed.

(248.) From the consideration of the structure of the eye, we may easily understand how notices of external objects are conveyed to the brain.

Let *PQR* be an object, towards which the axis of the eye is directed; then, the rays which diverge from any point *Q*, and fall upon the convex surface of the aqueous humour §, have a degree of convergency given

---

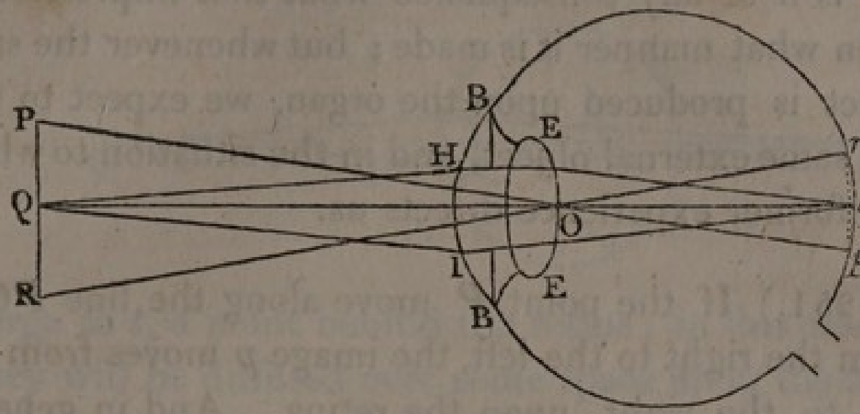
* This is manifest from the figure of the crystalline humour, and the circumstance that persons couched, (in which case the crystalline lens is taken out) are obliged to use convex glasses.

† The dimensions of the several parts of the eye may be seen in Harris's *Optics*, p. 94.

‡ Harris, p. 97.

§ The surfaces of the cornea are nearly parallel to each other, and therefore it produces little alteration in the divergency of rays which pass through it.

them; they are then refracted by a double convex lens, denser than the ambient mediums, which in-



creases the convergency; and if the extreme rays *QH*, *QI*, have a proper degree of divergency before incidence, the pencil will be again collected upon the retina, at *q*, and there form an image of *Q*. In the same manner, the rays which diverge from any other points, *P*, *R*, in the object, will be collected at the corresponding points *p*, *r*, of the retina, and a complete image, *pqr*, of the object *PQR*, will be formed there. The impression, thus made, is conveyed to the brain by the optic nerve, which originates there, and is evidently calculated to answer this purpose.

(249.) Since the axes of the several pencils cross each other within the eye, (see Art. 247), the image upon the retina is inverted with respect to the object*; and if, by any means, the image of an erect object, be erect upon the retina, that object appears inverted.

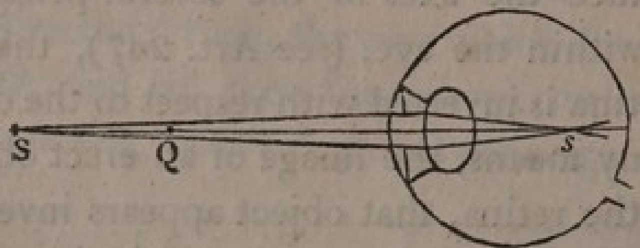
(250.) It has been objected, that if the images upon the retina be inverted, external objects ought to appear inverted. To which it may be answered, that experience

* If the outer coat be taken from an ox's eye, whilst it is warm, the images of external objects are observed to be inverted upon the retina.

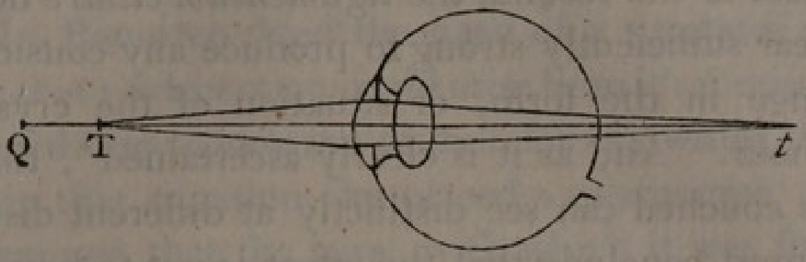
alone teaches us, what situation of the external object corresponds to a particular impression upon the retina; nor is it of any consequence what that impression is, or in what manner it is made; but whenever the same effect is produced upon the organ, we expect to find the same external object, and in the situation to which our former experience directs us.

(251.) If the point *P* move along the line *PQR*, from the right to the left, the image *p* moves from the left to the right, upon the retina. And in general, whenever the image, upon the retina, moves from the left to the right, we are led, by experience, to conclude that the object really moves from the right to the left.

(252.) If the form of the eye, the situation of the several humours, and their respective surfaces, remain unaltered, it is manifest that those rays only, which diverge from points at a particular distance, can be collected upon the retina. Thus, if the image of *Q* be formed exactly upon the retina, the image of *S*, a



point farther from the eye than *Q*, will be formed within the eye; therefore, the rays which proceed from this point, will be diffused over some space upon the retina; and, if they are mixed with the rays which diverge from other points in the object necessary to be distinguished from the former, the vision will be indistinct*. The rays which diverge from *T*, a point nearer to the eye than *Q*, will, after refraction, con-



verge to *t*, a point behind the retina; in this case also, they will be diffused over some space upon the retina, and the vision, as before, will be indistinct.

(253.) By what change in the conformation of the eye, we are enabled to see objects distinctly at different distances, is not fully ascertained. The fact itself is sufficiently manifest; but authors differ in opinion as to the manner in which the effect is produced. It is supposed by some, that the general figure of the eye is altered; that, when the object to be viewed is near, the length of the eye, measured along the axis, is increased by the lateral pressure of external muscles; and, on the contrary, when the object is remote, that the length of the eye is diminished, by the relaxation of that pressure. Others suppose the effect to be produced by a change in the place, or figure of the crystalline humour. Others, by an alteration in the diameter of the pupil. Others ascribe the effect to a change in the curvature of the cornea.

Much stress cannot be laid upon the first of these causes, as distinguished from the last, since it's existence

---

* In many cases it is not necessary to distinguish very nicely the adjacent parts of objects; as in reading large print, viewing trees, houses, mountains, &c. and though the rays are not exactly collected upon the retina, the image is sufficiently well defined for the purpose.

is not proved by experiment; and there is no necessity for recurring to a bare hypothesis of this kind. With respect to the second, the *ligamentum ciliare* does not appear sufficiently strong to produce any considerable change in the form, or situation of the crystalline humour. And as it is clearly ascertained*, that persons couched can see distinctly at different distances, we must conclude that the effect is not to be ascribed to any change in this humour.

A change in the aperture of the pupil has very little effect in rendering objects distinct at different distances.

It has been seen before (Art. 219), that if the position of an object, placed before a lens, be given, the image is formed at the same distance, whether the rays proceeding from the object pass through a greater or smaller portion of the lens. The same is manifestly true of the eye, so long as the several mediums of which it consists, retain their forms and positions; the expansion, therefore, or contraction of the iris cannot cause the images, of objects at different distances from the eye, to be formed on the retina. If, however, lateral rays are stopped by the iris, the indistinctness arising from the diffusion of the rays in each pencil, over some space on the retina, will be lessened; but this can only take place in a small degree, and when the objects are very near to the eye†.

---

* *Philos. Trans.* Vol. lxxxv. p. 6.

† It is on this account that a small hole in a thin plate enables us to view objects at a less distance than we could with the naked eye, as it answers the purpose of a farther contraction of the pupil, and excludes those rays in each pencil, which diverge too much. This assistance

The principal change by which the effect is produced, seems to be an alteration in the curvature of the cornea. In order to shew that such a change takes place, Mr. Ramsden fixed the head of a spectator so securely, that no deception could arise from it's motion, and directed him to look at a distant object; whilst the eye was in this situation, he placed a microscope, in such a manner, that the wire, with which it was furnished, apparently coincided with the outer surface of the cornea; and then directing the spectator to look at a nearer object, he found that the cornea immediately projected beyond the wire of the microscope *.

Now, when the distance of an object is diminished, supposing no alteration to take place in the eye, the divergency of the extreme rays of the pencils incident upon the pupil, is increased; and therefore, if the image of the object in the first situation, be formed upon the retina, in the latter it will be formed behind it (Art. 252); but an increase in the curvature of the cornea will increase the convergency of the refracted rays, or bring them sooner to a focus; and thus, by a proper change in this coat of the eye, the rays will again be brought to a focus upon the retina, and the object be still seen distinctly.

(254.) The least distance at which objects can be seen distinctly by common eyes, is about 7 or 8

assistance cannot be made use of to any great extent, because the image upon the retina will soon become indistinct for want of light; and the inflection of rays at the sides of the hole, will render it confused.

* This experiment is described by Mr. Home, in a very ingenious paper on the subject, *Philosoph. Trans.* Vol. lxxxv. p. 16.

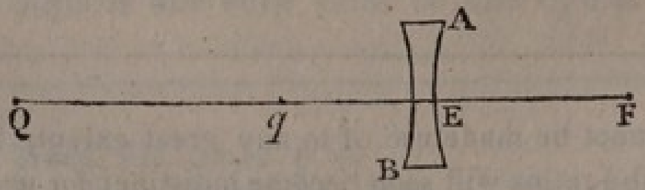
inches*. The greatest distance cannot be so easily, or accurately ascertained. It seems that the generality of eyes are capable of collecting parallel rays upon the retina, or so near to it as to produce distinct vision; and thus, the greatest distance at which objects can be distinctly viewed, is unlimited. For this reason, in adapting optical instruments to common eyes, and calculating their powers, we suppose the parts to be so arranged, that the rays in *each pencil* may, when they fall upon the cornea, be parallel.

(255.) If the humours of the eye be too convex, parallel rays, and such pencils as diverge from points at any considerable distance, are collected before they reach the retina (Art. 252); and objects, to be seen distinctly, must be brought nearer to the eye. This inconvenience may be remedied by a concave glass whose focal length is so adjusted as to give the rays, proceeding from a distant object, such a degree of divergency as the eye requires.

Prop. LV.

(256.) *Having given the distance at which a short sighted person can see distinctly, to find the focal length of a glass which will enable him to see distinctly at any other given distance.*

If qE be the distance at which he can see distinctly,



and QE a greater distance, at which he wishes to view

* *Harris*, p. 124.

objects; let AB be a concave lens, whose focal length is such, that the rays which are incident upon it, diverging from Q, may, after refraction, diverge from q; then they will have a proper degree of divergency for the eye of this spectator. Take F the principal focus of rays incident in the contrary direction; then, since Q and q are conjugate foci, $QF : QE :: QE : Qq$ (Art. 183); *dividendo*, $FE : QE :: Eq : Qq$; therefore $FE = \frac{QE \times Eq}{Qq}$.

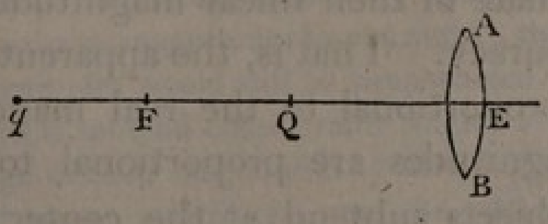
(257.) Cor. If QE be indefinitely great, $FE = Eq$.

(258.) When the humours of the eye are too flat, the rays which diverge from a point near the eye, converge to a point behind the retina. This imperfection may be remedied by a convex lens, whose focal length is adjusted to the distance at which objects are to be viewed, and the degree of convergency in the rays of each pencil, which the eye requires.

Prop. LVI.

(259.) *Having given the distance at which a long sighted person can see distinctly, to find the focal length of a glass which will enable him to see distinctly at any other given distance.*

If qE be the distance at which he can see distinctly, and QE the distance at which he wishes to view



objects, let AB be a convex lens whose focal length FE, is such, that the rays which diverge from Q, may, after refraction, diverge from q. Take F the principal focus of rays incident in the contrary direction; and since Q and q are conjugate foci, $QF : QE :: QE : Qq$; *componendo*, $FE : QE :: Eq : Qq$; and $FE = \frac{QE \times Eq}{Qq}$.

(260.) Cor. 1. If qE be indefinitely great, or the eye require parallel rays, $FE = QE$.

(261.) Cor. 2. If the eye require converging rays, q falls on the other side of the lens; in this case, FE is less than QE.

(262.) In the choice of glasses for long, or short sighted persons, care should be taken to select such as have the least refracting power that will answer the purpose. For, the eye has a tendency to retain that conformation to which it is accustomed; and therefore, by the use of improper glasses, it's imperfection may be increased.

Prop. LVII.

(263.) *If the apparent distance of an object be given, and the angle which it subtends at the center of the eye be small, it's apparent linear magnitude is nearly proportional to that angle.*

When objects are at the same distance from the eye, and appear to be so, we learn by experience to form an estimate of their linear magnitudes with considerable accuracy. That is, the apparent magnitudes are nearly proportional to the real magnitudes, and the *real* magnitudes are proportional to the angles which the objects subtend at the center of the eye, when those angles are small; therefore their *apparent* magnitudes are nearly in that ratio.

(264.) An object, and it's image upon the retina, subtend equal angles at the center of the eye (Art. 247); and supposing the center fixed, and the angles small, the linear magnitude of the image is nearly proportional to the angle which it subtends at that center; therefore the linear magnitude of an object at a given distance from the eye, is nearly proportional to the linear magnitude of it's picture upon the retina*.

(265.) The judgment we form of the magnitude of an object, depends very much upon the notion we have of it's distance; and since the apparent distance depends upon a variety of causes, which are subject to no calculation, in speaking of apparent magnitude, authors generally suppose the apparent distance to be given.

(266.) Def. By the *visual angle* of an object, we understand the angle which the axes of the extreme pencils coming from it, contain at the center of the eye; whether the object is viewed with the naked eye, or with the assistance of reflecting surfaces, or refracting mediums.

---

* On this account, perhaps, we learn to estimate the magnitudes of objects, at a given distance, more readily than we should otherwise be able to do; but, did the magnitude of the picture upon the retina vary according to any other law, we should still learn by experience to estimate magnitudes by the sight; that is, the apparent and real magnitudes would still be proportional.

When objects subtend considerable angles at the center of the eye, we judge of their magnitudes by carrying the optic axes over their several parts; and in this case also, the apparent, and real magnitudes are nearly proportional, if we have had sufficient experience in estimating magnitudes of this description.

Prop. LVIII.

(267.) *When a given object is viewed with the naked eye, the density of light in the image upon the retina, supposing none to be lost in it's passage through the air, and the diameter of the pupil to be invariable, is nearly the same at all distances of the eye from the object.*

The density of light, in the image of a small portion of the object, varies directly as the number of rays, and inversely as the space over which they are diffused. The number of rays which pass through the pupil, supposing it's diameter given, and that none are stopped in their progress, varies inversely as the square of the distance of the object from the eye (Art. 11). Also, the linear magnitude of the picture upon the retina, varies as the angle which the object subtends at the center of the eye, nearly (Art. 264); that is, nearly in the inverse ratio of the distance of the object from the eye; consequently, the area of the picture upon the retina, or space over which the rays are diffused, varies inversely as the square of that distance, nearly. Hence it follows, that the density of rays in the image, varies inversely as the square of the distance of the object from the eye, on one account, and directly as the square of that distance on the other; therefore, upon the whole, the density is invariable.

What has been proved of the image of one small portion of the object, may be proved of every other; consequently, the density, in every part of the image upon the retina, is invariable.

SCHOLIUM.

(268.) It may here be observed, that a considerable quantity of light is lost, or absorbed, in it's passage through the air; and that the quantity thus lost, *cæteris paribus*, increases as the distance between the object and the eye increases, though not in that ratio *. On this account, therefore, the brightness of an object decreases, as it's distance from the eye increases. As the distance of the object increases, however, the aperture of the pupil is enlarged; and therefore more rays are, by this means, received into the eye; and thus the former effect is, in some degree counteracted.

Did the density of rays in the picture upon the retina decrease considerably, as the distance of the object increases, bodies in the neighbourhood of the spectator would, by their superior brightness, overpower the impressions made by those which are more

* If the spaces, through which the light passes, increase in arithmetical progression, the quantity of light will decrease in geometrical progression.

Let the space be divided into equal portions; and let A, B, C, D, &c. represent the quantity of light which enters the 1st, 2d, 3d, 4th, &c. portion, respectively; also, suppose $\frac{1}{m}$th part of the whole light to be lost, or absorbed in it's passage through the 1st portion of space; then $\frac{1}{m}$th part of the remainder will be lost in passing through the 2d; and so on. Thus $A-\frac{A}{m}$, or $\frac{m-1}{m}\times A=B$. In the same manner, $\frac{m-1}{m}\times B=C$; $\frac{m-1}{m}\times C=D$, &c. That is, A, B, C, D, &c. form a decreasing geometrical progression, whose common ratio is $\frac{m-1}{m}$.

remote; and the latter would be discerned with great difficulty, or not at all. We are indeed able to distinguish objects in exceedingly different degrees of light, at different times; thus we are able to read a small print by moon-light, though it's intensity does not exceed $\frac{1}{90,000}$th part of the intensity of common day-light*. But this quantity of light is not sufficient to render such objects discernible as are surrounded by others much more luminous; for, the strong light proceeding from the latter bodies, by the powerful impression it makes upon the retina, overcomes the effect produced by the more delicate pencils which flow from the former, as weaker sounds are not distinguishable in a hurricane. This seems to be the reason that the flame of a candle is scarcely discernible in broad day-light; and that stars become visible at different times after sun-set, according to their different degrees of brightness.

(269.) The impressions made by rays of light upon the retina continue some time after the impulses cease, as appears by the experiment of a burning coal, whirled round in a circle, which was mentioned on a former occasion (Art. 8). Sir Isaac Newton accounts for this phenomenon by supposing that the impressions of light are conveyed to the brain by vibrations excited in the retina, and propagated, through the optic nerve, to the sensorium†; and that the vibrations once produced, continue some time, perhaps about 1″, after the exciting cause has ceased to act.

---

* This is Dr. Smith's *calculation*. M. Bouguer concludes from *experiment*, that the strength of moon-light is about $\frac{1}{300,000}$th part of that of day-light.

† *Optics*, Query 16.

(270.) In explaining the nature and circumstances of vision, we have only to attend to the structure of one eye; for, in whatever manner rays are refracted, and images formed by the humours of one eye, in the same manner will the same effects be produced by the humours of the other. The only question that can arise is, how it happens that in vision with both eyes, objects appear single. It is not easy to decide, whether this effect is produced by the similarity of corresponding parts of the optic nerves, and their union in the brain *, or by habit. In support of the latter opinion, we may be allowed to alledge the following fact, related by Mr. Chesselden: A person had one of his eyes distorted by a blow; and, for some time, every object, to him, appeared double; but by degrees the most familiar ones became single, and in time, all objects became so, though the distortion continued †. To this we may add, that children sometimes *learn* to squint; and by proper attention, this habit may again, in a great measure, be corrected. Under both circumstances, objects appear single, and it is manifest that the images cannot, in both cases, fall upon corresponding points of the retinas.

(271.) But, whatever be the cause of single vision, when an object is viewed attentively, the axes of both eyes are, in general ‡, directed to it. Thus, if *P* be the object, the eyes are moved till the optic axes, *AP*,

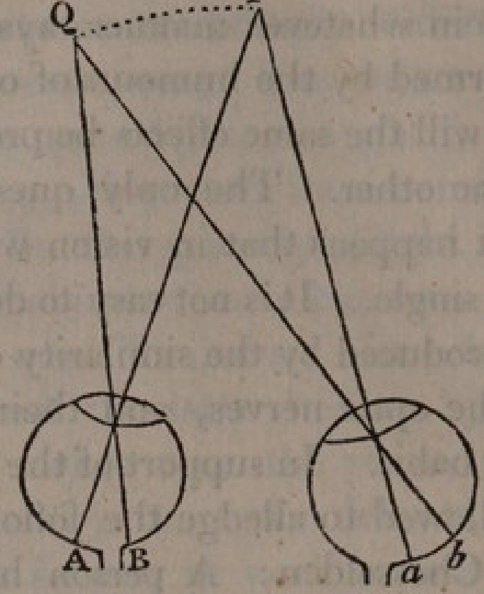
---

* *Optics*, Query 15.

† Smith's *Optics*, Art. 137.

‡ Persons who squint do not direct the optic axes to the object they are looking at.

aP, meet in *P*; and the images, *A*, *a*, are formed on

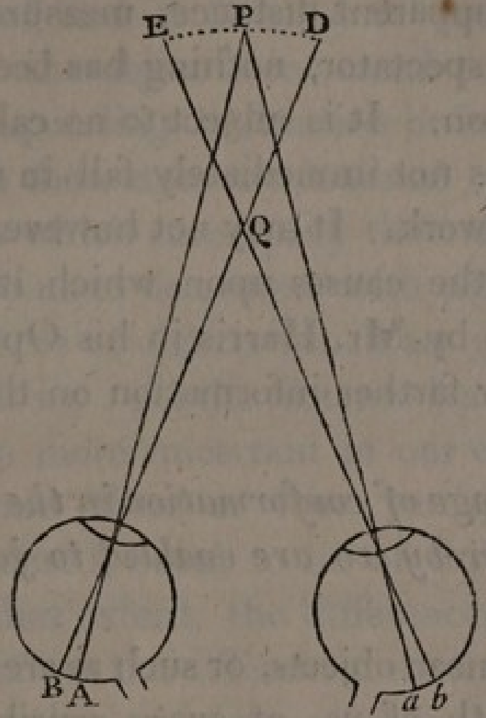


corresponding points of the retinas *. In this position of the images, whether from the correspondence of the nerves, or from experience, the idea of a single object is suggested to the mind; scarcely differing from the idea excited by one of the images alone, excepting that the object appears somewhat brighter when seen with both eyes, than when seen with one. Also, whilst the eyes remain in the same position, the images, *B*, *b*, of *Q*, an object near to *P*, and at the same distance from the eyes, will be formed on the retinas; the eyes having assumed a proper conformation for distinct vision at that distance; and *B*, *b*, which are

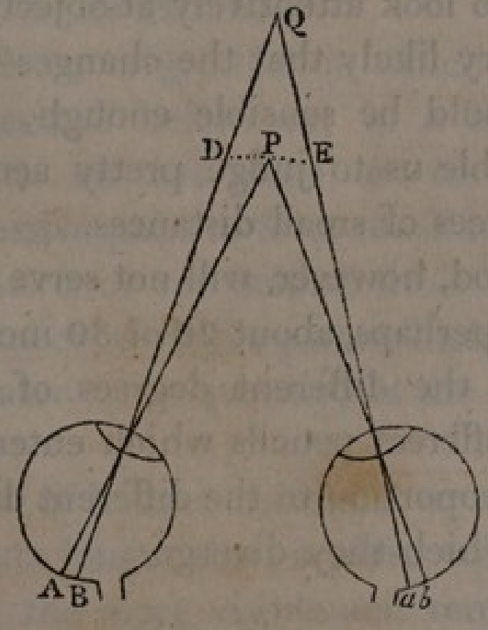
* The retina may perhaps be most susceptible of the impressions of light where the optic axis meets it; and the images formed near that part of the retina, will be less distorted and more regularly and distinctly defined, than when the rays pass more obliquely through the humours of the eye. For one or both of these reasons, we direct the axis of each eye to an object, when we wish to view it to the greatest advantage.

both on the right, or both on the left of the respective axes, are corresponding points, and suggest the idea of a single object, as in the former case.

But, if an object *Q* lie between the optic axes, or



those axes produced, it's images will be formed at *B*, *b*, on points upon the retinas which do not correspond ;



and thus they will excite the sensations, usually produced by different objects, *D*, *E*, at that distance to which the eyes are adapted for vision. However, when the attention is more particularly called to the object Q, the optic axes are directed to it, and the points, D, E, coincide.

(272.) Of apparent distance, measured in a direct line from the spectator, nothing has been said in the foregoing section. It is subject to no calculation, and therefore, does not immediately fall in with the plan of the present work. It may not however be improper to enumerate the causes upon which it depends, as they are given by Mr. Harris in his Optics, referring the reader, for farther information on the subject, to that work*.

1. *The change of conformation in the eye, is one of the means whereby we are enabled to judge of small distances.*

In viewing near objects, or such as are within about an arm's length of us, at every sensible change of distance, the eye must also change it's conformation for procuring distinct vision. And thus, if we were accustomed to look attentively at objects with one eye only, it is very likely that the changes made on these occasions, would be sensible enough, after repeated trials, to enable us to judge pretty accurately of the different degrees of small distances.

This method, however, will not serve us to a greater extent than, perhaps, about 20 or 30 inches. Beyond which limit, the different degrees of divergency of rays in the differect pencils which enter the eye, bear no sensible proportion to the different distances of the points from which they diverge.

---

* Page 154. . . . 168.

2. *Inclination of the optic axes, is another more certain mean of distinguishing degrees of small distances.*

When an object is viewed attentively with both eyes, the axis of each is directed towards it; and as the distance of the object is increased or diminished, there is a corresponding diminution or increase in the angle at which these axes are inclined to each other. The sensations which accompany these different inclinations, enable us to determine with considerable accuracy, the places of objects which are not above five or six feet from us. As the distance becomes greater, we begin to be more uncertain in our estimations of it; and beyond three or four yards, the means, hitherto considered, seem to be of little or no use. For, beyond that extent, the differences of the optic angles, arising from the different distances, are so very small as to become in a manner insensible.

3. *The length of the ground plane, or the number of intervening parts perceived in it, is another mean, by which we estimate distance.*

When the floor, or ground on which we stand, is uniformly extended before us, in a line produced directly from us on this plane, we can distinguish, that such successive parts as form sensible angles at the eye, are successive, or one behind the other; and the greater the number of visible parts which the line contains, the greater, consequently, is the visible extent of the whole.

