THE THEORY

OF THE

EQUILIBRIUM AND MOTION

OF

FLUIDS.

By THOMAS WEBSTER, M.A.

OF TRINITY COLLEGE.

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PREFACE.

In the "Principles of Hydrostatics," published a few months ago, I dwelt in detail on the phenomena which occur in considering the mechanical properties of fluids, and on the principles to which they lead, and I illustrated those principles by their applications in various machines. In the present treatise, which may be considered as a mathematical supplement to the former, taking these principles as established, I have endeavoured to develope them by the application of the Calculus. The two will, I hope, be found to contain the inductive and deductive reasoning which belongs to that department of natural philosophy of which they profess to treat.

The present treatise is compiled principally from the writings of Poisson and Challis, the well-known work of the former having furnished most of the propositions in the equilibrium, as the various papers of the latter have done those in the motion, of fluids; and I have endeavoured to bring before the student what has hitherto been done in this department of science, and to point out the difficulties which present themselves to its further progress. These difficulties are purely mathematical, and I venture to hope, that when it is fully understood that this science and that of Light are at a stand because of the imperfect state of our analysis, some vigorous efforts will be made by those who have time and talents for this pursuit, to remove this barrier, and to place these sciences in the same rank, as inductive and deductive sciences, with that of Gravitation. Much has been done in the last few years, much is almost within our grasp, but much still remains to be done.

The importance of the theory of fluid motion in the present state of science is very great; for the physical and mathematical phenomena of this department present many suggestions in the theories of Light and Heat: thus the way in which the crests of one set of waves in water may be superposed over the hollows of another, producing a level instead of an undulated surface, is strikingly analogous to the interference of the vibrations of two musical strings producing a momentary silence, of two waves of light producing absolute darkness; a complete theory of the one may be the means of leading to a complete theory of the others, and all will advance contemporaneously.

The obligations of this treatise to the published papers of Professor Challis are, as I have stated, considerable; but I am also deeply indebted to him for the assistance which he has afforded me on every occasion of difficulty throughout this work. When I commenced it, many points appeared to me involved in difficulty, and incapable of being explained in an elementary and distinct manner; the reverse is now however the case, as I hope the following pages, and especially Capillary Attraction, (which subject I had considered as hopeless, until he furnished me with the very simple and elementary propositions here given), will testify.

Notwithstanding that great care has been bestowed on the correction of the press, I can hardly hope that the errata will be either few or trivial; and as an author is generally the last person to detect them, I should be extremely obliged to any one who would forward to me or to the Publisher any which he may discover.

T. W.

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CHAPTER I.

ON THE GENERAL PROPERTIES OF FLUIDS.

1. A FLUID may be defined to be a collection of particles which can be moved amongst each other in every direction by any assignable force.

This definition will express the conception of a fluid mass as consisting of a collection of particles which have a connection with each other very different from the connection which subsists between the particles of a solid; and also, that while the particles yield to the least pressure, they yet require the exertion of *some* force to disturb their equilibrium, that is, to change their relative position, or the state of rest in which they exist. Hence it follows, that,

2. PROP. *A fluid may be divided in any direction.*

For the obstacle which prevents the division of a solid mass in any direction, namely, the cohesion of its particles, does not exist here; hence a collection of particles constituting a fluid, may be considered as capable of division in any direction.

3. Any small or elementary portion of a fluid mass may be considered as consisting of a very great number of constituent particles; and into whatever elementary portions we conceive the fluid to be divided, the conditions for the equilibrium and motion of the whole fluid, and of this elementary portion, will be precisely the same, so long as both retain their fluid character.

4. In treating of the equilibrium of fluids, the transmission of pressure is the contradistinguishing property between them and solids. A fluid transmits pressure in all directions, a solid only in one, namely, in the direction in which the pressure is exerted.

This characteristic property is involved in our conception of a fluid mass subjected to pressure and remaining in equilibrium; it may be considered as the necessary consequence of the application of pressure to such a collection of particles as constitute a fluid. For all the particles being equally free to move in all directions, if any number of these particles, that is, any portion of the fluid be subjected to pressure, the particles so acted on will be immediately put in motion, unless the action be counterbalanced by the action of the contiguous particles: this mutual action will extend throughout the whole fluid mass, that is, there will be a pressure transmitted in every direction. Hence it follows, that,

5. Prop. *The transmitted action is equal to the original action.*

The action or pressure which is transmitted will be transferred to every point of the containing surface. At any point in the containing surface, the transmitted action will be balanced by the reaction of the surface. We have then two forces of precisely the same kind impressed on the fluid, namely, the original action and the reaction of the surface. Now this reaction, whatever be its magnitude, will, since it is a pressure impressed on a fluid, give rise to a transmitted action in all directions; and at the point at which the original action is impressed, the original action becomes the reaction; for to suppose it either greater or less, involves an absurdity. The original action then and the reaction at any point being thus convertible and equal, and the reaction being, by the general law of the equality of action and reaction, equal to the transmitted action, the transmitted action is also equal to the original action.

Hence a pressure exerted on a fluid is transmitted equally in all directions; thus fluids *press* in all directions, they also *press equally* in all directions.

6. The preceding is also true for fluids whose particles have sensible tenacity or viscidity, the only difference being, that the pressure is not transmitted in all directions with the same velocity with which it is transmitted in the direction of the impressed action. This deviation is however only instantaneous, and when the equilibrium is established the equality of pressure obtains.

7. The action exerted on any portion of a fluid evidently depends on the number of particles which are acted on, that is, the pressure is proportional to the area pressed.

An area is measured by the number of units of area which it contains, or by the relation which it bears to that unit of area; hence the pressure at any point is most conveniently measured by the pressure which is or would be exerted on a unit of area situated at that point. The quantity (p) is the symbol used to denote the pressure so referred to a unit of area, and it is called the *unit of pressure*, and must be carefully distinguished from the pressure which is actually exerted on any portion of the surface. The quantity p does not represent any pressure actually produced by the fluid, but that which would be produced if the pressure at the point under consideration were uniformly applied to a unit of area.

When the pressure at any point in a fluid is simply the transmitted action from some pressure exerted upon it as at its surface, the pressure p will be the same at every point.

But when, as is generally the case, the impressed forces are different for every point, p will vary from one point to another, that is, it will be a function of x, y, z, the coordinates of the point; and the determination of it for different impressed forces is the object of the following pages.

The pressure exerted on any small elementary area (ω) is represented by $p\omega$; for we may conceive the unit of area sustaining throughout its extent the same pressure as this element; thus ω being the area of this element, the product $p\omega$ will be pressure upon it.

In elastic fluids the pressure bears a constant ratio to the density, the temperature being constant. This ratio is generally expressed by the quantity k, which depends on the nature and temperature of the fluid, being constant for the same fluid at the same temperature. If ρ be the density of the fluid, we have the equation $p = k\rho$. The pressure is in this case the measure of the elastic force of the fluid, and the same equation subsists.

8. Prop. *Any forces being in equilibrium on a fluid, the equation of virtual velocities holds.*

A fluid may be considered as a machine which possesses the property of transmitting equally in all directions the pressures to which it is subjected, hence the general conditions of equilibrium which apply to all other machines must be expected to apply here also; and it will be found that the equation of virtual velocities is true for a fluid in equilibrium, and subjected to pressure.

Now in the general proof of this principle, the absence of all change in the tensions or resistances of the parts of the system, is the supposition on which the whole demonstration rests, and the analogous supposition in a system of fluid particles is, that the *volume* of the fluid is invariable; for if the fluid change in volume, the mutual relation and dependence of the points of the system do not remain unaltered.

Let any forces P, P', P'',...applied to pistons whose areas are a, a', a'',...be in equilibrium on a fluid mass.

Let the points of application of the forces suffer a displacement, that is, let the pistons be moved through spaces h, h', h'',...

Then the new position is to be one of equilibrium, hence we must have as before the displacement,

$$P = pa,\ \ P' = pa',\ \ P'' = pa'',\ \&c. \ldots\ldots (1).$$

Also the volume of the fluid must be the same as before the displacement, or we must have

$$ah + a'h' + a''h'' + \ldots = 0 \ldots\ldots (2).$$

These two conditions obtaining, multiplying (2) by p and substituting from (1), we have

$$Ph + P'h' + P''h'' + \ldots = 0 \ldots\ldots (3);$$

which is the general equation of virtual velocities.

Thus the principle is true for all fluids compressible] viz, while the vol remains const or incompressible by virtue of the equation (2), which is the condition of the system. If this condition do not obtain, the fluid mass is no longer the same machine, being a collection of particles related by different internal forces.

9. The equation of virtual velocities, expresses the relation which may subsist between the external forces which act on a machine, independent of the internal forces or pressures. Assuming then this equation as a general truth, it may be worth while to apply it to a fluid.

Let P, P', P'',...be the forces which are in equilibrium on a fluid, and h, h', h'',...the virtual velocities of the points of application of the forces, then,

$$Ph + P'h' + P''h'' + \ldots = 0 \ldots\ldots\ldots\ldots (1).$$

But the equations of condition are in this case reduced to one, namely, the invariability of the volume of the fluid, which is expressed by the condition

$$ah + a'h' + a''h'' + \ldots = 0 \ldots\ldots\ldots\ldots (2);$$

where a, a', a'' are the areas of the pistons by which the forces are impressed, or they may be considered as expressing the points which suffer displacement.

Let there be two forces in equilibrium on the fluid,

then (1) $Ph + P'h' = 0$ and (2) $ah + a'h' = 0$.

Let $a = a'$,

then $h = -h'$, and $P - P' = 0$, or $P = P'$,

and since these forces may be applied at any part of the surface, the transmitted action is equal in all directions.

Again, let (2) be multiplied by a quantity (q), then subtracting from (1),

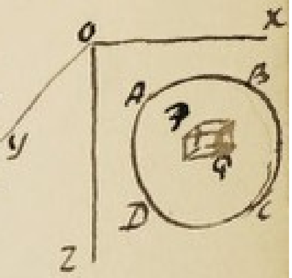
$$(P - qa)h + (P' - qa')h' + \dots = 0,$$

which cannot be satisfied in all cases unless $P - qa = 0$, &c.; whence $P = qa$, $P' = qa'$, &c.; or the pressures are proportional to the areas pressed.

CHAPTER II.

ON THE GENERAL EQUATION OF THE EQUILIBRIUM OF FLUIDS.

10. Prop. *To find the condition of equilibrium of a fluid mass, every particle of which is acted on by given forces.*



Let AB (Fig. 1.) represent any fluid mass in equilibrium, and let it be referred to rectangular co-ordinates whose origin is O. Let the plane of xy be horizontal, and the axis of z vertical; and let x, y, z be the co-ordinates of any point P in the interior of the mass.

Let any elementary parallelopiped PQ of the fluid be taken whose edges are dx, dy, dz, then, if dM be the mass of this element and ρ its density,

$$dM = \rho\, dx\, dy\, dz.$$

Let the impressed forces be resolved in the direction of the co-ordinate axes, and let X, Y, Z be the accelerating force in the directions of x, y, z, respectively, at the point P. Then XdM, YdM, ZdM, are the moving forces on the element dM in the direction of the axes.

The mass dM is then pressed from without to within, on its six faces, by the surrounding fluid, and equilibrium must subsist between the internal forces and the external pressures.

The pressure on the upper face of the element is $p\, dx\, dy$, (Art. 7.). But p is an unknown function of x, y, z, hence for the under face of the element, the co-ordinates of a point in which are x, y, $z + dz$, p will

Just take the □ as having sides dx dy diminishing without limit compared with δz so as to have the p uniform over δx δy (So says Routh) become $p + \frac{dp}{dz} . dz$. The pressure therefore on the under face $= (p + \frac{dp}{dz} . dz) \, dx \, dy$, and the difference of the pressures on the upper and under face is

$$(p + \frac{dp}{dz} dz) \, dx \, dy - p \, dx \, dy = \frac{dp}{dz} . dx \, dy \, dz.$$

The element dM is therefore pressed upwards by this force, and that it may move neither upwards nor downwards, but may remain in equilibrium, we must have

$$\frac{dp}{dz} . dx \, dy \, dz = Z \, dM.$$

But $dM = \rho \, dx \, dy \, dz$; when therefore there is equilibrium, we have

$$\frac{dp}{dz} = \rho Z.$$

Similary, if q, r be the pressures referred to a unit of area on the other faces, we have for the faces parallel to xz and yz respectively,

$$\frac{dq}{dy} = \rho Y \text{ and } \frac{dr}{dx} = \rho X.$$

If the element dM were solid, there would be no necessary connection between the unit of pressure on the faces which are not parallel, but the element might be in equilibrium if the pairs of forces on any two parallel faces were in equilibrium; but the element being a fluid element, and consisting of an indefinite number of fluid particles, it transmits the pressure on any one face to all the others, and the equilibrium cannot subsist unless $p = q = r$; the three preceding equations become therefore

$$\frac{dp}{dx} = \rho X, \quad \frac{dp}{dy} = \rho Y, \quad \frac{dp}{dz} = \rho Z \ldots\ldots (1);$$

which are the general conditions of fluid equilibrium.

Multiply these respectively by dx, dy, dz, adding and observing, that since p is a function of x, y, z, its complete differential is

$$dp = \frac{dp}{dx}dx + \frac{dp}{dy}dy + \frac{dp}{dz}dz$$

we have, whenever there is equilibrium,

$$dp = \rho\,(X dx + Y dy + Z dz) \ldots\ldots\ldots (2),$$

which is the general condition required, and from which the pressure at any point may be obtained.

11. In the preceding investigation we ought, in strict accuracy, to have taken notice of the moving force of the fluid contained in the element, which must be added to the transmitted pressure. If then γ be the pressure due to this force which is exerted on the face $dy\,dz$, we should have $p\,dy\,dz + \gamma$ for the whole pressure which takes place from within to without, or from the right to the left on this same face.

Now the pressure arising from the surrounding fluid, and exerted from without to within, that is, from the left to the right on this face $dy\,dz$, has been represented by $r\,dy\,dz$: this force is the resistance which the surrounding fluid opposes to the pressure transmitted from the interior, that is, to $p\,dy\,dz + \gamma$; hence we must have

$$r\,dy\,dz = p\,dy\,dz + \gamma.$$

But notwithstanding γ being unknown, we are certain that it must be a very small quantity of the third order, and may therefore be omitted in comparison with $p\,dy\,dz$; whence $r = p$, also $q = p$. The same conclusion would have been arrived at, if the element dM, instead of being taken rectangular, had been any polyhedron, indefinitely small in all its dimensions, and it had been shewn that the external pressure exerted perpendicularly on all its faces by the surrounding fluid, is proportional to their respective areas, and independent of the moving force of the polyhedron.

12. The conditions of equilibrium which we have obtained, require that p should be a function of x, y, z, such as will satisfy the three equations (1) at once, or satisfy (2). Hence, that the value of p may be possible, the product of ρ and $Xdx + Ydy + Zdz$ must be a complete differential of some function of the three independent variables x, y, z. Conversely, when this product is a complete differential of a functiou of these variables, the value of p can be found by integration, and the three equations (1) will be satisfied. This then being the case, we have

$$p = \int \rho\,(Xdx + Ydy + Zdz) \ldots\ldots (3).$$

This integral may be taken with regard to any series of consecutive values of x, y, z. Hence, the integral taken with regard to any such series of values, or in other words, the pressure of every line of fluid particles or canal leading to the same point, is the same; this was assumed by Newton as the basis of the theory of the equilibrium of fluids, and would lead at once to the preceding conditions.

The principle that we may integrate in regard to any line whatever of fluid particles drawn from the point to the free parts of the fluid admits of some important applications, as will be seen hereafter.

13. That the equilibrium may subsist, we must have

$$\rho\,(Xdx + Ydy + Zdz)$$

a complete differential, which it is when

$$\frac{d\,.\,\rho X}{dy} = \frac{d\,.\,\rho Y}{dx}, \quad \frac{d\,.\,\rho X}{dz} = \frac{d\,.\,\rho Z}{dx}, \quad \frac{d\,.\,\rho Y}{dz} = \frac{d\,.\,\rho Z}{dy},$$

$$\text{or } \rho\frac{dX}{dy} + X\frac{d\rho}{dy} = \rho\frac{dY}{dx} + Y\frac{d\rho}{dx},$$

$$\rho\frac{dX}{dz} + X\frac{d\rho}{dz} = \rho\frac{dZ}{dx} + Z\frac{d\rho}{dx},$$

$$\rho\frac{dY}{dz} + Y\frac{d\rho}{dz} = \rho\frac{dZ}{dy} + Z\frac{d\rho}{dy},$$

multiplying by Z, $-Y$, X, respectively, and adding,

$$Z\left(\frac{dX}{dy}-\frac{dY}{dx}\right)+Y\left(\frac{dZ}{dx}-\frac{dX}{dz}\right)+X\left(\frac{dY}{dz}-\frac{dZ}{dy}\right)=0. \quad (\lambda)$$

Whenever then such a relation subsists between the forces that this condition is satisfied, the mass of fluid subject to these forces will be in equilibrium. If the fluid be homogeneous or incompressible, ρ is constant, and the conditions become

$$\frac{dX}{dy}=\frac{dY}{dx},\quad \frac{dX}{dz}=\frac{dZ}{dx},\quad \frac{dY}{dz}=\frac{dZ}{dy}.$$

14. The preceding conditions are satisfied whenever the impressed forces are some function of the distance from fixed or moveable centres.

Let P be the law of force, and let it be directed to a centre whose co-ordinates are a, b, c, and r its distance from the point x, y, z. Then

$$X=P\cdot\frac{x-a}{r},\quad Y=P\cdot\frac{y-b}{r},\quad Z=P\cdot\frac{z-c}{r};$$

$$\text{and } r^2=(x-a)^2+(y-b)^2+(z-c)^2;$$

$$\text{then } \frac{dX}{dy}=\frac{dP}{dr}\cdot\frac{dr}{dy}\cdot\frac{x-a}{r}-P\cdot\frac{x-a}{r^2}\cdot\frac{dr}{dy}$$

$$=\frac{dP}{dr}\cdot\frac{y-b}{r}\cdot\frac{x-a}{r}-P\cdot\frac{x-a}{r^2}\cdot\frac{y-b}{r}$$

$$=\left\{\frac{dP}{dr}-\frac{P}{r}\right\}\frac{(x-a)(y-b)}{r^2}.$$

$$\text{Similarly, } \frac{dY}{dx}=\left\{\frac{dP}{dr}-\frac{P}{r}\right\}\frac{(x-a)(y-b)}{r^2}$$

the same quantity; therefore,

$$Z\left(\frac{dX}{dy}-\frac{dY}{dx}\right)=0.$$

1. Eqn will subsist if $\rho(Xdx + \&c)$ be perf. diff
2. Whn (here prove this) it is, if λ be true
3. Whn (λ) is true — as proof in 14

And the same is the case with the other two terms, or the equation of condition becomes identically null, and therefore is satisfied for all laws of force which are a function of the distance. Hence, the laws of gravity and of centrifugal force are evidently such as may produce equilibrium, the one varying as the inverse square, and the other as the direct distance.

15. Prop. *To find the equation to the surface of a free fluid, and to a surface of equal pressure.*

Substituting the co-ordinates of any point in the surface for x, y, z, in the value of p (3), we obtain the pressure which is exerted at this point on the side of the containing vessel; this pressure will always be destroyed by the reaction of the vessel, provided it be fixed and capable of sustaining it. But in those places where the vessel is open, or where the fluid is entirely free, there is no surface whose reaction can destroy the pressure p, consequently we must have this equal to nothing for all the points of the free surface of a fluid in equilibrium; whence

$$Xdx + Ydy + Zdz = 0 \ldots\ldots\ldots (4),$$

is the differential equation to that surface. This equation subsists also if a constant pressure be exerted at the free surface of the fluid.

For at all the points where the pressure is constant, $dp = 0$; hence, the preceding equation obtains for a surface subject to a constant pressure.

Also, if there be any series of points in the interior at which the pressure is constant, for all these points we have the same condition; hence (4) is the equation to a surface of equal pressure.

If the pressure vary from one point to another at the free surface of a fluid, and the pressure referred to a unit of surface at any point x, y, z, be represented by $f(x, y, z)$, the value of p obtained from (4) would coincide for all points on the free surface with this given function; hence the differential equation of the surface would give

$$\rho\,(X dx + Y dy + Z dz) = d\,.\,f(x, y, z).$$

In the following articles the external pressure is always supposed either nothing or constant at all points of the surface of a fluid in equilibrium.

The pressure p being proportional to the density in elastic fluids, it follows, that the pressure can never be nothing in an elastic fluid, unless the density also be nothing, that is, so long as the fluid exists and it has not lost by cold all its elastic force.

An elastic fluid, then, cannot be in equilibrium, unless it is contained in a close vessel or acted upon at every point of its surface by pressures from without to within.

Hence Const. Elastic fluids as bounded by a constant pressure — e.g. an atmosphere with constant press. hemming them in

16. Prop. *The resultant of the forces is perpendicular to the surface at all surfaces of equal pressure.*

At all surfaces of equal pressure

$$X dx + Y dy + Z dz = 0.$$

If now any curve be traced on this surface, and ds be the differential element of the curve, the cosines of the angle which the tangent line at any point x, y, z, makes with the axes of x, y, z, are respectively,

$$\frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds}.$$

Also, the cosines of the angles which the resultant R of the forces X, Y, Z makes with the axes x, y, z, are respectively, since $R = \sqrt{X^2 + Y^2 + Z^2}$,

$$\frac{X}{R},\ \frac{Y}{R},\ \frac{Z}{R}.$$

Hence, dividing the preceding equation by $R\,ds$, it becomes

$$\frac{X}{R}\cdot\frac{dx}{ds} + \frac{Y}{R}\cdot\frac{dy}{ds} + \frac{Z}{R}\cdot\frac{dz}{ds} = 0.$$

Let α, β, γ be the angles which the tangent line makes with the axis, and α', β', γ' the angles which the resultant R makes with the axis, then this equation becomes,

$$\cos\alpha \,.\, \cos\alpha' + \cos\beta \,.\, \cos\beta' + \cos\gamma \,.\, \cos\gamma' = 0$$

which is the condition that two lines should be at right angles to each other.

Hence, the resultant R is perpendicular to the tangent line, that is, it is a normal to the surface.

This force will in general act from without to within, but when the external pressure is not equal to zero, it may be directed from within to without.

17. PROP. *To find the equation to a level surface.*

Definition. The bounding surface of a free fluid, under whatever circumstances the equilibrium takes place, is a level surface; thus, in some cases, this surface may be ellipsoidal; in others, as in the figure of the earth, it will be spheroidal, or very nearly spherical; and, analytically speaking, any surfaces which possess the same properties as a bounding surface, that is, all surfaces of equal pressure are levels or level surfaces.

If we integrate the differential equation to a surface of equal pressure, and give to the arbitrary constant contained in the integral any particular values, the resulting equation will belong to as many surfaces as there are particular values, each of which will have the same differential equation; and, consequently, will possess the properties of equal pressure at all its points, and of being at right angles to the resultant of the forces X, Y, Z.

Those surfaces which are in the interior of the fluid, as determined by the value of the arbitrary constant, that is to say, those series of points within the fluid which are included in the integrated equation, some value being assigned to the arbitrary constant, are level surfaces or *levels*, for they are surfaces of equal pressure.

If the constant vary by very small quantities, the fluid mass is divided into a number of successive layers, or strata, each of which is comprised between levels; hence they are called *level strata*.

The value of the constant which belongs to the surface depends in each case on the given volume of the fluid, so that the external pressure has no influence on the form of equilibrium. The equilibrium will not be disturbed by supposing any part to become solid, hence any constant normal pressures exerted from without to within on all the elements of the surface of a solid or fluid body are destroyed and cannot impress on the body any motion either of translation or of rotation. This equilibrium between the external pressures results from the characteristic property of fluids of transmission in all directions of all pressures exerted on their surface.

18. Let us now suppose the fluid which is in equilibrium to be homogeneous, and of uniform density and temperature throughout. The quantity ρ then being constant, we must have $Xdx + Ydy + Zdz$ an exact differential of the three independent variables; for this is a necessary condition of the equilibrium, and without it the equilibrium cannot take place, whatever form is given to the fluid mass.

Now the condition of integrability is always fulfilled in all those forces which exist in nature, namely, attractions and repulsions; the intensities of which vary as some function of the distance of the centres from which they proceed, (Art. 14). The equilibrium then of a homogeneous liquid subject to these forces is always possible, and that it may really take place, we must give to the fluid a form such that every point of its surface may cut at right angles the resultant of the attractive and repulsive forces.

19. Prop. *The external surface and all internal surfaces of equal pressure are surfaces of equal density.*

Let the fluid be acted on by forces which are some function of the distance, that is, let $Xdx + Ydy + Zdz$ be a complete differential dq. Then the equation (2) becomes

$$dp = \rho dq.$$

That this may subsist when ρ is variable, we must have the density some function of q; and conversely, when this condition is fulfilled there is always a value of p which satisfies the equation of equilibrium.

Let $\rho = \phi(q)$, then integrating and taking q' for the value of q at the external surface,

$$p = \psi(q) - \psi(q').$$

Now this value of p must be the same whatever point on the surface is taken, that is, we must have $\psi(q')$ constant for all points on the surface; hence, at the external surface ρ also is constant, or the external surface is a surface of equal density.

Since $\psi(q')$ is constant, we have q some function of p; hence ρ is constant where p is constant; or surfaces of equal pressure are also surfaces of equal density.

If the fluid be homogeneous, that is, if ρ be constant it is no longer a function of q; and the preceding condition that p is the same where ρ is the same, does not hold.

When the fluid is incompressible, ρ may be any function whatever continuous or discontinuous of the quantity q; when this is given, we may obtain the value of p as a function of ρ by integration.

20. Prop. *In an elastic fluid, surfaces of equal pressure are surfaces of equal temperature.*

In an elastic fluid the density is connected with the pressure by a constant relation, and cannot be arbitrarily

$dp = \rho dq = \phi q\, dq$

$\therefore p = \Phi q + C'$ Hence for p const. q is constant, $\therefore \rho$ constant

Therefore all surfaces of equal pressure - including therefore the free surface - are surfaces of equal dens assigned as in a homogeneous or heterogeneous fluid. The relation betwixt them is expressed by the equation $p = k\rho$; hence, dividing $dp = \rho dq$, we have

$$\frac{dp}{p} = \frac{dq}{k} \text{ ....(5).}$$

If the temperature be constant, k is constant, whence

$$\log p = \frac{q}{k} + C.$$

Let p' be the value of p when $q = 0$, then $\log \frac{p}{p'} = \frac{q}{k}$,

$$\therefore\ p = p'\,\epsilon^{\frac{q}{k}} \text{ and } \rho = \frac{p'}{k}\,\epsilon^{\frac{q}{k}} \text{ ....(6).}$$

If the temperature varies from one point to another, k will not be constant, and it must be some arbitrary function of q. The temperature will also be a function of q, consequently the temperature must be constant throughout each surface of equal pressure of an elastic fluid in equilibrium. Hence, all levels are of equal temperature, and consequently strata of equal pressure are also strata of uniform temperature.

This condition being included, we must replace equations (6) by

$$p = p'\epsilon^{\int\frac{dq}{k}} \text{ and } \rho = \frac{p'}{k}\epsilon^{\int\frac{dq}{k}} \text{ .... (7).}$$

21. When the mass $ABCD$ (Fig. 1) is composed of many different gases, the conditions of equilibrium may be fufilled in two different ways; when the gases are perfectly mixed so as to form a homogeneous mass, and when they are superposed in strata so that the bounding surfaces are levels. The former is the case with the atmosphere, which is found to consist of the same component gases at all heights. This state of perfect mixture is that in which

the equilibrium is the most stable; and when two different gases are superposed in a vessel closed at all parts, they must after a time become perfectly mixed, unless we can insure the containing vessel from the slightest disturbance. This however is not a sufficient explanation of the diffusion of gases through each other, but is a condition which must obtain if they are to remain superposed.

CHAPTER III.

ON THE APPLICATION OF THE PRECEDING THEORY.

22. Prop. *A mass of fluid subjected to a constant pressure at its surface, is acted on by a force varying inversely as the square of the distance from a fixed centre; required the form of equilibrium and the pressure at any point.*

Let this fixed centre be taken as the origin of co-ordinates, let μ be the intensity of the force at distance unity from the fixed centre, then $\frac{\mu}{r^2}$ will be its intensity at a distance r; resolving it into the direction of the axes of co-ordinates, if x, y, z be the co-ordinates of the point acted on,

$$X = \frac{\mu}{r^2} \cdot \frac{x}{r}, \quad Y = \frac{\mu}{r^2} \cdot \frac{y}{r}, \quad Z = \frac{\mu}{r^2} \cdot \frac{z}{r}.$$

The equation to the surface of the fluid (Art. 15) becomes

$$x\,dx + y\,dy + z\,dz = 0;$$

which integrated, gives $x^2 + y^2 + z^2 = c^2$, or r = a constant; that is, the surface is spherical, and the centre of the fluid mass is at the fixed centre of force.

Again, supposing the forces to tend *to* this fixed centre,

$$dp = -\rho \frac{\mu}{r^2} \cdot \frac{x\,dx + y\,dy + z\,dz}{r} = -\frac{\mu \rho\, dr}{r^2};$$ How incompressible

$$\therefore\ p = \frac{\mu \rho}{r} + C.$$

Of course when in applying this formula to get surfaces of equal pressure, we are able to solve it, it must have been a complete differential, so that we do not require to apply the laborious condition of art. 13.

At the surface of the sphere p has a constant value; let p_1 be this value of p when $r = a$;

$$\therefore\ p_1 = \frac{\mu\rho}{a} + C;$$

and subtracting from the preceding

$$p = p_1 + \mu\rho\left\{\frac{1}{r} - \frac{1}{a}\right\}.$$

If the force tend *from* the fixed centre, that is, if it be repulsive instead of attractive, we have only to change the sign of μ in this equation, and

$$p = p_1 - \mu\rho\left\{\frac{1}{r} - \frac{1}{a}\right\}.$$

Suppose this fixed centre to be replaced by a sphere which acts on all the points of the fluid with a force varying as the inverse square of the distance from its centre. Let b be the radius of this sphere. The value of p then would be given by the preceding equation for the points included between $r = b$ and $r = a$.

If the sphere were repulsive, the least value of p would be that corresponding to $r = b$, namely,

$$p = p_1 - \mu\rho\left\{\frac{1}{b} - \frac{1}{a}\right\}.$$

If this expression become negative, the fluid will be detached from the solid sphere and be dispersed in space. Hence we must have p_1, that is the external pressure, greater than $\mu\rho\dfrac{a-b}{ab}$.

In general it is necessary, in the equilibrium of a fluid, that p have a positive value throughout the whole of the mass, so that the contiguous particles may everywhere be sustained one against the other, and the fluid be not separated. For wherever p becomes negative, it indicates a defect of continuity in the fluid.

P Q for a pressure pushing the layer P inwards & one pushing the layer Q outwards will evidently result in tearing the two layers asunder

When the radius is large, the attractive forces directed towards the centre of the sphere may be considered to have their directions parallel; the surface of the fluid will then be plane and perpendicular to the direction of this fluid for a considerable extent.

This is the case of the equilibrium of a heavy liquid, which we shall consider in the subsequent chapter.

23. When then a mass of fluid subject to a central force which varies as the inverse square of the distance from a fixed centre, is in equilibrium, it will consist of spherical layers concentric with this centre, and the resultant of the forces will be in the direction of the radius. If it be a heterogeneous liquid, it is a necessary condition of the equilibrium, that the mass be formed of spherical and concentric layers in which the density is constant throughout the same layer, but varies in any arbitrary manner from one layer to another. In the same manner, if any number of heavy liquids are contained in a vessel, it is a necessary condition of the equilibrium that each horizontal and indefinitely small slice contain only one fluid; and this condition will be fulfilled if the upper surface, which we suppose submitted to a constant pressure, and the surfaces which separate two consecutive liquids, are all plane and horizontal.

The stability of the fluid moreover requires that the densities of the superposed liquids decrease from the lower to the upper liquid, so that the centre of gravity of this system of heavy bodies may be the lowest possible.

24. Prop. *The atmosphere can never be in equilibrium.*

The centrifugal force and deviation from a spherical form of the earth being disregarded, the weight of the particles of air is directed towards the centre of the earth, and the level strata are spherical and concentric. In order therefore that the atmosphere may be in equilibrium, the temperature must be everywhere the same at the same height above the surface of the earth, and vary only with the elevation of successive concentric strata. This, however, is not the case, for the sun warms unequally different points of the surface of the earth and of the level strata of the atmosphere. The temperature depending on the latitude is sufficient to prevent equilibrium taking place in the atmosphere, and produces permanent winds, such as are known to exist near the equator. Moreover, the condition of equilibrium of the atmospheric strata cannot give us any information respecting the variation of temperature in the vertical direction, for the equation (5) of the preceding chapter subsists, whatever function k be of q, and consequently whatever be the law of this variation of temperature.

all round the Earth

25. Prop. *A mass of fluid revolves about an axis; required the form of equilibrium and pressure at any point.*

If a homogeneous or heterogeneous liquid turns uniformly round a fixed axis, the preceding formulæ give us the necessary and sufficient conditions for its preserving a permanent figure, and moving as a solid.

Let us take the axis of rotation for the axis of z, and let r be the distance of any point P from this line, then

$$r^2 = x^2 + y^2.$$

Let a be the angular velocity, which, since the motion is uniform, is constant and common for all points of the fluid mass; then ra is the absolute or linear velocity of the point P; and since it will describe a circle whose radius is r, the centrifugal force is ra^2.

The tendency of this force is to increase r, and its components in the direction of the axis of x and y are

$$ra^2 \cdot \frac{x}{r}, \text{ and } ra^2 \cdot \frac{y}{r}, \text{ or } a^2x, \text{ and } a^2y,$$

A Cylinder round the axis of rotation is the form for a fluid under centrifugal force only which being added to the forces X, Y, Z, the general equation (2) becomes

$$dp = \rho\,(X dx + Y dy + Z dz + a^2 x dx + a^2 y dy) \ldots (a).$$

The expression within brackets is an exact differential, namely, the differential of q increased by the differential of

$$\tfrac{1}{2} a^2 (x^2 + y^2), \text{ or } \tfrac{1}{2} a^2 r^2.$$

Consequently, the form will be one of permanent equilibrium; and if the free surface sustain a constant pressure throughout, the equation common to this surface, and to all other level surfaces, is,

$$X dx + Y dy + Z dz + a^2 (x dx + y dy) = 0 \ldots\ldots (b).$$

In the case of a homogeneous liquid, the free surface will be determined by the integral of this differential equation, the arbitrary constant being determined from the whole volume of the liquid, as we shall see presently.

In the case of a heterogeneous liquid it must be composed of homogeneous strata, the forms of which will also be determined by the integral of this same equation, and which differ from the bounding surface only in the value of the arbitrary constant.

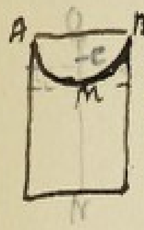
26. PROP. *A mass of liquid in an open vessel and subject to gravity revolves about a vertical axis; required the form of its surface and the pressure at any point.* parabola

Let g be the force of gravity, and let the positive values of z be measured upwards; let a be the angular velocity, then $X = a^2 x$, $Y = a^2 y$, $Z = -g$, and substituting in (4), the differential equation to the surface is

$$a^2 (x dx + y dy) - g dz = 0;$$

whence integrating and adding an arbitrary constant,

$$z = \frac{a^2}{2g}(x^2 + y^2) + c;$$



which is the equation to a paraboloid; hence the free surface of the fluid is that of a paraboloid whose axis is that of rotation, and whose latus rectum is $\frac{2g}{\alpha^2}$.

To determine the arbitrary constant c, let us suppose the vessel to be a vertical cylinder whose axis coincides with the axis of z or of rotation.

Let a be its radius, and h the height due to the absolute velocity $a\alpha$ of the surface, so that $a^2\alpha^2 = 2gh$, and consequently

$$z = \frac{h}{a^2} r^2 + c.$$

Let b be the height of the water before the motion commences, then $\pi a^2 b$ is the whole volume of the liquid which does not change during the rotation; hence dividing the paraboloid into infinitely small cylindrical shells having the axis of z for a common axis, we shall have $2\pi r dr$ for the base, and $2\pi z r dr$ for the volume of the cylindrical shell, whose radius is r and thickness dr. The total volume will then be found by integrating $2\pi z r dr$ from $r = 0$ to $r = a$; whence we may conclude, that

$$\pi a^2 b = \int_0^a 2\pi z r dr, \text{ or } a^2 b = 2\int_0^a z r dr.$$

Now substituting for z its value,

$$\int z r dr = \int\left(\frac{h}{a^2} r^2 + c\right) r dr$$

$$= \tfrac{1}{4}\frac{h}{a^2} r^4 + \tfrac{1}{2} c r^2 + C;$$

$$\therefore \int_0^a z r dr = \tfrac{1}{4}\frac{h}{a^2} a^4 + \tfrac{1}{2} c a^2 = \tfrac{1}{2} a^2 \left(\frac{h}{2} + c\right);$$

But we have seen that $a^2 b = 2\int_0^a z r dr$,

$$\therefore a^2 b = a^2\left(\frac{h}{2} + c\right), \text{ or } c = b - \tfrac{1}{2} h.$$

The equation then of the superior surface of the liquid is

$$z = \frac{h}{a^2} r^2 + b - \tfrac{1}{2} h.$$

The least and greatest value of z, are those which correspond to $r = 0$, and $r = a$, hence calling them $z_{,}$ and z' respectively, we have

$$z_{,} = b - \tfrac{1}{2} h, \quad z' = b + \tfrac{1}{2} h,$$

whence it appears that the depression of the fluid at the axis, and its elevation at the circumference due to the rotation, are each equal to half the height due to the velocity of the circumference.

To find the pressure on the side.

