from the author

A SHORT

ELEMENTARY TREATISE

ON

EXPERIMENTAL AND MATHEMATICAL OPTICS.

BY

THE REV. BADEN POWELL, M.A. F.R.S.

OXFORD: PRINTED BY TALBOYS AND BROWNE.

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AND

MATHEMATICAL

OPTICS

DESIGNED FOR THE USE OF STUDENTS
IN THE UNIVERSITY.

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PREFACE.

In offering a new treatise on any science, it may be considered encumbent on the author to point out wherein it differs from those already extant, and to explain the particular object in view in its publication.

In the present instance, the author's object is to facilitate the study of Elementary Optics, with immediate reference to the wants of the student, circumstanced as he at present is in the University of Oxford. To this particular case none of the existing treatises appear completely adapted; and though such is the discouragement under which physical studies labour in this place, that the mere character of the treatises produced on these branches, can, perhaps, very little affect their actual progress; yet that no obstacle may remain which is capable of being removed, he considers it worth while to try to remedy the complaints against existing works, whether as too large and too difficult on the one hand, or as incomplete and obscure on the other.

Of those works it is not his intention to speak in a tone of criticism, but merely to point out in what particulars he has been led to deviate from the methods adopted in them.

The treatise of Dr. Wood has long maintained its ground from the admirable soundness of its geometrical reasoning; but the character thus given to it has necessarily produced two results, that of excessive length in the establishment of the theorems of ordinary reflexion and refraction, and of limiting the range of subject to those few points which are susceptible of this mode of demonstration: and in the most important of these, in point of fact, the author is obliged to deviate from the geometrical method, and though still in the disguise of a geometrical dress, to introduce, what are in reality, analytical processes.

In the later treatises of Mr. Coddington, especially in his first and smaller work, these evils have been in some degree remedied. In his new and larger work, the complete development of the theory of reflexion, refraction, and optical instruments is highly valuable, but too elaborate for the limited range of academical reading; whilst at present it is confined to these branches of the science.

The valuable "Treatise on Light and Vision," by the Rev. H. Lloyd, F. T. C. D. Dublin, 1831, is perhaps better suited to the purposes of general study, but appears, in several parts unnecessarily lengthy: and does not extend to the higher properties of light.

On the other hand, the masterly work of Sir J. Herschel, even were it to be had in a separate form, is hardly of a description to suit the purposes of the academical student, at least until academical study shall have acquired a much higher character than it can at present boast: whilst the popular treatises by Sir D. Brewster, in the Cabinet Cyclopedia and Library of Useful Knowledge, from the professed avoidance of mathematics, are of course deprived of all that comprehensive brevity and perspicuity which results from the adoption of that valuable instrument of investigation; and in regard to the more recondite experimental results, the student would feel rather overwhelmed with the multitude of details, than enlightened by a view of general truths.

In the present work it has been the author's endeavour to preserve the utmost degree of brevity consistent with perspicuity, and to unite this with the greatest simplicity in the method of pursuing the investigation. To effect these objects he has adopted throughout the analytical method, as that which conducts the student most easily and rapidly to the comprehensive principles of the science, at the same time taking care to point out those propositions which admit of elegant geometrical constructions. The immense advantage gained in this way, in point of conciseness, and avoiding those repetitions which occur in the separate establishment of the individual cases, will clearly appear upon comparison: and it is presumed whatever tends thus to abridge the labour of acquiring elementary knowledge, must be regarded as no inconsiderable aid to the advancement of the study. The analytical methods made use of, never extend beyond the very first elements of the differential and integral calculus. And it is always an easier process for the student first to acquire such knowledge before he enters upon the study of the mixed branches, than to proceed with those parts which may be treated (though in a very disadvantageous form) by the application of geometry; when, after all, he must at a certain stage have recourse to analysis, with all the evils of a change of system, and an unnecessary loss of time and trouble in the first instance. One source of abridgement has been found in the omission of the steps of algebraical processes; but these are generally of such a kind as will be quite obvious, and are properly passed over in a work whose specific business is not to *teach* mathematical processes, but supposing them already familiar, to employ them as instruments of physical investigation.

The range of subject which it has been attempted to include in this treatise, is considerably wider than that to which elementary works of the same description have hitherto been usually confined. This appears to be demanded by the existing state of knowledge; and the introduction of the more attractive subjects of recent discovery will not fail to act as an additional stimulus to the learner, to master the difficulties of the preliminary parts. It has hitherto been customary to draw a broad line of distinction between what were called *mathematical, or common* optics, and *physical* optics: a distinction *wholly arbitrary*, and not a little repressive of the student's advance into the latter portion of the subject. Reflexion and refraction are as much *physical* properties of light, as polarization and double refraction. It so happens that from the simplicity of the laws which regulate the two former, a multitude of important consequences are deduced by mere mathematical reasoning, on the assumption of straight lines representing the directions of the rays; but this constitutes no essential distinction: and the only real division of the subject is that which would be occasioned by the necessity of introducing a physical theory of the nature of light; and a reference to such a theory has perhaps been regarded as inseparable from the discussion of polarization, coloured rings, &c. That these subjects, however, are quite capable of being stated in a way entirely unconnected with any physical hypothesis, will be evident to any one who has studied them. And that they no more necessarily involve a theory than any of the properties of lenses or reflectors, but in reality stand as completely on their own basis of experimental facts, reducible to mathematical laws, the author trusts will be rendered evident to the satisfaction of every reader in the following pages.

To the neglect of this consideration is probably owing much of the reluctance which has prevailed to the general introduction of the study of the more recondite and delicate investigations of the science: and the notion that they were identified with a gratuitous and doubtful theory, would seem actually to have thrown discredit on some of the most thoroughly established matters of fact which the science discloses, and considerably impeded its advance. The idea of a *physical polarity*, and the theory of *fits*, would appear to have been so viewed that the credit of the experimental conclusions seemed to be at stake along with the terms.

But these distinctions must now be generally recognized; and instead of making a division between classes of facts, where no essential difference can be pointed out, the more correct mode of proceeding will be, throughout the whole course of optics, to keep *separate* the statement of the experimental laws, and of the theoretical principles by which they may be accounted for. In the higher portions of the subject, this distinction has hardly perhaps entered into the views of any writer. It has been a main object, however, with the author of the present treatise, to keep the statement and investigation of the experimental facts, their mathematical laws, and the consequences deducible, entirely separate from what he conceives ought to constitute quite another department of the science,—the discussion of such a physical theory as shall provide forces adequate to produce the observed effects by the operation of established dynamical principles. To the latter class of investigations surely the term *physical optics* ought properly to be limited: the distinction being precisely the same as that between *plane* and *physical astronomy*.

The design, then, of the present work is limited entirely to the experimental and mathematical part of optics, without entering in the smallest degree upon physical optics, properly so called. It is intended to hold exactly the same place in relation to the development of the undulatory (or any other preferable) theory of light, as a treatise on plane astronomy does to one on the physical theory of the planetary motions.

The full development of the physical theory is to be found in Sir J. Herschel's treatise on Light, or in the more concise and more recent tract on the Undulatory Theory, appended to the second edition of Professor Airy's Mathematical Tracts, Cambridge, 1831. To these profound works the present is designed as *introductory*. It aims only at the humbler task of clearing the ground, and presenting the facts in a simple and systematic point of view, referring the student to these authorities both for the more enlarged details of many of the experimental parts, and for the entire theory by which the singular FACTS of the *intervals*, the *retardations*, the *interferences*,

&c. are shewn to be such as ought to result from a peculiar system of motions propagated according to given laws, consistent with acknowledged dynamical principles. For the establishment of some of the most important points of this theory the student must consult the Papers of Professor Airy, in the Cambridge Transactions. An able article on the subject will also be found in the Solutions of Cambridge Problems, 1831.

In extending his range of subject the author has been restricted in the selection of topics, from the desire of keeping the work within a small compass. His choice has therefore been always directed with a view to general results rather than to a multitude of details, which however valuable in themselves, and in their practical relations, might rather encumber than advance the study of principles. This must be his apology for the omission of many points which he would willingly have introduced or discussed more copiously. Scarcely any mention is made of the numerous experimental and practical applications of the theory, beyond the two important cases which refer to the principle of the telescope and microscope, and these are but very briefly discussed: equally destitute will the treatise be found to be of particular examples to the formulæ, or deductions from, and problems founded upon, the theorems. But the former class of illustrations are generally of such a nature as to be readily understood as soon as described or seen, by any one who is well possessed of the principles here laid down: whilst ample descriptions may be found in professedly practical treatises. To this class belong the various singular optical exhibitions; of which a remarkable case may be instanced as discussed in a recent tract by Mr. Horner, on A Forgotten Fact in Optics, Bath, 1832. As to those innumerable deductions and problems, which involve nothing but the solution of mathematical questions of more or less difficulty arising out of the theory, they are excluded by the avowed nature and limits of the work; whilst it appears more than questionable whether they afford any useful exercise to the mind; and whether the time and pains bestowed upon them, under the old system, has not tended to withdraw the student's attention from more useful researches, and greatly to check the advancement of the science.

It remains only for the author to acknowledge the assistance he has not scrupled to derive from all sources within his reach: he is principally indebted (as will indeed be apparent throughout) to the treatises of Sir J. Herschel and Mr. Coddington. The whole discussion § 49—53, is borrowed from Mr. Lloyd's treatise on Light and Vision; and the abstract of the systems of rays § 54—57, was kindly furnished by the Rev. A. Neate, M. A. of Trinity College, Cambridge and Oxford.

Oxford, Nov. 9th.

CONTENTS.

REFERENCES TO THE PLATES.

In the text no specific reference is made to diagrams, and the investigations are of such a nature as do not require any formal constructions. In order, however, to afford all facilities of illustration to the student, figures adapted to each of the principal subjects of discussion are annexed; the references to which are as follows:

A SHORT
ELEMENTARY TREATISE
ON THE THEORY OF OPTICS.

Preliminary notions of Light, and its propagation.

1. OUR notion of Light is one wholly and immediately derived from the sensation of Vision; which being one of the most primary and simple of our sensations, neither requires nor admits further definition. The Science of Optics is conversant about a vast assemblage of phenomena all disclosed to us through the medium of this sensation: its object is to assign the laws to which they are reducible; and to trace, as far as the facts can guide us, the nature and properties of that agent or cause whose existence or operation we suppose as that which produces the effects observed.

The slightest observation leads us to distinguish certain bodies which we call self-luminous, such as the sun, a flame, or a red-hot iron, whose presence is essential to exciting the sense of vision: they not only excite this sensation so as to convey to our eyes the impression of themselves as sources of light, but also are the means of enabling us to see all other objects, which do not possess this property but which yet thus excite in our organs the perception of themselves: we *see* bodies of both kinds; but we do not see those of the latter class without the presence of some of the former.

By the term *light* we shall for the present understand simply that which produces on our eyes the sensation of vision, without any supposition as to its nature or mode of action; all that we have to notice in regard to our present subject will be grounded on the following general facts, which are proved by universal and constant experience.

1st. The sensation of vision, as referred by us to the presence of those bodies which are the objects of sight, is *propagated* from them to our eyes in *straight lines:* that is, if we take the smallest point which is capable of being distinguished by our eyes, as the light admitted through a pinhole, and place another small aperture in a straight line between the first and the eye, we still see it; but if the second aperture be moved ever so little out of the straight line the first becomes invisible.

Also any impression of light made on the eye is always referred by an instinctive judgment to a point or object in the same rectilinear direction as that in which the impression arrives at the eye.

2nd. This rectilinear propagation continues without limit and without any alteration until the light meets with some external cause capable of acting upon it.

3rd. It takes place from any one visible or luminous point *in all possible directions.*

2. The sense in which the term *ray of light* is used, will require a little explanation. It denotes simply any one of those *rectilinear directions* in which the effect of light is conveyed. Regarding the *direction* simply, it is in fact purely a mathematical conception; but we can hardly help associating the idea with that of a physical portion of what we call light, of indefinitely small area; or what is perhaps a better mode of viewing the subject, if we regard the luminous point or source of light as the centre of a sphere of light emanating from it, the *radius of that*

sphere at the mathematical point where it impinges upon or touches any surface (such as our eye for example) is called *a ray of light*.

Light propagated or radiating in all directions from a single luminous point will be readily understood to diverge and become more diffuse: or in other words its intensity to diminish as the distance increases, and this (it is easily seen) in proportion to the square of the distance. The effect being, according to the above view, proportional to the segment of the spherical surface comprehended by radii forming a given angle, at different distances.

We shall for the present suppose light to be homogeneous; that is, that whatever modification one integrant part of it undergoes from the action of bodies upon it, the same will all the other integrant portions undergo. We shall afterwards see, that for light of ordinary kinds, such as that of the sun, &c., this is untrue; but so far as our present object is concerned, it will suffice, for the sake of simplicity, to reason about light supposed to be homogeneous.

Reflexion and Refraction of Light.

3. Light, as we familiarly know, can continue to be propagated only through certain sorts of media, which we call transparent; and among these there is great difference in the degree in which they possess this property: those which are imperfectly transparent allow only a portion of the light to pass, and those which are opaque intercept it altogether. The light in impinging upon the surface of any medium different from that in which it was originally proceeding, whether transparent or opaque, undergoes various modifications, whether in the act of impinging on the surface or in its transmission; these constitute the principal properties of light: and a vast variety of such properties are

presented to our examination and form the subjects of the various branches of optical science.

For the present our attention will be confined to some of the simplest of these properties, and in the first instance to the mere consideration of those which refer to the *direction* in which the light proceeds, which is found to undergo certain changes according to the nature of the body on whose surface the ray impinges.

Let us then suppose a simple ray of light proceeding in the first instance through a transparent medium of uniform or homogeneous nature, in a straight line; and that at a certain point it arrives at the surface of another medium of a different nature: in the first place, if the surface be smooth or polished we may see upon it, or rather as it were *in* it, an image of the luminous point from which the ray issues: in the second place, if the medium which is bounded by this surface be transparent, we may also find that if we view the luminous point through it (and this more conspicuously as the thickness is greater), it will appear to undergo a change in its position, or in other words, the ray of light suffers a deviation in its course.

The first of these effects is called the *reflexion* the latter the *refraction* of light. We find a vast variety in those properties of bodies on which these effects depend, some of which we shall have occasion to consider hereafter: for the present we merely notice the general fact with the view of entering upon those elementary investigations which regard simply the *direction* and the *change in direction* which light thus undergoes. We may imagine the surface to be a plane; for if curved, we shall have only to take into consideration that minute portion of it on which the ray of light impinges, and which we may therefore without error consider as if it were plane. Let us suppose the ray to be incident upon it in any oblique direction, and let a perpendicular or normal to the surface at the point of impact be conceived drawn, with which the incident ray forms an angle ϕ. This is called the *angle of incidence*. If now the eye be directed so as to see the image of the luminous point by reflexion from the surface, it is immediately found that there are only certain positions in which this can be done: or upon more precise examination we find, that if the relative position of the luminous point and the reflecting surface remain unaltered, or in other words, if the value of ϕ remain unaltered, there is only one direction in which the ray can be reflected, and from the constant result of innumerable experiments by which this direction is ascertained, it is found that the reflected ray is always in *the same plane* with that of the incident ray and the normal, and forms with the normal an *angle precisely equal* to the angle ϕ: or the law of reflexion is commonly expressed by saying, that the *angle of reflexion is* EQUAL *to the angle of incidence: and is formed in the same plane.*

4. When we speak of the reflexion of light from a plane surface, and the equal inclination of the incident and reflected rays, it must be understood, that such a surface is meant as is capable of reflecting back distinctly the ray from any one point, so as to give an image of that point: this is only the case with bodies whose surface has been *polished;* that is, there is a certain amount to which the roughness and irregularity unavoidably occurring on all surfaces must be reduced and equalized, before the surface can possess the property of distinct reflexion. The precise reason of this is probably connected with the intimate nature of light, but at all events when a body is not thus polished, another phenomenon is presented; we have no image seen by reflexion, but the light is broken and dispersed, as it were, by the numerous minute irregularities of the surface; and it is by light thus scattered, that the surface itself is rendered visible to us; and even the best polished reflectors still retain enough of

this irregularity to allow us to perceive distinctly their surface, besides giving us the regularly reflected images of objects. It is found that any tolerably smooth surface will, at very oblique incidences, give a more or less distinct regular reflexion; which has been ascribed to the inclined path of the ray with respect to the minute unevennesses among which it would otherwise become lost. The quantity of light reflected is in all cases found to be much increased as the angle of incidence is greater.

5. When a ray of light impinges on the smooth surface of any body, whether opaque or transparent, it undergoes reflexion as just described; but this applies only to a certain *portion* of the incident light: the most perfectly polished surfaces by no means reflect the whole of the light which falls upon them, and those less polished only a very small portion. What this depends upon is as yet but imperfectly understood, and we shall not here attempt to enter upon the subject. If the substance be *transparent,* that portion which is not reflected enters the medium and is propagated through it in a straight line, but with an altered direction. This change of direction, as already observed, is called *refraction*, and the amount of it varies very greatly in different transparent media; but the law by which it takes place in the same medium for a ray supposed homogeneous at different incidences is invariable, and has been very precisely deduced from numerous experiments. The most direct principle of such determinations is easy to conceive: the apparent position assumed by an object as seen through a medium, compared with that in which it is seen when viewed directly, is readily subject to exact measurement, which indicates the angular deviation to which the ray proceeding from that object or point has been subjected on entering the new medium. It will be obvious that this deviation is relative and refers to the direction of the ray in the surrounding medium in which we suppose it at first pro-

ceeding. In order to consider the subject most generally we must bear this in mind, and consider simply the *relative* deviation of a ray in passing *from one medium into another*. Supposing then these two media to remain the same, and taking different angles of incidence or values of ϕ, if the corresponding values of the angle which the refracted ray forms also with the normal, and which we will call ϕ, be ascertained, it is found that *the refracted ray lies always in the plane of the incident ray and the normal:* and further, upon comparing the magnitudes of the angles, though we find no direct relation between the *angles themselves,* yet there is an universal relation between their *sines:* and this is that of a *simple ratio, constant for all incidences,* at the surface separating the *same* two *media,* but differing greatly for different media. This law therefore, writing this constant ratio $= m$, is expressed thus:

$$\sin. \phi = m \sin. \phi, \qquad (1)$$

This important law, the foundation of the whole Science of Optics, was the discovery of Willebrod Snell, though claimed by and ascribed to Des Cartes. In practice there are various modes of verifying this law, less direct indeed, but more convenient than that referred to above.

6. The value of m has been experimentally determined for nearly all known transparent substances: but no general principle has been discovered which connects their refractive powers with their other properties: though speaking generally it is highest in the denser substances and in those which contain an inflammable principle.

The following numbers will convey some notion of the relative values of m or indices of refraction for a few of the most remarkable substances, taken from the extensive table of such values, as determined by various observers, given in Sir J. Herschel's treatise on Light, to which the reader must be referred for a variety of important observations on the subject.

Realgar	2.54.
Diamond	2.439.
Melted sulphur	2.148.
Flint glass, containing much lead . .	2.028.
Oil of Cassia	1.641.
Various kinds of flint glass . .	1.64. / 1.57.
Crown glass	1.525.
Plate glass	1.514.
Various oils, about	1.46.
Sulphuric and other acids . .	1.43. / 1.41.
Water	1.336.
Fluids in the cavities of crystals . .	1.21.
Ether, expanded to three times its volume	1.057.
Several gases, about	1.0004.
Atmospheric air	1.00029.
Hydrogen	1.00013.
Vacuum	1.00000.

Sir D. Brewster observes, that no values of this sort, founded on direct observation, give the real refractive powers of the *ultimate particles* of the different substances. For this purpose he takes into account their specific gravities, and obtains the absolute refractive power by reducing the observed indices in this ratio: and Sir J. Herschel has further remarked, that to obtain the result for the *elementary atoms* of bodies, these last values must be again reduced in the ratio of the atomic weights.

The quantity m is greater than unity when the ray enters a more refractive medium from one which is less so: when nothing is said to the contrary, this is generally assumed to be the case: if it be reversed, we have m less than unity, and it is evident that according as we suppose m greater or less than unity, we shall have sin. $\phi >$ or $<$ sin. ϕ, and the refracted ray will deviate from the original direction either *towards* or *from* the *normal* respectively.

7. We shall not in this place trace any further the physical results connected with the reflexion and refraction of light: we must here proceed to consider certain consequences directly derivable from these fundamental laws, of a purely mathematical kind, but which we shall afterwards find to bear most directly on the experimental facts and their applications.

Now in this point of view the first remark we have to make is one of great simplicity, and at the same time of primary importance. If we consider the case of refraction, it is obvious that the ray though deviated by a certain angle out of its rectilinear course, yet proceeds onwards *towards* the same region: in reflexion, on the contrary, its new direction is *turned back*, relatively to its original course. This difference in direction will obviously correspond to the mathematical condition of a change of *sign*; and if we assume the direction of the incident and consequently that of the refracted ray as *positive*, that of the *reflected* ray will be *negative*. The law of reflexion will therefore be expressed by writing

$$\phi = -\phi,$$

Guided by this simple consideration, we shall be able to include in one simple and general expression, the law both of reflexion and of refraction; for if we take the formula for refraction, and to make it more general suppose m to have a double sign, or write it thus,

$$\sin. \phi = \pm m \sin. \phi,$$

it is evident that this will include the law of refraction, expressed by the upper sign, as well as a number of *imaginable* laws of reflexion, if we use the lower sign, corresponding to the different values of m, among which we shall include the *actual* law of reflexion, if we suppose $m = -1$, which on substitution gives

$$\sin. \phi = -\sin. \phi,$$

or

$$\phi = -\phi,$$

which is the law of reflexion.

Adopting then for the present this view of what we may now call by the general name of OPTICAL DEVIATION, we may proceed to examine various consequences which will result, all of which we shall deduce on the general supposition of the value of m; and in all cases by substituting $m = -1$, shall be able to adapt the expressions to the case of reflexion. In fact it will be both allowable and convenient in most of our investigations to discard altogether all reference to the physical ideas of light or its modifications; and to confine our reasonings simply to the pure mathematical conception of a straight line incident upon a surface, and there deviated from its course according to the general law above expressed. And further, since we have observed that the effects we are considering take place in one plane we shall commence more simply by considering merely a *section* of the surface in the plane in which we suppose the direct and deviated rays or *straight lines* to lie.

By this method of proceeding it will be found that we shall save an immense quantity of repetition and prolixity. We shall speak of lines connected by this law of deviation as lines or rays indifferently, but it will be seen that our reasonings and results are wholly mathematical: and pursuing our inquiry first on the assumption of one line so deviated, and then of *systems* of such lines or rays connected according to some determined relation, we shall investigate numerous particulars relative to the positions, intersections, etc., of the deviated lines as resulting from the primary law, under different conditions as to the nature of the surface or curve on which they impinge.

We have already observed that m is assumed relatively to one surface: we shall extend the supposition by conceiving the ray after passing this first surface to fall on a second, whose position is given, where a new deviation takes place according to a similar law, but with a new value of m which we express by writing

$$\sin. \phi_{,,} = m_{,} \sin. \phi_{,,,}$$

and similarly in succession for any number of surfaces.

Reflexion and Refraction at plane surfaces.

8. If a ray impinge on a plane surface and we take u $u_{,}$ to represent the lengths of the incident and deviated rays between the points of incidence and where they are met by a perpendicular to the surface, then it will be evident that the angles thus formed by u and $u_{,}$ with the perpendicular, are respectively equal to ϕ and $\phi_{,}$; hence from the triangle we shall have the relation

$$\frac{\sin. \phi}{\sin. \phi_{,}} = \frac{u_{,}}{u} = m \qquad (2)$$

This expresses the physical result that the image of a point within a transparent medium bounded by a plane, will appear to the eye placed without it, in the direction of the refracted ray, and at a distance behind the surface less or greater than that of the object in the ratio expressed by m. Thus the bottom of a vessel of water appears raised, and a straight rod partly immersed appears bent.

9. A line or ray being incident on a *plane* surface is deviated agreeably to the fundamental law: let us now suppose it to

meet a second plane parallel to the former, at which the same law prevails, and where $m_{,} = \frac{1}{m}$: we shall of course have $\phi_{,,} = \phi_{,}$ and consequently deducing the value of sin. $\phi_{,,,}$ by the equation (1), it is evident that $\phi_{,,,} = \phi$ or the ray after passing the second plane is in a direction parallel to its first direction. The same will be true of small portions of curve surfaces if the ray pass at points where they are parallel.

Again, it is evident from the nature of the law (1) that when m is greater than unity, whatever value is given to ϕ, that of $\phi_{,}$ will be less, or there is a finite limit to the positions which the ray $u_{,}$ can assume, and which in the extreme case when $\phi = \frac{\pi}{2}$ is found by taking the corresponding value of the sine, or, expressing this extreme value by brackets, we readily deduce

$$[\phi_{,}] = \sin.^{-1}(m).$$

If we proceed to a second surface where we suppose $m_{,} = \frac{1}{m}$ conditions precisely reversed will prevail, and we shall find a certain limit to the value of $\phi_{,,}$, upon the same principle as in the last instance: or we find the value corresponding to $\phi_{,,,} = \frac{\pi}{2}$

$$[\phi_{,,}] = \sin.^{-1}\left(\frac{1}{m}\right)$$

which shows that when sin. $\phi_{,,} = \frac{1}{m}$ then the ray beyond the second surface emerges parallel to it or rather coincides with it. Beyond this limit, or if we have $\phi_{,,}$ greater than this value, we shall have sin. $\phi_{,,,} > 1$, an imaginary value, from which we can infer nothing as to the position which the ray may assume. There is however a remarkable physical effect corresponding to this case.

When a ray, refracted into a transparent medium reaches its second surface, in general a portion of it emerges agreeably to the law of refraction, whilst another portion undergoes internal reflexion. As the incidence increases the portion reflected increases; but when the angle arrives at the position to which we have just referred, where $\phi_{\prime\prime}$ has arrived at the limit of refrangibility, then the ray is no longer refracted: but suddenly the reflected portion receives the addition of all the remaining portion of the light, and thus an internal reflexion of great brilliancy takes place. At this position the value of $\phi_{\prime\prime}$ as stated above, is,

$$\phi_{\prime\prime} = \sin.^{-1}\left(\frac{1}{m}\right)$$

if therefore we observe the angle, we can thus determine m. Dr. Wollaston suggested a method of finding the refractive powers of substances, founded on this principle. It is on this principle also that the instrument called the Camera Lucida is constructed. A solid piece of glass is cut with plane sides at such an angle that rays of light entering it from an object shall be *totally reflected* at the internal surface upwards to the eye; which consequently refers the image to a horizontal plane below; as a table or paper on which the image appears painted, and may be traced out with a pencil.

From this consideration of the limit of refraction, it follows that an eye placed in a more refracting medium, as that of a fish under water, will receive the ray coming from an object on the horizon in a considerably inclined direction, and such an object will appear therefore elevated in the air. Thus all objects above the water will appear contracted into a circular space round the vertical point; and in the region immediately without this will be seen the objects at the bottom of the water by internal reflexion from the surface.

The atmosphere consists of air decreasing in density, and therefore in refractive power, as we ascend. Hence a ray of light coming from any of the heavenly bodies is refracted slightly on entering our atmosphere, and undergoes a greater deviation at every successive stratum of denser air: hence its ultimate direction, especially if incident very obliquely, that is, when the body is near the horizon, will be considerably deviated; and its apparent position much raised. This is the case with the sun at its rising and setting. Also the upper and lower parts of the sun's disk being unequally refracted, it sometimes presents a flattened appearance when near the horizon. In like manner various objects, such as ships, have been rendered visible when really much below the horizon.

Many curious atmospherical phenomena are explained on this principle; but the most remarkable of them are owing to a combination of this cause with the internal reflexion above explained: and which takes place at the boundary between two strata of air of unequal density. Thus the mirage or appearance, resembling an extensive surface of water over a level heated surface of sand, and accompanied by direct and inverted images of objects, is explained by the internal reflexion which takes place, where owing to the heating of a stratum of air near the surface its refractive power is very different from that of the denser stratum next to it.

Such appearances may be imitated, and the effect therefore explained experimentally, by looking at objects so that the rays pass immediately over the surface of a heated iron: or by looking at them through a glass vessel in which are contained several fluids of different refractive powers, which will remain one above the other without mixing, as water, oil, and spirits of wine.

10. If in the formula (2) we make $m = -1$ it becomes

$$\frac{u_{\prime}}{u} = -1.$$

This shows that in reflexion at a plane surface, the reflected rays appear to proceed from an image of the object at the same distance behind the reflector as the object is in front of it.

If an object be placed between two parallel plane reflectors, a continued series of images are produced by the reflexion of each preceding image as an object: and the last remark will also be true; they will appear at successively greater distances; all situated in the same straight line, which is perpendicular to the surface at the point of incidence: and becoming successively fainter from loss of light.

If the two reflectors are inclined, the first apparent image will lie as far behind the reflector as the object is before it, and the reflector bisecting the angle formed by the rays, the positions of the object and image will be such that a circle described about the concourse of the reflectors as a centre and passing through the object, will also pass through the image. This image becoming an object in turn, will be reflected at the second surface, and its image will be formed as far behind that surface as itself is before it. Here again the same conditions will hold, or this second image will lie in the same circle: and so on for all the successive images, or thus *all the images formed by successive reflexion between two plane surfaces inclined to each other, appear to lie in the circumference of a circle described about the intersection of the reflectors at a distance equal to that of the object.*

If the two plane reflectors are inclined to each other at an angle ι since the normals to them are inclined at the same angle, it will readily appear that if a ray be reflected at one surface at an angle ϕ and then again at the second, at $\phi_{,}$ we shall have

$$\phi - \phi_{,} = \iota$$

If again the ray thus falling back on the first surface be reflected thence again to the second, and so on successively, we shall have a series of similar equations

$$\phi_{,} - \phi_{,,} = \iota$$
$$\phi_{,,} - \phi_{,,,} = \iota$$
$$\cdots\cdots\cdots$$
$$\phi_{n-1} - \phi_{n} = \iota$$

Whence the ultimate deviation of the ray from its original direction will be obtained by adding these equations, which gives,

$$\phi - \phi_n = n\iota \qquad (3)$$

It is evident that some value of n will render $n\iota > \phi$, and therefore ϕ_n negative. When this happens the successive points of reflexion which have hitherto been advancing towards the concourse of the two reflectors will recede, and a similar set of reflexions will take place in reverse order, until the reflected ray emerges in such a position that it will not meet the opposite surface.

By pursuing the investigation we might, if it were worth while, calculate the number of images which will be formed for a given inclination of the reflectors. This includes the principle of the optical toy called the kaleidoscope: but the subject being of no importance we shall not pursue it further. A full discussion will be found in Wood's Optics.

11. But there is a result of great importance which follows immediately from the two first of the foregoing equations, from which we have the deviation after two reflexions

$$\phi - \phi_{\prime\prime} = 2\iota \qquad (4)$$

If then two plane reflectors have their inclination variable, we may by means of such a variable angle, measure the inclination of a ray coming directly from one distant object with that from another, by making this last coincide with the position of the twice reflected ray from the first object; in other words, 2ι measures the angle subtended by the two objects. This is the principle of Hadley's quadrant or sextant, or in general of what are termed circular *reflecting* instruments.

The instrument consists essentially of two plane reflectors facing each other, one of which is capable of being inclined to the first at a variable angle, measured on a graduated arc: one object is viewed directly, and the image of the other after two reflexions is made to coincide with the first by moving the revolving mirror into a proper position. If the two objects coincided in position the reflectors ought to be parallel, in order that the one object and the twice reflected image of the other might appear coincident. But if they are separated by an angular space, the mirror must revolve through *half* that angle (as above explained) in order that the image and object may still coincide. This being measured off on the graduated scale and doubled gives the angle sought. The details of the construction and adjustments are usually given in treatises on Astronomy, from which also the student will acquire an idea of the incalculable use of this instrument in Navigation. Though it was first constructed in its present form by the individual whose name it bears, it appears that the idea was originally suggested by Newton. The principle is frequently announced by saying, that *the angular velocity of the reflected ray is double that of the mirror thus revolving.*

12. The geometrical definition of a *prism* includes the idea of a solid having two opposite plane sides inclined to each other at a given angle. When a transparent substance has two plane sides so inclined, it constitutes what in *Optics* is termed a *prism:* we are concerned only with the passage of a ray through these two inclined sides, and do not consider the others. We therefore usually suppose a ray falling on the side of a triangular prism, and after refraction passing to an opposite side inclined to the first at an angle ι. The refracted ray then within the prism forms with the section of its sides in the same plane, a triangle whose angles are

$$\left(\frac{\pi}{2}-\phi_{,}\right),\ \left(\frac{\pi}{2}-\phi_{,,}\right),\ \text{and}\ \iota$$

whence we have $$\phi_{,,} = \iota - \phi_{,} \qquad (5)$$

The angle through which a ray is *deviated* at any surface being the difference $(\phi - \phi_{,})$ at that surface we have on the whole the *total deviation* in the prism, or inclination of the first incident to the last emergent ray expressed by

$$\delta = (\phi - \phi_{,}) + (\phi_{,,,} - \phi_{,,})$$

which on substituting the value of $\phi_{,,}$ becomes

$$\delta = \phi + \phi_{,,,} - \iota \qquad (6)$$

The deviation then being a variable we may investigate the value of ϕ which gives δ a *minimum* value. This is easily done as follows. We have by differentiating the last equation and equation (5)

$$d\,\delta = d\,\phi + d\,\phi_{,,,} \qquad (7)$$

Also $$d\,\phi_{,,} + d\,\phi_{,} = 0$$

Again $$\sin.\,\phi = m \sin.\,\phi_{,}$$

$$\sin.\,\varphi_{,,,} = m \sin.\,\varphi_{,,}$$

Whence $$\cos.\,\varphi\, d\,\varphi = m \cos.\,\varphi_{,}\, d\,\varphi_{,}$$

$$\cos.\,\varphi_{,,,}\, d\,\varphi_{,,,} = m \cos.\,\varphi_{,,}\, d\,\varphi_{,,}$$

Hence we can eliminate $d\,\varphi_{,}$ and $d\,\varphi_{,,}$ and obtain a value of $d\,\varphi_{,,,}$ which substituted in the equation (7) gives

$$d\,\delta = d\,\varphi \left\{ 1 - \frac{\cos.\,\varphi \cos.\,\varphi_{,,}}{\cos.\,\varphi_{,} \cos.\,\varphi_{,,,}} \right\}$$

To have this $= 0$ the second term within the brackets must be $= 1$ and since $\cos.\,\varphi$ cannot be equal to $\cos.\,\varphi_{,}$ nor $\cos.\,\varphi_{,,}$ to $\cos.\,\varphi_{,,,}$ the only value which can make this term $= 1$ will be when $\varphi_{,} = \varphi_{,,}$ and $\varphi = \varphi_{,,,}$ Thus when the first angle of incidence is equal to the last of emergence, the rays have their minimum deviation from their original position.

In this position of minimum deviation we may observe that the values (5) and (6) become

$$\delta = 2\,\varphi - \iota \qquad \varphi_{,} = \tfrac{1}{2}\,\iota$$

Whence we readily deduce

$$\sin.\ \tfrac{1}{2}\,(\delta + \iota) = m \sin.\ \tfrac{1}{2}\,\iota \qquad (8)$$

By means of which equation we can find experimentally the value of m, for the substance of which the prism is made, if ι be measured and δ observed: or again if m be known we can find δ.

Reflexion and Refraction at Spherical Surfaces.

