AN

ELEMENTARY TREATISE

ON

MECHANICS:

DESIGNED FOR THE USE OF STUDENTS
IN THE UNIVERSITY.

By W. WHEWELL, M.A. F.R.S. M.G.S.

FELLOW AND TUTOR OF TRINITY COLLEGE, AND PROFESSOR OF
MINERALOGY IN THE UNIVERSITY OF CAMBRIDGE.

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PREFACE.

In the following Treatise I have attempted to present the principles of Mechanics in such a manner that the reasoning by which they are proved and connected may be as strict and simple as the subject allows; and I have also endeavoured to give the elementary parts of the science such a form and arrangement that they may prepare the student for its higher applications and more difficult problems.

The science of Mechanics must be looked upon as beyond all doubt the most perfect of the inductive sciences; and as, among the divisions of mathematics, coming next to Geometry in the simplicity of its foundations and the closeness of its reasoning. It differs much however from Geometry in the discrepancies which exist among different authors with regard to the assumptions and principles on which it is to be established, and in the diversity of their opinions as to the extent to which it is dependent on observation. And it does not appear that even yet it is considered as an essential object in elementary treatises on this science to mark distinctly what truths are proved by an appeal to experiments: and to connect them and their consequences with logical accuracy.

One ground of confusion and unsatisfactoriness on this head in the writers of our own country is to be found in their neglect of the distinction between Statics and Dynamics. This division is not merely technical and arbitrary: the sciences, though closely connected, have not their principles, and it may be said not even their definitions in common. The two sisters were born in different ages and countries. The doctrine of Statics is as old as the time of Archimedes. Several problems belonging to this doctrine were solved by that great mathematician; and among them, some which, including their geometrical difficulties, can by no means be considered as easy. The theory of the lever, and of the oblique equilibrium of floating bodies, which he has treated of with admirable skill, involve the fundamental laws of Statics; and these laws, without the introduction of any new mechanical principle properly so called, might have led to a complete developement of the theory of equilibrium. From the time of Archimedes however, ages past away, and the world of science went through some of its greatest revolutions, before a worthy successor appeared to carry forwards this speculation. Pappus and Cardan and some others tried to follow the path which had been opened, and erred or failed. But, towards the end of the sixteenth century, Simon Stevin of Bruges made the decisive step. He succeeded in estimating in a manner demonstrably right the effects of forces acting obliquely; and it was, after this, only necessary to prosecute his views and methods in order to obtain a complete theory of equilibrium. This however was not the direction which such speculations took at that period. Galileo had already called into existence the science of the motion of bodies; and this, a new and fertile subject, soon attracted a large portion of the attention of mathematicians. It was manifest that the principles of this science were more extensive than those of equilibrium: it was easy to imagine that they included these. It was attempted to enunciate the elementary laws in such a manner that they should comprehend both cases; and the leading mechanical notions being thus thrown into some confusion, the ultimate analysis of the doctrines of motion and equilibrium remained imperfect from that time to our own days.

This imperfection may be recognized, for instance, in some of the modes often adopted of considering the composition of forces. The statical definition of *force* is in the highest degree precise and clear. Two forces are *equivalent* to a third when they produce an equilibrium in the same case in which *it* does so. The relations of three such forces are to be investigated from the properties of equilibrium only. This may be done in various ways; all of which depend on the consideration, that when several forces are in equilibrium we may conceive any parcel of the forces to have their equivalent force substituted for them. The method selected in the following treatise is that in which we first consider weights in equilibrium on a straight lever, the simplest and most ancient of mechanical problems; and from this is deduced the equilibrium of forces acting at a point, and consequently obliquely to each other. Other methods may be of equal statical strictness: but it is clear that any method which, in order to prove *this* proposition for equilibrium, introduces a new and irrelevant measure of forces depending upon the *motion* which they produce, cannot at least be the most simple course. Such a proof however must be defective, not only in simplicity, but in connexion: for if it be maintained that because the *motions* due to the two forces are communicated to the body at the same time, these motions are compounded, and therefore might be produced by a force compounded of the other two in the usual way: it is obvious that we must ask *how* we know that *these* motions, (viz. those which the forces would have produced separately,) *are* communicated when the forces act together: and how this applies in the case of equilibrium, when the motions are not communicated, and consequently not compounded. And if it be attempted to remedy this defect by speaking of the *tendencies* to motion instead of the motions themselves, we must again ask how the composition of such tendencies is established, inasmuch as the cases on which the composition of motions is founded, are no instances of the law so interpreted.

While, in consequence of mixing together heterogeneous elements, the demonstrations of Statics have thus been vitiated by introducing into them ideas unnecessarily complex and experimental; those of Dynamics have from the same cause, been rendered imperfect by attempting to reduce them to the simplicity and self-evidence of Statics. Thus the law that the *reaction is equal and opposite to the action*, has been applied to rest and to motion, as if it were true of both in the same sense: and as if we could, by selecting definitions, make any common reasoning apply to pressures, and to momentum gained or lost. And in this way it has been sometimes overlooked that the point really to be proved is, that action and reaction according to one definition are proportional to action and reaction according to the other: that is, that the momentum generated is proportional to the pressure. In the same manner the second law of motion, that *a force produces its whole effect in the direction in which it acts*, cannot, when it is applied to the case of a force acting on a body already in motion, be looked upon as merely prescribing a measure of forces; for this has already been fixed in speaking of forces acting on bodies from rest. In both these cases the real proof is a reference to experiment; and we confuse the subject rather than illustrate it, when we express the theorems in such a manner that this dependence is concealed.

I have therefore made Newton's THREE LAWS OF MOTION the foundation of the Doctrine of Motion, and of that only. I have slightly modified the form of them in order to obviate any mistakes on this head; and I have endeavoured to present the presumptions in their favour which arise from their simplicity, along with the proof which is afforded by experiment.

In stating *three* fundamental laws of motion, I am compelled to differ from all the French writers on this subject, who in their Treatises acknowledge only *two* such laws formed from experience. These laws are, *the law of inertia*, which coincides with our first law of motion; and the law that *the velocity is proportional to the force*. This latter principle agrees most immediately with our third law, but is by the writers to whom I refer made also to include our second. I have stated elsewhere* the reasons why I cannot consider as satisfactory the arguments by which this connexion of the two principles is attempted to be established. It seems quite unallowable in strict reasoning, to apply to impulsive forces the propositions which have been demonstrated for pressures only; and it is far from being obviously certain that we may every where substitute, instead of a velocity, an impulsive force by which the velocity is supposed to have been generated. Indeed with regard to instantaneous impulsive forces, it does not seem too much to maintain, that the simplification which has been aimed at by means of their introduction into the elementary reasonings of the science

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* Edinb. *Journal of Science*, Vol. viii. p. 27.

is altogether fallacious. It is acknowledged on all hands that no such forces exist in nature. That impulses are in fact short pressures; and that their laws are derived from the laws of pressure, instead of being the foundation of them. And it must unavoidably be also acknowledged that if there should occur a case of pressure strictly *instantaneous*, our received laws of motion could not be applied to it with any justice; because such a case is not included in the induction by which their authority was established. It must therefore be unprofitable to endeavour to make the seeming simplicity of the laws in such an imaginary instance, the ground of their reception in a different and heterogeneous one. Nor must we permit the high and merited eminence of the authors who have adopted this view, to make us neglect that scrupulousness of logic in this science from which one of its most valuable lessons is to be gathered.

If there be any truth in the opinion formed on the preceding reasons, that the three laws above-mentioned are the simplest principles on which the simplest problems of Mechanics can be treated, it is hoped that the mode of presenting the subject adopted in the following pages, will be found sufficiently elementary for readers entering upon the study of this science. The first five chapters of Statics and the five chapters of Dynamics, may be pointed out in particular as forming an introductory Treatise on the subject; and in these a knowledge is not presupposed of any branches of mathematics except the Elements of Geometry, and the simpler processes of Algebra and Trigonometry.

One or two instances must be excepted of detached propositions, where a knowledge of some properties of the Conic Sections is assumed.

In one or two cases, I have taken problems, (as the form of an arch, in Chap. V, of the Statics; the fall of a body vertically, in Chap. III, and on a cycloid, in Chap. V, of the Dynamics,) which would naturally require the use of the Differential Calculus: and have solved the Problems, by dividing the quantities concerned into small portions, and reasoning from the limits to which their properties tend, when these portions became indefinitely small. These problems, thus treated, may serve to prepare the mind for the reasonings which are employed in the Differential Calculus; and will, I think, be easily intelligible after what precedes them.

Beginning with the sixth Chapter of Statics, the subject is treated in a manner in which more use is made of generalizations and symbols. That these general methods are to be adopted by any person who would proceed to the more extensive and difficult problems of the science, and to the most celebrated and valuable books upon it, cannot be doubted. But there appear to be several advantages in not introducing these processes to the learner till some of the more simple cases have been separately considered. In the *first* place, these methods suppose a command of analysis which can only be acquired in time, and which, therefore, should not be taken for granted at an early period. Even where the student possesses the *knowledge* that is required, the *habit* of reasoning by means of abstract symbols, and upon general propositions, can only be acquired gradually and slowly. In the *second* place it will be found, I think, that when we reason in general terms, we do so by substituting in our mind some particular case as the representative of the general one; and that, therefore, we shall be much better able to prosecute our general reasonings, when we are acquainted with some particular results, in which we may exemplify and embody them. And *thirdly*, it is desirable, in many cases, that this science should be studied by persons who do not ever acquire a knowledge of more than the elementary parts of mathematics. In a system of study like that of our University, it would be a mistake to present the introductory subjects in a manner which supposes that the learner is necessarily to advance far in mathematical pursuits. We ought to lay our foundation so as to admit of such a superstructure; but not so as to be useless without it. If we insist upon a person going through the preparatory step of studying general methods, and general formulæ, before we offer particular problems to his notice, he will frequently acquire imperfectly an apparatus which he does not understand and cannot employ, and frequently he will be repelled and discouraged altogether. Considering, however, the practical importance of the science of Mechanics; its philosophical interest as the most perfect specimen of inductive reasoning, and of mixed mathematics; and its connexion with the Newtonian System and Physical Astronomy; it seems desirable that its principles should be familiarized at an early period of the student's progress, and presented to many who may never become dexterous analysts.

I have, therefore, placed the general methods in the situation, which, upon these considerations, seems to be most proper for them; viz. *after* the easier problems and *before* the more difficult ones of the science. The consequence of this arrangement has unavoidably been, that parts of the subject which are connected in a scientific point of view, have been separated. Thus, Chap. V. may be considered as a sequel to Chap. II: Chap. VI, to Chap. IV; and Chap. VII, to Chap. V: but as the mathematical reasonings in these Chapters are of different kinds, it is hoped this will be no inconvenience to the student.

In this edition the work is broken into propositions, of which each is enunciated previously to proving it. As a mode of instruction, I believe, as I have stated elsewhere, that this arrangement is always felt as a guide and a relief by the unexperienced reader. Among other numerous additions and alterations which have been introduced, may be mentioned some important problems added to those which are classified and solved in the fifth Chapter. These include examples of the difference of *stable and unstable equilibrium*; an investigation of the stress and equilibrium of beams forming a *roof*, and a simplification of the theory of *arches*. I have also added, in Chap. X, some important and interesting theorems on the *Elasticity and Compression of Solid Materials*, partly adapted from Dr. Young's Elements of Natural Philosophy. I would gladly have given a section on the strength and fracture of beams, had there been any mode of considering the subject, which combined simplicity with a correspondence to facts. The common theory which supposes the materials incapable of compression, is manifestly and completely false; and though Mr. Barlow's experiments and investigations give us much information, they do not appear to lead to any conclusions, sufficiently general and simple, to authorize us to present the subject as an elementary one. The discussion of the various *species* of the curve formed by a uniform *elastic lamina*, have been transferred to an Appendix, as not necessarily entering into the plan of such a Treatise. With the same view, I have also placed, in the Appendix, the investigation of the various forms of *bridges* upon different hypotheses.

The *motion* of a body *upon a cycloid* is here treated in a manner somewhat different from that generally employed, but apparently not less simple. It seemed desirable, for reasons which I have already mentioned, to treat the subject of the oscillations of pendulums, without the necessity of recurring to the Differential Calculus.

In this third edition I have added two Chapters to the Appendix: one on the effects of Friction in the Equilibrium of Bodies, and the other on the Connexion of Pressure and Impact. For the latter I am principally indebted to a communication of Professor Airy, made some time ago to the Philosophical Society of Cambridge.

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STATICS.

DYNAMICS.

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MECHANICS.

INTRODUCTION.

1. *Mechanics is the science which treats of the laws of the motion and rest of bodies.*

Any cause which moves or tends to move a body, or which changes or tends to change its motion, is called Force.

In the science of Mechanics, as in every other branch of Natural Philosophy, we assume that the material world is governed by constant and determinate laws; and our object is to discover these laws, and to trace their consequences.

The appearances and occurences of the material world which form the subject of this science in particular, are all the instances in which motion is produced or prevented by means natural or artificial. And to it belong all the reasonings by which we examine and measure the agencies thus exerted and the effects thus produced.

We shall for example have to investigate such subjects as the following; — the manner in which a stone falls to the earth; the velocity with which it moves, and the path which it describes when thrown in any direction; — the mode in which a heavy body may be supported by any instrument, simple or complicated; — the motion of any machine by which weights are raised or removed;—the requisite form and adjustment, or the possible strength and stability, of any material structure;—and in general, any case in which bodies are pulled, or pushed, or struck, or stopped, or supported by other bodies in contact with them; or in which they are attracted or repelled by bodies at a distance.

In all such instances the immediate cause determining the rest or motion which takes place, is called FORCE. Thus when a man supports a stone in his hand, his hand is said to exert force upon the stone: and in the same manner if he moves a machine by turning a winch, he is said to exert force on the winch, and, by this, on the machine. If the machine be moved by the weight of a heavy body, this heavy body is said to exert force. When a stone falls, it is said to be moved by the force of gravity, or of the earth's attraction.

The force thus exerted by any agent is considered as a *measurable* quantity; and we shall hereafter see how the force is measured in such cases as the above.

2. It has been said that we have to treat of the motion and rest of *Bodies*. Body or matter is the most general name which we give to every thing that is the object of our senses. In Geometry we conceive figures to possess extension only, without solidity; or to occupy space without excluding other figures from it. In Mechanics we take objects such as they occur in nature; viz. not only extended but *impenetrable:* that is, while they fill a portion of space, they are supposed to exclude from it all other bodies. Thus, a stone pressed between two objects prevents their meeting, and resists their admission into the place which it occupies, by a force, insurmountable so long as its particles retain their situations. Of the particles of matter no two can occupy the same point of space at the same time.

3. It has already been stated that the science of which we have to treat includes the laws of the *rest* and *motion* of bodies. A single force will necessarily produce motion, but two or more forces may be so combined as to destroy each other's effects, and to produce rest. Thus if a person standing on the bank of a canal, as at *P*, fig. 1, pull a boat *B* which is in the water, by means of a rope *BP*, he will cause it to move in that direction; but if there

be other persons, Q, R, also pulling the boat in the directions BQ, BR, it may happen, by properly adjusting the directions of the ropes and the strength exerted, that the boat shall remain at rest by the united action of the three forces.

In this case the forces thus exerted are said to *destroy* or to *balance* each other, or to be in *equilibrium*.

4. All bodies within our observation fall or tend to fall to the earth; and the force which they exert in consequence of this tendency is called their WEIGHT.

Weight is *measured* by its mechanical effects, and two bodies which produce the same effect are said to have equal weights. Thus let MA, fig. 2, be a steel spring in its natural position, and let a mass of lead P bend it into the position MB; then a mass of iron Q, which, suspended in the same manner, bends it into the same position MB, has the same weight as P. If P be one pound, Q is one pound. And the two together are two pounds. Let MC be the position into which the spring is bent by P and Q together, or by two such as P; then any weight which bends it into the position MC is called two pounds; and, in the same manner, if three weights such as P, bend it into the position MD, any weight which bends it to D is called three pounds; and so on for any number.

If instead of the bending of a spring we had taken any other mechanical effect, the mode of explaining the way in which weight is measured would have been the same.

It appears by experiment that the mechanical effect or weight of a body is not altered by altering its figure. It depends solely upon the magnitude and the material. Hence if $2\frac{1}{2}$ cubic inches of lead be one pound, 25 cubic inches of lead will be 10 pounds, whatever be its form

5. The *same* effect may be produced by masses of *different* magnitudes, when the materials are different. Thus $3\frac{1}{2}$ cubic inches of iron will produce the same effect by its weight as $2\frac{1}{2}$ cubic inches of lead. Now it is assumed that so long as the mechanical effect is the same, the *quantity of matter* is the same. Hence the quantity of matter in $3\frac{1}{2}$ cubic inches of iron is said

to be the same as the quantity of matter in $2\frac{1}{2}$ cubic inches of lead; and the difference of magnitude is supposed to result from the different spaces in the two bodies which are porous, or not occupied by matter.

The quantity of matter of a body is proportional to its weight at a given place: for weight is the simplest measure of the mechanical effect of the mass*.

6. When a weight is supported by the hand, the weight exerts a force downwards, and the hand exerts a force upwards, and these forces balance each other. In this case, the forces which act upwards and downwards are *equal*, and measure each other. And in the same way any *two forces* which act in opposite directions, in the same line, and balance each other, *are equal*.

7. A heavy body may act by means of a string, and if we suppose the string to have no weight, the effect of the weight will not be altered by the length of the string.

In fig. 3, let a weight P be supported by a hand at Q, by means of a string QP; the force exerted by the hand is equal to the weight itself: for instead of the weight P, let a hand, as at R, exert a force equal to the weight; then this force will be supported as before; because the force of the hand produces the same effect as was produced by the weight; but in this case it is clear that the forces exerted by Q and R must be equal, whatever be the length of the string, because they are similar forces and act upon it in exactly the same manner. Hence, the force exerted by Q is equal to the weight P, to which the force of R was assumed equal. And thus the force exerted at Q is the same, whatever be the length of the string.

8. A heavy body may act by means of a string which is capable of sliding over a fixed point, (O, fig. 4.); and if we suppose the string to have no weight, the heavy body will be supported

* The quantity of matter is sometimes said to be proportional to its *inertia:* that is, to its resistance to the communication of motion. But in practice the quantity of matter is always measured by the weight. It will be shewn in considering the third law of motion that the inertia is as the weight.

by the same force, whatever be the length or direction of the string.

Let a weight P be supported by a hand at Q, acting over a fixed point O; take $OR = OQ$, and let a hand act at R, with a force equal to the weight P; then by the last Article, this force will still be supported by Q; but the forces of Q and R must be equal, because they are similar forces and act on the string QOR in exactly the same manner; the only difference being that one of them is in a vertical direction, and the other not so; which difference cannot disturb their equilibrium, since neither of the forces depends at all upon gravity. Hence, the force exerted by Q is always equal to that exerted by R, and therefore to the weight P: it is therefore the same, whatever be the length or direction of the string OQ.

Instead of a fixed *point* at O, we may suppose a circular pulley moveable about its center, and the reasoning will be the same.

9. We shall divide our subject into the part which relates to the action of forces in equilibrium, and the part which relates to the action of force producing motion. The first part is called STATICS, the second DYNAMICS.

Force being considered as balancing force in one case, and as producing motion in the other, is differently measured in the two divisions.

10. *In Statics a force is measured by the weight which it would support.*

Whatever be the direction and magnitude of a force, we may suppose it, as Q, in fig. 4, to act by means of a string which, passing over a pulley O, supports a weight P; and the force will be measured by the weight so supported.

Statical forces are called *pressures*. Thus, when a heavy body is supported, it exerts a pressure downwards on its supports, and is sustained by their pressure upwards.

The pressure exerted upwards by each prop, is equal to the pressure downwards *upon* it; and the latter being called the *action*, the former is called the *re-action*.

The pressures exerted by strings pulled by any forces are called *tensions*.

In the following articles we have to reduce all cases of equilibrium to the simplest principles by which they can be reasoned upon. And this may be done by various considerations. Thus in the case which has been already mentioned where, in fig. 1, three forces, *BP*, *BQ*, *BR* keep each other in equilibrium, each may be considered as balancing the other two, acting at the same point. In fig. 17, a force *AP* supports a wheel *AC* against an obstacle *C* and may be considered as acting at *A* to balance the weight which presses the wheel and which acts at the same point. But in this instance we might also, by Art. 7, consider the force *AP* as acting at *M*, and the weight as acting at *N*; and we should thus have to reason concerning forces acting upon different points of a rigid body, moveable about a point *C*. In the same manner, in fig. 41, we may consider the strings *AK* and *BC*, with the pulley *AB* and the weight *W*, which they support, as all exerting their force at the same point *n*, where the lines of their directions meet. But we may also consider these forces as exerted at *A*, *B*, and *o*, points in the rigid body *AB*, and as keeping it in equilibrium. By similar views the most complicated cases of equilibrium can be reduced to simple principles.

The simplest principles to which the doctrine of Statics can be reduced, are, the equilibrium of forces on the same point, and the equilibrium of forces on the lever. And these two principles are so connected that one can be deduced from the other. The most simple course appears to be, first to establish the latter principle, which is done in the following chapter, and then to deduce the composition of forces from it, which is the object of chap. 2.

STATICS.

CHAP. I.

THE LEVER.

11. A **LEVER** is an inflexible rod, moveable, in one plane, about a point which is called the *fulcrum* or center of motion.

The portions on the two sides of the fulcrum are called the *arms*.

When the arms are in the same straight line, it is called a *straight lever;* otherwise it is called a *bent lever.*

The forces which act upon it are supposed to act in the plane in which the lever is moveable.

The lever is generally considered to be without weight.

Its properties will be deduced from the following Axioms:

12. AXIOM I. *Equal forces acting perpendicularly at the extremities of equal arms of a lever to turn it opposite ways, will keep each other in equilibrium.*

For the forces act in a manner perfectly similar, and hence there can be no reason why one of them should prevail rather than the other.

We find that if one of the forces be greater, the arms remaining equal; or if one of them act at a longer arm, the forces being equal; the greater force or the longer arm preponderates, and the equilibrium is destroyed.

Cor. 1. Hence, the converse propositions are true; namely,

If two equal and opposite forces, acting perpendicularly at the extremities of a lever, keep it at rest, the arms are equal. For if they were not, the longer would preponderate.

If two forces, acting oppositely and perpendicularly at the extremities of equal arms of a lever, keep it at rest, the forces are equal. For if they were not, the greater would preponderate.

Cor. 2. If a weight, as *W*, fig. 5, be supported on two fulcrums *A* and *B*, at equal distances from it, the pressures on the two fulcrums are equal.

Cor. 3. If two equal weights, *P*, *Q*, fig. 5, be supported on two fulcrums *A* and *B*, situated so that *PA*, *QB*, are equal, the pressures on *A* and *B* are equal.

13. Axiom II. *If two equal weights balance each other upon a straight lever, the pressure upon the fulcrum is equal to the sum of the weights, whatever be the length of the lever.*

The whole weight is supported at the fulcrum of the lever, and hence it appears manifest, that the pressure which is there supported must be equal to the whole weight, that is to the sum of the two weights.

Axiom III. *If a weight be supported upon a lever which rests on two fulcrums, the whole pressure upon the fulcrums is equal to the weight.*

For here, as in the preceding Axiom, the whole weight is supported by the pressures which it exerts on these fulcrums, and therefore the sum of these pressures must be equal to the weight.

14. Prop. *If two equal weights act perpendicularly on a straight lever, they may be kept in equilibrium round any fulcrum by the same force as if they were collected at the middle point between them.*

Let *P*, *Q*, fig. 5, or 6, be the two weights, *A* the fulcrum, and *W* the middle point. Take $WB = WA$, and suppose a fulcrum placed at *B*.

When weights P and Q are supported on the lever, the pressure on each of the fulcrums is half the whole pressure, by Cor. 3. to Axiom 1; and the whole pressure is $P + Q$ by Axiom 3; therefore the pressure on each fulcrum is half $P + Q$.

When a weight W, equal to $P + Q$, is placed at the middle point, the pressure on each of the fulcrums is, by Cor. 2. to Axiom 1, equal to half the whole pressure; but the whole is $P + Q$, by Axiom 3; therefore the pressure on each fulcrum is half $P + Q$.

Hence, the pressure on the fulcrum B is in each case equal to half $P + Q$: and therefore the lever will in both cases be kept in equilibrium by the same force applied at B.

Cor. Hence, a horizontal prism or cylinder of uniform thickness and material, will produce the same effect as if it were collected at its middle point.

Thus, a cylinder BD, fig. 7, will produce the same effect on a lever CB as if it were collected at its middle point N: for this cylinder may be considered as composed of pairs of equal small weights (as d and b) at equal distances from N, and each such pair will produce the same effect as if collected at N, and hence, the whole cylinder will produce the same effect as if it be collected there.

15. Prop. *If two weights, acting perpendicularly upon a lever, on opposite sides of the fulcrum, have their distances from the fulcrum inversely as the weights, they will balance each other.*

Let P and Q, fig. 7, be the weights; and let AB be a cylinder equal to the sum of the weights. Divide AB in D so that $AD : DB :: P : Q$. Then the weight of the portion AD of the cylinder will be equal to P, and the weight of the portion DB equal to Q. The cylinder AB will balance on its middle point C. Let M be the middle of AD, and N of BD: then, by last Article, the cylinder AD will produce the same effect as if it were collected at M, and the cylinder BD as if it were collected at N. Hence, if we suppose AB instead of being a cylinder, to be a rod without weight, and upon this rod a weight equal to AD to be placed at M, and a weight equal to BD to be placed at N, these weights will balance each other on the point C. That is, P placed at M, and Q placed at N, will balance each other on C. And hence, by Art. 7, they will balance when suspended by the strings MP, NQ.

Now, since DM is half DA, and DN half DB, MN is half AB.

$$\text{Also, } CM = CA - AM = \tfrac{1}{2}AB - \tfrac{1}{2}AD,$$
$$\text{and } DN = MN - DM = \tfrac{1}{2}AB - \tfrac{1}{2}AD;$$
$$\therefore CM = DN, \text{ and hence } DM = CN.$$
$$\text{Hence, } CM : CN :: DN : DM :: DB : DA$$
$$:: Q : P.$$

Hence, when the weights and distances have this proportion, they will balance each other.

Cor. 1. Conversely, if the weights P and Q balance each other on C, we have $P : Q :: CN : CM$; for if not, let $P : Q' :: CN : CM$, and by the proposition, P and Q' will balance each other; and hence Q and Q' produce the same effect at N, and therefore must be equal.

Cor. 2. The pressure on the fulcrum C will be the same as the weight of the whole cylinder; that is, it will be the sum of the weights P and Q. (Art. 13.)

Cor. 3. What has been proved of weights, is true of any forces whatever, for these may be represented by weights. For instance, it applies to forces acting in the directions CR and MP, fig. 8, about a fulcrum N.

16. Prop. *If two forces acting perpendicularly on a straight lever on the same side of the fulcrum, are inversely as their distances from the fulcrum, they will balance each other.*

In this case the forces must act in opposite directions, as P and Q, in figures 8 and 9, acting at M and N.

If we suppose that there is at the fulcrum C a force R, acting parallel to that at M, and such that $R : P :: NM : NC$, the forces P and R will produce equilibrium about a fulcrum N, and the pressure on N will be $P+R$, by last Article and Cor. 2 and 3. Hence, if at N a force $= P+R$ act in the opposite direction, it will produce the effect of the fulcrum. If therefore $Q = P + R$, the three forces P, Q, R, will keep the lever in equilibrium. And this is true, if

$$R : P :: NM : NC,$$
$$\text{or if } R + P : P :: NM + NC : NC,$$
$$\text{or if } Q : P :: MC : NC.$$

And therefore, if Q and P have this proportion, they will balance each other.

COR. 1. Conversely, if the weights balance on C,

$$P : Q :: CN : CM.$$

COR. 2. The pressure on the fulcrum C will be the difference of the forces, for it will be R, and since

$$P + R = Q, \quad R = Q - P.$$

Also, it will be in the direction of the greater force Q.

COR. 3. Multiplying extremes and means, we have, both in this Proposition and the last,

$$P \,.\, CM = Q \,.\, CN.$$

17. When a lever is used as a mechanical instrument, one of the forces, as Q, is a weight to be raised or supported, and the other, P, is employed to produce this effect. Hence P is called the *power*, and Q the *weight*.

Straight levers are divided into three kinds.

The lever of the first kind is when the power and the weight are on opposite sides of the fulcrum, as in fig. 7.

Scissars, Pincers, &c. are examples of double levers of this kind; the fulcrum being at the center of motion; the power being the force of the fingers; and the weight, the pressure exerted by the cutting or holding part.

The lever of the second kind is that in which the power and the weight are on the same side of the fulcrum, the power being more distant from it.

A stock-knife, a pair of nutcrackers, &c. are examples of this kind. *An oar* also belongs to this class, the fulcrum being that point of the blade of the oar which is for a moment stationary in the water while the boat is impelled forwards; the power being the pull of the rower; and the weight, the pressure of the oar upon the side of the boat.

In these cases, the lever always acts at a mechanical advantage, in consequence of the power acting at the longer arm; that is, the pressure produced is greater than the power exerted.

The lever of the third kind is that in which the power is on the same side as the weight, and nearer to the fulcrum, as in fig. 9.

In this case, the power always produces a pressure less than itself, and the instrument is employed not to obtain a mechanical advantage, but to enable the force to act at a greater distance.

Examples of this kind of lever are *a pair of tongs*, or *a pair of sheep-shears:* some *bones* of animals considered with respect to equilibrium only, are also levers of this kind, the *fulcrum* being the joint; the *power*, the muscle which is attached near the fulcrum; and the *weight*, the force exerted by the limb further from the joint.

18. EXAMPLES of the *Straight Lever*.

Ex. 1. On a lever of the first kind, 3 feet long, a weight of 100 pounds is suspended at the extremity, and $2\frac{2}{3}$ inches from this end is placed a fulcrum; what weight at the other end will preserve the equilibrium?

In fig. 7, $MN = 36$ inches; $CM = 2\frac{2}{3}$ inches; $\therefore CN = 33\frac{1}{3}$ inches,

$$P : Q :: 33\tfrac{1}{3} : 2\tfrac{2}{3} :: 100 : 8;$$
$$\text{and } P = 100\text{lbs.}; \therefore Q = 8\text{lbs.}$$

Ex. 2. On a straight lever MO, fig. 10, let MC, CN, NN', $N'N''$, &c., be all equal; then if a weight Q be slid along the arm CO, what are the weights at M, which it will balance when at N, N', N'', &c.?

Q at N balances Q at M; Q at N' balances $2\,Q$ at M; Q at N'' balances $3\,Q$, &c.

Hence, excluding the weight of the lever, the weight at M might be known from knowing the place of Q. We shall see hereafter how the weight of the lever itself may be taken into account.

If $CM = CN$, the weights at M and N are equal, and one of them may be used to measure the other. This is the case in the common balance, but when the arms are unequal, it is called a *false balance.*

Ex. 3. In a false balance, to find the true weight of the substance weighed.

Let CM, CN, be unequal, and let x be the weight to be determined. Let x at N be balanced by a ounces at M, and let x at M be balanced by b ounces at N. Therefore,

$$x : a :: CM : CN,$$
$$x : b :: CN : CM;$$
$$\therefore x^2 : ab :: 1 : 1,$$
$$x^2 = ab, \text{ and } a : x :: x : b.$$

$\therefore x$ is a mean proportional between a and b, the apparent weights in opposite scales.

Ex. 4. When a weight is supported on a lever at two points, to compare the pressures supported at the two points.

Let a weight R be supported on a lever MN, fig. 11, by forces P, Q. The same force is exerted at M as if N were a fulcrum: hence,

$$P : R :: NC : NM.$$
$$\text{So } R : Q :: NM : MC;$$
$$\therefore P : Q :: NC : MC.$$

Or, the pressures supported are inversely as the distances from the weight.

19. Prop. *If two forces acting perpendicularly on the arms of a bent lever are inversely as the arms, they will balance each other.*

Let forces P, Q, fig. 12, act perpendicularly on the arms CM, CN, and be such that

$$P : Q :: CN : CM;$$

they will balance each other.

Produce NC to D, so that $CD = CM$, and at D let a force $R = P$ act perpendicularly to CD: also take $CE = CM$, and at E let a force S also $= P$ act perpendicularly to CN.

The forces P and S would balance each other because they are equal, and act in a manner exactly similar upon the arms CM, CE. But the force R would balance S, because $CD = CE$. Therefore P and R produce the same effect on the lever.

Now, since $P : Q :: CN : CM$, we have

$$R : Q :: CN : CD;$$

therefore, by Art. 15, R would balance Q on the straight lever. Hence, P will balance Q on the bent lever.

Cor. 1. Conversely, if the forces act perpendicularly and balance each other, they are inversely as the arms.

Cor. 2. If the arm CM or CN were bent so as to have any other form CFM, its extremity being the same, the same proportions would be true.

For CFM being perfectly rigid, if we join CM, the effect produced will be the same if instead of the arm CFM we suppose the rigid surface $CFMC$. And in this surface if we remove any portion of the part towards F by a line parallel to CM, the effect will manifestly be the same as before. Hence, whether we have the rigid rod CFM, or CM, the effect will be the same.

20. Prop. *If two forces acting at any angles on the arms of any lever are inversely as the perpendiculars from the fulcrum upon their directions, they will balance each other.*

Let ACB, fig. 13, be the lever moveable about C; P, Q, two forces acting in the lines AP, BQ, and CM, CN perpendiculars on those lines. And let $P : Q :: CN : CM$; the forces will balance each other.

Let AM and CM be considered as rigid rods; then by Cor. 2, to last Art., the same effect will be produced whether we suppose the force P to act by means of the crooked arm CAM, or the straight one CM. In the same manner the force Q, acting at B, will produce the same effect as if it acted at N. But the forces P and Q, acting at M and N, would produce equilibrium, by last Article, because $P : Q :: CN : CM$. Hence, acting at A and B, they will produce equilibrium.

Cor. 1. $P . CM = Q . CN$.

Cor. 2. Conversely, if this proportion is true, or if these products are equal, the forces will balance.

Cor. 3. $P . CA . \sin. A = Q . CB \sin. B$.

Cor. 4. The proposition is true, whatever be the angle made of the arms CA, CB, and hence it is true when this angle vanishes, and the arms coincide. That is, if two forces P, Q, fig. 14, act in directions AP, BQ, at points A, B, of the *same straight lever*, when they are inversely as the perpendiculars on their directions, there will be an equilibrium.

Cor. 5. Hence, if two forces act at the *same point* of a lever to turn it in opposite directions, when they are inversely as the perpendiculars, there will be an equilibrium.

Examples of the *Bent Lever*.

Ex. 1. Fig. 13. P is 99 pounds, Q 100 pounds; $CA = 9$, $CB = 5$, and the angle $CAP = 30^0$; to find the angle CBN, that there may be an equilibrium;

$$P . CA . \sin. A = Q . CB . \sin. B;$$

$$\therefore \sin. B = \frac{P . CA}{Q . CB} . \sin. A = \frac{99 . 9}{100 . 5} . \frac{1}{2}$$

$$= .891 = \sin. 62^0,$$

as appears by the tables of sines.

Ex. 2. In a straight lever AB, fig. 15, acted on by weights P, Q; if there be an equilibrium when it is horizontal, there will be an equilibrium in every position.

Let AB be any position; MCN a horizontal line. And if there be an equilibrium in the horizontal position

$$P : Q :: CB : CA.$$

But, by similar triangles, $CB : CA :: CN : CM$; therefore

$$P : Q :: CN : CM;$$

and therefore the equilibrium subsists.

Ex. 3. In a bent lever ACB, (without weight) fig. 16, having given the lengths of the arms, the angle which they make, and the weights P, Q, appended to them; to find the position in which it will rest.

Draw MCN horizontal, meeting PA, QB, in M, N. Let $CA=a$, $CB=b$, $ACB=\omega$; and $ACM=\theta$, which is to be found. Therefore $BCN=\pi-\omega-\theta$, π being two right angles.

$$P \, . \, CA \cos. ACM = Q \, . \, CB \, . \cos. BCN,$$

$$Pa \cos. \theta = Qb \cos. (\pi-\omega-\theta) = -Qb \cos. (\omega+\theta)$$

$$= -Qb \{\cos. \omega \cos. \theta - \sin. \omega \sin. \theta\},$$

$$(Pa+Qb \cos. \omega) \cos. \theta = Qb \sin. \omega \sin. \theta;$$

$$\therefore \tan. \theta = \frac{Pa + Qb \cos. \omega}{Qb \sin. \omega}.$$

Ex. 4. In the same case, having given P, to find Q, that the arm CA may be horizontal.

In this case, $\theta=0$;

$$\therefore Pa+Qb \cos. \omega = 0; \; Q = -\frac{Pa}{b \cos. \omega}.$$

The problem will not be possible, except ω be greater than a right angle, in which case cos. ω is negative, and Q is positive.

Ex. 5. To find the force requisite to draw a carriage wheel over an obstacle, supposing the weight of the carriage collected at the axis of the wheel.

Let A, fig. 17, be the axis of the wheel, CD the obstacle. Then if the wheel turn over the obstacle, it must turn round the point C; and the force which moves it being supposed to act in the line AP, and the weight in the vertical line AE, it will be a lever such as that referred to in Cor. 5, Art. 20. Hence, in order that P may balance the weight Q,

$$P : Q :: CN : CM :: \sin. CAE : \sin. CAP,$$

$$P = Q \cdot \frac{\sin. CAE}{\sin. CAP}.$$

Hence, P is least when sin. CAP is greatest, or when CAP is a right angle. In this case, $P = Q \sin. CAE$.

If the wheel be made larger, the obstacle being the same, the versed sine NE, or CD, is the same; and the radius being increased, the angle CAE is diminished. Hence, *cæteris paribus*, P is diminished, and the larger the wheel, the smaller is the force requisite.

22. Prop. Fig. 18. *If any number of forces P, Q, &c. and P', Q', &c., acting upon the arms of a lever to turn it opposite ways, be such that*

$$P.CM + Q.CN + \&c. = P'.CM' + Q'.CN' + \&c.$$

there will be an equilibrium.

CM, CN, &c., and CM', CN', &c. are here the perpendiculars on the directions of the forces; and a lever is supposed to have any number of arms inflexibly connected.

Draw any line OO' through C, and at O and O' let forces X, Y, &c., and X', Y', &c., act, perpendicularly to OO', to turn the system opposite ways. Let these be such that

$$X.CO = P.CM, \quad Y.CO = Q.CN, \ \&c.,$$

$$X'.CO' = P'.CM', \quad Y'.CO' = Q'.CN', \ \&c.;$$

$$\therefore X.CO + Y.CO + \&c. = P.CM + Q.CN + \&c.,$$

$$X'.CO' + Y'.CO' + \&c. = P'.CM' + Q'.CN' + \&c.$$

But by supposition

$$P.CM + Q.CN + \&c. = P'.CM' + Q'.CN' + \&c.$$

therefore

$$X.CO + Y.CO + \&c. = X'.CO' + Y'.CO' + \&c.$$

$$\text{or, } (X + Y + \&c.)\, CO = (X' + Y' + \&c.)\, CO';$$

and hence, by Art. 16, Cor. 3, the force $X + Y + \&c.$ at O, and $X' + Y' + \&c.$ at O', produce equilibrium.

But, since $X \cdot CO = P \cdot CM$, P would balance X; similarly, Q would balance Y, &c.; and similarly, P' would balance X', Q', Y', &c.

Hence, P, Q, &c. would balance X, Y, &c.; and hence P, Q, &c. produce the same effect as X', Y', &c. which balance X, Y, &c. But, in the same manner, X', Y', would balance P', Q'; therefore, P, Q, &c. will balance P', Q', &c.

Cor. 1. Conversely, if there be an equilibrium,

$$P \cdot CM + Q \cdot CN + \&c. = P' \cdot CM' + Q' \cdot CN' + \&c.$$

For, making the same construction, P, Q, &c. will balance X, Y, &c., and P', Q', will balance X', Y', &c. Hence, since P, Q, &c. balance P', Q', &c. X, Y, &c. will balance X', Y', &c.; and therefore,

$$(X + Y + \&c.)\ CO = (X' + Y' + \&c.)\ CO'.$$

But the first side

$$= X \cdot CO + Y \cdot CO + \&c. = P \cdot CM + Q \cdot CN + \&c.$$

by supposition.

And the second side, in the same manner,

$$= X' \cdot CO' + Y' \cdot CO' + \&c. = P' \cdot CM' + Q' \cdot CN' + \&c.;$$

$$\therefore\ P \cdot CM + Q \cdot CN + \&c. = P' \cdot CM' + Q' \cdot CN' + \&c.$$

Cor. 2. If the forces be weights acting at points M, N, M', N', &c., on a horizontal lever, fig. 19, the equilibrium will subsist, when

$$P \cdot CM + Q \cdot CN + \&c. = P' \cdot CM' + Q' \cdot CN' + \&c.$$

If $P \cdot CM + Q \cdot CN + \&c.$ be greater than

$$P' \cdot CM' + Q' \cdot CN' + \&c.$$

the lever will turn in the direction in which P, Q, &c. would draw it, and *vice versâ*. Hence, the quantity $P \cdot CM + Q \cdot CN + \&c.$ may be considered as the measure of the power or energy to turn the system in that direction. This quantity, viz., the sum of the products of each force into its perpendicular, estimated in the same direction, is called the *moment* or *momentum* of the forces round C; and the product of one force P, by its perpendicular CM, is called the moment of that force.

23. Ex. 1. In fig. 19, let P, Q, P', Q', be weights of 3, 5, 7, 9 pounds, respectively, and MN, NM', $M'N'$, equal distances of one foot: to find the point on which the weights will balance.

Let $MN = NM' = M'N' = a$, and $MC = x$;

$$\therefore\ NC = x - a,\ CM' = 2a - x,\ CN' = 3a - x;$$

and therefore, by last Cor.

$$3x + 5(x - a) = 7(2a - x) + 9(3a - x);$$

$$\therefore\ x = \frac{5a + 7 . 2a + 9 . 3a}{3 + 5 + 7 + 9} = \frac{46a}{24} = \frac{23a}{12};$$

$\therefore\ x = 23$ inches, and C is one inch from M'.

Ex. 2. To shew how the steelyard must be graduated.

The steelyard is a lever AB, fig. 20, which is moveable about a center C, and on which substances to be weighed are suspended from the extremity B, as at Q. A known weight P, moveable along the arm CA, is placed at such a distance from C as to balance the body Q: then from the place of P we may know the weight Q: and, if at different points of CA we place figures to represent the corresponding weights of Q, the arm CA is *graduated.*

The lever is now supposed to have weight, and the arm CA being heavier and longer than the other, will preponderate. Suppose, that when Q and P are removed, a weight equal to P, placed at D, would keep the beam horizontal. If we then take $CO = CD$, it appears that the whole beam AB produces the same effect as a weight P placed at O, for either of the two would balance P, placed at D. Now let P and Q balance at B and M: therefore, Q balances P at M, together with the beam; that is, Q balances P at M, together with a weight which produces the same effect as P at O. Hence,

$$Q . CB = P . CM + P . CO = P . CM + P . CD$$
$$= P . DM.$$

Hence, if we make DE, DF, DG, &c. equal to CB, $2CB$, $3CB$, &c. we shall have, when P is at E, at F, at G, &c.

$$Q = P,\ Q = 2P,\ Q = 3P, \text{ \&c.}$$

And therefore, the beam is graduated by taking equal distances from the point D, and numbering them 1, 2, 3, &c.

24. The reasonings of this chapter will apply if the arms of the lever, instead of being all in the same plane, are in any planes perpendicular to the axis of motion, the forces being supposed to act in these planes. The perpendiculars in these planes from the axis upon the force must be taken instead of CM, CN, &c. in the preceding Articles, and the same propositions are true.

A *windlass* moved by handspikes, is an example of forces acting in this manner.

CHAP. II.

THE COMPOSITION AND RESOLUTION OF FORCES.

25. HAVING considered the action of forces, and their equilibrium, upon a lever, we now proceed to consider the effect of the combined action of two or more forces on a point. If two forces act on a point, as the forces in BQ, and BR, fig. 1, they will produce the same effect as a single force acting in some intermediate direction as BS.

In this case, the force in BS is called the ***Resultant*** of the forces in BQ, BR; and the forces in BQ, BR, are called the *Components* of the force in BS.

When two forces act in the same direction, the combined effect is equivalent to the *sum* of the forces. Thus, if a force of 2 pounds, and another of 3 pounds, act upon the same point, in the same direction, the point will be drawn with a force of 5 pounds. And, in the same manner, if two forces act in opposite directions, the resultant will be the *difference* of the two, and in the direction of the greater. Thus, if a point be acted on by a force of 7 pounds in one direction, and 4 pounds in the opposite, it will be drawn with a force of 3 pounds in the first direction. For the force of 7 may be considered as composed of 4 and 3; of which the 4 will be destroyed by the opposite force, and the 3 will remain effective.

26. Prop. *If any two forces act at the same point, the force which is equivalent to the two is in the* direction *of the diagonal of the parallelogram, of which the sides represent the magnitude and direction of the component forces.*

Let Ap, Aq, (fig. 21.) represent in magnitude and direction the forces P, Q: complete the parallelogram $ApCq$; and draw AC. Draw also CM, CN perpendicular upon Ap, Aq produced. Now suppose CA to be a lever moveable about a point C, and acted on by the forces P, Q, in the directions Ap, Aq. The triangles CpM, CqN, have right angles at M and N, and angles $CpM = qAp = CqN$. Hence they are similar, and

$$CM : CN :: Cp : Cq :: Aq : Ap$$
$$:: Q : P \text{ by supposition.}$$

Therefore the forces P, Q are inversely as the perpendiculars CM, CN, and would therefore together keep the lever CA at rest. (Art. 20, Cor. 5.)

And since the force equivalent to the two P, Q, would produce the same effect as they would together, this force also would keep the lever CA at rest. But manifestly no single force can keep the lever CA at rest, except it act in the direction AC*. For if it made an angle with CA on either side, it would turn CA round C in that direction. Hence the equivalent force acts in the direction AC; and AC is the diagonal of the parallelogram whose sides are Ap, Aq, which represent the forces.

Cor. 1. If a point acted on by two forces Ap, Aq, be kept at rest by a third force, this force must act in the direction CA.

Cor. 2. Hence if three forces act at a point and keep *each other* in equilibrium, each of them is in the direction of the parallelogram whose sides represent the other two.

* If it were to act in the opposite direction CA it would keep the lever at rest, but in that case it manifestly could not be equivalent to forces AP, AQ including an angle on the side towards C.

27. Prop. *If any two forces act at the same point, the force which is equivalent to the two is expressed in* magnitude *by the diagonal of the parallelogram, of which the sides represent the magnitude and direction of the component forces.*

Let Ap, Aq, fig. 22, represent the two forces in magnitude and direction. Complete the parallelogram pq, and, by last Article, Cor. 2, the two forces Ap, Aq will be kept at rest by a force in the direction rA. Let Ay represent this force in magnitude; and complete the parallelogram py. Since the forces Ap, Aq, Ay, keep each other in equilibrium, we may consider Aq as counteracting Ap, Ay. Hence, Aq will be in the direction of the diagonal of the parallelogram py, and xAq will be a straight line. Hence, in the triangles xAy, qAr, the angles $xAy=qAr$; as also $xyA = Arq$; and the triangles are equiangular. Also, $qr=Ap=yx$; therefore the triangles are equal: and $Ay=Ar$: that is, the force Ay is represented in magnitude by the diagonal Ar. And Ap, Aq would counteract Ay, and therefore their resultant in Ar would counteract Ay, and is therefore equal to it. Hence, the resultant is represented in magnitude by Ar.

Cor. If the components be represented by the sides of a parallelogram, the resultant is represented in magnitude and direction by the diagonal.

28. Prop. *Forces may be represented by lines parallel to their directions, and proportional to them in magnitude.*

For the direction of a line parallel to a force is the same as that of the force itself: and hence the force is properly represented in magnitude and direction.

Cor. 1. If AB, BC, fig. 23, represent two forces, AC will represent their resultant; for, completing the parallelogram $ABCD$, the force represented by BC is the same with the force represented by AD; and therefore AC the resultant of AB, AD, is the resultant of AB, BC.

Cor. 2. If two forces be represented by two sides of a triangle proceeding from the same angle, as AB, AD; their resultant may be found by doubling the line which joins the angle and the bisection of the opposite side. For if the parallelogram be completed, its diagonals bisect each other, and therefore AC is twice AE.

Cor. 3. If three forces, represented in magnitude and direction by the three sides of a triangle taken in order, act on a point, they will keep it at rest. Let ABC be the triangle; AB, BC are equivalent to AC by Cor. 1; therefore AB, BC, CA are equivalent to AC, CA, and therefore will keep the point at rest.

Cor. 4. Conversely, if three forces act on a point in the *directions* of the sides of a triangle, and keep it at rest, they are represented in *magnitude* by the sides of the triangle. For if one of these forces, as that in direction BC, be not represented by BC, let it be represented by BC'; then the two AB, BC' are equivalent to AC', and therefore cannot balance a force in direction CA: which is contrary to the supposition.

Cor. 5. Hence, knowing the directions of three forces which keep each other in equilibrium, we may find their relative magnitudes by making a triangle whose three sides are parallel to these directions; these sides will be in the proportion of the forces.

Cor. 6. If three forces keep a body in equilibrium, and three lines be drawn making with the directions of the forces three equal angles towards the same parts; these three lines will form a triangle whose sides will represent the three forces respectively.

Let AB, BC, CA, fig. 24, and 25, be the directions of the forces; DM, EN, FO three lines so that the angles ADM, BEN, CFO are eqaal; these lines, produced if necessary, form a triangle abc. In the triangles aEM, ADM, the angle aME equals AMD, and by supposition aEM equals ADM; hence the remaining angle MaE or bac equals MAD or BAC; similarly, the angle abc equals ABC, and bca equals BCA. Hence the triangles abc, ABC are equiangular, and therefore

$ab : bc :: AB : BC$

$::$ force in direction AB : force in direction BC, by Cor. 4.

and similarly of ca.

If therefore ab represent the force in direction AB; bc, ca will represent the forces in directions BC, CA.

Cor. 7. If three forces keep a point at rest, they are each as the sine of the angle contained by the other two.

Let P, Q, R, acting in Ap, Aq, Ay, fig. 22, keep a point A at rest. Produce yA and draw pr parallel to Aq, and Ap, pr, rA will be as the forces, (Cor. 4.) Now

$$P : Q :: Ap : pr :: \sin. Arp : \sin. pAr$$
$$:: \sin. qAr : \sin. pAr$$
$$:: \sin. qAy : \sin. pAy.$$

And similarly, we should have

$$R : P :: \sin. pAq : \sin. yAq,$$
$$\text{and } Q : R :: \sin. yAp : \sin. qAp.$$

Cor. 8. If the angle between two given forces be diminished, the resultant is increased.

Let two forces Ap, Aq, fig. 26, act at the angle pAq, pr being equal and parallel to Aq, Ar is the resultant.

Let Ap, AQ, the same forces, act at the angle pAQ; pR being equal and parallel to AQ, AR is the resultant.

pR is equal to pr, and if the angle $pAQ < pAq$, we have $ApR > Apr$ and therefore $AR > Ar$. (Euc. xxiv. 1.)

29. We have instances of the composition of forces in such cases as the following. Suppose a boat fastened to a fixed point by a rope, and acted on at the same time by the wind and the current. Then the direction of the rope will indicate the direction of the resultant of these actions.

In fig. 1, let BQ, BR be at right angles, and let the forces exerted be 48 pounds, and 20 pounds: to find the magnitude and direction of the resultant.

If we make BRS a right angle, and $BR = 48$, $RS = 20$, BS will be the resultant. And $BS^2 = 48^2 + 20^2 = 2704$; $\therefore BS = 52$, and the resulting force is 52 pounds.

Also to find the angle SBR, we have $\sin. SBR = \frac{20}{52} = \frac{5}{13}$; $\therefore SBR = 22^0\ 37'$ nearly.

We have many examples of the resolution of forces, where the force exerted being resolved into two, one of them is somehow lost or counteracted, and the remaining part only is effective. Thus, if we would drag an object along the ground by a rope attached to it, if we suppose this rope to be inclined to the horizon at an angle of 45^0, the force which we exert is effective only in part. If we thus exert a force of 17 pounds, this force is equivalent to two equal forces, one in a horizontal and one in a vertical direction. And if each of these be called x, we shall have

$$x^2+x^2=17^2,\ x=\frac{17}{\sqrt{2}}=12 \text{ nearly.}$$

Hence the force with which we draw the body horizontally is 12 pounds.

30. PROP. *To find the resultant of any number of forces AB, Ac, Ad, Ae, acting in the same plane at a point A,* fig. 27.

Complete the parallelogram Bc and draw the diagonal AC;
Complete the parallelogram Cd, and draw the diagonal AD;
Complete the parallelogram De, and draw the diagonal AE;
and so on, if there are more forces.

AB, Ac, are equivalent to AC;
$\therefore$ AB, Ac, Ad, are equivalent to AC, Ad, that is, to AD;
$\therefore$ AB, Ac, Ad, Ae, are equivalent to AD, Ae, that is, to AE.
That is, AE is the resultant of the forces AB, Ac, Ad, Ae.

COR. 1. If any number of forces be represented by sides of a polygon taken in order, as AB, BC, CD, DE, their resultant will be represented by the line AE which completes the polygon. (See Art. 28.)

COR. 2. A number of forces which are represented by *all* the sides of a polygon taken in order, as AB, BC, CD, DE, EA, acting upon a point, will keep it at rest.

For AB, BC, CD, DE are equivalent to AE: therefore, AB, BC, CD, DE, EA, are equivalent to AE, EA, and will keep a point at rest.

COR. 3. It does not follow conversely, as in the case of three forces, that if they act in the direction of the sides of the polygon and are in equilibrium, they are proportional to the sides. For the directions of the sides may remain the same while their proportions are altered. Thus, if we draw $D'E'$ parallel to DE, forces parallel to the sides of the polygon will keep a point at rest, if they be proportional to AB, BC, CD', $D'E'$, $E'A$, as well as if they be proportional to AB, BC, CD, DE, EA.

31. PROP. *To find the resultant of forces which are not in the same plane.*

Let AB, AC, AD, fig. 28, be three forces not in the same plane. Let the planes BC, BD, CD, be drawn, and the planes DG, CG, BG, parallel to them, completing the parallelepiped, whose sides will be parallelograms. Join AF, DG; DF will be a parallelogram, as is evident; and by Art. 27,

$$AB, AC, \text{ are equivalent to } AF;$$

$$\therefore AB, AC, AD, \text{ are equivalent to } AF, AD; \text{ that is, to } AG.$$

Hence, if the edges of a parallelepiped, drawn from the same point, represent the components, the diagonal will represent the resultant.

COR. 1. If $ABEG$ be any four-sided figure, not all in the same plane, and if AB, BE, EG, represent three forces, AG will represent their resultant.

COR. 2. If four forces acting upon a point, be represented by the sides of *any* four-sided figure, taken in order, they will keep the point at rest.

COR. 3. If any number of forces be represented by sides, taken in order, of a polygon, which is not in the same plane, their resultant will be represented by the line which completes the polygon.

COR. 4. If any number of forces be represented by all the sides, taken in order, of any polygon, they will keep a point at rest.

The three last Corollaries are proved from this Article, as those of Art. 30. are from Art. 30.

32. From the preceding principles, we may find the conditions under which a point will be kept in equilibrium, as will appear in the following Problems.

PROB. I. Fig. 29. *A, B are two points in the same horizontal line, and AC, BC, two strings from which at the knot C the weight W hangs: to find the forces exerted by the strings CA, CB.*

The point at which the equilibrium is produced is, in this case, the point C; and the forces which produce it are the forces of the strings CA, CB, and the weight W acting by the string CW. From any point d in the vertical line WC produced, draw db, da, parallel to CA, CB. In order to support the weight W the resultant of the forces of the strings CA, CB, must be in the direction Cd, and must be equal to the weight W. The forces must therefore be as Ca, Cb, and their resultant will then be as Cd by Art. 27. Hence if Cd represent the weight W, we have the forces of the strings represented by Ca, Cb. Or if P, Q represent the forces of the strings CA, CB; we have

$$\frac{P}{W} = \frac{Ca}{Cd} = \frac{\sin. Cda}{\sin. Cad} = \frac{\sin. dCb}{\sin. aCb} = \frac{\sin. DCB}{\sin. ACB};$$

$$\text{similarly, } \frac{Q}{W} = \frac{\sin. DCA}{\sin. ACB};$$

whence P and Q are known.

COR. 1. The forces of the strings measure their *tensions*, (see Art. 10.) and these again are measured by the pressures exerted on the immoveable points A, B. But if instead of supposing the strings fixed at the points A, B, we suppose them to pass over those points, or over pullies placed there, and to have appended to them weights equal to the forces P, Q; these weights will be just supported, that is, there will be an equilibrium. See Art. 8.

PROB. II. Fig. 30. *Two strings CAP and CBQ pass over pullies A and B, in the same horizontal line, and support a weight W by means of equal weights P and Q suspended at their other extremities: to find the position of the point C.*

Draw lines as in the preceding problem, and let Cd meet AB in E: then by the last corollary the weights W, P, Q will be as Cd, Ca, Cb; and since the weights P and Q are equal, $Ca = Cb = ad$; $\therefore \angle aCd = Cda = dCb$; $\therefore$ triangles ACE, BCE are equal, and $AE = EB$. Hence E bisects AB, and C will be in the vertical line passing through E.

Join ab meeting cd in e; aec, AEC are right angles.

$$\text{And } \frac{P}{W} = \frac{Ca}{Cd} = \frac{Ca}{2Ce} = \frac{CA}{2CE} \text{ by similar triangles.}$$

Let $AE = EB = a$, $EC = x$; $\therefore CA = (a^2 + x^2)^{\frac{1}{2}}$,

$$\frac{P}{W} = \frac{(a^2 + x^2)^{\frac{1}{2}}}{2x};$$

$$\therefore \frac{4P^2}{W^2} x^2 = a^2 + x^2; \quad \frac{4P^2 - W^2}{W^2} x^2 = a^2;$$

$$x = \frac{Wa}{(4P^2 - W^2)^{\frac{1}{2}}}.$$

Whence the position of C is known.

Cor. 1. In order that x may be possible, we must have the quantity under the radical sign positive, and therefore

$$W^2 < 4P^2,$$
$$\text{or } W < 2P;$$

if W be equal to or greater than $2P$, it will descend, drawing up both the weights, and will never find a place where it will rest.

Cor. 2. In order that the string ACB may be drawn so as to be in the horizontal line, we must have $x = 0$,

$$\text{or } \frac{Wa}{(4P^2 - W^2)^{\frac{1}{2}}} = 0;$$

which cannot be, except either W be indefinitely small or P indefinitely great. That is, no weights P, Q, however great, can draw up a weight W, so that the string ACB shall be a horizontal straight line. If ACB, instead of being a line without weight loaded with a weight at its middle, be a cord of which each part has weight, the same will be true.

PROB. III. Fig. 31. *P, Q, support W as in the last Problem, the values of P, Q, and the positions of the pullies A, B, being any whatever; to find the position of equilibrium of C.*

As before, let Cd be vertical, and da parallel to BC. Then P, Q, W, are as Ca, ad, dC. Hence, in the triangle Cad, we have the proportions of three sides given, to find the angles aCd, Cda.

$$\text{Also } BAC = BAP - CAP = BAP - aCd,$$
$$ABC = ABQ - CBQ = ABQ - bCd$$
$$= ABQ - Cda.$$

Hence, knowing the position of the points A, B, and therefore the angles BAP, ABQ, we know the angles BAC, ABC; and hence knowing the side AB, we may solve the triangle ABC, and calculate the position of C.

PROB. IV. *A string ACDEB, fig. 32, of which the extremities A, B, are fixed, is kept in a given position by weights P, Q, R, suspended at knots C, D, E; to compare the weights P, Q, R.*

Let the sides of the polygon AC, CD, DE, EB make with the horizontal line angles $\beta, \gamma, \delta, \epsilon$. Then it is easily seen that if AC be produced to c, $DCc = \beta - \gamma$. Similarly, $EDd = \gamma - \delta$, &c.

The point C is kept at rest by three forces; viz. the weight P, the *tension* of CA, and the *tension* of CD: let the latter be called C, and we shall have, by Cor. 7, Art. 28,

$$\frac{P}{C} = \frac{\sin.\, ACD}{\sin.\, ACP} = \frac{\sin.\, DCc}{\sin.\, ACx}$$
$$= \frac{\sin.\, (\beta - \gamma)}{\cos.\, \beta} = \frac{\sin.\, \beta \cos.\, \gamma - \cos.\, \beta \sin.\, \gamma}{\cos.\, \beta}$$
$$= \cos.\, \gamma\, (\tan.\, \beta - \tan.\, \gamma).$$

Also the point D is kept at rest by three forces; the weight Q, the tension of DE, and the tension of CD at D; and the last is the same as C the tension of CD at C, because the string must be kept at rest by equal and opposite forces, and therefore must exert equal and opposite tensions at it two extremities.

$$\text{Hence, } \frac{Q}{C} = \frac{\sin.(\gamma - \delta)}{\cos.\delta}$$

$$= \frac{\sin.\gamma\cos.\delta - \cos.\gamma\sin.\delta}{\cos.\delta}$$

$$= \cos.\gamma\,(\tan.\gamma - \tan.\delta).$$

Hence, we have

$$\frac{Q}{P} = \frac{\tan.\gamma - \tan.\delta}{\tan.\beta - \tan.\gamma}.$$

Similarly, we shall have the proportions of the forces at the other angles.

Hence, P, Q, R, are proportional to the differences of the tangents of the angles which the supporting strings make with the horizon.

If one of the strings, as EB, have that end higher which is farther from the origin A, the corresponding angle ϵ is to be taken negative, and we shall still have, ($-\tan.\epsilon$ being a positive quantity,)

$$\frac{Q}{R} = \frac{\tan.\gamma - \tan.\delta}{\tan.\delta - \tan.\epsilon}.$$

Cor. If the forces P, Q, R, instead of being parallel, were to make any angles with each other, we should be able to compare them by the application of Cor. 7, Art. 28.

A cord kept in equilibrium in such a manner is called a *Funicular Polygon.*

Prob. V. Fig. 33. *A cord PAQ, which passes round a fixed point A, is drawn in different directions by forces P, Q; to find the pressure upon the point A.*

In the first place, the forces P, Q must necessarily be equal, for, as the string passes freely round A, the forces will balance each other in the same manner as if they acted at the two ends of a string which was in a straight line, and therefore they will be equal. Now if we suppose A, instead of being immoveable, to be retained in its place by a force as AR, this force must manifestly, with the forces in AP and AQ, produce equilibrium at the point A. Hence, if we produce RA to any point r, and draw rp parallel to AQ; Ap, pr, rA will, by Art. 28, be proportional to the forces in AP, AQ, and AR. Also it has been shewn that the forces in AP, AQ are equal, and therefore Ap, pr are equal, and the angle $rAp = Arp = rAQ$. Hence Ar bisects the angle PAQ, and if po be perpendicular to Ar,

$$Ar = 2Ao = 2Ap \cos. pAr = 2Ap \cos. \tfrac{1}{2}PAQ.$$

Hence, if we put the forces P, Q, each $= P$, and the force in $AR = R$; also $\angle PAQ = A$,

$$\frac{R}{P} = \frac{Ar}{Ap} = \frac{2Ap \cos. \frac{1}{2}A}{Ap} = 2\cos. \tfrac{1}{2}A\,;\ \therefore R = 2P \cos. \tfrac{1}{2}A\,;$$

and R, the force which would keep A at rest, is evidently equal to the pressure upon that point produced by the cord PAQ: hence we have the pressure upon $A = 2P \cos. \frac{1}{2} A$.

Cor. 1. If A, instead of a point, be a pulley round which the cord passes, the pressure on the pulley will be the pressure at the center of the pulley. For in this case, fig. 34, the strings aP, bQ, touch the circle abd of the pulley, and would if produced meet in the line CA which passes through the center, and would make equal angles with it. Hence the resultant of the tensions in aP, bQ passes through the center, and is as before equal to $2P \cos. \frac{1}{2}A$.

Cor. 2. If a string pass over any number of fixed points $ABCD$, and be kept at rest by forces or weights P, Q drawing it in opposite directions, these forces or weights must be equal. And the pressure upon any one of the points, as B, will be $2P \cos. \frac{1}{2}ABC$.

Prob. VI. Fig. 35. *A given weight W is supported by two props AC, BC upon a horizontal plane AB. To find the pressure upon each, their lengths and the distance at which they stand being given.*

If we take Cd in the vertical line CD to represent the weight of the body, and draw da parallel to BC, Ca, ad will represent the pressures (Art. 31.); but, to prepare the student for the solution of succeeding problems, we shall obtain them by a different method.

Let the *re-actions* of the props in the directions AC, BC, be P, Q, (see Art. 10.) Let P be resolved in the horizontal and vertical directions AD, DC. Then

$$\frac{\text{horizontal part of } P}{P} = \frac{AD}{AC} = \cos. A;$$

$$\frac{\text{vertical part of } P}{P} = \frac{DC}{AC} = \sin. A;$$

and similarly for Q.

$\therefore$ horizontal force of AC at $C = P \,.\, \cos. A$; of $BC = Q \,.\, \cos. B$;
vertical force of AC at $C = P \,.\, \sin. A$; of $BA = Q \,.\, \sin. B$.

And, since these forces support the weight, the horizontal parts must counteract each other, and the vertical parts must together $= W$;

$$\therefore P \cos. A = Q \cos. B;$$

$$P \sin. A + Q \sin. B = W.$$

By the first, $Q = \dfrac{P \cos. A}{\cos. B}$; hence, by the second,

$$P \sin. A + \frac{P \cos. A}{\cos. B} \sin. B = W;$$

$$\therefore P (\sin. A \cos. B + \cos. A \,.\, \sin. B) = W \cos. B;$$

$$\text{or } P \,.\, \sin. (A + B) = W \cos. B;$$

$$\text{or } P \,.\, \sin. C = W \cos. B;$$

$$\text{and } P = \frac{W \cos. B}{\sin. C}.$$

$$\text{Similarly, } Q = \frac{W \cos. A}{\sin. C}.$$

Wherefore, as we can express $\cos. A$, $\cos. B$, $\sin. C$, in terms of AC, BC, AB, we can thus obtain the forces or re-actions P, Q. And the pressures upon the props are equal to these the re-actions which they exert.

Cor. 1. If we make AC, BC, AB equal to a, b, c, respectively, we shall have

$$\cos. B = \frac{b^2+c^2-a^2}{2bc}, \text{ (Bridge's } Trig. \text{ p. 58.)}$$

$$\sin. C = \frac{\sqrt{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}}{2ab};$$

$$\therefore P = \frac{Wa.(b^2+c^2-a^2)}{c\sqrt{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}};$$

$$\text{and } Q = \frac{Wb.(a^2+c^2-b^2)}{c\sqrt{(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}}.$$

Cor. 2. The props exert upon the plane at A and B pressures equal to those which are exerted on their upper extremities: these pressures at A and B may be resolved in directions perpendicular and parallel to the plane.

The parts perpendicular to the plane will be

$$\frac{W.\cos. B.\sin. A}{\sin. C} \text{ at } A, \text{ and } \frac{W.\cos. A.\sin. B}{\sin. C} \text{ at } B,$$

and these are counteracted by the re-action of the plane.

The parts parallel to the plane will be

$$\frac{W.\cos. B.\cos. A}{\sin. C} \text{ at } A, \text{ and } \frac{W.\cos. A.\cos. B}{\sin. C} \text{ at } B;$$

and these, if not counteracted, will make the props slide in opposite directions from A and B along the horizontal plane. They may be counteracted by immoveable obstacles placed behind the props at A and B. They will sometimes be counteracted by the friction of the plane.

Prob. VII. Fig. 36. *A weight W is supported by three props AW, BW, CW, upon a horizontal plane ABC. To find the pressure on each: the lengths of the props and the distances at which they stand being given.*

Draw WO perpendicular to the horizontal plane; join AO, and produce it to meet BC in K, and join WK.

The pressures of the three props in their own directions together support the weight, and therefore produce a pressure in the vertical direction OW; also the pressure of AW will not be altered if we substitute for the pressures of BW, CW a pressure equivalent to them both; and this equivalent pressure must, along with AW, produce a vertical pressure in OW; hence it must be in the plane AWO, and therefore in the line KW, for it must manifestly be in the plane BWC; hence the weight W may be supposed to be supported by two props AW, KW, and the pressure on AW found by the last problem. Let as before P, Q, R, represent the pressures of the props AW, BW, CW; then

$$P = W . \frac{\cos. AKW}{\sin. AWK}; \text{ and similarly,}$$

$$Q = W . \frac{\cos. BLW}{\sin. BWL},$$

$$R = W . \frac{\cos. CMW}{\sin. CWM}.$$

Cor. 1. Since WO is perpendicular to AK, we have

$$\cos. AKW = \frac{OK}{KW},$$

$$\sin. AWK = \frac{AK}{KW} . \sin. KAW = \frac{AK}{KW} . \frac{OW}{AW};$$

$$\text{hence, } P = W . \frac{OK}{KW} . \frac{KW}{AK} . \frac{AW}{OW}$$

$$= W . \frac{OK}{AK} . \frac{AW}{OW}; \text{ and similarly,}$$

$$Q = W . \frac{OL}{BL} . \frac{BW}{OW};$$

$$R = W . \frac{OM}{CM} . \frac{CW}{OW}.$$

Cor. 2. If we draw AD, OE perpendicular to BC, we shall have

$$\frac{OK}{AK} = \frac{OE}{AD}; \text{ and hence } P = W . \frac{OE}{AD} . \frac{AW}{OW}$$

$$= W . \frac{OE}{OW} . \frac{AW}{AD}$$

$$= W . \frac{1}{\text{tan. } OEW} . \frac{AW}{AD};$$

and similar expressions may be obtained for Q and R.

Cor. 3. It is easily shewn that WE is perpendicular to BC; hence OEW measures the inclination of the planes CBA, CBW. Hence if a sphere with radius $= 1$, and center C, cut the pyramid, and make a spherical triangle efg, the angle e will be equal to the angle OEW. And if the angles made by the lines CA, CB, CW are known, the sides ef, fg, ge are known, and e may be found.

Cor. 4. The horizontal pressures which are to be resisted by obstacles at the lower ends A, B, C, are in the directions OA, OB, OC, and are equal to

$$P . \frac{OA}{AW} = W . \frac{OK}{AK} . \frac{OA}{OW};$$

$$Q . \frac{OB}{BW} = W . \frac{OL}{BL} . \frac{OB}{OW};$$

$$R . \frac{OC}{CW} = W . \frac{OM}{CM} . \frac{OC}{OW}.$$

Cor. 5. If the point O fall without the triangle ABC, the weight W cannot be supported.

CHAP. III.

MACHINES.

33. *Machines*, or, as they are called in their simplest state, *the Mechanical Powers*, are contrivances to enable a smaller force to keep at rest, or to put in motion a larger weight, or to overcome a greater resistance. We shall at present only consider the case where *equilibrium* is produced; for, knowing the force which would, by means of any machine, just support a weight, it is manifest that a larger force would raise it.

In these cases, as in the case of the lever, the force applied is called *the power*, and the resistance overcome is called *the weight*, and is measured by a weight to which it is equivalent.

The mechanical powers may be reduced to THE LEVER, THE WHEEL AND AXLE, THE TOOTHED WHEEL, THE PULLY, THE INCLINED PLANE, THE WEDGE, AND THE SCREW.

The four first are, in the state of equilibrium, reducible to the lever. The screw may be reduced to the inclined plane, as may the wedge. The way in which the latter is considered is not immediately applicable to it in its common use: instead of being kept at rest by pressure, and put in motion by excess of pressure, as is supposed in our reasonings, it is practically kept at rest by friction, and put in motion by impact.

1. *The Lever.*

This instrument has already been considered in Chap. I.

2. *The Wheel and Axle.*

34. *The wheel and axle* consists of a cylinder or *axle AB*, fig. 37, and a concentric circle or *wheel EF*, joined together, so that the whole is moveable about the axis of the cylinder: the weight

W is attached to a cord NW, and will manifestly be raised or lowered as the wheel and axle are turned one way or the other. It is supported by a force applied at the circumference of the wheel EF, either by another weight P acting by means of a string wrapped the contrary way to that at N, or by some other force as P', acting at a point M' in the circumference of the wheel.

PROP. *In the wheel and axle the power is to the weight as the radius of the axle to the radius of the wheel**.

Let fig. 38 be a representation of the machine referred to the plane EF, which is perpendicular to the axis. It is evident that the equilibrium will continue to subsist, if we suppose P and W, retaining their distances from the axis, to act in this plane. Let them act in the vertical lines MP and NW, and let MCN be a horizontal line through the center. Hence, considering MCN as a lever,

$$P : W :: CN : CM \text{ by Art. 15.}$$
$$:: \text{rad. of axle} : \text{rad. of wheel.}$$

It is obvious, that in the state of equilibrium this is the same machine with the lever. When they are put in motion, the two machines differ. In the wheel and axle the weight W ascends or descends in a vertical line: in the lever it describes a circular arc.

COR. 1. The power may act by means of a bar CM', and the wheel may be removed; this is the case in the *capstan* and *windlass*.

COR. 2. If the direction of the power be not perpendicular to CM, we must draw a perpendicular upon it from C, and the proportion will be

$$P : W :: \text{rad. of axle} : \text{perp. on dir}^{n}\text{. of power.}$$

3. *Toothed Wheels.*

35. If two circles A, B, fig. 39, moveable about their centers, have their circumferences indented or cut into equal *teeth*, all the

* In this and the following Propositions of this Chapter, the machines are supposed to be in equilibrium.

way round, and be so placed that their edges touch, as at *Q*, the prominences of one of them at that part lying in the hollows of the other; then if one of them, as *A*, be turned round by any means, the other will be turned round also. Such circles are called *Toothed Wheels*.

If we suppose the two circles in fig. 39, to be in the same plane, and if, one of them *A* being turned by a power *P* acting on a winch *CE*, the other raise a weight *W* by means of an axle *DF*, we shall have the proportion of *P* and *W* by the following proposition.

PROP. *In toothed wheels, the moment of P about the center of the first wheel is to the moment of W about the center of the second wheel, as the perpendiculars from the centers of the wheels upon the line of direction of their mutual action.*

The edges of the teeth which act upon one another are conceived to be perfectly smooth; that is, they are supposed by their pressure to exert a force perpendicular to their surface; all the effect produced to resist motion along the surface being supposed to arise from a defect of smoothness. If the pressure exerted at the point of contact were not perpendicular to the surface pressed, this pressure might be resolved into two forces, one perpendicular to the tangent, and the other in the direction of the tangent, and the latter force is understood to arise from friction, &c. and is at present left out of consideration.

Let the wheel *A* act upon the wheel *B* at *Q*; the action there exerted will be perpendicular to the surfaces which are in contact at that point: and the action of *A* on *B*, and the re-action of *B* on *A* will be equal and opposite: let this action be a pressure *Q* in the direction *MQN*. Then the force *Q* acting on the wheel *B* supports the weight *W*, and the re-action opposite to *Q* is supported by the power *P*. Hence, if *CM*, *DN*, be perpendiculars on *MQN*, we shall have, by Art. 20,

$$P : Q :: CM : CE,$$
$$Q : W :: DF : DN;$$
$$\therefore P : W :: CM . DF : DN . CE.$$

Hence multiplying the first and third terms by CE, and the second and fourth by DF, we shall have

$$P.CE : W.DF :: CM : DN,$$

$$\text{or mom. of } P : \text{mom. of } W :: CM : DN.$$

Cor. 1. If CD meet MN in O, we have, by similar triangles,

$$CM : DN :: CO : DO;$$

$$\therefore \text{mom. of } P : \text{mom. of } W :: CO : DO.$$

Cor. 2. If the form of the teeth be such, that the point O is fixed while the wheels revolve, the force continues the same during the motion.

This is the case when the form of the teeth is the involute of a circle.

Cor. 3. If the teeth be small in comparison with the radii of the wheels, Q will nearly coincide with O; and CO, DO will be very nearly the radii of the wheels measured to the point at which the contact takes place. Hence

$$\text{mom. of } P : \text{mom. of } W :: \text{rad. of } A : \text{rad. of } B.$$

Cor. 4. In order that the two wheels may work during a whole revolution, the intervals of their teeth must be equal; hence the numbers of teeth in each will be as the circumferences, and therefore as the radii: hence

mom. of P : mom. of W :: number of teeth of A : number of teeth of B, nearly.

Cor. 5. The case in the figure is a combination of a winch, two toothed wheels, and an axle. If we suppose the radius of the axle DF and the winch CE to be equal, the whole of the mechanical advantage will be owing to the toothed wheels. In this case, we have

$$P : W :: CO : DO.$$

When the number of teeth in A is very small, A is called a *Pinion*, and its teeth are called *Leaves*.

The teeth in which those of the wheel A work may be distributed along the edge of a straight bar instead of the circumference of a circle, the bar being restrained to move in the direction of its length.

Wheels are sometimes turned by simple contact with each other; sometimes by the intervention of cords, straps, or chains, passing over them; and in these cases the minute protuberances of the surfaces, or whatever else may be the cause of friction, prevents their sliding on each other. And at the points of contact an action and re-action are exerted corresponding to those which are supposed in the Proposition.

4. *Pullies.* (1.) *The single moveable Pully.*

36. A pully has already been mentioned, Art. 8, &c., as a means of changing the direction of part of the cord by which force is exerted; it is a small wheel which is moveable about its axis, and along part of the circumference of which the cord passes. So long as its axis is immoveable, it can produce only a change of direction; but when its axis is fixed in a *block* or *sheaf* which is moveable, it may produce a mechanical advantage.

Prop. *In the single moveable pully, the strings being parallel,*

$$P : W :: 1 : 2.$$

Let $CBAP$, fig. 40, be the cord passing round the pully AB; and let the force P act by this cord. By Art. 8, the tension of the string is the same throughout, and equal to the power P. Hence AB is supported by two equal and parallel forces in AP, BC; each equal to P; and hence, by Art. 25, the force W, which acts in the opposite direction upon AB, must be equal to their sum. Therefore $W = 2P$.

Cor. 1. If the strings be not parallel, as KA, CB, fig. 41, let them be produced and meet in n; and join on, o being the center of the pully. Then oA and oB, drawn to the points where the string touches the pully, are equal, because the pully is circular. And on is common, and oAn, oBn, right angles. Hence onA, onB, are equal.

The strings AK, BC will produce the same effect as if they acted at n. And the forces or tensions exerted by them are equal, each being equal to P. Hence the resultant bisects the angle AnB, and is in the direction no: and since the forces of the strings support the weight, no must be opposite to the direction in which the weight acts; and therefore vertical.

Let a horizontal line meet the strings in p, q, and the vertical line nm in m. np, nq, will be equal, and may be taken to represent the tensions of the strings AK, BC. And np is equivalent to nm, mp, and nq to nm, mq. And of these, the parts mp, mq destroy each other; and hence the force acting upwards is $2nm$. Therefore,

$$P : W : np : 2nm$$
$$: \text{rad.} : 2\cos. pnm.$$

If rad. $= 1$, and $pnm = a$, $W = 2P \cos. a$.

If $a = 0$, the strings are vertical, $\cos. a = 1$, and $W = 2P$, as before.

If $a = 60^0$, or $AnB = 120^0$, $\cos. a = \frac{1}{2}$, and $W = P$.

Cor. 2. When a weight is supported on a moveable pully, the two portions of the string make equal angles with the direction in which the weight acts.

Cor. 3. We may deduce the relation of P to W, in fig. 40, by considering BA as a lever. For if we suppose the point B to be a fulcrum, and the weight W to be supported by a force P acting vertically at A; we have $P : W :: Bo : BA :: 1 : 2$, as before.

Hence, the pully, in the state of equilibrium, may be reduced to the lever.

(2.) *First system of Pullies. Each Pully hanging by a separate String.*

37. The first system of pullies, fig. 42, is merely a repetition of the single moveable pully. The weight W is supported by the pully A_1; the string which passes round A_1 is supported by A_2; the string which passes round A_2 by A_3, and so on; and at the last string (which may pass over a fixed pully B) the power P acts.

PROP. *In the first system of pullies, where all the strings are parallel, and the weights of the pullies inconsiderable,*

$$P : W :: 1 : 2^n;$$

n being the number of moveable pullies.

By last Article,

$$\text{tension of } A_1A_2 = \tfrac{1}{2} \text{ weight at } A_1 = \frac{W}{2},$$

$$\text{tension of } A_2A_3 = \tfrac{1}{2} \text{ weight at } A_2 = \tfrac{1}{2} \text{ tension of } A_1A_2 = \frac{W}{2^2},$$

$$\text{tension of } A_3B = \tfrac{1}{2} \text{ weight at } A_3 = \tfrac{1}{2} \text{ tension of } A_2A_3 = \frac{W}{2^3}.$$

And similarly, we should have, if A_n were the last of the moveable pullies,

$$\frac{W}{2^n} = \text{tension of } A_nB = \text{power } P,$$

for the tension of the string at which P acts is equal to P. Hence

$$W = 2^n P,$$

when n is the number of moveable pullies.

COR. 1. If A_1, A_2, A_3, &c. be the weights of the pullies (including the blocks, &c.) respectively, we may consider each as a weight appended at that point: hence

$$\text{weight at } A_1 = W + A_1,$$

$$\text{tension of } A_1A_2 = \tfrac{1}{2} \text{ weight at } A_1 = \frac{W}{2} + \frac{A_1}{2};$$

$$\therefore \text{ weight at } A_2 = \frac{W}{2} + \frac{A_1}{2} + A_2;$$

$$\therefore \text{ tension of } A_2A_3 = \tfrac{1}{2} \text{ weight at } A_2 = \frac{W}{2^2} + \frac{A_1}{2^2} + \frac{A_2}{2};$$

$$\therefore \text{ weight at } A_3 = \frac{W}{2^2} + \frac{A_1}{2^2} + \frac{A_2}{2} + A_3;$$

$$\therefore \text{ tension of } A_3B = \tfrac{1}{2} \text{ weight at } A_3 = \frac{W}{2^3} + \frac{A_1}{2^3} + \frac{A_2}{2^2} + \frac{A_3}{2} = P;$$

and so on; and if there be n moveable pullies,

$$\frac{W}{2^n} + \frac{A_1}{2^n} + \frac{A_2}{2^{n-1}} \ldots\ldots + \frac{A_n}{2} = P;$$

$$\therefore W + A_1 + 2A_2 + \ldots\ldots + 2^{n-1}A_n = 2^n P.$$

Cor. 2. If the pullies be all equal, and each equal to A,

$$W + A\,(1 + 2 \ldots\ldots + 2^{n-1}) = 2^n P,$$
$$W + A\,(2^n - 1) = 2^n P,$$
$$W = 2^n P - A\,(2^n - 1).$$

Cor. 3. Hence the weight W is less as A is greater. If we have $2^n P = A\,(2^n - 1)$, W will $= 0$, and the power will only just support the pullies.

Cor. 4. If the strings be not parallel, we must compare the tension of each with that of the preceding by Cor. 1, to last Article.

(3.) *Second system of Pullies. The same String passing round all the Pullies.*

36. This system, fig. 43, consists of two blocks; an upper one $B_1 B_2$, and a lower one $A_1 A_2$: each contains a certain number of pullies, and the string passes round them alternately. The weight is hung to the lower block, and the power acts at the loose extremity of the string.

Prop. *In the second system of pullies, if the strings be parallel,*

$$P : W :: 1 : n;$$

n being the number of strings at the lower block.

Since the same string passes round all the pullies, its tension will be every where the same, and equal to the power P. And n being the number of strings at the lower block, since each of them supports a weight P, they will altogether, supposing them parallel, support a weight nP: hence

$$W = nP.$$

Cor. 1. If we consider the weight of the pullies, it is manifestly only requisite to add the weight of the lower block to W; hence if A be this block,

$$W + A = nP \; : \; W = nP - A.$$

Cor. 2. If the strings be inclined to the vertical we must take the resolved part of the force. But the angle made with the vertical is generally so small that this correction may be omitted.

(4.) *Third system of Pullies. Each String attached to the Weight.*

37. In this system, fig. 45, each string, as PA_1C_1, supports the weight, partly by its action at C_1, where it is attached, and partly by its pressure on the next string, as A_1A_2.

Prop. *In the third system of pullies, the strings being parallel, and the weight of the pullies inconsiderable,*

$$P : W :: 1 : 2^n - 1;$$

n being the number of pullies.

For, tension of $PA_1 = P$;

$\therefore$ weight supported at $C_1 = P$;

tension of $A_1A_2 =$ pressure on $A_1 = 2P$;

$\therefore$ weight supported at $C_2 = 2P$;

tension of $A_2A_3 =$ pressure on $A_2 = 2^2P$;

$\therefore$ weight supported at $C_3 = 2^2P$;

and so on.

Hence the whole weight W, which is the sum of all those supported at C_1, C_2, C_3, &c. is $W = P + 2P + 2^2P + \ldots\ldots$ $= (1 + 2 + 2^2 \ldots\ldots + 2^{n-1}) . P$, if there be n pullies;

$$\therefore W = (2^n - 1) . P.$$

Cor. 1. If we consider the weights of the pullies, and call them A_1, A_2, $A_3 \ldots\ldots A_n$, we shall have

tension of $PA_1 =$ weight supported at $C_1 = P$;

$\therefore$ pressure on $A_1 = 2P$;

$\therefore$ tension of $A_1A_2 =$ weight supported at $C_2 = 2P + A_1$;

$\therefore$ pressure on $A_2 = 2^2P + 2A_1$;

$\therefore$ tension of $A_2A_3 =$ weight supported at $C_3 = 2^2P + 2A_1 + A_2$;

$\therefore$ pressure on $A_3 = 2^3P + 2^2A_1 + 2A_2 + A_3$;

and so on.

Hence, since the weight W (including the hook, &c. at C_1) is equal to the sum of all the weights supported, if n be the number of pullies;

$$\begin{aligned} W = & (1 + 2 + 2^2 \ldots . + 2^{n-1}) P \\ & + (1 + 2 + 2^2 \ldots . + 2^{n-2}) A_1 \\ & + (1 + 2 + 2^2 \ldots . + 2^{n-3}) A_2 \\ & \ldots\ldots \\ & + (1 + 2) A_{n-2} \\ & + A_{n-1} \\ = & (2^n - 1) P + (2^{n-1} - 1) A_1 + (2^{n-2} - 1) A_2 \ldots . + A_{n-1}. \end{aligned}$$

Cor. 2. Hence, contrary to the other cases, W becomes greater by giving weight to the pullies. If we make $P = 0$, we may find the weight which will be supported by the pullies alone.

Cor. 3. If all the pullies be equal and each $= A$,

$$W = (2^n - 1) . P + (2^{n-1} - 1 + 2^{n-2} - 1 \ldots . + 2 - 1) . A$$
$$= (2^n - 1) . P + [2^n - 2 - (n-1)] . A$$
$$= (2^n - 1) . P + (2^n - n - 1) . A = (2^n - 1)(P + A) - nA.$$

Cor. 4. If the strings be not parallel, we must use Cor. 1, of the single moveable pully.

5. *The Inclined Plane.*

38. An inclined plane, that is, a plane inclined to the horizon, is sometimes used as a mechanical power. Let a weight W, fig. 46, be supported on an inclined plane AC, (inclined to the horizon at an angle CAB) by a power P acting in the direction WK.

Prop. *In the inclined plane, if WK in the direction of the power and WN perpendicular to the plane, be intercepted by a vertical line KN;*

$$P : W :: WK : KN.$$

The effect of the plane AC on the weight W will be in a direction WR perpendicular to the plane; for the plane cannot produce any pressure on W in a direction parallel to AC or CA. (See Art. 35.) Hence the weight will be supported in the same manner as if, instead of the plane AC, it were sustained by a string in the direction WR. It may therefore be considered as supported by a string WR, exerting a force equal to the re-action of the plane ($=R$); a string WK exerting a force equal to the power P; and its weight ($= W$) acting in the vertical direction WD.