Let $\rho = 1$, then the general equation becomes

$$dp = \alpha^2 (x dx + y dy) - g dz,$$

$$\therefore \; p = \tfrac{1}{2} \alpha^2 (x^2 + y^2) - gz + C.$$

Now at the part of the surface immediately in contact with the sides of the vessel, that is at the highest part of the surface, $p = 0$, and $z = b + \frac{1}{2} h$;

$$\therefore \; 0 = \tfrac{1}{2} \alpha^2 (x^2 + y^2) - g (b + \tfrac{1}{2} h) + C,$$

which subtracted from the preceding gives

$$p = g \left\{ b + \tfrac{1}{2} h - z \right\}.$$

The pressure on any elementary annulus of the side of the cylinder will be $p \times 2\pi a dz$. Therefore the whole pressure on the sides

$$= \int 2\pi a g \left\{ b + \tfrac{1}{2} h - z \right\} dz$$

$$= 2\pi a g \left\{ b + \tfrac{1}{2} h - \tfrac{1}{2} z \right\} z + C$$

which taken between the limits $z = b + \frac{1}{2} h$, $z = 0$,

$$= \pi a g (b + \tfrac{1}{2} h)^2.$$

27. When the forces whereof X, Y, Z, are the components, proceed from the attractions of all the points of

the liquid varying as the inverse square of the distance, or according to other laws, the total values of X, Y, Z depend in general on the form of the liquid and its level surface, and conversely this form depends on the values of these components. This mutual dependence between the attractions of the fluid and its figure, renders the determination of the latter extremely difficult by means of the equation (b). When the liquid is homogeneous, the problem may be solved for the ordinary laws of attraction, that is, of the inverse square of the distance, by supposing the centrifugal force very small, so that the form of the fluid differs but little from the spherical form which it would take if this force were nothing, that is to say, if the fluid were at rest. It may be shewn, that the form of the fluid is necessarily an oblate spheroid, the flattening of which at the poles is determined from the ratio of the centrifugal force at the equator to the attraction on the fluid at the same point. The investigation, however, cannot be given here*.

28. There is an essential difference between the level surfaces traced in the interior of a fluid, subject to the mutual action of all its points, and those which are described in a fluid subject only to extraneous force, that is to say, which are acted on only by attractions or repulsions directed to or from fixed centres, and which are some functions of the distance.

Let $ABCD$ be the surface of a fluid in equilibrium, at rest or turning about a fixed axis. Let $EFGH$ be any level surface, or surface of equal pressure in the interior, and let R be the resultant of all the forces which act in any point M of this surface. In both cases this force will be in the direction of the normal, but in the second case its magnitude and direction not depending on the action of the points of the fluid, will be perpendicular to the surface $EFGH$, if the strata of fluid com-

* See Figure of the Earth, *Encyc. Metrop.* or AIRY's *Tracts.*

prised between the two surfaces be removed, so that after this has been removed, the fluid bounded by $EFGH$ will still be in equilibrium. But in the case of the action of the points of the system, the force R will depend on the action both of the internal fluid and of this intermediate stratum. It will change in general in magnitude and direction when the fluid comprised between $ABCD$ and $EFGH$ is removed, and the fluid bounded by $EFGH$ will no longer be in equilibrium, which can only subsist by the surface becoming perpendicular at each point to the remaining force.

$X\delta x + Y\delta y + Z\delta z = 0$ is necessary condn. of eqm

The action of the external layer comprised between $EFGH$ and $ABCD$, will be nothing on all points in the interior of the fluid and in the surface $EFGH$, when the mass of the fluid is homogeneous and differs but little from a sphere, and the points are only acted on by their mutual attractions, varying as the inverse square and by the centrifugal force. In fact, all surfaces of level are similar ellipsoids, and consequently the fluid comprised between $ABCD$ and $EFGH$ exerts no action on the fluid in the interior of $EFGH$, since the attraction of an ellipsoidal or spherical shell on a point in the interior is nothing.

centrifugal force supp. very small

But this, that the action of a stratum terminated by level surfaces on the interior of the fluid is equal to zero, is not a condition of the equilibrium of fluids; for the forces being such as we have supposed, it is not zero when the fluid is heterogeneous; from which cause the surfaces of levels are dissimilar, but still elliptical, and such that the ellipticity of any surface $EFGH$ depends on the thickness and constitution of the exterior layer.

29. Among the different laws of attraction, there is one which does not exist in nature, but possesses some remarkable properties; this law is that of a mutual action in the direct ratio of the distance, and the remarkable

$x\delta x + y\delta y + z\delta z = 0$ as the eqn of the level surfaces gives $f(xyz) = C$ now if a given volume is to be contained within one, this will fully determine the value of C, and therefore the surface — hence the amount of fluid external is never considered in getting the result. The result wd not be altered were this removed property is that the resultant of the actions of all the particles of a mass on any point is independent of the form and constitution of the body, whether homogeneous or heterogeneous, and the same as if the whole mass were collected at its centre of gravity.

Let x, y, z be the co-ordinates of an attracted point,
x', y', z' .... attracting ......
and μ the mass of this second point, r the distance between them, and $k\mu$ the accelerating force in the direction from the first point to the second, where k is a constant. The components of this force in the directions parallel to the co-ordinate axes, are

$$k\mu\,(x'-x),\;\; k\mu\,(y'-y),\;\; k\mu\,(z'-z).$$

Hence, if X, Y, Z be the resultants of these attractions,

$$X = k\Sigma\mu x' - kx\Sigma\mu,$$
$$Y = k\Sigma\mu y' - ky\Sigma\mu,$$
$$Z = k\Sigma\mu z' - kz\Sigma\mu,$$

the symbol Σ applying to the whole mass of the attracting body.

If m be the whole mass of the body, and $x_{,}$, $y_{,}$, $z_{,}$, the co-ordinates of its centre of gravity,

$$\Sigma\mu = m,\;\; \Sigma\mu x' = mx_{,},\;\; \Sigma\mu y_{,} = my_{,},\;\; \Sigma\mu z' = mz_{,},$$

whence substituting in the preceding,

$$X = km\,(x_{,} - x),$$
$$Y = km\,(y_{,} - y),$$
$$Z = km\,(z_{,} - z),$$

or the forces are the same as if the whole mass were collected at its centre of gravity. and attracting under same law

Substituting these values of X, Y, Z in (b), and making $\dfrac{a^2}{km} = e$, we have,

$$(x_{,}-x)\,dx+(y_{,}-y)\,dy+(z_{,}-z)\,dz+e(x\,dx+y\,dy)=0,$$

whence integrating and adding an arbitrary constant c,

$$(x-x_{,})^2+(y-y_{,})^2+(z-z_{,})^2-e\,(x^2+y^2)=c.$$

This equation is that of the levels of a fluid turning about an axis z, and attracted by a force varying directly as the distance; we can shew that all the surfaces are concentric, and of the second degree.

If the origin be transferred to their common centre, that is, to the centre of gravity of the fluid, the terms involving the first powers of $x_{,}$, $y_{,}$, $z_{,}$ must disappear, or

$$x_{,}=0,\quad y_{,}=0,\quad z_{,}=0;$$

the equation becomes therefore,

$$z^2+(1-e)\,(x^2+y^2)=c;$$

hence the levels are spheroids or hyperboloids, according as e is less or greater than unity, having in both cases the same axis, which is the axis of rotation.

The volume of the fluid being given, the hyperboloid is not possible unless the fluid is contained in a vessel, and then the equation applies only to the free surface of the fluid. When then e is > 1, the permanent figure of a free liquid subject to the laws which we have supposed, is impossible.

If e be < 1, all the levels are spheroidal differing according to the value of c. To determine the value of this quantity corresponding to the external surface, we must equate the volume of the spheroid, which is $\dfrac{4\pi c\sqrt{c}}{3\,(1-e)}$, to the given volume of the liquid.

It is remarkable that in this example the law of the densities of the strata has no influence on the external form, and on that of the levels.

CHAPTER IV.

ON THE PRESSURE OF FLUIDS SUBJECT TO GRAVITY.

30. In the preceding chapters the conditions of the equilibrium of a mass of fluid subject to any impressed forces, have been fully examined; and we have now to consider the application of these principles where the only impressed force is gravity, and where different fluids are in equilibrium with each other.

Prop. *To determine the pressure at any point.*

Let any vessel (Fig. 3.) having its base AB on a horizontal plane contain a mass of liquid whose surface is PQ.

If the liquid is at rest, the surface PQ is perpendicular to the direction of gravity, (Art. 16.), and consequently horizontal or parallel to the base AB of the vessel, since for small spaces the directions of gravity may be considered as parallel.

The equilibrium also of the surface will not be affected by the application of a constant pressure of any magnitude (Art. 15.), and the pressure referred to a unit of surface being the same throughout each level (Art. 17.) when a fluid is in equilibrium, the pressure in this case will be the same throughout each horizontal section of the liquid mass.

Let the surface of the fluid be taken for the plane of xy, then the axis of z will coincide with the direction of gravity.

Hence in the general equation, (Art. 10.), putting

$$X = 0,\ Y = 0, \text{ and } Z = g,$$

we have

$$dp = g\rho dz;$$

whence, if we consider ρ constant, we have by integration,

$$p = g\rho z + C \ldots\ldots\ldots (1).$$

Let the surface be subject to a constant pressure, and when $z = 0$ let $p = p_1$, substituting therefore in (1),

$$p_1 = C;$$

$$\therefore\ p - p_1 = g\rho z, \text{ or } p = g\rho z + p_1 \ldots\ldots (2).$$

If the surface be subject to no constant pressure, p_1 or $C = 0$ and $p = g\rho z$.

The equation (2) expresses the pressure on a unit of surface situated any where in the interior of the liquid, and it may be observed, that the pressure thus found is the sum of two pressures, whereof the one ($g\rho z$) is the weight of the superincumbent column of the fluid, and varies with every value of z, that is, for every point in the liquid, being in fact proportional to the depth of the point; and the other (p_1) is the same for every point, being transmitted equally in all directions throughout the fluid mass.

31. This latter pressure (p_1) being the same at every point, may, for the sake of simplicity when we are considering the pressure of the liquid at any point, be omitted, and the general expression is

$$p = g\rho z.$$

To find the pressure at any point in the base, since the whole base is at the same depth below the surface, putting $z = h$, we have

$$p = g\rho h.$$

Let A be the area of the base, then the whole pressure on the base

$$= pA = g\rho hA.$$

But $g\rho hA$ is the weight of a vertical column of the fluid whose base is A and height h. Hence, the whole pressure exerted by the fluid on the horizontal base of any vessel containing it, is the weight of the superincumbent column of the fluid.

Thus, the pressure exerted by any liquid on the base of the containing vessel, is independent of the forms of those vessels. Hence, if there be any number of vessels standing on the same horizontal plane and filled to the same height with the same liquid, the pressure on their bases if they be equal, or on equal portions of their bases if they be unequal, must be the same whatever be the shapes of the containing vessel, and all experiments shew most distinctly that this is the actual fact.

32. If several liquids be superposed one above another in the same vessel, the only condition requisite for the equilibrium is that the surface of each fluid must be a level, (Art. 17.), that is, in this case horizontal. Thus each fluid will exert a constant pressure on the surface of the one below it, which will be transmitted to all points below it, without in any way affecting the equilibrium of the lower fluids.

33. Prop. *To find the pressure at any point in the bottom of a vessel containing any number of fluids lying one above the other.*

Let $P'Q'$, $P''Q''$, (Fig. 3.) represent the surfaces of fluids lying above PQ, these surfaces being all horizontal, are parallel to each other and to the base AB of the vessel.

Then if h, h', h'', be the thicknesses of the fluids, and ρ, ρ', ρ'', their densities, the pressure on the base AB for the fluid whose surface is PQ, is

$$p = g\rho h. \text{ (Art. 30.)}$$

The pressure on the surface PQ for the fluid whose surface is $P'Q'$, is $p = g\rho' h'$.

The pressure on the surface $P'Q'$ for the fluid whose surface is $P''Q''$, is $p = g\rho'' h''$; and so on, whatever be the number of layers.

The pressure exerted on $P'Q'$ is transmitted to every point of PQ; hence, the pressure at any point in PQ is

$$p = g\rho'' h'' + g\rho' h'.$$

The pressure exerted on PQ is transmitted to every point in AB, hence, the pressure at any point in AB is

$$\begin{aligned} p &= (g\rho'' h'' + g\rho' h') + g\rho h \\ &= g(\rho h + \rho' h' + \rho'' h''). \end{aligned}$$

Hence, whatever be the number of fluids, we should have

$$p = g\Sigma(\rho h),$$

the whole pressure on AB, if A be its area $= p \times A$

$$= g(\rho h + \rho' h' + \dots) A \text{ or } gA\Sigma(\rho h).$$

Thus, whatever be the number of the fluids superposed above each other, the whole pressure which they exert on the base of the containing fluid depends on the magnitude of that base, the thickness and density of the different fluids.

When the vessel is cylindrical and vertical, the whole pressure is equal to the weight of all the fluids, and the pressure will not change however the form of the vessel be changed, provided that the base of the vessel, and the thickness and density of each layer or stratum of fluid are all invariable.

34. Hence, when the same vessel contains different fluids, they are superposed in horizontal layers, and the pressure on the base is the product of its area and the sum of the thickness of each layer multiplied by its density. This result will hold when the thickness of each layer is

indefinitely diminished, that is, when the density of the fluid mass varies continuously in the vertical direction; it is therefore true for compressible fluids. It is equally true when the weight varies from one stratum to another with the density, which is the case when the height of the fluid cannot be neglected, in comparison with the radius of the earth.

The same conclusion is deduced immediately from the equation $dp = \rho g dz$, which applies to the equilibrium of all fluids compressible or incompressible, in which we may suppose that the force of gravity g and the density ρ are functions of the vertical ordinate z.

35. We have now to consider the equilibrium of a liquid contained in several vessels which communicate with each other, so that the liquid may run from one to another. If the apertures be all closed at once the equilibrium will not be disturbed, but the surface of the liquid in each vessel will be horizontal; this condition is not, however, sufficient when the orifices are not closed, and there exists a certain ratio between the elevations of the liquid in the different vessels and their densities.

The conditions of equilibrium under these circumstances are determined in the following propositions.

36. Prop. *When a liquid is in equilibrium in any system of communicating vessels the surfaces of the liquid must be on the same level, that is, when the vessels are near each other, in the same horizontal plane.*

Let AB, CD, (Fig. 4.) be the bases of two vessels standing on the same horizontal plane and communicating with each other, and containing the same liquid.

Let the liquid stand at Pm in one vessel, and at Qn in the other, which are not in the same horizontal plane, but let Qn produced meet the other vessel in ab at a distance h below Pm.

If the equilibrium can exist in this state, it will not be disturbed by replacing the open section Qn by a fixed plane, or by supposing the surface to become rigid. The fluid between Pm and ab exerts on ab a pressure which will be transmitted by the intermediate fluid and impressed on the rigid surface Qn. Let K be the area of Qn, then the pressure thus exerted on this surface from below

$$= pK = g\rho hK.$$

The pressure upwards then on Qn being $g\rho hK$, the fluid cannot be at rest unless this vanishes, which it can only do by h becoming zero. And the preceding demonstration is independent of the forms of the containing vessels, which may be supposed any whatever.

Hence, when the vessels are contiguous, the surface of the liquid in each must be in the same horizontal plane. If the vessels are not contiguous, but at a considerable distance from each other, the preceding reasoning will apply, ab and Qn being taken on the same level, for all level surfaces are surfaces of equal pressure; it will follow therefore that the equilibrium is not possible unless Pm and Qn are on the same level.

This proposition is also evidently true for any number of vessels, for since the surface in *any* two will be on the same level, the surface in all must be on the same level.

37. PROP. *To determine the condition of equilibrium of several fluids contained in any communicating vessels.*

Let PQ, $P'Q'$,...and $P_{,}Q_{,}$, $P_{,,}Q_{,,}$,... (Fig. 5.) be the bounding surfaces of several liquids, which are contained in two vessels which communicate with each other.

Let PQ be the level surface in which the two fluids bounded by $P'Q'$ and $P_{,}Q_{,}$ meet, and let PQ produced intersect the other vessel in mn. Then if a fixed plane be supposed at mn, or if the particles in the surface mn

As in last insert a rigid plane
note this must be at lowest separat. level

become rigid, the transmitted pressure of the liquids superposed above PQ will be sustained by the rigidity of this surface.

Let K be the section of the vessel at mn, and let ρ', ρ''...be the densities of the liquid contained between PQ and $P'Q'$, $P'Q'$ and $P''Q''$, &c. respectively, and h', h''...the thickness of the layers.

Then the transmitted pressure on mn

$$= pK$$

$$= g\Sigma(\rho' h') K. \quad \text{(Art. 33.)}$$

But this upward pressure may be counterbalanced by strata of liquids superposed above mn.

Let $\rho_{,}, \rho_{,,}$...be the density, and $h_{,}, h_{,,}$, the thickness of the successive strata of superposed fluids which effect this equilibrium, their surfaces being at $P_{,}Q_{,}, P_{,,}Q_{,,}$...

Then the pressure on $mn = pK$

$$= g\Sigma(\rho_{,} h_{,}) K,$$

and the equilibrium will subsist if this equals the pressure on the under surface, that is, the condition required is

$$\Sigma(\rho' h') = \Sigma(\rho_{,} h_{,}).$$

If there are only two fluids the equation becomes

$$\rho' h' = \rho_{,} h_{,},$$

that is, their elevations above the point in which they meet are inversely as their densities.

38. The following method of arriving at the same condition is given on account of the beautiful example it affords of the application of the principle of Virtual Velocities.

Let any number of fluids whose surfaces are at P, P'... in one vessel be in equilibrium with the fluids whose surfaces are at $P_{,}, P_{,,}$ in another. (Fig. 5.)

Let PQ be the horizontal plane in which the two fluids whose surfaces are at P' and $P_{,}$ meet.

Then if the fluid is in equilibrium, we may consider the pressures of the superposed strata as forces impressed on every particle in the surfaces at P and $P_{,}$ of the fluid machine. Then if k be the area of PQ, and K of mn, we have, using the same notation as before,

$$\text{the impressed force on } PQ = pk = gk\Sigma(\rho' h'),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; P_{,}Q_{,} = pK = gK\Sigma(\rho_{,}h_{,}).$$

Now let a small displacement consistent with the conditions of the system be given to the points of the application of the forces; that is, let the surface PQ and $P_{,}Q_{,}$ move through the vertical spaces α, $-\beta$, respectively; then by the principle of virtual velocities,

$$g\Sigma(\rho' h') \cdot k\alpha - g\Sigma(\rho_{,}h_{,}) \cdot K\beta = 0.$$

But the system must be invariable, that is, we must have $k\alpha - K\beta = 0$,

$$\text{whence } \Sigma(\rho' h') - \Sigma(\rho_{,}h_{,}) = 0,$$

$$\text{or } \Sigma(\rho' h') = \Sigma(\rho_{,}h_{,}).$$

If there be but two liquids the condition becomes

$$\rho' h' = \rho_{,}h_{,},$$

or their elevation above the horizontal plane in which they meet is inversely as their density.

If there is but one liquid, or $\rho' = \rho_{,}$,

$$\text{then } \rho'(h' - h_{,}) = 0,$$

which can only be satisfied if $h' = h_{,}$, or the surfaces must be in the same level.

39. The general condition at which we have arrived for the equilibrium of several fluids in a system of communicating vessels, is

$$\Sigma(\rho' h') = \Sigma(\rho_{,}h_{,}).$$

Either of the columns may be supposed to consist of indefinitely thin strata, whose density is uniform throughout the extent of any one stratum, but differing much from one stratum to another.

Also it is quite immaterial, so far as the actual equilibrium is concerned, in what order the densities succeed each other; for if this equation be satisfied, the columns may be in equilibrium without any reference to the order of succession of the densities.

Thus a heavier fluid may lie above a lighter, a liquid above a gas; such an arrangement, however, is not one of stable equilibrium; for the equilibrium of a system of particles is only stable when the centre of gravity is the lowest possible.

Such an arrangement then, being one of unstable equilibrium is theoretically possible, and also practicable, if any means be taken to prevent the least disturbance of any of the particles of the system. If, however, such a disturbance be not guarded against, the different fluids will pass into a position of stable equilibrium, in which the heaviest fluids will occupy the lowest place.

Thus as we have seen (Art. 24.) the atmosphere can never be in a state of stable equilibrium, since the different parts of all vertical columns are subject to great variations in density.

40. In applying the preceding condition where the atmosphere is one of the fluids, it will be convenient to assume

$$\Sigma(\rho_{,}h_{,}) = \rho h.$$

For the effect produced is the same as may be produced by a homogeneous column. If g be the accelerating force of gravity, the weight of the atmospheric column may in the same manner be represented by $g\rho h$, the weight of a column of uniform density ρ, and given height h, and whose base is equal to the unit of area.

41. If the surface Qn (Fig. 4.) be supposed rigid, it will sustain a column of fluid whose surface Pm is of any height, and the pressure on its under surface will be

$$pK = g\rho hK.$$

If W then be the whole upward pressure on Qn, we have

$$W = g\rho hK,$$

which may be increased indefinitely by increasing K and h, the unit of area remaining the same. But we may conceive Qn replaced by a piston so loaded as to equal W. There will then be equilibrium as before. Thus an enormous downward pressure may be sustained by the upward pressure, which is transmitted from the weight of a column of fluid of small section but considerable height; thus it is evident that an exceedingly small quantity of water may be made to sustain or raise a weight however large.

42. When water is contained in any vessel or in a system of communicating tubes or vessels, and exposed at its upper surface to the atmospheric pressure, it will be in equilibrium when the whole surface or the surfaces in the different vessels are in the same horizontal plane, whatever be the pressure to which their surfaces are subject, provided it be the same for all the surfaces. But the equilibrium may also subsist under certain circumstances when the vessels are inverted, and the conditions requisite for equilibrium in these cases are supplied by the equation of the preceding articles.

Suppose a vessel, as for instance, a tumbler full of water to be inverted, then since we have only two fluids, air and water, the preceding equation of equilibrium reduces itself (Art. 40.) to

$$\rho' h' = \rho h,$$

where ρ' is the density of the water, and ρ of the air supposed homogeneous, and h', h are the elevations of the fluids above the horizontal plane in which the fluids meet; and as long as this equation is satisfied the equilibrium is possible.

The equilibrium however cannot under these circumstances actually take place, for this being an instance of unstable equilibrium, any disturbance which causes the least displacement in any part of the surface of the fluid will destroy the equilibrium. If a piece of paper be laid on the surface of the water, the vessel may then be inverted and the water will remain suspended, the particles of the water being insured from any displacement by the rigidity of the paper. If the vessel be of small diameter, as a capillary tube, the molecular action will insure the particles at the surface from displacement.

But the most usual way of effecting this in practice is to invert the heavy liquid over a basin containing the same liquid, which serves the double purpose of insuring the stability of the surface and of permitting the superincumbent column to vary in altitude, the equilibrium still remaining stable.

These conclusions will be sufficiently illustrated by an explanation of the siphon and barometer.

43. *The Siphon.* Let a tube *ABC* (Fig. 6.) be partly filled with water, and inverted; let *P* and *Q* be the surfaces of the water in the two branches, they will be in the same horizontal plane when there is equilibrium.

Then if ρ' be the density of the water and h' the height of *B* above *P* or *Q*, the condition for the equilibrium of these columns with the atmospheric column is

$$\rho' h' = \rho h.$$

But the equilibrium will also subsist if ρh be $> \rho' h'$, since it is the same for both columns, that is, if the pressure exerted by the atmosphere at P and Q be greater than the weights of the columns BP or BQ of water.

Let p_1 be the atmospheric pressure, then $g\rho' h'$ is the weight of the column of water whose section is unity; and we must have p_1 equal to or greater than $g\rho' h'$.

If $p_1 = g\rho' h'$, the pressure at B will be nothing.

If p_1 be $> g\rho' h'$, the pressure at $B = p_1 - g\rho' h'$.

If p_1 be $< g\rho' h'$, $p_1 - g\rho' h'$ becomes negative, or the liquid will separate at B (Art. 23.); and no means being taken to insure the stability of the surfaces, it will run out.

When the extremity Q of the water is a little below or a little above the extremity P, the excess of the atmospheric pressure above the pressure of the liquid, is greater or less at the point P than at the point Q, and the liquid runs out in one case by the branch BC and in the other by the branch BA of the siphon.

In practice the shorter leg of the siphon is placed in a vessel of water, and consequently as long as the surface P in the shorter leg is above the surface Q in the longer leg, the liquid will be discharged at C.

44. *The Barometer.* Any vessel full of a liquid and inverted over a basin of the same or a different liquid, so as to secure the equilibrium of the surface, becomes a barometer.

In general, the barometer consists of a straight tube filled with mercury and inverted over a basin of the same liquid, or a tube ABC (Fig. 7.) is bent at B, so that the legs AB, BC are parallel.

Let P be the surface of the mercury in AB, and Q its surface in BC.

Then if ρ' be the density of the mercury, and h' the height of the surface P above the surface Q, we have as the condition of equilibrium,

$$\rho' h' = \rho h.$$

The equilibrium being established, we may conceive the branch BC to be prolonged vertically to the extremity of the atmosphere; consequently, the atmospheric pressure which is in equilibrium with the column of mercury, is the weight of the air contained in a vertical cylinder whose base is equal to the unit of surface and height equal to the extent of the atmosphere. The weight of this column varies with the variations of gravity, it is therefore less as we ascend from the surface of the earth; it varies also with the density, the temperature, and the quantity of vapour which exists in the atmosphere.

If any other liquid be used in the barometer instead of mercury, the altitude of the column sustained will be inversely as the density, that is, as the specific gravity of the fluid; and if several fluids be used, superposed one above another, the equilibrium will subsist when the condition $\Sigma(\rho' h') = \rho h$ (Art. 23) is satisfied.

If the vacuum above the surface of the mercury at P be not perfect, as if a small quantity of air or other elastic fluid occupy the space AP, the effect of its elastic force estimated by the weight of the column which it would sustain (as will be seen in the following article) must be taken into the equation of equilibrium.

The pump is the same in principle as the barometer; a column of water is in equilibrium with the atmospheric column, the piston of the pump, which is air-tight, being the closed end of the inverted tube. When the pump is not full of water it is an instance of an imperfect barometer, where the elastic force of the air between the piston and the sustained column must be taken account of.

45. *The Manometer.* The equilibrium being established, we may conceive a close vessel as represented by the dotted lines (Fig. 7.) to be attached to the open tube C of the barometer. All communication being thus cut off with the external air, the column of mercury is no longer sustained by the weight of the superincumbent atmospheric column, but by the elastic force of the air which has been enclosed in this vessel. The variations of gravity may be observed by this instrument, as will hereafter be shewn. It is thus that the elastic force of different gases may be measured and expressed in terms of the height of a mercurial or other barometric column.

CHAPTER V.

ON THE PRESSURE OF FLUIDS ON SURFACES.

46. When a body is immersed in a fluid or when any vessel contains fluid, each portion of the surface of the body immersed in it and each portion of the containing vessel sustains a pressure, the magnitude of which may be determined.

Prop. *To find the pressure exerted by a fluid on a surface in contact with it.*

Let S be the surface the pressure on which is required.

Let dS be any elementary portion of the surface at a depth z below the surface of the fluid.

The whole pressure on this element $= p\,dS$

$$= g\rho z\,dS.$$

The whole pressure on the surface S will be the sum of the pressures on all such elements, and will therefore be found immediately by the Integral Calculus.

The whole pressure on $S = \int g\rho z\,dS$

$$= g\rho \int z\,dS.$$

But $\int z\,dS$ is the moment of the surface S about a line in the surface of the fluid at right angles to it, and therefore, by the property of the centre of gravity, is the same as of the whole body collected in its centre of gravity about this line.

Let Z be the depth of the centre of gravity of S;

$$\therefore \int z\,dS = Z \,.\, S,$$

or the whole pressure $= g\rho Z \,.\, S$.

But $g\rho ZS$ is the weight of a column of fluid whose height is Z and base equal to the surface S.

The whole pressure therefore on any surface is the weight of a column of the fluid whose base is equal to the area of the surface pressed, and whose height is equal to the depth of the centre of gravity of the surface pressed below the surface of the fluid.

Ex. 1. A cone with its base downwards and filled with fluid.

Let a be the height of the cone (Fig. 8.), and b the radius of its base. Then for the base, $Z = a$, $S = \pi b^2$;

$$\therefore \text{ the whole pressure on } AB = g\rho Z \,.\, S$$
$$= g\rho \,.\, a \,.\, \pi b^2$$
$$= g\rho\pi a b^2 \ldots\ldots\ldots\ldots\ldots(1).$$

For the surface, $Z = \frac{2}{3}a$, $S = \pi b\sqrt{a^2 + b^2}$;

therefore the whole pressure on the surface of the cone

$$= \tfrac{2}{3} g\rho\pi a b\sqrt{a^2 + b^2} \ldots(2).$$

The weight of the contained fluid

$$= g\rho \,.\, \tfrac{1}{3}\pi b^2 \,.\, a$$
$$= \tfrac{1}{3} g\rho\pi a b^2 \ldots\ldots\ldots\ldots(3).$$

Then comparing (1) and (3), it appears that the pressure on the base equals three times the weight of the contained fluid.

The pressure on the base AB is the same for all the vessels represented by the dotted lines, since the base and the depth of its centre of gravity is the same in all of them.

47. *Centre of pressure.* When the bottom of the vessel, or the plane immersed, is horizontal, the pressure on every point is the same; and the forces being all parallel, the centre of these forces will be the centre of gravity of the plane, and their resultant passing through this point will be the sum of all these forces.

But when the plane is inclined at any angle to the surface of the fluid the pressure is not the same at all points, but greater at the lower than at the upper points; and the resultant of these forces will not pass through the centre of gravity of the surface, but through a point below it, which is called the centre of pressure. This point will evidently be below the centre of gravity for all fluids in which the pressure varies as the depth.

Definition. The centre of pressure of any surface immersed in a fluid is the point in which the resultant of the pressures of the fluid meets the surface. See 54 to justify this definition

If then this surface form part of the containing vessel and be supposed moveable, it will be kept at rest by a pressure equal to the sum of these pressures applied at this point in an opposite direction.

48. Prop. *To find the centre of pressure of any plane surface.*

Let ABC (Fig. 9.) be any plane surface in a fluid, and DE the water-line, that is, the line in which the plane (supposed produced) cuts the surface of the fluid.

Let A be the origin of the rectangular co-ordinates to which the plane surface is referred, and through A, drawing OAx perpendicular to DE; take Ax for the axis of x, and Ay parallel to DE for the axis of y.

Let P be any point x, y in the plane surface, and PQ an element $dx\,dy$, then $AN = x$, $NP = y$.

Let $l = OA$ be the distance of the origin A from the water-line, and θ the inclination of the plane to the surface of the fluid.

Let X, Y be the rectangular co-ordinates of the centre of pressure, measuring from A.

Then since the pressure at each point is the weight of the superincumbent column, the pressures are all parallel forces; and taking the moments with respect to Ay and Ax,

X. (pressure on ABC) = moment of pressure on ABC. (1),

Y. (pressure on ABC) = moment of pressure on ABC. (2).

And the moment of the pressure on ABC equals the sum of the moments of the pressures on the small elements into which the plane is divided.

The pressure on the element $dx\,dy$ at $P = p\,dx\,dy$.

Hence, the pressure on $ABC = \iint p\,dx\,dy$.

Drawing PM perpendicular to the surface of the fluid,

$$p = g\rho \cdot PM = g\rho\,(x + l)\sin\theta\,;$$

$\therefore$ the pressure on $ABC = g\rho \sin\theta \iint (x + l)\,dx\,dy$.

Multiplying this by x to obtain the moment about Ay.

The moment of the pressure on ABC about Ay

$$= g\rho \sin\theta \iint (x^2 + lx)\,dx\,dy.$$

Similarly, multiplying by y to obtain the moment about Ax.

The moment of the pressure on ABC about Ax

$$= g\rho \sin\theta \iint (x + l)\,y\,dx\,dy.$$

Substituting in (1) and (2), and omitting the common factor $g\rho \sin\theta$, which will be constant for the same plane, the fluid being supposed homogeneous,

$$X \iint (x + l)\, dx\, dy = \iint (x^2 + lx)\, dx\, dy \ldots\ldots (3),$$

$$Y \iint (x + l)\, dx\, dy = \iint (x + l)\, y\, dx\, dy \ldots\ldots (4),$$

whence X and Y may be determined.

When the surface is bounded by a curve whose equation is given, we have y a function of x; hence, integrating with respect to y,

$$X \int (x + l)\, y\, dx = \int (x^2 + lx)\, y\, dx,$$

$$\text{or } X = \frac{\int (x^2 + lx)\, y\, dx}{\int (x + l)\, y\, dx} \ldots\ldots (5),$$

$$\text{and similarly, } Y = \tfrac{1}{2} \frac{\int (x + l)\, y^2\, dx}{\int (x + l)\, y\, dx} \ldots\ldots (6),$$

which are the formulæ generally used. For the value of y being substituted in terms of x they are immediately integrable, and the integrals being properly corrected, the co-ordinates will be given in terms of known quantities.

When the plane coincides in any point with the surface of the fluid, we have if this point be the origin of co-ordinates, $l = 0$; and the preceding equations become

$$X = \frac{\int x^2 y\, dx}{\int xy\, dx} \ldots\ldots (7), \quad Y = \tfrac{1}{2} \frac{\int x y^2\, dx}{\int xy\, dx} \ldots\ldots (8),$$

which would be obtained at once by the same steps as the more general case here given.

49. It will be observed that these expressions are independent of θ, the inclination of the plane to the surface of the fluid; the position therefore of the centre of pressure will remain the same if the plane revolve through any angle about DE as an axis, so long as l, that is, the distance OA is the same.

The equations (7) and (8) are those for determining the centre of percussion of a plane moveable about the axis Ay, hence the centres of pressure and percussion coincide when the plane meets the surface of the fluid.

50. If the plane be sunk to a great depth, the centres of gravity and pressure coincide.

In equations (3) and (4) let $l = \infty^{\text{ty}}$, then

$$X = \frac{\iint xy\,dx\,dy}{\iint dx\,dy} = \frac{\int xy\,dx}{\int y\,dx},$$

when the surface is bounded by a plane curve, and

$$Y = \frac{\iint xy\,dx\,dy}{\iint dx\,dy} = \frac{1}{2}\frac{\int xy^2\,dx}{\int y\,dx}:$$

which are the co-ordinates of the centre of gravity of a plane surface bounded by a plane curve: hence the centres of pressure and gravity coincide.

When the centres of pressure and gravity do not coincide, that which is above or below the other in any one position of the plane is always above or below it. The centre of pressure is above or below the centre of gravity, according as its vertical co-ordinate is less or greater than the vertical co-ordinate of the centre of gravity, that is, as

$$\frac{\int (x^2 + lx)\,y\,dx}{\int (x + l)\,y\,dx} \text{ is } < \text{ or } > \frac{\int xy\,dx}{\int y\,dx},$$

$$\text{or as } \left(\int x^2 y\,dx\right)\left(\int y\,dx\right) \text{ is } < \text{ or } > \left(\int xy\,dx\right)^2,$$

which expression being independent of l, it follows that the relative positions of these centres cannot change for different depths.

51. Prop. *To determine the centre of pressure of a plane made up of portions whose centres of pressure are known.*

Let A be the area of a plane consisting of portions $A_1, A_2, A_3, \ldots$ whose centres of pressure are known, then

$$A = A_1 + A_2 + A_3 + \ldots$$

Let x_1, x_2...and y_1, y_2...be the co-ordinates of the centres of pressure of these portions.

Let h_1, h_2...and k_1, k_2...be the co-ordinates of their centres of gravity.

$$\text{Then } x_1 = \frac{\int x^2 y\, dx}{\int xy\, dx} \text{ between proper limits.}$$

Now $\int xy\, dx = A_1 h_1$, by the property of the centre of gravity;

$$\therefore \int x^2 y\, dx = A_1 h_1 x_1,$$

and similarly for all the other portions. But if X, Y be the co-ordinates of the centre of pressure of the whole surface, since the pressure on each portion may be considered as a single force through its centre of pressure, we have by the general proposition for the centre of any number of parallel forces,

$$X = \frac{\Sigma \int x^2 y\, dx}{\Sigma \int xy\, dx}, \quad Y = \frac{\Sigma \frac{1}{2} \int xy^2\, dx}{\Sigma \int xy\, dx};$$

$$\therefore X = \frac{A_1 h_1 x_1 + A_2 h_2 x_2 + \ldots}{A_1 h_1 + A_2 h_2 + \ldots},$$

$$Y = \frac{1}{2} \frac{A_1 k_1 y_1 + A_2 k_2 y_2 + \ldots}{A_1 k_1 + A_2 k_2 + \ldots},$$

which determine the centre of pressure of this compound area.

52. The formulæ in the preceding articles are sufficient to determine the centre of pressure in all cases, in consequence, however of the difficulty of expressing y as a function of x in some cases, particular methods are more convenient than the application of the general formula, as will be seen in some of the following examples.

When the plane is symmetrical about the axis of x, the equation (5) or (7) is sufficient to determine the centre of pressure since it lies on the axis of x, the equation (6) or (8) being nothing in the case of a symmetrical plane.

Similarly, if the plane be symmetrical about the axis of y, the equation (6) or (8) will be sufficient.

53. Ex. 1. A semiparabola at a given depth below the surface of the fluid with its axis vertical.

The vertex being uppermost and the axis being taken for the axis of x, the ordinates y will be parallel to the intersection of its plane with the surface of the fluid.