13. Proceeding from the cases which we have just investigated where the surfaces concerned are planes, to those in which they are curved, we have a very extensive subject before us. Without attempting to go into it in its more general relations we shall for the present confine ourselves to the simplest conditions under which the problem can be taken, with a view to the applications we shall want to make of it, and we shall here (as we before observed) find a great source of simplification and avoid much repetition, in looking at the subject in a purely mathematical light; discarding wholly for the present all physical ideas, and confining ourselves to the conception of *straight lines subject to deviation according to the given law* when they meet other lines given in position which we will here suppose to be circular arcs, the sections of spherical surfaces.

Mathematical Theory of Rays deviated at Spherical Surfaces.

14. We will commence with the supposition of a single surface, the section of which is a circular arc whose radius is r,

placed on a given axis, the origin being at the curve; and our object is to trace the course of a single ray u incident upon this arc, at an angle φ with its normal in the same plane, and intercepting a portion of the arc, which we will call θ, measured from the axis: at the curve it is deviated agreeably to the assumed law, or the deviated ray $u_,$ forms a new angle $\varphi_,$ with the normal, such that we have

$$\sin. \varphi = m \sin. \varphi_,$$

m being a given quantity, constant for all values of φ. The line u may be supposed to meet the curve in such a position that produced it would intersect the axis either before or behind the curve; and if we take the point where the curve cuts the axis as the origin, and call f the distance at which u intersects the axis, this will be either positive or negative accordingly. But for the sake of simplicity we will conduct all our investigations on the supposition that it is positive; and we shall afterwards be easily able to accommodate the results when necessary to the case where it is negative.

Then $u_,$ produced will intersect the axis at a distance from the origin, which we will write $f_,$. The position of this point will again depend on whether we have $m >$ or < 1. We will suppose $m > 1$ and afterwards adapt the results when necessary to the supposition $m < 1$. From the triangles thus formed by the rays u and $u_,$ respectively with the normal and axis, we have directly the relations

$$\frac{\sin. \varphi}{\sin. \theta} = \frac{f - r}{u} \qquad \frac{\sin. \theta}{\sin. \varphi_,} = \frac{u_,}{f_, - r}$$

and combining these with

$$m = \frac{\sin. \varphi}{\sin. \varphi_,}$$

we have $$m\, u\, (f_, - r) = u_, (f - r) \qquad (9)$$

From the same triangles also (by Euc. ii, 13) we readily find the following values of u and $u_{,}$ as functions of θ.

$$u^2 = (f - r)^2 + r^2 + 2(f - r) r \cos. \theta$$

which is directly reducible to the form

$$u^2 = f^2 - 2r(f - r) \text{ versin. } \theta$$

and in precisely the same way we shall have

$$u_{,}^2 = f_{,}^2 - 2r(f_{,} - r) \text{ versin. } \theta$$

If we write versin. $\theta = z$ and substitute these values in equation (2) it becomes

$$(f - r)\sqrt{f_{,}^2 - 2r(f_{,} - r)z} = m(f_{,} - r)\sqrt{f^2 - 2r(f - r)z} \qquad (10)$$

This may be considered the fundamental equation from which all expressions for the intersections of deviated rays with the axis are derived. But it is evident from the form of the expression that it does not give us a simple value of $f_{,}$ without some sort of development, or the adoption of some approximation. We shall readily find that it admits of several degrees of such approximation, which we may successively adopt with a view to the several purposes to which the expressions are to be applied.

Direct Pencil. First Approximation.

15. The first of such approximate values may be obtained by observing that if θ be taken a *small* arc, its versed sine z is *extremely small:* if then we neglect the terms involving z we shall have an expression giving a simple value of $f_{,}$ which is sufficiently accurate for many applications, and which may be taken as a first step to more exact values. On this supposition the equation is reduced to the form

$$(f - r) f_{,} = m (f_{,} - r) f$$

And this readily gives us the value

$$f_{,} = \frac{m\, r\, f}{(m - 1) f + r} \tag{11}$$

But we have more frequent occasion to consider the reciprocal of this quantity, which is

$$\frac{1}{f_{,}} = \frac{m - 1}{m\, r} + \frac{1}{m\, f} \tag{12}$$

16. If we now advance to the supposition that a second surface, or circular arc, of radius $r_{,}$ is introduced, placed upon the same axis, at a distance t from the first, and that the deviated ray falls on this curve, and then undergoes a second deviation with a new ratio $m_{,}$ to extend our first expression to this case we should have to consider a function of two variables, z and $z_{,}$ in the expression involving $f_{,,}$. But the first approximation on the above principle, may be obtained directly by the mere substitution of the corresponding symbols in (12): since the ray $u_{,}$ falling upon the second curve, stands in the place of u in respect to it; and its distance of intersection $f_{,}$ stands in the place of f: but in order to be expressed as referred to the new curve as the origin, must obviously have t substracted from it: thus we have the following form analogous to (12):

$$\frac{1}{f_{,,}} = \frac{m_{,} - 1}{m_{,}\, r_{,}} + \frac{1}{m_{,} (f_{,} - t)} \tag{13}$$

If we then take the value of $(f_{,} - t)$ from the previous formula (11), and substitute it here, we shall have the expression in terms of f.

Keeping to this approximate value, in like manner if a third circular arc with radius $r_{,,}$ and ratio $m_{,,}$ be conceived placed in a similar manner on the axis at a distance $t_{,}$ from the last, we might proceed exactly in the same way to obtain the value of $f_{,,,}$ or we should have

$$\frac{1}{f_{,,,}} = \frac{m_{,,} - 1}{m_{,,} r_{,,}} + \frac{1}{m_{,,}(f_{,,} - t - t_{,})}$$

where substituting as before, we obtain the result in terms of f: and similarly we might advance to the introduction of a fourth curve, and so on to any number.

17. In all these cases we shall obtain a great simplification if the intervals $t\ t_{,}$ &c. are small compared with the distances $f\ f_{,}$ &c.; and still more if any simple relation subsist between m and $m_{,}$ &c. The approximate value of $(f_{,} - t)$ from the equation (11) will be

$$f_{,} - t = \frac{m\,r f - [(m-1)f + r]t}{(m-1)f + r}$$

which being substituted in equation (13) and making the supposition $m = \frac{1}{m_{,}}$ we shall have

$$\frac{1}{f_{,,}} = \frac{1-m}{r_{,}} + \frac{m[f(m-1) + r]}{m\,r f - [f(m-1) + r]t}$$

We might continue a similar series of forms which would of course become more complex; if, however, we neglect $t\ t_{,}$ &c. the substitutions are much more easily made, and we have the equations

$$\frac{1}{f_{,,}} = \frac{m_{,} - 1}{m_{,} r_{,}} + \frac{1}{m_{,}} \left\{ \frac{m-1}{m\,r} + \frac{1}{m f} \right\}$$

$$\frac{1}{f_{,,,}} = \frac{m_{,,} - 1}{m_{,,} r_{,,}} + \frac{1}{m_{,,}} \left\{ \frac{m_{,} - 1}{m_{,} r_{,}} + \frac{1}{m_{,}} \left(\frac{m-1}{m\,r} + \frac{1}{m f} \right) \right\}$$

and so on to any number of arcs.

If we add the supposition of $m = \frac{1}{m_{,}}$ $m_{,,} = \frac{1}{m_{,,,}}$, &c. we shall have yet more simplified forms,

$$\frac{1}{f_{,,}} = \frac{1-m}{r_{,}} + \frac{m}{f_{,}} \qquad (15)$$

or substituting for $f_{,}$ in terms of f

$$\frac{1}{f_{,,}} = \frac{1-m}{r_{,}} + \frac{m-1}{r} + \frac{1}{f} \qquad (16)$$

which may be written

$$\frac{1}{f_{,,}} = (m-1) \left\{ \frac{1}{r} - \frac{1}{r_{,}} \right\} + \frac{1}{f} \qquad (17)$$

And if for brevity we write $\frac{1}{r} - \frac{1}{r_{,}} = \frac{1}{\rho}$, this becomes

$$\frac{1}{f_{,,}} = \frac{m-1}{\rho} + \frac{1}{f} \qquad (18)$$

And in like manner we might proceed to the forms for $\frac{1}{f_{,,,}}$ $\frac{1}{f_{,,,,}}$, &c.

18. If we suppose the incident ray to become parallel to the axis, f becomes infinite, or $\frac{1}{f} = 0$, and writing $F_{,}$ $F_{,,}$, &c. to express these particular values of $f_{,}$ $f_{,,}$, &c. and adopting an analogous use of the symbols $\rho_{,}$ $\rho_{,,}$, &c. supposing $m_{,,,} = \frac{1}{m_{,,}}$, &c. we may trace out a series of the forms which the above expressions assume; or we have

$$\frac{1}{F_{,}} = \frac{m-1}{m\,r} \qquad (19)$$

$$\frac{1}{F_{,,}} = \frac{m-1}{\rho}$$

$$\frac{1}{F_{\prime\prime\prime}} = \frac{m_{\prime\prime} - 1}{m_{\prime\prime}\, r_{\prime\prime}} + \frac{1}{m_{\prime\prime}} \left(\frac{m-1}{\rho}\right)$$

$$\frac{1}{F_{\prime\prime\prime\prime}} = \frac{m_{\prime\prime} - 1}{\rho_{\prime\prime}} + \frac{m-1}{\rho}$$

$$\frac{1}{F_{\prime\prime\prime\prime\prime}} = \frac{m_{\prime\prime\prime\prime} - 1}{m_{\prime\prime\prime\prime}\, r_{\prime\prime\prime\prime}} + \frac{1}{m_{\prime\prime\prime\prime}} \left(\frac{m_{\prime\prime} - 1}{\rho_{\prime\prime}} + \frac{m-1}{\rho}\right)$$

$$\frac{1}{F_{\prime\prime\prime\prime\prime\prime}} = \frac{m_{\prime\prime\prime\prime} - 1}{\rho_{\prime\prime\prime\prime}} + \frac{m_{\prime\prime} - 1}{\rho_{\prime\prime}} + \frac{m-1}{\rho}$$

&c. = &c.

We have here supposed $m_{\prime} = \frac{1}{m}$ $m_{\prime\prime\prime} = \frac{1}{m_{\prime\prime}}$ &c. and thus we perceive a remarkable relation between the alternate values, or those of even orders. This relation may be expressed more concisely and the last result generalized by considering each pair of surfaces thus connected as one system, and reckoning n such pairs: in which case we may express the general result by writing F unaccented to signify the value corresponding to the effect of the whole system.

$$\frac{1}{F} = \left(\frac{m-1}{\rho}\right)_{\prime} + \left(\frac{m-1}{\rho}\right)_{\prime\prime} + \cdots \left(\frac{m-1}{\rho}\right)_{n} \quad (20)$$

or adopting a similar mode of expression by the term $\left(\frac{1}{F}\right)$ as applying to each pair separately

$$\frac{1}{F} = \left(\frac{1}{F}\right)_{\prime} + \left(\frac{1}{F}\right)_{\prime\prime} + \cdots \left(\frac{1}{F}\right)_{n}$$

which may also be expressed compendiously by writing

$$\frac{1}{F} = \Sigma\left(\frac{1}{F}\right)$$

The values thus assumed when we suppose the incident ray parallel to the axis, are remarkable in many points of view, and we may here notice the following obvious deductions with respect to them.

From the equation (19) by inverting the terms and subtracting r, we obviously obtain this remarkable relation

$$\frac{F_{\prime}}{F_{\prime}-r} = m \tag{21}$$

Again, from mere inspection of the forms (12) and (18) we perceive that the first term of each is in fact the value of $\frac{1}{F_{\prime}}$ and $\frac{1}{F_{\prime\prime}}$ and it is often convenient to express them with this value introduced, which gives

$$\frac{1}{f_{\prime}} = \frac{1}{F_{\prime}} + \frac{1}{mf} \tag{22}$$

$$\frac{1}{f_{\prime\prime}} = \frac{1}{F_{\prime\prime}} + \frac{1}{f} \tag{23}$$

And here continuing the above supposition as to the successive alternate values of m, we have

$$\frac{1}{f_{\prime\prime}} = \left(\frac{1}{F}\right)_{\prime} + \frac{1}{f} \tag{24}$$

$$\frac{1}{f_{\prime\prime\prime\prime}} = \left(\frac{1}{F}\right)_{\prime\prime} + \frac{1}{f}$$

and we may continue such a series of values which may be generalized as in the former case, by writing

$$\frac{1}{f_n} = \frac{1}{F} + \frac{1}{f}\ *$$

* These remarkable results will be found more generally investigated in Sir J. Herschel's treatise on light, in the Encyclopædia Metropolitana, to which the reader who is desirous of entering profoundly into this as well as other parts of the subject is referred.

It is right here to make a remark on the elegant and compendious *notation* introduced in that treatise. The quantities we have chiefly to consider are *reciprocals:* viz. the reciprocal of the radius expressing the *curvature* of the surface; those of f and $f_{\prime}$ expressing the *proximity* of the points of intersection of incident and deviated rays to the surfaces; and of F, the *power* (as it is termed) of the system of surfaces; (the meaning of which will appear in the

19. We shall here notice a few deductions from the preceding formulæ, which are sometimes referred to:

From the expressions (22) (23) we have

$$f_{,} = \frac{m F_{,} f}{m f + F_{,}} \tag{25}$$

And by subtraction

$$(F_{,} - f_{,}) \; (F_{,} + m f) = F_{,}^{2} \tag{26}$$

Again $$f_{,,} = \frac{f F_{,,}}{f + F_{,,}} \tag{27}$$

Whence $$(F_{,,} - f_{,,}) \, (F_{,,} + f) = F_{,,}^{2} \tag{28}$$

These equations are occasionally referred to. They are usually expressed by the older writers in the form of proportions.

It is sometimes proposed to find when the distance between the points of intersection for the incident and the deviated ray is a maximum: that is, to find $f - f_{,}$ a maximum, for which we must have

$$d f - d f_{,} = 0$$

and consequently $$\frac{d f_{,}}{d f} = 1$$

This value we obtain by differentiating equation (12), which gives

$$\frac{d f_{,}}{d f} = \frac{f_{,}^{2}}{m f^{2}}$$

which by formula (11) will be

$$= \frac{m r^{2}}{[(m-1) f + r]^{2}}$$

sequel). The distinguished author just named has therefore introduced *single* letters to represent these *reciprocals*; but expressive and useful as this notation is for the more extended purposes to which he applies it, it appears on careful consideration that the change of terms might perplex the student without adequate advantage in a discussion of such limited extent as the present.

And when this fulfils the condition of being equal to unity, it is easily seen that we must have

$$f = \frac{(\sqrt{m} - 1)\, r}{m - 1} \tag{29}$$

It is in some cases desirable to express these forms by referring to the *centre* of the spherical surface as *the origin:* in which case writing $c\ c_{,}$ for the distances *from the centre* to the points of intersection hitherto measured by $f\, f_{,}$ we have

$$f = r + c \qquad f_{,} = r + c_{,}$$

and substituting these values in equation (11) it is easily reduced to the form

$$\frac{1}{c_{,}} = \frac{m - 1}{r} + \frac{m}{c} \tag{30}$$

If we have a second surface whose *centre* is situated at a distance t from that of the first, we have, referring to the new centre as in (13)

$$\frac{1}{c_{,,}} = \frac{m_{,} - 1}{r_{,}} + \frac{m_{,}}{c_{,} - t} \tag{31}$$

which may undergo the same modifications.

20. We have hitherto traced the course of a single ray deviated at a spherical surface, and corresponding to a given value of θ. We may further suppose any system of rays so impinging on different parts of the curve, or corresponding to different values of θ, for each of which separately we might find the values of $f_{,}$ &c. We shall generally assume such a system as diverging from or converging to a given point: such a system is called a *pencil of rays,* and if we suppose the *point of convergence or the focus* (as it is termed) of incident rays to *lie in the axis,* it is then termed a *direct* pencil: and in this case we have to observe, that the value of f is common to all the rays, and

consequently the resulting expressions for $f_{,}$ &c. will differ only as far as they are affected by the values of z.

Now it has appeared that the expression (10) gives a *constant value for* $f_{,}$ so long as we suppose f to remain constant and *neglect* z*:* as also does equation (18) for the value of $f_{,,}$. Now as was before observed, if θ be small we may without error adopt this supposition of the neglect of z. In such a case then *a small pencil of rays originating from a given point in the axis, will, after deviation, converge* VERY NEARLY *to one point also in the axis,* and which is called THE FOCUS of deviated rays.

If the arc be supposed to revolve about the axis, and thus generate a small segment of a spherical surface, we shall have a pencil of deviated rays lying very nearly in the form of a cone, whose vertex constitutes the approximate focus.

Physical application of the preceding Theory.

21. After the observations made at the outset, the student will obviously perceive that what has been thus far delivered in purely mathematical language, is in fact a comprehensive statement of the first principles of the optical theory of the refraction or reflexion of a single ray, or of a direct and very small pencil of rays, at a spherical surface, or several such surfaces successively. We may here stop to notice the actual cases to which these principles apply.

The formulæ apply directly and without any modification to the case of *refraction.* When one surface alone is considered, we suppose a ray refracted into a new medium of indefinite extent bounded by a spherical segment. This case simply considered is one which we have little occasion to refer to. But when we add another surface, so as to suppose a portion of a given medium included and isolated from the surrounding medium, this constitutes what is called a *lens*. Unless the contrary is expressed, we usually suppose it of greater refractive power than the surrounding medium. To this case belongs the value of $f_{\prime\prime}$, on the supposition (which is here obviously true since the same medium surrounds the lens on each side) that we have $m_{\prime} = \frac{1}{m}$. If the lens be very thin we may neglect t.

The combination of a series of surfaces, to which the remarkable properties apply expressed by the formulæ (19, 20, &c.) obviously correspond to a system of *lenses* placed together on the same axis: and afford us an elegant and simple expression for their joint action upon a ray of light, or pencil of rays, as compounded of what would be their separate effects upon it.

Spherical Lenses.

22. The formulæ already established give us directly the distances of the foci of lenses in terms which indeed are correct only for infinitely small pencils, but afford approximations in other cases the more accurate as the pencil is smaller. These values are applicable for nearly all the purposes of general explanation, since we suppose, what is almost always the case in practice, that the lens consists of only a very small segment of a sphere, that its thickness is inconsiderable, and that the axis of the pencil coincides with that of the lens.

It is only necessary to distinguish the different cases which arise out of the different combinations of surfaces by which the lenses are formed, and the effect which these differences have on the convergence of the refracted pencils: all which follow directly from our formulæ by the mere consideration of the *signs* of the radii.

At the outset we assumed $+ r$ as corresponding to a surface whose convexity is turned towards the incident ray. In conformity with this assumption the different species of lenses may be enumerated and described as follows:

1. The double convex; or both surfaces convex; in which we have	$+r$ and $-r_{,}$
2. The plano-convex; or one surface convex, the other plane	$+r \quad -r_{,} = \infty$
3. The meniscus, or concavo-convex; in which the surfaces meet	$+r < +r_{,}$
4. The double concave; or both surfaces concave	$-r \quad +r_{,}$
5. The plano-concave; one surface concave, the other plane	$-r \quad +r_{,} = \infty$
6. The convexo-concave; in which the surfaces do not meet	$+r > +r_{,}$

The three first are included under the general term of "convex" lenses, and the three last under that of "concave."

Whenever the focus of refracted rays falls on that side of the lens *from* which we conceive the incident rays to proceed, it is obvious that the rays will actually be diverging. So that the focus will merely be the point from which the *directions* of the rays originate, and is therefore not a physical but a *geometrical* or *imaginary* focus.

23. The principles on which we can shew the effect of the different species of lenses on the convergency or divergency of the rays are these: according as the incident rays are converging or diverging we have $+f$ or $-f$. After deviation they will converge or diverge according as we have $+f_{,,}$ or $-f_{,,}$, and this will of course depend on the signs of the two terms of which the

value of $\frac{1}{f_{,,}}$ is composed, viz. upon the sign of f, which will be as above stated, and on that of $F_{,,}$ which is dependent on the signs and magnitudes of $r\ r_{,}$, that is on the species of the lens. Whenever we have $f_{,,}$ with the opposite sign to f, then diverging rays are made to converge, or converging to diverge. When we have $f_{,,}$ with the same sign as f, then we shall have converging rays, converging more or less than they did before, (that is, converging to a point nearer to or farther from the lens, and therefore at a greater or less angle) according as $f_{,,}$ is less or greater than f. And similarly diverging rays diverging more or less than before, according as $f_{,,}$ is greater or less than f, which will depend in each case on the signs of the terms.

It appears from the formula (17) that in each of the three first cases above enumerated we have the factor $\left(\frac{1}{r} - \frac{1}{r_{,}}\right)$ positive, and in the three last negative. Hence we may trace the results in each of the cases arising out of the general formula (17) which may be thus exhibited in a tabular form.

Lens.	Incident Pencil.	$\frac{1}{f_{,,}}$	Sign of $f_{,,}$	Refracted Pencil.
Convex $+F_{,,}$	Diverging $-f$	$\frac{1}{F_{,,}} - \frac{1}{f}$	$f > F_{,,}$: $+$	Converges.
Convex $+F_{,,}$	Diverging $-f$	$\frac{1}{F_{,,}} - \frac{1}{f}$	$f < F_{,,}$: $-$, $f_{,,} > f$	Diverges less.
Convex $+F_{,,}$	Converging $+f$	$\frac{1}{F_{,,}} + \frac{1}{f}$	$+$, $f_{,,} < f$	Converges more.
Concave $-F_{,,}$	Diverging $-f$	$-\frac{1}{F_{,,}} - \frac{1}{f}$	$-$, $f_{,,} < f$	Diverges more.
Concave $-F_{,,}$	Converging $+f$	$-\frac{1}{F_{,,}} + \frac{1}{f}$	$f > F_{,,}$: $-$	Diverges.
Concave $-F_{,,}$	Converging $+f$	$-\frac{1}{F_{,,}} + \frac{1}{f}$	$f < F_{,,}$: $+$, $f_{,,} > f$	Converges less.

Upon the whole we may state the results of the above synopsis by saying that the *convex* lenses TEND to give *convergence*, and the *concave, divergence* to the incident rays.

From the formula for the focus of parallel rays it follows, that the place of that focus is the same, whichever side of the lens is turned towards the incident light.

If we have a combination of lenses acting together, the formulæ (19, 20) apply directly: and upon determining the principal focal length, whose reciprocal is called the *power* of each lens, separately, we find the *power* of the combination expressed by the algebraical *sum* of the powers of the separate lenses.

If the double convex lens become a sphere, that is, if its two radii are equal and opposite and have the same centre, then taking the formula (31) substituting $m = \frac{1}{m_{,}}$ $r = -r_{,}$ and $t = 0$ and putting for $c_{,}$ its value from (30) it gives us

$$\frac{1}{c_{,,}} = \frac{2(m-1)}{m\,r} + \frac{1}{c} \qquad (32)$$

On differentiating the formula (18), we have

$$\frac{-d f_{,,}}{f_{,,}^{2}} = \frac{\mp d f}{f^{2}}$$

the upper sign corresponding to the cases when f and $f_{,,}$ have the same sign, the lower to those in which they are different. Hence, on the same side of the origin, as f increases $f_{,,}$ also increases; on opposite sides, as f increases $f_{,,}$ decreases, or in either case the foci of incident and refracted rays (which are called *conjugate* foci) move in the *same direction*.

Spherical Reflectors.

24. In the case of reflexion we have usually to consider only one surface. And the formulæ for spherical reflectors result immediately from the first of the preceding expressions, by the

mere substitution of the value $m = -1$. By this means we obtain the following expressions from the forms (19) and (12)

$$\frac{1}{F_,} = \frac{2}{r} \qquad (33)$$

$$\frac{1}{f_,} = \frac{2}{r} - \frac{1}{f} \qquad (34)$$

$$(F_, - f_,)\ (F_, - f) = F_,^2 \qquad (35)$$

We may observe that in equation (34) the terms are in arithmetical progression, and consequently their reciprocals $f_,\ r\ f$ will be a harmonical. From equation (31) we also get by the same substitution the forms referring to the centre,

$$\frac{1}{c_,} = -\frac{2}{r} - \frac{1}{c} \qquad (36)$$

And adopting an analogous notation, $C_,$ to distinguish the case of parallel rays,

$$\frac{1}{C_,} = -\frac{2}{r} \qquad (37)$$

Whence in general $$\frac{1}{c_,} = \frac{1}{C_,} - \frac{1}{c} \qquad (38)$$

25. Of spherical reflectors it is evident that we have only two species: the *convex*, which accords with the general case of our former investigation, and in which we have supposed $+ r$; and the *concave*, in which we consequently have $- r$.

It is evident from the formula (33) that according to the sign of r we have $F_,$ positive or negative. Hence taking the cases which arise out of the general equation (34) we have the following results exactly analogous to those in the case of lenses, and which may be best exhibited in the same tabular form :

Mirror.	Incident Pencil.	$\frac{1}{f_,}$	Sign of $f_,$	Reflected Pencil.
Convex $+F_,$	Diverging $-f$	$\frac{1}{F_,}+\frac{1}{f}$	$+$; $f_,<f$	Diverges more.
Convex $+F_,$	Converging $+f$	$\frac{1}{F_,}-\frac{1}{f}$	$f>F_,$: $+$	Diverges.
			$f<F_,$: $-$; $f_,>f$	Converges less.
Concave $-F_,$	Diverging $-f$	$-\frac{1}{F_,}+\frac{1}{f}$	$f>F_,$: $-$	Converges.
			$f<F_,$: $+$; $f_,>f$	Diverges less.
Concave $-F_,$	Converging $+f$	$-\frac{1}{F_,}-\frac{1}{f}$	$-$; $f_,<f$	Converges more.

Hence in general convex mirrors *tend* to give divergency, and concave, convergency to incident pencils.

On differentiating the formula (35) we have

$$\frac{-df_,}{f_,^2} = \frac{\pm df}{f^2}$$

The upper sign corresponding to f and $f_,$ with the same signs. Hence (on precisely the same principles as in refraction) the conjugate foci of spherical reflectors move in *opposite* directions.

Direct Pencil. Second Approximation.

26. We before remarked that the constancy of $f_,$ or of $f_{,,}$ and consequently the accuracy of convergence to a focus, depends upon our being able to neglect z. It is also evident if z be not neglected, yet if it be small compared with the other quantities, we shall still have for the different rays of a pencil extending over an arc measured by moderately small values of θ, the distances $f_,$ or $f_{,,}$ expressed by quantities which differ extremely

little, or the points at which the rays passing at different parts of the arc intersect the axis will differ little from being coincident; and this conclusion entirely depends upon the nature of the function in which we at first had the values of $f_{,}$ involved, and which is of such a form as to admit of a limit, or to have a finite value when $z = 0$.

If we conceive, agreeably to our former suppositions, rays falling upon all parts of a portion of the curve including an arc θ, the smaller θ is taken the more will the distance $f_{,}$ or $f_{,,}$ approach its limiting value, and will decrease as θ increases.

Thus for such systems of given extent, the points of intersection with the axis lie within certain limits, the rays successively crossing the axis and intersecting each other: so that when the arc is supposed to revolve about the axis and form a spherical surface, we shall thus have conical surfaces of rays corresponding to successive annuli, having their vertices in the axis; and as the arc θ is larger the vertices will lie at points further separated along the axis, the different conical surfaces successively intersecting each other, and the locus of such intersections being a sort of funnel-shaped surface.

27. The investigation of the limits within which these vertices or points of intersection of the rays $u_{,}$ with the axis lie, depends evidently on principles involved in our primary formula, that is on a further approximation towards the exact values of $f_{,}$ $f_{,,}$ &c. To proceed then to such further approximation we must consider the development of the expression (10), which we shall be able to obtain as that of $f_{,}$ a function of z, by means of Maclaurin's theorem, in terms of the limiting values of the original function and of its successive differential coefficients. The modification we have hitherto adopted has been equivalent to taking the first term of such a development or the value of $f_{,}$ when $z = 0$. The entire development will evidently be

$$f_{,} = (f_{,})_0 + \left(\frac{df_{,}}{dz}\right)_0 z + \left(\frac{d^2 f_{,}}{d z^2}\right)_0 \frac{z^2}{1.2} + \&c. \quad (39)$$

In executing the differentiation of equation (10), in order to obtain the second term, we meet with no other difficulty than what arises from the length of the expression: but after several reductions we obtain the value when $z = 0$ in the form

$$\left(\frac{d f_{,}}{d z}\right)_0 = \frac{(f - r)\, r}{(m - 1) f + r} \left(f_{,0} - r\right) \left\{ \frac{m}{f} - \frac{1}{f_{,0}} \right\}$$

This again admits of another simplification, since the first factor is evidently what would result from taking the value of $f_{,}$ given by equation (11), and subtracting r: thus we have

$$\left(\frac{d f_{,}}{d z}\right)_0 = \left(f_{,0} - r\right)^2 \left\{ \frac{m}{f} - \frac{1}{f_{,0}} \right\} \quad (40)$$

But while θ remains of moderate magnitude, the powers of z above the first may be neglected, ; and if we proceed only to this term, involving the first power of z, we shall have a second approximation sufficient for the purposes of our investigation.

28. If we proceed to introduce a second surface (and we may limit our investigation to the case when $m = \frac{1}{m_{,}}$) we shall have further considerations to attend to. We have hitherto been concerned with $f_{,}$ as a function of z: we have now to consider $f_{,,}$ whose value depends first on that of $f_{,}$ which determines the position in which the ray $u_{,}$ is incident on the second surface: and secondly, on the conditions introduced with the second surface; where in a way precisely analogous to what takes place at the first, $f_{,,}$ appears as a function of the new arc, or of its versed sine $z_{,}$: or simply we have to develope $f_{,,}$ as a function of two

variables, $z\ z_{\prime}$, which we can do by the formula corresponding to Maclaurin's theorem, or we must take

$$f_{\prime\prime} = f_{\prime\prime 0} + \left\{ \left(\frac{d f_{\prime\prime}}{d z}\right)_0 z + \left(\frac{d f_{\prime\prime}}{d z_{\prime}}\right)_0 z_{\prime} \right\} + \&c. \qquad (41)$$

For the formation of the first of these partial differential coefficients it will suffice to take the first approximation (15), considering $f_{\prime}$ as the variable, and on differentiating it we have directly

$$\frac{d f_{\prime\prime}}{d f_{\prime}} = \frac{m f_{\prime\prime}^2}{f_{\prime}^2}$$

Whence we obtain

$$\left(\frac{d f_{\prime\prime}}{d z}\right)_0 = \left(\frac{d f_{\prime\prime}}{d f_{\prime}} \cdot \frac{d f_{\prime}}{d z}\right)_0 = \left(\frac{m f_{\prime\prime}^2}{f_{\prime}^2}\right) \cdot \left(\frac{d f_{\prime}}{d z}\right)_0$$

The second is obtained at once by the formula (40), merely writing $f_{\prime}$ for f, $r_{\prime}$ for r, $z_{\prime}$ for z, and $m = \frac{1}{m_{\prime}}$ which gives us

$$\left(\frac{d f_{\prime\prime}}{d z_{\prime}}\right)_0 = (f_{\prime\prime 0} - r_{\prime})^2 \left(\frac{1}{m f_{\prime}} - \frac{1}{f_{\prime\prime 0}}\right)$$

Hence by substituting these expressions we have the approximate value of the development (41)

$$\left.\begin{aligned} f_{\prime\prime} = f_{\prime\prime 0} + \frac{m f_{\prime\prime}^2}{f_{\prime}^2} \left\{ (f_{\prime 0} - r)^2 \left(\frac{m}{f} - \frac{1}{f_{\prime 0}}\right) z \right\} \\ + (f_{\prime\prime 0} - r_{\prime})^2 \left(\frac{1}{m f_{\prime}} - \frac{1}{f_{\prime\prime 0}}\right) z_{\prime} \end{aligned}\right\} \qquad (42)$$

It has been usual in elementary optics to regard the first approximation as generally sufficient, and to introduce the second rather as a correction subsequently applied by transposing the first term of the development to the other side, and thus having the *difference* between the exact value and the first approximation expressed by the second term : or in other words, the distance between limiting position of the intersection of a ray with the axis, and the actual position of any ray corresponding to a given value of θ. In this point of view the second term is called the *aberration*, and written $= a$.

29. These expressions obviously include both reflexion and refraction. For the aberration in reflexion at one surface, we substitute as before $m = -1$, and we have by substituting the corresponding value of $\frac{1}{f_{,0}}$ in the second factor, the expression

$$a = (f_{,0} - r)^2 \frac{2z}{r} \tag{43}$$

which for parallel rays becomes

$$a = \frac{rz}{2} \tag{44}$$

In the general case the aberration in reflexion is frequently expressed as referred to the centre of the spherical surface for the origin : or we have,

$$(f_{,0} - r) = c_, = \frac{c\, C_,}{c - C_,} \qquad \text{also } C_, = -\frac{r}{2}$$

Hence we find on substituting in (43)

$$a = \frac{c^2\, C_,}{(c - C_,)^2} z \tag{45}$$

Some writers commence the investigation and deduce the whole development in these terms: but the method here followed appears at once simpler and more general.

30. From the above expression, for one surface evidently, and for two if we suppose $z = z_,$ nearly, it appears that the aberration will vary as z or versin. θ : that is, for a small arc, or when

we neglect z^2, by the property of the circle, as $\sin.^2 \theta$. When the arc is supposed to revolve about the axis, the sine becomes the radius of the circle, limiting the extent of the surface, and is termed the "*aperture.*" Hence we say that *the aberration varies as the square of the aperture, nearly.* (46)

A perpendicular to the axis at the distance $f_{,0}$ meeting the ray produced being written $= b$, we have directly

$$\frac{b}{a} = \frac{r \sin. \theta}{f_{,} - z} \tag{47}$$

But for small changes in the value of θ or $\sin. \theta$ the change in $f_{,}$ and in z will be insensibly small, so that if we regard the denominator as nearly constant, since a varies as $\sin.^2 \theta$, this shews that b varies as $\sin.^3 \theta$. b is frequently called the *lateral aberration,* as a is called the *longitudinal:* hence we say that *the lateral aberration varies as the cube of the aperture, nearly.* (48)

Returning now to the expression (47), if we take another arc $\theta_{,}$ and corresponding lines $b_{,}$ $a_{,}$, and, as just observed, the denominator being regarded as constant, on comparing the values we have very nearly

$$\frac{b\, a_{,}}{a\, b_{,}} = \frac{\sin. \theta}{\sin. \theta_{,}}$$

This will of course apply equally if the arc $\theta_{,}$ be taken on the other side of the axis; so that the two rays corresponding to these arcs being produced will intersect; and from the point of their intersection dropping a perpendicular y on the axis, it will divide the distance $a - a_{,}$ into two parts, of which that commencing from $f_{,}$ may be called x: then it will be evident from the similar triangles that we have the following ratios:

$$\frac{x}{y} = \frac{a}{b} \qquad \frac{a - a_{,} - x}{y} = \frac{a_{,}}{b_{,}}$$

And thence $$\frac{a - a_{,} - x}{x} = \frac{\sin.\ \theta}{\sin.\ \theta_{,}}$$

From which we deduce

$$a - a_{,} = x\ \frac{\sin.\ \theta + \sin.\ \theta_{,}}{\sin.\ \theta_{,}} \tag{49}$$

But from what has preceded we can express this in another form: for since on the principle (46) we have

$$\frac{a_{,}}{a} = \frac{\sin.^{2}\ \theta_{,}}{\sin.^{2}\ \theta}$$

this will give us

$$a - a_{,} = a\ \frac{\sin.^{2}\ \theta - \sin.^{2}\ \theta_{,}}{\sin.^{2}\ \theta} \tag{50}$$

And comparing these values we find

$$x = a\ \frac{\sin.\ \theta_{,}\ (\sin.\ \theta - \sin.\ \theta_{,})}{\sin.^{2}\ \theta} \tag{51}$$

From this form we may find the *maximum* value of x: which from a well-known principle will take place when the two factors in the numerator are equal; this will evidently be the case when we have $\sin.\ \theta_{,} = \frac{1}{2}\sin.\ \theta$; and the corresponding value of x will obviously be

$$x = \tfrac{1}{4}\, a \tag{52}$$

And since x and y increase together, we have at the same time the maximum value of y

$$y = \tfrac{1}{4}\, b \tag{53}$$

From the slightest consideration of the successive positions of the rays, it will be evident that this value of y is the greatest distance from the axis at which any ray corresponding to a variable value $\theta_{,}$ intersects the extreme ray, which we have supposed corresponding to a fixed value θ: all the rays therefore pass *within* the distance so determined; and it is the least perpendicular through which they all pass; or when the system revolves it becomes the radius of the *least circular section* of the funnel-shaped surface formed by the intersections of the cones of rays. This is called *the circle of least aberration, or least diffusion.*

31. We may here observe that (for one surface) the aberration will be nothing, or the rays converge accurately if the conditions be such that we have either

$$f_{,0} = r \text{ and therefore by (12) } f = r$$

or

$$\frac{1}{f_{,0}} = \frac{m}{f}$$

that is, by an easy deduction from (12), when we have

$$f = r\,(m + 1) \qquad (54)$$

In this last case the position of the focus being found, a circular arc of any convenient radius about it as a centre will cut all the converging rays at right angles, and therefore a second surface so placed will not alter their convergence; or we may thus have a *lens* possessing the same property.