Hence, since KN is parallel to WD,

$$P : W :: WK : KN.$$

Similarly, $R : P :: WN : WK.$

Cor. 1. If the force act parallel to the horizon in the direction WE, fig. 47, K coincides with E, E is a right angle, and

$P : W :: WE : EN :: BC : AB$, by similar triangles.

In the same manner,

$$R : W :: WN : EN :: AC : AB.$$

Cor. 2. If the force act parallel to the plane, K coincides with C,

$$P : W :: WC : NC :: BC : AC,$$
$$R : W :: WN : NC :: AB : AC.$$

Cor. 3. If the force act perpendicular to the horizon, in the direction WZ, we must suppose the point K removed to an infinite distance, so that WK, NK, are infinite and equal. Hence

$$P = W, \ R = 0.$$

Cor. 4. If the force act so as to make an angle CWY below the plane equal to CWZ above it, take away from the right angles CWN, CWR, equal angles CWY, CWZ, and we have $YWN = ZWR$; and this $= YNW$; therefore $YW = YN$; $\therefore P = W$.

$$\text{Also } YWC = YCW = CWZ; \ \therefore YW = YC,$$
$$\text{and } NC = 2YC = 2YN;$$
$$\therefore R : W :: WN : NY :: 2WN : NC :: 2AB : AC.$$

Cor. 5. If the force act in a direction WS, situate between WZ the perpendicular to the horizon, and WR the perpendicular to the plane, the point K will fall below N, as at M, and the weight, the power, and the re-action of the plane, are represented in magnitude and direction by NM, MW, WN. Hence the re-action of the plane is in the direction WN, and the body W is supported on the underside of the plane.

Cor. 6. In fig. 46, if we draw CV parallel to KW, we have

$$P : W :: KW : KN :: CV : CN.$$

Hence, W being given, P is least when CV is least, that is, when CV coincides with CW, or the force is in the direction of the plane.

Cor. 7. Let two weights W, W', fig. 48, support each other on two inclined planes AC, $A'C$, by means of a string which is parallel to the planes. Let P be the tension of the string, which will be the same on each, then, by Cor. 2,

$$P : W :: BC : AC,$$
$$W' : P :: A'C : BC;$$
$$\therefore W' : W :: A'C : AC.$$

Cor. 8. If a body is supported on a curve surface to which AC is a tangent at W, fig. 46, the effect will be the same as if it were supported on the plane AC. For the equilibrium depends only upon the direction of the surface at the point W, which is the same in the plane and the curve.

Cor. 9. If the weight W, instead of being in contact with the plane in one point only, touch it in a finite portion, or the whole, of its length AC, (as in fig. 48) the proportion of the power and weight will be the same as before, supposing the friction not to be considered. For the weight supported at each point will be in the same proportion to the part of the power which supports it; and hence the whole weight will have this proportion to the whole power.

Cor. 10. Let the angle of inclination of the plane to the horizon (CAB) be α; the angle which the string makes with the plane (KWC) be ϵ: then

$$\begin{aligned} WK : KN &:: \sin. WNK : \sin. KWN, \\ &:: \sin. CAB : \sin. KWR, \\ &:: \sin. CAB : \cos. KWC; \\ \text{or } P : W &:: \sin. \alpha : \cos. \epsilon. \end{aligned}$$

§ 6. *The Wedge.*

39. A Wedge is a triangular prism; and, when applied as a mechanical power, is generally used to separate obstacles, by introducing between them its edge and then thrusting it forwards. Thus, if ACc, fig. 49, be the end of a wedge, two objects EW, Ew, which have a tendency to rush together, may be separated by a force, (as a weight P,) applied at the back of the wedge, provided there be an immoveable obstacle at E. In the present Chapter we must consider the power as *in equilibrium* with the resistance, that is, P must be such a power as is just sufficient to prevent the wedge from being driven upwards, and not great enough to force it downwards.

In consequence of the immoveable obstacle at E, and of the nature of the object EW, the point W in the object, will, if it move at all, be compelled to move in a certain direction *WU**. Whatever force tends to produce motion in *W* will be effective only so far as it acts in this direction. Thus, if *WV* be a force acting on the object *W*, it will be equivalent to *WU*, and to *UV* perpendicular to *WU*: of which the latter is counteracted by the immoveable obstacle at *E*, and *WU* is effective in opposing the resistance.

It is manifest, that the weight or resistance at *W* must be measured by the force which must be applied *immediately* at *W* to balance it. That is, if *UW*, *uw*, be the directions in which the points *W*, *w* will move if they move at all, and if we suppose *W*, *w*, to represent the forces which must be applied at those points in the directions *WU*, *wu*, to keep the parts *EW*, *Ew* in their present position when the wedge is removed; *W*, *w*, will also represent the resistances which are to be balanced by the wedge.

If we suppose the sides of the wedge to be perfectly smooth, their action at *W*, *w*, will necessarily be perpendicular to their surfaces, (Art. 38.)

This being premised, we can find the proportion of the power, and the weight or resistance. We shall take the case in which the wedge is isosceles, that is, when *AW* is equal to *Aw*, and the angles *AWU*, *Awu*, as also the resistances *W*, *w*, are equal. In this case the direction *DA* in which the power acts, must pass through the point *A*, and bisect the angle *WAw*.

Prop. *In the wedge, to find the proportion of P and W.*

Draw *OWV* perpendicular to *AW*, and join *Ow*, which will be perpendicular to *Aw*, because the triangles *OAW* and *OAw* are equal. Join *Ww* meeting *AO* in *M*; therefore $WM = wM$.

Let *WV*, equal to *WO*, represent the action of the wedge perpendicular to its side; *WV* is equivalent to a force *WU* (which immediately opposes the resistance, and is therefore equal to it,) and a force *UV* perpendicular to *WU*, which is counteracted by the obstacle at *E*. Hence, *UW* representing the resistance, $WO = WV$ may represent the re-action on the side of the wedge. Similarly,

* *W* will move in a curve to which *WU* is a tangent at *W*.

the re-action on the wedge at w, arising from an equal resistance similarly applied, may be represented by wO. Also WO is equivalent to WM, MO; and wO, to wM, MO; of which WM, wM balance each other; and if the remaining forces MO, MO be balanced by a power $2P$ represented by $2MO$, the whole will be in equilibrium,

$$\therefore 2P : W :: 2MO : WU.$$

Cor. 1. If WU coincide with WV, or the resisting body be to be moved perpendicularly to AC, WMO, ADC will be similar triangles,

$$\therefore P : W :: MO : WV :: MO : WO :: DC : AC;$$
$$\therefore 2P : W :: Cc : AC.$$

Cor. 2. If WU be perpendicular to AD;

$$\therefore P : W :: MO : WU :: MO : MW :: DC : AD;$$
$$\therefore 2P : W :: Cc : AD.$$

Cor. 3. The action of the resistance upon the side AC is necessarily perpendicular to AC*. The reason why W does not move in that direction is that it is also acted on by the resistance of an immoveable obstacle E.

Cor. 4. Let CAD, half the angle of the wedge, $= \alpha$, and UWV, the angle contained between WV perpendicular to AC, and WU the direction of the resistance, $= \iota$; W the resistance on each side; $2P$ the power,

$$2P : W :: \frac{2MO}{OW} : \frac{WU}{WV} \text{ because } OW = WV;$$
$$:: 2 \sin. OWM : \cos. UWV,$$
$$:: 2 \sin. \alpha : \cos. \iota,$$
$$\text{and } 2P : 2W :: \sin. \alpha : \cos. \iota,$$

where $2W$ is the whole resistance.

* This is different from the way in which the wedge is sometimes considered, when the resistances are supposed to act in any direction, as for instance, parallel to AD. This is impossible; for if a body, as W, be pressed upon the side AC with a force parallel to AD, and with no lateral force, it will necessarily slide along AC, and the equilibrium cannot be established

7. *The Screw.*

40. The general form of a screw is well known. It consists of a cylinder, as CD, fig. 50, on the surface of which is a projecting rib or *thread* which runs round the cylinder, and at the same time proceeds uniformly along the cylinder lengthways. This part of the instrument is inserted into a similar hollow cylinder AB which, with its thread, it exactly fits. In fig. 50, half of the internal and half of the external screw, are supposed to be removed, for the purpose of shewing its construction.

It is manifest that if the external screw be fixed, the internal one can only move by turning on its axis, by which means it will also move lengthways. If we suppose the vertical cylinder DC to be urged in the direction of its length by a weight W, it will be clear, by considering the form of the machine, that DC will descend, each point of the thread which is in contact with the external screw descending upon the inclined surface of the external thread, as upon an inclined plane. And the weight may be prevented from descending by a force P acting at an arm CM, which prevents the screw from turning round.

The form of the screw is such that when its axis is vertical, the inclination of the thread to the horizon is at every point the same. Let in fig. 51, fhf' be a right-angled triangle, of which the base fof' is equal to the circumference of a horizontal section FoF of the cylinder. If then this triangle be wrapped round the cylinder so that fof' coincides with the circle FoF, the hypotenuse fnh will coincide with FnH, the thread of the screw. And FH will be parallel to the axis, and is called the *distance of two contiguous threads.*

PROP. *In a vertical screw, when a weight W is supported by a horizontal force P acting perpendicularly at the end of an arm CM, $P : W ::$ dist. of two threads : circumference of the circle whose radius is CM.*

In fig. 51, let the internal screw, which sustains the weight W, be supposed to be supported by its thread resting on the fixed thread FGH of the external screw. Then we may suppose a portion of the weight to be supported at each portion of the thread, and the whole weight will be the sum of these portions. Let a weight w be supported at n, by means of the arm CM; let cnm be an arm equal to CM, and let a force p, acting horizontally, and perpendicularly to cm, support w. Then w will be sustained in the same manner as if it were upon the inclined plane fnh, for this plane and the thread FnH are in the same direction at the point n. And the effect of the force p is to produce a horizontal pressure on n, which prevents it from descending; let this force be q. Then we have, by the property of the lever,

$$p : q :: cn : cm;$$

$$:: \text{circumf. to rad. } cn : \text{circumf. to rad. } cm;$$

and, by the property of the inclined plane,

$$q : w :: f'h : ff' :: FH : \text{circumf. to rad. } cn;$$

$$\therefore p : w :: FH : \text{circumf. to rad. } cm :: D : C, \text{ suppose};$$

$$\therefore p = \frac{D}{C} w.$$

In the same manner, let the weight w' be supported at any other point by p' acting at the end of an arm $= CM$; w'' by p'', &c. And we shall have

$$p' = \frac{D}{C} w', \quad p'' = \frac{D}{C} w'', \text{ \&c.}$$

$$\therefore (p + p' + p'' + \text{\&c.}) = \frac{D}{C} (w + w' + w'' + \text{\&c.}).$$

And the sum of all the partial weights will be the whole weight supported; and the power $p + p' + p'' + \text{\&c.}$ acting at M will produce the effect of the separate powers p at cm, &c. (Art. 24.) Hence,

$$p + p' + p'' + \text{\&c.} = P, \quad w + w' + w'' + \text{\&c.} = W;$$

$$\text{and } P = \frac{D}{C} W, \text{ or}$$

$$P : W :: D : C :: \text{dist. of two threads} : \text{circumf. to rad. } CM.$$

Cor. 1. Instead of supposing the screw to support a weight W acting vertically, we may suppose it employed to produce a pressure W in any direction, and the proportion will be the same as before.

Cor. 2. In fig. 50, the form of the thread which is wrapped round the cylinder is such that its section through the axis of the screw gives a rectangular profile, with sides parallel and perpendicular to the axis. But the mechanical advantage will be the same, whatever be the form or depth of this profile, so long as the inclination of the thread is the same.

Cor. 3. The proportion of the power and weight would be the same, if the internal screw were fixed, and the external one, carrying the weight, were moveable.

Cor. 4. The diameter of the cylinder does not affect the proportion of P to W, so long as the distance of the threads remains the same.

8. *Combination of Mechanical Powers.*

41. The *advantage* of a simple machine is the number expressing the multiple which the weight or effect produced is of the power or force producing it. The advantages of the different simple mechanical powers are as follows; (see the preceding Articles).

Of the Lever, advantage $= \dfrac{\text{arm of the power}}{\text{arm of the weight}}$,

Wheel and Axle.......... $\dfrac{\text{radius of wheel}}{\text{radius of axle}}$,

Toothed wheels.......... $\dfrac{\text{n}^{\text{r}}\text{. of teeth of wheel}}{\text{n}^{\text{r}}\text{. of teeth of pinion}}$ {nearly, when the teeth are small.}

Single moveable Pully............2
First System....................2^n
Second System...................n
Third System....................$2^n - 1$
{when the strings are parallel and the pullies without weight.}

Inclined Plane........	$\frac{\text{length of plane}}{\text{height of plane}}$	when the power acts parallel to the plane.
Wedge.............	$\frac{\text{side of wedge}}{\text{back of wedge}}$	when the resistance acts perpendicularly to the side.
Screw........	$\frac{\text{circumf. desc}^{d}\text{. by power}}{\text{distance of threads}}$	when the power acts in a plane perpendicular to the axis.

From these the *advantage* of compound machines may be found.

PROP. *The* advantage *of a combination is found by multiplying together the* advantages *of the separate machines.*

This may be shewn without difficulty in any particular case.

Fig. 52, represents a combination of the screw, the wheel and axle, the pully, and the inclined plane. A winch BC turns a cylinder CD, on which is the thread of a screw. This thread works in the teeth of a wheel ED, which has an axle EF. The cord which passes round this axis acts on a system of pullies of the second kind, attached to the fixed point G. This system draws a mass W up the inclined plane GH.

Let P be the power at B, acting perpendicularly to CB;

$$\frac{\text{pressure at } D}{P} = \frac{\text{circ. desc}^{d}\text{. by } B}{\text{dist. of threads}} = n \text{ suppose,}$$

$$\frac{\text{pressure at } F}{\text{pressure at } D} = \frac{\text{rad. of wheel } ED}{\text{rad. of axle } EF} = n',$$

$$\frac{\text{force at } H}{\text{tension at } F} = \text{number of strings at } H = n'',$$

$$\frac{W}{\text{force in } GH} = \frac{\text{length of plane}}{\text{height of plane}} = n''';$$

$$\therefore \frac{W}{P} = n n' n'' n'''.$$

For the sake of example, take the following numbers.

Let $CB = 18$ inches, distance of threads $= 1$ inch;

$\therefore$ circumf. by $B = 113$ inches nearly; $n = 113$.

$ED = 2$ feet, $EF = 6$ inches; $\therefore n' = 4$.

Number of strings at $H = 4$; $\therefore n'' = 4$.

Inclination of plane $= 30^0$; $\therefore n''' = 2$;

$$\therefore \frac{W}{P} = 113 . 4 . 4 . 2 = 3616 .$$

Hence, on such a machine a force of 3 pounds would raise a weight of 10,000 pounds.

The following example of a combination of levers has some remarkable properties.

In fig. 59, let CA, AB, BD be three bars moveable in the plane of the paper about centers at C and D, and about joints at A and B. A force acts at E in the direction EF, and produces a pressure at B. Let this pressure be exerted in the direction CB, against a body placed between B and the immoveable obstacle Q; and let it be required to determine the magnitude of the pressure. Draw CM perpendicular on EF; CN, DO on AB; DL on CB.

Let the force which acts in EF be P; and let W be the pressure produced at B in the direction CB. The lever CA communicates pressure to the lever DB by means of the bar AB; and the pressure thus communicated is in the direction of the length AB. Let Q be this pressure. The force Q acting on the lever CA in the direction BA balances the force P acting in EF: hence

$$\frac{P}{Q} = \frac{CN}{CM} .$$

Also the force Q acting in the direction AB on the lever DB produces the pressure W: hence

$$\frac{Q}{W} = \frac{DL}{DO} ;$$

$$\therefore \frac{P}{W} = \frac{CN . DL}{CM . DO} .$$

If we suppose CA, AB, to be in the same straight line, CN will vanish and W will be infinitely greater than P. And if, at the moment when the pressure W is exerted, CA, AB be nearly a straight line, the pressure will be very great in comparison of the force employed, and may be increased without limit.

A combination of this kind is used in the Stanhope and the Columbian printing presses, for the purpose of pressing together the types and the paper. The considerations by which its advantage appears belong partly to the following articles. It will there be seen that when W is very great compared with P, the velocity of B's motion must in the same proportion be small compared with P's. But by the contrivance above described, B's velocity is not small compared with P's, till CA, AB are nearly a straight line. Hence B moves with a convenient rapidity while it is going toward the position in which the great pressure is to be exerted, and then only moves very slow when it is come into this position and is actually exerting the pressure.

42. By means of machines a given force may be made to overcome any resistance, or to raise any weight whatever: but it will be shewn in the following Propositions that what is gained in power is lost in time: that is, in proportion as the force which we exert to move a weight is increased by machinery, the velocity with which the weight moves is diminished.

When bodies move through spaces which have always the same proportion, their velocities have this proportion also. But when the proportion of the space is variable, we may suppose the bodies to describe very small spaces, and the ratio of these will be the ratio of the velocities *ultimately*, that is, by supposing the spaces to be diminished without limit.

PROP. *To find the velocity of a body estimated in a given direction.*

Let a point W, fig. 56, move in a direction Ww. Let WP, wp be parallel lines drawn in any other direction; and let wn be perpendicular on WP. If Ww represent the body's velocity in the direction of its motion, Wn will represent its velocity estimated in the direction WP.

Also we have $Wn = Ww \cdot \cos. PWw$.

43. Prop. *In any of the mechanical powers, we shall have power : weight :: weight's velocity in the direction of its action : power's velocity in the direction of its action.*

We shall prove this by an enumeration of the cases of the different mechanical powers.

1. *The Lever.*

Let ACB, fig. 53, be a lever, acted on in directions AP, BW, by forces P, W: and let CM, CN, be perpendiculars on the directions of the forces. Let the lever move through a small angle into the position aCb. A and B will describe circular arcs Aa, Bb, which will be as the velocities of the points A and B, and being very small, may ultimately be taken for straight lines; and hence if am, bn be drawn perpendicular to AP, BW, Am, Bn will be as the velocities in the directions of the forces, by last Article.

Now, considering Aa as a straight line, CAa will ultimately be a right angle; hence,

$$CAM + aAm = \text{a right angle} = CAM + ACM,$$

and taking away CAM, $aAm = ACM$. Hence, the triangles CAM, Aam are similar. In the same way CBN and Bbn are similar. Also the angle aCb being equal to ACB, taking away aCB, we have $ACa = BCb$; and $CA = Ca$, $CB = Cb$, therefore the triangles ACa, BCb, are similar. Hence, we have these proportions,

$$Am : Aa :: CM : CA,$$
$$Aa : Bb :: CA : CB,$$
$$Bb : Bn :: CB : CN.$$

Hence, compounding the proportions,

$$Am : Bn :: CM : CN$$
$$:: W : P, \text{ by Art. 20.}$$

$\therefore$ P's velocity : W's velocity :: W : P.

2. *The Wheel and Axle.*

If the wheel and axle, fig. 38, turn through any angle, it is manifest that the arcs described by the points M and N are as CM and CN. But the arcs described are equal to the length of string wrapped at one point and unwrapped at the other, and are therefore as the velocities of P and W. Hence

$$P\text{'s velocity} : W\text{'s velocity} :: CM : CN$$
$$:: W : P, \text{ by Art. 34.}$$

3. *Toothed Wheels.*

Let A, B, fig. 54, be wheels which turn each other in any manner by means of their circumferences. If they are toothed wheels, we suppose the teeth small, so that the point of contact may be conceived to be at O, in the line joining their centers. We will suppose also that the power and weight hang from equal axles CE, DF. In this case $P : W :: CO : DO$, by Art. 35.

Now, let the wheels turn through a small angle, so that the points which were in contact at O, come to m and n. Om and On will be equal, because they have been applied to each other. And drawing meC meeting the circle CE in e, and nDf meeting the circle DF in f, Ee and Ff will be the spaces ascended and descended by P and W. And we have, by the similar sectors in the figure,

$$Ee : Om :: CE : CO,$$
$$On\ (=Om) : Ff :: DO : DF\ (=CE);$$
$$\therefore Ee : Ff :: DO : CO,$$
$$\text{or } P\text{'s velocity} : W\text{'s velocity} :: W : P.$$

Cor. Ee, Ff are as the angular velocities of the wheels A and B. Hence, in wheels which work in each other, the angular velocities are inversely as the radii. Hence also the number of revolutions in a given time will be inversely as the radii.

4. *Pullies.*

(1.) *In the single moveable pully with parallel strings,* if the weight W, fig. 40, be raised through any space, as 1 inch, each of the strings, AP, BC, will be shortened one inch at the lower end, and hence the power P will move upwards through 2 inches. Hence,

$$P\text{'s velocity} : W\text{'s velocity} :: 2 : 1 :: W : P.$$

(2.) *In the single moveable pully with strings not parallel;* fig. 55, let the pully at A be considered as a point. Let CAK be the position of the string, and let it be moved into the position CaK, so that W ascends through the small space Aa, and P descends through Pp. Take Km, Cn equal to Ka, Ca respectively; and $Am + An$ is the quantity by which the string CAK is shortened, and therefore the quantity by which KP is lengthened, or $Pp = Am + An$. Now when the angle AKa is very small, am may be considered as ultimately perpendicular on AK, and an on AC: hence

$$Am = Aa \cos.\ aAm = Aa \cos.\ \alpha, \text{ if } \alpha = KAa.$$

$$\text{Similarly, } An = Aa \cos.\ \alpha;$$

$$\therefore Pp = 2Aa \cos.\ \alpha;$$

$$\therefore Pp : Aa :: 2\cos.\ \alpha : 1;$$

$$\text{or } P\text{'s velocity} : W\text{'s velocity} :: W : P, \text{ by Art. 36.}$$

If the pully be of finite magnitude, as in fig. 41, since, when the change of position is small, the strings KA, CB, may be considered as remaining parallel to themselves, the part of the string AB which is wrapped round the pully is not altered; and hence the length of the space described by P is not altered on this account.

(3.) *In the first system of pullies,* fig. 42, if the weight W be raised through any space, as 1 inch, the pully A_2 is, as in the single moveable pully, raised 2 inches; hence, for the same reason, the pully A_3 is raised 2 . 2 inches; and similarly, a succeeding pully would be raised 2.2.2 inches; and so on to P, which will, by this reasoning be lowered 2^n inches: hence

$$P\text{'s velocity} : W\text{'s velocity} :: 2^n : 1 :: W : P.$$

(4.) *In the second system of pullies*, fig. 43, if the weight W be raised 1 inch, each of the strings by which the lower block hangs will be shortened 1 inch; and hence the whole length of the string between the blocks will be shortened n inches, and P will descend n inches;

$$P\text{'s velocity} : W\text{'s velocity} :: n : 1 :: W : P.$$

Cor. In this system, while 1 inch passes round the pully A_1, 2 inches pass round the pully B_1, 3 round A_2, 4 round B_2, &c.

Hence, if the radii of A_1, B_1, A_2, B_2, &c. be as 1, 2, 3, 4, the velocities of their circumferences will be as the radii, and therefore the angular velocities will be equal; and hence A_1, A_2 may be on the same axis, and may form one mass, and similarly B_1 and B_2 may be united on one axis, as in fig. 44.

(5.) *In the third system of pullies*, fig. 45, let the weight be raised 1 inch; then the pully A_2 will descend 1 inch: on this account the pully A_1 will descend 2 inches; and also on account of C_2 being raised 1 inch, A_1 will descend 1 inch; therefore it will descend $2+1$ inches. Again, on this account P will descend $2(2+1)$ or 2^2+2 inches, and 1 inch more in consequence of C_1 being raised 1 inch; hence, P will descend 2^2+2+1 inches$=2^3-1$ inches; hence,

$$P\text{'s velocity} : W\text{'s velocity} :: 2^3-1 : 1 :: W : P;$$

and similarly for any number of pullies.

5. *The Inclined Plane.*

Let W, fig. 56, be raised through a small space Ww, WP being supposed parallel to wp. Draw WE horizontal, and wm, wn perpendicular to WE, WP. Therefore Wn, wm are ultimately as the velocities in the directions of the power and weight. But if $CAB = wWm = a$, and $CWP = \epsilon$, we have

$$Wn : wm :: Ww \cos. \epsilon : Ww \sin. a$$
$$:: \cos. \epsilon : \sin. a;$$

or P's velocity : W's velocity :: $W : P$, Art. 38. Cor. 10.

6. *The Wedge.*

Let an isosceles wedge ADC, fig. 57, in which AD is the line bisecting the back, move in the direction of the line DA through a small space AQ. Let the point W move through a space Wn, in the direction WU, making an angle ι with WV, which is perpendicular to the side AC. Then we shall have

$$Wn = \frac{Wm}{\cos. \iota} = \frac{Aa \sin. a}{\cos. \iota}, \; a \text{ being} = DAC;$$

$$\therefore Aa \text{ or } Dd : Wn :: \cos. \iota : \sin. a,$$

or P's velocity : W's velocity :: $W : P$, by Art. 39. Cor.

7. *The Screw.*

If M, fig. 51, make a whole revolution with a uniform velocity, W will rise with a uniform velocity through the distance of two contiguous threads; and the space described by P, estimated in a horizontal direction (in which direction the force is supposed to act) is the circle whose radius is CM; hence

P's velocity : W's velocity :: circle (rad. $= DE$) : distance of threads
:: $W : P$.

8. *Any Combination of Machines.*

In any combination of these machines, the ratio of the power's velocity to the weight's velocity will be found by multiplying the ratios which obtain in the machines of which it is composed; and the ratio of the weight to the power is found by multiplying the ratios in each of the component machines, which ratios have been shewn to be the same as the former; hence the resulting ratios will be the same; and hence in all combinations of machines by which a power P sustains a weight W, if the machine be put in motion through a very small space,

P's velocity in its direction : W's velocity in its direction :: $W : P$.

Cor. 1. Hence we have P . P's velocity $= W$. W's velocity.

A weight multiplied into its velocity is called its *Momentum:* hence P's momentum $= W$'s momentum.

Cor. 2. If P . P's velocity $= W$. W's velocity, P and W will balance: for if not, let P and W' balance on the same machine: then P . P's velocity $= W'$. W''s velocity: and the velocity of W' is the same as that of W, so long as the machine remains the same. Hence $W' = W$, and therefore P and W balance.

CHAP. IV.

THE CENTER OF GRAVITY.

44. *The center of gravity of any body or system of bodies is a point upon which the body or system, acted upon only by the force of gravity, will balance itself in all positions.*

It will be made to appear that in every system there is such a point, by shewing how it may be found in every case. And it will also appear that there is only one point to which the definition is applicable.

Many of the properties of the point which we call the center of gravity do not depend upon the action of gravity, and might be enunciated and proved without supposing that force to exist. This point has been by some authors called the *center of magnitude,* and by others the *center of parallel forces.*

The definition given above supposes the particles of the system to be connected inflexibly; but the point may be conceived to exist where the particles are detached from one another.

It follows from our definition, that if a line or a plane which passes through the center of gravity be supported, the system will be supported in all positions.

45. PROP. *If a system balance itself upon a line in all positions, the center of gravity is in that line.*

If not, let the line be moved parallel to itself till it passes through the center of gravity; then we have, on one side of the line, increased both the quantity of matter and its distance from the line; and on the other side we have diminished both of these. Hence, if the system balanced itself before we moved the line, the tendency of one side to descend will, in some positions, be increased on both accounts, and, therefore, it cannot now balance round the same line in all positions. Hence the line about which the system balances itself in all positions cannot pass otherwise than through the center of gravity.

COR. 1. In the same manner it appears that a system cannot have more than one center of gravity.

COR. 2. If there be two lines, about each of which a system will balance in all positions, the center of gravity must be at their intersection.

COR. 3. If a system balance itself upon a line in one position, the center of gravity will be in the vertical plane which in that position passes through the line.

For if not, we might draw a line through the center of gravity, and the system would balance on this line. And hence it would balance on two lines in two different vertical planes, which is impossible, by the reasoning of the Proposition.

46. PROP. *To find the center of gravity of two bodies P, Q considered as points, fig. 59.*

Suppose PQ joined by an inflexible rod, and take $P+Q$: P :: $PQ : GQ$, and $\therefore Q : P :: PG : QG$; G will be the center of gravity. For let the horizontal line MGN meet the vertical lines PM, QN. And since $P : Q :: QG : PG :: GN : GM$ by similar triangles, we have $P \cdot GM = Q \cdot GN$; hence P and Q will balance on G, in every position. Therefore G is the center of gravity.

Cor. 1. The effect of the weights P, Q is the same as if they acted at the points M, N; but in this case, by Cor. 2, Art. 15, the pressure on the fulcrum G is $P+Q$; hence in every position of the two weights the pressure on the center of gravity is equal to their sum.

Cor. 2. To find the center of gravity of any number of bodies P_1, P_2, P_3, P_4, fig. 60, considered as points.

Suppose P_1P_2 joined by an inflexible rod, and take P_1P_2 : P_1g_1 :: $P_1+P_2 : P_2$, and as before it will appear that P_1, P_2 will balance on g_1 in every position. Also by Cor. 1, the pressure on g_1 is P_1+P_2.

Join g_1P_3, and take $g_1P_3 : g_1g_2 :: P_1+P_2+P_3 : P_3$; whence $g_2P_3 : g_1g_2 :: P_1+P_2 : P_3$; or $g_2P_3 : g_2g_1$ :: pressure at g_1 : pressure at P_3; whence, as in the beginning of this Article, g_1 and P_3, that is, P_1, P_2, P_3, will balance in every position on g_2, which is therefore the center of gravity of P_1, P_2, P_3. Also, in the same way, the pressure on g_2 is $P_1+P_2+P_3$.

Join g_2P_4, and take $g_2P_4 : g_2g_3 :: P_1+P_2+P_3+P_4 : P_4$; whence, as before, P_1, P_2, P_3, P_4 will balance in every position on g_3, which is therefore their center of gravity. Also the pressure on g_3 will be $P_1+P_2+P_3+P_4$.

And similarly, we might go on to any number of points.

This construction is applicable if the points be not in the same plane.

Cor. 3. If we take the points P_1, P_2, P_3, P_4, in any other order, we shall find the same point g_3. This appears from the last Article; for a system cannot have more than one center of gravity. It might also be shewn geometrically

Cor. 4. It appears from the demonstration of Cor. 2, that the pressure on the center of gravity, when it is supported, is equal to the whole weight of the system.

Cor. 5. In order that the parts may *balance each other*, it is necessary that they should be connected, as we have supposed them to be; but the point found as in Cor. 2, is called their *center of gravity* when they are unconnected, and even in motion. Also if the force which acts upon them be not gravity, but any other uniform force acting in parallel lines, this point retains the same denomination.

47. Prop. *To find the center of gravity of any number of bodies, P_1, P_2, P_3, P_4, considered as points, in the same straight line, fig.* 61.

Let G be the point on which they will balance in a horizontal position; by Art. 46, G will be the center of gravity. To find G, take any point A in the straight line; and since the weights P_1, P_2, P_3, P_4 balance, we have (Art. 22. Cor. 2.)

$$P_1 . P_1G + P_2 . P_2G = P_3 . P_3G + P_4 . P_4G;$$
$$\text{or } P_1 . (AG - AP_1) + P_2 (AG - AP_2)$$
$$= P_3 (AP_3 - AG) + P_4 (AP_4 - AG);$$
$$\text{or } P_1 . AG - P_1 . AP_1 + P_2 . AG - P_2 . AP_2$$
$$= P_3 . AP_3 - P_3 . AG + P_4 . AP_4 - P_4 . AG;$$
$$\therefore P_1 . AG + P_2 . AG + P_3 . AG + P_4 . AG$$
$$= P_1 . AP_1 + P_2 . AP_2 + P_3 . AP_3 + P_4 . AP_4;$$

and similarly, for any number of bodies;

$$\therefore AG = \frac{P_1 . AP_1 + P_2 . AP_2 + P_3 . AP_3 + P_4 . AP_4}{P_1 + P_2 + P_3 + P_4}.$$

Hence AG is known, and therefore G.

Cor. 1. If the center of gravity do not lie between P_2 and P_3, but, for instance, between P_1 and P_2; instead of having $P_2 (AG - AP_2)$ on the first side of the above equation, we shall have $P_2 (AP_2 - AG)$ on the second side, so that the result will be exactly the same.

Cor. 2. If any of the points be on the other side of A, their distances from A are to be reckoned negative; thus, in this case, instead of a term $P_5 (AG - AP_5)$, we shall have a term $P_5 (AG + AP_5)$, or $P_5 [AG - (-AP_5)]$.

48. Prop. *To find the center of gravity of any number of bodies, P_1, P_2, P_3, P_4...., considered as points, in the same plane, fig.* 60.

Let G be the center of gravity, found as in Art. 46, Cor. 2. Draw Ax any line in the plane, and draw on it perpendiculars $P_1 M_1$, $P M_2$, $P_3 M_3$, $P_4 M_4$...., $g_1 h_1$, $g_2 h_2$, $g_3 h_3$.... Also draw $m g_1 n$ parallel to Ax, meeting $P M_1$, $P_2 M_2$, in m, n. Then the triangles $P_1 g_1 m$, $P_2 g_1 n$ are similar; hence, by Art. 46,

$$\frac{P_1 m}{P_2 n} = \frac{P_1 g_1}{P_2 g_1} = \frac{P_2}{P_1}; \therefore P_1 . P_1 m = P_2 . P_2 n;$$

$$\text{or } P_1 . (M_1 m - M_1 P_1) = P_2 . (M_2 P_2 - M_2 n);$$

$$\text{or, since } M_1 m = M_2 n = g_1 h_1; \text{ transposing}$$

$$(P_1 + P_2) . g_1 h_1 = P_1 . P_1 M_1 + P_2 . P_2 M_2.$$

Similarly, since $(P_1 + P_2) g_1 g_2 = P_3 . P_3 g_2$, we should have

$$(P_1 + P_2 + P_3) . g_2 h_2 = (P_1 + P_2) g_1 h_1 + P_3 . P_3 M_3$$
$$= P_1 . P_1 M_1 + P_2 . P_2 M_2 + P_3 . P_3 M_3:$$

and

$$(P_1 + P_2 + P_3 + P_4) g_3 h_3 = (P_1 + P_2 + P_3) g_2 h_2 + P_4 . P_4 M_4$$
$$= P_1 . P_1 M_1 + P_2 . P_2 M_2 + P_3 . P_3 M_3 + P_4 . P_4 M_4.$$

And in like manner for any number of points,

$$(P_1 + P_2 + P_3 + ...) GH$$
$$= P_1 . P_1 M_1 + P_2 . P_2 M_2 + P_3 . P_3 M_3 + ...;$$

$$\therefore GH = \frac{P_1 . P_1 M_1 + P_2 . P_2 M_2 + P_3 . P_3 M_3 + ...}{P_1 + P_2 + P_3 + ...};$$

and is therefore known. Hence if we draw a line Kk parallel to Ax at this distance, G must be in this line. Similarly, if we draw any other line Ay, we may find the distance of G from Ay, and drawing a line Hh parallel to Ay at this distance, the intersection of Hh with Kk will give the point G.

Cor. If P_1M_1, P_2M_2, &c. and GH, instead of being drawn perpendicular to Ax, were drawn in any direction parallel to each other, and meeting Ax in M, M_2, &c. and H; we should have

$$GH = \frac{P_1 . P_1M_1 + P_2 . P_2M_2 + \dots .}{P_1 + P_2 + \dots .}.$$

49. Prop. *To find the center of gravity of any system of points whatever,* P_1, P_2, P_3....*Fig.* 62.

If we draw any plane yAx, and draw perpendiculars upon it P_1M_1, P_2M_2, P_3M_3....from the bodies, and g_1h_1, g_2h_2,.... from the centers of gravity of P_1, P_2, of P_1, P_2, P_3....; and GH from the center of gravity of the system: since P_1M_1, P_2M_2, g_1h_1 are in the same plane and perpendicular to M_1M_2, we shall have, by last Article,

$$(P_1 + P_2) . g_1h_1 = P_1 . P_1M_1 + P_2 . P_2M_2.$$

For the same reason, since g_1h_1, g_2h_2, P_3M_3 are in the same plane

$$(P_1 + P_2 + P_3)g_2h_2 = (P_1 + P_2)g_1h_1 + P_3 . P_3M_3$$
$$= P_1 . P_1M_1 + P_2 . P_2M_2 + P_3 . P_3M_3;$$

and, for any number of bodies,

$$(P_1 + P_2 + P_3 + \dots .)GH = P_1 . P_1M_1 + P_2 . P_2M_2 + P_3 . P_3M_3 + \dots .$$

$$\therefore GH = \frac{P_1 . P_1M_1 + P_2 . P_2M_2 + P_3 . P_3M_3 + \dots .}{P_1 + P_2 + P_3 + \dots .},$$

and is therefore known. Hence, if we draw a plane parallel to the plane xAy, at this known distance, G must be in this plane. Also if we take two other planes, as xAz and yAz, and find the distance of G from each of these, we shall be able to draw two other planes parallel to these, in each of which G must be: therefore it must be at the intersection of these three planes.

50. Prop. *The effect of any system* P_1, P_2, P_3....*to produce equilibrium is the same as if it were collected at its center of gravity. Fig.* 60.

Let the system produce equilibrium about a point, or a line, and let a vertical plane pass through the point or line; and let P_1M_1, P_2M_2, P_3M_3....be perpendiculars on this plane: then the effect in producing equilibrium about this plane will be the same so long as the *moment* $P_1 \,.\, P_1M_1 + P_2 \,.\, P_2M_2 + P_3 \,.\, P_3M_3 + \ldots\ldots$ remains the same. See Art. 22. But when all the system is collected at G this *moment* becomes $(P_1 + P_2 + P_3 + \ldots\ldots) \, GH$; and this, by Art. 49, is equal to the moment in the other case, however the plane be drawn. Therefore the effect remains the same as before.

It has been shewn (Cor. 4, Art. 46,) that when the system is supported at the center of gravity, the pressure there is the same as if the system were collected at that point.

Cor. Hence if P_1, P_2, P_3, P_4, fig. 60, instead of being points, be bodies of finite magnitude, we may find the center of gravity of the system, by supposing each body collected in its own center of gravity, and then proceeding as in Art. 46, 47, 48, or 49.

51. Examples *of finding the center of gravity.*

Ex. 1. To find the center of gravity of *a straight line:* supposed to be of uniform thickness and density.

A straight line will balance itself about its *middle point* in every position: this point is therefore the center of gravity.

Ex. 2. To find the center of gravity of *a parallelogram*, as $ABCD$, fig. 63.

Bisect the opposite sides AB and DC in E and F, and the opposite sides AD and BC in H and K; and let the lines EF, HK meet in G: G is the center of gravity.

For the parallelogram may be conceived to be made up of lines parallel to AB, as for instance PMQ; and since $PM = AE = EB = MQ$, each of these lines, as PQ, will balance on the point M, that is, on the line EF: hence the whole parallelogram will balance on the line EF. Similarly the whole parallelogram will balance on the line HK. Hence it will balance in every position on the point G; which is therefore the center of gravity.

Ex. 3. To find the center of gravity of *a triangle*; as ABC, fig. 64.

Bisect AB in E, and AC in F; join CE, BF; the intersection G is the center of gravity.

For the triangle may be conceived to be made up of lines parallel to AB, as PQ: and we have by similar triangles,

$$\frac{PM}{AE} = \frac{MC}{EC} = \frac{MQ}{EB}, \text{ and since } AE = EB,\ PM = MQ.$$

Hence each of the lines PQ will balance on the line CE in every position, and therefore the whole triangle will balance on that line. Similarly the whole triangle will balance on BF in every position; and hence it will balance in every position on the intersection G, which is therefore the center of gravity.

Join FE; and since $AE = \frac{1}{2}AB$, and $AF = \frac{1}{2}AC$, EF is parallel to BC; hence by similar triangles, AEF, ABC,

$$\frac{EF}{BC} = \frac{AE}{AB} = \frac{1}{2},$$

whence by similar triangles EFG, CBG,

$$\frac{EG}{GC} = \frac{EF}{BC} = \frac{1}{2}; \ \therefore GC = 2EG; \ \therefore EC = 3EG,$$

$$\text{hence } EG = \frac{1}{3}EC, \text{ and } GC = \frac{2}{3}EC.$$

Cor. 1. If we call the sides opposite to A, B, C, a, b, c respectively, and CE, e; since $AE = EB = \frac{c}{2}$,

$$e^2 = \frac{2a^2 + 2b^2 - c^2}{4} *.$$

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* In any triangle ABC, fig. 61, if a side AB be bisected in E; retaining the letters in the text, we have in

Hence $CG = \frac{2}{3}e = \frac{\{2(a^2+b^2)-c^2\}^{\frac{1}{2}}}{3}$.

Cor. 2. If we call GA, GB, GC, h, k, l respectively, we shall have $l = \frac{2}{3}e$; whence

$$3l = \{2(a^2+b^2)-c^2\}^{\frac{1}{2}}.$$

Hence $9l^2 = 2a^2+2b^2-c^2$;

similarly, $9h^2 = 2b^2+2c^2-a^2$;

$9k^2 = 2c^2+2a^2-b^2$; and, by addition,

$$9(h^2+k^2+l^2) = 3(a^2+b^2+c^2);$$

or $3(h^2+k^2+l^2) = a^2+b^2+c^2$.

Cor. 3. If three equal bodies be placed in the angles of a triangle, the center of gravity of these bodies is the same as the center of gravity of the triangle.

Cor. 4. To find the center of gravity of any *polygon*, divide it into triangles; and supposing each of these collected at its center of gravity, find the center of gravity of the whole; which, by Cor. to Art. 50, will be the center of gravity of the polygon.

Ex. 4. To find the center of gravity of *a quadrilateral* $ACBC'$, fig. 65, which has two adjacent sides equal, and also the two other adjacent sides equal: $AC = BC$, and $AC' = BC'$.

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in triangle ACE, $b^2 = \left(\frac{c}{2}\right)^2 + e^2 - 2 \cdot \frac{a}{2} \cdot e \cos. CEA$,

in triangle BCE, $a^2 = \left(\frac{c}{2}\right)^2 + e^2 + 2 \cdot \frac{a}{2} \cdot e \cos. CEA$,

add, and we have

$$a^2 + b^2 = \frac{c^2}{2} + 2e^2;$$

whence the formula in the text.

Join CC', which will bisect AB in D, and will be perpendicular to AB. Let E be the center of gravity of ABC and F of ABC; if we take G so that

$$EG : FG :: ABC' : ABC :: DC' : DC,$$

G will be the center of gravity:

and hence $EG : EF :: DC' : CC'$, and $EG = \dfrac{DC' . EF}{CC'}$.

Let $DC = c$, $DC' = c'$; $\therefore DE = \dfrac{c}{3}$, $DF = \dfrac{c'}{3}$; $\therefore EF = \dfrac{c + c'}{3}$;

$$\therefore EG = \frac{c'}{c+c'} \frac{c+c'}{3} = \frac{c'}{3}; \therefore DG = DE - EG = \frac{c}{3} - \frac{c'}{3} = \frac{c - c'}{3}.$$

Cor. Similarly if C and C' were both on the same side of AB, we should have

$$DG = \frac{c + c'}{3}.$$

Ex. 5. To find the center of gravity of *a quadrilateral* $ABDC$, fig. 66, of which two sides AB, CD are parallel.

Bisect AB, CD in H and K, and join HK; all lines parallel to AB will balance on HK, and therefore the center will be in that line. Join BC, CH, BK; and take $CE = \dfrac{2}{3} CH$, and $BF = \dfrac{2}{3} BK$; E and F will be the centers of gravity of the triangles ABC, DBC, which may, by Cor. to Art. 50, be considered as collected at those points. Hence, by Cor. to Art. 48, if EM, FN be parallel to HK, and G be the center of gravity,

$$GH = \frac{\text{triangle } ABC . EM + \text{triangle } BCD . FN}{\text{triangle } ABC + \text{triangle } BCD}.$$

Let CL be parallel to KH, CI perpendicular to AB; therefore, by similar triangles,

$$\frac{EM}{CL} = \frac{HE}{HC} = \frac{1}{3}, \quad \frac{FN}{KH} = \frac{DF}{DK} = \frac{2}{3},$$

$$\therefore EM = \tfrac{1}{3}CL = \tfrac{1}{3}KH,\ FN = \tfrac{2}{3}KH;$$

$$\therefore GH = \frac{\tfrac{1}{2}AB \,.\, CI \,.\, \tfrac{1}{3}KH + \tfrac{1}{2}CD \,.\, CI \,.\, \tfrac{2}{3}KH}{\tfrac{1}{2}AB \,.\, CI + \tfrac{1}{2}CD \,.\, CI}$$

$$= \frac{AB \,.\, KH + 2CD \,.\, KH}{3(AB + CD)}.$$

If $AB = a$, $CD = b$, $KH = c$,

$$GH = \frac{c}{3}\ \frac{a + 2b}{a + b}.$$

Cor. When $b = 0$, this gives $GH = \frac{c}{3}$, and the trapezium becomes a triangle.

Ex. 6. To find the center of gravity of *a pyramid whose base is a triangle* ABC, fig. 67, and whose vertex is O.

Bisect BC in D, join AD, OD, and take $DE = \frac{1}{3}DA$, $DF = \frac{1}{3}DO$; join OE, AF; their intersection G will be the center of gravity of the pyramid.

The pyramid may be conceived to be made up of planes parallel to ABC, as PQR; E is the center of gravity of the triangle ABC, and N, where OE meets PQR, will be the center of PQR; as may easily be shewn. Hence each of the triangles PQR will balance on the line OE, and hence the whole pyramid will balance in any position about OE. Similarly, the whole pyramid will balance on the line AF: hence it will balance in every position on the intersection G, which is therefore the center of gravity.

By similar triangles,

$$\frac{EF}{AO} = \frac{ED}{AD} = \frac{1}{3}; \text{ and } \frac{EG}{GO} = \frac{EF}{AO} = \frac{1}{3};$$

$$\text{hence } GO = 3\,EG;\ EG = \frac{1}{4}EO, \text{ and } GO = \frac{3}{4}EO.$$

Cor. 1. Bisect AO in H, and draw HK parallel to OE; hence by similar triangles,

since $AH = \frac{1}{2} AO$, $\therefore AK = \frac{1}{2} AE = DE$; $\therefore DK = 2DE$.

Also $HK = \frac{1}{2} OE$, and $GE = \frac{1}{4} OE$; $\therefore HK = 2GE$.

Hence $DE : DK :: GE : HK$, and DGH is a straight line bisected in G.

Hence we have this theorem: if in a triangular pyramid we bisect two edges which do not meet, and join the points of bisection, and bisect the joining line; the last bisection is the center of gravity of the pyramid.

Cor. 2. To find OG, let the edges of the pyramid adjacent to O, viz. OA, OB, OC be a, b, c; and the others BC, CA, AB, a', b', c' respectively: also let AD, OD, OE, be e, f, g.

Then we shall have*

$$g^2 = \frac{6f^2 + 3a^2 - 2e^2}{9}.$$

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* In any triangle AOD, fig. 67, if a side AD be divided so that DE is $\frac{1}{3}$ of DA; retaining the letters in the text, we have

in triangle DOE, $f^2 = \left(\frac{e}{3}\right)^2 + g^2 + 2 \cdot \frac{e}{3} \cdot g \cos. OEA$;

in triangle AOE, $a^2 = \left(\frac{2e}{3}\right)^2 + g^2 - 2 \cdot \frac{2e}{3} \cdot g \cos. OEA$.

Add twice the first to the second, and we have

$$2f^2 + a^2 = \frac{6e^2}{9} + 3g^2;$$

whence the formula in the text.

And by Cor. 1, to Ex. 3, we have

$$e^2 = \frac{2b'^2 + 2c'^2 - a'^2}{4},$$

$$f^2 = \frac{2b^2 + 2c^2 - a'^2}{4}.$$

Hence, by substitution,

$$g^2 = \frac{3(a^2 + b^2 + c^2) - (a'^2 + b'^2 + c'^2)}{9}.$$

And $OG = \frac{3}{4} \cdot g = \frac{1}{4} \{3(a^2 + b^2 + c^2) - (a'^2 + b'^2 + c'^2)\}^{\frac{1}{2}}$.

Cor. 3. If we join the center of gravity with each of the four angles O, A, B, C, and call the distances h, k, l, m, respectively, we shall have h^2, k^2, l^2, m^2, by formulæ easily derived from the preceding; and adding these together, we shall have

$$4(h^2 + k^2 + l^2 + m^2) = a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2.$$

Cor. 4. Since EG is $\frac{1}{4}$ of EO, it is manifest that if we draw parallel lines through G and O, meeting the base, the distance of G from this plane will be $\frac{1}{4}$ of the distance of O.

Ex. 7. To find the center of gravity of *any pyramid*, whose base is a polygon $ABCDE$, fig. 68, and vertex O.

The polygon may be divided into triangles by lines drawn from one angle to another; and if planes pass through these lines and through the vertex, the pyramid will be divided into triangular pyramids. If a plane be drawn parallel to the base, at a distance equal to $\frac{1}{4}$ of the altitude of the pyramid, by Cor. 4 to last Example, the center of gravity of each of the triangular pyramids, and therefore of the whole pyramid, will be in this plane. But if we join O with F the center of gravity of $ABCDE$, it will appear, as in the last Example, that the center of gravity will be in this line. Hence it will be in the point G where the line meets the plane. Also it is manifest that $FG = \frac{1}{4} FO$, and $OG = \frac{3}{4} OF$.

Cor. If the number of sides of the polygonal base of the pyramid be increased without limit, the method of finding the center of gravity remains the same. Hence it will be true in the case to which we thus approximate, that is, that of a *conical body with a curvilinear base.* In all such cases we must find the center of gravity by measuring from the vertex $\frac{3}{4}$ of the line which joins that point with the center of gravity of the base.

Ex. 8. To find the center of gravity of *a frustum of a pyramid;* cut off by a plane parallel to the base.

The two ends will be similar figures; let a, b, be homologous sides of the larger and smaller end. Also let the centers of gravity of the two ends be joined, and let the line which joins them be called the axis and be $= c$. Then the center of gravity will be in the axis, and it may be shewn, as in Ex. 5, that its distance from the larger end along this side will be

$$\frac{c}{4} \cdot \frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

Cor. The same will be true of the *frustum of a cone; a, b,* representing the radii, or any homologous lines, in the two ends.

52. Prop. *If in a system consisting of any number of particles, a point be taken, and if each particle be multiplied into the square of its distance from the point, the sum of these products will be the least when the point is the center of gravity.*

Let O, fig. 69, be the point, and G the center of gravity of P_1, P_2, &c. Join GO, and draw P_1M_1, P_2M_2, &c. perpendicular on it, and join P_1G, P_2G, &c. and P_1O, P_2O, &c.

Then

$$\overline{P_1O}^2 = \overline{P_1G}^2 + \overline{GO}^2 - 2GO \cdot GM_1,$$
$$\overline{P_2O}^2 = \overline{P_2G}^2 + \overline{GO}^2 - 2GO \cdot GM_2,$$
&c. = &c.

Hence $P_1 . \overline{P_1O}^2 + P_2 . \overline{P_2O}^2 + \&c.$

$$= P_1 . \overline{P_1G}^2 + P_2 . \overline{P_2G}^2 + \&c.$$
$$+ P_1 . \overline{GO}^2 + P_2 . \overline{GO}^2 + \&c.$$
$$- 2P_1 . GO . GM_1 - 2P_2 . GO . GM_2 - \&c.$$
$$= P_1 . \overline{P_1G}^2 + P_2 . \overline{P_2G}^2 + \&c.$$
$$+ (P_1 + P_2 + \&c.) \overline{GO}^2$$
$$- 2GO(P_1 . GM_1 + P_2 . GM_2 - P_3 . GM_3 - P_4 . GM_4.)$$

But by the property of the center of gravity,

$$P_1 . GM_1 + P_2 . GM_2 - P_3 . GM_3 - P_4 . GM_4 = 0;$$
$$\therefore P_1 . \overline{P_1O}^2 + P_2 . \overline{P_2O}^2 + P_3 . \overline{P_3O}^2 + P_4 . \overline{P_4O}^2$$
$$= P_1 . \overline{P_1G}^2 + P_2 . \overline{P_2G}^2 + P_3 . \overline{P_3G}^2 + P_4 . \overline{P_4G}^2$$
$$+ (P_1 + P_2 + P_3 + P_4) GO^2.$$

And it is manifest that the second side will diminish as GO diminishes, and will be least when GO is 0, or when O coincides with G.

Cor. If with center G and radius GO we describe a circle, the sum of each particle into the square of its distance from O will be the same in whatever part of the circumference O is.

For GO will be the same in all the situations of O.

53. Prop. *In any machine kept in equilibrium by the action of two weights, if an indefinitely small motion be given to it, the center of gravity of the weights will neither ascend nor descend.*

It is easy to shew this independently, in each of the mechanical powers.

In the straight lever, the center of gravity is at the fulcrum, and remains fixed however the lever be moved.

In the wheel and axle, fig. 38, the center of gravity of P and W is at G, in the vertical line passing through the center C, and if P descends, W ascends, and G remains fixed, as if PGW were a lever.

In the toothed wheels, fig. 54, if P ascends W descends; and the center of gravity G remains fixed in a point G, such that

$$PG : WG :: DO : CO.$$

In the systems of pullies, fig. 41, 42, 43, 44, 45, if we join P and W, and take $PG : WG :: W : P$, G will be the center of gravity; and if P descend W will ascend, so that P's descent : W's ascent :: $W : P :: PG : WG$; whence G remains fixed.

In the inclined plane, fig. 58, when the force is parallel to the plane, let P support W: and let P, W, be their situations when they are in the same horizontal line. Let P descend to p, and W ascend to W; $\therefore Pp = Ww$: join wp meeting WP in g; draw wm perpendicular on WP; now by similar triangles,

$$wg : pg :: wm : Pp :: wm : Ww :: BC : AC :: P : W;$$

therefore g is the center of gravity of p, w. Hence the center has moved in the horizontal line Gg; and this is true whatever be the space described.

The wedge and screw do not generally act by gravity; when they do, the same property is easily proved.

CHAP. V.

THE EQUILIBRIUM OF RIGID BODIES.

54. Bodies are hard or soft, rigid or flexible, extensible or inextensible, elastic or inelastic; and in all cases the conditions of their equilibrium may be deduced from the properties of the lever.

Prop. *A lever is kept at rest by any two forces; it is required to find the pressure on the fulcrum.*

Let ACB, fig. 70, be a lever acted on by two forces P and Q; the lever and the two forces will be in the same plane. Let a portion of this plane, as EF, including the lever, be supposed to be material and rigid, moveable about C in its own plane, and acted on by the forces P and Q. Then this plane will be kept at rest in the same manner as the lever was; and if CM, CN be perpendicular upon the directions of the forces, we shall have

$$P : Q :: CN : CM.$$

Let the directions of the forces meet in D, and let Cp, Cq be parallel to BD, AD; then the angle GqN is equal to CpM, and the triangles CpM, CqN are similar; and $CN : CM :: Cq : Cp$. Therefore $P : Q :: Cq : Cp$, that is, $P : Q :: Dp : Dq$. Hence Dp, Dq may represent the forces P and Q, and DC would, on the same scale, be their resultant, if they acted at D. But the force P produces the same effect as if it were applied at any other point of its direction AD, considering AD as a material line; and similarly of Q. Hence P and Q produce the same effect as if they acted at D; therefore they produce a pressure on C, which is equal to the resultant of the two forces.

Now the pressure on C will continue the same if any portion of the plane be removed. Suppose portions of the plane to be removed till nothing is left but the material line ACB composing the lever: then the pressure on C will be the same as before. Hence the pressure on the fulcrum of a lever agrees, in magnitude and direction, with the resultant of the two forces which act upon the lever, and keep it at rest.

This pressure acts in the direction of the line joining the intersection of the forces and the fulcrum.

Cor. 1. If the point C be acted upon by a force P, in the direction CD, and equal to the resultant of P and Q, the three forces P, Q, R, will keep the line ACB at rest, supposing no point of it to be fixed.

Cor. 2. If the forces be parallel, by Art. 18, the pressure on the fulcrum will be the sum of the forces, and also parallel to them. And by the same Article it appears that its distance from the two forces will be inversely as the forces.

Cor. 3. Hence it appears that two parallel forces produce the same effect as a force equal to their sum, acting in a direction parallel to them, and so situated that its distance from each force is inversely as the force.

This gives us the resultant of two parallel forces.

55. Prop. *In a lever acted on by any number of forces in the same plane, the pressure on the fulcrum is equal to the resultant of all the forces, supposing them applied at that point.*

Let any forces P, Q, and P', Q', &c. act on a lever CA, CB, &c. fig. 71.

Let P and P' meet in D, and let their resultant be R, in the direction DR: let DR meet Q in E, and let the resultant of R and Q be R', in the direction ER'; let ER' meet Q' in F, and let the resultant of R' and Q' be S; then S will be the pressure on the fulcrum. For since a force produces the same effect at whatever point of its direction it be supposed to act, P and P' produce the same effect as if they acted at D, and therefore the same effect as R; R and Q produce the same effect as if they acted at E, and therefore the same effect as R'; and R' and Q'

produce the same effect as if they acted at F, and therefore the same effect as S. Hence P, P', Q, Q' produce the same effect as S; but P, P', Q, Q' keep the system in equilibrium round C: therefore S does so; and therefore it passes through C; and hence it produces on C a pressure S; therefore P, P', Q, Q', produce on C a pressure S.

Also, since R at D is equivalent to P, P'; R at E is also equivalent to P, P'; therefore R and Q at E, or R', is equivalent to P, P', Q; therefore also at F, R' is equivalent to P, P', Q; therefore R', Q', or S, is equivalent to P, P', Q, Q' acting at the same point.

Hence the pressure on the fulcrum is the resultant of all the forces applied at one point.

It will be in the direction CF, for the force in FS produces the same effect as the forces P, Q, P', Q'. But these forces keep the lever at rest about C. Therefore the force in FS does not tend to turn the lever about C, and therefore passes through C.

Cor. Hence if $ABA'B'$ were a rigid body, and were acted on by the force P, Q, P', Q', and also by a force S acting in FC, the body would be kept at rest, supposing no point to be fixed.

For the force S acting thus would produce the same effect as a fulcrum.

56. Prop. *When three forces act upon any body and keep it at rest,* (1^0), *any one of them must be equal and opposite to the resultant of the other two;* (2^0), *and must pass through the intersection of the other two.*

Let a body EF, fig. 70, be kept at rest by three forces, P, Q, R. Take a point C in the direction of one of the forces R; and instead of a force R, suppose an immoveable fulcrum at C; then the re-action of this fulcrum will produce the same effect as the force R; but in this case, by Art, 54, the re-action will be equal to the resultant of the two forces P and Q, and will pass through their intersection. Hence the force R must fulfil these two conditions. And similarly, the Proposition is true of the forces P and Q.

COR. In the same manner, by Art. 55, if a rigid body be kept at rest by any number of forces, as P, Q, &c. and S, fig. 68; any one of them, as S, must be equal to the resultant of all the others. Also it must pass through the point F, found as in Art. 55; and its direction must be opposite to the direction of the resultant of the other forces.

We shall proceed to give examples of the manner in which we may determine, in particular problems, the conditions of equilibrium of a rigid body.

1. *Equilibrium on a Point.*

57. When a rigid body is moveable about a fixed point, its conditions of equilibrium are reducible immediately to those of a lever, of which the fixed point is the fulcrum; including, amongst the forces which act upon the lever, the weight of the body, supposed to be collected in its center of gravity.

PROB. *In the common balance, the weights being unequal, to find the position in which it will rest.*

The common balance consists of a beam AB, fig. 72, which is moveable about an axis C, and from which the scales are suspended at points A and B. The axis C is so placed, that, in the horizontal position, it is a little above the straight line AB joining the points of suspension. Let CD be perpendicular to AB; then the two arms DA, DB, must be equal in length and weight. Let $DA = DB = a$, $CD = b$. Also let G, a point in CD, be the center of gravity of the beam, and $CG = h$.

Draw MCN horizontal, meeting in H and E the vertical lines through G and D; and let θ be the angle which AB makes with the horizon, and therefore the angle which CD makes with the vertical. Then since $AD = DB$, $EM = EN = a \cos. \theta$; $CE =$

b sin. θ; $CH = h$ sin. θ. And if P and Q are the weights at A and B, and W the weight of the beam,

$$P.CM = Q.CN + W.CH;$$

or $P\ \{a \cos.\theta - b \sin.\theta\} = Q\ \{a \cos.\theta + b \sin.\theta\} + Wh \sin.\theta.$

$$\text{Hence, } \tan.\theta = \frac{(P-Q)\,a}{(P+Q)\,b + Wh}.$$

If we suppose D to be the difference of the weights, so that $P = Q + D$, we shall have

$$\frac{\tan.\theta}{D} = \frac{a}{(2Q+D)\,b + Wh}.$$

The requisites of a good balance are the following: 1. It should rest in a horizontal position when loaded with equal weights. 2. It should have great *sensibility;* that is, the addition of a small weight in either scale should disturb the equilibrium, and make the beam incline sensibly from the horizontal position. 3. It should have great *stability;* that is, when disturbed it should quickly return to a state of rest.

The first requisite will be obtained if the arms are equal, and the center of gravity lower than the point of suspension.

The sensibility is greater in proportion as, for a given value of D, tan θ is greater. It is greater also in proportion as for a given value of θ, D is less. It may therefore be conceived to be measured by $\frac{\tan.\theta}{D}$. Hence the sensibility of a balance is as

$$\frac{a}{(2Q+D)\,b + Wh}.$$

But D is small compared with $2Q$, and may be neglected. Hence the sensibility is as

$$\frac{a}{2Qb + Wh}.$$

Hence the sensibility of a balance is increased—by increasing the length of the arms (a)—by diminishing the weight of the beam (w)—by diminishing the distance between the center of motion and the center of gravity of the beam (h)—by diminishing the distance between the center of motion and the line joining the points of suspension (b).

The stability is as the force which at a given angle of inclination urges the balance to the position of equilibrium. Let the weights be equal, and this force is $2Q \, . \, CE + W \, . \, CH = (2Qb + Wh) \sin. \theta$. Hence the measure of the stability is $2Qb + Wh$.

Cor. By increasing the lengths of the arms we increase the sensibility without diminishing the stability.

Prob. II. *Fig.* 73. *From a given rectangle ABCD, of uniform thickness, to cut off a triangle CDO, so that the remainder ABCO, when suspended at O, shall hang with the sides AO, BC horizontal*.*

Let G be the center of gravity of BO, and H of CEO; OE, Gg, Hh, being vertical, and therefore perpendicular to AD.

$$\text{Hence } Og = \tfrac{1}{2} OA, \quad Oh = \tfrac{1}{3} OD.$$

Let $AD = a$, $AB = b$, $DO = x$; $\therefore AO = a - x$.

Now

$$Og \, . \, \text{rectangle } AE = Oh \, . \, \text{triangle } CEO,$$

$$\text{or } \frac{a-x}{2} \, . \, b \, . \, (a - x) = \frac{x}{3} \, . \, b \, . \, \frac{x}{2};$$

$$\therefore 3(a-x)^2 = x^2;$$

$$\therefore 2x^2 - 6ax = -3a^2;$$

$$\therefore x = \frac{a}{2}(3 \pm \sqrt{3}).$$

$$= .634a.$$

The negative sign is to be taken: the positive sign would place O beyond A.

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* This is Prop. 5, of Pappus's Mathematical Collections, Book 8.

2. *Equilibrium on a Surface.*

58. When a body rests on a given surface it will touch it either in one point, or in several points, or with a finite portion of its surface. In all these cases the body must be supposed to be acted on by forces perpendicular to the surface at the points where it is in contact; that is, by the re-action of the surface at those points.

PROB. III. *Fig.* 74. *A paraboloid DAd rests upon a horizontal plane; to find its position.*

If PK be a vertical line drawn through the point of contact, meeting the axis in K, this line must pass through the center of gravity; for the body may be supposed to be collected in its center of gravity, and it will then be supported by the re-action which acts in the line PK. And since the center of gravity is a point in the axis, K must be this center. Also, since PK is perpendicular to the tangent at P, PK is a normal; and hence if PN be perpendicular to the axis, by Conic Sections, $NK = \frac{1}{2}$ parameter $= \frac{1}{2}L$.

$$\text{And tan.} AKP = \frac{NP}{NK} = \frac{\sqrt{L.AN}}{\frac{1}{2}L} = 2\sqrt{\frac{AN}{L}} = 2\sqrt{\frac{AK-\frac{1}{2}L}{L}}$$

$$= 2\sqrt{\left\{\frac{AK}{L} - \frac{1}{2}\right\}}.$$

If AK be less than $\frac{1}{2}L$, this answer is impossible, that is, there will no longer be an oblique position of equilibrium, and the figure will not rest except when the axis is vertical.

It will be seen (Chap. 8,) that in a homogeneous paraboloid, if K be the center of gravity, $AK = \frac{2}{3}AC$. Hence the oblique position of equilibrium is possible so long as $\frac{2}{3}AC > \frac{1}{2}L$, or $AB > \frac{3}{4}L$.

PROB. IV. *Fig.* 75. *A solid composed of a cone and a hemisphere on the same base rests on a horizontal plane: to find its* *dimensions that it may rest on the hemispherical end in all positions.*

PC, the vertical line, will, in all positions, meet the axis in the center of the sphere; and hence this point must be the center of gravity of the whole figure. Let G be center of gravity of the hemisphere, and H of the cone; and we must have,

$$\text{mass of cone} \times CH = \text{mass of hemisphere} \times CG.$$

The cone is $\frac{1}{3}$, and the hemisphere is $\frac{2}{3}$, of the circumscribing cylinder. Hence cone = base $DE \times \frac{1}{3} BC$, and the hemisphere = base $DE \times \frac{2}{3} AC$. Also by Art. 51, $CH = \frac{1}{4} BC$; and by Chap. 8, $AG = \frac{5}{8} AC$, and $CG = \frac{3}{8} AC$. Hence

$$\text{base } DE \times \tfrac{1}{3} BC \times \tfrac{1}{4} BC = \text{base } DE \times \tfrac{2}{3} AC \times \tfrac{3}{8} AC;$$

$$\therefore BC^2 = 3AC^2.$$

Hence $BD^2 = BC^2 + CD^2 = 4AC^2$; and $BD = 2AC = DE$.

Hence the triangle DBE is equilateral.

PROB. V. *Fig.* 76. *When a body is supported on three vertical props* (A, B, C); *to find the pressure on each.*

Let G be the center of gravity of the body, Gg a vertical line meeting the plane ABC in g; join Ag, meeting BC in D; then, if we suppose the whole mass collected at the center of gravity, it may be considered as supported on a lever AD; and if W be the whole weight,

pressure at $A : W :: Dg : DA ::$ triangle BgC : triangle BAC.

In the same manner,

pressure at B (or C) : $W ::$ triangle AgC (or AgB) : triangle BAC.

Hence the pressure on each prop is as the triangle opposite to it, made by joining the angles of the triangle ABC with the point g.

COR. When a body is supported on four vertical props, as a table on its four legs, the pressures will be indeterminate, if we consider the body as perfectly rigid. For since it may be supported on three of these props, the fourth may support either nothing, or a finite portion of the weight. The only conditions are, that the pressures have their sum equal to the weight of the body, and that they be such, that if they be considered as weights, their center of gravity and the center of gravity of the body are in the same vertical line.

The same is true if the props be more than four.

PROB. VI. *Fig.* 77, 78. *A body ABCD rests with its base on a horizontal plane; to find when it will be supported.*

The effect of gravity will not be altered if the body be supposed collected at its center of gravity G; G being still supposed to be connected with the base AB, as for instance, by means of GA and GB. In this case, if the vertical line Gg fall within the base, AB, fig. 77, G will have no tendency to turn round A in the direction BG, or round B in the direction AG: and consequently the body will manifestly be supported.

If Gg fall without AB, fig. 78, the body will fall over on the side on which Gg falls. In order that this may be the case, G must evidently turn in the direction Gh round B, which is supposed to be prevented from sliding. Now if Gm represent the weight of the body at G, this force may be resolved into Gn, in the direction of the tangent to the circular arc Gh, which G would describe round B, and nm perpendicular to this, and therefore in the direction GB. Of these, the force Gn tends to cause motion round B, and is not at all counteracted: hence the point G will move in Gh, and the body will fall over.

The force nm is counteracted by the resistance at B, if B be prevented from sliding; but if the base AB and the plane on which it rests be supposed perfectly smooth, the body will slide as well as fall: and in fact, since there is no lateral force, G will descend in a vertical line whenever the body rests on a horizontal plane, as will be shewn when we consider the motion of such a body.

If we consider the base as an area, the same still holds; viz. that the body will be supported if the perpendicular from the center of gravity falls within, and will fall if this perpendicular falls without the base.

If the body be supported on several points, or on several portions of its surface, we may suppose a string to pass round all of them, and the area comprehended within this string is to be considered as the base.

3. *Equilibrium on a Point and a Surface.*

59. When a body rests with one part of it upon a point and another upon a surface, as in fig. 79, the forces by which it is supported are the re-actions of the point and of the surface. If A be the supporting point, the re-action will be in Ag perpendicular to the surface of the body. And if PB be the supporting surface, and P the point of contact, the re-action there will be in Pg perpendicular to both the surfaces. And by Art. 56, the point g of intersection of the two forces must be in the line in which the third force acts; that is, in the vertical line passing through G the center of gravity. Hence Gg is vertical; and from this property the position of equilibrium may be determined.

PROB. VII. *Fig.* 80. *A beam PQ, considered as a line, rests upon a point A, with its end against a vertical plane BC; to find the position in which it will rest.*

Let G be the center of gravity, Pg horizontal, Ag perpendicular to PA, Gg joined; and since the re-action of the plane is in Pg, and that of the point in gA, their point of intersection g must be in the vertical line through the center of gravity: hence Gg is vertical, and perpendicular to Pg. Draw AE vertical. By the Elements, the triangles PAE, PgA, PGg, are all similar; hence

$$\frac{PE}{PA} = \frac{PA}{Pg},$$

$$\frac{PE}{PA} = \frac{Pg}{PG}; \quad \therefore \frac{PE^2}{PA^2} = \frac{PA}{PG};$$

$\therefore PA^3 = PG \,.\, PE^2 = PG \,.\, AD^2$; $\therefore PA = \sqrt[3]{(PG \,.\, AD^2)}$, and is therefore known; for PG and AD are known.

COR. 1. To find the inclination of PQ, we have

$$\cos. PAD = \frac{AD}{PA} = \sqrt[3]{\frac{AD^3}{PG \,.\, AD^2}} = \sqrt[3]{\frac{AD}{PG}}.$$

Cor. 2. Hence if AD be greater than PG the equilibrium is impossible.

Prob. VIII. *Fig.* 81. *The same suppositions remaining, except that the plane BC is now inclined to the horizon at any angle, to find the position of the equilibrium.*

Let, as before, G be the center of gravity, Pg perpendicular to BC, and Ag to PA, and therefore Gg vertical, and also AEF vertical, and AD perpendicular to BC. We have by similar triangles,

$$\frac{FP}{FD} = \frac{PE}{AD},$$

$$\frac{PG}{Pg} = \frac{PA}{PE},$$

$$\frac{Pg}{PA} = \frac{PA}{AD}; \text{ and multiplying,}$$

$$\frac{FP \cdot PG}{FD \cdot PA} = \frac{PA^2}{AD^2}; \quad \therefore \frac{PA^3}{PF} = \frac{PG \cdot AD^2}{FD}.$$

Let $PG = a$, $AD = b$, $FD = c$; $PA = x$;

$$\therefore PD = (x^2 - b^2)^{\frac{1}{2}}, \quad PF = c - (x^2 - b^2)^{\frac{1}{2}};$$

hence we have

$$\frac{x^3}{c - (x^2 - b^2)^{\frac{1}{2}}} = \frac{ab^2}{c};$$

whence x must be found.

4. *Equilibrium on two Points.*

60. When a body is supported with its surface resting on two points, the re-action at each point will be in the direction of a perpendicular to the surface, and these perpendiculars must meet in the vertical line passing through the center of gravity as before.

Prob. IX. *Fig.* 82. *A plane figure, two contiguous sides of which are straight lines forming a right angle, rests in a vertical* *plane with these two sides on two given fixed points: to find its position.*

Let A, B, be the fixed points, CP, CQ two sides of the figure, G its center of gravity.

Let GH be perpendicular to PC; draw AD, HL horizontal, BD, CFL, GKE vertical. And if Ag and Bg be perpendicular to the sides CA, CB, they will be in the directions of the pressures exerted at A and B on their sides; and these directions will meet in the vertical line passing through G. Draw also DM perpendicular to BC.

Let $AD=a$, $BD=b$, $CH=h$, $HG=k$; and let the angle CAD be θ: then DBM, CHL, HGK also $=\theta$.

And since $AgBC$ is a rectangle, $Ag=CB$: and hence it appears that

$$AE=DF; \therefore EF=AD-2FD.$$

$$\text{Also } BC=CM-BM=a \sin. \theta-b \cos. \theta;$$

$$\therefore FD=BC \sin. \theta=a \sin.^2 \theta-b \cos. \theta \sin. \theta;$$

$$\therefore EF=AD-2FD=a-2a \sin.^2 \theta+2b \sin. \theta \cos. \theta.$$

$$\text{But } EF=KL=HL-HK=h \cos. \theta-k \sin. \theta;$$

$$\therefore a-2a \sin.^2 \theta+2b \sin. \theta \cos. \theta=h \cos. \theta-k \sin. \theta;$$

$$\text{or } a \cos. 2\theta+b \sin. 2\theta=h \cos. \theta-k \sin. \theta;$$

from which equation θ is to be determined.

5. *Equilibrium on two Surfaces.*

61. When a body rests on two surfaces, the re-actions at the points of support will take place in lines perpendicular to these surfaces; these lines must meet, for otherwise the body cannot be supported. And as before, the point of concourse will be in the vertical passing through the center of gravity.

PROB. X. *Fig.* 83. *A given beam PQ, considered as a line, is supported on two given inclined planes CP, CQ: to find the position in which it will rest.*

Let Pg, Qg, perpendicular to the planes, meet in g, and G being the center of gravity of PQ, Gg will be vertical. Let gG

meet the horizontal line drawn through C in H, and the plane PC in K. The angle PgK is the complement of PKg, as is also KCH. Therefore PgG is equal to KCH or PCA; similarly, QgG is equal to QCB.

Let PCA, the inclination of the plane $PA, = \iota$, $QCB = \iota'$; $\therefore PgG = \iota$, $QgG = \iota'$; also let $PG = a$, $QG = a'$, and let QP produced meet the horizontal plane in D, and $PDC = \delta$:

$$\text{hence } CPQ = PCD + CDP = \iota + \delta,$$
$$CQP = QCB - QDC = \iota' - \delta.$$

Now

$$\frac{PG}{Gg} = \frac{\sin. PgG}{\sin. GPg},$$

$$\frac{Gg}{QG} = \frac{\sin. GQg}{\sin. QgG};$$

$$\therefore \frac{PG}{QG} = \frac{\sin. PgG}{\sin. QgG} \cdot \frac{\sin. GQg}{\sin. GPg} = \frac{\sin. PgG}{\sin. QgG} \cdot \frac{\cos. PQC}{\cos. QPC},$$

$$\text{or } \frac{a}{a'} = \frac{\sin. \iota}{\sin. \iota'} \cdot \frac{\cos. (\iota' - \delta)}{\cos. (\iota + \delta)}$$

$$= \frac{\sin. \iota}{\sin. \iota'} \cdot \frac{\cos. \iota' . \cos. \delta + \sin. \iota' . \sin. \delta}{\cos. \iota . \cos. \delta - \sin. \iota . \sin. \delta}$$

$$= \frac{\tan. \iota}{\tan. \iota'} \cdot \frac{1 + \tan. \iota' . \tan. \delta}{1 - \tan. \iota' . \tan. \delta}.$$

Whence

$$a \tan. \iota' - a \tan. \iota . \tan. \iota' . \tan. \delta = a' . \tan. \iota + a' . \tan. \iota . \tan. \iota' . \tan. \delta;$$

$$\therefore \tan. \delta = \frac{a \tan. \iota' - a' \tan. \iota}{(a + a') \tan. \iota \tan. \iota'} = \frac{a \text{ cotan. } \iota - a' \text{ cotan. } \iota'}{a + a'};$$

whence we know the inclination of PQ to the horizon.

Cor. 1. If $a = a'$, which it will be if the line PQ be of uniform thickness and density;

$$\tan. \delta = \frac{\tan. \iota' - \tan. \iota}{2 \tan. \iota \tan. \iota'} = \frac{1}{2} \cdot \left(\frac{1}{\tan. \iota} - \frac{1}{\tan. \iota'} \right);$$

$$= \frac{\text{cotan. } \iota - \text{cotan. } \iota'}{2}.$$

Cor. 2. If $\iota' = \iota$, or the planes be equally inclined,

$$\tan. \delta = \frac{a - a'}{a + a'} . \cot an. \iota.$$

Cor. 3. In order that PQ may rest parallel to the horizon, we must have $\delta = 0$;

$$\therefore a \tan. \iota' - a' \tan. \iota = 0;$$

$$\therefore \frac{a}{a'} = \frac{\tan. \iota}{\tan. \iota'};$$

the segments GP, GQ must be as the tangents of the inclinations.

Prob. XI. *Fig.* 84. *Let p, q be two spheres, touching each other and resting on two inclined planes CP, CQ; to find their position.*

Join p, q, their centers. In every position the distance of their centers is equal to the sum of their radii; and hence they have no tendency to change their point of contact with each other, and may be considered as one mass. Also the re-action is perpendicular to the planes which touch the spheres, and will therefore pass through the centers p, q. Hence pq will be supported in the same way as if it rested at p and q on planes cp, cq parallel to CP and CQ. Hence we may find its position by the last problem.

Let r and r' be the radii of the spheres, p and q their weights, and δ the inclination of pq to the horizon. Let G be the center of gravity of the mass pq, therefore we shall have, retaining the notation of the last problem,

$$pq = r + r', \; pG = \frac{(r + r')q}{p + q} = a, \; qG = \frac{(r + r')p}{p + q} = a';$$

$$\text{hence } \tan. \delta = \frac{a \tan. \iota' - a' \tan. \iota}{(a + a') \tan. \iota \tan. \iota'},$$

$$\text{will} = \frac{q \tan. \iota' - p \tan. \iota}{(p + q) \tan. \iota \tan. \iota'}.$$

Cor. Hence it appears that the inclination of pq is independent of the radii r, r', and depends only upon the weights of the spheres.

The effect will be exactly the same whether the body be supported by the re-action of a surface, or by the tension of a string perpendicular to the surface. If any point of it hang by a string of given length, it will be confined to the surface of a sphere, and the case will be the same as if it rested on a spherical surface.

PROB. XII. *Fig.* 85. *A given beam PQ hangs by two strings of given lengths AP, BQ, from two given fixed points A, B: to find its position when it rests.*

Let AP, BQ meet in g; therefore gG through the center of gravity G is vertical; let this meet AB in E, and let PM, QN be parallel to it; and let QP meet BA in D.

Let $AB=c$, and its inclination to the vertical, $AEG=\epsilon$; $AP=p$, $BQ=q$, $GP=a$, $GQ=b$; $PAB=\alpha$, $QBA=\beta$, $PDA=\delta$. Hence

$$gPQ=APD=PAB-PDA=\alpha-\delta,$$
$$gQP=QBD+QDB=\beta+\delta,$$
$$AgB=PgQ=\pi-(\alpha+\beta);$$
$$\therefore Ag=AB\,.\frac{\sin.\,ABg}{\sin.\,AgB}=c\,.\frac{\sin.\,\beta}{\sin.\,(\alpha+\beta)},$$
$$Bg=AB\,.\frac{\sin.\,BAg}{\sin.\,BgA}=c\,.\frac{\sin.\,\alpha}{\sin.\,(\alpha+\beta)};$$
$$\therefore Pg=Ag-AP=c\,.\frac{\sin.\,\beta}{\sin.\,(\alpha+\beta)}-p;$$
$$Qg=Bg-BQ=c\,.\frac{\sin.\,\alpha}{\sin.\,(\alpha+\beta)}-q;$$

but

$$Pg=PQ\,.\frac{\sin.\,PQg}{\sin.\,PgQ}=(a+b)\frac{\sin.\,(\beta+\delta)}{\sin.\,(\alpha+\beta)};$$
$$Qg=QP\,.\frac{\sin.\,QPg}{\sin.\,QgP}=(a+b)\frac{\sin.\,(\alpha-\delta)}{\sin.\,(\alpha+\beta)};$$

hence
$$c\,.\frac{\sin.\,\beta}{\sin.\,(\alpha+\beta)}-p=(a+b)\,.\frac{\sin.\,(\beta+\delta)}{\sin.(\alpha+\beta)};$$
$$c\,.\frac{\sin.\,\alpha}{\sin.(\alpha+\beta)}-q=(a+b)\,.\frac{\sin.\,(\alpha-\delta)}{\sin.\,(\alpha+\beta)};$$

or $c \sin. \beta - p . \sin. (\alpha + \beta) = (a+b) . \sin. (\beta + \delta) \ldots\ldots (1),$

$c \sin. \alpha - q . \sin. (\alpha + \beta) = (a+b) . \sin. (\alpha - \delta) \ldots\ldots (2).$

To obtain α, β, δ, we must have a third equation; for this purpose we must find the tensions of the strings PA, QB; and as these tensions must be equivalent to the weight, which acts in a vertical direction, their components in a horizontal direction must destroy each other.

To find the tension of the string PA, we may suppose the point Q to be a fulcrum on which the beam PQ is sustained by the string PA; hence if we draw Qx and Qy perpendicular on Gg and Ag, we have

$$\frac{\text{tension of } PA}{\text{weight of } PQ} = \frac{Qx}{Qy}:$$

or, if we call the tensions of PA, QB, P, Q, and the weight of PQ, W; we shall have

$$\frac{P}{W} = \frac{Qx}{Qy} = \frac{QG . \sin. QGx}{QP . \sin. QPy}:$$

$$= \frac{QG . \sin. (GDE + GED)}{QP . \sin. (PAB - PDA)},$$

$$= \frac{b . \sin. (\delta + \epsilon)}{(a+b) . \sin. (\alpha - \delta)}.$$

Similarly, we should have

$$\frac{Q}{W} = \frac{a . \sin. (\delta + \epsilon)}{(a+b) . \sin. (\beta + \delta)}.$$

$$\text{Hence } \frac{P}{Q} = \frac{b . \sin. (\beta + \delta)}{a . \sin. (\alpha - \delta)}.$$

But the forces which draw the beam in the horizontal direction are the resolved parts of these tensions; that is, $P \sin. APM$, and $Q \sin. BQN$; $\therefore P \sin. APM = Q \sin. BQN$.

But $\sin. APM = \sin. (AMP + PAM) = \sin. (\epsilon + \alpha)$

$\sin. BQN = \sin. (ANQ - QBN) = \sin. (\epsilon - \beta)$;

$$\therefore \frac{P}{Q} = \frac{\sin. (\epsilon - \beta)}{\sin. (\epsilon + \alpha)};$$

$$\text{hence } \frac{b \sin.(\beta+\delta)}{a \sin.(\alpha-\delta)} = \frac{\sin.(\epsilon-\beta)}{\sin.(\epsilon+\alpha)} \ldots\ldots\ldots\ldots(3).$$

And the three equations (1), (2), (3), will give the three unknown quantities α, β, δ.

Cor. 1. If the center of gravity of PQ be in its middle point, which it will be if the beam be of uniform thickness and density, $a=b$; hence

$$\frac{P}{Q} = \frac{\sin.(\beta+\delta)}{\sin.(\alpha-\delta)} = \frac{\sin. BQP}{\sin. APQ},$$

or the tensions are inversely as the sines of the angles at P and Q.

Cor. 2. If A, B be in the same horizontal line, $\epsilon = \frac{\pi}{2}$, and equation (3) becomes

$$\frac{b \,.\, \sin.(\beta+\delta)}{a \,.\, \sin.(\alpha-\delta)} = \frac{\cos.\beta}{\cos.\alpha}.$$

Prob. XIII. *Fig.* 85. *A beam PQ is supported by strings which go over given pullies A, B and have given weights P and Q attached to them at p and q: to find its position.*

Let $PAB=\alpha$, $QBA=\beta$, and the rest of the notation as in the last Problem: the tensions of the strings Ap, Bq must be equal to the weights P, Q: hence, by the expressions there found for the tensions;

$$\frac{P}{W} = \frac{b}{(a+b)} \cdot \frac{\sin.(\delta+\epsilon)}{\sin.(\alpha-\delta)},$$

$$\frac{Q}{W} = \frac{a}{(a+b)} \cdot \frac{\sin.(\delta+\epsilon)}{\sin.(\beta+\delta)}.$$

Also, as before, the equation (3) of last Problem must be satisfied;

$$\therefore \frac{b \sin.(\beta+\delta)}{a \sin.(\alpha-\delta)} = \frac{\sin.(\epsilon-\beta)}{\sin.(\epsilon+\alpha)}:$$

from which three equations α, β, δ must be determined.

If a body be acted on by more than three forces in the same plane, we may suppose any two of them to be applied at their point of concourse. We may then suppose that at this point the resultant of the two forces is substituted for them: by this means the number of forces will be less by one than it was; and by successive operations of this kind we may reduce the forces to three, which is the case already considered.

6. *Stable and unstable Equilibrium.*

62. In some cases if a body be made to deviate slightly from the position of equilibrium, it has a tendency to return to it, in consequence of the action of the forces. In other cases if the position of the body be altered ever so little, it has a tendency to recede further and further from the position of equilibrium, and to assume some new position. In this latter case therefore the equilibrium would subsist only till some disturbing force, however slight, acted on the body; in the former case, if a slight disturbing force were to act, the body would come back to its position of equilibrium, and would rest there, if by any means the oscillatory motion, which would be produced by its returning, were put an end to.

The following problems will serve to illustrate this distinction.

PROB. XIV. *Fig.* 86. *A body, the lower surface of which is spherical, rests upon a sphere: to find in what case the equilibrium will be stable.*

In the position of equilibrium, the body must rest with its spherical surface touching the sphere at the highest point, and its center of gravity in the vertical line passing through the point of contact. Let A be this point, G the center of gravity, C the center of the sphere, and D the center of the spherical surface.

Let the body come into any other position touching the sphere in P, so that A, G come to A', G': the plane $PA'G'$ being vertical. Draw PR vertical, meeting $A'G'$ in R: and since the whole mass of the body may be supposed to be collected at G, it is manifest that if G' fall between R and A', the body will have a tendency to return to the position of equilibrium; and if G' fall beyond R, it will have a tendency to recede farther from it. Hence the equilibrium will be stable if $A'G'$ be less than $A'R$.

PA' is obviously equal to the arc PA, because, in moving from one position to the other, each point of PA' has been applied to each point of PA.

Hence,

$$\text{angle } PD'A' = \frac{\text{arc } A'P}{PD'} = \frac{AP}{AD}; \text{ and angle } ACP = \frac{AP}{AC}.$$

$$\text{Now } D'R : RP :: \sin. D'PR : \sin PD'R$$
$$:: \sin. PCA : \sin. PD'A'.$$

And when the angles become very small, the sines are as the angles;

$$\therefore D'R : RP :: PCA : PD'A' :: \frac{AP}{AC} : \frac{AP}{AD} :: AD : AC;$$

$$\therefore D'R + RP :: RP :: AC + AD : AC, \text{ ultimately.}$$

But ultimately, when the angle ACP is indefinitely diminished, $D'R + RP$ becomes $D'P$ or DA, and RP becomes RA';

$$\therefore DA : RA' :: AC + AD : AC; \therefore RA' = \frac{AD \cdot AC}{AC + AD},$$

and the equilibrium will be stable, if AG be less than this.

If AC be infinite, we have the case of a body with a spherical surface resting on a horizontal plane, and the equilibrium will be stable if AG be less than AD.

If AD be infinite, we have the case of a body with its lower surface plane, resting upon a sphere; and the equilibrium will be stable if AG be less than AC.