Again, a row of houses, columns, or trees regularly planted, appears longer than a plain wall of the same extent. For, the more visible and remarkable parts in the former case, enable us to correct the estimate we make when such objects do not intervene; and also, our previous knowledge that the several intermediate objects are disposed at equal intervals, tends to protract the apparent length of the whole chain still farther. A river, at first, looks not so broad, as after we have had a side view of the bridge across it: and indeed, a given extent of water, does not appear so long as the same extent of land; as it is more difficult to distinguish parts in the surface of the one, than it is in the surface of the other.

4. *Different degrees of apparent distance are suggested by the different appearances of known objects, or by the known magnitudes of their least visible parts.*

A building, none of whose parts are discernible, appears much farther off than another, whose windows and doors are visible; and this latter appears farther off than a house having visible parts which are known to be still smaller; as the bricks in the wall, tiles on the roof, &c. Objects of unusual magnitudes, detached as it were, from others, mislead us in our judgment of distances; the greater magnitude usually suggesting to the mind, the idea of less distance.

5. *All other things being the same, different colours and degrees of brightness of objects, cause a difference of apparent distance.*

As objects become more and more remote, the light, which arrives at the eye from their least visible parts, is continually diminished (Art. 268); and they appear more faint, languid, and obscure. Also, their colours not only gradually lose their lustre, but likewise degenerate from their native hue, and participate more of the blueish colour of the sky, as the rays have passed through a greater body of air, or as the images upon the retina are tinged with a greater proportion of sky light.

These different appearances are of use to us in judging of the real distances of known objects; and consequently affect the ideas of apparent distances; those objects that are brightest, and whose colours are most vivid, appearing nearest. Thus in foggy weather all objects appear farther off than ordinary; the diminution of light, in this case, producing the effect of that diminution which arises from greater distance.

(273.) When objects, which subtend small angles at the center of the eye, are of the same colour and brightness, and at the same distance, their apparent magnitudes are proportional to those angles (Art. 263). And when they are at different distances, and subtend equal angles at the center of the eye, since their *real* magnitudes are proportional to their *real* distances, it is probable that their *apparent* magnitudes are nearly proportional to their *apparent* distances*. And thus, in general, the apparent magnitudes are as the visual angles, and apparent distances, jointly.

Hence it follows, that any error in our estimate of apparent distance, will produce a proportional error in our estimate of magnitude. Thus, in foggy weather, at the same time that objects appear farther off, they appear larger; and the diameter of the sun, or moon, appears greater, or less, according as we are led by circumstances to suppose it's distance greater or less at one time than at another.

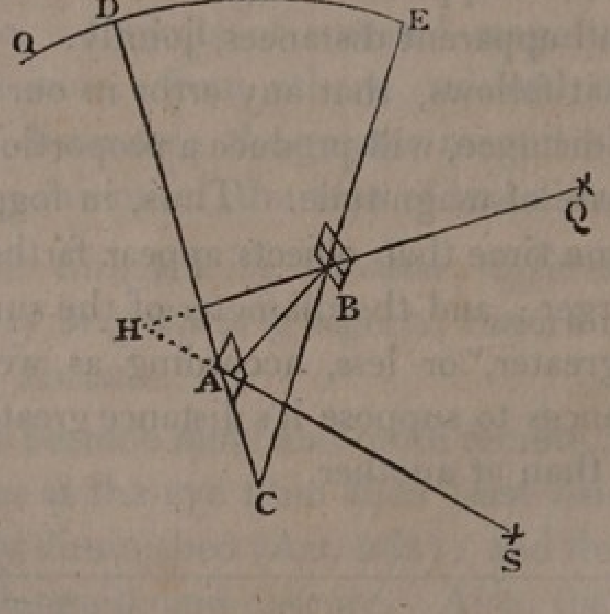
* Any error in this proposition must arise from the limited experience we have of the truth of the former. The magnitude of which we are here speaking, is the *linear* magnitude.

SECT. VII.

ON OPTICAL INSTRUMENTS.

On HADLEY'S *Quadrant*.

Art. (274.) UPON the radii *DC, EC,* of the quadrant *OEC,* and at right angles to it's plane, are fixed two plane reflectors *A, B,* whose surfaces are



parallel when the index *D,* on the moveable radius *CD,* is brought to *O*; and consequently, the arc *OD* will measure their inclination when the moveable radius *CD* is in any other situation. The whole surface of the glass *B* is not quicksilvered, a part of it being left transparent that objects may be seen directly through it, and by rays which pass close to the quicksilvered, or reflecting part.

When the angular distance of two objects, *S*, *Q*, is to be taken, the quadrant is held in such a position that it's plane passes through them both; and the radius *CD* is moved till one of them *S* is seen, after two reflections of the incident ray *SA*, in the direction *HQ*; and the other *Q*, by the direct ray *QH*, in the same line; that is, till the objects apparently coincide. Then, if *SA* be produced till it meets *QH* in *H*, the angle *SHQ*, contained between the first incident, and last reflected ray, is equal to twice the angle of inclination of the two reflectors (Art. 39); therefore, the angular distance of the two objects is measured by twice the arc *DO**.

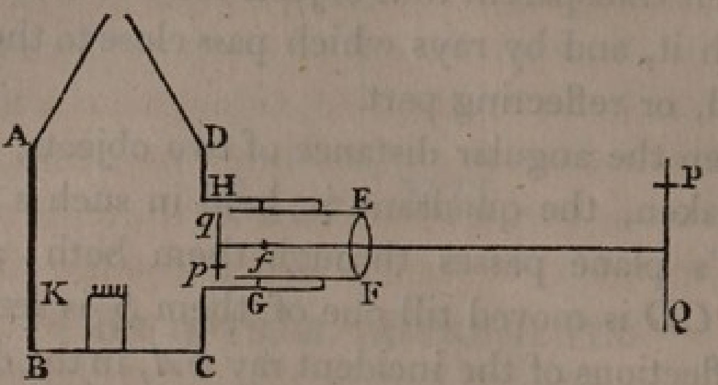
On the Magic Lantern.

(275.) The figure *ABCD* represents a tin box, or lantern, in the fore part of which is a sliding tube, furnished with a double convex lens *EF*. Between the lantern and the lens, a small space, *qp*, is left to admit a thin plate of glass, upon which inverted figures are painted in transparent colours.

When this instrument is used, the lamp *K* being lighted, and the room darkened, the tube is moved, till *qp* is farther from the lens than it's principal focus

* The method of adjusting this instrument may be seen in Mr. Vince's *Practical Astronomy*, p. 8.

f; and consequently PQ, an inverted image of pq, or an erect image of the figure intended to be represented,



is formed at some distance from the lens (Art. 221), and painted in it's proper colours upon a screen placed at the concourse of the refracted rays.

Sometimes a reflector is placed behind the lamp, or a convex lens before it, for the purpose of throwing a greater quantity of light upon pq.

(276.) Cor. If the screen and lantern be fixed, and their distance exceed four times the focal length of the lens, the image may be thrown upon the screen, by moving the lens nearer to, or farther from pq, as the case requires.

For, $qf : qE :: qE : qQ$ (Art. 183.), or $qE - fE : qE :: qE : qQ$; in which proportion there is only one unknown quantity, qE, which may be determined by the solution of a quadratic equation whose roots are possible, except the distance qQ be less than four times the focal length of the lens EF*.

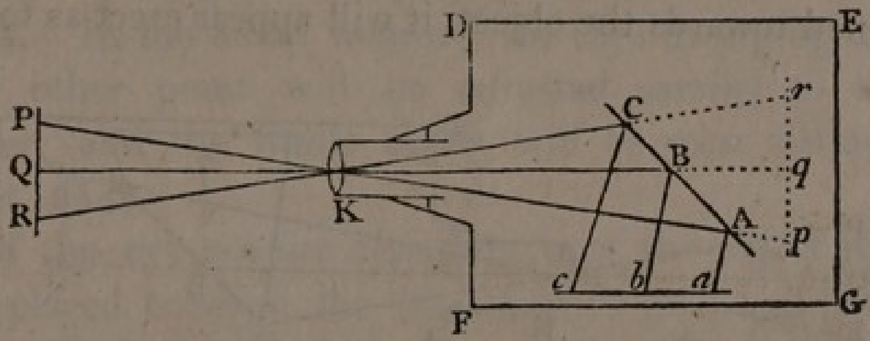
---

* Let $qE = x$; $fE = a$; $qQ = b$. Then, $x - a : x :: x : b$; therefore $x^2 = bx - ba$; or, $x^2 - bx = -ba$; hence, $x^2 - bx + \frac{b^2}{4} = \frac{b^2 - 4ba}{4}$; and $x = \frac{b}{2} \pm \frac{\sqrt{b^2 - 4ba}}{2}$. If $b = 4a$, we have $x = 2a$. If b be less than $4a$, the expression is impossible, which shews that the image cannot, in this case, be formed upon the screen.

On the Camera Obscura.

(277.) If light be admitted, through a convex lens, into a darkened chamber, or into a box from which all extraneous light is excluded, and the refracted rays be received upon a screen, placed at a proper distance, inverted images of external objects will be formed upon it. And if the lens be fixed in a sliding tube, the images of objects at different distances may successively be thrown upon the screen, by moving the lens backwards or forwards, as in the magic lantern.

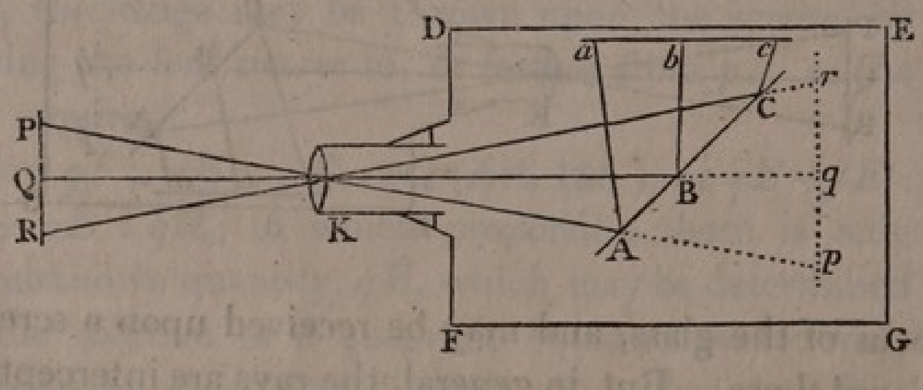
Let *PQR* be an object at a considerable distance from the lens, and at right angles to it's axis; the image *pqr*, will be formed, nearly in the principal



focus of the glass, and may be received upon a screen placed there. But, in general, the rays are intercepted before they form the image, by a plane reflector, *AC*, inclined at an angle of 45° to the axis of the lens; or, which is the same thing, at an angle of 45° to the image *pqr*; by this means, an image *abc* is formed, similar and equal to *pqr*, and inclined at an equal angle to the reflector (Art. 72), and consequently, parallel to the axis of the lens. If the lens be moved towards the reflector, the image *pqr*, (which is in the principal focus of the lens,) and consequently, *abc*, will move from the reflector; and the contrary. Thus, the image *abc*, may, by a proper adjustment of the lens, be thrown upon a table, or any surface prepared to receive it.

(278.) Cor. 1. When the image is thrown downwards, it will appear erect to a spectator between *DF* and the reflector *ABC*; since the point *a*, which corresponds to *p*, or to *P* in the object, is nearest to the reflector. Also, the axis of a pencil of rays which flows from a point to the right of *Q*, will cross the axis *Qq* at the center of the lens, and the rays will form an image which, to this spectator, is to the right of *b*.

(279.) Cor. 2. When the image is thrown upwards, if it be viewed by a spectator whose face is turned towards the object, it will appear erect as to the

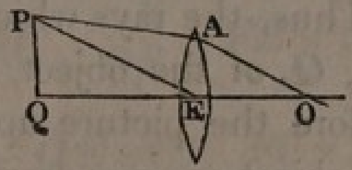


top and the bottom; for the point *a*, which corresponds to *p*, or to *P* in the object, is the farthest point in the image from the reflector; and therefore, the farthest point from the spectator. But, with respect to the right and the left, the image will be inverted. For, the axis of a pencil of rays which flows from a point in the object to the right of *Q*, will cross the axis *Qq* at the center of the lens, and the rays will be collected at a point which, to this spectator, is to the left of *b*.

Prop. LIX.

(280.) *An object may be seen distinctly through a convex lens.*

Let *AE* be a convex lens*; *E* it's center; *PQ* an object placed in it's principal focus. Then, the rays



which diverge from any point *P* will be refracted parallel to each other, and to *PE* (Art. 169); and therefore, they will be proper for vision to common eyes. In the same manner, the rays diverging from any other point will be refracted parallel to each other, and the whole object will be seen distinctly (Art. 254).

If the eye require diverging rays, the object must be placed between the lens and it's principal focus; for then, the rays which diverge from *P*, a point between the principal focus and the glass, will, after refraction diverge (Art. 190); and therefore be proper for distinct vision in this case.

If the eye require converging rays, the object must be placed beyond the principal focus; for then, the rays in each pencil will, after refraction, converge.

---

* Of this description are the double convex lens, the plano convex, and the meniscus.

Prop. LX.

(281.) *When an object is placed in the principal focus of a convex lens through which it is viewed, it appears erect.*

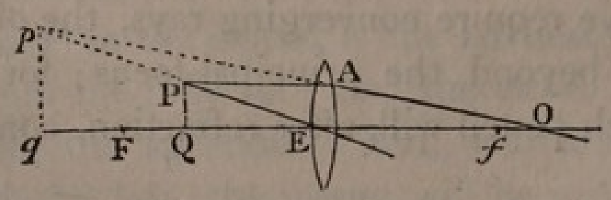
For, if AO be drawn parallel to PE (Fig. Art. 280), the rays which diverge from P, and are received by the eye, enter the pupil in the direction AO; and the rays which diverge from Q, enter the pupil in the direction EO. Thus, the rays which flow from the extreme points P, Q, of the object, cross each other at O, and therefore the picture upon the retina is inverted; or, the object appears erect (Art. 249).

(282.) Cor. In the same manner, if the object be *near* to the principal focus, and the eye not very distant from the glass, the image upon the retina will be inverted; and consequently, the object will appear erect.

Prop. LXI.

(283.) *To determine the variation of the angle which a given object subtends at the center of the eye, when viewed through a convex lens.*

Let E be the center of the lens; F and f it's principal foci; PQ the given object; pq it's image; O



the place of the eye; join Op, and let it meet the lens in A. Then, the rays which diverge from P, enter the eye in the direction AO; and those which diverge from Q, enter the eye in the direction EO; therefore, the $\angle AOE$, or pOq, is the angle which the object, thus seen, subtends at the center of the eye; and this angle varies as $\frac{pq}{Oq}$*. Now, $QF : FE$ (:: $QE : EQ$) :: $QP : qp$, therefore $qp = \frac{FE \times QP}{QF}$. Also, $QF : FE :: Ef : fq$ (Art. 186); therefore, $fq = \frac{FE \times Ef}{QF}$, and, when Q is between F and E, and O beyond f, $Oq = fq + Of = \frac{FE \times Ef}{QF} + Of = \frac{FE \times Ef + QF \times Of}{QF} = \frac{FE \times Ef + \overline{FE - QE} \times Of}{QF} = \frac{FE \times \overline{Ef + Of} - QE \times Of}{QF} = \frac{FE \times OE - QE \times Of}{QF}$. Consequently, $\frac{pq}{Oq} = \frac{FE \times QP}{FE \times OE - QE \times Of}$; therefore, the visual angle varies as $\frac{FE \times QP}{FE \times OE - QE \times Of}$; or, since FE and QP are invariable, inversely as $FE \times OE - QE \times Of$.

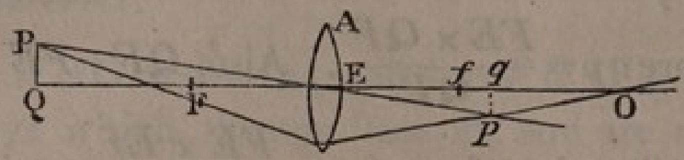
When O is between E and f, Of is negative, and the visual angle varies inversely as $FE \times EO + QE \times Of$.

When the image pq is between O and f, the expression, $FE \times EO - QE \times Of$, becomes negative; for, $Oq = Of - fq = \frac{QE \times Of - FE \times EO}{QF}$; and therefore

---

* In these, and other calculations of the same kind, the angles contained by the axes of the extreme pencils, at the center of the eye, are supposed to be small; and our conclusions, though not strictly true, are sufficiently accurate for the purposes to which they are applied.

$\frac{pq}{Oq} = \frac{FE \times QP}{QE \times Of - FE \times EO}$; and the visual angle varies inversely as $QE \times Of - FE \times EO$. This shews that the angle pOq lies on the other side of the axis



QO; and the image upon the retina, which was before inverted, will now be erect.

(284.) Cor. 1. When the eye is placed close to the glass, the expression, $FE \times EO + QE \times Of$, becomes $QE \times Ef$; therefore the visual angle varies inversely as QE. In this case, the $\angle pOq$ = the $\angle PEQ$.

(285.) Cor. 2. When O coincides with f, the expression becomes $FE \times Ef$, which is invariable. That is, when the eye is in the principal focus of the glass, the visual angle is the same, whatever be the distance of the object from the lens.

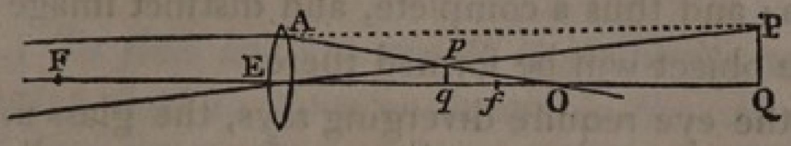
(286.) Cor. 3. When Q coincides with F, the expression becomes $FE \times EO \pm FE \times Of$, or $FE \times Ef$.

That is, the visual angle is the same, whatever be the distance of the eye from the glass; and it is equal to the visual angle when the eye is in the principal focus, and to the angle which the object subtends at the center of the lens.

(287.) Cor. 4. When the eye is farther from the glass than the principal focus f, as QE decreases, $FE \times EO - QE \times Of$ increases; and, therefore, the visual angle decreases, unless the image fall between the eye and the glass; in which case the visual angle varies inversely as $QE \times Of - FE \times EO$; and therefore it increases as QE decreases.

(288.) Cor. 5. When the eye is between the principal focus and the lens, as QE decreases, the expression $FE \times EO + QE \times Of$ decreases; and, therefore, the visual angle increases.

(289.) Cor. 6. When the rays, tending to form the image QP, are intercepted by the glass, and after-

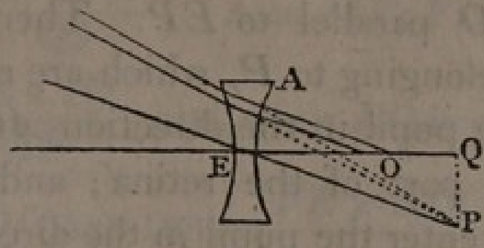


wards received by the eye, QE becomes negative; and the visual angle, when EO is greater than Ef, varies inversely as $FE \times EO + QE \times Of$. When EO is less than Ef, the visual angle varies inversely as $FE \times EO - QE \times Of$, or $QE \times Of - FE \times EO$, according as EO is greater, or less than Eq.

Prop. LXII.

(290.) *The rays which, after reflection or refraction, tend to form an image, may be refracted to the eye by a concave lens, in such a manner as to form a distinct image upon the retina.*

Let the rays which tend to form the image PQ, be received upon the concave lens AE*, whose focal



length is EQ. Then, since P is the principal focus

* Of this description are the double concave, the plano concave, and the concavo convex lenses.

of the lens, the rays which converge to P will, after refraction, be parallel to each other (Art. 169), and therefore proper for vision to common eyes; that is, a distinct image of P will be formed upon the retina. In the same manner it appears, that a distinct image of every other point in PQ will be formed upon the retina; and thus a complete, and distinct image of the whole object will be formed there.

If the eye require diverging rays, the glass must be moved farther from the image PQ. For, then the rays in each pencil converge to a point farther from the glass than the principal focus; and therefore, after refraction they will diverge (Art. 225), and be proper for vision in this case.

If the eye require converging rays, the lens must be moved the contrary way.

Prop. LXIII.

(291.) *When the image* QP *is in the principal focus of the concave lens, the picture upon the retina is erect with respect to* QP.

Let E be the center of the lens*; O the place of the eye; P the lowest point in the image; join PE; and draw AO parallel to EP. Then, those rays of the pencil belonging to P, which are received by the eye, enter the pupil in the direction AO, and proceed to the lower part of the retina; and those which belong to Q, enter the pupil in the direction EO, and proceed to the upper part; consequently, the picture upon the retina is erect with respect to QP.

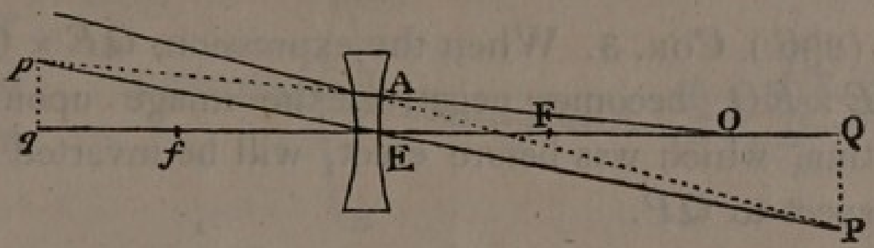
* See the last Figure.

(292.) Cor. In the same manner it appears, that when QP is *nearly* in the principal focus, and the eye not very distant from the glass, the image upon the retina is erect.

Prop. LXIV.

(293.) *To find how the visual angle varies, when rays, which tend to form an image in the axis of a concave lens, are refracted to an eye situate in that axis.*

Let rays which tend to form the image QP be intercepted by the concave lens AE, whose center is E, and axis qFQ; and after refraction let them be received



by an eye placed at O. Take qp the image of QP, and join pO. Then, the visual angle pOq varies as $\frac{qp}{Oq}$. Now, $QF : FE$ (:: $QE : Eq$) :: $QP : qp$; therefore, $qp = \frac{QP \times FE}{QF}$. Also, $QF : FE :: Ef : fq$; whence, $fq = \frac{FE \times Ef}{QF}$; and, when EQ is greater than EF, $Oq = fq + Of$ (Art. 188) $= \frac{FE \times Ef + QF \times Of}{QF} =$ $\frac{FE \times Ef + \overline{QE - FE} \times Of}{QF} = \frac{FE \times \overline{Ef - Of} + QE \times Of}{QF}$ $= \frac{QE \times Of - FE \times EO}{QF}$; therefore, $\frac{qp}{Oq} =$

$\frac{QP \times FE}{QE \times Of - FE \times EO}$; and since QP and FE are given, the visual angle varies inversely as $QE \times Of - FE \times EO$.

(294.) Cor. 1. When the eye is placed close to the glass, EO vanishes; therefore the visual angle varies inversely as $QE \times Of$; that is, inversely as QE. In this case, the angle qOp becomes equal to QEP.

(295.) Cor. 2. When Q coincides with F, the expression, $QE \times Of - FE \times EO$, becomes $FE \times \overline{Of - EO} = FE \times Ef$, which is invariable; in this case, therefore, the visual angle is the same, whatever be the distance of the eye from the glass.

(296.) Cor. 3. When the expression, $QE \times Of - FE \times EO$, becomes negative, the image upon the retina, which was before erect, will be inverted with respect to QP.

On the Astronomical Telescope.

(297.) The astronomical telescope consists of two convex lenses, whose axes are in the same right line, and whose distance is equal to the sum of their focal lengths.

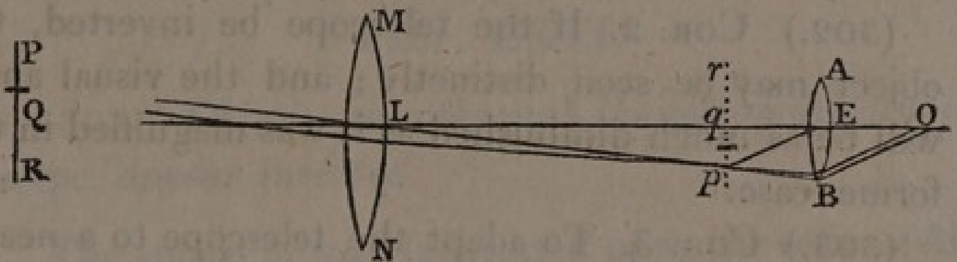
(298.) The common axis of the two lenses is called the *axis* of the telescope.

(299.) The lens which is turned towards the object to be viewed, and which has the greater focal length, is called the *object glass*. The lens to which the eye is applied, is called the *eye glass*.

Prop. LXV.

(300.) *A distant object may be seen distinctly through the astronomical telescope; and the angle which it subtends at the center of the eye, when thus seen, is to the angle which it subtends at the naked eye, as the focal length of the object glass, to the focal length of the eye glass.*

Let L and E be the centers of the two glasses; QP an object, towards which the axis of the telescope is directed; and so distant, that the rays which flow from any one point in it, and fall upon the object glass L,



may be considered as parallel. Then qp, an inverted image of QP, will be formed in the principal focus of the glass L, and contained between the lines QLq, and PLp (Art. 214); and, because LE is equal to the sum of the focal lengths of the two glasses, pq is in the principal focus of the glass AB; consequently, pq may be seen distinctly through this glass, if the eye of the observer be able to collect parallel rays upon the retina (Art. 280). Produce PLp till it meets the eye glass in B; join pE; and draw BO parallel to pE. Then, the rays which flow from P in the object, or p it's image, enter the eye, placed at O, in the direction BO. Also, the rays which flow from Q, enter the eye in the direction EO; thus, the angle which QP subtends at the center of the eye, when viewed through the telescope, is the angle BOE, which is equal to pEq. The angle which QP subtends, when viewed with the naked eye from L, is PLQ, which is equal to pLq. And, since the $\angle pEq$: the $\angle pLq$ (when these angles are small)* :: $Lq : Eq$, the angle which the object subtends at the center of the eye, when seen through the telescope : the angle which it subtends at the center of the naked eye :: Lq : Eq.