The formulæ here are

$$X = \frac{\int (x^2 + lx)\, y\, dx}{\int (x + l)\, y\, dx}, \quad \text{and } Y = \frac{1}{2} \frac{\int (x + l)\, y^2\, dx}{\int (x + l)\, y\, dx}.$$

In the parabola $y^2 = 4mx$;

$$\therefore \int (x^2 + lx)\, y\, dx = 2\sqrt{m} \int (x^{\frac{5}{2}} + l x^{\frac{3}{2}})\, dx$$

$$= 2\sqrt{m} \left(\tfrac{2}{7} x + \tfrac{2}{5} l\right) x^{\frac{5}{2}};$$

$$\int (x + l)\, y\, dx = 2\sqrt{m} \int (x^{\frac{3}{2}} + l x^{\frac{1}{2}})\, dx$$

$$= 2\sqrt{m} \left(\tfrac{2}{5} x + \tfrac{2}{3} l\right) x^{\frac{3}{2}};$$

$$\int (x + l)\, y^2\, dx = 4m \int (x^2 + lx)\, dx$$

$$= 4m \left(\tfrac{1}{3} x + \tfrac{1}{2} l\right) x^2;$$

no correction being requisite since the integrals begin with x and y;

$$\therefore X = \frac{\frac{2}{7} x + \frac{2}{5} l}{\frac{2}{5} x + \frac{2}{3} l} \cdot x = \frac{\frac{1}{7} x + \frac{1}{5} l}{\frac{1}{5} x + \frac{1}{3} l} \cdot x,$$

$$Y = \sqrt{m} \cdot \frac{\frac{1}{3} x + \frac{1}{2} l}{\frac{2}{5} x + \frac{2}{3} l} x^{\frac{1}{2}} = \frac{1}{2} \sqrt{mx} \cdot \frac{\frac{1}{3} x + \frac{1}{2} l}{\frac{1}{5} x + \frac{1}{3} l},$$

whence the position of the centre of pressure is known.

Let the vertex of the parabola be in the surface of the fluid, or $l = 0$, then

$$X = \tfrac{5}{7}\,x, \text{ and } Y = \tfrac{5}{6}\sqrt{mx} = \tfrac{5}{12}\,y,$$

as would have been determined at once from equations (7) and (8).

Let the parabola be sunk to a great depth, or $l = \infty^{ty}$, then

$$X = \tfrac{3}{5}\,x, \quad Y = \tfrac{3}{4}\sqrt{mx} = \tfrac{3}{8}\,y,$$

which are the expressions for the centre of gravity of a semiparabola, x and y being the extreme ordinates.

Ex. 2. A rectangular flood gate its upper side coinciding with the surface.

Bisect the upper side, and taking this point for the origin of co-ordinates, equation (7) is sufficient.

Let a be the vertical, and b the horizontal side of the gate, then

$$X = \frac{\int_0^a x^2 y\,dx}{\int_0^a xy\,dx} = \frac{\int_0^a x^2\,dx}{\int_0^a x\,dx}, \text{ since } y \text{ is constant,}$$

$$= \frac{\frac{1}{3}a^3}{\frac{1}{2}a^2} = \tfrac{2}{3}\,a.$$

The whole pressure on the gate $= g\rho \times \dfrac{a}{2} \times ab$ (Art. 46.)

$$= \tfrac{1}{2} g\rho a^2 b.$$

If then a force be applied at the centre of pressure in an opposite direction to the pressure of the fluid, and equal in magnitude to $\frac{1}{2}g\rho a^2 b$, the gate will be kept at rest by this single force.

Hence may be found the moment of the pressure to turn the gate about a given axis.

The moment to turn the gate about a vertical axis at the side

$$= \tfrac{1}{2} g\rho a^2 b \,.\, \frac{b}{2} = \tfrac{1}{4} g\rho a^2 b^2.$$

The moment about an axis in the surface

$$= \tfrac{1}{2} g\rho a^2 b \times \tfrac{2}{3} a$$

$$= \tfrac{1}{3} g\rho a^3 b.$$

The moment about an axis at the bottom of the gate

$$= \tfrac{1}{2} g\rho a^2 b \times \tfrac{1}{3} a = \tfrac{1}{6} g\rho a^3 b.$$

The flood gate may turn about a vertical axis on a single hinge at the depth of $\frac{2}{3}$ds the height of the gate, and then a force equal to $\frac{1}{2} g\rho a^2 b$ at the centre of pressure, or $\frac{1}{4} g\rho a^2 b$ at the opposite side will keep the gate at rest.

If it turn about an axis in the surface, then a force equal to $\frac{1}{3} g\rho a^2 b$ applied at the bottom of the gate will keep it at rest.

If it turn about an axis coinciding with its bottom, the force which must be applied at the top is $\frac{1}{6} g\rho a^2 b$.

In the preceding reasonings the height of the water is supposed to be invariable.

Ex. 3. The staves of a barrel held by a single hoop.

From the preceding example it appears that the depth of the centre of pressure of any rectangle is equal to $\frac{2}{3}$ds of its height. Hence if we conceive a barrel to be composed of a great number of similar staves, each of which differs insensibly from rectangular ones, they present a plane surface to the fluid, and the barrel when full of fluid will be kept together by a single hoop passing through the centre of pressure of all the staves. Hence the hoop must be at a distance equal to $\frac{2}{3}$ds of the height of the barrel from the top.

If the staves be prevented from revolving inwards by the bottom of the barrel, the hoop may be below the centre of pressure, but not above it; it will be best placed just below the line of pressure.

Ex. 4. A quadrant of a circle the radius coinciding with the surface of the fluid.

Let a be the radius of the circle, and the centre be the origin, then by (7) and (8), since $y = \sqrt{a^2 - x^2}$,

$$X = \frac{\int_0^a x^2\sqrt{a^2 - x^2}\,dx}{\int_0^a x\sqrt{a^2 - x^2}\,dx}, \quad Y = \frac{1}{2}\,\frac{\int_0^a x\,(a^2 - x^2)\,dx}{\int_0^a x\sqrt{a^2 - x^2}.dx}.$$

Now $\int x^2\sqrt{a^2 - x^2}\,dx$ may be found by the method of parts having multiplied and divided by $\sqrt{a^2 - x^2}$, but the following substitution is remarkably convenient in integrals of this form, and a similar one may be adopted in all integrals which involve circular functions. Assuming $x = a \sin\theta$, then $dx = a\cos\theta\,d\theta$, and $\sqrt{a^2 - x^2} = a\cos\theta$, therefore

$$\int_0^a x^2\sqrt{a^2 - x^2}\,dx = a^4\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta.$$

$$\text{Now } \int \sin^2\theta\cos^2\theta\,d\theta = \tfrac{1}{4}\int \sin^2 2\theta\,d\theta$$

$$= \tfrac{1}{8}\int (1 - \cos 4\theta)\,d\theta$$

$$= \tfrac{1}{8}(\theta - \tfrac{1}{4}\sin 4\theta) + C;$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta = \tfrac{1}{8}\,\frac{\pi}{2}.$$

$$\text{And } \int x\sqrt{a^2 - x^2}\,dx = -\tfrac{1}{3}(a^2 - x^2)^{\frac{3}{2}} + C;$$

$$\therefore \int_0^a x\sqrt{a^2 - x^2}\,dx = \tfrac{1}{3}a^3.$$

$$\text{And } \int x\,(a^2 - x^2)\,dx = (\tfrac{1}{2}a^2 - \tfrac{1}{4}x^2)\,x^2 + C;$$

$$\therefore \int_0^a x\,(a^2 - x^2)\,dx = \tfrac{1}{4}a^4;$$

$$\therefore X = \frac{\frac{a^4}{8}\,\frac{\pi}{2}}{\frac{1}{3}a^3} = \tfrac{3}{16}\pi a,$$

$$Y = \tfrac{1}{2}\,\frac{\frac{1}{4}a^4}{\frac{1}{3}a^3} = \tfrac{3}{8}a.$$

Ex. 5. A right-angled triangle with its base in the surface.

Let ABC (Fig. 10.) be the triangle, let $AB = a$, $BC = b$,

$$\text{then } X = \frac{\int_0^a x^2 y\, dx}{\int_0^a xy\, dx}, \quad Y = \frac{1}{2}\frac{\int_0^a xy^2\, dx}{\int_0^a xy\, dx}.$$

The relation betwixt y and x, or the equation to BC which cuts the co-ordinate axes at distances a and b from the origin is

$$\frac{y}{b} + \frac{x}{a} = 1,$$

then substituting for y,

$$\int x^2 y\, dx = \int b\left(1 - \frac{x}{a}\right) x^2\, dx = b\left(\frac{1}{3} - \frac{1}{4}\frac{x}{a}\right) x^3 + C;$$

$$\therefore \int_0^a x^2 y\, dx = \tfrac{1}{12} a^3 b.$$

$$\text{Similarly } \int_0^a xy\, dx = \tfrac{1}{6} a^2 b,$$

$$\int xy^2 dx = \int b^2 \left(1 - 2\frac{x}{a} + \frac{x^2}{a^2}\right) x\, dx$$

$$= b^2\left(\frac{1}{2} - \frac{2}{3}\frac{x}{a} + \frac{1}{4}\frac{x^2}{a^2}\right) x^2 + C;$$

$$\therefore \int_0^a x^2 y\, dx = b^2\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) a^2 = \tfrac{1}{12} a^2 b^2;$$

$$\therefore X = \tfrac{1}{2} a = AN, \; Y = \tfrac{1}{4} b = NP,$$

whence P the centre of pressure is fully determined.

Ex. 6. A sector of a circle, the centre being in the surface.

Let ABC (Fig. 11.) be the sector, and let the side AB make an angle α with the surface of the fluid, and let $BAC = \beta$ and $AB = a$.

Let $AP = r$ be any radius vector inclined at an angle θ to AB, and let PQ be the small element described by

dr, the radius vector $r + dr$ having revolved through an angle $d\theta$.

Then the element $PQ = dr \times r d\theta$

$$\text{and } p = g\rho \, . \, PN = g\rho r \sin(\alpha + \theta);$$

therefore, the pressure on PQ

$$= g\rho \, r^2 dr \sin(\alpha + \theta) \, d\theta.$$

Then, taking the moments about a vertical and horizontal line through A, we have if X, Y be the vertical and horizontal ordinates of the centre of pressure, since

$$NP = r \sin(\alpha + \theta) \text{ and } AN = r \cos(\alpha + \theta),$$

$$X \int_0^a \int_0^\beta r^2 \sin(\alpha + \theta) \, dr \, d\theta = \int_0^a \int_0^\beta r^3 \sin^2(\alpha + \theta) \, dr \, d\theta,$$

$$Y \int_0^a \int_0^\beta r^2 \sin(\alpha + \theta) \, dr \, d\theta = \int_0^a \int_0^\beta r^3 \sin(\alpha + \theta) \cos(\alpha + \theta) \, dr \, d\theta.$$

$$\iint r^3 \sin^2(\alpha + \theta) \, dr \, d\theta = \int \tfrac{1}{4} r^4 \sin^2(\alpha + \theta) \, d\theta + C;$$

$$\therefore \iint_0^a r^3 \sin^2(\alpha + \theta) \, dr \, d\theta = \frac{a^4}{4} \int \tfrac{1}{2} \{1 - \cos 2(\alpha + \theta)\} \, d\theta$$

$$= \frac{a^4}{8} \{\theta - \tfrac{1}{2} \sin 2(\alpha + \theta)\} + C;$$

therefore,

$$\int_0^a \int_0^\beta r^3 \sin^2(\alpha + \theta) \, dr \, d\theta = \frac{a^4}{8} \{\beta + \tfrac{1}{2}[\sin 2\alpha - \sin 2(\alpha + \beta)]\}.$$

Similarly,

$$\int_0^a \int_0^\beta r^2 \sin^2(\alpha + \theta) \, dr \, d\theta = \frac{a^3}{3} \{\cos \alpha - \cos(\alpha + \beta)\}.$$

Also,

$$\int_0^a \int_0^\beta r^3 \sin(\alpha + \theta) \cos(\alpha + \theta) \, dr \, d\theta = \int \frac{r^4}{4} \tfrac{1}{2} \sin 2(\alpha + \theta) \, d\theta + C$$

$$= \frac{a^4}{8} \{C - \tfrac{1}{2} \cos 2(\alpha + \theta)\}$$

$$= \frac{a^4}{16} \{\cos 2\alpha - \cos 2(\alpha + \beta)\};$$

$$\therefore \; X = \frac{3}{8} \frac{\beta + \frac{1}{2}\{\sin 2\alpha - \sin 2(\alpha+\beta)\}}{\cos\alpha - \cos(\alpha+\beta)} a,$$

$$Y = \frac{3}{16} \frac{\cos 2\alpha - \cos(\alpha+\beta)}{\cos\alpha - \cos(\alpha+\beta)} a,$$

whence the centre of pressure is fully determined.

Let the radius AB be perpendicular to the surface of the fluid, then $\alpha = \frac{\pi}{2}$;

$$\therefore \; X = \frac{3}{8} \frac{\beta - \frac{1}{2}\sin(\pi + 2\beta)}{-\cos\left(\frac{\pi}{2} + \beta\right)} a = \frac{3}{8} \frac{\beta + \sin\beta \, . \cos\beta}{\sin\beta} a$$

$$= \frac{3}{8}\left\{\frac{\beta}{\sin\beta} + \cos\beta\right\} a,$$

$$Y = \frac{3}{16} \frac{\cos\pi - \cos(\pi + 2\beta)}{-\cos\left(\frac{\pi}{2} + \beta\right)} a$$

$$= -\frac{3}{16} \frac{1 + \cos 2\beta}{\sin\beta} a = \frac{3}{8} \frac{\sin^2\beta}{\sin\beta} a$$

$$= \frac{3}{8} \sin\beta \, . \, a.$$

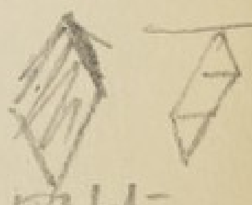
Let the sector be a quadrant, or $\beta = \frac{\pi}{2}$;

$$\therefore \; X = \frac{3}{8} \, . \, \frac{\pi}{2} \, . \, a = \frac{3}{16} \pi a, \quad Y = \frac{3}{8} a,$$

the same values as were obtained before, Ex. 4.

Ex. 7. An oblique parallelogram with one angle in the surface.

Let $ABDC$ (Fig. 12.) be the parallelogram, and let $AB = a$, $AC = b$, make angles α, β, with the vertical through A.



Let X, Y, be the vertical and horizontal co-ordinates of the centre of pressure.

Let $x_{,}$, $y_{,}$, be the vertical and horizontal co-ordinates AN, NP, of any point P whose co-ordinates referred to the axes AB, AC are x, y.

$$\text{Then, } x_{,} = x\cos\alpha + y\cos\beta,$$

$$y_{,} = x\sin\alpha - y\sin\beta.$$

The co-ordinates being oblique, an element $dx\,dy$ at P

$$= dx\,.\,dy\sin(\alpha+\beta),$$

$$\text{and } p = g\rho x,$$

we have therefore,

$$X\int_0^a\int_0^b x_{,}\,dx\,dy = \int_0^a\int_0^b x_{,}^2\,dx\,dy,$$

$$Y\int_0^a\int_0^b x_{,}\,dx\,dy = \int_0^a\int_0^b x_{,}y_{,}\,dx\,dy,$$

whence X and Y may be determined.

Now,

$$\iint x_{,}^2\,dx\,dy = \iint (x^2\cos^2\alpha + 2xy\cos\alpha\cos\beta + y^2\cos^2\beta)\,dx\,dy$$

$$= \int (x^2\cos^2\alpha + xy\cos\alpha\cos\beta + \tfrac{1}{3}y^2\cos^2\beta)\,y\,dx + C,$$

the integration being performed for y, then

$$\int\int_0^b x_{,}^2\,dx\,dy = \int (x^2\cos^2\alpha + xb\cos\alpha\cos\beta + \tfrac{1}{3}b^2\cos^2\beta)\,b\,dx$$

$$= (\tfrac{1}{3}x^2\cos^2\alpha + \tfrac{1}{2}xb\cos\alpha\cos\beta + \tfrac{1}{3}b^2\cos^2\beta)\,bx + C;$$

$$\therefore \int_0^a\int_0^b x_{,}^2\,dx\,dy = (\tfrac{1}{3}a^2\cos^2\alpha + \tfrac{1}{2}ab\cos\alpha\cos\beta + \tfrac{1}{3}b^2\cos^2\beta)\,ab.$$

Similarly,

$$\int_0^a\int_0^b x_{,}\,dx\,dy = \tfrac{1}{2}(a\cos\alpha + b\cos\beta)\,ab.$$

Also, $\iint x_, y_, dx\,dy = \iint \{\tfrac{1}{2}x^2 \sin 2\alpha + xy \sin(\alpha - \beta)$
$$- \tfrac{1}{2} y^2 \sin 2\beta\}\, dx\, dy$$
$$= \int \{\tfrac{1}{2}x^2 \sin 2\alpha + \tfrac{1}{2}xy \sin(\alpha - \beta)$$
$$- \tfrac{1}{6} y^2 \sin 2\beta\}\, y\, dx;$$
$$\therefore \iint_0^b x_, y_, dx\, dy = \int \{\tfrac{1}{2}x^2 \sin 2\alpha + \tfrac{1}{2}xb \sin(\alpha - \beta)$$
$$- \tfrac{1}{6} b^2 \sin^2 \beta\}\, b dx$$
$$= \{\tfrac{1}{6}x^2 \sin 2\alpha + \tfrac{1}{4}ab \sin(\alpha - \beta)$$
$$- \tfrac{1}{6} b^2 \sin 2\beta\}\, bx + C;$$
$$\therefore \int_0^a \int_0^b x_, y_, dx\, dy = \{\tfrac{1}{6}a^2 \sin 2\alpha + \tfrac{1}{4}ab \sin(\alpha - \beta)$$
$$- \tfrac{1}{6} b^2 \sin 2\beta\}\, ab;$$
$$\therefore X = \frac{\tfrac{1}{3}a^2 \cos^2 \alpha + \tfrac{1}{2}ab \cos\alpha \cos\beta + \tfrac{1}{3}b^2 \cos^2 \beta}{\tfrac{1}{2}(a \cos\alpha + b \cos\beta)},$$
$$Y = \frac{\tfrac{1}{6}a^2 \sin 2\alpha + \tfrac{1}{4}ab \sin(\alpha - \beta) - \tfrac{1}{6}b^2 \sin 2\beta}{\tfrac{1}{2}(a \cos\alpha + b \cos\beta)}.$$

Let the parallelogram be right-angled, the side AC being in the surface.

Then $\alpha = 0$, $\beta = \dfrac{\pi}{2}$,
$$X = \frac{\tfrac{1}{3}a^2}{\tfrac{1}{2}a} = \tfrac{2}{3}a, \quad Y = \frac{-\tfrac{1}{4}ab}{\tfrac{1}{2}a} = -\tfrac{1}{2}b,$$
as we have already determined them, Ex. 2.

The Pressures on Curved Surfaces.

54. The pressure on any portion of a curved surface is determined by resolving the normal force at each point into the directions of the three co-ordinate axes, and calculating by two integrations the total components in these directions. These components may always be reduced to two forces; these two forces however seldom admit of a single resultant. But when the pressure exerted on a curved surface is a fluid pressure, the pressures always admit of a single resultant whose direction and magnitude must be determined. To find then the resultant of the fluid pres-

sure on a curved surface, the pressure at any point being resolved into its components in the planes of the co-ordinates, it will be shewn that the horizontal pressures destroy each other, their resultant therefore is zero, and that the resultant of the pressures is therefore vertical.

55. Prop. *The horizontal pressures on the surface of any body immersed in a fluid are in equilibrium with each other.*

Let APB (Fig. 13.) be any body immersed in a fluid; let the body be referred to three rectangular axes, the surface of the fluid being taken for the plane of xy.

Let P be any point x, y, z, in the surface of the body, and let ω be a small element of the surface at this point, then $p\omega$ is the pressure on this element in the direction PG of the normal to the surface.

The value of p will be the same for all points which are at the same distance z from the surface of the fluid, that is, for all points which are in the same horizontal plane, whether the fluid be homogeneous, or whether it be composed of level strata of different densities.

Let a, b, c be the projections of the element ω on the co-ordinate planes yz, xz, xy, respectively; and let α, β, γ be the angles which the normal PQ makes with the axes of x, y, z. Then α, β, γ are also the angles which the tangent plane at the point P makes with the planes of yz, xz, xy, respectively.

Therefore,

$$a = \omega \cos \alpha, \quad b = \omega \cos \beta, \quad c = \omega \cos \gamma;$$

and multiplying these equations by p,

$$pa = p\omega \cos \alpha, \quad pb = p\omega \cos \beta, \quad pc = p\omega \cos \gamma.$$

But $p\omega \cos \alpha$ is the resolved part of the normal pressure $p\omega$ in the direction of x.

Hence pa, pb, pc, are the components of the normal pressure $p\omega$ in the directions of the axes of x, y, z; or the component in the direction perpendicular to any co-ordinate plane of the normal pressure on any portion of the surface, equals the product of that pressure and of the projection of the portion of the surface on the co-ordinate plane.

Whatever be the nature of the body, there must always be a portion of its surface opposite to ω, which will have the same projection on the co-ordinate plane. Let $PP_{,}M$ be drawn perpendicular to the plane of yz, meeting the side of the body opposite to P in $P_{,}$; and let $\omega_{,}$ be the element of the surface at this point, which has the same projection on the plane of yz. Then the pressure on $\omega_{,}$ resolved in the direction perpendicular to the plane of yz, being by what has just been stated equal to the product pa, will be the same whenever p is the same, that is, for all points in the same horizontal plane.

Thus the pressures on any portion ω in the direction of the axis x, destroy each other; and the same may be shewn for the direction of y, and for every point in the same horizontal plane.

Hence it appears that the horizontal components of the pressures exerted on the elements of the surface of any body immersed in a fluid, destroy each other in each horizontal section; and therefore the horizontal forces on its whole surface destroy each other.

There is then no force which can produce lateral motion, the resultant of the horizontal pressures being zero; it follows therefore that the only forces which are to have a resultant are the vertical pressures, and consequently that all the forces may be reduced to a single vertical force which is the resultant of the components perpendicular to the plane of xy, and which arises from the excess of the value of p for the lower parts of the body.

Cor. If the pressure p arises from a pressure exerted on the surface of the fluid, its value would be constant for every point in the surface of the body; and the components would destroy each other in the vertical as well as in the horizontal direction. Whatever then be the form of a solid or fluid mass, a constant normal pressure impressed at all points of its surface cannot produce any motion either of translation or rotation.

56. Prop. *To find the resultant of the vertical pressures acting on a body immersed in a fluid.*

From P draw a perpendicular to the plane of xy meeting the body in P', and let ω' be the element of the surface at this point corresponding to ω at P.

Their projections on the plane of xy are the same and equal to c, but the value of p is different. Let p' be the value of p at P'. Then the vertical line of particles terminated by these two elements will be pressed vertically from below upwards, with a pressure $pc - p'c$.

Let the fluid be homogeneous, then if $PP' = l$,

$$p - p' = g\rho l \text{ and } pc - p'c = g\rho cl.$$

But $g\rho cl$ is the weight of a column of fluid whose volume is lc, hence the vertical pressure equals the weight of a volume lc of the fluid, that is, it equals the weight of the fluid column whose place is occupied by that portion of the body; and the same being true for every other column of the body, the whole vertical pressure is the weight of a mass of the fluid equal in bulk to the body displaced. The resultant therefore of the fluid pressures is a vertical force applied at the centre of gravity of the body immersed in an opposite direction to gravity. When the body is homogeneous, the centres of gravity, of the body and of the fluid displaced, coincide.

If the body be not wholly immersed, we have $p' = 0$, and the resultant of the pressures is the weight of the volume of the fluid displaced by the portion of the body which is immersed, and is applied at the centre of gravity of this portion.

57. The preceding results are also true when the fluid is composed of horizontal strata of very different densities; as will also be evident from the following considerations.

The equilibrium once established will not be disturbed by supposing any part of the fluid to become solid, so that this part itself becomes a floating or immersed body. But in order that the normal pressures exerted on the surface of the body by the surrounding fluid may be in equilibrium with the weight of this solid part, their resultant must be a single force, and act in a direction contrary to the weight of the body; and if we replace the part of the fluid which is supposed to have become solid by another body having exactly the same surface, it is evident that no change can have taken place in the pressures of the surrounding fluid; consequently the pressures exerted on the surface of a body immersed, wholly or not, in a fluid at rest either homogeneous or heterogeneous, are always equivalent to a single force, which is equal to the whole weight of the successive strata of fluid whose place is occupied by the body, and which is applied in a direction contrary to gravity at the centre of gravity of these strata.

We may conclude then that a body totally immersed in a fluid will be in equilibrium when its mean density is equal to that of the fluid displaced, and when its centre of gravity and that of the fluid displaced are in the same vertical; which latter condition is always fulfilled when the body and liquid are both homogeneous.

The equilibrium of bodies which are not wholly immersed, but float at the surface of a fluid will be examined in the following chapter.

58. *Hydrostatic Balance.* The conclusion at which we have just arrived is generally enunciated by saying, that a body immersed in a fluid loses as much of its weight as is equal to the weight of the fluid displaced.

Hence it is evident that to obtain the true weight of a body it ought to be weighed in vacuo.

Two bodies weighed in air, or in water, or in any other liquid, and which are in equilibrium on a very exact balance, have really very different weights unless their volumes should be equivalent.

The greater weight is that of the body which has the greater volume, because having experienced a greater loss in the fluid, it is still in equilibrium with the other.

If the same body is weighed successively in vacuo and in water, and W be its weight in vacuo, and W' in water, W and $W-W'$ will be the absolute weight of the body, and of a quantity of water of the same volume. But when the volume is constant the weight varies as the density, hence W and $W-W'$ are as the densities of the solid and water. If then D be the density of the solid, that of water being unity, we shall have

$$D = \frac{W}{W-W'}.$$

It is thus that the densities or specific gravities of substances which can be weighed in water without being dissolved, are ascertained by means of the Hydrostatic Balance.

59. The reasonings in the preceding articles apply equally to the pressures exerted on the sides of vessels containing fluid; and the same result would be obtained; namely, that the horizontal pressures exerted from within to without on all the internal surface destroy each other, that is, they consist of pieces of equal and opposite forces; whence if a vessel is set on a horizontal plane, the action

of the fluid which it contains cannot put it in motion: this result is also a necessary consequence of the conservation of the motion of the centre of gravity. But if an aperture be made in one of the sides of the vessel below the surface of the fluid, the fluid will run out, and the pressure being no longer exerted on that part of the surface which is removed, the pressure which is exerted on the opposite side will not be destroyed. In this case then the sum of the horizontal forces are not zero for these points, and consequently the vessel can be put in motion on the side opposite to the issuing fluid.

This is the principle of all the machines whose motion depends on the reaction of a fluid, and which has been suggested by Bernouilli as applicable to the motion of vessels. The application of this principle is exhibited in the machine called Barker's Mill.

60. From the same reasoning it is also evident that the whole vertical pressure exerted on the bottom and on the sides of a vessel is always equal to the weight of the fluid contained and applied in the direction of gravity to the centre of gravity of the fluid.

Each vertical line of the fluid which extends without interruption from the surface to any point of the vessel, exerts at this point a normal pressure, which is equal to the weight of this line: that line which is interrupted and meets the internal surface of the vessel in more points than one, as for instance, at a point in the bottom and in one of the sides, exerts at these two points pressures whose vertical components are in opposite directions. The component which belongs to the lower point is in the direction of gravity, and exceeds the other by a quantity equal to the weight of this line; and the same is true for all the points in which this line meets the containing surface; thus the excess of the pressures downward over those upwards is equal to the weight of the contained fluid. The resultant then of all the vertical pressures of these lines of fluid is precisely the same as the weight of the fluid in question.

This pressure must be accurately distinguished from that which takes place simply on the bottom of the vessel, and which is only equal to the weight of the fluid when the vessel is a right cylinder. It is less than the weight when the vessel increases in size from the bottom to the top, as the frustrum of an inverted cone, because the vertical lines of fluid which extend from the surface, and are intercepted by the sides of the cone, do not press on the bottom of the vessel; on the other hand, it is greater than the weight of the fluid, when the vessel increases from the top to the bottom, because the vertical lines which extend from the bottom of the vessel, and are intercepted by the side, exert nevertheless the same pressure on the bottom of the vessel as if they extended to the surface of the fluid; the deficiency in the weight of each of these incomplete lines being made up by the reaction of the side by which they are terminated.

61. When fluid is contained in a flexible vessel, the pressure and the resultant of the tensions on a portion of any section of the vessel must be equal and opposite.

Let PQ (Fig. 14.) be a portion of any section of a flexible vessel which is full of fluid.

Let p be the pressure at any point, and t the tension, which must be uniform throughout each section.

Draw normals at P and Q meeting in O, and tangents at P and Q meeting in T, join TO.

Let γ be the radius of curvature at P, and $2ds$ the portion PQ of the curve,

$$\text{the pressure on } PQ = 2pds.$$

The tensions at P and Q compound a force in the direction TO

$$= 2t \,.\, \sin POT = 2t \,.\, \frac{ds}{\gamma}, \text{ nearly.}$$

Hence, since there is equilibrium,

$$2p\,ds = 2t\,\frac{ds}{\gamma},$$

$$\text{whence } p = \frac{t}{\gamma}, \text{ or } t = p\gamma.$$

This equation will serve to determine one of the quantities when the other two are assigned.

Other questions connected with the pressure of fluids, as the equilibrium of dykes, &c. the reader will find treated in Moseley's Hydrostatics, Arts. 49—54.

CHAPTER VI.

ON THE EQUILIBRIUM OF FLOATING BODIES.

62. When a body is placed in a fluid, its density, if it be homogeneous, or its mean density, if it be not homogeneous, being less than the density of the fluid, it sinks in the fluid until the weight of the fluid displaced becomes equal to the weight of the body; it then remains at rest, provided its centre of gravity and that of the fluid displaced are in the same vertical.

For the only forces which exist are the weight of the body and the resultant fluid pressure. And it has been shewn that the resultant of the fluid pressures is a vertical force, the horizontal forces destroying each other. The weight of the body therefore, and the resultant of the fluid pressure, must be in equilibrium with each other; and they are parallel forces, and may be applied at the same point, or at different points in the same vertical, or at points in different verticals.

In the first case, the body will either ascend or descend vertically, and when the weight of the fluid displaced equals the weight of the body, it will be absolutely at rest.

In the second case, the body cannot rest but may have a motion both of translation and of rotation communicated to it.

Now, these forces are applied at the centres of gravity of the respective masses; hence the conditions for the equilibrium which are both necessary and sufficient, are,

1. That the weight of the fluid displaced be equal to the weight of the body.

2. That the line joining the centre of gravity of the body and of the fluid displaced, be vertical.

63. *To find the positions of equilibrium of a floating body.*

If V be the volume of the fluid displaced and ρ its density, and V' the volume of the body and ρ' its density, then by the first condition

$$V\rho g = V'\rho' g'; \quad \therefore V\rho = V'\rho',$$

$$V : V' :: \rho' : \rho, \text{ or } V : V' - V :: \rho : \rho' - \rho;$$

that is, the body must be cut by a plane, so that the segments shall have to each other a given ratio.

Also, by the second condition the line joining the centres of gravity of the two portions must be vertical, or the cutting plane must be perpendicular to it. Hence the determination of the positions of equilibrium of a floating body is reduced to the following problem in Analytical Geometry:—"To cut any body by a plane so that the volume of one segment may be to that of the whole body in a given ratio, and that the line joining the centres of gravity of this segment and of the whole body may be perpendicular to the cutting plane." When the section of a body satisfying these two conditions has been determined, it must be placed coincident with the surface of the fluid; the segment whose volume has been considered being the segment which is immersed in the fluid, and the other segment being above the fluid: this position will be one of equilibrium.

These conditions may be expressed by equations, the complete solution of which will give all the positions of the equilibrium of a body; sometimes their number will be infinite, as is the case for a solid of revolution whose axis is horizontal; but it would be difficult to demonstrate *a priori* that there is always a position of equilibrium whatever be the form of the body. The method of proceeding in any case will be sufficiently illustrated by the following example.

Ex. *A triangular prism with its edges horizontal.*

The determination of the positions of the equilibrium of this body evidently reduces itself to the determination of the position of equilibrium of a triangle which is its generating section.

Here two cases present themselves according as one or two angles are immersed; we shall first consider the case when one angle is immersed, and shew how one case may be reduced from the other.

1°. Let ABC and PQ (Fig. 15.) be the sections of of the triangular prism and of the surface of the fluid made by a vertical plane.

Let a, b, c, be the sides of the triangle which are opposite to the angles A, B, C, and x, y the sides CP, CQ of the part which is immersed.

Let s be the ratio of the specific gravities of the body and of the fluid, that is, the ratio of their densities.

Let V be the area of the part immersed, and V' of the whole, then by the first condition, (Art. 62.)

$$V\rho g = V'\rho' g, \quad \text{or } V = V'\frac{\rho'}{\rho} = sV'.$$

But $V = \frac{1}{2}xy\sin C$, and $V' = \frac{1}{2}ab\sin C$, therefore

$$xy = sab \ldots\ldots (1).$$

Again, let G and F be the centres of gravity of the triangles ACB, PCQ. Then if D and E be the bisections of AB and PQ, $CG = \frac{2}{3} CD$ and $CF = \frac{2}{3} CE$; join GF, DE, DP, DQ.

Then by the second condition (Art. 62.) GF is vertical, and therefore perpendicular to PQ.

But DE is parallel to GF. It is therefore perpendicular to PQ; that is, the line joining the bisections of AB and PQ must be vertical. Hence also, since $PE = EQ$, and DE is perpendicular to PQ, $DP = DQ$.

Conversely, if $DP = DQ$, the line DE will be perpendicular to PQ, and therefore its parallel GF will be perpendicular to PQ. Hence that the line joining the centres of gravity of the body and of the fluid displaced may be perpendicular to the surface of the fluid, it is necessary and sufficient that DP should be equal to DQ. Then if $CD = h$ and α, β be the angles DCA, DCB, we have

$$DP^2 = h^2 + x^2 - 2hx\cos\alpha,$$

$$DQ^2 = h^2 + y^2 - 2hy\cos\beta;$$

$$\therefore\ x^2 - 2hx\cos\alpha = y^2 - 2hy\cos\beta \ldots\ldots\ldots\ldots\ldots (2).$$

But from (1) $y = \dfrac{sab}{x}$, eliminating y, we have

$$x^4 - 2h\cos\alpha\ x^3 + 2hsab\cos\beta\ x - s^2a^2b^2 = 0 \ldots\ldots (3);$$

whence having determined the four values of x, the corresponding values of y are given by the equation

$$y = \frac{sab}{x}.$$

The equation (3) is of even dimensions, and has its last term negative; it must have, therefore, two real roots of contrary signs.

The other two roots may be real or imaginary. If they are real, the rule of signs shews us that it has three positive and one negative root; for there must be three changes and one continuation, whatever be the sign of the term which is wanting.

The unknown quantities x and y, which are the sides of the triangle PCQ, can only be positive quantities less than the sides CA and CB respectively; the negative root, therefore, of the equation may be rejected as inapplicable.

There are, therefore, at the most but three positions of equilibrium when one angle only is immersed.

2°. Let two angles, as A and B of the triangle, be immersed.

Then if PQ be considered as the line of floatation, the centres of gravity of ACB and $APQB$ must be on the same vertical, and we must have, as before,

$$APQD : ACB :: \rho' : \rho$$
$$:: s : 1,$$

$$\text{whence } PCQ : ACB :: 1 - s : 1;$$

$$\therefore xy = (1 - s)ab \ldots\ldots\ldots(4).$$

Hence, eliminating y between this and (2), the equation is the same as before, $(1 - s)$ being in the place of s, or the equation required is

$$x^4 - 2h\cos\alpha\, x^3 + 2h(1-s)ab\cos\beta\, x - (1-s)^2a^2b^2 = 0 \ldots(5).$$

And from the same reasoning as was applied to (3) it appears that there are at the most but three positions of a triangle floating with two angles immersed.

There are, therefore, three positions of equilibrium of a triangular prism when one angle is immersed, and also three when that angle is the only one not immersed; that is, there are on the whole six possible positions of equilibrium for each angle, and therefore eighteen for the whole triangular prism.

64. Let the section of the prism be an isosceles triangle; then pursuing the same reasoning, we shall arrive

at an equation which admits of immediate solution. The equation of the preceding article may be adapted at once to this case.

Let $a = b$, then the triangles ACD, BCD, are right-angled and equal, whence,

$$\beta = a, \quad h^2 = a^2 - \tfrac{1}{4}c^2, \quad a \cos \alpha = h,$$

and equations (1) and (2) become

$$xy = sa^2, \text{ and } x^2 - y^2 - \frac{4a^2 - c^2}{2a}(x - y) = 0 \ldots\ldots (6).$$

This is satisfied by taking $x = y = a\sqrt{s}$, which is a possible value, since s is less than unity. Hence PQ must be parallel to AB, that is, AB is horizontal; and the same is true when C is out of the water. But there are other positions of equilibrium; for suppressing the factor $(x - y)$ we have

$$x + y = \frac{4a^2 - c^2}{2a},$$

which combined with $xy = sa^2$ gives for the two values of x and y

$$\frac{1}{4a}\left\{4a^2 - c^2 \pm \sqrt{(4a^2 - c^2)^2 - 16sa^4}\right\}.$$

Each of these being taken successively for x and y, if both be less than a, there are two new positions of equilibrium in which the base AB is out of the fluid.

Substituting $1 - s$ for s, there are two other positions when AB is immersed, provided both roots be less than a. When the two preceding roots are equal, the base AB is horizontal: these new positions ought to be identical with the former; in this case

$$4a^2 - c^2 = \pm 4a^2\sqrt{s},$$

whence $x = y = a\sqrt{s}$, as before.

65. Let the section be an equilateral triangle, then $a = b = c$, and the equations of the preceding articles give for the unequal values of x and y, when one angle is immersed,

$$\frac{a}{4}\{3 \pm \sqrt{9 - 16s}\},$$

and when two are immersed,

$$\frac{a}{4}\{3 \pm \sqrt{16s - 7}\}.$$

The value of the ratio s must be examined into, that these may be real and less than a.

If s be $< \frac{9}{16}$ and $> \frac{1}{2}$, the first expression is real and less than a; and if s be $< \frac{1}{2}$ and $> \frac{7}{16}$, the second expression is real and less than a.

Hence when one angle is immersed, the limits of s are between $\frac{1}{2}$ and $\frac{9}{16}$; and when two angles are immersed, the limits are $\frac{1}{2}$ and $\frac{7}{16}$; and between the values $\frac{7}{16}$ and $\frac{9}{16}$ the prism has no oblique position of equilibrium.

Since all the angles are equal, there may sometimes be eighteen and sometimes only six positions of equilibrium.