If the body be a hemisphere, $AG = \frac{5}{8} AD$, (Chap. 8.). Hence the equilibrium will be stable, if

$$\frac{5}{8} AD < \frac{AD \cdot AC}{AD + AC};$$
$$\text{if } 5AD + 5AC < 8AC;$$
$$\text{if } AD < \tfrac{3}{5} AC.$$

If the body rest on the concave surface of a sphere, we shall find in the same manner that the equilibrium will be stable, if

$$AG < \frac{AD \cdot AC}{AC - AD}.$$

If the lower part of the body and the surface upon which it rests, instead of being circular, have any other curvilinear form, the stability of the equilibrium will be the same as if the surfaces were both spherical, with radii equal respectively to the radius of curvature of the body and of the surface at the point of contact.

PROB. XV. *A homogeneous elliptical spheroid rests on its smaller end in a concave hemisphere; to find what the radius of the hemisphere must be that the equilibrium may be stable.*

Let the radius of the hemisphere $= c$; and let a, b, be the semi-axes major and minor of the ellipse. Then, by conics, the radius of curvature at the extremity of the major-axis is $\frac{b^2}{a}$; which must be put for AD in the formula. Also the center of gravity is at the center of the ellipse: hence AG is $= a$. And the equilibrium will be stable, if

$$a < \frac{\frac{b^2}{a} \cdot c}{c - \frac{b^2}{a}}; \text{ or if } a < \frac{b^2 c}{ac - b^2};$$

$$\text{if } a^2 c - ab^2 < b^2 c, \text{ or if } c < \frac{ab^2}{a^2 - b^2}, \text{ or } c < \frac{\frac{b^2}{a}}{1 - \frac{b^2}{a^2}}.$$

Also, that the spheroid may be within the hemisphere, the radius must be greater than the radius of curvature of the ellipse at the point of contact. Therefore

$$c > \frac{b^2}{a}.$$

$$\text{Let } b = \tfrac{1}{2}a; \therefore c > \frac{a}{4}, \text{ and } < \frac{a}{3}.$$

7. *The Equilibrium of Roofs.*

63. We shall consider a few Problems relating to the subject of the pressure or *thrust* which beams, combined so as to support themselves and other weights, exert in the direction of their length. The consideration of the strength of such structures, requires also an examination of the force which tends to produce fracture, and of the power which different materials and different forms have to resist this tendency; but this part of the subject does not belong to our present investigation.

PROP. *Fig.* 87. *A roof ACA', consisting of beams forming an isosceles triangle with its base horizontal, supports a given weight at C: the weights of the beams being also given, it is required to find the horizontal force at A and A'.*

Let G be the center of gravity of AC, and Gg a vertical line: and let Cg be the direction of the force at C, arising both from the weight at C, and from the beam $A'C$. Then Ag must be the direction of the force exerted at A; for it is requisite that the three forces which support the beam AC should meet in the same point.

In the same manner if G' be the center of gravity of $A'C$, and $G'g'$ vertical, Cg' and $A'g'$ will be the directions of the forces at C and A'; and if the beams AC, $A'C$ be exactly similar, gg' will be horizontal: and if Ag and $A'g'$ be produced, they will meet in N, a point in the vertical line NC.

NC, Cg, gN, which are in the directions of the forces which support the beam AC, are therefore as these forces. In the same way NC, Cg', $g'N$ are as the forces which act on BC. Hence the weight at C is supported by the two re-actions $gC, g'C$. Let gg' meet NC in M, and the two forces gC, $g'C$, are equivalent to a vertical force $2MC$. Also the force at A being represented by gN, the horizontal part of it is represented by gM. Hence NC representing the weight of the beam AC, $2CM$ represents the weight at C, and Mg represents the horizontal force at A or A', which stretches the beam AA'.

Let G bisect AC; $\therefore Gg = \frac{1}{2}CN$. Hence, if we bisect CN in O, $CO = Gg$, and gO will be parallel to AC. And by what has been said, if B be the weight of the beam AC, C the weight at C, and H the horizontal pressure at A,

$$\frac{H}{B+C} = \frac{Mg}{CN+2CM} = \frac{Mg}{2CO+2CM} = \frac{Mg}{2MO} = \frac{AD}{2DC};$$

by similar triangles.

If therefore a be the tangent of the angle which AC makes with the horizon,

$$\frac{H}{B+C} = \frac{1}{2 \tan. a}; \quad H = \frac{B+C}{2 \tan. a}.$$

If the beam AA' were not there, this horizontal pressure H must be counteracted by the supports on which the ends A, A', were placed.

If the roof ACA' support a covering of uniform thickness, the formula will still be true, including in the weight of B, the weight of that portion of the covering which rests upon the beam.

The weight C, at the point C, may arise from a longitudinal beam perpendicular to the plane $AA'C$.

64. Prop. *Any number of given beams, arranged as sides of a polygon, in a vertical plane, support each other, and support also given weights at the angles; it is required to find the horizontal pressure at the points of support.*

Let AC, fig. 88, be any one of the beams; and, G being its center of gravity, let Gg be a vertical line. Then the pressures at A and C will converge to some point in Gg, as g; and their directions will be Ag, Cg. Produce Ag, meeting in N the vertical through C. And since the beam AC is supported by three pressures in directions NC, Cg, gN, those forces are as these lines. Hence, NC representing the weight of the beam, Ng and gC represent its re-action at A and C. Also Ng is equivalent to NM, Mg, and gC to gM, MC. Hence Mg represents the horizontal pressure of the beam at A, and gM the equal horizontal pressure at C. NM its vertical pressure downwards at A, and MC its vertical pressure upwards at C.

Let O bisect CN; and suppose G to bisect AC; then $CO = Gg$, and therefore gO is parallel to GC; and OgM will be the angle which AC makes with the horizon: let this be called a, and let H be the horizontal pressure at A or C, and B the weight of the beam,

$$\frac{\text{pressure downwards at } A}{H} = \frac{NM}{Mg} = \frac{MO}{Mg} + \frac{ON}{Mg}$$

$$= \text{tan. } a + \frac{\frac{1}{2} B}{H};$$

$\therefore$ pressure downwards at $A = H \text{ tan. } a + \frac{1}{2} B$.

Similarly,

$$\frac{\text{pressure upwards at } C}{H} = \frac{MC}{Mg} = \frac{MO}{Mg} - \frac{OC}{Mg}$$

$$= \text{tan. } a - \frac{\frac{1}{2} B}{H};$$

$\therefore$ pressure upwards at $C = H \text{ tan. } a - \frac{1}{2} B$.

In the same manner we should find, calling the weight of $CD = B_1$, and the angle which it makes with the horizon $= a_1$;

pressure downwards at $C = H \text{ tan. } a_1 + \frac{1}{2} B_1$;

pressure upwards at $D = H \text{ tan. } a_1 - \frac{1}{2} B_1$;

and similarly for the other angles.

Now the pressure upwards at C must support the pressure downwards at C, together with the weight at C. Calling this weight C, we have

$$H \text{ tan. } a - \frac{1}{2} B = H \text{ tan. } a_1 + \frac{1}{2} B_1 + C;$$

$$\therefore H (\text{tan. } a - \text{tan. } a_1) = \frac{1}{2} (B + B_1) + C;$$

$$\therefore H = \frac{\frac{1}{2} (B + B_1) + C}{\text{tan. } a - \text{tan. } a_1},$$

whence the horizontal pressure is known.

It appears from the proof that the horizontal pressure is the same at each angle.

Cor. 1. If we suppose the weights of the beams $=0$, we have

$$H=\frac{C}{\tan. a-\tan. a_1}.$$

Cor. 2. If we suppose no weights except the beams, we have

$$H=\frac{\frac{1}{2}(B+B_1)}{\tan. a-\tan. a_1}.$$

65. Prob. *To find the positions of the beams, having given their weights B_1, B_2, B_3, &c. the weights C_1, C_2, &c. and the position of two of them.*

By the last Proposition, we have the following equations; a_1, a_2, a_3, &c. being the angles which the beams make with the horizon,

$$H(\tan. a_1-\tan. a_2)=\tfrac{1}{2}(B_1+B_2)+C_1,$$
$$H(\tan. a_2-\tan. a_3)=\tfrac{1}{2}(B_2+B_3)+C_2,$$
$$\&c.=\&c.$$

If there be n beams, there will be $n-1$ weights C_1, C_2, &c. and $n-1$ equations. The number of unknown quantities is $n+1$; viz. the n tangents $\tan. a_1$, $\tan. a_2$, &c. and the pressure H. Hence if we know two of the angles a_1, a_2, &c. we can find the rest.

In this investigation, if any one of the beams have its farther end (beginning from A) lower than the other, it makes an angle below the horizon, and the corresponding value of a will be negative.

Cor. 1. If the weights of the beams be 0, we shall have

$$H=\frac{C_1}{\tan. a_1-\tan. a_2}=\frac{C_2}{\tan. a_2-\tan. a_3}=\&c.$$

Hence it appears that the weights C_1, C_2, &c. are as $\tan. a_1-\tan. a_2$, $\tan. a_2-\tan. a_3$, &c.; which agrees with the proportion of the weights on a funicular polygon (Art. 32. p. 30.); as it should do. For if each side of the funicular polygon were supposed to be rigid, and if the polygon were inverted, so that the vertical lines should remain vertical, the angles being upwards, it is clear that all the forces would act in the directions opposite to their former directions, and the equilibrium would continue to subsist.

Cor. 2. If we suppose the weights C_1, C_2, &c. to be each 0, we have

$$H = \frac{\frac{1}{2}(B_1 + B_2)}{\tan. a_1 - \tan. a_2} = \frac{\frac{1}{2}(B_2 + B_3)}{\tan. a_2 - \tan. a_3} = \&c.$$

Hence $\frac{1}{2}(B_1 + B_2)$, $\frac{1}{2}(B_2 + B_3)$, &c., are as the differences of the tangents of the angles which the beams make with the horizon.

Cor. 3. If the positions of the beams be all unknown, their lengths b_1, b_2, b_3, &c., and the positions of the extreme points being given, we shall have, in addition to the above $n - 1$ equations, these two, from which we must determine the $n + 1$ unknown quantities,

$$b_1 \cos. a_1 + b_2 \cos. a_2 + b_3 \cos. a_3 + \&c. = h,$$
$$b_1 \sin. a_1 + b_2 \sin. a_2 + b_3 \sin. a_3 + \&c. = k;$$

h and k being the horizontal and vertical co-ordinates AH, HB of B measured from A. For it is easily seen that the part of AH which corresponds to b_2 is $b_2 \cos. a_2$, and so of the rest.

66. Prop. *Four beams of equal length and weight are to be placed with their extreme points in the same horizontal line, so that they, with the horizontal line, may form an irregular pentagon, and may balance each other: to find their position, fig.* 89.

It is manifest that they must be placed so that the two halves of the pentagon are symmetrical. Let a, a_1 be the two angles made by the lower and upper beam on one side with the horizon. Then, by Prop. 64,

$$H(\tan. a - \tan. a_1) = B.$$

And there is no weight at the highest point, therefore the pressure upwards at that point must $= 0$. Therefore, by Prop. 62, also,

$$H \tan. a_1 - \tfrac{1}{2} B = 0, \text{ or } H \tan. a_1 = \tfrac{1}{2} B.$$

Dividing the former equation by this, $\frac{\tan. a}{\tan. a_1} - 1 = 2$;

$$\tan. a = 3 \tan. a_1.$$

If the extremities A, A', and the vertex B be given, we may find the figure by the following construction.

BH, perpendicular to AA', bisects it. Bisect AH in E, and erect EF perpendicular to AH, meeting in F the circle passing through A, B, A'. Join AF, and this will be the position of the lower beam; and if we take BG equal to AF, meeting AF in C, ACB will be the position of the two beams on one side of DH; and $A'C'B$, their position on the other side, will be a figure exactly similar to ACB.

For supposing BC' to meet the circle in G'.

$$\begin{aligned} AG' &= A'B - BG' \\ &= AB - BG \\ &= AB - AF = BF; \end{aligned}$$

$\therefore$ BG' is parallel to AF;

$\therefore$ angle $BCK = BC'K = FA'E$.

Hence

$$\tan.\ CAD = \frac{FE}{EA} = \frac{3\,FE}{EA'} = 3 \tan.\ FA'E = 3 \tan.\ BCK.$$

Cor. If the beams be not in the position of equilibrium, there will be a horizontal pressure, which may be resisted by a horizontal beam CC', fastened at C and C'.

If it be not resisted, the beams will fall, B descending or ascending as it is too low or too high. The position is one of unstable equilibrium.

8. *The Equilibrium of Arches.*

67. Suppose a number of bodies of the form of wedges with the points truncated, as C_3, C_2, C_1, C, c, c_1, c_2, c_3, fig. 90, to be arranged with their lateral planes PQ, P_1Q_1, &c. in contact, and all perpendicular to the same vertical plane, which may be supposed to be the plane of the paper. Let these wedges be pressed downwards by their gravity or any other forces: then, if their

lateral planes be supposed perfectly smooth, they will have a tendency to slide past each other in consequence of the action of these forces; and if their efforts do not balance each other, (and if the friction be not considered) those which have the stronger tendency will descend, pushing up the others out of their places. But it is possible so to adjust the magnitudes and forms of these wedges that they shall exactly balance, and that the combination shall remain supported in its present situation by the mutual action of its parts. In this case it is called an *Arch*. The wedges C, C_1, C_2, &c. are called *Voussoirs:* and the voussoir which is at the top or *Crown* of the arch, is called the *Key-Stone*. The surfaces PQ, P_1Q_1, P_2Q_2, &c. which seperate the voussoirs, are called *Joints*.

68. Prop. *It is required to find what must be the proportion of the weights of the voussoirs that there may be an equilibrium.*

In order that there may be an equilibrium, each voussoir must be kept at rest by the forces which act upon it. Now these are, besides its own weight, the pressures of the two voussoirs with which it is in contact on each side. These pressures are necessarily perpendicular to the surfaces which act on each other. At each joint the pressure may be supposed to act over the whole surface in contact, but it will be equivalent to a single force, acting at a certain point of the surface. The point of application of this force is determined by the condition that the forces at two successive joints must meet in the vertical passing through the center of gravity of the voussoir which is between them. (Art. 56.)

Now since the body C_2 is kept at rest by three forces, (its weight and the two pressures,) these forces must have the same proportion as if they acted on a point. Hence they will be as the sides of a triangle which are perpendicular to their directions, (Art. 28.) Let OT be in the line PQ or parallel to it, and therefore perpendicular to the pressure on PQ; OT_1 parallel to P_1Q_1, and therefore perpendicular to the pressure on P_1Q_1; TT_1 horizontal, and therefore perpendicular to the direction of gravity. The triangle OTT_1 will therefore have its sides as the three forces; hence

$$\frac{\text{weight of } C_1}{\text{pressure on } P_1Q_1} = \frac{TT_1}{OT_1}.$$

Similarly if OT_2, parallel to P_2Q_2, meet the horizontal line TT_1 in T_2, the sides of the triangle OT_1T_2 will be as the forces which act on M_2; hence

$$\frac{\text{pressure on } P_1Q_1}{\text{weight of } C_2} = \frac{OT_1}{T_1T_2}.$$

But the pressures at the joint P_1Q_1 on M_1 and on M_2, arising from the action and re-action of the voussoirs, are equal. Hence, multiplying the above equations,

$$\frac{\text{weight of } C_1}{\text{weight of } C_2} = \frac{TT_1}{T_1T_2}.$$

Similarly, if OT_3 be parallel to P_3Q_3, another joint, we have

$$\frac{\text{weight of } C_1}{\text{weight of } C_2} = \frac{TT_1}{T_1T_2}, \text{ and } \frac{\text{weight of } C_2}{\text{weight of } C_3} = \frac{T_1T_2}{T_2T_3}.$$

Hence *the weights of the voussoirs are as the portions* TT_1, T_1T_2, T_2T_3 *of a horizontal line, which are intercepted by lines drawn from any point O parallel to the joints.*

COR. 1. If we draw a vertical line OX meeting the horizontal line in X, OX being made radius, XT, XT_1, XT_2, XT_3, are, to radius OX, the tangents of the angles which the joints PQ, P_1Q_1, P_2Q_2, P_3Q_3 make with the vertical. Hence we have this theorem.

In an arch which is in equilibrium, the weights of the voussoirs are as the differences of the tangents of the angles which their joints make with the vertical.

COR. 2. This agrees with what was proved of a system of beams, Art. 63, Cor. 2. For suppose each beam in fig. 88 to be bisected, and suppose the two halves contiguous to C to be collected at C, the two halves contiguous to D to be collected at D, and so on. And instead of a line CD, suppose a joint perpendicular to CD; then the pressure on this joint will be in the direction of CD, and therefore the equilibrium will subsist if we consider the system as an arch, with weights $\frac{1}{2}(B_1 + B_2)$, $\frac{1}{2}(B_2 + B_3)$, &c., at the points C, D, &c. But these weights are as $\tan. \alpha_1 - \tan. \alpha_2$, $\tan. \alpha_2 - \tan. \alpha_3$, &c.: and α_1, α_2, &c., the angles made by the beams with the horizon, are the angles made by the joints perpendicular to them with the vertical. Hence this agrees with last Corollary.

Cor. 3. Let the pressure at any joint, as P_1Q_1, be resolved into forces parallel and perpendicular to the horizon; and since the pressure and its resolved parts are perpendicular respectively to OT_1, OX, XT_1, these forces will be as these lines. Hence the horizontal force is represented by OX, and is the same at each joint.

If H be the horizontal pressure at DE, H is the horizontal pressure at each joint.

Cor. 4. The pressures at the joints are as OT, OT_1, &c.; hence it appears that the pressures are as the secants of the angles which the joints make with the vertical.

If θ be the angle of any joint with the vertical, H sec. θ is the pressure at that joint.

Cor. 5. Since the weights of the voussoirs C, C_1, C_2, C_3, &c. are as XT, TT_1, T_1T_2, &c.; the line XT_2 will represent the whole weight of the mass between DE and P_2Q_2, and similarly for any other joint. Hence the weight of any portion, beginning from the vertex, is as the tangent of the angle made by the joint which bounds it with the vertical.

H tan. θ is the weight of any portion.

Cor. 6. If the joint be horizontal, the weight of the arch must become infinite, in order that it may be exactly in equilibrium. The conclusions here obtained are greatly modified by introducing the continuation of friction, as will be seen hereafter.

The *pier* or *abutment*, is the solid mass which forms the lower part of an arch on each side, and on which the lowest voussoirs rest.

69. Prop. *To find the horizontal pressure exerted on the pier of an arch.*

The pressure at the surface RS, fig. 90, acts at all its points; but it is equivalent to a single force acting at a certain point. Let this point be A, and let the force act in the direction KA. Also the pressure at the vertical joint DE is equivalent to a horizontal force acting at a certain point B: (B may be supposed to be in the middle of DE.) Let KH be the vertical line (passing through the center of gravity of the half arch) in which the weight of the half arch acts. Then the weight of the half arch acting in the line KH is supported by the two pressures in AK and BK. And if AH be horizontal, the forces are as KH, AK and HA. And KH representing the weight A of the half arch, KA will represent the pressure at A, and therefore HA the part of it which acts in a horizontal direction. Hence

$$\text{horizontal pressure at } A = A \cdot \frac{HA}{KH}.$$

This is true whether friction act or not; if the half arch be supported by a pressure in the direction AK.

If the joint RS be perfectly smooth, and make an angle β with the vertical, the pressure will be perpendicular to the surface RS;

$$\text{and horizontal pressure} = A \frac{HA}{KA} = A \cos. \beta.$$

70. PROP. *To find the pressure exerted to overturn the pier of an arch.*

If the pier AF be overturned by the pressure of the arch, it will turn about the point F of its base. The force to overturn it is the pressure (see last Article) acting at A. This pressure may be supposed to act at L, in the direction AL. Let it be resolved in AN, NL; the former part will be A the weight; the latter force, in NL, will not tend to turn the pier round F. Hence the force to overturn the pier is A, acting perpendicularly at an arm FL.

The force which opposes this is the weight of the pier, which may be supposed to be collected at its center of gravity G, and to act in the vertical line GM. Let B be the weight of the pier; then, in order that it may stand, we must have $B \cdot FM > A \cdot FL$,

$$\text{also } FL = NL - NF.$$

If we draw AH horizontal, meeting the vertical line KH, $NL = AN \cdot \frac{AH}{KH}$.

By introducing this value for NL, the expression for the force becomes independent of the angle which RS makes with the horizon, provided we suppose that no sliding can take place.

71. Prop. *The line of pressure must not fall without the voussoirs.*

If, instead of supposing the pressure to be distributed over the surface of each joint, we suppose an equivalent pressure to act at a single point of the surface in the same direction, the polygon formed by all the lines in which these forces act is called here *the line of pressure*.

Each voussoir, and the extraneous materials which act upon it, produce a vertical force which acts in a vertical line passing through the voussoir. Let ZC, Z_1C_1, Z_2C_2, &c. be these lines. Now in order that each voussoir may be kept at rest, the three forces which act upon it must meet in one point (Art. 56.). Hence we shall obtain the lines in which the pressures at each joint must act, in the following manner. Let B be the point in DE at which the pressure may be supposed to act. Draw BC horizontal, meeting the first vertical line in C. Draw CC_1 perpendicular to the joint PQ, meeting the next vertical line in C_1. Draw C_1C_2 perpendicular to P_1Q_1, meeting the next vertical line in C_2; and so on. Then BCC_1C_2, &c., is the line of pressure. And at any joint, as for instance, P_1Q_1, the pressures at different points of the surface P_1Q_1 must be such that their resultant may be in the line C_1C_2. And this is impossible if the line C_1C_2 do not fall between P_2 and Q_2. Hence the line of pressure must every where fall within the voussoirs.

This will be the case, when the voussoirs are small, if the lower surfaces of the voussoirs be perpendicular to the joints, and if the vertical forces pass through the centers of gravity of the voussoirs.

If the first condition of the equilibrium of an arch, (Art. 68.) be not satisfied, the voussoirs will tend to slide past one another. If the second condition (contained in this Article) be not satisfied, the voussoirs will turn round the inner or outer edges of the joints.

In the proofs of the preceding Propositions we have supposed a joint at the highest point of the arch D. In general there is not such a joint; but the reasoning is the same as if there were, because the line of pressure will there be necessarily perpendicular to the vertical line DE.

72. Prob. *The intrados being a circle with the joints in the direction of the radii, to find the extrados so that the voussoirs may be in equilibrium.*

The *intrados* is the curve which bounds the arch internally, as DP; the *extrados* is the curve which bounds it externally, as EQ, fig. 91.

Let P be any point of the intrados, O its center; POP' a small angle $=\phi$; and let the whole arch be made up of voussoirs, as $PQQ'P$, the angle of each being $=\phi$. Let there be n of these between DO and PO; therefore $DOP = n\phi = \theta$, $DOP' = (n+1)\phi$. Also let $OD = OP = l$, $OQ = r$, $OE = k$. And if, with center O and radius OQ, we describe an arc Qq, the area $PQqP' = \frac{1}{2}(r^2 - l^2)\phi$. Also if we take $DOD' = \phi$, and describe Ee with radius OE, the area $DEeD' = \frac{1}{2}(k^2 - l^2)\phi$. And we have, by Art. 66, considering $PQqP'$, $DEeD'$ as voussoirs;

$$\frac{\frac{1}{2}(r^2 - l^2)\phi}{\frac{1}{2}(k^2 - l^2)\phi} = \frac{\tan.\,(n+1)\phi - \tan.\,n\phi}{\tan.\,\phi};$$

$$\therefore \frac{r^2 - l^2}{k^2 - l^2} = \frac{\sin.\,\phi}{\tan.\phi\,.\,\cos.(n+1)\phi\,.\,\cos.\,n\phi} = \frac{\cos.\,\phi}{\cos.(n+1)\phi\,.\,\cos.n\phi}.$$

Now if we make ϕ indefinitely small, $DEeD'$, $PQqP'$ will approach indefinitely near to the portions $DEE'D'$, $PQQ'P'$, of the area contained between the curves. But in this case, $n\phi$ and $(n+1)\phi$ are indefinitely near to equality, and each equal to $DOP = \theta$; also $\cos.\,\phi$ approaches to 1. Hence we shall have

$$\frac{r^2 - l^2}{k^2 - l^2} = \frac{1}{\cos.^2\theta};$$

$$\therefore r^2 = l^2 + (k^2 - l^2)\sec.^2\theta.$$

Cor. 1. We have the following construction:

Make OR horizontal, $RF = OE$, FG horizontal. Let OP meet FG in S; draw ST vertical, and take $OQ = ET$; the *locus* of Q will be the extrados.

For $OF^2 = RF^2 - RO^2 = k^2 - l^2$; therefore $FS^2 = (k^2 - l^2)$ tan.$^2\,\theta$ and $ET^2 = EO^2 + OT^2 = EO^2 + FS^2 = k^2 + (k^2 - l^2)$ tan.$^2\theta$ $= l^2 + (k^2 - l^2)$ sec.$^2\,\theta = r^2$.

Cor. 2. ET is always greater than FT or OS; hence P is always above FG; and the extrados has FG for an asymptote.

Prob. *To find the conditions of the equilibrium of voussoirs terminated by a horizontal line above and below, as in fig.* 92.

The weights of the voussoirs must be as the differences of the tangents of the angles: and the weight of each, as PQQ_1P_1, will be as the surface PQQ_1P_1, &c. Now if ODE be vertical, the surface $PQQ_1P_1 = \frac{1}{2}DE\,(PP_1 + QQ_1)$. And $QQ_1 = PP_1 \,.\, \dfrac{OQ}{OP}$;

$$\therefore\ PQQ_1P_1 = \tfrac{1}{2}DE\,.\,PP_1\left(1 + \frac{OQ}{OP}\right).$$

Also the difference of the tangents is as $\dfrac{P_1D}{DO} - \dfrac{PD}{DO} = \dfrac{PP_1}{DO}$;

$$\therefore \frac{PP_1}{DO} \propto \tfrac{1}{2}DE\,.\,PP_1\left(1 + \frac{OQ}{OP}\right).$$

And since DE is the same for all the voussoirs,

$$\frac{1}{DO} \propto 1 + \frac{OQ}{OP} \propto 1 + \frac{OE}{OD} \propto 1 + \frac{OD + DE}{OD};$$

$$\therefore \propto 2 + \frac{DE}{OD},$$

$$\text{or } 1 \propto 2OD + DE;$$

and since DE is constant, OD is constant.

Hence it appears that the point O is constant, and all the joints must converge to the same point. If this be the case the weights of the voussoirs will be such as to produce equilibrium.

It is also requisite that the line of pressure DCC_1C_2A should cut the joints within the limits of the voussoirs.

The preceding are the mathematical results of the problem of the equilibrium of an arch, when it is required that it shall not fall in consequence of the *voussoirs sliding* along each other. As however, in point of fact, an arch never does fall in this manner, the preceding theory cannot be considered as a foundation for practical rules. For some observations on the mode in which we may consider cases more approaching those which really occur, the student is referred to the appendix on friction.

CHAP. VI.

THE CONDITIONS OF EQUILIBRIUM OF A POINT.

73. In this and the following Chapter we shall express the conditions which are requisite that a point or a body may be in equilibrium, by means of equations among the symbols which the forces and their positions introduce; and we shall thus obtain the means of reducing to the solution of equations, all problems whatever relative to equilibrium.

Prop. *To find the resultant of two forces acting at a point, as AP, AQ, fig.* 94.

If we suppose a line, as Ax, the position of which is known, to pass through the point A, we may determine the positions, both of the components, and of the resultant, by the angles which they make with this line.

Let p, q, be the forces in AP, AQ; α, β, the angles which they make with Ax. If p be resolved into two forces, one in the direction Ax, and the other in the direction Ay perpendicular to Ax, it is easily seen that these resolved parts will be $p \cos. \alpha$, $p \sin. \alpha$. In the same manner q is equivalent to forces $q \cos. \beta$ in the direction Ax, and $q \sin. \beta$ in the direction Ay. Hence the forces p, q are equivalent to

$$p \cos. \alpha,\ q \cos. \beta \text{ in } Ax,$$
$$p \sin. \alpha,\ q \sin. \beta \text{ in } Ay.$$

And the resultant of p and q will be the resultant of these four forces. If we put

$$p \cos. \alpha + q \cos. \beta = X,$$
$$p \sin. \alpha + q \sin. \beta = Y;$$

and take in Ax, Ay, $AM=X$, $AN=Y$, and complete the rectangle $AMRN$, AR will be the resultant of p and q. And if r be this resultant, and θ the angle which it makes with Ax, we have

$$r=\sqrt{(X^2+Y^2)},\ \tan.\ \theta=\frac{Y}{X},$$

whence the magnitude and position of the resultant are known.

Cor. 1. By putting the values of X and Y in the expression for r, we find

$$r=\sqrt{\begin{Bmatrix}p^2\cos.^2\alpha+2pq\cos.\alpha\cos.\beta+q^2\cos.^2\beta\\+p^2\sin.^2\alpha+2pq\sin.\alpha\sin.\beta+q^2\sin.^2\beta\end{Bmatrix}},$$

and since $\cos.^2\alpha+\sin.^2\alpha=1$;

and $\cos.\alpha\cos.\beta+\sin.\alpha\sin.\beta=\cos.(\alpha-\beta)$,

$$r=\sqrt{\{p^2+2pq\cos.(\alpha-\beta)+q^2\}}.$$

Cor. 2. This agrees with the result obtained in Chap. ii.; for if AR be found by completing the parallelogram $APRQ$, we shall have

$$AR^2=AP^2+PR^2+2AP.PR.\cos.RPE,$$

$$\text{or } =p^2+q^2+2pq\cos.(\alpha-\beta),$$

because $RPE=QAP=PAx-QAx$.

Cor. 3. If we call the angles PAR and QAR, ϕ and ψ respectively, we shall have

$$\frac{\sin.PAR}{\sin.APR}=\frac{PR}{AR},\ \text{or}$$

$$\frac{\sin.PAR}{\sin.EPR}=\frac{AQ}{AR};$$

$$\therefore\ \frac{\sin.\phi}{\sin.(\alpha-\beta)}=\frac{q}{r},$$

$$\sin.\phi=\frac{q\sin.(\alpha-\beta)}{r}=\frac{q\sin.(\alpha-\beta)}{\sqrt{[p^2+2pq\cos.(\alpha-\beta)+q^2]}}.$$

Similarly $\sin.\psi=\dfrac{p\sin.(\alpha-\beta)}{r}=\dfrac{p\sin.(\alpha-\beta)}{\sqrt{[p^2+2pq\cos.(\alpha-\beta)+q^2]}}$.

74. Prop. *To find the resultant of any number of forces $p_1, p_2, p_3 \dots . p_n$ in the same plane; their directions making with the line Ax, angles $a_1, a_2, a_3, \dots . a_n$ respectively.*

As in last Article, Ay being perpendicular to Ax, the forces may be shewn to be equivalent to

$$p_1 \cos. a_1, p_2 \cos. a_2, p_3 \cos. a_3 \dots p_n \cos. a_n \text{ in direction } Ax,$$
$$p_1 \sin. a_1, p_2 \sin. a_2, p_3 \sin. a_3 \dots p_n \sin. a_n \text{ in direction } Ay.$$

Hence, if r be the resultant, and θ the angle which it makes with Ax, r and θ will be given by the equations

$$p_1 \cos. a_1 + p_2 \cos. a_2 + p_3 \cos. a_3 \dots + p_n \cos. a_n = X,$$
$$p_1 \sin. a_1 + p_2 \sin. a_2 + p_3 \sin. a_3 \dots + p_n \sin. a_n = Y,$$
$$r = \sqrt{(X^2 + Y^2)}; \ \tan. \theta = \frac{Y}{X}.$$

We have considered the forces as lying within the angle yAx and *pulling* the body. In this case the resolved parts will be in the directions Ax and Ay; but if one of the forces act in the direction AP', fig. 95, situated in the angle yAx', the resolved part AM' will act in the direction xA, and the corresponding term in the sum $p_1 \cos. a_1 + p_2 \cos. a_2 +$ &c. should be negative. And if $p' \cos. a'$ be this term, it will be negative, because $a' = P'Ax$, and $p' \cos. a' = p' \cos. P'Ax = -p' \cos. P'Ax'$, which is a negative quantity. In this case $p' \sin. a'$ will be positive, which agrees with the direction of the resolved force $M'P'$.

In the same manner, if a force p'' act in the direction AP'', in the quadrant $y'Ax$, the term $p'' \sin. a''$ will be negative, and $p'' \cos. a''$ will be positive.

And if a force p''' act in the direction AP''' in the quadrant $y'Ax'$, the terms $p''' \cos. a'''$, $p''' \sin. a'''$, will both be negative. And these changes of sign agree with the changes of direction of the resolved parts.

Also if the force, instead of being a *pulling* force in the direction AP, be a *pushing* force in the direction PA, we must make p negative; and the resolved parts $p \cos. a$ and $p \sin. a$ will both be negative. In the same manner if the force in $P'A$ be a pushing force, we must make p' negative. And similarly in the other quadrants.

75. PROP. *To find the resultant of forces whose directions are not all in the same plane.*

We have in the preceding case resolved forces in the directions of two lines at right angles to each other. In this case we shall resolve them in the directions of three lines, each at right angles to the other two. The nature of space admits of three such lines, or *axes*, and no more. Let xAy, fig. 96, be conceived to be a horizontal plane, in which Ax and Ay are at right angles; and let Az be vertical. Then Ax, Ay, Az are all at right angles to each other; and the planes xAy, xAz, yAz are also at right angles to each other. For (Euc. XI. Def. 6.), yAx measures the inclination of zAy, zAx. And similarly of the others.

Let P be any point in space; and through P let three planes be drawn, $PmOn$, $PoNm$, $PoMn$, parallel respectively to xAy, xAz, yAz. Hence Mm will be a rectangular parallelopiped; and therefore the plane nMo is perpendicular to AMo, AMn. Therefore AM is perpendicular to the plane nMo (Euc. XI. 19.), and therefore to the line PM (Euc. XI. 4.).

If AP represent any force acting at A, AP may be resolved into forces represented by AM, MP. Also MP may be resolved into Mo, oP; and hence the force AP is equivalent to AM, Mo, oP; or to AM, AN, AO.

Since PM is perpendicular to AM, $AM = AP \,.\, \cos.\, PAx$. And similarly $AN = AP \,.\, \cos.\, PAy$ and $AO = AP \,.\, \cos.\, PAz$. Hence if p be the force AP, and α, β, γ, the angles which it makes with Ax, Ay, Az, the force will be equivalent to three forces

$$p \cos.\, \alpha \text{ in } Ax,\quad p \cos.\, \beta \text{ in } Ay,\quad p \cos.\, \gamma \text{ in } Az^*.$$

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* *Two* of the angles α, β, γ are sufficient to determine the position of the line AP, for they are connected by the equation

$$\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma = 1;$$

so that two of them being known, the third may be found.

This appears thus;

$$\begin{aligned} AP^2 &= AM^2 + MP^2 = AM^2 + Mo + oP^2 \\ &= AM^2 + AN^2 + PO^2 \\ &= AP^2 \cos.^2 \alpha + AP^2 \cos.^2 \beta + AP^2 \cos.^2 \gamma; \end{aligned}$$

$$\therefore\ 1 = \cos.\, \alpha^2 + \cos.^2 \beta + \cos.^2 \gamma.$$

Hence if we have forces $p_1, p_2, p_3, \dots p_n$, acting at a point A

making with Ax_1 angles $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n$;

with Ay_1 angles $\beta_1, \beta_2, \beta_3, \dots \beta_n$;

with Az_1 angles $\gamma_1, \gamma_2, \gamma_3, \dots \gamma_n$;

and if we make

$$p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 \dots + p_n \cos. \alpha_n = X;$$

$$p_1 \cos. \beta_1 + p_2 \cos. \beta_2 + p_3 \cos. \beta_3 \dots + p_n \cos. \beta_n = Y;$$

$$p_1 \cos. \gamma_1 + p_2 \cos. \gamma_2 + p_3 \cos. \gamma_3 \dots + p_n \cos. \gamma_n = Z;$$

the forces will be equivalent to X in Ax, Y in Ay, and Z in Az.

If R be the resultant, and θ, η, ζ the angles which it makes with Ax, Ay, Az respectively, we shall have

$$R = \sqrt{(X^2 + Y^2 + Z^2)},$$

$$\cos. \theta = \frac{X}{R}, \quad \cos. \eta = \frac{Y}{R}, \quad \cos. \zeta = \frac{Z}{R}.$$

For if AM, AN, AO now represent X, Y, Z, AP will represent R; and $AP^2 = AM^2 + AN^2 + AO^2$ (see note last page).

Also $AM = AP \cos. PAM$, &c.

One of the three last equations is superfluous, as was observed before.

As in last Article, the resolved forces may become negative when the angles α_1, β_1, γ_1, &c. pass beyond the first quadrant. Also the forces are negative when they push instead of pulling.

76. Prop. *When a point is acted upon by any forces, to find the conditions of equilibrium*.*

* The conditions of the equilibrium of any number of points may be deduced from the conditions belonging to one point. In the state of equilibrium, each point, by means of the rods, strings, &c. which connect it with the other points, exerts and suffers a certain pressure. And this pressure may be introduced as one of the forces at each point, and then eliminated by considering that it is equal at each two points so connected.

Any

In order that there may be an equilibrium, the resultant of all the forces must be 0. And in order that this may be the case it is evident that we must have, in Art. 74, $X=0$, $Y=0$; and, in Art. 75, $X=0$, $Y=0$, $Z=0$. Hence we have for the conditions of equilibrium in the former case,

$$p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 + \ldots = 0;$$

$$p_1 \sin. \alpha_1 + p_2 \sin. \alpha_2 + p_3 \sin. \alpha_3 + \ldots = 0.$$

And in the latter case

$$p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + p_3 \cos. \alpha_3 + \ldots = 0;$$

$$p_1 \cos. \beta_1 + p_2 \cos. \beta_2 + p_3 \cos. \beta_3 + \ldots = 0;$$

$$p_1 \cos. \gamma_1 + p_2 \cos. \gamma_2 + p_3 \cos. \gamma_3 + \ldots = 0.$$

77. PROP. *When a body is supported upon a curve (the curve being in a vertical plane); to find the conditions of equilibrium.*

Let AM, MP, fig. 97, be the vertical abscissa and the horizontal ordinate of the curve; and let $AM=x$, $MP=y$, $BP=s$. Let the forces which act on the body be resolved in the directions parallel to x and to y, and let the resolved parts thus obtained be called X and Y: X and Y being considered positive when they tend to increase x and y. Also let R be the re-action of the curve in the direction of the normal, or what is the same thing, the pressure of the body on the curve. Then, in order to obtain the conditions of equilibrium, resolve R in the directions parallel to x and to y;

Any number of points any how connected, and acted upon by any forces, may be considered as a machine. And the proposition which was proved in Art. 42, admits of an extension to this case. This proposition thus extended is called the Principle of *Virtual Velocities*. See Poisson *Traité de Mec.* Chap. VII.

The proposition which was proved in Art. 53, admits also of a similar extension, and may be thus stated. *When any number of points, any how connected, are acted upon by gravity, the equilibrium will take place when the center of gravity of the system is either in the highest or in the lowest position which the nature of the system allows it to assume.* See Poisson, Art. 176.

$\therefore$ resolved part of R in direction $PX = R$ cos. RPX

$$= R \text{ sin. } XPT = R \,.\, \frac{dy}{ds},$$

resolved part of R in direction $PY = R$ cos. RPY

$$= - R \text{ cos. } RPM = - R \,.\, \frac{dx}{ds}.$$

Hence, by Art. 40, the equilibrium will subsist if

$$X + R\frac{dy}{ds} = 0;$$

$$Y - R\frac{dx}{ds} = 0.$$

Multiply the first by dx, and the second by dy, and add;

$$\therefore \; Xdx + Ydy = 0,$$

which is the equation of equilibrium. If we know the curve, that is, the relation between x and y, this will give us the relation between X and Y; and if we know this also, it will enable us to find the actual values of x and y, or the point when the body will be supported. This will be illustrated by the problems which follow.

If the weight P, instead of resting upon a material surface BP, fig. 97, be suspended by a string KP which confines it to the curve BP, the conditions of equilibrium will be the same as before. The re-action which was before supplied by the resistance of the surface is now produced by the tension of the string. This re-action will as before be perpendicular to the curve: it will also manifestly be in the direction of the string, and this agrees with what is collected from the way in which the curve is described; for when a curve is traced out by one end of a string of which the other is fixed, the string will at every point be perpendicular to the curve. Hence the formulæ which we are about to give for the former case apply also to this.

78. Prop. *A body is supported upon a curve by a weight acting over a fixed pully* K, *fig.* 98; *to find the conditions of equilibrium.*

Take the vertical line KM, passing through the pully, for the line on which x is measured downwards.

Let $KM=x$, $MP=y$, $KP=r=(x^2+y^2)^{\frac{1}{2}}$; and if the weight which acts by means of KP be $=q$, the parts which act parallel to MK and PM are

$$q\frac{x}{r}, \text{ and } q\frac{y}{r};$$

$$\text{hence } X=p-q\frac{x}{r}; \quad Y=-q\frac{y}{r};.$$

hence the equation of Art. 77, namely, $Xdx+Ydy=0$, becomes

$$pdx-q\frac{xdx}{r}-q\frac{ydy}{r}=0,$$

$$\text{or } pdx-q\left(\frac{xdx}{r}+\frac{ydy}{r}\right)=0;$$

$$\text{but } x^2+y^2=r^2; \therefore xdx+ydy=rdr,$$

$$\text{and } \frac{xdx}{r}+\frac{ydy}{r}=dr;$$

$$\therefore pdx-qdr=0,$$

and this, combined with the relation between x and r which is given by the nature of the curve, gives the position of equilibrium.

Om. 79. PROB. I. *Let AP, fig.* 98, *be a hyperbola with its axis vertical, on which a given weight P is supported by another given weight Q by means of a string passing over a pully at the center; to find the position of equilibrium.*

Let as before KM, MP, AP, be x, y, r; the semi-axes of the hyperbola a and b: the given weights p and q.

$$y^2=\frac{b^2}{a^2}(x^2-a^2);$$

$$\therefore r=(x^2+y^2)^{\frac{1}{2}}=\left(x^2+\frac{b^2x^2}{a^2}-b^2\right)^{\frac{1}{2}}$$

$$=\left(\frac{a^2+b^2}{a^2}x^2-b^2\right)^{\frac{1}{2}}$$

$$=(e^2x^2-b^2)^{\frac{1}{2}};$$

making $e=\dfrac{(a^2+b^2)^{\frac{1}{2}}}{a}$ which is called the *eccentricity* of the hyperbola;

$$\therefore\ dr=\frac{e^2x\,dx}{(e^2x^2-b^2)^{\frac{1}{2}}};\ \text{hence } p\,dx-q\,dr=0 \text{ becomes}$$

$$p\,dx-q\,\frac{e^2x\,dx}{(e^2x^2-b^2)^{\frac{1}{2}}}=0;$$

$$\therefore\ p^2(e^2x^2-b^2)=q^2e^4x^2;$$

$$\therefore\ x^2=\frac{p^2b^2}{e^2(p^2-q^2e^2)};$$

$$\therefore\ x=\frac{pb}{e(p^2-q^2e^2)^{\frac{1}{2}}};\ \text{and hence we may find } y,\ r.$$

Cor. 1. If $q^2e^2>p^2$, or $qe>p$ the equilibrium is impossible.

Cor. 2. If $qe=p$, $x=\infty$: the body would in this case be supported upon the asymptote.

80. Prob. II. *It is required to find a curve such that a given weight $=q$ hanging over the pully may balance another given weight $=p$ at every point of it.*

We must have at every point

$$p\,dx-q\,dr=0;\ \text{hence, integrating,}$$

$$px-qr+c=0;$$

$$\therefore\ px+c=qr=q\,(x^2+y^2)^{\frac{1}{2}};$$

$$\therefore\ p^2x^2+2pcx+c^2=q^2x^2+q^2y^2;$$

$$\therefore\ y^2=\frac{p^2-q^2}{q^2}x^2+\frac{2pc}{q^2}x+\frac{c^2}{q^2}.$$

Let $x+\dfrac{pc}{p^2-q^2}=t$;

$$\therefore\ (p^2-q^2)\,x^2+2pcx+\frac{p^2c^2}{p^2-q^2}=(p^2-q^2)\,t^2,$$

$$y^2=\frac{1}{q^2}\left((p^2-q^2)\,t^2-\frac{p^2c^2}{p^2-q^2}+c^2\right);$$

$$=\frac{p^2-q^2}{q^2}\left(t^2-\frac{q^2c^2}{(p^2-q^2)^2}\right).$$

But if t, y, be the abscissa and ordinate of a hyperbola in which the semi-axes are a, b,

$$y^2 = \frac{b^2}{a^2}(t^2 - a^2); \text{ which agrees with our equation, if}$$

$$\frac{p^2 - q^2}{q^2} = \frac{b^2}{a^2}, \text{ and } \frac{q^2c^2}{(p^2-q^2)^2} = a^2:$$

$$\text{hence } \frac{c^2}{p^2 - q^2} = b^2.$$

Hence the curve required is a hyperbola in which $KM = x$, and $CM = t$; fig. 99; and in which the semi-axes are

$$a = \frac{qc}{(p^2 - q^2)}, \text{ and } b = \frac{c}{(p^2 - q^2)^{\frac{1}{2}}};$$

$$CK \text{ is } = \frac{pc}{p^2 - q^2} = \frac{pa}{q}.$$

$$(a^2 + b^2)^{\frac{1}{2}} = \left(\frac{q^2c^2}{(p^2 - q^2)^2} + \frac{c^2}{(p^2 - q^2)}\right)^{\frac{1}{2}} = \frac{pc}{p^2 - q^2} = CK;$$

$\therefore K$ is the focus.

If we call AK, k, we have

$$k = CK - CA = \frac{pc}{p^2 - q^2} - \frac{qc}{p^2 - q^2} = \frac{c}{p+q};$$

$\therefore c = (p+q)k$, and putting this value for c the semi-axes become

$$a = \frac{q}{p-q}k, \text{ and } b = \left(\frac{p+q}{p-q}\right)^{\frac{1}{2}} k.$$

81. Prop. *Two given weights P, P′, connected by a string of given length* ($= b$) *passing over a given pully K, fig.* 100, *are supported on two curves. Having given one curve, to find the other so that they may balance in every position.*

Let the weights be p, p', and the tension of the string q. And let x, x', r, r', be the values of the abscissæ KM, KM', and of KP, KP'. Then since q must be equal to a weight which, hanging freely, would support either P or P', we have, by the last Problem,

$$p\,dx - q\,dr = 0, \text{ and } p'\,dx' - q\,dr' = 0.$$

Also $r+r'=b$; $\therefore dr'=-dr$, whence the second equation becomes $p'dx'+qdr=0$, which added to the first gives

$pdx+pdx'=0$;

$\therefore px+p'x'=c$; this equation, along with the one

$r+r'=b$, enables us to find x' and r' in terms of x and r: and as we know the nature of the curve $A'P'$, we have the relation between r' and x', which we may represent thus $r'=f(x')$, $f(x')$ representing a function of x'; and by substituting the values of x' and r' we have a relation between x and r which determines the curve required.

82. Prob. III. *As an example, suppose the curve $A'P'$, fig.* 101, *to be a circle and CK a vertical line through its center:* and let $KC=k$, $A'C$ the radius of the circle $=a$; then

$$KP'^2=KC^2+CP'^2-2KC\,.\,CM',\text{ or}$$

$$r'^2=k^2+a^2-2k(k-x')=a^2-k^2+2kx',$$

$$\text{or since } r'=b-r,\text{ and } x'=\frac{c-px}{p'},\text{ by last Article;}$$

$$\therefore (b-r)^2=a^2-k^2+\frac{2k}{p'}(c-px),$$

$$\text{or } b-(x^2+y^2)^{\frac{1}{2}}=\left(a^2-k^2+\frac{2k}{p'}(c-px)\right)^{\frac{1}{2}}.$$

Cor. This is an equation to an epicycloid, as might be shewn. We shall, however, instead of this, shew geometrically that an epicycloid will satisfy the conditions. An epicycloid is the figure described by a point in one circle which *rolls* upon the circumference of another which is fixed.

Let CP', fig. 102, be the radius of the given circle, and K the pully in the vertical line CK. In this line produced take a point O, so that $CK : KO ::$ weight P' : weight P; and in the same line take Oq equal to the length of the string $P'KP$. Take qs equal to qO, and qp equal to qK; and describe a circle qr with center s and radius sq. Let this circle, carrying along with it the point p in the radius sq, produced if necessary, roll along the circle described with center O and radius Oq: the point p will describe a curve pKP, which will possess the property required.

For let qr come into the position QR, so that the describing point p may come to P: if T be the point where the circles are in contact, the circle SQ may, for an instant, be supposed to revolve about the point T, so that the curve will be perpendicular to TP; hence the re-action of the curve will be in the direction PT.

Let SP produced meet CK in y, and let KP be joined: and since, by the description of the curve, the arc TQ is equal to the arc Tq, the angle TSQ is equal to the angle TOq, and therefore yO equal to yS. Also SQ equals sq or Oq, and QP equals qp or qK; hence SP equals OK, and therefore yP equals yK. Hence KP is parallel to OS; and hence if PV be parallel to KO, PV will equal KO; also OV will equal KP.

We made $CK : KO :: P' : P$; hence if KC represent the weight P', OK or VP will represent the weight P. Now the weight P' is kept at rest by three forces, gravity, re-action, and tension, in the directions KC, CP', $P'K$; hence, on this supposition $P'K$ represents the tension of KP'. And the weight P is kept at rest by three forces, gravity, re-action, and tension, in the directions VP, PT, TV; hence TV represents the tension of KP.

Now OT equals KP and KP', and OV equals KP; therefore TV equals KP'; and hence the tensions of KP and of KP' are equal, and the bodies will balance each other*.

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* If instead of supposing a weight P' to rest on the circumference of a circle, we suppose CP' a heavy mass, (as a draw-bridge,) moveable about a hinge at C, and it be required to find the curve on which P must rest so as always to balance it, the question will easily be seen to be the same. Under this form the Problem was solved by the Marquis de l'Hopital in the Leipzig Acts for February 1695. The curve, which was at first called *the Curve of Equilibration*, was shewn by John Bernoulli to be such an epicycloid as we have proved it to be. From the construction it appears, that if Oq the length of the string be to KC as P to P', the curve is the common epicycloid, in which the describing point is in the circumference of the rolling circle or *rota*: if the former ratio be less, as in fig. 102, the describing point is without the circumference of the *rota*: if greater, the describing point is within the circumference of the *rota* and the curve has a point of contrary flexure, as in fig. 101.

The

83. PROB. IV. *Fig.* 103. *A body P which hangs by a string CP, without weight, and consequently must be somewhere in the circumference whose center is C, is sustained at the point P by a*

The Problems we have solved in the text suggest the following:

PROB. V. *Two weights connected by a string passing over a fixed pully rest on the same curve; to find the nature of the curve that they may in all positions balance.*

By Art. 81, we must have

$$px + p'x' = c,\ r + r' = b;$$

also r and r' must be the same function of x and x'; that is, if f represent this function, $r = f(x)$, $r' = f(x')$.

Let $x = t + \frac{c}{p+p'}$, $x' = t' + \frac{c}{p+p'}$, $r = u + \frac{b}{2}$, $r' = u' + \frac{b}{2}$;

whence our equations become

$$pt + p't' = 0,\ u + u' = 0;\ \therefore t' = -\frac{pt}{p'};\ u' = -u.$$

Also u and u' will be the same function of t and t';

$$\therefore \phi(t) = -\phi(t') = -\phi\left(-\frac{pt}{p'}\right);$$

from which the form of the function ϕ must be determined, whence the form of f and the nature of the curve will be known.

It does not appear that there exists a solution to this equation, when p and p' are unequal. If $p' = p$, it becomes

$$\phi(t) = -\phi(-t);$$

that is, ϕ must be such a function that it only changes its sign by putting $-t$ for t. This condition will manifestly be satisfied by the functions,

$\phi(t) = mt$,

$\phi(t) =$ any rational function composed of odd powers of t,

$\phi(t) = m \,.\, \sin.\, nt$, &c.

If we make $\phi(t) = mt$, we have $u = mt$;

$$\text{or } r - \frac{b}{2} = mx - \frac{mc}{2p};$$

$$\text{or } (x^2 + y^2)^{\frac{1}{2}} = mx + \frac{b}{2} - \frac{mc}{2p}$$

which

repulsion acting from the lowest point A. The repulsive force is directly proportional to the intensity of the repulsive power in A and inversely proportional to the square of the distance. Knowing the repulsive power, to find the position; and conversely, from the position, to find the intensity of the repulsive power.

An instrument of this kind is used to measure the intensity of the electrical repulsions which exist between two bodies A and P, in the same state of electricity; and it is then called the *Electrometer.*

Let $CA = CP = a$, $AP = r$; NP, perpendicular on $CA, = y$; $AN = x$. And let f represent the intensity of the repulsive power of A: then the force which it exerts at the distance r will be proportional to $\frac{f}{r^2}$; and if f be equal to the force at a distance $= 1$, $\frac{f}{r^2}$ will be *equal* to the force at P, in the direction AP. Resolve this force in the directions AN, NP, or PX, PY, and the forces will be

$$\frac{f}{r^2} \cdot \frac{x}{r}, \text{ and } \frac{f}{r^2} \cdot \frac{y}{r}.$$

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which will give a hyperbola as in Prob. V. In fact, it is manifest that since equal weights in two positions P and P', would each support a weight Q, they will support each other; and this in every situation.

If we make $\phi(t) = m \cdot \sin. nt$, we have $u = m \cdot \sin. nt$;

$$\text{or } r - \frac{b}{2} = m \cdot \sin. n \left(x - \frac{c}{2p}\right):$$

and if we now suppose, that when $x' = 0$, x is $= 2h$, we have $c = 2ph$; and hence

$$r = \frac{b}{2} + m \sin. n (x - h).$$

When $x = h$, $r = \frac{b}{2}$; $\therefore r' = \frac{b}{2}$, and $x' = h$. Hence if with radius $KB = \frac{b}{2}$, fig. 104, we describe a circle, and take $KH = h$ in the vertical line, and draw DH horizontal, D, D, will be corresponding positions of the weights. When one is at E the other will be at F; and in other positions P, P' they will rest on such a curve as is represented in the figure.

Hence, considering also the action of gravity $=p$, we have, (see Art. 77.)

$$X=\frac{fx}{r^3}-p, \quad Y=\frac{fy}{r^3}.$$

But, in the circle, $y^2=2ax-x^2$; $\therefore ydy=(a-x)\,dx$.

Hence the formula $Xdx+Ydy=0$, or $Xydx+Yydy=0$, becomes

$Xy+Y(a-x)=0$. And putting for X and Y their values;

$$\frac{fxy}{r^3}-py+\frac{fay}{r^3}-\frac{fyx}{r^3}=0,$$

or $\frac{fa}{r^3}=p$; $\therefore r=\left(\frac{fa}{p}\right)^{\frac{1}{3}}$; whence the position is known.

And $\frac{f}{p}=\frac{r^3}{a}$, whence the ratio of the force f to the weight p is known.

Cor. If another force of the same kind $(=f')$ balance the same body P, at a distance r' from A, we have also

$$\frac{f'}{p}=\frac{r'^3}{a};\ \therefore \frac{f'}{f}=\frac{r'^3}{r^3};$$

or the forces are as the cubes of the distances from A at which the body is supported.

Read Vist. Vels. where you can find them

CHAP. VII.

THE CONDITIONS OF EQUILIBRIUM OF A RIGID BODY.

84. Prop. *To find the resultant of any number of parallel forces acting on a rigid body. Fig.* 105.

Let any number of parallel forces p_1, p_2, p_3, &c. act upon a rigid body. Let a plane yAx, be drawn perpendicular to these forces; and let two lines, Ax, and Ay, be drawn in this plane at right angles to each other. Let P_1M_1 be parallel to Ay, and let x_1, y_1 be AM_1, M_1P_1, the *co-ordinates* of the point P_1, where p_1 meets the plane yAx. Similarly let x_2, y_2, be the co-ordinates of P_2, where p_2 meets the plane; x_3, y_3 the same quantities for P_3, &c. And let R be the resultant of the forces, and α, β, the co-ordinates of the point where it meets the plane.

The two forces p_1, p_2, produce the same effect as if they acted at P_1, P_2. And if we consider them as weights, they will balance each other upon their center of gravity, and produce at that point a pressure $=p_1+p_2$. (Art. 15.) Hence their effect upon the rigid body is the same as that of a force p_1+p_2 acting at the center of gravity of P_1, P_2, in a direction parallel to these forces. Similarly it will appear that this force p_1+p_2, along with p_3, that is, p_1, p_2, p_3, will produce the same effect as $p_1+p_2+p_3$, acting at the center of P_1, P_2, P_3. And in the same manner it may be shewn that any number of forces p_1, p_2, p_3, &c. will produce the same effect as $p_1+p_2+p_3+$ &c. acting parallel to the forces, at the center of gravity of P_1, P_2, P_3, &c.

Hence we shall have, by Art. 48.

$$R = p_1 + p_2 + p_3 + \text{\&c.}$$
$$R\alpha = p_1x_1 + p_2x_2 + p_3x_3 + \text{\&c.}$$
$$R\beta = p_1y_1 + p_2y_2 + p_3y_3 + \text{\&c.}$$

Whence R is known, and α, β, which determine the position of the resultant.

Cor. 1. If any of the forces act in the opposite direction they must be considered as negative.

Hence it appears that we may have $p_1+p_2+p_3+\&c.=0$. In this case, if $p_1x_1+p_2x_2+\&c.$ be finite, α will be infinite. And similarly for β.

Cor. 2. For example, let two forces, each $=p$, act in opposite directions at points in the line Ax, distant from each other by a distance a. Hence we shall have

$$R=0; \quad Ra=p(x_1+a)-px_1=pa;$$

$$\beta=0; \quad \alpha=\frac{pa}{R}=\frac{pa}{0}.$$

Therefore α is infinite, and the forces are equivalent to a force $=0$, acting at any infinite distance.

In this case no single force could produce the effect of the two. Their tendency is to turn the system round in the plane in which they are, without producing any motion except a rotatory one.

Cor. 3. Hence it is not true that parallel forces can in all cases be reduced to a single finite force. If $p_1+p_2+p_3+\&c.=0$, they can not.

In this case the forces can be reduced to two, equal to each other, and acting at two different points in opposite directions. For since $p_1+p_2+\&c.=0$, these forces may be divided into two groups, of which one is equal to the other with a negative sign. And hence if we take the resultants of these groups separately, we shall obtain two equal forces in opposite directions.

Cor. 4. Let one of these groups consist of p_1, p_2, &c. and the other of p', p'', &c. Then

$$p_1+p_2+\&c.+p'+p''+\&c.=0,$$
$$p_1+p_2+\&c.=-p'-p''-\&c.$$

And if R be the resultant of p_1, p_2, &c., $-R$ will also be the resultant of p', p'', &c. Let α, β, be the co-ordinates of the point where the first resultant meets the plane, α', β', the corresponding co-ordinates for the second resultant. Then

$$R\alpha = p_1 x_1 + p_2 x_2 + \&c.; \quad R\beta = p_1 y_1 + p_2 y_2 + \&c.$$
$$-R\alpha' = p'x' + p''x'' + \&c.; - R\beta' = p'y' + p''y'' + \&c.$$
$$\therefore R(\alpha - \alpha') = p_1 x_1 + p_2 x_2 + \&c. + p'x' + p''x'' + \&c.$$
$$R(\beta - \beta') = p_1 y_1 + p_2 y_2 + \&c. + p'y' + p''y'' + \&c.$$

And these quantities are the same whatever be R, that is, however the groups are selected. Hence if l be the line which joins the two points of application, λ the angle which it makes with Ax,

$$Rl = R\sqrt{\{(\alpha - \alpha')^2 + (\beta - \beta')^2\}}; \text{ tan. } \lambda = \frac{\beta - \beta'}{\alpha - \alpha'};$$

which quantities are the same whatever R be.

Hence the position of the line l, and the moment of the pair of forces to turn the system in the plane in which are R and l, are found to be the same, however the two groups are selected.

85. Prop. *To find the conditions of equilibrium of parallel forces acting upon a rigid body.*

In order that the equilibrium may subsist, one of the forces, as p_1, must be equal and opposite to the resultant of all the others. Hence $-p_1$ must be the resultant of the forces p_2, p_3, &c. And therefore by last Article, x_1, y_1, x_2, y_2, &c. being the co-ordinates, as before,

$$-p_1 = p_2 + p_3 + \&c.$$
$$-p_1 x_1 = p_2 x_2 + p_3 x_3 + \&c.$$
$$-p_1 y_1 = p_2 y_2 + p_3 y_3 + \&c.;$$
$$\therefore p_1 + p_2 + p_3 + \&c. = 0 \ldots\ldots\ldots\ldots\ldots (1);$$
$$\left.\begin{array}{l} p_1 x_1 + p_2 x_2 + p_3 x_3 + \&c. = 0 \\ p_1 y_1 + p_2 y_2 + p_3 y_3 + \&c. = 0 \end{array}\right\} \ldots\ldots\ldots (2).$$

Cor. 1. If the rigid body have one point fixed, let this point be the origin of co-ordinates. And it is manifest that the equilibrium will subsist if the resultant pass through this point; for it will be counteracted by the resistance of the fixed point. Hence in last Article $\alpha = 0$, $\beta = 0$. Therefore, by that Article,

$$p_1 x_1 + p_2 x_2 + p_3 x_3 + \&c. = 0;$$
$$p_1 y_1 + p_2 y_2 + p_3 y_3 + \&c. = 0.$$

86. Prop. *To find the resultant of any number of forces acting in the same plane upon a rigid body. Fig.* 106.

Let p_1, p_2, p_3, &c. be the forces acting in the plane yAx, at the points P_1, P_2, P_3, &c. Let Ax, Ay be at right angles in this plane; and let x_1, y_1; x_2, y_2; x_3, y_3, &c. be the co-ordinates of these points parallel to Ax, Ay. Let P_1D_1, P_1E_1 be lines parallel to Ax, Ay, and let a_1, be the angle which P_1p_1 makes with P_1D_1. If the force p_1 be resolved in these directions, the components will be p_1 cos. a_1, p_1 sin. a_1. In the same manner p_2 cos. a_2, p_2 sin. a_2 will be the components of p_2; and similarly for the others. Hence the forces are resolved into two sets of parallel forces, acting at the points P_1, P_2, P_3, &c. and parallel to Ax and to Ay respectively. Let X be the resultant of the first set; Y, of the second. Also let X meet Ay in K, and let $AK=t$; and let Y meet Ax in H, and let $AH=s$. Then we may suppose X to act at K, and Y at H. Therefore, by Art. 84,

$$X=p_1 \cos. a_1 + p_2 \cos. a_2 + p_3 \cos. a_3 + \&c.$$
$$Y=p_1 \sin. a_1 + p_2 \sin. a_2 + p_3 \sin. a_3 + \&c.$$
$$Ys=p_1x_1 \sin. a_1 + p_2x_2 \sin. a_2 + p_3x_3 \sin. a_3 + \&c.$$
$$Xt=p_1y_1 \cos. a_1 + p_2y_2 \cos. a_2 + p_3y_3 \cos. a_3 + \&c.$$

Hence we know X, Y, s, t. And hence, knowing AK, AH, if we draw KG, HG parallel to Ax, Ay, the forces X, Y may be supposed to act at G; and will then produce a resultant R, which is the resultant of the whole system. Also if a be the angle which this resultant makes with Ax, we shall have

$$R=\sqrt{(X^2+Y^2)}; \ \tan. a=\frac{Y}{X}.$$

Hence we know the magnitude and position of R. The co-ordinates s, t, of its point of application have already been found.

Cor. 1. To find the equation of the straight line in which R acts.

Let the equation to the straight line be $y=Ax+B$. In this case A is the tangent of the angle a; therefore $A=\frac{Y}{X}$, and

$y = \frac{Y}{X}x + B$. Also, since G is a point in this line, when $x = s$, $y = t$. Therefore $t = \frac{Y}{X}s + B$; and $B = t - \frac{Y}{X}s$. Hence

$$y = \frac{Y}{X}x + t - \frac{Y}{X}s; \text{ and } Xy - Yx = Xt - Ys,$$

which is the equation to the line.

COR. 2. Putting for Xt and Ys their values, we have

$$Xy - Yx = p_1 (y_1 \cos. a_1 - x_1 \sin. a_1)$$
$$+ p_2 (y_2 \cos. a_2 - x_2 \sin. a_2) + \&c.$$

call this quantity L. Then the equation is

$$Xy - Yx = L.$$

COR. 3. In this case, when *one* of the sets of parallel forces, as that parallel to Ax, is not reducible to a single force, (see Cor. 3, Art. 84,) we shall have $X = 0$, $t = \text{inf.}$ Hence

$$R = Y, \ \cos. a = 0, \ s = \frac{p_1 x_1 \sin. a_1 + p_2 x_2 \sin. a_2 + \&c.}{Y}.$$

Hence the resultant will be a single force parallel to Ay, and determined in position by the above equations.

COR. 4. But in the case when *both* the sets of parallel forces are incapable of being reduced to single forces, we shall have $X = 0$, $Y = 0$, $s = \text{inf.}$ $t = \text{inf.}$ Hence $R = 0$; and we have a resultant $= 0$, acting at an infinite distance.

In this case the forces are equal to two, equal and opposite, but not in the same line, as in Cor. 3, Art. 84.

87. PROP. *To find the conditions of equilibrium of any number of forces acting in the same plane upon a rigid body.*

In order that there may be an equilibrium, p_1 must be equal and opposite to the resultant of p_2, p_3, &c. Hence $-p_1$ must be the resultant of p_2, p_3, &c. And $-p_1 \cos. a_1$, $-p_2 \sin. a_1$ will be the parts of the resultant which are parallel to Ax, Ay. Hence, by the equations of Art. 45,

$$-p_1 \cos. a_1 = p_2 \cos. a_2 + p_3 \cos. a_3 + \&c.$$
$$-p_1 \sin. a_1 = p_2 \sin. a_2 + p_3 \sin. a_3 + \&c.$$

Hence

$$\left.\begin{array}{l} p_1 \cos. a_1 + p_2 \cos. a_2 + p_3 \cos. a_3 + \&c. = 0 \\ p_1 \sin. a_1 + p_2 \sin. a_2 + p_3 \sin. a_3 + \&c. = 0 \end{array}\right\} \ldots\ldots (1).$$

Also the equation of the line in which $-p_1$ acts must be the same as that in which the resultant of p_2, p_3, &c. acts. And the latter equation is, (Cor. 2, Art. 86.) putting for X, $-p_1 \cos. a_1$, and for Y, $-p_1 \sin. a_1$,

$$-p_1 y \cos. a_1 + p_1 x \sin a_1 = p_2 (y_2 \cos. a_2 - x_2 \sin. a_2) + p_3 (y_3 \cos. a_3 - x_3 \sin. a_3) + \&c.$$

And this, which is true for every point of p_1's direction, must be true for the point P_1, where $x = x_1$, $y = y_1$. Hence, putting these values for x and y, and transposing,

$$p_1 (y_1 \cos. a_1 - x_1 \sin. a_1) + p_2 (y_2 \cos. a_2 - x_2 \sin. a_2) + p_3 (y_3 \cos. a_3 - x_3 \sin. a_3) + \&c. = 0 \ldots\ldots\ldots (2).$$

This equation (2) and the two found above (1) are the equations of condition for the equilibrium of the forces.

Cor. 1. If a point of the rigid body in the plane in which the forces act be a fixed point, the equilibrium will subsist if the resultant of the forces pass through this point.

Let A be the fixed point. Then, in order that the resultant may pass through A, we must have $x = 0$, $y = 0$, at the same time; $\therefore 0 = Xt - Ys$,

or $0 = p_1 (y_1 \cos. a_1 - x_1 \sin. a_1) + p_2 (y_2 \cos. a_2 - x_2 \sin. a_2) + \&c.$ which is in this case the condition of equilibrium.

88. Prop. *Any number of forces being given, acting in any directions upon a rigid body, to reduce them to two sets of forces, one set being in a given plane, and the other perpendicular to it.*

Let Ax, Ay, Az, fig. 107, be three lines at right angles to each other. Let P be a point of the system, at which one of the forces acts, in the direction Pp. Let PD, PE, PF be three lines parallel to Ax, Ay, Az, and let α, β, γ be the angles which Pp makes with PD, PE, PF. Then p being the force, $p \cos. \alpha$, $p \cos. \beta$, $p \cos. \gamma$ will be the components in PD, PE, PF. Let FP meet the plane yAx in O, and let OM and ON be parallel to Ay and Ax.

If we suppose, at the point P, two equal forces in opposite directions to be added to the system, these will counteract each other, and the effect of the forces such as p will be the same as before. Let two forces, g in the direction PF, and g in the direction FP, act at P. Then the forces which act at P may be grouped thus,

$$p \cos. \alpha \text{ and } g; \; p \cos. \beta \text{ and } -g; \; p \cos. \gamma.$$

Let $p \cos. \alpha$ in PD, and g in PF have a resultant Ph: hP will be in the plane DPF. Let it meet ON in H. Ph acting at P is equivalent to Ph acting at H. And Ph at H may be resolved into two forces, $p \cos. \alpha$ parallel to Ax, and g parallel to Az.

Since Ph is compounded of $p \cos. \alpha$ in the direction HO, and g in the direction OP, we shall have, calling the co-ordinates of p, x, y and z,

$$HO : OP\,(=z) :: p \cos. \alpha : g;$$

$$\therefore HO = \frac{pz \cos. \alpha}{g}. \quad \text{And } NH = NO - HO = x - \frac{pz \cos. \alpha}{g}.$$

Hence $p \cos. \alpha$ and g at P, are equivalent to g parallel to Az, and $p \cos. \alpha$ parallel to Ax; both acting at a point H of which the co-ordinates parallel to Ax and Ay are $x - \dfrac{pz \cos. \alpha}{g}$, and y.

In the same manner $p \cos. \beta$ and $-g$ are equivalent to a force Pk in the plane EPF, and this produces the same effect as if it acted at K. And at K it may be resolved into $p \cos. \beta$ and $-g$. Also as before, $OK = \dfrac{pz \cos. \beta}{g}$; and $MK = y + \dfrac{pz \cos. \beta}{g}$.

Hence $p \cos. \beta$ and $-g$ are equivalent to $p \cos. \beta$ parallel to Ay and $-g$ parallel to Az; both acting at a point K of which the co-ordinates parallel to Ax and Ay are x and $y + \dfrac{pz \cos. \beta}{g}$.

$p \cos. \gamma$ is parallel to Az, and produces the same effect as if it acted at O, of which the co-ordinates are x, y.

Hence if p_1 is a force acting at a point of which the co-ordinates are x_1, y_1, z_1, making with the three co-ordinates angles a_1, β_1, γ_1; and if p_2, p_3, &c. be other forces; x_2, y_2, z_2; x_3, y_3, z_3, &c. the corresponding co-ordinates; a_2, β_2, γ_2; a_3, β_3, γ_3, &c. the corresponding angles; the forces p_1, p_2, p_3, &c. will be equivalent to the following forces in the plane yAx;

$p_1 \cos. a_1$, par. to Ax, with co-ordinates $x_1 - \dfrac{p_1 z_1 \cos. a_1}{g_1}$ and y_1;

$p_1 \cos. \beta_1$, par. to Ay, with co-ordinates x_1 and $y_1 + \dfrac{p_1 z_1 \cos. \beta_1}{g_1}$;

and to the following forces parallel to Az,

$p_1 \cos. \gamma_1$ with co-ordinates x_1, y_1;

g_1 with co-ordinates $x_1 - \dfrac{p_1 z_1 \cos. a_1}{g_1}$ and y_1;

$-g_1$ with co-ordinates x_1 and $y_1 + \dfrac{p_1 z_1 \cos. \beta_1}{g_1}$.

And to similar forces with the exponents 2, 3, &c. instead of 1.

89. PROP. *To find the conditions of equilibrium of any number of forces p_1, p_2, p_3, &c. acting in any directions upon a rigid body.*

The forces being resolved as in the last Article, the equilibrium will subsist if the forces in the plane yAx, and the forces parallel to Az be in equilibrium separately. Hence we shall have by Art. 87, these three equations for the equilibrium of the forces in the plane yAx:

$$\left.\begin{array}{l} p_1 \cos. a_1 + p_2 \cos. a_2 + p_3 \cos. a_3 + \&c. = 0 \\ p_1 \cos. \beta_1 + p_2 \cos. \beta_2 + p_3 \cos. \beta_3 + \&c. = 0 \end{array}\right\} \ldots\ldots (1);$$

$$p_1 (y_1 \cos. a_1 - x_1 \cos. \beta_1) + p_2 (y_2 \cos. a_2 - x_2 \cos. \beta_2)$$
$$+ p_3 (y_3 \cos. a - x_3 \cos. \beta_3) + \&c. = 0 \ldots\ldots\ldots (2).$$

Also by Art. 85, we have, for the equilibrium of the forces parallel to Az,

$$p_1 \cos. \gamma_1 + \&c. = 0;$$

$$p_1 x_1 \cos. \gamma_1 + g_1 \left(x_1 - \frac{p_1 z_1 \cos. a_1}{g_1}\right) - g_1 x_1 + \&c. = 0;$$

$$p_1 y_1 \cos. \gamma_1 + g_1 y_1 - g_1 \left(y_1 + \frac{p_1 z_1 \cos. \beta_1}{g_1}\right) + \&c. = 0;$$

with other similar terms corresponding to p_2, g_2, p_3, g_3, &c. And these three equations become

$$p_1 \cos. \gamma_1 + p_2 \cos. \gamma_2 + p_3 \cos. \gamma_3 + \&c. = 0 \ldots\ldots\ldots\ldots (1);$$

$$\left.\begin{aligned} p_1(x_1 \cos. \gamma_1 - z_1 \cos. \alpha_1) + p_2(x_2 \cos. \gamma_2 - z_2 \cos. \alpha_2) \\ + p_3(x_3 \cos. \gamma_3 - z_3 \cos. \alpha_3) + \&c. = 0; \\ p_1(y_1 \cos. \gamma_1 - z_1 \cos. \beta_1) + p_2(y_2 \cos. \gamma_2 - z_2 \cos. \beta_2) \\ + p_3(y_3 \cos. \gamma_3 - z_3 \cos. \beta_3) + \&c. = 0. \end{aligned}\right\} \ldots\ldots (2).$$

And these three equations, with the former three, are the conditions of equilibrium.

Cor. 1. It has been proved that these equations are *sufficient;* that is, that if they are satisfied the equilibrium subsists. They are also *necessary;* for except both sets are satisfied the equilibrium does not subsist.

If possible let the equilibrium subsist when the forces parallel to z are not separately in equilibrium. The equilibrium will still subsist if we suppose any line in the plane yAx to be fixed. But in that case all the forces in the plane yAx will be counteracted by the resistance of this line. And the forces parallel to Az will turn the system about this line in some of its positions. Hence the equilibrium will not subsist.

And since the forces parallel to Az are in equilibrium separately, the other forces must also be in equilibrium separately.

Cor. 2. Let the rigid body be moveable about a fixed point. Let this point be made the origin of co-ordinates A. Then the forces may be resolved as in Art. 88. and the equilibrium will subsist if the forces in the plane yAx have a resultant which passes through the point A, and if the forces parallel to Az have also a resultant which passes through A. Hence by Art. 87, Cor. 1, and by Art. 85, Cor. 1, we have

$$p_1(y_1 \cos. \alpha_1 - x_1 \cos. \beta_1) + p_2(y_2 \cos. \alpha_2 - x_2 \cos. \beta_2) + \&c. = 0;$$
$$p_1(x_1 \cos. \gamma_1 - z_1 \cos. \alpha_1) + p_2(x_2 \cos. \gamma_2 - z_2 \cos. \alpha_2) + \&c. = 0;$$
$$p_1(y_1 \cos. \gamma_1 - z_1 \cos. \beta_1) + p_2(y_2 \cos. \gamma_2 - z_2 \cos. \beta_2) + \&c. = 0.$$

It appears from this that the forces are to be such as to keep each other in equilibrium about three axes at right angles to each other passing through the fixed point.

90. Prop. *To find the condition which is requisite in order that a system of forces acting any how in space may have a single resultant.*

Retaining the notation of Art. 88, we may reduce the forces to the two sets mentioned in that Article. The resultants of these sets may be found by Articles 84 and 86; and if these resultants intersect each other, they may be compounded into a single force which will be the resultant of the whole. If the two resultants do not intersect each other, this will be impossible.

$$\text{Let } p_1 \cos. \alpha_1 + p_2 \cos. \alpha_2 + \&\text{c.} = X;$$
$$p_1 \cos. \beta_1 + p_2 \cos. \beta_2 + \&\text{c.} = Y;$$
$$p_1 \cos. \gamma_1 + p_2 \cos. \gamma_2 + \&\text{c.} = Z.$$
$$p_1 (y_1 \cos. \alpha_1 - x_1 \cos. \beta_1) + p_2 (y_2 \cos. \alpha_2 - x_2 \cos. \beta_2) + \&\text{c.} = L;$$
$$p_1 (x_1 \cos. \gamma_1 - z_1 \cos. \alpha_1) + p_2 (x_2 \cos. \gamma_2 - z_2 \cos. \alpha_2) + \&\text{c.} = M;$$
$$p_1 (z_1 \cos. \beta_1 - y_1 \cos. \gamma_1) + p_2 (z_2 \cos. \beta_2 - y_2 \cos. \gamma_2) + \&\text{c.} = N^*.$$

Then we shall have for the equation of the line in which the force acts, which is the resultant of those in the plane yAx,

$$Xy - Yx = L; \text{ (Cor. 2, Art. 86.)}$$

and for the force which is the resultant of those parallel to Az we shall have, by Art. 84,

$$Zx = M; \; Zy = -N.$$

And that these two forces may intersect, the point in which the latter meets the plane yAx must be in the direction of the former. Hence the equation $Xy - Yx = L$ must be satisfied by the values of x and y in $Zx = M$, $Zy = -N$; and substituting, we find

$$LZ + MY + NX = 0 \ldots\ldots (a),$$

which is the equation of condition required.

91. Prop. *In the case where it is possible, to find the resultant of any number of forces acting any how in space.*

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* The quantities L, M, N are the *moments* of the forces p_1, p_2, &c. *projected* on the planes xy, xz, yz respectively. These projected moments give rise to some remarkable propositions. See Poison. *Traitè de Mec.* Chap. III.

The force in the plane yAx will be composed of X and Y, and the force at the same point parallel to Az will be Z. Hence, if R be the resultant, and a, b, c, the angles which it makes with lines parallel to Ax, Ay, Az, we shall have, as in Art. 75,

$$R = \sqrt{(X^2 + Y^2 + Z^2)};$$

$$\cos. a = \frac{X}{R}; \ \cos. b = \frac{Y}{R}; \ \cos. c = \frac{Z}{R}.$$

And the point where it cuts the plane yAx is known by the equations $Zx = M$, $Zy = -N$.

Cor. 1. It may be easily shewn that the equations to the line in which the resultant acts are

$$Xy - Yx = L, \ Zx - Xz = M, \ Yz - Zy = N \ldots\ldots (b),$$

of which two only are necessary, the third being included in them in consequence of the equation of condition (a) of last Article.

92. Prop. *In the case where a number of forces are not reducible to one force, they are always reducible to two.*

Without altering the conditions of the system, we may suppose, in addition to the forces of the system, two new forces S, $-S$, acting at the origin A, and making angles a, b, c, with the axes. And these forces and their angles may be so taken that the force S, along with p_1, p_2, &c., shall satisfy the equation (a), and have a single resultant. Thus the forces are reduced to this resultant, and to the force $-S$ acting at the point A.

Cor. In this case the two forces to which the system is reduced are not determined in magnitude and direction.

93. The following example may serve to illustrate the preceding Articles.

ABCDEFG, fig. 108, is a rectangular parallelepiped acted on by forces, which have their directions in the edges BE, CF, DG of the parallelepiped, taken so that none of them pass through A, and no two of them are in the same plane: to shew when there is a single resultant, and to find it.

Let AD, AB, AC be in the directions of Ax, Ay, Az; let $AD=a$, $AB=b$, $AC=c$: and let the forces be p_1, p_2, p_3. Then we shall have

$$p_1 \cos. a_1 = p_1, \quad p_1 \cos. \beta_1 = 0_1, \quad p_1 \cos. \gamma_1 = 0;$$
$$p_2 \cos. a_2 = 0, \quad p_2 \cos. \beta_2 = p_2, \quad p_2 \cos. \gamma_2 = 0;$$
$$p_3 \cos. a_3 = 0, \quad p_3 \cos. \beta_3 = 0, \quad p_3 \cos. \gamma_3 = p_3.$$
$$x_1 = 0, \quad y_1 = b, \quad z_1 = 0;$$
$$x_2 = 0, \quad y_2 = 0, \quad z_2 = c;$$
$$x_3 = a, \quad y_3 = 0, \quad z_3 = 0;$$

Hence we have $X=p_1$, $Y=p_2$, $Z=p_3$

$$L=p_1 b; \quad M=p_3 a, \quad N=p_2 c.$$

And the equation of condition (a) becomes

$$p_1 p_3 b + p_2 p_3 a + p_1 p_2 c = 0;$$

$$\text{or } \frac{a}{p_1} + \frac{b}{p_2} + \frac{c}{p_3} = 0.$$

If this equation be not satisfied, the forces are not reducible to a single force.

Let p_1, p_2, be as the edges BE, CF. Then

$$\frac{b}{p_2} = \frac{a}{p_1}; \therefore \frac{c}{p_3} = -\frac{2a}{p_1} \text{ when the reduction is possible;}$$

$$\therefore L=p_1 b; \; M=p_3 a = -\tfrac{1}{2} p_1 c; \; N=p_2 c = p_1 \frac{bc}{a}.$$

Hence, by Cor. 1. to Art. 91. the equations to the line of direction of the force will be

$$\left.\begin{aligned} p_1 y - p_2 x = p_1 b, \text{ or } y - \frac{b}{a} x = b \\ p_3 x - p_1 z = -\tfrac{1}{2} p_1 c, \text{ or } \frac{cx}{a} + 2z = c \end{aligned}\right\}.$$

These two equations determine the position of the force; and its magnitude is known, being

$$= \sqrt{(p_1^2 + p_2^2 + p_3^2)} = p_1 \frac{\sqrt{(a^2 + b^2 + \frac{1}{4}c^2)}}{a}.$$

If we produce DE to H, making $EH=ED$, BH will be the line to which the first equation belongs. And when $x=0$, $z=\frac{1}{2}c$. Hence, if we bisect BF in K, KH is the direction in which the resultant of the three forces acts.

CHAP. VIII.

THE APPLICATION OF THE INTEGRAL CALCULUS TO FINDING THE CENTER OF GRAVITY.

94. Prop. *To find the center of gravity of any curvilinear body.*

Let $P'PBQQ'$ (fig. 109.) be any body: Ax the axis of x: and let the body be cut by planes PQ, $P'Q'$, perpendicular to Ax.

Let G, G', K, be the centers of gravity of the portions of the body PBQ, $P'BQ'$, and $PQQ'P'$: and let GH, $G'H'$, KL, be perpendiculars upon a plane Ay parallel to PM.

Let the mass $PBQ=m$, $P'BQ'=m'$: therefore we have $PQQ'P'=m'-m$.

Also let $GH=h$, $G'H'=h'$, $KL=k$.

Now we may suppose the masses PBQ, $PQQ'P'$, to be collected at their respective centers of gravity G, K; (Art. 50, Cor.) and since G' is the center of gravity of the whole mass, we have, (Art. 49.)

$$G'H' = \frac{PBQ\,.\,GH + PQQ'P'\,.\,KL}{PBQ + PQQ'P'}:$$

$$\text{or } h' = \frac{m\,.\,h + (m'-m)\,k}{m'}.$$

$$\text{Hence, } k = \frac{m'h' - mh}{m' - m}.$$

Now if we suppose the plane $P'Q'$ to come indefinitely near to PQ, so that $PQQ'P'$ may become an indefinitely thin slice, K will ultimately be in PQ, and k ultimately $=AM$ or x.

Also in this case $\frac{m'h'-mh}{m'-m}$, which is the ratio of the *increments* of mh and of m', will, by the principles of the differential calculus, ultimately become the ratio of their *differentials*. Hence taking the ultimate limits on both sides, which will necessarily be equal, we have

$$x = \frac{d \,.\, mh}{d \,.\, m}; \quad d \,.\, mh = x \,.\, dm;$$

$$\text{Integrating,} \quad mh = \int x\,dm;$$

$$\therefore h = \frac{\int x\,dm}{m}.$$

We may thus find the distance of the center of gravity from the known plane Ay.

If Ax, perpendicular to Ay, be a line along which abscissas are measured; and if $AM = x$, $MP = y$, the curve may be defined by a relation between x and y, if the body be a plane figure, or a figure of revolution round Ax, and hence dm may be found.

In other cases we may suppose two planes, at right angles to each other, passing through Ax; and if y and z be the distances of P from these planes, the surface of the body may be defined by an equation between x, y, and z, whence dm may be found.

If the body be symmetrical on the two sides of Ax, supposing it to lie in a plane; or if its section by every plane passing through Ax be symmetrical to Ax, supposing it extended in three dimensions; its center of gravity will be in the line Ax: and hence, to determine its position, it is sufficient to find the value of GH.

If the body be not thus symmetrical with respect to Ax, its center of gravity will not necessarily be in that line. In this case it will be necessary, if the body lie in a plane, to find the distance of the center of gravity from some other line besides Ay; for instance, GF its distance from Ax: this may be found in the same way as GH. If the body have three dimensions, it will be necessary to find, by similar methods, the distances of the center of gravity from three known planes, which will determine its position.

We shall consider the cases separately.

1. *A Symmetrical Area.*

95. Let PAp, fig. 110, be a curvilinear area symmetrical with regard to AM. We may suppose it to be a lamina of matter whose thickness may be neglected; or, if the thickness be supposed finite and constant, the position of G will be the same. It is manifest that the weight or quantity of matter of any part, supposing the density uniform, will be as the magnitude, or as the area of that part, and may be represented by the area.

Now if the abscissa AM (fig. 110,) $= x$, and the ordinate $MP = y$, the differential of the area PAp is $2y dx$; hence $dm = 2y\,dx$; and if $GA = h$,

$$\therefore h = \frac{\int 2y\,dx.x}{\int 2y\,dx} = \frac{\int xy\,dx}{\int y\,dx}.$$

We shall give some instances of the application of this formula.

Ex. 1. The equation to a curve is $y = \frac{x(a^2 - x^2)^{\frac{1}{2}}}{a}$; to find the center of gravity of its area.