Recurring to the general theory, its application to the aberration of lenses and mirrors is sufficiently obvious. So long as we suppose very small portions of spherical surfaces, or small pencils of rays incident about the axis, the first approximation suffices for the position of the focus. For larger arcs the point of maximum condensation, or least aberration above determined, is to be taken as the sensible focus. The aberration when applied to reflectors is most usually expressed in terms referring to the centre.

In estimating the precise effect and amount of aberration, and following out the different cases, there is however some complexity. These points, together with the investigation of the general conditions which shall make the aberration vanish, or give accurate convergence by combinations of spherical surfaces (of which we just mentioned one very simple case), and which are called *aplanatic* lenses; and when this cannot be effected, the determination of that combination which with a given power shall have the *least possible* aberration:—will all be found fully discussed in Sir J. Herschel's treatise on light: Art. 294, seq.; but it would be unsuitable to the limits of this work to enter upon them.

Oblique Pencil, passing through the Centre.

32. We have thus far considered an incident pencil of *rays whose origin is situated in that radius of the spherical surface which we have taken as the axis,* and upon which also we suppose the other surfaces, if such are introduced, to be similarly situated. And we have investigated the distances along that axis at which the deviated rays will successively intersect it, and by the concurrence of those intersections accurately, or within certain limits, give rise to foci situated in the axis.

If, however, we suppose the *origin of the incident rays not to be restricted to lying in the axis,* then we must have recourse to other considerations.

We shall at present consider one case in which the conditions are remarkably simplified, and which involves an important property.

Taking one surface, if we conceive a system of rays diverging in all directions from a given point *out of the axis,* it is evident that among all these rays there is one whose direction passes through the *centre* of the spherical surface, and which consequently passes the surface *undeviated.* This may be taken as the axis of *a small pencil surrounding it,* and for this small pencil the distance $f_{,}$ may be found along this new axis by the same formulæ as before.

33. If we suppose *two surfaces* this will no longer hold good: but we find that there are positions here in which oblique rays may pass without deviation, or nearly so. This will readily appear from simple geometrical considerations, since corresponding points may be found in the two circular arcs at which the tangents or small portions of the surfaces are parallel.

Now let any two circular arcs with radii $r\ r_{,}$ be placed on the same axis at an interval t, and points be taken in each where the tangents are parallel, *the line joining those points* cuts the axis at a distance e from the first arc. Then since the radii at these two points are necessarily parallel, we have from the similar triangles the proportion

$$\frac{r}{r-e} = \frac{r_{,}}{r_{,}+t-e}$$

Whence
$$e = \frac{t\,r}{r-r_{,}} \qquad (55)$$

Since this value involves none but quantities *constant* for all parts of the same arcs, it follows that *all such lines cut the axis in the same point*. This investigation applies to all the cases of lenses by giving r its proper sign; and it appears that the point thus determined may lie either within or without the lens, according to the conditions of the different cases.

Now to apply this to our purpose we have only to observe, that if a ray be incident at such an angle that *its direction after passing the first surface* pass through the point thus determined, it will emerge *parallel* to its original direction: and there may be an indefinite number of rays so incident at all parts of the first surface. This point has been termed the "*centre*" of the lens; or to distinguish the word used in this sense from its geometrical meaning, it may be called the *optical centre*. If the distance t be inconsiderable the ray will pass without any sensible deviation, or *coincide* with its original direction.

Any such ray therefore may (as in the former case) be taken as the *axis of a small pencil,* which (unless the obliquity be very great) will converge very nearly to a focus *measured along it* by the original formulæ. The distance e is obviously the particular value of $f_{,}$ belonging to such a ray.

Where only one surface is concerned, the analogous property which we have just noticed referring to its centre, may be considered as a particular case of this more comprehensive property. The distance e in that case becomes the radius of the sphere.

34. An important consequence arises if we suppose several systems of rays incident in this way, the positions of their origins being given, in which case among all the rays of each set there must be some one in each, and a small pencil about it to which the above conditions apply.

If we conceive the origins to lie in the plane of the section in a given form, or a number of them contiguous so as to form a locus whose nature is known, then the points corresponding to the respective values of $f_{,,}$ for each pencil, will form another locus whose nature may be determined from that of the given locus.

If the lens be such that the optical centre falls within it, then, if the given locus be a circular arc whose centre coincides with the optical centre, it will be readily seen that the focal locus will be also a circular arc similar to it and concentric with it.

If it be a *straight line perpendicular to the axis,* then, in the case of two surfaces (t being disregarded) and designating the values f &c. on the oblique axes by f^1 &c. θ being the angle which any of these oblique axes passing through the "centre" form with the original axis, we have

$$f = f^1 \cos. \theta.$$

Hence in the formula for $f_{,,}$ (27) applied to the oblique axis on substituting the value of f' and dividing, since $F_{,,}$ is constant, we have

$$f_{,,}^1 = \frac{F_{,,}}{1 + \frac{F_{,,}}{f} \cos. \theta} \tag{56}$$

And in the same way for one surface, observing that e or $f_,$ is here equal to the radius of the spherical surface and the angle θ is formed at the centre, we have by formula (25)

$$f_,^1 = \frac{m\,F_,}{m + \frac{F_,}{f} \cos. \theta} \tag{57}$$

and if $m = -1$ this becomes

$$f_,^1 = \frac{F_,}{1 - \frac{F_,}{f} \cos. \theta} \tag{58}$$

Thus we find that in all the cases the equation of the locus of the focal points is *the polar equation of the second degree,* the species of the curve being determined by the values of $F_,$ and f.

An investigation of the same point might also be made by referring to rectangular coordinates; but the method here followed is simpler.

Optical Images.

35. The optical application of these principles is of importance. Since from every point in a luminous or illuminated object

there originates a system of rays diverging in all directions, some small pencil of each of these systems will fall on a lens or spherical reflector placed at a given distance in such a position that the *direction* of the ray after *reflexion*, or after the *first refraction*, will pass through the *optical centre* of the reflector, or of the lens, which is the condition of the above theory. Hence the foci of these small pencils will form a corresponding locus of bright points, giving rise to an *image* or picture of the original object : these are usually called *optical images*.

We have considered certain cases in which we have found the locus of these focal points *in the plane of refraction or reflexion*, and the revolution of the system would thence produce a focal luminous surface; and if we conceive some material surface made to coincide exactly with this optical surface, it would have painted, as it were upon it, a representation of the original object, which in the case (for example) of a spherical segment, would be a similar segment: if a straight line, or object occupying a plane surface, the image will lie in the surface of revolution of a conic section. If therefore in this case the image were received upon a *plane* screen, it would appear distorted and confused towards the edges.

If we suppose the extent of the object to be but small, and regard only its *linear* dimension in the plane of any section of the lens or reflector, and compare this with the *linear* dimension of the image in the same plane at right angles to the axis, it is obvious that since the two extreme rays including that linear space cross on the axis, the ratio of the linear magnitudes will be simply that of the distances from the point of intersection at which the object and image are respectively situated.

In the case of a reflector these will be the focal distances referred to the centre.

In the case of a lens, when the optical centre falls *within* the lens, and when the thickness is neglected, the distances will be the conjugate focal distances referred to the surface. These are the cases most usually considered.

36. The formation of optical images will be better understood if we here consider the simplest case in which it takes place. If a small aperture be made in the shutter of a dark room, among the rays which proceed in all directions from every point in external illuminated objects within the range of the visible space opposite, some from every such point will fall upon the aperture, and crossing there will pass into the room, and may be received on a screen or on the wall opposite: here then there will be a corresponding point of light received from every point outside, and of the same proportional intensity and colour: so that in fact a picture of the external objects in their relative position, but all *inverted*, will be formed on the screen, and may be seen by an eye situated anywhere in the room by means of the irregular reflexion or dispersion of the light so incident from the surface of the screen. The clearness of the image depends upon the minuteness of the aperture, or on the small diameter of each pencil when it falls on the screen.

If instead of the small aperture, a *lens* were fixed in the shutter, every one of the incident small central pencils would be brought to a focus (as just explained), and being thus not only reduced to a mere point, but a greater quantity of light being concentrated, a much brighter and more distinct image is produced, and may be seen painted as it were on a screen with great beauty and accuracy towards the central part, though distorted towards the edges. This is the principle of the Camera Obscura.

On the same principle, if the direct light of the sun be received on the small aperture, the rays from the different parts of its disk which cross at the aperture, will give a corresponding circular image on the screen properly placed. In this case if the aperture be very small, its *figure* is immaterial, for suppose it *triangular*, a number of minute triangular pencils will cross and proceed to form an image made up of a number of small triangular luminous spaces partially superposed, but arranged so as to cover a circular space resembling the figure of the sun's disk, and which if the triangles be very small will be sensibly circular. This fact was noticed by Aristotle. In a solar eclipse the image has a part similarly cut off.

Oblique Pencil, not passing through the centre.

37. In order to proceed to the more general discussion of oblique rays, we must first establish as a fundamental position, the limiting ratio of the increments of the incident and deviated rays subjected to the general law at any surface. And we here consider u to represent the length of the incident ray from the radiant point to the surface, and $u_,$ that of the deviated ray to *its point of intersection with a contiguous ray* at a very small interval.

Corresponding to the very small increment of the incident ray $d\,u$, we will suppose a small increment of the arc, which is the section of the given surface, and which we will write $d\,s$. Then in the usual construction of the incremental triangle, it will readily appear that the angle formed with the arc by the small perpendicular dropped on the ray will be equal to φ: thus we shall have

$$d\,u = d\,s\,.\,\sin.\,\varphi$$

And in precisely the same way with regard to $u_,$

$$d\,u_, = d\,s\,.\,\sin.\,\phi_,$$

Whence, introducing the value m from the fundamental law, we have the equation

$$d\,u - m\,d\,u_, = 0 \qquad (59)$$

38. Let us suppose a small pencil converging to a point out of the axis incident upon a portion of a curve whose radius of curvature is γ, and that the pencil after passing the surface converges accurately or approximately to a focus, whose distance is measured along the oblique axis of the pencil. Let the original point of convergence be joined with the centre of curvature by a line which we will call k, and the focus of the oblique pencil with the same centre by a line $k_{,}$. The length $u_{,}$ here expresses the approximate focal distance. Then by the oblique triangles thus formed, we have

$$k^2 = u^2 - \gamma^2 + 2\, u\, \gamma \cos.\, \phi$$
$$k_{,}^2 = u_{,}^2 - \gamma^2 + 2\, u_{,}\, \gamma \cos.\, \varphi_{,}$$

And since a small variation in the arc intercepted makes an insensibly small change in $k_{,}$ and k is constant, we may differentiate the two equations for the variables, $u\ \phi\ u_{,}\ \phi_{,}$ respectively, which gives,

$$0 = u\, d\, u + \gamma \cos.\, \phi\, d\, u - u\, \gamma \sin.\, \phi\, d\, \varphi$$

$$0 = u_{,}\, d\, u_{,} + \gamma \cos.\, \phi_{,}\, d\, u_{,} - u_{,}\, \gamma \sin.\, \phi_{,}\, d\, \phi_{,}$$

From these expressions we can eliminate the differentials by means of the relation established in equation (59), and of

$$\sin.\, \phi = m \sin.\, \phi_{,}$$

which gives $\quad d\, \phi \cos.\, \phi = m\, d\, \phi_{,} \cos.\, \phi_{,}$

Hence by getting the equations above into such a form that we can readily avail ourselves of these substitutions, we shall obtain an expression for $u_{,}$ in terms of u and the other quantities.

There are several forms into which such an expression may be successively brought dependent merely on trigonometrical changes: for our present purpose it will suffice to observe that we may easily deduce the following:

$$u_{,} = \frac{u\, \gamma \cos.\, \phi_{,} \tan.\, \phi}{u \tan.\, \phi - (u + \gamma \cos.\, \phi) \tan.\, \phi_{,}} \qquad (60)$$

It will be readily seen that by means of such an expression for a small pencil supposed to be brought to its focus at the distance $u_{,}$ we thus determine its position, that of the focus of the incident pencil being supposed given. We shall have occasion afterwards to refer more particularly to this form. For the present it may be observed that we have hitherto always investigated the positions, &c., of rays in *one plane,* and then found that by supposing the whole construction to *revolve* we had similar conclusions applying to the *surface* and to the solid pencil so generated. This mode of proceeding however is necessarily limited to the case where the axis of the pencil coincides with the axis of the curve.

In the case of an oblique pencil we must suppose in the preceding construction that the ray or axis $u_{,}$ of the small deviated pencil, is produced to meet the line k, and its length so intercepted we will call $v_{,}$; also let the angle which the line k forms with the radius be χ, then it will be evident that we have the ratios

$$\frac{\gamma}{u} = \frac{\sin.\ (\phi + \chi)}{\sin.\ \chi} \qquad (61)$$

$$\frac{\gamma}{v_{,}} = \frac{\sin.\ (\phi_{,} + \chi)}{\sin.\ \chi} \qquad (62)$$

39. The use of introducing this last construction will now appear. We suppose a small arc of a curve which may be taken as coincident with an arc of the circle of curvature, and the plane in which any one deviation takes place, and to which the foregoing formula applies, must pass through the origin of the incident rays and the centre of the circle of curvature.

If we suppose another section taken anywhere to the right or left of the plane we have hitherto assumed, which we will call the *primary* plane still passing through the line k, or what is the same thing, if we suppose the whole to revolve about k through a very small angle, it is evident that the focus for each section retains its position determined by $u_,$ in that plane; and that we shall in consequence have a series of such focal points lying in a small circular arc corresponding to the arc through which the section revolves, having its centre in the line k, and in the point where $u_,$ produced meets it. This small circular arc, then, lies in a plane at right angles to the primary plane, and which we will therefore call the *secondary* plane. All the rays in each section will pass through their focal point in this small arc, and will all meet in the line k. Thus considering this small arc as very nearly a portion of a straight line, it is said *that in a small oblique pencil after deviation all the rays pass through two lines, in planes at right angles to each other,* which, as the rays do not pass through any one focal point, are called *focal lines:* and this property has been denominated *astigmatism.*

The form which the deviated pencil takes is worthy of notice. If the original incident pencil be supposed to have its transverse section a circle, whose diameter is λ, when it falls obliquely on the surface its section will be an ellipse, whose principal diameters are λ and λ sec. ϕ. It will thus take the form of a conoidal solid whose base is an ellipse, and its transverse sections will in *general* be ellipses. Let us suppose a section made at any distance x from the surface measured along $v_,$: calling the diameter in the primary plane l, that in the secondary h, we evidently have their lengths expressed by the forms

$$l = \lambda \sec. \phi \frac{u_, - x}{u_,} \quad (63)$$

$$h = \gamma \frac{v_, - x}{v_,} \quad (64)$$

as x increases both l and h decrease: at one point they become equal or the section is a circle.

When $x = u_{,}$ we have $l = 0$ and

$$(h) = \lambda \frac{v_{,} - u_{,}}{v_{,}}$$

or the ellipse merges in the secondary focal line, whose length is thus given.

Proceeding onwards, x being greater than $u_{,}$, or the numerator being $(x - u_{,})$ we have elliptic sections in which l increases and h decreases, till at a certain distance they become again equal, and the section a circle. When this happens we have, by equating the above values

$$u_{,} (v_{,} - x) = v_{,} (x - u_{,}) \text{ sec. } \phi$$

Whence by an easy reduction we obtain

$$x = \frac{u_{,} (1 + \text{cos. } \phi)}{1 + \frac{u_{,}}{v_{,}} \text{ cos. } \phi}$$

for the distance at which this circle is formed. Its diameters are found by substituting this value of x, and are each equal to

$$(h) = \lambda \frac{v_{,} - u_{,}}{v_{,} + u_{,} \text{cos. } \phi}$$

Here the rays approach most nearly to convergence, and it is called the *circle of least confusion*.

Increasing x further we have h decreasing and l increasing till $x = v_{,}$ when $h = 0$, and

$$(l) = \lambda \text{ sec. } \phi \frac{(v_{,} - u_{,})}{u_{,}}$$

The ellipse here merges in the primary focal line, whose length is thus given. After this l and h both continue to increase indefinitely.

The subject of oblique pencils if further pursued would lead us into very complex details: what has been here given must suffice for the purpose of elementary explanation, and will serve to shew how greatly a system of reflected or refracted rays, if it include pencils of any considerable obliquity, will deviate from accuracy of convergence. The effect thus produced as well by the aberration as by the astigmatism upon the formation of images will be readily apparent, and the distortion and indistinctness it will occasion towards the edges, or indeed at all parts except just those formed by the most central portion of the rays. It will thus be seen that the full development of this part of the subject must be of considerable importance in its practical application to the construction of optical instruments, where nice determinations are required of the focal lengths, and exact estimates of the amount of deviation corresponding to given apertures and obliquities. But for full information on these points the student must refer to Coddington's treatise on Reflexion and Refraction.

40. It is worth noticing that the general supposition of the formula (60), is such as to include under it the case of a small pencil coincident with the axis; for dividing by tan. $\phi_{,}$ it becomes,

$$u_{,} = \frac{u \, \gamma \cos. \varphi_{,} \dfrac{\tan. \varphi}{\tan. \varphi_{,}}}{u \dfrac{\tan. \phi}{\tan \phi_{,}} - (u + \gamma \cos. \varphi)}$$

Now if we suppose the rays u and $u_{,}$ (which may be taken as the axes of small pencils), to coincide with the axis, or make $\phi = 0$, $\varphi_{,} = 0$, we shall have,

$$\text{The limit of } \frac{\tan. \varphi}{\tan. \varphi_{,}} = \frac{\sin. \varphi}{\sin. \varphi_{,}} = m$$

Or, since the values $u \; u_{,}$ when measured along the axis become $f, f_{,}$ and $\gamma = r$ we have the formula

$$f_{,} = \frac{f \, r \, m}{(m - 1) f + r}$$

The same as that found by our first investigation.

Loci of the Intersections of deviated Rays.

41. Recurring now to the equation (60) we may make use of it to determine the nature of the locus of successive intersections of contiguous rays corresponding to any extent of the given curve *in the primary plane.* For it is evident, from the very nature of the formula, that it will enable us to exhibit an equation to such a curve; the radius γ being expressed in terms of the coordinates of the given curve, and the angles ϕ and consequently $\phi_{\prime}$ being also determined agreeably to some known property of that curve, and the given position of the origin of incident rays: *such loci are called* CAUSTICS.

If we consider the surface instead of its section, in general, the loci of the intersections of the deviated rays become very complicated, as will appear from the mere consideration of the focal lines. In the case of a surface of revolution, the origin of incident rays lying in the axis, it will be evident that the focal line in the primary plane coincides with the axis: and this line, or the locus of its centre, is to be considered as one of the caustic loci. The focal line in the secondary plane is evidently a small segment of a transverse section of the surface formed by the locus of successive intersections in successive positions of the primary plane revolving about the line k, which now coincides with the axis. We shall confine our enquiries therefore to the caustic in the primary plane.

42. The general equation (60) will receive modifications in its form according as the following suppositions are successively adopted.

1st. If the incident rays are parallel, the terms which do not involve u will vanish in comparison with those which do, and we shall have

$$u_{,} = \frac{u\,\gamma \cos.\,\phi_{,} \tan.\,\phi}{u \tan.\,\phi - u \tan.\,\phi_{,}}$$

Which is easily reducible to the form

$$u_{,} = \frac{\gamma \cos.^{2} \phi_{,} \sin.\,\phi}{\sin.\,(\phi - \phi_{,})} \tag{65}$$

2nd. For the case of reflexion, taking the general form and making $m = -1$ we have $\phi = -\phi_{,}$, and observing that we must substitute $-\tan.\,\phi$ for $\tan.\,\phi_{,}$ there results

$$u_{,} = \frac{u\,\gamma \cos.\,\phi}{2\,u - \gamma \cos.\,\phi} \tag{66}$$

3rd. In this case, with parallel rays, we find directly

$$u_{,} = \tfrac{1}{2}\,\gamma \cos.\,\phi \tag{67}$$

43. The subject of caustics when different curves are assumed as the section of the given surface, is one which has been extensively treated by several writers. It is purely a matter of curiosity, and this chiefly in a geometrical point of view. We will however consider a few of the most striking instances which will serve to illustrate the theory.

It may first be observed, that if we could suppose a medium whose refractive power is infinite, or where $m = \infty$ then, in the fundamental equation

$$m = \frac{\sin.\,\phi}{\sin.\,\phi_{,}}$$

(Supposing ϕ to remain finite) we must have $\phi_{,} = 0$, or the deviated ray would coincide with the normal, and the equation (60) would be reduced to

$$u_{,} = \gamma$$

Hence in this case the *caustic would coincide with* the locus of the centre of the circle of curvature, that is, with *the evolute* of the given curve.

But returning to finite values of m, let the given curve bounding the refracting medium be a *logarithmic spiral* and the focus of incident rays at its pole. From the nature of the curve it is evident that ϕ is the complement of the angle formed by the radius with the tangent (and which is designated by ψ in the theory of curves), ϕ therefore is equal to the angle designated by ϕ in that theory. Hence * we have

$$u = \gamma \cos. \phi$$

And since from the nature of the curve, ϕ, and therefore also $\phi_{\prime}$ are constant for all points in the curve, the general formula (60) assumes a form which is directly reducible to

$$u_{\prime} = \frac{u}{\cos. \phi} \cdot \frac{\cos. \phi_{\prime} \tan. \phi}{\tan. \phi - 2 \tan. \phi_{\prime}}$$

or $u_{\prime}$ is in a constant ratio to u.

Now from the constancy of the angles ϕ $\phi_{\prime}$ their difference or that contained between u and $u_{\prime}$ is constant, and consequently that contained between $u_{\prime}$ and the line joining the pole and the extremity of $u_{\prime}$ or the point in the sought locus: but since $u_{\prime}$ is always a tangent to this locus, it follows that it has the property that its radius forms a constant angle with its tangent, or it is itself a *logarithmic spiral:* whose species is determined by the ratio of the constant angles, which is readily formed.

In the case of reflexion, sin. $m = -1$ or $\phi = \phi_{\prime}$ the above expression for $u_{\prime}$ becomes simply

$$u_{\prime} = u$$

And on the same considerations we find that the new locus cuts its radii at the same angle as the given spiral does; or it is a logarithmic spiral *similar* and therefore *equal* to the former.

* See The Geometry of Curves and Curved Surfaces, investigated by the application of the Differential and Integral Calculus; by the author of this work, *Oxford*, 1830, p. 132.

44. If the surface be a plane, we might take the general formula as modified by the supposition of $\gamma = \infty$ and so proceed to obtain an equation for the caustic. But we may in this case adopt a simpler method founded on geometrical considerations. We have before established the relation (2)

$$\frac{u}{u_{,}} = \frac{1}{m}$$

supposing the rays u $u_{,}$ limited by meeting a perpendicular to the surface. Now let us suppose that where u meets this perpendicular is the radiant point, and let us conceive *an ellipse or hyperbola* having its major axis coinciding with this perpendicular, and its minor axis with the surface, its focus at the radiant point, and its ratio of eccentricity $= m$; then from the point of incidence we have a line u drawn to the focus, and if also from the same point a second line be drawn cutting the major axis, and such that its direction shall coincide with a normal to the curve at some opposite point, its segment intercepted by the major axis being called v, then, from the properties of the conic sections, it will be readily seen that we have

$$\frac{u}{v} = \frac{1}{m}$$

Comparing this with the relation above stated, we see that $v = u_{,}$ and the *deviated ray coincides in position with the normal of the conic section* thus constructed. And this being the case for successive rays which intersect each other, it follows that the locus of their intersections will coincide with the locus of the *intersections of the normals* of the conic section: but this locus is the *evolute* of the conic section, or *the caustic to a plane surface is the evolote to a conic section* described as above: and which will be an ellipse or hyperbola as we have m less or greater than unity.

45. In the case of reflexion when the given surface is *spherical,*

the caustic in the primary plane is easily found by geometrical considerations, of which we will merely give the outline: 1st. When the incident rays are parallel to the axis; 2nd. When they diverge from the extremity of the diameter.

In the 1st case we have directly

$$u_{,} = \frac{r}{2} \cos. \phi$$

In the 2nd since $u = 2\,r \cos. \phi$ the form (66) becomes

$$u_{,} = \frac{u}{3}$$

In either case the incident and deviated rays within the circle will form equal chords; and since a right angle at the extremity of the chord formed by $u_{,}$ lies in the semicircle, a perpendicular at any point will divide the diameter through the point of incidence, in the same ratio.

Hence in the first case a perpendicular at the extremity of the value given to $u_{,}$ will cut off half the radius. In the second case one-third of the diameter.

In either case the extremity of $u_{,}$ will thus lie in the circumference of a small circle on the segment of the diameter so determined. This point will *revolve* in the small circle which itself moves along with the radius of the original circle, *and will therefore trace out an epicycloidal curve.*

In the first case the radius of the revolving circle is half that of the base. In the second it is equal to it. In the first case therefore the locus is a *common epicycloid;* in the second it is the species called the *cardioid.*

We can find the locus on principles very similar when the reflecting curve is a cycloid, and the incident rays parallel to its axis.

In this curve the normal is a chord of the generating circle, and forms with its diameter, perpendicular to the base, an angle equal to the angle φ which the ray $u_{\prime}$ forms with it. Hence (from the nature of the circle) the ray $u_{\prime}$ passes through the centre of the circle.

But we have in this curve the normal $n = \frac{\gamma}{2}$. Hence by the formula for the caustic

$$u_{\prime} = \frac{\gamma}{2} \cos. \varphi = n \cos. \varphi$$

At the extremity of $u_{\prime}$ therefore a right angle is formed with a line drawn to the foot of the normal: it therefore lies in the circle, on half the diameter of the generating circle.

As the diameter of this circle moves along the base parallel to itself, the point $u_{\prime}$ revolves in it: and thus traces out the caustic, which is *another cycloid of half the linear dimensions of the first.*

We might give other examples: but enough has probably been said to exemplify the principle. We will add a few remarks.

46. From the nature of the investigation by which we deduce the theory of caustics it will readily appear, that the property of the assumed curve or surface which is made use of in order to deduce the caustic, is solely such as refers to the angles of incidence at that surface; and it will readily appear that several different curves might give the same directions to the deviated rays, and so generate the same caustic. This will be understood in a more general point of view by only recollecting that the differential equation (59), which is involved in the discussion, gives the connexion between the increments only of the rays, and its integration introduces an arbitrary constant, and whether it be applied to rays $u_{\prime}$ converging in one point, or successively intersecting, and so becoming tangents to a new curve, there may be an infinite number of loci which shall fulfil the conditions according to the values given to the constant.

We may also observe in general that *for a very small arc* of a spherical surface, and by consequence of any other whose section may be considered coincident with its circle of curvature, and for rays whose origin lies in the axis, since the aberration varies as the square of the aperture, if ω be the angle which a ray touching the caustic forms with the axis, taking the tangent instead of the sine of its inclination as determining the aperture, and assuming a constant k, we have the aberration

$$a = k \tan.^2 \omega$$

or (taking the ultimate focus as the origin) in terms of the co-ordinates $x\,y$ of the point in the caustic

$$a = k \frac{d\,y^2}{d\,x^2} = k\,p^2$$

Also from the curve we have

$$a = x - y \frac{d\,x}{d\,y} = x - y \frac{1}{p}$$

Combining these equations

$$y = p\,x - k\,p^3 \qquad (68)$$

Differentiating this equation and observing that $p\,d\,x = d\,y$ we obtain

$$(x - 3\,k\,p^2)\;d\,p = 0$$

writing the first factor $= 0$ and substituting in (68) we have

$$y = 2\,k\,p^3$$

restoring the value of p, transposing and integrating we obtain

$$x = \frac{3}{2}\,(2\,k)^{\frac{1}{3}}\;y^{\frac{2}{3}}$$

or within small limits near the ultimate focus, or cusp, every caustic approaches to coincidence with a semi-cubical parabola.

Rays after forming a caustic may intersect again and form a new caustic, and so on successively. On this point the reader is referred to Herschel on Light, p. 362. Examples of simple caustics will be found in Coddington's short treatise, p. 17 and 80. The subject is almost entirely one of geometrical curiosity; and the different cases may be investigated either independently by geometrical methods, as in Wood's Optics, p. 240, seq. or as we have taken them; or yet more generally by means of an equation which we shall give presently. The loci are easily traced out by accurate drawing; and may be readily exhibited experimentally by light reflected from any polished curved surfaces, as from the inside of a cup, when the caustic is formed on the surface of the liquid in it; or by means of a slip of polished metal which may be bent into any concave form, and held upon a card so that the plane of the card is nearly in the plane of reflexion, when the curves are beautifully formed upon it. The caustics by reflexion were formerly called *catacaustics,* as those by refraction *diacaustics:* these last may also be shewn by placing a card nearly in the plane of refraction, and receiving on it the rays which have passed through a glass vessel full of water &c., or by other similar means.

Surfaces of accurate Convergence.

47. We have hitherto all along supposed the nature of a curve or surface given, and our object has been to find the position assumed by the deviated rays in consequence of the conditions supposed. We will terminate this part of our subject by an investigation of a reverse description: where the problem is *to find the nature of a surface such that the deviated rays shall converge accurately to one point;* the discussion of which was originally pursued by Newton and Des Cartes.

Joining the given point with the focus of incident rays by a line $= c$, and taking this as the axis X to which the locus is to be referred, the given point of convergence being the origin, we shall have,

$$u = \sqrt{(c - x)^2 + y^2}$$

$$u_{,} = \sqrt{x^2 + y^2}$$

If we also take the fundamental formula

$$du - m\,du_{,} = 0$$

and integrate it adding a constant n and substituting the above values of u and $u_{,}$ we obtain

$$\sqrt{(c - x)^2 + y^2} - m\sqrt{x^2 + y^2} + n = 0 \qquad (69)$$

If we proceed to remove the radical sign it is evident that there must result an equation of the fourth degree, the locus of which is the curve sought, and which by its revolution will generate a surface fulfilling the required condition.

Without stopping to discuss this curve we may consider one or two cases in which the results assume particular forms worthy of notice.

If we suppose the incident rays parallel, the increment of u will be equal to that of x or $du = dx$ and the formula (59) becomes,

$$dx - m\,du_{,} = 0$$

and integrating and substituting as before, we have

$$x - m\sqrt{x^2 + y^2} + n = 0$$

In this case by transposing and removing the radical sign the resulting equation is of the second degree

$$m^2 y^2 + (m^2 - 1)\,x^2 - 2nx - n^2 = 0 \qquad (70)$$

the species of the curve being dependent on the value of m^2 compared with unity, if greater an ellipse, if less an hyperbola.

In the general case if $m = -1$ the equation (69) become

$$\sqrt{(c - x)^2 + y^2} + \sqrt{x^2 + y^2} + n = 0$$

and in this instance it will be found that on transposing and squaring, certain terms will destroy each other, and we easily deduce

$$4 n^2 y^2 - (c^2 - n^2)^2 - 4 (c^2 - n^2) x^2 + 4 (c^2 - n^2) c x = 0$$

the species of which depends on the value of c^2 compared with n^2, and will be an hyperbola if $c^2 > n^2$, an ellipse if $c^2 < n^2$.

In the case of parallel rays with $m = -1$ the equation (70) becomes

$$y^2 - 2 n x - n^2 = 0$$

the equation to a parabola.

These last results might easily be established on separate grounds by well-known geometrical properties of the Conic Sections.

The relation expressed by the general equation (69) between u and $u_{\prime}$ is such as affords an easy construction of the curve, one of the intersecting radii together with a constant being in a given ratio to the other.

If we suppose the constant $n = 0$ we have the intersecting radii u and $u_{\prime}$ in the constant ratio m, a condition which shews (by a well-known property) that the locus is in this case a circle: as indeed also appears from the equation which thus is at once cleared of roots, and becomes

$$(c - x)^2 + y^2 = m^2 (x^2 + y^2)$$

Thus we can determine a spherical surface, such that in a determinate position it shall bring to an accurate focus rays whose point of convergence is given.

In the construction of this circle by means of the constant ratio above stated, it appears that writing b for the value of x at which the locus cuts the axis and $2 r$ for the diameter, we have the same ratio subsisting at each extremity of the diameter, or

$$\frac{c + b}{b} = m = \frac{c + b - 2 r}{2 r - b}$$

whence we find

$$c + b = r(m + 1)$$

And observing that from the assumption here made, as to the origin, $(c + b)$ is the distance of the incident focus from the surface, we see that this case coincides with that deduced before on another principle (54).

In any of these cases we may introduce a second surface, which in order that the convergency may not be affected, must be such as shall be perpendicular to all the rays, or a spherical surface of any convenient radius having the point of convergence as its centre.

More general view of the preceding Theory.

48. In the preceding theory we have pursued mathematical investigations of the positions, intersections, &c. of rays having undergone optical deviation at one or more surfaces, usually spherical, by methods sufficiently general to include all the cases we had occasion to consider, though far from possessing all the generality of which the subject is susceptible. We will now proceed to a brief view of a more comprehensive principle: and without entering into considerations of too abstruse a nature for the ordinary student, we shall be able to present all the leading points hitherto discussed connected together by their common derivation from *the equation to a deviated ray:* whence also we shall deduce some further properties.

Equations of deviated Rays.