(301.) Cor. 1. If the angle which the object subtends at the center of the naked eye be given, the angle which it subtends at the center of the eye, when seen through the telescope, varies as $\frac{Lq}{Eq}$. This quantity is usually called the *magnifying power* of the telescope.

(302.) Cor. 2. If the telescope be inverted, the object may be seen distinctly; and the visual angle will be as much diminished as it was magnified in the former case.

(303.) Cor. 3. To adapt the telescope to a nearer object, the eye glass must be moved farther from the object glass.

For, if QP be brought nearer to the glass L, the image qp will be formed at a greater distance from it (Art. 189); and therefore, in order that qp may still be in the principal focus of the glass E, this glass must be moved farther from L.

(304.) Cor. 4. When QP is brought nearer to L, the magnifying power is increased. For, Lq is increased, and Eq remains the same; therefore $\frac{Lq}{Eq}$ is increased.

---

* Proportions of this kind are to be considered only as approximations, which become more accurate as the angles decrease.

(305.) Cor. 5. To adjust the telescope to the eye of a short sighted person, the eye glass must be moved nearer to the object glass. If the eye require converging rays, the eye glass must be moved the contrary way (Art. 280).

(306.) Cor. 6. If we suppose the eye to be placed between the glass *AB* and it's principal focus, the visual angle is increased by adjusting the telescope to the eye of a short sighted person (Art. 288). If the eye be farther from the glass than it's principal focus, that angle is diminished (Art. 287). The contrary effects are produced when the telescope is adjusted to the eye of a long sighted person.

Prop. LXVI.

(307.) *Objects, seen through the astronomical telescope, appear inverted.*

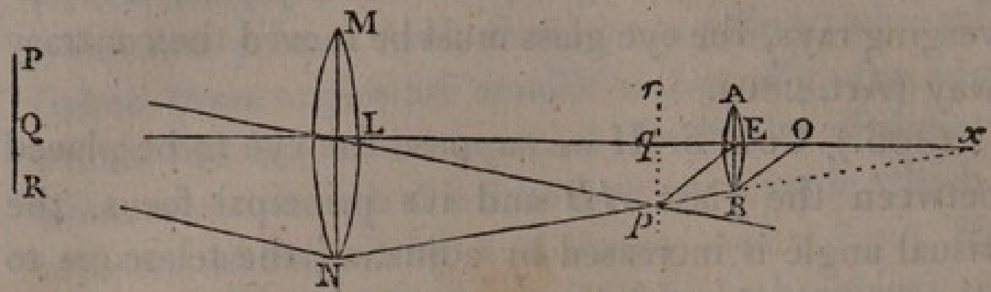
The image of *pqr* is inverted upon the retina (Art. 281); and *pqr* is inverted with respect to ***PQR*** (Art. 221); therefore the image upon the retina is erect with respect to ***PQR***; and consequently the object appears inverted (Art. 249).

(308.) Cor. An object moving across the field of the telescope from the right to the left, appears to move from the left to the right.

Prop. LXVII.

(309.) *In the astronomical telescope, the field of view is the greatest, when the eye is placed between the eye glass and it's principal focus.*

The field of view is the greatest, when the eye is so placed as to receive the extreme rays, refracted from the object glass to the eye glass. Let MN be the diameter of the object glass; AB, the diameter of the

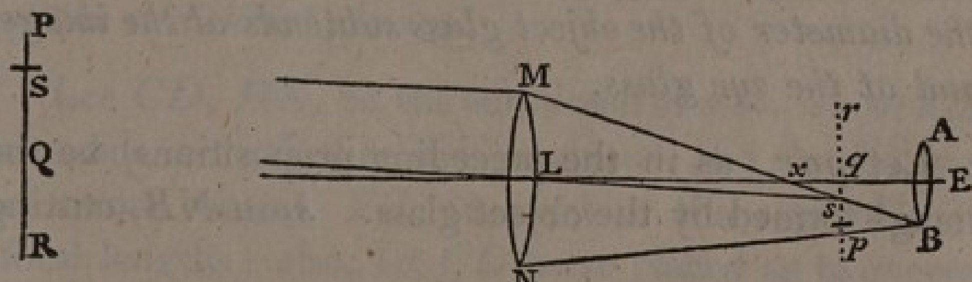


eye glass; join N, B, the corresponding extremities of the glasses, and let NB cut the image pqr in p; join pL, and suppose pL, EL to be produced till they meet the object QP, in P and Q; join also, pE, NP; and draw BO parallel to pE. Then, if the eye be placed at O, it will receive the ray NBO. And this is the ray which comes from a point in the object, at the greatest visible distance from the axis of the telescope; for, the rays which flow from any point in the object, above P, will, after the first refraction, converge to a point below p, and all of them fall below the glass AB; therefore P is the extreme point visible in the object, and QP is half the linear magnitude of the visible area. Now, when the diameter of the object glass is greater than that of the eye glass, which is the case in the astronomical telescope, the rays NB, LE converge to some point x beyond the eye glass; and therefore, the point O to which they converge after the second refraction, and where the eye must be placed to receive them, is between the glass E and it's principal focus (See Arts. 186. 188).

(310.) Cor. 1. If NB and LE be parallel, O is the principal focus of the glass E; and if they diverge, O lies beyond the principal focus.

(311.) Cor. 2. It appears from the preceding figure, that *NBO* is the only ray which comes to the eye from *P*; for, any other ray of the pencil, as *PLp*, after refraction at the object glass, crosses *NB* in *p*, and falls below the eye glass. Hence it follows, that the extremity of the visible area is very faint. If a point be taken in the object, nearer to the center of the field of view, more of the rays which flow from it will be refracted to the eye; and thus, the brightness will continually increase till all the rays in each pencil, incident upon *MN*, are received by the glass *AB*; when this takes place, the brightness will become uniform; because the same number of rays, or very nearly so, is received by the glass *MN* from every point in the object*.

(312.) Cor. 3. To determine the bright part of the field of view, join *MB*, cutting *pqr* in *s*, and the axis *LE* in *x*; join also *sL*, and produce it till it meets the



object in *S*. Then, if *xE* be greater than *qE*, all the rays which flow from *S*, and fall upon *MN*, will be refracted to the eye glass; for, *SM* is refracted in the direction *MsB*; and any other ray of the pencil, as *SL*, crosses *MB* at *s*, and falls somewhere between *A*

* Here it is supposed that the pupil is properly placed, and so large as to receive all the rays refracted by the lens *AB*.

and *B*. In the same manner, the rays which flow from any point between *S* and *Q*, and fall upon *MN*, will be refracted to the eye glass *AB*.

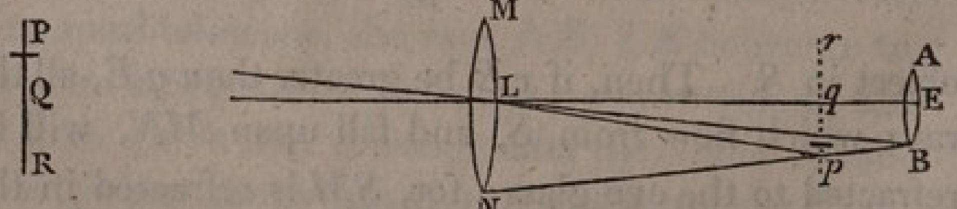
(313.) Cor. 4. If q and x coincide, that is, if the linear apertures, *MN*, *AB*, of the glasses, be proportional to their focal lengths, the brightness of the field increases to the center.

(314.) Cor. 5. If q fall between *L* and x, the whole of the rays belonging to any one pencil incident upon *MN*, will not be received by the eye glass. In this case, a less aperture of the object glass would produce the same brightness, with less aberration *.

Prop. LXVIII.

(315.) *The linear magnitude of the greatest visible area is measured by the angle which the diameter of the eye glass subtends at the center of the object glass, increased by the difference between the angles which the diameter of the object glass subtends at the image, and at the eye glass.*

Let pqr, as in the preceding propositions, be the image formed by the object glass. Join *NB*, cutting



pqr in p; join also, *LB*, *Lp*; and suppose pL, *EL* to be produced till they meet the object in *P* and *Q*. Then, *QP*, which is half the linear magnitude of the

* See Sect. viii.

greatest visible area (Art. 309), is measured by the angle PLQ, or it's equal pLq; that is, by $BLE+BLp$; or, $BLE+LpN-LBN$; therefore, 2 QP is measured by $2\,BLE+2\,LpN-2\,LBN$.

(316.) Cor. In the same manner it may be shewn, that the linear magnitude of the brightest part of the visible area, is measured by the angle which the diameter of the eye glass subtends at the center of the object glass, diminished by the difference between the angles which the diameter of the object glass subtends at the image, and at the eye glass.

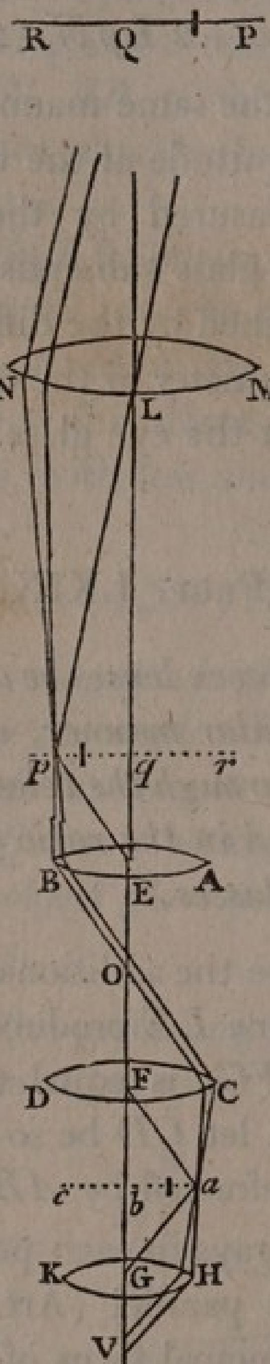
Prop. LXIX.

(317.) *If two convex lenses be added to the former, and placed in a similar manner, a distant object may be seen distinctly through the telescope, and the visual angle will be altered in the ratio of the focal lengths of the additional glasses.*

Let CD, HK, be the additional glasses, whose axes coincide with the line LE produced, and the distance of whose centers, FG, is equal to the sum of their focal lengths; also, let CD be so placed as to receive the extreme rays refracted by AB.

Then, since the rays in each pencil, after refraction at the glass E, are parallel (Art. 300), they will be collected in the principal focus of the glass CD; that is, in the principal focus of the glass HK; and an image, abc, will be formed there, which may be seen distinctly through the lens HK (Art. 280).

Again, let *BC* be the extreme pencil of rays refracted by *AB*; draw *Fa* parallel to *BC*, and let it meet *abc*



in *a*; join *aG*, *Ca*; and produce *Ca* till it meets the lens *HK* in *H*; draw *HV* parallel to *aG*; and at *V* let the eye be placed. Then, *NBCHV* being the course of the pencil of rays which flows from *P*; and *LEFGV* the course of the pencil which flows from *Q*, the angle which QP subtends at the center of the eye, when seen through the four glasses : the angle which it subtends there, when seen through the first two :: the $\angle HVG$: the $\angle EOB$ :: the $\angle aGb$: the $\angle aFb$ :: $Fb : Gb$.

(318.) Cor. 1. Since the angle which QP subtends at the center of the eye, when it is seen through the two first glasses : the angle which it subtends at the center of the naked eye :: $Lq : Eq$ (Art. 300), by compounding this proportion with the last, we have, the angle which the object subtends at the center of the eye, when viewed through the four glasses : the angle which it subtends at the naked eye :: $Lq \times Fb$: $Eq \times Gb$.

(319.) Cor. 2. If Gb and Fb be equal, the visual angle is not altered.

(320.) Cor. 3. The object, when viewed through the glasses thus combined, appears erect. For, the axes of the several pencils of rays which flow from the image pqr, cross each other at E; therefore, the image abc is inverted with respect to pqr; or, erect with respect to the object PQR; and consequently, the image upon the retina is inverted (Art. 281); that is, the object appears erect*.

(321.) Cor. 4. If the apertures of the additional glasses be properly adjusted, the field of view will not be altered. For, the extreme rays, refracted by AB, may be received by CD, and refracted to HK.

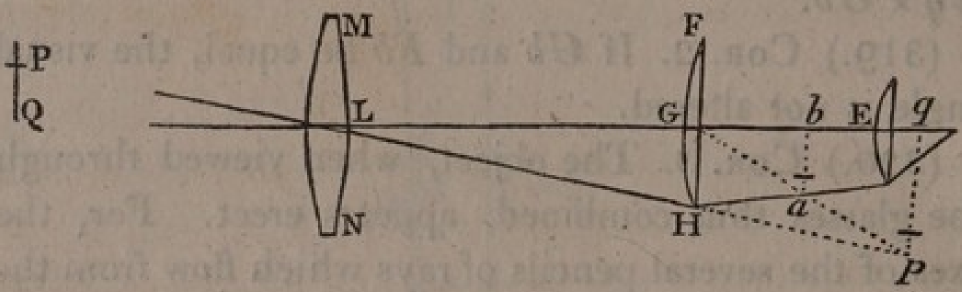
---

* This is one advantage gained by the additional glasses. The inconvenience with which they are attended is, that they render objects more faint, by increasing the quantity of the refracting medium through which the rays must pass, before they arrive at the eye (See Note, p. 137).

(322.) Cor. 5. This telescope may be adjusted to the eye of a short sighted person, by moving the glass HK nearer to CD (Art. 280); or, if E, F, G, be connected, by moving these three glasses nearer to L. If the eye of the observer require converging rays, the glasses must be moved the contrary way.

(323.) Sometimes a convex lens is interposed between the object glass and it's principal focus, in the astronomical telescope, for the purpose of increasing the field of view, and diminishing the aberration of the lateral rays (Sect. viii).

Let FGH be such a lens, whose axis is coincident with the axis of the telescope. Then, the rays which



tend to form the image qp, after refraction at the glass FH, form the image ab, between G and q (Art. 223); and this image is viewed through the eye glass E, whose focal length is bE*.

On Galileo's Telescope.

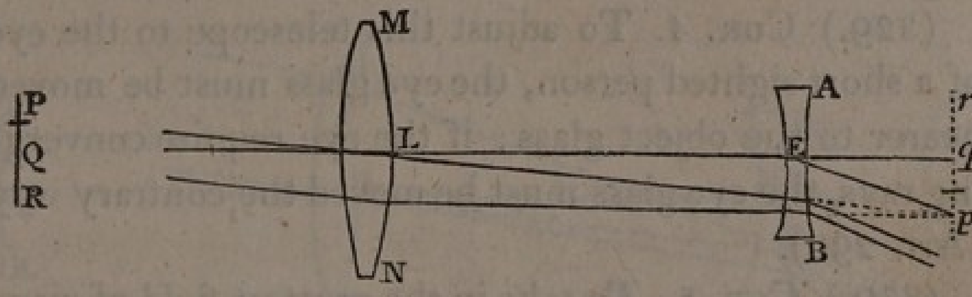
(324.) Galileo's telescope consists of a convex and a concave lens, whose axes are in the same line, and whose distance is equal to the difference of their focal lengths.

* Other constructions, with the explanation of their advantages, may be seen in the *Encyclopædia Britannica*, Article *Telescope*.

Prop. LXX.

(325.) *A distant object may be seen distinctly through Galileo's telescope; and the angle which it subtends at the center of the eye when thus seen, is to the angle which it subtends at the center of the naked eye, as the focal length of the object glass, to the focal length of the eye glass.*

Let L and E be the centers of the glasses; PQR a distant object, towards which the axis of the telescope



is directed; pqr it's image in the principal focus of the glass L, and therefore in the principal focus of the glass E; then, since the rays tend to form an image in the principal focus of the concave lens E, after refraction at that lens, they will be proper for vision (Art. 290); or, a distinct image of the object PQR, will be formed upon the retina of a common eye.

Also, the angle under which the object QP is seen through the telescope, is equal to the angle qEp (Art. 294); and the angle under which it is seen with the naked eye from L, is QLP, which is equal to qLp; therefore, the visual angle in the former case : the visual angle in the latter :: $Lq : Eq$.

(326.) Cor. 1. The magnifying power of the telescope is measured by $\frac{Lq}{Eq}$ (See Art. 301).

(327.) Cor. 2. To adapt this telescope to a nearer object, the eye glass must be moved to a greater distance from the object glass.

For, as QL decreases, Lq increases (Art. 189); therefore, in order that the principal focus of the glass E may coincide with q, LE must be increased.

This is the construction of a common opera glass.

(328.) Cor. 3. When the telescope is adjusted to a nearer object, the magnifying power is increased; for Lq is increased, and Eq remains the same; therefore $\frac{Lq}{Eq}$ is increased.

(329.) Cor. 4. To adjust this telescope to the eye of a short sighted person, the eye glass must be moved nearer to the object glass; if the eye require converging rays, the eye glass must be moved the contrary way (Art. 290).

(330.) Cor. 5. To take in the greatest field of view, the eye must be placed close to the glass AB, as will be shewn in a subsequent article; and therefore, by adjusting the telescope to the eye of a short sighted person, the visual angle of a given object is diminished (Art. 294); and on the contrary, by adjusting it to the eye of a long sighted person, this angle is increased.

Prop. LXXI.

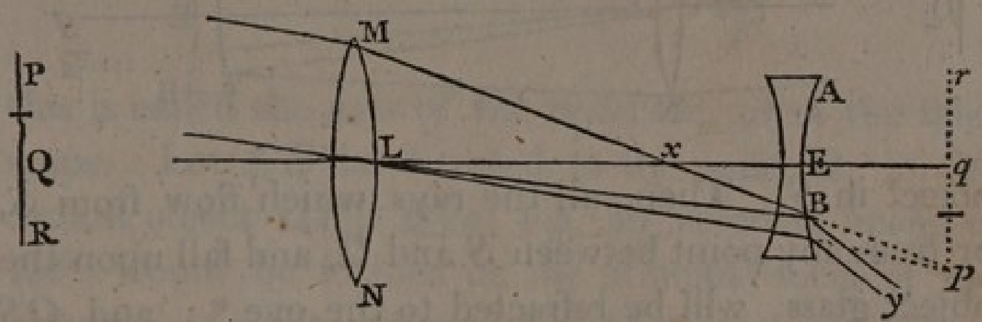
(331.) *Objects, seen through Galileo's telescope, appear erect.*

For, the image upon the retina is erect, with respect to pqr (Art. 291); therefore it is inverted with respect to the object; or the object appears erect (Art. 249).

Prop. LXXII.

(332.) *The linear magnitude of the field of view, when the eye is placed close to the concave lens in Galileo's telescope, is measured by the angle which the diameter of the pupil subtends at the center of the object glass, increased by the difference between the angles which the diameter of the object glass subtends at the pupil, and at the image.*

Let MN be the diameter of the object glass; AB, the diameter of the pupil, whose center is in the axis

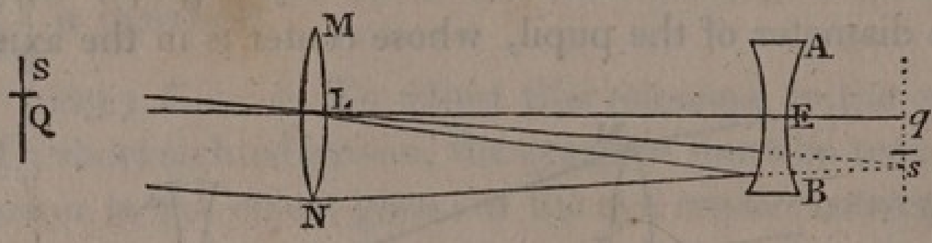


of the telescope; join M, B, the opposite extremities of these diameters; and let MB meet the axis in x, and the image in p; join LB, Lp; and suppose pL, qL to be produced till they meet the object in P and Q; also, suppose MP to be joined. Then will PM be refracted to the pupil in the direction MB; but any other ray in the pencil, as PL, and every ray which flows from a point more distant from the axis of the telescope, and is refracted by MN, will fall below the pupil; therefore QP is half the linear magnitude of the greatest visible area. Also, QP is measured by the angle QLP, or it's equal $pLq = ELB +$

$BLp = ELB + LBM - LpM$; and by doubling these quantities, $2\ QP$ is measured by $2\ ELB + 2\ LBM - 2\ LpM$.

(333.) Cor. 1. The rays MxB, LxE, which are incident upon the glass E, diverging from x, after refraction will diverge more (Art. 190); therefore, if the eye be moved to any other point in the axis of the telescope, the ray xBy will not enter the pupil; and consequently the visible area will be diminished.

(334.) Cor. 2. To determine the brightest part of the visible area, join NB, and let it meet the image in s; join BL, sL, and produce sL till it meets the



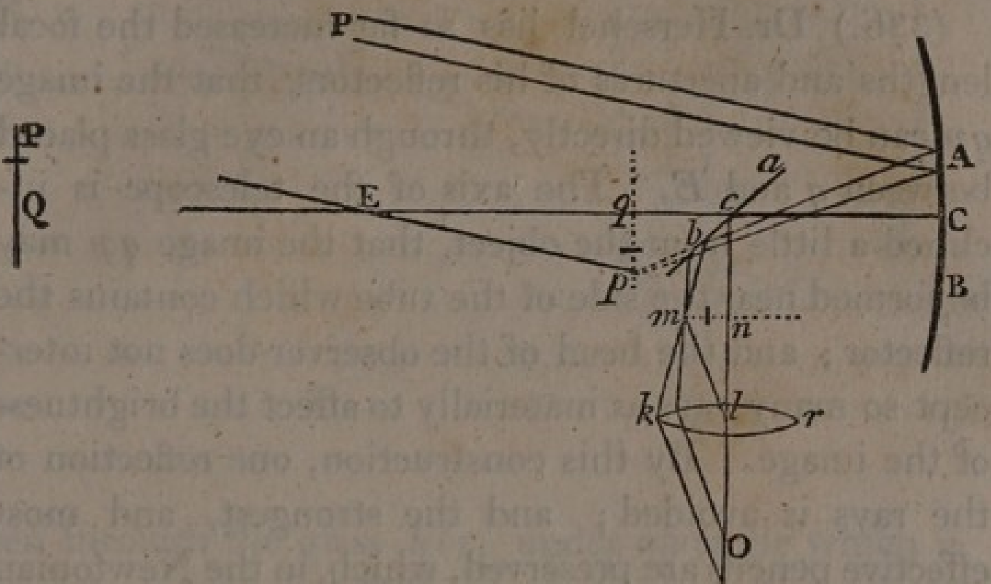
object in S. Then, all the rays which flow from S, or from any point between S and Q, and fall upon the object glass, will be refracted to the eye *; and QS will be the linear magnitude of half the bright part of the field of view. Also, QS is measured by the angle SLQ, which is equal to $sLq = BLE - BLs = BLE - LBN + LsN$; and $2\ QS$ is measured by $2\ BLE - 2\ LBN + 2\ LsN$.

---

* See Art. 311.

On Sir Isaac Newton's *Telescope.*

(335.) Let *ACB* be a concave spherical reflector, whose middle point is *C*, and center *E*. Join *CE*,



this is called the *axis of the reflector*, or of the telescope. Let *CE* be directed to the point *Q*, in the distant object *QP*; then, *qp*, an inverted image of *QP*, would be formed in the principal focus of the reflector, at right angles to *CE* (Art. 47), and terminated by the lines *PEp*, *QEq*, were the reflected rays suffered to proceed thither; but, before they arrive at the focus, they are received upon a plane reflector *acb*, inclined at an angle of 45° to the axis *CE*; and thus an image, *mn*, is formed, similar and equal to *pq*, and equally inclined to the plane reflector (Art. 72); and consequently, *mn* is parallel to *EC*. This image is viewed through a convex eye glass *klr*, whose axis is perpendicular to *EC*, and whose distance from the image *mn*, is equal to it's focal length *.

* See Art. 280.

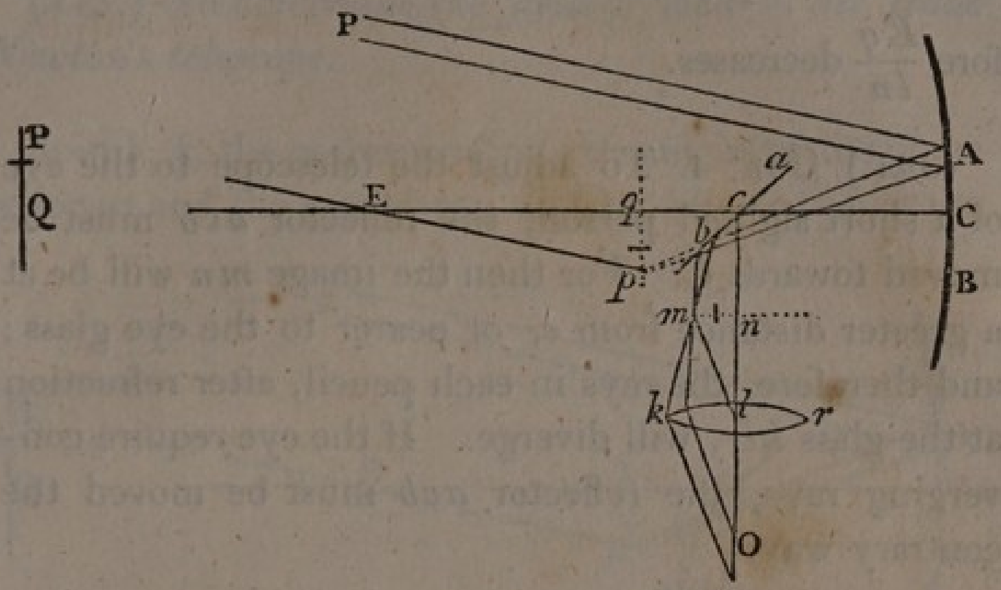
If the reflection be made at *c*, by a small right angled prism, one of whose sides is perpendicular, and the other parallel to the axis, much less light will be lost, than if the reflection be made by a plane speculum (See Art. 101).