$\int y\,dx = \int \frac{x\,dx\,(a^2 - x^2)^{\frac{1}{2}}}{a} = C - \frac{(a^2 - x^2)^{\frac{3}{2}}}{3a}$; C being an arbitrary constant; and if we suppose the area to be taken from the point when x and $y = 0$, $\int y\,dx = \frac{a^3 - (a^2 - x^2)^{\frac{3}{2}}}{3a}$.

$$\int xy\,dx = \int \frac{x^2\,dx.(a^2 - x^2)^{\frac{1}{2}}}{a} = \frac{\int x.x\,dx.(a^2 - x^2)^{\frac{1}{2}}}{a}.$$

But $\int x.x\,dx\,(a^2 - x^2)^{\frac{1}{2}} = -\frac{x.(a^2 - x^2)^{\frac{3}{2}}}{3} + \int \frac{(a^2 - x^2)^{\frac{3}{2}}dx}{3}$ *

$$= -\frac{x.(a^2 - x^2)^{\frac{3}{2}}}{3} + \int \frac{a^2.(a^2 - x^2)^{\frac{1}{2}}.dx}{3} - \int \frac{x^2.(a^2 - x^2)^{\frac{1}{2}}dx}{3};$$

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* By the formula $\int u\,dv = uv - \int v\,du$: see Lacroix's Elementary Treatise, &c. No. 148.

$$\therefore\ 4\int x^2 dx\,(a^2-x^2)^{\frac{1}{2}} = -x\,.\,(a^2-x^2)^{\frac{3}{2}} + a^2\,.\int(a^2-x^2)^{\frac{1}{2}}\,.\,dx$$

$$= -x\,(a^2-x^2)^{\frac{3}{2}} + \frac{a^2}{2}\,.\left\{x\,.\,(a^2-x^2)^{\frac{1}{2}} + a^2\,.\,\text{arc}\left(\sin. = \frac{x}{a}\right)\right\} + C;$$

$$\therefore \int \frac{x^2 dx\,(a^2-x^2)^{\frac{1}{2}}}{a}$$

$$= \frac{1}{8a}\,.\left\{-2x(a^2-x^2)^{\frac{3}{2}} + a^2x\,.(a^2-x^2)^{\frac{1}{2}} + a^4.\text{arc}\left(\sin. = \frac{x}{a}\right)\right\} + C$$

$$= \frac{1}{8a}\,.\left\{(a^2-x^2)^{\frac{1}{2}}\,.\,x\left[a^2 - 2(a^2-x^2)\right] + a^4.\,\text{arc}\left(\sin. = \frac{x}{a}\right)\right\} + C$$

$$= \frac{1}{8a}\,.\left\{(a^2-x^2)^{\frac{1}{2}}\,.\,(2x^2-a^2)\,.\,x + a^4\,.\,\text{arc}\left(\sin. = \frac{x}{a}\right)\right\} + C:$$

and the integral being taken from $x=0$, and $y=0$, we have $C=0$.

$$\text{Hence } h = \frac{3}{8}\,.\,\frac{(a^2-x^2)^{\frac{1}{2}}.\,(2x^2-a^2)\,.\,x + a^4.\,\text{arc}\left(\sin. = \frac{x}{a}\right)}{a^3-(a^2-x^2)^{\frac{3}{2}}}.$$

If we take the whole curve, that is, make $x=a$, we have

$$h = \frac{3}{8}\,.\,\frac{a^4\,.\,\frac{\pi}{2}}{a^3} = \frac{3\pi}{16}\,.\,a.$$

Similarly, we should obtain the following results:

Ex. 2. If PAp, fig. 110, be the common parabola,

$$AG = \frac{3}{5}\,AM.$$

Ex. 3. If PAp be any parabola whose equation is $y^{m+n} = a^m x^n$,

$$AG = \frac{m+2n}{2m+3n}AM.$$

Ex. 4. If PAp be a segment of a circle whose center is C,

$$CG = \frac{PM^3}{3\ \text{area}\ AMP}.$$

Ex. 5. If BAb be a semi-circle; center C; (fig. 110.)

$$CG = \frac{4}{3\pi} AC.$$

Ex. 6. If PAp be any segment of an ellipse, whose semi-axes are $CA=a$, and $CB=b$;

$$CG = \frac{a^2}{b^2} \cdot \frac{PM^3}{3 \text{ area } AMP}.$$

Ex. 7. If BAb be a semi-ellipse; center C;

$$CG = \frac{4}{3\pi} AC.$$

Ex. 8. If PAp, fig. 112, be any segment of a hyperbola whose semi-axes are $CA=a$, $CB=b$;

$$CG = \frac{a^2}{b^2} \cdot \frac{PM^3}{3 \text{ area } AMP}.$$

Ex. 9. If $P'Bbp'$ be a segment of the area contained between the two hyperbolas which are conjugate to PAp;

$$CG' = \frac{a^2}{b^2} \cdot \frac{P'M'^3 - CB^3}{3P'Bbp'}.$$

Ex. 10. If BP' be a rectangular hyperbola whose asymptotes are CE and Ce, and if we complete the parallelogram $P'O$, we have for the area $PP'Q'Q$;

$$Cg = \frac{\text{area } P'O}{\text{area } P'Q'} \cdot mn.$$

Ex. 11. If PAp, fig. 110, be a cycloid; axis AC; we have for the whole cycloid,

$$AG = \frac{7}{12} AC.$$

Ex. 12. If $PApC$ be a sector of a circle; center C;

$$CG = \frac{2}{3} \cdot \frac{AC \cdot Pp}{\text{arc } Pp} = \frac{2}{3} \cdot \frac{\text{rad. chord}}{\text{arc}}.$$

2. *A Curvilinear Area not symmetrical.*

96. Let $BCQP$, fig. 111, be a curvilinear area bounded by two curves BP, CQ, and their ordinates. Let G be its center of gravity, and GK, GH the co-ordinates of this point parallel to the known lines Ax and Ay. Then we may find GH as before, by the formula

$$h = \frac{\int x\,dm}{\int dm};$$

when the value of dm is the differential of the area $BCQP$. If $AM = x$, $MP = y$, $MQ = y'$, we have $dm = (y - y')\,dx$;

$$\therefore h = \frac{\int (y - y')\,x\,dx}{\int (y - y')\,dx} \ldots\ldots\ldots (1).$$

But if we take the integral $\int dy\,dx$ with respect to dy, considering dx as constant, and supposing the integral to begin when y is y', it will be $y\,dx - y'dx$; or $(y - y')\,dx = \int dy\,dx$. Hence we have for h

$$h = \frac{\iint x\,dy\,dx}{\iint dy\,dx} \ldots\ldots\ldots (2).$$

When an integral is to be found by two integrations, thus indicated by the double sign $\iint$, the first integration is to be performed considering y as the variable quantity, and x and dx as constant. We must then substitute for y the value which it has as a function of x, according to the manner in which the integral is limited, and must integrate the resulting expression considering x as the variable quantity.

If we take any point p in the area, and p' near to it, and draw pM, pN, $p'M'$, $p'N'$ parallel to the co-ordinates, we shall have a small rectangle pp' which, when we consider it as tending to its limit by the approach of p and p', will correspond to $dx\,dy$. And by integrating the expression which involves $dx\,dy$, with respect to y, we obtain that element of the integral which answers to the elementary space $PQQ'P'$. And again by integrating the element so found with respect to x, we find the integral corresponding to the whole space $BCPQ$.

Now if we perform the integrations with respect to y and x in a reverse order, we shall evidently obtain the same result. For the first integration with respect to x will give us the element corresponding to the elementary space $oo'p'q$; and this integrated with respect to y, will give us the integral corresponding to the whole space $BCPQ$.

Hence, instead of $\int\int x\,dy\,dx$, we may use $\int\int x\,dx\,dy$; and $\int\int dx\,dy$ instead of $\int\int dy\,dx$.

$$\text{But } \int\int x\,dx\,dy = \int dy \int x\,dx$$
$$= \int dy \frac{x^2 - x'^2}{2}$$

where x' is the value of x when the area begins, and x the value where it ends, corresponding to a constant value of y. Hence

$$h = \frac{\int dy\,(x^2 - x'^2)}{2\int\int dx\,dy} \ldots\ldots\ldots (3).$$

In the same manner if k be the ordinate GK, we shall have

$$k = \frac{\int dx\,(y^2 - y'^2)}{2\int\int dy\,dx} \ldots\ldots\ldots (4),$$

where y' and y are the first and last values of the ordinate corresponding to a given value of x.

The value of k may be found by formulæ corresponding to any of those which we have obtained for h. And by taking a value of h and a value of k, we determine the position of the center of gravity.

Thus we may take formulæ (1) and (4), putting $\int (y - y')\,dx$ for $\int\int dy\,dx$.

$$\left.\begin{array}{l} h = \dfrac{\int (y-y')\,x\,dx}{\int (y-y')\,dx} \\[2ex] k = \dfrac{\int (y^2 - y'^2)\,dx}{2\int (y-y')\,dx} \end{array}\right\} \ldots\ldots\ldots (5).$$

In these formulæ the value of y in terms of x is to be substituted, after which the integration is to be performed with respect to x, and to be taken between the limits corresponding to the extremities of the curve.

If the curvilinear space be bounded, as ADB, fig. 113, by the abscissa, the ordinate and the curve, we shall have $y'=0$,

$$h=\frac{\int yx\,dx}{\int y\,dx};\ k=\frac{\int y^2\,dx}{2\int y\,dx}\ldots\ldots\ldots(6).$$

If the curvilinear space be bounded as ACD, fig. 114, by the lines Ax, Ay, and by the curve, we might use the formulæ

$$h=\frac{\int xy\,dx}{\int y\,dx},\ k=\frac{\int yx\,dy}{\int x\,dy}\ldots\ldots\ldots(7),$$

the integrations in the first beginning when $x=0$, and in the second when $y=0$.

Ex. 13. Let AB, fig. 113, be a parabola whose axis is AD: to find the center of gravity of the space ADB.

Let $y^2=cx$; and by formula (6),

$$h=\frac{\int yx\,dx}{\int y\,dx};\ k=\frac{\int y^2\,dx}{2\int y\,dx}.$$

$$\text{Now }\int y\,dx=\int c^{\frac{1}{2}}x^{\frac{1}{2}}dx=\tfrac{2}{3}c^{\frac{1}{2}}x^{\frac{3}{2}},$$

$$\int yx\,dx=\int c^{\frac{1}{2}}x^{\frac{3}{2}}dx=\tfrac{2}{5}c^{\frac{1}{2}}x^{\frac{5}{2}},$$

$$\int y^2\,dx=\int cx\,dx=\tfrac{1}{2}cx^2;$$

$$\therefore h=\tfrac{3}{5}x,\ k=\tfrac{3}{8}c^{\frac{1}{2}}x^{\frac{1}{2}}=\tfrac{3}{8}y.$$

Hence, if we take $AH=\frac{3}{5}AD$, and $AK=\frac{3}{8}AC$, by completing the parallelogram, we have G the center of gravity of ADB.

3. *A Solid of Revolution.*

97. In a solid of revolution, whose axis is Ax, $dm=\pi y^2dx$; hence

$$h=\frac{\int\pi y^2x\,dx}{\int\pi y^2\,dx}=\frac{\int y^2x\,dx}{\int y^2\,dx}.$$

And the center of gravity will be in the axis Ax: hence if we measure this value of h along Ax, we have the center of gravity.

Ex. 14. Let PAp, fig. 110, be a segment of a sphere whose center is C; $AC=a$, $AM=x$, $AG=h$,

$$h=\frac{8ax-3x^2}{4(3a-x)}.$$

And for the whole hemisphere, when $x=a$,

$$h=\frac{5a}{8}.$$

Ex. 15. Let the body be a segment of a spheroid generated by an elliptical segment PAp; the center of gravity will be the same as that of a segment of a sphere with the same axis and center C; or, as before,

$$h=\frac{8ax-3x^2}{4(3a-x)}.$$

And for the hemispheroid $h=\frac{5a}{8}$

Ex. 16. Let PAp, fig. 112, be a hyperboloid with center C; $CA=a$, $AM=x$, $AG=h$;

$$h=\frac{8ax-3x^2}{4(3a+x)}.$$

As x becomes very great the value to which this tends is

$$h=\frac{3x}{4};$$

which agrees with the expression for a cone.

Ex. 17. If PAp be a paraboloid;

$$h=\frac{2x}{3}.$$

Ex. 18. If the figure be a frustum of a paraboloid, of which the radii of the less and of the greater ends are a and b, and the length of the axis x, the distance of the center from the smaller end, h,

$$h = \frac{a^2 + 2b^2}{a^2 + b^2} \cdot \frac{x}{3}.$$

Ex. 19. If PAp be a solid by the revolution of any parabola whose equation is

$$y^{m+n} = a^m x^n;$$

$$h = \frac{m + 3n}{2m + 4n} \cdot x.$$

4. *Any Solid.*

98. Let PBQ, fig. 115, represent any solid bounded by a surface to which we have an equation in terms of three rectangular co-ordinates x, y, z. Let Ax be the direction of one of the co-ordinates, and let the body h be cut by a plane PM perpendicular to Ax. Let A be the area of the section of the body made by this plane. Then dm will $= A dx$.

Now the boundaries of the plane A perpendicular to AM will be determined by the co-ordinates y and z, which are perpendicular to AM, and in the plane A. Hence we shall have $A = \int z\, dy$, or as before $A = \iint dy\, dz$. And $dm = dx \iint dy\, dz$;

$$\therefore h = \frac{\int x\, dx \iint dy\, dz}{\int dx \iint dy\, dz}.$$

Or, since, as in last Article, the order of the integrations is indifferent*,

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* The expression for the solidity of a body is $\iint z\, dy\, dx$. Similarly it is $\iint x\, dy\, dx$, and $\iint y\, dx\, dz$. The expression $\iiint dx\, dy\, dz$ comprehends all these three. For the order of the integrations is indifferent (by reasoning similar to that in p. 144.); and if we make the first integration with respect to z, we obtain $\iint z\, dx\, dy$: if with respect to x, we have $\iint x\, dy\, dz$;

$$h = \frac{\iiint x\,dx\,dy\,dz}{\iiint dx\,dy\,dz};$$

similarly, $$k = \frac{\iiint y\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

$$l = \frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz}:$$

k, l being the co-ordinates of the center of gravity parallel respectively to y and to z.

If we suppose the integration in z to be performed, we shall have

$$h = \frac{\iint xz\,dx\,dy}{\iint z\,dx\,dy},$$

$$k = \frac{\iint yz\,dx\,dy}{\iint z\,dx\,dy},$$

$$l = \frac{\iint z^2\,dx\,dy}{2\iint z\,dx\,dy}.$$

And z being known in terms of x, y, its value may be substituted, and the integrations in y performed, between the proper limits. Then the value of y in terms of x may be substituted; and the integrations performed with respect to x will give the value of h.

$\iint x\,dy\,dz$; if with respect to y, we have $\iint y\,dx\,dz$. Similarly $\int x\,dx \iint dy\,dz$ is the same as $\iiint x\,dx\,dy\,dz$; and so of the rest.

In the same way in which $\iint dx\,dy$ is the area of a curve (Art. 96), $\iiint dx\,dy\,dz$ is the content of a solid. The product $dx\,dy$ may be considered as an evanescent rectangle whose sides are dx and dy, and whose position is determined by the co-ordinates x and y; the whole area being supposed to be made up of such rectangles: and similarly $dx\,dy\,dz$ may be considered as an evanescent rectangular parallelepiped. And the body being conceived to be composed of such particles, the center of gravity may be found by substituting them for the points P_1, P_2, &c. in Art. 49, and by putting the integral for the sum.

Ex. 20. Let the body be a fourth part of a paraboloid of revolution; as *ABCD*, fig. 115; cut off by a plane *BAC* perpendicular to the axis, and by two planes *BAD*, *CAD*, perpendicular to the preceding and to each other: to find its center of gravity.

Let *A* be the origin, and *AB*, *AC*, *AD* the axes of the rectangular co-ordinates x, y, z, respectively. If $AM = x$, $AN = MO = y$, $OP = z$, the equation to the surface will be

$$x^2 + y^2 + bz = a^2,$$

where AB or $AC = a$, and the axis $AD = \frac{a^2}{b}$.

$$\iint z\,dx\,dy = \iint \left(\frac{a^2 - x^2 - y^2}{b}\right) dx\,dy = \frac{1}{b}\int dx \int (a^2 - x^2 - y^2)\,dy.$$

And, integrating first for y,

$$= \frac{1}{b}\int dx \left(a^2 y - x^2 y - \frac{y^3}{3}\right).$$

The limits of the integration for y are determined by the nature of the part considered; if it is to be bounded by a plane *RNO* parallel to the plane of xz at a distance AN, y must be taken from 0 to AN; and in the next integration must be supposed constant. Hence we have

$$\iint z\,dx\,dy = \frac{1}{b}\left(a^2 xy - \frac{x^3 y}{3} - \frac{y^3 x}{3}\right).$$

Where the limits of the integration for x are determined in the same way as for y. If the solid be bounded by a plane *QMO* parallel to the plane of yz, at a distance $x = AM$, the quantity now found expresses the solid; or

$$\text{solid } AP = \frac{xy}{b}\left(a^2 - \frac{x^2 + y^2}{3}\right) = \frac{xy\,.\,(3\,a^2 - x^2 - y^2)}{3\,b}.$$

If the solid be not bounded by a plane *RNO*, but continued till its surface meets the plane *CAB* in *Cm*, we must, after the integration for y, put for y the value which it assumes by making $z = 0$, or $x^2 + y^2 = a^2$, whence $y^2 = a^2 - x^2$.

Hence

$$\iint z\,dx\,dy = \frac{1}{b}\int dx \left(a^2 y - x^2 y - \frac{y^3}{3}\right)$$

$$= \frac{1}{b} \int dx \left(a^2 - x^2 - \frac{y^2}{3} \right) \cdot y$$

$$= \frac{2}{3\,b} \int (a^2 - x^2)^{\frac{3}{2}} \cdot dx,$$

which (taken from $x = 0$) gives the solid $ACmQD$

$$= \frac{2}{3.8.b} \left\{ 2x\,(a^2 - x^2)^{\frac{3}{2}} + 3a^2 x\,(a^2 - x^2)^{\frac{1}{2}} + 3a^4 . \text{arc} \left(\sin. = \frac{x}{a} \right) \right\}.$$

And if we take the whole solid, x must be taken $= a$; in this case the arc will become $\frac{\pi}{2}$, and we shall have

$$\text{whole solid } ABCD = \frac{2}{3.8.b} . 3a^4 . \frac{\pi}{2} = \frac{\pi a^4}{8\,b}.$$

The solidity of the whole might be more simply found; for it will manifestly be $\frac{1}{4}$ of the whole paraboloid; and a paraboloid is $\frac{1}{2}$ the cylinder on the same base (rad. $= a$), and with the same altitude; hence

$$\text{whole solid} = \frac{1}{4} . \frac{1}{2} \pi a^2 . \frac{a^2}{b} = \frac{\pi a^4}{8b}, \text{ as before.}$$

We now proceed to find $\iint xzdxdy$,

$$\iint xzdxdy = \iint x \left(\frac{a^2 - x^2 - y^2}{b} \right) dxdy;$$

$$= \frac{1}{b} \int xdx \int (a^2 - x^2 - y^2)\, dy;$$

$$= \frac{1}{b} \int xdx \left(a^2 y - x^2 y - \frac{y^3}{3} \right);$$

the limits of y as before. On the first supposition, that the body is bounded by planes RNO, QMO, we have

$$\iint xzdxdy = \frac{1}{b} \left(\frac{a^2 x^2 y}{2} - \frac{x^4 y}{4} - \frac{x^2 y^3}{6} \right)$$

$$= \frac{x^2 y\,(6a^2 - 3x^2 - 2y^2)}{12\,b};$$

hence, for AP,

$$h = \frac{b \,.\, \{x^2 y \,.\, (6a^2 - 3x^2 - 2y^2)\}}{4b \,.\, \{xy \,.\, (3a^2 - x^2 - y^2)\}}$$

$$= \frac{x}{4} \cdot \frac{6a^2 - 3x^2 - 2y^2}{3a^2 - x^2 - y^2}.$$

If we suppose $AMON$, the base of the figure, to be a square, or $y = x$; this becomes

$$h = \frac{x}{4} \cdot \frac{6a^2 - 5x^2}{3a^2 - 2x^2}.$$

If we suppose the quadrilateral curve surface $DQPR$ to have its angle P in the circumference of the base BC, as at m, we shall have $z = 0$; and hence

$x^2 + y^2 = a^2$, or $y^2 = a^2 - x^2$; and hence for the solid Mr,

$$h = \frac{x}{4} \cdot \frac{4a^2 - x^2}{2a^2}.$$

On the second supposition, that the surface of the solid is to be continued till it meets the plane ABC, we must, after the integration for y, substitute for y its value in that plane, that is, $y = \sqrt{(a^2 - x^2)}$; hence we have

$$\iint xz\,dx\,dy = \frac{1}{b}\int x\,dx \left(a^2 y - x^2 y - \frac{y^3}{3}\right)$$

$$= \frac{1}{b}\int x\,dx \left(a^2 - x^2 - \frac{a^2 - x^2}{3}\right) . \sqrt{(a^2 - x^2)}$$

$$= \frac{2}{3b} . \int x\,dx \,.\, (a^2 - x^2)^{\frac{3}{2}}$$

$$= \frac{2}{3b} \cdot \left(\frac{a^5}{5} - \frac{(a^2 - x^2)^{\frac{5}{2}}}{5}\right),$$

taking the integral from $x = 0$; and for the whole solid $ABCD$, or when $x = 0$, it becomes

$$= \frac{2}{3b} \cdot \frac{a^5}{5} = \frac{2a^5}{15b}.$$

Hence, for the whole solid $ABCD$,

$$h = \frac{2a^5}{15b} \cdot \frac{8b}{\pi a^4} = \frac{16a}{15\pi} = \frac{a}{3}, \text{ nearly.}$$

In the same way it might be shewn that, for the part of the solid bounded by planes parallel to the planes of xz and xy, we have

$$k = \frac{y}{4} \cdot \frac{6a^2 - 3y^2 - 2x^2}{3a^2 - y^2 - x^2};$$

$$l = \frac{3\left\{a^4 - \frac{2}{3}a^2(x^2+y^2) + \frac{2}{9}x^2y^2 + \frac{1}{5}(x^4+y^4)\right\}}{2b \cdot (3a^2 - x^2 - y^2)}.$$

And for the whole solid

$$k = \frac{16a}{15\pi},$$

$$l = \frac{b^2}{3a} = \frac{\text{axis}}{3},$$

which last result also follows from Ex. 17.

5. *A Plane Curve.*

99. When the body is a curve lying in one plane, if we suppose it to be a physical line of inconsiderable thickness, ds being the differential of its length, dm will be as ds; we may suppose dm to represent ds. Hence

$$h = \frac{\int x\,ds}{\int ds}.$$

But we have $ds = dx\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$ (Lacroix, *Elementary Treatise*, Art. 75.). Therefore

$$h = \frac{\int x\,dx\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{\int dx\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}.$$

Similarly, $k = \frac{\int y\,ds}{\int ds}$;

$$\text{or, } k = \frac{\int y\,dx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{\int dx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}.$$

If the curve be symmetrical with respect to Ax, it will be sufficient to find h, since the center of gravity will be in Ax.

Ex. 21. Let PAp, fig. 110, be a circular arc, center C, radius $= a$.

Let arc $AP = s$; $\therefore CM = x = a \,.\, \cos \frac{s}{a}$;

$$\therefore \int x\,ds = a \int \cos. \frac{s}{a} \,.\, ds = a^2 \sin. \frac{s}{a}.$$

And if the whole arc be $2l$, and its middle point A, the integral must be taken from $s = -l$ to $s = l$; $\therefore \int x\,ds = 2a^2 \,.\, \sin. \frac{l}{a}$.

$$\text{Hence } CG = h = \frac{2a^2 \,.\, \sin. \frac{l}{a}}{2l} = \frac{a \,.\, 2 \sin. l \; (\text{rad.} = a)}{2l}$$

$$= \frac{\text{radius . chord}}{\text{arc}}.$$

Cor. Hence for the semi-circle, $h = \frac{2a}{\pi}$.

Ex. 22. Let APB, fig. 113, be a semi-cycloid with axis AD.

$$AH = h = \frac{AD}{3}, \quad HG = k = DB - \frac{2}{3} AD.$$

Cor. Hence $CK = DH$.

Ex. 23. Let PAp, fig. 110, be a catenary of which A is the lowest point. Take AD vertical and equal to a length of the string equivalent to the tension; then

$$DG = \tfrac{1}{2} DM + \frac{DA \,.\, MP}{PAp}.$$

6. *A Curve of double Curvature.*

100. Let s be the length of the curve; and, as before, $dm=ds$. Then, if x, y, z be the rectangular co-ordinates to the curve,

$$ds = dx\sqrt{\left(1+\frac{dy^2}{dx^2}+\frac{dz^2}{dx^2}\right)}, \text{ and}$$

$$h=\frac{\int x ds}{\int ds},\ k=\frac{\int y ds}{\int ds},\ l=\frac{\int z ds}{\int ds}.$$

Ex. 24. Let the curve be the thread of a screw, of which the axis is Az.

This thread projected on the plane xy will become a circle; and if a be the radius of this circle, $a^2=x^2+y^2$. Also $\frac{x}{a}$ will be the cosine of the arc of this circle, corresponding to any point in the curve, and z will be proportional to this arc. Hence the equations to the curve are

$$y = \sqrt{(a^2-x^2)},$$

$$z = m \,.\, \text{arc}\left(\cos. = \frac{x}{a}\right),$$

m being a constant quantity, and the thread of the screw being supposed to begin in the line Ax.

$$\text{Hence } \frac{dy}{dx} = -\frac{x}{\sqrt{(a^2-x^2)}},$$

$$\frac{dz}{dx} = -\frac{m}{\sqrt{(a^2-x^2)}};$$

$$\therefore\ ds = dx\sqrt{\left(1+\frac{dy^2}{dx^2}+\frac{dz^2}{dx^2}\right)}$$

$$= -dx\frac{\sqrt{(a^2+m^2)}}{\sqrt{(a^2-x^2)}}.$$

$$\text{Hence also } ds = \frac{\sqrt{(a^2+m^2)}}{m}dz,$$

$$\text{and } s = \frac{\sqrt{(a^2+m^2)}}{m}z.$$

$$\text{Now} \int x\,ds = \int -x\,dx\,\frac{\sqrt{(a^2+m^2)}}{\sqrt{(a^2-x^2)}}$$

$$= \sqrt{(a^2+m^2)(a^2-x^2)};$$

which begins when $x = a$.

$$\text{Also} \int y\,ds = \int -\sqrt{(a^2+m^2)}\,dx$$

$$= (a-x)\sqrt{(a^2+m^2)},$$

which also begins when $x = a$.

$$\text{And} \int z\,ds = \int \frac{\sqrt{(a^2+m^2)}}{m}\,z\,dz$$

$$= \frac{\sqrt{(a^2+m^2)}}{m} \cdot \frac{z^2}{2}.$$

Hence

$$h = \frac{m\sqrt{(a^2-x^2)}}{z};$$

$$k = \frac{m(a-x)}{z};$$

$$l = \frac{z}{2}.$$

If $x = a$, that is, if the spiral consist of a complete number of revolutions, $h = 0$, $k = 0$. In this case the center of gravity is in the axis, and in the middle of its length.

If $x = 0$, $h = \frac{ma}{z}$, $k = \frac{ma}{z}$: in this case the spiral consists of a complete number of revolutions together with a quarter of a revolution.

7. *A Surface of Revolution.*

101. If s be the length of the curve, $2\pi y\,ds$ is the differential of the surface, and, as before, this may be put for dm. Also the center of gravity will be in the axis of revolution. Hence

$$h = \frac{\int xy\,ds}{\int y\,ds} = \frac{\int yx\,dx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}{\int y\,dx \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}}.$$

Ex. 25. If the surface be a cone, and h the distance from the vertex,

$$h = \frac{2x}{3}.$$

Ex. 26. If the surface be a sphere, and h measured from the vertex,

$$h = \frac{x}{2}.$$

8. *Any Surface.*

102. Let the surface be defined by an equation $u = 0$, between x, y, z; whence we may find z in terms of x and y.

Let $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$; h, k, l as in Art. 98. The differential of the surface is $(1 + p^2 + q^2)^{\frac{1}{2}} dx\,dy$; hence as before,

$$h = \frac{\iint x(1 + p^2 + q^2)^{\frac{1}{2}} dx\,dy}{\iint (1 + p^2 + q^2)^{\frac{1}{2}} dx\,dy};$$

$$k = \frac{\iint y(1 + p^2 + q^2)^{\frac{1}{2}} dx\,dy}{\iint (1 + p^2 + q^2)^{\frac{1}{2}} dx\,dy};$$

$$l = \frac{\iint z(1 + p^2 + q^2)^{\frac{1}{2}} dx\,dy}{\iint (1 + p^2 + q^2)^{\frac{1}{2}} dx\,dy}.$$

Ex. 27. Let a conical surface be divided into four parts by two planes perpendicular to each other, passing through the axis. To find the center of gravity of one of these parts: as BCD, fig. 115; DB and DC being supposed to be straight lines.

If we make the vertex D the origin of co-ordinates, the axis the line of z, and measure x, y, parallel to AB, AC, respectively, we have

$$z = m\sqrt{(x^2 + y^2)};$$

Where m is the tangent of the angle which the slant side makes with the base.

Hence

$$p = \frac{dz}{dx} = \frac{mx}{\sqrt{(x^2+y^2)}},$$

$$q = \frac{dz}{dx} = \frac{my}{\sqrt{(x^2+y^2)}};$$

$$\therefore (1+p^2+q^2)^{\frac{1}{2}} dx dy = (1+m^2)^{\frac{1}{2}} dx dy;$$

$$\therefore \iint (1+p^2+q^2)^{\frac{1}{2}} dx dy = (1+m^2)^{\frac{1}{2}} . \int y dx.$$

And if the curve surface be a quadrilateral figure $DQPR$ bounded by planes parallel to those of the co-ordinates,

$$\text{this} = (1+m^2)^{\frac{1}{2}} xy.$$

But if we take the surface as bounded by a plane BAC perpendicular to the axis at the distance $a = DA$, we must have, after the integration for y, $z = a$, the axis;

$$\text{Now } a^2 - m^2x^2 = m^2y^2,\ y = \frac{\sqrt{(a^2-m^2x^2)}}{m},$$

$$\therefore \iint (1+p^2+q^2)^{\frac{1}{2}} dx dy = \frac{(1+m^2)^{\frac{1}{2}}}{m} . \int (a^2-m^2x^2)^{\frac{1}{2}} dx$$

$$= \frac{(1+m^2)^{\frac{1}{2}}}{2m} \left\{ (a^2-m^2x^2)^{\frac{1}{2}} . x + \frac{a^2}{m} . \text{arc} \left(\sin. = \frac{mx}{a} \right) \right\} + \text{const.}$$

and, from $x = 0$ to $x = AB = \frac{a}{m}$

$$= \frac{(1+m^2)^{\frac{1}{2}}}{2m} . \frac{a^2 . \pi}{2m} = \frac{(1+m^2)^{\frac{1}{2}} . \pi a^2}{4m^2},$$

which might be deduced also from the known method of finding the surface of a cone.

To find the numerator of h, we have it

$$= (1+m^2)^{\frac{1}{2}} \iint x dy dx = (1+m^2)^{\frac{1}{2}} \int yx dx,$$

and, for the quadrilateral $DPQR$, $= \dfrac{(1+m^2)^{\frac{1}{2}}}{2} yx^2.$

But, for the whole surface DBC,

$$= \frac{(1+m^2)^{\frac{1}{2}}}{m} \int (a^2 - m^2 x^2)^{\frac{1}{2}} x \, dx$$

$$= -\frac{(1+m^2)^{\frac{1}{2}}}{3m^3} (a^2 - m^2 x^2)^{\frac{3}{2}} + \text{constant}.$$

And, taken from $x=0$ to $x = AB = \frac{a}{m}$

$$= \frac{(1+m^2)^{\frac{1}{2}} a^3}{3m^3}.$$

The numerator of k will manifestly be the same.

Similarly, for the numerator of l,

$$\iint (1+p^2+q^2)^{\frac{1}{2}} z \, dx \, dy = (1+m^2)^{\frac{1}{2}} \iint z \, dx \, dy$$

$$= (1+m^2)^{\frac{1}{2}} \iint m (x^2+y^2)^{\frac{1}{2}} dx \, dy$$

$$= (1+m^2)^{\frac{1}{2}} \, . \, m \int \left\{ \frac{(x^2+y^2)^{\frac{1}{2}} y}{2} + \frac{x^2}{2} \, \mathrm{l} \, \frac{y + \sqrt{(x^2+y^2)}}{x} \right\} dx,$$

in which the denominator x is given to the quantity under the logarithmic sign that the integral may begin when $y=0$. For the quadrilateral surface $DPQR$ we must now integrate for x supposing y constant; and the double integral becomes,

$$= \frac{(1+m^2)^{\frac{1}{2}} \, . \, m}{6} \left\{ 2xy \, . \, (x^2+y^2)^{\frac{1}{2}} + x^3 \, \mathrm{l} \, \frac{y + \sqrt{(x^2+y^2)}}{x} \right.$$

$$\left. + \, y^3 \, \mathrm{l} \, \frac{x + \sqrt{(x^2+y^2)}}{y} \right\}.$$

For the whole surface DBC, we must, after integrating for y, put for y the value $\frac{\sqrt{(a^2 - m^2 x^2)}}{m}$; and it becomes,

$$= \frac{(1+m^2)^{\frac{1}{2}} \, . \, m}{2} \int \left\{ \frac{a \sqrt{(a^2 - m^2 x^2)}}{m^2} + x^2 . \mathrm{l} \, \frac{\sqrt{(a^2 - m^2 x^2)} + a}{mx} \right\} dx:$$

and the integral being taken from $x=0$ to $x = \frac{a}{m}$, this becomes

$$= \frac{(1+m^2)^{\frac{1}{2}} \, . \, \pi a^3}{6m^2}.$$

Hence, for the whole conical surface,

$$h = k = \frac{(1+m^2)^{\frac{1}{2}}.a^3}{3m^3} \cdot \frac{4m^2}{(1+m^2)^{\frac{1}{2}}.\pi a^2} = \frac{4a}{3\pi m},$$

$$l = \frac{(1+m^2)^{\frac{1}{2}}.\pi a^3}{6m^2} \cdot \frac{4m^2}{(1+m^2)^{\frac{1}{2}}.\pi a^2} = \frac{2a}{3},$$

which last agrees with Ex. 33, as it should.

For the center of gravity of the quadrilateral surface $DPQR$, where $AM=x$, $AN=y$, we have for h, k, l, other expressions which are easily deduced from the results given above.

9. *Guldinus's Properties**.

107. Prop. *If any plane figure revolve about an axis in its own plane, the content of the solid generated by this figure in its revolution is equal to a prism whose base is the revolving figure, and its height the length of the path described by the center of gravity of the plane figure.*

The figure may either be bounded by straight lines, or curves; or by a combination of the two; and the revolution may take place either through a whole circumference or any part of it.

We shall suppose the whole of the revolving figure to be on one side of the axis.

Let AB, fig. 116, be the axis of revolution, PQR the figure; G its center of gravity; GK, PQM, ordinates perpendicular to AB. And let the figure revolve into the position $P'Q'R'$; the angle PMP' being $=\theta$. Also let $AM=x$, $PM=y$, $MQ=y'$, $GK=k$.

The sector $PMP' = \frac{1}{2}y^2\theta$, and $QMQ' = \frac{1}{2}y'^2\theta$. Hence $PQQ'P' = \frac{1}{2}(y^2-y'^2)\theta$; and hence the differential of the solid PR' corresponding to dx is $\frac{1}{2}(y^2-y'^2)\theta dx$. Hence the solid $=\frac{\theta}{2}\int(y^2-y'^2)dx$.

* The propositions known by this name were discovered by Pappus, and republished about 1640, by Guldin or Guldinus, a Jesuit, who was professor of Mathematics at Rome.

Also the prism whose base is PQR, and altitude the arc GG', is $= PQR \, . \, GG'$; and area $PQR = \int (y - y') \, dx$, $GG' = k\theta$. Hence this prism $= k\theta \int (y - y') \, dx$.

But by formula (5) for k, in Art. 96, we have

$$k\int (y - y') dx = \tfrac{1}{2} \int (y^2 - y'^2) dx.$$

Therefore the figure described by the revolution of PQR is equal to the prism mentioned in the Proposition.

If the figure be composed of several curves, or of straight lines, both the numerator and denominator of k will consist of several integrals added together, corresponding to the different parts of the figure. Also both the area of the figure, and the content of the solid will consist of parts corresponding to these; and the solid and the prism will be found to be equal in the same manner as before.

108. Prop. *If any plane figure revolve about any axis in its own plane, the area of the surface generated by the perimeter of this figure in its revolution is equal to a rectangle, one of whose sides is the perimeter and the other the length of the path described by the center of gravity of the perimeter.*

The denominations remaining as in last Article, let ds be the differential of the length of the curve corresponding to dx; and since $y\theta$ is the length of the path described by P, $y\theta ds$ is the differential of the surface described by the revolution; and $\theta \int y ds$ is the whole surface.

Also the whole perimeter is $\int ds$, and if G be now its center of gravity, $k\theta$ is the path described by the center of gravity of the perimeter; and $k\theta \int ds$ is the rectangle mentioned in the proposition.

But by Art. 99,

$$k\int ds = \int y ds,$$

whence the proposition is manifest.

109. Hence we may find the contents and areas of surfaces of revolution whenever we can find the area or perimeter of the revolving figure and its center of gravity.

Ex. 34. Let the figure be a circle which, revolving round an axis without it, generates a solid, resembling a cylinder bent so as to return into itself, or the ring of an anchor. The center of the circle will be the center of gravity both of the area and of the perimeter. Hence, by Article 107, the solid content of such a ring is equal to the cylinder whose base is the revolving circle and its length the circle described by the center of the circle. Also by Article 108, the surface of the ring is equal to the rectangle contained by the circumference of the revolving circle and the path of its center; that is, it is equal to the surface of the cylinder beforementioned. Hence if we could suppose that the ring was cut through in some part, and unrolled into a cylinder so that its axis should remain of the same length as before, both the solidity and the surface would continue unaltered.

CHAP. IX.

Cut — M.S

THE EQUILIBRIUM OF A FLEXIBLE BODY.

110. The equilibrium of a flexible body depends upon the same conditions as that of a rigid one, and may be deduced from the principles already laid down. These principles may be applied by observing that, in all cases, *a flexible body may be supposed to become rigid after the equilibrium is established:* or, that the forces which keep a flexible body at rest would keep at rest a rigid body of the same form. For after the body has assumed the form which the forces produce in it, there is no tendency to change the form; hence it makes no difference whether we suppose the body to have the property of resisting a change of form or not: that is, it makes no difference whether we suppose the body to be rigid or flexible.

We shall suppose the bodies to be *perfectly* flexible, that is, to offer *no* resistance to any change of figure.

The *Tension* of a string or chain is the force exerted by one part upon another contiguous one in the direction of its length. Every point of the string must be acted upon by equal and opposite forces of this kind: and a force of the same kind is exerted upon any fixed point to which the string is attached.

We shall consider the equilibrium of a flexible *line**, acted on by various forces. This line may be supposed to be a *cord*, indefinitely slender and perfectly void of stiffness; or a *chain* composed of indefinitely small links. On this account the curve formed by the line is called *the Catenary.*

1 *The Catenary, when a uniform Chain is acted on by Gravity.*

111. PROP. *To find the equation to the catenary between x and s, beginning at the lowest point.*

Let AB, fig. 117, represent the catenary. Let C be the lowest point, CM vertical $=x$, MP horizontal $=y$, $CP=s$. The portion CP may be supposed to become rigid after it has assumed the form of equilibrium: and since its weight and figure remain the same as before, it will be supported in the same manner. Now the forces which act upon the portion CP are, besides its own gravity, the tension at C and the tension at P: and these three forces must keep CP in equilibrium. Also the tensions are in the directions of the tangents RC and RP at C and P.

Let PR meet MC in T, PM will be parallel to RC, and hence the three lines MT, PM, TP are parallel to the directions of the three forces (gravity, tension at C, tension at P), which keep CP at rest, and hence (Art. 56 and 28) the forces will be as those three lines. Hence

$$\frac{\text{tension at } C}{\text{weight of } CP} = \frac{MP}{TM}.$$

* Flexible bodies may be lines, surfaces, or solids. A flexible *line* can in all cases be extended into a straight line. A flexible surface is not necessarily susceptible of being unrolled into a plane without stretching or tearing; if it be capable of this it is called a *developable surface.*

Let the tension of the string at C be equal to the weight of a length c of the string; the weight of the length CP will be as CP or s; and the first side of the above equation will be $\frac{c}{s}$. Also $\frac{dy}{dx}$ will be equal to the second side.

$$\text{Hence } \frac{c}{s} = \frac{dy}{dx} \dots\dots\dots\dots (1);$$

from which equation the properties of the curve may be deduced.

If we square both sides of equation (1) and add unity to them, we have

$$\frac{c^2 + s^2}{s^2} = \frac{dy^2 + dx^2}{dx} = \frac{ds^2}{dx^2};$$

$$\therefore dx = \frac{s\,ds}{\sqrt{(c^2 + s^2)}} \dots\dots (2),$$

and integrating, supposing $s = 0$ when $x = 0$,

$$x + c = \sqrt{(c^2 + s^2)} \dots\dots\dots (3).$$

Hence also we find

$$s = \sqrt{(x^2 + 2cx)} \dots\dots\dots (4).$$

COR. 1. If the angle which the curve makes with the vertical be called a, we have

$$\tan. a = \frac{dy}{ds} = \frac{c}{s};$$

$$\cos. a = \frac{dx}{ds} = \frac{s}{\sqrt{(c^2+s^2)}}.$$

COR. 2. For the tension at any point P,

$$\frac{\text{tension at } P}{\text{weight of } CP} = \frac{TP}{TM} = \frac{ds}{dx} = \frac{\sqrt{(c^2+s^2)}}{s};$$

$$\therefore \text{tension at } P = \sqrt{(c^2 + s^2)} = x + c.$$

Hence and from the last Corollary it appears that

$$\text{tension at } P = \frac{s}{\cos. a}.$$

Cor. 3. If we take $CD=c$ and draw DQ horizontal and PQ vertical, $PQ=DM=x+c=$ tension at P.

Cor. 4. If we put $DM=u$, $x=u-c$ and hence

$$s=\sqrt{(u^2-c^2)},$$
$$u=\sqrt{(s^2+c^2)}.$$

112. Prop. *To find the equation between y and s.*

As in last Article,

$$\frac{s}{c}=\frac{dx}{dy}\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\frac{s^2+c^2}{c^2}=\frac{dx^2+dy^2}{dy^2}=\frac{ds^2}{dy^2};$$

$$\therefore\ \frac{dy}{c}=\frac{ds}{\sqrt{(s^2+c^2)}}\ldots\ldots\ldots(2).$$

Integrating, supposing $y=0$ when $s=0$,

$$\frac{y}{c}=l\,\frac{s+\sqrt{(s^2+c^2)}}{c}\ldots\ldots(3).$$

$$\text{Hence, }\epsilon^{\frac{y}{c}}=\frac{s+\sqrt{(s^2+c^2)}}{c},$$

$$\therefore\ \epsilon^{-\frac{y}{c}}=\frac{c}{\sqrt{(s^2+c^2)}+s}=\frac{\sqrt{(s^2+c^2)}-s}{c}.$$

Subtracting, and reducing,

$$s=\frac{c}{2}\left(\epsilon^{\frac{y}{c}}-\epsilon^{-\frac{y}{c}}\right)\ldots\ldots\ldots(4).$$

Cor. We have

$$\frac{\text{tension at }P}{\text{tension at }C}=\frac{TP}{TM}=\frac{ds}{dy}=\frac{\sqrt{(s^2+c^2)}}{c}.$$

Also as before, tension at $P=\dfrac{c}{\sin.\alpha}$.

113. PROP. *To find the equation between x and y.*

If in the equation (1) of last Article we put the value of s from (4), we have

$$\frac{dx}{dy} = \frac{1}{2}\left(\epsilon^{\frac{y}{c}} - \epsilon^{-\frac{y}{c}}\right) \dots\dots\dots\dots (1).$$

Multiply by dy and integrate, ($x = 0$ when $y = 0$),

$$x + c = \frac{c}{2}\left(\epsilon^{\frac{y}{c}} + \epsilon^{-\frac{y}{c}}\right) \dots\dots\dots\dots (2).$$

Again, if in equation (1) of Art. 111, we put for s its value from (4) of that Article, we have

$$\frac{dy}{dx} = \frac{c}{\sqrt{(x^2 + 2cx)}} \dots\dots\dots\dots (3).$$

Multiply by dx and integrate,

$$\therefore\ y = c\,\mathrm{l}\,\frac{x + c + \sqrt{(x^2 + 2cx)}}{c} \dots . (4).$$

114. PROP. *To find the equations to the catenary beginning from any point.*

Let A, fig. 118, be a point which is considered as the beginning of the catenary, AP any arc. Let the curve of equilibrium be continued if necessary, and let C be its lowest point. Let AN vertical $= x$, NP horizontal $= y$, $AP = s$.

The portion AP will be kept in equilibrium in the same form whether we suppose it to be acted on at A by the tension of CA, or by the re-action of a fixed point. But if we suppose AP to be a portion of CAP its form will be determined as in the preceding Articles. Let c be the tension at the lowest point C, $CP = s'$, and we have as before,

$$\frac{c}{s'} = \frac{dy}{dx} \dots\dots\dots\dots (1),$$

for dx is the same whether x be CM or AN; and dy is the same whether y be MP or NP.

Let the tension at $A = a$, and the angle which the curve makes with the vertical $= \alpha$. Also let $CA = m$. Then by Art. 111, Cor. 2, a cos. $\alpha = m$. Also $s' = m + s = a$ cos. $\alpha + s$, and $ds = ds'$. And by Cor. to Art. 112, a sin. $\alpha = c$.

By equation (1),

$$\frac{c^2 + s'^2}{s'^2} = \frac{dy^2 + dx^2}{dx^2} = \frac{ds^2}{dx^2} = \frac{ds'^2}{dx^2};$$

$$\therefore\ dx = \frac{s'ds'}{\sqrt{(c^2 + s'^2)}} \ldots\ldots\ldots\ldots\ldots (2),$$

$x + C = \sqrt{(c + s'^2)}$, and putting for c and s' their values, observing that $x = 0$ when $s = 0$.

$$x + a = \sqrt{(a^2 + 2as \text{ cos. } \alpha + s^2)} \ldots\ldots\ldots (3).$$

Hence we find

$$s = \pm \sqrt{(x^2 + 2ax + a^2 \text{ cos.}^2\, \alpha)} - a \,.\, \text{cos. } \alpha \ldots\ldots (4).$$

The double sign indicates that there are two arcs corresponding to the same value of x, as AP and AP' in the figure.

In the same manner if we take

$$\frac{s'}{c} = \frac{dx}{dy},$$

we shall find

$$\frac{dy}{c} = \frac{ds'}{\sqrt{(c^2 + s'^2)}} \ldots\ldots\ldots\ldots\ldots (5),$$

$$\frac{y}{c} = \text{l}\, \frac{s' + \sqrt{(c^2 + s'^2)}}{c},$$

$$y = a \text{ sin } \alpha \text{ l } \frac{s + a \text{ cos. } \alpha + \sqrt{(s^2 + 2as \text{ cos. } \alpha + a^2)}}{a\ (1 + \text{cos. } \alpha)} \ldots (6).$$

And hence

$$a\,(1 + \text{cos. } \alpha)\, \epsilon^{\frac{y}{a \text{ sin. } \alpha}} = \sqrt{(s^2 + 2as \text{ cos. } \alpha + a^2)} + s + a \text{ cos. } \alpha,$$

whence

$$a\,(1 - \text{cos. } \alpha)\, \epsilon^{-\frac{y}{a \text{ sin. } \alpha}} = \sqrt{(s^2 + 2as \text{ cos. } \alpha + a^2)} - (s + a \text{ cos. } \alpha),$$

as appears by multiplying the equations. Hence, subtracting and dividing by 2,

$$s + a\cos.\alpha = \frac{a}{2}\left\{(1+\cos.\alpha)\,\epsilon^{\frac{y}{a\sin.\alpha}} - (1-\cos.\alpha)\,\epsilon^{-\frac{y}{a\sin.\alpha}}\right\}\ldots(7).$$

Multiply by dy, observing that

$$(s + a\cos.\alpha)\,dy = s'dy = c\,dx = a\sin.\alpha dx,$$

by (3). Hence integrating

$$x + a = \frac{a}{2}\left\{(1+\cos.\alpha)\,\epsilon^{\frac{y}{a\sin.\alpha}} + (1-\cos.\alpha)\,\epsilon^{-\frac{y}{a\sin.\alpha}}\right\}\ldots\ldots(8).$$

And in nearly the same manner as before we should find

$$y = a\cos.\alpha\ \mathrm{l}\,\frac{x + a \pm \sqrt{(x^2 + 2ax + a^2\cos.^2\alpha)}}{a\,(1 \pm \cos.\alpha)}\ldots\ (9).$$

Cor. It appears that $\dfrac{dy}{dx} = \dfrac{a\sin.\alpha}{a\cos.\alpha + s}$.

115. By means of the formulæ thus obtained we may solve the following Problems.

Prob. I. *A chain of given length $BCB' = 2l$, fig. 118, hangs from two given points B, B' in the same horizontal line, of which the distance $BB' = 2h$ is given; to find its position.*

The middle point will here be the lowest, and the chain will form a symmetrical figure with respect to the axis CE;

$$CB = CB' = l, \quad EB = EB' = h.$$

Let α be the angle which the curve at B makes with the vertical line; and by equation (3) Art. 112,

$$\frac{y}{c} = \mathrm{l}\left(\frac{s}{c} + \frac{\sqrt{(s^2 + c^2)}}{c}\right)$$

$$= \mathrm{l}\left\{\frac{dx}{dy} + \frac{ds}{dy}\right\}$$

$$= \mathrm{l}\left\{\frac{\cos.\alpha}{\sin.\alpha} + \frac{1}{\sin.\alpha}\right\}.$$

$$\text{Also } c = s\,\frac{dy}{dx} = l\tan.\alpha, \text{ and } y = h;$$

$$\therefore h = l\tan.\alpha\ \mathrm{l}\,\frac{1+\cos.\alpha}{\sin.\alpha} = l\tan.\alpha\ \mathrm{l}\ \text{co-tan.}\,\frac{\alpha}{2};$$

$$\therefore \frac{h}{l} = -\tan. \alpha . 1 \tan. \frac{\alpha}{2}.$$

From this equation we have to determine α. This cannot be done directly, but it is easy to approximate to it with sufficient rapidity. For this purpose it will be proper to adapt the formula last found, which is calculated with Napierian logarithms and a radius $= 1$, to the common tables. Let Tan. be the tangent and R the radius of the tables; the Napierian logarithm of $10 = 2.3025851 = M$; and making log. signify the logarithm to base 10, we have

$$\frac{h}{l} = - \frac{\text{Tan.}\,\alpha}{R} . M . \log. \frac{\text{Tan.}\frac{\alpha}{2}}{R};$$

$$= - \frac{\text{Tan.}\,\alpha}{R} . M . \log. \frac{R}{\text{Co-tan.}\frac{\alpha}{2}}$$

$$= \frac{\text{Tan.}\,\alpha}{R} . M . \left(\log. \text{Co-tan.}\frac{\alpha}{2} - \log. R\right);$$

$$\therefore \log. \frac{h}{l} = \log. \text{Tan.}\,\alpha + \log.\left(\log. \text{Co-tan.}\frac{\alpha}{2} - \log. R\right) + \log. M - \log. R,$$

where $\log. R = 10$, $\log. M = .3622157$.

Assuming values of α, we may calculate $\frac{h}{l}$, and by observing the error of the result obtain a more accurate value of α.

Ex. Thus let the string $BCB' = 2BB'$, to find the position.

We have $\log. \frac{h}{l} = \log. \frac{1}{2} = \bar{1}.6989700$.

By a few trials we shall find that $\alpha = 13^0$ will nearly give this value by the formula.

13^0 would give $\log. \frac{h}{l} = \bar{1}.7002484$; therefore 13^0 is too large:

$12^0.30'$..........$\log. \frac{h}{l} = \bar{1}.6904752$; therefore $12^0.30'$ is too small.

Hence, since the differences of the results, when very small, are nearly proportional to the differences of the suppositions;

$$7002484 - 6904752 : 7002484 - 6989700 :: 30' : 4', \text{ nearly.}$$

Therefore $\alpha = 13^0 - 4' = 12^0 . 56'$ very nearly; and by repeating the process we might obtain it still more accurately.

Knowing α, we know $a = \dfrac{l}{\cos. \alpha} = l \sec. \alpha$.

To find the depth EC to which the vertex hangs, we have, by Art. 114,

$$EC = BF - CD = a - a \sin. \alpha = l . (\sec. \alpha - \tan. \alpha).$$

For c, the tension at the point C, we have, by the same Article,

$$c = a \sin. \alpha = l \tan. \alpha.$$

In the case just mentioned when $l = 2h$, we shall have

$$a = 2.152h;$$

$$c = .459h;$$

$$EC = 1.693.$$

If it were required to find the form of the curve when α is 45^0, it might be obtained directly from the formula; which gives in this case

$$\frac{l}{h} = 1.1346.$$

116. Prob. II. *A chain of given length APB, fig.* 118, *is suspended from two given points A, B, not in the same horizontal line; to find its position.*

Let s represent the whole length of the chain, and x and y the ordinates of the point B from A; and therefore given quantities. By equation (3), Art. 114, we have

$$a^2 + 2as . \cos. \alpha + s^2 = (a + x)^2 = a^2 + 2ax + x^2;$$

$$\therefore a = \frac{s^2 - x^2}{2(x - s \cos. \alpha)}.$$

Hence $\sqrt{(a^2+2as.\cos.\alpha+s^2)}=a+x=\dfrac{s^2-2sx.\cos.\alpha+x^2}{2(x-s\cos.\alpha)}$;

$$a.\cos.\alpha+s=\frac{2sx-s^2.\cos.\alpha-x^2\cos.\alpha}{2(x-s\cos.\alpha)}.$$

But by (6) Art. 114,

$$y=a.\sin.\alpha\ \mathrm{l}\ \frac{\sqrt{(a^2+2as.\cos.\alpha+s^2)}+(a.\cos.\alpha+s)}{a(1+\cos.\alpha)},$$

$$=\frac{(s^2-x^2).\sin.\alpha}{2(x-s.\cos.\alpha)}\mathrm{l}\frac{(s^2+2sx+x^2).(1-\cos.\alpha)}{(s^2-x^2).(1+\cos.\alpha)},$$

$$=\frac{(s^2-x^2).\sin.\alpha}{2(x-s.\cos.\alpha)}\mathrm{l}\frac{s+x}{s-x}.\frac{1-\cos.\alpha}{1+\cos.\alpha};$$

whence α must be determined by approximation, as in the last problem. The approximation may be facilitated by the following artifice. Let $x=s.\cos.\beta$; hence

$$\frac{y}{s}=\frac{\sin.^2\beta.\sin.\alpha}{2(\cos.\beta-\cos.\alpha)}.\mathrm{l}\frac{1+\cos.\beta}{1-\cos.\beta}.\frac{1-\cos.\alpha}{1+\cos.\alpha}$$

$$=\frac{\sin^2\beta.\sin.\alpha}{4\sin.\frac{\alpha+\beta}{2}.\sin.\frac{\alpha-\beta}{2}}\mathrm{l}\frac{\tan.\frac{\alpha}{2}}{\tan.\frac{\beta}{2}};$$

$$=\frac{\text{Sin.}^2\beta.\text{Sin.}\alpha}{4R.\text{Sin.}\frac{\alpha+\beta}{2}.\text{Sin.}\frac{\alpha-\beta}{2}}.M.\left\{\log.\text{Tan.}\frac{\alpha}{2}-\log.\text{Tan.}\frac{\beta}{2}\right\},$$

R being the radius, and Sin. &c. the sine, &c. of the tables.

Hence

$$\log.\frac{y}{s}=2\log.\text{Sin.}\beta+\log.\text{Sin.}\alpha-\log.\text{Sin.}\frac{\alpha+\beta}{2}-\log.\text{Sin.}\frac{\alpha-\beta}{2}$$

$$+\log.\left\{\log.\text{Tan.}\frac{\alpha}{2}-\log.\text{Tan.}\frac{\beta}{2}\right\}+\log M-\log 4-10,$$

and by assuming values of α, and comparing the resulting with the true values of log. $\frac{y}{s}$, we may obtain as before an answer nearly correct.

117. Prob. III. *A chain of given length $FBCB'F' = 2l$, hangs freely over two given points B, B', in the same horizontal line, its ends BF, $B'F'$ hanging vertically: to find the position in which it will support itself.*

It is manifest that there cannot be an equilibrium except the two vertical parts BF, $B'F'$, are equal. Also each of these must be equal to the length which expresses the tension at B or B'; that is, $BF' = BF = \frac{s}{\cos. \alpha}$.

Let $BB' = 2h$, $CB = CB' = s$, angle at $B = \alpha$, and we have, as before, Art. 115,

$$h = s \tan. \alpha \,\mathrm{l}\, \frac{1 + \cos. \alpha}{\sin. \alpha}.$$

Also by Cor. 2. Art. 111, tension at $B = \frac{s}{\cos. \alpha} = BF$;

$$\therefore\ l = CB + BF = s + \frac{s}{\cos. \alpha} = s\frac{1 + \cos. \alpha}{\cos. \alpha};$$

$$\therefore\ \frac{h}{l} = \frac{\sin. \alpha}{1 + \cos. \alpha}\,\mathrm{l}\,\frac{1 + \cos. \alpha}{\sin. \alpha};$$

$$\text{or, } \frac{h}{l} = \tan. \frac{\alpha}{2} . \,\mathrm{l}\, \text{co-tan.} \frac{\alpha}{2} = -\tan. \frac{\alpha}{2} \,\mathrm{l}\, \tan. \frac{\alpha}{2};$$

whence tan. $\frac{\alpha}{2}$ must be found. And α being known, we know

$$s = l . \frac{\cos. \alpha}{1 + \cos. \alpha}: \text{ and } h = s \tan. \alpha \,\mathrm{l}\, \frac{1 + \cos. \alpha}{\sin. \alpha};$$

and hence the curve is known.

118. Prob. IV. *In the last case, to find when the equilibrium is possible.*

In the equation $\frac{h}{l} = \quad \tan. \frac{a}{2} \,\mathrm{l}\, \tan. \frac{a}{2}$, making $\frac{l}{h} = u$, and $\tan. \frac{a}{2} = t$, we have $u = -\frac{1}{t \,\mathrm{l}\, t}$. And the relative changes of magnitude of t and u will be seen most easily by constructing a curve of which these shall be the abscissa and ordinate. Let bm, fig. 119, be always taken $= \tan. \frac{a}{2} = t$, and $mp = u$, and let us consider the locus of p.

The value of a will be between 0 and $\frac{1}{2}\pi$, and hence the value of t will be between 0 and 1; and hence $\mathrm{l}\, t$ will be negative, and u will always be positive.

When $t=0$, $t \,\mathrm{l}\, t=0$, and u is infinite.

As t increases u decreases; we have

$$\frac{du}{dt} = \frac{1 + \mathrm{l}\, t}{(t \,\mathrm{l}\, t)^2};$$

which is negative so long as $-\mathrm{l}\, t > 1$.

When $1 + \mathrm{l}\, t = 0$, or $t = \frac{1}{e}$, $\frac{du}{dt} = 0$, and u is a minimum; at this point $u = \frac{l}{e}$, or $l = eh$. Afterwards u increases continually till $t = 1$, when u is infinite.

Hence the curve is of the form pqp', with asymptotes at b and c, bc being $= 1$. If $bn = .368$, &c. nq will be the minimum ordinate $= 2.718$, &c.

For every value, as bo, of u or $\frac{l}{h}$, there are two values of t, which may be found by drawing opp' parallel to bc. Hence there are *two* positions of equilibrium*, for given values of h and

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* In the case when the equilibrium is possible, the higher position is one of *stable*, the lower one of *unstable* equilibrium. If the chain be placed with

l. If we make bA perpendicular and equal to bc, and join Am, Am', the angles a for these two positions will be the doubles of bAm, bAm', respectively.

Thus, if $l = 10h$ the values of a are $3^0.\ 12'$, and $83^0.36'$.

The least value of u or $\frac{l}{h}$ for which the equilibrium is possible, is when $u = e$, or $l = he$, which gives the minimum ordinate nq. In this case we have

$$t = \frac{1}{e};\ \therefore \text{co-tan.}\ \frac{a}{2} = \frac{1}{t} = 2.718281824, \&c.;$$

$$\therefore\ \frac{a}{2} = 20^0.\ 12', \text{ and } a = 40^0.\ 24'.$$

If $a = BF$, fig. 118,

$$a = \frac{s}{\cos.\ a} = \frac{l}{1 + \cos.\ a} = \frac{l}{2\cos.^2 \frac{a}{2}} = \frac{l}{2}\left(1 + \frac{1}{e^2}\right);$$

$$s = l - 2a = \frac{l}{2}\left(1 - \frac{1}{e^2}\right);$$

and by Art. 114, if k be the depth of the vertex below the horizontal line,

$$k = a - a\,.\sin.\ a = \frac{l}{2}\,.\,\frac{1 - \sin.\ a}{\cos.^2 \frac{a}{2}} = \frac{l}{2}\left(\sec.^2 \frac{a}{2} - 2\tan.\ \frac{a}{2}\right);$$

$$= \frac{l}{2}\,.\left(1 + \frac{1}{e^2} - \frac{2}{e}\right) = \frac{l}{2}\left(1 - \frac{1}{e}\right)^2.$$

$$\frac{s}{k} = \frac{e + 1}{e - 1}.$$

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with its vertex above the higher of the two positions of equilibrium, it will descend towards that: if it be placed any where between the two positions, it will ascend towards the upper. If it be placed below the lower it will descend and never come to another position of equilibrium.

If the chain be so short, compared with the distance, that $\frac{l}{h}$ is less than e, it cannot be supported: the middle part will descend and draw up the ends.

119. Prob. V. *To find the center of gravity of the catenary AP*, fig. 118.

For this purpose we must find $\int x ds$ and $\int y ds$. By Cor. Art. 114,

$$\frac{dy}{dx} = \frac{c}{s'} = \frac{a \sin. \alpha}{a \cos. \alpha + s}.$$

$$\therefore a \cos. \alpha . dy + s dy = a \sin. \alpha . dx;$$

$$\therefore s dy = a (\sin. \alpha . dx - \cos. \alpha . dy)$$

$$\int s dy = a (x \sin. \alpha - y \cos. \alpha);$$

$$\therefore \int y ds = ys - \int s dy = ys - a (x \sin. \alpha - y \cos. \alpha).$$

Again, since by (2) and (5), Art. 114,

$$dx = \frac{s' ds'}{\sqrt{(c^2 + s'^2)}},\ dy = \frac{c ds'}{\sqrt{(c^2 + s'^2)}};$$

$$\sqrt{(c^2 + s'^2)}\, ds' = s' dx + c dy;$$

or, putting for c and s' their values,

$$\sqrt{(a^2 + 2as \cos. \alpha + s^2)}\, ds = (a \cos. \alpha + s)\, dx + a \sin. \alpha dy.$$

Hence by (3), Art. 114,

$$\int x ds = \int \{\sqrt{(a^2 + 2as \cos. \alpha + s^2)} - a\}\, ds$$

$$= \int \{(a \cos. \alpha + s)\, dx + a \sin. \alpha dy - a ds\}.$$

Add $\int x ds$ to both sides, and integrate, observing that

$$\int (s dx + x ds) = xs;$$

$$\therefore 2 \int x ds = ax \cos. \alpha + xs + ay \sin. \alpha - as.$$

Hence, by the formulæ Art. 99,

$$h = \frac{\int x ds}{s} = \frac{a (x \cos. \alpha + y \sin. \alpha)}{2 s} - \frac{a - x}{2},$$

$$k=\frac{\int y\,ds}{s}=y-\frac{a\,(x\sin.\,a-y\cos.\,a)}{s}.$$

Cor. 1. It may be observed that if we draw PO perpendicular on the tangent at P,

$$AO=x\cos.\,a+y\sin.\,a,$$

$$PO=x\sin.\,a-y\cos.\,a.$$

Cor. 2. If we suppose A to be the lowest point, we have a a right angle. Hence

$$a+h=\frac{ay}{2s}+\frac{a+x}{2}.$$

which agrees with Art. 99, Ex. 23.

Cor. 3. Let the tangents at A and P meet in T, and let TU be vertical; $PU=u$; $\therefore$ $NU=y-u$, and if $PTU=\theta$, we shall have

$$u\ \text{co-tan.}\,\theta+(y-u)\ \text{co-tan.}\,a=x.$$

$$\text{But co-tan.}\,\theta=\frac{dx}{dy}=\frac{a\cos.\,a+s}{a\sin.\,a}$$

$$=\text{co-tan.}\,a+\frac{s}{a\sin.\,a};$$

$$\therefore\ \frac{us}{a\sin.\,a}+\frac{y\cos.\,a}{\sin.\,a}=x,$$

$$u=\frac{a\,(x\sin.\,a-y\cos.\,a)}{s};$$

$\therefore$ $k+u=y$, and the center of gravity is in the vertical line TU which passes through T.

2. *The Catenary when the force acts in parallel lines and the Chain is not uniform.*

120. We may consider the thickness of the chain or cord to be variable, supposing it still to be so small throughout that we may consider the flexible body as a physical line. Or we may

conceive the catenary to be a surface of unequal breadth, resembling a ribbon, its breadth being parallel to the horizon; so that it may be a portion of a cylindrical surface, the curve of the cylinder being the catenary. We may also suppose the density to be variable. Or we may conceive the force which acts upon the chain and gives weight to it to be different in different parts.

Upon any of these suppositions the weight of equal portions of the curve taken in different parts of it will be different. Let ds be the differential of the curve, and let wds be the differential of the weight: w being the quantity (thickness, breadth, density or force) to which the weight of a given element of length is proportional.

Hence $\int wds$ taken between proper limits is the weight of any portion.

Prop. *To find the curve when the law of the thickness is given and conversely.*

In fig. 117, let C be the lowest point, and let ma be the tension there. Then x and y being CM and MP as before, we shall have, as in Art. 111,

$$\frac{dx}{dy} = \frac{\int wds}{ma} \dots\dots\dots\dots (1).$$

If we differentiate this, considering dy as constant, we have

$$\frac{d^2x}{dy^2} = \frac{w}{ma} \cdot \frac{ds}{dy} \dots\dots\dots\dots (2).$$

And w being known in terms of the other variable quantities, we shall, by integrating, have the equation to the curve.

$$\text{Also } w = ma \cdot \frac{d^2x}{dy^2} \cdot \frac{dy}{ds} \dots\dots (3);$$

whence, if the curve be known, we may by differentiating find w.

If the curve be no where perpendicular to the direction of the forces, the relation of the differentials will still be the same, and the above formulæ will be true.

121. PROB. VI. *A string whose thickness at every point is inversely as the square root of the length measured from the lowest point, is acted upon by gravity; to find its form.*

Let m be the thickness at a length c from the lowest point; hence, at the end of a length s,

$$w = m\frac{\sqrt{c}}{\sqrt{s}}; \int w\,ds = m\int\frac{\sqrt{c}}{\sqrt{s}}\,ds = 2m\sqrt{(cs)};$$

$$\therefore \text{ by (1) } \frac{dx}{dy} = \frac{2\sqrt{(cs)}}{a}; \quad \frac{ds^2}{dy^2} = \frac{dx^2 + dy^2}{dy^2} = \frac{a^2 + 4cs}{a^2};$$

$$\frac{dy}{a} = \frac{ds}{\sqrt{(a^2+4cs)}}; \quad \frac{y}{a} = \frac{\sqrt{(a^2+4cs)}}{2c} - \frac{a}{2c};$$

$$\left(\frac{y}{a} + \frac{a}{2c}\right)^2 = \frac{a^2}{4c^2} + \frac{s}{c}; \quad \frac{s}{c} = \frac{y^2}{a^2} + \frac{y}{c};$$

$$\frac{ds}{c} = \left(\frac{2y}{a^2} + \frac{1}{c}\right)dy; \quad \frac{dx^2+dy^2}{c^2} = \left(\frac{2y}{a^2} + \frac{1}{c}\right)^2 dy^2;$$

$$\frac{dx^2}{c^2} = \left(\frac{4y^2}{a^4} + \frac{4y}{a^2c}\right)dy^2; \quad dx = \frac{2c}{a^2}\sqrt{\left(y^2 + \frac{1}{c}\,y\right)}dy;$$

whence y is easily found by integrating; and hence the curve is known.

122. PROB. VII. *A string is acted on by a force which is, at every point, as the height above the lowest point: to find its form.*

The origin as before: and at the height c above the lowest point let the force be m; hence, at the height x, since *cæteris paribus* the weight of any portion will be as the force,

$$w = \frac{mx}{c}; \quad \therefore \text{ by (1) } \frac{dx}{dy} = \frac{\int x\,ds}{ca}; \quad \frac{d^2x}{dy} = \frac{x\,ds}{ca};$$

$$\frac{d^2x}{dy\sqrt{(dx^2+dy^2)}} = \frac{x}{ca}; \quad \frac{dx\,d^2x}{dy\sqrt{(dx^2+dy^2)}} = \frac{x\,dx}{ca};$$

$$\frac{\sqrt{(dx^2+dy^2)}}{dy}=\frac{x^2}{2ca}+1;\ \frac{dx}{dy}=\sqrt{\left(\frac{x^4}{4c^2a^2}+\frac{x^2}{ca}\right)};$$

$$\frac{2ac\,.\,dx}{x\sqrt{(x^2+4ac)}}=dy;\ y=\frac{\sqrt{(ac)}}{2}\,\mathrm{l}\,\frac{\sqrt{(x^2+4ac)}-2\sqrt{(ac)}}{\sqrt{(x^2+4ac)}+2\sqrt{(ac)}}+\text{const.}$$

When $x=0$, y is infinite and negative; when x is infinite y is equal to the constant. Hence the curve has a vertical and a horizontal asymptote, and never meets the horizontal line in which the force is $=0$.

123. Prob. VIII. *To find the law of thickness of a string that it may hang in the form of a semi-circle.*

Placing the origin at the lowest point, as before, we must have, calling the radius of the circle c,

$$y=\sqrt{(2cx-x^2)};\ \therefore\ \frac{dx}{dy}=\frac{\sqrt{(2cx-x^2)}}{c-x};$$

$$\therefore\ \frac{\sqrt{(2cx-x^2)}}{c-x}=\frac{\int w\,ds}{ma};\ \text{and differentiating}$$

$$\frac{c^2dx}{(c-x)^2\sqrt{(2cx-x^2)}}=\frac{w\,ds}{ma}=\frac{w\,.\,cdx}{ma\sqrt{(2cx-x^2)}},$$

$$\therefore\ w=\frac{mac}{(c-x)^2};$$

hence the thickness must vary inversely as the square of the depth below the horizontal diameter.

The tension will be found, as before, by the equation

$$\text{tension}=ma\,.\,\frac{ds}{dy}=\frac{mac}{c-x}.$$

Hence at the extremities of the horizontal diameter it is infinite,

If, instead of supposing the thickness of the string to vary, we suppose to be hung to each point of it vertical strings of uniform thickness whose lengths are proportional to

$$\frac{mac}{(c-x)^2},$$

the curve which it will form will be the same. And this also is applicable to all the cases of this section.

124. PROB. IX. *To find the law of thickness of a string that it may hang in the form of a parabola with its axis vertical.*

The origin is at the lowest point as before: $w = ma \cdot \frac{d^2x}{dy^2} \cdot \frac{dy}{ds}$,

$$y^2 = 4cx; \therefore \frac{dy}{dx} = \frac{\sqrt{c}}{\sqrt{x}}; \frac{ds}{dy} = \frac{\sqrt{(x+c)}}{\sqrt{+c}};$$

$$\frac{d^2x}{dy^2} = \frac{1}{2\sqrt{(cx)}} \cdot \frac{dx}{dy} = \frac{1}{2c};$$

$$\therefore w = \frac{ma}{2\sqrt{(cx+c^2)}}.$$

When $x = 0$, $w = \frac{ma}{2c}$; and if m be the thickness at the lowest point, $a = 2c$.

$$w = \frac{m\sqrt{c}}{\sqrt{(x+c)}}.$$

So long as x is small, this is nearly constant. Hence, conversely, if the thickness be constant, the catenary, within a small distance of the vertex, nearly coincides with a parabola. This is a conclusion to which Galileo was led by experiment.

3. *The Catenary when the Chain is acted upon by a central attractive or repulsive force*.*

125. PROP. *To find the equation to the catenary when the force tends to a center.*

Let S, fig. 120, be the center of attractive force, and at any distance $SP = r$, let the force be $= f$, f being a function of r. Let $AP = s$ be the chain or cord, and at the point P let the mass of a particle ds be μds, μ depending upon the thickness, density, &c.

* The force spoken of here and in the last Article is the attractive force which produces weight or pressure in the bodies on which it acts. If other things remain the same, such attractive forces are as the weight which they produce in a given particle of matter.

Let A be the point at which the curve is perpendicular to SA. Make SA a line of abscissas, and let MP be an ordinate perpendicular to it: Py a tangent at P, and Sy perpendicular on it.

Put $Sy=p$; tension at $A=a$, tension at $P=t$; angle $ASP=\theta$, $ATP=\phi$. The weight of a particle ds at P will be $f\mu ds$ in the direction PS: (see Note:) and if we resolve this force in the directions parallel and perpendicular to AS, the components will be $f\mu \cos.\theta . ds$ and $f\mu \sin.\theta . ds$: and hence the whole effects of the weight in those directions will be $\int f\mu \cos.\theta . ds$ and $\int f\mu \sin.\theta . ds$. The other forces which act on the cord AP, are the tension at $A=a$; and the tension at $P=t$, which may be resolved into the parts $t . \cos.\phi$ and $t . \sin.\phi$, in the rectangular directions. As before, the forces which keep AP at rest must be subject to the conditions of Art. 87. Hence,

$$\left.\begin{aligned}\int f\mu \cos.\theta . ds &= t \cos.\phi \\ \int f\mu \sin.\theta . ds &= t \sin.\phi - a\end{aligned}\right\}\dots\dots\dots(1).$$

Differentiating

$$f\mu \cos.\theta . ds = dt . \cos.\phi - t \sin.\phi . d\phi$$
$$f\mu \sin.\theta . ds = dt . \sin.\phi + t \cos.\phi . d\phi.$$

Multiply the first by cos. ϕ, and the second by sin. ϕ, and add; and we have

$$f\mu ds \cos.(\phi-\theta) = dt\dots\dots\dots(2).$$

Multiply the first by sin. ϕ, and the second by cos. ϕ, and subtract; and we have

$$f\mu ds \sin.(\phi-\theta) = -td\phi\dots\dots(3).$$

But $\phi-\theta=ATP-TSP=SPy$. Also if we take PQ a small arc and draw Qn perpendicular on SP, PQ, Pn, Qn are ultimately as ds, dr, $rd\theta$. Hence

$$ds \cos.(\phi-\theta)=dr,\ ds \sin.(\phi-\theta)=rd\theta,$$

and (2) and (3) become

$$f\mu dr = dt\dots\dots\dots\dots\dots(4)$$
$$f\mu rd\theta = -td\phi\dots\dots\dots\dots(5).$$

$$\text{Now } \frac{rd\theta}{dr} = \frac{Qn}{Pn} = \frac{p}{\sqrt{(r^2-p^2)}};$$

$$\therefore d\theta = \frac{p\,dr}{r\sqrt{(r^2-p^2)}}.$$

$$\text{Also } \phi - \theta = \text{arc}\left(\sin. = \frac{p}{r}\right);$$

$$\therefore d\phi - d\theta = \frac{r\,dp - p\,dr}{r\sqrt{(r^2-p^2)}};$$

$$\therefore \text{adding, } d\phi = \frac{dp}{\sqrt{(r^2-p^2)}}.$$

And putting the values of $d\theta$ and $d\phi$ in (5) it becomes

$$f\mu p\,dr = -t\,dp \ldots\ldots\ldots\ldots (6).$$

Dividing (6) by (4) we have

$$p = -\frac{t\,dp}{dt};$$

$$\therefore p\,dt + t\,dp = 0; \text{ and integrating}$$

$$pt = C^{*} \ldots\ldots\ldots\ldots (7),$$

C being a constant quantity, to be determined by the conditions of the frustum.

Also we have, by (4)

$$t = \int f\mu\,dr.$$

$$\text{Hence } p = \frac{C}{\int f\mu\,dr}.$$

When f is known in terms of r, this equation gives the curve AP by an equation between the distance SP and the perpendicular Sy upon the tangent. And from this equation we may determine

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* The property, that the perpendicular is inversely as the tension, appears also from this, that AP is acted on by the tensions at A and P, and also by central forces all tending to S. Hence the result of these latter forces will also tend to S; and hence we may suppose AP retained by a lever passing through S as a fulcrum, and the two forces at A and at P will be inversely as the perpendiculars or their directions; therefore tension at $P \,.\, Sy$ = tension at $A \,.\, SA$ = a constant quantity.

the relation between r and θ; and between x and y; unless this is rendered impossible by the difficulty of integrating.

The tension $t = \int f\mu dr$; if the thickness and density be constant we may make $\mu = 1$, and $t = \int f dr$; hence the tension depends only on the distance r, and is not affected by the form of the curve. If we suppose the end Pp to hang freely over the point P, and thus to produce the equilibrium, its weight must be $\int f dr$; which is also the weight of a string extending from a point p, at a given distance from S, up to P. Hence at every point P the string Pp will hang to the same distance Sp from S; and the ends of all the strings will be in a circle with center S; in which circle also is the point a, Aa being the length whose weight is requisite to produce the tension at A.

126. Prob. X. *The force varying inversely as the square of the distance from S, it is required to find the form of the catenary.*

Let SA, fig. 120, $= c$, and the force at $A = k$; hence $f = \frac{kc^2}{r^2}$;

$$t = \int f dr = \int \frac{kc^2 dr}{r^2} = \text{constant} - \frac{kc^2}{r} = a + kc - \frac{kc^2}{r};$$

for when $r = c$, $t = a$.

Hence $p = \dfrac{C}{a + kc - \frac{kc^2}{r}} = \dfrac{ac}{a + kc - \frac{kc^2}{r}}$, for when $r = c$, $p = c$;

$$= \frac{acr}{(a + kc)\, r - kc^2}.$$

Let $a = nkc$; kc being the weight of a length of string AS, acted on by a constant force equal to that at A; hence

$$p = \frac{ncr}{(n+1)\, r - c}.$$

To determine the nature of the curve, we have

$$d\theta = \frac{dr}{r\sqrt{\left(\frac{r^2}{p^2} - 1\right)}} = \frac{ncdr}{r\sqrt{[(n+1)^2 r^2 - 2(n+1)\, cr + c^2 - n^2c^2]}};$$

which will give different forms as n is greater than, equal to, or less than unity.

(1.) Let $n > 1$; therefore

$$d\theta = \frac{ncdr}{r^2\sqrt{\left((n+1)^2 - 2(n+1)\cdot\frac{c}{r} - (n^2-1)\frac{c^2}{r^2}\right)}};$$

which may be integrated by making $\frac{n-1}{n}\cdot\frac{c}{r} + \frac{1}{n} = u$, and gives

$$\theta = \frac{n}{\sqrt{(n^2-1)}}\cdot\text{arc}\left(\cos. = \frac{(n-1)c+r}{nr}\right);$$

θ being measured from the line SA.

If we make r infinite, we have

$$\theta = \frac{n}{\sqrt{(n^2-1)}}\cdot\text{arc}\left(\cos. = \frac{1}{n}\right);$$

which gives the position of the asymptotes of the curve.

The angle which the asymptotes make with SA is greater, as n, and consequently the tension at A, is greater. When n is infinite, it is a right angle, and the curve becomes a straight line perpendicular to SA. As n diminishes to unity, θ diminishes to the value which it has in the next case.

(2.) Let $n = 1$: hence

$$d\theta = \frac{cdr}{2r\sqrt{(r^2-cr)}} = \frac{cdr}{2r^2\sqrt{\left(1-\frac{c}{r}\right)}};$$

which gives, by integrating,

$$\theta = \sqrt{\left(1-\frac{c}{r}\right)} + \text{const.} = \sqrt{\left(1-\frac{c}{r}\right)};$$

because $\theta = 0$ when $r = c$.

When r is infinite $\theta = 1$. Hence the angle which the asymptotes makes with SA is that whose arc is equal to the radius; or, if RO be the asymptote, $ARO = 57^\circ\ 14'\ 44''\ 48'''$.

In every case we may find the position of the asymptote by making r infinite in the value of p; which will give $Sz = \frac{nc}{n+1}$.

(3.) Let $n < 1$: hence

$$d\theta = \frac{ncdr}{r^2\sqrt{\left((1+n)^2 - 2(1+n)\frac{c}{r} + (1-n^2)\frac{c^2}{r^2}\right)}};$$

which may be integrated by making $1-(1-n)\frac{c}{r} = u$; and gives $\theta =$

$$\frac{n}{\sqrt{(1-n^2)}} \, l \, \frac{r-(1-n)c + \sqrt{\{(1-n^2)r^2 - 2(1-n)cr + (1-n)^2c^2\}}}{nr};$$

the integral being corrected so as to vanish when $r = c$.

When r is infinite, $\theta = \frac{n}{\sqrt{(1-n^2)}} \cdot l \frac{1+\sqrt{(1-n^2)}}{n}$, which gives the position of the asymptote. When $n = 1$, $\theta = 1$, as may easily be shewn, agreeably to the last case. As n diminishes, the angle which the asymptotes make with SA diminishes, and when n becomes 0 this angle vanishes.

The tension at A is equal to the weight of a string whose length is nc, acted upon by a constant force equal to that at A. But if $Sa = b$, the weight of the portion Aa, acted on by the variable force (which weight expresses the tension at A) will be

$$= \int \frac{kc^2 dr}{r^2}, \text{ the integral taken from } r = b, \text{ to } r = c$$

$$= \frac{kc^2}{b} - \frac{kc^2}{c} = nkc, \text{ by supposition;}$$

$$\therefore b = \frac{c}{1+n}.$$

Hence if a circle were described with a radius $Aa = b$, the string, hanging down from any point of the curve, must, in order to produce the tension at that point, reach to the circumference of this circle.

127. **Prob. XI.** *Let the force vary as the m^{th} power of the distance from S: to find the curve.*

Retaining the notation of the last Problem, we have

$$\text{force at } P = \frac{kr^m}{c^m};\ \therefore t = \int \frac{kr^m dr}{c^m} = \frac{kr^{m+1}}{(m+1)c^m} + \text{const.}$$

$$= a - \frac{kc}{m+1} + \frac{kr^{m+1}}{(m+1)c^m};$$

a being the tension at A. Let $a = \frac{nkc}{m+1}$; therefore

$$t = \frac{kc}{m+1}\left(n - 1 + \frac{r^{m+1}}{c^{m+1}}\right).$$

Hence

$$p = \frac{nc}{n - 1 + \frac{r^{m+1}}{c^{m+1}}} = \frac{nc^{m+2}}{(n-1)c^{m+1} + r^{m+1}}:$$

and

$$d\theta = \frac{dr}{r\sqrt{\left(\frac{r^2}{p^2} - 1\right)}} = \frac{nc^{m+2}dr}{r\sqrt{\{(r^{m+2} + (n-1)c^{m+1}r)^2 - n^2c^{2m+4}\}}},$$

which cannot be integrated generally except $n = 1$.

In the case of $n = 1$,

$$d\theta = \frac{c^{m+2}dr}{r\sqrt{\{r^{2m+4} - c^{2m+4}\}}};$$

which may be integrated by making $c^{m+2}u = r^{m+2}$: this substitution gives

$$d\theta = \frac{du}{(m+2)u\sqrt{(u^2-1)}};$$

$$\therefore \theta = \frac{1}{m+2}.\text{arc. (sec.} = u);$$

$$= \frac{1}{(m+2)}\text{arc}\left(\text{sec.} = \frac{r^{m+2}}{c^{m+2}}\right);$$

$$\frac{r^{m+2}}{c^{m+2}} = \sec.\,(m+2)\,\theta\,;\;\; r^{m+2}\cos.\,(m+2)\,\theta = c^{m+2}.$$

If we make r infinite, we have for the inclination of the asymptotes to SA,

$$\theta = \frac{\pi}{2m+4}.$$

128. We may find the equation between $SM = x$ and $MP = y$.

For
$$r = \surd(x^2+y^2),$$

$$\cos.\,\theta = \frac{x}{\surd(x^2+y^2)},\;\; \tan.\,\theta = \frac{y}{x}.$$

Hence

$$c^{m+2} = r^{m+2}\cos.\,(m+2)\,\theta$$

$$= r^{m+2}.\left(\cos.^{m+2}\theta - \frac{(m+2)(m+1)}{1\,.\,2}\cos.^{m}\theta\,.\,\sin.^2\theta\right.$$

$$\left.+\frac{(m+2)(m+1)m(m-1)}{1.2.3.4}\cos.^{m-2}\theta\,\sin.^4\theta\ldots.\right)$$

$$= r^{m+2}\cos.^{m+2}\theta\,.\left(1 - \frac{(m+2)(m+1)}{1.2}\tan.^2\theta\right.$$

$$\left.+\frac{(m+2)\ldots.(m-1)}{1.2.3.4}\tan.^4\theta - \ldots.\right)$$

$$= x^{m+2}\left(1 - \frac{(m+2)(m+1)}{1\,.\,2}.\frac{y^2}{x^2} + \frac{(m+2)\ldots(m-1)}{1.2.3.4}.\frac{y^4}{x^4} - \ldots\right).$$

Cor. 1. If $m=0$, or the force be constant,

$$c^2 = x^2\left(1 - \frac{y^2}{x^2}\right) = x^2 - y^2.$$

Hence the curve is the rectangular hyperbola. The asymptotes make angles of 45^0 with SA.

Cor. 2. If $m=1$, or the force be as the distance,

$$c^3 = x^3\left(1 - \frac{3y^3}{x^3}\right) = x^3 - 3y^3.$$

In this case the angle which the asymptotes make with SA is $34^0\ 44'$.

129. Prop. *To find the catenary when the central force is repulsive.*

The process for finding the curve of equilibrium in this case will be nearly the same as before, with the exception of the signs of some of the quantities, and the results will be

$$-f\mu\, dr = dt;$$

$$f\mu\, dr \,.\, p = t dp:$$

$$\therefore \text{dividing}, \ -\frac{dp}{p} = \frac{dt}{t}; \ \therefore p = \frac{C}{t},$$

$$\text{and } t = -\int f\mu\, dr; \ \therefore p = \frac{C}{-\int f\mu\, dr}.$$

130. Prob. XII. *S, fig. 121, is a center of repulsive force varying inversely as the square of the distance from S: to find the form of the curve AP, formed by a flexible string.*

Retaining the notation of Prob. 10,

$$t = -\int \frac{kc^2\, dr}{r^2} = a - kc + \frac{kc^2}{r};$$

a being the tension at A; let $a = nkc$;

$$\therefore t = kc\left((n-1) + \frac{c}{r}\right).$$

Hence $p = \dfrac{ncr}{(n-1)r + c}$, supposing the curve at A perpendicular to SA;

$$\therefore d\theta = -\frac{dr}{r\sqrt{\left(\frac{r^2}{p^2} - 1\right)}}$$

$$= -\frac{ncdr}{r\sqrt{\{(n-1)^2 r^2 + 2(n-1)cr + c^2 - n^2c^2\}}},$$

which may be integrated nearly as in Prob. 10.

If we suppose the curve not to be perpendicular to SA, but to make with it an angle a, we shall have at that point $p = c . \sin. a$;

$$\therefore p = \frac{ncr . \sin. a}{(n-1)r + c}.$$

If $n = 1$, this becomes $p = r . \sin. a$, and the curve is the *logarithmic spiral.*

131. Prob. XIII. *Let the force be inversely as the m^{th} power of the distance: to find the curve.*

$$t = -\int \frac{kc^m dr}{r^m} = a - \frac{kc}{m-1} + \frac{kc^m}{(m-1)r^{m-1}};$$

$$= \frac{kc}{m-1} . \left(n - 1 + \frac{c^{m-1}}{r^{m-1}}\right);$$

$$\text{putting } a = \frac{nkc}{m-1}.$$

Take the case when $n = 1$, and we have

$$t = \frac{kc^{m-1}}{(m-1)r^{m-1}}; \text{ and let } p = c . \sin. a, \text{ at } A;$$

$$\therefore p = \frac{C}{t} = \frac{r^{m-1} . \sin. a}{c^{m-2}};$$

$$d\theta = -\frac{dr}{r\sqrt{\left(\frac{r^2}{p^2} - 1\right)}} = -\frac{r^{m-2} \sin. a . dr}{r\sqrt{(c^{2m-4} - r^{2m-4} \sin.^2 a)}}$$

$$= -\frac{r^{m-3} dr . \sin. a}{\sqrt{(c^{2m-4} - r^{2m-4} \sin.^2 a)}}.$$

To integrate, put $\dfrac{r^{m-2} . \sin. a}{c^{m-2}} = u$;

$$\therefore d\theta = -\frac{du}{(m-2)\sqrt{(1-u^2)}},$$

$$\theta = \frac{1}{m-2} \text{ arc } (\cos. = u) = \frac{1}{m-2} \text{ arc } \left(\cos. = \frac{r^{m-2} \sin. a}{c^{m-2}}\right).$$

Or, if $\dfrac{c^{m-2}}{\sin. a} = a^{m-2}$, $(m-2)\,\theta = \text{arc } \left(\cos. = \dfrac{r^{m-2}}{a^{m-2}}\right)$.