66. Besides the horizontal positions of equilibrium which we have just treated of, prisms and cylinders may float in a vertical position with their bases parallel to the surface of the fluid; there are two positions of equilibrium for each body when they float in this manner, for there is one for each base of the solid.

The line joining the centres of gravity of a vertical prism and of the part immersed must be perpendicular to the surface of the fluid; the ratio of their volumes is the same as that of their heights, and consequently the height of the part immersed is to that of the whole prism as the density of the body is to the density of the fluid; this one condition then determines the depth to which the body sinks, and is the solution of the problem.

Solids of revolution, and all bodies which are symmetrical about a given axis, have two positions of equilibrium for each axis, which may be determined as in the following example.

Ex. *An ellipsoid with an axis vertical.*

Let a, b, c, be the semiaxes of the ellipsoid, and let the axis $2c$ be vertical.

Let $z_{\prime}$ be the distance of the plane of floatation from the centre of the ellipsoid, which will be positive or negative according as this section is above or below the centre of the ellipsoid.

Let K be the area of any section at a distance z from the centre of the ellipsoid.

The volume of the semiellipsoid $= \frac{2}{3}\pi abc$; the volume of the part between the plane of xy and the surface of the ellipsoid is $\int_0^{z'} K\,dz$.

Hence the volume of the part immersed

$$= \tfrac{2}{3}\pi abc - \int_0^{z'} K\,dz.$$

Hence when there is equilibrium

$$\tfrac{2}{3}\pi abc - \int_0^{z'} K\,dz = \tfrac{4}{3}\pi abc \, . \, s.$$

Now the area K is an ellipse, and the equation to the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and the semiaxes of a section at a distance z from the centre being obtained by putting y and x successively equal to nothing, are

$$\frac{a}{c}\sqrt{c^2 - z^2} \text{ and } \frac{b}{c}\sqrt{c^2 - z^2}.$$

The area therefore required is

$$K = \frac{\pi ab}{c^2}(c^2 - z^2);$$

$$\therefore\ \tfrac{2}{3}\pi abc - \int_0^{z_i} \frac{\pi ab}{c^2}(c^2 - z^2)\, dz = \tfrac{4}{3}\pi abcs.$$

But $\int_0^{z_i}(c^2 - z^2)\,dz = (c^2 - \frac{1}{3}z_i^2)z_i$, substituting and omitting the common factor πabc, we have

$$\frac{2}{3} - \frac{1}{c^3}(c^2 - \tfrac{1}{3}z_i^2)z_i = \tfrac{4}{3}s,$$

$$\text{or } z_i^3 - 3c^2 z_i - 2(2s-1)c^3 = 0,$$

whence the distance of the plane of floatation from the centre of the ellipsoid is known. This equation being independent of a and b, the values of z_i are independent, or are the same for the ellipsoid, the spheroid, and the sphere.

It must have one real root, which will lie between $\pm c$, being positive or negative according as s is $>$ or $< \frac{1}{2}$. In the extreme cases when $s = 0$ and $s = 1$ this root is $z_i = c$ and $z_i = -c$.

The other two roots will be found greater than c, and are therefore excluded.

The Stability of a Floating Body.

67. The conditions that a body may rest in a fluid are, as we have seen, two; it remains now to consider what is the nature of the equilibrium in which it exists, that is, if the body be slightly disturbed from that position, by being moved through a small angle about some axis, and then left to itself, whether it will have a tendency to return to its original position, or to recede from it, or to move neither way, but rests in that new position.

These distinctions in the nature of the equilibrium have, as is well known, received the corresponding terms of stable, unstable, and indifferent or neutral equilibrium, they apply to the nature of the equilibrium of any system acted on by any forces and are therefore applicable to a floating body. The general proposition therefore that the positions of stable and unstable equilibrium recur alternately will obtain here.

In general when a floating body is disturbed there will be a motion both of translation and of rotation about some axis; as these however may be independent, we shall consider only the motion of rotation, and suppose that the body is disturbed by being caused to revolve about some axis, and then left to itself; it is the motion after the disturbance has ceased which we have to examine.

It will always be supposed that the volume of the fluid displaced is unaltered by the disturbance, for the impressed force being the weight of the fluid displaced, if there be any variation in its volume a motion of translation must occur as well as a motion of rotation.

It will also be supposed that the body is symmetrical about a vertical plane, and that the disturbance leaves the plane of symmetry vertical.

Under these circumstances it will be found that in some cases, where the body is left to itself, its motions will be vertical and angular simultaneously. To illustrate this, we shall premise the following proposition.

68. Prop. *If a body be turned through a small angle about an axis through the centre of gravity of the plane of floatation, the fluid displaced is unaltered.*

Let ADB (Fig. 16.), represent the body in its new position, having been turned through a small angle about an axis through C, the centre of gravity of the plane of floatation AB.

Let the axis of rotation be taken for the axis of y, and let θ be the angle of displacement.

The wedges ACa, BCb may be considered as generated by the revolution of CA and CB, hence if $dxdy$ be an element of the plane CA at a distance x, y, from C the elementary prism which is generated $= x\theta \,.\, dxdy$, therefore,

$$\text{the wedge } ACa = \iint x\theta dxdy = \theta \iint x dxdy.$$

Similarly, if $dx'dy'$ be an element in the plane CB at the distance $x'y'$ from C,

$$\text{the wedge } BCb = \theta \iint x' dx' dy'.$$

But by the property of the centre of gravity

$$\iint x dxdy = \iint x' dx' dy';$$

$$\therefore \text{ the wedge } ACa = \text{the wedge } BCb.$$

Hence $aDb = ADB$ or the volume of the body immersed or of the fluid displaced is unaltered.

69. Prop. *To explain the connection between the vertical and angular motions of a floating body.*

A body immersed in a fluid is acted on by two forces, its own weight applied at its centre of gravity, and the weight of the fluid displaced applied at its centre of gravity. Now the motion of the centre of gravity of a body will be the same as if these forces were applied at that point, and motion of rotation round the centre of gravity will be the same as if that point were fixed and the same forces applied. Suppose the body to have been disturbed from its position of equilibrium by being made to revolve about an axis through the centre of gravity of the plane of floatation, then as has been shewn the volume of the fluid displaced will not be altered. The centre of gravity of the body will describe a small circular arc which may be considered as a straight line. If the centre of gravity of the body be (in the position of equilibrium) vertically beneath the centre of gravity of the plane of floatation, this small straight line will be horizontal. There is no force at present tending to move the centre of gravity of the body, and if the equilibrium be stable, so that the angular motion round the centre of gravity brings the body back to its original position of equilibrium, the centre of gravity of the original plane of floatation will remain in the surface of the fluid. In this case then the small angular motions will be unattended with any vertical ones, or there will be motion of rotation simply.

But if in the position of equilibrium the centre of gravity of the body is not vertically beneath the centre of gravity of the plane of floatation, when the body is disturbed as before, its centre of gravity will be raised or lowered, and though there is no force in consequence of such a disturbance tending to produce a motion of translation in the centre of gravity but only of rotation about it, yet in consequence of this angular motion round that point the centre of gravity of the original plane of floatation will be raised above or lowered beneath the surface of the fluid. The volume of the fluid displaced will therefore be altered, and the weights of the body and of this fluid thus becoming unequal, a force will be generated which tends to produce a vertical motion in the centre of gravity of the body.

Hence there must be simultaneous motions of translation and of rotation.

In bodies which are symmetrical with respect to the vertical line through their centre of gravity, it is evident that the centre of gravity of the plane of floatation will be in this line. But there are cases in which it is not, as for instance, in a scalene triangle with one angle immersed.

70. The connection which thus subsists between the motions of translation and rotation, when the volume of the fluid displaced remains unaltered, having been shewn, we shall now suppose the floating body to assume a new position, and consider simply the force which exists in

consequence of this new position to move it about an axis through its centre of gravity.

The body is supposed to be symmetrical about a vertical plane both before and after the disturbance, that is, in its old and new position, and the volume of the fluid displaced is constant. It will be convenient to premise the following proposition.

71. PROP. *The intersection of the two planes of floatation is a line passing through their common centre of gravity.*

Let ADB (Fig. 16.), be the section of the floating body by the plane of symmetry and let the two planes of floatation AB and ab intersect in C, then C is their common centre of gravity.

Since the fluid displaced is invariable, subtracting the common part aDB, the wedge ACa = the wedge BCb.

The wedges may be divided into elementary prisms the base of one of which $= dxdy$ and its height $= x\theta$, therefore,

$$\text{the wedge } ACa = \iint x\theta\, dx\, dy = \theta \iint x\, dx\, dy = Ah,$$

if A be the area of the portion Ca of the plane of floatation and h the distance of its centre of gravity from C.

Similarly the wedge $BCb = A'h'$, therefore $Ah = A'h'$, for the wedges are equal.

The distance of the centre of gravity of the plane ab from $C = \dfrac{Ah - A'h'}{A + A'} = 0$, or is at C.

The same reasoning would apply to AB, for the base $dxdy$ of the elementary prisms may be taken in either plane, hence the centre of gravity of the two planes is in the line of their intersection and is therefore at C, since the solid is symmetrical about the vertical plane.

72. Prop. *To determine the nature of the equilibrium of a floating body.*

This as we have seen depends on the tendency of its motion when left to itself after an angular disturbance; hence the moment of the impressed force, that is, of the fluid displaced about the centre of gravity of the body, is the quantity to be discovered.

Let AB, ab (Fig. 16.), be the planes of floatation of the old and new position; they will intersect in C their common centre of gravity (Art. 71).

Let G, H be the centres of gravity of the body and of the fluid displaced before the disturbance, and H' the centre of gravity of the fluid displaced in the new position.

Through H' draw the vertical $H'M$ and draw GN horizontal.

The fluid displaced acts upwards in $H'M$, and if W be the weight of the body, that is, of the fluid displaced, the moment of the impressed force about $G = W.GN$ and the equilibrium will be stable, unstable, or indifferent, according as this force diminishes or increases, or does not affect the angle GMH': to determine these different cases for any given body we must express GN in terms of assigned or assignable quantities. This may be readily effected by taking the moments of the whole body made up in two different ways.

The moments may be taken with respect to any vertical plane; let them be taken with respect to the vertical through C.

Let g and g' be the centres of gravity of the wedges, Cm and Cm' their horizontal distances; then, since

$$ACa + aDb = ADB + BCb$$

and the moment of each of these equals the moment of

ADb, we have drawing the horizontals GE, HF, $H'F'$,

$$ACa.Cm + aCb.H'F' = ADB.HF + (-BCb.Cm'),$$

or, V being the volume of the part immersed and observing that $HF - H'F' = GN + HL$,

$$V.(GN + HL) = ACa.Cm + BCb.Cm' \text{.........(1)}.$$

But $ACa.Cm = \theta \iint x\,dx\,dy \times Cm$.........(Art. 71.)

$$= \theta \iint x^2 dx\,dy$$

by the property of the centre of gravity.

Similarly $BCb.Cm' = \theta \iint x'\,dx'\,dy' \times Cm'$

$$= \theta \iint x'^2 dx'\,dy'.$$

And the sum of these two double integrals is the moment of inertia of the plane of floatation ab about an axis through C, let this be I;

$$\therefore\ V(GN + HL) = \theta I, \quad \text{or } GN = \theta \frac{I}{V} - HL.$$

Let $GH = \lambda$, then $HL = \lambda \sin\theta = \lambda\theta$ nearly;

$$\therefore\ GN = \theta\left(\frac{I}{V} - \lambda\right),$$

whence GN is known for any given body.

The moment of the impressed force $= W\theta\left(\frac{I}{V} - \lambda\right)$.

When $\frac{I}{V}$ is greater than λ this is always positive, or the moment of the fluid displaced brings the body back to its original position; the equilibrium is therefore stable.

When $\frac{I}{V} = \lambda$ the moment is zero, or the body has no tendency to move; the equilibrium is therefore neutral or indifferent.

When $\frac{I}{V}$ is less than λ the quantity becomes negative, or the moment evidently moves the body farther from its original position: the equilibrium is therefore unstable.

This moment is a *measure of the stability* of the floating body, and depends entirely on the quantity $\frac{I}{V} - \lambda$, that is, on the moment of inertia of the plane of floatation, the quantity of the fluid displaced, and the relative position of the centre of gravity of the body and of the fluid displaced. Hence the equilibrium will always be stable if this is always positive, which will be the case if G is below H, for then, as will be seen at once by retracing the steps of the investigation, the term $(-\lambda)$ will be positive. The equilibrium then is stable, unstable, or indifferent, according as $\left(\frac{I}{V} \mp \lambda\right)$ is positive, negative, or zero.

73. PROP. *To determine the metacentre of a floating body.*

The nature of the equilibrium evidently depends on the position of M the metacentre. Then, pursuing the investigation as in the preceding article, we have by (1),

$$GN + HL = \theta \frac{I}{V}.$$

But $GN + HL = HM . \sin\theta = HM . \theta$, nearly;

$$\therefore HM . \theta = \theta \frac{I}{V}, \quad \text{or } HM = \frac{I}{V},$$

whence M is fully known.

Also, $GN = GM \sin\theta = GM . \theta$, and $HG = \lambda\theta$.

Let $GM = \mu$, then

$$\mu = \frac{I}{V} \mp \lambda,$$

according as G is above or below H. The determination of the nature of the equilibrium of a floating body depends on the value of μ.

If μ be positive, the equilibrium is stable and M is above G.

If μ be negative, the equilibrium is unstable and M is below G.

If $\mu = 0$, the equilibrium is neutral and M coincides with G.

Hence the equilibrium is stable, unstable, and indifferent, as M is above, below, or coincident with G.

From an inspection of the figure, it is evident that when M is above, below, or coincident with G, the equilibrium will be of the character just assigned to it. Hence, if a body be ballasted so that M can never come below G, the equilibrium cannot be unstable.

Ex. *A cone floating vertically.*

The plane of floatation will be a circle, and let the cone float with its vertex downwards.

Let x, y be the height and radius of the base of the cone which is immersed, and a, b of the whole cone. Then

$$I = \frac{1}{4}\pi y^4, \quad \lambda = \frac{3}{4}(a - x), \quad V = \frac{1}{3}\pi y^2 x.$$

Making these substitutions in the value of μ, we have

$$\mu = \frac{3}{4}\left\{\frac{y^2}{x} - (a - x)\right\}$$

$$= \frac{3}{4}\left\{\frac{y^2}{x} - a\left(1 - \frac{x}{a}\right)\right\}.$$

But the part immersed bears a constant ratio to the whole body, (1° Art. 63.), and the cones being similar solids are to each other as the cubes of their height or of the radii of their bases, hence,

$$x^3 = sa^3 \text{ and } y^3 = sb^3;$$

$$\therefore \frac{y^2}{x} = s^{\frac{1}{3}} \frac{b^2}{a} \text{ and } \mu = \frac{3}{4} \left\{ s^{\frac{1}{3}} \frac{b}{a} - (1 - s^{\frac{1}{3}}) \right\} b,$$

whence the stability for particular values of s and of the ratio $\frac{b}{a}$ may be determined.

If the cone float with its base immersed, we must express V and λ in terms of the proper quantities, and replace s by $(1 - s)$, as in 2°. Art. 63.

The Oscillations of Floating Bodies.

74. In the preceding propositions, the conditions of the equilibrium and stability of floating bodies have been fully considered; we have now to consider the vertical and angular oscillations consequent on a body being disturbed and then left to itself.

The body when left to itself will make oscillations about its original position of equilibrium, until by the action of the fluid it is reduced to rest.

In a complete solution of this problem the motion of the fluid ought to be taken into the account, as this however would be an investigation of extreme difficulty, we shall consider simply the vertical and angular oscillations about the centre of gravity of a body symmetrical about a vertical plane.

In determining the *time* of an oscillation, no account need be taken of the resistance of the fluid, for this resistance is a disturbing force which affects the extent but not the time of each oscillation; and if the disturbing force ceased at any instant to act, the body would go on for ever oscillating in an arc of equal extent to that which it had the instant at which the disturbing force ceased to act*.

Hence, when the oscillations are small the time may be found very accurately.

* AIRY's *Planetary Theory*. Art. 104.

75. Prop. *To determine the time of the small vertical oscillations of a body floating in a fluid.*

The vertical motions of the body are the same as the motions of its centre of gravity, and the motion of the centre of gravity is the same as if the whole mass were collected in it, and the forces applied immediately to it.

Now the resultant of all the forces acting on a body floating in a fluid, is a single force equal to the weight of the fluid displaced. If, therefore, the body floating in the fluid be depressed through any space and then left to itself, the force applied to the body will be the whole weight of the fluid displaced, the resultant of which being a single force in a vertical direction, the motion of the centre of gravity, and therefore the motion of the body will be wholly in a vertical direction.

The body not being wholly immersed, let V be the volume of the fluid displaced when the body is at rest, and V' that of the whole body. Then (Art. 62.)

$$V\rho = V'\rho' \ldots\ldots(1).$$

Let the centre of gravity be the origin of co-ordinates, and a plane parallel to the surface of the fluid be the plane of xy.

Let z be the distance of the centre of gravity from its original position at any time (t), then $-\dfrac{d^2z}{dt^2}$ is the effective accelerating force on the body to bring it back to its original position.

The impressed force is the weight of the fluid displaced by the motion of the body.

Let U be the volume of the fluid displaced by the depression of the body, its weight $= U\rho g$; then since $V'\rho'$ is the mass moved,

$$\text{the accelerating force} = \frac{U\rho g}{V'\rho'} = \frac{Ug}{V} \text{ by (1).}$$

The impressed and effective forces are equivalent, therefore

$$\frac{Ug}{V} = -\frac{d^2z}{dt^2}.$$

But since the motion is small, we may assume $U = Kz$ where K is the area of the plane of floatation;

$$\therefore \frac{d^2z}{dt^2} = -\frac{Kg}{V}z, \qquad \therefore t = 2\pi\sqrt{\frac{V}{Kg}}$$

whence multiplying by $2dz$ and integrating

$$\left(\frac{dz}{dt}\right)^2 = C - \frac{Kg}{V}z^2.$$

When the body is at its lowest point, that is, when the motion commences let $z = a$;

$$\therefore 0 = C - \frac{Kg}{V}a^2,$$

and subtracting from the preceding,

$$\left(\frac{dz}{dt}\right)^2 = \frac{Kg}{V}(a^2 - z^2),$$

whence the velocity is known. To determine the time

$$t = \left(\frac{V}{Kg}\right)^{\frac{1}{2}} \int_0^a \frac{-dz}{\sqrt{a^2 - z^2}}$$

$$= \left(\frac{V}{Kg}\right)^{\frac{1}{2}} \frac{\pi}{2},$$

this therefore is the time of the body returning to its original position of rest, and it will go on oscillating till reduced to rest, the time of each whole oscillation being $\pi\sqrt{\frac{V}{Kg}}$, that is, the motions are isochronous with those of a cycloidal pendulum whose length is $\sqrt{\frac{V}{K}}$.

76. Prop. *To determine the time of the angular oscillations of a body about its centre of gravity.*

The impressed force acting on the body to turn it about its centre of gravity, is the moment of the weight of the fluid displaced.

Let V be the fluid displaced, and ρ its density; then (Art. 72.) the

$$\text{impressed force} = V\rho g\left(\frac{I}{V} \mp \lambda\right).$$

Let k be radius of the circle of gyration, then if V' be the volume of body and ρ' its density,

$$\text{the moment of inertia about } G = V'\rho' k^2.$$

Dividing the impressed force by this, we have, since $V\rho = V'\rho'$,

$$\text{the effective force} = \frac{g}{k^2}\left(\frac{I}{V} \mp \lambda\right)\theta.$$

And the equilibrium being stable, this tends to diminish θ, therefore, the effective force being $-\dfrac{d^2\theta}{dt^2}$, we have

$$\frac{d^2\theta}{dt^2} = -\frac{g}{k^2}\left(\frac{I}{V} \mp \lambda\right)\theta.$$

Whence, multiplying by $2d\theta$ and integrating and supposing that when t commences $\theta = \alpha$ or that α is the amplitude of the oscillation, we have

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{g}{k^2}\left(\frac{I}{V} \mp \lambda\right)(\alpha^2 - \theta^2),$$

for the square of the velocity, and

$$t = \frac{1}{\sqrt{\frac{g}{k^2}\left(\frac{I}{V} \mp \lambda\right)}} \int_0^a \frac{-d\theta}{\sqrt{a^2 - \theta^2}}$$

$$= \frac{1}{\sqrt{\frac{g}{k^2}\left(\frac{I}{V} \mp \lambda\right)}} \frac{\pi}{2}$$

for the time of the body's returning to its original position. Thus the body will perform isochronous oscillations, the time of each of which is $\pi \frac{k}{\sqrt{g\left(\frac{I}{V} \mp \lambda\right)}}$, and the length of the isochronous cycloidal pendulum is $\frac{k}{\sqrt{\frac{I}{V} \mp \lambda}}$.

Ex. *A cylinder floating vertically.*

Let a be its height and b the radius of its base.

Then $I = \frac{1}{4}\pi b^4$, $k^2 = \frac{1}{12}a^2 + \frac{1}{4}b^2$.

Also $V = s \,.\, \pi a b^2$, $\lambda = \frac{1}{2}a - \frac{1}{2}sa = \frac{1}{2}(1-s)a$, (Art. 62.); therefore the time of an oscillation

$$= \pi \sqrt{\frac{\frac{1}{12}a^2 + \frac{1}{4}b^2}{g\left\{\frac{1}{4s}\frac{b^2}{a} - \frac{1}{2}(1-s)\,a\right\}}}$$

$$= \pi \sqrt{\frac{s\,(a^2 + 3b^2)\,a}{g\,\{3b^2 - 6s\,(1-s)\,a^2\}}}.$$

For a more general investigation, see Moseley's Hydrostatics, (Art. 88.)

CHAPTER VII.

ON THE APPLICATIONS OF THE BAROMETER.

77. From the explanation which has been already given of the barometer (Art. 44.) it appears that the atmospheric pressure is in equilibrium with the weight of the mercury in the barometer tube: hence, if m be the density of mercury, g the accelerating force of gravity, h the difference of level in the two branches of the tube, and p_1 the atmospheric pressure, we have

$$mgh = p_1.$$

It must be supposed that the barometer is accurately filled, so that there is no sensible pressure above the mercury at the closed end of the tube. Now the open end of the tube may be considered as produced to the limit of the atmosphere; then p_1 is the weight of the vertical and cylindrical atmospheric column whose base is equal to the unit of area. This weight, as appears from the preceding condition, is equal to the weight of a column of mercury of the same base, and whose height, as appears from observations on the barometer, to be very accurately 29.92 inches as its mean value. The pressure then of the atmosphere on each square inch of the earth's surface will be 29.92 × the weight of a cubic inch of mercury = 29.92 × 7.85 ounces* = 14.7 pounds avoirdupois.

As we ascend above the surface of the earth, the height and consequently the weight of the superincumbent column of the atmosphere diminishes; the height therefore, and

* *Hydrostatics.* Art. 68. Ex. 6.

consequently the weight of the sustained barometric column must diminish also; there must then exist some relation between the height which has been ascended, and the height of the sustained column, and it is the object of the present chapter to ascertain this relation.

78. The mass of the atmosphere may be compared with the mass of the earth by the preceding article.

Let S be the surface of the earth expressed in square inches, then the mass of the atmosphere may be considered as equal to mSh.

The mass of the earth (considered spherical) $= \frac{1}{3}\rho S r$, if ρ be its mean density, and r its radius. Then

the mass of the atmosphere : mass of the earth

$$:: mSh : \tfrac{1}{3}\rho S r$$

$$:: 3\frac{mh}{\rho r} : 1.$$

But the mean density of the earth is about $5\frac{1}{2}$ times that of water, and the density of mercury is about $13\frac{1}{2}$ that of water at the same temperature and pressure;

$\therefore \dfrac{m}{\rho} = 2.5$, nearly; also $r = 4000$ miles, and $h = 29.9$ inches,

from which data it will be found that the mass of the atmosphere is a little less than one millionth part of the mass of the earth.

79. If the air had the same density throughout the atmospheric column, the height of this column, and the height (h) of the barometric column would be inversely proportional to the densities of the air and the mercury.

Let H be the height of the atmospheric column, and ρ its mean density, then

$$H : h :: \frac{1}{\rho} : \frac{1}{m}; \quad \therefore H = \frac{m}{\rho} h.$$

$$\text{But } \frac{m}{\rho} = \frac{13.58^*}{.001299} = 1045;$$

$$\therefore H = 1045 \times 29.9 = 4.9 \text{ miles, nearly.}$$

The atmosphere must evidently extend much higher than this, since the density and weight of its strata diminish as we ascend above the surface of the earth. We shall fix a limit to which it cannot reach by determining the point at which the centrifugal force is equal to gravity; for from that point the centrifugal force would disperse the molecules of air in space. This limit is less elevated at the equator than at any other place.

At the equator the centrifugal force $= \frac{g}{289}$†.

At a height z above the surface it becomes

$$\frac{g(r+z)}{289r},$$

and the intensity of gravity at that point is $\frac{gr^2}{(r+z)^2}$;

the limit proposed is given therefore by the equation

$$\frac{r+z}{289r} = \frac{gr^2}{(r+z)^2};$$

$$\text{whence } \left(1 + \frac{z}{r}\right)^3 = 289, \quad \text{or } z = \left\{\sqrt[3]{289} - 1\right\} r,$$

that is, about five times the radius of the earth. This might be near the truth were the temperature invariable as we ascend, but the repulsive power of the particles is so rapidly diminished by the cold, that a limit is soon fixed to the extent of the atmosphere.

The general equation $p = k\rho$ is true in all ordinary cases, but evidently fails in extreme cases, as when the condensation or rarefaction is extreme. The accurate equation must be of the form $p = k(\rho - \delta)$, where δ is a

* *Table of Specific Gravities.* † *Figure of the Earth.*

very small quantity, which may generally be omitted, but at the limit of the atmosphere it is appreciable.

80. The force of gravity may also be measured by observations on the barometric column: for if the manometer (Art. 45.) be observed at different places on the earth's surface, the temperature and density of the air contained in the vessel C (Fig. 7.) remaining the same, the height of the mercury must vary inversely as the gravity, in order that the weight of the column may remain the same.

In order to make these observations with accuracy, the variation in the volume of the air contained in the vessel at C, in connection with the height of the mercury in the closed tube must be taken into the account.

Let g be the force of gravity, and h the height of the column at one station, the surfaces of the mercury in the manometer being at P and Q. When the manometer is transferred to another place, let g', h' be the values of g and h, the surfaces of the mercury being at P' and Q'.

The pressures of the barometric columns in the two cases will be as gh and $g'h'$; these will be proportional to the density of the air in the manometer, and consequently in the inverse ratio of its volume.

Let V and V' be its volume at the two stations, then

$$\frac{gh}{g'h'} = \frac{V'}{V}.$$

Let k be the area of a horizontal section of the tube at P, then the volume of mercury contained between P and P' will equal $(h' - h)k$. But the mercury being incompressible, this must be equal to the variation $V' - V$ of the volume of the elastic fluid, therefore

$$V' = V + (h' - h)k;$$

whence substituting for V' in the preceding

$$\frac{g'}{g} = \frac{Vh}{\{V + (h' - h)k\}h'},$$

which gives the ratio of the intensities of gravity at the two places. This method is precarious, however carefully the observations are made, and the accuracy of the results cannot be compared with the accuracy of those derived from experiments with a pendulum.

81. It is found by experiment that the air and all other gases when subject to the same and a constant pressure dilate equally for equal increments of temperature, and this increment of bulk is found to be equal to $\frac{1}{480}$th of its volume for each degree of Fahrenheit*.

If then the volume of any gas be constant, its elastic force will increase, and if the elastic force be constant, that is, if it be subject to the same pressure, its volume will increase for every increase of temperature. It is therefore of the greatest importance to connect these quantities by an equation.

Prop. *To express the elastic force of any gas as a function of its density and temperature.*

Let V be the given volume of a gas at the standard temperature, e its elastic force, and D its density.

The elastic force e, that is, the pressure on a unit of surface remaining the same, let the temperature be increased by θ^0, let V' be the volume, and D' the density of the gas, then if a be the increment of bulk for each degree of temperature,

$$V' = V(1 + a\theta).$$

But the density varies inversely as the volume;

$$\therefore \frac{D'}{D} = \frac{V}{V'} = \frac{1}{1 + a\theta}; \quad \therefore D' = \frac{D}{1 + a\theta}.$$

* *Hydrostatics.* Art. 73.

Now suppose that the pressure is changed, the temperature remaining constant, namely, let p be the value of e, and ρ the value of D', then by Mariotte's law,

$$\frac{p}{e} = \frac{\rho}{D'}; \quad \therefore \; p = \frac{\rho e}{D'} = \rho \frac{e}{D}(1 + \alpha\theta).$$

Let $\frac{e}{D} = k$ a constant quantity which expresses the ratio between the elastic force and the density at a given temperature, therefore

$$p = k\rho(1 + \alpha\theta).$$

This formula is applicable to all gases, vapours, or their mixtures.

82. PROP. *To find the difference of the altitude of two stations by means of the barometer.*

The general equation between the pressure at any point of a fluid mass, and the impressed forces is (Art. 10.)

$$dp = \rho(Xdx + Ydy + Zdz).$$

In the atmosphere, gravity being the only force

$$X = 0, \; Y = 0, \text{ and } Z = \frac{gr^2}{(r+z)^2}$$

for a point at a height z above the surface of the earth, g being the gravity at the surface, and r the radius of the earth. Then since this force tends to diminish z it is negative; the equation becomes therefore

$$\frac{dp}{\rho} = -\frac{gr^2dz}{(r+z)^2}.$$

But $p = k\rho(1 + \alpha\theta)$ (Art. 81.) hence dividing,

$$\frac{dp}{p} = -\frac{gr^2dz}{k(1+\alpha\theta)(r+z)^2}.$$

It is impossible to integrate this expression, since θ is an unknown function of z; and the exact law of the variation of the temperature being unknown, we shall consider θ as constant; integrating therefore on this hypothesis,

$$\log p = \frac{gr^2}{k(1+\alpha\theta)} \cdot \frac{1}{r+z} + C.$$

In order to determine the constant, let $p_{,}$ be the value of p when $z = 0$, that is, the pressure at the surface of the earth, then

$$\log p_{,} = \frac{grz}{k(1+\alpha\theta)} \cdot \frac{1}{r} + C,$$

and subtracting

$$\log \frac{p}{p_{,}} = -\frac{gr}{k(1+\alpha\theta} \cdot \frac{z}{(r+z)} \text{ ....(1).}$$

To apply this formula to determine the distance of any point above the surface of the earth.

Let z' be the height of the upper station, and let p' be the value of p at that point.

Let $\tau_{,}$ be the number of degrees by which the temperature at the surface of the earth exceeds the standard temperature, and τ' the number of degrees for the point at the height z'.

Now the change of temperature as we ascend from the surface of the earth is gradual and nearly uniform for small elevations, hence there will be no great error in assuming the quantity $\theta = \frac{\tau_{,} + \tau'}{z}$.

Let $h_{,}$, h' be the observed heights of the barometric column at the lower and upper station, then, since (Art. 77.) $p_{,} = mgh_{,}$, $p' = mgh'$,

$$\frac{p'}{p_{,}} = \frac{h'}{h_{,}}.$$

Making then these substitutions in the equation (1) and changing the sign, since

$$\log \frac{p_{\prime}}{p'} = -\log \frac{p'}{p_{\prime}} \text{ we have}$$

$$\log \frac{h_{\prime}}{h'} = \frac{g r z'}{k\left(1 + a\,\dfrac{\tau_{\prime} + \tau'}{z}\right)(r + z')} \ldots\ldots\ldots\ldots (2),$$

whence z' the height above the surface of the earth may be found, since all the other quantities are known, and the height of any other station being ascertained in the same manner, the elevation of one above the other is determined.

83. The preceding equation will require several corrections.

1°. The temperature will be different at the station whose height is required, and at the surface of the earth, and the mercury will be denser at the colder place than at the other, and consequently the same atmospheric pressure sustains a less column than it would have sustained had the temperature remained unchanged. Hence to compare the pressures at the station and at the surface of the earth, the barometric column must be reduced to the same density; and the column at the colder place must be increased by the quantity by which it would expand at the temperature of the warmer.

Let β be the coefficient expressing the change in bulk which each unit of volume undergoes for each degree of temperature.

Then since $(\tau_{\prime} - \tau')$ is the difference of temperature of the two places (the upper being taken as the colder), each unit of bulk of the barometric column is diminished by $\beta(\tau_{\prime} - \tau')$; and this correction may be considered as due to the height simply, no correction being necessary for the diameter of the column, since glass and mercury expand and contract equally at ordinary temperatures. Instead therefore of using the observed height h', we must use

$$h' \{1 + \beta(\tau_{,} - \tau')\}.$$

Hence $\log h'$ is to be replaced by

$$\log h' \{1 + \beta(\tau_{,} - \tau')\} = \log h' + \log \{1 + \beta(\tau_{,} - \tau')\}$$
$$= \log h' + M\beta(\tau_{,} - \tau'),$$

nearly, where M is the modulus of the system of logarithms.

2^0. The force of gravity varies with the latitude, hence g is not constant for all places on the earth's surface; and the general expression for gravity in terms of the latitude is

$$g = E(1 + n\sin^2\lambda)^*,$$

where E is the equatorial gravity, and n a known quantity.

Then if G be the force of gravity at latitude 45^0,

$$G = E(1 + n),$$

$$\therefore\ g = G\frac{1 + n\sin^2\lambda}{1 + n} = G\left\{1 - \frac{n}{2}(1 - 2\sin^2\lambda)\right\} \text{ nearly}$$
$$= G\left(1 - \frac{n}{2}\cos 2\lambda\right).$$

3^0. The coefficient α will require some correction, and also the constant k.

In determining the values of these quantities, the air was either supposed to be dry, that is, not to contain any aqueous vapour, or that the quantity of that vapour is constant. But as the temperature increases, the quantity of vapour increases also in the atmosphere, and the elastic force of the vapour being added to the elastic force of the air, the increment of volume for a given volume of air must be greater for air which contains vapour than for dry air, hence α must be increased by a small quantity.

* *Figure of the Earth.*

For the same reason k will require a small correction, since it expresses the ratio of the elastic force to the density at a given temperature of air that is dry, or contains a constant quantity of vapour.

84. For practical purposes an approximate value of the general equation (2), (Art. 82.), may be found. Multiplying up it becomes

$$\frac{rz'}{r+z'} = \frac{k}{g}\left(1 + a\,\frac{\tau_{,} + \tau'}{z}\right)\log\frac{h_{,}}{h'}.$$

$$\text{But } \frac{rz'}{r+z'} = z'\left(1 - \frac{z'}{r}\right)^{-1} = z'$$

very nearly, in all cases to which the barometer can generally be applied;

$$\therefore\; z' = \frac{k}{g}\left(1 + a\,\frac{\tau_{,} + \tau'}{z}\right)\log\frac{h_{,}}{h'}$$

$$= \frac{k}{g}M\left(1 + a\,\frac{\tau_{,} + \tau'}{z}\right)\log\frac{h_{,}}{h'},$$

where M is the modulus of the common system of logarithms.

Now $\frac{k}{g}M$ may be taken equal to 20117 yards, and $\frac{a}{2}$ equal to $\frac{1}{900}$, as mean values including the corrections. Whence

$$z' = 20117\left\{1 + \frac{\tau_{,} + \tau'}{900}\right\}\log\frac{h_{,}}{h'},$$

which will be found a convenient formula for determining in yards the elevation of one station above another, the temperatures being the number of degrees above 32° F.

CHAPTER VIII.

ON CAPILLARY ATTRACTION.

85. The centres of the attractive or repulsive forces which act on a fluid mass, may be all the other points of the fluid. In this case, the components of X, Y, Z, of the accelerating force acting on the point P, will consist of an infinite number of terms; these may be certain functions of x, y, z, common to all points of the fluid, if we suppose that the principle of the equality of action and reaction obtain in their mutual attractions and repulsions, and that all the points are besides submitted to the same extraneous forces.

In nature, these forces are of two kinds, the one varying according to the inverse square of the distance, and the intensities of the other are expressed by functions which decrease with extreme rapidity and are not sensible except at insensible distances.

The components of the former of these may be calculated by dividing the fluid into small elementary masses, and obtaining by the integral calculus the sum of the attractive or repulsive forces in each direction. The other class of forces, which are *molecular forces*, and which are either attractive or repulsive according as the attraction of the ponderable matter is greater or less than the repulsive power which is due to heat, cannot be taken any account of in the calculation of the forces X, Y, Z, for any point in the interior of the fluid mass. For these molecular forces are those which produce the pressure p equal on all sides of the point, and which we have already considered in forming the equations of equilibrium.

It follows from this latter consideration, that the equations (1), (Art. 10.) which we have obtained, are the necessary and sufficient conditions of equilibrium of all the forces, and that the molecular forces which act on any element of a fluid mass are comprised in them; so that the equilibrium most certainly subsists when there is a value of p which satisfies these equations for all the points of the fluid, which coincides with the value given directly of the pressure at a free surface, and which does not become negative at any point so long as the particles of the fluid remain contiguous.

If the law of these molecular forces were given as a function of the distance, and we could deduce from these forces the expression for p as a function of the mean interval between the molecules, it might be substituted in the equations (1). One of them would determine the magnitude of this interval which exists in a state of equilibrium about the point P, and the other two would express the conditions of that equilibrium.

The numerical value of p would be afterwards found from that of the mean interval or from the corresponding value of the density, and the method in which this pressure p may vary very much, for the very small variations of the density which we observe in fluids is explained by Poisson.* But the direct determination of the pressure p being impossible, we are obliged to deduce its value from the conditions of equilibrium themselves, or from the equation (2) which results from these.

When the point P is situated at the surface of the fluid or is distant from it by a less quantity than the radius of the action of the molecular forces, we must take account of these forces, and also of the rapid variation of the density at the surface, in the calculation of their components X, Y, Z, and consequently of the value of the pressure p deduced from (2). Thence there arises an

* *Journal de l'Ecole Polytechnique*, 20 *cahier*.

influence of molecular forces on the figure of a fluid in equilibrium, which is not in general sensible, and which cannot be so except in capillary spaces.