(336.) Dr. Herschel has so far increased the focal lengths and apertures of his reflectors, that the image *qp* can be viewed directly, through an eye glass placed between *q* and *E*. The axis of the telescope is inclined a little from the object, that the image *qp* may be formed near the side of the tube which contains the reflector; and the head of the observer does not intercept so many rays as materially to affect the brightness of the image. By this construction, one reflection of the rays is avoided; and the strongest, and most effective pencils are preserved, which, in the Newtonian telescope, are stopped by the plane reflector.

Prop. LXXIII.

(337.) *When a distant object is viewed with Sir Isaac Newton's reflecting telescope, the angle which it subtends at the center of the eye, is to the angle which it subtends at the naked eye, as the focal length of the reflector, to the focal length of the eye glass.*

Let *QP* be the object; *qp* it's image, in the principal focus of the reflector; *QEC* the axis of the telescope, cutting *acb* in *c*. Draw *cnlO* perpendicular to *CE*; make *cn* equal to *cq*, draw *nm* at right angles to *cn*, and make *nm* equal to *qp*; then *nm* is the image of *qp*, or *QP*. At *l* let a convex eye glass *klr* be placed, whose focal length is *ln*, and whose axis coincides with that line; join ml. Then the image nm, which corresponds to QP in the object, is



seen through the glass klr, under an angle which is equal to mln; and QP is seen, with the naked eye placed at E, under an angle which is equal to qEp; and since these angles have equal subtenses nm, and qp, they are to each other inversely as the radii ln, qE; therefore the angle which the object subtends at the center of the eye, when viewed with the telescope : the angle which it subtends at the naked eye :: $Eq : ln$.

(338.) Cor. 1. The magnifying power of this telescope is measured by $\frac{Eq}{ln}$.

(339.) Cor. 2. To adapt the telescope to nearer objects, the reflector acb, to which the eye glass is attached, must be moved towards E. For, as QE decreases, qE decreases (Art. 59); and therefore, that qc, or cn may remain of the same magnitude, acb must be moved nearer to E.

(340.) Cor. 3. As the object is brought nearer to E, the magnifying power of the telescope is diminished. For, Eq decreases, and ln is invariable; therefore $\frac{Eq}{ln}$ decreases.

(341.) Cor. 4. To adjust the telescope to the eye of a short sighted person, the reflector acb must be moved towards C. For then the image mn will be at a greater distance from c, or nearer to the eye glass; and therefore, the rays in each pencil, after refraction at the glass klr, will diverge. If the eye require converging rays, the reflector acb must be moved the contrary way.

Prop. LXXIV.

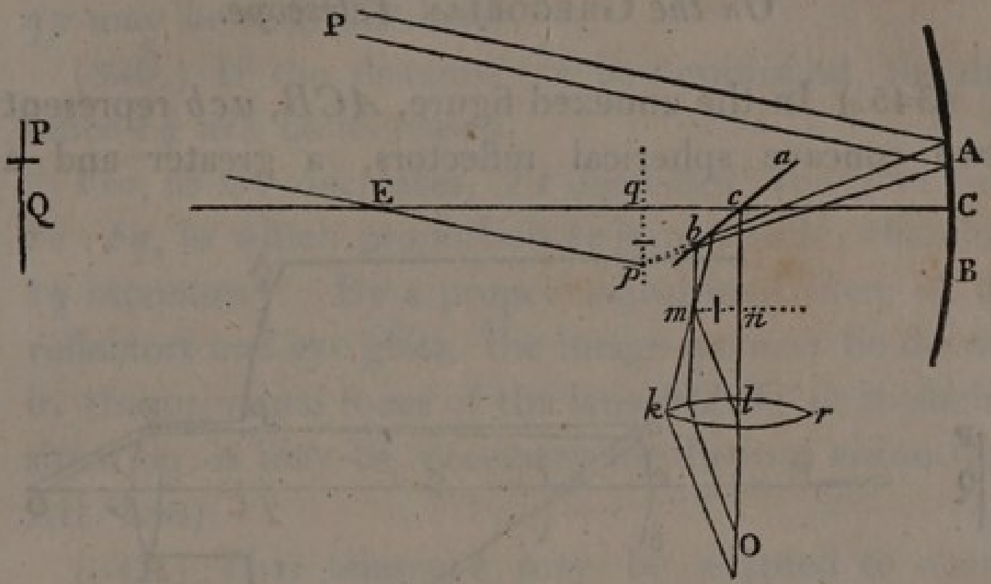
(342.) *Objects, viewed with Sir Isaac Newton's telescope, appear inverted.*

Let QP lie in the plane of the paper; and, when the eye of the observer is applied to the glass kr, let the plane which passes through both his eyes, also coincide with the plane of the paper. Then, the rays which flow from P, the right side of the object, converge to m, the left side of the image. Also, the rays which flow from a point in the object, above the plane of the paper, converge to a point in the image, below that plane; thus, the image mn is inverted; and since it is in, or near to, the principal focus of the convex lens through which it is viewed, it appears inverted (Art. 281).

Prop. LXXV.

(343.) *To determine the field of view in Sir Isaac Newton's telescope.*

Join b, k, the corresponding extremities of the plane reflector and the eye glass; and let bk cut the image



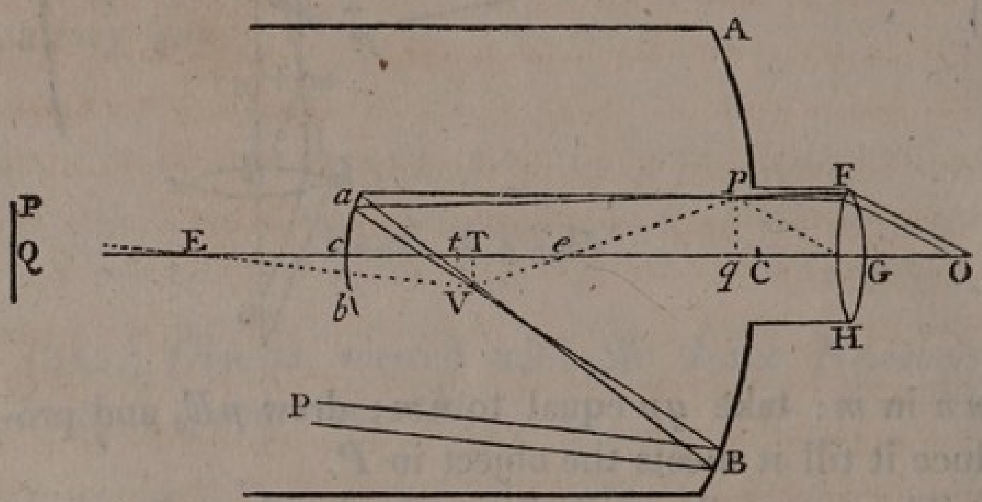
mn in m: take qp equal to nm; draw pE, and produce it till it meets the object in P.

Then, a small pencil of rays flowing from P, will, if acb be not so large as to intercept them before they are incident upon the reflector ACB, be reflected at b, to the extremity of the eye glass, and refracted thence to the eye at O; the point P will therefore be visible. Also, this point is the extremity of the field of view. For, if a point be taken in nm, farther from the axis of the lens than m, any straight line drawn through it, will either fall without ab, or without kr; that is, no ray, belonging to a point in the object QP, beyond P, can be reflected from acb to the eye glass.

(344.) Cor. In order that the field of view may be circular, *acb* must be the transverse section of a cone, or cylinder, generated by the revolution of *bk*, about the axis *cl*; and therefore it must be an ellipse, whose major and minor axes depend upon the nature of this solid.

On the Gregorian *Telescope.*

(345.) In the annexed figure, *ACB*, *acb* represent two concave spherical reflectors, a greater and a



smaller, whose axes are coincident, and whose concave surfaces are turned towards each other; *E* and *e*, their centers; *T*, *t*, their principal foci, of which, *T* lies between *e* and *t*. In the middle of the larger reflector is an opening, nearly of the same dimensions with the aperture of the smaller, to admit a moveable tube containing a convex eye glass *FGH*.

When the axis of the telescope is directed to the point *Q* in a distant object *QP*, an inverted image *TV*, of this object, terminated by the lines *QET* *PEV*, is formed in the principal focus of the reflector

AB. The rays which diverge thence, and fall upon the concave reflector *acb*, after reflection, form an image *qp*, terminated by the lines *Teq*, *Vep*; which, because *T* is between *e* and *t*, is inverted with respect to *TV* (Art. 88); or erect, with respect to *QP*.

The rays are then received by the eye glass *FGH*, whose axis coincides with the axis of the telescope, and whose focal length is *Gq*; and therefore the image *qp* may be seen distinctly.

(346.) If the distance *Cc* be diminished, the distance *tq* will be increased.

For, as *Cc* decreases, *Tt* decreases, and $Tt : te :: te : tq$, in which proportion *te* is invariable; therefore *tq* increases*. By a proper adjustment then, of the reflectors and eye glass, the image *qp* may be formed in the principal focus of the lens *FGH*; or in such a situation as may be necessary for distinct vision (See Art. 280).

(347.) This telescope may be adapted to nearer objects, by increasing the distance *Cc*. For, as *QP* approaches towards *E*, *TV* also approaches towards *E*, or towards *t*; therefore the distance *tq* increases; or *q* is nearer to the eye glass than before; which inconvenience may be remedied by increasing the distance *Cc* (Art. 346).

(348.) Objects seen through this telescope, appear erect. For, *qp* is an erect image of *QP*, and in, or

---

* Having given the focal lengths of the reflectors, and the distance of *q* from *T*, the distance of the reflectors may be found.

Let $Tt = x$; $te = b$; $TC = a$; $Tq = c$. Then, $x : b :: b : x + c$; therefore, $x^2 + cx = b^2$; from which equation we obtain $x = \frac{\sqrt{4b^2 + c^2} - c}{2}$; and $Cc = a + b + x$.

near to the principal focus of the convex lens through which it is viewed; therefore it appears erect (Art. 281).

(349.) As so much has been said upon the field of view in other telescopes, the reader will find no difficulty in determining it in this case. Join *F*, *a*, corresponding extremities of the small reflector, and the eye glass; let *Fa* cut *qp*, in *p*. Draw *pe*, and produce it till it meets *TV* in *V*; join *VE*, and produce this line till it meets the object *QP* in *P*; then is *P* the extremity of the field of view. For, if *aVB* be drawn, the rays which flow from *P*, and fall upon the large mirror at *B*, after reflection converge to *V*, and fall upon the reflector *acb* at *a*; thence they proceed in the direction *apF*, and are refracted to the eye in the direction *FO*, which is parallel to *pG**. But no ray, which belongs to a point in the object above *P*, can be reflected from *acb* to the eye glass *FH*.

Since *qp* is much nearer to the eye glass than to the reflector *acb*, the field of view will depend more upon the aperture of the former, than of the latter.

Prop. LXXVI.

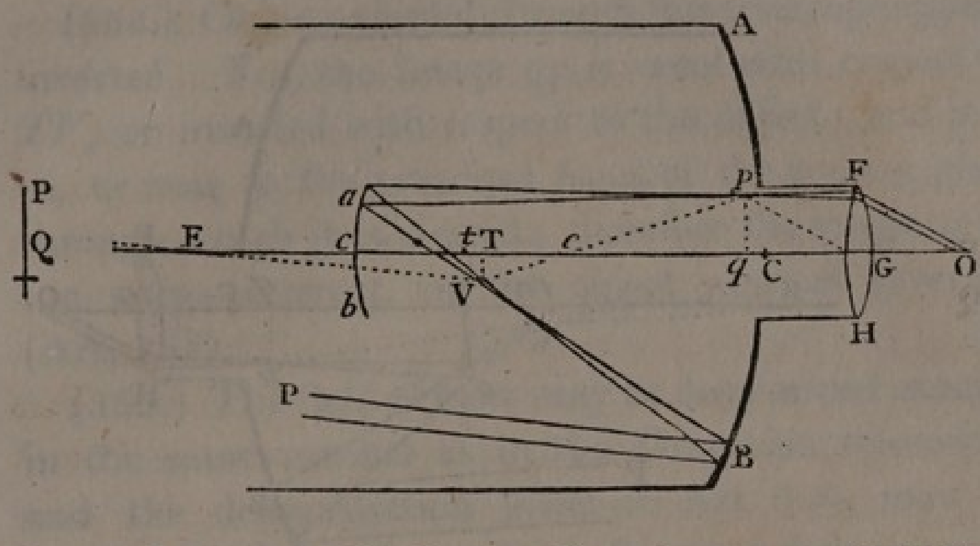
(350.) *To determine the angle which an object subtends at the center of the eye, when seen through Gregorie's telescope.*

The construction being made as before, the angle under which *QP* is seen through the telescope, is

---

* This is on supposition that *aVB* meets the reflector *AB*; and that the ray *PB*, which is parallel to *EV*, is not intercepted by the reflector *ab*.

equal to pGq; and the angle under which it is seen with the naked eye, is equal to TEV. Now, when



the angles are small, the $\angle pGq$: the $\angle peq$ (TeV) :: $eq : qG$; and also, the $\angle TeV$: the $\angle TEV$ :: ET : eT; by comp. the $\angle pGq$: the $\angle TEV$ :: $eq \times ET$: $qG \times eT$; that is, the visual angle, when the object is seen through the telescope : the visual angle, when it is seen with the naked eye :: $eq \times ET : qG \times eT$.

(351.) Cor. Since $Tt : te :: eT : eq$ (Art. 54); inversely, $te : Tt :: eq : eT$; therefore $te \times ET : Tt \times qG :: eq \times ET : qG \times eT$ (Alg. Art. 185); and the visual angle, when the object is seen through the telescope : the visual angle, when it is seen with the naked eye :: $te \times ET : Tt \times qG :: \frac{te \times ET}{qG} : Tt$.

On Cassegrain's *Telescope.*

(352.) In this telescope, the smaller reflector, acb, is convex, and so placed, that T falls between t and c. In other respects, it is similar to the Gregorian telescope (See Art. 345).

Let *GCE*, the axis of this telescope, be directed to the point *Q* in a distant object *QP*; then, an inverted

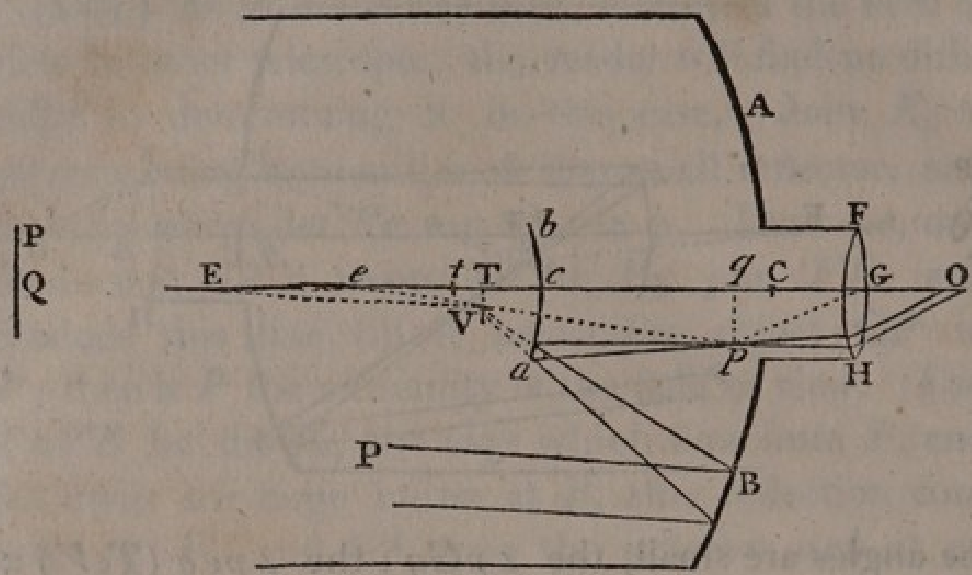


image *TV*, of this object, would be formed in the principal focus of the large reflector, and terminated by the lines *QET*, *PEV*, if the rays were suffered to proceed thither. But, before they reach the focus, they are received upon the convex reflector *bca*; and since, by the construction, *TV* falls between the surface *c*, and principal focus *t*, of this reflector, an erect image *qp*, of *TV*, is formed, and terminated by the lines *eTq*, *eVp*. This image is viewed through the lens *FGH*, whose axis coincides with the axis of the telescope, and whose focal length is *Gq*.

(353.) The distance of the image *qp*, from *t*, may be determined by the proportion $Tt : et :: et : tq$. Also, by diminishing the distance *Cc*, the distance *Tt* is diminished, and since *te* is given, *tq* is increased.

Hence it is manifest, that by a proper adjustment of the reflectors and the eye glass, the image *qp* may be formed in the principal focus of the lens *FGH*;

or in such a situation, that the emergent rays may have a proper degree of divergency, or convergency, for the eye of the observer.

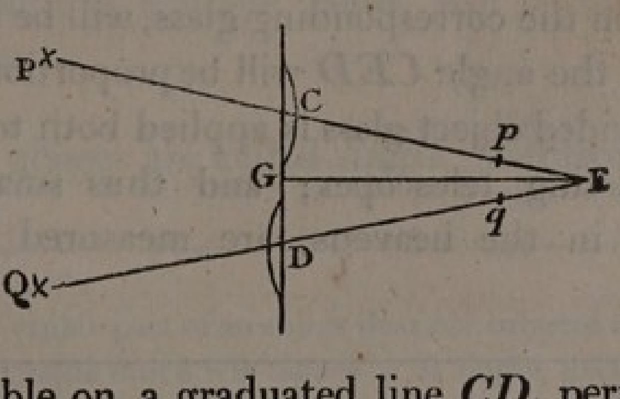
(354.) Objects viewed through this telescope appear inverted. For, the image *qp* is erect with respect to *TV*, or inverted with respect to the object; and it is in, or near to the principal focus of the convex glass through which it is viewed; therefore the image upon the retina is erect, or the object appears inverted (Art. 281).

(355.) The field of view may be determined exactly in the same manner as in the Gregorian telescope; and the demonstration given in Art. 349, may be applied, in the same words, to the preceding figure.

This telescope may also be adapted, in the same manner, to nearer objects; and the visual angle is expressed in the same terms (See Arts. 347. 350).

On the divided object glass Micrometer.

(356.) This micrometer consists of a convex lens divided into two equal parts, *C*, *D*, by a plane which passes through it's axis; and the segments



are moveable on a graduated line *CD*, perpendicular to that axis. Let *C*, *D*, be the centers of the segments; and *P*, *Q*, two remote objects, images of which will be formed in the lines *PCE*, *QDE*, and also in the principal foci of the segments (Art. 219). Let the glasses be moved till these images coincide*, as at *E*; then, the angle *PEQ*, which the objects subtend at *E* the principal focus of *C*, or *D*, is equal to the angle which *CD*, the distance of the centers of the two segments, subtends at the same point; and therefore, by calculating this angle, we may determine the angular distance of the bodies *P* and *Q*, as seen from *E*. Draw *EG* perpendicular to *CD*; and, because the triangle *CED* is isosceles, $CG = GD$, and the $\angle CEG =$ the $\angle GED$; also, *GD* is the sine of the angle *GED*, to the radius *ED*; therefore, knowing *ED*, and *GD*, the angle *GED* may be found by the tables; and consequently $2GED$, or *CED* may be determined.

(357.) The angle *CED* is in general so small, that it may, without sensible error, be considered as proportional to the subtense *CD*. And being determined in one case by observation, it may be found in any other, by a single proportion.

(358.) If the objects be at a *given finite* distance, the angle *PEQ* will still be proportional to *CD*; for, on this supposition, the distance *CE*, or *DE*, of either image from the corresponding glass, will be invariable; therefore, the angle *CED* will be proportional to *CD*.

The divided object glass is applied both to reflecting and refracting telescopes; and thus small angular distances in the heavens, are measured with great accuracy.

---

* Two other images are formed, an image of *P* by the segment *D*, and an image of *Q* by the segment *C*; but, as *CD* increases, these images always *recede* from each other.

On single Microscopes.

(359.) If the angle which an object subtends at the center of the eye, when at a proper distance for distinct vision, be diminished beyond a certain limit, the image upon the retina is so small as to convey to the mind only the idea of a single physical point, not distinguishable into parts; respecting which, therefore, no judgment can be formed by the sight, except what relates to it's colour*. If we endeavour to increase the image upon the retina by bringing the object nearer to the eye, the extreme rays which enter the pupil will diverge too much, and the image become confused. If the extreme rays be stopped, to lessen the indistinctness produced by the lateral rays, the image will be indistinct for want of light†. But if the object be placed in the principal focus of a glass spherule, or lens whose focal length is short, it may be seen distinctly; the visual angle, as well as the quantity of light admitted into the eye, will be increased; and thus, the several parts, of what before appeared only as a single point, will be subjected to examination.

These glasses are called *single Microscopes.*

---

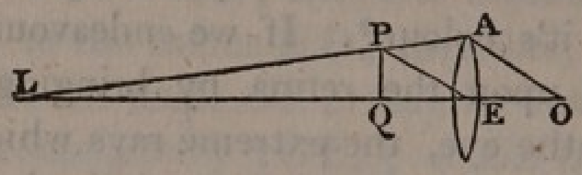
* The least visible part of an object does not subtend at the center of the eye, an angle much less than 4′. A single, detached object is perceivable under an angle of about 2′. Harris's *Optics*, p. 121. Jurin's *Essay*, in Smith's *Optics*, Art. 164.

† See Note, p. 130.

Prop. LXXVII.

(360.) *The visual angle of an object when seen through a single microscope, is to it's visual angle when viewed with the naked eye at the least distance of distinct vision, as that least distance, to the focal length of the glass.*

Let QP be an object placed in the principal focus of the lens, or spherule AE, whose center is E; LQ



the least distance at which it can be seen distinctly with the naked eye. Join LP, PE. Then, the angle under which the object is seen through the glass, is equal to PEQ; and the angle under which it is seen with the naked eye, is QLP; also, when these angles are small, since they have a common subtense QP, they are nearly in the inverse ratio of the radii EQ, LQ; that is, the visual angle when the object is seen through the glass : the visual angle when it is seen with the naked eye at the distance $LQ :: LQ : EQ$.

Ex. If the focal length of the glass be $\frac{1}{50}$ of an inch, and the least distance of distinct vision, eight inches, the visual angle of the object when viewed through the glass : the visual angle when it is seen with the naked eye $:: 8 : \frac{1}{50} :: 400 : 1$.

In this microscope, the object appears erect (Art. 281).

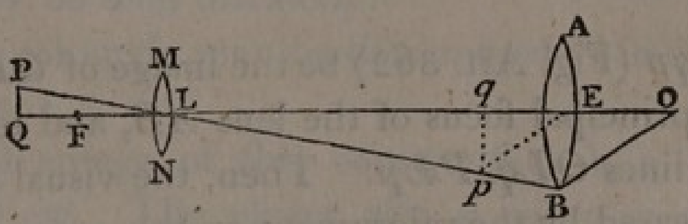
(361.) The solar microscope is a single convex lens, used in the same manner as in the magic lantern (Art. 275). The moveable tube is adjusted to a hole in the window shutter of a darkened chamber; and the object to be examined, is strongly illuminated, and placed a little farther from the lens than it's principal focus; an inverted image of the object is thus formed, at a considerable distance from the lens, and received upon a screen placed at the concourse of the refracted rays.

The angles which the image and object subtend at the center of the eye, when viewed at the least distance of distinct vision, are proportional to their linear magnitudes, that is, to their distances from the center of the glass *.

On the double Microscope.

(362.) The astronomical telescope, when adapted to near objects, becomes a double microscope.

QP is an object, placed a little farther from the lens MN than it's principal focus F; qp the image of QP,



formed on the other side of the lens, and at a considerable distance from L it's center. AEB is a convex eye glass, whose axis coincides with the axis of

* On the construction, and use of the solar microscope, the reader may consult Mr. Adam's work, entitled '*Essays on the Microscope.*'

the lens MN, and whose distance from L is equal to the sum of Lq, and it's own focal length qE; consequently, the image qp is in the principal focus of the eye glass, and it may therefore be seen distinctly by a spectator whose eyes are able to collect parallel rays.

(363.) Since the conjugate foci, Q and q, move in the same direction upon the indefinite line QLO (Art. 189), if the glasses be fixed in a tube, or attached to each other in any other way, by moving the object QP, the image qp may be brought into the principal focus of the eye glass; or into such a situation, that the rays may, after the latter refraction, have a proper degree of divergency, or convergency for the eye of the spectator (See Art. 280).

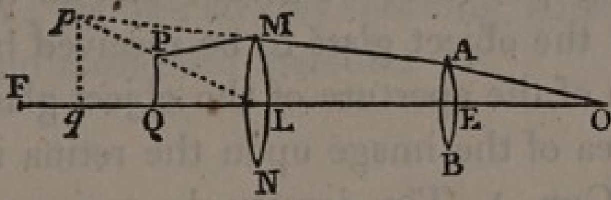
Prop. LXXVIII.