49. Taking as the section of the surface an arc of any curve if the angles which the normal to the curve forms with the

axes be written X, Y, respectively, and we take $d\,s$ the small increment of the arc, then it will be readily seen that we have

$$\sin. X = \cos. Y = \frac{d\,x}{d\,s} \qquad \cos. X = \sin. Y = \frac{d\,y}{d\,s}$$

And if the incident ray form with X the angle ω, it is evident that we also have from this triangle whose three angles are φ, ω, and X, or their supplements

$$\sin. \varphi = \sin. (\omega + X)$$

or
$$\sin. \varphi = \sin. \omega \frac{d\,y}{d\,s} + \cos. \omega \frac{d\,x}{d\,s} \tag{71}$$

In precisely the same way we have for the deviated ray

$$\sin. \varphi_{\prime} = \sin. \omega_{\prime} \frac{d\,y}{d\,s} + \cos. \omega_{\prime} \frac{d\,x}{d\,s} \tag{72}$$

Substituting these values in the fundamental equation which may be written in general

$$\sin. \varphi + m \sin. \varphi_{\prime} = 0$$

$$\sin. \omega\, d\,y + \cos. \omega\, d\,x + m\,(\sin. \omega_{\prime}\, d\,y + \cos. \omega_{\prime}\, d\,x) = 0 \tag{73}$$

Let $x\,y$ be the coordinates of the point of incidence, $\alpha\,\beta$ those of any point in the incident ray, and $\alpha_{\prime}\,\beta_{\prime}$ those of the deviated ray: then, since both rays pass through the point $x\,y$, we shall have their respective equations

$$\left.\begin{array}{l} \beta - y = \tan. \omega\,(\alpha - x) \\ \beta_{\prime} - y = \tan. \omega_{\prime}(\alpha_{\prime} - x) \end{array}\right\} \tag{74}$$

Between the equations (73) and (74) we can eliminate ω and $\omega_{\prime}$ and get the relations of $\alpha\,\beta\;\alpha_{\prime}\,\beta_{\prime}$. And we thus obtain

$$\cos. \omega = \frac{\alpha - x}{\sqrt{(\alpha - x)^2 + (\beta - y)^2}}$$

$$\sin. \omega = \frac{\beta - y}{\sqrt{(\alpha - x)^2 + (\beta - y)^2}}$$

$$\cos.\,\omega_{,} = \frac{\alpha_{,} - x}{\sqrt{(\alpha_{,} - x)^2 + (\beta_{,} - y)^2}}$$

$$\sin.\,\omega_{,} = \frac{\beta_{,} - y}{\sqrt{(\alpha_{,} - x)^2 + (\beta_{,} - y)^2}}$$

Hence we deduce

$$\frac{(\alpha - x)\,dx + (\beta - y)\,dy}{\sqrt{(\alpha - x)^2 + (\beta - y)^2}} + m\,\frac{(\alpha_{,} - x)\,dx + (\beta_{,} - y)\,dy}{\sqrt{(\alpha_{,} - x)^2 + (\beta_{,} - y)^2}} = 0 \quad (75)$$

An equation which since the coordinates of the point of incidence and their differentials are known from the given nature of the curve, and $\alpha\ \beta$ from the given position of the incident ray, gives us the relation between $\alpha_{,}\ \beta_{,}$ or contains *the equation of the deviated ray.*

From this fundamental equation we may proceed to make various deductions.

50. In the first place we may remark that the denominators of the two fractions in (75) will represent the *lengths* of the portions of the rays intercepted between the point of incidence and any points $\alpha\beta\ \alpha_{,}\beta_{,}$ in them respectively. We will therefore express those denominators by u and $u_{,}$

When the incident ray is parallel to the axis $\beta - y = 0$, whence $u = \alpha - x$ and the equation (75) is reduced to

$$u_{,}\,dx + m\left\{(\alpha_{,} - x)\,dx + (\beta_{,} - y)\,dy\right\} = 0 \qquad (76)$$

Resuming the general equation (75). If we differentiate the value of $u^2\ u_{,}^2$ relatively to x and y only, we obtain

$$(\alpha - x)\,dx + (\beta - y)\,dy = -u\,du$$
$$(\alpha_{,} - x)\,dx + (\beta_{,} - y)\,dy = -u_{,}du_{,}$$

Substituting these values in equation (75) we have

$$du + m\,du_{,} = 0 \qquad (77)$$

when the incident ray is parallel to the axis $du = -dx$, and this equation becomes

$$m\,d\,u_{,} - d\,x = 0 \qquad (78)$$

It hence appears that the function $(u + m u_{,})$ is a minimum when taken from any assumed point in the incident to any in the deviated ray.

We hence obtain the surface of accurate convergence as before.

51. If the given surface be a plane, we may take the normal as the axis X, which gives $\beta = 0$ and if we take the point where this normal meets it as the origin, we shall have for any point in the plane which may be taken for the point of incidence $x = 0$, and therefore $d\,x = 0$. Hence the equation (75) gives for this case,

$$m\,u\,(\beta_{,} - y) - u_{,}y = 0 \qquad (79)$$

or if we transpose and square both sides, substituting for u and $u_{,}$ their values, it becomes

$$(\beta_{,} - y)\left\{m^2\,\alpha^2 + (m^2 - 1)\,y^2\right\}^{\frac{1}{2}} = \alpha_{,}y \qquad (80)$$

52. If the given surface be spherical, taking the line joining the radiant point and the centre as the axis X, we have $\beta = 0$, and from differentiating the equation to the circle (the origin at the centre) we have

$$p = \frac{d\,y}{d\,x} = -\frac{x}{y}$$

Substituting in (75) we have

$$u_{,}\,\alpha\,y + m\,u\,(\alpha_{,}y - \beta_{,}x) = 0 \qquad (81)$$

which gives the equation of the deviated ray.

For the point at which the deviated ray meets the axis $\beta_{,} = 0$ and we have

$$u_{,}\,\alpha + m\,u\,\alpha_{,} = 0 \qquad (82)$$

which on squaring and substituting for u and $u_{,}$ gives

$$\alpha^2 \left\{ (\alpha_{,} - x)^2 + y^2 \right\} = m^2 \alpha_{,}^2 \left\{ (\alpha - x)^2 + y^2 \right\} \quad (83)$$

and for rays indefinitely near the axis $y = 0 \therefore x = r$ and the equation becomes

$$\alpha (\alpha_{,} - r) = m \alpha_{,} (\alpha - r) \quad (84)$$

or

$$\frac{m}{\alpha} - \frac{1}{\alpha_{,}} = \frac{m - 1}{r} \quad (85)$$

These equations obviously coincide with those from which we before derived the whole theory of foci, aberrations, etc.

53. If we make the radiant point the origin, $\alpha = 0$ $\beta = 0$ and the equation (75) becomes (omitting the accents of $\alpha_{,}$ $\beta_{,}$)

$$m u \left\{ (\alpha - x) + p (\beta - y) \right\} - u_{,} (x + p y) = 0$$

If we consider $\alpha \beta$ as constant, and $x y$ as variable, this will correspond to the condition of two deviated rays meeting at the point $\alpha \beta$, which are nearly contiguous by virtue of a small increment in the arc, or change in x and y. Differentiating then this equation in respect to x and y we shall have an expression which is without difficulty reduced to

$$\left\{ m u (\beta - y) - u_{,} y \right\} q = (m u + u_{,}) (1 + p^2) + (u + m u_{,}) \frac{d u}{d x} \cdot \frac{d u_{,}}{d x}$$

Here again we can substitute for $u u_{,}$ and their differentials, and so ultimately obtain

$$m^2 (x^2 + y^2) (\beta x - \alpha y) q = m^2 (p x - y)^2 (\alpha + p \beta) + (m^2 - 1) (x + p y)^3 \quad (86)$$

By this equation then eliminating from the given equation of the curve the terms $x y p q$, we shall have an equation between α

and β, which gives *the locus of the points of intersection of contiguous deviated rays* or the equation to the *caustic*.

If the rays become parallel, or the radiant point be infinitely distant, we shall have $x = \infty$ $\alpha = \infty$ and the terms not involving these factors will vanish in comparison of the others—on this consideration it will be seen that the equation (86) becomes

$$m^2 (\beta - y)\, q = m^2 (1 + p^2) - 1$$

If $m = -1$ the general equation (86) becomes

$$(x^2 + y^2)(\beta x - \alpha y)\, q = (\alpha + p\beta)\,(p x - y)^2$$

and for parallel rays

$$(\beta - y)\, q = p^2$$

The deviated ray being a tangent to the caustic forms with the axis X an angle ω, determined by

$$\tan.\ \omega_{,} = \frac{d\beta_{,}}{d\alpha_{,}}$$

or $$d\beta_{,} \cos.\ \omega_{,} - d\alpha_{,} \sin.\ \omega_{,} = 0 \qquad (87)$$

Again, if we differentiate the two equations which consist in writing u $u_{,}$ equal to their values in (75), and substitute the values $(\alpha - x) = u \cos.\ \omega$, &c. from the equations (74), introducing the conditions of equation (73), we shall obtain

$$d\beta_{,} \sin.\ \omega_{,} + d\alpha_{,} \cos.\ \omega_{,} = \frac{du}{m} + du_{,}$$

Squaring and adding this to the equation above, (87), we have

$$d\beta_{,}^2 + d\alpha_{,}^2 = \left(\frac{du}{m} + du_{,}\right)^2$$

But the first side of this equation is the square of the differential of the arc of the caustic; or we have

$$ds = \frac{du}{m} + du_{,}$$

and therefore $$s = \frac{u}{m} + u_{,} + \text{const.}$$

The caustic curve therefore is always *rectifiable* if the given curve be an algebraic one.

Let us take as an example, the problem to find the caustic to a plane. We have already found the equation of the deviated ray in this case

$$(\beta_{,} - y) \left\{ m^2 \alpha^2 + (m^2 - 1) y^2 \right\}^{\frac{1}{2}} = \alpha_{,} y \qquad (88)$$

dividing by y and differentiating in respect to y we find

$$m^2 \alpha^2 \beta_{,} + (m^2 - 1) y^3 = 0 \qquad (89)$$

and if we multiply the equation (88) by $m^2 \alpha^2$ and from (89) obtain by subtraction

$$m^2 \alpha^2 (\beta_{,} - y) = - y \left\{ m^2 \alpha^2 + (m^2 - 1) y^2 \right\}$$

we deduce

$$m^2 \alpha^2 \alpha_{,} = - \left\{ m^2 \alpha^2 + (m^2 - 1) y^2 \right\}^{\frac{3}{2}} \qquad (90)$$

Again, finding from (88) a value of y in terms of $\beta_{,}$ substituting it in (89) and raising both sides to the power $\frac{2}{3}$, we get

$$\alpha^{\frac{2}{3}} + \left(\sqrt{1 - m^2}\, \beta_{,} \right)^{\frac{2}{3}} = (m\alpha)^{\frac{2}{3}}$$

which is the equation to the *evolute of a conic section*, having its centre at the origin, and focus at the radiant point. It will be an ellipse or hyperbola according as m is less or greater than unity.

General Systems of Rays.

54. Before quitting the purely theoretical part of our subject we may properly introduce a very slight mention of the yet more

extended point of view in which theories of optical rays have been investigated.

Such a theory has been given by Malus in his Traité d'Optique, prefixed to his prize memoir on Double Refraction, *Paris*, 1810.

In this investigation he conceives a system of rays to undergo *optical deviation* at any number of surfaces of any given kind successively, and on their emergence at the last surface he finds expressions from which are deduced the laws which regulate the positions they assume, and the loci formed by their intersections. In order to convey a somewhat more distinct idea of the nature of these investigations we will endeavour to describe, though very briefly and generally, the first principle of them.

55. If in the last curve surface, which we will call M, we suppose any points $P\ P'$ at which rays $R\ R'$ emerge, very near together, these two rays will not in general intersect or even lie in the same plane. But taking P as a fixed point, it is shewn that certain positions of P' may be found in which the rays $R\ R'$ will intersect. These positions are such that a series of rays $R\ R'\ R''$ &c. will successively intersect emerging from points $P\ P'\ P''$ &c. which are found to lie in a certain curve which we will call S, traced upon the curve surface M, and having its nature dependent on that of the surface and the law of deviation to which the rays have been subjected. And this curve continues in like manner through a similar series of points on the other side of P.

Again, it is found that a corresponding set of points, $P\ p'\ p''$ &c. from which emergent rays form another series of successive intersections, lie in another curve S' on the surface M, which continued also the other way through P intersects the curve S where it passes through P.

Thus for every point in the surface M there will be sets of curves S accompanied by others S' crossing them.

From each of the curves S we shall have a corresponding curve s formed by the consecutive intersections of the rays $R\ R'\ R''$ &c. and the assemblage of these for all the curves S will give a curve surface which we will call Σ.

Similarly the assemblage of all the curves s' formed by the intersections of rays from the points along the curve S' will give another curve surface Σ'. These are termed caustic surfaces.

The nature and form of such surfaces admit of infinite diversity. And any surface being assumed as a caustic it may be shewn that there are an infinite number of surfaces which might generate it.

A small pencil of the rays $R\ R'$ &c. thus intersecting being produced and supposed to enter the eye, enter there and produce vision, having the same arrangement as rays would have coming from a real surface coinciding with Σ. The same is the case with the rays forming the surface Σ'. Hence the combined effect of the two together is to produce a confused and indistinct impression on the eye. With real surfaces (supposing them, or one at least of them, transparent, so as to allow the eye to receive rays from both at once in the same direction,) this confusion would not exist, because they would send also at the same time other rays irregularly dispersed, which would give ideas of their distance, solidity, &c.

The curves and surfaces thus generated will of course undergo great modifications according to the particular limitations which we may successively adopt in the conditions of the general theory.

56. To take one instance, which illustrates well the foregoing remarks, and which we have before considered in part; when the rays pass only one surface, and that a plane, the medium

from which they emerge being the *denser*, it is found that one of the surfaces Σ reduces to a straight line passing through the radiating point or origin of light, and perpendicular to the plane surface. The other surface Σ′ is found to be that generated by the revolution of the evolute of the ellipse about its axis, having its focus at the radiating point. Hence an eye receiving a small emergent pencil consisting of rays from a small portion of the curve Σ′, on producing the tangent to meet the line Σ, it appears that one point in that line will also furnish a ray coinciding in direction with the former pencil, and reaching the eye along with it; hence there will be a certain degree of confusion in the impression produced. But if the eye be placed perpendicularly over the point so as to receive the rays belonging to the caustic line Σ, as well as those from the cusp of the surface Σ′, it will receive a very distinct impression.

If the surface be spherical there will be a line Σ as in the last case, and the surface Σ′ of a regular funnel shape.

And in a plane reflector the whole is reduced to one point, through which the directions of all the reflected rays pass behind the reflector at a distance equal to that of the radiating point before it.

But a more remarkable case is that afforded by the Iceland crystal, which, as is well known, separates a ray traversing it into two pencils, the one undergoing ordinary refraction and on emergence entering the eye composed of a small assemblage of rays, each of which of course has in reality a slightly different direction with respect to the axis of the crystal. But for such difference in direction in the rays of the emergent pencil, by virtue of the law which regulates the extraordinary refraction, will the corresponding rays of the *extraordinary* pencil assume *directions deviating more from each other* and intersecting in two surfaces Σ Σ′ neither of which reduces to a straight line, and which have no points in common; hence the extraordinary image results from the rays of *two* surfaces which, as before stated, should cause it to be somewhat indistinct. And it is found experimentally to be less distinct than the ordinary image.

57. The fundamental principle, that at every point there are four positions in which contiguous rays will intersect, may be shewn as follows: Let us assume as the equations of a straight line referred to three rectangular coordinate axes

$$z = a x + \alpha \quad z = b y + \beta \tag{91}$$

If in these equations the coefficients $a\,\alpha\;b\,\beta$ be considered as variable independently of each other they will represent all possible straight lines. If these coefficients be made to vary according to a certain law, then the above equations will represent a certain definite system of straight lines, and conversely any assignable system of straight lines may be represented by them by introducing a proper relation between the coefficients. Let it be required to introduce such a relation between them as that the equations (91) shall represent a system of rays after undergoing optical deviation at any number of surfaces successively, supposing the luminous body to be a point, and the equation of the last surface to be

$$F\,(x\,y\,z) = 0$$

Let $x'\,y'\,z'$ be the coordinates of a point of the surface, and let us consider the equations (91) as those of the deviated ray passing through that point. It is clear that the position and direction of the deviated ray depend entirely on the coordinates of the point of incidence and the inclination of the plane tangent to the surface of that point. Therefore the coefficients $a\,\alpha\;b\,\beta$ are each a certain function of $x'\,y'\,z'\;\frac{d\,z'}{d\,x'}\,\frac{d\,z'}{d\,y'}$ or since $\frac{d\,z'}{d\,x'}\,\frac{d\,z'}{d\,y'}$ may be supposed given by the equation $F\,(x'\,y'\,z')$ corresponding to the point of incidence, it follows that all the coefficients $a\ \alpha\ b\ \beta$ are given functions of $(x'\ y'\ z')$ and that we shall have

$$\left.\begin{array}{ll} a = \phi_1\ (x'\ y'\ z') & \alpha = \phi_2\ (x'\ y'\ z')\ . \\ b = \phi_3\ (x'\ y'\ z') & \beta = \phi_4\ (x'\ y'\ z')\ . \end{array}\right\} \tag{92}$$

in which $\phi_1\ \phi_2\ \phi_3\ \phi_4$ denote known functions.

Eliminating $x'\ y'\ z'$ and a from the four equations (92), and $F\ (x'\ y'\ z') = 0$ we shall get an equation between $\alpha\ b$ and β; eliminating α from the same equations we shall get an equation between $a\ b$ and β; and these two equations will constitute the required relation that must be introduced between the coefficients of equations (91) in order to make them represent the system of deviated rays. We might in the same way eliminate b and β or any two of the coefficients $a\ \alpha\ b\ \beta$; but the resulting equations would only be equivalent to those already obtained. The required relation, in fact, amounts to two equations between $a\ \alpha\ b\ \beta$; one equation containing any three, and the other any three including the coefficient left out in the first. They may be represented thus,

$$a = \chi\ (\alpha\ \beta) \qquad b = \psi\ (\alpha\ \beta) \tag{93}$$

Suppose now we want to know when two *consecutive* deviated rays will intersect each other. If x, y, z, be the coordinates of their intersection, we shall have

$$z = a\,x + \alpha \qquad z = b\,y + \beta$$

because this point belongs to the one ray, and supposing $a\ b\ \alpha\ \beta$ to take the increments $\delta a\ \delta b\ \delta\alpha\ \delta\beta$ respectively, they become $a + \delta a\ \ b + \delta b$ &c. Substituting these values in the original equation of the straight line, and supposing $\delta\alpha$, &c. very small, we get the equations of a line contiguous to the original one.

Hence we shall also have

$$z = (a + \delta a)\,x + \alpha + \delta\alpha \quad z = (b + \delta b)\,y + \beta + \delta\beta$$

because the same point belongs to the second.

These four equations give, when x, y, z, are eliminated,

$$(\alpha\, \delta a - a\, \delta \alpha)\, \delta b = (\beta\, \delta b - b\, \delta \beta)\, \delta a \qquad (94)$$

Now by equations (93)

$$\delta a = \frac{d\,a}{d\,\alpha}\, \delta \alpha + \frac{d\,a}{d\,\beta}\, \delta \beta,$$

$$\delta b = \frac{d\,b}{d\,\alpha}\, \delta \alpha + \frac{d\,b}{d\,\beta}\, \delta \beta.$$

Substituting these values in equation (94), we get an equation containing $\delta\alpha$ and $\delta\beta$ raised to the square, consequently there are two different directions in every system of deviated rays in which we may pass from one ray to a contiguous one meeting the first. The *two directions* are indicated by the *double value* of $\frac{\delta\alpha}{\delta\beta}$· and we may observe that if $\delta\alpha$ and $\delta\beta$ change signs both at once, the value of their ratio is not altered, but they will change signs when $dx'\, dy'\, dz'$ do so.

Therefore, setting out from the point of incidence, we may go in either *sense* on the tangent to the reflecting surface given by each value of $\frac{\delta\alpha}{\delta\beta}$.

Thus it is shewn that every point P in the surface M has four points circumjacent and very near to it from which the emergent rays respectively intersect with R.

58. This subject has been more recently taken up by Professor Hamilton of Dublin, who has published in the Transactions of the Royal Irish Academy, vol. xv, his " Essay on the Theory of Systems of Rays:" to which also he has appended a supplement in vol. xvi, part i, and a second supplement in part ii, of the same volume.

Professor Hamilton has in these profound papers treated the subject with the most advantageous adoption of all the resources of the higher analysis; and has taken it in its utmost generality.

He supposes homogeneous rays to diverge from a given origin and undergo any number of successive changes of direction, according to the law of optical deviation, at surfaces having any given figures and positions, and enclosing media of any given refractive indices. The position of the final ray referred to three rectangular axes gives an expression which is equal to the differential of a certain function, which he calls the *characteristic function.*

Thus, according to this view the geometrical properties of an optical system of rays may be deduced by analytic methods from the form of this one *characteristic function;* of which the partial differential coefficients of the first order, taken with respect to the three rectangular coordinates of any proposed point of the system, are, in the case of ordinary light, equal to the index of refraction of the medium, multiplied by the cosines of the angles which the ray, passing through the point, makes with the axes of coordinates: and as these cosines are connected by the known relation that the sum of their squares is unity, there results a corresponding connexion between the partial differential coefficients to which they are proportional.

In the memoir the demonstration is only partially given; but in the first supplement it is exhibited in the most general form; and is deduced by the method of variations.

In the second supplement the author effects the integration of the partial differential equations by a new method: taking several cases of the assumed system of rays.

In the case of rays contained in one plane, or symmetric about one axis, the partial differential equation takes simpler forms of which he has assigned the integrals, and has given an example of their optical use, by briefly deducing from his principles the formula for the longitudinal aberration in the case of spherical surfaces.

With this very brief notice the limits of this treatise oblige us to quit this portion of the subject; and we can only recommend the researches just named to the careful study of those who are desirous of advancing into the more extended fields of mathematical optics.

Theory of Optical Instruments.

59. In order to apply the preceding theory of lenses and reflectors to explaining the construction and principle of the various kinds of optical instruments which are of such important use to us, it will be necessary to commence with a brief examination of the process of ordinary vision, and the structure of the eye, in which we shall find the same principles beautifully exemplified.

The Eye and Process of Vision.

60. The eye in man and the superior animals consists of:—

1st. An external coat, which at the front part is transparent and projects beyond the general spherical form of the eye. This part is called the cornea, and is of the form of a segment of a prolate spheroid.

2nd. Enclosed behind this and filling up a small cavity is a transparent liquid like water, called the aqueous humour: the cavity is bounded behind by,

3rd. A diaphragm, in the centre of which is a circular aperture called the pupil, capable of enlargement or contraction.

4th. Next to this a soft transparent mass formed into an *exact lens,* having both its surfaces convex, but the front least curved, called the crystalline lens. It is found to be denser towards its centre than at the edges.

5th. Behind this the whole remaining segment of the spherical cavity of the eye is filled up with a transparent and somewhat viscous fluid, called the vitreous humour; and,

6th. The surface of the cavity is lined with a blackened coat called the nigrum pigmentum, covered with a delicate reticulated expansion of the end of the optic nerve, called the retina.

The optical effects produced are as follows: The pupil limits the area of the rays admitted, the elliptical form of the front of the eye tends to give accurate convergence, and the crystalline lens has its focal length for parallel rays such that with reference to the refractive power of the two media between which it is placed, it *converges rays incident parallel, or nearly so, upon the surface of the retina;* and its greater density at the centre tends to counteract the aberration. Hence since among the rays issuing in all directions from all parts of an object, a small pencil of nearly parallel rays will arrive from each point in such directions that on entering the pupil they will cross at the optical centre of the lens, their respective foci will fall on corresponding points of the retina, where thus the images of external objects are painted in an inverted position. In what way the light acts upon the retina or the sensation is subsequently communicated is wholly unknown. There also exists a power in the eye of slightly altering its curvature so as to accommodate itself to near as well as remote objects. The means by which this is effected are not yet understood.

The defects of what are called *short* or *long* sighted eyes depend on the circumstance that the crystalline lens and humours of the eye are so formed as to bring the incident pencils to their foci at distances which do not fall accurately on the retina, but are *before* or *behind* it respectively. Hence in the former case to produce distinct vision the rays require to be made less convergent, and in the latter more so. This is done to a certain extent by bringing an object nearer than the ordinary distance for correct vision in the one case, and removing it further in the other. But this applies only to near and moveable objects: and to remedy the evil for vision at all distances a lens must be used, such as will slightly *diminish* the convergency in the first case, and *increase* it in the second: or a *concave* or *convex* lens respectively.

61. By means of the impressions on the retina the eye judges not only of the forms and colours of objects, but of the relative *intensity* of illumination. Of this last point we have no precise standard of measurement whatever. The impressions on the eye are vague, and cannot be compared with each other or with any standard. The only case in which any degree of accuracy appears to be attainable, is when we have the means of gradually bringing two lights to an equality. If the colours be the same, and the illuminated spaces adjacent, the eye can determine with very considerable precision when an exact equalisation takes place. It is on this principle alone that any real determinations of *Photometry* can be made. The method proposed by Count Rumford was to throw the shadows of an opaque body formed by different lights on the same screen near each other, to equalise them by altering the *distances* of the lights, and thence to infer the relative intensity from the squares of the distances so found. A method proposed by Mr. Ritchie consists in viewing together the light proceeding from two plane surfaces, each inclined at half right angles to the direction of vision, and illuminated by different lights, when again by varying the distances and measuring them we obtain an estimate of the relative illumination.

62. The idea we form of the relative magnitudes of objects is

determined by the relative space which their images occupy upon the retina. This will depend on the angle at which the small pencils from the two extreme points of the object cross at the optical centre of the lens of the eye, and this angle will be as the actual linear magnitude of the object directly, and its distance from the eye inversely.

A luminous body, as we observed at first, has the intensity of its light decreasing as the square of the distance at which we view it. But its *linear* apparent diameter decreases as that distance simply: and consequently its apparent *area* as the square. The apparent area, therefore, and quantity of light at all distances bear the same proportion; or, in other words, from the same extent of surface the same number of rays appear to emanate, or *a luminous body appears equally bright at all distances.* This of course supposes that the light does not undergo any diminution by its passage through the air or other medium. This is not really the case for great distances, and hence very distant bodies become dim, and at length invisible: but within moderate distances the proposition, holds good.

There is also a curious fact observable, that a luminous surface (as that of a plate of red-hot iron) appears equally bright at all inclinations to the line of sight. Hence it follows that the copiousness of emission must be greatest in a direction perpendicular to the surface; and varies as the sine of the angle which the emitted ray forms with the surface.

63. If we take the smallest visible point, the rays from which may be considered a parallel pencil at moderate distances, and bring it very near the eye, the pencil will cease to possess this condition of parallelism, and the object will be indistinct. The same will be true of an extended object, which in fact we see by means of the small pencils of parallel rays emanating from each of the points into which its length or surface may be con-

ceived to be divided. And there is thus found to be a certain distance varying a little for different individual eyes, which we call the *nearest distance of correct vision.*

If this limit to the distance for distinct vision did not exist, there would be no limit to the degree of magnification under which we might view small objects by bringing them close to the eye.

At such distances, however, correct vision, and consequently magnification, may to a certain small extent be obtained by placing an opaque screen with a minute aperture in it close to the eye, between it and the object. This allowing only small pencils to pass and fall on the pupil, enables us to see the object distinctly at a shorter distance than we could without it, and therefore somewhat magnified.

But more perfect means of doing this are furnished by lenses, the use of which may be thus explained.

From the manner in which (as we have seen) the image is formed at the focus, it is evident that the rays forming the several small pencils, which at their points of convergence give the several points in the image, crossing at those points continue to diverge: and if the eye be beyond a certain distance, from each point some rays will reach it so as to cross at the centre of the crystalline lens, and thus produce a distinct image on the retina, as if they proceeded from a real object. This image however may, according to circumstances, have a greater or less angular magnitude than the object would have if seen directly. But it will always be magnified if the object be placed at the principal focus of a convex lens, and the parallel rays emerging be received by the eye. In this case it will easily be understood that of the rays diverging in all directions from any point in the object, one small pencil will, after refraction at the lens, emerge in a parallel state, and in such a position as to cross the axis (on which we suppose the eye situated) at the optical centre of the lens of the eye; and consequently to form a distinct focus on the retina. Since all the emergent rays after passing the lens, which came from one point in the object are parallel to each other, and therefore to the central ray forming the axis of that oblique pencil, it follows that any of them (such as those we have just conceived entering the eye) cut the axis at the same angle as the central ray does at the lens. Hence the emergent rays from the *extreme* points of the object will cross at the eye at an angle equal to that at which the incident central rays from the same points cross at the lens: or, in other words, the *angular magnitude of the image at the eye is equal to the angle subtended by the object at the centre of the lens.*

Telescopes and Microscopes.

64. The angular magnitude of an object is equal to the absolute linear magnitude divided by the distance from the eye. Hence if $\mu\ \mu_{\prime}$ &c. are the angular magnitudes of objects, whose linear magnitudes are $l\ l_{\prime}$ &c., referred to distances $\epsilon\ \epsilon_{\prime}$ &c., and supposing the first to be the object before a lens at a distance f from the lens, of which the second is the image formed at the focal length $f_{\prime\prime}$ from that lens: let us now suppose a second lens placed so that the *focal image* of the last becomes in turn the *object* to this second lens, and of which an image is again formed at the proper focal distance: these two last distances, as referred to the second lens separately we will express by (f) $(f_{\prime\prime})$; and similarly, if more lenses were introduced; but for the present we will suppose only two. Then for the first object and image, if the lens be such that according to Art. (35) the linear magnitudes are as the focal distances, we have,

$$\frac{\mu}{\mu_{\prime}} = \frac{l\,\epsilon_{\prime}}{l_{\prime}\,\epsilon} = \frac{f\,\epsilon_{\prime}}{f_{\prime\prime}\,\epsilon} \qquad (95)$$

Again, for the second object and image in like manner, we have,

$$\frac{\mu_{,}}{\mu_{,,}} = \frac{l_{,}\epsilon_{,,}}{l_{,,}\epsilon_{,}} = \frac{(f)\epsilon_{,,}}{(f_{,,})\epsilon_{,}} \tag{96}$$

Whence we obtain for the ultimate comparison of the magnitudes, which we will call M,

$$M = \frac{\mu_{,,}}{\mu} = \frac{f_{,,}(f_{,,})\epsilon}{f(f)\epsilon_{,,}} \tag{97}$$

Such is the general expression for the ratio of the angular magnitudes of the original object and final image formed by a combination of two lenses as just described. To apply these results to practical purposes we suppose the ultimate image so formed as to be fit for vision: that is, the rays must emerge parallel, or the position of the image which constitutes the object of the second lens must be at its principal focus, that is, $(f) = (F_{,,})$, and the quantities $(f_{,,})$ and $\epsilon_{,,}$ may be considered as equal: hence the general expression (97) becomes

$$M = \frac{f_{,,}\epsilon}{f(F_{,,})} \tag{98}$$

The second lens is in this case called the *eye lens*.

We have here retained the general supposition with regard to the first, or as it is termed, the *object lens;* and this construction consequently applies to an object placed at any distance from it; and it is evident from the above expression, that the image thus received by the eye from the eye lens will be *magnified* if the distances be such that we have the numerator greater than the denominator : that is, if we form such a combination as shall have f and $(F_{,,})$ as small as possible compared with $f_{,,}$ and ϵ. Any such combination being suited for obtaining magnified representations of small objects which are within our reach and can be placed at convenient distances, is the essential part of the various instruments called *microscopes*.

If the object lens also receive parallel rays, or the first image be formed at its principal focus, then the general expression will be modified by the circumstance, that we have $f_{\prime\prime} = F_{\prime\prime}$, and that ϵ and f may be considered as equal, which gives the result

$$M = \frac{F_{\prime\prime}}{(F_{\prime\prime})} \tag{99}$$

In this case rays being received from *very distant objects*, the final image will be magnified in the proportion of the principal focal length of the object-glass to that of the eye-glass. Such a combination gives the essential principle of the *telescope*.

If instead of an object *lens* we suppose a *spherical reflector* substituted, a precisely similar investigation will apply; but in comparing the magnitudes, we must recollect what was before remarked (Art. 35), that we have here taken the focal distances of the object and image *as referred to the centre*; or, in other words, the form (95) becomes by this substitution

$$\frac{\mu}{\mu_{\prime}} = \frac{c \ \epsilon_{\prime}}{c_{\prime} \ \epsilon}$$

and the resulting expression (98) will likewise be,

$$M = \frac{c_{\prime} \ \epsilon}{c\,(F_{\prime\prime})}$$

Again, in the case of parallel rays, the form (99) would on the same principle require the substitution of $C_{\prime}$ for $F_{\prime\prime}$; but since these quantities are equal, and the difference of sign is here of no consequence, that formula will apply here without alteration.

Such will therefore be the principles of the investigation of *reflecting* telescopes and microscopes.

65. The subject of telescopes is one of immense practical importance, and at the same time of great extent and complexity to follow into all its details. We can here attempt nothing more

than to put the student in possession of the general and fundamental principle applying to all the constructions. It sufficiently appears from the foregoing investigation, that the essential points are these:

From a distant object rays are emanating in all directions, but only a very small pencil of them enters the pupil of the unassisted eye: the object-glass receives a much larger quantity of light from the object; and thus in proportion to its aperture collecting these rays at its focus, it there produces a very *bright* image: this is then *magnified* by the eye-glass in proportion to the *shortness* of its own focal distance compared with the *length* of that of the object-glass.

The extent to which we can succeed in obtaining a bright and magnified image of a distant object, is limited in practice by a variety of causes: those connected with the theory are principally the spherical aberration, and another species of aberration which we shall consider in the sequel. In the eye-glass, again, the pencils must necessarily, in some measure, fall on it obliquely without passing through its centre: hence an inaccuracy resulting from what we have already shewn respecting the convergence of oblique pencils.

66. The emergent parallel pencils from the eye-glass, originating from different parts of the image, cross the axis at different successive points, more distant as they come from points more towards the extremities of the image.

If the object-glass be of greater diameter than the eye-glass, the most extreme ray which falls upon the eye-glass, if produced, would meet the axis beyond it, and hence the refracted ray must cut the axis at a distance nearer to the lens than its principal focus.

Hence the eye must be placed within that distance to receive rays from all parts of the portion of the object limited by these extreme rays.

The axis of the pencil to which this extreme ray belongs having crossed the axis at the centre of the object-glass, thus determines the magnitude of the image limited by this ray: or in other words, the measure of the image so limited (which is called the *field of view*) is the angle which it subtends at the centre of the object-glass.

But towards the extremities so few rays compose the pencil, that they produce no sensible effect. Hence the actual field of view in practice is somewhat less than as thus determined, and the angle subtended by the bright portion at the eye-glass, is usually taken as very nearly equal to that *subtended by the object-glass at the eye-glass.*

67. Since the use of the object-glass is simply to collect a greater quantity of light, and rays from every part of the object or visible space fall upon every part of the lens, any opaque obstacle placed here only diminishes the light, but does not intercept any part of the image. By means of thus bringing so much more light into the image the telescope is applied for shewing objects, such as very faint stars, otherwise invisible from the diffuseness of their light owing to their distance, as well as objects invisible from the absence of sufficient light to shew them, as at night; a telescope adapted to this last purpose, by collecting as much light as possible, whilst the magnifying power is of secondary importance, is called a *night glass.*

In the case we have considered, supposing both the lenses to be convex, it is evident that the image is inverted. This being of no consequence in astronomical observations, such a telescope is designated as an *astronomical telescope.* For other purposes an additional eye lens is introduced to restore the proper position, and sometimes further combinations are added. These are called *eye pieces,* and admit of some variety according to the purpose for which they are introduced.

The refracting telescope was invented by Galileo in 1609, though in a form somewhat different from that just described.

In Galileo's telescope the object-glass being a double convex lens, the rays converging to its focus are received on a *double concave* eye lens *before* they reach the focus, and to that the principal foci of both coincide: hence the focus of incident rays for the eye-glass being an imaginary one, they emerge parallel on the same side on which this imaginary focus lies, and are consequently adapted to produce vision, and give a magnified image. This construction is used in the common opera glass. There is here no inversion.

68. The simplest of *reflecting telescopes*, the Herschelian, is in its theory precisely the same as the simple astronomical among refracting telescopes. The practical difference is, that in this case the focus of reflected rays necessarily lies on the same side of the mirror as that on which the light is incident: hence the eye lens placed to receive it and the head of the observer must intercept a portion of the light. This construction can therefore only be used for instruments of very large size, where the portion of the light thus intercepted is small compared with the whole: the mirror is also adjusted so that the focus may be thrown towards the side of the tube, by which means less light is intercepted.