86. If a fluid be regarded as composed of atoms held in places of equilibrium by attractive and repulsive forces proceeding from the atoms, it will necessarily follow that every change of pressure is accompanied by a change of density, and that at their surfaces there will be a rapid change of density within a small, and, as experience shews, insensible extent, depending on the sphere of sensible activity of the molecular forces. In strictness, this superficial variation of density should, as we have just said, be taken into account in treating of capillary action, as Poisson has done in his *New Theory of Capillary Action;* but as neither theory nor experiment has hitherto determined to what degree it affects capillary phænomena, and, considering the great repulsive and feeble attractive molecular action of fluids, the effect is probably of small magnitude, we shall therefore neglect it in the following propositions, and suppose the fluid to be perfectly incompressible and to be acted upon, in addition to gravity, only by the molecular *attraction* of its own particles and of those of the solid with which it is in contact. The law of the attraction is unknown, but as experience teaches, must be considered sensible only at insensible distances from the attracting centres. With this limitation Problems in Capillary Attraction are to be treated as any other questions in Hydrostatics, with the modifications that the peculiar nature of the forces introduces.

87. Let AB and CD (Fig. 17.) be the sides of a solid between which fluid is drawn up, and abc the capillary surface of the fluid. Let P and Q be points in the capillary surface and in the horizontal surface of the external fluid, both points being beyond the sphere of the molecular action of the particles of the solid. Let an indefinitely small canal be drawn from P to Q, its extre-

mities being perpendicular to the surfaces at P and Q, and having every point beyond the sphere of the molecular attractions of the particles of the solid.

PROP. *To find the condition of equilibrium of this canal.*

The forces that sustain it are the molecular attractions of the surrounding fluid and gravity; the molecular attractions of the solid being by the hypothesis laid out of the case, every point of the canal being beyond the sphere of these actions. If we find the resolved parts of these forces in the direction of the axis of the canal and equate their sum to zero since the canal is in equilibrium, we shall have an equation for determining the form of the surface abc.

Take any element as R at a sensible distance from the extremities of the canal; the molecular attractions on this element must be equal in opposite directions, and therefore destroy each other. This will not be the case at the extremity Q, where the attractions of the surrounding fluid on an element Qq *downwards* will not be counteracted by an upward attraction; and consequently, the canal will be urged in a direction from Q towards R.

If a tangent plane be drawn to the capillary surface at P, the fluid below this plane will urge an element at that extremity of the canal in the same manner and to the same degree as the fluid below Q urges an element at that extremity. These forces acting in opposite directions along the canal will destroy each other.

The only remaining molecular attraction, is that of the fluid contained between the capillary surface aPc and the tangent plane TP. Let the moving force on the canal due to this attraction for the present be considered equal to $P\rho k^2$, where ρ is the density of the fluid, k^2 the section of the canal, and P is to be determined. Opposed to this force is the action of gravity, tending to depress the part PS which rises above the external horizontal surface.

The action of gravity on the part QRS of the canal has plainly no tendency either to elevate or to depress the fluid. Now if z be the vertical height of the point P above the horizontal surface, the action of gravity on PS resolved in the direction of its length is to produce a weight equal to that of a column of height z and base k^2. Hence

$$P\rho k^2 = g\rho z k^2, \quad \text{or} \quad P = gz.$$

88. PROP. *To find an expression for* P *in terms of the principal radii of curvature of the surface.*

Let the tangent plane at any point O (Fig. 18.) in the surface be taken for the plane of xy, then Oz a normal to the surface will be the axis of z.

Let a plane passing through the normal, and making an angle θ with the plane of xz, intersect the capillary surface and the tangent plane at O in OP and OT. Then zOx, zOy are the planes of maximum and minimum curvature.

Let $Ot = r$, $pt = z$, and R be the radius of curvature of OpP at O, then

$$R = \frac{Ot^2}{2pt}, \quad \text{whence } z = \frac{r^2}{2R}.$$

Let R_1, R_2, be the greatest and least radii of curvature; then, since

$$\frac{1}{R} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2},$$

we have

$$z = \frac{r^2}{2}\left\{\frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2}\right\}.$$

Let another plane be conceived drawn through the normal Oz, making an angle $\theta + d\theta$ with the plane of xz; then we shall have a small pyramid or wedge whose four edges are the sections of the normal planes with the capillary surface and the tangent plane. Let an elementary column of this pyramid be taken as at pt.

The column thus taken is such as $pbcd$ (Fig. 19.), where $apbc$ is the portion of the capillary surface, and $tdef$ of the tangent plane; whence it will be seen at once that

$$\text{the content of this column } pt = z\,r\,d\theta\,dr.$$

As the attraction is sensible only for very small distances from the canal whose axis will coincide with Oz, the height z of the column may be considered to be always very small, and its attraction to be the same as if collected at its middle point o.

Let $OG = z_1$, $GO = r_1$, and $\rho k^2 dz_1$ be the element of the canal at G; the attraction then of the column at pt upon it is, if $\phi(r_1)$ be the law of the molecular attraction,

$$z\,r\,d\theta\,dr\,\phi(r_1) \times k^2 \rho\,dz_1.$$

Resolving this force in the direction GO, the part required is

$$k^2 \rho \phi(r_1) z\,r\,dr\,d\theta\,dz_1 \times \frac{z_1}{r_1},$$

which, substituting the preceding value of z, becomes

$$\tfrac{1}{2} k^2 \rho \phi(r_1) \cdot \frac{1}{r^2} \cdot r^3 dr \cdot z_1 dz_1 \left(\frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2}\right) d\theta;$$

$$\therefore\; P = \iiint \tfrac{1}{2} \phi(r_1) \frac{1}{r_1} r^3\,dr\,z_1\,dz_1 \left(\frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2}\right) d\theta.$$

Integrating with respect to θ, from $\theta = 0$ to $\theta = 2\pi$,

$$P = \tfrac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \iint r_1\,\phi(r_1)\,r^3 dr\,z_1\,dz_1.$$

Now $r_1^2 = z_1^2 + r^2$, and as z_1 is to vary independently of r, $r_1 dr_1 = z_1 dz_1$;

$$\therefore \iint \phi(r_1) \frac{1}{r_1} z_1 dz_1 r^3 dr = \iint \phi(r_1) dr_1 r^3 dr.$$

Here the integration is to be performed with respect to r considering r_1 as constant. Hence the limits of this integration must be from $r = 0$ when $GO = r_1$ to $r = r_1$ when $GO = 0$.

The integral therefore is equal to $\frac{1}{4}\int \phi(r_1) r_1^4 dr_1$ from $r_1 = 0$ to $r_1 = \infty^{\text{ty}}$, as its value is not generally increased by increasing r_1, on account of the form of the function $\phi(r_1)$. The last integration cannot be performed, since the form of the function ϕ is unknown. Let us assume, however,

$$\tfrac{1}{4}\int_\infty^0 \phi(r_1) r_1^4 dr_1 = H.$$

$$\text{Then} \quad P = \tfrac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) H;$$

$$\therefore \tfrac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) H = gz \ldots\ldots (A).$$

The equation thus obtained is the differential equation of the capillary surface, by integrating which and determining the values of the arbitrary constants, the form of the surface will become known.

We shall proceed in the following articles to apply the preceding theory to some of the known instances of capillary attraction.

89. Ex. 1. *A cylindrical tube of small diameter.*

Let $ACDB$ (Fig. 20.) be the section of the cylindrical tube, and abc the section of the capillary surface by a vertical plane through the axis of the tube. Since every point is symmetrical with respect to this axis, the capillary surface will be one of revolution.

Let x, y be the vertical and horizontal co-ordinates MN, NP of a point P in the section abc.

Thus the radii of greatest and least curvature at the point P are the normal and radius of curvature at that point; hence

$$R_1 = y\sqrt{1+\frac{dy^2}{dx^2}}, \quad R_2 = \frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}};$$

or, writing p and q for the first and second differential coefficients, and substituting in (A), we have

$$\frac{1}{2}\left\{\frac{1}{y(1+p^2)^{\frac{1}{2}}} - \frac{q}{(1+p^2)^{\frac{3}{2}}}\right\}H = gx,$$

$$\text{or} \quad \frac{H}{2g}\left\{\frac{p}{(1+p^2)^{\frac{1}{2}}} - \frac{ypq}{(1+p^2)^{\frac{3}{2}}}\right\} = xyp.$$

Let $Mb = h$, $bN = x'$, then

$$x = h + x', \quad p = \frac{dy}{dx'}, \quad q = \frac{d^2y}{dx'^2},$$

and the preceding becomes

$$\frac{H}{2g}\left\{\frac{p}{(1+p^2)^{\frac{1}{2}}} - \frac{ypq}{(1+p^2)^{\frac{3}{2}}}\right\} = hyp + x'yp.$$

But the first side of the equation is the same as

$$\frac{H}{2g}\frac{d}{dx'}\cdot\frac{y}{\sqrt{1+p^2}};$$

integrating, therefore, and adding an arbitrary constant,

$$\frac{H}{g}\frac{y}{\sqrt{1+p^2}} = hy^2 + 2\int x'yp + C.$$

When the diameter of the tube is very small, x' will be small compared with h; hence, neglecting the term involving x', and supposing the integral to begin when $y=0$, so that the constant becomes nothing, we have

$$y = \frac{H}{gh}\frac{1}{\sqrt{1+p^2}}.$$

Let $\dfrac{H}{gh} = a$; then

$$1 + p^2 = \frac{a^2}{y^2}, \quad \text{and } p^2 = \frac{a^2}{y^2} - 1;$$

$$\therefore \frac{1}{p} = \frac{y}{\sqrt{a^2 - y^2}}, \quad \text{or } dx' = \frac{y\,dy}{\sqrt{a^2 - y^2}};$$

and, integrating,

$$x' = -\sqrt{a^2 - y^2} + C.$$

When $y = 0$, $x' = 0$;

$$\therefore 0 = -\sqrt{a^2} + C;$$

and, subtracting from the preceding,

$$x' = a - \sqrt{a^2 - y^2};$$

that is,

$$y^2 = 2ax' - x'^2,$$

the equation of a circle whose radius is a. The capillary surface is therefore very nearly spherical.

Since $a = \dfrac{H}{gh}$, the radius of the capillary surface varies inversely as h.

Ex. 2. *Two parallel plates.*

Let AB, CD as before (Fig. 20.) be the sections of the plates, and abc of the capillary surface by a vertical plane perpendicular to the plates. Then BD is the distance of the plates from each other.

The capillary surface will here be cylindrical, and one of the radii of curvature will be infinite at every point.

Hence if $R_1 = \propto^{\text{ty}}$, and $R_2 = \dfrac{(1+p^2)^{\frac{3}{2}}}{-q}$,

the equation (A) becomes

$$-\frac{H}{2g}\frac{q}{(1+p^2)^{\frac{3}{2}}} = x = h + x',$$

if $Mb = h$ and $bN = x'$, as in the preceding example.

Separating this into two terms by adding and subtracting p^2 in the numerator,

$$-\frac{H}{2g}\left\{\frac{q}{(1+p^2)^{\frac{1}{2}}} - \frac{p^2 q}{(1+p^2)^{\frac{3}{2}}}\right\} = h + x'.$$

Whence, integrating,

$$-\frac{H}{2g}\frac{p}{\sqrt{1+p^2}} = hx' + \tfrac{1}{2}x'^2 + C.$$

Now $p = \propto^{\text{ty}}$ when $x = 0$,

$$\therefore\ -\frac{H}{2g} = C;$$

whence, by subtraction,

$$\frac{H}{2g}\left\{1 - \frac{p}{(1+p^2)^{\frac{1}{2}}}\right\} = hx' + \tfrac{1}{2}x'^2 = (h + \tfrac{1}{2}x')x'.$$

Since x' is small compared with h, we may omit $\frac{1}{2}x'$ compared with h, and assuming $\dfrac{H}{2gh} = \beta$, we have

$$1 - \frac{p}{(1+p^2)^{\frac{1}{2}}} = \frac{x'}{\beta};$$

$$\therefore\ \frac{p}{\sqrt{1+p^2}} = 1 - \frac{x'}{\beta} = \frac{\beta - x'}{\beta}.$$

Whence $1 + \dfrac{1}{p^2} = \dfrac{\beta^2}{(\beta - x')^2}$, or, $\dfrac{1}{p^2} = \dfrac{2\beta x' - x'^2}{(\beta - x')^2}$;

$$\therefore\ p \text{ or } \frac{dy}{dx'} = \frac{\beta - x'}{\sqrt{2\beta x' - x'^2}},$$

and integrating,

$$y = \sqrt{2\beta x' - x'^2},$$

the equation to the section, no correction being requisite, since y and x' begin together.

The section of the capillary surface is therefore a semicircle whose radius is β. The whole surface is therefore cylindrical.

We assumed $\frac{H}{2gh} = \beta$, and in the preceding example we assumed $\frac{H}{gh} = a$. It follows, therefore, that if h be the same both for the parallel plates and for the cylindrical tube, $2\beta = a$.

90. In the preceding investigation every point of the canal was supposed to be beyond the sphere of the molecular attraction of the particles of the solid.

Let AB, CD (Fig. 21.) represent the bounding surfaces of the solid, and let $A'B'$, $C'D'$, drawn parallel to these through the points b, d, of the capillary surface, be the limits of the sensible molecular attraction of the sides of the solid.

Draw tangents at these points, meeting the axis MN in m and n.

The angle which the fluid in the capillary surface makes with the surface of the solid is called the *actual* angle of contact, or, the angle of actual contact.

The angle which the fluid in the capillary surface makes with the line drawn at the limit of the molecular attraction of the particles of the solid at the point where its surface meets this line, is the *angle of contact*, or, the theoretical angle of contact.

In the figure amN is the angle of actual contact, and bnN the angle of contact.

91. Prop. *To determine the law of ascent of a fluid in different capillary tubes, or between parallel plates separated by different intervals.*

The equation (A) having been obtained on the hypothesis that every part of the canal was beyond the sphere of the molecular attraction of the particles of the solid, that is, that the canal was wholly without the portions of the fluid between the side AB and the line through b parallel to it, and between the side CD and the line through c parallel to it, cannot be applied to the fluid contained between these portions.

The above equation applies therefore to the fluid bounded by a surface $A'B'C'D'$, similar to the surface $ABCD$ but not to the fluid which is included between these two surfaces.

The angles in which the fluid meets these two surfaces as well as the forms of the portions ab, cd, of the surface will depend on the law of the molecular attractions, and their relative intensities for the solid and the fluid: they cannot therefore be determined since these elements are at present unknown: We may however assert, that, considering the small distances to which the molecular attractions are sensible, the portion ab, of the curve, and the angles which the tangents at a and b make with the vertical are no ways dependant on the diameter of the capillary tube, they will be the same for instance, in a tube one-twentieth of an inch in diameter, as when the fluid ascends against a plane surface.

92. Let us apply these considerations to the preceding examples.

Let O be the centre of the circular arc bd, and ω the angle of contact, that is, the angle which the tangent bn makes with the vertical through b. Let $Ob = Od = r$, and the chord $bd = 2b$. Then $\omega = bnO$, $bN = b$, and since Obn is a right-angled triangle and bN is perpendicular on its base, the angle $ObN =$ the angle bnN;

$$\therefore\ b = r\cos\omega, \quad \text{or } r = b\sec\omega.$$

Now since $A'B'$ is exceedingly near to AB, bN in the case of a capillary tube differs by a very small quantity from the radius of the tube.

But, it was shewn (Ex. 1.) that $r = \dfrac{H}{gh}$;

$$\therefore\ \frac{H}{gh} = b\sec\omega, \quad \text{or } h = \frac{H\cos\omega}{g}\frac{1}{b}.$$

Consequently, as ω is the same for tubes of different diameters;

$$h \propto \frac{1}{b},$$

that is, the height of ascent of the fluid in the capillary tube is inversely as the radius. And experiments confirm this result.

Cor. As h may be taken for the mean height of ascent, the weight of fluid raised is $\pi b^2 h\rho$ very nearly. This quantity, by substituting for h the above value, is equal to $\dfrac{\pi bH\rho}{g}\cos\omega$. Hence for a given tube, the weight of fluid raised varies as $\cos\omega$. Although the angle ω is not affected by the magnitude of the radius of the tube, it greatly depends on the matter of which it is composed, and the state of the internal surface as to polish or greasiness. The way in which the solid tube affects the height of ascent, is by determing the magnitude of the angle ω. The *immediate* action of the tube is on the aqeous cylindrical shell, included between the surfaces $ABDC$, $A'B'D'C'$, and by the intervention of this, it supports the rest of the fluid. The vertical action of the tube on the aqueous cylindrical shell is very nearly the same as the vertical action of the shell on the rest of the fluid, since the weight of the shell by reason of its thinness is exceedingly small. If the latter action be calculated and equated to the weight of fluid raised, this weight will be found to be proportional to $\cos\omega$, in confirmation of the result obtained above. This calculation however, which is given in Art. 18. of Poisson's Treatise, is too long to be inserted here.

93. When the fluid rises between two plates, $2b$ is very nearly the interval between the two plates, and (Ex. 2.), $r = \frac{H}{2gh}$. Hence $\frac{H}{2gh} = b \sec\omega$,

$$\therefore\ h = \frac{H\cos\omega}{g}\frac{1}{2b} \propto \frac{1}{2b},$$

or the height of the fluid varies inversely as the interval between the plates, and is the same as in a tube where the *radius* is equal to the interval between the plates.

If b be given, the height of ascent is greatest when $\omega = 0$.

When ω is > 90, the fluid is depressed below the level of the external fluid as is the case with mercury.

It follows immediately from the law of ascent between parallel plates above determined, that if two plates inclined at a very small angle be dipped in a fluid, with the line of their junction vertical, the fluid will ascend between them in the form of a rectangular hyperbola, the asymptotes of which are the line of junction and the intersection of either plate with the horizontal surface of the fluid. For any two opposite elements of the surfaces of the plates may be considered as parallel, and the rise between these elements will consequently be inversely proportional to their distances from each other, and therefore inversely proportional to their common distance from the vertical asymptote, which indicates that the boundary of the fluid surface will be a rectangular hyperbola.

94. PROP. *To determine the angle of actual contact, with the capillary surface.*

The condition of equilibrium requires, that the resultant of the forces which act at any point of the surface should be perpendicular to the surface, and this will enable us to determine something about the angle of actual contact.

We proceed to determine the direction of the resultant of the forces which act on a particle situated at a or c. Draw a tangent am, and let $amN = \phi$.

Conceive a plane perpendicular to the plane of the paper to pass through a, making an angle θ with AB, the dotted line representing its section. Let another plane be drawn through the same point, making an angle $\theta + d\theta$ with AB.

Then $d\theta$ being indefinitely small, the attraction of the fluid between the planes on the particle at a will vary as $d\theta$. Let it be equal to $qd\theta$. The parts of this, in the vertical and horizontal direction respectively, are $qd\theta\cos\theta$, and $qd\theta\sin\theta$, hence,

$$\text{the total vertical attraction} = \int_0^\phi q\cos\theta d\theta = q\sin\phi,$$

$$\text{the total horizontal attraction} = \int_0^\phi q\sin\theta d\theta = q(1-\cos\phi).$$

The total action of the solid, which will be wholly in the horizontal direction, will be found by putting q' for q, and 180^0 for ϕ in the last expression, therefore

$$\text{the total action of the solid} = 2q'.$$

The resulting attraction of the fluid between the surface ab, and the tangent plane am, cannot be calculated as the form of the surface is unknown. It will in general be small, and its direction will very nearly coincide with am; let its value be μ, then g being the force of gravity, we have, the total force

$$\text{in the vertical direction} = g + q \sin \phi + \mu \cos \phi,$$

$$\text{in the horizontal direction} = 2q' - q(1 - \cos \phi) - \mu \sin \phi.$$

The resultant of these is perpendicular to the surface at a, that is, to am, their ratio must equal $\tan \phi$; or calling then X and Y respectively, and R their resultant, we have $X = R \sin \phi$, $Y = R \cos \phi$, whence

$$\frac{g + q \sin \phi + \mu \cos \phi}{2q' - q(1 - \cos \phi) - \mu \sin \phi} = \tan \phi;$$

$$\therefore\ g \cos \phi + q \sin \phi \cos \phi + \mu \cos^2 \phi$$

$$= 2q' \sin \phi - q \sin \phi (1 - \cos \phi) - \mu \sin^2 \phi,$$

$$\text{or} \quad (2q' - q) \sin \phi = g \cos \phi + \mu.$$

Now, with respect to all fluids which are capable of hanging in drops of sensible thickness from the horizontal surface of the solid, $2q'$ is greater than $2q$, and both these quantities are exceedingly greater than gravity.

Also, if ϕ were an angle of considerable magnitude, μ must be exceedingly smaller than the terms on the left side of the equation. Hence this equation cannot in general be satisfied, except for a very small value of ϕ.

For mercury, which is not capable of suspension from a solid, that angle is not small. The smallness of this angle is a necessary condition, that a fluid may be capable of *wetting* a solid.

In experiments with capillary tubes, it is usual to moisten the interior of the tube as much as possible before the ascent of the fluid in them. In this case, the ascent is occasioned by the molecular attraction of the particles of the coating of fluid which lines the cylinder.

To apply the preceding equation to these cases, we must put $q' = q$, and ϕ will still be a very small angle in consequence of the largeness of q in comparsion of g. The angle ϕ, will in this case be the same as that called ω, and thus, as ω will be very small, the capillary surface will be very nearly a hemisphere, and the height of ascent the greatest possible.

If $2q' = q$, $\phi = 90^0$ and $\mu = 0$. Hence also $\omega = 90^0$, and $h = 0$, or there is no ascent of the fluid.

95. Prop. *A drop of water placed in a conical tube of very small vertical angle will run towards the vertex.*

Let $abdc$ (Fig. 22.) be a drop of water in a conical surface, and aeb, cfd the bounding surfaces.

Let $ab = 2b$, and $cd = 2b'$, the capillary attraction at c will sustain a column, whose height equals $\frac{H \cos \omega}{g} \frac{1}{b}$,

and at f, will sustain a column whose height $= \frac{H \cos \omega}{g} \frac{1}{b}$,

and they act in opposite directions, that at f acting towards c.

Hence the resulting action towards C

$$= \frac{H \cos \omega}{g} \left(\frac{1}{b'} - \frac{1}{b} \right) = \frac{H \cos \omega}{g} \frac{b - b'}{b' b}.$$

This is the force which causes the drop to run, which, when the drop is small, and $b - b'$ nearly constant, varies as $\frac{1}{b^2}$.

CHAPTER IX.

ON THE SPECIFIC HEAT OF GASES, AND ON THE LAWS OF COOLING.

96. THE law of Mariotte, that the elastic force is proportional to the density, is true only on the supposition that a fluid has had time after condensation or rarefaction to return to its original temperature. If this be not the case, the temperature increases or diminishes with the density, and the elastic force increasing or decreasing by reason of the increase or decrease both of the density and temperature, ought for the same fluid to vary in a greater ratio than the density simply. When the fluid is contained in a vessel whose sides are impermeable to heat, it preserves all its caloric during condensation and rarefaction, and consequently, the temperature increases or diminishes. The same takes place when the variations in the density are so sudden that no transfer of heat can take place, that is, in the case of condensation the heat has not had time to escape by radiation, or to communicate itself by contact to the neighbouring substances; and in the case of dilatation, the surrounding bodies have not had time to communicate to the fluid, either by radiation or contact, any sensible quantity of caloric. This is the supposition made, as will be hereafter seen in the case of the variations of density which take place in the waves of air which produce sound, the duration of these variations being some thousandths part of a second.

In this and many other questions it is important to know the expression for the elastic force of a gas in terms of the density, and the corresponding elevation or depression of the temperature, the actual quantity of heat or caloric which the fluid mass contains remaining constant. In the present state of our knowledge however, we have not the requisite data for the complete solution of the problem, and the following chapter will contain what is principally at present known from calculation and experiment on this important subject.

All gases expand equally for equal increments of temperature, and we have a relation subsisting between the elastic force, the density and the temperature, which is given by the general equation,

$$p = k\rho(1 + a\theta)\ldots\ldots\ldots(1),$$

where a is the same for all gases, and k is different for different gases.

The absolute quantity of heat which a given weight, as a pound of any substance, contains cannot be calculated, but it is supposed to be inexhaustible, since experiment shews that all substances, however apparently devoid of heat, may be made to give some out; it is also supposed extremely great as compared with the quantities by which it is increased or diminished, when the body changes its density or temperature; it is these variations, that is, the quantities added and subtracted, which have to be compared together and submitted to calculation. This variation is evidently a function of the elastic force, the density, and the temperature, or of any two of them, by virtue of the equation (1), which subsists between these three quantities.

97. Prop. *To express the variation in the quantity of heat.*

Let q be the excess of heat which a given quantity of any gas, whose elastic force is p, density ρ, and temperature θ, contains above the quantity of heat which the same portion of gas contains at the standard pressure and temperature. Then q is a function of p, ρ, θ, or by virtue of the equation $p = k\rho(1 + a\theta)$, we have

$$q = f(p\rho),$$

where the form of the function must be determined.

The specific heat of the fluid is the quantity of heat which must be added to raise its temperature one degree, or, it is the rate of increase of q with respect to θ, and will therefore be expressed by $\frac{dq}{d\theta}$.

Now two cases present themselves; first, we may consider the pressure constant, and that the gas has the liberty of expanding; and secondly, we may consider the volume constant, and that the pressure varies with the temperature.

In the first case p being constant, and ρ the dependent, and θ the independent variable, we have from (1), (Art. 96.);

$$0 = k\frac{d\rho}{d\theta}(1 + a\theta) + ka\rho; \quad \therefore \frac{d\rho}{d\theta} = -\frac{a\rho}{1 + a\theta}.$$

In the second case ρ being constant, and p the dependent, and θ the independent variable, we have from the same equation,

$$\frac{dp}{d\theta} = ka\rho = \frac{ap}{1 + a\theta}.$$

Let c be the specific heat of the gas when the pressure is constant, and $c_{,}$ its specific heat when the density is constant, hence since $\frac{dq}{d\theta}$ is the general expression for the specific heat,

$$c = \frac{dq}{d\theta} = \frac{dq}{d\rho}\frac{d\rho}{d\theta}, \quad \text{and} \quad c_{,} = \frac{dq}{d\theta} = \frac{dq}{dp}\frac{dp}{d\theta}.$$

Substituting from the preceding equations,

$$c = -\frac{dq}{d\rho}\,\frac{a\rho}{1+a\theta}, \quad \text{and } c_{,} = \frac{dq}{dp}\,\frac{ap}{1+a\theta} \ldots\ldots\ldots(2),$$

whence, dividing so as to eliminate $\frac{a}{1+a\theta}$,

$$\rho\frac{dq}{d\rho} + p\frac{c}{c_{,}}\frac{dq}{dp} = 0.$$

Let γ express the ratio of the specific heat of the gas at a constant pressure to its specific heat at a constant volume, or $\gamma = \frac{c}{c_{,}}$,

$$\rho\frac{dq}{d\rho} + \gamma p\frac{dq}{dp} = 0.$$

The value of γ can only be known by experiment, but it is evident that its value must be greater than unity, for it must require a greater quantity of heat to augment the temperature of the gas and dilate it at the same time, than only to augment its temperature, without removing the particles from each other. We shall see hereafter the method of determining it.

98. Prop. *To determine the increment of temperature for a small condensation.*

Let θ be the temperature of the gas, and $\theta + \omega$ be its temperature when the density of the fluid has been increased by a very sudden condensation in the ratio of $1 + s : 1$, where s is a very small fraction.

If the loss of heat during the compression is insensible, the increase ω of the temperature, corresponding to the increase s of the density, is the quantity which has to be determined in the following experiment.

For this purpose, suppose the atmospheric air to be the gas in question, and let it be contained in a closed vessel, the pressure, density, and temperature being the same both at the exterior and interior, which, for the external air, we shall suppose represented by p, ρ, θ, during the whole experiment.

Let a small portion of the atmospheric air be removed, and when the air has acquired its original temperature, let p', ρ' be its elastic force and density.

Let a communication be again opened with the external air; the elastic force, the density, and the temperature will increase together, so that in a very short time the internal and external pressures will be equal to the external pressure. At this instant let the communication be cut off, and let ρ'' be the density, and $\theta + \omega$ the temperature of the internal air. Very soon, the increment ω of the temperature will vanish, and without the density ρ'' having undergone any change the pressure will be diminished to p'' suppose.

The density of the internal air having passed very rapidly from ρ' to ρ'', if we take $s = \frac{\rho'' - \rho'}{\rho'}$, and neglect the small quantity of heat which is absorbed by the vessel during the small time of this passage, the increase ω of the temperature is that which corresponds to the condensation s, and is the quantity which is required. The change in the thermometer is too slow to indicate this increment of temperature, which exists only for a very short period, but its value may be inferred from the three pressures p, p', p'', as indicated by the heights of a barometric column at the time of the experiment.

Now it is to be observed in the preceding experiment that there are two epochs, so to speak, at which the same temperature θ corresponds to the densities ρ' and ρ'', and to the pressures p' and p''.

Hence, the temperature being constant, we have by Mariotte's law

$$\frac{\rho''}{\rho'} = \frac{p''}{p'};$$

$$\therefore\ s = \frac{p'' - p'}{p'};$$

whence the condensation s is known.

Again, there are two epochs at which the same density ρ'' corresponds to the temperatures $\theta + \omega$, and θ, the pressures being p and p''.

Hence, since

$$p = k\rho'' \{1 + \alpha(\theta + \omega)\},\ p'' = k\rho''(1 + \alpha\theta),$$

$$\frac{p}{p''} = \frac{1 + \alpha(\theta + \omega)}{1 + \alpha\theta} \ \ldots\ldots\ldots (4),$$

whence the value of ω corresponding to the condensation s may be determined.

Experimental Determination.*

In an experiment made by Desormes and Clements, when the change of the density from ρ' to ρ'' took place in less than half a second, they observed,

$$p = 0^{\text{m}}.7665,\quad p' = 0^{\text{m}}.7527,\quad p'' = 0^{\text{m}}.7629,$$

$$\text{whence } s = 0{\cdot}0133.$$

The temperature θ was $12^0{\cdot}5$ *C.*, and since α always $= 0{\cdot}00375$, we deduce from equation (4),

$$\omega = 1^0{\cdot}3173\ C.$$

Hence for a condensation 0·0133 without loss of heat, the temperature of the air is augmented by $1^0{\cdot}3173$ *C.*, or the temperature of the air would be raised 1^0 *C.* for

* In these experiments I have retained the measures of the original experimenters.

a condensation $s = \dfrac{0{\cdot}01331}{1{\cdot}3173} = 0{\cdot}0101.$

The increase of temperature may, as we shall see hereafter, be deduced from the velocity of sound.

99. Prop. *To determine the ratio of the specific heat of a gas at a constant pressure to its specific heat at a constant volume.*

Let us suppose, as in the preceding article, that the elastic force and temperature of any gas are p and θ; the condensation s may be equivalent to that which the fluid experiences when the temperature is slightly diminished, the pressure remaining unaltered.

Let ϵ be this slight variation in temperature and ρ' the value of ρ.

Dividing the one by the other, we have, since $p = p'$,

$$\frac{\rho}{\rho'} = \frac{1 + a(\theta - \epsilon)}{1 + a\theta} = 1 - \frac{a\epsilon}{1 + a\theta},$$

whence $\dfrac{\rho'}{\rho} = 1 + \dfrac{a\epsilon}{1 + a\theta}$ very nearly, and therefore,

$$\frac{\rho' - \rho}{\rho} = \frac{a\epsilon}{1 + a\theta} = s.$$

Let q' be the quantity of heat which must be communicated to the given quantity of gas to raise its temperature from $\theta - \epsilon$ to θ without changing the pressure p, then if c be the specific heat at a constant pressure,

$$q' = c\epsilon.$$

After this communication of heat let the fluid be suddenly compressed so as to resume its former volume, it will then undergo a condensation s, and if there be no loss of heat, its temperature being augmented by ω will become $\theta + \omega$. Under these circumstances the pressure will be greater than p, but if without changing its volume the temperature be allowed to sink as far as $\theta - \epsilon$, this pressure will diminish at the same time and become p.

During this fall of temperature the gas will lose a quantity of heat proportional to the small diminution $\epsilon + \omega$ of temperature, which may be expressed by $c_{\prime}(\epsilon + \omega)$, since $c_{\prime}$ is its specific heat at a constant volume. The volume, the temperature, and the pressure being all the same after this loss of heat as they were before the quantity q' of heat was communicated to the fluid, the loss $c_{\prime}(\epsilon + \omega)$ must be equal to q', hence

$$c\epsilon = c_{\prime}(\epsilon + \omega);$$

$$\therefore \frac{c}{c_{\prime}} = 1 + \frac{\omega}{\epsilon}.$$

$$\text{But } \gamma = \frac{c}{c_{\prime}}; \quad \therefore \gamma = 1 + \frac{\omega}{\epsilon}, \text{ and } s = \frac{a\epsilon}{1 + a\theta};$$

$$\therefore \gamma = 1 + \frac{a\omega}{(1 + a\theta)\, s} \ldots\ldots(5).$$

$$\text{Now } s = \frac{p'' - p'}{p'}, \text{ (Art. 98.) and from (4), } \frac{\omega}{1 + a\theta} = \frac{p - p''}{p''};$$

$$\therefore \gamma = 1 + \frac{(p - p'')\, p'}{(p'' - p')\, p''},$$

which is an expression for γ in terms of quantities capable of being observed. If we take the data furnished by the experiment detailed in the last example, we shall find

$$\gamma = 1 + \frac{\dfrac{p}{p''} - 1}{\dfrac{p''}{p'} - 1} = 1 + 0{\cdot}3482 = 1{\cdot}3482,$$

for the value of the ratio of the specific heat of air at a constant pressure to its specific heat at a constant volume.

By an analogous proceeding, Gay-Lussac and Walter have obtained $\gamma = 1 \cdot 3748$, and Dulong has obtained by a different proceeding $\gamma = 1 \cdot 421$ for air perfectly dry. These results differ but very slightly, and their small differences do not prevent us from considering γ as constant.

100. Considering then γ as constant, the integral of the partial differential equation

$$\rho \frac{dq}{d\rho} + \gamma p \frac{dq}{dp} = 0$$

is, if f be the form of the arbitrary function,

$$q = f\left(\frac{1}{\rho} p^{\frac{1}{\gamma}}\right).$$

Hence, $p^{\frac{1}{\gamma}} = \rho f^{-1}(q)$, or $p = \rho^{\gamma} \phi(q) \ldots\ldots\ldots(1)$,

where ϕ is an inverse function of f.

But since $p = k\rho(1 + \alpha\theta)$, we have

$$\theta = \frac{p}{k\rho} - \frac{1}{a} = \frac{1}{a} \rho^{\gamma-1} \phi(q) - \frac{1}{a} \ldots\ldots\ldots(2).$$

If now q remain the same, and p, ρ, θ become p', ρ', θ', respectively,

$$p' = \rho'^{\gamma} \phi(q) \ldots(3), \text{ and } \theta' = \frac{1}{k} \rho'^{\gamma-1} \phi(q) - \frac{1}{a} \ldots(4).$$

Eliminating $\phi(q)$ between (1) and (3),

$$p' = p\left(\frac{\rho'}{\rho}\right)^{\gamma},$$

and eliminating it also between (2) and (4),

$$\theta' = \left(\frac{1}{a} + \theta\right)\left(\frac{\rho'}{\rho}\right)^{\gamma-1} - \frac{1}{a}.$$

These two equations express the laws of the elastic force and temperature of a gas compressed or dilated without any variation in the quantity of the heat; but it must be observed, that they depend on the fact of γ being constant, which, from what has been said, may be considered as established for common air.

The form of the arbitrary function may be determined by supposing that under a constant pressure a gas dilates equally for equal increments of temperature, as is shewn by Poisson*, to whom the reader must refer for further applications of this theory.

On Cooling.

101. When a body cools suspended in air, the heat is transferred by conduction, that is, by transmission through particles in immediate contact, by convection, that is, by the motion of the warm particles which are replaced by colder ones, and by radiation. But when a body cools in vacuo it is by the latter method, namely, by radiation that the cooling takes place; and it is the laws of cooling as depending on this radiation that we are now about to consider.

we only consider mathematically the laws of cooling as dep. on rad.

The principle arrived at by observation and which may be made the basis of the mathematical theory, is, that the temperature of a body is in the excess of the sensible heat which it gives out above that of the surrounding bodies, and the cooling of a body is the excess of its radiation above the radiation of the surrounding bodies†.

Prop. *To obtain the law of cooling in vacuo.*

If then τ be the excess of the temperature of the body cooling in vacuo, above the surrounding substances whose temperatures are θ, $\tau + \theta$ will be the temperature of the body, and the velocity of cooling may be expressed by $F(\tau + \theta) - F(\theta)$, where the form of the function F is to

* *Traité de Mécanique*, Art. 639.

† *Principles of Hydrostatics*, Art. 147.

be determined. If then v be this velocity of cooling, which will be nothing when τ is nothing,

$$v = F(\tau + \theta) - F(\theta).$$

Now, Newton and all succeeding philosophers have from their observations been led to assign some geometric progression as expressing the velocity of cooling. Let us suppose, therefore, that the velocity of cooling may be expressed by $\phi(\tau)\, a^\theta$, where a is some constant and ϕ has to be determined. Then

$$\phi(\tau)\, a^\theta = F(\tau + \theta) - F(\theta)\,;$$

and expanding by Taylor's series, we have

$$\phi(\tau) = \frac{F'(\theta)}{a^\theta}\frac{\tau}{1} + \frac{F''(\theta)}{a^\theta}\frac{\tau^2}{1\,.\,2} + \ldots$$

Now this equation must subsist for all values of τ, and since when τ is nothing, the temperature would be that of the surrounding bodies or constant, we must have in this case or when τ is small, $\dfrac{\phi(\tau)}{\tau} = n$, some constant. But

$$\frac{\phi(\tau)}{\tau} = \frac{F'(\theta)}{a^\theta} + \frac{F''(\theta)}{a^\theta}\frac{\tau}{1\,.\,2} + \ldots$$

the right side of which when τ is small is reduced to the first term, hence,

$$n = \frac{F'(\theta)}{a^\theta}, \quad \text{or} \quad F'\theta = n a^\theta.$$

Integrating and adding an arbitrary constant,

$$F(\theta) = \frac{n}{\log a} a^\theta + C = m a^\theta + C,$$

if $\dfrac{n}{\log a} = m$. Hence,

$$F(\tau + \theta) = m a^{\tau + \theta} + C.$$

φ(τ) disappears with τ, therefore φτ = A₀ + A₁τ + &c of Taylor becomes φτ = 0 + A₁τ + &c or approx.ly φτ = Aτ, φτ/τ = A

Substituting then for these quantities, we have

$$v = ma^{\tau+\theta} - ma^{\theta}$$
$$= ma^{\theta}(a^{\tau} - 1).$$

If then the theory of exchanges on which the preceding reasoning is founded be true, we arrive at the following law: "that when a body cools in vacuo in a vessel whose temperature is constant, the velocity of cooling for excesses of temperature in arithmetic progression increases as the terms of a geometric progression diminished by a constant quantity."