Here a is the value of r at the point A.

To find the angle $2ASO$ which comprehends the whole curve, make $r=0$:

$$\therefore\ \theta = \frac{\pi}{2m-4}; \quad \therefore\ 2ASO = 2\theta = \frac{\pi}{m-2}.$$

We may find the equation between $SM=x$, and $MP=y$, as before,

For $r^{m-2}=a^{m-2}\cos.(m-2)\theta$

$$=a^{m-2}\left(\cos.^{m-2}\theta - \frac{(m-2)(m-3)}{1.2}\cos.^{m-4}\theta.\sin.^2\theta + \ldots\right);$$

$$\therefore\ r^{2m-4}=a^{m-2}r^{m-2}\cos.^{m-2}\theta\left(1-\frac{(m-2)(m-3)}{1.2}\tan.^2\theta = \ldots\right);$$

$$\text{or } (x^2+y^2)^{m-2}=a^{m-2}x^{m-2}\left(1-\frac{(m-2)(m-3)}{1.2}.\frac{y^2}{x^2}+\ldots\right).$$

COR. 1. If $m=3$, $\theta = \text{arc}\left(\cos. = \frac{r}{a}\right)$; hence APS is a circle on the diameter AS.

COR. 2. If $m=4$, $2\theta = \text{arc}\left(\cos. = \frac{r^2}{a^2}\right)$; hence APS is the lemniscata with its knot at S.

COR. 3. Hence if there be a centre of repulsive force which varies inversely as the cube of the distance, and if the two ends of a string be fastened at this center, it will form itself into a circle. If the force vary inversely as the fourth power, the curve will be a lemniscata, and so on.

4. *The Catenary when the Chain is acted upon by any Forces.*

132. PROP. *Let forces to act upon the flexible body AP, Fig. 122, in the same plane, according to any law whatever; it is required to find its form.*

Let the force at any point P be represented by f, and act in the direction PF, which makes with the line of abscissas AM an angle ψ. The reasoning is exactly the same as in Art. 125. The effect of the force f at P is fds, and this, resolved parallel and perpendicular to AM, gives $f.\cos.\psi$, and $f.\sin.\psi$. Hence the whole effects on AP are $\int f ds \cos.\psi$, and $\int f ds \sin.\psi$. The remaining forces are the tension at P, which is represented by t, and makes with AM an angle ϕ, and the tension at A, which is represented by a, and is supposed to be perpendicular to AM. Hence the conditions of Art. 87, give

$$\int f \cos.\psi\, ds - t \cos.\phi = 0,$$
$$\int f \sin.\psi\, ds + t \sin.\phi = a.$$

Differentiating,

$$f \cos.\psi\, ds - dt \cos.\phi + t \sin.\phi\,.\,d\phi = 0,$$
$$f \sin.\psi\, ds + dt \sin.\phi + t \cos.\phi\,.\,d\phi = 0.$$

Multiply the first by $\cos.\phi$ and the second by $\sin.\phi$, and subtract: also multiply the first by $\sin.\phi$ and the second by $\cos.\phi$, and add: we shall thus get

$$fds\,(\cos.\phi\,.\cos.\psi - \sin.\phi\,.\sin.\psi) - dt = 0;$$
$$fds\,(\sin.\phi\,.\cos\psi + \cos.\phi\,.\sin.\psi) + t d\phi = 0;$$
$$\text{or } fds\,.\cos.(\phi+\psi) = dt,$$
$$fds\,.\sin.(\phi+\psi) = -t d\phi.$$

The angle $\phi + \psi$ is FPT, the angle which the force makes with the tangent. This angle and the force f being expressed in terms of x and y and their differentials, t is known from the first equation: and this and $d\phi$ being substituted in the second, we have the equation to the curve. For $d\phi$, we have

$$\phi = \text{arc}\left(\tan. = \frac{dy}{dx}\right);$$

$$\therefore\ d\phi = \frac{dx d^2y - dy d^2x}{dx^2 + dy^2} = \frac{dx d^2y}{dx^2 + dy^2}, \text{ if } dx \text{ be constant.}$$

133. Prop. *If the force to be at every point perpendicular to the curve: to find the form.*

We shall have $\phi + \psi = \frac{1}{2}\pi$; hence $\cos.(\phi+\psi) = 0$, $\sin.(\phi+\psi) = 1$; and our equations become

$$0 = dt;$$
$$fds = -td\phi:$$
$$\text{or } t = a;$$
$$f = -a \cdot \frac{dxd^2y}{ds^3} = \frac{a}{\rho}; \ \therefore a = f\rho,$$

ρ being the radius of curvature.

Hence *when the force is perpendicular to the curve, the tension is constant;* and is at every point equal to the weight of a portion of the cord whose length is the radius of curvature, acted on by the force at that point. If curvature be supposed inversely proportional to the radius, *the curvature at every point will be as the force.*

134. PROB. XIV. *A flexible line AP, fig.* 122, *is acted upon at every point P by a force f perpendicular to the line, and which is as the square of the sine of the angle EPV; to find the curve AP.*

The sine of $EPV = \sin. \phi = \frac{dy}{ds}$; hence the force $f = k \cdot \frac{dy^2}{ds^2}$, k being its value at A;

$$\therefore k \cdot \frac{dy^2}{ds^2} = -a \cdot \frac{dxd^2y}{ds^3};$$

$$\therefore kds = -a \cdot \frac{dxd^2y}{dy^2};$$

$\therefore ks + \text{const.} = a \cdot \frac{dx}{dy}$; also at A, $s = 0$, $\frac{dx}{dy} = 0$; $\therefore$ const. $= 0$;

$$\therefore \frac{ks}{a} = \frac{dx}{dy};$$

which coincides with the equation to the common catenary when the origin is placed at the lowest point and x taken vertical. Hence this is the same curve as when the force is parallel and constant*.

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* Soon after the time (1691) when the Problem of the figure of a chain acted upon by gravity was proposed and solved by the Bernoullis and Leibnitz, the attention of these geometers was directed to other curves which flexible bodies may assume under various circumstances. In particular

135. Prob. XV. *AP is acted upon by forces which are every where perpendicular to the curve, and which are, at every point P, proportional to the distance PE of P from a given line BE; to find the curve.*

Let BE be perpendicular to AB, $AB = c$; $PE = x$; $k =$ the force at A;

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ticular the action of a fluid, whether by elasticity, weight, or impact, was considered; and as this action must be perpendicular to the surface on which it acts, this case comes under Art. 133. of the text. One of their problems was to find the figure of a rectangular *sail*, with two opposite sides fixed, inflated by the wind: and as the figure of a chain or cord had been called the *Catenaria* or *Funicularia*, this was called the *Velaria*. The weight of the sail itself being neglected, the problem may be solved on either of the following hypotheses:

1st, That the air which immediately presses the sail is, relatively to the sail, at rest; and of course kept in its place by the pressure produced by the wind behind. On this supposition it is the elasticity of the air which acts upon the curve; and since this force is the same at every point, the radius of curvature will be constant, and the curve will be a circular arc; consequently the surface will be a portion of a common cylinder.

2nd, That the air acts by impact, and produces no effect by pressure after the first impulse. This may be nearly the case when a single thread is stretched by a current of fluid, which can after the impact escape past it. In this case the force is as the square of the sine of the angle of impact, as appears from hydrodynamical principles. Hence this is the case of Prob. 14. of the text, in which, as is shewn, the curve is the common *Catenaria*.

It appears to have been supposed that the actual curve of the sail would be something compounded of both these forms.

Another problem of the same kind was, to find the form of a rectangle of cloth, &c. which having two opposite sides supported parallel to the horizon, is pressed by the weight of a fluid which is contained in it, and of course supposed to be prevented from running out at the ends. The curve of this problem was called the *Lintearia*; if BC, fig. 122, be the surface of the fluid, the pressure on any point P will be as the depth EP; hence the curve is the one found in Prob. 15; which, as is mentioned in the text, is the same with the *Elastica*.

$$\therefore \frac{kx}{c} = -a \cdot \frac{dxd^2y}{ds^3}:$$

which coincides with the equation to the elastic curve, as will be seen in the next Chapter where that curve is considered.

We might now proceed to consider more complicated cases, as for instance when the flexible string rests upon any curve surface or surfaces. We might also investigate the conditions of equilibrium of a flexible *surface* acted upon by gravity or by any forces. The mechanical principles of such problems would not present much difficulty after what has preceded, but the analytical results to which they would lead would in most cases be too complicated for an elementary work like the present.

CHAP. X.

THE EQUILIBRIUM OF AN ELASTIC BODY.

136. BODIES are said to be *elastic* when they admit of a certain change of figure and dimensions, but possess a force which resists this change, which makes it depend upon the power applied, and which restores the bodies to their original dimensions and figure, when the power which altered them is removed. This restitutive energy acts in various ways.

1. *The Elasticity of Extension and Compression.* A string may be stretched by a force applied lengthways to it, and an elastic surface or solid may be considered as a collection of elastic fibres.

It is found by experiment, that when a string is stretched, the increase of length is proportional to the force which produces it; that is, the *extension is as the tension**. We may also suppose the

* See s'Gravesande's *Elem. Physices*, lib. i. c. 26.

same law to extend to compression; but in order that a string may be susceptible of compression lengthways, it must be supposed to be inflexible.

2. *The Elasticity of Flexure.* Wires and lamina of different metals and other substances exert a force to unbend themselves when forcibly bent. In the flexure of elastic rods and laminæ, it appears by experiment that the deflexion, and consequently the curvature, is nearly as the force*. This also follows from supposing an elastic rod to be composed of fibres which have elasticity of extension, as will be seen.

3. *The Elasticity of Torsion.* Threads of metal, &c. when twisted exert a force to untwist themselves. It appears from experiment†, that when very fine threads of metal are twisted by means of levers transverse to them, the force by which they tend to resume their natural state is very accurately as the angle of torsion.

1 *Elasticity of Extension.*

137. PROP. *When an elastic string of given length is stretched by a given force, to find its length.*

In a given elastic string the length added is, as we have said, proportional to the tension. If the tension be the same, it will, in different lengths of the same string, be proportional to the length; for it is manifest that a string two feet long will be twice as much extended by the same tension as a string one foot long; since the tension will be the same throughout, and therefore each of the halves of the first string will be as much stretched as the second string. In strings which differ in material, thickness, &c. the extension for a given length and tension, will be different for different substances; and will in each be proportional to a certain quantity which may be considered as the measure of the *extensibility*

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* See Biot's *Traité de Physique*, tom. I. p. 509.

† For the experiments of Colomb, see Biot, *Traité de Physique*, tom. I. p. 492.

of the particular substance which is to be taken. If ϵ be this quantity for a certain string whose length at first (that is, when not stretched by any force) is a, when this string is stretched by a force or weight t, which will of course measure the tension, its increase of length will be proportional to $a\epsilon t$, and may be *equal* to this expression by properly assuming ϵ. Hence the length under these circumstances will be $a + a\epsilon t$, or $a(1+\epsilon t)$.

We may determine ϵ if we know the original length of the string and its length for any given value of t. It may be convenient to know it in terms of the force which will draw out the string to *double* its length. Let E be this force; hence

$$a(1+\epsilon E) = 2a; \ \therefore \epsilon E = 1, \text{ and } \epsilon = \frac{1}{E}.$$

Hence the length of the string under a tension t becomes

$$= a\left(1 + \frac{t}{E}\right).$$

E may be expressed by a length of the given string whose weight would draw the string a to double its length. E is then called the *Modulus of Elasticity.*

If the tension be not the same throughout the string, this formula is not applicable. In this case we may suppose the string divided into indefinitely small portions; and in each of these portions the tension may be supposed constant, and the extension of that part found; and by combining all these, we get the extension of the whole.

138. Knowing thus the relation of the length and tension of such lines, we can easily express the conditions required by the solution of problems in which they occur, as will appear by the following examples.

Prob. I. *Fig.* 30. *AC, BC, are two given equal and similar elastic strings fixed at two points A, B, in the same horizontal line, and supporting at C a weight W: knowing the extensibility of the strings, to find where W will be supported; the strings themselves being supposed without weight.*

It is manifest that the vertical line CE will bisect AB. Let $AE = b$, angle $CAE = \alpha$, weight at $W = w$, tension of AC or $BC = t$, extensibility of $AC = \epsilon$, original length of $AC = a$, hence $AC = a(1 + \epsilon t)$.

Since W is supported by the tensions of AC, BC, in those directions, we have

$$w = 2t \sin.\alpha; \text{ also } AE = AC.\cos.\alpha, \text{ or}$$
$$b = a(1 + \epsilon t)\cos.\alpha.$$

Eliminating t,

$$\frac{b}{a} = \cos.\alpha + \frac{\epsilon w}{2}\cot.\alpha \ldots\ldots(1)$$

If we should attempt to obtain α from this equation, we should arrive at an equation of four dimensions: and by solving this, we should find the position of equilibrium. But for the most common case, that is, when the extensibility is small, and the weight w not very large, we may easily deduce from our equation an approximation to the situation. For we have

$$\alpha = A + A'\epsilon + A''.\frac{\epsilon^2}{1.2} + \ldots.$$

when A, A', A'', ... are the values which α, $\frac{d\alpha}{d\epsilon}$, $\frac{d^2\alpha}{d\epsilon^2}$, ... assume by making $\epsilon = 0$, (Lacroix, *Elem. Treat.* Art. 21.)

Hence, putting 0 for ϵ in the fundamental equation, (1), and in its differentials, we obtain

$$\frac{b}{a} = \cos.A;$$

$$0 = -\sin.\alpha.d\alpha - \frac{\epsilon w}{2}.\frac{d\alpha}{\sin.^2\alpha} + \frac{d\epsilon.w}{2}\cot.\alpha;$$

$$\therefore A' = \frac{w}{2}.\frac{\cot.A}{\sin.A} = \frac{w}{2}.\frac{ab}{a^2 - b^2}, \text{ \&c.} \ldots.$$

Therefore

$$\alpha = A + \frac{w\epsilon}{2}.\frac{ab}{a^2 - b^2} + \text{\&c.} \ldots.$$

Here A is the angle BAC on the supposition that the strings were inextensible: hence $\frac{w\epsilon}{2} \cdot \frac{ab}{a^2 - b^2}$ is, when ϵ is small, very nearly the quantity by which this angle is increased by supposing the strings extensible.

Cor. 1. If $a = b$, that is, if the string ACB be just equal to AB when not stretched, we have from (1)

$$1 = \cos.\, \alpha + \frac{\epsilon w}{2} \cdot \cot \alpha; \text{ and multiplying by tan. } \alpha,$$

$$\tan.\, \alpha - \sin.\, \alpha = \frac{\epsilon w}{2}; \text{ and expanding } \tan.\, \alpha = \sin.\, \alpha\, (1 - \sin.^2 \alpha)^{-\frac{1}{2}},$$

$$\frac{1}{2} \sin.^3 \alpha + \frac{1 \cdot 3}{2 \cdot 4} \sin.^5 \alpha + \ldots\ldots = \frac{\epsilon w}{2}.$$

If ϵ be small, α will be small; hence, neglecting the higher powers of sin. α,

$$\sin.^3 \alpha = \epsilon w; \ \sin.\, \alpha = \sqrt[3]{(\epsilon w)} = \sqrt[3]{\frac{w}{E}}.$$

E being the tension which would double the string. Hence for the same string, fixed horizontally and not stretched, the small deflection produced by a weight hung at the middle point is as the cube root of the weight.

Cor. 2. If $a < b$, the string would not reach from A to B horizontally without being stretched.

In this case, the equation becomes, multiplying by tan. α,

$$\frac{b}{a} \tan.\, \alpha - \sin.\, \alpha = \frac{\epsilon w}{2}.$$

And when α is small, neglecting its higher powers, we may put α both for its sine and tangent; hence

$$\frac{b - a}{a} \cdot \alpha = \frac{\epsilon w}{2} = \frac{w}{2E}; \ \alpha = \frac{w}{2E} \cdot \frac{a}{b - a}.$$

Therefore in this case the deflection varies as the weight w, if it be supposed small.

139. **Prob. II.** *A uniform elastic string hangs vertically, stretched by its own weight: to find its length.*

Let ϵ, as before, be its extensibility when its weight is not supposed to act. Let a be its length when it is supposed not stretched; and x the distance, on the same supposition, of any element dx from the upper extremity, by which it is suspended. The part below the element dx is $a-x$, when it is not stretched; and as the quantity of matter is not altered by extension, the weight of this part when stretched is as $a-x$; and may be represented by $a-x$, if we represent weights by the corresponding lengths of the unstretched string. Hence the element dx will become $dx\,[1+\epsilon(a-x)]$; or if z be the distance from the upper extremity to a point whose distance in the unstretched state was x,

$$dz = dx\,[1+\epsilon(a-x)];$$

$$\therefore\ z = x - \frac{\epsilon(a-x)^2}{2} + \text{constant};$$

and at the upper extremity where $x=0$, $z=0$;

$$\therefore\ z = x + \frac{\epsilon(2ax-x^2)}{2}.$$

At the lower extremity, $x=a$; let the stretched length $=l$;

$$\therefore\ l = a + \frac{\epsilon a^2}{2}.$$

Hence, $\frac{\epsilon a^2}{2}$ is the quantity by which the length of the string is increased when it is hung up. If E be a length of the string whose weight alone would be sufficient to stretch any part to twice its length, $\epsilon = \frac{1}{E}$, and $\frac{a^2}{2E}$ is the increment of length.

Cor. 1. If we had $a=E$, we should have the length when stretched $= E + \frac{E^2}{2E} = \frac{3E}{2}$.

Cor. 2. Since $l = a\left(1+\frac{\epsilon a}{2}\right)$; it appears that the weight of the string stretches it half as much as if it were all collected at the lowest point.

140. Prob. III. *To find the catenary when the chain is extensible.*

Let the chain or cord be of uniform thickness and density, and let, as before, the elasticity be such that a length a becomes $a(1+\epsilon t)$ by a tension t.

Let C, fig. 123, be the lowest point; and let the tension at C be equal to the weight of a length $CA=c$ of the unstretched string: $AN=x$, $NP=y$ the horizontal and vertical co-ordinates: $s=$ the arc CP, and $s'=$ the length of CP before it was stretched, which may therefore represent the weight of CP; $t=$ the tension at P.

If ds, ds' be corresponding elements of s, s', we have

$$ds=ds'(1+\epsilon t);\ \therefore\ ds'=\frac{ds}{1+\epsilon t}.$$

The forces which keep CP at rest are the tension t at P, the tension c at C, and the weight s'. Hence these forces are as the sides of a triangle which are parallel to them; for instance, the elementary triangle at P whose sides would be the elements dx, dy, ds: hence

$$\frac{t}{c}=\frac{ds}{dx},\ \text{ and } \frac{s'}{c}=\frac{dy}{dx}.$$

By the second of these equations $\dfrac{d^2y}{dx^2}=\dfrac{1}{c}\dfrac{ds'}{dx}=\dfrac{\frac{ds}{dx}}{c+c\epsilon t}$;

$$\therefore\ \frac{d^2y}{dx^2}=\frac{\frac{ds}{dx}}{c+c^2\epsilon\frac{ds}{dx}}\ \text{by the first.}$$

If we make $\dfrac{dy}{dx}=p$, $\dfrac{ds}{dx}=\sqrt{(1+p^2)}$; and supposing dx constant; our last equation becomes,

$$\frac{dp}{dx} = \frac{\sqrt{(1+p^2)}}{c + c^2\epsilon\sqrt{(1+p^2)}};$$

$$\therefore dx = \frac{c\,dp}{\sqrt{(1+p^2)}} + c^2\epsilon\,dp;$$

$$\text{and } dy = p\,dx = \frac{cp\,dp}{\sqrt{(1+p^2)}} + c^2\epsilon p\,dp.$$

Integrating these equations, we obtain

$$x = c\,\mathrm{l}\,[p + \sqrt{(1+p^2)}] + c^2\epsilon p;$$

$$y = c\sqrt{(1+p^2)} + \tfrac{1}{2}c^2\epsilon p^2;$$

the integrals being taken so that at C, where $p = 0$, we may have $x = 0$, and $y = c$.

By eliminating p we should have the relation between x and y: and p is the tangent of the angle which the curve at P makes with the horizon.

$$ds = \sqrt{(1+p^2)}\,dx = c\,dp + c^2\epsilon\sqrt{(1+p^2)}\,.\,dp;$$

$$\therefore s = cp + \tfrac{1}{2}c^2\epsilon\,\{p\sqrt{(1+p^2)} + \mathrm{l}\,[p + \sqrt{(1+p^2)}]\},$$

$$t = c\frac{ds}{dx} = c\sqrt{(1+p^2)}.$$

It appears that the values of x, y, and s, consist of two parts; terms independent of ϵ, which are the same as they would be in a cord not extensible; and terms which involve ϵ. Hence if CP and CP' be arcs of an extensible and an inextensible catenary for which the value of c, that is, the tension at C, is the same; and the values of p the same, that is, the tangents, PT, $P'T'$ parallel; $P'O$ and OP being horizontal and vertical, we have

$$P'O = c^2\epsilon p, \quad OP = \tfrac{1}{2}c^2\epsilon p^2.$$

The tension t is the same in both cases, and CP' is the length of CP not stretched.

Cor. If PT meet OP' in Q, $OQ = \dfrac{OP}{p} = \tfrac{1}{2}c^2\epsilon p = \tfrac{1}{2}OP'$.

From these few examples it will be seen how problems involving extensible lines may be reduced to calculation.

2. *Elasticity and Resistance of Solid Materials.*

141. All solid substances, as wood, stone, metals, &c., are susceptible of some compression and extension. This compression and extension are greater as the forces producing them are greater, and when the forces produce a compression or extension greater than the texture of the substance can bear, the bodies are crushed or broken. We shall here find the change of figure of such bodies when they are compressed under given circumstances.

We shall suppose that all solid bodies may be considered as made up of elastic fibres, capable of extension and compression. We shall also suppose, as in the last Section, that the resistance to extension is proportional to the extension in each fibre, and the same of compression. We shall further assume, that the resistance to extension and to compression are the same in the same fibre.

These principles would follow if we were to suppose the particles of bodies to be kept in equilibrium by their mutual forces in the natural state of the body; and the change to be small, which they undergo by the action of any force. In this case it might be proved that the displacement of a given particle would be ultimately as the force which produces it.

When a solid body is acted on by any force, it may be partly extended and partly compressed. Thus let a mass $ABQP$, fig. 124, be acted upon by a force F, compressing it in the direction EF. The surface PNQ may be brought into the direction pNq; in this case all the fibres RR' which are on one side of N are shortened; all those on the other side of N are lengthened. NN' remains the same as in the natural state. N is called the *neutral point*, and the line which separates the parts of the body which are compressed from those which are elongated is called the *neutral line.*

142. Prop. *When a rectangular prismatic mass is compressed by a force parallel to the direction of the axis; to find the neutral line.*

Let AB, fig 124, be the rectangular base of the mass, MM' its axis. And let the slice $UTPQ$ be compressed so as to assume the form $UTpq$, N being the neutral point. Then any fibre parallel to the axis, as VR, is compressed so that its length becomes Vr. And by the supposition, if t be the force compressing it, E the modulus of elasticity, as in last Article; we shall have

$$Rr = VR\frac{t}{E}; \text{ and hence } t = E.\frac{Rr}{VR}.$$

Let $PM = MQ = a$, $MF = h$, $MR = x$, and breadth of the beam perpendicular to $AB = b$; $MN = n$; $\therefore RN = n + x$. Force at $F = f$.

Also let UT and QP meet in O, and let $OK = \rho$. $MK = NL = k$. Hence

$$\frac{Rr}{VR} = \frac{Rr}{NL} = \frac{NR}{OL} = \frac{n+x}{n+\rho}.$$

And the force of VR, supposing its breadth and thickness each 1, is

$$t = E.\frac{Rr}{VR} = E.\frac{n+x}{n+\rho}.$$

Hence if we take a portion of which the thickness is dx and breadth b, its force is $E.\frac{n+x}{n+\rho}.bdx$, and this is the differential of the force exerted at R corresponding to dx. When x is negative and greater than n, this is negative; and accordingly the compression for that part becomes extension.

The forces which keep each other in equilibrium are the force f acting at F, and the elementary forces of all the fibres VR. And hence, by Art. 87, we must have, 1st, the force f equal to all the forces $E.\frac{n+x}{n+\rho}bdx$; and 2d, the moment of the force f about N equal to the moments of all the forces $E.\frac{n+x}{n+\rho}bdx$ about N.

Also the aggregate of all the forces will be found by taking the integrals of the differential expressions from $x = -a$, to $x = a$.

Hence we have

$$f = \int E.\frac{n+x}{\rho+n}bdx,$$

$$f(h+n)=\int E\,\frac{(n+x)^2 b\,dx}{\rho+n}.$$

And integrating between the proper limits,

$$f=E\,.\frac{2nab}{\rho+n},$$

$$f(h+n)=E\,.\,\frac{2n^2ab+\frac{2}{3}a^3b}{\rho+n}.$$

Dividing, we have

$$h+n=n+\frac{a^2}{3n};$$

$$\therefore n=\frac{a^2}{3h}=\frac{(2a^2)}{12h}.\quad \text{And } MN=\frac{PQ^2}{12\,MF}.$$

Cor. 1. If $MF=\frac{1}{3}MP$, or $h=\frac{1}{3}a$, $n=a$, the neutral point is in the surface, and the whole beam is compressed.

If $MF>\frac{1}{3}MP$, the neutral point is beyond the surface.

Cor. 2. From the above equations we have

$$\rho+n=\frac{E}{f}2nab=\frac{E}{f}.\frac{2a^3b}{3h}.$$

And $\rho+n$ is the radius of curvature of the neutral line NN' at N. Let the force f be equivalent to a length F of the prism; then $f=2Fab$; and we have

$$\rho+n=\frac{E}{F}\frac{a^2}{3h};\ \text{or } NO=\frac{E}{F}.\frac{PQ^2}{12\,MF}.$$

143. Prop. *When a rectangular prism is acted upon by any force in any direction; to find the neutral point at any part.*

Let a force f, fig. 124, act in the line yF on a prism $ABPQ$. The force will produce the same effect as if it acted at F, a point in QP. Let the angle MFy at $F=\alpha$. The force may be resolved into f cos. α in QP, and f sin. α perpendicular to QP. Of these the former is resisted by the lateral cohesion of the materials, and produces no compression. The latter produces a compression as in the last Article. Hence, retaining the denominations of last Article, calling MF, h, and putting f sin. a for f, we have

$$f \sin. a = E \,.\, \frac{2nab}{\rho + n};$$

$$f(h+n) \sin. a = E \,.\, \frac{2n^2 ab + \frac{2}{3} a^3 b}{\rho + n}.$$

And hence $h + n = n + \dfrac{a^2}{3h}$; $\therefore n = \dfrac{a^2}{3h}$.

Cor. 1. We have also

$$\rho + n = \frac{E}{f \sin. a} 2nab = \frac{E}{f \sin. a} \cdot \frac{2a^3 b}{3h}$$

$$= \frac{E}{f} \cdot \frac{2a^3 b}{3h \sin. a} = \frac{2Ea^3 b}{3fk},$$

if $k = My = h$ sin. a, the perpendicular on the direction of the force from the axis.

Cor. 2. If as before $f = 2Fab$,

$$\text{rad. of curv.} = \rho + n = \frac{Ea^2}{3Fk}.$$

Cor. 3. If the force act perpendicularly to the axis, h is infinite, $n = 0$, and the neutral point is in the axis.

144. Prop. *When a rectangular prismatic beam is made to deviate a little from a straight line by the action of a given force perpendicular to it, to find the deflexion.*

Since the force is perpendicular to the beam, and the beam is nearly a straight line, we may, by Cor. 3, of last Art., suppose the neutral point to be every where coincident with the axis. Let AME, fig. 125, represent the axis bent by a force acting perpendicularly to AD its original position. And let XM be the ordinate at any point, also perpendicular to AD. $AX = x$, $XM = y$.

And since the curve is nearly a straight line, $\frac{dy}{dx}$ is small: hence the radius of curvature

$$= \frac{dx^2\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{d^2y} \text{ is } = \frac{dx^2}{d^2y}, \text{ nearly.}$$

But by Cor. 2. to last Art. if $AD=l$, $k=DX=l-x$,

$$\text{rad. of curv.} = \frac{E}{F}\cdot\frac{a^2}{3(l-x)};$$

$$\therefore \frac{d^2y}{dx^2} = \frac{F}{E}\cdot\frac{3(l-x)}{a^2}.$$

Multiply by dx and integrate, observing that $\frac{dy}{dx}=0$ when $x=0$,

$$\therefore \frac{dy}{dx} = \frac{F}{E}\cdot\frac{3lx-\frac{3}{2}x^2}{a^2}.$$

Multiply by dx and integrate again, observing that $y=0$, when $x=0$.

$$y=\frac{F}{E}\cdot\frac{\frac{3}{2}lx^2-\frac{1}{2}x^3}{a^2}.$$

And if the whole deflexion $DE=\delta$, making $x=l$,

$$\delta=\frac{F}{E}\cdot\frac{l^3}{a^3}.$$

Cor. 1. If we put for F its value $\frac{f}{2ab}$, we have

$$\delta=\frac{fl^3}{2Ea^3b}.$$

Hence it appears that for a given breadth and thickness the deflexion is as the force and cube of the length.

And for a given weight and length the deflexion is inversely as the breadth and cube of the thickness.

Cor. 2. Let the direction of the tangent at E make an angle θ with the tangent at A. Then θ may be called the *angular deflexion*.

And $\frac{dy}{dx} = \tan.\theta$; hence, putting l for x in the value of $\frac{dy}{dx}$,

$$\tan.\theta = \frac{F}{E}\cdot\frac{3l^2}{2a^2} = \frac{3fl^2}{4a^3b}.$$

The angular deflexion is as the force and square of the length.

145. Prop. *When a rectangular prismatic beam, in a horizontal position, is bent by its own weight, (its thickness being vertical) to find the deflexion.*

In Art. 143, Cor. 2; put Fk, the moment of the force which bends the beam $= (l-x)\frac{l-x}{2} = \frac{1}{2}(l-x)^2$; and for the rad. of curv. $\frac{dx^2}{d^2y}$. Hence we have

$$\frac{d^2y}{dx^2} = \frac{3(l-x)^2}{2Ea^2};\quad \frac{dy}{dx} = \frac{l^3-(l-x)^3}{2Ea^2};$$

$$y = \frac{l^3x + \frac{1}{4}(l-x)^4 - \frac{1}{4}l^4}{2Ea^2};$$

and the whole deflexion $\delta = \frac{3l^4}{8Ea^4}$.

Cor. In this and the last Article, δ being observed, E may be found.

146. Prop. *When an isosceles triangular prism is acted upon by any force in any direction, to find the neutral point at any part.*

The force is supposed to act in the plane which bisects the vertical angle of the isosceles triangle. Let $ABQP$, fig. 124, be this plane, the vertex of the triangle being at P, and its base at Q.

Let $OT=\rho$, $TV=x$, $TL=n$, $TU=a$, $PF=h$, $MFy=\alpha$, and the force $=f$, modulus of elasticity $=E$.

As before, in Art. 143, we shall have the force of a single fibre at $R=E\frac{NR}{OL}=E\frac{n-x}{\rho+n}$.

And whatever be the form of the section perpendicular to the plane $ABQP$, if y be the ordinate of this section perpendicular to the line PQ, we shall have for the differential force exerted at R,

$$E\frac{n-x}{\rho+n}y\,dx.$$

And by the same reasoning as in Art. 143,

$$f\sin.\alpha=E\int\frac{n-x}{\rho+n}y\,dx,$$

$$f(h+n)\sin.\alpha=E\int\frac{(n-x)^2}{\rho+n}y\,dx.$$

In the case of the triangle $y=mx$, m being a constant quantity. And integrating from $x=0$ to $x=a$,

$$f\sin.\alpha=\frac{Em}{\rho+n}\cdot\left(\frac{1}{2}na^2-\frac{1}{3}a^3\right),$$

$$f(h+n)\sin.\alpha=\frac{Em}{\rho+n}\left(\frac{1}{2}n^2a^2-\frac{2}{3}na^3+\frac{a^4}{4}\right);$$

$$\therefore\ h+n=\frac{6n^2-8na+3a^2}{6n-4a}$$

$$=\frac{3a^2-4an}{6n-4a}+n;$$

$$\therefore\ h=\frac{3a^2-4an}{6n-4a};$$

$$\therefore\ n=\frac{3a^2+4ah}{4a+6h}.$$

COR. 1. If $h=0$, or the force act at P, $n=\frac{3}{4}a$.

COR. 2. If the force act perpendicularly to the prism, h is infinite, $n=\frac{2a}{3}$.

COR. 3. If the force act above P, h will be negative. Thus if the force act at Q, $h=-a$, $h=\frac{a}{2}$.

COR. 4. To find the radius of curvature of the neutral line, we have

$$\text{rad. curv.} = \rho+n=\frac{Em}{f\sin.\alpha}\left(\frac{1}{2}na^2-\frac{1}{3}a^3\right);$$

and putting for n its value

$$\text{rad. curv.} = \frac{Em}{f\sin.\alpha}\cdot\frac{a^4}{6(4a+6h)}=\frac{Ema^4}{36f\left(h+\frac{2a}{3}\right)\sin.\alpha}.$$

And if we take a point distant from P by $\frac{2}{3}$ PQ, and from this point draw a perpendicular on line of direction of the force; if this perpendicular $=k$,

$$k=\left(h+\frac{2a}{3}\right)\sin.\alpha;\ \text{rad. curv.} = \rho+n=\frac{Ema^4}{36fk};$$

or if b be the base of the triangle, $ma=b$, $\rho+n=\frac{Ea^3b}{36fk}$.

COR. 5. If f be the weight of a length F of the prism, $f=\frac{1}{2}Fab$;

$$\therefore\ \rho+n=\frac{Ea^2}{18Fk}.$$

In the same manner we might find the neutral point for prismatic beams of other figures. And the deflexion when they are acted on by given weights would be found in the same manner as before.

Also if the beams are not prismatic, a will be variable; and by putting for it the expression belonging to each case, we may find the deflexion in beams of other forms.

147. Prop. *A rectangular prismatic beam is compressed by a given force acting in a direction parallel to the axis; to find the deflexion.*

Let $ABA'B'$, fig. 126, be the beam, FF' the line in which the force acts. P any point in the axis. And since the deflexion is supposed to be small, PM, which is perpendicular to FF', may be considered as perpendicular also to the axis. Hence if a be half the thickness of the beam ($=\frac{1}{2}AB$) and n the distance of the neutral point above P, $EM=x$, $PM=y$, we have, by Art. 143, $n=\dfrac{a^2}{3y}$.

Also if ρ be the radius of curvature of the axis CP, by Cor. 2, of the same Article,

$$\rho+n=\frac{E}{F}\cdot\frac{a^2}{3y};\ \therefore\ \rho=\left\{\frac{E}{F}-1\right\}\frac{a^2}{3y}=\frac{c^2}{y},\ \text{suppose.}$$

Now $\rho=-\dfrac{dx^2}{d^2y}$ nearly, because the deflexion is small.

$$\therefore\ \frac{d^2y}{dx^2}=-\frac{y}{c^2}.\quad \text{Multiply by } 2dy \text{ and integrate;}$$

$$\therefore\ \frac{dy^2}{dx^2}=C-\frac{y^2}{c^2}.$$

And if k be EV the greatest ordinate, $y=k$ where $\dfrac{dy}{dx}=0$;

$$\therefore\ \frac{dy^2}{dx^2}=\frac{k^2-y^2}{c^2};\quad \frac{dy}{\sqrt{(k^2-y^2)}}=-\frac{dx}{c};$$

$$\therefore\ \text{arc}\left(\cos.=\frac{y}{k}\right)=\frac{x}{c};\ x \text{ being measured from } E,$$

$$y=k\cos.\frac{x}{c}.$$

Let $l=EF=$ half the length of the beam. And let $h=CF$ the distance of the force from the axis. Therefore when $x=l$, $y=h$,

$$h = k \cos. \frac{l}{c}; \quad y = h \cdot \frac{\cos. \frac{x}{c}}{\cos. \frac{l}{c}}.$$

Hence $EV = h \sec. \frac{l}{c}$; and DV the deflection $= EV - FC$;

$$\therefore \text{ deflection} = h \left\{ \sec. \frac{l}{c} - 1 \right\}.$$

$$\text{But } c^2 = \frac{a^2}{3} \left\{ \frac{E}{F} - 1 \right\}; \therefore \frac{l}{c} = \frac{l}{a} \frac{\sqrt{3F}}{\sqrt{E-F}}.$$

Cor. 1. If E be very large compared with F, we shall have the deflexion $= h \left\{ \sec. \frac{l\sqrt{3F}}{a\sqrt{E}} - 1 \right\}$.

Cor. 2. The radius of curvature at V

$$= \frac{c^2}{k} = \frac{c^2 \cos. \frac{l}{c}}{h} = \frac{a^2 \cos. \frac{l}{c}}{3h} \left\{ \frac{E}{F} - 1 \right\}.$$

And when E is very large compared with F,

$$\text{rad. curv. at } V = \frac{Ea^2}{3Fh} \cos. \frac{l\sqrt{3F}}{a\sqrt{E}}.$$

Cor. 3. The deflection will be greater as the secant, in Cor. 1, is greater; and when the secant is infinite, the formula will fail; in this case the prism will either be crushed, or will bend so much that the above reasoning is no longer applicable. And this will be the case if the arc be a quadrant. Hence in order that the prism may support a weight with a small deflexion, the weight acting on one side of the axis, we must have

$$\frac{l\sqrt{3F}}{a\sqrt{E}} < \frac{\pi}{2},$$

$$\frac{l^2}{a^2} < \frac{\pi^2 E}{12F}.$$

Cor. 4. If the force act at the extremities of the axis, $h=0$; and there will be no deviation except the secant of the arc be infinite; that is, except

$$\frac{l^2}{a^2}=\frac{\pi^2 E}{12 F}=.8225\frac{E}{F}.$$

Hence we may find the weights which columns of given materials will support. Thus, if in fir-wood the modulus E be 10,000,000 feet, a bar an inch square and 10 feet long may begin to bend when

$$F=.8225\times\left(\frac{1}{120}\right)^2\times 10{,}000{,}000=571 \text{ feet.}$$

that is, it will bend when pressed by the weight of 571 feet of the same bar, or about 120 pounds, neglecting the pressure arising from weight of the bar itself.

The modulus of elasticity for iron or steel is about 9,000,000 feet; for wood, from 4,000,000 to 10,000,000; and for stone, probably about 5,000,000.

Cor. 5. In the same manner we might find the deflexion of a triangular prismatic beam acted on by a longitudinal force. For in this case, supposing E large with respect to F, $\rho=\frac{Ea^2}{18Fy}$.

3. *The Curves formed by Elastic Laminæ.*

148. If we consider the thickness of the elastic bodies in Art. 143, to be small, we may neglect n, and we have, when the section of the body is a rectangle,

$$\rho=\frac{2Ea^3b}{3fk};$$

and in all cases $\rho=\frac{\mathrm{E}}{fk}$; when E is a constant quantity depending upon the size and form of the section of the elastic body, and upon its elasticity. If we suppose the body to be a lamina of uniform thickness, the value of a will be constant, and E will be proportional to b.

149. PROP. *An elastic lamina of uniform breadth and thickness is fixed at one end and acted upon by a given force; it is required to determine the form of the curve.*

Let BA, fig. 127, be the lamina, fixed at B; f the force, which acts at A or E in the direction AE; $CM=x$, $MP=y$, coordinates perpendicular and parallel to the direction of the force AE; $AP=s$. Making dx constant, the radius of curvature at P is $-\dfrac{(dx^2+dy^2)^{\frac{3}{2}}}{dx\,d^2y}$. Now it will manifestly make no alteration in the curvature at any point, as P, whether, after the equilibrium is established, we suppose the part PA rigid or not, or of one form or another. Hence the force f may be supposed to act on a straight rigid arm $PK=x$; and we have by the last Article,

$$fk=\frac{\text{E}}{\rho}, \text{ or } fx=-\frac{\text{E}\,dx\,d^2y}{(dx^2+dy^2)^{\frac{3}{2}}} \ldots\ldots\ldots\ldots(1).$$

If $\dfrac{dy}{dx}=p$, this becomes $fx\,dx=-\dfrac{\text{E}\,dp}{(1+p^2)^{\frac{3}{2}}}$; and integrating,

$$\frac{f}{2}(b^2+x^2)=-\frac{\text{E}p}{\sqrt{(1+p^2)}} \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

b^2 being an arbitrary constant, to be determined. Hence, obtaining p^2,

$$p^2=\frac{f^2(b^2+x^2)^2}{4\text{E}^2-f^2(b^2+x^2)^2}; \text{ and, making } a^2=\frac{2\text{E}}{f},$$

$$p^2=\frac{(b^2+x^2)^2}{a^4-(b^2+x^2)^2}=\frac{(b^2+x^2)^2}{(a^2-b^2-x^2)(a^2+b^2+x^2)};$$

$$\therefore \frac{dy}{dx}=\pm\frac{b^2+x^2}{\sqrt{(a^2-b^2-x^2)(a^2+b^2+x^2)}};$$

$$\text{also } ds=dx\sqrt{(1+p^2)}=\pm\frac{a^2\,dx}{\sqrt{(a^2-b^2-x^2)(a^2+b^2+x^2)}}.$$

We must determine b^2 from known circumstances in the problem. If the curve BP be continued to meet the line AE, and at the point of intersection make with the line of abscissas an angle α, we shall easily determine b^2. Since at that point $x=0$ and $p=\tan.\ \alpha$, equation (2) becomes

$$\frac{fb^2}{2} = -\frac{\text{E} \tan.\ \alpha}{\sec.\ \alpha};\ \therefore\ b^2 = -\frac{2\,\text{E} \sin.\ \alpha}{f} = -a^2 . \sin.\ \alpha.$$

If the curve do not meet the line AE, b^2 must be otherwise determined, as will be seen hereafter.

150. Making $a^2 - b^2 = c^2$, whence $a^2 + b^2 = 2a^2 - c^2$, our equations become

$$\frac{dy}{dx} = \pm \frac{a^2 - c^2 + x^2}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}} \text{.........(3)},$$

$$\frac{ds}{dx} = \pm \frac{a^2}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}} \text{.........(4)}.$$

When $x=0$, as at A, the curve makes an angle α with the abscissa. When $a^2 - c^2 + x^2 = 0$, or $x^2 = c^2 - a^2 = -b^2 = a^2 \sin.\ \alpha$, or $x = a \sin.^{\frac{1}{2}} \alpha$, we have $\frac{dy}{dx} = 0$ and the curve is parallel to the abscissa.

When $x=c$, $\frac{dy}{dx}$ becomes infinite, and the curve is perpendicular to the axis. When x is greater than this, the expression is impossible. Hence $c = ED$. Beyond this point the curve turns back with an arc DC, fig. 127, similar to the arc AD before this point: and these two arcs correspond to the double sign of $\frac{dy}{dx}$ in (3).

If we find the radius of curvature we shall obtain it $= \frac{a^2}{2x}$.

Hence the radius of curvature at the points A, C, A', &c. where $x=0$, is infinite. These are points of contrary flexure, and the curve between each successive two of them consists of similar arcs placed alternately. The curve, as determined from the equation, may be continued indefinitely in this form.

151. To obtain the values of y and s we should have to multiply the right hand sides of equations (3) and (4) by dx, and to integrate. The expressions, however, cannot be integrated in finite terms*. We may easily integrate them in series, by making $\sqrt{(c^2-x^2)}=u$; whence we have, neglecting the signs,

$$ds=\frac{a^2dx}{u\sqrt{(2a^2-u^2)}},\quad dy=\frac{(a^2-u^2)\,.\,dx}{u\sqrt{(2a^2-u^2)}},$$

$$ds-dy=\frac{udx}{\sqrt{(2a^2-u^2)}}.$$

Expanding $\frac{1}{\sqrt{(2a^2-u^2)}}$ by the binomial theorem, these equations become

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* These expressions are of the kind which have been called *Elliptical Transcendentals*, from their connexion with the functions on which the rectification of elliptical arcs depends. Though the integration cannot be effected rigorously, many properties and relations of them have been discovered, and methods of finding the integrals within any requisite degree of approximation. The Student will find these very completely treated of in the *Exercices de Calcul Integral* of Legendre; to whom, along with Euler and Lagrange, we are indebted for the discoveries made in this province of analysis.

If we make $x=c\,.\,\sin.\,\phi$, we shall find $s=\pm\frac{a}{\sqrt{2}}.\int\frac{d\phi}{\sqrt{(1-m^2\,.\,\sin.^2\phi)}}$, putting $\frac{c^2}{2a^2}=m^2$: which is what Legendre calls an elliptical function of the first order, and designates by F. Similarly, y is reducible to elliptical functions. It appears from the work above-mentioned, that though we cannot find the length of an arc s, we can determine arcs double, treble, &c. or the halves, thirds, &c. of given arcs; with many other properties, for which the reader is referred to the work itself. We can also obtain very converging series for the integrals; both when m is small, (which we have given in the text,) and when m is nearly $=1$; and likewise for other cases, in which the calculation is facilitated by the Tables given by Legendre.

$$ds = \frac{dx}{\sqrt{2}} \cdot \left\{ \frac{a}{u} + \frac{1}{4} \cdot \frac{u}{a} + \frac{1 \cdot 3}{4 \cdot 8} \cdot \frac{u^3}{a^3} + \&c. \right\},$$

$$ds - dy = \frac{dx}{\sqrt{2}} \cdot \left\{ \frac{u}{a} + \frac{1}{4} \cdot \frac{u^3}{a^3} + \frac{1 \cdot 3}{4 \cdot 8} \cdot \frac{u^5}{a^5} + \&c. \right\}.$$

It is only necessary to take the integrals from $x = 0$, to $x = c$, which give AD and ED, fig. 127. Now, since $u = \sqrt{(c^2 - x^2)}$, we shall have, between these limits, $\int \frac{dx}{u} = \int \frac{dx}{\sqrt{(c^2 - x^2)}} = \frac{\pi}{2}$; and by the known methods of finding $\int (c^2 - x^2)^{\frac{2n+1}{2}} dx$, (Lacroix, *Elem. Treat.* Art. 171.), we shall find between the same limits,

$$\int u\,dx = \frac{1}{2} \cdot \frac{\pi}{2} \cdot c^2;$$

$$\int u^3 dx = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2} \cdot c^4; \text{ and so on.}$$

Hence if the length $ADC = l$, and the height $AC = h$;

$$\frac{l}{2} = \frac{\pi a}{2\sqrt{2}} \cdot \left\{ 1 + \frac{1^2}{2^2} \cdot \frac{c^2}{2a^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{c^4}{4a^4} + \dots \right\}.$$

$$\frac{l - h}{2} = \frac{\pi a}{2\sqrt{2}} \cdot \left\{ 1 - \frac{1 \cdot c^2}{2a^2} + \frac{1^2 \cdot 3}{2^2 \cdot 4} \cdot \frac{c^4}{2a^4} + \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 6} \cdot \frac{c^6}{4a^6} + \dots \right\};$$

$$\therefore l = \frac{\pi a}{\sqrt{2}} \cdot \left\{ 1 + \frac{1^2}{2^2} \cdot \frac{c^2}{2a^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{c^4}{4a^4} + \dots \right\} \dots \dots (5)$$

$$h = \frac{\pi a}{\sqrt{2}} \cdot \left\{ 1 - \frac{1^2}{2^2} \cdot \frac{3}{1} \cdot \frac{c^2}{2a^2} - \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} \cdot \frac{5}{3} \cdot \frac{c^4}{4a^2} - \dots \right\} \dots (6)$$

and knowing

$$\frac{a}{\sqrt{2}} = \sqrt{\frac{\mathrm{E}}{f}}, \text{ and } \frac{c^2}{2a^2} = \frac{a^2 - b^2}{2a^2} = \frac{1 + \sin. a}{2} = \cos.^2 \left(\tfrac{1}{4}\pi - \tfrac{1}{2}a\right)$$

we may calculate l and h approximately.

From equation (3) we must determine the species of the curve. They will depend on the value of c compared with a. This will be the subject of an Appendix.

152. Prop. *When the elasticity is variable, to determine the curve having given the elasticity, and conversely.*

We have supposed the moments of the forces which tend to bend the lamina at any point to be equal to $\frac{E}{\rho}$, where E is the measure of the elasticity, and is the same for every point. We shall now suppose E to be a function of the curve or its coordinates. As before, let a force f act on the lamina and let the abscissa be perpendicular to its direction. Hence

$$fx = \frac{E}{r}; \therefore E = fx\rho.$$

If E be given in terms of s, or of x and y, we may substitute and integrate. If E be to be found, it will be had from the formula,

$$E = -\frac{fx.(dx^2+dy^2)^{\frac{3}{2}}}{dx\,d^2y}, \; dx \text{ being constant.}$$

E, as appears from Art. 148, may be supposed proportional to the breadth, when the thickness is constant, and to the cube of the thickness, when the breadth is constant.

Prob. *To find how the breadth of a uniform elastic lamina must vary, that by a weight hung at the end of it, it may be bent into the form of a quadrant. Fig.* 128.

In this case r is constant; therefore E, $=f\rho x$, is as x. Hence the lamina must be such that its projection ACc on a horizontal plane is a triangle.

153. Hitherto we have supposed that the elastic rod or lamina in its natural state, when it is not acted on by any forces, is a straight line. But we may suppose that it is naturally of any form whatever, and that it is deflected from this natural form by the same laws by which we before supposed it deflected from a straight line.

PROP. *In an elastic rod which is naturally a given curve, the curvature produced by any force at any point is equal to the natural curvature, together with the curvature which the same force would produce in a rectilinear rod of the same elasticity, acting in the same manner.*

Let Pq, fig. 129, be a small given arc whose natural curvature is Pq, and its center of curvature o; and be bent into the position PQ, with its center of curvature at O, by means of a force acting at the arm QE. Then the deflection Qq, of Q from its natural position, is the same which it would be if Pq were a straight line.

Now ultimately, when PQ or Pq is indefinitely small, Qq may be considered as perpendicular to the tangent at P, and will therefore be equal to the difference of the perpendiculars QR and qr upon the tangent. Hence (Newt. *Princ.* Lem. XI.)

$$Qq = QR - qr = \frac{PQ^2}{2PO} - \frac{Pq^2}{2Po} = \frac{PQ^2}{2} \cdot \left\{\frac{1}{PO} - \frac{1}{Po}\right\}.$$

But if Pq were a straight line, Po would be infinite; and if $Q'q'$ be the deflexion in this case for an arc PQ', and PO' the radius of curvature for the same force;

$$Q'q' = \frac{PQ'^2}{2} \cdot \frac{1}{PO'},$$

And by supposition the deflexion from the natural form is the same in the two cases for the same arc: or $Q'q' = Qq$, PQ' being equal to PQ. Hence

$$\frac{1}{PO} - \frac{1}{Po} = \frac{1}{PO'},$$

$$\text{and } \frac{1}{PO} = \frac{1}{PO'} + \frac{1}{Po};$$

and the curvature being inversely as the radius, the Proposition is manifest.

COR. Since by Art. 150, $\dfrac{1}{PO'} = \dfrac{\text{E}}{fk}$,

$$\text{we have } \frac{1}{PO} = \frac{fk}{\text{E}} + \frac{1}{Po}.$$

E being, as before, a quantity which measures the elasticity of the rod PA; and fk the moment of the force which acts.

154. PROP. *A uniform elastic rod, which is naturally a given curve, is fixed at one end and acted on by a given force: it is required to find the form which it will assume.*

Let BA, fig. 127, be the curve when the force f is applied. And as before, CM perpendicular to $AE=x$, $MP=y$, $AP=s$. And let the radius of curvature of any point P be, in the original form, $=r$, and in the form which it assumes, $=\rho$. Hence

$$\frac{1}{\rho}=\frac{fx}{\text{E}}+\frac{1}{r}, \text{ or } fx=\frac{\text{E}}{\rho}-\frac{\text{E}}{r};$$

and r being given in terms of s, we have a differential equation to the curve AB.

155. PROB. *Fig. 130. The curve Ba, being originally a quadrant, fixed at its lowest point B, it is required to find the curve BA, when it is acted on by the force F.*

Let FA meet the horizontal line BD in D: $DM=x'$, $MP=y$; radius of $Ba=r$; and since the original curvature is in a direction contrary to that which the force would produce, r must be made negative in the formula. Hence it becomes

$$fx'=\frac{\text{E}}{\rho}+\frac{\text{E}}{r}; \text{ or if we make } x'-\frac{\text{E}}{fr}=x,$$

$$fx=\frac{\text{E}}{\rho}; \text{ and, putting for } \rho \text{ its value,}$$

$$fx=-\frac{\text{E}\,dx\,dy}{(dx^2+dy^2)^{\frac{3}{2}}}; \text{ which agrees with equation (1), Art. 124,}$$

for the common elastic curve.

Hence the curves into which the circular rod can be bent are the same as in the case of the straight lamina.

If we take $DE=\dfrac{\text{E}}{fr}$, we shall have $EM=x$, and hence if EC, perpendicular to DE, meet the curve, it will cut it in a point of contrary flexure C.

156. Prob. *Fig.* 131. *To find what must be the natural form of a lamina aB, that force F, acting perpendicularly at its extremity, may deflect it into a straight line AB.*

For the same reason as before r must be negative. Also ρ is infinite. And if a point p be, by the action of the force, brought to P, we have $AP = ap = s$, suppose; hence,

$$fs = \frac{E}{r}, \text{ or } rs = \frac{E}{f}; \text{ or, making } \frac{2E}{f} = a^2$$

$$rs = \frac{a^2}{2}; \text{ which equation contains the property of the curve.}$$

If pn be perpendicular on an, and if we make $an = x$, $np = y$, and the angle $ptn = \phi$, we shall have,

$$dx = ds \,.\, \cos.\, \phi, \; dy = ds \,.\, \sin.\, \phi, \; r = \frac{ds}{d\phi}.$$

Hence $d\phi = \frac{ds}{r} = \frac{2sds}{a^2}$; $\phi = \frac{s^2}{a^2}$, the arbitrary constant being $= 0$ if an be a tangent at a.

$$\therefore\; dx = ds \,.\, \cos.\, \frac{s^2}{a^2}, \; dy = ds \,.\, \sin.\, \frac{s^2}{a^2};$$

and by integrating these expressions, we should have the values of x and y in terms of s.

We may integrate by expanding $\cos.\, \frac{s^2}{a^2}$ and $\sin.\, \frac{s^2}{a^2}$, and thus we obtain

$$x = s - \frac{s^5}{1\,.\,2\,.\,5a^4} + \frac{s^9}{1\,.\,2\,.\,3\,.\,4\,.\,9a^8} - \&c.$$

$$y = \frac{s^3}{1\,.\,3a^2} - \frac{s^7}{1\,.\,2\,.\,3\,.\,7a^6} + \frac{s^{11}}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,11\,.\,a^{10}} - \&c.$$

which converge rapidly, except when s is very large.

Since the curvature increases in proportion to the distance from a, it is manifest that the curve will be a kind of spiral, which will tend to a point C with an infinite number of revolutions. The co-ordinates of this point C would be found, if we could find the values of x and y when s is infinite, which cannot be obtained from the series given above.

It makes no difference what point of the spiral we take for the point B. If we suppose that point and its tangent to be fixed, the portion of the curve Ba may always be bent into a straight line.

3. *Elasticity of Torsion.*

157. When a slender thread of metal, &c. is twisted, it tends to resume its natural condition, and would communicate angular motion to any body to which it is annexed, for instance, to a straight rod or rigid line fastened across it at right angles. A force acting on this rod may resist this tendency to motion and produce equilibrium. The force necessary for this purpose is, as has been already mentioned, proportional to the angle through which the thread is twisted. Let there be a thread, perpendicular at C, fig. 132, to the plane of the paper. Let its upper extremity be fixed, and let Bb be a bar suspended at its lower extremity in a horizontal position. If this needle be turned out of the position Bb in which it would naturally hang, into any other Pp, the force which, acting at P in a horizontal plane and perpendicular to CP, would retain it in this position, will be as the arc BP, or as the angle BCP. If BC vary, the equilibrium will be preserved so long as the product of the force $(=F)$ and distance BC remains the same; hence $F.BC \propto BCP$. If we call the angle BCP, θ, and the distance $CB=CP$, a, we shall have $Fa \propto \theta$, and $Fa = \epsilon\theta$, by properly assuming ϵ. The quantity ϵ is manifestly the value of the force F when the arm $BC=1$, and the angle $\theta=1$; it is different for different substances and masses, and may be considered as measuring the *elasticity of torsion*.

Problems in which elasticity of torsion enters present few difficulties; especially as there is no change of figure in the bodies which are concerned. We shall therefore only give one instance of their solution.

158. PROB. *Fig.* 132. *The extremity P of the bar whose natural position is Bb, is acted on by a repulsive force which varies inversely*

as the square of the distance from the center of force A, and is kept in its place by torsion; given its position, to find the force at A.

Let the force of repulsion exerted by A be $\frac{f}{z^2}$; z being the distance AP. This force acts in the direction AP. Let it be resolved into two, one in the direction MP, of the lever CP, and the other in TP, perpendicular to CP. The former of these produces no effect to turn the lever CP, and the latter only is balanced by the torsion. Let $ACP = \theta$, and

$$APT = ApP = \tfrac{1}{2}ACP = \tfrac{1}{2}\theta.$$

Hence the force which balances the torsion is $\frac{f}{z^2}\cos.\tfrac{1}{2}\theta$.

Let $CA = a$, and we have manifestly $z = AP = 2a\sin.\tfrac{1}{2}\theta$. Hence the force which balances the torsion is $\frac{f\cos.\frac{1}{2}\theta}{4a^2\sin^2\frac{1}{2}\theta}$.

Let now $ACB = \beta$, and the angle BCP, to which the torsion is proportional, will be $\theta+\beta$. The force of torsion will be $\epsilon(\theta+\beta)$ acting at P, perpendicular to CP; as is stated in last Article. Hence.

$$\frac{f\cos.\frac{1}{2}\theta}{4a^2\sin.^2\frac{1}{2}\theta} = \epsilon(\theta+\beta);$$

whence

$$f = 4a^2\epsilon(\theta+\beta)\sin.\tfrac{1}{2}\theta\tan.\tfrac{1}{2}\theta.$$

If θ correspond to another position of Pp, f' being the force which retains it there, we have

$$f' = 4a^2\epsilon(\theta'+\beta)\sin.\tfrac{1}{2}\theta'\tan.\tfrac{1}{2}\theta',$$

whence

$$\frac{f}{f'} = \frac{(\theta+\beta)\sin.\frac{1}{2}\theta\tan.\frac{1}{2}\theta}{(\theta'+\beta)\sin.\frac{1}{2}\theta'\tan.\frac{1}{2}\theta'}.$$

If the arcs θ, θ' be very small, we may put the arc for its sine and tangent; and hence

$$\frac{f}{f'} = \frac{(\theta+\beta)\theta^2}{(\theta'+\beta)\theta'^2}.$$

If the points B and A coincide, $\beta = 0$,

$$\frac{f}{f'} = \frac{\theta \sin. \frac{1}{2}\theta \tan. \frac{1}{2}\theta}{\theta' \sin. \frac{1}{2}\theta' \tan. \frac{1}{2}\theta'};$$

and when θ, θ' are small,

$$\frac{f}{f'} = \frac{\theta^3}{\theta'^3}.$$

The combination supposed in this proposition is the *Torsion Balance* of Coulomb, which has been employed for the purpose of measuring very small repulsive and attractive forces. In some cases the instrument was constructed with so much delicacy, that each degree of torsion required a force of only $\frac{1}{122400}$ of a grain.

DYNAMICS.

CHAP. I.

DEFINITIONS AND PRINCIPLES.

159. DYNAMICS (see Art. 9,) is the part of Mechanics which relates to the action of force producing motion. In any of the machines and mechanical combinations which we have described, if the forces be not in that relation which is requisite for equilibrium, the excess of force will produce motion. Thus, in fig. 29, if the weights P, Q and W had not the proportion which their position makes requisite to the equilibrium, they would move in a manner depending upon their magnitudes: and the laws which we are about to lay down, are those by which their motions are to be calculated.

160. The *pressure* which produces motion is understood here in the same sense as in Statics. Let two equal bodies A, B, fig. 136, hang over a pully E. They will balance each other, exerting equal and opposite pressures on the string AEB, and no motion will be produced. Let now a weight P be added to A, and the pressure of P will cause A to descend and B to ascend. In this case the string exerts its tension upon A and B equally and in opposite directions. The mass $A+P$ is pressed downwards by its weight $A+P$, and upwards by the tension of the string, and descends by the difference of these forces. And in the same manner B is drawn upwards by the tension, and downwards by its weight, and ascends by the difference of these forces.

In this case the bodies move in the directions of the forces; but if we suppose one of the weights to be compelled to move obliquely to the force, as would be the case with P in fig. 98, if P and Q were to move, the pressures will act obliquely, and must be resolved by the rules given for resolution of force in Art. 27, in order to obtain the pressure which produces the motion.

In order to obtain the effect of pressures in producing motion under given circumstances, we shall establish the three laws of motion, defining also the measures of velocity, accelerating force, and moving force.

161. All motion is performed in *time:* and the time employed is measured by the number of units of time which it contains. The passage of time is marked by the events which take place in it, and those intervals in which there is no discoverable reason why they should be unequal, are supposed equal. The intervals thus taken as a standard are in all countries the natural day and its divisions. The unit of time may be any portion we choose: in Mechanics a second is generally taken for the unit.

162. VELOCITY is the measure of the degree in which a body moves quickly or slowly: that is, one body is said to have a greater velocity than another when it moves over a greater space in the same time, or an equal space in a less time.

When a body moves over equal spaces in equal successive times the motion is said to be *uniform.* And the velocity is *measured* by the space described in a unit of time, as for instance, in one second.

In variable motions it will be seen hereafter that the velocity is measured by the space which *would be* described in a unit of time, if the velocity were uniform.

163. PROP. *In uniform motions the space described in any time is equal to the product of the velocity and the time.*

Let v be the velocity; then by the last article, v is the space described in one second. And since the motion is uniform, $2v$ is the space described in two seconds; $3v$ in three seconds; and generally, tv in t seconds. If s be the space $s = tv$.

If we suppose the space described in equal fractions of a second to be equal, this equation will also be true when t is a fractional or mixed number.

COR. Since $s = tv$, $v = \frac{s}{t}$.

Hence in uniform motions the quotient of the space by the time is constant, and measures the velocity.

Thus if a ship, sailing uniformly, move 10 miles in 1 hour, the velocity, measured by the space described in a second, is

$$\frac{10 \times 5280}{60 \times 60} = 14\tfrac{2}{3} \text{ feet.}$$

164. When the velocity is not uniform, it can no longer be measured by the quotient of the space divided by the time; for these quotients will be different for different times. Thus if we suppose P, A and B, fig. 136, to be such that A shall fall from rest 16 feet in the 4 first seconds, A will move not with a uniform but with an increasing velocity. And if we then measure the space described by this body in the 4 seconds succeeding, we shall find it 48 feet; in three seconds from the end of the first four, the space would be 33 feet; in two seconds 20; in one second 9; in the half second immediately following the fourth it will be $4\frac{1}{4}$ feet, and in the quarter second after the fourth it will be $2\frac{1}{16}$. Hence we shall have the following values of the quotient of the space by the time measuring from the beginning of the fifth second.

Values of t,	4″	3″	2″	1″	$\frac{1}{2}$″	$\frac{1}{4}$″
of s,	48	33	20	9	$4\frac{1}{4}$	$2\frac{1}{16}$
of $\frac{s}{t}$,	12	11	10	9	$8\frac{1}{2}$	$8\frac{1}{4}$.

The quotients, commencing at the beginning of the fifth second, go on increasing, and are larger as we take the time larger. And this must be the case with an increasing velocity. For the space described beginning from any time will depend upon the velocity at that time, and upon the augmentation of velocity which takes place afterwards.

Also the portion of the space which is due to this augmentation is smaller as the time of the motion is smaller. And if we approach nearer and nearer to the initial point of time, we approach nearer and nearer also to the velocity at that point of time.

Hence the VELOCITY *at any point is measured by the* LIMIT *of the quotient of the space by the time beginning from that point;*

the limit being taken by supposing the space and the time indefinitely diminished.

Thus in the above instance, if we were to suppose more minute values of t to be taken, as $\frac{1}{8}$, $\frac{1}{16}$, it would appear that the value of $\frac{s}{t}$ would always be greater than 8. But the excess above 8 might be diminished, by diminishing s and t sufficiently, so as to be made smaller than any assigned quantity. Hence 8 is the *limit* of the fraction $\frac{s}{t}$, and 8 feet measures the *velocity* of the body at the beginning of the 5th second.

Instead of taking the time immediately after the point considered, we may take the time immediately before it, and we shall have analogous results.

165. PROP. *In any motion the velocity is measured by the space which would have been described in a unit of time, if the velocity had continued constant.*

Let the velocity be increasing, and let s' be the space from the given point, which would be described in the time t if the velocity were to continue constant for that time; $s' + s''$ the space which is actually described in t. Then, by last Article, the limit of $\frac{s' + s''}{t}$ is the measure of the velocity. In this expression s'' is the part which arises from the augmentation of the velocity *after* the body leaves the given point, and its effect diminishes as t diminishes. Hence in taking the limit, the effect of s'' cannot appear. Therefore the limit of $\frac{s' + s''}{t}$ is the same as the limit of $\frac{s'}{t}$; and $\frac{s'}{t}$ measures the velocity.

When $t = 1$, s' is the space described in a unit of time, supposing the velocity to become constant. Hence this space measures the velocity.

And similarly for a decreasing velocity.

166. PROP. *If s be the space, v the velocity, t the time,*

$$v = \frac{ds}{dt}.$$

For by the definition of a differential coefficient, the differential coefficient of s with respect to t is the limit of the quotient of the increments of s and t. Hence, by Art. 164, the differential coefficient, or $\frac{ds}{dt}$, is equal to the velocity.

167. Def. *Momentum is the product of the velocity and quantity of matter.*

168. The velocity and direction of a body's motion are regulated by the forces which act upon it, and the simplest principles to which the relation of these quantities can be reduced, are called the Laws of Motion. In proceeding to these, we must first establish the law of the motion of a body when it is not acted upon by any force, but left to itself.

First Law of Motion. *A body in motion, not acted on by any force, will move on in a straight line, with a uniform velocity.*

This law consists of two parts: first, that the body will go on in a straight line, which is demonstrated from the nature of the case; and second, that it will move with a uniform velocity, which is proved from observation.

First, it will move in a straight line. The body is supposed not to be acted on by any force; that is, its motion is not influenced by nor related to any external objects. But the only law which can govern its motion so that this may be true, is, that the body shall move in a straight line. For if the law were that its path should be any curve line, we should have to determine towards which side the convexity must lie, and how great the curvature must be; and manifestly there is no way in which we can conceive this to be determined, except by some reference to external objects. But external objects cannot determine the quantity and direction of the curvature, for if they did they would be said to exert force upon the body: hence it must move in a path without curvature, that is, its course must be a straight line.

Secondly, it will move with a uniform velocity. This is not, like the former, a necessary truth. Though the most simple law, perhaps, which can be supposed is, that a body in motion and acted on by no external cause, should proceed describing equal spaces in equal successive times; yet it would also be by no means difficult to conceive that its velocity should go on increasing or diminishing in an infinite number of ways. We might, for instance, without any inconsistency, suppose the law of nature to be, that every body in motion, when left to itself, should go on describing in every second $\frac{9}{10}$ of the space which it had described in the preceding second: so that its velocity should perpetually decrease. Indeed it would seem that our first bias is, from the appearances which we perpetually behold, to imagine that there does exist in motion a disposition continually to retard itself; as if from a kind of inertness in matter: and a tendency of velocity to waste away till the whole is exhausted and the body reduced to a state of rest. For all the bodies which we see in motion, we observe to move more and more slowly, losing their velocity by different degrees according to the nature of the case, and finally to stop.

But in all the cases in which motion is thus perpetually retarded, the body is not left entirely to itself: it is in each instance acted on by some cause which tends to destroy its velocity. If a ball be thrown along a level surface, as a bowling-green; or if a wheel, supported by its axis, have a rotatory motion given it; or if a pendulum, hanging freely, be made to oscillate*; these motions will, after a short time, cease. But this extinction of the motion arises from external causes which act in these cases. Thus the ball is resisted by the friction of the surface along which it runs; the wheel by the friction of the axis; and the bodies in all the cases, by the resistance of the air.

Hence the retardation which such bodies experience is what might be expected, supposing the law of motion above enunciated to be true. And we find that in proportion as we remove the impediments to the continuance of the motion, we diminish the re-

* It is easy to see that in the first of these three cases the action of gravity does not, except by producing friction against the plane, tend to retard the velocity: and that in the other two cases, though the motion is not rectilinear, it would go on for ever if rectilinear motion would do so. In the case of the wheel, the actions of gravity in the different parts counteract each other, so that there is no force to retard the velocity; and in the case of the pendulum the quantities of velocity alternately generated and destroyed, would, if it were not for the impediments mentioned in the text, be perpetually the same.

tardation; and this seems to be true without limit; so that it may be inferred that if we could entirely remove the external causes of retardation, the motion would continue uniform and go on for ever.

Thus the ball will soon stop if thrown along rough ground; its motion will continue longer if it be projected along a smooth pavement; and if it be thrown along a sheet of smooth ice it will lose its velocity very slowly, and move a long way before it stops, though it is still retarded by the resistance of the air, and by the ice, which is never, mathematically speaking, devoid of friction.

In the same manner, if in a wheel we diminish friction by the employment of friction wheels, we cause its motion to continue longer. And if we remove also the resistance of the air by making the experiment in a vacuum, the motion will continue apparently unabated for a great length of time*.

In the same manner a pendulum, which in the air ceases to oscillate after a short time, will in a vacuum continue its oscillations for a long time, and longer as the vacuum is more perfect.

From these and similar instances we infer that if we could entirely remove the external causes which retard the motions of bodies, the velocity could continue undiminished for ever; and therefore, that the first law of motion is true.

169. We may also consider the proof of this first law of motion in the following manner. If, when bodies in motion are not acted

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* Experiments shewing how rotatory motion tends to become uniform by removing the resistance of the air, may be seen in accounts of the effects of the air-pump. An account of the effects of friction-wheels may be found in the *Phil. Trans.* vol. LIII.

The undisputed authority which is now allowed to the laws of motion mentioned in this Chapter, is the result of innumerable experiments never recorded and discussions now forgotten, to which they were subjected during the seventeenth century. Great numbers of trials were made, both by individuals and before learned bodies, to prove almost every one of the propositions which are now considered as nearly self-evident. An experiment for proving this first law was made before the Royal Society by Hooke in 1669; of which there is an account in Birch's History of the Royal Society, vol. II. p. 342.

upon by any external force, they have a tendency to retard their own motion, then, when they are acted upon by forces which retard them, the retardation will be caused in part by their own tendency, and in part by the retarding forces. Now if this be the case, we can, by investigating the retardation which is produced by the external causes, and by comparing it with the whole retardation, discover the part which belongs to the motion of a body left to itself: but if we make the experiment we shall find that in each case this latter part of the retardation is equal to nothing, and the whole depends on the external retarding forces.

This process supposes that we know the law according to which forces that act upon a body influence its motion; and this knowledge in some measure depends on this first law which we are proving; but there is no law according to which we can assume the action of force to take place, consistently with observation, which will allow us to suppose the natural motion of a body any other than uniform.

170. This first law of motion being proved, it follows, that if a body, considered as a point, move either in a curve line, or in a straight line with a velocity not constant, it is acted upon by some external force: and the deviation from rectilinear and uniform motion depends upon the direction and magnitude of the force which acts upon it.

The *Direction* of a force is the straight line in which the force would cause a body to move if it acted on the body at rest. When it acts on a body already in motion, it will combine the motion towards this part, with the other motion which the body has, according to laws which will be mentioned hereafter. If a force act upon a body in motion, so that its direction coincides with the direction of the body's motion, the body manifestly will not be made to deviate on one side or the other, but will go on in the straight line with an altered velocity. If a force act so as to make an angle with the direction of the motion of the body, it will cause it to describe a curvilinear path with its concavity on the side towards which the force tends.

171. The *Magnitude* of forces is measured by their effects, and the effect which we consider in Dynamics is *Velocity*. Hence

forces are greater or less as they produce a greater or less velocity in the same time.

ACCELERATING FORCE is measured by the velocity which it would add in a given time.

MOVING FORCE is measured by the momentum which it would add in a given time.

When a body is acted upon by a continuous force, as pressure or attraction, the velocity communicated to the body goes on increasing as the force acts for a longer time. Thus, if a stone fall from rest during one second, and another fall during two seconds, the velocity of the latter, upon which gravity has acted for a longer time, will be the greater. Similarly, if we produce velocity by the continued action of the hand, as in a fly-wheel, or by means of a spring, as in a bow, the velocity goes on increasing so long as the operation of the force continues. Now we may, at any point of time, suppose the action of the force to cease; and, by the first law of motion, the body would then go on with the velocity already acquired: if, after this, we suppose the force again to begin to act in the direction of the motion, an additional velocity will be communicated. Thus force *produces* velocity in a body at rest, and *adds* velocity to a body already in motion; and if it be supposed to act for any time, it is adding velocity during the whole of that time, and the velocity produced at last is the aggregate of all the successive additions.

If, under these circumstances, the velocity *added* be equal in equal times, the force is said to be *uniform* or constant. Thus if it be found that a certain velocity is at the end of one second communicated by gravity to a body falling from rest, and twice the velocity at the end of two seconds, thrice the velocity at the end of three, and so on; that is, if the velocity *added* in the second second to the motion be equal to the velocity given in the first; and the velocity added in the third, fourth, &c. seconds also equal to this; the force of gravity, which in these equal times generates these equal velocities, is said to be uniform. Similarly, if the force retard, instead of accelerating, the body's motion, it is uniform when the velocity *subtracted* in equal successive times is equal.

In this case, the uniform force is *measured* by the velocity added (or subtracted) in a given time, as for instance, one second.

Thus gravity, which every second generates in a body moving vertically downwards a velocity of $32\frac{1}{5}$ feet, may be represented by this velocity (or by $32\frac{1}{5}$ feet); and then any other uniform force, as for instance, one which would generate a velocity of 1 foot in a second, will be measured by this its velocity, and its proportion to gravity will be that of 10 to 322.

172. PROP. *With uniform accelerating forces, the velocity generated in any time is equal to the product of the force and the time.*

Let f be the accelerating force; then f is the velocity generated in one second. And since the force is uniform, f will also be the velocity added in the next second; and $2f$ will be the velocity at the end of 2 seconds. In the same manner $3f$ will be the velocity at the end of 3 seconds; and generally tf will be the velocity at the end of t seconds. If v be the velocity, $v = tf$.

If we suppose the velocity generated in equal fractions of a second to be equal, this equation will also be true when t is a fractional or mixed number.

COR. Since $v = tf$, $f = \frac{v}{t}$.

Hence in uniform forces the quotient of the velocity generated by the time in which it is generated, is constant, and measures the force.

Thus if, as in Art. 164, a velocity of 8 feet be generated in 4 seconds, the accelerating force is $\frac{8}{4}$ or 2.

The velocity generated by gravity in one second is $32\frac{1}{5}$ feet. Hence the accelerating force of gravity is $32\frac{1}{5}$*.

173. When the accelerating force is not uniform, it can no longer be measured by the quotient which results from dividing *any* velocity by the time in which it is generated. This quotient

* According to some writers forces are *proportional* to the velocity generated in 1″, but not equal to it. They are measured so that gravity is = 1. If we call $32\frac{1}{5}$ feet $= g$, and if F be any other force which generates a velocity f in 1″, measured on the scale now mentioned,

$$F : 1 \text{ (gravity)} :: f : g; \therefore F = \frac{f}{g}.$$

$$\text{Also } \frac{dv}{dt} = f = Fg.$$

will vary with the time during which the force is supposed to act, in the same manner as the quotient of the space by the time in Art. 164 was variable; and the quotient of the velocity by the time will have a limit, in the same manner as the quotient of the space by the time in that case was shewn to have a limit. And this limit will measure the *accelerating force* at the given point in the same manner as the limit of the quotient in Art. 164 measured the velocity. For by taking the value of the quotient of the velocity generated by the time in which it is generated; and by taking the whole of this time nearer and nearer to the given point; we approximate to the measure of the force *at* the given point. And hence by taking the ultimate limit of this quotient, we obtain the exact measure of the force at the point which is considered.

Hence the ACCELERATING FORCE *at any point is measured by the limit of the quotient of the velocity generated, (beginning from that point,) divided by the time in which it is generated.*

If the velocity be diminished by the force, the force is a retarding force, and the same is true.

174. PROP. *In any motion the force is measured by the velocity which would have been generated in a unit of time, if the force had continued constant.*

This is proved in the same manner as the corresponding proposition with regard to the velocity in Art. 165. For if the force be an increasing one, the augmentation of velocity in any time, beginning from a given point, will be due partly to the force *at* that point, and partly to the increase of force *after* that point. And the latter portion of the augmented velocity must disappear when we consider only what belongs to the given point itself. Hence the force is to be measured as if it had continued constant from the given point; that is, it is measured by the velocity generated in a unit of time.

175. PROP. *If f be the force, v the velocity, t the time,*

$$f = \frac{dv}{dt}.$$

For $\frac{dv}{dt}$ is the limit of the quotient of the increments of v and t, and therefore, by Art. 173, it is equal to the force.

COR. 1. If s be the space $\frac{ds}{dt} = v$; hence multiplying,

$$v\frac{dv}{dt} = f\frac{ds}{dt}. \text{ And (Lac. D. C. Art. 9.) } \frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt}.$$

Hence substituting and omitting $\frac{ds}{dt}$ on both sides $v\frac{dv}{ds} = f$.

COR. 2. We have also $f = \frac{dv}{dt} = \frac{d\left(\frac{ds}{dt}\right)}{dt} = \frac{d^2 s}{dt^2}$.

176. DEF. MOVING FORCE *is the product of the accelerating force by the quantity of matter.*

Moving force is measured by momentum generated, in the same way as accelerating force is measured by velocity generated. Hence if we multiply the velocity, which measures the accelerating force, by the quantity of matter, we have the measure of the moving force.

Hence accelerating force is equal to the quotient of the moving force by the quantity of matter.

177. We have already seen that when a body in motion is acted on by a force which is not in the direction of its motion, it will no longer describe a straight line. We proceed to consider the law according to which the action of the force takes place in this case.

SECOND LAW OF MOTION. *When any force acts upon a body in motion, the change of motion which it produces is in the direction and proportional to the magnitude of the force which acts.*

This may also be thus expressed. When any force is exerted upon a body already in motion, the motion which the force would produce in a body at rest, is compounded with the previous motion, in such a way, that both produce their full effects parallel to their own directions.

Thus, suppose a body, considered as a point, to be moving in the direction AB, fig. 135, with such a velocity that it may describe AB uniformly in 1″. Then by the first law of motion it would in the next 1″ describe Bb in the same straight line equal to AB. But when it comes to B, let a force in the direction BM begin to act and act uniformly upon it for 1″; the force being of such a magnitude that it would in 1″ cause the body to describe *BM* from rest. Then at the end of 1″ from the time when the body is at *B*, it will be found at *C*, so that *MC* and *bC* are equal and parallel to *Bb* and *BM*.

If when the body comes to *C* the force were to cease to act, it would go on moving in the direction and with the velocity which it has at *C*. Let *Cc* be the space it would thus describe in 1″. But now suppose a force to begin to act at *C*, which by its uniform action for 1″ would carry the body through *CN*. Then its place at the end of 1″ from *C*, will be *D*, *DN* and *Dc* being parallel and equal to *cC* and *NC*.

Similarly, if other forces act uniformly for successive seconds, we may find their effects. If the forces be not such that they can be considered as constant in magnitude and direction for 1″, we must apply this law of motion to them for any small time during which they may be considered as constant. If they vary continuously, we must consider the limits of *Bb* and *BM*, &c. as will be seen hereafter.

178. The proof of this law of motion depends upon experiment, and may be inferred from such facts as the following.

If we are in a ship, moving equably, any force which we exert will produce the same motion relative to the vessel whether it be or be not exerted in the direction of the vessel's motion. If we stand on the deck, (which is supposed to be level,) and roll a body along it, the same effort will produce the same velocity along the deck whether the motion be from head to stern, or from stern to head, or across the vessel. Also a body dropped from the top of the mast will not be left behind by the motion of the ship, but will fall along the mast as it would if the vessel were at rest, and will reach the foot of the mast in the same time as it would have done in that case. If a body be thrown perpendicularly upwards, it will rise directly over the hand and fall perpendicularly upon it again; and if it be thrown in any other direction, the path and motion relative to the person who throws it will be the same as they would have been if he had been at rest*.

* For an account of experiments of this kind made for this purpose, see Gassendi's Works, tom. III. p. 478. "*De motu impresso a motore translato.*"

Let PQ, fig. 134, be a boat moving uniformly in the direction Pp: let B be a ball at rest upon the deck, carried along by the motion of the boat, (the deck being supposed to be horizontal). Let BD be a line drawn upon the deck, and by the motion of the boat in any time, let BD come into the position bd. At the instant when the boat is in the situation PQ and the ball at B, let the ball be struck, so as to receive an impulse in the direction BD. Then it is found that the ball moves uniformly in the line BD, so that when B comes to b the ball is at some point C in bd, moving, relatively to the boat, in the line BD.

Now since the ball, if no impulse had been communicated to it, would have moved from B to b, and since it is found in C at the end of the same time, it appears that the effect of the impulse has been to compound a motion in bC with the motion in Bb*; which is agreeable to the second law of motion.

Again, let BK, fig. 135, be the mast of a vessel, and let this in one second be transferred by the motion of the ship into the position bk. When it is at BK suppose a body to be let fall from B, and let BM be the space through which a body would fall from rest in $1''$. Then it is found by experiment that at the end of $1''$ the body has fallen down the mast through a space bC, bC being equal to BM. Now at B the body had the velocity Bb, and was then acted on by a force which would carry it over BM; and it appears that these motions are compounded so that $BMCb$ is a parallelogram, as by the law which we have enunciated it should be.

It appears that according to this second law of motion, all motions are compounded so as not to disturb each other; each remaining, relatively, the same as if there were no others.

If this law of motion were not true, it would follow that bodies placed upon a horizontal plane which is in motion, and struck in a given direction, would not move, relatively to the plane, in the direction of the impact, except when the impact was in the direction

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* Also since the motion in bd is uniform, bC varies as the time from B; but Bb varies as this time; therefore bC varies as Bb, and therefore the motion of the body from B to C is in a straight line.

And BC varies as Bb, that is, as the time; therefore the motion in BC is uniform.

of the plane's motion. Hence if we suppose the earth to revolve on its axis, a body struck in a direction north and south, would not move in that direction, but would deviate to the east or west: which is not found to be the case. In the same manner the oscillations of a pendulum would be performed in different times, accordingly as it oscillated in a north and south, or in an east and west plane*: and similarly in other cases.

Besides the motion of the earth on its axis, which is combined with all terrestrial motions according to this second law of motion, the motion of the earth round the sun, which is much more considerable, is also combined with them; and if the whole solar system be in motion, the motion of the solar system is also similarly combined.

179. In order that this law may be fully proved from experience, we ought to have an indefinite number of experiments made with all possible velocities. But there are some facts which supersede the necessity of these experiments. The laws of motion are the same in all parts of the earth and at all times; and every where the same forces produce the same relative effects. But in different places and at different times the velocity of the surface of the earth, in which all bodies partake, is very different. Now it may be shewn as above, that in order that the relative velocities may correspond to the forces, the second law of motion must be true at *these* places and times. Hence it is either true for all velocities, or it is true for a great number of different velocities, (and these precisely the ones which happen to come under our observation) and it is false for the rest: but the latter supposition is manifestly highly improbable; hence the law is true for all velocities†.

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* According to Laplace, *Mec. Cel.* tom. I. liv. i. No. 5, this observation would indicate a deviation from the above law of motion if any existed, though it should be very small.

† When any bodies which are situated in a moving space are acted on by forces, their relative motions are the same as if the space were at rest. If the forces or agents be confined *within* that space, (and be not extraneous forces, as gravity is to moveable space enclosed in a vessel,) this truth is evident independently of experiment. For our mechanical ideas are all relative; our ideas of motion relative to space; our ideas of force relative to motion. Hence, so long as the relative situations and connexions of

bodies

180. THIRD LAW OF MOTION. *When pressure communicates motion to a body, the moving force is as the pressure.*

This will be proved by means of the following Propositions.

PROP. *When different constant pressures act separately for equal times on* the same body, *the velocities which they produce are proportional to the pressures.*

The most obvious and universal experience shews us, that by increasing the pressure which moves a body, we increase its motion. Thus, if a body, laid on a smooth table, so as to be easily moveable, be pushed sideways by the hand, or by a spring resting its other end against something fixed; the more violent the exertion we make, or the stronger the spring that we use, and the greater will be the velocity communicated. The same would be true if the body were drawn sideways by a weight hanging over the edge of the table by means of a string: the greater the weight was, and the greater would be the velocity communicated, in a given time, for instance, in one second.

This being considered, it is perhaps the *most simple* supposition that this increased velocity is directly *proportional* to the pressure which produces it. But in order to prove satisfactorily this proportionality, experiments are necessary; and some consideration is requisite in order to fix upon such experiments as shall be at the same time practicable and decisive.

If, as before, we suppose a body A to be laid upon a smooth horizontal table, and another body B, connected with it by a string, to hang over the edge of the table*, B will descend, and will

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bodies in a finite space are the same, the forces which they exert, and the relative motion in that space, must be the same, without considering whether the space move or not, or with what velocity it moves. The same principles of Dynamics must of course be discovered, whether they are investigated by one who does or does not acknowledge the motion of the earth: and if it should be proved that the solar system is moving, our laws of motion will still necessarily remain unaltered.

* It is supposed that the table is perfectly smooth, and the body A so small that the string may be considered to be parallel to the table.

communicate a velocity to *A*; and this velocity will be greater as *B* is greater. In this case the *pressure* which produces motion is the weight of *B**. If *A* be 3 pounds, and *B* 1 pound, there will in one second be generated a velocity of 8 feet a second.

Now let *A* remain 3 pounds, and let *B* become also 3 pounds; the student might perhaps at first imagine that, since the weight *B* which produces velocity is *tripled,* the velocity should be 3 times as much as before. In fact however it will only be 16 feet, or *twice* as much; and a little consideration will shew the reason.—The body *B*, which moves *A*, has also to move *itself* with the same velocity, and hence the pressure arising from the weight of *B* is not *all* employed in moving *A*. The masses moved in the two cases are 4 and 6, and therefore the rule which we are considering does not immediately apply.

Suppose *A* and *B* to be 3 and 1 in the first case, and 2 and 2 in the next, so that the whole mass may be the same (viz. 4,) in the two cases. Now since the weights which produce motion are 1 and 2, the velocities ought to be in that ratio; and accordingly they will be found, (neglecting the errors which arise from friction, &c.) to be 8 feet and 16 feet per second.

Experiments of this kind, if they could be made with accuracy, would establish the proposition which we are considering. Instead of supposing the body *A* to rest on a horizontal plane, we might suppose *A* and *P* to hang over a pully, (fig. 136,) in which case it would be only the excess of *B* above *A* which would produce motion. A machine invented by Atwood enables us in this case to reduce the magnitudes of the velocities while we retain their law, so that we can more easily measure their quantities. The experiments being made, are found uniformly to agree with this law†. See Atwood on Rect. and Rot. Motion, Sect. 7. Also Mr. Smeaton's Experiments, Phil. Trans. Vol. LXVI.