The Newtonian telescope is the same in its general theory: but the difficulty just mentioned is overcome by placing a small plane reflector in the tube at half right angles, which throws the focus out without alteration at an aperture in the side, where the eye-glass is placed.

Sometimes instead of the plane reflector the total internal reflexion from a glass prism, properly placed, is used: in this way much less light is lost.

The Gregorian construction obviates the difficulty by placing in the tube a small concave reflector facing the large one, and beyond the focus, from which by a second reflexion the rays return to the object lens, and passing through an aperture in its centre are received on the eye lens placed at the back of it.

The Cassegrainian differs from the last construction only in this, that the small mirror is *convex*, and is placed before the focus.

The principle of the compound *Microscope* has been given in the general theory. A variety of differences in the details of the construction prevail in practice, into which we cannot here enter.

Refracting microscopes are by far the most commonly used. Reflecting microscopes have been constructed by Dr. Smith on the principle of Cassegrain's telescope inverted: and by Professor Amici on that of the Newtonian, the rays from the object being let into a hole in the side of the tube, and by an inclined plane reflector thrown on the object mirror, whence the image is received by an eye lens opposite.

Single lenses or small spheres are found to afford microscopes of very high power.

In regard to the various constructions of these instruments and their several adjustments, &c., the fullest sources of information are accessible to the student in the larger works already often referred to, and especially Coddington's treatise on optical instruments, forming the second part of his enlarged treatise on optics.

Unequal Refrangibility of Light.

69. We have hitherto all along reasoned about light considered as homogeneous; or supposing that every integrant part of a ray of light was subject to precisely the same laws of optical deviation. It is found however that in regard to refraction this is not

the case, and the proof of it is rendered most conspicuous by the successive refractions at the two surfaces of a prism.

We must here recur to what was shewn in Art. 12, relative to the position of minimum deviation; and we may remark that this position is easily found in practice without any measurement of the angles; for near this position the motion of the image corresponding to a small motion of the prism about its axis will be insensible, but a little out of it increases rapidly in proportion to the revolution of the prism. We have therefore only to observe when the image becomes for a moment stationary, and it is then in the required position.

Now there is an important consequence from what is established in Art. 12. If we conceive a small homogeneous pencil to impinge on the side of a prism, so that the central ray of the pencil is at the incidence of minimum deviation, then the emergent ray forms the same angle at the other side of the prism. If then the pencil be so small that we can consider this to be true for each of the extreme rays of the pencil, it follows that we shall have for one of these rays $\phi = \phi_{,,,}$

and for the other also $\phi' = \phi'_{,,,}$

whence $\phi - \phi' = \phi_{,,,} - \phi'_{,,,}$

But these differences express the inclinations of the rays to each other; this inclination therefore remains the same after emergence as it was before.

Thus, then, for an object of small angular magnitude, the angle subtended by the image formed by passing through a prism, ought to be the same as that subtended by the object.

70. Now when we try the experiment, taking an object of small angular magnitude, the light from which is of any ordinary kind, such as the sun, and cause the small pencil of rays to be refracted through a triangular prism in the position of minimum

deviation, on receiving the emergent rays on a screen, instead of the circular image subtending the same angle as the sun which should be produced on the above principle, we find it *elongated* to several times its breadth, in the direction perpendicular to the axis of the prism. It follows then that *some of the rays undergo greater refraction than others,* or for different integrant parts of the incident beam the value of m must be different.

But this is not the whole of the observed results: the image is not only elongated, but, whereas the light was at first white, it is now *separated into a succession of colours,* which, throughout the whole length of the image (or spectrum as it is called) are continually shading off from one into another; beginning from a deep red at the least refracted extremity, the tints pass through successive shades of light red and orange to a bright yellow, and this again into green, which, acquiring a more blue tint, at length passes into blue, beyond which it terminates in a faint violet light at the most refracted end. It is uncertain when these phenomena were first observed, but they were first accurately explained and the legitimate conclusion drawn from them by Newton, in 1671.

The unequal refrangibility of light opens to us a new field of optical research; but, as throughout the preceding articles of this treatise, we have as yet confined ourselves to those points connected with the theory of light which regard its *direction* and *change of direction* in different cases, so we shall here first examine those results of the principle of unequal refrangibility which involve simply the consideration of the *direction* taken by the different rays; and afterwards recur to those which relate to other properties, or the nature and constitution of light.

The principle of Newton's experiments was precisely that which we have already explained. He found that the different component parts of white light obeyed different laws of refraction, and the parts into which the refracted beam was thus separated retained their distinct degree of refrangibility, whatever subsequent modifications they were made to undergo.

Any one coloured ray of the spectrum being insulated by letting it pass through a hole in the screen which stopped the rest, could be made to exhibit no further elongation, separation, or change, by being again subjected to more prismatic refractions. The red space formed by one prism being thrown on a screen level with the violet of another, and both being viewed through a third prism, they appeared to separate from each other by the effect of the different refrangibilities of the rays which formed them.

71. Conversely also he found that when the light had been thus decomposed the spectrum might be received through another prism in an inverted position, and the coloured rays thus be *recompounded* into white light. This could also be done by a lens.

But the most simple and elegant form of the experiment is that afforded by *one prism within itself;* supposing it equilateral and the rays incident in the position of minimum deviation. The ray at its first refraction being formed into an initial coloured spectrum, when it reaches the second surface is separated into two portions, one emerging and forming the spectrum, another reflected internally which meets the third side where it is partly reflected again, but from the equality of the angles of incidence and emergence in this position, it forms with this side an angle equal to that which the first refracted ray formed with the first surface; the rest of it therefore emerges in a beam of white light.

72. When a certain position is given to the prism it is evident, from Art. 9, that the rays cannot emerge to form the spectrum but undergo total reflexion: this is seen to take place succes-

sively for the different rays in the order of their refrangibility, until the reflected ray becomes white.

A remarkable appearance owing to this cause is presented if we look into a prism laid on one of its sides, and exposed to the sky. That part of the base away from the spectator gives a bright reflexion of the sky, which at a certain point is bounded by a coloured edge, first greenish blue and then violet, within which the rest of the surface appears dark. This is easily explained from considering that rays coming from different parts of the sky fall on the first surface, and are refracted to the base at different degrees of obliquity; all those which come *within* the limit are wholly reflected; *at the limit*, first the red ceases to be reflected, or the reflected portion consists of the remaining rays, having therefore a preponderance of green and blue; successively other rays go out, till only violet is reflected, and beyond this the reflexion ceases.

This remark also applies to the parallel case we before considered, of vision under water (p. 13). At the limit between reflexion and refraction there mentioned, the different primary rays will disappear in succession, and the circle of vision will be bounded by a ring of colours similar to those just mentioned as seen by reflexion in the prism.

Dispersive Powers.

73. It appears then decisively established by the experiments of Newton, that, for the *same refracting medium, a different refractive index*, or value of m, inseparably belongs to each of the different rays, or *parts into which the length of the spectrum can be distinguished*; or, if we recur to what we before established respecting the deviation of a ray, and observe the deviation of any one given ray or point in the spectrum we may choose to fix

upon, and substitute the value of δ thus obtained in the formula (8), we shall have the value

$$m = \frac{\sin. \frac{1}{2} (\delta + \iota)}{\sin. \frac{1}{2} \iota} \qquad (100)$$

which will be different for each different ray or point in the spectrum: least in the red, greatest in the violet, and about the yellow nearly equal to the mean.

There is however considerable difficulty in *defining the particular point* of the spectrum for which this observation is made merely by means of its colour, but if great accuracy be not desired it is sufficient to take a part near each of the extremes, and comparing the refractive indices for these points with that which belongs to a mean or compound white ray, we may express what is called the *dispersive power* of the medium, by which term is understood its effect in producing this separation of rays, or the difference of the deviations for the extreme rays compared with that of the mean.

74. If we try the effects with prisms formed of different substances, we find the degree of separation not only of the extreme rays, but of any two rays which we may compare in the spectrum, very different; or, in other words, the whole spectrum extends to a different length, and the colours severally occupy different proportional parts of that length, in different media. Newton took for granted that there was no material difference between substances in this respect: its existence, however, was observed by Clairault, Boscovich, and others: and it has been fully investigated by subsequent philosophers. This difference, both in the amount and character of the dispersion in different media, is a remarkable physical fact, the cause of which is wholly unknown.

75. Let us suppose any definite rays or points in the spectrum

fixed upon, and that for each we observe the particular value of δ, and thence deduce that of m, which we will distinguish as belonging to the red, violet, &c., by subjoining the initial letters, as m_r, m_v, &c., m simply standing for the index of a mean ray. Then the *dispersive power* of a medium is usually estimated by taking, first, for each respective ray the difference of its index from unity (which corresponds to its deviation from the direction of incidence); then, the difference of these quantities for the two extreme rays; and then, comparing this difference with the same quantity for a mean ray. Thus we have the dispersive power expressed by

$$D = \frac{(m_v - 1) - (m_r - 1)}{m - 1}$$

$$= \frac{m_v - m_r}{m - 1} \qquad (101)$$

In this way tables have been formed of the dispersive powers of different media, these determinations referring, as we have observed, to the two extreme rays of the spectrum, those rays being but ill defined, and their degree of separation being in no way proportional to the action of the media on the other rays, such results are of no great precision in a philosophical point of view, though affording approximations which are valuable in practice. The following numbers, from the same authority as cited for the refractive powers, may suffice to give some notion of the differences among transparent media in this respect; and it will be apparent that the dispersive follows no proportion to the refractive power.

Chromate of lead	0.4.
Realgar	0.26.
Oil of Cassia	0.139.
Sulphur	0.13.

Several oils, about	0.06.
Flint glass	0.05.
Other oils	0.04.
Several acids	{ 0.04. / 0.03.
Diamond	0.038.
Water	0.035.
Crown glass	{ 0.03. / 0.027.
Alcohol	0.029.
Fluor Spar	0.022.

Chromatic Aberration.

76. It is evident that the unequal refrangibility of the different integrant rays will affect their convergence when subjected to the action of a lens, and that the focal length for red rays will be greater than that for the violet. Thus continuing an analogous notation, or dropping the accents and writing f_r f_v for the distances of the foci of the red, violet, &c. rays, we have f_r greater than f_v : and this circumstance will create a deviation from exact convergence, (independent of the spherical aberration,) which is called the *chromatic aberration*, and which is measured by the interval $f_r - f_v$.

We may easily perceive that in any one pencil supposed (abstracting from this cause) to converge accurately to a point, we shall have, from the crossing of the differently refracted rays at different points, *a circle of least chromatic aberration*, whose position is easily found, as follows.

If we conceive the extreme red ray from one side of the axis, intersecting the violet from the other, we may drop a perpendicular, y, from their intersection upon the axis, which shall divide the distance $f_r - f_v$ into two parts, w and x, and calling the semi-aperture of the lens α, we shall obviously have,

$$\frac{y}{\alpha} = \frac{w}{f_r} = \frac{x}{f_v}$$

whence we deduce, $$y = \alpha \frac{f_r - f_v}{f_r + f_v} \tag{102}$$

The denominator of this expression is evidently equal to twice the mean value of $f_{\prime\prime}$ or

$$2y = \alpha (f_r - f_v) \frac{1}{f_{\prime\prime}}$$

Again, we have from the original formula (18)

$$\frac{1}{f_v} - \frac{1}{f_r} = \frac{(m_v - 1)}{\rho} - \frac{(m_r - 1)}{\rho}$$

Which on the above consideration easily leads to the expression,

$$f_r - f_v = \frac{m_v - m_r}{\rho} f_{\prime\prime}^2$$

Or, substituting the value of ρ, this becomes

$$= \frac{m_v - m_r}{m - 1} \cdot \frac{f_{\prime\prime}^2}{F_{\prime\prime}}$$

Whence we obtain

$$2y = \alpha \frac{m_v - m_r}{m - 1} \frac{f_{\prime\prime}}{F_{\prime\prime}} \tag{103}$$

$$= \alpha D \frac{f_{\prime\prime}}{F_{\prime\prime}}$$

In the case of parallel rays the last factor is unity, and the diameter of the least circle of chromatic aberration is equal to the semi-aperture multiplied by the dispersive index.

The position of this circle is easily found from the forms above, (102), whence by adding and reducing we find

$$f_v + x = \frac{2 f_r f_v}{f_r + f_v} \tag{104}$$

77. The effect of the chromatic aberration is not merely that of giving a coloured appearance to the image, but of creating a confusion and indistinctness owing to the different degree of convergence in the different rays at the same distance.

But the vertices of all the cones are the most strongly illuminated points; and these all lying in the axis, the centre of any circular section will be the brightest point, from which the illumination decreases to the edges. The yellow rays converge nearly to the *mean* focus, and these have by far the greatest illuminating power.

These circumstances tend to compensate the bad effects of this aberration. It is diminished also by reducing the aperture, though not in the same ratio as the spherical aberration; the latter being as the square, the former as the simple power of the aperture.

Achromatism.

78. Such is the case with a single lens, the aberration being greater or less according to the dispersive power of the substance. But by means of the differences found in the dispersive powers of different media, the chromatic aberration of a single lens may be counteracted by combining it with another: and the principle on which this may be effected follows immediately from what has been already explained.

Let us suppose two lenses of different dispersive powers combined close together, so that we may express the power of the combination by the form (20); supposing the value of m to refer first to the red and then the violet extremity, we shall have successively

$$\frac{1}{F_r} = \frac{m_r = 1}{\varrho} + \frac{m_{r1} - 1}{\varrho_1}$$

$$\frac{1}{F_v} = \frac{m_v - 1}{\rho} + \frac{m_{v1} - 1}{\rho_1}$$

Now if these focal lengths were equal, the *chromatic aberration would be destroyed:* and they may be made so by properly assuming ρ ρ_1 or such values may be found by equating the expressions

$$(m_r - 1 \rho_1) + (m_{r1} - 1) \rho = (m_v - 1) \rho_1 + (m_{v1} - 1) \rho$$

Whence $$\frac{\rho}{\rho_1} \frac{(m_v - 1) - (m_r - 1)}{(m_{r1} - 1) - (m_{v1} - 1)}$$

But since m_r is less than m_v the denominator will be negative, or we have

$$\frac{\rho}{\rho_1} = - \frac{(m_v - m_r)}{(m_{v1} - m_{r1})}$$

We may thus find lenses which will fulfil the condition by recollecting that ρ ρ_1 give us the *principal focal lengths* for mean rays: and these must be *in the ratio of the dispersive powers, and of opposite signs:* or the one lens convex, the other concave.

But it is evident that such a mode of correction can refer only to those two rays whose indices we introduce.

The same values of the radii which produce the complete counteraction for these rays will not produce it for others. Hence further means must be employed; an additional lens or lenses are sometimes introduced to correct the colour which the first combination leaves uncorrected. But the investigation becomes more complicated, and in practice opticians are generally obliged to satisfy themselves by approximate corrections. Lenses of crown and flint glass are usually combined. Sometimes fluid lenses have been employed; a liquid being enclosed between glasses: this method possesses many advantages.

The principle of *achromatic* combinations of lenses, has been here directly deduced from the consideration of the formulæ: it admits, however, of the following more familiar illustration. If we suppose a prism which separates the extreme (or any two given rays) through a given angular space: another prism of the same substance and with the same angle, placed in an inverted position with its sides parallel to those of the first, and receiving the rays as they emerge, will of course recombine them and make them emerge in a white ray, and in a direction parallel to their first direction. If it be of a different substance, and have a *higher* dispersive power, it will recombine them with a smaller refringent angle, in which case its second surface will not be parallel to the first surface of the first prism, and therefore the emergent white ray will *deviate* from its original direction, and we shall thus have refraction without colour.

If we conceive these prisms to be portions of lenses, we shall thus have rays refracted out of their first direction, and consequently capable of being brought to a focus, without chromatic aberration.

79. It follows from the formulæ (103) for parallel rays, that the circle of least chromatic aberration, or of least colour, has the same absolute magnitude whatever be the focal length of the lens, provided the aperture remain the same. Now since in a telescope (with a *given* eye-glass) the image is magnified in proportion to the focal length of the object-glass, it follows, that by increasing the focal length the magnitude of the image increases, while that of the coloured border remains the same: by continuing therefore to increase the focal length, we get an image so much magnified that the colour bears an insensible proportion to it.

Hence as long as simple lenses only were used, in order to correct the aberrations and secure a due quantity of light, it was necessary to have telescopes of very unmanageable length. Some of those constructed by Huyghens were of one hundred and one hundred and fifty feet focal length. The introduction of the compound achromatic object-glass renders such great lengths unnecessary, and reduces to convenient limits instruments of greater power than any of those formerly made with single lenses.

The principle of achromatic combinations appears to have been first suggested by Mr. Hall, in 1729; but was neglected until rediscovered and applied with such eminent success, by Dollond, towards the end of the last century. Mr. Barlow of Woolwich has lately carried the construction of fluid lenses (originally suggested by Dr. Blair) to great perfection. [Philos. Trans. 1831, i.]

We have here supposed the correcting lenses to be placed close together, so that we could apply the formula for the power of a compound lens. They may however be separated by any interval, and sometimes such an arrangement is made by which certain practical advantages are gained. In this case we must employ formulæ similar to those for a compound lens, which are deducible in the same manner, only introducing the quantity t which expresses the distance between the surfaces.

Upon a principle somewhat similar it is possible to construct an eye piece of two lenses of the *same* material, which shall be achromatic. But into these details the limits of this treatise forbid our entering: the student must refer for a full discussion of the subject to the complete treatises so often named, Herschel on Light, and Coddington on Optical Instuments, and the last mentioned author's Treatise on Reflexion and Refraction, p. 257.

We shall afterwards recur to the subject of the prismatic analysis of light in connexion with other properties: for the present, still confining ourselves to those which involve merely the *directions* of the different rays owing to reflexion and refraction, we shall be enabled, by the help of the simplest deductions from principles already established, to explain one of the most splendid phenomena in nature, the rainbow.

The Rainbow.

80. If a ray fall upon a sphere of greater refractive power than the surrounding medium, on being refracted into it (confining ourselves to one circular section) it will proceed in the direction of a chord till it meets the circumference again, where, agreeably to what has been already explained, a portion of it will emerge and another portion will be *reflected internally:* and this describing another chord equal to the former will meet the surface again, when the same thing will take place; and this successively a certain number of times dependent on the first incidence; after which the ray will finally emerge. If the arcs subtended by these chords be written $= \chi$ radii to the points of reflexion will form with the chords, angles $= \phi_{\prime}$ and we shall have

$$\chi = \pi - 2\,\phi_{\prime}$$

Let there be n such reflexions; also let θ be the arc intercepted between the first incidence and the last emergence, then we find

$$\begin{aligned} \theta &= 2\,\pi - (n+1)\,\chi \\ &= 2\,(n+1)\,\phi_{\prime} - (n-1)\,\pi \end{aligned}$$

It is also evident that according to the number of reflexions will the ray emerge in such a position as to meet the direction of the incident ray before or behind the sphere. Thus if there be only one reflexion, the rays cross behind; if two, before, &c. Let the angle at which they meet be δ: then, the upper sign corresponding to the case when they cross before and the lower when behind, we have

$$\tfrac{1}{2}\,\delta = \pm \left\{ \phi - \tfrac{1}{2}\,\theta \right\}$$

$$\delta = \pm \left\{ 2\,\phi - 2\,(n+1)\,\phi_{\prime} + (n-1)\,\pi \right\} \quad (106)$$

It is also evident, that since the internal angles are all equal, the *last angle of emergence* is equal to the *first angle of incidence.*

81. If now instead of a single ray we suppose a small pencil of parallel rays, it will be evident that, the small arc presenting different incidences to the different rays, they will in *general* take different directions, and the final emergent pencil will consist of *diverging* rays. But they may under certain conditions emerge in a *parallel* pencil. These conditions may be thus investigated: corresponding to the small arc occupied by the breadth of the incident pencil there is a small increase of the angle ϕ; and consequently a similar small variation in $\phi_{,}$. Now if the rays ultimately emerge parallel, the angle δ is the same for all the rays in the pencil, or its value undergoes no alteration corresponding to these small variations, which we may express analytically by writing

$$\frac{d\,\delta}{d\,\phi} = 0$$

Now having an expression (106) for δ as a function of ϕ, we can easily find this value of the differential coefficient: we have directly

$$\frac{d\,\delta}{d\,\phi} = \pm \left\{ 2 - 2\,(n+1)\,\frac{d\,\phi_{,}}{d\,\phi} \right\}$$

$$= \pm \left\{ 2 - 2\,(n+1)\,\frac{\cos.\,\phi}{m\,\cos.\,\phi_{,}} \right\}$$

Hence, in order that this may be $= 0$ we must have

$$\frac{1}{n+1} = \frac{\cos.\,\phi}{m\,\cos.\,\phi_{,}}$$

or $$(n+1)^2\,\cos.^2\,\phi = m^2\,\cos.^2\,\phi_{,}$$

and adding $$\sin.^2\,\phi = m^2\,\sin.^2\,\phi_{,}$$

we easily deduce the value,

$$\cos.\,\phi = \sqrt{\frac{m^2 - 1}{n^2 + 2n}} \qquad (107)$$

for the *incidence at which after* n *internal reflexions parallel rays emerge parallel.*

Proceeding to a second differentiation we find, after an obvious reduction,

$$\frac{d^2 \delta}{d\,\phi^2} = \pm \left\{ \frac{-2(n+1)}{m} . \frac{\sin.\,(\phi_{,} - \phi)}{\cos.^2\,\phi_{,}} \right\}$$

But since $\phi_{,} < \phi$ the sign within the brackets is always positive. Hence the value above obtained corresponds to a minimum or maximum value of δ according as the sign of δ is + or —; that is, according as the emergent ray meets the direction of the incident ray before or behind the sphere.

82. A pencil emerging at the angle above determined, therefore, emerges parallel: this forms a limiting position, on one side of which no rays emerge: on the other we have rays more and more divergent. In the case of a maximum these diverging rays must meet the incident ray at a less angle and behind the sphere: in the case of a minimum, at a greater angle and before. Thus, in either case, they emerge at a greater angle than that of the parallel pencil.

In the same sphere different parts of the incident beam falling on different parts of the circumference will, according to the incidence, enter the sphere in such a position as to emerge after one, two, three, &c. internal reflexions. Supposing the rays to come *downwards* upon the sphere, they will *emerge* after *one* reflexion at the *lower part* of the sphere, and diverge *above* the parallel pencil.

For *two* reflexions, the rays are incident on the *lower* part and emerge *above*, and the diffuse rays below the parallel pencil.

We have here supposed m to remain the same. If we now consider different values of m, for any order of reflexion, it is evident from (107), that as m is supposed greater, cos. ϕ will be greater, and therefore ϕ less. But it is also apparent, that the pencils of the different primary rays do not emerge at the same point. Thus (with *one* reflexion) the violet ray separated at the first incidence will trace a longer chord, and consequently emerge at a greater distance (measured round the circumference) than the red. The parallel violet pencil, therefore, will have a less angle of emergence, but will lie in a direction less inclined to the incident ray than the parallel red pencil. Or, if we suppose the rays to come *downwards*, the red pencil will emerge below the violet, and will be more inclined downwards: and the diffuse rays will diverge *above* their respective parallel pencils.

With *two* reflexions, in like manner, the violet parallel pencil will be found to have a less angle of emergence, but a greater inclination to the incident ray than the red; or here the violet parallel pencil emerges below the red, and is inclined more downwards: the diffuse rays diverge below the parallel pencils.

If we now conceive a number of spheres arranged in a vertical line, and parallel rays coming downwards upon them all, we shall have emerging from each several sets of rays under the above conditions. And if *we assume any point* on which the emergent rays fall, there will pass through it, the parallel pencil after one reflexion from one sphere, that after two from another, &c., together with some rays of each of the diverging pencils from all the spheres.

83. These conditions are fulfilled in the circumstances of the *Rainbow*. The spheres which we have considered may represent the drops of rain: and for the sake of simplicity we will regard

first only a set of drops situated in a vertical line: upon these parallel rays come downwards from the sun. Let the assumed point be the eye of a spectator on which the rays are received after reflexion. Supposing the sun's light to consist of *homogeneous* rays of any kind, as red for instance, then, according to the preceding theory, the spectator would see a *vertical line of red light*, very diffuse at its lower part, till at a certain height it rapidly increases in brightness and then terminates. After an interval it reappears above at a point of maximum intensity, though much less bright than the last, and continuing upwards fades away: and so on successively.

What has been said of drops in a vertical line, will apply equally to those in a line of any inclination. The same series of reflexions and refractions will go on at the same angles in a *plane passing through the sun, the eye of the observer, and any set of drops*. Thus the rays reaching the eye after emergence at the same angle in all the different planes, will lie in a *conical surface*, the eye being at the vertex, and the resulting appearance will be that of a circular space occupied by faint red light with a bright red edge, succeeded by a dark zone; and then another less bright red edge followed by a very faint red space; and so on.

The same things will hold with homogeneous rays of any other colour: but the diameters of the first circle will be successively *less* for the yellow, green, blue, and violet rays; those of the second *greater* in the same order.

These being all *superposed* we shall have, within the first circle, first a faint white light, whilst towards the edges the violet and blue will first predominate at their respective maxima; next the green, yellow, &c., and then the red, forming circular arcs of their respective colours, each more pure, since the lower one terminates at its bright edge, and there is successively less mixture: this will constitute a first or inner bow; to this will succeed a space comparatively dark. Then, in the next bow, (formed by two reflexions) the red circle will be innermost, and the other coloured circles follow in reversed order, till beyond the violet there occurs a space occupied by very faint white light: other orders might succeed, but from loss of light all beyond the second will be extremely faint.

These results of our theory are, in fact, an exact account of what is observed to take place in nature. The first bow is commonly seen, sometimes the second, but never any more from the great loss of light and also from proximity to the sun.

The altitude of the sun being supposed given, it is evident that the altitude of the part of any of the bows directly opposite, measured with reference to any definite point (such as the maximum of any one colour) is found directly by taking the value of cos. ϕ corresponding to any assumed value of m or n, thence finding ϕ and consequently $\phi_{\prime}$ and substituting them in the expression for δ. The arc of a great circle measured by δ will evidently give the radius of the circle of which the bow is a segment, and if α be the sun's altitude above the horizon, that of the highest point of the bow will be $(\delta - \alpha)$.

The cause of the rainbow was first pointed out in a general way by Antonio de Dominis, about 1590: it was more closely examined by Des Cartes, but not fully till Newton applied to it his accurate conclusions respecting the different refrangibility of the primary rays.

Halos.

84. Some other natural phenonema are explicable by the simple consideration of the unequal refrangibility of light. The term *Halo* is usually applied to luminous circles sometimes seen round the sun and moon: but these are not all of the same nature. Those which are called *coronæ,* surrounding the luminaries

at small distances, when light clouds or mists are about them, exhibit colours which are not the same, nor in the same order, as those of the prism or rainbow. These are owing to another cause, which will be considered hereafter.

Sometimes larger circles are seen, usually two, the inner having a diameter subtending an arc of a circle of about 45°, the outer about 90°, the sun or moon being their common centre. These exhibit the prismatic colours, the red being nearest the centre: but they are often faint, and sometimes none but the brightest, viz. the yellow, are visible; or, according to the state of the atmosphere, the halo may appear nearly colourless. Sometimes these circles are intersected by others parallel to the horizon, one of which passes through the sun or moon: at the intersections there are points of more intense brightness, giving images of the luminary called *parhelia* and *paraselenæ*.

These phenomena are very rare, but accurate descriptions have been published of those which have been observed; and various theories proposed to account for them. Some of these assume the existence of causes of which there is no independent evidence, and are therefore merely hypothetical.

85. An explanation grounded on a *true cause* will perfectly account for the concentric circles, as follows: the phenomenon is seen most commonly in cold climates; ice is known to crystallize in minute prisms, having angles of 60° and sometimes of 90°. These being extremely small are known to float in the air like a mist, and must have their axes in all possible directions. Ice has a refractive power of about 1.31. In any plane passing the eye, the sun, and a section of a crystal transverse to its axis, a prismatic image will be formed, which it may be easily calculated from the formula (8), will give on the above data $2\delta = 45°$ nearly in one case, and $= 90°$ in the other: thus the concentric

circles with prismatic colours, if sufficiently bright to be seen, are accounted for.

This explanation is confirmed by an experimental imitation, effected by looking through a plate of glass, on which a saturated solution of alum has been allowed to evaporate and crystallize, at the sun or a candle, when it it is seen surrounded by halos.

It does not appear that this principle will account for the other part of the phenomenon, the horizontal circles and luminous images.

Impressions of Light on the Eye, &c.

86. The eye has a power of retaining the impression of light for some seconds after the source of light has been actually removed. This is exemplified in the familiar appearance of a luminous continuous circle, formed by whirling round a piece of glowing ember, and the various optical illusions occasioned by rapid rotation, as the thaumatrope, where figures drawn on the opposite sides of a card, whirled rapidly about an axis in its own plane, are both seen together; various appearances assumed by wheels revolving in opposite directions, or of the spokes of a wheel in progressive motion, seen through a series of vertical bars. The image of the sun sometimes remains impressed on the eye for a considerable time, and even recurs after a temporary disappearance.

87. The different parts of the prismatic spectrum differ greatly in their *illuminating* effect on the eye; the intensity is very small towards either of the extremities, and has its maximum nearly in the yellow space. It is probable that rays of too deep a tint to be visible, extend to some distance beyond the boundary as ordinarily seen. When we look at the spectrum through a

deep blue glass, this may be seen distinctly with regard to the red end of the spectrum. Some eyes are partially or wholly insensible to red light: hence to them all colours into which red enters appear different.

Irradiation, or the fact that a white body on a dark ground appears larger than reality, that the fixed stars appear to have some sensible magnitude, &c.; these and various other facts of the same description are referable rather to physiological than optical causes.

88. It has been ascertained that light is propagated from luminous sources not instantaneously but in time, though with a velocity quite inconceivable. This fact was first established by Roemer from observing that the eclipses of Jupiter's satellites are seen later when the earth is in a part of its orbit furthest from Jupiter, than when nearest to it. It was hence calculated that light travels at the rate of about 192,500 miles in a second. A similar result has been also deduced from other astronomical facts.

Internal Reflexion.

89. When we look at light reflected by a piece of plate glass, whose surfaces are not exactly parallel, besides the image formed at its upper surface, we see another formed by internal reflexion at the second surface, which at considerable obliquities is nearly as bright. This takes place when the surrounding medium is air: or is produced by reflexion at the bounding surface *between glass and air:* and between these media there is, as is obvious, a great difference in refractive power.

Let us now suppose, that to the lower surface is applied a drop of water: this has a refractive power approaching nearer to that of the glass; and we now find the brightness of this image much diminished.

Again, olive oil, in which the *difference is less,* diminishes the brightness still more. And if we use pitch, which is very nearly *equal* to glass in refractive power, *the image is totally obliterated.*

If again we apply oil of cassia, which has its refractive index *greater* than that of glass, the image is *restored.* If sulphur, whose index is greater still, the image is more bright. If an amalgam of mercury, (as in a common looking-glass,) this image is far brighter than that at the first surface.

Thus from all these instances we collect, that *the more two adjacent media differ in their refractive powers, the greater will be the intensity of the light reflected at their bounding surface; and when they are exactly equal in this respect the reflexion is entirely destroyed.* It is hence inferred by analogy that *mercury* has a high refractive index.

90. All this applies to the *mean* refractive index; but since, as we have just observed, substances differ materially in their relative indices for the different primary rays, it may happen that two substances shall have the *same index* for *some one ray,* and different for the others. In this case, according to the foregoing statement, the result will obviously be, that the image reflected at the bounding surface will be totally wanting in that ray for which the two media have an equal refractive power; or, in other words, it will appear of a tint compounded of all the other rays of the spectrum. For example, in oil of cassia and flint glass, the index for the red ray is nearly the same, whilst for others the oil of cassia has a much greater index than the glass; hence the reflected image has a predominance of blue. And thus in all cases of reflexion, unless in the instance (as yet unknown) of two media with absolutely the same refractive and dispersive powers, the reflected ray must always have a tint different from

that of the incident ray. This subject has been investigated by Sir D. Brewster, and he has pursued it into a variety of highly curious results, for which the student must refer to his paper, (Phil. Trans. 1829, vol. i).

Constitution of the Prismatic Spectrum.

91. In what has been said respecting the unequal refrangibility of light, our conclusions, strictly speaking, must be understood to apply to *a single ray* incident on the prism, and which is there separated into its component parts. In practice, however, where we must employ a pencil of finite diameter, in proportion as that diameter is larger we shall have a less perfect separation of the component parts, or a greater degree of *superposition*. If, for instance, we expose the prism to the rays of the sun without any limitation by an aperture, and receive the spectrum on a screen *very near* the prism, it presents the appearance of a white space, having a narrow border of red and yellow at one side, and of blue and violet at the other: as we recede the white diminishes, and at greater distances disappears, and the succession of tints becomes such as we have already described. This shews that from any one point in the incident pencil, there arise rays of all colours: these crossing and intermixing through all the central part, produce the white, whilst the unmixed portions only appear at the extremities.

In order therefore to make the experiment satisfactory, we should employ as small a pencil as possible: and the best method is to admit the light through a very narrow rectilinear slit in the screen or window shutter, the edge of the prism being parallel to the slit. It is also better, instead of using the direct rays of the sun, and forming the spectrum on a screen, to use the white reflected light of the clouds, and to receive the spectrum *on the eye*;

that is, to view the aperture, or narrow line of light, directly through the prism. It is only when the rays are thus rendered as pure as possible, that we can advance towards any precise knowledge of the constitution of the spectrum, respecting which indeed but little is yet absolutely ascertained.

92. One remarkable class of facts bearing on this point was first noticed by Dr. Wollaston, (Phil. Trans. 1802, p. 378,) and since observed with extreme accuracy by M. Fraunhofer, of Munich, (Edinb. Phil. Journ. No. xviii, p. 288, and xix, p. 26). It consists in this: When the origin of the light is a very narrow line, and the spectrum formed by a prism of very pure glass, in the position of minimum deviation, the whole spectrum appears marked by innumerable *bright and dark lines ;* all parallel to the original line, some better defined, broader, and more conspicuous than others. With an ordinary prism of flint glass the eye distinguishes perhaps twelve of them: Fraunhofer, with an exquisite prism of his own glass, and by means of a telescope, distinguished six hundred. Certain of these lines are well marked, and easily recognised: they are at unequal intervals, which also differ for different media, though the lines are in the same order, and in the same coloured spaces. They differ essentially with the species of light employed: the light of the clouds, of the moon, and of Venus, shew them exactly as in the direct light of the sun. The brighter fixed stars have lines peculiar to themselves: as also has electric light. The light of flames shew none, or at least only certain dark intervals under particular circumstances.

These lines supply the desideratum alluded to before, viz. certain definite points in the spectrum to which our measurements can apply; and thus we have *precise* determinations of *dispersion*. The values of m at the principal lines, for several different media, have been determined with great accuracy by Fraunhofer.

93. A question often asked, as to the *number* of primary colours, can be answered only with reference to the *sense* in which it is asked. If it be meant to apply to the number of tints distinguishable in the spectrum, this will be a matter of individual judgment to different eyes. Newton distinguished seven, others four, or three: but perhaps most observers would admit that it is impossible to fix on any number, since the light appears to go through every possible shade of colour between the deep red and faint violet. If we understand the question as applying to the number of definite points or rays, to each of which a different refrangibility belongs, their number must be considered as infinite. Newton concluded, that "to the same colour ever belongs the same refrangibility." But this must evidently be understood in the sense and to the extent of the conditions of his experiments; that is, that to the same assumed point or ray in the spectrum, inseparably belongs the same index, and that it cannot be further altered by prismatic analysis. But it is in no way at variance with this conclusion, to admit that such a ray may be subject to other kinds of analysis, or to modification from other causes.