(364.) *To compare the angle which an object subtends at the center of the eye when seen through the double microscope, with the angle which it subtends at the naked eye when viewed at the least distance of distinct vision.*

Let qp (Fig. Art. 362) be the image of QP, formed in the principal focus of the lens AB, and terminated by the lines QLq, PLp. Then, the visual angle will be increased by the microscope on two accounts; first, because the image qp is greater than the object; and secondly, because this image is seen under a greater angle when viewed through the glass, than when viewed with the naked eye. Now, supposing QP and qp to be viewed with the naked eye, at the least distance of distinct vision, the visual angle of QP : the visual angle of qp ($:: QP : qp$) $:: QL : Lq$. Also, the visual angle of qp, when thus viewed, : it's visual angle when seen through the glass AB $:: qE$: the least distance of distinct vision (Art. 360); therefore, by compounding these two proportions, the visual angle of QP, when viewed with the naked eye at the least distance of distinct vision, : the visual angle, when it is viewed with the microscope, $:: QL \times qE$: $Lq \times$ the least distance of distinct vision.

(365.) When the glasses are thus combined, the object appears inverted (Art. 307).

(366.) When a great magnifying power is not required, the object is placed between the glass MN



and it's principal focus; thus an erect image qp is formed, on the same side of the lens with the object; and if Eq be the focal length of the eye glass AB, the image may be seen distinctly.

The visual angle may be determined as in the preceding case.

One advantage of this construction is a greater field of view. The object also appears erect; and less confusion is produced by the spherical surfaces of the glasses, than would be caused by a single glass with the same magnifying power.

Prop. LXXIX.

(367.) *The density of rays in the bright part of the image of a given object, formed upon the retina by a refracting telescope, or double microscope, varies, nearly, as the aperture of the object glass directly, and the area of the picture upon the retina inversely.*

The density of rays in the image, varies directly as their number and inversely as the area over which they are diffused (Art. 7); that is, supposing the transmitting power to be given, and all the rays refracted at the object glass to be received by the eye, as the area of the aperture of the object glass directly, and the area of the image upon the retina inversely.

(368.) Cor. 1. The density also varies according to the same law, when reflectors are used; but the effects of reflectors and refractors are not here compared, because a much greater quantity of light is lost in reflection than in refraction.

(369.) Cor. 2. If F and f be the focal lengths of the object glass and eye glass, and A the linear aperture of the object glass, the density of rays in the picture upon the retina varies as $\frac{A^2 f^2}{F^2}$.

For, the visual angle of the object when seen with the naked eye : it's visual angle when seen through the telescope :: $f : F$; and since the object is given, the first term in the proportion is invariable; therefore the visual angle when the object is seen through the telescope, or the linear magnitude of the image upon the retina, varies as $\frac{F}{f}$; and the area of the image, as $\frac{F^2}{f^2}$; therefore, the density of rays in that image, varies as $\frac{A^2 f^2}{F^2}$.

(370.) Cor 3. In the same manner, if F be the focal length of the object metal in the Newtonian telescope, A it's linear aperture, and f the focal length of the eye glass, the density of rays in the picture upon the retina, if the reflecting power be given, varies as $\frac{A^2 f^2}{F^2}$.

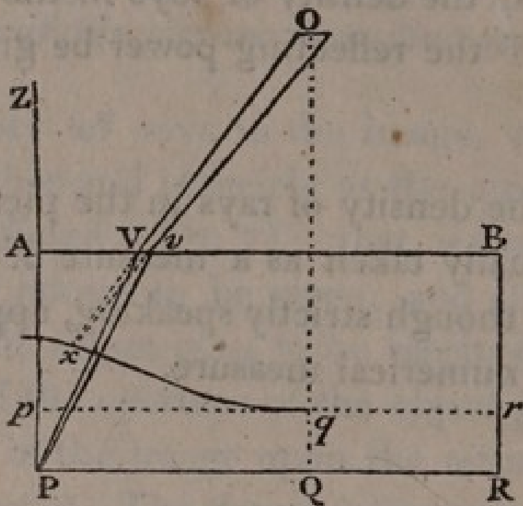
(371.) The density of rays in the picture upon the retina is usually taken as a measure of the apparent brightness; though strictly speaking, apparent brightness has no numerical measure.

SCHOLIUM.

(372.) In explaining the construction and effects of optical instruments, we have supposed the images to be similar to the objects, and accurately formed in the geometrical foci of refracted or reflected rays. Were these suppositions true, telescopes and microscopes would be perfect; no limit could be set to their magnifying powers, but such as arise from the difficulty of forming spherical surfaces of proper dimensions; and by their assistance, objects might be seen as distinctly as if they were viewed in plane reflectors. The imperfections to which they are subject, arise from two causes; The spherical figure of the reflecting and refracting surfaces; and the unequal refrangibility of the different rays which constitute the body of light by which objects are seen.

1. When any points in an object are seen by oblique pencils, that is, by such pencils as are not nearly perpendicular to the reflecting, or refracting surfaces, those points do not appear in the places determined by the constructions and calculations, hitherto given*.

This will be easily understood by considering the most simple case of refraction. Suppose *AB* to be a



plane refracting surface; *PQR* a straight line parallel to it; *pqr* the geometrical image of *PQR*, as determined Art. 203; *O* the place of the eye. Then, the rays which flow from *P*, after refraction at the surface *AB*, are diffused through all parts of the medium in which the eye is placed; and if those rays of the pencil, which pass through *O*, diverge accurately from *p*, they enter the eye as if they came from a real object there; and *p* is the *visible* image of *P*; but, when the eye is at any considerable distance from the perpendicular *PAZ*, the point *P* is seen by an oblique portion

* The case of objects seen by rays reflected at any plane surfaces is excepted.

of the general pencil, as PVO. Those oblique rays do not diverge accurately, from *any* point; and therefore the visible image will be indistinct. But, if x be the place where, if produced backwards, they occupy the smallest space, we may suppose this to be the visible image of P; and the curve which is the locus of x, to be the visible image of the whole line PR*. Again, if a distorted image be formed by the object glass, or reflector of a telescope, different parts of it lie at different distances from the principal focus of the eye glass; and, if one part can be seen distinctly, the rest will appear confused. Instead of spherical surfaces, it has been proposed to adopt such as are generated by the revolution of the ellipse, parabola, or hyperbola; but, independent of the difficulty of grinding these surfaces, little advantage can be expected from them, as each surface, will only reflect, or refract, those rays accurately, which belong to one particular focus, and the aberrations, in other cases, will generally be greater than those produced by such surfaces as are of a spherical form.

2. Another, and more considerable cause of imperfection in optical instruments, is the unequal refrangibility of differently coloured rays. We have, in all our calculations, supposed light to be homogeneal;

---

* There is considerable difficulty in determining the *visible* image of an object, when seen by reflected, or refracted rays. In the preceding case, if we suppose the impression to be made by those rays which are incident in the plane perpendicular to the refracting surface, the equation to the curve which is the locus of x, rises to eight dimensions. If we suppose the impression to be made by those rays which are equally inclined to the refracting surface, the equation rises to four dimensions. See Newt. *Lect. Opt.* p. 1, Prop. viii.

that, whilst it passes out of one given medium into another, the sine of incidence bears the same invariable ratio to the sine of refraction. But, Sir Isaac Newton discovered that the common light by which objects are viewed, consists of rays which differ both in *colour* and *refrangibility*; and, that those rays which differ in colour, always differ in refrangibility; that is, if the sines of incidence be equal, the sines of refraction are different, though the mediums remain the same. Hence it follows, that if the image of an object be distinctly formed by the red, which are the least refrangible rays, at one particular distance from a refractor, a distinct image will be formed by rays which have a different degree of refrangibility, as the blue rays, at a different distance from it; thus, the rays of different colours, which flow from the same point, being collected at different distances from the refractors, a confused, and coloured image of that point, is necessarily produced upon the retina; or upon any screen which receives the refracted rays.

We are now to consider, in what manner these imperfections may, in some degree, be remedied. And we shall begin with the latter, which is of greater importance, as the errors it produces are much more considerable than those which arise from the spherical form of the surfaces; we may add moreover, that it's theory is more easily explained, and it's effects more likely to be corrected in practice.

Sect. VIII.

ON THE ABERRATIONS PRODUCED BY THE UNEQUAL REFRANGIBILITY OF THE RAYS OF LIGHT; AND BY THE SPHERICAL FORM OF REFLECTING AND REFRACTING SURFACES.

Prop. LXXX.

Art. (373.) *THE sun's light consists of rays which differ in refrangibility and colour.*

This important discovery was made by Sir Isaac Newton, who describes the experiment by which it is established, in the following words *.

"In a very dark chamber, at a round hole, about one third part of an inch broad, made in the shut of a window, I placed a glass prism, whereby the beam of the sun's light which came in at the hole, might be refracted upwards toward the opposite wall of the chamber, and there form a coloured image of the sun. The axis of the prism was perpendicular to the incident

---

* *Optics*, B. I. P. I. Prop. ii.

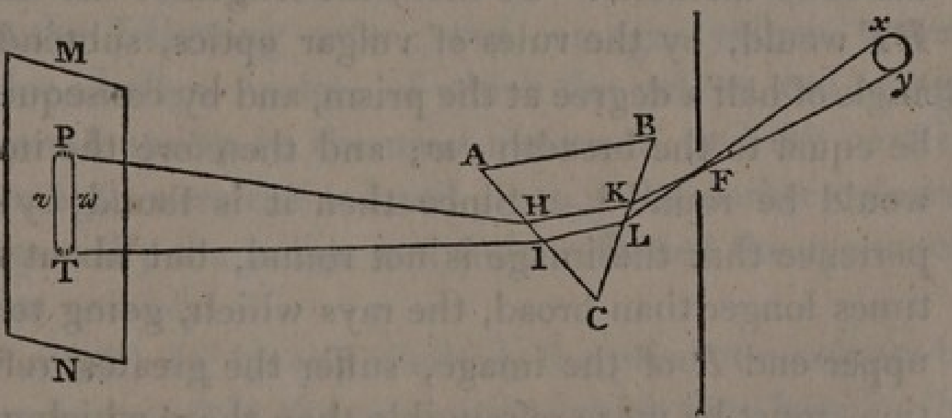
rays. About this axis I turned the prism slowly, and saw the refracted light on the wall, or coloured image of the sun, first to descend, and then to ascend. Between the descent and ascent, when the image seemed stationary, I stopped the prism, and fixed it in that posture, that it should be moved no more. For in that posture, the refractions of the light at the two sides of the refracting angle, that is, at the entrance of the rays into the prism, and at their going out of it, were equal to one another*. The prism therefore being placed in this posture, I let the refracted light fall perpendicularly upon a sheet of white paper at the opposite wall of the chamber, and observed the figure and dimensions of the solar image formed on the paper by that light. This image was oblong and not oval, but terminated with two rectilinear and parallel sides, and two semicircular ends. On it's sides it was bounded pretty distinctly, but on it's ends very confusedly and indistinctly, the light there decaying and vanishing by degrees. The breadth of this image answered to the sun's diameter, and was about two inches and the eighth part of an inch, including the penumbra. For the image was eighteen feet and an half distant from the prism, and at this distance, that breadth, if diminished by the diameter of the hole in the window-shut, that is by a quarter of an inch, subtended an angle at the prism of about half a degree, which is the sun's apparent diameter. But the length of the image was about ten inches and a quarter, and the length of the rectilinear sides about eight inches; and the refracting angle of the prism,

---

* *Lect. Opt.* P. I. Prop. xxv.

whereby so great a length was made, was 64 degrees. With a less angle the length of the image was less, the breadth remaining the same. It is farther to be observed, that the rays went on in right lines from the prism to the image; and therefore, at their very going out of the prism, had all that inclination to one another from which the length of the image proceeded, that is the inclination of more than two degrees and an half. And yet, according to the laws of optics vulgarly received, they could not possibly be so much inclined to one another.

"For let *F* represent the hole made in the window-shut, through which a beam of the sun's light was transmitted into the darkened chamber, and *ABC* a



triangular imaginary plane, whereby the prism is feigned to be cut transversely through the middle of the light; and let xy be the sun, *MN* the paper upon which the solar image or spectrum is cast, and *PT* the image itself; whose sides, towards v and w, are rectilinear and parallel, and ends, toward, *P* and *T*, semicircular. $yKHP$, and $xLIT$ are two rays; the first of which comes from the lower part of the sun to the higher part of the image, and is refracted in the prism at *K* and *H*; and the latter comes from the higher part of the sun to the lower part of the image, and is refracted at *L* and *I*. Since the refractions on both sides the prism are equal to one another, that is the refraction at *K* equal to the refraction at *I*, and the refraction at *L* equal to the refraction at *H*, so that the refractions of the incident rays at *K* and *L* taken together, are equal to the refractions of the emergent rays at *H* and *I* taken together; it follows, by adding equal things to equal things, that the refractions at *K* and *H* taken together, are equal to the refractions at *I* and *L* taken together; and therefore the two rays, being equally refracted, have the same inclination to one another after refraction, which they had before; that is, the inclination of half a degree, answering to the sun's diameter. So then, the length of the image *PT* would, by the rules of vulgar optics, subtend an angle of half a degree at the prism, and by consequence be equal to the breadth *vw*; and therefore the image would be round*. Since then it is found by experience that the image is not round, but about five times longer than broad, the rays which, going to the upper end *P* of the image, suffer the greatest refraction, must be more refrangible than those which go to the lower end *T*.

"The image or spectrum *PT* was coloured, being red at it's least refracted end *T*, and violet at it's most refracted end *P*, and yellow, green and blue in the intermediate spaces."

(374.) To shew that the unequal refrangibility of the rays in this experiment, is not accidental, or owing

---

* See *Lect. Opt.* P. I. Sect. iv, &c.

to any new modification produced by the medium through which they pass, Sir Isaac Newton refracted the rays of each colour separately, and found that they ever after retained both their colour and peculiar degree of refrangibility *.

(375.) Nearly in the same manner, it may be shewn that common day-light consists of rays which differ in colour and refrangibility.

For, if the round hole in the shutter receive only light from the clouds, and the eye be applied to the prism, the image is observed to be oblong, and coloured as in the former case.

(376.) Sir Isaac Newton, with the assistance of a person who had a more critical eye than himself, distinguished the spectrum into seven principal colours, proceeding from the less to the more refrangible rays, in the following order; red, orange, yellow, green, blue, indigo, violet; of which the yellow and orange were found to be the most luminous, and the next in strength were the red and green; the darker colours, especially the indigo and violet, affected the eye much less sensibly.

(377.) If, by any method, the prismatic colours be again united in the proportion which they have in the spectrum, they compound a white sun light; and by the mixture of different sorts of rays, in different proportions, various colours are produced, according to the quantity and nature of the rays united.

Thus, a mixture of red and yellow produces an orange; yellow and blue form a green †, &c.

---

* Newt. *Optics*, Part I. Exp. 6.

† *Ibid*, Part I. Prop. iv.

(378.) From the former part of the last article we may conclude, that if a ray of white light be refracted through a medium contained by parallel planes whose distance is inconsiderable, it will not, as to sense, be separated into distinct colours.

For, the ray of each particular colour emerges parallel to the incident white ray; consequently, the emergent rays of different colours are parallel to each other; and since the thickness of the medium is inconsiderable, they emerge nearly at the same point, and therefore excite only the sensation of whiteness.

Thus it happens, that objects seen through common window glass do not appear coloured *.

(379.) The same may be said, if the emergent rays, after several refractions, be parallel to the incident white ray, and the points of emergence nearly coincide.

Prop. LXXXI.

(380.) *The more refrangible rays are more reflexible.*

A ray of light cannot, consistently with the general law of refraction, pass out of a denser medium into a rarer when the sine of incidence exceeds the limit determined by this proportion, sin. refraction : sin. incidence :: radius : sin. incidence, which is the limit sought (Art. 101); therefore, the greater the ratio of the sine of refraction to the sine of incidence, the less

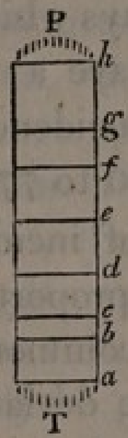
* Newt. *Lect. Opt.* P. II. Sect. iv.

will this limit be; and, consequently, the sooner will the rays be reflected *.

Prop. LXXXII.

(381.) *If a small cylindrical beam of white light pass nearly perpendicularly out of common glass into air, the dispersion of the differently coloured rays is about* $\frac{2}{55}$ *of the mean refraction.*

Let a small beam of the sun's light be refracted by a glass prism, in the manner described (Art. 373); and let *PT* represent the spectrum, divided by lines



which are perpendicular to it's parallel sides, and drawn through the confines of the several colours. Also, let *ab*, *bc*, *cd*, *de*, *ef*, *fg*, *gh*, be the spaces occupied by the red, orange, yellow, green, blue, indigo and violet rays, respectively; then if the whole length *ah* be represented by unity, *ab* is found to be $\frac{1}{8}$; $ac = \frac{1}{5}$; $ad = \frac{1}{3}$; $ae = \frac{1}{2}$; $af = \frac{2}{3}$; $ag = \frac{7}{9}$; and these are nearly

* This proposition may also be proved by experiment. Newt. *Optics*, B. I. P. I. Prop. iii.

proportional to the differences of the sines of refraction of the differently coloured rays, to a common sine of incidence.

Now, when the rays pass out of glass into air, if the common sine of incidence be represented by 50, the sines of refraction of the extreme red and violet rays are found to be 77 and 78 respectively*; therefore the sines of refraction of the other rays, are $77\frac{1}{8}$, $77\frac{1}{5}$, $77\frac{1}{3}$, $77\frac{1}{2}$, $77\frac{2}{3}$, $77\frac{7}{9}$. That is, the sine of incidence of any red ray, is to the sine of refraction, in a ratio not greater than that of 50 : 77, nor less than that of 50 : $77\frac{1}{8}$; but varying, in different shades of red, through all the intermediate ratios. In the same manner, the sines of refraction of all the orange rays extend from $77\frac{1}{8}$ to $77\frac{1}{5}$, &c. the rays which are in the confines of the green and blue, have a mean degree of refrangibility, and the sine of incidence of these rays, is to the sine of refraction, as 50 to $77\frac{1}{2}$.

When the angles of incidence and refraction are small, they are nearly proportional to their sines; and consequently, if the common angle of incidence be represented by 50, the deviation of the violet rays is $78-50$, or 28; the deviation of the red rays is $77-50$, or 27; therefore the difference of these, or the angle through which the rays of different colours are dispersed, is $\frac{1}{27}$ of the deviation of the red rays, $\frac{1}{28}$ of the deviation of the violet rays, and $\frac{1}{27\frac{1}{2}}$, or $\frac{2}{55}$ of the deviation of the rays of mean refrangibility, from their original course.

* Newt. *Optics*, P. I. Prop. vii.

(382.) The angle through which all the red rays are dispersed is $\frac{1}{8}$ of $\frac{2}{55}$, or $\frac{1}{220}$ of the mean refraction, &c.

(383.) In general, if the sines of refraction of the red and violet rays, in their passage out of any given medium into air, be $1+m$ and $1+n$, to the common sine of incidence 1, then, when the angles of incidence and refraction are small, the dispersion of the rays is an $\frac{n-m}{m}$th part of the refraction of the red rays; and since m and n are invariable, this expression may be properly taken as the measure of the *dispersing* power of the medium.

(384.) Whilst the refracting mediums are the same, a given refraction of the mean rays is always attended with the same dispersion, which may be destroyed by an equal refraction in the opposite direction (Art. 29). But if the latter refraction fall short of the former, the dispersion will not be wholly corrected; if it exceed the former, the dispersion will be the contrary way; that is, the order of the colours will be changed; and no refraction can finally be produced by mediums of the same kind, without colour.

(385.) Mr. Dolland, an eminent optician in London, discovered*, about the year 1757, that different substances have different dispersing powers; that the same dispersion may be produced, or corrected, by a less refraction of the mean rays in one case, than in another; and thus refraction may, upon the whole, be produced without colour.

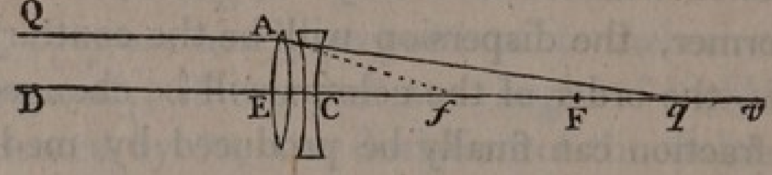
---

* The discovery has been ascribed to others; Mr. Dolland was the first who made it public.

Prop. LXXXIII.

(386.) *Having given the refracting powers of two mediums, to find the ratio of the focal lengths of a convex and concave lens, formed of these substances, which, when united, produce images nearly free from colour.*

Let $1+m:1$, and $1+n:1$ be the ratios of the sines of incidence and refraction of the red and violet rays out of air into the convex lens; $1+p:1$, and $1+q:1$, the ratios of those sines, out of air into the concave lens; F and f the focal lengths of the lenses, for red rays. Then $\frac{1}{m}:\frac{1}{n}::F:$ the focal length of the convex lens for violet rays (Art. 173); therefore,



the focal length of the convex lens for violet rays is $\frac{mF}{n}$; in the same manner it appears, that $\frac{pf}{q}$ is the focal length of the concave lens for violet rays. Let AC be the compound lens; Eq it's focal length for red rays; Ev it's focal length for violet rays. Then $f-F:f::F:Eq$; and $\frac{pf}{q}-\frac{mF}{n}:\frac{pf}{q}::\frac{mF}{n}:Ev$ (Art. 192); hence $Eq=\frac{Ff}{f-F}$, and $Ev=\frac{mpFf}{npf-mqF}$; and when $Ev=Eq$, the red and violet rays, after both refractions, are collected at q, or v. In this case, $\frac{Ff}{f-F}=\frac{mpFf}{npf-mqF}$; or $npf-mqF=mpf-mpF$; whence $p.\overline{n-m}.f=m.\overline{q-p}.F$; and $F:f::$ $p.\overline{n-m}:m.\overline{q-p}::\frac{n-m}{m}:\frac{q-p}{p}$. That is, the focal lengths are proportional to the dispersing powers of the two mediums. If the intermediate rays be dispersed according to the same law by the two mediums, it is manifest that the focal length of the compound lens, for these colours, will be Eq or Ev; and thus the image of a distant object will be formed in q or v, free from colour (Art. 377).

When the rays of different colours proceed from a point at a finite distance from this compound lens, after refraction they will converge to, or diverge from a common focus. For, the distance of the focus of refracted rays of any colour from the lens, depends upon the focal length of the lens, and the distance of the focus of incident rays from it (Art. 184); and since the latter quantities, by the supposition, are the same for rays of all colours, the distance of the focus of refracted rays from the lens, is the same; and thus, the image of an object at any finite distance from the compound lens, will be free from colour.

(387.) Ex. In crown, or common glass, $1+m=1.54$; and $1+n=1.56$. In flint glass, $1+p=1.565$; and $1+q=1.595$ *; therefore the dis-

---

* The refracting and dispersing powers of different kinds of glass are exceedingly various; and the causes upon which these powers depend are but imperfectly understood. See Dr. Blair's experiments on this subject, in the *Edinburgh Transactions*, vol. iii.

persing power of common glass : the dispersing power of flint glass :: $\frac{.02}{.54} : \frac{.03}{.565}$ :: $2 \times 565 : 3 \times 540$. To form a compound lens of these substances which shall produce a *real* image of a distant object, nearly free from colour, the convex lens must have the greater refracting power; and therefore it must be made of common glass, which has the less dispersing power. In this case, $F : f :: 2 \times 565 : 3 \times 540 :: 7 : 10$, nearly.

The focal length of the compound lens, $\frac{Ff}{f-F} = \frac{10F}{3}$.

(388.) Cor. 1. If the greater refraction be produced by the concave lens, it's focal length : the focal length of the convex lens :: 7 : 10, nearly; and the refracting power of the compound lens corresponds to that of a single concave glass.

(389.) Cor. 2. It is found by experience, that the extreme and intermediate rays are not dispersed by crown and flint glass, according to the same law; therefore, though the red and violet rays are united by the compound lens above described, yet the intermediate rays are not collected at the same point; and consequently, the images formed are not entirely free from colour.

The discovery of two sorts of glass, which shall disperse the extreme and intermediate rays in the same proportion, is still a desideratum in optics.

To form the most distinct image, the lenses ought to be so adjusted as to collect the brightest, and strongest colours, the yellow and orange.

(390.) Cor. 3. By a method similar to that employed in the proposition, two *compound* lenses, which collect the extreme rays, but disperse the intermediate rays in different proportions, might be so adjusted as to collect rays of three different colours, exactly; but the advantage thus gained, would probably not compensate for the loss of light.

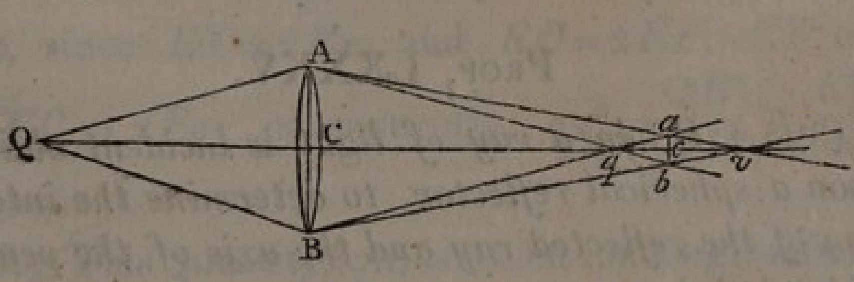
(391.) Cor. 4. Instead of a single convex lens, two are frequently employed, one on each side of the concave lens, which, when combined, have the same focal length with the single lens for which they are substituted. This construction lessens the aberration arising from the spherical form of the refracting surfaces.

Prop. LXXXIV.

(392.) *Having given the aperture of any lens, single or compound, and the foci to which rays of different colours, belonging to the same pencil, converge, to find the least circle of aberration through which these rays pass.*

Let QCv be the axis of the lens; AB or $2\,AC$ it's linear aperture; q and v the foci of differently coloured rays. Draw Av, Bv; Aqb, Bqa; join a, b the points of their intersection, and let ab cut the axis QCv, in c.