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* It is manifest that the action of gravity on the body *A*, which moves on a horizontal plane, cannot produce any effect in accelerating or retarding its motion.

† To compare the velocities observed in such machines as this, with the results of the third law, we must take into account the *rotatory inertia* of

If we should communicate motion to A by means of the pressure of the hand or of a spring, as was before supposed, it would be more difficult to illustrate this law by experiments. In the case of pressure by the hand it is impossible to ascertain whether in two cases the effort be exactly the same, and still more impracticable to determine its ratio when different. In a spring this ratio might be ascertained by observing the weights which bend it to given curvatures. But in both cases the pressure would not be uniform, because it is perpetually diminished as the body acted upon recedes and moves away from the agent: and a part of the force, the quantity of which it is not easy to ascertain, is employed in moving the hand itself in one case and the spring in the other.

The above proposition being established by experiment, we proceed to prove the following ones.

181. Prop. *If constant pressures which act upon different masses be* proportional *to the masses, the velocities generated in a given time will be equal.*

For instance, a pressure of 1 pound, acting on a mass of 4 pounds, will produce the same velocity as a pressure of 2 pounds acting on a mass of 8.

We may prove this by considering the former case. Let a body $A = 3$ be drawn along a horizontal table by the weight of a body $B = 1$, connected with A by means of a string, and hanging freely over the edge. We have here a pressure 1 communicating motion to a mass 4 (viz. $A + B$). Let another body A', also $= 3$,

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of the machine, which is calculated upon principles belonging to a higher branch of Mechanics, but depending also upon this third law of motion.

It is here assumed that gravity, which by its action upon B produces the pressure, and consequently the motion, acts with the same intensity whatever be the velocity with which B is moving. This is proved hereafter. (Chap. III.)

The result of Mr. Smeaton's experiments was that when pressure or weight, which he calls *Impulsive Force* or *Impelling Power*, produces velocity in a given mass, the velocity produced in a given time is as the pressure, consideration being had of the mechanical advantage at which it acts.

have another body $B' = 1$, annexed to it in the same manner; and let this system be placed contiguous to the former one; (the two bodies A, A' touching each other, and the strings being parallel;) and let them begin to move at the same time in the same direction. Then it is manifest that at the end of any time, A, A' will have moved through the same space, and will therefore still touch each other. Hence if we now suppose A, A' to form one mass, and to be acted on in the same manner by B, B' together, the motion will still be the same. That is if a pressure 2 communicate motion to a mass 8, the velocity will be the same as before.

In the same manner, by the addition of other systems A'', B'', &c. we may increase the mass and the pressure in a common ratio, and the velocity will be the same in all the cases.

And generally, whenever we have two pressures, whether weights, springs, animal forces, or any other, acting upon two proportional masses, we may, (if the masses are commensurable,) divide the two masses into equal parts, and the pressures into the same number of corresponding equal pressures. Each of the equal pressures will communicate to one of the parts of the mass an equal velocity; and therefore, all being supposed to act together, they will produce in the whole masses the same velocity.

If the masses are not commensurable, the proof may be extended to them by reasoning *ex absurdo*, as in similar cases.

182. Prop. *When constant pressures produce motion, the velocities produced in a given time are directly as the pressures and inversely as the masses moved.*

Let constant pressures P and Q produce motion in masses A and B: the velocities produced in a given time will be as $\frac{P}{A}$ to $\frac{Q}{B}$.

Let X be a pressure such that $X : Q :: A : B$; and let X act on A. Then since two pressures P and X act on A, the velocities generated by them in A, will be as $P : X$ by Art. 180. Also since $X : Q :: A : B$, X and Q are proportional to A and B, and therefore, by Art. 181, the velocity which X generates in A is equal to that which Q generates in B. Hence,

$$\text{vel}^y.\text{ which } P \text{ generates in } A : \text{vel}^y.\text{ which } Q \text{ generates in } B :: P : X$$

$$:: \frac{P}{A} : \frac{X}{A}$$

$$\left(\text{because } \frac{X}{A} = \frac{Q}{B}\right) \qquad :: \frac{P}{A} : \frac{Q}{B}$$

Hence the velocity generated is *as* the pressure directly and the mass inversely.

Cor. We have

$$A \times \text{velocity of } A : B \times \text{velocity of } B :: P : Q;$$

or, momentum which P generates in A : momentum which Q generates in B :: P : Q.

We can now prove the third law of motion as above enunciated. For the momentum generated in a time 1 is the measure of the moving force when the force is constant. Hence

$$\text{moving force on } A : \text{moving force on } B :: P : Q.$$

Also if the force be not constant, it will be measured by the momentum which would have been generated, if it had continued constant for the time 1. And this momentum will be proportional to the pressure at the beginning of the time. Hence the moving force is as the pressure.

183. Prop. *The accelerating force is as the pressure directly, and the quantity of matter inversely.*

This follows from the third law of motion. For the accelerating force is as the moving force divided by the quantity of matter.

Ex. 1. To find the moving force of a body P which falls freely by gravity.

Let g be the velocity generated in a time 1 by gravity. Then (Art. 175.) g represents the accelerating force on P. Also the moving force is the product of the accelerating force by the quantity of matter. Therefore the moving force $= Pg$.

Ex. 2. To find the accelerating force when two equal bodies A, B are caused to move over a pully by a body P, fig. 136.

When a body falls freely by gravity let the accelerating force $=g$. In this case the pressure which produces motion is the weight of the body, and the quantity of matter moved is also the mass of the body itself.

When P produces motion in P, A, and B, since A and B balance each other by their equal pressures in opposite directions, the weight of P only is the pressure *which produces motion*. Also in this case the mass moved is P, A, and B, which all move with the same velocity, and therefore are moved in the same manner as if they were one mass*. Hence $P+A+B$, or $P+2A$ is the mass moved. And

accelerating force in fig. 136 : accelerating force of P falling freely (g) :: $\frac{\text{pressure}}{\text{mass moved}}$ in first case : $\frac{\text{pressure}}{\text{mass moved}}$ in second case

$$:: \frac{P}{P+2A} : \frac{P}{P} \text{ or } 1;$$

$$\therefore \text{ accelerating force in fig. 136, } = \frac{Pg}{P+2A}.$$

Ex. 3. The velocity generated by a gun in a bullet of 1 oz. is 1000 feet per second: supposing that the bullet described the length of the barrel in $\frac{1}{10}$ of a second, and that the force is uniform, to find the moving force.

Since the velocity generated in $\frac{1}{10}$ of a second is 1000 feet, if the force were uniform the velocity generated in one second would be 10000. Hence

Moving force : force of gravity (1 oz.) :: 10000 : g. And

* If, instead of supposing the mass B to hang over the pully E, we suppose the string AE to be continued in a straight line, and B', equal to B, to be annexed to the string; and if we suppose A and B' to be destitute of gravity, so as to have no tendency to move, and to be put in motion by the pressure of P; the case will manifestly be the same as that in the text; because it will make no difference whether A and B' be kept at rest by the absence of gravity, or by their opposite actions balancing each other. But in the case thus supposed, it is clear that motion is produced in A, P and B' as if it were one mass at A. Hence the truth of the reasoning above is manifest.

putting $32\frac{1}{5}$ for g, the moving force is equal to that of a weight $= \dfrac{10000}{32\frac{1}{5}}$ ounces $= 310$ ounces.

It appears from the first example that the *moving force* of a *weight* P is Pg. In Statics we represented a weight P by the quantity of matter P. But in Dynamics it is requisite also to introduce the force of gravity g, and Pg represents the force of the weight.

184. DEF. *The inertia of a body is its quantity of matter, considered as resisting the communication of motion.*

If a force Pg produce motion in a mass A, the accelerating force is $\dfrac{Pg}{A}$, and therefore the velocity produced in a given time is proportional to $\dfrac{Pg}{A}$. Hence the greater A is, and the less is the velocity produced. And hence A is sometimes considered as measuring the resistance or disinclination of the body to motion, and is called its *inertia*.

It appears from what has been said, that this term implies a law of motion rather than a property of matter.

185. Besides pressure, *impact or collision* is a mode in which bodies act upon each other, and the laws of motion are applicable to this case also.

PROP. *Impact is really a pressure of short duration.*

All bodies are susceptible of a change of figure sensible or insensible; and this change occupies the time, apparently infinitely small, which bodies employ in changing their motions by impact. Thus, if an ivory ball in motion strike another at rest, they appear to separate as soon as they touch, and the second ball appears to have a certain finite velocity communicated to it instantaneously. Similarly, if there be two balls which do not separate when one strikes the other, either from their want of elasticity, as in balls of lead or clay, or from their adhering to each other when they come in contact; the alteration of velocity which is produced by the impact will appear to take place in an instant.

But in all these cases, if it were not for the rapidity of the change, we should see that the communication of motion was gradual, and that the ball which was at rest was brought into motion by insensible degrees of velocity. As soon as one ball comes in contact with the other, their surfaces are compressed, and motion is, by the pressure, communicated to the ball at rest. The change of figure increases so long as the impinging ball has a tendency to move faster than the other, and during the whole of this time, the one is gaining and the other is losing velocity. When this action ceases, the bodies move on together if inelastic; or if they are elastic, they separate by their elasticity; recovering their globular figure and producing a further change of velocity by the pressure they thus exert upon each other.

That this change of figure takes place in impact, is evident in soft bodies which do not recover their shape; and may be made manifest in elastic balls by covering one of them with some substance, as ink, which will, in the impact, stain those parts of the other with which it is in contact. The spot thus produced is found to be of a finite magnitude, which could not be if the balls retained their globular shape, and it is found to be larger as the force of the impact is greater.

That the communication of motion is thus gradual, is obvious also by considering that the action is *of the same kind*, whether the bodies, which undergo this change of shape in impact, are compressible easily or with difficulty. But in the case of a body which easily admits a certain change of figure, as for instance, a balloon filled with air, it would be manifest to the senses that any motion produced by impact would be generated by degrees, and the change and restitution of figure would employ a finite time. Hence in other cases where the magnitude and elasticity are different, the same is true.

186. Prop. *The third law of motion is true in the case of impact.*

Impact is a pressure continued for a short time; increasing from nothing to a finite magnitude, and then decreasing to nothing again. And hence, if the third law of motion be true for pressure, it will be true for impact.

We can easily see from this the effects that impact would produce in generating, and consequently also in destroying, velocity: for the same force which would generate any velocity, would also, applied in the opposite direction, destroy it. Now when two bodies impinge on each other, the pressures on each, arising from the contact of their surfaces, must be *equal*, and in *opposite directions*. Hence, by this third law, the momenta which it would generate (and consequently destroy) in each, must be equal. Hence it appears, that if two bodies move in opposite directions with equal momenta, and meet, (being supposed not to separate after the impact,) the impact will destroy both their velocities, and the mass will remain at rest. Now this is found to agree with experiment*.

187. PROP. *In all cases of the direct mutual action of bodies, the momenta gained by one and lost by the other in the same direction are equal.*

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* The way in which the experiment may be made is described by Newton, (Scholium to the laws of motion, *Principia*, Book I.). Two pieces of wood A, B, fig. 137, are hung up by equal strings from two points C, D, in the same horizontal line, so that they can swing in arcs MA, NB. One of them, A, has a steel point, which, when it strikes against the other, sticks into it, so that the two move on together. They are drawn aside into positions M, N, and let fall so as to meet at the lowest points A, B. The weights may be made to bear any proportion to each other by loading one of them with lead; and the velocities may be made to bear the inverse proportion of the weights, by taking properly the arcs AM, BN, as will be seen hereafter. If this be the case, when they meet at A, B, they will stop each other.

Newton makes the experiment in a form a little different. He draws one of the weights, as A, into the position M, and letting it impinge on the other, which is at rest, he examines the velocities after impact, which he can do by observing the arcs through which the bodies rise. These velocities he finds to be that which, in the Chapter on Collision, will be shewn to result from the third law of motion.

In the place referred to, Newton shews how allowance may be made for the small errors which arise in this experiment from the resistance of the air.

The action of bodies is said to be *direct* when it takes place in the direction of the line joining their centers of gravity.

The mutual action of the two bodies may be considered as a pressure, which will act with equal intensity and in opposite directions upon the two bodies. And hence the moving forces on the two bodies will be equal: and the momenta added to one, in one direction, and to the other in the other, will be equal. And these are called the momenta gained and lost.

Thus if P, fig. 136, were to fall freely, it would fall quicker than it does, its momentum would be greater than it is, and the difference from what it would be is the momentum lost. A and B on the contrary would remain at rest if not acted on by P, and therefore their whole momentum is momentum gained.

Momentum gained and lost are sometimes called *Action and Re-action.* And in that case, this proposition is true;

PROP. *Re-action is equal and opposite to Action.*

This Proposition includes the third law of motion.

188. The following considerations may serve to shew that the third law of motion as above stated, though not demonstrable *a priori*, is agreeable to the *most simple* suppositions.

Let two inelastic bodies A, B, fig. 138, approach each other with velocities which are inversely as their quantities of matter. If C be taken so that $A : B :: BC : AC$, C will be their center of gravity. At the end of a certain time suppose that by their motions they come into the positions a, b. Then since the velocities are inversely as the bodies, we have $A : B :: Aa : Bb$. From this and the former proportion we have $A : B :: aC : bC$. Hence C is still the center of gravity. Hence it appears that during the whole time in which the bodies approach each other, the center of gravity remains at rest. But if they do not destroy each others motions, they will move together after impact, and therefore their center of gravity will also move. Hence, if this third law be not true, it follows that the center of gravity, having remained at rest during their separate movements, will start into motion as soon as the impact takes place. But it is more simple, and therefore more probable, to suppose that the center of gravity will continue at rest, and therefore that the third law of motion is true.

The same reasoning may be applied to cases of continued pressure. Thus let A and B be a boat and a ship afloat, and let a person in one of them pull the other by means of a rope AB. The force on each is the same, namely, the tension of the rope; and hence the velocities produced should be inversely as the quantities of matter in A and B; in which case the center of gravity will remain at rest all the time they are moving towards each other, and they will meet in this point. This is the most natural supposition, for after they have met, if we suppose the tension to continue, it is manifest that the center of gravity must remain at rest, because the tension will produce only a statical action and re-action which balance each other.

This may be applied also to attractions. If we suppose A, B, to be two bodies, as a magnet and a piece of iron, which are at liberty to approach each other*, their attraction will act in exactly the same manner as the tension of a cord by which one should be pulled to the other. Hence the pressure on each arising from the attraction is equal; and therefore by this third law the velocities will be inversely as the quantities of matter. The bodies will approach, their center of gravity remaining at rest all the while, and will meet in this point. And this agrees with experiment.

189. The preceding laws of motion are, it would seem, the fewest and most simple principles to which mechanical phænomena

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* This experiment may be made by placing each on a piece of cork and setting them to float in water.

There is another consideration which has sometimes been brought forwards as a confirmation of this third law of motion. It has been seen (Art. 43,) that if two bodies A, B balance each other on any machine, as for instance a straight lever, when they are supposed to move through small spaces, their velocities are inversely as their masses. Since therefore, when the velocities of bodies acting *on a machine* would be inversely as the masses, they keep each other at rest; it is considered as agreeable to the uniformity of nature, that when bodies *meet* with velocities which are inversely as their masses, they will reduce each other to rest. This is nearly the same illustration as the one in the text, and like that, is only an analogy.

can be reduced. It appears that the principles which we obtain from experiment, and which are the foundation of our demonstrations, are the following:

1st. Motion undisturbed by external force, is uniform.

2d. The effect of force on a body in motion, is the same, relatively, as on a body at rest.

3d. The velocity communicated by pressure is directly as the pressure.

And these can neither be dispensed with, nor deduced one from another, nor from any more elementary truth. In fact, since the effects of force in producing and varying motion are known to us only by experience; and since force and motion are, to us, at least, only notions suggested by a succession of observed facts, it is manifest that we cannot deduce the laws of these phenomena except from observation. Many attempts have been made to establish these laws by *a priori* arguments, but in all such cases there is either a confusion in the use of terms, or a latent assumption of facts.

Thus these laws rest for the greater part of their proof upon experience; but, independently of any express experiments, they are, *a priori*, from their simplicity and their agreement with the general train of appearances which suggest our mechanical ideas, much more probable than any others could be. Innumerable results have been deduced from them, of all kinds and of all degrees of complexity; and these have, in every case, been found consistent with observation; thus confirming beyond a doubt the solidity of the foundations of the reasoning.

There are other propositions, some of which may occur hereafter, which have been called Mechanical *Principles:* and some which have been brought forwards as elementary. Many of these are valuable, both as remarkable propositions in Dynamics, and as convenient steps in the solution of extensive and difficult classes of problems; but when distinctly stated and examined, they will be found, so far as they are true, to be consequences and combinations of the preceding three laws of motion.

CHAP. II.

UNIFORM MOTION AND COLLISION.

190. When two bodies which are in motion, acted upon by no extraneous forces, impinge upon each other, and then move on, together or separate, their motions before the impact, and also after it, will, by the first law of motion, be uniform and rectilinear. The change which takes place in consequence of the impact will depend upon the third law of motion, as we shall see hereafter. This is, in some respects, the most simple case of dynamical action. We have only to consider two states of each body with respect to velocity and direction; that *before*, and that *after* the collision. We have not, as in most other instances, a perpetual and continuous change of velocity, or a curvilinear path. The alteration which the concourse of the bodies produces, is supposed to be abrupt and instantaneous; so that, between their condition before and after that event, there is no intermediate one which requires to be contemplated.

It is true, as has already been observed, that, in fact, the change which collision produces in the motions of bodies does occupy a finite time; that the velocity is increased or diminished not at once but by degrees; and that no body passes from one state to another without going through all the intermediate states. But in the cases to which we shall apply our reasonings, the time which this change employs, that is, the time during which two impinging bodies continue in contact, is so small that it may be neglected as inconsiderable, and all that we attend to are the preceding and succeeding states, which it divides. During the mutual action, however, of the two bodies, they exert a certain pressure upon each other, which will produce and destroy velocity according to the third law of motion.

191. If the bodies be not, as we have supposed them to be, of inconsiderable magnitude, the point in each which is taken to represent it, is its center of gravity; and its path is the line described by this point. The bodies are supposed in general to be homogeneous, and bounded by spherical surfaces, or, at least, by such convex surfaces that their contact may only take place in a point. The action which takes place at this point of contact will necessarily be exerted in the line which, passing through that point, is perpendicular to the surfaces which there touch; and this line is supposed to pass through the center of gravity of each body. That this may be the case in every position of the bodies, they must necessarily be spherical; but for a particular position it may happen with innumerable different forms.

If the line of the action of the bodies upon one another did not pass through the center of gravity of one of the bodies, it would communicate to that body a rotatory motion; which is a case that we do not consider here.

If the line in which the action of the bodies takes place be the line in which they are moving, the impact is called *direct*. If either or both of them be moving in any other line, their impact is said to be *oblique*.

The *relative velocity* of two bodies is the velocity with which they approach to or recede from each other; and is therefore the difference of their velocities when they move in the same, and the sum when they move in opposite directions.

192. Now in the direct impact of two bodies which move in opposite directions *with equal momenta*, as we have already said, the velocity of each is destroyed during the compression, and a certain velocity is again generated during the restitution of the figures. The ratio of the velocities destroyed and generated may be taken as the measure of the proportion of the forces of compression and restitution, and we must then examine by experiment how these forces are related. The result which we shall obtain is this;

Prop. *When two bodies meet, moving in the same straight line, with equal momenta, in opposite directions, their velocities are destroyed by the force of compression, and new momenta, opposite and equal to each other, are generated by the force of restitution.*

It has been seen in the proof of the third law of motion, (Art. 186,) that if two inelastic bodies moving in opposite directions in the same straight line, meet with equal momenta, the collision will destroy the motions of both, and they will remain at rest. If they be not inelastic, they will after the impact separate with velocities different according to the nature of the bodies.

In this case, the action between them will manifestly consist of two parts. The compression, or change of figure which their concourse and mutual pressure produce; and the restitution of figure which takes place in consequence of the elasticity, and makes them again rebound from each other.

It is obvious that the effect in the former part of the process is the same as if the bodies were inelastic; for when the compression is completed, and just before they begin to recover their figures, if we suppose the internal constitution of the bodies to undergo a sudden change by which they lose all their elasticity, they will remain in contact, and the laws of their motion will be the same as those of inelastic bodies. Hence in this first part of the collision they will lose their whole velocities, as inelastic bodies would have done.

When they separate by their elasticity, the momenta which are communicated to them afresh by their mutual pressure, will be, by the third law of motion, equal. The actual magnitudes of the velocities, and consequently the relative velocity with which the bodies separate, depend upon the elasticity of the bodies, and the laws which regulate these circumstances are to be determined from experiments as we shall shew.

193. Prop. *In the direct impact of elastic bodies, the force of restitution is to the force of compression in a ratio which is constant for bodies of the same nature.*

That is, whatever be the velocities, magnitudes, and figures of the bodies in question, so long as the material continues the same, the ratio of the velocity of each after impact to the velocity before impact, is the same. Hence also, since the velocity of each body after impact bears a certain proportion to its velocity before, the sum of those velocities, that is, the relative velocity with which the bodies separate, ought to have a given ratio to the velocity with which they approach; and this is found to be the case.

The experiments necessary for the proof of this proposition may be made in the manner already described in proving the third law of motion: see Note, Art. 186. Two balls A, B, fig. 137, are hung by vertical strings CA, DB; and being drawn out of their vertical position, are allowed to fall so as to come together at the lowest point, where they meet and recoil. The arcs down which they fall, and up which they rise, afford the means, as will be shewn hereafter, of knowing the velocities before and after the impact. And thus it was found by Newton and others that the relative velocities before and after impact are always in a given ratio*.

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* Newton, *Principia,* Book I, Scholium to the Laws of Motion. "Propterea quod vis illa (elastica) certa ac determinata sit (quantum sentio) faciatque ut corpora redeant ab invicem cum velocitate relativâ quæ sit ad velocitatem relativam concursus in datâ ratione." He then goes on to mention in what manner and on what substances he made his observations. Experiments upon elastic bodies were made by Wren and Hooke before the Royal Society about 1670. Mariott, a French mathematician, also made experiments, of which he has given an account in his *Traité de Percussion.* Mr. Smeaton (see *Phil. Trans.* vol. LXXII.) repeated the fundamental experiments upon elastic bodies with an ingenious apparatus for separating the effects on soft and on elastic bodies. But all these mechanicians, except Newton, have considered their experiments as made on bodies *perfectly* elastic; and have taken, as approximations to such bodies, the most elastic bodies which occurred. The theory of imperfect elasticity has, it would appear, been taken for granted on the authority of Newton; and, if it were necessary to rest upon authority, there is none on which we might rely with less scruple. But, that elasticity, depending upon the internal constitution of bodies so completely different, metals, stones, wood, ivory, cork, &c., should in all instances obey one general law, is, though not improbable, highly curious and, if it be really and exactly true, worth-establishing by repeated trials. And

If the bodies be made to meet with velocities which are not in the inverse proportion of their masses, we may find, as we shall shortly shew, what their motions ought to be after impact; and these are found to coincide with the results of observation. The case in which the experiment is most easily made, is when one of the bodies is at rest and is struck by the other.

Bodies are called *perfectly elastic* when the force of restitution is equal to the force of compression. When the force of restitution is less, the bodies are said to be *imperfectly elastic.*

The *elasticity* of imperfectly elastic bodies is the fraction which the force of restitution is of the force of compression.

Thus it appears from Newton's experiments that in the collision of balls of worsted, the relative velocity after impact is to that before as 5 to 9. Hence the fraction $\frac{5}{9}$ expresses the elasticity in this case. In balls of steel the ratio was nearly the same; in cork it was a little less; in ivory, it is 8 to 9; in glass, 15 to 16. According to this way of measuring, perfect elasticity will be represented by 1. In every case the value of the elasticity may be ascertained by a single experiment; and represented by a fraction e, which expresses the portion that the force of restitution is of the force of compression.

We now proceed to determine from these principles the motions of bodies in every case of direct impact.

And even if further observation should prove the truth of Newton's results, there are still several obvious questions to which his experiments do not enable us to give any answer whatever. For instance, if two bodies of *different* degrees of elasticity impinge upon each other, how are their motions to be determined? Manifestly this and similar problems can only be resolved by obtaining from new experiments the principles on which they depend.

The authors of the common Theory of Collision, were Wren, Wallis, and Huyghens, who about the same time (1669) sent to the Royal Society papers on the subject. Wren appears to have confirmed his results by experiment; the attempts to establish the doctrine upon axioms independent of observation, are, as they must be, very unsatisfactory.

1. *Direct Impact.*

194. Prop. *Two inelastic* bodies, moving with given velocities, impinge directly upon each other: it is required to determine their common velocity after impact.*

Let A, B represent the magnitudes, and a, b the velocities of the bodies. And first, let them move in opposite directions, and let a and b be inversely as A and B; that is, let $A a = B b$, or the momenta be equal. In this case, as has been seen in considering the third law of motion, the bodies will destroy each others velocities by the impact; and they are supposed to be inelastic, and

* The bodies here mentioned may be either soft bodies, which change their figure without recovering it, or bodies of any kind, which, when they meet, are prevented from separating by some such contrivance as that mentioned in the note to Art. 186. In the former case the effect will be the same, whatever be the quantity of compression which the bodies suffer, and the time employed in producing it, provided there be no elasticity. The compression, and the time during which it is produced, will be less as the inelastic body is harder; but we cannot conceive any action taking place between two bodies, which does not occupy some portion of time. Hence we cannot conceive the action of two bodies which are perfectly hard, that is, which are not susceptible of the smallest change of figure. If there were such substances, the actions which would take place between them would be of a nature entirely different from any thing with which we are acquainted, and therefore we should have no right to extend our laws of motion to them. Accordingly, those who have reasoned concerning such bodies have gone upon arbitrary or inconsistent assumptions. Wallis makes the motions of hard bodies to be such as we have shewn those of soft bodies to be. Huyghens and Wren suppose these motions to be the same as those of perfectly elastic bodies. Succeeding authors have generally followed the former theory with respect to the motions of two hard bodies, though the congress of a perfectly hard and a perfectly elastic body is supposed to follow the laws of two elastic bodies. It is clear, that since we do not find in nature any facts from which the principles of such cases can be deduced, the problems can only be solved by gratuitous hypotheses, and do not form any portion of mechanics considered as the science of the laws by which motions are *actually* regulated.

have therefore no tendency to separate; they will therefore remain at rest in contact.

Now if the motions and collision of these bodies take place in a limited space which is not at rest, but which is moving uniformly in any direction, these motions will, by the second law of motion, still continue to be the same as before, relatively to the parts of that space. Let the space move in the direction of A's motion with a velocity v; and in this space, let A and B meet with velocities α and β relative to it, such that $A\,\alpha = B\,\beta$. Hence they will, after the impact, be at rest relatively to this space.

But since the space which includes the two bodies is carried in the direction of A's motion with the velocity v, A has, before the impact, an absolute velocity $\alpha + v$; and B, which is carried in the same direction with a velocity v, and in the opposite direction by its own motion with a velocity β, has in the former direction an absolute velocity $v - \beta$, supposing v greater than β. Also after the impact the bodies are at rest in the space, and are carried by its motion with a common velocity v.

Hence it appears that if a body A with a velocity $a = \alpha + v$ overtake a body B moving with a velocity $b = v - \beta$ they will after the impact move on with a common velocity v. Now since

$$a = \alpha + v, \text{ and } b = v - \beta,$$

$$\text{we have } \alpha = a - v, \text{ and } \beta = v - b;$$

and the equation $A\,\alpha = B\beta$ becomes $A\ (a - v) = B\,(v - b)$:

$$\text{hence } A a + B b = A v + B v;$$

$$\text{and } v = \frac{A a + B b}{A + B}.$$

If v be not greater than β, the absolute velocity of B before the impact is $\beta - v$ in a direction opposite to that of A. If this be called b, we have

$$\beta = b + v, \text{ and } A(a - v) = B\,(b + v);$$

$$\text{whence } v = \frac{A\,a - B\,b}{A + B}:$$

which differs from the former case only in having the sign of b negative.

Hence if a and b represent the velocities *in the same direction,* and if velocities in the opposite direction be considered to be *negative;* the first expression for v is general for all cases.

Cor. 1. If the resulting value of v be negative, the bodies will, after impact, move in a direction contrary to that which was supposed positive.

Cor. 2. If $b=0$, or if A impinge on B at rest, we have for their velocity after impact;

$$v=\frac{Aa}{A+B}.$$

Cor. 3. The velocity lost by A is

$$a-v=a-\frac{Aa+Bb}{A+B}=\frac{B(a-b)}{A+B}.$$

The velocity gained by B in the direction of A's motion is

$$v-b=\frac{Aa+Bb}{A+B}-b=\frac{A(a-b)}{A+B};$$

$a-b$ is obviously the velocity with which A approaches B; that is, the relative velocity.

Cor. 4. If B before the impact be moving in a direction opposite to A, the velocity *gained* by B in the direction of A's motion, is not the *excess* of the velocity after impact above the velocity before, but the *sum* of the velocity destroyed in the opposite direction and of the velocity communicated in A's direction. This is also the result of the expressions in the last Corollary, paying proper attention to the signs. The same is applicable to the velocity lost by A, when it moves in the opposite direction after the impact.

Cor. 5. The *momentum* lost by A is

$$Aa-Av=\frac{AB(a-b)}{A+B}.$$

The momentum gained by B is

$$Bv-Bb=\frac{BA(a-b)}{A+B}.$$

Hence the momentum gained by B and the momentum lost by A are equal. This is what is meant by the equality of action and re-action in this case.

This equality of action and re-action is sometimes made the principle on which the theory is established. See Art. 187.

195. Prop. *Two bodies of which the common elasticity is e, moving with given velocities, impinge directly upon each other; it is required to determine their velocities after impact.*

Let A, B be the bodies, and a, b their velocities. And first, let their velocities be inversely as their masses, and opposite: that is, let $Aa = Bb$. As before, in the first part of the collision the velocities will be destroyed; and then, by the elasticity, will be generated new velocities in the opposite directions, with which the bodies will separate. By Art. 193, these velocities will be in the ratio to the velocities before impact of e to 1. That is, A will return with a velocity ea, and B with a velocity eb, and thenceforth the bodies will move uniformly with these velocities.

Now let the same actions take place in a space which is moving with a velocity x in the direction of A's motion. Let A and B meet with velocities α, β, relative to this space, and such that $A\alpha = B\beta$. They will then separate with velocities $e\alpha$ and $e\beta$ relative to this moveable space. Hence if a, b, be the absolute velocities in the same direction before impact, and u, v the velocities after it; since A's velocity will be its velocity in the space together with the velocity of the space,

$$a = \alpha + x; \text{ similarly, } b = x - \beta; \ \therefore \alpha = a - x, \ \beta = x - b:$$

$$\text{also, } A\alpha = B\beta; \text{ hence } A(a - x) = B(x - b);$$

$$\therefore x = \frac{Aa + Bb}{A + B}, \ \alpha = a - x = \frac{B(a-b)}{A+B}, \ \beta = x - b = \frac{A(a-b)}{A+B}.$$

And since after impact A is carried forwards with the velocity x, and backwards with the velocity $e\alpha$;

$$u = x - e\alpha. \quad \text{Similarly, } v = x + e\beta.$$

Hence

$$u = \frac{Aa + Bb - eB(a-b)}{A+B};$$

$$v = \frac{Aa + Bb + eA(a-b)}{A+B}.$$

If the bodies are not moving in the same direction before impact, attention to the signs of the velocities will preserve the truth of the formulæ.

Cor. 1. The velocity lost by A is

$$a - u = a + x - (x - ea) = a + ea = \frac{(1+e).B.(a-b)}{A+B}.$$

The velocity gained by B is

$$v - b = x + e\beta - (x - \beta) = \beta + e\beta = \frac{(1+e).A.(a-b)}{A+B}.$$

Both these are greater than the velocities gained and lost in the case of inelastic bodies, in the ratio $1 + e : 1$.

Cor. 2. The momentum lost by A and that gained by B are each

$$\frac{(1+e)AB(a-b)}{A+B}.$$

Hence the sum of the momenta is the same before and after impact; or $Aa + Bb = Au + Bv$.

Cor. 3. The relative velocity after impact is

$$v - u = x + e\beta - (x - ea) = e(a + \beta)$$
$$= e.\frac{B(a-b) + A(a-b)}{A+B} = e(a-b).$$

Hence for the same bodies it is in a given ratio to the relative velocity before impact.

Cor. 4. If the bodies be equal, or $A = B$, we have for the velocities after impact,

$$u = \tfrac{1}{2}\{(1-e)a + (1+e)b\};$$
$$v = \tfrac{1}{2}\{(1+e)a + (1-e)b\}.$$

Cor. 5. If B be at rest before the impact, $b = 0$, and

$$u = \frac{(A - eB)a}{A+B};$$

$$v = \frac{(A + eA)a}{A+B}.$$

196. Prop. *When the elasticity is perfect, to determine the motions.*

We must here make $e = 1$ in the preceding expressions. Hence,

$$\text{the velocity lost by } A = \frac{2B(a-b)}{A+B}:$$

$$\text{the velocity gained by } B = \frac{2A(a-b)}{A+B}.$$

The relative velocity after impact $= a - b =$ the relative velocity before impact.

Cor. 1. If the bodies be equal, $B = A$. Hence

velocity lost by $A = a - b$; velocity gained by $B = a - b$.

Therefore A's velocity after impact $= a - (a - b) = b$,

B's .... $= b + (a - b) = a$.

Hence the bodies in this case *interchange* velocities.

Cor. 2. If B be at rest when it is struck by A, $b = 0$. Hence

$$\text{velocity lost by } A = \frac{2Ba}{A+B};$$

$$\text{therefore } A\text{'s velocity after impact } = a - \frac{2Ba}{A+B} = \frac{(A-B)a}{A+B},$$

$$B\text{'s} \ldots\ldots\ldots\ldots = \frac{2Aa}{A+B}.$$

Hence if the bodies be equal, after the impact A will stop, and B will move on with A's velocity.

If A be the less, it will move backwards, and B will move forwards with less than A's original velocity. If A be greater, B

will move forwards with a velocity greater than A's original one, and A will follow it more slowly.

Cor. 3. Hence if there be a row of perfectly elastic bodies at rest,

$$A, B, C, D, E, \ldots$$

and if the first A be made to impinge on the second B with a certain velocity; and B, with the motion thus acquired, on C; C on D; and so on; we see what will become of the bodies. If they be all equal, each will stop after impact, and the last will move off with the original velocity. If they go on increasing, each will, after it is struck, move forwards with a velocity less than the preceding; and after it has struck the next, will move backwards. If they are a decreasing series, each will move faster than the preceding, and after the impacts they will all move forwards.

197. Prop. *To compare the velocity communicated immediately from A to C with that communicated by the intervention of B as in the last Corollary.*

Let a be A's velocity. Then, by Cor. 2, Art. 196,

$$\text{the velocity communicated by } A \text{ to } C = \frac{2Aa}{A+C}.$$

$$\text{Also the velocity communicated by } A \text{ to } B = \frac{2Aa}{A+B} = b \text{ suppose};$$

$$\therefore \text{ the velocity communicated by } B \text{ to } C = \frac{2Bb}{B+C} = \frac{4ABa}{(A+B)(B+C)}.$$

The latter of these velocities communicated to C is greater than the former,

$$\text{if } \frac{2Aa}{A+C} < \frac{4ABa}{(A+B)(B+C)};$$

$$\text{if } (A+B)(B+C) < 2B(A+C);$$

$$\text{if } B^2 + AB + CB + AC < 2AB + 2CB;$$

$$\text{if } B^2 - AB - CB + AC < 0;$$

$$\text{if } (B-A)(B-C) < 0;$$

which will be the case if one of the factors $B-A$, $B-C$, be positive, and one negative; that is, if the body B be greater than one and less than the other of the bodies A, C. In this case the velocity communicated by the mediation of B is greater than that communicated immediately from A to C.

The velocity communicated through B is the greatest when B is a mean proportional between the other two bodies, as may easily be shewn*.

COR. If we interpose in the same manner a body which is a mean proportional between A and B, or between B and C, the velocity communicated to C will be increased. By increasing perpetually the number of mean proportionals between the first and last body, we increase the velocity communicated to the last, and make it approach to a certain limit, which we shall find.

198. PROP. *To find the limit of the velocity communicated in the last Corollary.*

Let there be $n+1$ perfectly elastic bodies $A, B, C, D \ldots Z$, their magnitudes being in a geometrical progression of which the common ratio is $1+r$. Therefore $B=(1+r)A$, $C=(1+r)^2 A$, &c. $Z=(1+r)^n A$. Let A impinge with a velocity a upon B, and communicate a velocity b; and let B communicate to C a velocity c, and so on; z being the velocity of Z. Hence,

$$b=\frac{2Aa}{A+B},\ c=\frac{2Bb}{B+C},\ \&c.$$

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* To find what must be the magnitude of B that the velocity communicated by its interposition may be the greatest possible, we must make the expression for the velocity of C a maximum, or its reciprocal a minimum; that is,

$$\frac{(A+B)(B+C)}{4ABa} = \text{min.};$$ and, omitting constant factors,

$$B+A+C+\frac{AC}{B} = \text{min.};$$ and, differentiating with respect to the variable B, and putting the differential coefficient $=0$:

$$1-\frac{AC}{B^2}=0;$$

$$\therefore B^2=AC,\ B=\sqrt{(AC)}:$$

or B is a mean proportional between A and C.

$$\text{or } b = \frac{2a}{2+r},\ c = \frac{4a}{(2+r)^2},\ \&c.$$

$$\text{and } z = \frac{2^n a}{(2+r)^n} = \frac{a}{\left(1+\frac{r}{2}\right)^n}.$$

But since $Z=(1+r)^n A$, $\sqrt{Z}=(1+r)^{\frac{n}{2}}\sqrt{A}$; and multiplying this equation by the former one,

$$z\sqrt{Z} = a\sqrt{A}\,.\,\frac{(1+r)^{\frac{n}{2}}}{\left(1+\frac{r}{2}\right)^n}.$$

$$\text{Now } \mathrm{l}\left(1+\frac{r}{2}\right)^n = n\,\mathrm{l}\left(1+\frac{r}{2}\right) = n\left\{\frac{r}{2} - \frac{1}{2}\cdot\frac{r^2}{4} + \&c.\right\}.$$

$$\mathrm{l}(1+r)^{\frac{n}{2}} = \frac{n}{2}\,\mathrm{l}(1+r) = \frac{n}{2}\left\{r - \frac{1}{2}r^2 + \&c.\right\};$$

$$\therefore \frac{\mathrm{l}\left(1+\frac{r}{2}\right)^n}{\mathrm{l}(1+r)^{\frac{n}{2}}} = \frac{1-\frac{1}{4}r+\&c.}{1-\frac{1}{2}r+\&c.}.$$

And as n becomes very large, r becomes very small, and ultimately may be neglected in comparison with 1. Hence the second side of this equation becomes 1, when n becomes indefinitely great. Therefore, ultimately,

$$\mathrm{l}\left(1+\frac{r}{2}\right)^n = \mathrm{l}(1+r)^{\frac{n}{2}},$$

$$\left(1+\frac{r}{2}\right)^n = (1+r)^{\frac{n}{2}};$$

$$\therefore z\sqrt{Z} = a\sqrt{A};$$

$\therefore \frac{a}{z} = \sqrt{\frac{Z}{A}}$ and $z = \frac{\sqrt{A}}{\sqrt{Z}}.a$; which is the value to which the velocity approximates, by increasing indefinitely the number of mean proportionals between A and Z.

199. PROP. *In the direct impact of perfectly elastic bodies the sum of each body into the square of its velocity is the same before and after impact.*

We have, by Cor. 2, Art. 195,

$$Aa + Bb = Au + Bv;$$
$$\therefore A(a-u) = B(v-b).$$

Also $a - b = v - u$, or $a + u = v + b$; (Art. 196.)

hence, multiplying the equations,

$$A(a^2 - u^2) = B(v^2 - b^2),$$
$$\text{or } Aa^2 + Bb^2 = Au^2 + Bv^2.$$

200. PROP. *In the direct impact of imperfectly elastic bodies, to compare the sum of each body into the square of its velocity before and after impact.*

By Cor. 3, Art. 195,

$$e(a-b) = v - u;$$
$$\therefore ea + u = eb + v;$$
$$\text{and } 2ea + 2u = 2eb + 2v;$$

$$\text{or } (1+e)(a+u) - (1-e)(a-u) = (1+e)(v+b) + (1-e)(v-b).$$

Also by Cor. 2, Art. 195,

$$Aa + Bb = Au + Bv,$$
$$\text{or } A(a-u) = B(v-b).$$

Multiply the former equation by this, and we have

$$(1+e)A(a^2-u^2) - (1-e)A(a-u)^2 = (1+e)B(v^2-b^2) + (1-e)B(v-b)^2.$$

Hence

$$Aa^2 - Au^2 + Bb^2 - Bv^2 = \frac{1-e}{1+e}\{A(a-u)^2 + B(v-b)^2\}.$$

Let the velocity lost by $A = a - u = p$, and the velocity gained by $B = v - b = q$, and we have

$$Aa^2 + Bb^2 = Au^2 + Bv^2 + \frac{1-e}{1+e}(Ap^2 + Bq^2).$$

2. *Oblique Impact.*

201. Prop. *In oblique impact the mutual action of the bodies is perpendicular to the surfaces at the point of contact; and it affects only the velocities resolved in this direction.*

Suppose a ball B, fig. 139, moving uniformly in a straight line bB, to be struck by a ball A, which is moving in a direction aA at right angles to bB. The impact is supposed to take place in such a way, that the line of A's motion passes through the center of B, and is perpendicular to the surfaces at the point of contact. Therefore, neglecting the effects of friction*, the action of each of the bodies upon the other is entirely in this line. The quantity of the compression will not be altered by the lateral motion of B, and hence the action in this direction will be the same as if A impinged on B at rest. The motion of B in the direction bB will not be affected by this action; and the motion impressed by A will be combined with this original motion by the second law of motion. Let Bn represent the velocity which A would communicate to B at rest; and let Bm on the same scale represent the original velocity of B. Then if we complete the parallelogram mn, and draw the diagonal Bq; by Art. 177, the velocity of B after impact will be represented by Bq.

In the same way it will appear in every other case, that the mutual action of the bodies is wholly in a direction perpendicular to the surfaces in contact. The motion at right angles to this direction will not be affected; the motion in this direction will be regulated by the same laws as if the bodies had no lateral motion; and these motions combined, according to the second law of motion, give the motion after impact.

202. Prop. *Two given bodies of given elasticity meet with given velocities in given directions; it is required to find their motion after impact.*

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* During the time that the bodies are in contact, the surface of B must by its lateral motion slide along the surface of A, and hence, if we suppose the surfaces not to be perfectly smooth, its motion in that direction will be, in some measure, retarded.

It is supposed, as before, that at the instant of collision the line joining the centers of gravity of the two bodies passes through the point of contact of the surfaces and is there perpendicular to them.

Let PA, QB, fig. 140, represent the velocities of the bodies which meet at A and B. Draw AB joining their centers, which will be perpendicular to their surfaces in the point of contact c. Produce AB and draw PM, QN perpendicular upon it. The velocities PA, QB may be considered as compounded of PM, MA and of QN, NB; and by what has just been said, it appears that the lateral velocities PM, QN are not affected by the collision, and continue the same after the impact. Also the action in the direction AB is the same as if the bodies had only the velocities MA, NB.

Suppose, therefore, the bodies to impinge with the velocities MA, NB; and let Am, Bn be their velocities after impact, found by Art. 195. Draw mp, nq perpendicular to AB, and equal to PM, QN respectively. Join Ap, Bq: these will be the velocities of A and B after impact. For they are compounded of the velocities PM, QN, which are not affected by the collision, and of Am, Bn, which will be the velocities in that direction.

Hence if we can find when the bodies meet, we can determine all the circumstances of the collision.

203. Prop. *When two spherical bodies move in the same straight line, to determine where they will meet.*

When A is at M, fig. 141, let B be at N, and let MO, NQ represent their volocities. Let A, B be the positions of the centers at the time of concourse; therefore AB is the sum of the radii of the spheres, and is known.

And since MA, NB, are described in the same time,

$$MA : NB :: MO : NQ;$$

$$\therefore MA : MA - NB :: MO : MO - NQ;$$

and $MA - NB = MN + NA - (NA + AB) = MN - AB$.

And hence the three last terms of the proportion are known, and therefore MA; which gives the position of A, and hence of B, at the time of concourse.

204. PROP. *When two spherical bodies move uniformly in any two straight lines in the same plane, to determine their concourse.*

In fig. 142, let MO be taken to represent A's velocity, and let NQ on the same scale represent the velocity of B. Join MN, and complete the parallelogram MP, and join PQ. With center O and radius equal to the distance of the centers of A and B when in contact, describe a circle meeting PQ in D. Join DO; draw DB parallel to OM, and BA parallel to DO; A, B will be the positions of the centers at the instant of concourse.

By similar triangles,

$$NP : BD :: NQ : BQ;$$
$$\text{or } MO : AO :: NQ : BQ;$$
$$\therefore MO : MA :: NQ : NB;$$
$$\text{or } MA : NB :: MO : NQ :: \text{vel. of } A : \text{vel. of } B;$$

therefore when one body comes to A, the other comes to B. Also $AB = OD =$ the distance of the centers; therefore they will then be in contact. And if we divide AB in c, so that Ac and Bc may be the distances of the surfaces from the centers, a plane perpendicular to AB in c will touch both the surfaces at the instant of contact.

The circle described with center O will meet PQ in two points; of these we must take the one which is nearest to P.

When the two directions are not in the same plane, the problem may be solved in a manner nearly similar.

205. We may also find the position thus.

Let MO, NO, fig. 142, be the lines; M, N the positions at the beginning of the time t; a, b, the velocities; θ the angle MON; and c the distance AB of the centers when the bodies meet; which will be the sum of the radii if the bodies are spherical. The bodies are supposed to move towards the point O;

hence if $OM = m$, and $ON = n$, at the end of the time t the distances from O will be $OA = m - ta$, $OB = n - tb$. And we shall have

$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cdot \cos.\theta;$$

or at the point of concourse,

$$c^2 = (m - ta)^2 + (n - tb)^2 - 2(m - ta)(n - tb)\cos.\theta;$$

which would enable us to determine t by means of a quadratic. Of the two roots, we must manifestly take the less for the value of the time; for that will give the first time that the surfaces come in contact; and after that they will no longer go on in the same lines; so that the second root will not be applicable.

These processes will give the solution of the problem when it is possible. It is impossible when the centers of the bodies in the course of their motions never approach within a distance sufficiently small to bring their surfaces into contact. This will happen when the roots of the quadratic become impossible; or when the circle in the construction does not meet the line PQ.

206. PROP. *When the bodies do not meet, to find at what instant they approach nearest to each other.*

It appears from the demonstration in Art. 204, that if any line OD be drawn, and DB parallel to OM, and BA parallel to DO; A, B, will be the positions of the bodies when their distance is equal to OD. Hence their distance will be least when OD is least. Therefore if we draw Od *perpendicular* to PQ, and db, ba, parallel to OM, OD; we shall have b and a the positions of the bodies when their distance is Od, the smallest possible.

207. When bodies are in motion, influenced by no forces but their mutual action, there are some remarkable properties of their center of gravity; which we now proceed to demonstrate.

PROP. *When two bodies move uniformly in straight lines, their center of gravity also moves uniformly, and in a straight line.*

Let MO, NO, fig. 143, be the paths; when the bodies are in any positions A, B, let G be the center of gravity: when A is at O, let B be at Q, and the center of gravity at R. Take $Bb = QO$; $\therefore bO = BQ$, hence

$$OA : Ob :: OA : QB :: \text{velocity of } A : \text{velocity of } B;$$

a constant ratio: hence the triangle AOb is similar in all positions of A, B. Take g so that $Ag : bg :: B : A$, a constant ratio; therefore the triangle AOg is similar in all positions and the locus of g is a straight line. But

$$Ag : bg :: B : A :: AG : BG; \therefore Gg \text{ is parallel to } Bb;$$

and

$$Gg : Bb :: AG : AB, \text{ or}$$
$$Gg : OQ :: B : A + B :: OR : OQ;$$

therefore Gg is equal and parallel to OR; and OG is a parallelogram, RG being parallel to Og. Hence the locus of G is a straight line. Also RG, $= Og$, is in a given ratio to OA, and therefore RG increases uniformly as OA does, and G moves with a uniform velocity.

It is easy to extend this demonstration to the case when the bodies are in different planes.

208. The proposition in last Article may also be proved in the following manner.

First, let the bodies A, B, move in the same straight line; and at any time t let x, x' be their distances from a given point O. Then the center of gravity will be in this line; and if g be its distance from O, we have, (Art. 47,)

$$g = \frac{Ax + Bx'}{A + B}.$$

Now in the uniform motion of A and B, if a and b be their velocities, and m and m' their distances from O at the beginning of the time t, we have, by Art. 163,

$$x = m + ta \quad x' = m' + tb;$$

hence $g = \dfrac{Am + Ata + Bm' + Btb}{A + B} = \dfrac{Am + Bm'}{A + B} + \dfrac{t(Aa + Bb)}{A + B}$;

whence it appears that the center of gravity is a point whose distance from O at the beginning of the time t is $\frac{Am + Bm'}{A+B}$, and whose velocity is $\frac{Aa+Bb}{A+B}$.

209. Now, let A and B move in any straight lines in the same plane, and as before let a and b be their velocities. Through a point O in this plane draw two lines at right angles, to represent the axes of x and y. Let the direction of A's motion make with the axis of x an angle α, and the direction of B's motion an angle β with the same line: consequently the angles which these directions make with the axis of y will be $\frac{1}{2}\pi - \alpha$ and $\frac{1}{2}\pi - \beta$. Now if x, y be the co-ordinates of A, x', y' of B, and g, h of the center of gravity; we shall have, by Art. 48;

$$g = \frac{Ax + Bx'}{A+B};\ h = \frac{Ay + By'}{A+B}.$$

But if the velocity of A be resolved in the directions of x and y, it will be uniform in these directions; and its component parts will be $a . \cos. \alpha$, $a . \sin. \alpha$. Hence if m and n be the co-ordinates of A when $t = 0$, we have

$$x = m + ta . \cos. \alpha;\ y = n + ta . \sin. \alpha;$$

similarly, $x' = m' + tb . \cos \beta$; $y' = n' + tb . \sin. \beta$;

m', n', being the corresponding quantities for B. Hence

$$g = \frac{Am + Bm'}{A+B} + t\frac{(Aa . \cos. \alpha + Bb . \cos. \beta)}{A+B}:$$

$$h = \frac{An + Bn'}{A+B} + t\frac{(Aa . \sin. \alpha + Bb . \sin. \beta)}{A+B}.$$

From which it appears that g and h are spaces described with uniform velocities,

$$\frac{Aa . \cos. \alpha + Bb : \cos. \beta}{A+B}, \text{ and } \frac{Aa . \sin. \alpha + Bb . \sin. \beta}{A+B};$$

which velocities are the components, in those directions, of the velocity of the center of gravity; which is therefore, uniform and its motion rectilinear, as would easily be shewn by compounding the two uniform velocities.

If the paths be not in the same plane, we must take *three* rectangular co-ordinates to each of the bodies, and it will easily be shewn in the same way as before that the motions of the center of gravity resolved in these three directions, and of course its whole motion, are uniform.

Cor. 1. The angle which the path of the center of gravity makes with the axis of x has its tangent

$$= \frac{Aa \,.\, \sin.\, \alpha + Bb \,.\, \sin.\, \beta}{Aa \,.\, \cos.\, \alpha + Bb \,.\, \cos.\, \beta}.$$

Cor. 2. It may be shewn, in the same way, that the motion of the center of gravity of any number of bodies is uniform.

210. Prop. *The direction and velocity of the motion of the center of gravity are not altered by the impact of the bodies.*

First, let the bodies move in the same straight line. The center of gravity will, both before and after impact, move in this line; and, by Article 208, the velocity of this center will be

$$\text{before impact } \frac{Aa + Bb}{A + B}\text{; after impact } \frac{Au + Bv}{A + B}.$$

And by Cor. 2, Art. 195, $Aa + Bb = Au - Bv$. Hence the velocity will not be affected by the impact.

Next, let the bodies move in different straight lines in the same plane. Let them be referred to rectangular co-ordinates as in the last Article; and let these be so taken that the axis of x is parallel to the surfaces which are in contact in the collision. Then the motion in the direction of x will not be altered, by Art. 202. Also the motions in the direction perpendicular to this will be affected as if there were no other motions; and therefore, by what has just been shewn, the motion of the center of gravity will not be altered. Since therefore the motion of the center of gravity in these two directions at right angles to each other remains the same, this point will manifestly go on describing the same straight line, and with the same velocity, as before the impact.

If the paths be not in the same plane, the demonstration is easily extended to that case as before.

3. *Impact on Planes.*

211. When an elastic body impinges perpendicularly upon an immoveable surface, so as to be reflected, its elasticity is supposed to act according to the same laws as in the former case; that is, the velocity with which it is reflected is supposed to bear a certain constant ratio to that with which it impinges, which ratio is independent both of the magnitude and of the velocity of the elastic body. Hence we shall be able to determine the result of oblique impact.

Prop. *A body of given elasticity impinges in a given direction upon a plane CD: to find the direction in which it will be reflected.*

It is supposed, as before, that the perpendicular to the surface at the point of contact passes through the center of gravity of the impinging body.

Let PA, fig. 144, represent the velocity of the body before impact; and let CA be perpendicular to the surface at the point of impact. Draw PM perpendicular to CA. Then the velocity PA may be supposed to be resolved into the two PM, MA: and, as in the oblique impact of bodies, the first of these will not be affected, if we leave out of consideration the momentary friction during the contact. The body will be impelled against the plane by the other part of the velocity MA, and will rebound with a velocity Am, which is to AM in the ratio of e to 1; e being the elasticity between the body and the plane. Hence if we take mp perpendicular to Am and equal to PM, the velocity after impact will be compounded of Am and mp, and will therefore be represented in quantity and direction by Ap.

Cor. 1. When $e = 1$, or the elasticity is perfect, produce PM to Q so that $MQ = PM$, and AQ will be the motion after impact.

Cor. 2. The angles which the directions of the body before and after impact make with the perpendicular to the plane are called the *Angles of Incidence* and *of Reflexion*. In the case of perfect elasticity these angles are PAM, QAM, which are manifestly equal.

Cor. 3. In any other case PAM, pAm, are the angles of incidence and reflexion. Now we have

$$\tan. PAM : \tan. pAm :: \frac{PM}{AM} : \frac{pm}{Am}$$

$$:: Am : AM, \text{ because } pm = PM;$$

or tan. ang. incidence : tan. ang. reflexion :: e : 1.

Cor. 4. If we join Qp it will be parallel to AM; hence

$$QA : Ap :: \sin. QpA : \sin. AQp$$

$$:: \sin. pAM : \sin. QAM,$$

or since $QA = PA$, and $QAM = PAM$,

vel. before impact : vel. after :: sin. ang. reflexion : sin. ang. incid.

Cor. 5. When the body is perfectly inelastic and perfectly smooth, it will, after impact, move along the plane with its lateral velocity; and

vel. before impact : vel. after impact :: PA : PM

:: 1 : cos. APM,

and APM is equal to the angle which the direction of incidence makes with the plane.

From the principles of this Chapter we can without difficulty solve such problems as the following.

212. Prob. I. *A and B are two bodies whose elasticity is e: to find their proportion so that A, impinging directly upon B, may be at rest after the impact.*

By Cor. 5. to Art. 195, the velocity of A after the impact is $\frac{(A - eB)a}{A + B}$; and that this may be nothing, we must have $A = eB$; or A less than B in the ratio of e to 1.

This conclusion is independent of the velocity of A, and might therefore be made a means of trying the accuracy of the common hypothesis concerning elasticity, by observing whether, experimentally, in the collision of two bodies which have this proportion, A remains at rest.

213. Prob. II. *A perfectly elastic ball A, strikes an equal and perfectly elastic ball B at rest; to find the conditions under which it is possible that after impact they may strike two given points P and Q respectively. Fig.* 145.

Join Q with B, the center of the second ball, and let QB produced meet the surface in c. In order that B may move in the direction BQ after collision, the contact must manifestly take place in the point c: and hence the center A, of the other ball, must be in the line QB produced. Let aA be its velocity before impact; draw am perpendicular to QA; then, since the bodies are equal, the part mA of the velocity will, by Cor. 2. to Art. 196. be entirely destroyed. Hence the body will retain only its velocity parallel to am; and, if AP be perpendicular to AQ, will move in the direction AP.

Since PAQ is a right angle, the locus of the possible positions of A at the moment of contact is a semicircle on PQ. If the balls be small, the locus of the positions of B for which the problem is possible, is nearly a semicircle.

214. Prob. III. *The elasticity of the balls being imperfect and* $= e$, *to find the conditions necessary to make the balls after impact strike two given points* P' *and* Q. *Fig.* 145.

Let aA be the velocity of A before impact; as before, the center of A must, at the moment of contact, be in QB. And mA being the velocity in this direction before impact, we shall find the velocity An after impact in the same direction by Cor. 4, Art. 195. By that Corollary we have, making $b = o$, $An = \frac{1}{2}(1 - e) \cdot Am$. and drawing np perpendicular to An and equal to am, Ap will be the direction of A's motion after impact, which, by the question, is to pass through P'.

Hence conversely, if we join AP', and from any point p in

AP' draw pn perpendicular on AQ; and take Am so that $An = \frac{1}{2}(1-e)\,Am$, or $Am = \dfrac{2\,An}{1-e}$; and draw ma perpendicular to Am and equal to np; aA is the direction in which A must impinge.

In this case the problem is always possible when the point A is without the semicircle described on $P'Q$.

215. PROB. IV. *To find in what direction a perfectly elastic ball must be projected from a given point P, that after reflexion at a given plane DE, it may strike a given point Q. Fig.* 146.

Draw QN perpendicular on the given plane; produce it and make $Nq = NQ$; join Pq meeting DE in A; PA is the direction required.

Join AQ. The triangles QNA, qNA are manifestly equal; hence $PAD = NAq = NAQ$; and since in this case the angle of incidence is equal to the angle of reflexion; the ball projected in the direction PA, will be reflected in the direction AQ, and will strike the point Q.

We have here supposed the ball to be a point. If its magnitude be not inconsiderable, let de be the plane, and draw DE parallel to it, at a distance equal to the radius of the ball: DE will be the plane at which the reflexion of the *center* of the ball may be supposed to take place.

216. PROB. V. *The same things being given, and the ball being imperfectly elastic, to find the direction in which it must be projected to strike the point Q. Fig.* 147.

Draw QN perpendicular to the plane, and produce it to q, so that $Nq : NQ :: 1 : e$; e being the fraction which expresses the elasticity. Join Pq, and this will be the direction in which the body must be projected. For let Pq meet DE in A, and join AQ; and draw Az perpendicular to the plane. Then

$$\tan.\, PAz : \tan.\, QAz :: \tan.\, AqN : \tan.\, AQN$$

$$:: \frac{AN}{Nq} : \frac{AN}{NQ} :: NQ : Nq :: e : 1.$$

Hence by Cor. 3, Art. 211, if the body impinge in the direction PA, it will be reflected in the direction AQ, and will strike the point Q.

If the ball be of finite magnitude, the construction must be modified as in the last problem.

Cor. If PM perpendicular to the plane meet QA in p, we have, as may easily be shewn,

$$PM : Mp :: qN : NQ :: 1 : e.$$

Hence the point A may be found by taking $PM : Mp ::$ $1 : e$ and joining pQ.

217. Prob. VI. *A ball of given elasticity, perfect or imperfect, is to be projected from a given point P, so that, being reflected at any number of given planes in a given order, it may afterwards strike a given point Q. Fig.* 148.

Let DE, EF, FG be the planes, and let them be produced when necessary.

Draw QN perpendicular on the last plane, and take $QN : Nq :: e : 1$.

Draw qn perpendicular on the next plane, and take $qn :$ $nq' :: e : 1$.

Draw $q'n'$ perpendicular on the next plane, and take $q'n'$: $n'q'' :: e : 1$.

And so on if there be more planes.

Join Pq'', and this will be the direction in which the ball must be projected.

For let Pq'' meet the first plane in A; join Aq' meeting the second plane in A'; join $A'q$ meeting the third plane in A''; join $A''Q$.

It may be shewn, as in the last problem, that if the body strike the plane in the direction PA, it will be reflected in the direction Aq'; that if it strike the second plane in the direction AA' it will be reflected in the direction $A'q$; and so on. Hence its path will be $PAA'A''Q$, and it will strike the point Q as required.

If the elasticity be perfect, QN, Nq; qn, nq', &c. must be equal respectively.

Cor. If we had begun the construction from P, and made $PM : Mp :: 1 : e$, &c. we should have got the same result. Also if we had drawn perpendiculars on some of the planes beginning from Q, and on others beginning from P, so as to comprehend them all between the two, we should still obtain the same solution, and the proof would be nearly the same.

CHAP. III.

UNIFORMLY ACCELERATED MOTION AND GRAVITY.

218. The simplest case of the action of a continuous force, is when the force is uniform, and acts in the straight line in which the body moves. We shall consider in the first place what will be the motion of a body under these circumstances; and in the next place to what cases these conclusions are applicable.

1. *Uniformly Accelerated Motion.*

Prop. *When a body is accelerated in a straight line by a uniform force, the velocity is as the time from the beginning of the motion.*

By Art. 171, a force is said to be uniform when it always generates the same velocity in a body in equal successive times, whatever velocity the body may have already in that direction. Let the force be such as would in 1 second generate a velocity f in a body at rest. Then in the next second, the force, acting upon the body which is moving with this velocity, will add to it a velocity f, so that at the end of 2 seconds its velocity will be $2f$. Similarly in the third, fourth, &c. seconds, equal additions of velocity will be made, and at the end of 3, 4, &c. seconds the velocities will be $3f$, $4f$, &c.; and in the same way it will appear, that at the end of any number, t, of seconds, the velocity will be tf. Also the additions of velocity are equal in equal portions of seconds, however small. Hence the velocity will be tf when t consists of any fractional parts, as well as when it is a whole number. Therefore in all cases the velocity is as the time.

If v represent the velocity, $v = ft$.

If the forces be measured by the velocity generated by them in 1 second, f represents the force*.

219. PROP. *On the same supposition, the space from the beginning of the motion is as the square of the time.*

During the time t, the velocity of a body acted on by a force f begins from 0, and increases incessantly up to a certain finite magnitude v. It is manifest therefore that the space described in any portion of the time, with this increasing velocity, is less than the space which would have been described in the same portion of time, if the velocity had been, during the whole portion, as great as it is at the end of that portion. Let the time t be divided into n equal portions τ, τ, &c. so that $n\tau = t$. Then we can, by the last Article, find the velocities at the end of each of these portions of time; and the space which would have been described if these velocities had been respectively continued uniform through each

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* If gravity be called 1, of course a force which is F times gravity will be called F. Let g be the velocity generated by gravity in 1″, then Fg will be the velocity generated by the force F in 1″; and we shall have

$$v = Fgt.$$

portion of time will be found, by Art. 172, by multiplying the velocity by the time in each portion. Thus, at the end of

the 1st, 2d, 3d, 4th. . . . nth of the portions τ,
the velocities are $f\tau$, $2f\tau$, $3f\tau$, $4f\tau$, ... $nf\tau$.

Hence, *if these velocities had been uniform* through their respective times τ, the space described would have been

in the 1st, 2d, 3d, 4th. . . . nth,

$$f\tau^2,\ 2f\tau^2,\ 3f\tau^2,\ 4f\tau^2 \ldots\ldots nf\tau^2;$$

and the sum of all these is

$$f\tau^2(1+2+3+\ldots\ldots n) = f\tau^2 \cdot \frac{n\,.\,(n+1)}{2}$$

$$=\frac{f\tau^2 n^2}{2}+\frac{f\tau^2 n^2}{2n}=\frac{ft^2}{2}+\frac{ft^2}{2n}, \text{ because } \tau n = t.$$

Now the space which is actually described by the uniformly accelerated body, is, as has been said, in each of these portions, *less* than the corresponding space just found. Hence the whole space described, which we shall call s, is *less* than the sum of all these spaces; that is,

$$s<\frac{ft^2}{2}+\frac{ft^2}{2n}.$$

But the sum of all the spaces described with the uniform velocities will differ less from the actual space, as the portions of time are made smaller, and of course their number larger; and by increasing this number indefinitely, the aggregate of the spaces described with the successive velocities, approaches indefinitely near to the space described with the accelerated motion; that is

$$s=\frac{ft^2}{2}+\frac{ft^2}{2n} \text{ when } n \text{ is indefinitely large.}$$

Or $s=\frac{ft^2}{2}$*, because the fraction $\frac{ft^2}{2n}$ becomes indefinitely small. Hence s varies as t^2.

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* That this is the accurate value of s may perhaps be made more evident as follows.

The

COR. 1. In the two equations $v = ft$, $s = \frac{1}{2}ft^2$, we have four quantities, any two of which serve to determine the other two. By simple eliminations we obtain the following results;

The velocities at the *beginning* of the

1st, 2d, 3d. . . . . . . *n*th of the portions τ,

are 0, $f\tau$, $2f\tau$. . . . . .$(n-1)f\tau$.

Hence if these initial velocities had been continued uniform through these portions respectively, the spaces described would have been

$$0,\ f\tau^2,\ 2f\tau^2 \ldots\ldots (n-1)f\tau^2.$$

And the sum of these (an arithmetical progression) is,

$$f\tau^2 \cdot \frac{n\,.\,(n-1)}{2} = \frac{f\tau^2 n^2}{2} - \frac{f\tau^2 n^2}{2n} = \frac{ft^2}{2} - \frac{ft^2}{2n}.$$

Now the space described (s) when the velocity increases continually, is *greater* than this. Hence, (combining this with what is said in the text,) whatever be n,

$$s > \frac{ft^2}{2} - \frac{ft^2}{2n},\ \text{and}\ s < \frac{ft^2}{2} + \frac{ft^2}{2n}.$$

And as the fraction $\frac{ft^2}{2n}$ may become smaller than any assigned quantity by increasing n, this cannot be true except $s = \frac{ft^2}{2}$.

If we had taken gravity for our unit of force, the formula would have been

$s = \frac{1}{2}Fgt^2$; or, if m be the space through which a body would fall in 1″, $s = mFt^2$.

The equations in the text may be immediately obtained from the equations $\frac{dv}{dt} = f$, and $\frac{ds}{dt} = v$, putting g for f, and integrating. We have thus

$$dv = g\,dt;\ \therefore v = gt;$$
$$ds = v\,dt = gt\,dt;\ \therefore s = \tfrac{1}{2}gt^2.$$

There are no corrections required in these integrations, for when $t = 0$, $v = 0$, and $s = 0$.

$$\left.\begin{aligned} s &= \tfrac{1}{2}ft^2 = \tfrac{1}{2}tv = \frac{v^2}{2f}; \\ v &= ft = \frac{2s}{t} = \sqrt{(2fs)}; \\ t &= \frac{v}{f} = \frac{2s}{v} = \sqrt{\frac{2s}{f}}; \\ f &= \frac{v}{t} = \frac{v^2}{2s} = \frac{2s}{t^2}. \end{aligned}\right\} \ldots\ldots\ldots\ldots (A).$$

COR. 2. Since $s = \frac{1}{2}tv$, and tv is the space described in the time t with the velocity v, it appears, that *the space described by a body uniformly accelerated from rest, is half the space described in the same time with the last acquired velocity.*

Hence also the space through which the body moves in the first second is the half of f, because f is the velocity acquired in $1''$.

COR. 3. The space described in t seconds $= \frac{1}{2}ft^2$;

. . . . in $t - 1$ seconds $= \frac{1}{2}f(t-1)^2 = \frac{1}{2}f(t^2 - 2t + 1)$;

therefore, subtracting, we have

the space in the t^{th} second $= \frac{1}{2}f(2t - 1)$.

Hence the spaces in the 1st, 2d, 3d, 4th, &c. seconds are $\frac{1}{2}f.1$, $\frac{1}{2}f.3$, $\frac{1}{2}f.5$, $\frac{1}{2}f.7$, &c. and are as the odd numbers 1, 3, 5, 7, &c.

220. PROP. *Let a body be projected with a given velocity u, and acted on in the same direction by a constant force f; it is required to determine the relation of the space, time, and velocity.*

It is manifest that if the body is, at a certain point, moving with a certain velocity, its motion after that point will be the same, however we suppose the velocity to have been acquired. Hence the motion will be the same, if we suppose that velocity to have been generated by the force accelerating the body from rest. Let the force f generate the velocity u by acting for a time t', through a space s'. Hence $u = ft'$. Let the body afterwards continue to be acted on by the same force, and describe a space s in a time t; so as to describe a space $s' + s$ from rest in a time $t' + t$. Hence we have, by the last Article,

$$s' + s = \tfrac{1}{2}f(t' + t)^2 = \tfrac{1}{2}f(t'^2 + 2t't + t^2),$$
$$\text{and, } s' = \tfrac{1}{2}ft'^2;$$
$$\therefore s = \tfrac{1}{2}f(2t't + t^2) = ft't + \tfrac{1}{2}ft^2;$$
$$\text{but } u = ft'; \ \therefore s = tu + \tfrac{1}{2}ft^2.$$

Cor. 1. Since tu is the space which the body would have described in the time t, with the uniform velocity u, and $\frac{1}{2}ft^2$ the space through which the force would have drawn it in the same time; it apears that *the space in any time is equal to the space described with the velocity of projection, plus the space described from rest by the action of the force.*

Cor. 2. If v be the velocity at the end of the time t,

$$v = f(t' + t) = ft' + ft = u + ft.$$

Hence *the velocity after any time is equal to the velocity of projection plus the velocity generated by the force:* as is also manifest from the definition of uniform force.

Cor. 3. We have also, by equations (A),

$$v^2 = 2f(s' + s); \ u^2 = 2fs':$$
$$\text{hence } v^2 - u^2 = 2fs.$$

221. Prop. *When a body is projected in a direction opposite to that in which the force acts, the same formulæ will be true as in the last two Articles, s being the space, and t the time, from the end of the motion.*

In this case the force will diminish the velocity; and, since it is uniform, will produce equal decrements in equal times. In a certain time the body will be reduced to rest, and during this time the velocity will go on decreasing by exactly the same degrees by which it increased when a body was accelerated from rest. Hence the spaces reckoned from the end of this motion will be the same as from the beginning of the former one.

Cor. 1. Let the body be projected with the velocity u, and let t' be the time and s' the space in which the whole of the velocity would be destroyed by the action of the force in the opposite direction. In a time t let a space s be described; then in the remaining time $t' - t$ from the end of the motion in a direction opposite to the force, there would be described a space $s' - s$. Hence we have

$$s' = \tfrac{1}{2} f t'^2 ;$$
$$s' - s = \tfrac{1}{2} f (t' - t)^2 = \tfrac{1}{2} f (t'^2 - 2t't + t^2) ;$$
$$\therefore s = \tfrac{1}{2} f (2t't - t^2) = f t' t - \tfrac{1}{2} f t^2,$$

and since $u = ft'$, $s = tu - \frac{1}{2} f t^2$.

Cor. 2. Hence, as in the last Article, *the space in any time is equal to the space described with the velocity of projection, minus the space described from rest by the action of the force.*

Cor. 3. Similarly, *the velocity after any time is equal to the velocity of projection, minus the velocity generated by the force.*

2. *Vertical Motion by Gravity.*

222. Prop. *Gravity, near the earth's surface, is a uniform force.*

When a stone falls from rest by the action of gravity, its velocity goes on perpetually increasing so long as it falls freely. The law according to which this acceleration takes place, is, of course, to be determined from experiment; and it is found, that whatever be the material and mass of the falling body, and the other circumstances of the fall, if we make allowance for the effects produced by the resistance of the air and other impediments, the velocity generated by gravity is as the time, and consequently, from what has been said, that the force of gravity is constant.

This was first asserted by Galileo; some facts were adduced by him to prove the hypothesis; and all the experiments which were made afterwards, tended to confirm it. The motions of bodies which fall freely, are so rapid, that they cannot be observed with sufficient accuracy; and hence some contrivance is necessary which may diminish the velocity while it preserves the law of the acceleration. This effect may be obtained in different ways. Instead of allowing the body the velocity of which we observe, to fall unconnected with any other, we may cause it to descend, drawing up another body, or producing rotatory motion in a mass fixed upon an axis; by which means the motion will be so much retarded, that it may be measured. Or we may make the body descend down a very smooth inclined plane, or other inclined surface; and by making the inclination small, the velocity will become sufficiently slow to be observed. Or we may cause bodies to descend down circular arcs by hanging them to strings of given lengths, and making them swing; which will be equivalent to letting them descend down perfectly smooth circular surfaces. This last method is susceptible of great accuracy. The times of oscillation of the pendulums depend upon the velocities with which the bodies move in the circular arcs; and these velocities have a known relation to the velocities with which the bodies would fall perpendicularly. Hence, since the times of the oscillations of pendulums agree with the theory as deduced from the supposition of a constant gravity, that supposition is proved to be true. In the same way the other experiments confirm the proposition that *gravity is a constant force**.

* The contrivance first mentioned is the principle of the machine invented by Atwood, and an account of experiments made with it may be found in his Treatise of Rectilinear and Rotatory Motion, Sect. 7. The descent of bodies down inclined planes was the method used by Galileo. It does not admit of much accuracy, on account of the effects of friction, which causes the bodies to roll instead of sliding, and otherwise affects their motion.

The quantities on which we might expect the variation of gravity to depend, if it were not constant, might be the situation of places upon the earth's surface, and their elevation; the velocity of the body on which the force acts, and the size and substance of the body. Accurately speaking, it does vary with some of these. Gravity is a force arising from the attraction of the earth, tending at every point nearly to its center, and dependent on the distance from that center; at different distances from the equator, and at different altitudes, it is, if the intervals be taken of sufficient magnitude, perceptibly different; and it is only in consequence of the smallness of this variation in any spaces with which we are here concerned, that we may suppose it constant and in parallel lines. There are also other variations still more inconsiderable, arising from the irregularities of the form and materials in the structure of the earth, which in some measure influence its attraction, from which gravity arises.

In producing the same effect upon a body whatever be its previous velocity, gravity differs remarkably from all the mechanical powers which

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223. Prop. *Gravity is the same in all bodies, whatever be the difference of material or magnitude.*

In bodies of different material this was proved by Newton from experiments upon pendulums: (see *Principia*, Book III. Prop. 6:) from which he inferred that all substances would descend to the earth with equal velocities.

That bodies of the *same* material and different magnitudes would descend with the same velocity, is easily seen. For if one body be 10 times the other, let the first be divided into 10 bodies each equal to the second. If these were all to fall at the same time from the same point, but separate, they would descend each with the same velocity as the second body. Hence if they were supposed to be connected and united, they would not accelerate or retard each other's motions; and therefore the whole mass would still descend with the same velocity.

224. The intensity of gravity, or the space through which a body would fall in 1″, must be determined by experiment. The most accurate observations for this purpose are those that are made upon pendulums. It will be shewn hereafter, that, knowing

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we can exert. For instance, supposing that by turning a winch with a certain muscular exertion we could communicate to a wheel a certain angular velocity in 1″, we should not add an *equal* velocity, if we were to exert the same effort for the next second, because part of the muscular power must be employed in moving the hand so as to keep up with the winch. In the same way, if motion were communicated by a spring, the action of the spring would be less as the body receded faster from it; and the body might move with such a velocity that the effect of the spring should be only just sufficient to enable it to keep up with the body, and should not at all increase its motion. Gravity, on the contrary, acts with the same energy, whether the body acted on be at rest, or moving in the direction of its action, or in a contrary direction.

It was formerly supposed that heavier bodies descended faster than lighter ones in proportion to their weight. The falsity of this was shewn by Galileo from experiment. So far as gravity is concerned, the same velocity is communicated to all bodies, whatever be their mass; but in consequence of the resistance of the air, which is proportionally greater on smaller bodies, heavy ones do, in the atmosphere, descend with greater celerity.

the length of a pendulum which oscillates once in a second, we can find the space through which a body would fall in the same time. By the latest experiments of this kind it appears that in the latitude of London, and at the surface of the earth, a body would, *in vacuo*, fall through a space of 193.14 English inches, or $16\frac{1}{10}$ feet nearly. Consequently, (Art. 219. Cor. 2,) the velocity generated in that time would in the same time carry it through 386.28 inches; and thus this space measures the velocity generated in 1″ by gravity, and is therefore the value of that force, according to the way of measuring accelerating forces (Art. 171). This quantity will generally be represented by g.

Hence we can easily solve all questions relating to the fall of bodies *in vacuo* by gravity. We have only to substitute the known quantity g for f in the formulæ (A) of Art. 219: as is seen in the following examples*.

Ex. 1. *To find how far a body will fall in vacuo in* $2\frac{1}{2}''$, *and the velocity acquired.*

By the first expression for s in (A), putting g for f;

$$s=\tfrac{1}{2}gt^2=\tfrac{1}{2}\times 32.2\times\left(\frac{5}{2}\right)^2 \text{ feet} = 100.6 \text{ feet}.$$

By the first expression for the velocity;

$$v=gt=32.2\times\frac{5}{2}\text{ feet}=80.5\text{ feet}.$$

Ex. 2. *A body is projected upwards with a velocity of* 100 *feet; to find how high it will rise, and in what time it will reach its greatest height.*

By Art. 221, the height to which it will rise will be the same as the height down which it must fall to acquire the velocity. Hence, by the third expression for s in (A),

$$s=\frac{v^2}{2g}=\frac{100^2}{2\times 32.2}=\frac{10000}{64.4}=155.28\text{ feet}.$$

* The descent of a body in the atmosphere will be nearly the same as *in vacuo*, so long as the velocity is small. But when bodies fall freely through great heights, the resistance becomes very large, and the effect arising from this may be made to bear any ratio to the whole.

Similarly the time of the ascent is equal to the time of the descent; hence, by the first expression for t,

$$t = \frac{v}{g} = \frac{100}{32.2} = 3.1'.$$

Ex. 3. *On the same supposition, to find how high the body will ascend in 2''.*

Putting g for f in Art. 180, we have

$s = tu - \frac{1}{2}gt^2 = 2 \times 100 - \frac{1}{2} \times 32.2 \times 4 = 200 - 64.4 = 135.6$ feet.

By means of our formulæ we can easily solve such problems as the following:

225. Prob. I. *A person drops a stone into a well, and after t'' hears it strike the water; to find the depth to the surface of the water.*

We neglect the resistance of the air. The velocity of sound, as appears by experiment, is uniform, and equal to 1130 feet in a second. Now the time between dropping the stone and hearing the sound, is equal to the time of the stone falling the depth of the well, together with the time of the sound rising through the same distance. Let x be this depth, and n the velocity of sound. Then

$$\text{time of falling through } x = \sqrt{\frac{2x}{g}};$$

$$\text{time of sound's passage} = \frac{x}{n};$$

$$\therefore \frac{x}{n} + \sqrt{\frac{2x}{g}} = t;$$

$$\therefore \frac{2x}{g} = t^2 - \frac{2tx}{n} + \frac{x^2}{n^2};$$

$$\therefore x^2 - 2\left(tn + \frac{n^2}{g}\right)x = -t^2n^2;$$

$$x = tn + \frac{n^2}{g} \pm \sqrt{\left(\frac{2tn^3}{g} + \frac{n^4}{g^2}\right)}.$$

The negative sign must be taken*. Also it will be found that $\frac{n}{g} = 35$ nearly†.

$$\therefore x = n . \left\{t + \frac{n}{g} - \sqrt{\left(\frac{2tn}{g} + \frac{n^2}{g^2}\right)}\right\}$$
$$= n . \{t + 35 - \sqrt{(70t + 1225)}\}.$$

Thus, let the time t be $3''$; then

$$x = n . \{38 - \sqrt{1435}\} = n . \{38 - 37.88\}$$
$$= .12n = 135.6 \text{ feet.}$$

226. Prop. *When two bodies hang over a fixed pully, to determine their motion, neglecting the inertia of the pully and the string.*

Thus let two unequal bodies p and q, hang over a pully as in fig. 136; (p corresponding to $P + A$ and q to B). Let p be greater than q. If p were equal to q, it would just balance q, and there would be no motion; the weight which is employed in producing motion is the excess of p above q, or $p - q$. Also the two bodies move with equal velocities, and hence the mass in which motion is produced is $p + q$. The accelerating force is therefore equal to $\frac{p-q}{p+q}$ multiplied into some constant quantity. (Art. 183.)

Let the accelerating force $f = \frac{p-q}{p+q} c$: and when $q = 0$, or p

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* For t, the whole time, must be greater than the time of the sound's motion, which is

$$\frac{x}{n}, \text{ or } t + \frac{n}{g} \pm \sqrt{\left(\frac{2tn}{g} + \frac{n^2}{g^2}\right)}.$$

But t is not greater than $t + \frac{n}{g} + \sqrt{\left(\frac{2tn}{g} + \frac{n^2}{g^2}\right)}$ with the positive sign.

† If n be equal to 1127 feet, $\frac{n}{g}$ is accurately equal to 35. The values generally taken for n have been 1142 and 1130 feet. Recent experiments would seem to shew that at the usual temperature of the air, the velocity is less; but the determinations are too various to entitle us to fix upon any particular value. See *Trans. of Camb. Phil. Soc.* vol. II. Part I. p. 120.

descends freely, it is g. Therefore $c = g$. Hence in other cases $f = \frac{p-q}{p+q} \cdot g$: which agrees with Ex. 2. Art. 183.

By substituting this value for f in equations (A) we can find the circumstances of the motion in any given case.

Ex. *Supposing $p = 81$ ounces and $q = 80$; to find the space descended by p in 1″, and the velocity acquired.*

$$s = \frac{g}{2} \cdot \frac{p-q}{p+q} t^2 = 16.1 \times \frac{1}{161} \times 1^2 = .1, \text{ or } \tfrac{1}{10} \text{ of a foot.}$$

$$v = g \cdot \frac{p-q}{p+q} = 32.2 \times \frac{1}{161} = .2, \text{ or } \tfrac{1}{5} \text{ of a foot.}$$

If instead of p drawing q vertically upwards, p draw q along a horizontal plane; as, for instance, if q be laid upon a perfectly smooth table, and p, connected with it by a string, hang over the edge of the table; the whole weight of p is employed in producing motion; and, as before, the two bodies move with the same velocity, and therefore may be considered as one mass $p + q$. Hence the accelerating force

$$f = \frac{pg}{p+q}.$$

3. *Motion on Inclined Planes.*

227. Prop. *To find the force which accelerates a body down an inclined plane.*

When a body q is *supported* on an inclined plane whose height and length are h and l respectively, by a force p, acting parallel to the plane, we have, by Art. 38, Cor. 2,

$$p : q :: h : l; \therefore p = \frac{qh}{l}.$$

Hence, when q is not supported, $\frac{qh}{l}$ is the pressure which it exerts *along* the plane, and which is employed in producing motion. Also if the body be suffered to descend by its weight, the mass moved is the body q itself*. Therefore, by Art. 183, the accelerating force will be proportional to $\frac{p}{q}$ or $\frac{h}{l}$; and, as in last Article, equal to $\frac{hg}{l}$. By substituting this value for f in equations (A), we obtain the circumstances of the descent of bodies down inclined planes.

Also if the body descend down the whole length of the plane, l may be put for s.

By this means from the first value of s in (A), we have

$$l = \frac{1}{2} \cdot \frac{gh}{l} \cdot t^2; \therefore t^2 = \frac{2l^2}{gh}; t = \sqrt{\frac{2l^2}{gh}}.$$

Also by the third expression for the velocity,

$$v = \sqrt{(2fs)} = \sqrt{\left(\frac{2gh}{l} \cdot l\right)} = \sqrt{(2gh)}.$$

Since this expression for the velocity is independent of the length, it appears that *the velocity acquired down all planes whose perpendicular heights are equal, will be the same; and equal to the velocity acquired by falling down the perpendicular height.* This principle was *assumed* by Galileo as the basis of his reasonings on inclined planes.

Cor. If α be the inclination of the plane to the horizon, $\frac{h}{l} = \sin. \alpha$, and $g \sin. \alpha$ is the accelerating force upon the plane, which may be substituted for f in the formulæ (A).

Ex. *A smooth plane* 10 *feet long has one end* 1 *foot higher than the other: to find the time of a body sliding down it, and the velocity acquired.*

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* The plane is supposed to be *perfectly* smooth, so as to exert no resistance to motion along it; in which case the body p will slide, and not roll, down the plane, even if it be spherical in its form. In actual cases the friction is almost always great enough to produce rotatory motion in round bodies. This circumstance changes the value of the accelerative force, as may be shewn hereafter.

By the formula just obtained,

$$t = \sqrt{\frac{2l^2}{gh}} = \sqrt{\frac{200}{32.2}} = \frac{10}{4}, \text{ nearly, } = 2\tfrac{1}{2}''.$$

$$\text{Also } v = \sqrt{(2gh)} = \sqrt{(64.4)} = 8.02 \text{ feet.}$$

228. Prob. II. *A right-angled triangle being placed with its two sides horizontal and vertical respectively, it is required to determine their proportion, so that the time of the body falling down the perpendicular and describing the base with the velocity acquired, may be equal to the time of descent down the hypotenuse.*

Let x and y be the vertical and horizontal sides respectively; therefore the hypotenuse will be $= \sqrt{(x^2+y^2)}$. And by formula (A),

$$\text{time down } x = \sqrt{\frac{2x}{g}}; \text{ velocity acquired} = \sqrt{(2gx)};$$

$$\therefore \text{ time through } y = \frac{y}{\sqrt{(2gx)}}.$$

Also by last Article, x and $\sqrt{(x^2+y^2)}$ being the height and length on an inclined plane, time down the hyp. $= \sqrt{\dfrac{2(x^2+y^2)}{gx}}$.

$$\text{Hence } \sqrt{\frac{2(x^2+y^2)}{gx}} = \sqrt{\frac{2x}{g}} + \frac{y}{\sqrt{(2gx)}};$$

$$\text{squaring, } \frac{2(x^2+y^2)}{gx} = \frac{2x}{g} + \frac{2y}{g} + \frac{y^2}{2gx};$$

$$\therefore 4x^2 + 4y^2 = 4x^2 + 4xy + y^2;$$

$$\therefore 3y = 4x, \text{ or } \frac{x}{y} = \frac{3}{4}.$$

Hence also $\dfrac{\sqrt{(x^2+y^2)}}{y} = \dfrac{5}{4}$; and the sides of the triangle are as 3, 4, and 5.

229. Prop. *To find the accelerating force when a heavy body draws another along an inclined plane.*

Let, in fig. 46, the weight W $(=q)$ be fixed to a string WC, which is parallel to the inclined plane on which the weight rests, and passing over a fixed pully at C has a weight p appended to it hanging freely. If $p = \frac{qh}{l}$, p and q will balance. And if p be greater than this value, it will descend and draw q up the inclined plane. If p be less than the value just mentioned, q will descend down the inclined plane and draw up p. In both cases the accelerating force will be constant.

When p draws up q, the part $\frac{qh}{l}$ of p is employed in supporting q, and the remainder, $p - \frac{qh}{l}$, is the pressure which produces motion in the two bodies p, q. And these two bodies move with the same velocity. Hence the accelerating force in this case is

$$\frac{p - \frac{qh}{l}}{p+q} \cdot g = \frac{pl - gh}{pl + ql} \cdot g;$$

and by substituting this value for f we can apply our formulæ.