These questions are connected with the mode in which the spectrum is actually formed, respecting which we have two hypotheses for examination: 1st, There may be a great number of distinct images of the original luminous point or line, each of a definite tint and refrangibility, which by *juxtaposition* form the spectrum. 2nd, There may be several spectra, each of the same colour throughout, but having maxima at different points, which by *superposition* give all the observed tints. All that has been hitherto said is equally applicable to each hypothesis. We shall now mention a class of facts which bear more directly on these questions.

Absorption of Light.

94. It has been already observed that media differ in their degree of transparency: and not only is this true as to the relative *quantity* of light which they transmit, but they also differ in transmitting *more of some particular coloured rays* and less of others, whilst some are wholly stopped. Numerical estimates of the powers of different media in this respect may be called their *indices of transparency*, which like their indices of refraction will differ both for different media, and for different rays in the same medium. This power is also found to vary remarkably with *the thickness* of the medium. The phenomena are easily observed by viewing the prismatic spectrum through plates of different transparent media, when it will be found that the different parts of the spectrum will be very unequally transmitted; and in certain cases different portions will be entirely wanting. The various common coloured glasses, and coloured liquids contained between two plates of colourless glass, give a variety of highly curious results.

The whole effects are expressed analytically by the following formula.

If we take as unity the number of rays, or intensity of any one colour in the incident light, and y the number transmitted by a given medium, whose thickness is t; then, if c $c_{\prime}$ $c_{\prime\prime}$ &c. be the number of equally illuminating rays of red, yellow, &c., we shall have for the incident white light,

$$c + c_{\prime} + c_{\prime\prime} + \&c.$$

Then the transmitted light through a thickness t, will be expressed by the formula

$$c y^t + c_{\prime} y_{\prime}^t + c_{\prime\prime} y_{\prime\prime}^t + \&c. \qquad (108)$$

in which each term expresses the intensity of each primary ray transmitted.

Here it is evident, that as t is increased, since y is a quantity less than unity, the quantity of that ray transmitted will diminish in geometrical progression as t increases in arithmetical.

This formula explains the change of tint in white light, after transmission through different thicknesses of a given medium, as well as in the prismatic spectrum. If for any ray y be a small quantity, a small increase in t will give an almost total interception.

95. Sir J. Herschel, to whom this formula is due, has also employed a method of illustration by means of loci, which represent the intensities at different points. Assuming rectangular axes, a straight line parallel to X, at an ordinate $= 1$, represents the intensity for a perfectly unabsorptive medium, the abscissæ being the indices of refraction for the different rays of the spectrum: ordinates proportional to the values y $y_{\prime}$ $y_{\prime\prime}$ &c. will give a curve expressing the intensities of the transmitted rays, or "type" of the spectrum, for a given thickness of a given medium. The dark lines in the spectrum are considered, by the same philosopher, to be analogous to the dark intervals thus occurring when any particular ray is intercepted: and which must in this case be stopped in their passage through some medium exterior to our atmosphere; probably the atmosphere of the sun.

The spectra formed by different coloured lights, as flames, &c. may be represented by such curves.

Opaque substances reduced to a great degree of tenuity, become partially transparent: gold leaf transmits a bluish-green light. Substances in one form opaque, may, by a different mode of aggregation, or by chemical combination become transparent;

as carbon and diamond, &c. Many substances appear of a totally different colour by reflected and by transmitted light. Heat is found to diminish the absorptive power in some cases, and to increase it in others. Some coloured media, not only alter the intensity of particular rays, but change the colour of the different spaces in the spectrum.

96. By means of the absorptive powers of different media for the different rays, Sir D. Brewster has applied a new species of analysis to the prismatic spectrum; and has hence deduced a theory of its constitution, of the same nature as the second of the hypotheses before stated (Art. 93).

He has found that certain media exhibit *red* rays in the *blue* and *indigo* spaces, and therefore infers their existence in the *green*, which is composed of *blue* and yellow: they also exist in the violet, which is a compound of blue and red.

Yellow is shewn in the red space by several media: it exists evidently in the orange and green: and other media shew it in the blue: no absolute proof, however, appears of its existence in the violet space, but here the light is much too faint to allow of any conclusion of an opposite kind.

Blue light is shewn to exist in the red space by the media which give this part a yellow tint, that is, absorb *blue* from it. It consequently exists in the *orange*, which contains *red:* in the green which is composed of blue and yellow; and evidently in the violet.

Again, by the action of various media, the same philosopher has succeeded in transmitting *white* light at almost all parts of the spectrum; that is, in absorbing the excess of those particular rays which predominate, and shewing therefore that there remains a certain proportion of each to form white. Hence he infers that *the spectrum consists of three separate spectra of red, yellow, and blue light, extending the whole length, but each* *having a maximum of intensity at a different point. These by their superposition give rise to all the observed colours.* [Edinb. Journ. of Science, No. x, p. 197.]

Double Refraction.

97. We have hitherto considered a ray of light on entering a transparent medium as subjected to one simple and general law of refraction. There are, however, numerous cases where, in addition to this, another effect is produced: the ray on its entrance into the medium being *divided into two portions*, one of which follows the ordinary law, whilst the other undergoes a separate refraction according to a different law. The substances in which this takes place are those possessing a crystalline structure, and such as belong to certain crystallographic classes. The property is developed in different substances in very different degrees; in some, as in the familiar instance of the Iceland crystal, or rhomboidal carbonate of lime, it is very conspicuous; and is observed by placing a dot or a pinhole against one surface, which seen through the crystal *appears double*: in others the separation of the two images is so small that it cannot be rendered perceptible except by indirect methods. The subject of double refraction is one which is intimately connected with a new class of the properties of light: we shall, however, for the present consider it simply in regard to the *directions* taken by the two rays into which the incident ray is separated, and the experimental laws which regulate their positions.

For the sake of illustration we will suppose the case of the Iceland crystal just mentioned. This substance occurs in rhomboidal masses, and is always reducible by natural cleavage into exact rhomboids, having each of their faces equal and similar rhombs. These are the forms of the ultimate molecules into which the mass can be separated by continued subdivision; in every one of these rhomboids the *short diagonal* is called the *optical axis.* Similarly in other doubly refracting crystals, other lines, in certain positions, according to the primitive form of the crystal, are the optical axes.

98. The observed facts are as follows: In the first place, if we conceive a ray to traverse the crystal *along the axis,* no double image is formed; the ray undergoes only the ordinary refraction, with a certain value of m, which may be determined as in ordinary cases; or, in other words, the *ordinary* and *extraordinary* rays (as they are termed) *coincide* when the incidence is such that the ordinary ray is refracted in the direction of the axis.

A plane passing through the axis is called a *principal section:* and if a ray be incident so that the ordinary refraction take place *in the plane of a principal section,* then for all incidences the ordinary ray having its constant index of refraction m_o the *extra-ordinary* ray will also be *in the same plane,* though with an index m_e which *varies* according to its position.

If the ordinary refraction be in a plane *perpendicular* to the axis, the extraordinary ray will also in this case be in the *same plane,* and the value of m_o remaining of course constant, that of m_e will also be *constant:* and this particular value, which we will distinguish by brackets (m_e), is found to be its *maximum* value in this crystal: in general it will be a maximum or a minimum according to the nature of the crystal.

99. In general, the position of the incident ray being determined by the angle ϕ and the azimuth ϖ, or inclination of the plane of incidence to the principal section; in all cases the ray o has the same azimuth ϖ, and an angle of refraction $\phi_,$ agreeably to the ordinary law: whilst the ray e is found to have a variable value of $\varpi_,$ as well as of $\phi_{,}$, that is, of m_e, which can be expressed

only by a complex law. This law was fully investigated by Huyghens; and from it, in order to represent the position assumed in general by the extraordinary ray, he deduced the following geometrical method.

If we conceive a *spheroid* having its axis of revolution coincident with the axis of the crystal, and its semiaxes a, b, in the ratio of the *ordinary index* and the *maximum or minimum extraordinary index*, or such that we have,

$$\frac{a}{b} = \frac{m_0}{(m_e)}$$

this will be $>$ or $<$ 1, according to the nature of the crystal; and a being the axis of revolution, the spheroid will be prolate or oblate accordingly: and for the better conception of the case, if we imagine the point of incidence on any surface of the crystal taken as the centre of the spheroid, then *the position of the ray e will always coincide with a radius of the spheroid,* determined by the following construction: from the point of incidence, in the ray u produced, take a distance k, which shall be the value of *unity in the same scale as that in which the values of m* are measured. A perpendicular from the extremity of k (in the plane of incidence), will give a point in the surface, through which let a line l be drawn in the surface at right angles to the projection of u upon the surface. If a plane be conceived to revolve about l as an axis till it touch the spheroid, its point of contact will be the point to which the radius ρ is to be drawn, in order that it may represent the extraordinary refracted ray.

100. The radius ρ forming an angle θ with the axis, we shall have, by the nature of the solid,

$$\rho = \frac{a\,b}{\sqrt{b^2 \sin.^2 \theta + a^2 \cos.^2 \theta}}$$

Huyghens found that in all cases *the value of* m_e *could be ex-*

pressed by the reciprocal of this quantity; which is easily put into the form,

$$m_e = \frac{1}{\rho} = \sqrt{\frac{1}{b^2} + \left\{\frac{1}{a^2} - \frac{1}{b^2}\right\} \sin.^2 \theta} \qquad (109)$$

The sign of the second term being + or — according to the values of a and b, or the nature of the crystals, which are thus distinguished into two species, called from this circumstance crystals with *positive* or *negative* axes respectively.

101. The geometrical construction for the extraordinary ray might be translated into analytical language, but the formula is somewhat complex: it has however been given by Malus in his Théorie de Double Refraction; and it is necessary in order to compare the experimental law with the result of theory.

Malus found by very careful determinations in the case of carbonate of lime,

$$m_o = 1.6543 \qquad (m_e) = 1.4833$$

Whence in the construction of the spheroid and the extraordinary ray,

$$k = 1 \quad b = \frac{1}{m_o} = 0.60449 \qquad a = \frac{1}{(m_e)} = 0.67417$$

Here we have $b < a$, consequently the sign of the second term, under the radical sign in the expression (109) becomes negative: or the crystal of carbonate of lime belongs to the negative class: to which are also found to belong tourmaline, beryl, emerald, apatite, &c.

Of those which belong to the positive class, we may mention as examples, quartz, ice, zircon, &c.

102. A very large class of crystals has been also found to possess *two axes* of double refraction, and the law becomes more

complex. They have been examined by Sir D. Brewster: and M. Fresnel has discovered, that in this case neither of the rays is, properly speaking, an ordinary ray; both being subject to variations in their values of m according to the position of incidence: he has given a complete mathematical investigation of the theory. But for a full account of this as well as the former theory, the student must refer to Herschel on Light, Art. 779 et seq., and 997 et seq.

Interferences of Light.

103. If we conceive two pencils diverging from single points near each other, it is evident that each of the rays of one pencil will cross each of the other at some point in its course; and that if at any distance we receive the light on a screen, it will consist of a central portion formed of the joint light of the two pencils, together with an external portion of each single pencil. According to ordinary suppositions we might expect that this central part would consist of uniform light of double the intensity of either of the pencils singly. Such, however, is not the case: for when the divergence is small, or the two luminous points subtend a very small angular interval at the distance at which we view them, it is found that the central space is *luminous at the exact central point; but at equal distances on each side shews alternate spaces of light and of total darkness.* This takes place whenever the above-mentioned very simple conditions are fulfilled; and it is quite indifferent by what means this is accomplished. It may be done in several different ways, but since in all the effect is precisely the same, this will shew that the result is independent of the particular means employed, and depends entirely on some principle or affection of the light itself; and of

the mutual action, as it were, of the rays one upon another. To give a more distinct notion, before reasoning upon the mode in which the effect is produced, we will mention the simplest methods by which the experiment is tried. Light diverging from a single point is obtained by throwing the sun's rays into a darkened room by means of a plane inclined mirror outside the shutter, through a minute aperture; or still better, through a small lens of short focus, when the rays cross almost at a single point, and thus give a diverging beam. If now, at a distance of several feet, we place flat on a table two small pieces of glass of *exactly* the same thickness (halves of the same piece) close together, and in a line from the point of light, and look at the image of the luminous point reflected from them, it will almost always happen, owing to the inequalities of the surface of the table, that they will not be precisely in the same plane, and we shall see *two images:* this may at all events be obtained by slightly pressing one of them, or placing a slip of paper under its edge. We have thus *two pencils diverging as if from the two images, with a very small angular separation;* this is easily increased or diminished at pleasure by altering the pressure, or slip of paper. If we now look at the double image through a small eye lens, at about six inches distance, it becomes easy to adjust the glasses till at the part where the two pencils cross we perceive, with the aid of the lens, a beautiful and clearly defined set of *alternating black and bright stripes,* which are always parallel to the intersection of the planes of the reflecting surfaces.

Precisely the same effect may be produced, if instead of reflexion from two planes, we use the same diverging beam transmitted through a glass having one side plane and the other cut into two planes, inclined at an extremely large angle; or, in other words, an extremely obtuse prism: by this means we have two images of the original point by refraction, from which with a very small angle of separation diverging beams originate and cross;

and by an eye lens at a few inches distance, as before, we see the same set of stripes occupying the central part of the mixed light.

104. If we proceed to examine more precisely the laws of the phenomenon, it will appear in the first place, that the stripes become closer and extend over a smaller space as the eye approaches the reflectors or the prism. Again, they become closer as the inclination of the mirrors or refracting planes, that is, of the two crossing pencils, is increased; and if this be increased beyond a very small angle, they become too narrow to be perceptible.

If we consider the paths of the crossing rays it will be evident that the bright stripe which is the centre at all distances occurs at a point where two rays cross, whose *lengths* from the two luminous points are precisely *equal:* the other bright points on each side are those at which two rays cross of unequal lengths: and at each such point the *difference of length* of the two crossing rays will be successively greater as that point is more distant from the centre. If now the two pencils were, each, at the mirror or prism, separated into a certain definite number of rays, with dark intervals, the crossing of these might be imagined to give rise to the appearance observed; but we know by direct observation that each ray separately is *not* thus divided; and therefore rays of *light* approach and cross at the *dark* points just the same as at the bright points, but with *difference of length exactly intermediate* to those of the rays giving the bright points. Hence it is the unavoidable conclusion, *that any ray divided along its length into intervals equal to these differences, must at the alternate points be somehow in a different state; such that if two rays cross at points where they are in the* SAME *condition, they conspire to form a* BRIGHT *point; and if in* DIFFERENT *conditions, they neutralise and destroy each other, and leave a point of* DARKNESS. The existence, then, of such INTERVALS along the length of a ray of light is established as an experimental fact entirely independent of any theory: the phenomenon arising from the crossing of the rays in the way thus explained, is appropriately called the INTERFERENCE of light.

There are numerous other cases in which similar effects are observed: but these experiments are the most direct and unequivocal; and will further supply the means of *measuring the lengths of the intervals.*

105. The principle on which this is done is sufficiently obvious. The distances from each other at which the stripes occur are such as are susceptible of measurement by a micrometer to any degree of accuracy: the angular separation of the two images may also be subjected to direct measurement. And supposing (as is the case) that this angle is small, and that only small portions of the diverging pencils are concerned, the concentric circular arcs in which the corresponding *intervals* on all the rays in each diverging pencil would lie, may be considered as straight lines at right angles to the rays: and we shall thus have two sets of parallel straight lines crossing at the same small angle, which we will call $2\,\psi$; then if c be the observed distance between two bright stripes, and λ the *interval* along the ray between two points where the light is in the *same condition,* and which we will call *similar points,* we shall evidently have the relation between them expressed by

$$\lambda = 2\,c \tan.\,\psi \qquad (110)$$

That a ray of light is constituted so as to possess *certain different affections* at these *intervals* along its length, may therefore be taken as a *true cause* to explain numerous other phenomena, to which we shall find it most easily apply. The fact, as well as the whole physical theory with which it is connected, but on which we forbear to enter as foreign to our immediate design, was first established by Dr. Young, and has since been success-fully applied in explaining the most recondite phenomena by M. Fresnel and Professor Airy.

106. We have here supposed the case of two sets of rays coming from origins at equal, and great distances, so that the portions of concentric circles marking the corresponding intervals upon the diverging rays of the same pencil, might be taken as sensibly parallel straight lines. If we suppose the distances less, we could not in this way find the value of λ; though the intervals c would still be equal. If to the last supposition we add that of *unequal* distances to the origins, we might still have a similar construction of arcs intersecting; but the values of c would be unequal. In the former cases also, if we take any series of the points of intersection, they will lie in diverging straight lines; or, if the origins be very distant, these lines will be nearly parallel; and the light received on a screen, or the eye, at successive distances, will give the dark and bright stripes at equal distances along the screen, which will contract gradually and uniformly as we approach the source. If we suppose the origins at unequal distances, the construction would give stripes not only at unequal intervals along the screen, but those intervals varying rapidly as we approach the origin. If we consider any one set of consecutive points, it will be evident in this case, upon the simplest geometrical principles, that as they are the intersections of two radii originally differing by a given quantity, and constantly increasing by equal increments, their locus will be an *hyperbola,* of which the two origins are the foci: and if one be infinitely remote the hyperbola will approach to a parabola. We shall presently recur to this case.

107. The existence of *similar and dissimilar points at given equal intervals* along a ray being established as an experimental fact, we may observe as a consequence from it, that if we con-

ceive two rays *superposed* throughout their course, or *coinciding* in direction, if owing to any cause they have *similar* points of each *coinciding*, they will produce an impression of double intensity; but if *dissimilar* points *coincide*, the effect of each will be wholly neutralised, and darkness will result. If intermediate points coincide, a more or less faint light will be produced.

108. Other methods of determining the values of λ have been deduced from different experiments; and in point of fact these have been more usually resorted to: the values are found to *differ for the different primary rays*; and the following table gives the result of some very accurate measurements for the mean and extreme rays:

Red	$\lambda = 0.0000266$ inch.
Yellow	0.0000227
Violet	0.0000167

This circumstance limits the extent within which the stripes are formed in the experiments before described, when common light is employed. If homogeneous light, red for example, were made use of, we should have a series of stripes whose intervals are given by the preceding formulæ, extending through the whole mixed light. Yellow and violet would have intervals successively less: and these being *superposed* to produce the actual phenomena in white light, the distances will coincide only near the central part, and becoming more and more different towards the edges, the stripes will gradually cease to be distinctly formed, will become coloured, and beyond a short distance none will be seen.

109. We have supposed the interference to take place in air: in any denser medium it is found that the *stripes* are *closer*; or, in other words, *the values of* λ *becomes less:* that is, we must

regard the recurrence of these intervals along the ray of light as modified by the medium through which it is propagated: they become *shorter* in the more refractive, or denser medium: and this, as is found by very accurate observations, precisely in *proportion to the refractive power*. This will be readily apprehended and will prepare the student for another result which may be less obvious.

110. If in the path of *one* of the interfering pencils an opaque substance be placed, that pencil is of course stopped, and the stripes disappear: this affords an obvious proof that the effect is due to the *mutual* action of *both* the pencils. But it is a more remarkable fact that if a plate of glass, or any *transparent* substance (of moderate thickness) be similarly interposed, the *stripes likewise disappear*. If both pencils, however, are similarly intercepted by the glass, the stripes remain. If the transparent plate be *very thin*, the stripes appear; but the whole body of them is *shifted* within the boundaries of the luminous space, towards that side on which the interception takes place. If the two pencils are intercepted by glasses of precisely the same thickness, and a slight inclination be given to one of them, so that it presents a slightly greater thickness, the stripes can be made to shift towards that side, in proportion as the thickness is increased, and at length move entirely out of the bright space, and thus disappear.

If different transparent substances be interposed, it has been found that the degree of shifting *increases* with the *refractive power* of the body (the thickness being the same); and M. Arago has found by very accurate experiments, that it is *precisely proportional to it*, and has even applied measurements of the displacement occasioned by different media to deduce their refractive powers. Now, in accordance with what was stated above, it appears that in this case the lengths of the intervals in that part of the ray within the glass are shortened, or there are more in a given length.

It is also to be observed, that when the stripes are thus shifted, the *lengths of route of two rays* going to form *any one stripe* (as the central stripe, for example, or the first, second, &c. from the centre), are no longer the same as before: their difference is necessarily increased by the new position the stripes have assumed. In other words, since by the plane parallel surfaces of the glass the rays undergo no deviation, the same two rays which before formed any one stripe do not now interfere at all at the same distance; but that one of them which has passed through the glass interferes with another ray of the non-intercepted pencil which lies more towards that side on which the glass is, that is, with a ray which is more oblique, or has a longer route from the origin to the point of intersection: whilst at the same time the shorter ray has a greater number of intervals in proportion to the thickness of the glass it has passed through. And this increase in the number of intervals in the one ray, by its transmission, is found to be exactly equal to that in the other, owing to increased length of route. This is strikingly shewn by the following experiment.

111. If a *prism with a small angle* (4° or 5°) be interposed over both pencils, the whole bright space is of course deviated by the prismatic refraction, but the *stripes retain the same relative position* (except a trifling extension towards the violet end of the spectrum, and a slight degree of colour obviously due simply to prismatic refraction). Abstracting from this, they have *undergone no shifting;* and yet the two rays forming any one stripe, have passed through very different thicknesses of the prism: but it may also be observed from the course of the refracted rays, that the ray which passes the *thinner* part of the prism has gone through a *longer* route before it meets the other.

Now let the angle of the prism $= \iota$, and we may consider the two rays incident upon it in directions nearly perpendicular, since $2\,\psi$ is by supposition very small: hence their paths within the prism will be still more nearly perpendicular to the first surface: at their emergence let them be distant from each other by a space b measured on the second surface: then we shall have *the difference of their routes within the prism* very nearly expressed by

$$d = b . \sin. \iota$$

If we trace the course of one of the rays through its several refractions, we shall easily perceive that owing to the conditions assumed, we have very nearly

$$\phi_{\prime\prime} = \iota$$

and thence $$\sin. \phi_{\prime\prime\prime} = m \sin. \iota$$

From the smallness of the angle $2\,\psi$, we may take the *difference of the lengths of the rays after emergence till they meet*, by dropping a perpendicular on the longer, and thus we shall easily deduce the value of that difference

$$d_{\prime} = b\, m \sin. \iota$$

But on comparing this with the difference *within* the prism, we obviously have

$$d_{\prime} = m\, d$$

Or *the differences of the lengths of route of the two rays which meet to form any given stripe, are in the ratio of the refractive powers of the prism and the air.*

In this case the whole difference in the lengths of route of the two rays will be

$$d_{\prime} - d = d\,(m - 1)$$

Or in general, returning to the case of any parallel intercepting medium of thickness t, we shall have

$$d = t\,(m - 1)$$

And substituting this difference for the particular multiple of λ, which expresses the difference of routes of two rays forming any given stripe in the formula (110), we shall have the displacement of that stripe from the expression

$$c = t\,(m - 1)\;\frac{\text{cot.}\,\psi}{2}$$

Whence follows, as a general experimental fact, that if two rays are so situated that they interfere and form any given stripe, with a given difference of route, according to the foregoing conditions, then two other rays will form the *same* stripe, though with a much greater difference of route as measured by the length traversed, provided the shorter ray pass through a denser medium of such thickness and refractive power, that if the length be increased in the ratio of the refractive power and thickness, the difference of route shall remain the same as at first. Or, in other words, two rays will form the same stripe when their difference of routes *as measured by the number of intervals* is the same as before.

This fact of more intervals occurring in the same space in the passage of light through a medium precisely in proportion to its refractive power, is called the *retardation* of the intervals. As we shall have occasion to use this term, it must be carefully borne in mind that we attach no other meaning to it than what is implied in this definition.

Divergence of Light.

112. Before proceeding to discuss several other phenomena dependent on the principle of interferences, it will be necessary to mention a very simple but important property of light, which has not yet been adverted to.

When, by the means already described, we have a pencil diverging from a single point, it is found that if any portion of this pencil be intercepted, or it be any how caused to terminate, or a boundary be formed between parts more and less bright, the portion on the brighter side of the boundary has always a *tendency to diverge anew from that boundary as a fresh origin*. This is found to take place on examining either the shadows of opaque bodies placed in the diverging pencil, or the portion of this pencil transmitted through an aperture in a screen, or a space of double light, as in the last experiments: in either case it is found either by receiving the light on a ground glass screen simply, or more accurately, by examining it with an eye lens, that *the light diverges into the shadow* beyond the position in which it ought to be confined, if it proceeded strictly in a rectilinear course past the edge or boundary. The appearances are, in point of fact, complicated by other phenomena, which we shall describe presently: but the simple *fact* of this *new divergence* is one which is matter of distinct measurement and observation. It is a *real exception* to the primary law of the *rectilinear* propagation of light: which must always be understood as limited by this exception. All our previous investigations are indeed grounded on the truth of this law. But since it is only a portion of the light which is thus modified, and this not rendered perceptible except under peculiar circumstances, none of those results will be invalidated by this consideration. The simple fact is best established by the method of Maraldi, who observed that the shadows of cylinders terminated at a shorter distance from the origin than they should do, on the supposition of rectilinear rays.

Coloured Fringes of Shadows and Apertures.

113. We before stated that a series of stripes will be formed when two diverging pencils cross at a small angle, by *whatever*

means such a condition is brought about: now, besides the direct methods already described, this may be effected by admitting the light (originally diverging from a single point as before) through two minute apertures near together. From each of these, agreeably to what has just been stated, the light diverges anew, and in the mixed light of the two pencils, dark and bright stripes are observed, exactly as in the former cases, though less brilliant and distinct. They extend to a short distance both ways, in a direction perpendicular to the line joining the apertures.

If the apertures be larger, so long as the intervening opaque space remains the same, the stripes remain unaltered. If they be indefinitely increased, so that merely an opaque body remains, the stripes are unchanged, and extend parallel to the length of the body. If it be cut into the form of a small circle (the edge being *extremely* well defined) concentric circles are produced, and the *centre is a bright spot*. As the diameter of the opaque body is greater (its distance from the origin and from the eye remaining the same), the stripes increase in number and decrease in breadth: beyond a certain diameter they become too numerous and fine to be distinguished. If the breadth be diminished, they decrease in number and increase in breadth. With a very narrow body there is only one central white stripe. If the distance of the body from the origin be diminished (the eye remaining at the same distance from the body), it merely intercepts rays diverging at a greater angle, and acts as a broader body would do at its original distance. If this distance remain, but the *eye* be nearer to the body, it merely receives rays crossing at a greater angle, and the effect is the same as if it were placed at the original distance from a broader body. Thus in all cases the appearance of the stripes depends simply on *the angle subtended by the opaque body at the origin and at the eye jointly*. The experiment is most simply tried by a slip of metal or card, which can be turned about an axis in its own plane, so as to present any *effective* breadth, on a stand which can be slid along a table to various distances. If it be cut in a triangular form, it exhibits at once the variation in the stripes due to different breadths: and this is more remarkable when the vertical angle is considerable.

114. But the effect in all these cases is accompanied by others apparently more complex, though equally dependent on the same simple principles.

If we regard merely one *edge* of the opaque body, it will be seen that on the *outside* of it there are *several parallel bands of colours,* usually *three* can be distinguished, at successively decreasing distances from each other. In homogeneous light these consist of alternate bright and dark stripes, and by superposition of such alternations of different breadths, for the different colours, the observed tints are accounted for. These bands follow the course of the sides of the body, and at an angle cut *into* the body assume a more complex appearance from overlapping: whilst at *projecting* angles they are curved round it. They do not belong exclusively to the edges of the shadows of *opaque bodies,* but are produced in a diverging pencil wherever there is *a boundary,* any how produced, between a more and a less illuminated space. Thus we see them within the edges of the doubly illuminated space in the experiment of Art. 103. When the edges of two opaque bodies approach, these bands *overlap,* are partially superposed, and give rise to a beautiful variety of tints, according to the degree of proximity of the edges extending into the shadows: or, in other words, we have the case of *apertures* of different diameters; if rectilinear they are beautifully striped; if of irregular shapes a singular complexity of colours is produced; if circular, concentric rings are formed: all these varying with the distance; the *narrower* the aperture the more the two sets of colours overlap, and therefore the *wider* are the apparent fringes of colours.

This is strikingly seen in a narrow triangular aperture, where the coloured image assumes almost a reversed form. But all the varieties of form, especially those arising from apertures near together, &c., must be seen to be understood; and we shall not pursue detailed descriptions.

115. In order to proceed to an accurate examination of the simple case of the fringes formed at an edge, (from which all other cases may be derived,) we will suppose a narrow body placed vertically, and a horizontal line drawn to the centre of its breadth in which the eye is situated, and which we will call the axis.

The first point to be noticed is, that the bright bands commence together upon the very edge; and that if the screen or eye be removed to successive small distances from the body along the axis, it is found that the lateral distance from the axis to any bright band, increases faster than in simple proportion to the distance along the axis: so that the locus of the successive positions would be a curve resembling one branch of an hyperbola. Also the edge of the shadow is altogether undefined, the light shading off gradually into the dark space, in the central part of which the internal stripes are formed, as before described.

The lateral distances of the bright points from the axis, or a parallel to it passing through the edge, have been subjected to most exact measurement by M. Fresnel, for homogeneous light; and have been found to agree in a singularly precise manner with the numerical results of a formula expressing the distances of those maxima, as well as their intensities, (arising from the conspiring or counteracting effects of the concurrence of *similar or dissimilar intervals* of the same values as those before stated,) deduced from the physical theory of those forces or motions by which the similar and dissimilar points are explained. The explanation of the mode in which the interferences take place in this instance depends simply on the fact before stated, that *at the edge a new origin of divergence is given* to rays which pass close to it, while the rest of the pencil passes near it unaltered.

Thus, in the first place, the diffused rays diverging into the shadow render it ill defined towards its edge: 2nd. For the interferences we shall have just the case before referred to, (Art. 106), of diverging pencils from origins at unequal distances. Rays of the more divergent pencil meeting those of the less divergent, will encounter them successively with lengths of route differing by successive *intervals* or multiples of such intervals as in the case before considered: these will evidently occur at distances from each other decreasing *outwards* from the axis, and as we proceed *onwards* along the axis the locus of the intersections will be the hyperbola before spoken of. The alternate stripes of homogeneous light thus produced will, by superposition, give the colours actually seen.

These phenomena were originally called the *inflexion* or *diffraction* of light. This was naturally, and as far as was then known, correctly supposed to be a *distinct property* of light, depending upon some peculiar action which the edge of a body exercised on rays passing near it, until it was shewn to be explained by the *general facts* of the interferences, and the formation of a new origin of divergence at the edge.

Some of the phenomena were observed by Hooke and Grimaldi. The external fringes were examined with great precision by Newton, who framed a theory of the sort of action necessary to be supposed to produce them. Dr. Young suggested the application of the principle of interference: but the explanation was not complete until Fresnel gave the full investigation of it in his Mémoire sur la Diffraction de la Lumière. If these experiments be conducted in any other medium than air it is found exactly as before explained, that the intervals diminish in proportion as the refractive power increases.

Colours of thin Plates.

116. One of the most celebrated phenomena in optics is that known by the name of Newton's coloured rings, or the *colours of thin plates.* It may be exhibited under various forms: one of the most simple and common is that of the thin films of water formed by blowing soap bubbles; which when brought to a certain degree of tenuity, are seen to exhibit beautiful colours by reflected light; that is, at a certain thickness the film of water is incapable of reflecting to the eye all the rays of the white light incident upon it, and will give only a particular tint; which varies according to the thickness; and in no case consists of any simple prismatic colour.

If we use homogeneous light it is found that as the thickness is gradually and continually diminished, the film appears alternately bright and black, and beyond a certain tenuity continues black.

If we receive the *transmitted* light this is always found exactly the reverse; bright when the film by reflexion appears black, and black when it is bright: this is the simplest mode of stating the phenomenon. When compound light is used the reflected and transmitted tints at a given thickness are *complementary*, or such as together would make white. There are several other ways in which the phenomenon is seen: a very small drop of oil placed on a surface of water will spread itself over the surface till it is reduced to such tenuity as to shew the colours: or again, the lamina of air contained between two plates of glass, when pressed hard together, will produce the same effect in a more stationary form: if we use a convex lens of small curvature placed upon a plane glass, or still better upon a concave lens of a radius slightly greater, the colours will appear arranged in the form of exact rings about a central spot; which, if the pressure be sufficient, will be totally black: this forms the limit beyond which no diminution of thickness will produce any other tint. If we look at the light through such a combination, we see a central *white* spot surrounded by rings complementary to the former. It was in this way that the facts were examined by Newton, who first investigated them with accuracy, though they had been observed by Boyle and Hooke.

In this form of the experiment it is easy to understand the composition of the tints observed. We have only to conceive concentric circles of each of the primitive colours alternately dark and bright, the intervals being greatest in the red rays; and the first, or central circle in all being black: these being superposed will explain all the observed compound tints. These resulting tints or orders of colours are commonly called *Newton's Scale*, and are as follows:

1st order.—Black, very faint blue, brilliant white, yellow, orange, red.

2nd. Dark violet, blue, yellow-green, bright yellow, crimson, red.

3rd. Purple, blue, rich green, fine yellow, pink, crimson.

4th. Dull blue-green, pale yellow-pink, red.

5th. Pale blue-green, white, pink.

6th. Pale blue-green, pale pink.

7th. The same, very faint. The remaining orders are so faint as not to be distinguishable.

117. It is of importance to the explanation of the phenomenon to ascertain the *thicknesses* at which the several tints or the several points of maximum and minimum occur: this is readily done to a great degree of accuracy when we use this form of the experiment. This was in fact the method pursued by Newton. He formed the rings between two spherical surfaces of great radius,

one convex and the other concave; and found the diameters of the darkest rings to be as the square roots of the even numbers 0, 2, 4, 6, &c.; and those of the brightest, as the square roots of the odd numbers, 1, 3, 5, 7. The radii of curvature being very great in proportion to the diameter of the rings, it follows, that the intervals between the surfaces, or the values of t at the alternate points of greatest obscurity and illumination, are as the natural numbers themselves, 0, 1, 2, 3, 4, &c. Now if the actual values of the radii are known, the thickness is easily found for any one point, and deduced by the above proportion for any other.

If the diameter of any ring $= d$, and the radii of the two spherical surfaces $r\ r_{,}$, it is evident that we have a very small arc of each sphere with the same chord d, and the difference of their versed sines will be the thickness t. Now these versed sines being written $v\ v_{,}$ we shall have by the property of the circle for a very small value of v,

$$v = \frac{\left(\frac{d}{2}\right)^2}{2\,r}$$

And in like manner in the other circle

$$v_{,} = \frac{\left(\frac{d}{2}\right)^2}{2\,r_{,}}$$

Whence we have

$$t = v_{,} - v = \frac{d^2}{8}\left(\frac{1}{r_{,}} - \frac{1}{r}\right)$$

In this way Newton found the thickness at the brightest part of the first ring after the central black spot $t = .00000561$ inch.: this number multiplied in the ratios above, will give the thicknesses at the other points.