Then, in the similar and equal triangles ACv, BCv, $Av = Bv$; and the $\angle AvC =$ the $\angle CvB$. In the



same manner, the $\angle AqC =$ the $\angle BqC$, or the $\angle bqc =$

the $\angle aqc$; therefore, in the triangles aqv, bqv, the angles avq, aqv are respectively equal to the angles bvq, bqv, and qv is common to both triangles, consequently aq is equal to bq. Hence it follows, that the triangles AqB, aqb, as also the triangles AqC, bqc, are similar, and that ab is perpendicular to Qcv; therefore ab is the diameter of the least circle of aberration, into which the rays converging to q and v are collected.

Now, from the similar triangles AqB, aqb, $AB : ab :: Aq : qb$; and from the similar triangles ACq, bqc, $Aq : bq :: Cq : cq$; therefore $AB : ab :: Cq : cq$. In the same manner, $AB : ab :: Cv : cv$; therefore $AB : ab :: Cv+Cq : cv+cq$ $(Cv-Cq)$.

(393.) Cor. 1. When the ratio of Cv to Cq is given, ab varies as AB; and the area of the least circle of aberration varies as AB^2.

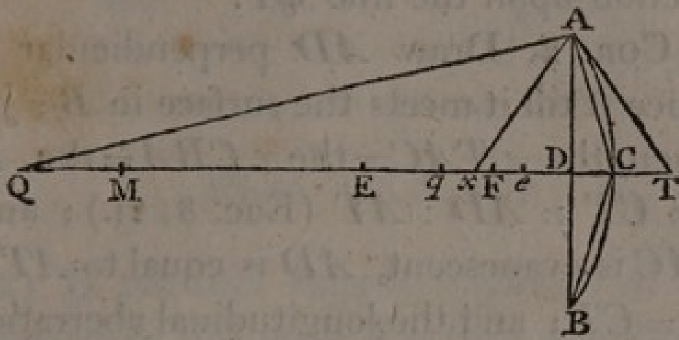
(394.) Cor. 2. Let parallel rays fall upon a single lens of crown glass, to compare the linear aperture of the lens, with the diameter of the least circle into which all the rays, of different colours, are collected.

Here $1+m : 1 :: 1.54 : 1$; and $1+n : 1 :: 1.56 : 1$ (Art. 387); and $Cv : Cq :: 56 : 54$ (Art. 173); therefore, $Cv+Cq : Cv-Cq :: 110 : 2 :: 55 : 1$; that is, $AB : ab :: 55 : 1$; or the diameter of the least circle of aberration into which the extreme rays, and consequently all the intermediate rays, are collected, is $\frac{1}{55}$ part of the linear aperture.

Prop. LXXXV.

(395.) *When a ray of light is incident obliquely upon a spherical reflector, to determine the intersection of the reflected ray and the axis of the pencil to which it belongs.*

Let ACB be the spherical reflector; E it's center; QA a ray incident obliquely upon it; QET the axis



of the pencil to which the ray belongs. Draw AT, touching the arc ACB in A, and let it meet QT in T; bisect ET, EC, in e and F; take q the geometrical focus, conjugate to Q; and let QA be reflected in the direction Ax; also, let Fe be small when compared with QF.

Then, $QF : FE :: QE : Eq$;

and $QF \pm Fe : FE + Fe :: QE : Ex$ (Art. 54);

therefore $Eq = \frac{QE \times FE}{QF}$; and $Ex = \frac{QE \times \overline{FE + Fe}}{QF \pm Fe}$,

whence, by actual division, $Ex = \frac{QE \times FE}{QF} + \frac{QE^2 \times Fe}{QF \times \overline{QF \pm Fe}} = \frac{QE \times FE}{QF} + \frac{QE^2}{QF^2} \times Fe$, nearly; therefore $Ex - Eq = qx = \frac{QE^2}{QF^2} \times Fe$, nearly.

Also, since $ET = 2Ee$, and $EC = 2EF$, CT or $ET - EC = 2Fe$; consequently, $qx = \frac{QE^2}{QF^2} \times \frac{CT}{2}$ nearly.

(396.) This quantity, qx, is called the *longitudinal* aberration of the oblique ray QA.

(397.) Cor. 1. Since the expression $\frac{QE^2}{QF^2} \times Fe$ has always the same sign, qx is always measured in the same direction upon the line QT.

(398.) Cor. 2. Draw AD perpendicular to QC, and produce it till it meets the surface in B; join AC, CB. Then, the $\angle TAC$ = the $\angle CBA$ = the $\angle DAC$, and $CD : CT :: AD : AT$ (Euc. 3. vi.); and when the arc AC is evanescent, AD is equal to AT; therefore, $CD = CT$; and the longitudinal aberration $qx = \frac{QE^2}{QF^2} \times \frac{CD}{2}$.

(399.) Cor. 3. When QA is parallel to QC, QE becomes equal to QF, and $qx = \frac{CD}{2}$.

(400.) Cor. 4. If Q, when in FE, or in FE produced, approach to E, the ratio of QE to QF decreases; and therefore, if CD be given, the aberration decreases. If Q be in FC, or FC produced, as QF decreases, the aberration increases.

(401.) Cor. 5. If the distances QE and QF be invariable, the aberration varies as CD.

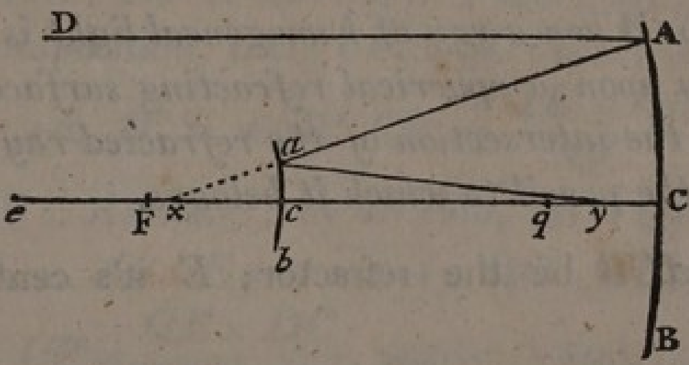
(402.) Cor. 6. Let CM be the diameter of the reflector; then, by the property of the circle, $CD : DA :: DA : DM$, and $CD = \frac{DA^2}{DM}$; therefore, when CD is very small, or DM nearly equal to the diameter of the given reflector, CD varies as DA^2, nearly; and consequently, when QE, QF are given, and the arc AC is very small, the longitudinal aberration varies as DA^2.

(403.) Cor. 7. When parallel rays are incident upon the reflector, the longitudinal aberration is ultimately equal to $\frac{CD}{2}=\frac{DA^2}{2DM}=\frac{DA^2}{4EC}=\frac{DA^2}{8EF}$; and therefore it varies as $\frac{DA^2}{EF}$.

Prop. LXXXVI.

(404.) *If parallel rays be reflected at a concave, and afterwards fall upon a convex spherical reflector, converging to a point between it's surface and principal focus, as in* Cassegrain's *telescope, the aberration of the lateral rays produced by the first reflection, will, in some measure, be corrected by the latter.*

Let ecC be the axis of the telescope; x the intersection of the axis and lateral ray after the first reflec-



tion; y their intersection after the second reflection; q the geometrical focus after both reflections.

The place of y, with respect to q, will be affected by two causes: 1st, x is nearer to c than F, the geometrical focus after the first reflection (Art. 397); and therefore, on this account, yc is less than qc (Art. 59); 2dly, in consequence of the aberration arising from the form of the reflector acb, yc is greater than

qc (Art. 397); therefore these causes counteract each other; and, by a proper adjustment of the reflectors, the aberration qy may, in a great measure, be destroyed.

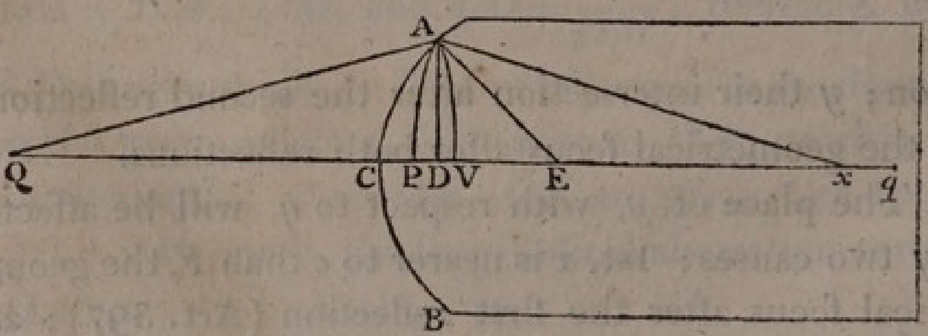
(405.) Cor. 1. Though the aberration qy, of the extreme ray DA, should be wholly destroyed, the aberration of the intermediate rays will not be entirely corrected; and this seems to be an insuperable obstacle to the perfection of reflecting telescopes.

(406.) Cor. 2. If the reflectors be both concave, as in Gregorie's telescope, the aberrations produced by the two reflections are in the same direction; that is, the second reflection increases the aberration produced by the first.

Prop. LXXXVII.

(407.) *When a ray of homogeneal light is incident obliquely upon a spherical refracting surface, to determine the intersection of the refracted ray and the axis of the pencil to which it belongs.*

Let ACB be the refractor; E it's center; QA



a ray incident obliquely upon it; QCq the axis of the pencil to which QA belongs; Ax the refracted ray;

q the geometrical focus, conjugate to Q. Draw AD perpendicular to the axis; and from the centers Q, x, with the radii QA, xA, describe the circular arcs AV, AP, cutting the axis in V and P. Take $m+1 : 1 ::$ sin. incidence : sin. refraction.

Then, in the triangle QAE, $QE : QA ::$ sin. incidence : sin. $\angle AEQ$; also, in the triangle AEx, $Ax : Ex ::$ sin. $\angle AEx$ (sin. $\angle AEQ$) : sin. refraction; therefore, by compounding these two proportions, $QE \times Ax : QA \times Ex ::$ sin. incidence : sin. refraction $:: m+1 : 1$; hence $\overline{m+1} \,.\, QA : QE :: Ax : Ex$; by division, $\overline{m+1} \,.\, QA - QE : QE :: Ax - Ex : Ex$*; that is, $\overline{m+1} \,.\, QV - QE : QE :: EP : Ex$; or $\overline{m+1} \,.\, QC + \overline{m+1} \,.\, CV - QE : QE :: EC - CP : Ex$; or $m \,.\, QC - EC + \overline{m+1} \,.\, CV : QE :: EC - CP : Ex$.

Now, $DC : DV :: QV : EC :: QC : EC$, nearly; and by composition, $DC : CV :: QC : QE$; therefore, when the arc AC is small, $CV = \frac{QE \times DC}{QC}$. Also $DC : DP :: Ax : EC$; by division, $DC : CP :: Ax : Ax - EC :: Ax : Ex :: \overline{m+1} \,.\, QC : QE$, nearly; therefore $CP = \frac{QE \times DC}{\overline{m+1} \,.\, QC}$, nearly. And, by substituting these values of CV and CP in the former proportion, we obtain $m \,.\, QC - EC + \frac{\overline{m+1} \,.\, QE \times DC}{QC}$:

* In this investigation of the aberration, diverging rays are supposed to fall upon a convex spherical surface of a denser medium, and to converge after refraction. The other cases may be derived from this, by a proper attention to the symbols.

$QE :: EC - \frac{QE \times DC}{\overline{m+1} . QC} : Ex$; hence, $Ex = QE \times$

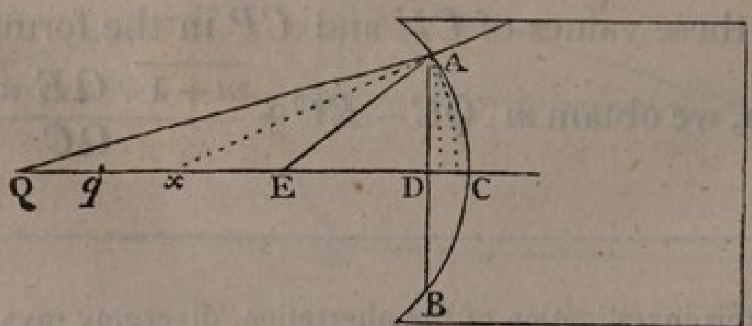
$$\frac{EC - \frac{QE \times DC}{\overline{m+1} . QC}}{m . QC - EC + \frac{\overline{m+1} . QE \times DC}{QC}}$$; and by actually

dividing, and taking the remainder, $Ex =$
$\frac{QE \times EC}{m . QC - EC} - \frac{m . QE^2}{\overline{m+1} . QC} \times \frac{QC + \overline{m+2} . EC}{\overline{m . QC - EC}|^2} \times DC$,
nearly. When DC vanishes, $Ex = Eq = \frac{QE \times EC}{m . QC - EC}$;
therefore the aberration $qx = \frac{m . QE^2}{\overline{m+1} . QC} \times$
$\frac{QC + \overline{m+2} . EC}{\overline{m . QC - EC}|^2} \times DC$.

(408.) Cor. 1. If the refractor be given, and the situation of the focus of incident rays, the aberration varies as DC, the versed sine of the arc AC.

(409.) Cor. 2. When the incident rays are parallel, QC becomes equal to QE; and $qx = \frac{DC}{m . \overline{m+1}}$.

(410.) Cor. 3. When diverging rays are incident upon a concave spherical refracting surface of a denser

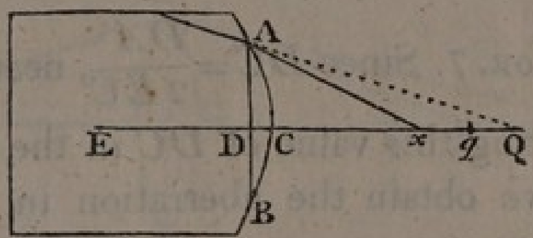


medium, the construction being made as before, EC

and DC are negative; whence, $Ex = -\frac{QE \times EC}{m.QC+EC} + \frac{m \cdot QE^2}{\overline{m+1} \cdot QC} \times \frac{QC - \overline{m+2} \cdot EC}{\overline{m \cdot QC+EC}|^2} \times DC$; and $qx = -\frac{m \cdot QE^2}{\overline{m+1} \cdot QC} \times \frac{QC - \overline{m+2} \cdot EC}{\overline{m \cdot QC+EC}|^2} \times DC$; this aberration, therefore, is to be measured in an opposite direction to the former.

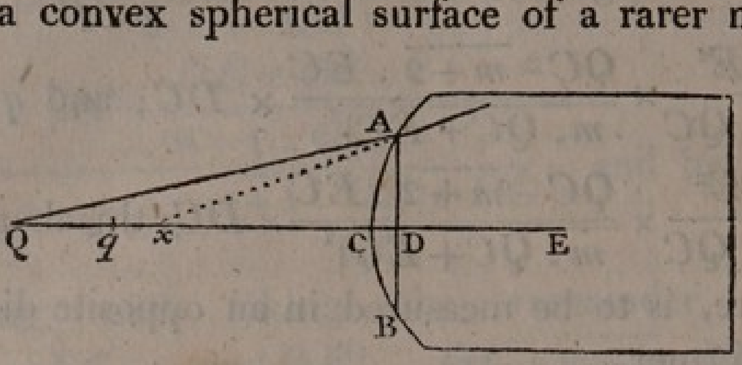
(411.) Cor. 4. In this last case, if $QC = \overline{m+2} \cdot EC$, the aberration vanishes; that is, if $QC : EC :: m+2 : 1$, or $QE : EC :: m+1 : 1$ :: sin. incidence : sin. refraction (Art. 145).

(412.) Cor. 5. When converging rays are incident upon a concave spherical surface of a rarer medium,



QC, QE, EC and DC are negative. Also, if $1-\mu : 1$ :: sin. incidence : sin. refraction, $-\mu$ must be substituted for m, and $Ex = \frac{QE \times EC}{\mu \cdot QC+EC} - \frac{\mu \cdot QE^2}{\overline{1-\mu} \cdot QC} \times \frac{QC + \overline{2-\mu} \cdot EC}{\overline{\mu \cdot QC+EC}|^2} \times DC$; hence, $qx = \frac{\mu \cdot QE^2}{\overline{1-\mu} \cdot QC} \times \frac{QC + \overline{2-\mu} \cdot EC}{\overline{\mu \cdot QC+EC}|^2} \times DC$.

(413.) Cor. 6. When diverging rays are incident upon a convex spherical surface of a rarer medium,



$$Ex = -\frac{QE \times EC}{\mu . QC + EC} + \frac{\mu . QE^2}{1-\mu . QC} \times \frac{QC + \overline{2-\mu} . EC}{\overline{\mu . QC + EC}|^2} \times$$

DC; therefore the aberration $qx = -\frac{\mu . QE^2}{1-\mu . QC} \times$

$$\frac{QC + \overline{2-\mu} . EC}{\overline{\mu . QC + EC}|^2} \times DC.$$

In the same manner, the aberration may be found in the other cases.

(414.) Cor. 7. Since $DC = \frac{DA^2}{2EC}$, nearly (Art. 402), by substituting this value of DC in the foregoing expressions, we obtain the aberration in terms of the semi-aperture.

(415.) Cor. 8. Since $Eq = \frac{QE \times EC}{m . QC - EC}$, if QE be diminished by the small quantity x, Eq will be increased by the quantity $\frac{\overline{m+1} . EC^2 \times x}{\overline{m . QC - EC}|^2}$. For on this supposition, Eq becomes $\frac{\overline{QE-x} . EC}{m . \overline{QC-x} - EC} =$

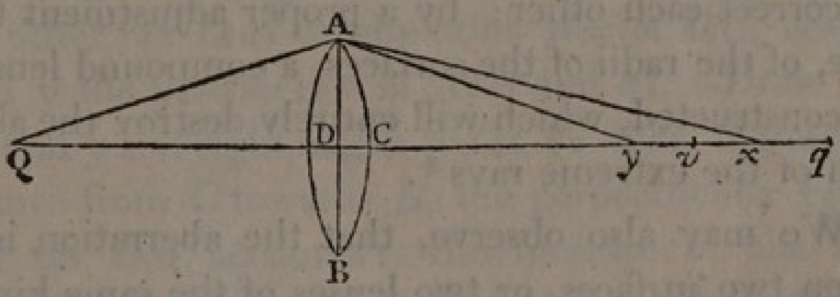
$$\frac{\overline{QE-x} . EC}{m . QC - EC - mx} = \frac{QE \times EC}{m . QC - EC} \times \frac{\overline{m+1} . EC^2 \times x}{\overline{m . QC - EC}|^2}$$

nearly; and therefore $\frac{\overline{m+1}\,.\,EC^2 \times x}{m\,.\,\overline{QC-EC}|^2}$, where x is the decrement of QE, is the increment of Eq, nearly.

(416.) Cor. 9. If x vary as DC, the increment of Eq, when the radius of the refractor and the situation of the focus of incident rays are given, will also vary as DC.

(417.) Cor. 10. By a proper application of the foregoing rules, the longitudinal aberration, arising from the spherical form of refracting surfaces may be found in all cases where the apertures are small.

Ex. Let Qq be the axis of a lens; Q the focus of incident rays; q the geometrical focus after the first refraction, determined by Art. 137; v the geometrical



focus of emergent rays (Art. 194). Also, let QA be refracted, at the first surface, in the direction Ax, and emergent in the direction Ay. Then, the aberration vy arises from two causes; 1st, x does not coincide with the geometrical focus q (Art. 407); and since v is determined on supposition that q is the focus of rays incident upon the second surface, an aberration will be produced, which may be determined by Art. 415. 2dly, Ax is incident obliquely upon the latter surface, and the aberration arising from this cause may be determined by Art. 412; therefore the whole aberration vy may be found.

(418.) Cor. 11. If the lens and place of the focus of incident rays be given, the aberration arising from each of these causes will vary nearly as AD^2 (Arts. 407. 416. 414); and therefore the final aberration vy, which is the sum or difference of the former, will also vary nearly as AD^2.

(419.) It is not consistent with the plan of this work to enter farther into these calculations; perhaps too much has been said already. The reader will find little difficulty in the application of the principles, if he wish to deduce practical rules for the construction of object glasses. Thus much it may be proper to observe, that the aberrations produced by a convex and concave lens are of contrary affections, and tend to correct each other; by a proper adjustment therefore, of the radii of the surfaces, a compound lens may be constructed, which will entirely destroy the aberration of the extreme rays*.

We may also observe, that the aberration is less, when two surfaces, or two lenses of the same kind are employed, than when the *same refraction* is produced by a single surface, or lens of the same description, and equal aperture.

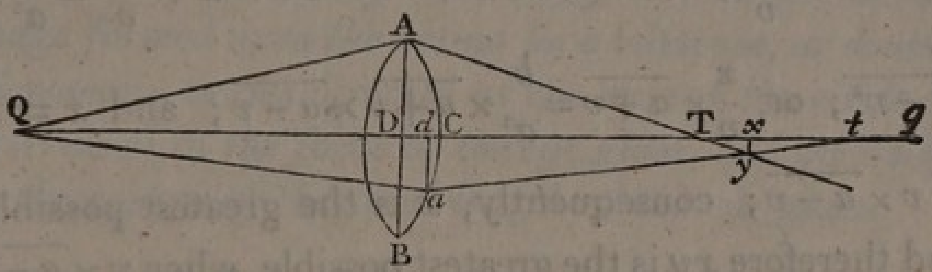
---

* This subject is treated with great ability in the Encyclopædia Britannica, under the head Telescopes.

Prop. LXXXVIII.

(420.) *To find the least circle of aberration into which all the homogeneal rays of the same pencil, refracted by a lens or single surface, are collected.*

Let AB be the refractor; QCq it's axis; Q the focus of incident rays; T the intersection of the ex-



treme ray QAT, and the axis; t the intersection of any other ray Qat on the other side of QC, and the axis; y the intersection of ATy and at. Draw AD, ad, and yx at right angles to QCq; then, if the point a move from C towards B, the perpendicular xy will vary on two accounts: the increase of the angle Cta, and the decrease of the distance Tt; and when xy is a maximum, all the rays incident upon the same side of QC with Qa, will pass through it; and if the figure revolve about the axis Qq; all the rays incident upon the lens will pass through the circle generated by xy. It is also manifest, that the circle thus generated, is less than any other circle through which all the refracted rays pass. To find when xy is the greatest possible, let $Tx=x$; $ad=v$; $AD=a$; $DT=f$; $Tq=b$. Then, since AD, ad, when the lens is thin, are the semi-apertures through which the rays QAT, Qat pass, $AD^2 : ad^2 :: qT : qt$ (Art.

418); or, $a^2 : v^2 :: b : qt$; whence $qt = \frac{bv^2}{a^2}$; therefore $Tq - qt = Tt = b - \frac{bv^2}{a^2} = \frac{b}{a^2} \times \overline{a^2 - v^2}$. Again, $DT : DA :: Tx : xy$; or, $f : a :: x : xy$; consequently, $xy = \frac{ax}{f}$; also, $da : dt :: xy : tx$; or, $v : f :: \frac{ax}{f} : tx$; therefore $tx = \frac{ax}{v}$; hence, $Tx + xt = Tt = x + \frac{ax}{v} = \frac{b}{a^2} \times \overline{a^2 - v^2}$; or, $\frac{x}{v} \times \overline{a + v} = \frac{b}{a^2} \times \overline{a+v} \times \overline{a - v}$; and $x = \frac{b}{a^2} \times v \times \overline{a - v}$; consequently, x is the greatest possible, and therefore xy is the greatest possible, when $v \times \overline{a - v}$ is the greatest possible; or when $v = \frac{1}{2}a$. Hence it follows, that the greatest value of x is $\frac{b}{4}$; and the corresponding value of $xy = \frac{ab}{4f} = \frac{DA \times qT}{4\,DT}$.

(421.) Cor. 1. If the focal length of the refractor, and the focus of incidence, be given, DT is given, and $xy \propto qT \times DA \propto DA^3$ (Art. 418).

(422.) Cor. 2. On the same supposition, the area of the least circle of aberration varies as DA^6.

(423.) Cor. 3. Exactly in the same manner, we may find the least circle into which a pencil of rays, reflected by a spherical surface, is collected.

(424.) Cor. 4. When parallel rays are incident upon a spherical reflector, the longitudinal aberration varies directly as the square of the semi-aperture, and inversely as the focal length (Art. 403); therefore, xy, the radius of the least circle of aberration, varies directly as the cube of the semi-aperture, and inversely as the square of the focal length of the reflector.

Prop. LXXXIX.

(425.) *The area of a circle of aberration in the image formed upon the retina by a telescope, or double microscope, varies directly as the area of the circle of aberration in the focus of the eye glass, and inversely as the square of the focal length of the eye glass.*

When the circle of aberration is in the principal focus of the glass through which it is viewed, it's visual angle is equal to the angle which it subtends at the center of the glass; and therefore, the *linear magnitude* of the circle of aberration upon the retina, varies as this angle (Art. 264); that is, it varies directly as the linear magnitude of the circle of aberration in the principal focus of the glass, and inversely as the focal length of the glass; consequently, the *area* of the circle of aberration on the retina, varies directly as the area of the circle of aberration in the focus of the eye glass, and inversely as the square of the focal length of the eye glass.

(426.) Cor. 1. In a reflecting telescope of Sir Isaac Newton's construction, if F be the focal length of the reflector, A it's semi-aperture, f the focal length of the eye glass, the radius of the circle of aberration in it's principal focus, varies as $\frac{A^3}{F^2}$ (Art. 424); and therefore the area of this circle varies as $\frac{A^6}{F^4}$; consequently, the area of the circle of aberration on the retina, varies as $\frac{A^6}{F^4 f^2}$.