PROB. III. *The notation remaining, to find the time in which p will draw q up the given plane whose length is l and height h.*

We have as before $l = \frac{1}{2} f t^2 = \frac{pl - qh}{pl + ql} \cdot \frac{t^2 g}{2}$,

$$\therefore t = \sqrt{\frac{2(p+q)l^2}{g(pl - qh)}} *.$$

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* PROP. *On the same supposition, h, p, and q being given, to find l so that the time of drawing q up it may be the least possible.*

We must have the above expression for t a minimum, and therefore its square a minimum; and it will also be a minimum if we omit the constant factors $2(p+q)$ and g. Hence

$$\frac{l^2}{pl - qh} = \text{min.} \therefore \frac{pl - qh}{l^2} = \text{max.}$$

230. Prop. *If a circle be placed with its plane vertical, the times of descent down all chords drawn through the highest or lowest points are equal.*

Let ABP, fig. 149, be a circle, and AB a vertical diameter. Let PA be any chord drawn through A. By the Article 227, we have

$$\text{time down } AP = \sqrt{\frac{2AP^2}{g.AM}};$$

but by similar triangles $\dfrac{AP}{AM} = \dfrac{AB}{AP}$; $\therefore \dfrac{AP^2}{AM} = AB$;

$$\therefore \text{time down } AP = \sqrt{\frac{2AB}{g}}.$$

This is independent of the position of P: and hence the times down all chords AP, Ap, &c. are equal: and of course equal to the time of falling freely down AB.

In the same way it may be shewn that times down all chords PB, pB, &c. are equal.

Cor. Also we have

velocity acquired down $AP = \sqrt{(2g.AM)}$; (Art. 227.)

$$\text{and as before } AP^2 = AM.AB; \therefore AM = \frac{AP^2}{AB};$$

$$\therefore \text{velocity} = \sqrt{\frac{2g.AP^2}{AB}} = AP\sqrt{\frac{2g}{AB}}.$$

And as g and AB are constant, *the velocities acquired down planes AP, Ap, &c. are as the lengths AP, Ap, &c.*

$$\therefore \frac{p}{l} - \frac{qh}{l^2} = \text{max.} \quad \therefore -\frac{p}{l^2} + \frac{2qh}{l^3} = 0;$$

$$\therefore l = \frac{2qh}{p}.$$

Here $p = \dfrac{2.qh}{l}$; that is, p is twice as great as it is for equilibrium.

231. PROP. *Let APB, AQC, fig. 149, be two circles with their diameters in the same vertical line AB, and with the highest point common. Apq, APQ any chords: the times down PQ, pq from rest at P and p are equal.*

On BC describe a semi-circle; join CQ meeting this semi-circle in R: join BR. The angles APB, AQC, BRC are right angles, and therefore AQ, BR, and PB, QC, are parallel. Hence $PBRQ$ is a parallelogram, and BR is equal and parallel to PQ; and hence the time down PQ is equal to the time down BR. Similarly if Br be drawn parallel to pq, the time down pq will be equal to the time down Br. But the times down BR, Br are equal; therefore the times down PQ, pq are equal.

COR. Similarly, if two circles touch each other at the lowest point, and chords be drawn through this point; it may be shewn that the times down those portions of the chords which are intercepted between the circles are all equal.

4. *Planes of Quickest and Slowest Descent.*

232. There are a number of problems concerning the planes in which bodies would descend between given points, lines, and circles, so as to employ in their descent the longest or the shortest time possible. The constructions and demonstrations are very nearly similar for all of them, and the student will have no difficulty, after one or two specimens, in making out the rest. We shall give the Problems with their constructions, and the demonstrations in some of the most important cases, which will suggest them in the others.

It is required to find the plane of shortest descent,

PROB. IV. *From a given point P to a given straight line AB, fig.* 150.

From the given point P draw a horizontal line meeting the given line in A. Take AQ *downwards* along the given line equal to AP: PQ will be the plane required.

PROB. V. *From a given straight line AB to a given point P*, *fig*. 150.

Draw PA as before; and along the given line measure a distance *upwards* from A, equal to AP; the line joining the extremity of this distance with the point P is the plane required.

PROB. VI. *From a given point without a given circle to the circle.*

Join the given point with the *lowest* point of the given circle: the part of the joining line which lies without the circle is the plane required.

PROB. VII. *From a given circle to a given point without it.*

Join the given point with the *highest* point of the given circle: the part of the joining line which lies without the circle is the plane required.

PROB. VIII. *From a given point within a given circle to the circle.*

Join the given point and the *highest* point of the circle: the part of the joining line produced which is between the point and the circle is the plane required.

PROB. IX. *From a given circle to a given point within it.*

Join the given point and the *lowest* point of the circle: the part of the line produced which is between the circle and the point is the plane required.

PROB. X. *From a given straight line* (RM, *fig*. 151,) *without a given circle* (ASB) *to the circle.*

Through B, the *lowest* point of the circle, draw BM horizontal. Take MR *upwards* equal to MB, and join RB: RS is the plane required.

PROB. XI. *From a given circle to a given straight line without it.*

Draw a horizontal line through the *highest* point, terminated by the given line, and take *downwards* along the given line a distance equal to this line. Join the extremity of this distance with the highest point: a part of this joining line is the plane required.

PROB. XII. *From a given circle to another given circle without it.*

Join the *highest* point of the first circle with the *lowest* point of the second: the portion of the joining line which is between the circles is the plane required.

PROB. XIII. *From a given circle to another given circle within it.*

Join the *lowest* point of the first circle with the *lowest* point of the second: the part of the joining line produced which lies between the two circles is the plane required.

PROB. XIV. *From a given circle within another given circle to the other circle.*

Join the *highest* point of the first circle with the *highest* point of the second: the part of the joining line produced which lies between the two circles is the plane required.

It may also be required to find the plane of *longest* descent;—

PROB. XV. *From a given point without a given circle to the circle.*

Join the given point and the *highest* point of the circle: this joining line, produced till it again meets the circle, is the plane required.

PROB. XVI. *From a given circle to a given point without it.*

Join the given point and the *lowest* point of the circle: this joining line, produced till it again meets the circle, is the plane required.

PROB. XVII. *From a given circle to another given circle without it.*

Join the *lowest* point of the first circle with the *highest* point of the second: the joining line, produced both ways till it again meets the circumferences, is the plane required.

The plane of longest descent cannot be determined in any case when there is a possibility of drawing a horizontal plane under the conditions: for as the plane approaches to this position, the time of descent increases without limit. Also the plane of shortest descent cannot be determined in any case when the circles, &c. between which it is to be drawn, intersect each other: for by bringing the extremities of the plane near this point, we may diminish the plane and the time down it indefinitely.

233. We shall now give the demonstrations of Prob. 4, Prob. 7, and Prob. 12.

Demonstration for Prob. 4. Draw PO vertical and QO perpendicular to AQ. Since AQ was taken equal to AP, the angles AQP, APQ are equal. Also APO, AQO are equal, being right angles. Hence OPQ, OQP are equal, and therefore OP, OQ. With center O and radius OP describe a circle, which will pass through Q, and there touch AQ: also P will be the highest point. Draw any line Pr, meeting the circle in q. Now by the last Article the time down Pq is equal to that down PQ; hence the time down Pr, which is greater than that down Pq, is greater than that down PQ: and as this is true for every line Pr which does not coincide with PQ, the time down PQ is the shortest.

Demonstration for Prob. 7. Fig. 151. P being the given point and AB a vertical diameter of the given circle, AP is joined, and QP is the plane required. For C being the center of AQB, let CQ meet a vertical line PO in O. The angles OPQ, QAC, CQA, OQP are equal: hence OP, OQ are equal. With center O describe a circle PQ, which will touch AQB. As before, the time down QP may be shewn to be less than the time down any other plane rP.

Demonstration for Prob. 12. Fig. 152. AB, ab being vertical diameters of the given circles, Ab is joined, and PQ is the plane required. It appears from the demonstration for Prob. 7, that whatever be the point *to* which the plane is drawn, it must pass through the highest point A of the first circle, in order that the time may be less than down any other plane from the first circle to the point Q in the second. Hence, we have to determine down which of the planes pr, which produced pass through the point A, the time is least. The center of aQb being c, let cQ meet AB in O, and as before OA, OQ are equal. With center O describe a circle AQ. Then it follows from Cor. 1. to Art. 186, that the time down PQ is equal to the time down pq, and therefore less than the time down pr. Hence PQ is the plane of shortest descent of all that pass through A; and hence, by what has been said, of all that can be drawn from one circle to the other.

CHAP. IV.

THE MOTION OF PROJECTILES.

234. WHEN a body is projected in any direction, not vertical, and acted upon by gravity, it will describe a curve line. The nature of this curve may be deduced from the principles laid down in Chap. I. It follows from the second law of motion, that if a body be projected in the direction AR, fig. 153, with any velocity, and if, in the time in which it would describe AR with this velocity continued uniform, it would by the action of gravity fall through the space Am from rest; its place at the end of this time will be P, so situated that RP is equal and parallel to Am.

It appears from this that the motion of the body will be in a vertical plane.

235. PROP. *A body is projected from a given point, in a given direction, with a given velocity; it is required to find where it will strike the horizontal plane passing through the point of projection.*

Let A be the point, AT the direction of projection, and APH the path; AH being horizontal. Let TAH, the angle of projection $= a$; the velocity of projection $= V$; and the time of describing $APH = T$. Then for the reasons mentioned in the last article, in the time T, AT would have been described uniformly, and TH would have been fallen through by the force of gravity. Therefore

$$AT = TV, \text{ and } TH = \tfrac{1}{2}gT^2, \text{ (Art. 219.)}$$

$$\text{Also } TH = AT \,.\, \sin. a, \text{ or } \tfrac{1}{2}gT^2 = TV \sin. a;$$

$$\therefore T = \frac{2V \sin. a}{g}.$$

$$\text{Hence } AT = TV = \frac{2V^2 \sin. a}{g};$$

$$AH = AT \,.\, \cos. a = \frac{2V^2 \sin. a \cos. a}{g} = \frac{V^2}{g} \sin. 2a.$$

The distance AH is called the *range*, and T is called the *time of flight* of the projectile.

COR. 1. If t be any other time in which the arc AP is described, and if RPM be vertical; $AR = Vt$, $RP = \frac{1}{2}gt^2$, and

$$PM = Vt \sin. a - \tfrac{1}{2}gt^2.$$

$$\text{Also } AM = Vt \cos. a;$$

and therefore the point M moves uniformly in AM.

COR. 2. Let $t = \frac{1}{2}T = \dfrac{V \sin. a}{g}$, and let V be the corresponding place of the body; and we have

$$PM \text{ or } VG = \frac{V^2 \sin.^2 a}{g} - \frac{V^2 \sin.^2 a}{2g} = \frac{V^2 \sin.^2 a}{2g}.$$

COR. 3. Let t be greater or less than $\frac{1}{2}T$. Suppose

$$t = \tfrac{1}{2}T(1 \pm m) = \frac{V \sin. a}{g}(1 \pm m).$$

$$\text{Then } PM = \frac{V^2 \sin.^2 a}{g}(1 \pm m) - \frac{V^2 \sin.^2 a}{2g}(1 \pm m)^2$$

$$= \frac{V^2 \sin.^2 a}{2g}(1 - m^2).$$

Hence it appears that PM is greatest when $m=0$, that is, when $t=\frac{1}{2}T$; or the greatest height VG occurs in the middle of the time of flight.

Cor. 4. It appears also that for equal values of m, whether they be positive or negative, we have the same value of PM; hence on the two sides of the highest point V, the points P, P', corresponding to equal times from V, are at the same height above the horizontal plane.

Also equal distances GM, GM', correspond to equal times from V (Cor. 1.). Hence the curve consists of two equal and similar arcs from V to A, and from V to H.

236. Prop. *On the same suppositions, it is required to find where the body will strike any given plane passing through the point of projection.*

Let the body be projected in the direction AT, fig. 154; and let AQ be the line in which the vertical plane passing through AT meets the given plane; AH horizontal. Let

$$TAH=\alpha,\ QAH=\iota;\ \therefore\ TAQ=\alpha-\iota.$$

Also let T be the time of flight in AQ; R the *range* or distance AQ: V the velocity of projection. Hence, as in last Article,

$$AT=TV,\ TQ=\tfrac{1}{2}gT^2.$$

$$\text{But by Trig. } QT : AT :: \text{sin. } QAT : \text{sin. } AQT,$$

$$\text{and sin } AQT=\text{sin. } AQH=\text{cos. } QAH.$$

$$\text{Hence } QT=AT.\frac{\text{sin. } QAT}{\text{sin. } AQT}=AT.\frac{\text{sin. } QAT}{\text{cos. } QAH};$$

$$\text{or } \tfrac{1}{2}gT^2=TV.\frac{\text{sin. }(\alpha-\iota)}{\text{cos. }\iota};\ \therefore\ T=\frac{2V}{g}.\frac{\text{sin.}(\alpha-\iota)}{\text{cos. }\iota}.$$

$$AT=TV=\frac{2V^2}{g}.\frac{\text{sin. }(\alpha-\iota)}{\text{cos. }\iota}.$$

$$\text{Again } AQ : AT :: \text{sin. } ATQ : \text{sin. } ATQ.$$

And sin. $ATQ = $ cos. TAH; sin. $AQT = $ cos. QAH, as before;

$$\therefore AQ = AT . \frac{\text{sin. } ATQ}{\text{sin. } AQT} = AT . \frac{\text{cos. } TAH}{\text{cos. } QAH};$$

$$\text{or } R = \frac{2V^2}{g} . \frac{\text{sin. } (\alpha - \iota)}{\text{cos. } \iota} . \frac{\text{cos. } \alpha}{\text{cos. } \iota} = \frac{2V^2}{g} . \frac{\text{sin. } (\alpha - \iota) \text{ cos. } \alpha}{\text{cos.}^2 \iota} .$$

237. Prop. *To find in what direction a body must be projected with a given velocity, that its range upon a given plane may be the greatest possible.*

On a horizontal plane, the range $= \frac{V^2}{g}$. sin. 2α.

This will be greatest when sin. 2α is greatest, that is, when $2\alpha = $ a right angle, and α, the angle of projection, = half a right angle.

On an inclined plane the range $= \frac{V^2}{g} . \frac{2 \text{ sin. } (\alpha - \iota) \text{ cos. } \alpha}{\text{cos.}^2 \iota}$.

Since ι is constant, the range will be greatest when

$$2 \text{ sin. } (\alpha - \iota) \text{ cos. } \alpha$$

is greatest. But

$$2 \text{ sin. } (\alpha - \iota) \text{ cos. } \alpha = \text{sin. } \{\alpha + (\alpha - \iota)\} - \text{sin. } \{\alpha - (\alpha - \iota)\}$$
$$= \text{sin. } (2\alpha - \iota) - \text{sin. } \iota :$$

which, since ι is constant, is greatest when sin. $2\alpha - \iota$ is greatest; that is, when $2\alpha - \iota$ is a right angle: or,

$$2\alpha - \iota = \frac{\pi}{2}; \quad \therefore \alpha = \tfrac{1}{2}\left(\frac{\pi}{2} + \iota\right);$$

$$\therefore \alpha - \iota = \tfrac{1}{2}\left(\frac{\pi}{2} - \iota\right) \text{ or } TAQ = \tfrac{1}{2} ZAQ; \ AZ \text{ being vertical.}$$

Hence in this case AT bisects the angle ZAQ.

Cor. Hence, on the inclined plane, the greatest range is

$$= \frac{V^2}{g} \cdot \frac{2 \sin.(\alpha - \iota) \cos. \alpha}{\cos.^2 \iota} = \frac{V^2}{g \cos.^2 \iota} \{\sin.(2\alpha - \iota) - \sin. \iota\}$$

$$= \frac{V^2}{g \cos.^2 \iota} \left\{\sin. \frac{\pi}{2} - \sin. \iota\right\} = \frac{V^2 (1 - \sin. \iota)}{g (1 - \sin.^2 \iota)} = \frac{V^2}{g (1 + \sin. \iota)}.$$

238. Prop. *To express the formulæ for projectiles in terms of the height due to the velocity of projection.*

The *height due* to the velocity of projection is the height down which a body must fall so as to acquire that velocity. Let h be this height: then $V^2 = 2gh$; $h = \frac{V^2}{2g}$, $\frac{V^2}{g} = 2h$; and by the preceding Article we shall find,

On a horizontal plane,

$$\text{Range} \ldots\ldots = 2h \sin. 2\alpha.$$

$$\text{Time of flight} = \sqrt{\frac{2h}{g}} \cdot 2 \sin. \alpha.$$

$$\text{Greatest height} = h \sin.^2 \alpha.$$

$$\text{Greatest range} = 2h.$$

On an inclined plane,

$$\text{Range} \ldots\ldots = 4h \frac{\sin.(\alpha - \iota) \cos. \alpha}{\cos.^2 \iota}$$

$$= \frac{2h}{\cos.^2 \iota} \{\sin.(2\alpha - \iota) - \sin. \iota\}.$$

$$\text{Time of flight} = \sqrt{\frac{2h}{g}} \cdot \frac{2 \sin.(\alpha - \iota)}{\cos. \iota}.$$

$$\text{Greatest range} = \frac{2h}{1 + \sin. \iota}.$$

239. Prop. *The curve described by a projectile is a parabola, and the velocity at any point is that acquired by falling from the directrix.*

In fig. 153, $AR = Vt$, and $RP = \frac{1}{2} g t^2$.

$$\therefore \frac{AR^2}{RP} = \frac{2V^2}{g} = 4h : \text{ and } AR^2 = 4h . RP;$$

or, Am being vertical, and mP parallel to AR,

$$mP^2 = 4h . Am.$$

Hence the curve AP is a parabola, of which Am is the abscissa, mP the ordinate, and $4h$ the parameter.

If AC be taken in mA produced, = one-fourth the parameter at A, and CK drawn at right angles to it, CK will be the directrix of the parabola. And one-fourth the parameter at A is h, the height due to the velocity.

Hence the velocity at the point of projection A is equal to the velocity acquired in falling from the directrix. Also the velocity at any point P will be the same as if P were considered as the point of projection. Hence at any point the velocity is equal to that acquired in falling down DP, the distance from the directrix.

Cor. 1. It is manifest that AR will be a tangent to the curve at the point A. Now the tangent to the parabola makes equal angles with two lines, one drawn to the focus, and the other perpendicular to the directrix. Hence if we make the angle $RAS = RAC$, the focus will be in the line AS.

Also the distance of a point from the focus is equal to its distance from the directrix. Hence if we take $AS = AC$, S will be the focus.

Cor. 2. To find the *principal parameter* or *latus rectum* of the parabola.

If we draw SK perpendicular to the directrix, and bisect SK in V, V will be the vertex of the parabola: and $4SV$ or $2SK$ will be the parameter. Now

$$SK = GK \pm GS = AC + AS . \cos. SAm = AC - AS . \cos. SAC$$
$$= AC - AS . \cos. 2TAC = AC\{1 - \cos. 2TAC\}$$
$$= AC . 2\sin.^2 TAC, \text{ by Trigonometry};$$

$$\therefore 2\,SK = 4\,AC \,.\, \sin.^2 TAC = 4\,h \cos.^2 \alpha$$
$$= \text{the principal parameter.}$$

240. Prop. *To find an equation to the curve referred to horizontal and vertical co-ordinates.*

In fig. 153, let $AM = x$, $MP = y$; t any time; the rest of the notation as before.

$$AM = AR \,.\, \cos.\,\alpha; \text{ or } x = Vt \cos.\,\alpha; \therefore t = \frac{x}{V \cos.\,\alpha};$$

$$\therefore RP = \tfrac{1}{2} g t^2 = \frac{g x^2}{2\,V^2 \cos.^2 \alpha}.$$

Also $MR = AM \,.\, \tan.\,\alpha$. And $MP = MR - RP$;

$$\therefore y = x \tan.\,\alpha - \frac{g x^2}{2\,V^2 \cos.^2 \alpha},$$

the equation to the curve.

Cor. 1. If, as before, $h = \dfrac{V^2}{2g}$,

$$y = x \tan.\,\alpha - \frac{x^2}{4\,h \cos.^2 \alpha}.$$

Cor. 2. To find where the curve meets the horizontal plane.

For this point we must have $y = 0$;

$$\therefore x \tan.\,\alpha - \frac{x^2}{4\,h \cos.^2 \alpha} = 0.$$

This gives two values; viz. $x = 0$, which belongs to the point A; and

$$\tan.\,\alpha - \frac{x}{4\,h \cos.^2 \alpha} = 0, \text{ whence}$$

$$x = 4\,h \tan.\,\alpha \cos.^2 \alpha = 4\,h \sin.\,\alpha \cos.\,\alpha = 2\,h \sin.\,2\,\alpha:$$

which agrees with Article 235.

241. Prop. *To find the angle which the curve makes with the horizon at any point.*

If ϕ be this angle, tan. $\phi = \frac{dy}{dx}$; and differentiating the value of y,

$$\tan. \phi = \tan. \alpha - \frac{x}{2h \cos.^2 \alpha}.$$

Cor. To find the point V when the height of the projectile above a given plane AQ is the greatest. Fig. 154.

At this point it is evident that the direction of the motion must be parallel to AQ; hence tan. ϕ = tan. ι;

$$\therefore \tan. \iota = \tan. \alpha - \frac{x}{2h \cos.^2 \alpha};$$

$$\therefore \frac{x}{2h \cos.^2 \alpha} = \tan. \alpha - \tan. \iota = \frac{\sin. \alpha}{\cos. \alpha} - \frac{\sin. \iota}{\cos. \iota}$$

$$= \frac{\sin. (\alpha - \iota)}{\cos. \alpha . \cos. \iota};$$

$$AL = x = 2h \frac{\sin. (\alpha - \iota) \cos. \alpha}{\cos. \iota}.$$

$$\text{Hence } AG = \frac{AL}{\cos. \iota} = 2h \frac{\sin. (\alpha - \iota) \cos. \alpha}{\cos.^2 \iota}.$$

By comparing this with the value of AQ the range, Art. 238, it will be seen that $AG = \frac{1}{2} AQ$.

242. Prob. I. *A body is to be projected from a given point with a given velocity so as to strike another given point: to find the direction of projection. Fig.* 154.

Let Q be the point to be struck; then AQ and the angle QAH are known as before. Let $AQ = R$, $QAH = \iota$. Then by Art. 236 and 238;

$$R = \frac{2V^2}{g} . \frac{\sin. (\alpha - \iota) . \cos. \alpha}{\cos.^2 \iota} = \frac{2h}{\cos.^2 \iota} . \{\sin. (2\alpha - \iota) - \sin. \iota\};$$

$$\therefore \sin. (2\alpha - \iota) = \frac{R \cos.^2 \iota}{2h} + \sin. \iota.$$

When this is possible, it will necessarily give a value of $2\alpha - \iota$ less than $\frac{1}{2}\pi$; let this be θ: then since the sine of $\pi - \theta$ is the same as the sine of θ, the equation will also be satisfied if $\pi - \theta$ be the value of $2\alpha - \iota$. Let α', α'' be the two values of α; that is, let

$$2\alpha' - \iota = \theta; \; 2\alpha'' - \iota = \pi - \theta;$$

$$\text{or } \alpha' = \frac{\theta + \iota}{2}, \; \alpha'' = \frac{\pi - \theta + \iota}{2}.$$

Both these values are comprehended in the formula

$$\frac{1}{2}\left(\frac{\pi}{2} + \iota\right) \pm \frac{1}{2}\left(\frac{\pi}{2} - \theta\right).$$

If AI bisect the angle QAZ, $IAH = \frac{1}{2}\left(\frac{\pi}{2} + \iota\right)$. Hence, if, in fig. 154, TAH, tAH be the two values of α given by the formula, AT and At, which are the required directions of projection, make equal angles with AI.

We can easily find the limits within which this problem is possible. It is impossible if sin. $(2\alpha - \iota)$ be greater than 1; that is,

$$\text{if } \frac{R\cos.^2\iota}{2h} + \sin.\iota > 1;$$

which may happen either from R becoming too large, or V, and therefore h, becoming too small.

This problem might likewise have been solved by putting the known values of AN, NQ for x and y in the equation to the curve, Art. 240; by which means α, which determines the direction of projection, will be the only unknown quantity, and may be found.

It is easy also to obtain, from the properties of the parabola, geometrical constructions, which shall satisfy the question.

243. Prob. II. *A body is projected in a given direction with given velocity from the summit of a hill whose form is an upright paraboloid: to find where the projectile will strike it. Fig.* 155.

Let AQ be that section of the hill which is in the vertical plane of projection; AQ is a parabola; and if we refer the curve AQ to horizontal and vertical co-ordinates x, y, its equation will be, if b be the parameter,

$$y = -\frac{x^2}{b};$$

y being negative, because ordinates measured upwards are positive. At the point Q where the projectile strikes the hill, the parabola AQ and the curve of the projectile must have the same co-ordinates. Hence, equating this value of y with that of the ordinate to the curve APQ, Art. 240, we have

$$-\frac{x^2}{b} = x \tan. a - \frac{x^2}{4h \cos.^2 a};$$

$$\frac{x}{4h \cos.^2 a} - \frac{x}{b} = \tan. a;$$

$$x = \frac{4bh \sin. a . \cos. a}{b - 4h \cos.^2 a}$$

$$= \frac{2bh \sin. 2a}{b - 4h \cos.^2 a}.$$

If $4h \cos.^2 a = b$, or $2V^2 \cos.^2 a = bg$, x is infinite, and the projectile never meets the parabola. In this case, by Art. 239, the two parabolas have equal parameters, and are parallel. If b be less than this value, the parabolas diverge, and never meet.

In the same way we may find where a body, projected under given circumstances, meets any curve, given by an equation between its co-ordinates.

244. Prob. III. *A body is projected from a given point with a given velocity: to find the direction, that it may just touch a given plane. Fig.* 155.

Let Bb be the intersection of the given plane with the vertical plane of projection; Bb must necessarily be above A. Let $AB = b$, and the angle $ABb = \beta$. Hence the equation to the line Bb is

$y = (x+b)\,.\,\text{tan.}\,\beta$. Also for the path of the projectile,

$$y = x\,.\,\text{tan.}\,\alpha - \frac{x^2}{4h\,\text{cos.}^2\,\alpha}.$$

And since at the point P, where the projectile touches the plane, we must have the co-ordinates common, we have

$$(x+b)\,\text{tan.}\,\beta = x\,.\,\text{tan.}\,\alpha - \frac{x^2}{4h\,\text{cos.}^2\,\alpha}.$$

Also since the curve and the line *touch* at P, we must have $\frac{dy}{dx}$ the same for both. Now for the line,

$$\frac{dy}{dx} = \text{tan.}\,\beta;$$

$$\text{and for the curve, } \frac{dy}{dx} = \text{tan.}\,\alpha - \frac{x}{2h\,\text{cos.}^2\,\alpha}.$$

$$\text{Hence tan.}\,\beta = \text{tan.}\,\alpha - \frac{x}{2h\,\text{cos.}^2\,\alpha}.$$

From this equation, $x = 2h\cos^2\alpha\,(\text{tan.}\,\alpha - \text{tan.}\,\beta)$,

$$= \frac{2h\,.\,\text{cos.}\,\alpha\,.\,\text{sin.}\,(\alpha-\beta)}{\text{cos.}\,\beta},$$

which substituted in the former equation, or in

$$b\,.\,\text{tan.}\,\beta = x(\text{tan.}\,\alpha - \text{tan.}\,\beta) - \frac{x^2}{4h\,\text{cos.}^2\alpha} = \frac{x\,.\,\text{sin.}\,(\alpha-\beta)}{\text{cos.}\,\alpha\,\text{cos.}\,\beta} - \frac{x^2}{4h\,\text{cos.}^2\alpha},$$

$$\text{gives } b\,.\,\text{tan.}\,\beta = \frac{2h}{\text{cos.}^2\beta}\,.\,\text{sin.}^2\,(\alpha-\beta) - \frac{h}{\text{cos.}^2\beta}\,.\,\text{sin.}^2\,(\alpha-\beta)$$

$$= \frac{h}{\text{cos.}^2\beta}\,\text{sin.}^2\,(\alpha-\beta).$$

$$\text{Hence sin.}\,(\alpha-\beta) = \pm\frac{(b\,\text{sin.}\,\beta\,\text{cos.}\,\beta)^{\frac{1}{2}}}{h^{\frac{1}{2}}} = \pm\left\{\frac{b\,\text{sin.}\,2\beta}{2h}\right\}^{\frac{1}{2}}.$$

It appears from this, that there are *two* directions which answer the condition: and if Aa be drawn parallel to Bb, these directions make with Aa equal angles aAT, aAt.

In the same way it may be shewn how to make the path of the projectile touch any given curve.

245. PROB. IV. *Several bodies being projected in different directions with the same velocity from the same point A: to find the locus of them all at the end of a given time. Fig.* 156.

Let AR, AR', AR'' be any directions in which the bodies are projected. If $AR = AR'$, $= AR''$, be the space which the bodies would describe in any time with the uniform velocity of projection, and $RP = R'P' = R''P''$ the space through which a body would fall by gravity in the same time; it appears by what has been said, that P, P', P'' will be the places of the bodies at the end of this time.

Let AM be vertical and $= RP$. Hence $MP = AR$, $MP' = AR'$, $MP'' = AR''$. Therefore $MP = MP' = MP''$. Hence P, P', P'' are all in the circumference of a circle whose center is M and radius MP. The center of this circle descends according to the laws of a falling body, and the radius increases uniformly.

COR. If the projections take place in different vertical planes, the bodies will, at any time, be situated in the surface of a sphere.

246. PROB. V. *On the same supposition, to find the locus of the vertices of the parabolas described, fig.* 157.

Let V be the vertex of any one of the parabolas; $AU = p$, $VU = q$, vertical and horizontal co-ordinates. It is manifest that $UV = AG = \frac{1}{2}AH$, AH being the horizontal range; and $AU = GV$, the greatest height above the horizontal plane.

Hence, if h be the height due to the velocity of projection, a the angle which the direction of projection makes with the horizon, we have, by Art. 238;

$$q = h \,.\, \text{sin. } 2a = 2h \,.\, \text{sin. } a \,.\, \text{cos. } a;\ p = h \,.\, \text{sin.}^2\, a;$$

$$\therefore \text{sin.}^2\, a = \frac{p}{h};\ q^2 = 4h^2 \,.\, \text{sin.}^2\, a \,.\, \text{cos.}^2\, a$$

$$= 4h^2 \,.\, \text{sin.}^2\, a \,.\, (1 - \text{sin.}^2\, a)$$

$$= 4h^2 \,.\, \frac{p}{h} \,.\, \left(1 - \frac{p}{h}\right)$$

$$= 4 \,.\, (hp - p^2).$$

And if $k = \frac{h}{2}$; we shall have $h = 2k$, $4 = \frac{h^2}{k^2}$; whence

$$q^2 = \frac{h^2}{k^2}(2kp - p^2):$$

which is the equation to an ellipse whose minor axis is $2k = AC$, the height due to the velocity of projection; and whose major axis is $2h$, horizontal, and double the former.

247. Prob. VI. *Planes AQ, AQ', AQ'', being drawn in every direction from the point A, and bodies projected from A with a given velocity, at such angles that the ranges on each of these planes shall be the greatest: to find the locus of all the extreme points, Q, Q', Q''. Fig.* 157.

By Art. 283, if ι be the angle HAQ we have

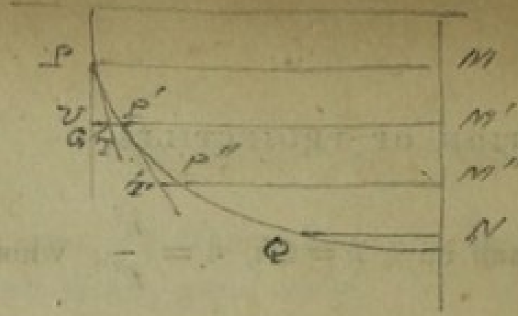
$$AQ = \frac{2h}{1 + \sin.\iota} = \frac{2h}{1 + \cos.\theta}, \text{ putting } CAQ = \theta.$$

And this is the equation to a parabola whose parameter is $4h$. Hence Q, Q', Q'' is a parabola with focus A.

Cor. 1. This parabola circumscribes all those described by projectiles from the point A with the velocity $V = \sqrt{(2gh)}$. For these parabolas will meet any point in this curve; and they will reach no point without it, as would be clear by joining that point with the point A.

Cor. 2. It has been seen that AR bisects the angle CAQ: hence the focus S, of AVQ, is in AQ; and hence the parabolas AVQ, CQ have a common tangent at Q, for the tangent must bisect the angle AQq, Qq being parallel to the axis. Hence the parabola $CQQ'Q''$ *touches* all those described by projectiles from A with the velocity V.

Cor. 3. If the bodies be projected in different vertical planes, their paths will all be circumscribed by a paraboloid, formed by the revolution of QA round CA.



CHAP. V.

MOTION UPON A CURVE.

248. When a body is compelled to move along a curve, it is acted on at every point by the re-action, which is perpendicular to the curve, and therefore to the direction of the body's motion. Hence this force neither accelerates nor retards the body. To determine the velocity, we must take the force in the direction of the curve, and consider the effect produced by it.

Prop. *If a body descend down any curve by the action of gravity, the velocity acquired at any point will be the same as if the body had descended down the same vertical space falling freely.*

In fig. 158, let a body which has any velocity at P descend down the curve PQ to Q. Let PM, QN be horizontal lines meeting a vertical line in M and N; and let a body, which has at M the same velocity as the body at P, fall to N. The velocity at N will be equal to the velocity at Q.

Let the arc PQ be divided into small portions PP', $P'P''$, &c. and let MM', $M'M''$, &c. be the corresponding portions of MN. Suppose the force which accelerates the body down the curve to be throughout PP' uniform, and equal to its value at P; uniform through $P'P''$ and equal to its value at P'; and so on. Then the velocity thus acquired at Q will approach to that acquired by the real action of the force down PQ, as the portions PP', &c. become smaller and more numerous.

Let PT be a tangent at P meeting $M'P'$ in T, and TG a perpendicular at T meeting the vertical line PG. If PG represent the force of gravity at P, this force may be resolved into PT, TG. Of these TG is counteracted by the re-action of the curve, and PT is the force which accelerates the body's motion along the curve at P. If g be the force of gravity, $g \cdot \frac{PT}{PG}$ will be the force at P. Also if MT meet PG in U, we shall have

$$PU : PT :: PT : PG;$$

therefore $g \cdot \frac{PU}{PT}$, or $g \cdot \frac{MM'}{PT}$, is the force at P, in the direction of the curve.

When the force acts uniformly in PP', $P'P''$, &c., let u be the velocity at P, u' at P', u'' at P'', &c. and v at Q. By Art. 220, Cor. 3, we have, if the force f act uniformly from P to P',

$$u'^2 - u^2 = 2f \cdot PP' = 2g \cdot \frac{PP' \cdot MM'}{PT};$$

$$\text{similarly, } u''^2 - u'^2 = 2f' \cdot P'P'' = 2g \cdot \frac{P'P'' \cdot M'M''}{P'T'}, \text{ \&c.}$$

And adding all these equations together, observing that v^2 is the last of the values u, u', u'', &c.

$$v^2 - u^2 = 2g \left\{ MM' \frac{PP'}{PT} + M'M' \frac{P'P''}{P'T'} + \text{\&c.} \right\}.$$

Also as the portions PP', P' P'', &c. are taken smaller and smaller, the velocity thus generated approaches to that actually acquired in the curve. And on the same supposition each of the fractions $\frac{PP'}{PT}$, $\frac{P'P''}{P'T'}$, &c. approaches to unity. Hence taking the limits, and considering v and u as the actual velocities at Q and P,

$$v^2 - u^2 = 2g \{ MM' + M'M'' + \text{\&c.} \} = 2g \cdot MN,$$

$$\text{and } v^2 = u^2 + 2g \cdot MN.$$

And this is (by Art. 220,) the value of the square of the velocity at N, on the supposition of the body falling from M, with the velocity u. Hence the velocities acquired down MN and down PQ are the same.

Cor. 1. If a body begin to move up the curve QP from Q with the same velocity with which another body is projected vertically upwards at N, they will have the same velocities when they arrive at P and M. For the velocity destroyed in ascending QP is equal to that generated in descending PQ, that is, to that acquired down MN, and therefore to that destroyed up NM.

Cor. 2. If a body begin to ascend a curve surface with a certain velocity, it will rise to the same height above a given horizontal line, whatever be the form of the curve.

For in each case it will ascend to the same height as if it had been projected vertically upwards.

249. Prop. *To find the time of falling down any arc of an inverted cycloid**.

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* If a circle EPF, fig. 160, 161, roll along a straight line CBc, a point P in the circumference of this circle will describe a curve which is called a *Cycloid*.

When the circle has made one complete revolution, the describing point which was in contact with the straight line at C, will return to it again at c, having described the curve $CPAc$.

If we bisect Cc in B, and draw BA at right angles to it, A will be the position of the describing point when the circle has made half a revolution; and the two branches AC, Ac will be equal and similar.

AB is called the axis of the cycloid; Cc its base; A its vertex; and the circle AQB is called the generating circle.

Prop. I. *Fig.* 160, *if an ordinate MQP be drawn perpendicular to the axis*, QP = *arc* QA.

Let EPF be the position of the generating circle at the time when the describing point is at P. Then the arc PF has been applied to CF, so that all the points of each have successively coincided: therefore the two are equal, that is, CF=arc PF=arc QB. For the same reason CB = semicircle AQB. Hence, taking away equals, FB=arc AQ. But evidently $PN = QM$, and therefore $PQ = NM = FB$. Hence PQ = arc AQ.

Prop. II. *Fig.* 160. *The tangent to the cycloid at the point P is parallel to the chord AQ.*

If

Let A, fig. 159, be the vertex of the cycloid, and L the point from which the body begins to fall: LH horizontal, meeting the axis in H; and PM horizontal meeting in q a semi-circle described on AH.

If the circle EPF be supposed for an instant to turn round a fixed point F, instead of rolling along FB, the motion of P will be ultimately in the same direction on either supposition. But on this supposition the motion of P will evidently be perpendicular to FP, or in the direction PE. Hence the direction of the curve CP at P is PE, and therefore PE is a tangent. And PE is parallel to QA: hence the tangent at P is parallel to QA.

PROP. III. *Fig.* 160. *The length of the arc of the cycloid AP, beginning from the vertex, is double of the chord of the circular arc AQ cut off by the same ordinate.*

Let $M'Q'P'$ be an ordinate very near to MQP. Let AQ meet $M'Q'$ in R, and draw at Q a tangent to the circle meeting $M'Q'$ in S, and meeting a tangent at A in T. Also draw SO perpendicular upon QR.

Since $TA = TQ$, angle $TAQ = TQA$. And $TAQ = QRS$, and $TQA = RQS$; therefore $QRS = RQS$ and $SQ = SR$. Hence the triangles SOQ and SOR are equal; $QO = RO$, and $QR = 2\,QO$.

Now when Q' approaches indefinitely near to Q, S approaches to Q', and OS coincides ultimately with a circular arc to radius AQ' and center A. Hence QO is ultimately the excess of AQ' above AQ or the quantity by which AQ is increased.

Also QR is parallel to the tangent at P, and hence QR is ultimately equal to PP', the quantity by which AP is increased.

Hence it appears, that, for corresponding points, AP is increased by a quantity twice as great as the increase of AQ; and, therefore, as AP and AQ begin together, AP will always be twice as great as the chord AQ.

COR. $AP = 2\,PE$.

PROP. IV. *To make a pendulum oscillate in a given cycloid.*

Let APC, fig. 161, be a given semi-cycloid, AB being its axis. Produce AB to S, making $SB = AB$: complete the rectangle $SBCD$, and with an axis CD, and base DS, describe a semi-cycloid CS.

Draw any line EFG parallel to ABS; and on opposite sides of this line describe the two semi-circles EPF, FOG, of the generating circles of the

The velocity at P acquired down LP, is equal to the veloeity acquired down HM. Hence

$$\text{velocity at } P = \sqrt{(2g \cdot HM)} = \sqrt{\left(2g \cdot \frac{Hq^2}{AH}\right)} = Hq\sqrt{\frac{2g}{AH}}.$$

Also if AQB be a semi-circle described on AB, $AP = 2$ chord AQ, (see Note)

$$\therefore AP = 2\sqrt{(AB \cdot AM)}$$

$$= 2\sqrt{\left(\frac{AB}{AH} \cdot AH \cdot AM\right)} = 2\sqrt{\left(\frac{AB}{AH} \cdot Aq^2\right)}$$

$$= 2Aq\sqrt{\frac{AB}{AH}}.$$

And similarly, if $P'M'$ be near and parallel to PM,

$$AP' = 2Aq'\sqrt{\frac{AB}{AH}}.$$

$$\text{Therefore } PP' = 2(Aq - Aq')\sqrt{\frac{AB}{AH}}.$$

Hence, if PP' be described uniformly with the velocity at P,

$$\text{time in } PP' = \frac{PP'}{\text{vel. at } P} = \frac{Aq - Aq'}{Hq}\sqrt{\frac{2AB}{g}}.$$

---

the cycloids AC, CS. Join OF, FP. Then arc $FP = FC$, and arc $FPE = BFC$; therefore arc $PE = BF$, and $BF = SG =$ arc GO. Hence $PE = GO$; and therefore the angles EFP, GFO are equal. Hence OFP is a straight line. Hence also $OF = FP$; therefore $OP = 2\,OF =$ arc OC, by Cor. to Prop. III. And by Prop. II, OP is a tangent to the cycloid at O.

Hence it appears, that if a string SOC, fixed at S, and wrapped along the semi-cycloid SOC, be unwrapped, beginning at C, its extremity will describe a semi-cycloid CPA. And if an equal and similar semi-cycloid Sc be placed with its base Sd in the same line with DS, the same string fixed at S and wrapping upon the semi-cycloid Sc, will, with its extremity, describe the semi-cycloid Ac, thus completing the cycloid CAc. Hence a body P, suspended by a string SOP between two such semi-cycloids in a vertical plane, will oscillate in an inverted cycloid.

And if we take the times, supposing PP' indefinitely diminished, the sum of all such intervals from L to P will approximate to the actual time of describing LP.

Join Hq' meeting Aq in o. And since AoH approximates to AqH, a right angle, oq approximates to $Aq - Aq'$. Hence taking the limit, we have

$$\text{time in } PP' = \frac{oq}{Hq}\sqrt{\frac{2AB}{g}}$$

$$= \text{angle } qHo \sqrt{\frac{2AB}{g}};$$

$$\text{and angle } qHo = qHq' = \tfrac{1}{2} qCq';$$

$$\therefore \text{ time in } PP' = \tfrac{1}{2} qCq' \sqrt{\frac{2AB}{g}}.$$

And the whole time in LP will be found if we take the sum of all such intervals from L to P. Now the sum of all the angles qCq' is evidently HCq. Hence

$$\text{time in } LP = \tfrac{1}{2} HCq \sqrt{\frac{2AB}{g}}.$$

Cor. 1. To find the time of descending through the *whole* arc LA, we must put for the angle HCq its value for that case, which is two right angles. Hence we have

$$\text{time in } LA = \frac{\pi}{2}\sqrt{\frac{2AB}{g}}.$$

Cor. 2. If we suppose the body, after coming to the vertex A, to go on and to ascend the opposite semi-cycloid, (Ac, fig. 161.) it will ascend to a point l, having described an arc Al equal to AL. And the time of ascending through Al will be equal to that of descending through LA. Hence we shall have

$$\text{time in } LAl = 2 \text{ time in } LA = \pi\sqrt{\frac{2AB}{g}}.$$

Cor. 3. Since the time of descending down LA is independent of the position of the point L, it appears that the times down all arcs LA are the same, whatever be their magnitude. Hence the curve CA is said to be *isochronous.*

250. Prop. *When a body oscillates in a cycloid; to determine the time of oscillation.*

If two equal inverted semi-cycloids SC, Sc, fig. 161, be placed in contact at S, in the same vertical plane, and if a string SOP equal in length to either of them, be suspended from S and oscillate between them, its extremity P will describe a cycloid CAc. And if a body be suspended by this string, it will move in the same manner as if it moved upon a curve PAp. After descending down LA, it will, with the velocity acquired, ascend up Al, describing an arc equal and similar to AL. And after coming to l it will again descend through lA and ascend through AL, and so on continually.

Let l be the length of the pendulum SA. Then $l = 2AB$; and we shall have the time of oscillation $= \pi\sqrt{\dfrac{l}{g}}$.

Cor. 1. Since $\sqrt{\dfrac{l}{g}} = \sqrt{\dfrac{2 \times \frac{1}{2}l}{g}} =$ time of falling down $\frac{1}{2}l$, (Art. 219,) we shall have time of oscillation $= \pi \times$ time of falling down $\frac{1}{2}$ pendulum.

Cor. 2. For very small distances from the point A, the cycloid will very nearly coincide with the circle whose center is S. Hence the motion of the cycloidal pendulum will very nearly coincide with the motion of a body suspended by a string SA and oscillating *freely* through very small arcs. We shall suppose the times of oscillation of these two pendulums to be equal.

Hence we may determine all the circumstances of the small oscillations of pendulums in circular arcs from the expression above,

$$t = \pi\sqrt{\frac{l}{g}},$$

where t is the time of oscillation.

Cor. 3. It is manifest that t varies as the root of l, when g is constant.

Also that t varies inversely as the root of g, when l remains the same. And that g varies as l when t remains the same.

Hence if L be the length of the pendulum which oscillates seconds, and t the time in seconds of the oscillation of a pendulum whose length is l,

$$t = \sqrt{\frac{l}{L}}; \text{ hence } l = Lt^2.$$

The value of L, the length of the seconds pendulum in the latitude of London, (*in vacuo,*) is found by experiment to be 39.1386 inches.

From this value of L we can find the value of g; for making $t = 1$, we have

$$1 = \pi\sqrt{\frac{L}{g}}; \ \therefore g = \pi^2 L = 386.28 \text{ inches.}$$

Ex. 1. To find the time of oscillation of a pendulum 20 feet long,

$$t = \sqrt{\frac{240}{39.1386}} = 2.5'', \text{ nearly.}$$

Ex. 2. To find the length of a pendulum which shall make its oscillations in half minutes,

$$l = L.(30)^2 = 39.1386 \times 900 \text{ inches} = 978.4 \text{ yards.}$$

251. Prop. *If a pendulum be slightly altered in length, to find the number of oscillations gained or lost in a day.*

If n be the daily number of oscillations of the pendulum in the latitude of London, (*in vacuo,*) and $N = 24 \times 60 \times 60 = 86400$, the number of seconds in 24 hours; we have

$$t = \frac{N}{n}; \ \therefore \frac{N}{n} = \sqrt{\frac{l}{L}}; \ \therefore l = \frac{LN^2}{n^2}.$$

If n and l be nearly equal to N and L, we may obtain approximations for the differences.

Suppose the length of the pendulum L to be increased by a small quantity p: to find q, the number of seconds it will lose in a day.

Here $L+p=\frac{LN^2}{(N-q)^2}=L\left(1+\frac{2q}{N}\right)$; omitting powers of $\frac{q}{N}$;

$$\therefore p=\frac{2qL}{N}; \text{ and } q=\frac{pN}{2L}.$$

The same formula will apply when L is diminished, and consequently N is increased.

Ex. A seconds pendulum is lengthened $\frac{1}{100}$ of an inch: to find the number of seconds it will lose per day.

Here $p=.01$; $\therefore q=\frac{.01\times 86400}{2\times 39.13}=\frac{43200}{3913}=11''$ nearly.

252. Prop. *If the force of gravity be slightly altered, to find the number of seconds gained or lost in a day by a seconds pendulum.*

Let G be the value of g at a given place; if l remains the same, t varies inversely as the root of g; hence

$$\frac{t}{1}=\frac{\sqrt{G}}{\sqrt{g}};\ g=\frac{G}{t^2}=\frac{Gn^2}{N^2}.$$

Hence if a seconds pendulum is taken to a place where the gravity is greater, n will be greater than N, and the pendulum will gain, and *vice versâ*. The increase of gravity is generally small, and hence we may approximate as before. Let $g=G(1+h)$, and let the pendulum gain q seconds a day;

$$\therefore G(1+h)=\frac{G(N+q)^2}{N^2}=\frac{G(N^2+2qN)}{N^2}, \text{ omitting } q^2;$$

$$\therefore h=\frac{2q}{N}.$$

Ex. 1. A pendulum which would oscillate seconds at the equator, would, if carried to the pole, gain 5 minutes a day: to find the proportion of the polar and equatoreal gravity,

$$h=\frac{2\times 300''}{86400}=\frac{1}{144};$$

hence gravity at equator : gravity at pole :: 144 : 145.

Ex. 2. A pendulum which oscillates seconds, is carried to the top of a mountain whose height is m: to find the number of seconds which it will lose per day; gravity being supposed to vary inversely as the square of the distance from the center.

Let r be the distance from the center of the earth to the first station, and G the gravity there. Therefore $r+m$ is the distance of the second station, and gravity is

$$\frac{Gr^2}{(r+m)^2} = G\left(1 - \frac{2m}{r}\right), \text{ omitting } \frac{m^2}{r^2}, \text{ \&c.}$$

Hence, putting $\frac{2m}{r}$ for h in the formula, which will be the same for the diminution as for the increase of gravity,

$$\frac{2m}{r} = \frac{2q}{N}, \text{ and } q = \frac{Nm}{r}.$$

If the radius of the earth be 4000 miles, and the height of the mountain 1 mile,

$$q = \frac{86400}{4000} = 21''.6, \text{ the number of seconds lost per day.}$$

APPENDIX.

CHAP. I.

ON THE MATHEMATICAL FORMS OF ARCHES.

1. In Art. 70, we have found the *extrados* of an arch upon the supposition that the *intrados* is a circular arc. By means of the differential calculus, we can deduce general formulæ by which we may, in any case, find the extrados from the intrados, and *vice versâ*.

The equilibrium of an arch will require different considerations according to the difference of the forces which we suppose to act upon the voussoirs. These may be, 1st, The *weights* of the voussoirs themselves alone. 2d, Forces acting *vertically* upon the voussoirs. 3d, Forces acting upon the voussoirs *in any direction;* as, for instance, in the direction of the joints.

The case of actual practice may, in some measure, belong to any of these three hypotheses according to circumstances. If the arch consist of the voussoirs alone; or of distinct members, (as, for instance, frames of wood or iron,) which may be considered as voussoirs; or of concentric rings of voussoirs with their joints diverging in nearly the same directions; its equilibrium is to be calculated on the *first* supposition. If each of the voussoirs have above it a mass which may be supposed to press vertically upon it, as, for instance, if squared stones rest upon the horizontal tops of the voussoirs, as in fig. 164; we must take the *second* hypothesis. If the part above the voussoirs be filled with a fluid; or with loose materials, which will act like a fluid; or if pressure be communicated from adjoining parts by means of walls or small arches resting on the voussoirs; the pressure may be perpendicular to the arch, and the equilibrium will belong to the *third* mentioned case.

In all cases it is to be observed, that the consequence of fulfilling the conditions of equilibrium which we shall obtain, is, that the voussoirs will not have any tendency to *slide past* each other; and that it requires other considerations to ascertain that the line of pressure does not fall without the proper limits, which is the way in which the stability of the arch would be more frequently endangered. See Art. 69.

1. *The Equilibrium pressed by the Weights of the Voussoirs.*

2. The arch is supposed to be bounded by the plane of the figure, (fig. 91,) and by another plane parallel to this: the materials are supposed to be of uniform density, and hence the weight of any portion, as that contained between the planes PQ and $P'Q'$, is proportional to the area $PQQ'P'$, which corresponds to it.

The intrados and extrados are supposed to be continuous curves, and of such a nature that the joints may be taken at any point whatever of them. This leads to the same considerations as if the arch were supposed to consist of voussoirs indefinitely thin. If the equilibrium subsist on this supposition, it will manifestly still exist, if we suppose any number of these small voussoirs to cohere and become one larger one.

In general also the joints are supposed to be perpendicular to the intrados.

Prop. *Given the extrados, to find the length of the voussoirs.*

In Art. 66, it is shewn that the requisite condition, in order that the equilibrium of any voussoirs may subsist, is, that the weight of each voussoir be as the difference of the tangents of the angles which the joints make with the vertical.

Let DP, EQ, fig. 91, and fig. 163, be the intrados and extrados of an arch: PQ a joint, and $P'Q'$ another indefinitely near it, drawn so that they meet in O. Let the angles which PQ and $P'Q'$ make with the vertical be θ and θ': $OP = r$, $OQ = R$, $PQ = v$. We may consider the angle POP' as $d\theta$: hence we have

$$\text{area } PQQ'P' = \text{tri. } OQQ' - \text{tri. } OPP'$$
$$= \tfrac{1}{2}\, OQ^2 \,.\, d\theta - \tfrac{1}{2}\, OP^2 \,.\, d\theta$$
$$= \tfrac{1}{2}\,\{(r+v)^2 - r^2\}\, d\theta$$
$$= \tfrac{1}{2}\,\{2\,rv + v^2\}\, d\theta.$$

Also the difference of the tangents, tan. θ' — tan. θ, is d . tan. θ, since $\theta' - \theta$ is $d\theta$. And hence, by Art. 66, $\frac{1}{2}\{2\,rv + v^2\}\,d\theta$ is proportional to d . tan. θ. Therefore, c being a constant quantity, we shall have

$$\{2\,rv + v^2\}\, d\theta = c^2 \,.\, d \,.\, \text{tan. } \theta = \frac{c^2\, d\theta}{\text{cos.}^2\, \theta};$$

$$\therefore v^2 + 2\,rv = \frac{c^2}{\text{cos.}^2\, \theta};$$

$$(v + r)^2 = \frac{c^2}{\text{cos.}^2\, \theta} + r^2,$$

$$R = \sqrt{\left\{\frac{c^2}{\text{cos.}^2\, \theta} + r^2\right\}};$$

$$v = R - r = \sqrt{\left\{\frac{c^2}{\text{cos.}^2\, \theta} + r^2\right\}} - r.$$

If the intrados be given, and the position of the joints, r is known at every point, and therefore v, the length of the voussoir; and hence the form of the extrados.

Cor. 1. If the extrados be given to find the intrados, we have

$$r = \sqrt{\left\{R^2 - \frac{c^2}{\text{cos.}^2\, \theta}\right\}};$$

whence r is known; from which the intrados may be found.

Cor. 2. By giving different values to c, we obtain different curves of extrados for the same intrados. If k be the depth of the key-stone DE, let l be the value of OP at D, and we have, since $\theta = 0$ at that point,

$$k = \sqrt{(c^2 + l^2)} - l;$$
$$\therefore c = \sqrt{(k^2 + 2\,kl)}.$$

3. PROP. *If the joints all pass through the same given point when produced; to find the extrados.*

Let the given point be made the origin of co-ordinates; and let x be a horizontal abscissa, and y a vertical ordinate to the point P; x', y' corresponding co-ordinates to the point Q. Then

$$r = \sqrt{(x^2 + y^2)}, \text{ cos. } \theta = \frac{y}{r};$$

and hence, by last Proposition,

$$= R\sqrt{\left\{\frac{c^2 r^2}{y^2} + r^2\right\}}$$

$$= \frac{r}{y}\sqrt{(c^2 + y^2)}.$$

Also we have manifestly

$$x' = \frac{R}{r}x, \; y' = \frac{R}{r}y;$$

$$\text{therefore } y' = \sqrt{(c^2 + y^2)};$$

$$x' = \frac{x}{y}\sqrt{(c^2 + y^2)} = \frac{\sqrt{\{(r^2 - y^2)(c^2 + y^2)\}}}{y}.$$

And eliminating y, we have the equation between x' and y'.

PROB. I. *The intrados being a circle, and the joints being in the direction of radii, to find the extrados.*

In this case $r^2 = x^2 + y^2$ is constant,

$$y = \sqrt{(y'^2 - c^2)}; \; \sqrt{(r^2 - y^2)} = \sqrt{(r^2 + c^2 - y'^2)};$$

$$\therefore x' = \frac{y'\sqrt{(r^2 + c^2 - y'^2)}}{\sqrt{(y'^2 - c^2)}};$$

which is the equation to the curve.

It might easily be shewn that this agrees with the construction, Art. 72.

PROB. II. *The intrados of an arch being composed of two equal circular arcs, ABa, fig. 162, and the joints being perpen-*

dicular to the curve, to find the extrados, that there may be an equilibrium.

Let O be the center of AB; then all the joints of AB pass through O. Let ECQ be the extrados, (as in Prob. I,) supposing the curve complete to the vertex D. Let BC be the joint passing through B; then, if we suppose the part $BCED$ to be removed, and a pressure to be exerted on BC equal to that which $BCED$ exerted, the part $ABCQ$ will still remain in equilibrium. In the same manner, if ecq be the extrados of dBa, (de being equal to DE) $aBcq$ will be in equilibrium, if the pressure upon BC be equal to the pressure exerted by $Bced$.

Hence the arch ABa, $QCcq$ will be in equilibrium if we introduce at the vertex B a voussoir VB, which shall exert on BC and Bc the same pressures which the complete arches would have exerted. And these pressures are equal.

By Cor. 4, Art. 66, if H be the horizontal pressure which would exist at DE, and θ the angle which BC makes with the vertical, H sec. θ is the pressure on BC. Now if W be the weight of the wedge VB, $\frac{\frac{1}{2}W}{\sin.\theta}$ is the pressure on each of the surfaces BC, Bc. (Art. 39.) Hence we must have

$$\frac{\frac{1}{2}W}{\sin.\theta} = H\sec.\theta;$$

$$\text{or } W = 2H\tan.\theta.$$

4. Prop. *If the joints be all perpendicular to the intrados, to find the extrados, and vice versâ.*

In this case OP will be the radius of curvature. And, in fig. 163, let AM, horizontal, $=x$, MP, vertical, $=y$; and we shall have

$$r = -\frac{ds^3}{dx\,d^2y};\ \cos.\theta = \frac{dx}{ds};\ \sin.\theta = -\frac{dy}{ds}.$$

Hence we know

$$R = \sqrt{\left(\frac{c^2 ds^2}{dx^2} + r^2\right)};$$

$$\text{and } v = R - r = \sqrt{\left(\frac{c^2 ds^2}{dx^2} + r^2\right)} - r.$$

If x', y' be the co-ordinates of Q, we have

$$x' = x + v \sin.\theta = x - \sqrt{\left(\frac{c^2 dy^2}{dx^2} + \frac{r^2 dy^2}{ds^2}\right)} + \frac{r\,dy}{ds},$$

$$y' = y + v \cos.\theta = y + \sqrt{\left(c^2 + \frac{r^2 dx^2}{ds^2}\right)} - \frac{r\,dx}{ds}.$$

and eliminating x and y, we have an equation between x' and y'.

Prob. III. *The intrados being a cycloid with its axis vertical, it is required to find the lengths of the voussoirs.*

In this case, making the lower end of the axis the origin, we have

$$\frac{dx}{dy} = -\frac{\sqrt{y}}{\sqrt{(2a-y)}};\ \frac{ds}{dx} = \frac{\sqrt{2a}}{\sqrt{y}};\ r = 2\sqrt{(2ay)}.$$

$$\text{Hence } R = \sqrt{\left(\frac{2ac^2}{y} + 8ay\right)} = \sqrt{\left\{\frac{2a}{y}(c^2 + 4y^2)\right\}};$$

$$v = R - r = \sqrt{\frac{2a}{y}}\left\{\sqrt{(c^2 + 4y^2)} - 2y\right\}.$$

When $y = 2a$, that is, at the vertex, $v = k$;

$$\therefore k = \sqrt{(c^2 + 16a^2)} - 4a;\ \therefore c^2 = k^2 + 8ak.$$

When $y = 0$, PQ is infinite; hence AB is an asymptote to the curve.

Prob. IV. *To find the intrados so that the extrados may be a horizontal straight line.*

Let the height of the horizontal line above the origin be b; then we shall have

$$b = y + v\cos.\theta = y + \sqrt{\left(c^2 + \frac{r^2 dx^2}{ds^2}\right)} - \frac{r\,dx}{ds};$$

$$\therefore \left(b - y + \frac{r\,dx}{ds}\right)^2 = c^2 + \frac{r^2 dx^2}{ds^2};$$

$$\therefore (b-y)^2 + 2(b-y)\frac{r\,dx}{ds} = c^2,$$

$$\frac{r\,dx}{ds} = \frac{c^2-(b-y)^2}{2(b-y)};$$ and putting for r its value $-\dfrac{ds^3}{dx\,d^2y}$;

$$\frac{d^2y}{ds^2} = -\frac{2(b-y)}{c^2-(b-y)^2};$$ and multiplying by dy,

$$\frac{dy\,d^2y}{dx^2+dy^2} = -\frac{2(b-y)\,dy}{c^2-(b-y)^2};$$

$$\therefore \mathrm{l}\,\frac{\sqrt{(dx^2+dy^2)}}{dx} = \mathrm{l}\,\frac{c^2-k^2}{c^2-(b-y)^2};$$ the integrals being taken, so that at D, where $b-y = DE = k$, we have $dy = 0$;

$$\therefore \frac{dy^2}{dx^2} = \frac{(c^2-k^2)^2 - \{c^2-(b-y)^2\}^2}{\{c^2-(b-y)^2\}^2}.$$

The curve is perpendicular to the vertical axis in D, when $y = b-k$. As y decreases, the angle which the curve makes with the horizon becomes greater, and when $c = b-y$, or $y = b-c$, it is vertical. Here the arch must end.

2. *The Equilibrium preserved by vertical Pressures upon the Voussoirs.*

5. If, in fig. 164, we suppose the mass perpendicularly over each voussoir not to be affected by lateral pressure from the adjacent parts, but to be wholly employed in pressing vertically upon the voussoir, it is manifest that this pressure may be considered simply as an addition of weight to the voussoir. Hence, the pressures on the voussoirs, including in each its own weight, must be subject to the conditions of the weights in the last case; that is, they must be as the differences of the tangents of the angles of the joints.

We may conceive the vertical pressure to be produced by the weight of the part of the uniform wall which is above the voussoir. Thus the voussoir $PQQ'P'$ is supposed to be pressed vertically by the column $QSS'Q'$

6. Prop. *To find the height of the wall above the intrados that it may preserve equilibrium by its vertical pressure.*

We shall suppose the length PQ of the voussoir to be small compared with QS. In this case the weight of the column $PQSS'Q'P'$ will be nearly equal to that of $PRR'P'$. Therefore when the voussoirs are supposed to be very small, the weight of this column will be nearly as $PR \times MM'$.

Let, as before, $AM = x$, $MP = y$, $PR = z$; hence the pressure on P is as $PR \, . \, MM'$, or as zdx.

Now if θ be the angle which PQ makes with AD, we have by Art. 66, the pressure or weight on P, as $d \, . \tan. \theta$; hence, c^2 being a certain constant quantity,

$$zdx = c^2 \, . \, d \, . \tan. \theta.$$

If we suppose, as before, the joints to be perpendicular to the intrados, we have

$$\tan. \theta = -\frac{dy}{dx}; \;\therefore d \, . \tan. \theta = -\frac{d^2 y}{dx};$$

$$\therefore zdx = -\frac{c^2 d^2 y}{dx}, \quad z = -\frac{c^2 d^2 y}{dx^2}.$$

Prob. V. *The intrados being an ellipse with its major axis horizontal, it is required to find the height of the wall over each point of the arch.*

If a, b, be the major and minor semi-axes, we have

$$y = \frac{b}{a} \sqrt{(a^2 - x^2)}; \; \frac{dy}{dx} = -\frac{b}{a} \cdot \frac{x}{\sqrt{(a^2 - x^2)}};$$

$$\frac{d^2 y}{dx^2} = -\frac{ab}{(a^2 - x^2)^{\frac{3}{2}}} = -\frac{b^4}{a^2 y^3}.$$

$$\text{Hence } z = \frac{b^4 c^2}{a^2 y^3}.$$

Hence the height PR is inversely as the cube of MP, and is therefore infinite for the point B at the extremity of the major axis. No finite height at B could preserve the equilibrium.

When $y = b$, $z = DE = k$; $\therefore\ k = \dfrac{bc^2}{a^2}$, and $c^2 = \dfrac{a^2 k}{b}$.

To find the locus of R, let $MR = y'$; $\therefore\ y' = y + z = y + \dfrac{b^4 c^2}{a^2 y^3}$:

and by substituting for y its value in x, we should have an equation between y' and x.

We have $\dfrac{dy'}{dx} = \dfrac{dy'}{dy} \cdot \dfrac{dy}{dx} = -\left(1 - \dfrac{3b^4c^2}{a^2y^4}\right) \dfrac{bx}{a\sqrt{(a^2 - x^2)}}$.

Hence when $x = 0$, $\dfrac{dy'}{dx} = 0$, and the curve is perpendicular to AE at E. It then falls below the horizontal line. To find the point where it is again horizontal, we have

$$1 - \frac{3^2 b^4 c}{a^2 y^4} = 0; \ y^4 = \frac{3b^4c^2}{a^2} = 3b^3k.$$

If $k = \frac{1}{3}b$, this gives $y = b$; if k have this or a greater value, the curve ER does not fall below the horizontal line.

If the intrados be an ellipse with the *minor* axis horizontal, a and b will change places; and in every other respect the result will be similar to the preceding.

If $b = a$ the ellipse becomes a circle. In this case $z = \dfrac{a^2c^2}{y^3}$;

$$y' = y + \frac{a^2c^2}{y^3} = \frac{y^4 + a^2c^2}{y^3} = \frac{(a^2 - x^2)^2 + a^2c^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

PROB. VI. *To find the intrados so that the extrados bounding the wall which presses vertically on the voussoirs may be a horizontal line.*

Let b be the height of this horizontal line above the origin;

$$\therefore\ z = b - y;\ b - y = -c^2 \cdot \frac{d^2y}{dx^2};$$

$$-(b - y)\, dy = c^2 \cdot \frac{dy\, d^2y}{dx^2};$$

$$(b-y)^2 - k^2 = c^2 \cdot \frac{dy^2}{dx^2};$$

k being the value of DE or $b-y$ at D, where $\frac{dy}{dx} = 0$.

$$\text{Hence } \frac{dx}{c} = \frac{dy}{\sqrt{\{(b-y)^2 - k^2\}}};$$

$$\frac{x}{c} = \text{l}\,\frac{b-y-\sqrt{\{(b-y)^2-k^2\}}}{k}; \text{ for when } x=0,\ b-y=k.$$

Hence we may obtain, ϵ being the base of Napierian logarithms,

$$b-y = \frac{k}{2} \cdot \left\{\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right\}.$$

$$\frac{dy}{dx} = -\frac{k}{2c} \cdot \left\{\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}\right\}.$$

Therefore the curve has its concavity towards A, and tends perpetually to become perpendicular to the horizon as we recede from the point D.

7. Prop. *To compare the vertical pressures on the voussoirs when their angles are equal.*

Since we may suppose these angles indefinitely small, we shall here have to make $d\theta$ constant. Now on any voussoir

$$\text{pressure} = c^2 d \,.\, \tan.\, \theta = \frac{c^2 d\theta}{\cos.^2 \theta} = c^2 d\theta \,.\, \frac{ds^2}{dx^2}.$$

Hence the pressure on each is as the value of $\frac{ds^2}{dx^2}$ at the point. And if the voussoirs be of finite magnitude, they will be nearly as the value of this quantity for the middle point of the arc occupied by each.

Prob. VII. *A cycloidal arch is formed of voussoirs whose angles are equal: to compare the vertical pressures on them.*

In this case making a the radius of the generating circle, and measuring y vertical and x horizontal from the middle of the base of the cycloid;

$$\frac{dx}{dy} = -\sqrt{\frac{y}{2a-y}};\ \frac{ds}{dy} = \sqrt{\frac{2a}{2a-y}};$$

$\therefore \frac{ds^2}{dx^2} = \frac{2a}{y}$; hence the pressure is inversely as the height above the base.

8. Prop. *To compare the vertical pressures on the voussoirs when their breadths are all equal.*

We must here make ds constant and find how $d\,.\,\tan.\,\theta$ varies. Now

$$d\,.\,\tan.\,\theta = d\left(-\frac{dy}{dx}\right) = -\frac{d\left(\frac{dy}{dx}\right)}{ds}\,.\,ds.$$

Hence the pressures on the voussoirs are $-c^2\,.\,d\frac{\left(\frac{dy}{dx}\right)}{ds}$; and if in the differentiation of this expression we consider dx as constant, the pressure will be $-\frac{c^2 d^2 y}{dx\,ds}$, and will vary as $-\frac{d^2 y}{dx\,ds}$.

Prob. VIII. *An elliptical arch is composed of voussoirs of equal breadth: to find the law of the pressure.*

Supposing $2a$, the major axis, to be horizontal, and b, the minor semi-axis, to be vertical, we have, measuring from the center,

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)};\ \frac{dy}{dx} = -\frac{b}{a}\,.\,\frac{x}{\sqrt{(a^2 - x^2)}},$$

$$\frac{ds}{dx} = \frac{\sqrt{(a^2 - e^2 x^2)}}{\sqrt{(a^2 - x^2)}};\ \frac{d^2 y}{dx^2} = -\frac{ab}{(a^2 - x^2)^{\frac{3}{2}}};$$

hence

$$-\frac{c^2 d^2 y}{dx\,ds} = \frac{abc^2}{(a^2 - x^2)\sqrt{(a^2 - e^2 x^2)}} = \frac{b^4 c^2}{ay^2\sqrt{(b^4 + a^2 e^2 y^2)}}.$$

Making $y = b$, we have the pressure on the middle of the arch

$$= \frac{bc^2}{a^2} = k.$$

Hence if v be the pressure on any voussoir,

$$v = \frac{ab^3k}{y^2\sqrt{(b^4 + a^2e^2y^2)}}.$$

PROB. IX. *The intrados being an inverted catenary with the axis vertical, it is required to find the pressures on the voussoirs, their breadths being supposed equal.*

Let $AD = b$, $PM = y$, $DN = b - y$, which corresponds to x in the equations to the catenary, p. 167: and $AM = x$, which corresponds to y in the same equations. We have therefore instead of the equation in Art. 114, Cor.

$$-\frac{dy}{dx} = \frac{a \cos. \alpha + s}{a \sin. \alpha}.$$

Differentiating, supposing dx constant, and dividing by ds,

$$-\frac{d^2y}{dxds} = \frac{1}{a \sin. \alpha}.$$

Hence the pressure on each of the voussoirs is the same, as it will be seen that it should be.

9. If in the expression (3), p. 177, for the thickness of a string or chain which hangs in a curve of which the ordinates are x and y, we put, as in last Problem, $b - y$ for x, and x for y, it becomes $-\frac{mad^2y}{dxds}$; which agrees, so far as the variable part is concerned, with $-\frac{c^2d^2y}{dxds}$, the presssure on the voussoir in last Article.

It is easily seen that these problems ought to give the same result. For when a chain, hanging between two points, has formed itself into any curve; if we suppose it to be inverted, retaining its form, so that the lines which before ran vertically upwards shall now pass vertically down, and the curve rest upon the two points with its convexity upwards; it is manifest that, mathematically speaking, it will support itself. For the forces on each point are the same and in the same lines as before, but in opposite directions. Instead of the weights of the links acting towards the convexity of the curve, we have the same weights acting towards the concavity; instead of the tensions pulling each point, we shall have the pressures of the contiguous links pushing the point. And as these forces are in the same lines as before, they will have the same proportion, and preserve the equilibrium under the same circumstances. Now the case of such a chain agrees exactly with that of an arch whose joints are perpendicular to the curve. For the tension or pressure in the direction of the curve is the same as the pressure perpendicular to the joints. Hence in order to preserve the equilibrium of the arch, the weights on the voussoirs and their breadths must be as the weights and lengths of the links; and if the breadths of the voussoirs be equal, and the lengths of the links equal, the pressures on the voussoirs will be as the thickness of the chain, as above mentioned. Hence it appears that if the curve be the catenary, the pressures on the voussoirs will be equal, as it was found in Prob. IX. they should be.

3. *The Equilibrium preserved by Pressures in the directions of the Joints.*

10. Prop. *When an arch is kept at rest by pressures in the directions of the joints, to find the pressures on the voussoirs.*

If, in fig. 90, we suppose the voussoirs to be acted on by forces perpendicular to the lines BC, CC_1, C_1C_2, &c. we may find, as in Art. 66, the proportion of the forces which act upon them. If we draw XT, TV_1, V_1V_2, &c. perpendicular to the forces which act upon C, C_1, C_2, &c. we shall have triangles OXT, OTV_1, OV_1V_2, &c. whose sides are as the forces and pressures which act on C, C_1, C_2, &c. On the present supposition, XT, TV_1, V_1V_2, &c. will be perpendicular to OX, OT, OV_1, &c. respectively. And XT, TV_1, V_1V_2, &c. will be as the forces which act on C, C_1, C_2, &c.

When we suppose the voussoirs to become indefinitely thin, the polygon XTV_1V_2, &c. becomes a curve; which is parallel to the intrados at every point. If the joints be perpendicular to the intrados, OA, OT, OV_1, OV_2, &c. will all be perpendicular to the curve, which will therefore be a circle with center O.

Hence when the angles XOT, TOV_1, V_1OV_2, become indefinitely small so that each may be represented by $d\theta$, the forces on the voussoirs, which are as the lines XT, TV_1, V_1V_2, &c. are as these angles, and therefore as $d\theta$.

If the angles of the voussoirs be the same, $d\theta$ is constant; and hence the pressure on each voussoir, perpendicular to the intrados, will be the same.

If the breadths of the voussoirs be the same, we have, taking horizontal and vertical co-ordinates x, y, as before,

$$\frac{d\theta}{ds} = \frac{d\,.\,\text{arc}\left(\tan. = -\dfrac{dy}{dx}\right)}{ds} = \frac{-\,dx\,d^2y}{ds\,(dx^2+dy^2)} = -\,\frac{dx\,d^2y}{ds^3}.$$

And as ds is the same throughout, this is as the force on each voussoir.

This agrees with the expression, p. 192, for the form of a chain to which a force acts perpendicular. They ought to coincide, as appears by the reasoning in last Article.

PROB. X. *To find the form of the arch that the equilibrium may be preserved by the pressure of a fluid (whose surface is horizontal) which is supported by the voussoirs.*

The voussoirs are supposed very small, so that their length PQ may be neglected. Let PP', fig. 164, be an indefinitely small portion of the arch. The force upon this is the pressure of column PY of fluid extending to the horizontal surface EY. This pressure is, by hydrostatical principles, equal to the weight of a column of height PY and breadth PP'; and is perpendicular to DP at P.

Let $EN=x$, $NP=y$, be vertical and horizontal co-ordinates to P; $CP=s$. The pressure on ds is as $x\,ds$. Hence by the preceding reasoning $x\,ds$ is as $d\theta$. Let $x\,ds=c^2d\theta$; therefore

$$x = c^2\,.\,\frac{d\theta}{ds} = c^2\,.\,\frac{d\,.\,\text{arc}\left(\tan. = \dfrac{dx}{dy}\right)}{ds};$$

$$x = -\,\frac{c^2\,dx\,d^2y}{ds\,(dx^2+dy^2)};$$

which, compared with Art. 144, shews that the curve is the same with the elastic curve, the line of abscissas being vertical.

By referring to the following Chapter of this Appendix, it will be seen that the curve belongs to Species 8, fig. 172, A being the crown of the arch. By making the depth DE, fig. 164, indefinitely small, it approximates to Species 7, and by making DE indefinitely large, it approximates to Species 9, which is a circle.

CHAP. II.

ON THE SPECIES OF THE ELASTIC CURVE.

1. WHEN a uniform elastic rod is acted upon by any force, it is bent into a curve of which the equation is given in Art. 149. According to the magnitude of the force and other circumstances, the forms of this curve will be different, and these forms may be classed under nine species. We shall examine these species in order.

In fig. 127, we have $CM = x$, $MP = y$ horizontal and vertical, $a^2 = \frac{2\text{E}}{f}$, b^2 a quantity depending upon the angle at A, $c^2 = a^2 - b^2$, and

$$\frac{dy}{dx} = \pm \frac{a^2 - c^2 + x^2}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}} \ldots\ldots (3),$$

$$\frac{ds}{dx} = \pm \frac{a^2}{\sqrt{(c^2 - x^2)(2a^2 - c^2 + x^2)}} \ldots\ldots (4).$$

Also if α be the angle which the curve makes with the abscissa when $x = 0$,

$$b^2 = -a^2 \sin.\alpha;\ c^2 = a^2 - b^2 = a^2(1 + \sin.\alpha).$$

SPECIES I. *Let* $\frac{c}{a}$ *be very small. Fig.* 165.

That this may be the case we must have sin. α negative, and nearly $= -1$, because $c^2 = a^2(1 + \sin. \alpha)$. Hence the direction of the power must make a very acute angle with the curve, as at A.

Since x is never greater than c, we may neglect c^2 and x^2 in comparison with a^2; and making $\sin. \alpha = -1$, we have

$$\frac{dy}{dx} = \pm \frac{a}{\sqrt{\{2(c^2 - x^2)\}}};$$

$$\therefore y = \text{const.} \pm \frac{a}{\sqrt{2}} \text{ arc } \left(\sin. = \frac{x}{c}\right).$$

If we measure from C, where AP again meets the vertical line through A, we shall have $x = 0$, and $y = 0$;

$\therefore y = \dfrac{a}{\sqrt{2}} \text{ arc } \left(\sin. = \dfrac{x}{c}\right)$; omitting the $-$ sign, which does not give any values different from the $+$ sign,

$$x = c \,.\, \sin. \frac{y\sqrt{2}}{a}.$$

We have $x = 0$, when $y = 0$, that is, at C. Again, $x = 0$, when the arc $\dfrac{y\sqrt{2}}{a} = \pi$, that is, when $y = \dfrac{\pi a}{\sqrt{2}}$. Also $x = 0$, when the arc $\dfrac{y\sqrt{2}}{a} = i\pi$, i being any whole number: whence $y = \dfrac{i\pi a}{\sqrt{2}}$. Hence the height CA may be equal to $\dfrac{i\pi a}{\sqrt{2}}$, where i is any whole number. If $AP'C$ be the curve, it will cut the line AC, $i - 1$ times between A and C. If we take the simplest value $i = 1$, we have the curve APD.

The greatest value of x, or the deflexion DE, is c.

The length APC is very nearly equal to AC, but greater. If $l = APC$,

$$l > \frac{\pi a}{\sqrt{2}}.$$

If the elasticity be such that a force f, acting at an arm k, would produce a radius of curvature k, $\text{E} = fk^2$, by Art. 148.

Hence $a^2 = \frac{2\text{E}}{f} = 2k^2$, and $\frac{a}{\sqrt{2}} = k$;

$$\therefore\ l > \pi k.$$

Whence we have this result; that if l be less than πk, or f less than $\frac{\pi^2 \text{E}}{l^2}$, the rod will not be bent at all, but will support the weight at A in a perfectly vertical position. This agrees with Cor. 4, Art. 147.

The weight $\frac{\pi^2 \text{E}}{l^2}$, which the rod will support without bending, is, *cæteris paribus*, proportional to E, the elasticity of the rod; and when the elasticity is given, it is inversely as the square of the length l.

Also since $h = \frac{\pi a}{\sqrt{2}} = \frac{\pi \sqrt{\text{E}}}{\sqrt{f}}$, it appears that if l be a little longer than the length which would support f without bending, it will bend till the height AC is equal to this length.

To find the length, if any arc $CP = s$, we have

$$s = \int \sqrt{(dy^2 + dx^2)} = \int dy \cdot \left(1 + \frac{1}{2} \cdot \frac{dx^2}{dy^2} + \ldots\ldots\right);$$

and as $\frac{dx}{dy}$ is small, we may neglect the higher powers of it.

$$\text{Also } \frac{dx}{dy} = \frac{c\sqrt{2}}{a} \cos. \frac{y\sqrt{2}}{a}; \text{ and}$$

$$\frac{dx^2}{dy^2} = \frac{2c^2}{a^2} \cos.^2 \frac{y\sqrt{2}}{a} = \frac{c^2}{a^2} \cdot \left(1 + \cos. \frac{2\sqrt{2}.y}{a}\right);$$

$$\therefore\ ds = dy + \frac{c^2}{2a^2} \cdot \left(1 + \cos. \frac{2\sqrt{2}.y}{a}\right) dy;$$

$$s = y + \frac{c^2}{2a^2} y + \frac{c^2}{4\sqrt{2}.a} \sin. \frac{2\sqrt{2}.y}{a}.$$

And for the whole length APC, taking the integral from $y = 0$ to $y = \frac{\pi a}{\sqrt{2}} = h$;

$$l = h + \frac{c^2}{2a^2}h = h.\left(1 + \frac{c^2 f}{4\text{E}}\right):$$

$$\text{or since } h = \frac{\pi a}{\sqrt{2}} = \frac{\pi\sqrt{\text{E}}}{\sqrt{f}},\ l = \frac{\pi\sqrt{\text{E}}}{\sqrt{f}}.\left(1 + \frac{c^2 f}{4\text{E}}\right).$$

It is only where the excess of l above $\frac{\pi\sqrt{\text{E}}}{\sqrt{f}}$, or of f above $\frac{\pi^2\text{E}}{l^2}$, is very small, that these formulæ are applicable. In this case, it is easy to find c. Let $l = \frac{\pi\sqrt{\text{E}}}{\sqrt{f}}(1+\delta)$, when δ is small;

$$\therefore \frac{\pi\sqrt{\text{E}}}{\sqrt{f}}.\left(1 + \frac{c^2 f}{4\text{E}}\right) = \frac{\pi\sqrt{\text{E}}}{\sqrt{f}}(1+\delta);$$

$\therefore c = \frac{2\sqrt{(\delta\text{E})}}{\sqrt{f}} = \frac{2h\sqrt{\delta}}{\pi}$, h being the greatest length which will not bend with a force f.

If $\sqrt{f} = \frac{\pi\sqrt{\text{E}}}{l}(1+\delta)$, $l = \frac{\pi\sqrt{\text{E}}}{\sqrt{f}}(1+\delta)$, and the result is the same.

Thus, if there be a spring of such a strength, that 1 foot of it standing vertically will just support a given weight without flexure; suppose a length of $1\frac{1}{100}$ feet to support the same weight; to find its flexure.

Here $\delta = \frac{1}{100}$, $h = 1$, $\therefore c = \frac{2}{\pi}.\frac{1}{10} = \frac{1}{15.7079}$ of a foot; which is the greatest deflexion DE.

When δ ceases to be very small, the curve passes into the next species.

Species II. *Let* $\frac{c}{a} < 1$. *Fig.* 166.

That this may be the case, we must have sin. α negative; that is, the direction of the power must make an acute angle with the curve, as at A. This will happen whenever the weight is $> \frac{\pi^2 E}{l^2}$, and less than that which is requisite for the next species. It comprehends the last case, which was only an approximation. The accurate values of AC and ADC can only be obtained by series. DE is always $= c = a\sqrt{(1 - \sin. \alpha)}$, if α represent the angle which the curve at A makes *below* the horizontal line.

Since $\frac{c^2}{2a^2}$ is less than $\frac{1}{2}$, the series in equations (5) and (6), p. 216, will converge, and l and h may be found from them.

As the force increases, the other conditions remaining, the curve is more bent, the angle α increases, and we approach the next case.

SPECIES III. *Let* $\frac{c}{a} = 1$. *Fig.* 167.

In this case $\sin. \alpha = 0$, and $\alpha = 0$: therefore the force acts at right angles to the curve. The curve in this case is called the *Rectangular Elastica.*

By equation (3), omitting the $-$ sign, which refers to a point beyond A' or D, we have, in the arc CD,

$$\frac{dy}{dx} = \frac{x^2}{\sqrt{(a^2 - x^2)(a^2 + x^2)}} = \frac{x^2}{\sqrt{(a^4 - x^4)}};$$

$$\therefore dy = \frac{x^2 dx}{\sqrt{(a^4 - x^4)}}; \quad ds = \frac{a^2 dx}{\sqrt{(a^4 - x^4)}}.$$

We will suppose $x = 0$, $y = 0$, $s = 0$, at the point C. When $x = 0$, $\frac{dy}{dx} = 0$, $\therefore$ the curve touches the line of abscissas at C. When $x = a$, $\frac{dy}{dx} = \text{inf.}$; $\therefore DE = a$, the curve at D being parallel to AC.

By putting a for c, the equations (5) and (6) of page 216 give

$$l = \frac{\pi a}{\sqrt{2}} \cdot \left\{ 1 + \frac{1^2}{2^2} \cdot \frac{1}{2} + \frac{1^2 . 3^2}{2^2 . 4^2} \cdot \frac{1}{4} + \&c. \right\},$$

$$h = \frac{\pi a}{\sqrt{2}} \cdot \left\{ 1 - \frac{1^2}{2^2} \cdot \frac{3}{1} \cdot \frac{1}{2} - \frac{1^2 . 3^2}{2^2 . 4^2} \cdot \frac{5}{3} \cdot \frac{1}{4} - \&c. \right\}.$$

There is a property of certain integrals, discovered by Euler, (*Inst. Calc. Int.* vol. I. Chap. 8, Art. 334.) which connects l and h. The property is this: if the integrals be taken between $x = 0$ and $x = a$, we shall have

$$\int \frac{a^2 dx}{\sqrt{(a^4 - x^4)}} \cdot \int \frac{x^2 dx}{\sqrt{(a^4 - x^4)}} = \frac{\pi a^2}{4}.$$

Hence

$$\frac{l}{2} \cdot \frac{h}{2} = \frac{\pi a^2}{4}; \text{ or } lh = \pi a^2.$$

Hence $h = \frac{\pi a^2}{l}$; and l being known, h is known.

By taking a proper number of terms of the above series, we may obtain l and h with sufficient accuracy. Thus we shall have*,

$$\frac{l}{a} = \frac{5\pi}{6} \text{ nearly, } = 2.622012 \text{ more nearly;}$$

$$\frac{h}{a} = \frac{\pi a}{l} = \frac{6}{5} \text{ nearly, } = 1.19814 \text{ more nearly.}$$

We have $a^2 = \frac{2E}{f}$; $\therefore f = \frac{2E}{a^2} = \frac{25}{18} \cdot \frac{\pi^2 E}{l^2}$ nearly.

Species IV. *Let* $\frac{c}{a} > 1$, *and* $h > 0$, (*viz.* $\frac{c^2}{a^2} < 1.651868$:) *Fig.* 168.

In this case sin. α is positive, and the angle which the curve and the power make at A is obtuse. For this purpose the force

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* All the numerical results in this Article may be most easily and accurately obtained from Legendre's Tables.

must be greater than $\frac{25\pi^2 E}{18 l^2}$, and less than the value which it has in the next species. The angle which CO, the tangent at C, makes with Cx is less than $40^0\ 41'$, as will be seen.

To find the point P where the curve is parallel to the abscissa, we have $\frac{dy}{dx} = 0$;

$$\therefore a^2 - c^2 + x^2 = 0,\ x^2 = c^2 - a^2 = -b^2 = a^2 \,.\, \sin.\, \alpha.$$

As the force increases, the angle α becomes greater; AC diminishes, and at last is bent down to C, which is the next case.

Species V. *Let* $\frac{c}{a}$ *be such that* $h = 0$: $\left(viz.\ \frac{c^2}{a^2} = 1.651868,\right)$ *Fig.* 169.

This requires that the series for h, Art. 151, should $= 0$; or, if $\frac{c^2}{2a^2} = v$,

$$1 - \frac{1^2}{2^2}\cdot\frac{3}{1}v - \frac{1^2 \,.\, 3^2}{2^2 \,.\, 4^2}\cdot\frac{5}{3}v^2 - \&c. = 0.$$

And we shall find that the value of v is .825934, &c. Hence $c^2 = a^2 \times 1.651868$.

In this case A coincides with C, and however we suppose the curve prolonged, all its branches will coincide with $CPpC$, $CqQC$.

To find the angle OCD we have $\sin.\, \alpha = \frac{c^2 - a^2}{a^2} = .651868$; $\therefore \alpha = 40^0\ 41'$. Hence, when the two extremities of a uniform elastic lamina are made to meet, the angle OCO' which they make is $81^0\ 22'$.

To find the value of x for the point P, where y is a maximum; as before, $x^2 = a^2 \sin.\, \alpha = a^2 \times .651868$.

Hence $CD = a \times 1.285$, $CM = a \times .807$; and by taking the value of l from the series we should have $l = a \times 3.28178$.