118. In proceeding to examine the laws of this phenomenon we shall first establish this important circumstance, viz. *that* BOTH THE SURFACES *of the thin lamina are concerned in producing the effect*: this is proved by availing ourselves of another property of light of which nothing has yet been said, but which we will now merely assume as an experimental fact belonging to a part of the subject to be explained hereafter. The fact in question is this: When light reflected from a surface of *glass* is viewed through a certain mineral called tourmaline, in a particular direction, *at a certain incidence* it appears as if *totally extinguished*, or no light is transmitted through the mineral: whereas with light reflected from polished *metal* this is *not the case.*

Now if we form the colours between *two glasses,* viewing them through the tourmaline, we observe them disappear at the proper incidence: if for the lower glass we substitute a surface of polished *metal,* and then repeat the observation, we find that they *disappear exactly as before:* hence it is an unavoidable conclusion that the reflexion from the *upper* surface, which is of glass, is concerned in *producing the colours.*

Again, if in this last case we continue to depress the eye and the tourmaline, so as to receive the light at greater obliquities than that at which the disappearance takes place, we see the rings reappear; but with this remarkable change, that the centre which before was black is now white, and the rings complementary. When the lower surface is of glass this is not the case. This additional circumstance supplies an equally decisive proof that the reflexion from the *lower* surface is also concerned in producing the rings.

This very simple but important experiment was devised by Professor Airy: his reasoning, in fact, extends much further, but this suffices for our present purpose. The student who would follow up the complete theoretical explanation, must refer to the original paper in the Cambridge Transactions, 1831.

In the production of the rings seen by reflexion, then, there is in some way concerned the light reflected from the lower surface of the lamina, (that is, the upper surface of the lower glass), and also that reflected from the upper surface of the lamina of air, (that is, the internal reflexion from the lower surface of the upper glass). The *interference* of these two portions of light is a *true cause*, and will be found sufficient to explain the phenomena.

119. To take the simplest view of the case let us suppose a ray incident perpendicularly on the first surface, at a part where the thickness of the lamina is t. This ray will be partly reflected at the first surface; and this portion will retrace its former path, or coincide with the incident ray: the remaining portion traverses the thickness t. At the second surface it is again partly reflected, and repassing the thickness t, emerges at the first surface and coincides in position with the first portion.

The superposition may take place according to any of the conditions expressed in Art. 107, as regards the *intervals*. If dissimilar intervals are superposed, a point of darkness will result; and in order to produce this, their difference of route (as measured by these intervals) must be equal to $\frac{\lambda}{2}$ or some odd multiple of it: the difference actually traversed is $2\,t$.

But this portion, which after passing $2\,t$ emerges at the first surface, also undergoes a partial internal reflexion at that surface, and traversing t again emerges below: or upon the whole emerges having a length of route within the lamina $= 3\,t$; or differing from that of the portion transmitted directly by $2\,t$, and consequently if we are to take this as equal to $\frac{\lambda}{2}$, we ought to have a point of darkness, as in the last case.

This, however, is not the fact: for, as was just now stated, we always observe by *transmission* a *bright* point at the same thickness where by *reflexion* we have a *dark* one. Hence it is a necessary conclusion that, owing to some cause, there must be *half an interval added to or subtracted from the length of route at each internal reflexion.*

This will give for the interfering reflected rays, a difference of route expressed by,

$$2t + \frac{\lambda}{2}$$

And for the transmitted rays,

$$2t + \lambda$$

If we proceed to other values of t it will be evident, that as we suppose them equal to successive multiples of $\frac{\lambda}{4}$ we shall have for the reflected rings the even multiples, giving the successive values $\frac{\lambda}{2}$ $\frac{3}{2}\lambda$ $\frac{5}{2}\lambda$ &c. or points of darkness; whilst the odd multiples give λ, 2λ, 3λ, &c. or points of brightness. The transmitted rings will evidently be complementary, since they always differ from the former by $\frac{\lambda}{2}$.

It will be manifest on comparing the value of t as above given from Newton's measurement for the first bright ring, with the values of λ before stated for the different rays, that this value .00000561 inch. is exactly equal to that of $\frac{\lambda}{4}$ for the yellow ray, or that which is the most predominant in the composition of the white ring. It was from measurements of this kind that the values of λ were originally determined; though Newton designated them agreeably to the theory which he first framed to account for these rings: and in which, since the fact of interferences had not then been established, nor the joint action of the *two* surfaces recognised, he was necessitated to suppose that at the dark points no light was reflected, or that at alternate intervals the light had an incapability of being reflected: hence he called the intervals *fits* of easy transmission and reflexion: and the experiments fairly led to this as a new and peculiar property of light, until it was shewn to merge in a more general property.

120. It will be apparent from what has been already observed, that the phenomena are equally produced of whatever transparent medium the thin plate may consist. They are observed if the glasses are placed in a vacuum: or if any liquid be allowed to insinuate itself between them. But the *greater the refractive power of the medium the less will be the diameters of the rings;* or, in other words, the less will be the thickness corresponding to the same bright or dark point; the values of λ being diminished in the same ratio.

The experiment has been tried by Mr. Talbot, by means of glass films, formed by blowing glass bulbs till they burst. These when exposed to homogeneous light appeared streaked with distinct bright and dark lines, following the directions in which the thickness was the same. These films were estimated to be about one thousandth of an inch in thickness, which would correspond to the 89th order of rings. In theory there is no limit to the increase of thickness at which the alternations might not be produced; but in practice such a limitation is found in the circumstance, that even in perfectly homogeneous light the lines would be too fine and too close to be seen, and that we cannot procure strictly homogeneous light.

121. We have thus far considered the case of perpendicular incidence: if we now suppose the rays to be inclined by depressing the eye it will be evident that the path of any ray after its first reflexion will no longer coincide with its course, on emergence after the second reflexion. This last ray, however, will coincide with some other ray reflected from the first surface, whose point of incidence coincides with its point of emergence; and thus the same superposition, and the same correspondence or opposition in

the intervals, may take place. If t be the oblique length which the ray has traversed between the two surfaces, and if $t_{,}$ be the perpendicular thickness at the point of emergence, and ϕ the angle of incidence upon the second surface, we have,

$$t = t_{,} \text{ sec. } \phi$$

Now since at the perpendicular incidence the diameter of any given ring, or the thickness at which it is formed, depends entirely on the condition that the *increment of the thickness* from that which gave the preceding ring is precisely equal to $\frac{1}{4}$ *of an interval* of the homogeneous light we refer to; and since we have seen that $t > t_{,}$ it follows, that if the inclination of the surfaces be considered uniform, this increment of t is also greater than the corresponding increment of $t_{,}$: in order, therefore, that this increment of $t_{,}$ may be equal to $\frac{\lambda}{4}$ we must take a point at a greater distance along the surface, or *a ring of greater diameter*, to give the *same tint*. This agrees with what we observe, viz. that on depressing the eye the rings of the same tint enlarge in diameter, or, in other words, to the same thickness belongs a tint *higher* in the scale.

All this must be understood as applying only to the case of *moderate* obliquities. When the inclination becomes very great, it is found that there is a limit to the dilatation of the rings. The cause of this is not well understood. Newton investigated an empirical formula for the tints at different obliquities as modified by this limitation, but the subject being somewhat complex we shall not here enter upon it.

The rings produced at thicknesses and obliquities considerably greater than those at which, under ordinary circumstances, they cease to be distinguishable, may be seen by means of a prism laid on a plane glass with scarcely any pressure. On looking into it, within the limit of total reflexion from the base (Art. 72), there appears a series of coloured bands formed by rays, which emerging from the prism at the great obliquities very near the limit of refraction, are reflected again at the surface of the plane glass, and reenter the prism so as to interfere with some of the rays reflected internally from the base. The tints, however, are greatly modified by the dispersion of the prism.

Colours between inclined Glasses.

122. If a luminous object be viewed through two plates of glass of precisely equal thickness slightly inclined to each other, it will be evident that besides the transmitted image, we shall see a number of images formed by the successive reflexions between the glasses accompanying it. The first or brightest of these is formed by an assemblage of rays which have all undergone *two* reflexions, though at *different pairs* of the four surfaces. On entering the first plate they undergo a partial reflexion at every surface they successively encounter, each of the reflected rays again undergoing a similar series of partial reflexions at each surface. Thus it will be readily evident, that these different portions into which the ray has been separated, must go through lengths of route differing by the lengths of the interval between the glasses and the thicknesses of the glasses, or the different multiples of those which they have respectively traversed: they will therefore *in general* emerge after traversing routes which differ by considerable quantities.

Among these portions, however, there are two which (if we abstract from the very small difference in the interval between the glasses at the two points where they respectively pass) will have gone through *different routes of precisely equal length.* These two rays will be, 1st, one which passes directly through the first plate or thickness t, and through the interval i, is then reflected at the first surface of the second plate; retraces i, and the first plate t; at the first surface it is reflected again, and passes the whole system $2t + i$: or upon the whole it has gone through $4t + 3i$.

2ndly. Another portion proceeds directly through the whole, or $2t + i$ is reflected at the last surface; retraces $t + i$, is reflected at the second surface of the first glass, and retracing $i + t$ emerges, after having on the whole passed through $4t + 3i$; or a route exactly equal to the former: neglecting the difference of i.

It will readily be seen that out of all the possible combinations of different successive reflexions, these two are the only ones which will give rays with precisely equal routes, all the others will differ by quantities amounting to some multiples of t or i. If we now recur to the small difference in the values of i for the points at which the two rays respectively pass, it is obvious that by slightly altering the inclination of the plates we can diminish the difference of the routes to any amount; and can consequently make them differ by *half an interval*, or any multiple of an *interval:* and we shall thus have a *dark* or *bright point*, or rather band, parallel to the intersection of the planes of the glasses. The same thing will take place for other rays passing at different parts of the glass, or forming different parts of the reflected image, under the same conditions; and the image will thus appear, in homogeneous light, crossed by dark and bright bands; or in white light, by coloured bands.

The experiment may be tried with the flame of a candle, or still better with a portion of the light of the clouds, limited by an aperture of an inch in diameter; two halves of the same plate of glass being used. This experiment was devised by Sir D. Brewster.

Colours of Thick Plates.

123. Another phenomenon, originally observed by Newton, and referable to precisely the same principles as the foregoing, was denominated by him the colours of *thick plates*, and will be readily understood after what we have seen above.

The effect is observed thus: Light being transmitted through a small hole in a screen, so as to be incident on a spherical concave reflector of glass with concentric surfaces, the back being silvered, and the aperture situated at the centre of the spherical surfaces, on the screen surrounding it are seen *coloured rings*; or, in homogeneous light, alternate dark and bright circles. They become faint and disappear if the distance of the screen be increased or diminished beyond a small difference from the original position. They diminish in diameter as the glass is *thicker*. The reflexion from the back of the mirror is *essential* to their production, as they are rendered faint if the silvering be removed; and disappear if a substance of a refractive power nearly equal to glass is applied: they are also not produced in metallic reflectors.

They are explained thus: Besides the regular reflexions the incident pencil is partly *scattered* at each surface; that is, some rays of it are reflected thence in every direction: these rays, irregularly reflected from the second surface, meet at a small angle some of those irregularly reflected from the first, and interfere with them at the screen; which is obviously in the position of the focus: the alternations produced would be very faint for a single pencil, but at this point, where by the nature of the spherical reflexion many of such rays differing little in their direction from those regularly reflected are concentrated, they become visible.

124. The incidence being very nearly central, and y being the distance on the screen from the hole to any bright or dark point, or the diameter of a ring, it will be evident that the length of the ray irregularly reflected from the first surface, whose radius is r, will be equal to

$$\sqrt{r^2 + y^2}$$

That of the ray which meets it after reflexion from the second surface, reckoned from the point behind the second surface, at which, if produced, it meets the radius produced, (and whose distance from the first surface we will call a,) will be equal to

$$\sqrt{(a + r)^2 + y^2}$$

Consequently the difference of length of the two *reflected* rays which interfere, will be obtained by taking the difference of the above quantities, after adding a to the former; and since this difference must be a multiple of half an interval, we shall have

$$a + \sqrt{r^2 + y^2} - \sqrt{(a + r)^2 + y^2} = n \frac{\lambda}{2}$$

And if we solve this equation, neglecting the square of y, we shall obtain for the diameter of the ring,

$$y = \sqrt{\frac{n \lambda}{a} r (a + r)}$$

Or, if the thickness of the plate be small compared with the distance of the screen, a will also be small, and the expression becomes,

$$y = r \sqrt{\frac{n \lambda}{a}}$$

This formula accords precisely with the most exact measurements made by Newton in the cases he tried.

Colours of Dew, Striæ, &c.

125. When light is transmitted through a glass covered with a fine *dew*, by breathing on it, &c., colours are produced, which Dr. Young explained on the principle of interference, by applying the consideration of the *retardation* already mentioned. A ray passing through a drop of water would be more *retarded* than that passing very near it through the interstice of air, in proportion to their difference of refractive power: on this principle Dr. Young calculated the intervals, &c., of the colours, and found them agree exactly with the experiment. To a similar cause are ascribed the coronæ occasionally seen surrounding the sun and moon.

Fine fibres, and striæ, also give beautiful colours by interference, when single, between the rays reflected from their opposite sides; and when many are placed together more complex colours are produced by their combined interferences. A striking example of this kind is seen in the iris buttons, invented by Mr. Barton, the surface of which is covered with minutely engraved parallel lines, in some instances not more than one 10,000th of an inch apart. A phenomenon very similar is that of the colours exhibited by the surface of mother of pearl. This substance, when examined by a powerful microscope, is found to present a surface covered with minute striæ arranged in parallel waving lines.

126. Various other phenomena, of a kind extremely similar to those just mentioned, have been observed, and are readily accounted for by interferences. Beautiful sets of colours are seen on viewing a candle, or line of light, by very oblique reflexion from any moderately polished plane surface, as ivory, ebony, &c., held close to the eye. The images of a narrow line of light

formed by the successive internal reflexions of a piece of glass slightly prismatic, may be seen divided into several dark and bright bands.

A variety of remarkable phenomena attending the formation of colours by grooved surfaces, &c., have been investigated by Sir D. Brewster, Phil. Trans. 1829.

Colours of Gratings.

127. When the origin is a narrow line of light, and is viewed through a telescope whose object-glass is covered by a *fine grating of wires* parallel to the line of light, the several pencils which diverge from the apertures of this grating as new origins, interfere with each other, and produce some highly remarkable appearances; which were observed with great accuracy by M. Fraunhofer, with an extremely delicate apparatus. The principal phenomena are as follow:

In the centre there appears a simple colourless image of the line of light, somewhat less bright than when there is no grating. On each side of this occurs a perfectly dark space; then an image brilliantly coloured, according to the order of the prismatic spectrum, the violet being nearest the centre, and the colours perfectly pure; so much so that Fraunhofer observed in them the dark lines before described. Then succeeds a dark interval; then a second spectrum: and to this a third, though this mixes a little with the preceding, and the interval is not absolutely black, but sombre purple: in the succeeding spectra this is more and more the case, till they become superposed. Fraunhofer, however, was able to distinguish not less than thirteen spectra.

He subjected the appearances to extremely precise measurement by means of the well defined lines in these spectra: and deduced results for a number of different conditions as to the size of the wires and intervals of the gratings, &c. But we shall

Polarization of Light.

128. We have hitherto been engaged in examining the properties of light, which concerned in the first place, simply the *directions* which it takes under certain conditions, considered as homogeneous; next we were led to recognise distinctions among the *integrant parts* of which a ray, as produced from any ordinary source, is *composed*, in regard to the different laws which they follow: then we found distinctive properties to mark the recurrence of certain regular though minute *intervals* along its length: we now come to consider other distinctions which prevail with regard to *directions transverse to the length of the ray*. Thus without any particular supposition as to the physical nature, or even as to the imaginable diameter of a ray, if we merely confine ourselves to the supposition of a mathematical line, and conceive a plane to which it is perpendicular, crossing it at any point, and in that plane two rectangular directions assumed, passing through the ray, these will point, as it were, to four parts of space with respect to the ray, and which for distinction we will name in the order of succession in a circumference round the ray, *a*, *b*, *c*, *d*.

Under ordinary circumstances it is wholly a matter of indifference in what position with respect to these directions the ray encounters any surface, medium, or body at which it undergoes any of the modifications hitherto spoken of. But there are conditions under which a material difference in this respect is observed.

Polarization of Light by Reflexion.

129. One of the simplest cases may be described as follows: If a ray of light be reflected from a surface of glass at an angle of about 56° and the reflected ray be then received on another plate of glass at the same angle, the planes of the two reflexions being coincident, the ray will be reflected again as usual: but if the second glass be turned round, so that the angle of incidence upon it remain the same, but that the plane of the second reflexion is at right angles to that of the first, the ray will no longer be reflected, or the image will wholly disappear. When the planes form intermediate angles, the image will be seen with intermediate degrees of brightness: and if the angle of reflexion be any other than 56° the effect will be produced in a less degree; this being the incidence at which the complete or maximum effect takes place.

If we consider this experiment it evidently consists essentially of two parts; the first reflexion, which puts the light into a certain state; and the second, by which the nature of the property it has acquired is exhibited. Now this property evidently has a relation to the parts of space in rectangular directions transverse to the ray before supposed: recurring, then, to the illustration there imagined, the ray after its first reflexion has acquired a property (which it does not possess in its natural state,) of being reflected or not, according to which of the rectangular directions, taken at the point of its course where it meets the second glass, coincides with the plane of its incidence on that glass. And this relation is the same throughout its whole length to any extent after the first reflexion. Thus the directions *a, b, c, d,* taken at any one point in the ray, are the same as at any other; or they lie in fixed planes, whose intersection is in the ray, and one of which coincides with the plane of the first reflexion. These directions being thus constant, the case may be illustrated by the imaginary resemblance of them to the cardinal points of the compass about the ray: hence the application of the term *polarity* to this property, the ray being said to be *polarized*: in the case we have supposed, it is *polarized by reflexion:* and the plane in which the first reflexion takes place, is called the *plane of polarization*, or it is said to be *polarized in this plane*.

The difference between polarized and common light may then be stated thus: in a common ray, at successive points along its length, the four points are turned in all possible directions; in a polarized ray they lie in the same planes throughout its whole length.

Or, again, if we conceive a pencil of a certain imaginable diameter, and that its section at any point has a certain figure, square for example,—if the light be polarized, the form of the ray is *prismatic;* it acquires plane *sides* throughout its length: whereas in unpolarized light, the sections at successive points have their sides in all possible directions.

This remarkable property, which has opened an entirely new field of optical research, was the discovery of Malus, in 1810. Two glasses are easily arranged so as to be inclined at the requisite angle to the common axis of two tubes, which can turn one in the other, so as to give different inclinations or azimuths to the two planes of reflexion. It is most convenient to use the white reflected light of the clouds.

130. This property is found to be communicated by a reflexion at the surfaces of all *transparent* bodies; in its complete degree, at a particular incidence, constant for the same subtance, but differing for different bodies; and in a less degree at all other incidences.

When a ray is incident on a plate of glass with parallel surfaces at the maximum polarizing angle, the part which enters the glass, and is of course reflected from its second surface at an angle equal to the angle of refraction at the first, will also emerge completely polarized.

We have supposed the case of glass *in air*, where the refractive powers are very different. It has been found that when two adjacent media differ very little in refractive power, the angle of complete polarization at their bounding surface approaches to 45°. This leads us to consider the variation of the polarizing angle for different substances; and a simple and comprehensive law has been discovered from very numerous observations by Sir D. Brewster, which includes the above. If m be the relative index of refraction of the two media, then, if we call the maximum polarizing angle Φ, the law is expressed by

$$m = \tan. \Phi$$

This gives us a remarkable geometrical result, for since we have also $m = \frac{\sin. \Phi}{\sin. \Phi_{,}}$ this gives $\sin. \Phi_{,} = \cos. \Phi$, or, at the incidence of complete polarization, *the refracted ray is perpendicular to the reflected ray.*

From this law we obviously deduce the result stated above: for if $m = 1$ we have $\Phi = 45°$.

The following are some of the values according to this formula for different substances; the adjacent medium being air:

Water . . .	Φ = 53.°11′
Crown glass . . .	56.55
Plate glass . . .	57.45
Oil of Cassia . . .	58.39
Diamond	68.6

It will also follow, that since m is different for the different primary rays, there will also be a slight difference in Φ for the same substance: or, there will never, strictly speaking, be a total absence of light at the second reflexion; but a certain tint will remain, varying with the dispersive character of the medium.

The law applies in general to transparent bodies, though there is some doubt as to the case of diamond. With metallic surfaces no complete polarization takes place, though certain effects are produced which will be considered hereafter.

131. This law also applies to the case of the ray reflected at the second surface of a parallel plate before mentioned. For in this case we have obviously $\Phi_{,,,} = \Phi$ and $\Phi_{,,} = \Phi_{,}$

Whence $$\sin. \Phi_{,,} = \frac{1}{m} \sin. \Phi_{,,,}$$

And since, as before observed, we have $\sin. \Phi_{,} = \cos. \Phi$

we have also $$\sin. \Phi_{,,,} = \cos. \Phi_{,,}$$

and thence we deduce, $$\tan. \Phi_{,,} = \frac{1}{m}$$

Or, since $\Phi_{,,}$ becomes the angle of reflexion at the second surface, the law applies in this case also.

If we suppose the second surface bounding another medium, when we have not $\frac{1}{m}$, but a new index $m_{,}$, then for complete polarization at the bounding surface

$$\tan. \Phi_{,,} = m_{,}$$

But since $\Phi_{,,} = \Phi_{,}$ it will depend upon the nature of the first medium whether this can take place.

If the back of a glass be silvered, since the principal reflexion is from this second surface, we cannot use it to polarize light.

132. The angles of incidence at both the glasses being those of complete polarization, the intensity I of the light reflected at any azimuth α of the second glass, has been conceived by Malus as represented by this formula, where A is the absolute intensity of the light employed,

$$I = A \cos.^2 \alpha$$

On this principle a common or unpolarized ray may be conceived as composed of two rays, polarized in planes at right angles, and of equal intensity. Such a compound ray being incident on a reflecting surface at the polarizing angle, one portion of it having its plane of polarization inclined to that of reflexion by an angle α, the other will be inclined $(90 - \alpha)$, and we shall have,

$$A \cos.^2 \alpha + A \cos.^2 (90 - \alpha) = A$$

or the intensity of the reflected ray will be unaltered in whatever azimuth it is incident.

When the polarized ray is not incident on the second glass at the polarizing angle, but at any angle, the law of intensity of the reflected ray is expressed by a more complex formula investigated by M. Fresnel.

Polarization by other Methods.

133. Sir D. Brewster found, that if a ray be made to undergo a number of successive reflexions between two parallel glass plates, at angles differing from the complete polarizing angle, it at length becomes completely polarized.

134. Malus and Biot also discovered, that if a ray be incident on a *pile of parallel glasses,* the *transmitted* portion is partially polarized; and more completely so as the number of glasses is increased: and if the *incidence be at the angle* Φ, the portion transmitted by the first glass penetrates the subsequent ones without any loss by reflexion; and if the number of glasses be considerable, the emergent ray *is wholly polarized in a plane perpendicular to that of refraction.* The same effect may be produced by piles of plates of mica.

135. M. Arago discovered a general law, that at all incidences of unpolarized light upon a plate of glass, the *reflected and transmitted portions contain equal quantities of polarized light, the planes of polarization being at right angles to each other.*

On this subject some curious researches have been made by Sir D. Brewster, for which the reader must refer to his paper in the Phil. Trans. 1830.

136. A well known mineral called tourmaline, which crystallizes in prisms, when of a brown or purplish colour, it is found to possess the remarkable property that a plate of it cut *parallel to the axis* of the prism about $\frac{1}{20}$ inch thick, will *polarize* the whole of the light which traverses it in a plane perpendicular to the axis of the crystal.

It appears to be a law that a substance possessing this property will only transmit light so polarized. Hence, if the light be *previously* polarized in a given plane, and be incident on the plate of tourmaline, if the plane of polarization be *coincident* with that perpendicular to the axis of the crystal, the light will be *wholly transmitted,* but if *at right angles to it, wholly intercepted.*

Hence two plates of tourmaline form a very convenient apparatus when set in cells so as to be capable of turning each in its own plane about a common axis. The one polarizes the incident light, the other analyzes it: performing analogous parts to the two plane reflectors in Malus's experiment.

A similar property is observed in some other minerals, as in some specimens of rock crystal which have a brown tinge. Also in plates of agate, cut perpendicular to the laminæ of which it is composed, about $\frac{1}{15}$ inch thick. [See Brewster on Philos. Instruments, etc. p. 329.]

In general it is to be observed, that taking any of the pieces of apparatus thus described singly, we may combine each with any other, and use either of the two in any such combination indifferently as the *polarizing* or as the *analyzing* part of the apparatus: and in whatever way we operate we have always analogous results conformable to the principles at first established.

Polarization by Double Refraction.

137. If two rhombs of Iceland spar be placed upon one another, or still better if fixed in cells capable of turning in a tube about a common axis, the cover at one end being in contact with the surface of one of the crystals, and containing a minute aperture, then the light transmitted through this hole and the two crystals to the eye at the other end, will *in general* be divided into two portions, O and E at the first crystal, and each of these into two again at the second crystal *Oo Oe*, and *Eo Ee*, or the eye will see four images of the hole. If we turn one crystal about in its cell we shall find one position, and one only, in which the four images are all of equal intensity: in every other, two of them will appear to diminish in brightness and the other two to increase, till at length the first two vanish altogether: then these reappearing and increasing the other two diminish, and at length disappear, and so on. The appearances are somewhat complex: but to analyze them let us take only *one crystal*, and use light previously polarized by any of the methods just described. It will here be seen that one of the two images vanishes at every quadrant, or is in the same predicament as the image reflected at the second glass in Malus's experiment. If we reverse this experiment, and look at the two images by reflexion from glass at 56°, we shall find one of them vanish at each quadrant; or, if we stop one of the rays from the first crystal, and examine the other by the second, it will exhibit the same results. *The two images, then, are polarized: and in planes at right angles to each other.* Hence, if we resume the experiment with the four images,

the same thing takes place for each one of the two first images O and E, by the action of the second crystal, as by that of the reflector. The planes in which the two images are polarized are, one parallel, and the other at right angles to the principal section of the crystal. Either of the above experiments may be repeated with the substitution of a tourmaline, a pile of glasses, or any other polarizing apparatus; which may also be used in the reverse manner as an analyzer.

These properties of the Iceland spar were originally investigated by Bartholinus, Newton, and Huyghens.

In the double refraction of a ray, previously polarized, if α be the azimuth of the plane of polarization to the principal section, and A the intensity of the light which enters the crystal, and using the letters $O\ E$ to signify the intensities of light in the two rays, they may be expressed by,

$$O = A \cos.^2 \alpha \qquad E = A \sin.^2 \alpha$$

whence,

$$O + E = A$$

If we have two crystals superposed, we shall have in like manner the intensities of the four emergent pencils; since the planes of polarization of the first two images O, E, are at right angles, those of Oo and Ee will be in the same azimuth, and of Oe and Eo in azimuths complementary: thus we shall have

$$Oo = \tfrac{1}{2} A \cos.^2 \alpha = Ee$$

$$Oe = \tfrac{1}{2} A \sin.^2 \alpha = Eo$$

and

$$Oo + Oe + Ee + Eo = A$$

These formulæ express the changes before described.

Polarized Rings.

138. We have already observed, that in all the experiments on polarized light there are two essential parts of whatever apparatus we employ; the one to produce polarization, the other to analyze the light so modified; or to act as a test of the properties it has acquired. If, now, in any combination, we suppose the analyzing part in the position where the ray disappears, and that between the two parts we interpose a doubly refracting substance, in such a position that its optical axis is traversed by the polarized ray, then, in this particular position, no effect is produced; the analyzer still giving a defalcation of light as before: but if the ray pass in *any other direction, a portion of the light is restored;* and this is found to depend on the thickness of the crystal traversed. In homogeneous light, at successively different thicknesses, alternations of light and darkness are produced; and thence in white light, compound tints.

139. The easiest way of observing these phenomena is by interposing a plate of mica between the two parts of the apparatus. This mineral naturally splits into plates, and its two axes lie in a plane perpendicular to that of the laminæ, inclined at 45° to each other, and each $22\frac{1}{2}°$ to the perpendicular.

Let the intersection of the plane containing these axes, with the surface, be called *A*, and a line perpendicular to it in the surface be called *B*.

Then, the lamina being perpendicular to the ray, if we take such a position that *A* coincides with the plane of polarization, the lamina produces no change in the light: if the lamina revolve in its own plane and *A* be inclined, light is transmitted; and if the inclination be 45° it is at a maximum.

If the thickness of the lamina be greater than about $\frac{1}{30}$ inch, the light transmitted is white. If less, it is coloured; and the tint is the same as that of the *thin plates* corresponding to the *same difference* of thickness. The tint merely changes in *intensity* with the revolution of the lamina in its own plane.

If we *incline* the lamina to the ray, we alter the *thickness* traversed: suppose *A* in the position of maximum transmission, and that we make the lamina revolve about *B*, there will be a succession of tints till we arrive at such a virtual thickness as gives white in the order of Newton's tints: beyond this no further change can occur.

If we make the lamina revolve round *A*, then the tints change till we come to black; but beyond this, if the inclination be continued, we have the tints recurring again in reverse order, till we arrive at white.

140. But the most complete view of the whole of these phenomena is obtained if we apply the eye and the analyzing part of the apparatus close to the crystal, so as to receive a cone of polarized rays which have traversed it, the optical axis lying in the axis of the cone: in this case, the rays as they cross it further from the axis, will successively traverse increasing lengths of the medium, and thus, in the position of the axis, a *black spot will be seen surrounded by rings of colours.* If there be two axes, we shall have two such cones or sets of coloured rings, which may be either completely distinct from each other, if the axes are inclined at a considerable angle, and the lamina of some thickness; or, if at a smaller angle, or the lamina be thin (as in mica about $\frac{1}{40}$ inch thick), they will be partly mixed with each other; but this takes place in a very remarkable manner—they are not in the former case perfect circles, but of an oval form; the points corresponding to the axes, forming foci towards the outer ends. If the axes be close, these ovals form into a compound figure with

two foci, about which the innermost rings are formed as before; whilst the middle ones coalesce at the part where they touch into a form resembling a figure of 8, and the outer bands assume the form of a single oval enclosing all the others. Through each of the poles there passes a black band, ill defined, and in a curved form, resembling an hyperbola; the convexities, in the two sets, being towards each other. If there be only one axis these bands will become straight lines crossing at right angles.

If the crystallized plate be made to revolve about the ray, the black band will shift its place with respect to the rings; and when the plate has moved through 45° the black band will have gone through 90°, and will now assume the form of a straight line, in the plane of polarization: it will also be prolonged to meet the corresponding line belonging to the other set, and they will be crossed at their centre by another black line at right angles.

The whole of these phenomena are best seen by the combination of two tourmalines, above described: on which principle an instrument is made fitted up with several specimens of crystals, called the polariscope, which will afford the best means of becoming acquainted with these beautiful appearances. Plates of the crystals of nitre cut across the axis of the prism about $\frac{1}{5}$th inch thick, give the best instance of the rings about two axes: those of mica are too far separated to be conveniently exhibited. Carbonate of lime affords a good specimen of the rings about one axis; by grinding down the obtuse angles of the rhomb, so as to form two new surfaces perpendicular to the axis: clear ice also shews them well if about one inch thick.

141. The whole series of curves formed by the coalescing rings about the two axes have been examined with great accuracy by Sir J. Herschel, and found to coincide exactly with a series of *lemniscates,* whose equation is

$$(x^2 + y^2 + a^2)^2 = a^2 (b^2 + 4x^2)$$

Where the perimeter b varies from 0 to ∞; and $2a$ represents the constant distance between the poles: b is found to increase by equal differences from one ring to another, for the same thickness of the plate; and is inversely as the thickness in different plates.

These curves are distinguished by the property, (which easily follows from their equation,) that the product of the radii drawn from the two poles to any point in the curve, is equal to the constant rectangle ab.

If we draw a line through either of the poles, perpendicular to the line joining them, this will cut all the rings at points where they appear to follow almost exactly the same order of tints as Newton's coloured rings: in other positions they differ a little: but, for the present disregarding this small discrepancy, we may consider each tint or ring as distinguished, or measured by its own particular value of the product of its radii above-mentioned. These radii being the measures of *angular* separation, we must consider them as the sines of arcs $\theta\ \theta_{,}$, and the order of the tint will in general be proportional to this product, and to the thickness of the crystal traversed by the ray jointly; or, since the thickness traversed is equal to t sec. $\phi_{,}$, it will upon the whole be expressed by

$$ab = \sin.\ \theta \sin.\ \theta_{,} .\ t \sec.\ \phi_{,}$$

Or, if n be the number of periods, or order of the ring, and $h = \frac{ab}{n}$, or the unit whose multiples determine the order of the rings, we shall have

$$h = \frac{t}{n \cos.\ \phi_{,}} .\ \sin.\ \theta \sin.\ \theta_{,}$$

This function, then, is invariable in whatever direction the ray penetrates the crystal.

The truth of this law has been verified by very accurate observations on a plate of mica, made to revolve about B, (as before described,) which corresponds to a section of the rings through both the poles, in homogeneous light. The angles of incidence at which the successive bright and dark intervals were produced, being measured with the index for mica, 1.5, the values of $\phi_{,}$ were computed; from which, those of θ (being the differences from the first value of $\phi_{,}$) were found, and those of $\theta_{,}$ (the differences of these last from 45°, the inclination of the axes): and on substituting these successively in the formula, the value thus given to h, was *absolutely constant.* Similar determinations have been made for all other directions across plates of a great variety of crystals.

It is evident that if we conceive the crystal formed into a *sphere,* the thickness traversed would be constant, and the formula would become simply *the product of the sines.* The discovery and verification of this law is due to the united researches of Sir D. Brewster, M. Biot, and Sir J. Herschel.

When the two axes unite in one, the lemniscates become circles, we have $\theta = \theta_{,}$, and the tint is represented by

$$\frac{t \sin.^2 \theta}{\cos. \phi_{,}} = t \sec. \phi_{,} \sin.^2 \theta$$

If the rings are close, cos. $\phi_{,}$ is nearly constant, and the *arc* θ may be taken instead of its sine.

142. We may here make an observation which is important to the explanation of these phenomena. We have before seen that in uniaxial crystals, when the ray passes in any other direction than that of the axis, the extraordinary ray, having a variable value of m, has, agreeably to what was remarked in Art. 111, a corresponding *retardation* different from that of the ordinary ray. If, now, continuing the same analogy, we compare the *velocities*

as referring to such retardation, and call those of the two rays respectively υ and $\upsilon_{,}$, it will be evident from the expression (109) that the value of υ being constant, that of $\upsilon_{,}$ is given by that formula, and we shall have the difference of the squares (writing k for the constant factor),

$$\upsilon_{,}^2 - \upsilon^2 = k \sin.^2 \theta$$

But we have just seen that the order of the tint is also proportional to $\sin.^2 \theta$; hence we shall have that tint proportional to

$$t \sec. \phi_{,} (\upsilon_{,} + \upsilon) (\upsilon_{,} - \upsilon)$$

Now since a small change in the direction of the extraordinary ray, or in $\upsilon_{,}$ will make a change in $\upsilon_{,} + \upsilon$, which is insensible compared with that made in $\upsilon_{,} - \upsilon$, we may for such small changes as occur within limits very near the axis, consider the factor $(\upsilon_{,} + \upsilon)$ as constant; and we shall thus have *the tint very nearly proportional* to the simple difference of velocities $\upsilon_{,} - \upsilon$, or *to the relative retardation* of the two rays.

In by far the largest number of crystals with one axis, the tints follow very nearly those of Newton's scale: that is, agreeably to the last observation, the difference in the retardation of the ordinary and extraordinary rays (of any one primary colour) is in the simple proportion of their respective values of λ.

There are many instances, however, where we find a deviation from the Newtonian scale. In some the diameters of the rings for different colours differ less, in others more. Or we have the retardation in a ratio less or greater than that of a simple proportion to the values of λ.