The experimental verification of this law is most remarkably exact, and the Memoir* of Dulong and Petit, from which the preceding is taken, is a most beautiful example of the plan that must be pursued in these and similar researches. The remarkable accuracy of the results obtained from the preceding formula for all temperatures, removes all doubt respecting the truth both of the law and of the principles on which it is founded.

102. The total radiation of the surrounding medium is $F(\theta)$, and its value is $ma^{\theta} + C$. But the point for the commencement of the absolute temperatures being arbitrary, it may be chosen so that the constant will vanish; hence, the absolute radiation may be expressed by $F(\theta) = ma^{\theta}$, simply without any constant. If then it were possible to observe the cooling of a body in vacuo, so that there was no interchange of radiation, that is, no portion of heat being restored from the surrounding bodies, we should have for the velocity of cooling,

$$v = ma^{\tau+\theta} = ma^{\theta}a^{\tau} = Ma^{\tau},$$

if $ma^{\theta} = M$, or the velocities of cooling would increase in geometric, the temperatures increasing, in arithmetic progression.

* *Annales de Chimie*, VII. 1817. See *Encyc. Metrop.* Art. *Heat.*

But will this constant be the same for two values of θ. Thus $F\theta = ma^{\theta} + C = ma^{\theta+\theta_0}$; $F\theta' = ma^{\theta'} + C$. This ought to be equal to $ma^{\theta'+\theta_0}$ ... In fact $ma^{\theta'}(a^{\theta_0}-1) = ma^{\theta}(a^{\theta_0}-1)$ which requires $\theta = \theta_1$ and shows that the origin of temperature must change for every value of θ — Hence the method and results only hold approximately for a small space of temperature.

The real velocity of cooling in vacuo in any case may then be expressed by $v = M(a^{\tau} - 1)$, where M is the quantity to be taken from the terms of the geometric progression, and depends on the temperature of the surrounding bodies. When the temperatures are low $V = Ma^{\tau}$ nearly, which is the Newtonian law.

103. Prop. *To find the time of cooling in vacuo.*

The time of cooling may be readily deduced from the velocity with respect to the time, for generally we have $V = \frac{d\tau}{dt} = -\frac{d\tau}{dt}$ in this case, since the excess of temperature diminishes with the time;

$$\therefore\ -\frac{d\tau}{dt} = M(a^{\tau} - 1), \text{ and } dt = -\frac{d\tau}{M(a^{\tau} - 1)};$$

$$\therefore\ t = -\int \frac{d\tau}{M(a^{\tau} - 1)} = \frac{1}{M \log a} \int \frac{d \cdot a^{-\tau}}{1 - a^{-\tau}}$$

$$= -\frac{1}{M \log a} \log(1 - a^{-\tau}) + C.$$

When $t = t_{,}$ let $\tau = \tau_{,}$, then

$$t_{,} = -\frac{1}{M \log a} \log(1 - a^{-\tau_{,}}) + C;$$

$$\therefore\ t - t_{,} = \frac{1}{M \log a} \log \frac{1 - a^{-\tau}}{1 - a^{-\tau}} \ldots\ldots (1).$$

When $t = t'$ let $\tau = \tau'$,

$$\therefore\ t' - t_{,} = \frac{1}{M \log a} \log \frac{1 - a^{-\tau_{,}}}{1 - a^{-\tau'}} \ldots\ldots (2),$$

and the coefficient $\frac{1}{M \log a}$ being eliminated between these two equations, the time of cooling is fully known.

104. The laws of cooling in vacuo being known, it will be easy to deduce from them and observation the cooling which is due to the contact of any gas. For we have only to subtract from the actual velocities of cooling those quantities which would be the velocities of cooling if the body, *cæteris paribus*, were placed in vacuo. Thus we can determine the energy of cooling due to the sole contact of fluids, and such as would be observed directly if the body could be deprived of its property of radiating.

From a series of most careful experiments, Dulong and Petit are led to infer that the state of the surface of the body has no influence on the quantity of heat which is carried away by the contact of the gas, and that the density and temperature of the gas do not affect the cooling, except by the variation which they cause in the elastic force of the gas. So that the cooling power of a gas may be considered as depending simply on its elastic force. The velocity of cooling of a body due to the contact of a gas depends on its excess of temperature and on the elastic force of the gas; and if v' be the velocity, τ the excess of temperature, and p the elastic force, we have as the result of experiment,

$$v' = m\tau^b p^c,$$

where b is the same for all substances, c the same for all bodies, but varies for different gases; and m varies with the nature of the gas and with the dimensions of the solid.

When a given body cools from the contact of any gas, mp^c is constant for that body and gas; hence we may have

$$v' = N\tau^b.$$

Prop. *To determine the complete law of cooling.*

Let the body be suspended in air, then the velocity of cooling due to radiation is $v = M(a^\tau - 1)$.

The velocity of cooling due to the contact of the gas is $v' = N\tau^b$. The total velocity of cooling then being the sum of these, is

$$V = M(a^\tau - 1) + N\tau^b.$$

Now the velocity of cooling due to the radiation depends very much on the state of the surface of the body, but that due to the contact of the gas depends simply on the elastic force of the gas. Hence, if V' and M' be the corresponding values of V and M for a change in the surface of the body,

$$V' = M'(a^\tau - 1) + N\tau^b.$$

Then the ratio of the velocities of cooling is

$$\frac{V}{V'} = \frac{M(a^\tau - 1) + N\tau^b}{M'(a^\tau - 1) + N\tau^b}.$$

Suppose M greater than M', that is, let M belong to the body which radiates best; and let the value of this ratio be ascertained for different values of τ.

Now when $\tau = 0$ or $\tau = \propto^{\text{ty}}$, this ratio becomes $\frac{0}{0}$, which must be determined in the usual manner; hence, differentiating the numerator and denominator,

$$\frac{V}{V'} = \frac{M \log a\, a^\tau + Nb\tau^{b-1}}{M' \log a\, a^\tau + Nb\tau^{b-1}} = \frac{M}{M'},$$

when $\tau = 0$ or $\tau = \propto^{\text{ty}}$. Thus for very small or any large excesses of temperature, the ratio of the velocities of cooling depends simply on the nature of the cooling body.

For other values of τ we have

$$\frac{V}{V'} = \frac{M + N\dfrac{\tau^b}{a^\tau - 1}}{M' + N\dfrac{\tau^b}{a^\tau - 1}}.$$

Then, since $\frac{M}{M'}$ is by hypothesis a ratio of greater inequality, it is diminished by the additional term $N\frac{\tau^b}{a^\tau - 1}$; but the less diminished the greater τ becomes, so long as this quantity is a proper fraction.

Thus it appears that for small excesses of temperature the velocities of cooling are less rapid for the surface which radiates most, and for large excesses of temperature are more rapid.

Many other conclusions may be drawn from the preceding equations and compared with experiment, but recourse must be had to the Memoir from which the preceding propositions have been taken, or to the Article Heat, in the Encyclopædia Metropolitana.

CHAPTER X.

ON THE GENERAL EQUATIONS OF THE MOTION OF FLUIDS.

105. The general equation of the equilibrium of fluids was obtained from the property which all fluids possess of transmitting pressure equally in all directions, so that, it is impressed on every particle throughout its mass.

This property is conceived by Poisson* to arise from the fact, that the particles of any fluid after compression or dilation return to a similar relative state, so that the fluid is a system of material points, similar to itself and existing on a smaller or a larger scale. The time of the fluid passing into a similar state, produces no influence on the laws of the equilibrium, which are only observed after it has obtained that state. But this time, however small, must influence the laws of the motion of fluids, so that the equal transmission of pressure does not obtain so accurately in the motion as in the equilibrium of fluids.

Another distinction must be remarked with respect to Marriotte's law. This law requires that the temperature of the fluid should be the same before and after the compression or dilation. This distinction is of no importance in liquids, but in gases, where the vibrations of the particles are very rapid, the equality of pressure is considerably modified.

* *Journal de l'Ecole Polytechnique*, 20 cahier.

Laws failing The Equal transmission of pressure
$p = k\rho$ — this supposes the temperature the same
not true for gases where the temperature
alters much before & after condensation

motions of perfect fluids. Then the law of
equal transmission of pressure does hold

These circumstances introduce conditions of great importance but of extreme difficulty, and in the following articles we shall suppose that the equal transmission of pressure obtains equally in a fluid at rest and in motion.

106. Prop. *To find the pressure at any point of a fluid mass in motion.*

Let x, y, z, be the co-ordinates of any point P of a fluid mass at the time t, and let dM be any elementary mass of the fluid at the same point.

Let ρ be the density of the fluid at that point, and X, Y, Z, the impressed forces in the directions of the three co-ordinate axes. These quantities will be given functions of x, y, z, when the forces are directed to or from a fixed centre, and these functions will contain the time explicitly when the centres are moveable. When the centres are within the fluid they will be functions of x, y, z, t.

Let u, v, w, be the velocity of the particle at the same time resolved in the same directions; these are unknown functions of x, y, z, t, because for the same value of t the velocity varies from one point to another, both in magnitude and direction, and for the same value of x, y, z, it changes from one instant to another. Now,

$$\frac{d(u)}{dt}, \quad \frac{d(v)}{dt}, \quad \frac{d(w)}{dt},$$

are the effective accelerating forces in the directions of the three axes at the time t.

Hence $X - \frac{d(u)}{dt}$, $Y - \frac{d(v)}{dt}$, $Z - \frac{d(w)}{dt}$, are the forces lost during the time dt by the particle submitted to the action of the forces X, Y, Z.

But by D'Alembert's principle the impressed and effective forces are in equilibrium with each other, or the forces lost are precisely such as would preserve the system in equilibrium. Hence the general equation of equilibrium will be satisfied by these forces, and we have

$$d(p)=\rho\left\{\left(X-\frac{d(u)}{dt}\right)dx+\left(Y-\frac{d(v)}{dt}\right)dy+\left(Z-\frac{d(w)}{dt}\right)dz\right\},$$

or,

$$\frac{d(p)}{\rho}=Xdx+Ydy+Zdz-\left(\frac{d(u)}{dt}dx+\frac{d(v)}{dt}dy+\frac{d(w)}{dt}dz\right) \quad (1).$$

We have seen that $u=f(x, y, z, t)$, and the increments of x, y, z within the time dt will be udt, vdt, wdt, respectively; therefore

$$u'=f(x+udt,\ y+vdt,\ z+wdt,\ t+dt).$$

Whence,

$$\frac{d(u)}{dt}=\frac{du}{dx}u+\frac{du}{dy}v+\frac{du}{dz}w+\frac{du}{dt},$$

and similarly for the quantities v and w,

$$\frac{d(v)}{dt}=\frac{dv}{dx}u+\frac{dv}{dy}v+\frac{dv}{dz}w+\frac{dv}{dt},$$

$$\frac{d(w)}{dt}=\frac{dw}{dx}u+\frac{dw}{dy}v+\frac{dw}{dz}w+\frac{dw}{dt}.$$

Before these quantities are substituted in (1), let us assume that $Xdx+Ydy+Zdz$ is a complete differential of dP, and also that $udx+vdy+wdz=d\phi$; we shall see hereafter to what circumstances these analytical facts have reference. Then,

$$u=\frac{d\phi}{dx},\quad \frac{du}{dx}=\frac{d^2\phi}{dx^2},\quad \frac{du}{dy}=\frac{d^2\phi}{dxdy},\quad \frac{du}{dz}=\frac{d^2\phi}{dxdz},$$

(the quantities being written in the denominators *in* the order of the differentiation) and similarly for the other quantities.

Then

$$\frac{d(u)}{dt} = \frac{d^2\phi}{dx^2}\frac{d\phi}{dx} + \frac{d^2\phi}{dx\,dy}\frac{d\phi}{dy} + \frac{d^2\phi}{dx\,dz}\frac{d\phi}{dz} + \frac{d^2\phi}{dx\,dt},$$

$$\frac{d(v)}{dt} = \frac{d^2\phi}{dy\,dx}\frac{d\phi}{dx} + \frac{d^2\phi}{dy^2}\frac{d\phi}{dy} + \frac{d^2\phi}{dy\,dz}\frac{d\phi}{dz} + \frac{d^2\phi}{dy\,dt},$$

$$\frac{d(w)}{dt} = \frac{d^2\phi}{dz\,dx}\frac{d\phi}{dx} + \frac{d^2\phi}{dz\,dy}\frac{d\phi}{dy} + \frac{d^2\phi}{dz^2}\frac{d\phi}{dz} + \frac{d^2\phi}{dz\,dt},$$

these being multiplied respectively by dx, dy, dz, and added, and observing that

$$\frac{d^2\phi}{dx^2}\frac{d\phi}{dx}dx + \frac{d^2\phi}{dy\,dx}\frac{d\phi}{dx}dy + \frac{d^2\phi}{dz\,dx}\frac{d\phi}{dx}dz = \tfrac{1}{2}d\,.\left(\frac{d\phi^2}{dx^2}\right),$$

and that there is a similar expression for the sums of the second, third, and fourth terms of these equations so multiplied, we have, by substitution in (1),

$$\frac{d(p)}{\rho} = dP - \tfrac{1}{2}d\,.\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2 + \left(\frac{d\phi}{dz}\right)^2\right\} - d\,.\left(\frac{d\phi}{dt}\right)$$

$$= dP - \tfrac{1}{2}d\,.\left(\frac{ds}{dt}\right)^2 - d\,.\left(\frac{d\phi}{dt}\right),$$

if ds be the space which the particle describes in dt; integrating, therefore,

$$\int\frac{d(p)}{\rho} = P - \tfrac{1}{2}\left(\frac{ds}{dt}\right)^2 - \frac{d\phi}{dt}.$$

The quantity $\int\frac{d(p)}{\rho}$ being properly determined, the pressure p at the point required will be known. This equation contains five unknown quantities; hence for the solution of the problem five equations will be necessary and sufficient.

107. Prop. *To form the equations for the determination of the five unknown quantities in the general equation of fluid motion.*

Three equations are supplied at once from the general equation (1), which has been obtained by applying D'Alembert's principle to pass from the general equation of equilibrium to that of motion.

Let $\frac{dp}{dx}$, $\frac{dp}{dy}$, $\frac{dp}{dz}$ be the partial differential of p with respect to x, y, z; then

$$\frac{1}{\rho}\frac{dp}{dx} = X - \frac{d(u)}{dt},\ \frac{1}{\rho}\frac{dp}{dy} = Y - \frac{d(v)}{dt},\ \frac{1}{\rho}\frac{dp}{dz} = Z - \frac{d(w)}{dt} \ldots (A),$$

where

$$\frac{d(u)}{dt} = \frac{du}{dx}u + \frac{du}{dy}v + \frac{du}{dz}w + \frac{du}{dt}, \text{ and } \frac{d(v)}{dt},\ \frac{d(w)}{dt}$$

have similar values.

One equation, and in some cases two, may be formed out of the condition that the mass of any element of the fluid continues the same during the time dt; hence this equation is called the *equation of continuity*. It is formed as follows.

Since during the motion the element of fluid will change both in form and density, but its mass is always to remain the same, the difference of the product of the volume and density at the time $t + dt$ and at the time t will be zero.

At the time t the co-ordinates of any point P are x, y, z, and the values of x for the two ends of the edge dx of the element are x and $x + dx$. Let $u_{,}$ be the value of u for the point whose co-ordinates are $x + dx$, y, z; then at the time $t + dt$, these are

Consider the particle x y z. In the course of the motion dx is enlarged by the amount wh. the two particles at the beginning and end of dx separate wh. is dx + du/dx dx − u (relative vel?) = du/dx dx
x alone varying in these conditions
∴ dx dy dz becomes (dx + du/dx dx dt)(sim. terms)

$$x + udt, \text{ and } x + u_{,}dt + dx;$$

$$\therefore \text{ the length of the edge} = (u_{,} - u)dt + dx.$$

Now $u = f(x, y, z, t)$, then $u_{,}$ being the value of u for the variation only of x,

$$u_{,} = f(x + dx, y, z, t)$$

$$= u + \frac{du}{dx}dx, \text{ very nearly.}$$

The length, therefore, of the edge $dx = dx + \frac{du}{dx}dxdt.$

Similarly, the length of the edge $dy = dy + \frac{dv}{dy}dydt,$

and .... $dz = dz + \frac{dw}{dz}dzdt.$

Again, the density ρ is a function of x, y, z, t;

$$\therefore \rho' = f(x + udt, \; y + vdt, \; z + wdt, \; t + dt)$$

$$= \rho + \frac{d\rho}{dx}udt + \frac{d\rho}{dy}vdt + \frac{d\rho}{dz}wdt + \frac{d\rho}{dt}dt \ldots$$

The new element will therefore

$$= \rho'\left(dx + \frac{du}{dx}dxdt\right)\left(dy + \frac{dv}{dy}dydt\right)\left(dz + \frac{dw}{dz}dzdt\right)$$

$$= \rho'\left(1 + \frac{du}{dx}dt\right)\left(1 + \frac{dv}{dy}dt\right)\left(1 + \frac{dw}{dt}dt\right)dxdydz,$$

the change of the element from rectangular to oblique, introducing only quantities which may be omitted. Now the variation of this element is to be zero; hence, subtracting from this product $\rho dxdydz$, and omitting all terms above the fifth order, and all common factors, we have

$$\frac{d\rho}{dx}u + \frac{d\rho}{dy}v + \frac{d\rho}{dz}w + \frac{d\rho}{dt} + \rho\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0 \ldots\ldots (B),$$

$$\text{or} \quad \frac{d \,.\, \rho u}{dx} + \frac{d \,.\, \rho v}{dy} + \frac{d \,.\, \rho w}{dz} + \frac{d\rho}{dt} = 0.$$

Another equation is $p = k\rho$, provided the motion is of such a nature that this equation can subsist; that is, if there be no change of temperature during the motion. Thus we have five equations which are sufficient to determine p, ρ, u, v, w, the five unknown quantities.

108. In the preceding article the *fifth* equation was furnished by Mariotte's law, but when the fluid is incompressible this equation does not obtain. We have, therefore, only *four* equations. But in this case the equation of continuity will be sufficient. For let the fluid be incompressible and *heterogeneous*.

Then $\dfrac{d(\rho)}{dt} = 0$, or,

$$\frac{d\rho}{dx}u + \frac{d\rho}{dy}v + \frac{d\rho}{dz}w + \frac{d\rho}{dt} = 0,$$

which is one equation, and (B) becomes

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

which is another equation. In this case, then, we have the proper number, or five equations.

Again, let the fluid be incompressible and *homogeneous*.

Then $\dfrac{d\rho}{dx}$, &c. are all nothing, and (B) becomes

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

Here, then, we have but *four* equations; there are but four unknown quantities; hence we have in each case the proper number of equations.

If $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, $\frac{d\phi}{dz}$ be the partial differential coefficients with respect to x, y, z, derived from the supposition that $udx + vdy + wdz$ is a complete differential, the preceding equation, substituting for u, v, w, their values $\frac{d\phi}{dx}$, $\frac{d\phi}{dy}$, $\frac{d\phi}{dz}$, becomes

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0,$$

a partial differential equation of the second order.

Thus when the fluid is incompressible the equation of continuity (B) and the equations (A) are sufficient for the solution of the problem; and when the fluid is compressible, they furnish four equations, the fifth being given by Mariotte's law, which may be considered to hold if the temperature be the same throughout the whole mass during the motion as it was during a state of equilibrium.

But variation in density is always attended with change in temperature; hence the pressure is no longer proportional to the density simply. When the motion is rapid, the development of heat gives rise to a great increase in the elastic force of the fluid, as will be seen in the theory of sound. But if the motion be slow, so that the variation in the density and elastic force is small, the equation $p = k\rho(1 + \alpha\theta)$ may subsist, and will furnish the fifth equation.

109. In the preceding investigations it was assumed that $udx + vdy + wdz$ is a complete differential $d\phi$, and consequently the general equation obtained on this hypo-

thesis can only be applied to the cases in which that condition obtains.

Hitherto no general determination has been given of the cases in which $udx + vdy + wdz$ is a complete differential. Particular cases have been indicated by Poisson*, and Professor Challis† has shewn that when in incompressible fluids the motion at each point of any element is directed to fixed or moveable *focal lines*, the equation $\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$, depending on the above condition, is satisfied. It seems probable that ultimately it will be found that in every instance of fluid motion for which $udx + vdy + wdz$ is a complete differential, the character of the elementary motions will be of the same description. When the parts of the fluid do not move *inter se*, that is, do not change their relative positions, but move as if they were rigidly connected, the quantity ϕ has no existence, and the pressure is determined by the equation

$$p = \int \rho (Xdx + Ydy + Zdz),$$

the forces X, Y, Z including those which arise from the rotation; as for instance, when a mass of fluid revolves about an axis without changing its form.

It was also assumed that $Xdx + Ydy + Zdz$ is a complete differential dP. This, as is well known, is the case whenever the forces are directed to fixed centres, or when they are directed to moveable centres‡.

The integrals of partial differential equations contain arbitrary functions, and the existence of these arbitrary functions in the equations of the motion of fluids is an analytical fact which shews that in their application to physical questions any motion whatever may be given to the particles, which is an evident consequence of the

* *Traité de Mecanique*, Art. 654.

† *Camb. Phil. Trans.* Vol. V. VIII.

‡ *Ibid.* Vol. III. XVIII.

fundamental principle of the perfect mobility of the particles.

In the following articles we shall proceed to illustrate fully the circumstances of the motion in space of two dimensions, and state what is at present known of motion in three dimensions.

Motion in two Dimensions.

110. PROP. *The motion being in space of two dimensions, to obtain an integral of the equation of continuity.*

Let the motion be in the plane of xy; then the equation of continuity becomes

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0.$$

Let ϕ be a function of t, and $x^2 + y^2$, or r^2.

$$\text{Then } \frac{d\phi}{dx} = \frac{d\phi}{dr}\frac{dr}{dx} = \frac{d\phi}{dr}\frac{x}{r},$$

$$\frac{d^2\phi}{dx^2} = \frac{d^2\phi}{dr^2}\frac{dr}{dx}\frac{x}{r} + \frac{d\phi}{dr}\frac{1}{r} - \frac{d\phi}{dr}\frac{dr}{dx}\frac{x}{r^2}$$

$$= \frac{d^2\phi}{dr^2}\frac{x^2}{r^2} + \frac{d\phi}{dr}\left(\frac{1}{r} - \frac{x^2}{r^3}\right).$$

$$\text{Also } \frac{d^2\phi}{dy^2} = \frac{d^2\phi}{dr^2}\frac{y^2}{r^2} + \frac{d\phi}{dr}\left(\frac{1}{r} - \frac{y^2}{r^3}\right);$$

$$\therefore \frac{d^2\phi}{dr^2}\frac{x^2 + y^2}{r^2} + \frac{d\phi}{dr}\left(\frac{2}{r} - \frac{x^2 + y^2}{r^3}\right) = 0,$$

$$\text{or } \frac{d^2\phi}{dr^2} + \frac{d\phi}{dr}\frac{1}{r} = 0.$$

But $\dfrac{d \cdot \dfrac{r d\phi}{dr}}{dr} = \dfrac{d\phi}{dr} + \dfrac{r d^2\phi}{dr^2}$;

$$\therefore \frac{d \cdot \dfrac{r d\phi}{dr}}{r dr} = 0.$$

Then integrating and adding an arbitrary function of t,

$$\frac{r d\phi}{dr} = f(t), \therefore \frac{d\phi}{dr} = \frac{f(t)}{r} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\text{and } \phi = f(t) \log . r + F(t) \ldots\ldots (2).$$

The velocity, or $\dfrac{d\phi}{dr}$, thus appears equal to $\dfrac{f(t)}{r}$. The physical meaning of this result may be illustrated by supposing fluid to be contained in a cylinder capable of expanding in the direction of its radius, and a very slender cylinder of solid matter to be inserted with its axis coincident with the axis of the cylinder. The fluid particles, by the insertion of this solid cylinder, will be moved through spaces which vary inversely as the distances from the axis.

The distance of the point under consideration from the origin of co-ordinates being (r), we see by (1) that it is moving in such a manner that its velocity varies inversely as its distance from some point, and its motion is directed either to or from this point.

111. But it is of great importance to obtain also the integral of this equation of continuity independently of any hypothesis respecting ϕ, and to shew the connection which subsists between this and the preceding integral. The following investigation is given by Professor Challis*. The usual method of finding the integral of

* *Camb. Phil. Trans.* Vol. V. VIII.

a linear partial differential equation of the second order between *two* variables leads in the present case to the integral

$$\phi = F(x + y\sqrt{-1}) + f(x - y\sqrt{-1}).$$

To ascertain its *general* signification, let the forms of the functions F and f be determined independently of any hypothesis respecting the mode in which the fluid was put in motion. The quantity ϕ is subject to the condition $d\phi = u\,dx + v\,dy$, whence $\frac{d\phi}{dx} = u$, $\frac{d\phi}{dy} = v$. Then

$$u = F'(x + y\sqrt{-1}) + f'(x - y\sqrt{-1}),$$

$$v = \sqrt{-1}\,F'(x + y\sqrt{-1}) - \sqrt{-1}\,f'(x - y\sqrt{-1}).$$

First, it may be observed that u and v are not both possible for *any* values of x and y, unless the functions F' and f' be the same. Again, as the form of F' we are seeking for is to be independent of all that is arbitrary, it will remain the same whatever direction we arbitrarily assign to the axes of co-ordinates. Let therefore the axis of y pass through the point to which the velocities u, v, belong. Then

$$y = 0, \quad u = 2F'(x), \quad v = 0.$$

If now the axes be supposed to take any other position, the origin remaining the same, u will be equal to

$$\frac{2x}{\sqrt{x^2 + y^2}} F'(\sqrt{x^2 + y^2}).$$

Hence

$$F'(x + y\sqrt{-1}) + F'(x - y\sqrt{-1}) = \frac{2x}{\sqrt{x^2 + y^2}} \cdot F'(\sqrt{x^2 + y^2}),$$

a functional equation for determining the form of F'.

Let

$$x + y\sqrt{-1} = m, \quad \text{and} \quad x - y\sqrt{-1} = n;$$

then

$$2x = m + n, \quad \text{and} \quad \sqrt{x^2 + y^2} = \sqrt{mn}.$$

Therefore,

$$F'(m) + F'(n) = \frac{m + n}{\sqrt{mn}} F'(\sqrt{mn})$$

$$= \frac{m}{\sqrt{mn}} F'(\sqrt{mn}) + \frac{n}{\sqrt{mn}} F'(\sqrt{mn}).$$

It is easily seen that if $F'(\sqrt{mn}) = \dfrac{C}{\sqrt{mn}}$, the equation is satisfied. Hence

$$\frac{d\phi}{dx} = \frac{C}{x + y\sqrt{-1}} + \frac{C}{x - y\sqrt{-1}} = \frac{2Cx}{x^2 + y^2},$$

$$\text{and} \quad \frac{d\phi}{dy} = \frac{2Cy}{x^2 + y^2};$$

and consequently the velocity at xy, or

$$\sqrt{u^2 + v^2} = \frac{2C}{\sqrt{x^2 + y^2}}.$$

These results shew that the velocity is directed to or from the origin of co-ordinates, and varies inversely as the distance from it. But we must observe that this limitation as to the point to which the velocity is directed, is owing to the particular forms, $x + y\sqrt{-1}$, $x - y\sqrt{-1}$, of the quantities which the function F' involves. For the differential equation is also satisfied by a more general value of these quantities, as is there shewn; and the result shews that the velocity is directed to a certain point, and varies inversely as the distance from it. And this result having been arrived at without considering any circumstances under which the fluid was caused to move, the inference to be drawn is, that such is the general character of the motion.

Also the co-ordinates of the point to which the motion is directed may be constant, or functions of the time and the given conditions of the motion.

112. The following considerations are added in confirmation of the foregoing reasoning. In whatever manner the fluid is put in motion, we may conceive a line, commencing at any point, to be continually drawn in a direction perpendicular to the directions of the motions at a given instant of the particles through which it passes. This line may be of any arbitrary and irregular shape, not defined by a single equation between x and y. But it must be composed of parts either finite or indefinitely small, which obey the law of continuity. Consequently the motion, being at all the points of the line in the directions of the normals, must tend to or from the centres of curvature, and vary, in at least elementary portions of the fluid, inversely as the distances from those centres. An unlimited number of such lines may be drawn through the whole extent of the fluid mass in motion.

Motion in Space.

113. An integral of the equation of continuity for incompressible fluids may be obtained on the particular supposition that ϕ is a function of r and t, when

$$r^2 = x^2 + y^2 + z^2. \quad \text{Then,}$$

$$\frac{d\phi}{dx} = \frac{d\phi}{dr}\frac{dr}{dx} = \frac{d\phi}{dr}\frac{x}{r},$$

$$\frac{d^2\phi}{dx^2} = \frac{d^2\phi}{dr^2}\frac{dr}{dx}\frac{x}{r} + \frac{d\phi}{dr}\frac{1}{r} - \frac{d\phi}{dr}\frac{dr}{dx}\frac{x}{r^2}$$

$$= \frac{d^2\phi}{dr^2}\frac{x^2}{r^2} + \frac{d\phi}{dr}\left(\frac{1}{r} - \frac{x^2}{r^3}\right).$$

$$\frac{d^2\phi}{dy^2} = \frac{d^2\phi}{dr^2}\frac{y^2}{r^2} + \frac{d\phi}{dr}\left(\frac{1}{r} - \frac{y^2}{r^3}\right),$$

$$\frac{d^2\phi}{dz^2} = \frac{d^2\phi}{dr^2}\frac{z^2}{r^2} + \frac{d\phi}{dr}\left(\frac{1}{r} - \frac{z^2}{r^3}\right).$$

And substituting in the equation of continuity,

$$\frac{d^2\phi}{dr^2}\frac{x^2+y^2+z^2}{r^2} + \frac{d\phi}{dr}\left(\frac{3}{r} - \frac{x^2+y^2+z^2}{r^3}\right) = 0.$$

$$\therefore \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = 0.$$

But $\dfrac{d^2.r\phi}{rdr^2} = \dfrac{d^2\phi}{dr^2} + \dfrac{2}{r}\dfrac{d\phi}{dr}$; $\quad \therefore \dfrac{d^2.r\phi}{rdr^2} = 0.$

Integrating and adding an arbitrary function of t,

$$\frac{d.r\phi}{dr} = f(t),$$

and integrating again and adding another arbitrary function,

$$r\phi = f(t) + F(t);$$

$$\therefore \phi = f(t) + \frac{F(t)}{r} = f(t) + \frac{F(t)}{\sqrt{x^2+y^2+z^2}} \text{......(1).}$$

The velocity of the fluid $= \sqrt{u^2+v^2+w^2}$

$$= \sqrt{\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}} = \frac{d\phi}{dr};$$

$\therefore$ the velocity $= -\dfrac{F(t)}{r^2}$ ....(2).

The preceding result may be illustrated as before by conceiving a spherical mass of fluid enclosed in an extensible envelope, and that a small sphere being placed at the centre of this mass, the particles will be made to move from their original places through spaces which vary inversely as the square of their distances from the centre.

114. A general integral of the preceding equation of continuity cannot be obtained, and the general law of the motion of the parts of the fluid amongst each other so as to fill always the same space, cannot in this case be found in the same manner as in space of two dimensions by subjecting the general integral to a similar discussion, but by reasoning analagous to that contained in (Art. 112.), Professor Challis infers that the elementary motions are every where directed to *focal lines*. But the reader must have recourse to the original paper*, where it is proved that the equation of continuity is satisfied by the kind of motion there supposed. Also the general conclusion arrived at by this reasoning is, that the law of variation or the velocity from any point to another indefinitely near in the direction of the motion at a given instant, may be expressed by $\frac{C}{r(r+l)}$, where C and l are constant at a given instant, and r is the distance of the point under consideration from a line whose position is fixed at a given instant.

If $C = 0$, we have as a particular case the velocity $= \frac{C}{r^2}$, which represents the law of the variation of the velocity under these particular circumstances, and agrees with the particular case of the integral just treated of.

Since r is ultimately in the direction in which the velocity V takes place, if a line commencing at a given

* *Cam. Phil. Tran.* Vol. V. Part II.

point be drawn constantly in the direction of the motion, at a given instant, of the point through which it passes, dr may be considered as the increment of this line.

Hence, if s be its length reckoned from the fixed point

$$\frac{d\phi}{dr} = \frac{d\phi}{ds} = V.$$

Then integrating $\phi = \int V ds + f(t)$, and differentiating under the sign of integration,

$$\frac{d\phi}{dt} = \int \frac{dV}{dt} ds + f'(t).$$

Substituting this value in the general expression for p,

$$p = \int (X dx + Y dy + Z dz) - \int \frac{dV}{dt} ds - \tfrac{1}{2} V^2 - f'(t).$$

If V be always the same in quantity and direction at the same point,

$$\frac{dV}{dt} = 0; \quad \therefore \ p = \int (X dx + Y dy + Z dz) - \tfrac{1}{2} V^2 - f'(t).$$

This equation, which may be considered as strictly deduced from the general equations of fluid motion, is the equation of *steady* motion, as we shall see presently.

115. When the fluid is compressible or elastic, the equation of continuity is not as we have seen, resolved into two, and the general equation is of too complicated a nature to be readily treated.

In one case, however, when the motions of the particles are small, and no extraneous force acts, the equation of continuity admits of simplification, and the general equation can be treated.

The motions being small, $\frac{d\rho}{dx}$, $\frac{d\rho}{dy}$, $\frac{d\rho}{dz}$, may be omitted, and the equation of continuity becomes

$$\frac{d\rho}{dt} + \rho\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0,$$

$$\text{or} \quad \frac{d.\log\rho}{dt} + \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0 \ldots\ldots\ldots(1).$$

And since no extraneous force acts, the general equation becomes

$$\int\frac{dp}{\rho} = -\frac{d\phi}{dt}.$$

Let $p = a^2\rho$, then $\int\frac{dp}{\rho} = a^2\int\frac{dp}{\rho} = a^2\int d.\log\rho$;

$$\therefore\ a^2\int d.\log\rho = -\frac{d\phi}{dt},$$

$$a^2\frac{d.\log\rho}{dt} = -\frac{d^2\phi}{dt^2}, \text{ and substituting in (1),}$$

$$\frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right).$$

If as before, ϕ be supposed a function of rt, an integral may be found, and the equation becomes

$$\frac{d^2.r\phi}{dt^2} = a^2\frac{d^2\phi}{dx^2},$$

which will be treated in the chapter on Sound.

CHAPTER XI.

ON THE MOTION OF FLUIDS ON PARTICULAR HYPOTHESES.

116. THE general equation of the motion of fluids is not readily applicable to practice, and in a case of such great difficulty as the present, recourse must be had to particular cases of the motion. Fortunately, indeed, the cases which most commonly occur in practice, are such as can be brought under the equation of motion, either directly or by particular hypotheses which conduct to results very nearly true.

One large class of questions is where the motion is *steady*, and we have already seen that the general equation admits of great simplification in this case; we shall shew how the same equation may be deduced at once from the general equation to which we are led by the application of D'Alembert's principle.

Steady Motion.

117. *Definition.* The motion of a fluid is said to be *steady*, when the velocity at all points in space is constantly the same both in magnitude and direction; that is, when the accelerating force on each particle is the same as it passes through the same point in space.

PROP. *The motion being steady to find the pressure at any point.*

Let x, y, z be the co-ordinates of any point in motion at the time (t); X, Y, Z the impressed accelerating forces, and u, v, w the velocities in the direction of the co-ordinate axes.

Then $\frac{d(u)}{dt}$, $\frac{d(v)}{dt}$, $\frac{d(w)}{dt}$ are the effective accelerating forces, and

$$X - \frac{d(u)}{dt},\ \ Y - \frac{d(v)}{dt},\ \ Z - \frac{d(w)}{dt},$$

are the forces lost, hence the fluid would be in equilibrium if these forces acted on it, or we have as before

$$dp = \rho\left\{\left(X - \frac{d(u)}{dt}\right)dx + \left(Y - \frac{d(v)}{dt}\right)dy + \left(Z - \frac{d(w)}{dt}\right)dz\right\};$$

$$\therefore\ p = \int\rho(Xdx + Ydy + Zdz) - \int\rho\left(\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz\right).$$

This integral is generally to be taken between the limiting values of x, y, z, which belong to some point at the surface, and at some arbitrary point at which the pressure is required; t being constant, and x, y, z being entirely independent, or having such relations as may belong to any line of particles arbitrarily chosen betwixt the above limits.