SPECIES VI. *Let* $\frac{c}{a}$ *be such that h is negative: viz.* $\frac{c^2}{a^2} > 1.651868$ *and* < 2, *fig.* 121.

In this case A will fall below B, and the branches of the curve will cross each other. The elastic rod must be supposed a mere line, that the two parts which cross may not interfere with each other. If C be still the point where it crosses the vertical AB, the curve at C will make with the abscissa the angle α greater than $40^0\, 41'$. The greatest abscissa DE is c, as before, and passes through the intersection O. The height AC and the length $APDC$ are to be determined by series as before*.

Having found $h = AC$ we may find the line CE which is $\frac{1}{2}h$; and EO will be the value of x corresponding to this value of y, and may be found by reverting the series for y. The angle which the two branches make at O is greater than $81^0\, 22'$ and less than $112^0\, 56'$, as will be seen in the next species.

In this case the force tends to draw A *from* C. As it increases, the distance of A from C increases without limit, as will be seen.

SPECIES VII. *Let* $\frac{c}{a} = \sqrt{2}$, or $\frac{c^2}{2a^2} = 1$, *fig.* 170.

In this case we have sin. $\alpha = 1$ and $\alpha = \frac{1}{2}\pi$. Therefore the curve, where it meets the axis, touches it. Also the series in (5) and (6) become

$$l = \frac{\pi a}{\sqrt{2}} \cdot \left\{1 + \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \&c.\right\} = \text{infinity}.$$

$$h = \frac{\pi a}{\sqrt{2}} \cdot \left\{1 - \frac{1^2}{2^2} \cdot \frac{3}{1} - \frac{1^2 \cdot 3^2 \cdot 5}{2^2 \cdot 4^2 \cdot 3} - \&c.\right\} = -\text{ infinity};$$

And hence the curve never meets the vertical line, but has it for an asymptote.

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* The series in (5) and (6) converge very slowly as $\frac{c^2}{2a^2}$ approaches unity. For series which converge rapidly, in this case, see Legendre, *Exerc. de Calc. Int.* p. 68. Tom. I.

But in this case we can integrate; for since $c^2 = 2a^2$, equation (3) becomes

$$\frac{dy}{dx} = \frac{x^2 - a^2}{x\sqrt{(2a^2 - x^2)}} = \frac{x^2 - \frac{1}{2}c^2}{x\sqrt{(c^2 - x^2)}},$$

$$dy = \frac{x\,dx}{\sqrt{(c^2 - x^2)}} - \frac{\frac{1}{2}c^2\,dx}{x\sqrt{(c^2 - x^2)}};$$

$$\therefore y = -\sqrt{(c^2 - x^2)} + \frac{c}{2}\,\mathrm{l}\,\frac{c + \sqrt{(c^2 - x^2)}}{x};$$

supposing that at D, (where $x = c$,) we have $y = 0$: that is, making DE the line of absissas. $DE = c$ as before.

To find the point O, make $y = 0$;

$$\therefore \sqrt{(c^2 - x^2)} = \frac{c}{2}\,\mathrm{l}\,\frac{c + \sqrt{(c^2 - x^2)}}{x};\text{ or, making } x = c \sin.\theta,$$

$$c \cdot \cos.\theta = \frac{c}{2}\,\mathrm{l}\,\frac{1 + \cos.\theta}{\sin.\theta} = \frac{c}{2}\,\mathrm{l}\cot.\frac{\theta}{2};$$

$$\therefore 2\cos.\theta = \mathrm{l}\cot.\frac{\theta}{2}.$$

Or, supposing Sin. &c. and log. to belong to the common Tables, and M to be the modulus of the Tables,

$$\log.2 + \log.\mathrm{Cos}.\theta - 10 + \log.M = \log.\left(\log.\mathrm{Cot}.\frac{\theta}{2} - 10\right),$$

$$\text{or } \log.\mathrm{Cos}.\theta - \log.\left(\log.\mathrm{Cot}.\frac{\theta}{2} - 10\right) = .0611856 + 10,$$

and by trial we shall find $\theta = 16^0\,46'$ nearly. Hence $\frac{x}{c} = .2884191$.

At O,

$$-\frac{dy}{dx} = \frac{\frac{1}{2}c^2 - x^2}{x\sqrt{(c^2 - x^2)}} = \frac{\frac{1}{2} - \sin.^2\theta}{\sin.\theta\cos.\theta} = \frac{\cos.2\theta}{\sin.2\theta}$$

$$= \cot.2\theta = \cot.33^0\,32' = \tan.56^0\,28',$$

and the angle which the curve at O makes with the abscissa is

$56^0\ 28'$. Hence the two branches of the curve make with each other an angle of $112^0\ 56'$. And hence in Species 6, the angle is less than $112^0\ 56'$.

Also by (4),

$$ds = -\frac{\frac{1}{2}c^2\,dx}{x\sqrt{(c^2-x^2)}},$$

$$s = \frac{c}{2}\,\mathrm{l}\,\frac{c+\sqrt{(c^2-x^2)}}{x},$$

the arc being measured from D.

To find the arc DPO, we must put for x its value $c\,.\,\sin.\,\theta$, θ being $16^0\ 46'$. Hence

$$DPO = \frac{c}{2}\,\mathrm{l}\,\frac{1+\cos.\,\theta}{\sin.\,\theta} = \frac{c}{2}\,\mathrm{l}\ \text{co-tan.}\,\frac{\theta}{2} = \frac{c}{2}\times 1.9148 = c\times .9574.$$

The force which is requisite for the formation of this curve is $f = \frac{2\,\mathrm{E}}{a^2} = \frac{4\,\mathrm{E}}{c^2}$. It must either be supposed to act at an infinite distance in the curve, or it must act by means of an arm, as AC.

Species VIII. *Let* $\frac{c}{a} > \sqrt{2}$; *or* $\frac{c^2}{2a^2} > 1$. *Fig.* 172.

Suppose $c^2 = 2a^2 + g^2$; hence (3) becomes

$$\frac{dy}{dx} = \frac{(x^2 - \frac{1}{2}c^2 - \frac{1}{2}g^2)\,dx}{\sqrt{(c^2-x^2)(x^2-g^2)}}.$$

Hence y is impossible when $x < g$. When $x = g$, $\frac{dy}{dx} = $ inf. as at A. When $x^2 = \frac{1}{2}(c^2 + g^2)$, $\frac{dy}{dx} = 0$, as at P. And when $x = c$, $\frac{dy}{dx} = $ inf. as at D. Beyond this y is again impossible.

In this case the force must necessarily act by means of an arm, as AC.

Species IX. *Let* $\frac{c}{a}$ *be infinite, fig.* 173.

In this case $EO\,(=g)$ and $ED\,(=c)$ (see last species) become indefinitely large. OD remains finite. Let $OD = 2r$, and, C being the bisection of OD, let $CM = t$. That is, let $g = c - 2r$, and $x = c - r - t$;

$$\therefore c^2 + g^2 = c^2 + c^2 - 4cr + 4r^2,$$

$$\text{or } \tfrac{1}{2}(c^2 + g^2) = c^2 - 2cr + 2r^2;$$

$$\therefore\ x^2 - \tfrac{1}{2}(c^2 + g^2) = -2ct + 2rt - r^2 + t^2,$$

$$c^2 - x^2 = 2c(r+t) - (r+t)^2;\ x^2 - g^2 = 2c(r-t) - 3r^2 + 2rt + t^2.$$

Also $dx = -dt$. Substituting, and omitting the terms which do not involve c, as indefinitely small in comparison;

$$\frac{dy}{dt} = \frac{2ct}{\sqrt{2c(r-t)\,.\,2c(r+t)}} = \frac{t}{\sqrt{(r^2 - t^2)}}:$$

hence OPD is a circle with center C.

2. Prop. *When the weight is appended to the curve, to determine in what case each species will occur.*

Let a lamina BA, fig. 127, be fixed firmly at B, in an immoveable obstacle, so that its direction at B is given; and bent by a weight F, hung at A.

Experimentally the species of the curve might be determined immediately by observing the angle FAP, which the force makes with the curve. For we have, when this angle is

very small,.........Species 1,	$130^0\ 41'$,....Species 5,
between 0^0 and 90^0,....... 2,	between $130^0\ 41'$ and 180^0...6,
90^0,....... 3,	180^0..................7,
between 90^0 and $130^0\ 41'$, . 4,	

Species 8 and 9 cannot be formed by a weight immediately appended to the lamina, as will be seen shortly.

But to determine *a priori* the nature of the curve, we must consider the relation between the elasticity, the weight, the length

PA, and the angle at *P*. Thus, suppose the lamina to have its fixed extremity *vertical*, and a weight $= f$ appended at the other extremity, its length being given $= \frac{1}{2} l$. It is manifest that it must form the portion *DPA* in some one of the curves, fig. 165 to 173. Therefore here, as before, l represents the length *APDC*. Also a is known from the given elasticity of the lamina. And if we have

$\frac{l}{a} < \frac{\pi}{\sqrt{2}}$, the form will be a straight line $\left(\frac{\pi}{\sqrt{2}} = 2.22144\right)$,

$\frac{l}{a} = \frac{\pi}{\sqrt{2}}(1 + \delta)$, where δ is very small, it will be Species 1,

$\frac{l}{a} > 2.22144$ and < 2.62201 2,

$\frac{l}{a} = 2.62201$ 3,

$\frac{l}{a} > 2.62201$ and < 3.28178 4,

$\frac{l}{a} = 3.28178$ 5,

$\frac{l}{a} > 3.28178$ 6,

$\frac{l}{a}$ infinite..7.

As l becomes larger with regard to a, the curve approximates in species to 7. But in order that it may actually assume this form, as well as 8 and 9, it is necessary that the weight should act by an arm.

The quantity a depends upon the elasticity of the lamina. It has been seen, in considering Species 1, that if the elasticity be such that the force f acting at an arm k to bend the lamina, would produce in it a flexure whose radius of curvature is also k, we have $2k^2 = a^2$. Hence a is proportional to k. The greater the elasticity and the less is a, and consequently the greater is $\frac{l}{a}$. Also for a given lamina, the greater f, and the less is a, and in the subduplicate proportion. Hence in determining the species of the curve, the same effect is produced whether we double the weight appended, or quadruple the length of the lamina; and similarly for other proportions.

3. PROP. *When the weight hangs by an inflexible horizontal arm, to determine in what case each species will occur.*

Let the weight f act at the end of the horizontal arm MN, fig. 172. Let $MN = m$, and let α be the angle which the curve at N makes with MN. Then by equation (4), observing that at N, x is equal to m, and $\frac{dx}{ds}$ to $\cos \alpha$,

$$\cos.^2 \alpha = \frac{(c^2 - m^2)(2a^2 - c^2 + m^2)}{a^4} = \frac{2a^2(c^2 - m^2) - (c^2 - m^2)^2}{a^4};$$

$$\therefore \sin.^2 \alpha = 1 - \cos.^2 \alpha = \frac{\{a^2 - (c^2 - m^2)\}^2}{a^4}$$

$$\pm \sin. \alpha = \frac{a^2 - c^2 + m^2}{a^2}, \quad \frac{c^2}{a^2} = 1 \pm \sin. \alpha + \frac{m^2}{a^2}.$$

And as the values of α and m vary, the corresponding value of $\frac{c^2}{a^2}$ will determine the species of the curve by what has preceded.

The curve in this case may be any one of the first eight species, and tends to Species 9, as m increases indefinitely.

The double sign of $\sin. \alpha$ gives two values of $\frac{c^2}{a^2}$, which shew that either portion of the curve, NPD or nPD may be taken, Cn being equal to CN.

If instead of a single force acting downwards, we have two, R and S, acting upwards and downwards, at G, H, so that $R \,.\, GM = S \,.\, HM$, they ar equivalent to a force $S - R$ at M; and the species of the curve will be the same as if such a force acted.

If $R = S$, M will be at an infinite distance, and we shall have Species 9.

CHAP. III.

ON FRICTION.

1. When a body tends to slide along a material surface or tends to move so as to rub against the surface, there is a resistance exerted to this tendency. This resistance is called *Friction*. It may be measured and reasoned upon in the same way as other forces, but in the problems of equilibrium treated in the preceding pages its effect is neglected. We shall here consider its consequences in some of the cases of equilibrium.

2. Different surfaces exert more and less of this resistance or friction. When we suppose it to vanish, the surface is said to be *perfectly smooth*.

Prop. *When a body passes a perfectly smooth surface, the effect of the surface will be exerted in a direction perpendicular to the surface.*

A surface is perfectly smooth, as has been said, when it exerts no resistance in consequence of the tendency of a body to move *along* it. Hence the direction of the resistance which it does exert, must be similarly related to all the directions in which a body can move along the surface; that is, it must be equally inclined to all these directions; or it must be perpendicular to the surface. See Art. 38, and its 8th Cor.

3. When a surface is not perfectly smooth, it exerts, besides the resistance perpendicular to the surface, another resistance along the surface, or in the direction of a tangent, and directly opposed to the direction in which the body tends to move along the surface. This resistance is the friction. And when it is employed in

producing equilibrium, it is to be considered as a pressure like any other statical force, and its effect with regard to the equilibrium is to be estimated in the same manner in which the effect of other forces has been estimated in the chapters on equilibrium. Hence

The friction of a body which is moveable along any surface, is *measured* by the force which would just put the body in motion, the force being supposed to act in the direction in which the body must move if it does move.

Let a weight W, fig. 174, rest on a horizontal table AB, and let it be drawn horizontally by a force P. If the table be not perfectly smooth, P may be so small that its pressure will not overcome the friction, and W will remain at rest notwithstanding P's action. If P be gradually increased, W will at last begin to move, the friction being overcome. Hence there is a force P, such that any smaller force will not put W in motion, and any larger force will do so. This value of the force P measures the friction of W upon the table.

4. The friction in this case depends upon the *materials* of which the *table* AB and the body W consist; upon the degree of roughness of each; and upon the *weight* of W. It is not much affected by the *form* of W, or by the magnitude or form of the *surface* MN, which is *in contact* with the table. These laws of friction are proved by experiment.

5. The following experiment is referred to, for the purpose of shewing that the friction is not altered by altering the surface of contact, so long as the pressure continues the same.

Let LMN be a rectangular parallelepiped, and let it be placed on its broader side MN on the horizontal table AB, and let the force P be ascertained, which acting horizontally will put it in motion. Again, let it be placed on its narrower side LM, and put in motion as before. It will be found that the force Q requisite for this purpose is the same as P which was requisite before. And in this case it is manifest that the pressure is the same, viz. the weight of the body, while the surface in contact is different.

6. In what follows we shall suppose the friction to be proportional to the pressure. This is not exactly true; for the friction corresponding to large pressures is less than it would be according to this law.

Hence if R be the pressure of the body against the surface, fR may represent the friction, f being a friction varying according to the nature of the substances, &c.

7. We shall here consider only the *statical* consequences of friction, or its effect upon the conditions of equilibrium. Friction also acts between bodies in motion, and influences their motions. In these cases likewise the friction is as the pressure, and may be represented by fR; but f is less for the case of bodies in motion, than it is for the same bodies in equilibrium. As soon as the force has overcome the friction and put the body in motion, the friction is instantly diminished. Thus in the case of wood sliding on wood, the friction is *half* the pressure when the body is at rest, and is immediately reduced to *one-eighth* the pressure as soon as the motion begins.

The friction of bodies in motion is a *uniform* retarding force.

Friction operates in various ways, some of which are the following.

1. *Friction between finite Surfaces in contact.*

8. Let a weight W, fig. 174 or 175, of which the side MN is plane, be placed in contact with a plane AB. If R be its pressure on the plane, and fR the friction, the surfaces AB and LM being supposed to be made smooth by the usual processes of art, we shall have the value of f as follows.

When the surfaces are wood the *grains* being in the same directions, $f = \frac{1}{2}$.

When the grains are placed across each other, $f = \frac{1}{4}$.

When one surface is wood and the other metal, $f = \frac{1}{5}$.

When both surfaces are metal, $f = \frac{1}{4}$.

Friction is diminished by greasing or oiling the surfaces in contact. Fresh tallow is said to diminish the resistance by one half.

9. Prop. *To determine by experiment the magnitude of the friction.*

Let AB, fig. 175, be an inclined plane, the angle of which can be altered at will. Let the weight W be placed upon it, and let the plane be gradually raised from a horizontal position CB, to a greater and greater angle, till W begins to slide down the plane. Let the inclination of the plane be measured at which this just does *not* take place, and let this be the position represented in the figure.

In this position the force of the friction, which necessarily acts along the inclined plane, just keeps in equilibrium the body upon the inclined plane. Hence if we draw CD perpendicular to AB, the body W is kept at rest by three forces, its weight, the resistance of the plane, and the friction, which are perpendicular respectively to CB, BD, and DC. Those forces are therefore as these lines: and, (as in Art. 38,)

$$\text{friction} : \text{pressure} :: CD : DB.$$

Cor. 1. If we draw CA vertical we have also

$$\text{friction} : \text{pressure} :: AC : CB.$$

Cor. 2. Using the same notation as before,

$$fR : R :: AC : CB \text{ and } f = \frac{AC}{BC}.$$

Cor. 3. If β be the angle ABC at which the body begins to slide, $f = \tan. \beta$.

Cor. 4. If the friction be proportional to the pressure, the angle β, at which sliding takes place, is the same for all weights.

Cor. 5. Hence we may ascertain experimentally whether the friction is proportional to the pressure, by observing whether this angle remains the same when we alter the weight.

Since the friction is proportionally somewhat less for larger pressures than for small ones, the angle at which sliding takes place will be less where the weight is large.

When a ship is launched, it slides upon planes of which the inclination is very small, and along which a small weight would not slide.

10. One property by which friction differs from other forces, is, that it is not exerted except there be a tendency to motion which it has to resist. And it increases as the tendency increases. Thus when the plane AB in fig. 175 is horizontal, the friction is nothing: it increases as the inclination increases, being always of such a magnitude as is requisite to prevent the body from sliding, till it reaches its limit. And if the body had a tendency to slide in the opposite direction, the friction would also be exerted in the opposite direction.

This property of friction makes it modify in a remarkable manner all the conditions of equilibrium of bodies in actual practice. For it results from this existence of friction that the equilibrium will not be destroyed when the conditions investigated in the preceding pages are violated. The force of friction, to a certain extent, will enter the system as far as it is wanted, and supply the deficiency which occurs. So that the balance will not be broken till the conditions of equilibrium have been transgressed to an amount deviating considerably from the calculated state.

We will take one remarkable example of this.

11. In deducing the conditions of equilibrium of an arch, we have (Art. 66,) found the weights which its parts must have in order that they may have no tendency to *slide* past each other. But in point of fact the destruction of the equilibrium in this manner is what can never or scarcely ever happen.

Prop. *When two plane surfaces are in contact, with friction, to find what the direction of the surface of contact must be to produce the same effect without friction.*

Let two bodies of which the centers are C and C_1, fig. 176, be in contact by the surface PQ. Let M be the point at which the mutual action of the surfaces may be supposed to take place. And let Mn be taken in MQ to represent the friction, and nc perpendicular to Mn to represent the pressure; so that $Mn : nc :: f : 1$. Then if the whole friction in the direction PQ be called into action, the force which acts at M will be compounded of Mn and nc, and will be represented in magnitude and direction by MC. And if Mc' be drawn similarly the other way from M, Mc' will be the direction of the force which acts at M, when the whole friction in the direction QP be called into action.

The degree of friction which will really be exerted will depend upon what is wanted to preserve the equilibrium; and according to this event the resulting force may have any direction between Mc and Mc'.

If we draw pq perpendicular to Mc, and suppose that pq is the plane of contact, the effect of the contact without friction would be a force in the direction Mc. And similarly if $p'q'$ be perpendicular to Mc', the effect of a surface $p'q'$ would be in Mc'.

Hence we may suppose the surface of contact to have any position between pq and $p'q'$. If any of these positions *without friction* will preserve the equilibrium, the surface PQ will preserve it with friction.

12. Prop. *To explain the effect of friction in supporting an arch.*

Let fig. 176 be constructed in the same manner as fig. 90, Art. 66. And let PQ be any one of the joints separating the half arch ER into two parts. OT, OU are drawn parallel to PQ, RS, and XTU is horizontal, OX being vertical. Then if we suppose the action at the surface PQ to be perpendicular to this surface, the weights of the portions C, C_1 of the arch must be as $XT : TU$, as is shewn in Art. 66.

Let Mc be the limiting direction of the force at M, including friction, as in the last article, and pq perpendicular to cM. Let ot be parallel to pq. And if we suppose pq to be the joint in the manner there explained, we shall have, by the reasoning in Art. 66, the weights of C and C_1 in the proportional of $Xt : tU$.

In the same manner if Mc' be the limiting position of the force at M when the friction acts in the other direction, and if $p'q'$ be perpendicular to Mc', and Ot' parallel to $p'q'$, when $p'q'$ is supposed to be the joint, we have the weights of C and C_1 in the proportion of $Xt' : t'U$.

Hence where friction is supposed to act, the proportion of the weights of C and C_1 may be any whatever between the limits

$$Xt : tU \text{ and } Xt' : t'U.$$

And so long as the ratio $C : C_1$ is not greater than the first of these limits nor less than the second, there will be no sliding at the joint PQ.

Suppose, for instance, in the figure, that tU and $t'X$ are each $\frac{1}{11}$ of XU. Then the ratios

$$Xt : tU \text{ and } Xt' : t'U \text{ are respectively}$$

$$10 : 1 \text{ and } 1 : 10;$$

and hence, without disturbing the equilibrium, C may vary from 10 times C_1, to $\frac{1}{10}$ of C_1.

Hence it appears that in most cases the arch would stand whatever were the weights of the voussoirs.

Cor. If pq were parallel to OU and $p'q'$ to OX, tU and $t'X$ would vanish, and no possible increase or diminution of the weight of either part of the arch could destroy the equilibrium. The same would be true if t fall to the left of U in the figure, or t' to the right of X.

In this reasoning we have supposed the joint RS to be perfectly smooth; but the same consideration being applied to it as to PQ, the conclusions will be of the same kind, the possibility of equilibrium being still further extended.

Hence an arch may stand though the conditions of equilibrium explained in Articles 66, &c. be not satisfied. In fact it would be scarcely possible with common materials that an arch should fall, in consequence of the voussoirs *sliding* past each other. And the practical considerations requisite for the stability of such a system, are more nearly obtained by considering it on the principles of a roof as investigated in Art. 62, &c.

13. The following is the manner in which the late Professor Robinson has illustrated this view of the subject.

An arch, when exposed to a great overload on the crown (or indeed on any part) divides of itself into a number of parts, each of which contains as many voussoirs as can be pierced by one straight line; and it may then be considered as nearly in the same situation with a polygonal arch of long stones or beams abutting on each other. Thus the arch ABA', fig. 93, may be divided into four portions AC, CB, BC', $C'A'$ by straight lines AC, &c. drawn so as not to fall within the inner curve nor without the outer one. It may then be considered as resembling the roof $ACBC'A$, fig. 89. When pressed from above at B, it tends to break at the angles A, C, B, C', A'; and it is not sufficiently resisted there, because the materials with which the flanks are filled up have so little cohesion that the angle feels no load except what is immediately above it. Hence the arch will fail, the part B falling inwards, and the part C and C' bursting outwards.

In confirmation of this view of the subject, it was observed that an arch in which the arch-stones were too short, fell in this manner: splitting in several points; viz. in the middle B; at another point C, intermediate between the crown and the springing of the arch; and at the springing A. Also, about a fortnight before it fell, chips were observed to be dropping off from the joints of the arch-stones, at two points E and F, between A and B, and between B and C. This splintering may be conceived to have arisen from the lines of pressure BC, CA passing near the lower ends of the voussoirs at C and F so as to throw almost all the pressure upon the inner edges. Upon making the experiment by means of models of arches in chalk, Professor Robinson found that in all cases the overloaded arch always broke at some place C, considerably beyond another point F where the first splintering was observed (Robinson's System of Mech. Phil. Edinb. 1822. Vol. I. p. 639).

2. *Friction of Cylindrical Surfaces in Contact.*

14. Let two circles cn, dn, fig. 177, be in contact internally in the point n. If these two circles be the ends of two cylinders, the cylindrical surfaces will touch each other in a straight

line. And if one of the cylinders turn so as to slide upon the other, there will be a friction between them; which will be proportional to the pressure as in the preceding case, that is, friction $= fR$. But the fraction (f) of the pressure which expresses the friction will be different from what it was in that case.

Thus if both surfaces be of wood $f = \frac{1}{12}$.

If iron turn in contact with brass $f = \frac{1}{7}$.

This is the kind of friction which takes place at the axle of a lever, pully, &c. And we shall consider its effect in such a case.

15. Prop. *To investigate the limits within which friction will preserve the equilibrium of the lever.*

Let a lever AB consist of a bar pierced with a cylindrical hole yn, by means of which it turns upon a solid cylinder xn, which is something smaller than the hole. The friction takes place at the point n, and is there in the direction of a common tangent to the two surfaces of the circles xn, yn.

Let the lever be acted upon by forces P, Q, acting at A, B, and let it tend to turn in the direction AP. Then the friction will act in the direction nF. All the forces P, Q which act upon the lever are equivalent to a single force acting in some direction as HG; and this force combined with the friction nF keeps the lever at rest at the point n on the surface of the cylinder xn. Hence the force arising from the composition of these forces must pass through the point n, and therefore HG must pass through n.

Also the resultant of the forces nF and nG must be perpendicular to the surface of the cylinder xn, and must therefore pass through the center c. Hence if GK be perpendicular to nc, the forces nF and nG must be as GK, nG.

But on the same scale the pressure on the cylinder xn must be as nK. Hence we have $GK = f . nK$, and $nG = \sqrt{1+f^2}\, nK$.

If S be the resultant of all the forces P, Q, and R the pressure on the cylinder, we have

$$S = \sqrt{1+f^2}\,R, \text{ and } R = \frac{S}{\sqrt{1+f^2}}.$$

Let ch be perpendicular upon HQ, and let $ch = s$. Also let r be the radius cn. Then we have by similar triangles

$$ch : cn :: GK : Gn,$$

$$\text{or } s : r :: fR : \sqrt{1+f^2}\,R;$$

$$\therefore s = \frac{fr}{\sqrt{1+f^2}}.$$

This is the condition for the *limiting* case of equilibrium.

Hence if the resultant of the forces P, Q pass through c, there will be an equilibrium without the action of friction. But if this resultant pass at a distance s from c, the friction will preserve the equilibrium so long as s is not greater than $\frac{fr}{\sqrt{1+f^2}}$.

3. *Friction of a Body capable of revolving upon a Point.*

16. If a cup be inverted and placed on a sharp point, and be made to revolve horizontally while it is thus suspended, it will experience a certain degree of friction. This friction will affect the motion of rotation. It would also resist the tendency of a force to put the cup in motion in this manner, and would thus affect the conditions of equilibrium. But this friction is so small and of so little practical significance as a statical consideration, that we shall not dwell upon it.

As a retarding force operating upon a body in motion, this friction is a uniform force.

4. *Friction of a Rolling Body.*

17. When a cylinder rolls upon a horizontal plane, the resistance to its motion is much less than if it were to slide, but it does not absolutely vanish. It would seem that in this case the resistance must arise rather from the cohesion of the surfaces than from friction properly so called, since the surfaces do not at all *rub* upon each other. Like the last kind of friction this may be neglected in statical problems. In the case of motion it is found that the friction of rolling cylinders is as the pressure *directly,* and as the diameter of the cylinder *inversely.* Hence we see one of the mechanical advantages of large wheels.

When a cylinder of mahogany, of diameter 3 inches, rolled on a plane of oak, the friction was $\frac{1}{16}$ the pressure; when the same cylinder rolled on a plane of elm the friction was $\frac{1}{100}$ the pressure.

CHAP. IV.

ON THE CONNEXION OF PRESSURE AND IMPACT.

In Art. 185, it was observed that impact is really a pressure of short duration. The duration of the pressure depends, *cæteris paribus,* upon the materials of which the impinging bodies are composed, viz. upon their *hardness.* Various results are connected with the changes of this element, some of which we shall here consider.

"Suppose a hammer moving with a considerable velocity impinges on a hard block. The block sustains a very great force, and the magnitude of this force depends *cæteris paribus* on its own hardness, and the hardness of the hammer. An iron hammer produces a greater effect than a soft ball of worsted; and an iron anvil sustains a greater force than a soft pillow. How is this to be accounted for? The answer is not difficult. The momentum of the hammer must be destroyed by a finite force continued for a finite time; and the shorter the time, the greater must be the force. But the time will evidently be shortest with those bodies which undergo the least compression from a given force, since the time in which the momentum of the hammer is destroyed, begins at the instant of the first contact, and terminates when the center of the impinging body is nearest to that of the body struck, that is, when the sum of their compressions has attained its greatest value. The less then that this compression is, the less will be the space that the impinging body describes before its momentum is destroyed, and therefore the greater will be the force which will have resisted it."

"To subject this to mathematical calculation, it will be convenient to assume some law connecting the compression of a body with the force with which it endeavours to return to its former state. As one of the most probable hypotheses, suppose this force to be proportional to the compression: then if x be the space which the surface of the struck body has yielded, $\frac{x}{a}$ will be the force necessary to keep it in that state, or the force which it is then sustaining, a being a constant coefficient which is different for every different body."

In the same manner if b be the constant in the impinging body, and y the space through which its surface has yielded, $\frac{y}{b}$ will be the force which it is exerting.

Prop. *When one body impinges on another, the force exerted is greater in proportion as the bodies are harder.*

Let as above $\frac{x}{a}$ and $\frac{y}{b}$ be the forces exerted by the striking body, and the body struck. These must be equal, or $\frac{x}{a} = \frac{y}{b}$.

Let H be the weight of the striking body or *hammer*, V its velocity at the instant of first contact: s the space described by its center, since that time. This space is described in consequence of the compression of the two bodies, therefore $s = x + y$.

But since $\frac{x}{a} = \frac{y}{b}$, $y = \frac{bx}{a}$, $\frac{a+b}{a} x = x + y = s$;

$$\therefore x = \frac{as}{a+b}; \text{ similarly } y = \frac{bs}{a+b}.$$

Also if g be the force of gravity, $\frac{gy}{b}$ is the *pressure* on H; and the force which retards H is $\frac{gy}{Hb}$, or $\frac{1}{a+b} \cdot \frac{gs}{H}$.

Hence by the equation $v \frac{dv}{ds} = f$, Art. 175, we have (v being the velocity of H),

$$v \frac{dv}{ds} = \frac{1}{a+b} \cdot \frac{gs}{H}; \quad v^2 = C - \frac{1}{a+b} \cdot \frac{gs^2}{H};$$

and since $v = V$ when $s = 0$, $v^2 = V^2 - \frac{gs^2}{H(a+b)}$.

$$\text{When } v = 0, \quad s = V \sqrt{\frac{(a+b)H}{g}};$$

and the pressure at this moment $= \frac{y}{b} = \frac{s}{a+b} = \frac{V\sqrt{H}}{\sqrt{(a+b)g}}$.

At this moment the whole motion of the hammer is destroyed, and the compression is greatest and the force greatest; i. e. the force increases from the first contact, when it is 0, till it is $\frac{V\sqrt{H}}{\sqrt{(a+b)g}}$.

Now the harder the bodies are, the greater is the force for a given compression, and the smaller are a and b. Hence this force is greater as the bodies are harder.

Cor. Upon the law here assumed it appears that the force exerted is, *cæteris paribus*, as the square root of the weight of the hammer.

"Since the body struck was in the preceding case supposed to be kept at rest by an immoveable obstacle, this is the greatest pressure which can take place from the impact of H upon the other body. If the other body can move, it is evident that the force will not be so great. We will consider the effect of impact in moving a body in opposition to a uniform force; and first we will consider the body struck as being so small that its weight and inertia may be neglected. This will be nearly the case of a hammer driving a nail, the friction being supposed uniform."

4. PROP. *To find how far a given hammer will drive a nail.*

"Taking the same letters as before, and putting F for the friction, it is evident that the nail will not stir till the compression of the nail and hammer is sufficient to cause a force $f = F$. This will be the case when $\frac{y}{b} = \frac{1}{a+b} s = F$, or $s = (a+b) F$. Hence by the last Article, if v be the velocity of the hammer, when the nail begins to move,

$$v^2 = V^2 - \frac{gs^2}{H(a+b)} = V^2 - \frac{g}{H}(a+b)F^2.$$

"We must now consider that the hammer and nail move on, resisted by the uniform force (pressure) F, till the momentum of the hammer is destroyed. Let s' be the space through which they move: the retarding force being $\frac{Fg}{H}$,

$$\therefore \text{(Art. 219)}\ s' = \frac{v^2}{2\frac{Fg}{H}} = \frac{H}{2g}\left(\frac{V^2}{F} - \frac{g}{H}.(a+b)F\right).$$

This space will $= 0$, or the blow will not move the nail at all, if

$$V^2 = \frac{g}{H}.(a+b)F^2 \text{ or less.}$$

"If V be very great, the space through which the nail moves will be almost independent of the hardness: but if V be barely sufficient to move the nail, a small increase in the hardness of the hammer or nail will much increase the space through which it moves. If while friction prevents the nail from moving *in* the block into which it is driven, it move *with* the block on account of the yielding of the block, this must evidently produce the same effect as the softness of the nail, that is, b will be increased; hence it will be driven farther (*cæteris paribus*) into a substance that does not yield than into one that does.

"Since at the limit of motion $V^2 H = g F^2 (a+b)$, when the bodies are very hard, or a and b very small, a small hammer and small velocity will produce the effect of a very great pressure F. Thus it is found that a sledge hammer driving hard oak pegs produces as great an effect as a pressure of 70 tons.

"Suppose now we consider the weight of the body moved, as in this example.

5. Prop. "*A pile, of weight W, is driven by a hammer, or a ram H, impinging with a velocity V: the friction being represented by F, to find the motion.*"

We will here take account of the weight as well as the momentum of the ram. Let s' be the space through which the aggregate compression takes place before the pile moves. Then, as in last Article, this will occur when the force downwards = the resistance, or $\frac{y}{b} + W = F$. Hence

$$\frac{s'}{a+b} = \frac{y}{b} = F - W, \quad s' = (a+b)(F-W).$$

Also as before

$$v\frac{dv}{ds} = -\frac{1}{a+b}\frac{gs}{H} + g,$$

$$\therefore v^2 = V^2 - \frac{1}{a+b}\frac{gs^2}{H} + 2gs,$$

because when $s = 0$, $v = V$.

And if V' be the velocity of the ram when the pile begins to move, putting s' for s,

$$V'^2 = V^2 - \frac{g}{H}(a+b)(F-W)^2 + 2g(a+b)(F-W)\ldots(1).$$

Now as the inertia of the pile resists the communication of motion, the compression will still go on increasing after this time. Let r be the space through which the ram has moved since the first contact; p that through which the pile has moved. Then $p=r-s$, and $s=r-p$.

For the motion of the ram, (Art. 175.)

$$\frac{d^2r}{dt^2}=g-\frac{s}{a+b}\cdot\frac{g}{H};$$

and for that of the pile,

$$\frac{d^2p}{dt^2}=g+\frac{s}{a+b}\,\frac{g}{W}-\frac{Fg}{W};$$

subtracting,

$$\frac{d^2s}{dt^2}=-\frac{gs}{a+b}\left(\frac{1}{H}+\frac{1}{W}\right)+\frac{Fg}{W}$$

$$=-n^2s+\frac{Fg}{W},\ \text{where } n^2=\frac{g}{a+b}\cdot\frac{H+W}{HW};$$

$$\therefore\ s=\frac{Fg}{n^2W}+A\cos.nt+B\sin.nt.$$

Suppose $t=0$ when the pile begins to move: then, as has already been shewn, at this point of time $s'=(a+b)(F-W)$; hence

$$(a+b)(F-W)=\frac{Fg}{n^2W}+A=(a+b)\frac{FH}{H+W}+A;$$

$$\therefore\ A=(a+b)\left(\frac{FW}{H+W}-W\right).$$

Again, when $t=0$, the velocity of the pile $=0$ and that of the ram $=V'$. Therefore

$$\frac{ds}{dt}=\frac{dr}{dt}-\frac{dp}{dt}=V'-0=V'.$$

But generally

$$\frac{ds}{dt}=-An\sin nt+Bn\cos nt,$$

$$\therefore\ Bn=V',\ B=\frac{V'}{n};$$

Hence, substituting for A, B and n^2,

$$s=(a+b)\frac{FH}{H+W}+(a+b)\left(\frac{FW}{H+W}-W\right)\cos. nt+\frac{V'}{n}\sin. nt.$$

And substituting this,

$$\frac{d^2p}{dt^2}=g\left(1-\frac{F}{H+W}\right)(1-\cos. nt)+\frac{V'g}{(a+b)\,n\,W}\sin. nt$$

$$\text{or, putting } g\left(\frac{F}{H+W}-1\right)=m,$$

$$\frac{d^2p}{dt^2}=-m+m\cos. nt+\frac{V'g}{(a+b)\,n\,W}\sin. nt.$$

Integrating twice, making $\frac{dp}{dt}$ and p both $=0$ when $t=0$, we have

$$\frac{dp}{dt}=-\frac{m}{n}(nt-\sin. nt)+\frac{V'g}{(a+b)\,n^2\,W}(1-\cos. nt)$$

$$p=\frac{m}{n^2}(1-\tfrac{1}{2}n^2t^2-\cos. nt)+\frac{V'g}{(a+b)n^3W}(nt-\sin. nt)\text{.......}(2).$$

Now when the pile ceases to move, we shall have $\frac{dp}{dt}=0$.

$$\text{Hence } \frac{V'g}{(a+b)mnW}(1-\cos. nt)+\sin. nt-nt=0\text{.........}(3),$$

whence t is determined. Also t being known, we have p, the whole space through which the pile is driven.

6. If we suppose V' to be small, that is, the velocity to be such as only just to stir the pile, we may approximate. In this case t, the time during which the pile moves will be very small. Hence, expanding $\sin. nt$ and $\cos. nt$ in (3) and taking only the lowest two terms;

$$\frac{V'g}{(a+b)\,mnW}\cdot\frac{n^2t^2}{2}+\frac{n^3t^3}{2.3}=0;$$

$$\therefore t=\frac{3V'g}{(a+b)\,n^2\,W},$$

or putting for $n^2(a+b)$ and m their values,

$$t = \frac{3V'H}{g(F-H-W)}.$$

Hence find p from (2), again taking the lowest term only;

$$p = \frac{V'g}{(a+b)n^3 W} \cdot \frac{n^3 t^3}{2.3}$$

$$= \frac{gH^3}{2(a+b)W(F-H-W)} \cdot \frac{V'^4}{g^2}.$$

"From what precedes we may draw the following conclusions:

1st. "If the ram will just stir the pile, V' in (1) is small, and a slight increase of the hardness of the ram or pile (i. e. of a or b); or of the weight of the ram (i. e. of H) will very much increase V' and thus increase very considerably the distance p to which the pile is driven.

2d. "The resistance F being supposed great, the space p will be very nearly inversely as W the weight of the pile; consequently the lighter the pile is *cæteris paribus* the faster it will be driven.

3d. "The space p varies directly as the cube of the weight of the ram, the velocity with which the pile begins to move being given. And since this velocity itself is much increased by increasing the ram, there is on both accounts a great advantage in making the ram heavy.

7. The quantities a and b in the preceding investigations depend upon the hardness and elasticity of the substances striking and struck. If we suppose that a body of which the diameter is D would, by the action of a force E, undergo a contraction or expansion D, in linear dimensions (the law by which the contraction x takes place being supposed to extend to this case;) we shall have $E = \frac{D}{a}$, and $a = \frac{D}{E}$. Hence the force for a compression x is $\frac{Ex}{D}$. And if the force E be expressed by means of the weight of a column of the substance itself of height E, E will be

the same thing as the *modulus of elasticity* in Art. 142. The harder the bodies, the greater is this modulus, and the less is a.

The modulus of elasticity of iron or steel is about 9,000,000 feet (see p. 212.) By means of this value we can obtain numerical values in the preceding propositions.

PROB. I. An iron hammer strikes an anvil with a velocity acquired down a height H; to find the compression.

The weight of the hammer H may be expressed by means of a column of iron of which the base is the surface in contact with the anvil. If the hammer be a parallelepiped, H will be its height. Also $V^2 = 2gh$. And if A be the diameter of the anvil in the direction of the stroke, we have by Art. 3 of this Chapter

$$\left(\text{since } a = \frac{A}{E},\ b = \frac{H}{E}\right),$$

$$s = \sqrt{2h}\sqrt{\frac{(A+H)\,H}{E}}.$$

Thus if the hammer be $\frac{1}{4}$ foot high, and fall on an anvil 2 feet high from a height of 8 feet,

$$s = \sqrt{16}\sqrt{\frac{\frac{9}{4}\cdot\frac{1}{4}}{9{,}000{,}000}}$$

$$= \frac{1}{1000} \text{ of a foot,}$$

which is the space through which both have been compressed; and the hammer and anvil share this compression in the proportion of 1 : 8.

Also to find the greatest pressure in this case, we have by the formulæ in p. 361,

$$\text{pressure} = \frac{\sqrt{2n}\,\sqrt{HE}}{\sqrt{A+H}}$$

$$= \frac{\sqrt{16}\sqrt{\frac{1}{4}.9000000}}{\sqrt{\frac{9}{4}}}$$

$$= 4000.$$

Hence in this case the pressure is equal to that of a column of iron 4000 feet high.

If we suppose the face of the hammer to a square inch, the pressure will be 48000 inches of iron, which is above 12000 pounds.

PROB. II. The friction of a nail in wood being equal to the weight of a column of iron having its base the surface of the nail-head in contact with the hammer, and its height F; to find the space through which the nail is driven.

Using the same letters as in the last problem (A being now the length of the nail) we have by Art. 4 of this Chapter

$$s' = \frac{Hh}{F} - \frac{1}{2}\frac{(A+H)F}{E};$$

or since A is small in comparison with H,

$$s' = \frac{Hh}{F} - \frac{1}{2}\frac{HF}{E}.$$

Hence the space through which the nail is driven is, by the defect from absolute hardness, diminished by the quantity $\frac{1}{2}\frac{HF}{E}$.

8. "The same principles give easy and curious results in such problems as these. To find the effect of impact on the wedge, friction being taken into account. To find the weight which dropped from a given height into the empty scale of a balance, will raise any weight in the other," &c.

"It is evident that impact cannot be conveniently used to overcome any continuous force, except that force cease as soon as the motion ceases. This is the case with the resistance of friction, and some others: and in these instances the effect of impact is greater than that of any pressure which it would be practicable to employ. A construction which is rather a modification than a direct application of this principle is used in punching iron plates for fenders, and a similar one is employed in coining. The long lever which turns a screw is loaded at its ends with heavy weights; a man gives it a considerable velocity; and when the screw is suddenly stopped by the punches, the force impressed is enormous. In the same way (though by a construction rather different) the holes in the nuts of screw-bolts are punched in iron bars sometimes $\frac{3}{4}$ inch thick. All these, as well as the simplest cases of force produced by impact, depend on the same principle, viz. that to destroy momentum in a short time a great force is necessary."

"The same principle may be used to explain some facts observed by workmen, which at first sight appear very strange. It is found that if a cylindrical hole be made in a block of granite, and an iron rod driven into it, of such a size that a few blows with a hammer will overcome the friction, the block may be raised by this rod: but if the same process be used with a block of soft stone, the block cannot be raised by it. The reason appears to be this; the granite yields so little that a blow of the hammer overcomes a very great friction; whereas in the soft stone, except the friction be very small, the iron yields with the stone to the blow of the hammer, and the friction which is really overcome in the granite, and which sustains the granite when it is raised is much greater than that which the same blows overcome in the soft stone."

ERRATA.

Page Line

48 11, *for* as *read* in.

70 1 and 2 of note, *for* $\frac{a}{2}$ *read* $\frac{c}{2}$.

71 last line, *for* $\frac{DF}{DK}$ *read* $\frac{BF}{BK}$.

90 16, in denominator, *for* tan. ι' tan. δ *read* tan. ι tan. δ.

105 3 and 5, *for* M_1 and M_2 *read* C_1 and C_2.

115 7 of note, *for* PO^2 *read* AO^2.

131 5 from bottom, *for* p_2 *read* p_1.

— 3 ————, *for* 45 *read* 85.

147 16, in numerator, *for* $-$ *read* $+$.

152 2 from bottom, *for* O *read* a.

158 5, *for* $\frac{dz}{dx}$ *read* $\frac{dz}{dy}$.

165 lines 3, 4 and 5, *for* a *read* c.

167 6, *for* c *read* C^2.

— 17, in denominator after $=$, *for* c *read* C.

173 14, *for* $u =$ &c. *read* $u = e$, $h = \frac{l}{e}$ or $l = eh$.

174 12, *for* $2a$ *read* a.

178 11, *for* $\frac{1}{c}$ *read* $\frac{a^2}{c}$.

180 6, *for* $\sqrt{+c}$ *read* $\sqrt{c}$.

182 2 from bottom, *for* or *read* on.

189 13, *for* c^{m-1} *read* c^m.

209 5, *for* $h = \frac{a}{2}$ *read* $n = \frac{a}{2}$.

216 12, dele $1 -$.

217 12 and 23, *for* r *read* ϱ.

218 2 from bottom, *read* Art. 148. $PO' = \frac{E}{fk}$.

219 22, *for* $dxdy$ *read* dxd^2y.

222 20, *for* θ *read* θ'.

248 26, *for* $Aa : Bb$ *read* $Bb : Aa$.

— 27, *for* $aC : bC$ *read* $bC : aC$.

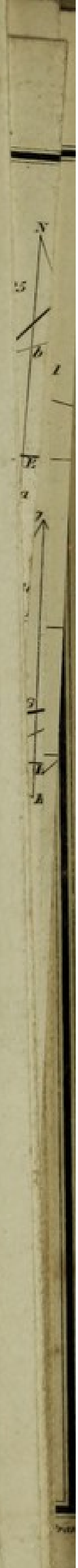
288 6, *for* 180 *read* 221.

291 2 of note, *for* p *read* q.

293 11, *for* gh *read* qh.

315 8, *for* FB *read* FP.

— 9, *for* PC *read* PE.



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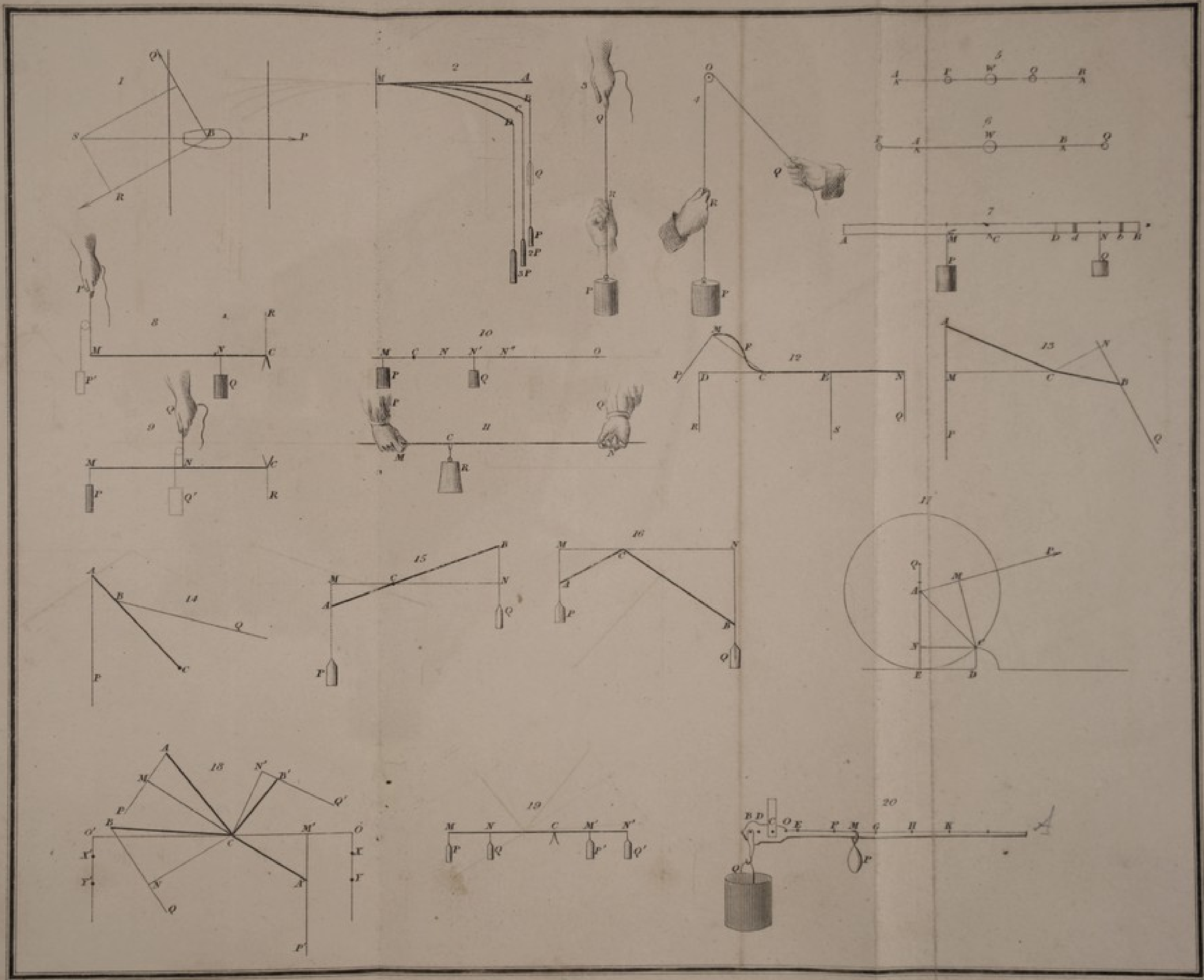

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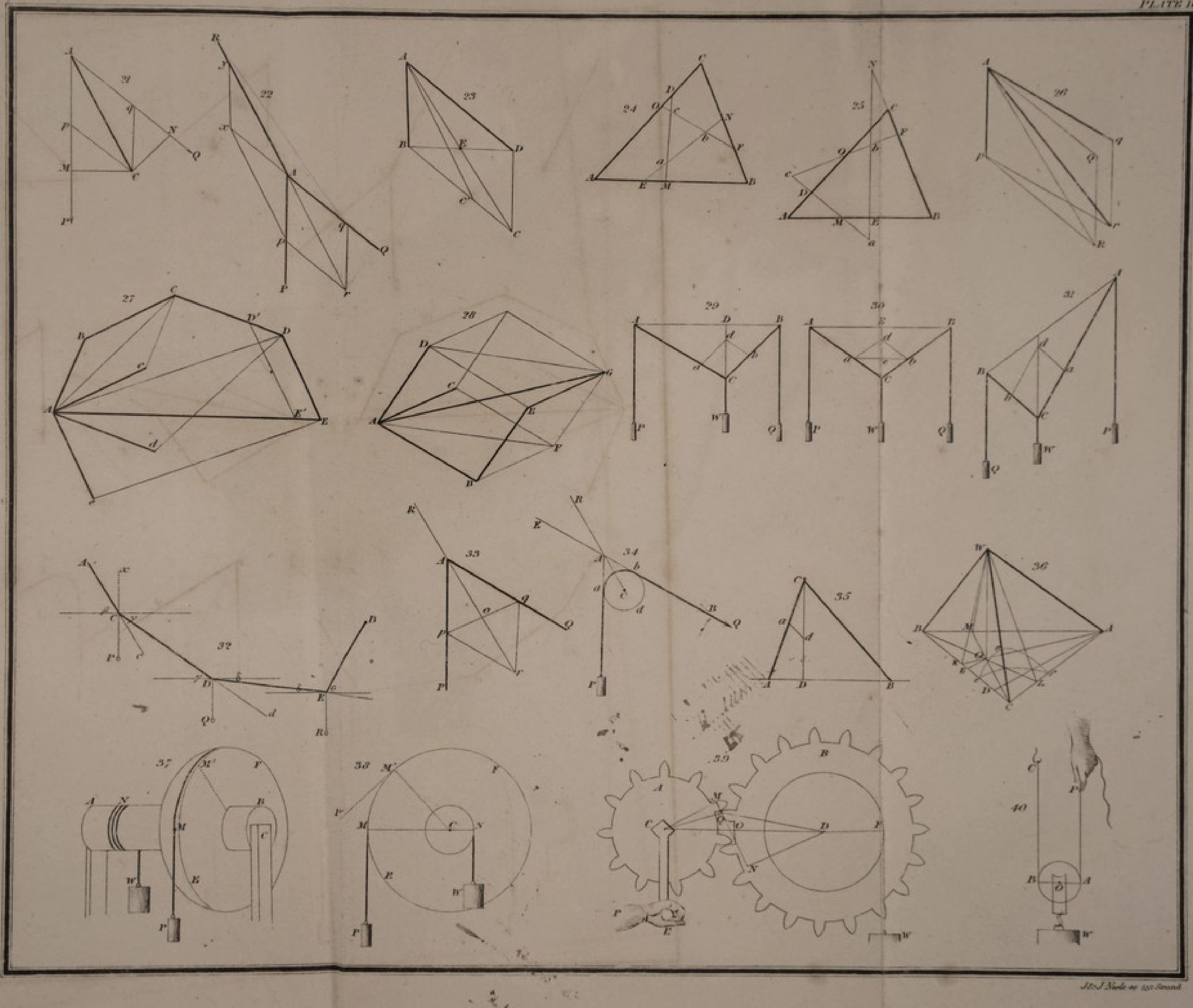

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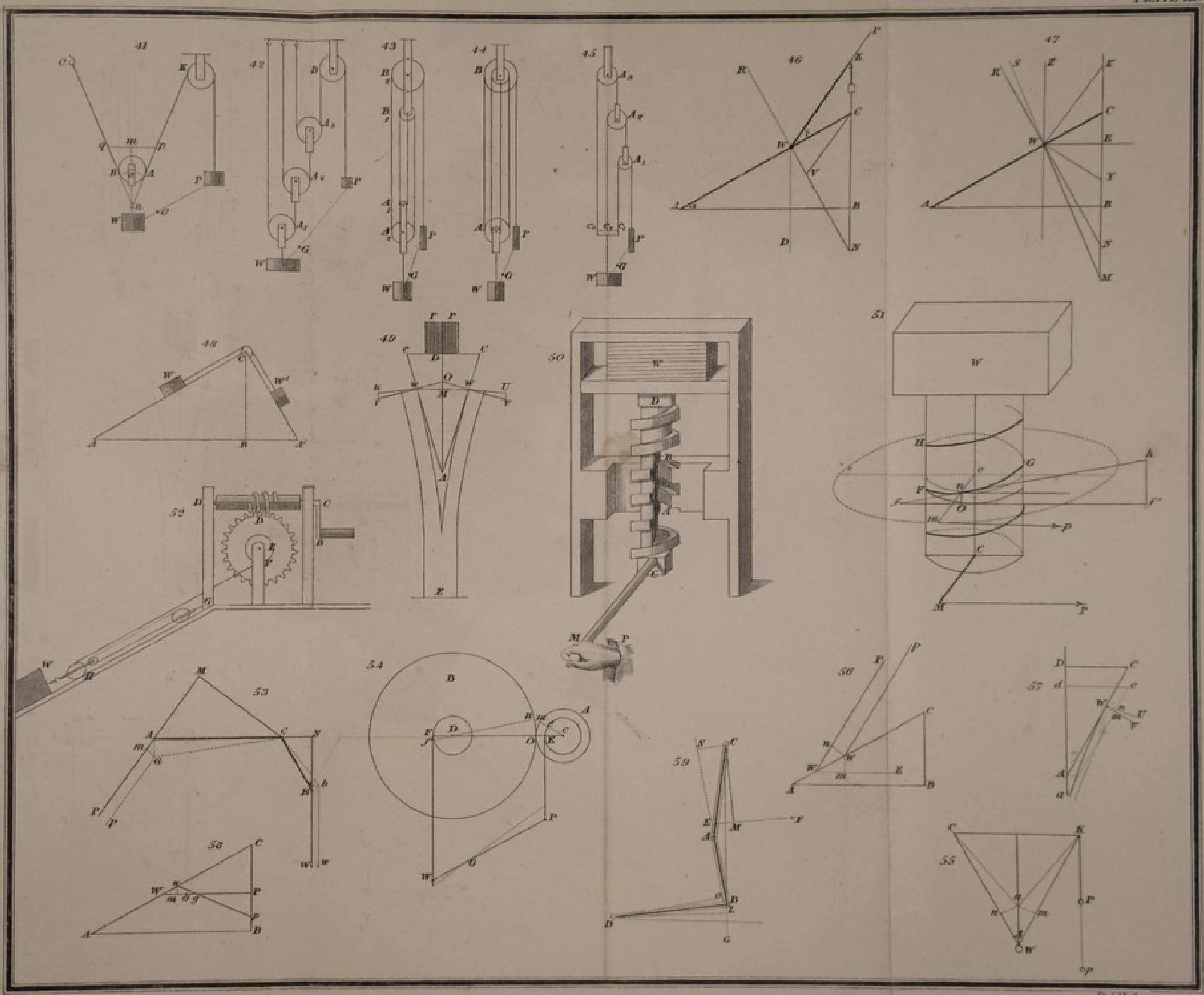

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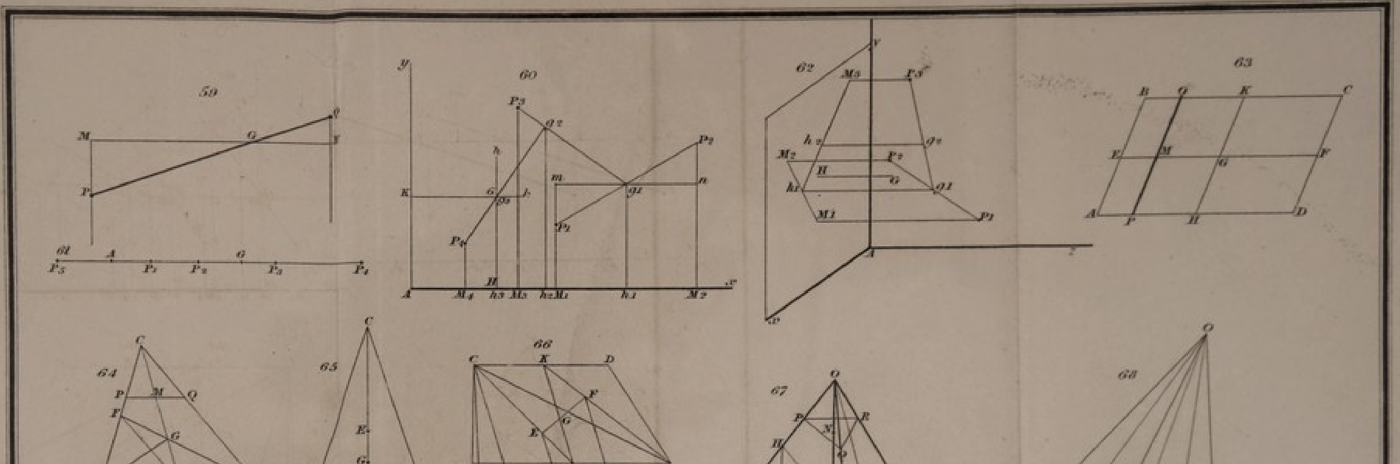

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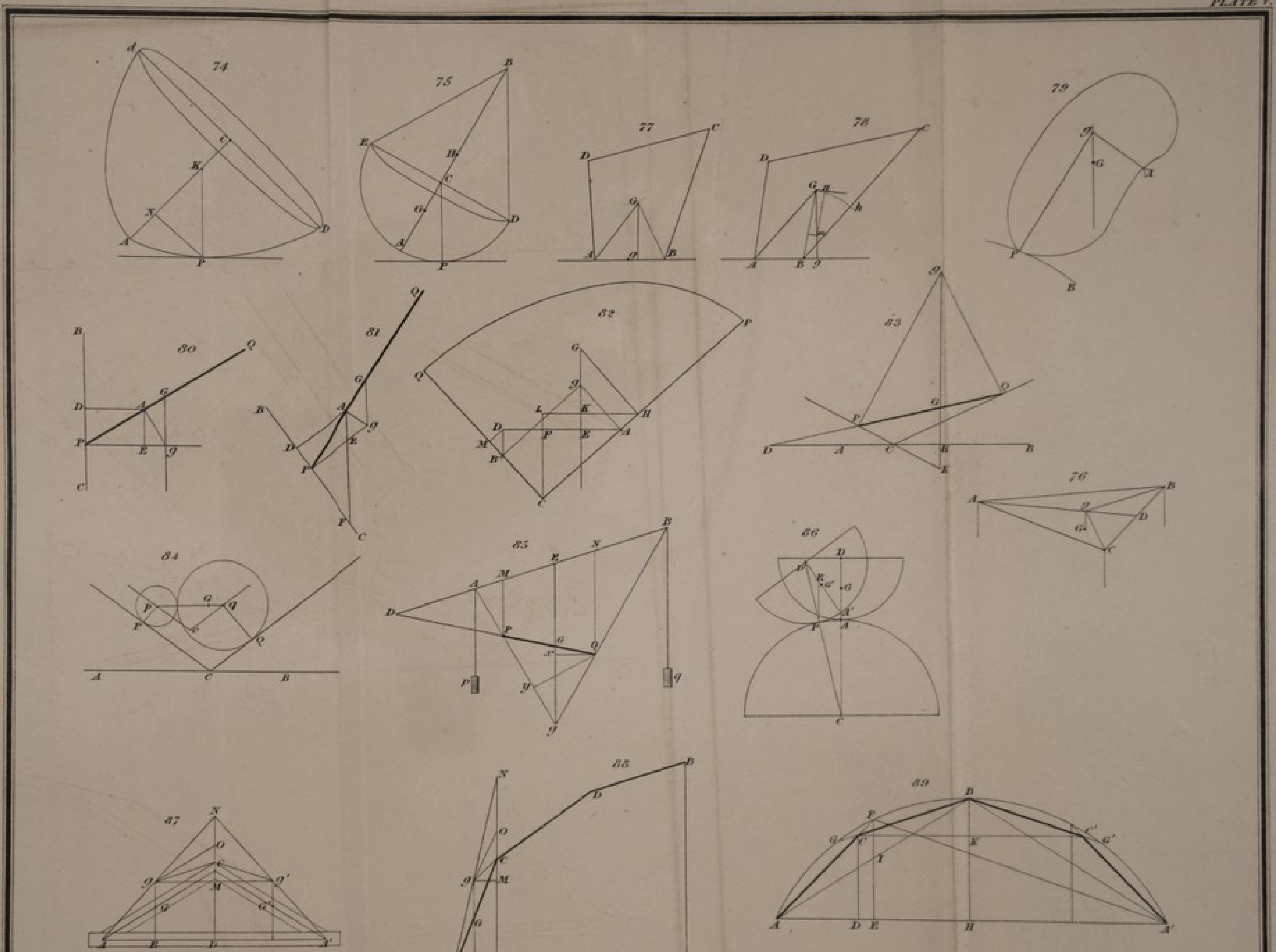
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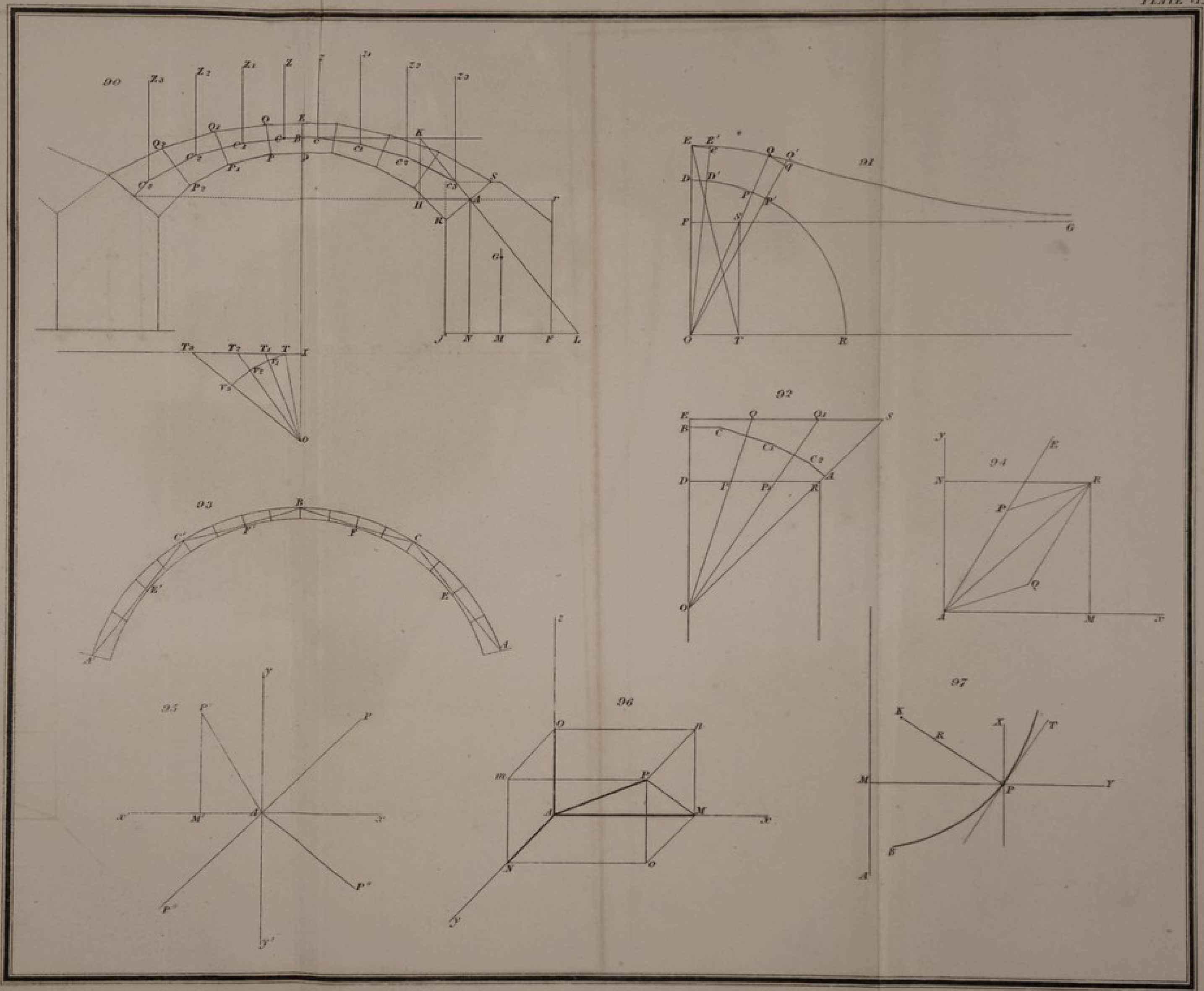

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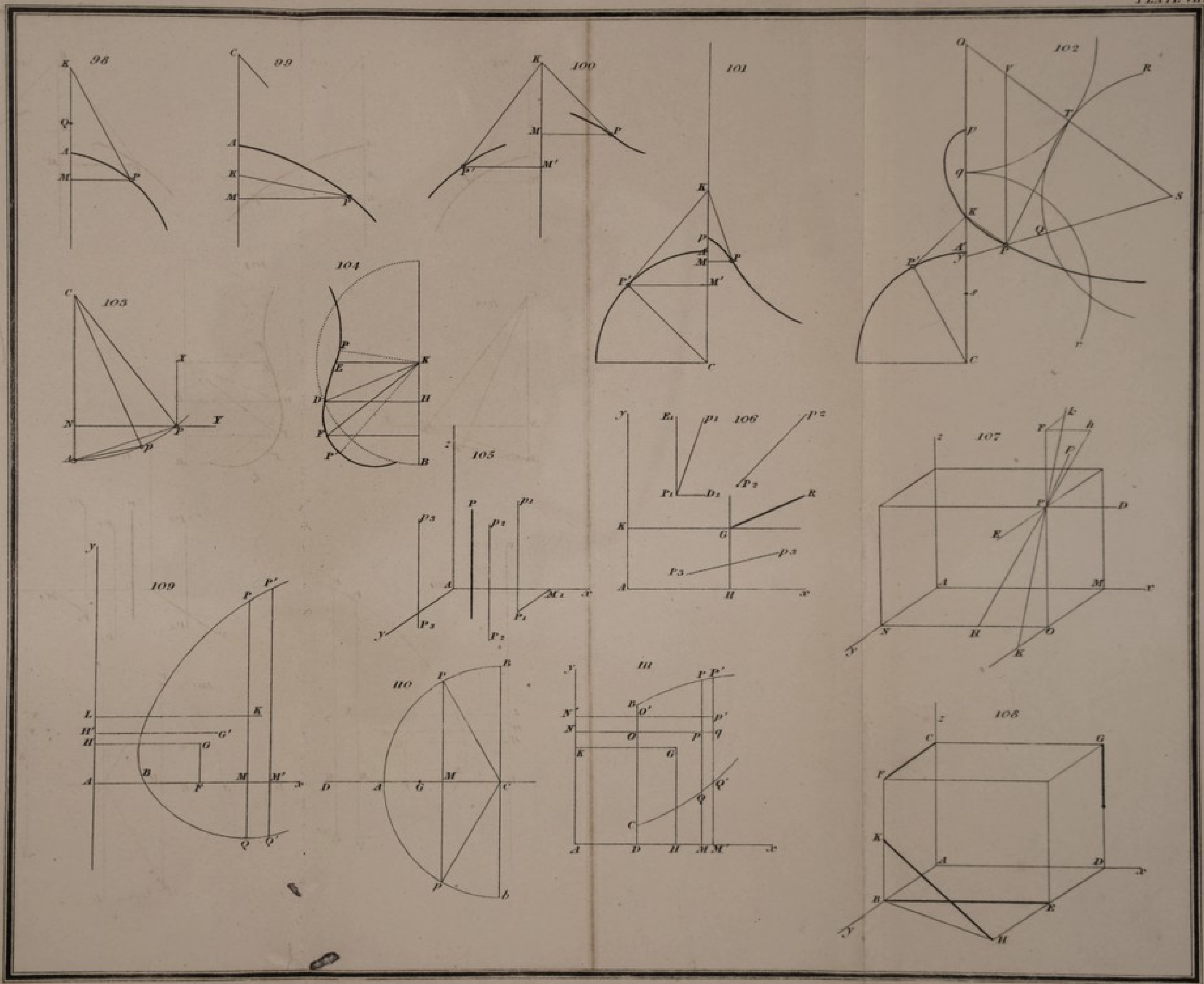

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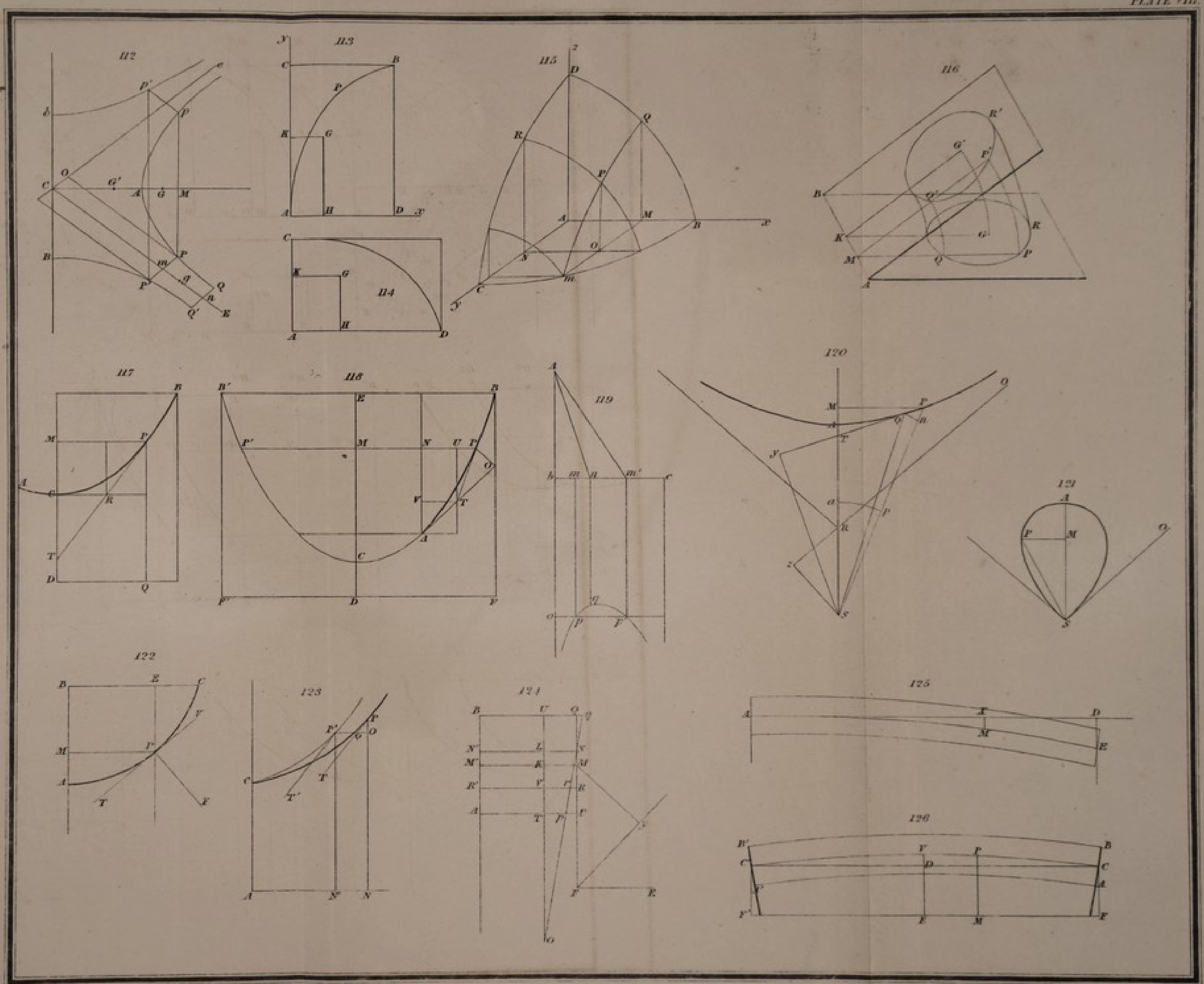

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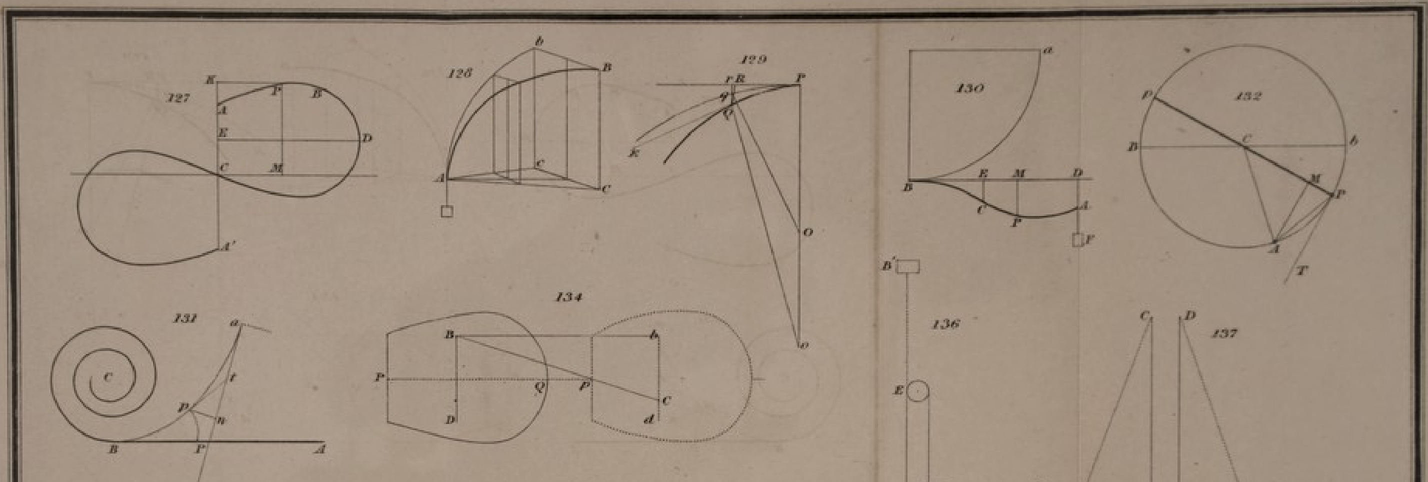

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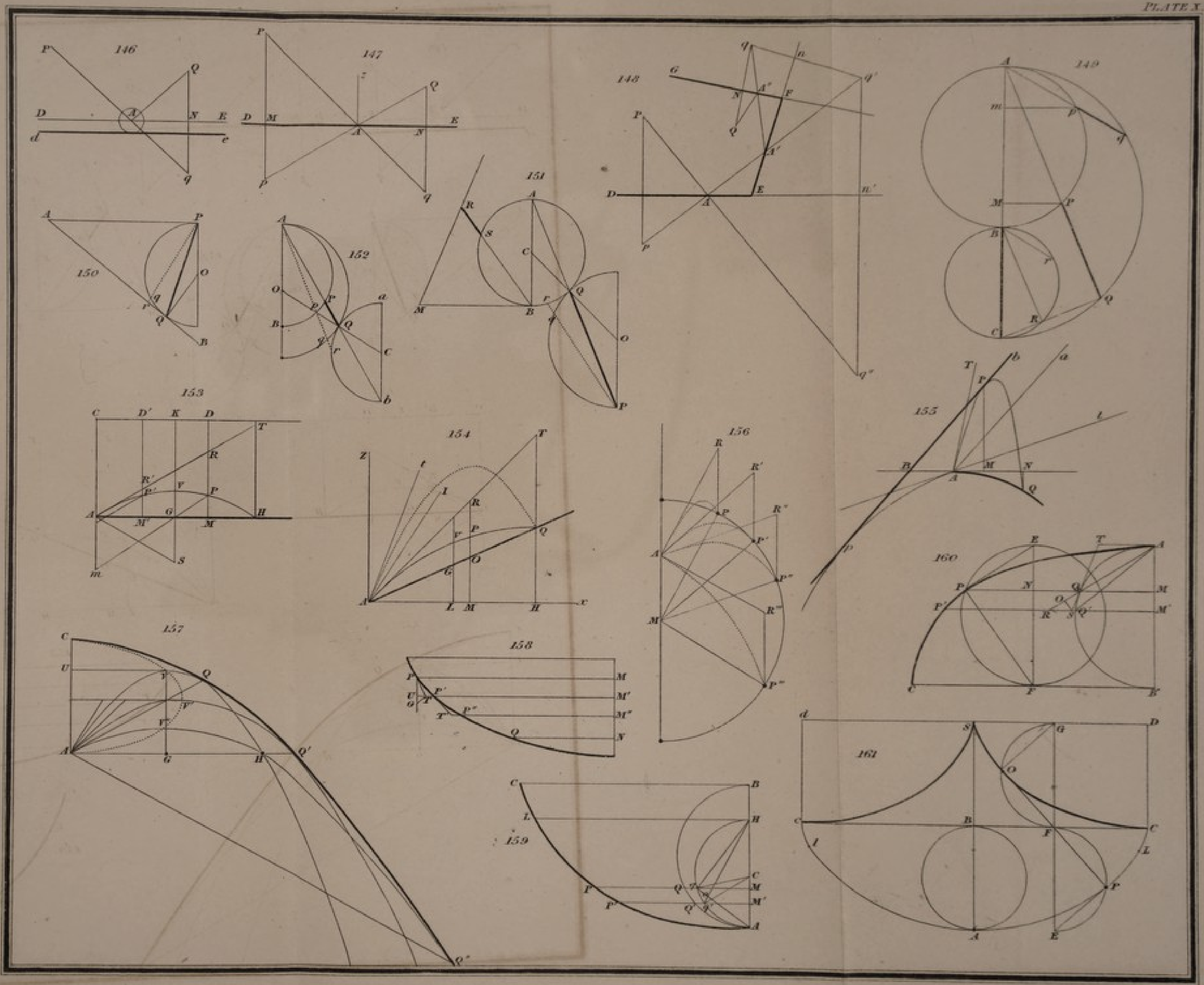

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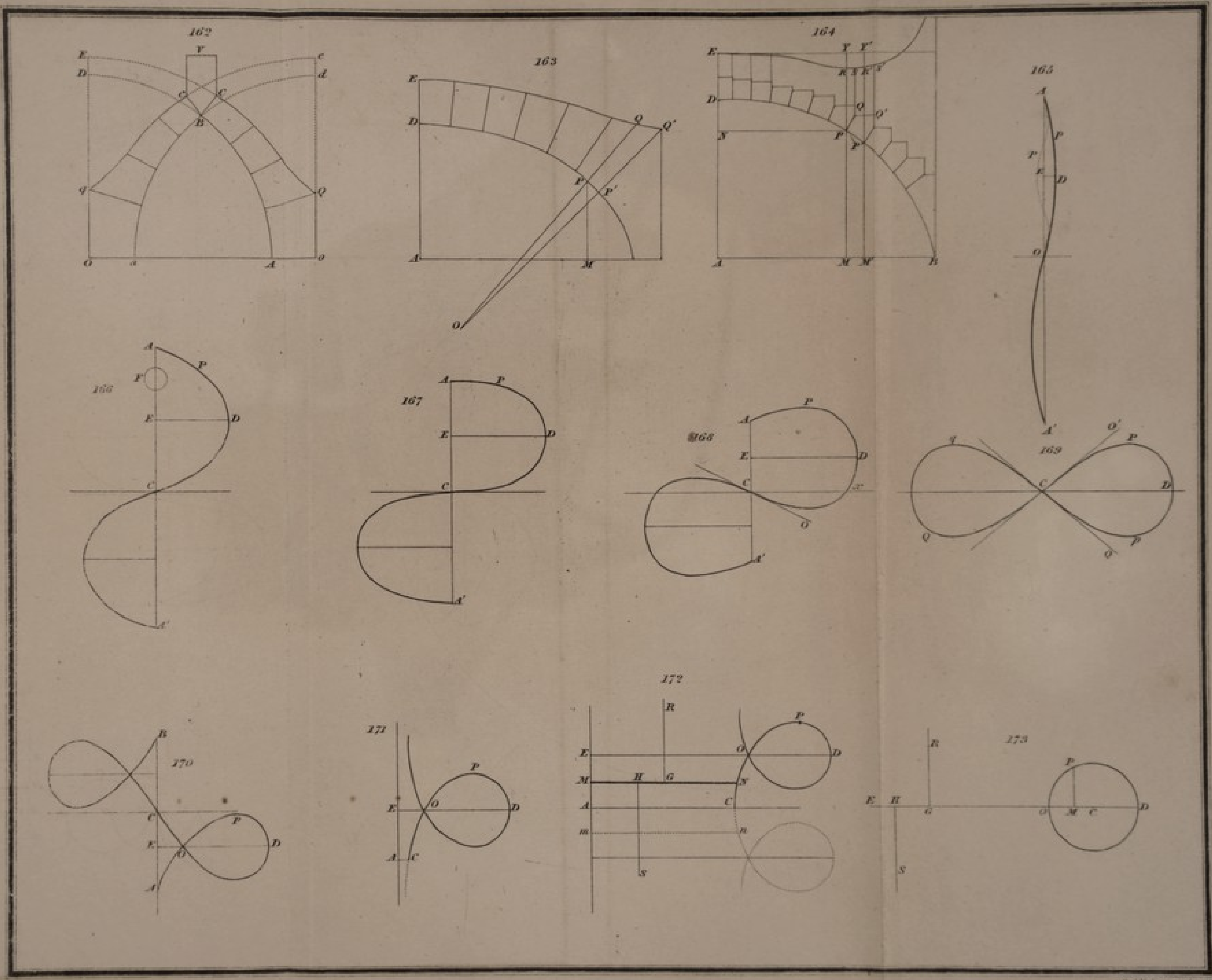
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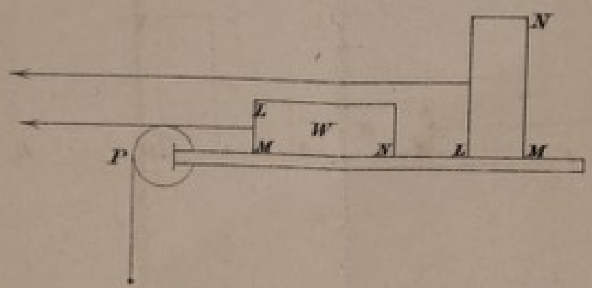

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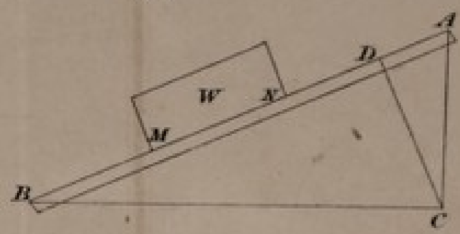

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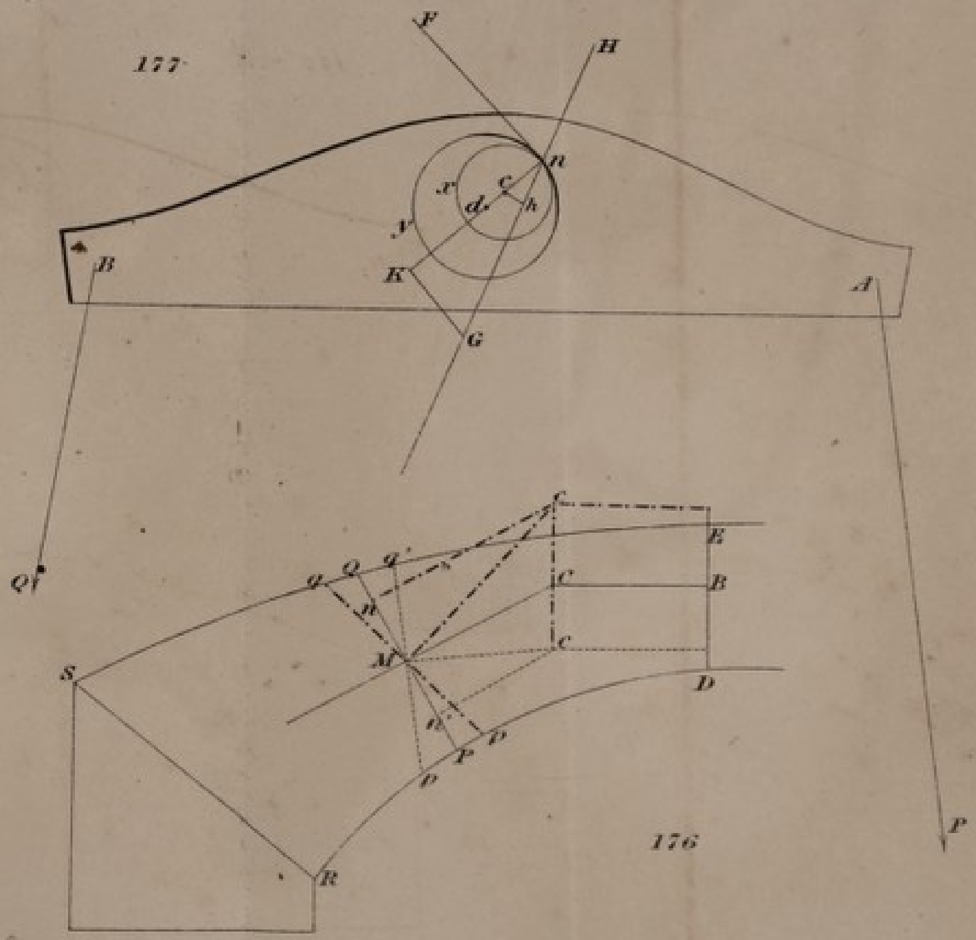

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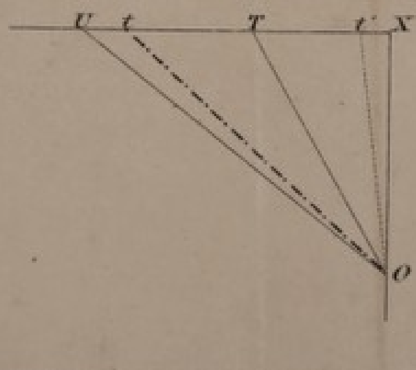

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