Other deviations are frequently caused by the simple circumstance of the imperfect, or distorted structure of the crystals, &c.

In biaxial crystals there is another source of apparent irregularity, dependent on the remarkable fact (to which it was traced by Sir J. Herschel), that the *situation of the axes differs for the different primary rays.*

The *axes* (in any case) *as related to these coloured rings,* being found by observation, we know in many cases that they are identical with those of double refraction: but they are found in many crystals in which the double refraction is in no other way apparent; its existence is inferred from analogy.

There is, however, one part of the phenomena of which nothing has yet been said; viz. the central black cross. This is easily understood in uniaxial crystals; since a ray which passes the crystal in the plane of polarization, when it coincides with the principal section of the crystal, furnishes only an ordinary ray in the crystal, and therefore at the analyzer is suppressed; or we have a dark line corresponding to its position. In precisely the same way, in a plane at right angles to this, only an extraordinary ray passes, which therefore gives also a dark line at right angles to the former.

143. In biaxial crystals, M. Biot has given the following theorem for the position of the plane of polarization, which he established from experimental results: it is also found to coincide precisely with the formula deduced from the physical theory by M. Fresnel. The construction is simply this: if two planes be drawn through the course of a ray within a crystal, and through the two optical axes, and a third plane bisecting the angle included between the two former, this will be the plane of polarization, if the ray be an ordinary one,—but one perpendicular to it if extraordinary.

This construction enables us also to determine the form assumed by the black lines, which we have observed before assume a curved form resembling hyperbolas, and passing through the poles. If we suppose the separation to be small, the arcs θ $\theta_{\prime}$, which measure the angular distance of any ray from the axes, may be considered as rectilinear. And thus, on the plane where the rings are traced, the projections of the plane of polarization will be lines at right angles; and a line parallel to one of them through a point in the dark curve, will bisect the angle contained between the lines θ $\theta_{\prime}$ for that point. The problem will then be reduced to one of plane geometry, viz. to find the nature of the locus traced out by an ordinate to given rectangular axes, which always bisects the angle formed by lines to the tracing point, from two points given in position, whose line of junction passes through the origin, and which are at equal distances on each side. The solution of this being a simple process of analysis, we shall not here enter upon it, but merely observe that it brings us to the equation of an *hyperbola, whose asymptotes are the axes.*

144. Hitherto we have described the rings as they appear when the polarizing and analyzing parts are in their rectangular position. If we now suppose the analyzer to be inclined from this position, at the commencement of the rotation, the arms of the black cross appear to dilate; they grow at the same time fainter, and segments of complementary rings appear in them. The junction of the two sets is marked by a faint white. As the rotation proceeds, the primary segments contract and become more dilute with white; while the secondary or new tints extend and become more decided: at the same time the centre of the system grows gradually bright, and when the quadrant is completed, the whole of the space before occupied by the black cross is now white, and the quadrants of rings all complementary. The phenomena are precisely analogous in biaxial crystals. The tints change as above, and we have a pair of white hyperbolas passing through the poles.

We have supposed the thickness of the plate such that the whole system of rings was of small angular extent, so as to be taken in by the eye at once. If the plate be much diminished in thickness, this will no longer be the case; and instead of rings of a distinguishable form, we shall see only broad bands of colour extending to great distances from the poles, and even visible when the axes themselves are so much inclined to the surface of the plate as to be quite out of sight: or even when the axes lie in the plane of the plate. This last circumstance actually occurs in the natural laminæ of sulphate of lime, which consequently to see the rings, must be cut and polished in a direction perpendicular to its laminæ. In any such case, however, by attending to the same considerations as those adverted to in the case of mica, we shall readily be able to analyze the phenomena presented. If we call the plane containing the two axes section *A*, a plane perpendicular to this, and passing through the line bisecting the angle, they form *B*; and a third perpendicular to both, through their point of concourse *C*, then, at a perpendicular incidence, if the plane of polarization coincide with either *B* or *C*, the polarization will be undisturbed; but if the plate be turned round in its own plane, the extraordinary image will reappear, and become of maximum intensity at every 45°. And if the plate be sufficiently thin, it will exhibit some one of the colours, and the tints will descend regularly in Newton's scale (i. e. *from* the black,) as the thickness is increased.

145. When two such plates are superposed, the sections *B* and *C* coinciding, they are evidently circumstanced as if merely parts of one plate of the sum of their thicknesses. But if they be *crossed*, i. e. so that *B* of the one coincides with *C* of the other, *the tint* produced is that which would be due to a single plate equal to the *difference* of their thicknesses. If the plates therefore be halves of the same plate, they will in this way exactly neutralize each other.

If the incidence be not perpendicular, the colours produced appear very complex; but they are easily understood if we use the tourmaline apparatus. We then see a central black cross, and in the quadrants hyperbolic branches in the order of Newton's tints: this is when the tourmalines are crossed. If parallel, we have a white cross and complementary tints. If the compound crystal be turned in its own plane, the tints only change in intensity.

If any number of plates of the same crystal (the incidences being perpendicular) are thus superposed, whose thicknesses are t, $t_{,}$, $t_{,,}$, &c., regarding that plate as negative whose sections B C are crossed with respect to those of any other plate, the tint polarized by the system will be that due to the algebraical sum, or to the thickness $t + t_{,} + t_{,,}$, &c. And if we have plates of different crystals cut in the same manner, or containing the axes, we shall have to multiply the thickness of each by a peculiar constant k, and the resulting effect will be

$$T = k\,t + k_{,}t_{,} + k_{,,}t_{,,} + \text{\&c.}$$

k being positive or negative, according as the crystal belongs to the positive or negative class before distinguished.

146. But this is only a particular case of a more general law, which though in the form in which it is announced has a direct reference to theory, may yet be considered apart from all physical hypothesis, if we carefully confine the use of the terms "retardation" and "acceleration" to the experimental sense in which we originally defined them. The law in question is stated thus:

The tint ultimately produced is proportional to the interval of acceleration or retardation of the ordinary ray, on the extraordinary, after traversing the whole system: the partial acceleration or retardation in each plate, being proportional to the length of the path described within the plate, multiplied by the square of the sine of the angle which the transmitted ray makes, internally with the optic axis of the plate, if it have but one axis, or to the product of the sines of its inclination to either, if it have two.

147. We have considered the effect produced on the interposition of *crystallized* bodies between the polarizing and analyzing apparatus. It has been discovered by Sir D. Brewster, that similar effects are produced if instead of crystallized substances, we place in the same manner various transparent bodies, whose *particles have been made to assume a peculiar state of aggregation*, by the rapid and unequal effect of sudden heating or cooling, or by mechanical compression or bending. Pieces of glass slightly bent by the action of screws, so that the parts towards one side were dilated, and towards the other compressed, whilst in a line separating the two, the opposite actions were neutralized, exhibited no action on the polarized light transmitted through the neutral part, whilst on each side of it coloured bands were produced. Glass heated and then applied on one side to a cold body, or cold and applied to hot iron, exhibits a succession of colours, as the heating or cooling process is transmitted along it. Unannealed glass retains permanently a structure of this kind. A similar arrangement is produced in various jellies, and other transparent bodies, by compression, or by the process of induration.

In all these cases, if the mechanical effects are examined, it is easily seen that there will be a *strain* upon certain parts of the body; and the particles will be brought into a state of compression in certain directions. The full account of these curious researches will be found in the Phil. Trans. 1816.

The existence of this action on polarized light, would by analogy, lead us to expect that a doubly refracting structure had been communicated to the particles of the transparent substance. And that such a property can be communicated to glass by simple *pressure*, has been shewn by M. Fresnel; who used highly ingenious means to render the separation of two pencils sufficiently great to be sensible.

148. Sir D. Brewster also found that the property of double refraction might be artificially communicated by the application of *mechanical pressure* to the particles of certain substances. A mixture of white wax and resin melted together and left to cool, is destitute of this property: but if pressed violently between glass plates, it acquires it. He applies this fact as affording an explanation of the mode in which it is produced in natural crystals. It is not owing to any cause inherent in their ultimate particles, because it is lost by solution, fusion, &c. It must therefore depend on something in their mode of aggregation. The particles are united into a crystalline mass, by powerful attraction. This must occasion a violent *compression* in each individual molecule, in certain constant directions: this, then, is a cause competent to produce *double refraction* in them: and the property will be referred to certain axes, to which also the forces are referable: there may be one or more, agreeably to certain conditions in the primitive forms. He has pursued this subject into the details necessary to support this conclusion, for which the reader must refer to his paper, Phil. Trans. 1829. Professor Mitscherlich has found, that while heat expands crystals in the direction of their axis, they contract in a direction perpendicular to it: and that their double refraction is at the same time diminished. This is remarkably in accordance with the above view.

149. A remarkable appearance is presented by some specimens of Iceland spar, in which *films of a different structure* interrupt the regular crystal. We cannot here enter into the detail of the phenomena; it must suffice to observe, that they are all explicable on the principles just discussed, when we consider the two

portions of the regular crystal as simply the polarizing and analyzing parts of an apparatus, with a thin lamina interposed. The effects have been imitated artificially by Sir D. Brewster. Again, there are other natural specimens of crystals, in which light being polarized by internal reflexion in the crystal, traverses it along its axis, and thus gives rings at the analyzer; or if previously polarized, the rings are seen directly. There are also other structures producing similar effects. Such crystals are called *idiocyclophanous.*

Another highly curious subject connected with the double refraction of light, is the property possessed by some crystals of giving two images of different colour: or of absorbing different rays, according to the direction of the incident light; a property called dichroism. But for the details of all these, and many other interesting points of the like description, we must refer the student to the larger works already often cited: or to Sir D. Brewster's Optics, in the Cabinet Encyclopedia.

150. It would appear at first sight (in reference to the explanation of these phenomena), that since the interposition of the crystallized plate restores the light, which when it is away is absolutely deficient, it has in fact given it the power of being reflected at the second glass, or transmitted through the tourmaline, as if it were no longer polarized; or, in a word, it might be thought that the polarized light was by these means *depolarized:* and that different portions being more or less affected, according to the thickness of the medium, there resulted the varied degrees of restoration which we perceive, and the formation of coloured rings. And, in fact, this supposition, and the name of *depolarization,* have been adopted by some philosophers: a complex theory, however, is necessary to account for the transmission of the different tints, and we shall find that they admit of simpler explanation grounded on facts.

Interferences of Polarized Light.

151. The state of *polarization* of two rays is found to have an influence on that property which, marked (as we have seen) by certain *intervals*, gives rise to the phenomena of interference. This curious and delicate enquiry has been prosecuted by MM. Arago and Fresnel, with the utmost care and precision; and we will now proceed to a brief account of their results.

In the first instance they verified what is indeed almost obvious, that two rays polarized, by any means, *in the same plane*, will produce stripes by interference exactly as common light.

They found, however, that *two rays polarized in planes at right angles to each other*, when brought under those conditions in which common rays or rays similarly polarized would interfere, *do not produce any stripes*. The experiment, simple in theory, is difficult in practice, from the circumstance that in any mode of producing the requisite polarization it is difficult to secure the exact equality of routes necessary for interference, if it can be produced, or for assuring ourselves that it is not. Several methods were devised with this object; but the simplest and most direct appears to be this: In the course of two interfering rays, small piles of glass plates or mica (as before described) were placed, care being taken to ensure their perfectly *equal thickness*; this, in fact, is the difficult part of the operation: then, one of these being capable of turning about the transmitted ray as an axis, when they were in the position to give *similar* polarization, *stripes* were produced: when *opposite, none*. Halves of the same plate of tourmaline might be similarly employed, but great exactness in the adjustment is requisite.

152. Another method is as follows: In the path of two interfering rays, which we will call R and L, is placed a thin plate of

sulphate of lime, which possesses double refraction : but for a thin plate the separation of the rays will be insensible, and they will emerge superposed. Thus there will be produced two images of each ray, or *Ro, Re, Lo, Le*: of these *Ro* and *Lo* will have passed through equal thicknesses of the crystal; as also will *Re* and *Le*; but their paths will be longer. Hence, agreeably to what we observed before respecting the lengths of routes, the portions *Ro Lo* will form stripes in the same central position as if no crystal were interposed, as also will *Re Le*; and these two sets of stripes occurring exactly at the same points will be superposed, and only one set will appear.

But *Ro* ought also to interfere with *Le*, and *Re* with *Lo*, and these being respectively of unequal routes the stripes of each would be shifted towards that side where the greater thickness is traversed; or we should have a set of stripes on each side of the central one, and if the thickness of the plate be sufficient, shifted entirely out of the central part.

No such lateral sets of stripes however can be observed: hence we conclude that they are not formed, because the rays which should produce them are *oppositely polarized*.

But if we cut the plate in half and turn one half round 90° in its own plane, these rays are then reduced to the *same* state of polarization; and the rays *Ro Lo, Re* and *Le*, which in the former case produced the central fringes, are now reduced to *opposite* states of polarization: and we find the *central stripes now disappear*, whilst the *two lateral sets are formed.*

153. The philosophers before mentioned also found, that if a pencil primitively polarized in one plane be separated into two in opposite planes, and then reduced to one, they will interfere like unpolarized rays.

If as before a thin plate of sulphate of lime be placed in the path of two interfering rays of light originally polarized, the emergent ordinary and extraordinary rays of each pencil are superposed, but consist of two rays differing in ROUTE by some interval d, and each pair polarized in planes at right angles.

If now a rhomboid of carbonate of lime be placed to receive these rays with its principal section 45° from the plane of polarization, so as to form two images considerably separated, these will each be composed of four rays superposed. Of the four in the *final ordinary* image each combination of two may interfere; that is, six interferences would result, of which two sets are formed by rays which do not differ in route, and therefore form two sets of stripes superposed in the centre: two other sets are composed of rays differing by d in favour of one pencil, and therefore form stripes on one side of the centre, and the other two for the like reason on the other side, or we should see three sets of stripes in the ordinary, and three in the extraordinary image: this is agreeable to observation. It is also evident that the rays which form the lateral sets of stripes are precisely those which, at their leaving the plate, had opposite polarizations; but have been afterwards reduced to similar polarization by the action of the rhomboid.

154. By a modification of this experiment another very important result is established.

If instead of the carbonate of lime, which causes a large separation of the two pencils, we interpose another thin plate of sulphate of lime, or of rock crystal, in which the two finally emergent rays shall be superposed, the two final sets consisting each of three sets of stripes, as in the last experiment, we should expect would here simply be superposed, or we should see three sets of double intensity.

Instead of this *the central set alone are seen*: this proves that the rays going to form either of the lateral sets must consist of two portions, which give complementary periods; or that the *final ordinary pencil consists of rays which differ by half an interval λ, from those of the final extraordinary pencil.*

155. This last remark, combined with what was before observed in Art. 142, enables us to explain the polarized rings.

A polarized ray, incident on a crystallized plate, is divided into two portions, which for small obliquities emerge superposed: the principal section of the crystal being supposed at half right angles with the plane of original polarization, the two emergent rays, O and E, will be oppositely polarized, and will also (Art. 142) have a difference of retardation d, (which may be either on even or odd multiple of half an interval) corresponding to the tint, and dependent on the value θ. Also, by Art. 154, to this difference must be added a half interval, or on the whole, they differ by $d \pm \frac{\lambda}{2}$. These rays, however, agreeably to what has just been established, being oppositely polarized cannot interfere, though, otherwise, in a condition to do so.

Now, introducing the analyzer, if it be a reflector or tourmaline with its plane of reflexion or transmission at right angles to the original plane, and therefore at half right angles the opposite way to the principal section of the crystal, its effect will be to restore that portion of both O and E, which it reflects or transmits to the same plane of polarization; they will therefore now interfere and produce the observed tints.

If we suppose the analyzer a doubly refracting crystal, having its principal section in the plane of polarization, in this position we shall have four rays emerging of equal intensity; the rays respectively formed from O and E continue to differ by the interval $d \pm \frac{\lambda}{2}$, as before: but here again, in consequence of the same laws, those which belong to the ordinary and extraordinay pencils formed by this crystal, will also in addition, differ by $d, \pm \frac{\lambda}{2}$, or we shall have first, Oe Ee, differing by $d \pm \frac{\lambda}{2}$; and secondly, *Oo Eo* differing by $d \pm \lambda$; and each pair being similarly polarized, the rays of the first will *interfere* together, as will also those of the second pencil, but at periods complementary to those of the first, and therefore in compound light giving complementary tint.

Circular Polarization.

156. We have observed that in general, when a polarized ray is passed along the axis of a doubly refracting crystal, it undergoes no change, but the analyzer presents a dark spot, as before.

In some substances, however, there is a remarkable exception to this law. It is most conspicuous in rock crystal, or quartz; in which if a polarized ray be transmitted along the axis, and on emergence analyzed by Iceland spar, in all positions, there are still two images, and those of different colour. To examine the case more simply, however, let us suppose that homogeneous light is used. When the analyzer is in the position in which no light should be produced, the interposition of the rock crystal allows a certain portion to appear, and in order to make the light vanish, the analyzer must be turned through *a certain angle,* from its rectangular position, dependent on the *thickness of the quartz*. As the thickness is increased the rotation of the analyzer must be continued, in order to make the light disappear: and we may thus go on through a complete revolution, or any number of revolutions. In other words, the *plane of polarization* of the ray on emerging from the quartz, is turned through a certain angle, and this angle continues to increase with the thickness of the plate, or the *plane of polarization* is, as it were, *twisted* into a *surface of double curvature,* like that which is formed by a cork-screw staircase, supposing the steps indefinitely increased in number, and diminished in depth; but at each point the ray remains *completely polarized,* as is shewn by its vanishing when analyzed.

157. M. Biot, who most accurately investigated this subject, found, that for a given thickness, the arc of rotation requisite to bring the plane of polarization into the position of evanescence, was different for the different primary rays, and the proportional to the reciprocal of the square of the interval λ for the particular ray. Thus k being a constant, determined by experiment, and t the thickness, he found the arc of rotation r, expressed by,

$$r = \frac{k\,t}{\lambda^2}$$

Thus for rock crystal of $\frac{1}{25}$ inch thick, the following were some of the values:

RAY.	ARC OF ROTATION.
Red	17° 50′
Yellow	24°
Violet	44°

He also observed this remarkable fact, that in some specimens of quartz it was necessary to turn the analyzer to the *right hand*, in others, to the *left*. Sir J. Herschel observed a singular coincidence between this property and the right or left-handed direction in which certain small faces of the crystal *lean* round the summit: these varieties are called plagiedral quartz.

Various substances have been examined by different philosophers in which this property exists. It is found even in certain liquids: oil of turpentine, oil of lemon, and syrup of sugar, ae instances.

If two crystals be superposed, the arc of rotation is that due to the sum or difference of their thickness, according as they are of the same or opposite kinds of circular polarization. Thus the resulting effect is expressed by

$$R\,T = r\,t + r_{,}t_{,} + r_{,,}t_{,,}\ \&c.$$

The capitals representing the thickness and rotation of the combination, the small letters those of the separate substances, each having its proper sign.

This has been found to hold good not only with crystals superposed, but with liquids when mixed together. The value of *r*, or *index of rotation*, is very small in these fluids compared with its value in the crystals.

Some specimens of amethyst present remarkable combinations of tints when polarized light is transmitted along the axis, arising from a compound structure of quartz.

158. To analyze the nature of the circularly polarized ray, Fresnel devised an experiment, simple in theory but of great delicacy in practice, to which he was led by theoretical considerations.

He took a prism of quartz, having its base parallel to the axis, and an angle of 150°. A second similar prism, but formed of quartz of an opposite rotatory character, was divided into two by a plane perpendicular to the base bisecting the angle. These halves were respectively applied with their hypothenusal sides to the oblique sides of the first prism, so as to form a compound rectangular solid, having its two ends perpendicular to the axis.

A ray entering it in the direction of the axis, on emergence was found *divided into two*, which were *separated by a sensible angle*. Without entering into theoretical considerations, it is evident that this must be occasioned by some different refractive action exercised on the two portions of which the ray consists, at the successive surfaces where they encountered a change in the rotatory character.

These two rays are not in the condition of common doubly refracted rays, for either of them being examined by double refraction gives two images of equal intensity in all azimuths; in this respect resembling common light. But that they differ from common light is shewn by another experiment. This is effected by means of a parallelopiped of crown glass, whose index is 1.51, and its acute angles 54½° which is the essential apparatus for experiments on circular polarization, and is called *Fresnel's rhomb*.

Upon the end of this rhomb a ray incident perpendicularly will be internally reflected at one surface, and thence again at the other, so that it shall emerge perpendicularly to the other end.

If either of the rays from quartz just spoken of, be made to undergo these reflexions in the rhomb, it emerges *completely polarized* in a plane of 45° inclined to that of reflexion : and the two rays on opposite sides of that plane, or in planes of 90° inclined to each other.

159. The reverse of this experiment is remarkable: for if a *polarized* ray be incident on the rhomb, so placed that the plane of reflexion be 45° inclined to that of polarization, it will emerge *circularly polarized*. We have thus the means of readily procuring this kind of light, and the properties it displays are highly curious.

One of the most striking is, that when we repeat the experiments on the colours of crystallized plates, or the polarized rings, with *circular,* instead of *plane, polarized light,* though tints are similarly produced, yet they are not the same, but *differ by a quarter of a period or tint.* And this, with uniaxial crystals, is in excess and defect in alternate quadrants, so that the rings appear *dislocated,* those in one quadrant being pushed out, as it were, and those in the adjacent, pushed in, by a quarter of a tint. In biaxial crystals the same thing applies to the alternate semicircles.

Fresnel found this modification communicated by internal reflexion at other angles, if the number of reflexions was increased.

When the rhomb is placed in any other position than having the plane of reflexion at half right angles to the plane of polarization, the light acquires a modification similar to circular polarization, which from certain analogies into which we cannot here enter, is called elliptic polarization.

160. Sir D. Brewster has made a number of curious and valuable researches on the effect impressed upon light by reflexion from metallic surfaces, and which are considered to have a close connexion with circular or elliptic polarization.

He conceives that metallic reflexion polarizes light in two superposed pencils in opposite planes, but of unequal intensity; and viewing this as analogous to circular polarization in which they are of equal intensity, denominates this elliptic polarization.

He also shews, that whereas after circular polarization a ray is restored to a state of plane polarization by successive internal reflexions (as above mentioned) *at the same angle,* whatever be the inclination of the planes of the first and the second reflexions, on the other hand in successive reflexions from metallic surfaces *the angle varies* with the inclination of the planes of reflexion.

Professor Airy has shewn that the light which constitutes the two rays produced by the double refraction of quartz, consists of light elliptically polarized according to the first meaning above alluded to, the positions corresponding to those which the analogy would assign to the major axes, being in planes at right angles: and the one ray right, the other left-handed.

But it would be unsuitable to the plan of this work to enter further into these delicate and recondite enquiries: more especially since the relation between these results and those of Sir D. Brewster seems yet open to considerable question. The reader must be referred to the paper of the last author in the Phil. Trans. 1830, and to those of Professor Airy in the Cambridge Transactions, 1831, and 1832; as also to his Tract on the Undulatory Theory of Optics, appended to the second edition of his Mathematical Tracts, Cambridge, 1831.

THE END.

ADDENDUM

Page 132, at the end of Art. 111.

The shifting of the stripes is precisely the same thing in principle as their non-appearance in the simple experiment beyond a short distance from the centre. A distinct stripe will cease to be formed at a distance $c_{\prime}$ from the centre corresponding to a difference $n \lambda$, when *from any cause* n is so increased that the difference in distance of the same bright point for the different primary rays

$$c_r - c_v = \tfrac{1}{2} n (\lambda_r - \lambda_v) \cot. \psi$$

exceeds the mean breadth of a stripe: when this happens, the stripes become confounded together into an uniform white light.

In the simple experiment, this increase of n depends merely on the difference of the *actual lengths in space* of the two rays which meet. In the retardation experiment, it depends on the difference of length of route *as measured by the number of intervals*. At the central point, in the simple experiment $n = 0$, at the same point on the screen, when one ray has been retarded, n is a considerable number, and we must advance much more towards the side on which the interception takes place, to find a point where two rays meet whose difference of route is either 0 or a number within the limits required.

ERRATA.

Page 21, l. 9, For equation (2), *read* equation (9).
23, l. 15, At the end of the equation add, (14)
26, The note belongs to the preceding page.
89, l. 5, For to, *read* so.
92, l. 19, For explained, *read* examined.
101, l. 3, rays) *read*) rays.
119, l. 15, &c. For rhomboidal, *read* rhombohedral.
164, l. 2, For perimeter, *read* parameter.

WORKS BY THE SAME AUTHOR.

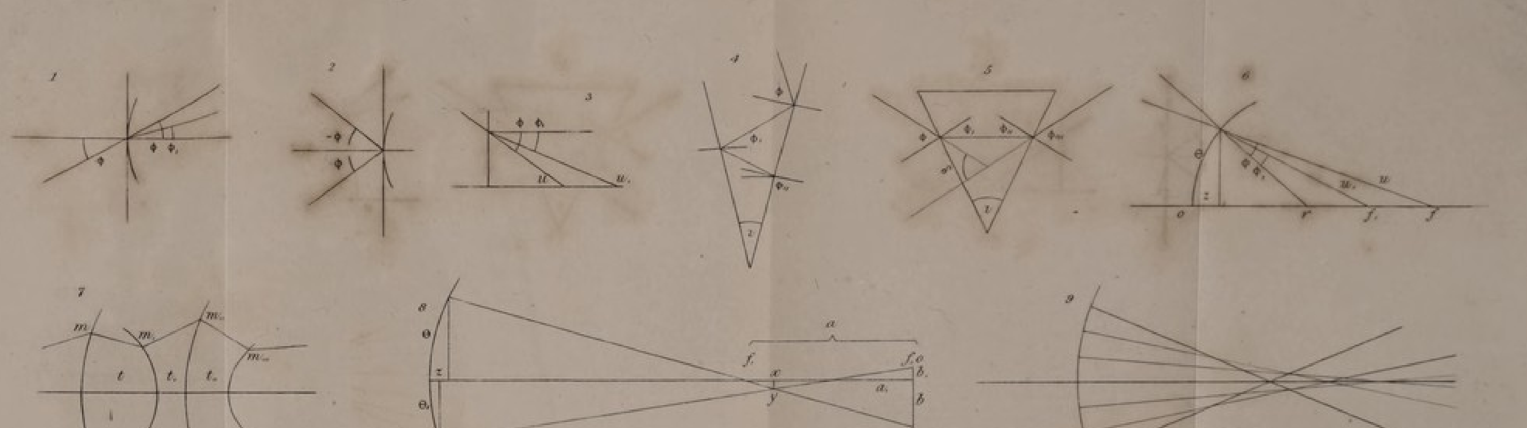
I. THE ELEMENTS OF CURVES, containing 1. A Geometrical Treatise on the Conic Sections. 2. The Algebraic Theory of Curves. Oxford, 1828.

N. B. This volume is not complete without a Supplement, given at the end of the Differential Calculus, Part I. and some supplementary pages at the end of Differential Calculus, Part II.

II. A SHORT TREATISE ON THE DIFFERENTIAL AND INTEGRAL CALCULUS, Part I, 1829; Part II, 1830.

III. THE APPLICATION OF THE DIFFERENTIAL CALCULUS to the Geometry of Curves and Curved Surfaces, 1830.

IV. THE PRESENT STATE AND FUTURE PROSPECTS of MATHEMATICAL and PHYSICAL STUDIES in the University of Oxford, Considered in a Public Lecture. 1832.



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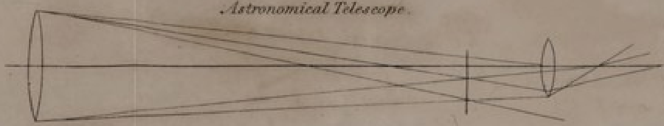
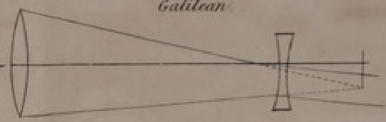
Astronomical Telescope



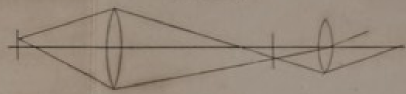
21

Galilean



22

Microscope



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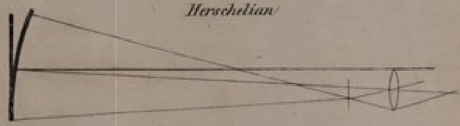
Herschelian



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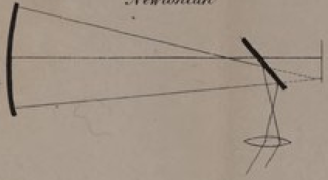
Newtonian



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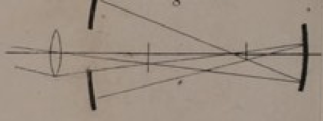
Gregorian



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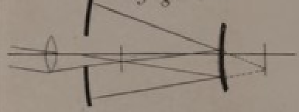
Cassegrainian



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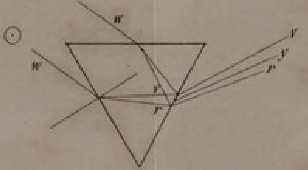

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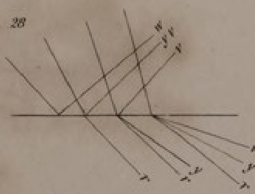

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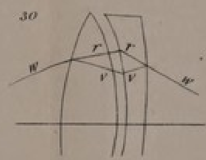

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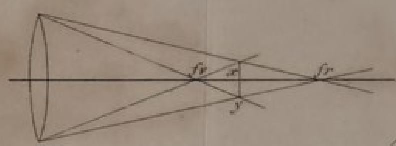

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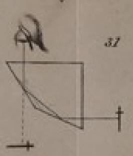

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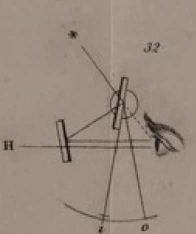

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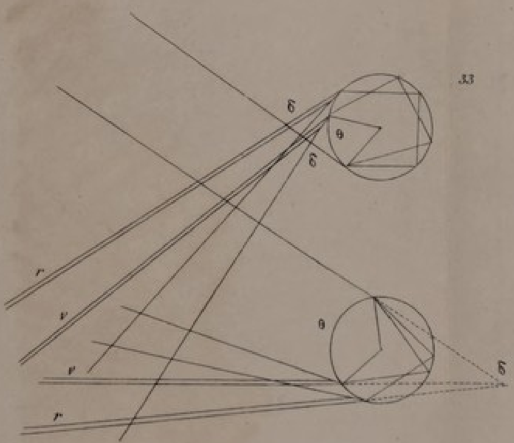

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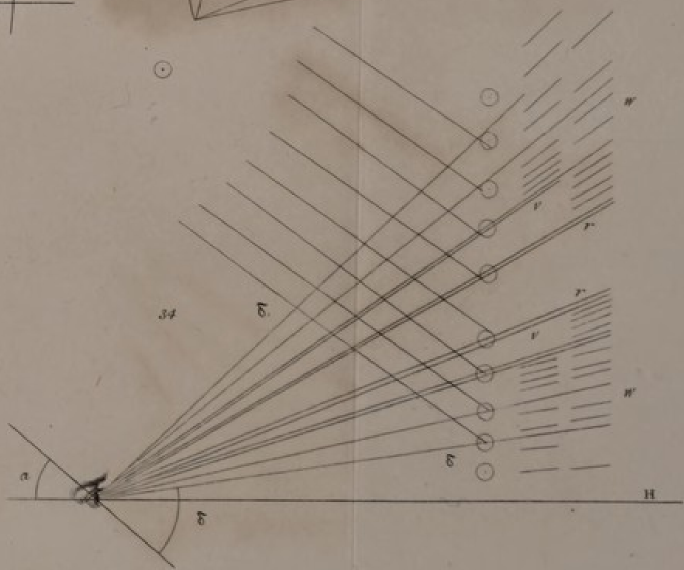

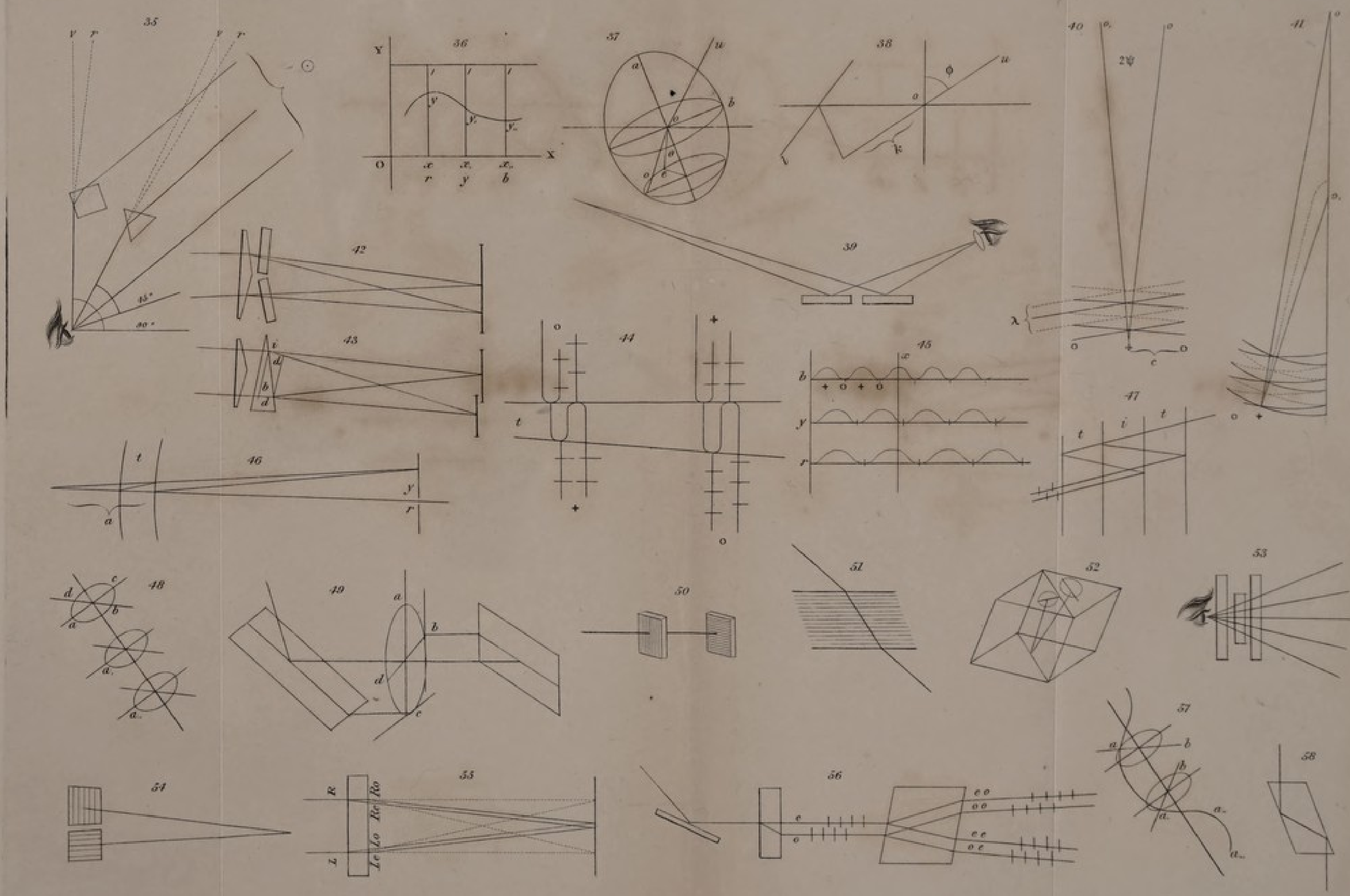
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