Let the integral be taken with regard to any line of particles which terminate in a given particle, as for instance, with regard to the line which this very particle has traversed in coming to the point under consideration, that is, to the point at which the pressure is required. Then since each particle is moving with the same velocity as it passes through the same point in space, the particles in the line thus traversed will all be moving with the same velocity, or acted on by precisely the same force at any given instant as the given particle was whilst traversing it; hence the values of $\frac{d(u)}{dt}$, $\frac{d(v)}{dt}$, $\frac{d(w)}{dt}$ for successive values of x, y, z, corresponding to successive values of t, will be precisely the same as the values of these quantities for the same successive values of x, y, z corresponding to any one value of t; and therefore in the case of steady motion we may integrate the above equation considering x, y, z as functions of t, and we shall obtain the same value as if we integrated on the supposition of t being constant. But x, y, z being functions of t, we may write

$$\frac{d^2 x}{dt^2} dx \quad \text{for} \quad \frac{d(u)}{dt} dx,$$

and similarly for the other quantities;

$$\therefore \int\left(\frac{d(u)}{dt} dx + \frac{d(v)}{dt} dy + \frac{d(w)}{dt} dz\right)$$

$$= \int \frac{dx\, d^2x + dy\, d^2y + dz\, d^2z}{dt^2}$$

$$= \tfrac{1}{2}\left(\frac{ds}{dt}\right)^2 + C,$$

if ds be the space described by the particle in the time dt, and C be the arbitrary constant which may be a function of the time; then for convenience calling v the absolute velocity $\dfrac{ds}{dt}$, we have

$$p = \int \rho\,(X dx + Y dy + Z dz) - \tfrac{1}{2}\rho v^2 + C,$$

which is the equation of steady motion.

To determine the value of the arbitrary constant, let v_1 be the velocity which the particle whose path has been considered had at the surface, and let p_1 be the value of p, then if the surface of the fluid be taken for the plane of xy,

$$p_1 = -\tfrac{1}{2}\rho v_1^2 + C,$$

which being subtracted from the preceding, we have

$$p - p_1 = \int \rho\,(X dx + Y dy + Z dz) - \tfrac{1}{2}\rho\,(v^2 - v_1^2) \ldots (C).$$

Let gravity be the only force which acts, and let $\rho = 1$, then

$$p - p_1 = gz - \tfrac{1}{2}(v^2 - v_1^2).$$

Let α be the ratio of the velocity at the upper surface to that of the issuing fluid, then

$$p - p_1 = gz - \tfrac{1}{2}v^2(1 - \alpha^2).$$

This equation is applicable to the issuing of water retained at a constant elevation in any vessel through any small orifice or adjutage fitted to the orifice. We shall proceed to illustrate its application.

118. PROP. *To determine the velocity of a fluid issuing through a small orifice.*

Let the fluid be constantly supplied so that the surface is retained at a constant elevation, the motion will then be steady, and the equation of steady motion may be applied.

Let k be the area of the orifice, and K the area of the surface of the fluid which will be constant, the fluid being retained at a constant elevation.

The velocity of the fluid at the orifice, and the surface is inversely as these sections, for the fluid being incompressible, the quantity contained in the vessel is constant, hence the same quantity must flow out and in during a given time, that is, the product of the area of any section and of the velocity of the fluid passing through it is invariable.

Let v, V be the velocities of the fluid issuing through the sections k and K, then

$$vk = VK, \quad \text{or } \frac{k}{K} = \frac{V}{v} = \alpha,$$

and the general equation is

$$p - p_1 = gz - \tfrac{1}{2}v^2\left(1 - \frac{k^2}{K^2}\right).$$

Now since k is small, the velocity at every point of the issuing stream which is in immediate contact with the air will be very nearly the same. But it appears from experiment, that when water issues through an orifice into the air, the stream converges for some distance, when it acquires a constant permanent form, neither converging nor diverging, which is called the *vena contracta*.

This form of the issuing stream shews that the surface of the converging stream is moving with a greater velocity as far as the vena contracta than any point in the interior; hence the pressure at the surface, is as the equation shews, less than at any point in the interior, and consequently the pressure in the interior is greater than the atmospheric pressure. But at the vena contracta there is no tendency to diverge or converge, every point of the section is moving therefore with the same velocity, and the pressure is every where the same as the atmospheric. At this part then of the stream we have $p = p_1$, and the equation becomes

$$0 = gz - \tfrac{1}{2}v^2\left(1 - \frac{k^2}{K^2}\right);$$

$$\therefore\ v = \sqrt{\frac{2gz}{1 - \dfrac{k^2}{K^2}}}.$$

The section k then is the section of the vena contracta and not of the actual orifice, for it is through a section of the vena contracta that the efflux really takes place, since the stream has not acquired its greatest velocity before reaching this point, and the converging part of the issuing stream must be considered as a continuation of the containing vessel.

The section of the vena contracta may be taken as equal to $\frac{5}{8}$ths the actual orifice*.

* See RENNIE'S *Report to British Association*, 1834.

119. If the constant surface of the fluid be large compared with the orifice, the ratio $\frac{k}{K}$ will be small, and $\frac{k^2}{K^2}$ exceedingly small, hence if h be the depth of the commencement of the vena contracta below the surface of the fluid, we have for the issuing stream $v = \sqrt{2gh}$, or the velocity of the issuing fluid is that due to the height through which it has descended supposing it to fall freely.

If the fluid be not supplied at the same rate as that with which it escapes, the surface is no longer stationary, and the hypothesis of the steadiness of the motion is not fulfilled. When, however, the orifice is exceedingly small, the true velocity will differ from $\sqrt{2gh}$ by a quantity which is not assignable, and no appreciable error will be introduced by using this value.

120. Prop. *To find the time of a vessel emptying itself through a small orifice in the base.*

At any time (t) let K be the area of the surface of the fluid, and z the depth of the effective orifice k below the surface of the fluid, and v the velocity of the issuing stream.

Then $-dz$ being the descent of the surface in the time dt, we shall have

$$kvdt = -Kdz.$$

$$\text{But } v = \sqrt{2gz}; \quad \therefore dt = -\frac{Kdz}{k\sqrt{2gz}},$$

$$\text{and } t = \frac{1}{k\sqrt{2g}}\int\frac{-Kdz}{\sqrt{z}}.$$

Let the vessel be prismatic, then K is constant, and

$$t = \frac{2K}{k\sqrt{2g}}\{C - \sqrt{z}\}.$$

When the motion commences let $z = a$;

$$\therefore\ 0 = \frac{2K}{k\sqrt{2g}}\{C - \sqrt{a}\},$$

and subtracting,

$$t = \frac{2K}{k\sqrt{2g}}\{\sqrt{a} - \sqrt{z}\}.$$

The whole time of efflux is $\frac{2K\sqrt{a}}{k\sqrt{2g}}$, which is double the time in which the same quantity would run out, if the fluid were retained at a constant elevation. If the vessel be not prismatic, the surface K must be expressed as a function of z, and the time can be found as before.

121. The effect of adjutages in increasing the expenditure of a given orifice is known practically to be very considerable; we shall apply the preceding equation of steady motion to some of those which were employed by Venturi in his experiments on issuing fluids.

Ex. 1. *An adjutage consisting of a conical and cylindrical tube of the form of the issuing stream.*

Let $abge$ (Fig. 23.) represent this adjutage.

Let k be the section at cd or the effective orifice, and h, h' the depths of ef, cd, below the constant surface of the fluid.

The velocity of the fluid issuing into the air without the adjutage would be $= \sqrt{\frac{2gh'}{1-a^2}}$, if a be the ratio of the effective orifice to the surface of the fluid.

The velocity of the fluid issuing at $eg = \sqrt{\frac{2gh}{1-a^2}}$.

The increase of velocity due to the adjutage is therefore

$$= \sqrt{\frac{2g}{1-a^2}}\,(\sqrt{h} - \sqrt{h'}).$$

The pressure at $cd = p_1 + gh' - \frac{1}{2}v^2(1 - a^2)$.

The pressure at the orifice $= p_1 + gh - \frac{1}{2}v^2(1 - a^2)$, hence the pressure at the commencement of the vena contracta is less than the atmospheric by $g(h - h')$, or the weight of the column in the cylindrical tube. This agrees very nearly with the results obtained by Venturi.

Ex. 2. *A cylindrical tube with its axis horizontal.*

When the fluid *fills* the tube, the velocity of rushing into the air will be $\sqrt{\frac{2gh}{1-a^2}}$, and the expenditure will consequently be increased by the adjutage in the ratio of the area of the orifice to the section of the vena contracta in air; for in this case the actual orifice is rendered the effective orifice.

Ex. 3. *A horizontal adjutage converging to the vena contracta and then diverging.*

An adjutage consisting of two conical portions having their smaller ends united at the commencement of the vena contracta, was found by Venturi to give a large expenditure.

The equation $p = p_{,} + gz - \frac{1}{2}v^2(1 - a^2)$ shews that as the velocity will decrease in passing from the minimum section towards the mouth of the adjutage, the pressure will increase; and this is confirmed by experiment. In such an adjutage as this, the stream is divergent when it leaves the adjutage and the velocity of that portion of it which is in immediate contact with the air, being nearly the same, it must consequently be less than the velocity in the interior of the stream at a small distance from the aperture. At a small distance from the adjutage there must be a section where the stream ceases to be divergent, and at which consequently the velocity is the same for every point and the pressure equal to the atmospheric. The velocity at this point will be $\sqrt{\dfrac{2gh}{1-a^2}}$, and as this section, the stream having being divergent, is larger than the aperture, there must be a greater expenditure from a conical diverging adjutage than from a cylindrical tube of the same aperture*.

The Variable Motion of Fluids.

122. Prop. *The motion not being steady, to determine the pressure on the hypothesis of parallel sections.*

When the motion is not *steady*, returning to the general equation, we have

$$p = \int\rho\,(Xdx + Ydy + Zdz) - \int\rho\left\{\frac{d(u)}{dt}dx + \frac{d(v)}{dt}dy + \frac{d(w)}{dt}dz\right\}.$$

Here since the particles are moving with different velocities as they pass through the same point in space, the quantities

$$\frac{d(u)}{dt},\ \frac{d(v)}{dt},\ \frac{d(w)}{dt},$$

are no longer the same for successive values of x, y, z and successive values of t, as for the successive values of x, y, z and the same value of t; but the integral must be taken *at a given instant* for a line of particles terminating in the given point, that is, the quantities must be integrated exclusively with reference to x, y, z.

To effect the integration in the most general manner, without any special hypothesis respecting the direction of

* See Challis, *Camb. Phil. Trans.* xviii. 1830.

the motions of the particles, let ds be the space described in time dt, by a particle whose co-ordinates are x, y, z, and let ϕ be its velocity.

Then $\frac{d\phi}{dt}$ is the effective accelerating force upon it in the direction of its motion, and

$$\frac{du}{dt} = \frac{d\phi}{dt}\frac{dx}{ds}, \quad \frac{dv}{dt} = \frac{d\phi}{dt}\frac{dy}{ds}, \quad \frac{dw}{dt} = \frac{d\phi}{dt}\frac{dz}{ds};$$

$$\therefore \int\rho\left(\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz\right) = \int\rho\frac{dx^2 + dy^2 + dz^2}{ds}\frac{d\phi}{dt}$$

$$= \int\rho\frac{d\phi}{dt}ds.$$

The quantity ϕ is a function both of the position of the whole mass of the fluid and of the given particle within it, or it is a function of t as well as of x, y, z; but since $\int\rho\frac{d\phi}{dt}ds$ is to be taken exclusively with reference to x, y, z, we must express $\frac{d\phi}{dt}$ in terms of the variables on which it depends, to effect which recourse must be had to some particular hypothesis.

The one chosen is the hypothesis of parallel sections, which supposes, that if any section be taken at any instant perpendicular to the motion of any particle, then all the other particles are moving with the same velocities and in the same direction.

This hypothesis amounts to supposing that a fluid, as water, descends in parallel slices, so that a portion which is at any instant included between two planes, will always be between planes parallel to these. It may be shewn from the general equations, that for some cases the motion is in strict conformity with this hypothesis*.

* Challis, *Phil. Mag.* Jan. 1831.

Then all the particles being supposed to move with the same velocity in the same section, let κ be the section of the fluid, every particle in which is moving with the velocity ϕ.

Also let k be any section of the vessel through which the particles are moving with a velocity v, then the fluid being incompressible,

$$\kappa\phi = kv, \text{ or } \phi = \frac{kv}{\kappa}; \text{ whence, differentiating,}$$

$$\frac{d\phi}{dt} = \frac{k}{\kappa}\frac{dv}{dt} - \frac{kv}{\kappa^2}\frac{d\kappa}{dt};$$

but since κ varies with the motion of the fluid,

$$\frac{d\kappa}{dt} = \frac{d\kappa}{ds}\frac{ds}{dt} \text{ and } \frac{ds}{dt} = \phi = \frac{kv}{\kappa};$$

$$\therefore \frac{d\phi}{dt} = \frac{k}{\kappa}\frac{dv}{dt} - \frac{kv}{\kappa^2}\cdot\frac{d\kappa}{ds}\cdot\frac{kv}{\kappa},$$

$$\text{or } \frac{d\phi}{dt}ds = k\frac{dv}{dt}\cdot\frac{ds}{\kappa} - k^2v^2\frac{d\kappa}{\kappa^3}.$$

Now v, and therefore $\frac{dv}{dt}$, is a function exclusively of the time, the position of k being given, and $d\kappa$ and $\frac{ds}{\kappa}$ are functions of x, y, z; hence, integrating with respect to these quantities,

$$\int\frac{d\phi}{dt}ds = k\frac{dv}{dt}\int\frac{ds}{\kappa} + \frac{1}{2}\frac{k^2v^2}{\kappa^2}.$$

Substituting and supposing that gravity is the only force and that $\rho = 1$,

$$p = gz - k\frac{dv}{dt}\int\frac{ds}{\kappa} - \frac{1}{2}\frac{k^2v^2}{\kappa^2} + C,$$

where C may be a function of the time. To determine the arbitrary constant, let us suppose that $k = K$ at the surface of the fluid, let $p_{,}$, $z_{,}$ be the values of p and z at the surface, and let $\int_{z_{,}}^{z}\frac{ds}{\kappa} = N$, then

$$p - p_{,} = g(z - z_{,}) - k\frac{dv}{dt}N - \frac{1}{2}k^2v^2\left\{\frac{1}{\kappa^2} - \frac{1}{K^2}\right\};$$

whence the pressure at any point below the surface of the fluid may be determined.

123. The preceding equation may be applied to an issuing fluid, and the results compared with those which have been already obtained.

Prop. *To find the velocity of a fluid issuing through a given orifice.*

Let water issue into free air, then the pressure at the orifice will be the same as the atmospheric pressure, or $p = p_{,}$. Let k be the orifice and v the velocity of the issuing fluid, then, since κ may be situated anywhere, let it coincide with the orifice, and let h be the depth of the orifice, then $h = z - z_{,}$;

$$\therefore\ 0 = gh - k\frac{dv}{dt}N - \frac{1}{2}k^2v^2\left(\frac{1}{k^2} - \frac{1}{K^2}\right)$$

$$= gh - k\frac{dv}{dt}N - \frac{1}{2}v^2(1 - a^2),$$

if $a = \dfrac{k}{K}$.

Now since $(1 - a^2)$ is essentially positive, we may assume it $= \beta^2$.

Let the fluid be retained at a constant height, then h and N are constant; we have therefore

$$dt = \frac{2kNdv}{2gh - \beta^2 v^2},$$

$$\therefore\ t = \frac{kN}{\beta\sqrt{2gh}} \log . \frac{\sqrt{2gh} + \beta v}{\sqrt{2gh} - \beta v}.$$

No correction is requisite, since v and t begin together. The quantities β and $\sqrt{2gh}$ may also be positive or negative. Taking them as positive, we have, assuming $\frac{kN}{\beta\sqrt{2gh}} = \frac{1}{\lambda}$,

$$\frac{\sqrt{2gh} + \beta v}{\sqrt{2gh} - \beta v} = \epsilon^{\lambda t},$$

$$\text{whence } \frac{\beta v}{\sqrt{2gh}} = \frac{\epsilon^{\lambda t} - 1}{\epsilon^{\lambda t} + 1}.$$

Substituting for β its value,

$$v = \left(\frac{2gh}{1 - \frac{k^2}{K^2}}\right)^{\frac{1}{2}} \frac{\epsilon^{\lambda t} - 1}{\epsilon^{\lambda t} + 1}.$$

As t increases, $\frac{\epsilon^{\lambda t} - 1}{\epsilon^{\lambda t} + 1} = \frac{1 - \epsilon^{-\lambda t}}{1 + \epsilon^{-\lambda t}}$ approaches rapidly to unity as its limit, and at this limit the velocity is independent of the time,

$$\text{or } v = \left(\frac{2gh}{1 - \frac{k^2}{K^2}}\right)^{\frac{1}{2}},$$

the velocity when the motion is steady, as was before shewn.

It appears then that on the hypothesis of parallel sections, the motion can never be strictly steady; if however the ratio $\frac{k}{K}$ be small, and h be not exceedingly small, λ is large, and the motion may be considered as steady, after a very small time.

Steady Motion of an Elastic Fluid.

124. The variation of the temperature being neglected, which may be done without sensible error when the motions are not very rapid, our general equation of steady motion becomes, when applied to elastic fluids, if $p = a^2\rho$,

$$a^2 \log p = gz - \tfrac{1}{2} v^2 + C.$$

Ex. *Let air be driven by a constant pressure through a small orifice out of a vessel into the atmosphere.*

The effect of gravity may here be neglected, and if P be the constant pressure and V the velocity where the pressure has this value, then

$$a^2 \log P = -\tfrac{1}{2} V^2 + C;$$

subtracting from the preceding,

$$\therefore\ a^2 \log \frac{p}{P} = \tfrac{1}{2}(V^2 - v^2);$$

hence the pressure is less as v is greater. Also,

$$v^2 = V^2 - 2a^2 \log \frac{p}{P} = V^2 + 2a^2 \log \frac{P}{p},$$

$$\text{or } v^2 - V^2 = 2a^2 \log \frac{P}{p}.$$

Let α be the ratio of the velocity at the surface subject to the constant pressure, to the velocity at the orifice, then

$$v^2 (1 - \alpha^2) = 2a^2 \log \frac{P}{p}.$$

When the orifice is very small, a^2 may be omitted,

$$\text{and } v = \sqrt{2a^2 \log \frac{P}{p}}.$$

The equation $a^2 \log \frac{p}{P} = \frac{1}{2}(V^2 - v^2)$, shews that points of equal pressure are also points of equal velocity, since $p = P$ and $v = V$ simultaneously; consequently, that as the pressure at every point of the surface must be nearly the atmospheric pressure, if the stream contract like water, as there is good reason to conclude it does, there will be a converging surface at every point of which the velocity is nearly the same and greater than the velocity at all points within it, so that the pressure within the surface is greater than the pressure of the atmosphere.

CHAPTER XII.

ON THE THEORY OF SOUND.

125. Sound is caused by the vibrations arising from some disturbance to which the particles that constitute an elastic fluid have been subjected, and the theory of sound consists in applying the preceding general equations to the motion consequent on such disturbance. A single disturbance is not sufficient to produce the vibrations necessary for the production of sound, but they arise from the constant repetition of such disturbances, and the velocity of sound is the rapidity with which these disturbances are propagated through the elastic medium.

The air is the fluid to which we shall now proceed to apply the general equations (Art. 106.); and we shall suppose that it is perfectly elastic and homogeneous, having, when at rest, every where the same density and temperature; and also, that it is so slightly disturbed from its state of rest, that during the motion which arises from this disturbance, the velocities of the particles are exceedingly small, in consequence whereof the accompanying condensations and rarefactions will also be exceedingly small quantities. Hence the squares and products of these small quantities may be omitted, the effect of which will be to render the general equation linear, and therefore integrable under a finite form.

This linearity of the equation is a point of the greatest importance, since, otherwise, the general equation is absolutely intractable; for it is evident that if no hypothesis be made limiting the extent of the motions of the particles from their points of quiescence, the case to which we should be about to apply the equations would involve all the possible motions of elastic fluids.

We shall suppose also that no extraneous force acts, or that X, Y, Z, are all equal to zero, in which case the density will, in a state of equilibrium, be constant and uniform throughout, as we have already supposed it to be.

126. Prop. *To form the equations for the small vibrations of an elastic fluid.*

Let D be the density of the air when at rest, and ρ its density at a point x, y, z, and after a time t from the commencement of the motion; then

$$\rho = D(1 + s),$$

where s is a small fraction either positive or negative.

Let h and gmh be the height and pressure of the barometric column corresponding to the density D, m being the density of the mercury. Then when the fluid is in motion, the pressure p which corresponds to the density ρ is $gmh\ (1 + s)$, provided the temperature remains invariable; this however is not the case, since the temperature is increased or diminished according as the density increases or diminishes, that is, according as the fraction s is positive or negative. Suppose then that we have

$$p = gmh(1 + s + \sigma),$$

where σ is a small quantity of the same sign as s, of which it is some function; and since s is small, let us assume

$$\sigma = \beta s,$$

where β is a positive quantity and independent of s.

We have then, making the substitutions and differentiating,

$$dp = gmh(1 + \beta)\,ds;$$

and dividing by $\rho = D(1+s)$ and representing the coefficient $\frac{gmh(1+\beta)}{D}$ by a^2,

$$\frac{dp}{\rho} = a^2 \frac{ds}{1+s}, \quad \therefore \int \frac{dp}{\rho} = a^2 \log(1+s);$$

no constant being requisite if the integral be supposed to vanish when $s = 0$.

But $\log(1+s) = s - \frac{1}{2}s^2 + \ldots = s$, very nearly;

$$\therefore \int \frac{dp}{\rho} = a^2 s.$$

But the squares of the velocities

$$\frac{d\phi}{dx}, \quad \frac{d\phi}{dy}, \quad \frac{d\phi}{dz},$$

are to be omitted here; hence the general equation becomes

$$s = -\frac{1}{a^2}\frac{d\phi}{dt} \ldots (1).$$

This together with the three equations

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz} \ldots (2),$$

gives four equations for determining the condensation and the velocity of the fluid at the time t and the point x, y, z, the function ϕ having been determined.

The displacement of the particles of the fluid being small, the products of s and $\frac{d\phi}{dx}, \frac{d\phi}{dy}, \frac{d\phi}{dz}$, are to be omitted, and the equation of continuity (Art. 107.) becomes, substituting for s its preceding value,

$$\frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right) \ldots (3).$$

These three equations are those of the theory of sound propagated in air of uniform temperature and density: they depend on the hypothesis that $u\,dx + v\,dy + w\,dz$ is a complete differential, which it is in the cases to which we shall apply them.

Rectilinear Motion.

127. Let a small quantity of air be enclosed in a cylindrical tube with its axis horizontal, and let the motion be in the direction of this axis; if then this axis be taken for the axis of x, we have $v = 0$, $w = 0$, and ϕ will be a function only of x and t; the equation (3) becomes

$$\frac{d^2\phi}{dt^2} = a^2 \frac{d^2\phi}{dx^2}.$$

This equation, which belongs to the simplest case of propagated motion, is the one which we shall employ. A different one, which will also serve to determine the motion, may be obtained from the conditions of the problem in an independent manner, as we shall proceed to do in the following proposition.

128. Prop. *To find the differential equation for a disturbance propagated in a small cylindrical column of air.*

Let the axis of the column be horizontal, and let it be supposed that the temperature is uniform throughout the motion, and that no extraneous force acts.

Let the section of the column be unity, and PQ (Fig. 24.) an element dx of the fluid at a distance x from the origin O of co-ordinates. In the time dt let PQ be transferred to $P'Q'$, let $PP' = X$ and ρ, ρ' the densities of the air in PQ and $P'Q'$. Then,

$$OP' = x + X,$$

and since X will vary for dx in passing from Q to Q',

$$OQ' = x + dx + X + \frac{dX}{dx}\,dx$$

$$= x + X + \left(1 + \frac{dX}{dx}\right) dx;$$

$$\therefore\ P'Q' = \left(1 + \frac{dX}{dx}\right) dx.$$

But the mass of the fluid being the same,

$$\rho\, dx = \rho' \left(1 + \frac{dX}{dx}\right) dx, \text{ or, } \rho = \rho' \left(1 + \frac{dX}{dx}\right).$$

The motion of the element dx is owing to the difference of pressures at P and Q; hence, if p be the pressure at P and p' at Q,

$$p' = p + \frac{dp}{dx}\,dx.$$

The impressed moving force is

$$p - p' = -\frac{dp}{dx}\,dx,$$

and the mass moved is $\rho\, dx$, therefore

$$\text{the accelerating force} = -\frac{1}{\rho}\frac{dp}{dx}.$$

But the effective accelerating force $= \dfrac{d^2X}{dt^2}$,

$$\therefore\ \frac{d^2X}{dt^2} = -\frac{1}{\rho}\frac{dp}{dx}$$

is the general equation of motion, and $\dfrac{1}{\rho}\dfrac{dp}{dx}$ must be expressed differently, according to the nature of the fluid in question.

Now the general law of elastic fluids is, that the pressure is proportional to the density; hence, since there is a change in the density, there will be a corresponding change in the elastic force.

Let e, e' be the elastic force corresponding to the densities ρ, ρ', then the temperature being supposed constant,

$$e = k\rho, \text{ and } e' = k\rho';$$

$$\therefore \frac{e}{e'} = \frac{\rho}{\rho'} = 1 + \frac{dX}{dx}, \text{ and } e' = e\left(1 - \frac{dX}{dx}\right),$$

the powers of $\dfrac{dX}{dx}$ above the first being omitted, since the motions are small.

But e' is the pressure exerted at an instant $t + dt$,

$$\therefore p = e\left(1 - \frac{dX}{dx}\right), \text{ and } \frac{dp}{dx} = -e\frac{d^2X}{dx^2},$$

and $e = k\rho$;

$$\therefore -\frac{1}{\rho}\frac{dp}{dx} = k\frac{d^2X}{dx^2};$$

and substituting this value,

$$\frac{d^2X}{dt^2} = k\frac{d^2X}{dx^2};$$

which is the partial differential equation required.

If the change of elastic force consequent on the sudden variations of density had been taken into account, a like equation would have been obtained with a^2 in the place of k.

129. The equations (1), (2), (3), become then in the case of rectilinear motion,

$$s = -\frac{1}{a^2}\frac{d\phi}{dt}, \quad u = \frac{d\phi}{dx}, \quad \frac{d^2\phi}{dt^2} = a^2\frac{d^2\phi}{dx^2}.$$

The integral of this partial differential equation of the second order is

$$\phi = F_{,}(x - at) + f_{,}(x + at) \ldots\ldots (4)$$

where $F_{,}$ and $f_{,}$ are arbitrary functions; hence,

$$u = F(x - at) + f(x + at) \ldots\ldots (5),$$

$$\text{and } as = F(x - at) - f(x + at) \ldots\ldots (6).$$

The discussion of these equations will shew distinctly the nature of the motions which we have to consider.

And first, the differential equation is linear. Now the linearity of this equation is a remarkable analytical fact, and arose from the omission of the terms which consisted of the products of the condensation and velocity of the particles; and this omission was allowable because of the hypothesis, that the motions of the particles from their state of quiesence were small. The physical fact in the propagation of sound through air is known to be in accurate accordance with this hypothesis, the agitation of each particle being so minute as not to move it *sensibly* from the state of rest; for when sounds are transmitted through a smoky or dusty atmosphere, there is no visible motion in the smoke or floating dust, unless the source of the sound be so near as to cause a wind*.

If then the motions of the particles be exceedingly small, the differences of these motions for two consecutive particles, that is, the amount of condensation or rarefaction undergone, must also be exceedingly small; the products then or squares of these small quantities will be quantities of a higher order than those of the other terms, and may therefore be omitted in comparison with them.

When motion therefore takes place in a medium under these circumstances and conditions, the equation will be linear.

* *Encyc. Met.* Art. *Sound*, 54.

Now it is the property of all linear equations of the first degree, that if any number of functions satisfy the equation independently, their sum will also satisfy it; and conversely, if the sum of any number of functions satisfy it, then each function will separately satisfy it. And this property of the linear equation of the first degree is a necessary consequence of its *linearity;* for the equation involves no powers of the differential coefficient but the first, and since any differential coefficient of the sum of any number of functions is the same as the sum of their differential coefficients, the substitution of a sum of functions is the same as their substitution separately and conversely; therefore if the sum of any number of functions satisfy the equation, each function will satisfy it separately. This property then that the equation may be broken up into several parts similar to each other and to the whole, is an analytical fact and the necessary consequence of its linearity.

130. From equation (5) we have the velocity given by the sum of two functions, but from what has preceded it is evident that we may have

$$u = F_1(x - at) + F_2(x - at) + \ldots\ldots$$
$$+ f_1(x + at) + f_2(x + at) + \ldots\ldots$$

If the disturbance be propagated only in one direction, there will be (as we shall see presently) but one of these two lines of functions to be taken, the form of each function being determined by the initial circumstances of the disturbance.

If there be but one original cause of the disturbance not resolvable into component disturbances, there will be but one function as F_1 to be considered; but if there be several original disturbances, there will be one function corresponding to each, and the whole disturbance will be the algebraical sum of these functions. And it is particularly to be remarked, that the whole disturbance thus found as the effect of all the original causes together, is precisely the algebraical sum of a number of disturbances, each of which would have been produced by one of the original causes acting separately.

Hence, whenever a particle is affected by several disturbances simultaneously, the motion it receives is compounded of all the different motions it would have received had each disturbance acted separately. Thus the velocity at any point is the resultant of several velocities produced by different causes, and any given cause will have the same effect in producing velocity at a given point, whether or not other causes are operating to produce velocities at the same point.

The preceding is the general theoretical proof of the co-existence of small vibrations in rectilinear propagation; an experimental confirmation of which may be derived from the well-known fact, that an ear is sensible of the effect of every instrument at a concert, which could not be the case, except on the hypothesis of the simultaneous transmission of different disturbances.

It must also be remarked with respect to these equations, that the origins of x and t are perfectly arbitrary; and that as the equations were investigated without any reference to the manner in which the particles were put in motion, all the results derived prior to any hypothesis about the mode of disturbance must be perfectly general; that is, these results must obtain whatever be the nature of the disturbance, or in whatever way the particles have been caused to move, provided always that u be small compared with a.

131. The determination of the velocity with which a disturbance is propagated in transmitted motion had been incorrectly treated by analysts, till Professor Challis directed their attention to it. In his Report* he expresses

* *Trans. Brit. Assoc.* 1834.

his doubts as to whether the arbitrary functions obtained by the integration of the differential equation can be immediately applied to any but the initial state of the fluid, and whether previous to their application at any subsequent epoch, the law of transmission must not be first deduced by means of the quantities which the arbitrary functions involve; that is, if the differential equations be applied to one, as for instance, to the initial state of the fluid, can they or can they not be applied to another state without first determining the law of transmission?

Now previous to an examination into the law of transmission, we cannot know whether the form of the functions may not change with the time; and if this be the case there will be an evident fallacy in determining the velocity of propagation, on the supposition that the same form which expresses the state of disturbance at a time (t) will also express it at the time $t + t'$.

This determination of the velocity of propagation leads, however, to no erroneous results in the case of sound, because the velocity of propagation happens to be uniform; and the fact that the functions do not change form is the *consequence* of the uniform transmission. Whenever then the velocity of propagation happens to be uniform, this method leads to no erroneous results, because it rests on a supposition which implies the uniformity of propagation. It could not, however be applied without error to an instance of propagation like that which obtains in waves at the surface of water, where the forms of the propagated waves, though dependent on the initial state of the fluid, are continually changing with the time.

The method of obtaining a general expression for the velocity of propagation is given in the following proposition*.

132. PROP. *To determine generally the velocity of propagation in transmitted motion.*

* See CHALLIS, *Ed. Phil. Mag.* April 1835.

Let $y = \phi(x, t)$ express the state of the particles at a distance x from the origin at the time t.

Suppose any given value of the ordinate (if we suppose the state of the particles to be represented by a curve) to be carried through space with the velocity v during the time t'; in general v will be a function of x and t, but it may be supposed constant during the small time t', and for a small increment of x, that is, for a portion dx of the axis of abscissæ. For on this supposition quantities of the order t'^2 and $t'dx$ only will be omitted.

Hence, for the small interval of time so far as it relates only to a portion dx of the axis of abscissæ, the function ϕ may be considered invariable, consequently,

$$\phi(x, t) = \phi(x + vt', t + t')$$

$$= \phi(x, t) + \frac{d\phi}{dx} vt' + \frac{d\phi}{dt} t' + \ldots\ldots$$

$$\therefore 0 = \frac{d\phi}{dx} v + \frac{d\phi}{dt} \text{ very nearly, and}$$

$$v = -\frac{\dfrac{d\phi}{dt}}{\dfrac{d\phi}{dx}};$$

which is a general expression for the velocity of propagation.

This formula is of extensive application, and will serve either to find v in terms of x and t when F is given, or to determine F by integration when v is given.

133. *To determine the velocity of propagation in a cylindrical column of air, and the nature of the motion.*

We have seen that either of the functions will satisfy the equation separately, then for the function F we have, since

$$\frac{dF}{dt} = -aF'(x - at), \text{ and } \frac{dF}{dx} = F'(x - at),$$

$$\therefore \text{ the velocity of propagation} = -\frac{-aF'(x - at)}{F'(x - at)} = a.$$

For $f(x + at)$ in the same manner,

the velocity of propagation $= -a$.

Thus it appears, that whatever be the initial disturbance, the velocity of propagation is constant; hence, we may consider that the ordinate of the curve representing the state of the particles is transferred with a uniform motion through space, and consequently the functions F and f do not change with the time.

It appears also that the functions apply to propagations in opposite directions; the function F to propagation in the positive direction, and f to propagation in the negative direction, whether on the positive or negative side of the origin.

Since the two kinds of functions F and f, which separately satisfy the differential equation, satisfy it conjointly, the inference from this analytical fact is, that the most general character of the motion is such as results from two simultaneous propagations in opposite directions. The velocity and condensation of the particles, whether at the instant of original disturbance or at any subsequent period, are such as are consistent with two motions transmitted in opposite directions with the uniform velocity a.

134. Prop. *To determine the nature of the transmitted motions.*

Since each of the functions satisfy the equation, let us first consider $F(x - at)$. This, as we have seen, refers to propagation in the positive direction with a uniform velocity a. The motion represented by one of the functions expresses a possible motion, but is not the most general which can obtain.

Then $u = as = F(x - at)$.

This equation shews that the velocity is always proportional to the condensation. Also, since

$$u = as, \quad \text{and} \quad s = \frac{1}{a} F(x - at),$$

if at any instant, that is, for any value of t, an ordinate, as determined from the equation $s = \frac{1}{a} F(x - at)$, be erected at each point in any line taken as the axis of abscissa, these ordinates will be proportional to the condensation, and the bounding curve will give at once the law of the density and the velocity; the positive ordinates corresponding to velocities in the positive direction ABE (Fig. 25.), or to condensations, and conversely; the negative ordinates to velocities in the negative direction EBA, or to rarefactions, and conversely.

The state and motion of the particles then at any time t, may be accurately represented by some curve, the exact form of which is of no importance.

This being the state of things at a time t, let us enquire what is the state at a time $t + t'$, that is, let t be supposed variable, and x constant.

$$\begin{aligned} \text{Then } u = as &= F\{x - a(t + t')\} \\ &= F\{(x - at') - at\} \\ &= F(x' - at), \text{ suppose,} \end{aligned}$$

which is of the same form as $F(x - at)$, that is, the state of the particles at a distance x' from the origin, and at a time t', is precisely the same as at a distance x

from the origin and a time t; or we shall have the same velocities and condensations of the particles when x is constant and t becomes $t+t'$, as when t is constant and x becomes $x-at'$.

Hence the velocities and condensations which the particles at a given point undergo during the time t' are the same as those which the particles in a space at' measured from the given point towards the origin of co-ordinates are undergoing at the instant t' commences.

The motion is therefore such as will be understood by imagining a curve, which gives the velocities and condensations, to move without undergoing any alteration along the axis and from the origin; but it must particularly be remarked, that this conceived transfer is a transfer of *form*, and not of *matter*, and that it is more properly expressed by saying that the particles at a distance $x+at'$ from the origin at the time $t+t'$ are in the same *relative state* as were the particles at a distance x from the origin at the time t.

The general conclusion then to which we are led by the preceding is, that each particle (taken successively in order of space) is successively, in order of time, in a similar state of displacement, which may be represented by conceiving a peculiar form of curve to advance from the origin with a uniform velocity.

The same remarks apply to the function f, the only difference being that the propagation is in the opposite direction, or towards the origin.

If there be several initial disturbances, there will be a function corresponding to each, and the ordinate expressing the velocity and condensation at any point will be the algebraical sum of the ordinates which would obtain by virtue of each disturbance considered separately. Thus the corresponding modes of vibration, when co-existing, will produce a compound curve very different from the curve which would be traced for any one of the disturbances acting singly.

135. PROP. *To express the disturbance at any epoch in terms of the initial disturbance.*

Let t be dated from the commencement of the motion, and let the initial disturbance extend through the limits $\pm l$, and let $\psi(x)$ be the function which represents the initial values of the condensation, the velocity being supposed nothing when $t = 0$. Then, (Art. 129.),

$$0 = F(x) + f(x),$$

$$\psi(x) = aF(x) - af(x),$$

and $F = -f$; therefore, since $F(x) = -f(x)$,

$$\psi(x) = 2F(x).$$

In the same manner $\psi(x) = -2f(x)$; we have, therefore, since the functions do not change with the time,

$$F(x - at) = \tfrac{1}{2}\psi(x - at),$$

$$f(x + at) = -\tfrac{1}{2}\psi(x + at).$$

Then

$$u = \tfrac{1}{2}\psi(x - at) - \tfrac{1}{2}\psi(x + at),$$

$$at = \tfrac{1}{2}\psi(x - at) + \tfrac{1}{2}\psi(x + at).$$

These equations apply to the motion for any values whatever of x and t, with the single limitation that the function ψ must become evanescent for any values of $x + at$ and $x - at$ not included between the limits $\pm l$. It will hence appear that from the first instant of the motion the initial disturbance is divided into two equal propagations, one in the positive and the other in the negative direction, and that as soon as these two parts are completely separated from each other by the propagation, the function $\psi(x - at)$ only applies to the former, and the function $\psi(x + at)$ only to the latter.

The motion then at any point on the positive side of the disturbance commences when $x - at = l$, and ends when $x - at = -l$, that is, it begins to move when $t = \frac{x-l}{a}$, and ceases when $t = \frac{x+l}{a}$; the duration, therefore, of its motion is $\frac{2l}{a}$, which is the time of vibration of a particle.