(427.) Cor. 2. The area of the circle of aberration on the retina, has usually been considered as a measure of the apparent indistinctness of vision. And, though it is manifest that indistinctness admits of no numerical representation*, yet if the circle of aberration be the same in two cases, *cæteris paribus*, the indistinctness will be the same; and if the circle of aberration be greater in one case than in another, the indistinctness will also, *cæteris paribus*, be greater. For, the rays which proceed from one point in the object, are diffused over the circle of aberration, and consequently they are mixed with the rays which belong to as many different foci as there are sensible points in that circle; therefore, the greater the area of the circle, the greater must be the confusion, or indistinctness arising from this dispersion of the rays.

Prop. XC.

(428.) *To find on what supposition a given distant object appears equally bright, and equally distinct, when viewed with different reflecting telescopes of Sir Isaac Newton's construction.*

The notation in Art. 426 being retained; since

---

* One degree of indistinctness can no more be said to be a multiple or part of another, than one degree of taste, or smell can be said to be the double, or half of another.

the brightness is given, $\frac{4A^2f^2}{F^2} \propto 1$ (Art. 370); or, $\frac{A^6f^6}{F^6} \propto 1$. Also, since the indistinctness is given $\frac{A^6}{F^4f^2} \propto 1$; therefore $\frac{A^6f^6}{F^6} \propto \frac{A^6}{F^4f^2}$; and $f^8 \propto F^2$; or, $f \propto F^{\frac{1}{4}}$.

Again, $\frac{A^4f^4}{F^4} \propto 1$; that is, $\frac{A^4F}{F^4} \propto 1$; or, $A^4 \propto F^3$; and $A \propto F^{\frac{3}{4}}$.

(429.) Cor. The magnifying power $\propto \frac{F}{f}$ (Art. 338); that is, as $F^{\frac{3}{4}}$.

Sect. IX.

ON THE RAINBOW.

Prop. XCI.

Art. (430.) *IF two quantities bear an invariable ratio to each other, their corresponding increments are in the same ratio.*

Let X and Y be the two quantities; x and y their corresponding increments. Then, by the supposition, $X : Y :: X+x : Y+y$; and alternately, $X : X+x :: Y : Y+y$; by division, $X : x :: Y : y$; therefore, alternately, $X : Y :: x : y$.

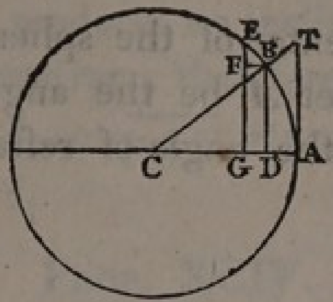
Cor. Hence the increment of nx is n times the increment of x.

Prop. XCII.

(431.) *If the sines of two arcs be always in a given ratio, the evanescent increments of the arcs are proportional to the tangents of those arcs.*

Let AB be a circular arc whose radius is CA, sine BD, and tangent AT; draw EG parallel, and indefinitely near to BD, BF parallel to DG, and join EB.

Then, the triangle *CBD* is similar to the triangle *EBF*, formed by *EF*, *FB*, and the chord *BE*; for, the angles *CDB*, *BFE* are right angles; and the



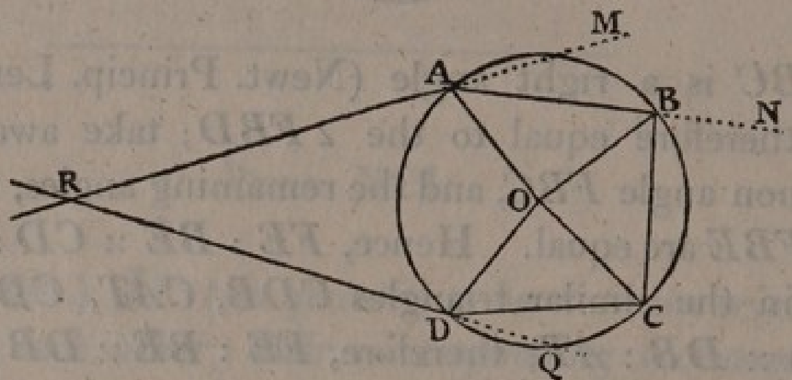
$\angle EBC$ is a right angle (Newt. Princip. Lem. 6), and therefore equal to the $\angle FBD$; take away the common angle *FBC*, and the remaining angles, *CBD* and *FBE* are equal. Hence, $FE : BE :: CD : CB$; and in the similar triangles *CDB*, *CAT*, $CD : CA$ $(CB) :: DB : AT$; therefore, $FE : BE :: DB : AT$; whence $BE = \frac{FE \times AT}{DB}$; and *BE* is ultimately equal to the increment of the arc *AB* (Newt. Princip. Lem. 7); consequently, *BE*, the increment of the arc, $= \frac{FE \times AT}{DB}$; and since *FE*, the increment of the sine, varies as *DB* the sine (Art. 430), *BE* varies as *AT*.

Prop. XCIII.

(432.) *If a ray of light refracted into a sphere, emerge from it after any given number of reflections, to determine the deviation of the ray, and the angle contained between the directions in which it is incident and emergent.*

Let a ray of light *RA*, incident upon the sphere *ABCD* at *A*, be refracted in the direction *AB*; at *B* let it be reflected in the direction BC; and at C, in the direction CD; at D let it be refracted out of the sphere, in the direction DR; produce RA, RD, to M, Q.

Take O the center of the sphere; join OA, OB, OC, OD; and let A be the angle of incidence of the ray RA; B the angle of refraction; R, a right angle.



Then the $\angle OAM = A$; the $\angle OAB =$ the $\angle OBA$ (Euc. 5. 1) $=$ the $\angle OBC$ (Art. 18) $=$ the $\angle OCB =$ the $\angle OCD =$ the $\angle ODC = B$. Also, the angles of deviation at A and D are equal; for if BA be supposed to be incident at A, the angle of incidence BAO, is equal to the angle of incidence CDO, of the ray CD; therefore the angles of deviation are equal*; and since the angle of deviation at A, is $A - B$, the whole deviation arising from the two refractions, is $2A - 2B$. Again, the angle of deviation at B is $2R - 2B$; and the angle of deviation, at every other reflection, is the same; therefore, if there be p reflections, the whole deviation, arising from this cause, is $2pR - 2pB$. To

---

* See Art. 25.

this, let the deviation arising from the refractions be added, and the whole deviation of the ray from it's original direction, is $2pR - 2 . \overline{p+1} . B + 2A$.

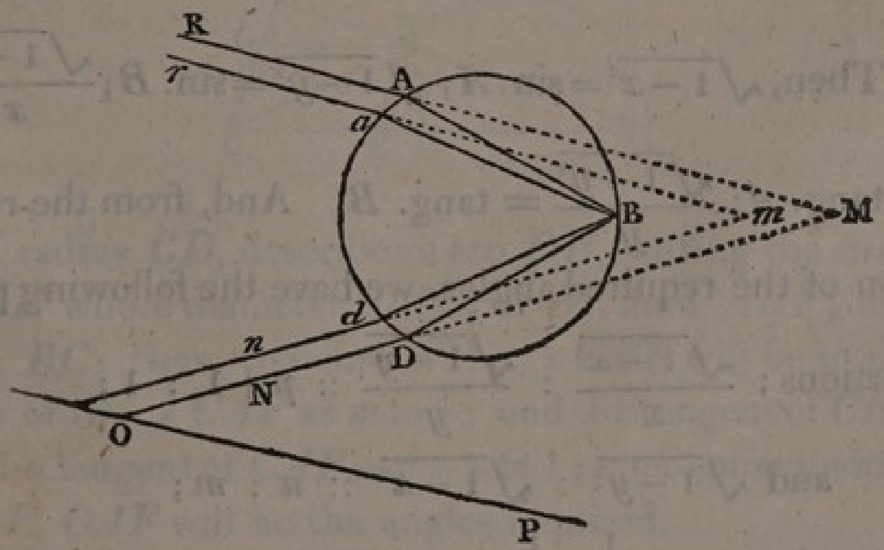
Also, a deviation through the angle $2pR$, which is a multiple of 180°, produces no inclination of the emergent to the incident ray; therefore, the inclination is represented by $2A - 2 . \overline{p+1} . B$; or $2 . \overline{p+1} . B - 2A$.

Prop. XCIV.

(433.) *If a small pencil of parallel homogeneal rays be refracted into a sphere, and the ratio of the sine of incidence to the sine of refraction be known, to find at what angle the rays must be incident, that they may emerge parallel after any given number of reflections within the sphere.*

Let RAM, ram, be the directions of the incident, DN, dn, the directions of the emergent rays; produce ND, nd, if necessary, till they meet RM, rm, in M and m.

Then, since AM, am, as also DN, dn, are parallel



by the supposition, the angles at M, and m are equal;

therefore, when the rays are incident at, or near to *A*, the angle *RMN*, contained between the incident and emergent ray, ceases to increase, or decrease; and therefore, the notation in the last article being retained, $2 . \overline{p+1} . B - 2A$, and consequently $\overline{p+1} . B - A$, ceases to increase, or decrease; that is, the increment of $\overline{p+1} . B$, is equal to the corresponding increment of *A*. Also, since sin. *A* is in a given ratio to sin. *B*, the increment of *B* : the increment of *A* :: tang. *B* : tang. *A* (Art. 431); or, multiplying the first and third terms by $p+1$, $\overline{p+1} \times$ increment of *B* : increment of $A :: \overline{p+1}$. tang. *B* : tang. *A*; and $\overline{p+1} \times$ increment of $B =$ increment of $\overline{p+1} . B$ (Art. 430); therefore, the increment of $\overline{p+1} . B$: the increment of *A* :: $\overline{p+1}$. tang. *B* : tang. *A*; and since the increment of $\overline{p+1} . B$ is equal to the increment of *A*, when the rays emerge parallel, $\overline{p+1}$. tang. $B =$ tang. *A*; or, tang. *A* : tang. *B* :: $p+1 : 1$.

To determine the angles *A* and *B*, suppose x and y to be their cosines, the radius being unity; and let sin. *A* : sin. *B* :: $m : n$.

Then, $\sqrt{1-x^2} =$ sin. *A*; $\sqrt{1-y^2} =$ sin. *B*; $\dfrac{\sqrt{1-x^2}}{x}$ $=$ tang. *A*; $\dfrac{\sqrt{1-y^2}}{y} =$ tang. *B*. And, from the relation of the required angles, we have the following proportions; $\dfrac{\sqrt{1-x^2}}{x} : \dfrac{\sqrt{1-y^2}}{y} :: p+1 : 1$;

and $\sqrt{1-y^2} : \sqrt{1-x^2} :: n : m$;

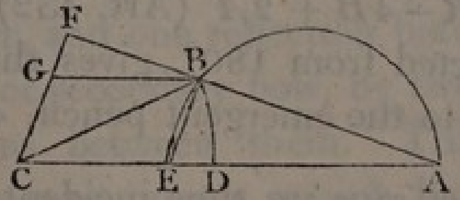
by composition, $\dfrac{1}{x} : \dfrac{1}{y} :: n . \overline{p+1} : m$; hence, $y =$

$\frac{\overline{p+1} \,.\, nx}{m}$; and $y^2 = \frac{\overline{p+1}|^2 \,.\, n^2x^2}{m^2}$; therefore, $1-y^2$: $1-x^2$ :: $1 - \frac{\overline{p+1}|^2 \,.\, n^2x^2}{m^2}$: $1-x^2$ :: $n^2 : m^2$; and by multiplying extremes and means, $n^2 - n^2x^2 = m^2 - \overline{p+1}|^2 \,.\, n^2x^2$; hence, $\overline{\overline{p+1}|^2 - 1} \,.\, n^2x^2 = m^2 - n^2$; or $\sqrt{p^2+2p} \,.\, nx = \sqrt{m^2-n^2}$; consequently, $1 : x$ :: $\sqrt{p^2+2p} \,.\, n : \sqrt{m^2-n^2}$. The cosine of A being determined by this proportion, the angle itself may be found from the tables.

Also, $m : n$ :: sin. A : sin. B; and the three first terms in the proportion being known, the fourth is known; that is, sin. B is known; and therefore the angle B may also be found from the tables.

The angles A and B may also be determined by the following construction:

In the straight line $CEDA$, take CA to CD as m to n, and CA to CE as $p+1$ to 1; with the center C



and radius CD, describe an arc DB, cutting the circle ABE whose diameter is AE, in B; draw ABF; and join BC; then, the sine of the $\angle\, CBF$ will be to the sine of the $\angle\, CAF$ as m to n; and the tangent of CBF to the tangent of CAF as $p+1$ to 1; and consequently CBF, CAF will be the angles required.

Join BE, and complete the parallelogram $CEBG$, produce CG till it meets ABF in F. Then, in the triangle CAB, sin. CBA (sin. CBF) : sin. CAB :: $CA : CB :: CA : CD :: m : n$. Again, since CF is parallel to EB, the $\angle BFG$, is equal to the $\angle EBA$, and is therefore a right angle; consequently, the lines FC, FG are tangents of the angles CBF, $GBF(CAF)$ to the radius BF; and, in the similar triangles FCA, FGB, $FC : FG :: CA : GB :: CA : CE :: p+1 : 1$.

(434.) Ex. 1. If a small pencil of parallel red rays be incident upon a sphere of water, at an angle of about 59°. 23′, and suffer two refractions and one reflection, the rays will emerge parallel.

Here, $p=1$; and $m : n :: 108 : 81 :: 4 : 3$; therefore, $1 : x :: \sqrt{27} : \sqrt{7}$; or $x=\sqrt{\frac{7}{27}}$; and the angle whose cosine, to the radius unity, is $\sqrt{\frac{7}{27}}$, is 59°. 23′, nearly.

The angle of refraction B, whose sine is to the sine of 59°. 23′ :: 3 : 4, is 40°. 12′. Hence, the whole deviation, $2R-4B+2A$ (Art. 432), is 137°. 58′; which subtracted from 180°, gives the inclination of the incident, to the emergent pencil, 42°. 2′.

When violet rays are thus incident and emergent, $m : n :: 109 : 81$, and in this case, $A=58°. 40′$; $B=39°. 24′$; hence, $2R-4B+2A$ is 139°. 44′, and the inclination of the emergent, to the incident pencil, 40°. 16′.

(435.) Ex. 2. If parallel red rays fall upon a sphere of water, they will emerge parallel, after two refractions and two intermediate reflections, when the angle of incidence is about 71°. 50′.

In this case, $p=2$; and $1 : x :: \sqrt{72} : \sqrt{7}$; therefore the cosine of the angle of incidence is $\sqrt{\frac{7}{72}}$, which corresponds to an angle of 71°. 50′, nearly.

Also, $B=45°. 27'$; and the whole deviation, $4R-6B+2A,=230°. 58'$; hence, the inclination of the emergent, to the incident pencil, which is the excess of the whole deviation above 180°, = 50°. 58′, nearly.

When violet rays are thus incident and emergent, $A=71°. 26'$; $B=44°.47'$; $4R-6B+2A=234°.10'$; and the inclination of the emergent, to the incident pencil = 54°. 10′, nearly.

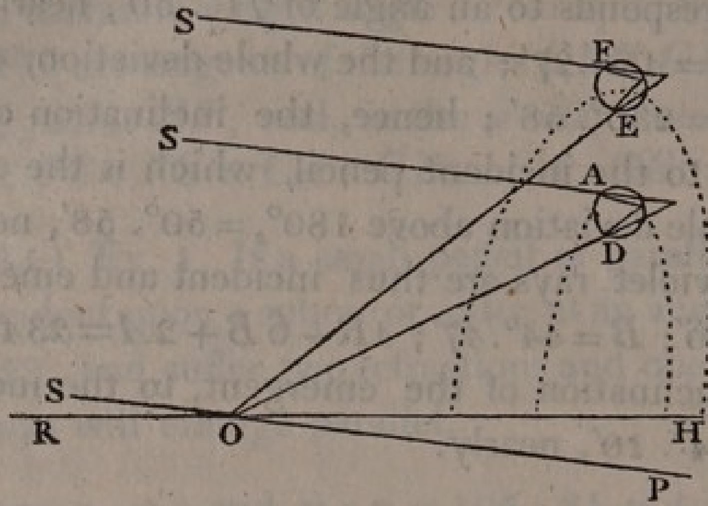
On the Formation of the Rainbow.

(436.) It has long been known that the rainbow is owing to the refraction and reflection of the sun's light by drops of rain. Antonius de Dominis first discovered that the interior, or primary bow, is caused by two refractions of the rays of light at each drop of water, and one reflection between them; and the exterior, or secondary bow, by two refractions and two reflections between them. This discovery he confirmed by experiments, which have been successfully repeated by more modern writers. If glass globes, filled with water, be placed in the sun's light, they may be elevated or depressed till they successively transmit to the eye, the colours of each bow, in their proper order*.

---

* Newton's *Optics*, Book I. Prop. ix.

(437.) To understand how the interior bow is formed, let *O* be the eye of a spectator; *SOP* a line passing through the eye and the sun. At the point



O, in the line *PO*, make the angle $POE = 42°. 2'$; then, when a drop of rain, *FE*, is in such a situation that the angle which *OE* makes with a perpendicular to it's surface at *E* is $59°. 23'$, a small pencil of parallel red rays will emerge from it at *E*, and enter the eye in the direction *EO*. For, if *OE* be considered as the incident pencil, it will emerge, after two refractions and one reflection, in the direction *FS*, which makes an angle of $137°. 58'$ with *OE* produced (Art. 434), or, an angle of $42°. 2'$ with *OE*, and is therefore parallel to *OS*; thus *FS* will pass through the sun*. Conversely, out of the beam of light proceeding from *S*, which falls upon the drop, the red rays incident at, and near to *F*, will, after two refractions and one reflection, emerge parallel, and entering the eye in the direction *EO* (Art. 29), will excite the sensation of their proper colour.

---

* The distance of the sun is so great, that two lines drawn from any points upon the surface of the earth, to a point in his disc, may be considered as parallel, in these calculations.

In the same manner, if *OE* revolve about the axis *OP*, every drop of water in the surface of the cone thus described, will transmit to the eye a small parallel pencil of red rays; and thus a red arc, whose radius, *measured by the angle which it subtends at the eye*, is 42°. 2′, will appear in the falling rain, opposite to the sun.

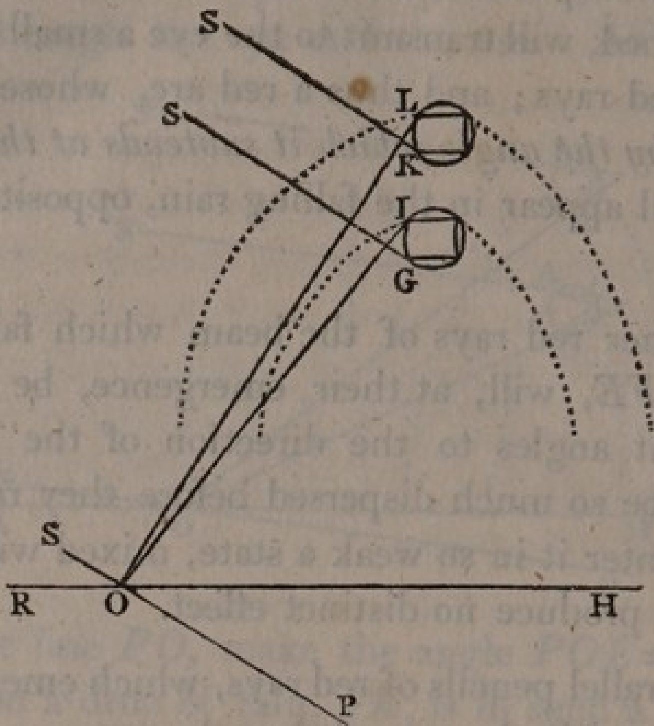
The other red rays of the beam which falls upon the drop *FE*, will, at their emergence, be inclined at different angles to the direction of the incident rays, and be so much dispersed before they reach the eye, and enter it in so weak a state, mixed with other rays, as to produce no distinct effect.

The parallel pencils of red rays, which emerge from other drops, fall above, or below the eye.

If the angle *POD* be 40°. 16′, and *OD* revolve about the axis *OP*, every drop of rain in the surface of the cone thus described, will transmit to the eye a parallel pencil of violet rays; and thus a violet arc will be formed, whose radius is 40°. 16′.

The drops between *E* and *D* will transmit to the eye parallel pencils of rays of different colours, orange, yellow, green, blue, indigo, in the order which they have in the prismatic spectrum (Art. 376); and the radii of the arcs of these respective colours may be calculated by the method employed in the 434th Article.

(438.) Again, let the angle *POI*=50°. 58′; and the angle *POL*=54°. 10′. Also, let *OI*, *OL* revolve about the axis *OP*. Then, it may be shewn as in the preceding case, that every drop of rain in the conical surface generated by *OI*, will transmit to the eye a small parallel pencil of red rays, which has suffered



two refractions and two reflections, but sufficiently strong to excite the sensation of it's proper colour.

Also, every drop in the conical surface generated by *OL* will transmit to the eye a small pencil of parallel violet rays; and the intermediate drops, parallel pencils of rays of the intermediate colours. Thus the exterior bow is formed, in which the radii of the red and violet arcs are, respectively, 50°. 58′, and 54°. 10′.

The radii of the intermediate arcs may be determined by the method employed in the 435th Article.

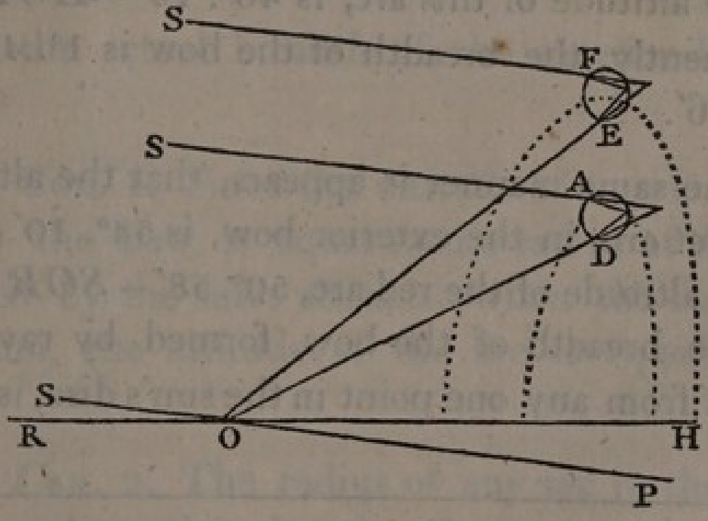
(439.) Cor. 1. The colours in the two bows lie in a contrary order; the red forming the exterior ring of the primary, and the interior ring of the secondary bow.

(440.) Cor. 2. Were the pencils sufficiently strong, a third bow, formed by two refractions and three reflections of the sun's rays in drops of rain, might be seen. But, when the rays which are refracted into a drop of water, reach the farther surface, some of them pass out of the drop, and others are reflected within it. When these reflected rays again meet the surface, some of them pass out of the drop, and others suffer another reflection; and so on*. Thus the pencil becomes weaker at every reflection; and at length it contains so few rays as not to make a distinct impression upon the retina.

Prop. XCV.

(441.) *To find the altitude of the highest point of the rainbow, above the horizon, and the breadth of the colours.*

The construction being made as in the 437th Article, through *O* draw *HOR* parallel to the horizon.



Then the angle *ROS*, or *HOP*, measures the altitude of the point *S* above the horizon; and the altitude

* This is a fact, the cause of which has not been satisfactorily explained. Sir Isaac Newton supposes that rays of light, when they arrive at the surface of a medium, are sometimes in a state to be reflected, and sometimes to be transmitted; these states he calls fits

of the highest point of the red arc above the horizon, in the primary bow, is measured by *EOH*, or *EOP* – *HOP*, which is equal to 42°. 2′ – *HOP*. Also, the altitude of the highest point of the violet arc is measured by *DOP* – *HOP*, or 40°. 16′ – *HOP*. Hence it follows, that the breadth of the bow, supposing it to be formed by the rays which come from one point *S*, in the sun's disc, is 42°. 2′—40°. 16′, or, 1°. 46′.

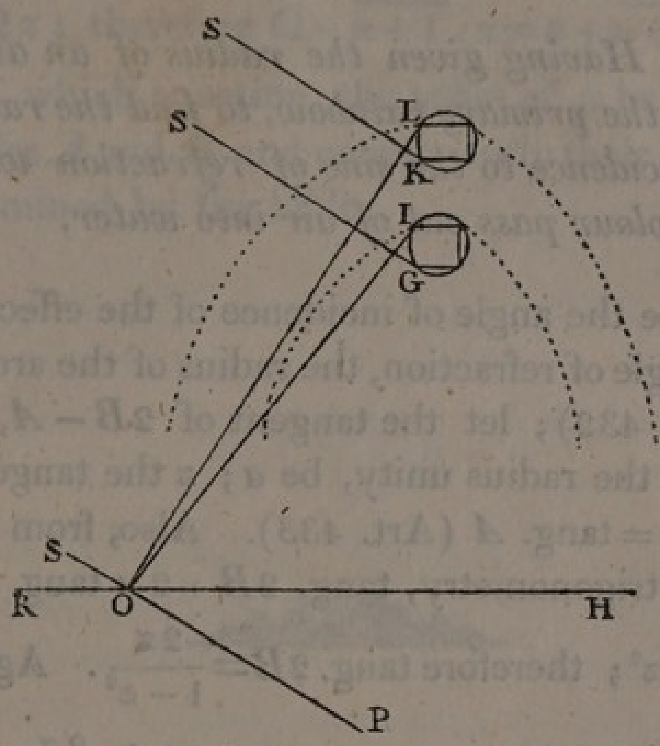
The breadth, thus determined, must be increased by 30′, the sun's apparent diameter; for, the highest red arc is produced by the rays which flow from the lowest point in the sun's disc, and if *ROS*, or *HOP*, measure the altitude of the sun's center, the altitude of the highest red arc is 42°. 2′ – *HOP* + 15′; also, the lowest violet arc is produced by the rays which flow from the highest point in the sun's disc, and therefore the altitude of this arc, is 40°. 16′ – *HOP* – 15′; consequently, the breadth of the bow is 1°. 46′ + 30′, or 2°. 16′.

In the same manner it appears, that the altitude of the violet arc, in the exterior bow, is 54°. 10′ – *SOR*; and the altitude of the red arc, 50°. 58′ – *SOR*; therefore, the breadth of the bow, formed by rays which proceed from any one point in the sun's disc, is 3°. 12′.

---

fits of easy reflection, and transmission; and accounts for them in the following manner: "Nothing more is requisite for putting the rays of light into fits of easy reflection, and easy transmission, than that they be small bodies, which by their attractive powers, or some other force, stir up vibrations in what they act upon; which vibrations being swifter than the rays, overtake them successively, and agitate them, so as by turns to increase and diminish their velocities, and thereby put them into those fits." *Opt.* Query 29.

If to this we add 30′, the sun's apparent diameter,



we have the actual breadth of the exterior bow = 3°. 42′.

(442.) Cor. 1. Since the altitude of an arc of any colour in the bow is equal to the radius of this arc diminished by the sun's altitude, when the sun is in the horizon, the altitude of the arc is equal to it's radius.

(443.) Cor. 2. The radius of any arc in the rainbow is equal to the altitude of the arc above the horizon, together with the sun's altitude.

(444.) Cor. 3. When the sun's altitude above the horizon, is equal to, or exceeds 42°. 2′, the primary bow cannot be seen; nor the secondary, when his altitude is equal to, or exceeds 54°. 10′.

Prop. XCVI.

(445.) *Having given the radius of an arc of any colour in the primary rainbow, to find the ratio of the sine of incidence to the sine of refraction when rays of that colour pass out of air into water.*

If A be the angle of incidence of the effective rays, B the angle of refraction, the radius of the arc is $4B-2A$ (Art. 432); let the tangent of $2B-A$, half this angle, to the radius unity, be a; z the tangent of B. Then $2z=$ tang. A (Art. 433). Also, from the principles of trigonometry, tang. $2B : 2\times$ tang. B $(2z)$ :: $1^2 : 1^2-z^2$; therefore tang. $2B=\frac{2z}{1-z^2}$. Again, tang. $\overline{2B-A}$ (a) : tang. $2B-$ tang. A $\left(\frac{2z}{1-z^2}-2z\right)$ :: $1^2 : 1^2+\frac{4z^2}{1-z^2}$*; hence, $\frac{2z}{1-z^2}-2z=a+\frac{4az^2}{1-z^2}$; and by reduction, $2z^3-3az^2-a=0$. The value of z being obtained† from this equation, the angles B and A, and consequently their sines, may be found from the tables.

(446.) Cor. In the same manner, if p be the number of reflections within the drop, z the tangent of B, Q

---

* The propositions here referred to are the following;

1st, The tangent of the sum of two arcs, is to the sum of their tangents, as the square of radius, is to the square of radius diminished by the rectangle under the two tangents. 2d, The tangent of the difference of two arcs, is to the difference of their tangents, as the square of radius, to the square of radius increased by the rectangle under the two tangents. Mr. Vince's *Trig.* Art. 117.

† This equation has two impossible roots. See *Alg.* Art. 361.

the tangent of $\overline{p+1}\,.B$, a the tangent of $\overline{p+1}\,.B-A$, then $\overline{p+1}\,.z=$ tang. A; and $a : Q-\overline{p+1}\,.z :: 1^2 : 1^2 + \overline{p+1}\,.Qz$; therefore $Q-\overline{p+1}\,.z=a+\overline{p+1}\,.aQz$.

From which equation, the value of z being found, the angles A and B, and consequently their sines, may be determined by the tables.

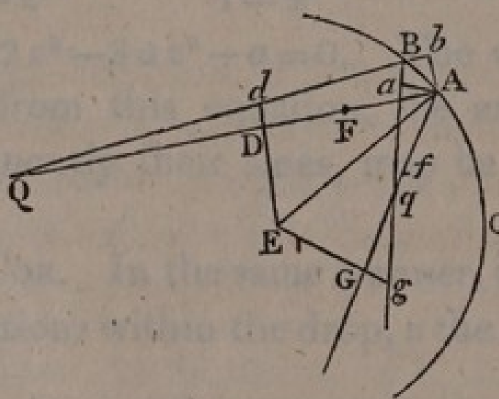
Sect. X.

ON CAUSTICS.

Prop. XCVII.

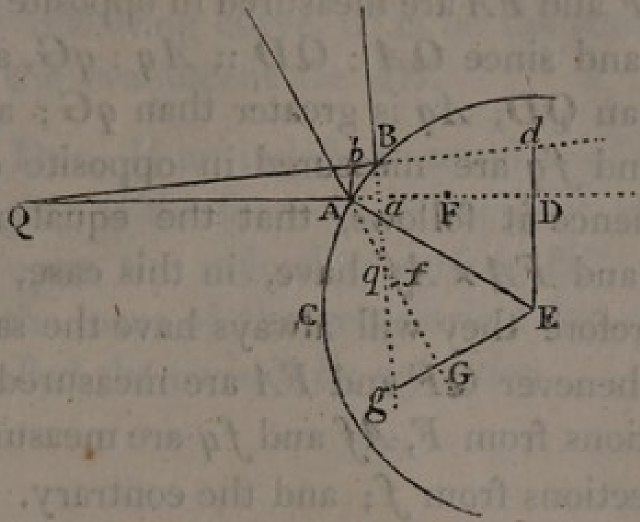
(Art. 447.) *WHEN a small pencil of diverging, or converging rays is incident obliquely upon a spherical reflector, in a plane which passes through it's center, to find the geometrical focus of reflected rays.*

Let *BC* be a spherical reflector whose center is *E*; *QA*, *QB* two rays of a small pencil incident obliquely



upon it, in the plane *QAE*; *AG*, *Bg* the reflected rays, or those rays produced backwards; *q* their intersection. From *E*, draw *EDd*, *EGg* at right angles to *QA*, *AG*; and when the arc *AB* is diminished without limit, they are also at right angles to *QB*, *Bg*; join *EA*, *AB*; from *A*, draw *Ab*, *Aa* at right angles to *QB*, *qB*, produced if necessary; bisect *AD*,

AG in F, f. Then, the angles EAD, EAG are the angles of incidence and reflection of the ray QA, or



equal to them; therefore they are equal to each other; the angles EDA, EGA are right angles; and the side EA is common to the two triangles EAD, EAG; consequently $AD = AG$; and $ED = EG$. In the same manner, $Ed = Eg$; whence, $Dd = Gg$. Again, in the evanescent triangles ABa, ABb, the angles ABb, ABa are complements to the angles of incidence and reflection of the ray QB, or equal to those complements; therefore they are equal to each other; also, the angles AbB, AaB are right angles; and AB is common to the two triangles; consequently, $Ab = Aa$.

Now, in the similar triangles QDd, QAb, $QA : QD :: Ab : Dd :: Aa : Gg$; and in the similar triangles qAa, qGg, $Aa : Gg :: Aq : qG$; therefore, $QA : QD :: Aq : qG$; whence, by composition, and division, $QA + QD : QA \sim QD :: Aq + qG : Aq \sim$

qG; that is, $2QF : 2FA :: 2Af : \overline{Af+fq} - \overline{Af-fq}$ $(2qf)$*; or, $QF : FA :: Af : fq$.

(448.) Cor. 1. In the case represented by the first figure, QF and FA are measured in opposite directions from F; and since $QA : QD :: Aq : qG$, and QA is greater than QD, Aq is greater than qG; and therefore Af and fq are measured in opposite directions from f; hence it follows, that the equal rectangles $QF \times fq$ and $FA \times Af$ have, in this case, the same sign; therefore they will always have the same sign; that is, whenever QF and FA are measured in opposite directions from F, Af and fq are measured in opposite directions from f; and the contrary.

(449.) Cor. 2. When the incident rays are parallel, FA is evanescent with respect to QF; therefore fq is evanescent with respect to Af; or, q coincides with f. Here, $Aq = \frac{1}{2} AG = \frac{1}{2} AD$.

(450.) Cor. 3. If D be the focus of incident rays, G will be the focus of reflected rays. In this case, $QF = FA$; therefore $Af = fq$; and since QF and FA are measured in opposite directions from F, Af and fq must be measured in opposite directions from f; consequently, q coincides with G.

---

* This conclusion depends upon the supposition that when QA and QD are measured in the same direction from Q, qA and qG are measured in opposite directions from q. If this be not the case, $Aq + qG = 2qf$; and $Aq \sim qG = 2fA = 2FA$; therefore $2qf = 2QF$, and $qf = QF$. Now let the rays be incident nearly perpendicularly upon the reflector, and F and f coincide with the principal focus; therefore Q and q are always equally distant from the principal focus, which is absurd (See Art. 50).

(451.) Cor. 4. If Q be a point in the circumference of the circle BC, $FA=\frac{1}{3}QF$; therefore $fq=\frac{1}{3}Af=\frac{1}{12}QA$; hence, $Aq=\frac{1}{4}QA+\frac{1}{12}QA=\frac{1}{3}QA$.

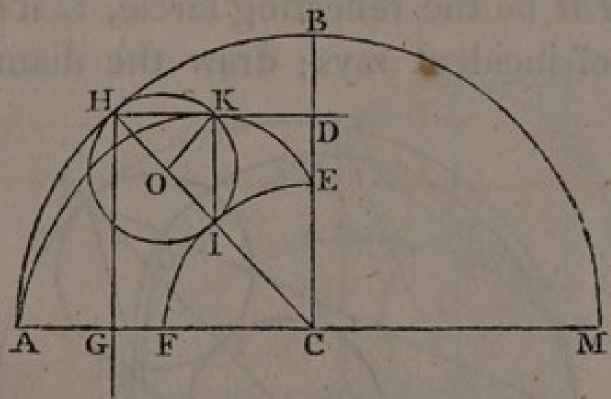
(452.) Cor. 5. The same propositions are true of any other reflecting curve, if E be the center of curvature of the evanescent arc AB.

(453.) Def. If an indefinite number of small pencils belonging to the focus Q, be incident, in the same manner, upon the reflecting surface BC, the curve which is the locus of the geometrical foci of reflected rays, is called the *caustic by reflection.*

Prop. XCVIII.

(454.) *To determine the form of the caustic, when the reflecting curve is a circular arc, and parallel rays are incident in the plane of the circle.*

Let C be the center of the proposed arc; CB, that radius of the circle which is parallel to the incident



rays; and ACM the diameter which is perpendicular to CB. Suppose GH, one of the incident rays, to be reflected in the direction HD; join CH; bisect CH

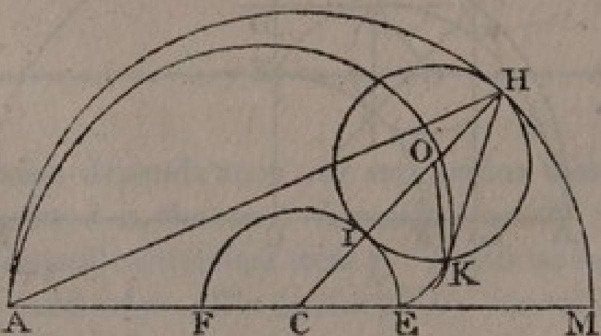
in I, and HI in O; with the centers C, O and radii CI, OH, describe the circles EIF, HKI; and let K be the intersection of HKI and HD; join OK, IK; and from C draw CD perpendicular to HD.

Then, since the angle HKI, in a semi-circle, is a right angle, the triangles HKI, HDC are similar; whence, $HD : HK :: HC : HI :: 2 : 1$; therefore K is a point in the caustic (Art. 449). Also, the $\angle KOI = 2 \angle IHK = 2 \angle CHG$ (Art. 18) $= 2 \angle ICE$ (Euc. 29. i.); and since circular arcs are as the angles which they subtend at their respective centers, and their radii jointly, the arc EI : the arc $IK :: 1 \times 2 : 2 \times 1$. Hence it follows, that the locus of the point K is an epicycloid, generated by the rotation of the circle HKI upon the circle EIF, in the plane of incidence AHM.

Prop. XCIX.

(455.) *To find the nature of the caustic, when the reflecting curve is a circular arc, and the focus of incident rays is in the circumference of the circle.*

Let AHM be the reflecting circle, C it's center, A the focus of incident rays; draw the diameter AM;

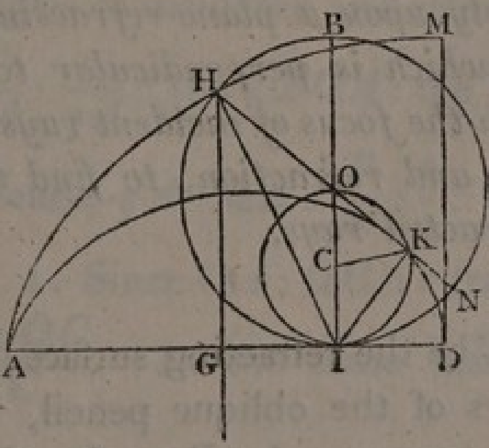


and let the ray AH be reflected in the direction HK; join CH, and divide it into three equal parts CI, IO, *OH*; with the centers *C* and *O*, and radii *CI*, *OI*, describe the circles *EIF*, *IKH*; let *K* be the intersection of the reflected ray *HK*, and the circle *HKI*; join *OK*. Then, since the $\angle OKH$ = the $\angle OHK$ = the $\angle CHA$ = the $\angle CAH$, the triangles *HCA*, *HOK* are similar, and $HA : HC :: HK : HO$; alternately, $HA : HK :: HC : HO :: 3 : 1$; therefore *K* is a point in the caustic (Art. 451). Also, since the $\angle HOK$ = the $\angle HCA$, the $\angle ICE$ = the $\angle KOI$; and the radii *CI*, *OI* are equal; therefore the arcs *EI*, *IK*, are equal; and the locus of the point *K* is an epicycloid, generated by the rotation of the circle *HKI* upon the circle *EIF*, in the plane of the reflecting arc *AHM*.

Prop. C.

(456.) *To find the nature of the caustic, when the reflecting curve is a common cycloid, and the rays are incident parallel to it's axis.*

Let *AHM* be the reflecting semi-cycloid, whose base is *AD*, and axis *DM*; *GH* a ray of light incident

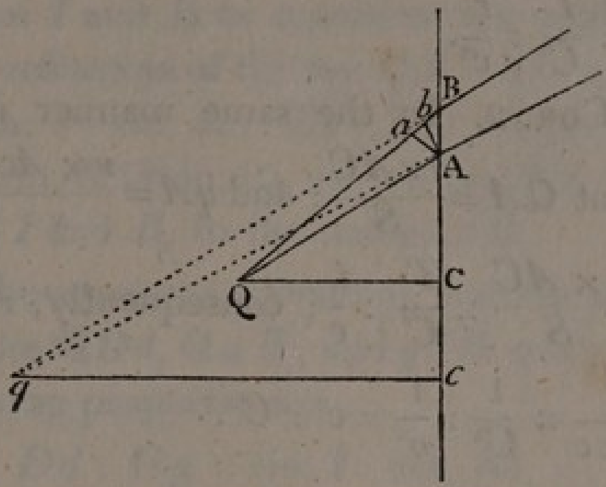


upon it at *H*; *BHI* the situation of the generating circle, when the point which traces out the cycloid is at H; and let I be the point in contact with the base. Take O the center of this circle, and draw the diameter HON; join OI; bisect the line OI in C, and with the center C and radius CI, describe the circle OKI, cutting HN in K; join IK, CK. Then, since OI is perpendicular to AD, or parallel to HG, the $\angle OIH =$ the $\angle GHI$; and the $\angle OIH =$ the $\angle OHI$; therefore, the $\angle OHI =$ the $\angle GHI$, and the ray GH is reflected in the direction HOK. Also, since IK is perpendicular to HK, and IH is half the radius of curvature of the cycloid at H (*Mech.* Art. 287), HK is one fourth part of the chord of curvature in the direction of the reflected ray; and therefore K is a point in the caustic (Art. 449). Again, since the $\angle KCI = 2 \angle NOI$, and $OI = 2\, IC$, the arc $IK =$ the arc $IN = ID$; therefore, the locus of the point K is a common cycloid, whose base is AD, and generating circle OKI.

Prop. CI.

(457.) *When a small pencil of homogeneal rays falls obliquely upon a plane refracting surface, and in a plane which is perpendicular to that surface, having given the focus of incident rays, and the angles of incidence and refraction, to find the geometrical focus of refracted rays.*

Let BAC be the refracting surface; QA, QB, the extreme rays of the oblique pencil, incident in the plane of the paper; qA, qB produced, the directions in which they are refracted; q the intersection of the refracted rays. From Q and q draw QC, qc at right angles to BC; and from A, draw Aa, Ab, at right



angles to QB, qB. Take S and s to represent the sines of incidence and refraction of the ray QA; C and c their cosines; T and t their tangents. Then, since the angles AQC, BQC, are equal to the angles of incidence, and Aqc, Bqc, to the angles of refraction of the rays QA, QB, BQA and BqA are contemporaneous increments of the angles of incidence and refraction of the ray QA; and therefore, the $\angle BQA$: the $\angle BqA : T : t$* $:: \frac{S}{C} : \frac{s}{c}$. Also, the $\angle BQA$: the $\angle BqA :: \frac{Aa}{QA} : \frac{Ab}{qA}$; and since Aa, Ab are the cosines of the angles of incidence and refraction, to the radius BA, $\frac{C}{QA} : \frac{c}{qA} ::$ the $\angle BQA$: the $\angle BqA :: T : t :: \frac{S}{C} : \frac{s}{c}$; whence, $qA : QA :: \frac{T}{C} : \frac{t}{c} :: \frac{S}{C^2} : \frac{s}{c^2}$.

(458.) Cor. 1. Since $QA : QC :: r$ (radius) : C, we have $QA = \frac{r \times QC}{C}$. In the same manner, $qA =$

* Art. 431.

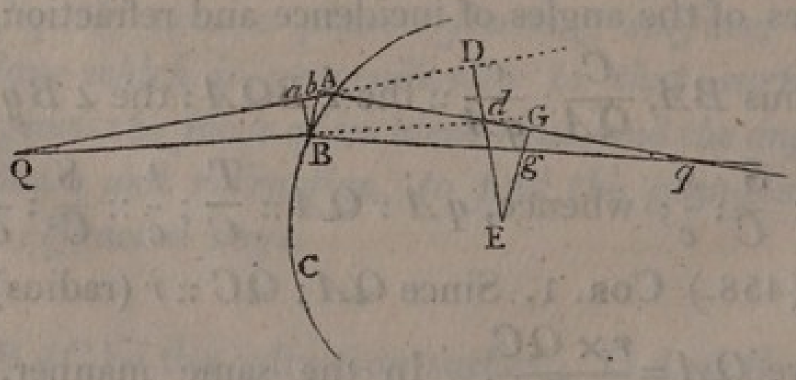
$\frac{r\times qc}{c}$; therefore $\frac{r\times qc}{c}:\frac{r\times QC}{C}::\frac{T}{C}:\frac{t}{c}$; whence, $qc:QC::\frac{T}{C^2}:\frac{t}{c^2}$.

(459.) Cor. 2. In the same manner it may be proved, that $QA=\frac{r\times AC}{S}$; and $qA=\frac{r\times Ac}{s}$; whence, $\frac{r\times Ac}{s}:\frac{r\times AC}{S}::\frac{T}{C}:\frac{t}{c}$; consequently, $Ac:AC::\frac{T}{S\times C}:\frac{t}{s\times c}::\frac{1}{C^2}:\frac{1}{c^2}::c^2:C^2$.

Prop. CII.

(460.) *When a small pencil of homogeneal rays falls obliquely upon a spherical refractor, in a plane which passes through it's center, having given the focus of incident rays, and the angles of incidence and refraction, to determine the geometrical focus of refracted rays.*

Let QA, QB, be the extreme rays of a small pencil incident obliquely upon the spherical refractor ABC,



in a plane which passes through it's center E; and let Aq, Bq, be the refracted rays. Draw EdD, EgG, and Ba, Bb, at right angles to QA, qA, or to those lines produced; and when the arc AB is diminished without limit, Ed and Eg, are at right angles to Qd, Bq. Take I and R to represent the angles of incidence and refraction of the ray QA. Then, since $ED : EG :: \sin. I : \sin. R :: Ed : Eg$, we have $Dd : Gg :: \sin. I : \sin. R$ (Euc. 19. v). Also, Ba, Bb are the cosines of I and R, to the radius AB.

From these two considerations, and the similarity of the triangles QDd, QaB; and qbB, qGg; we obtain the following proportions;

$$Dd : Gg :: \sin. I : \sin. R;$$
$$Gg : Bb :: Gq : bq\ (Aq);$$
$$Bb : Ba :: \cos. R : \cos. I;$$
$$Ba : Dd :: Qa\ (QA) : QD;$$

by compounding which proportions, we have $\sin. I \times Gq \times \cos. R \times QA = \sin. R \times Aq \times \cos. I \times QD$; and therefore, $Aq : Gq :: \frac{\sin. I}{\cos. I} \times QA : \frac{\sin. R}{\cos. R} \times QD ::$ $\text{tang. } I \times QA : \text{tang. } R \times QD$.

(461.) Cor. 1. The distances qA, qG, must be measured in the *same*, or *opposite* directions from q, according as QA, QD, are measured in the *same*, or *opposite* directions from Q*.

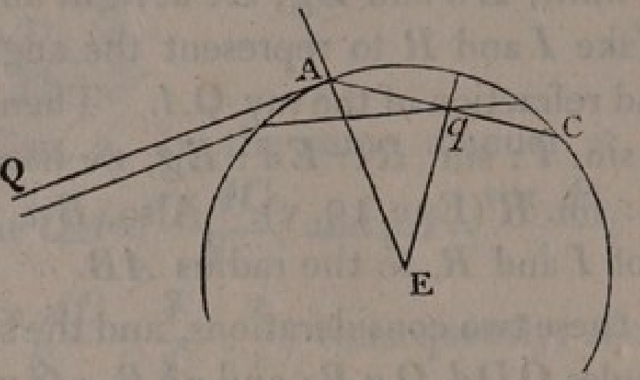
(462.) Cor. 2. When the incident rays are parallel, $QA = QD$, and therefore $Aq : Gq :: \text{tang. } I : \text{tang. } R$.

(463.) Cor. 3. On the foregoing supposition, when the rays pass out of a rarer medium into a denser, and the angle of incidence becomes nearly a right angle,

---

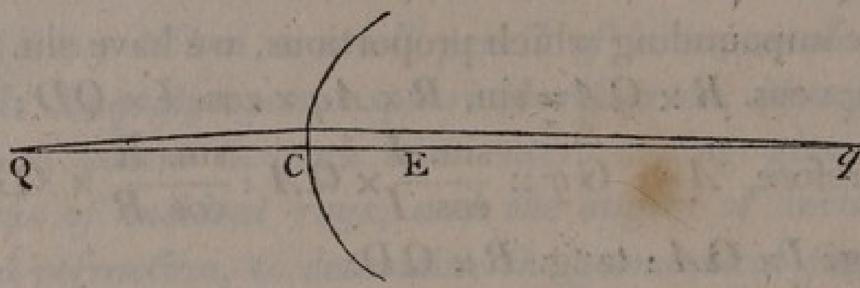
* See Art. 448, and Note, p. 242.

tang. I is indefinitely greater than tang. R; therefore



qG vanishes; or q bisects the chord of the arc, cut off by the refracted ray.

(464.) Cor. 4. When the rays are incident nearly perpendicularly upon the refracting surface, tang. I :



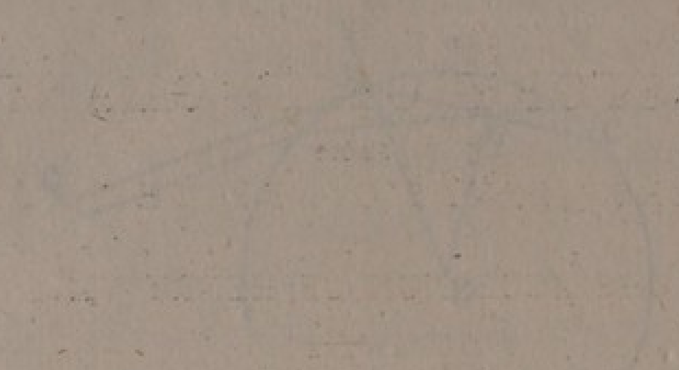
tang. R :: sin. I : sin. R; also, D and G coincide with E; therefore $qC : qE ::$ sin. $I \times QC$: sin. $R \times QE$.

(465.) Cor. 5. Similar conclusions may be drawn respecting the refraction of a small pencil of rays at any other surface, if E be the center of curvature of the refractor at the point of incidence.

On this subject, the reader may consult Hayes's *Fluxions*, Sect. ix. x. Smith's *Optics*, Book II. Chap. ix.

THE END.

cosp. I is nothing greater than R; therefore



gG varies as very nearly the versed sine of the arc, cut off by the refracted ray.

(484.) Cor. 4. When the rays are incident nearly perpendicularly upon the refracting surface, tang. I : [illegible]

tang. R : sin. I : sin. R, also D and G coincide with E, therefore gC : gE :: sin. I × DC : sin. R × EC.

(485.) Cor. 5. Similar conclusions may be drawn respecting the refraction of a small pencil of rays at any other surface, if E be the centre of curvature of the surface at the point of incidence.

On this subject, the reader may consult Maclaurin's Fluxions, Sect. [illegible] and Smith's Optics, Book II. Chap. [illegible]

THE END.