Now, since a is constant, the time of vibration depends simply on the extent of the disturbance; hence, if τ be the time of vibration and λ the length of the wave, we have, as in light*,

$$\text{the velocity of the wave} = \frac{\lambda}{\tau} = \frac{2l}{\frac{2l}{a}} = a,$$

which is independent of the length of the wave, or the velocity of transmission is constant, as it was before determined to be.

136. The periodicity of the motions of the particles leads us to assign some trigonometrical form to the arbitrary function, and the simplest which suggests itself, from the analogy which subsists between the motion of the particles of air and the oscillatory motion of a pendulum, is

$$b \sin \frac{n}{a}(x - at + C),$$

which will express a possible motion, but not the most general one.

This function goes through all its values when nt increases by 2π, that is, when t increases by $\frac{2\pi}{n}$, which is therefore the time of vibration of a particle.

* AIRY'S *Undulatory Theory*, Art. 5.

But by the last article the time of vibration $= \frac{\lambda}{a}$;

$$\therefore \frac{\lambda}{a} = \frac{2\pi}{n}, \text{ whence } \frac{n}{a} = \frac{2\pi}{\lambda}.$$

If, therefore, the origin either of t or x be so assumed that the arbitrary constant C is nothing, we have

$$b \sin \frac{2\pi}{\lambda}(x - at)$$

as the form of the function for a single disturbance; and if there be several such disturbances, the sum of a number of such functions will indicate a possible motion.

137. Prop. *To explain the reflexion of sound.*

We have generally for the disturbance of a particle of air in a cylindrical column

$$u = F(x - at) + f(x + at).$$

Suppose that $u = 0$ when $x = l$ for *all* values of t; then

$$f(l + at) = -F(l - at),$$

and since this holds for all values of t, we have

$$f(l + at') = -F(l - at').$$

Let t and t' be so connected together that

$$x + at = l + at';$$

$$\therefore 2l - x - at = l - at';$$

$$\text{and } f(l + at') = -F(l - at');$$

$$\therefore f(x + at) = -F(2l - x - at);$$

$$\therefore u = F(x - at) - F(2l - x - at)$$

$$= F(x - at) - F\{2l - (x + at)\}.$$

Hence it appears that when in a cylindrical column of air any particle at a distance l from the origin is always at rest, the motion of a particle at any point less than l will be such as results from two equal and opposite propagations having their origins equidistant from the point at rest, and commencing at the same instant.

Now the effect will not be altered by supposing a rigid partition at any point, provided it be endued with the motion of the particles at that point. Let such a partition be placed at a distance l from the origin, that is, at the point of rest; the partition will consequently be always at rest. Under these circumstances the air on one side of the partition does not act on the air at the other side; if one portion be removed, the motion will take place as before, but in this case the partition becomes a rigid reflecting surface, thus a sound is reflected back again.

138. In a preceding proposition (Art. 135.) we supposed that the velocity was originally nothing, the condensation being expressed by a given function. We will now consider the case when both the initial velocity and the condensation are given by separate arbitrary functions.

The equations for the velocity and condensation in a cylindrical column are

$$u = F(x - at) + f(x + at),$$

$$as = F(x - at) - f(x + at).$$

Suppose that when $t = 0$ we have $u = \psi(x)$ and $s = \chi(x)$; then

$$\psi(x) = F(x) + f(x), \quad \text{and} \quad a\chi(x) = F(x) - f(x),$$

whence,

$$F(x) = \tfrac{1}{2}\{\psi x + a\chi(x)\},$$

$$f(x) = \tfrac{1}{2}\{\psi x - a\chi(x)\};$$

$$\therefore\ u = \tfrac{1}{2}\{\psi(x-at)+\psi(x+at)\} + \frac{a}{2}\{\chi(x-at)-\chi(x+at)\},$$

$$s = \frac{1}{2a}\{\psi(x-at)-\psi(x+at)\} + \tfrac{1}{2}\{\chi(x-at)+\chi(x+at)\}.$$

Let the initial disturbance extend through a distance $\pm l$; then for all other values the initial functions are nothing; therefore $\psi(x)=0$ and $\chi(x)=0$ from $x=l$ to $x=\infty^{\text{ty}}$, and from $x=-l$ to $x=-\infty^{\text{ty}}$.

If now x be $>l$, we shall have

$$\psi(x+at)=0, \quad \chi(x+at)=0,$$

$$\text{and } u = \tfrac{1}{2}\{\psi(x-at)+a\chi(x-at)\},$$

$$s = \frac{1}{2a}\{\psi(x-at)+a\chi(x-at)\};$$

$$\therefore\ u = as.$$

Again, if x be $<-l$,

$$\psi(x-at)=0, \quad \text{and } \chi(x-at)=0,$$

$$\text{and } u = \tfrac{1}{2}\{\psi(x+at)-a\chi(x+at)\},$$

$$s = -\frac{1}{2a}\{\psi(x+at)-\chi(x+at)\};$$

$$\therefore\ u = as.$$

Hence, beyond the limits of the primitive disturbance on each side of it, the condensations of the particles are proportional to their actual velocities, and the particles are in a state of condensation or rarefaction according as their motion is in the same or contrary direction to that of the propagation.

This relation, which does not necessarily hold within the limits of the primitive disturbance, establishes a marked distinction between the primary and the propagated waves, the former being subject to no law, but to the arbitrary one which we assign to the function, the latter being subject to this condition. Any impulse in which this condition is not satisfied will immediately divide itself into two pulses running opposite ways, in each of which the preceding condition holds, and so long as this condition holds, no subdivision takes place. Hence we see the reason why every propagated wave does not divide itself into two, but is propagated only in one direction.

Suppose this condition to obtain in the primitive impulse, then,

$$\psi(x) = a\chi(x),$$

and the preceding equations give

$$u = \psi(x - at), \quad s = \frac{1}{a}\psi(x - at),$$

whence it appears, that for all negative values of x greater than $-l$, we shall have $u = 0$, $s = 0$, which shew that on the negative side the motion is not propagated beyond the limits of the primitive disturbance.

Whenever then in passing through a medium a wave receives from extraneous causes any modification, such as disturbs the preceding relation, it will be subdivided, and a portion reflected. Similarly, this portion may be again subdivided, and so on; this subdivision being always accompanied with reflection, will give rise to a continued series of repetitions of the original sound as echoes.

Motion in three Dimensions.

139. Having fully discussed rectilinear motion, we shall proceed to apply the general equations to a mass of air of indefinite extent, and in which the disturbance extends itself in all directions from a centre.

PROP. *To find the propagated motion in a mass of air of indefinite extent.*

Let the centre of disturbance be taken for the origin of co-ordinates, and r be the distance of any point x, y, z, at the time t, and ζ its velocity, which will be in the direction of the radius r, and a function of r and t and of the condensation s; for during the whole motion every thing must be symmetrical about the origin of co-ordinates.

Then we have

$$u = \frac{\zeta x}{r}, \quad v = \frac{\zeta y}{r}, \quad w = \frac{\zeta z}{r};$$

but $x^2 + y^2 + z^2 = r^2$; $\quad \therefore \; x\,dx + y\,dy + z\,dz = r\,dr$,

and $u\,dx + y\,dy + w\,dz = \dfrac{\zeta}{r}(x\,dx + y\,dy + z\,dz) = \zeta\,dr$;

or, $u\,dx + y\,dy + w\,dz$ is a complete differential of some function of r and t. This function being the quantity ϕ determined from the equation (3) (Art. 126.), we have

$$\zeta = \sqrt{u^2 + v^2 + w^2} = \frac{d\phi}{dr}$$

as the resultant of the velocities u, v, w.

Differentiating ϕ with respect to x, y, z, we have

$$\frac{d\phi}{dx} = \frac{d\phi}{dr}\frac{dr}{dx}, \quad \frac{d\phi}{dy} = \frac{d\phi}{dr}\frac{dr}{dy}, \quad \frac{d\phi}{dz} = \frac{d\phi}{dr}\frac{dr}{dz};$$

differentiating again and substituting,

$$\frac{d^2\phi}{dx^2} = \frac{d^2\phi}{dr^2}\frac{x^2}{r^2} + \frac{d\phi}{dr}\left(1 - \frac{x^2}{r^2}\right)\frac{1}{r}$$

$$= \frac{d^2\phi}{dr^2}\frac{x^2}{r^2} + \frac{d\phi}{dr}\frac{y^2 + z^2}{r^3},$$

$$\frac{d^2\phi}{dy^2} = \frac{d^2\phi}{dr^2}\frac{y^2}{r^2} + \frac{d\phi}{dr}\frac{x^2+z^2}{r^3},$$

$$\frac{d^2\phi}{dz^2} = \frac{d^2\phi}{dr^2}\frac{z^2}{r^2} + \frac{d\phi}{dr}\frac{x^2+y^2}{r^3},$$

and the equation (3), becomes

$$\frac{d^2\phi}{dt^2} = a^2\left(\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr}\right),$$

which may be put under the equivalent form,

$$\frac{d^2.r\phi}{dt^2} = a^2\frac{d^2.r\phi}{dr^2} \text{ ....(4).}$$

The complete integral of which is

$$r\phi = F(r-at) + f(r+at).$$

But $\zeta = \dfrac{d\phi}{dr}$ indicating therefore by the differential coefficients by accents,

$$\zeta = \frac{1}{r}\{F'(r-at)+f'(r+at)\} - \frac{1}{r^2}\{F(r-at)+f(r+at)\}\text{...(5).}$$

Also $s = -\dfrac{1}{a^2}\dfrac{d\phi}{dt}$;

$$\therefore\ s = \frac{1}{ar}\{F'(r-at) - f'(r+at)\}\text{.........(6),}$$

and these formulæ will determine the velocity and condensation at any instant when the functions F, F' have been determined for all the positive or negative values of $(r-at)$, and f, f' for all the positive values of $(r+at)$.

140. The remarks in Art. 129, on the linear equation for rectilinear motion, apply here also, and it is

therefore unnecessary to repeat them. The complete discussion of this equation has not as yet been effected, and for what is at present known respecting it, the reader must have recourse to the researches of Poisson* and Challis†. It appears, as in rectilinear motion, that the velocity and density are propagated uniformly, the velocity of propagation being equal to a; that the function F applies to propagation from a centre, and f to propagation towards a centre; and if the equations involve but one arbitrary function, they apply to a *single* disturbance. In this case, when r is very small, the second term of the equation (5), which involves r^2 in its denominator, may become much greater than that involving r; for expanding the functions, supposing r to be very small,

$$\zeta = \frac{F'(-at)}{r} + F''(-at) + \&c.$$

$$-\frac{F(-at)}{r^2} - \frac{F'(-at)}{r} - \frac{F''(-at)}{2} - \&c.$$

$$= -\frac{F(-at)}{r^2} \text{ nearly,}$$

$$= \frac{\psi(t)}{r^2}.$$

When, therefore, the disturbance is made by a sphere of small radius, the motion is transmitted from its surface to other parts of the fluid nearly as if the fluid were incompressible.

At a great distance from the centre of the disturbance we may neglect the term involving $\frac{1}{r^2}$ in (5) and

* Art. 660. *Traité de Mécanique.*

† *Camb. Phil. Trans.* Vol. III. *and* Vol. V.

(6) in comparison with the term involving $\frac{1}{r}$; we have then during the whole motion

$$\zeta = as,$$

as in rectilinear propagation.

The velocity of the particles decreases in the inverse ratio of r, hence, since the intensity of sound is proportional to the square of its velocity, its intensity at a considerable distance from the centre of the primitive disturbance will decrease inversely as the square of the distance; and experience confirms this conclusion.

141. We may here also determine the manner in which the motion of the fluid is affected when the rectilinear transmission of an impulse tending from any centre is interrupted by a plane surface. For suppose two impulses tending from two centres to be of equal magnitude and in every respect alike; then if the straight line joining these centres be bisected at right angles by a plane, there will be no motion of the particles contiguous to the plane in a direction perpendicular to it, because the resultant of the velocities from the two causes must lie wholly in the plane. Hence, since the division of fluids may be effected without the application of any force (Art. 2), nothing will be altered if we suppose the plane to become rigid and to intercept the communication of the fluid on one side with that on the other. The motion on each side will then be reflected, and the angle of incidence will be equal to the angle of reflection.

142. Prop. *To determine numerically the velocity of sound.*

The velocity of propagation of a disturbance through an elastic fluid is a. Now by assumption (Art. 126.),

$$a = \sqrt{\frac{gmh(1+\beta)}{D}},$$

and we must determine the value of β. From the same article, we have

$$p = \frac{gmh\rho(1+s+\beta s)}{D(1+s)} = \frac{gmh\rho}{D}\left(1+\frac{\beta s}{1+s}\right)$$

$$= \frac{gmh}{D}\rho(1+\beta s)\ldots\ldots\ldots\ldots(1)$$

very nearly.

Let η be the increase of temperature corresponding to this value of s, so that the temperature which was θ when the fluid was at rest becomes $\theta+\eta$ at the time t when the fluid is in motion. At this instant, p, ρ, $\theta+\eta$, being the simultaneous values of the elastic force, the density, and the temperature, we have the equation

$$p = k\rho\{1+\alpha(\theta+\eta)\}.$$

But the fluid being at rest, we have

$$p = gmh, \quad \rho = D, \quad \eta = 0,$$

and the preceding becomes

$$gmh = kD(1+\alpha\theta);$$

hence, the fluid being in motion,

$$p = \frac{gmh}{D}\rho\,\frac{1+\alpha(\theta+\eta)}{1+\alpha\theta} = \frac{gmh}{D}\rho\left(1+\frac{\alpha\eta}{1+\alpha\theta}\right);$$

and comparing this with the preceding equation (1) we have,

$$\beta = \frac{\alpha\eta}{(1+\alpha\theta)s}.$$

But the vibrations of the air being extremely rapid, so that the condensation s takes place without any loss of heat, we may substitute s and η for δ and ω in (5), (Art. 99.), hence,

$$1 + \beta = \gamma;$$

where γ is the ratio of the specific heat of air under a constant pressure to its specific heat under a constant volume.

The value of a becomes therefore

$$a = \sqrt{\frac{gmh\gamma}{D}}.$$

Let Δ be the density of the air under a constant pressure gmh at the standard temperature, then (Art. 81),

$$D = \frac{\Delta}{1 + \alpha\theta},$$

and consequently

$$a = \sqrt{\frac{gmh\gamma}{\Delta}(1 + \alpha\theta)} \ldots\ldots\ldots (2),$$

the expression whence the velocity of sound may be calculated numerically.

The value of γ, as determined by experiment (Art. 99.), was considered as independent both of the pressure and temperature; it appears then, 1^0, that the velocity of sound increases with the absolute temperature θ in the ratio of $\sqrt{1 + \alpha\theta}$ to unity; 2^0, that it does not vary with the barometric column, since h and Δ vary at the same time and in the same manner, so that their ratio is constant.

The hygrometric state of the air must produce a slight influence on the velocity, this however may in general be omitted, since the total variation for the extremes of dryness and of moisture will not amount to $\frac{1}{250}$th of the velocity of sound.

Numerical Determination.

The values of the constants which Poisson* has taken, are

$$g = 9^m.80896,\quad h = 0^m.76,\quad \frac{m}{D} = 10.462,$$

$$\alpha = 0.00375,\quad \theta = 15^0.9C.,\ \gamma = 1.3748,$$

whence he deduces

$$a = 337^m.07 = 1105 \text{ feet.}$$

This value is a little less than what he considers as its value according to the best observers, namely,

$$a = 340^m.89 = 1115 \text{ feet.}$$

We have seen (Art. 98.) that different experimentalists have assigned different values to γ; if we take its larger value, namely, $\gamma = 1.421$, we shall obtain, using the preceding data,

$$a = 342^m.69 = 1124 \text{ feet.}$$

Thus it appears that this value exceeds the observed value by nearly the same as the other falls short of it.

143. The determination of the ratio of the specific heats of an elastic fluid is a most important inquiry; we have already seen (Art. 99.) how this is to be determined experimentally, and that its value is essential to the numerical determination of the velocity of sound; and we shall now shew how its value may be determined from the observed velocity of sound. The circumstances under which sound is propagated, are far more favourable to the full production of the whole effect due to the cause in question, than the experiment with closed vessels; and the whole circumstances of the two cases are so widely different, that while a considerable deviation in the results would be insufficient to falsify the theory, a close agreement in the

* *Traité de Mécanique*, Art. 664.

results affords an evidence almost conclusive. Here then the results agree so nearly, that there can be no doubt of the truth of the hypotheses on which they rest.

The observed value of the velocity of sound is

$$a = 340^{m}.89.$$

Substituting in the formula

$$\gamma = \frac{a^2 \Delta}{gmh\,(1 + a\theta)},$$

the data of the preceding article will give

$$\gamma = 1.4061,$$

a most remarkable result, being nearly the mean of the smaller value of Gay-Lussac and Walter, and the larger one of Dulong.

In comparing this value of γ with the preceding experimental ones, it must be remembered that the condensation and rarefaction were supposed to take place so rapidly, that the quantity of heat which the fluid contained had not time to vary sensibly. But in the propagation of sound in free air, it is possible that the heat may escape or return more readily by radiation than in the propagation of sound in confined air, as in a closed tube, where the heat of each stratum of air can vary but little except by contact with the sides of the tube, and the large value of γ is the one which experiment assigns to the confined air. This remark may explain the difference of the two experimental results, and inclines to the larger value of γ as the more exact.

144. The velocity of sound as determined simply from the formula $\sqrt{\frac{gmh}{D}}$, by neglecting altogether the change of temperature consequent on the alternate rapid condensations and rarefactions, is less by one-sixth than the

observed velocity, and the accurate agreement of the theoretical with the observed velocity can leave no doubt of the truth of this theory, which is due entirely to Laplace.

The propagation of sound in the vapour of water at its *maximum* density is due to the same cause.

If a vibration be excited in a close vessel full of vapour and not mixed with air, sound will be generated and propagated without. But if the temperature of the stratum of vapour contiguous to the vibrating body was not augmented, the condensation consequent on the vibration would reduce the vapour to water, which would be precipitated on the surface of the vibrating body, since by the hypothesis the density is at the maximum, that is, the quantity of vapour is that which is due to the temperature under a given pressure. But heat being developed by the compression, the temperature of the condensed contiguous stratum is raised, and can consequently continue in a state of vapour. The condensation and increased temperature is propagated from stratum to stratum, and sound is produced just as in a vessel of permanently elastic fluid.

The rarefactions of the strata are accompanied with a diminution of temperature, but then the density being diminished at the same time, the vapour is not reduced to water, but descends to the *maximum* which is due to the relative temperature of the space it occupies.

The preceding is an *experimentum crucis* for deciding on the validity of the explanation above stated as given by Laplace, of the excess of the observed above the theoretical velocity of sound as determined without any regard to the developement of heat.

If the instantaneous condensations and rarefactions of an elastic fluid *do* (as is supposed in that explanation) give out and absorb heat, sound will be freely propagated in a *saturated* vapour, that is, in a vapour in contact with a liquid, or under a pressure it can just sustain. If not, no sound can be transmitted through it. The experiments are decisive*.

145. Water being considered as a fluid slightly compressible and elastic, sound will be propagated in it according to the same laws through any other elastic medium. The sound, when it reaches the surface of the water, will be partly transmitted to the external air and partly reflected; and the direction of the transmitted and reflected waves will follow the same laws as those of light. The velocity of the reflected sound will be the same as that of the direct sound, and the ratios of the intensities of the transmitted and reflected sound to the direct sound, will depend on the ratio of the velocities of propagation of sound in air and water.

When a given column of water suffers condensation, there does not appear to be any development of heat, so that there seems reason to conclude that the velocity of sound propagated in water is not influenced by any variation in temperature. Theory and observation give a velocity of propagation about quadruple the velocity in air.

On Musical Sounds.

146. The equations which express the nature of the disturbance produced in a cylindrical column of air have been discussed, and the sounds arising from the vibrations excited by blowing *across* the open end of a pipe or an aperture at its side, may be explained by the preceding equations. The current must be directed not *into* but *across* the aperture, so as to graze the opposite edge; a small portion will then be caught by the edge and turned aside down the pipe, thus giving an impulse to the contained air, and propagating along it a pulse in which the air is slightly condensed; this will be reflected at the end

* *Encycl. Met.* Art. *Sound*, 88. Mém. d'Arcueil, II. 99.

of the pipe as an echo, and return to the aperture where the condensation vanishes, since the density is the mean, that is, the same as in the undisturbed state. Thus a musical note is produced.

PROP. *To determine the note produced from a cylindrical pipe.*

Let l be the length of the tube closed at one end, the open end being the origin of co-ordinates. Then the equations are (Art. 137.)

$$u = F(x - at) - F(2l - x - at),$$

$$as = F(x - at) + F(2l - x - at).$$

We must assign some values to the form of the arbitrary function, let that value be taken which we have already seen (Art. 136.) may indicate a possible motion, then the preceding become,

$$u = m \sin\frac{2\pi}{\lambda}(x - at) - m \sin\frac{2\pi}{\lambda}(2l - x - at),$$

$$= 2m \cos\frac{2\pi}{\lambda}(l - at) \sin\frac{2\pi}{\lambda}(l - x)$$

$$as = m \sin\frac{2\pi}{\lambda}(x - at) + m \sin\frac{2\pi}{\lambda}(2l - x - at).$$

$$= 2m \sin\frac{2\pi}{\lambda}(l - at) \cos\frac{2\pi}{\lambda}(l - x).$$

Now in this case the condensation at the orifice is nothing, that is, $s = 0$ when $x = 0$;

$$\therefore\ u = 2m \sin\frac{2\pi l}{\lambda} \cos\frac{2\pi}{\lambda}(l - at)\ldots\ldots(1).$$

$$0 = 2m \cos\frac{2\pi l}{\lambda} \sin\frac{2\pi}{\lambda}(l - at)\ldots\ldots(2).$$

These equations hold for all values of t, hence, when $u = 0$, we must have $\sin\frac{2\pi l}{\lambda} = 0$, and when (2) is satisfied, $\cos\frac{2\pi l}{\lambda} = 0$; we shall proceed to discuss these cases.

147. *Nodes.* When a disturbance is propagated along a column of air, the column may at any instant be divided into several portions, in each of which the corresponding particles are in a similar state of displacement and motion.

These portions are termed *nodal* sections, and the points in which the axis of abscissa would cut the curve which gives the condensations and rarefactions at any instant, are the *nodes*. At these points the velocities of the particles are nothing, hence their position is determined by the equation

$$0 = 2m\sin\frac{2\pi l}{\lambda}\cos\frac{2\pi}{\lambda}(l - at),$$

which is satisfied when $\sin\frac{2\pi l}{\lambda} = 0$, that is, at a node

$$\frac{2\pi l}{\lambda} = n\pi, \text{ or } l = n\frac{\lambda}{2},$$

where n is any term of the series 0, 1, 2, &c.

Hence the interval between two consecutive nodes is half the length of a wave.

The closed end of the pipe is a node, or the extremity of a nodal section, since at this point the velocity of the particles is nothing.

Loops. Half way between two nodes the condensations are rarefactions, are evanescent, and the amplitudes of the molecular excursions are at a maximum; these points are called loops, and are given by the condition that $s = 0$;

hence, in the preceding case the closed end being a node, we have a loop when (2) is satisfied, that is, when $\cos \frac{2\pi l}{\lambda} = 0$.

For a loop then,

$$\frac{2\pi l}{\lambda} = (2n + 1)\frac{\pi}{2}, \text{ or, } l = (n + \tfrac{1}{2})\frac{\lambda}{2},$$

where n is any term of the series 0, 1, 2, &c.

At any point, then, in a cylindrical column of air, at which $l = n\frac{\lambda}{2}$, there is a node; and $l = (n + \frac{1}{2})\frac{\lambda}{2}$, there is a loop.

148. When a musical note is produced from a tube whose length is l closed at one end, by blowing across the open end, experience shews that when the lowest or the *fundamental* note is sounded, $\frac{\lambda}{4}$ is equal to the length of the tube, hence $n = 0$ for this note; also the condensation at the orifice is nothing, which will be the case if the orifice be the place of a loop.

If $n = 0$, then we have from $l = (n + \frac{1}{2})\frac{\lambda}{2}$,

$$l = \tfrac{1}{2}\frac{\lambda}{2}, \text{ or, } \lambda = 4l.$$

Let $n = 1$, then $l = \frac{3\lambda}{4}$, or $\lambda = \frac{4l}{3}$ and so on.

Hence, if the fundamental note be called 1, the others will be 3, 5, &c. being inversely as λ, the breadth of the wave.

It is found in fact, that 1, 3, 5, &c. are the only notes which can be sounded.

In the case above considered, the existence of the nodes and loops depends on the reflection at the closed end. If the tube be open at both ends, and the disturbance be made as before, the same cause does not operate to produce nodes and loops; yet is found by experiment, that there are places of maximum and minimum velocity at regular intervals, and that the two ends are nearly positions of maximum velocity. Assuming the ends to be positions of loops, we shall proceed to apply the equations to this case.

Let l' be the length of the tube, then

$$as = F(x - at) - F(2l' - x - at)$$

$$= m \sin \frac{2\pi}{\lambda}(x - at) - m \sin \frac{2\pi}{\lambda}(2l' - x - at)$$

$$= 2m \cos \frac{2\pi}{\lambda}(l' - at) \sin \frac{2\pi}{\lambda}(l' - x).$$

And $s = 0$, when $x = 0$, and $x = l'$;

$$\therefore\ 0 = \cos \frac{2\pi}{\lambda}(l' - at) \sin \frac{2\pi}{\lambda} l',$$

which is satisfied when $\dfrac{2\pi l'}{\lambda} = n\pi$; $\therefore\ nl' = \dfrac{n\lambda}{2}$.

But for the fundamental note $n = 1$;

$$\therefore\ \lambda = 2l';$$

therefore for the *same* note as in the preceding case we must have

$$2l' = 4l, \text{ or } l' = 2l,$$

that is, a tube open at both ends gives the same note as one of half the length closed at one end; and experience confirms this result.

149. Prop. *To find the time of vibration for any note.*

1°. Let the tube be closed at one end and $= l$, then

$$u = 2m \cos \frac{2\pi}{\lambda}(l - at) \sin \frac{2\pi}{\lambda}(l - x),$$

the period of which depends on that of $\cos \frac{2\pi}{\lambda}(l - at)$.

$$\text{Now } \cos \frac{2\pi}{\lambda}(l - at) = \cos 2\pi \left(\frac{l}{\lambda} - \frac{at}{\lambda}\right)$$

$$= \cos 2\pi \left(\frac{2n+1}{4} - \frac{at}{\lambda}\right)$$

$$= \pm \cos \left(\frac{\pi}{2} - \frac{2\pi at}{\lambda}\right)$$

($\pm$ according as n is even or odd)

$$= \pm \sin \frac{2\pi at}{\lambda},$$

which equals nothing when

$$\frac{2\pi at}{\lambda} = n\pi, \text{ or } t = \frac{1}{a} n \frac{\lambda}{2},$$

that is, for any multiple of half a wave.

Let n equal 1, 2, 3, &c. successively, and t_1, t_2, t_3, &c. be the corresponding values of t. Then the time of vibration is the interval $t_3 - t_1$, calling it τ,

$$\tau = \frac{1}{a} 3 \frac{\lambda}{2} - \frac{1}{a}\frac{\lambda}{2} = \frac{\lambda}{a} = \frac{4l}{a}$$

for the fundamental note.

And the number of vibrations in $1'' = \frac{1}{\tau} = \frac{a}{4l}$.

2°. Let the tube be open at both ends and in length $l' = 2l$, then

$$u = m \sin \frac{2\pi}{\lambda}(x - at) + m \sin \frac{2\pi}{\lambda}(2l - x - at)$$

$$= m \sin \frac{2\pi}{\lambda}(l - at) \cos \frac{2\pi}{\lambda}(l - x),$$

the period of which depends on that of $\sin \frac{2\pi}{\lambda}(l - at)$.

$$\text{Now } \sin \frac{2\pi}{\lambda}(l - at) = \sin 2\pi \left(\frac{l}{\lambda} - \frac{at}{\lambda}\right)$$

$$= \sin 2\pi \left(\frac{n}{2} - \frac{at}{\lambda}\right)$$

$$= \sin \frac{2\pi a t}{\lambda};$$

which is nothing when $t = \frac{1}{a} n \frac{\lambda}{2}$, for any multiple of half a wave. Whence as before

$$\tau = \frac{\lambda}{a} = \frac{2l'}{a} = \frac{4l}{a}$$

for the fundamental note.

150. *Pitch.* The pitch of a note is determined solely by the frequency of repetition of the impulse, so that all sounds, whatever be their *intensity* or *quality*, in which the elementary impulses occur with the same frequency, are pronounced by the ear to have the same pitch.

The intensity of sound depends on the violence of the impulses, the quality on the greater or less abruptness of these impulses.

The pitch then of a note depending on the number of waves which impinge on the ear during a given time varies inversely as the time of vibration of a particle; and the time of vibration $= \frac{\lambda}{a}$, therefore

$$\text{the pitch} \propto \frac{1}{\frac{\lambda}{a}}, \propto \frac{a}{\lambda}.$$

151. We have generally in the closed tube

$$u = 2m \cos \frac{2\pi}{\lambda}(l - at) \sin \frac{2\pi}{\lambda}(l - x),$$

which is nothing for all values of t, when $\sin \frac{2\pi}{\lambda}(l-x) = 0$,

$$\text{that is, when } l - x = n\frac{\lambda}{2}, \text{ or } x = l - n\frac{\lambda}{2}.$$

If therefore λ be given, we have nodal points corresponding to the values of x, which arise from giving different values to n.

$$\text{Also } as = 2m \sin \frac{2\pi}{\lambda}(l - at) \cos \frac{2\pi}{\lambda}(l - x),$$

and $s = 0$ for all values of t, when $\cos \frac{2\pi}{\lambda}(l - x) = 0$,

or $l - x = (2n + 1)\frac{\lambda}{4}$. Therefore $x = l - (n + \frac{1}{2})\frac{\lambda}{2}$ gives the position of the loops, which evidently occur at equal intervals with the nodes.

The maximum vibration depends on $2m \sin \frac{2\pi}{\lambda}(l-x)$ in the preceding value of u, which takes place when

$$\frac{2\pi}{\lambda}(l - x) = (2n + 1)\frac{\pi}{2},$$

and its maximum value is $2m$; and obtains at points where the condensation is nothing.

For further information on this subject, see a paper by Mr Hopkins*.

* *Camb. Phil. Trans.* Vol. v. p. 10.

CHAPTER XIII.

ON RESISTANCES.

152. When a solid is moved through a fluid its motion is resisted, and this resistance arises partly from friction and the tenacity of the fluid, but principally from the inertia of the fluid, that is, from the force which the body moving through the medium, necessarily exerts in putting the fluid particles in motion. Hence it may be considered as the reaction of the fluid particles, and *cæteris paribus*, if the velocity be increased, the resistance also will be increased, for the body will strike more particles and with greater violence. The *law*, according to which this resistance varies with the velocity must be deduced from experiment, and the square of the velocity is the power according to which it appears to vary; but no formula has hitherto been discovered which expresses with sufficient accuracy the absolute amount of the resistance for different velocities. In the following propositions we shall see how the subject is to be treated theoretically on the hypothesis that the resistance is as the square of the velocity, and what conclusions may thence be deduced.

153. *Definition.* The resistance of a fluid on a solid moving in it is the resultant of the excess of the pressure of the fluid on the solid in motion, above the pressure of the fluid on the solid at rest.

This resistance being a pressure, is of the nature of a moving force, and may be represented by weight. Its effect then on the body, or the *retarding force* of the resistance, is the resistance divided by the mass.

154. PROP. *To find the resistance on a plane moving perpendicularly to itself with a given velocity in a fluid.*

Let us suppose that the plane and fluid are moving *steadily* with the given velocity v, the plane being immersed perpendicularly to the motion of the stream.

Let the plane be stopped at any instant, then the motion of the fluid is resisted, and the mutual action between the fluid particles and the plane is precisely the same as the resistance on the plane moving with the given velocity through the fluid at rest. For we may suppose a velocity v to be impressed on every particle of the system in the direction opposite to the motion of the stream, the consequence of which will be that the fluid is reduced to rest, and the plane moves through it with the given velocity.

Now the pressure at any instant during the motion on a unit of the plane is

$$p = g\rho z - \frac{1}{2}\rho v^2 + C.$$

At the instant the plane is stopped, let p' be the value of p, then $v = 0$, and

$$p' = g\rho z + C;$$

$$\therefore\ p' - p = \frac{1}{2}\rho v^2.$$

But $p' - p$ is the resistance on a unit of surface, hence the resistance on an area A is $\frac{1}{2}\rho v^2 \times A$.

Let h be the height due to the velocity v, then $v^2 = 2gh$,

$\therefore$ the resistance on the plane $= g\rho h A$,

or the resistance on a plane moving perpendicularly to itself, is the weight of a column of fluid whose altitude is the height due to the velocity and base the area resisted.

Cor. If both the solid and fluid are in motion, the resistance on the solid is as the square of the relative velocity of the plane and of the fluid.

155. In the preceding proposition the resistance depends simply on the equality of action and reaction at the anterior surface of the plane, and no account is taken of the variation in the pressure which results from the disturbance at the posterior surface of the plane. This is doubtlessly one source of the discrepancy between the results of theory and experiment.

Again, no account is taken in the preceding of the fluid which collects in a quiescent state before the plane, the *instantaneous* effect will be such as is there stated, but the plane being moved through the fluid, the particles which have lost their velocity will constitute a conoidal mass of fluid, *quiescent* relatively to the plane, and bounded by a corresponding hollow conoid of *moving* fluid. The action between the surfaces of these two conoids will cause a pressure on the plane essentially different from the instantaneous action of the particles on the plane. These remarks are sufficient to point out the imperfections in the preceding theory.

156. Prop. *A plane moves obliquely in a fluid, required the resistance in the direction of its motion.*

Let P (Fig. 26.) be any point in the plane, PA the direction of its motion, and PB perpendicular to the plane.

Then we may either consider the plane as moving with a given velocity v in the direction PA, or the stream as impinging on the plane in the direction AP with the given velocity, the effect on the plane being in both cases the same; and the resistance on the plane in the direction of its motion is the same as the impelling force of the stream in the direction of its motion.

Let θ be the angle of incidence, that is, the angle APB, and R the resistance, or the force with which the stream impels the plane.

Let K be the area of the plane, and R the resistance upon it moving with the given velocity v, that is, the force with which the stream impels the plane in the direction perpendicular to the plane.

Then the velocity of the stream resolved in the direction $PB = v\cos\theta$; hence

$$R = \tfrac{1}{2}\rho v^2 \cos^2\theta K.$$

And the part of R in the direction of the plane's motion, that is, in the direction of the stream,

$$= R\cos\theta = \tfrac{1}{2}\rho v^2 \cos^3\theta K.$$

And the part in the direction perpendicular to the motion of the plane

$$= R\sin\theta = \tfrac{1}{2}\rho v^2 \cos^2\theta \sin\theta K.$$

Hence the resistance on any plane moving obliquely is as the cube of the cosine of the angle of incidence, that is, as the cube of the sine of its angle of inclination to the stream.

157. Prop. *To find the resistance on a solid of revolution moving in the direction of its axis.*

Let BAC (Fig. 27.) be a solid of revolution moving in the direction DA of its axis. Let x, y be the coordinates of any point P, and PQ an element ds of the generating curve, and mn the corresponding element dy of the base, and θ the angle which the tangent at P makes with the axis; then v being the velocity, the resistance on $PQ = \frac{1}{2}\rho v^2 \sin^3\theta \times PQ$, and the resistance on $mn = \frac{1}{2}\rho v^2 \times mn$, by the preceding article;

$\therefore$ resistance on PQ : resistance on mn :: $PQ \sin^3\theta : mn$

$$:: ds \frac{dy^3}{ds^3} : dy$$

$$:: \frac{dy^2}{ds^2} : 1.$$

And the same holds for every element of the annulus whose breadth is PQ, and for the corresponding portion of the base; it is therefore true for the whole annulus. The annular portion of the base corresponding to the portion of the surface $= 2\pi y dy$, and the resistance upon it $= \frac{1}{2}\rho v^2 \times 2\pi y dy$;

$\therefore$ the resistance on the surface $= 2\pi\rho v^2 \int y dy \dfrac{dy^2}{ds^2}$.

The mass of the solid of revolution $= \pi\rho' \int y^2 dx$, if ρ' be its density, and dividing,

$$\text{the retarding force of resistance} = \frac{\rho}{\rho'} v^2 \frac{\int y dy \frac{dy^2}{ds^2}}{\int y^2 dx}.$$

Ex. *Resistance and retarding force on a sphere.*

Let a be the radius of the sphere, the centre being the origin of co-ordinates; then

$$\text{the resistance} = \pi\rho v^2 \int y dy \frac{dy^2}{ds^2}.$$

Now $y^2 = a^2 - x^2$, and $\dfrac{ds}{dy} = \sqrt{1 + \dfrac{dx^2}{dy^2}} = \sqrt{1 + \dfrac{y^2}{x^2}} = \dfrac{a}{x}$;

$$\therefore \int y dy \frac{dy^2}{ds^2} = \int y dy \frac{x^2}{a^2} = \int y \left(1 - \frac{y^2}{a^2}\right) dy$$

$$= \left(\frac{1}{2} - \frac{1}{4}\frac{y^2}{a^2}\right) y^2 + C,$$

which, taken between the limits $y = 0$ and $y = a$,

$$= \tfrac{1}{4} a^2;$$

therefore the resistance on the sphere

$$= \tfrac{1}{4} \pi \rho v^2 a^2$$

$$= \tfrac{1}{2} \text{ the resistance on a great circle.}$$

And the mass of the sphere $= \frac{4}{3} \pi \rho' a^3$, if ρ' be its density;

$$\therefore \text{ the retarding force} = \frac{3\rho v^2}{16\rho' a}.$$

158. Prop. *A heavy sphere descends vertically in a fluid, required its velocity.*

Let a be the radius of the sphere and ρ' its density, and ρ the density of the fluid.

Then, as we have seen, the moving force of resistance upon a sphere is $\frac{1}{2}$ the resistance on one of its great circles moving perpendicularly to itself.

The resistance on one of its great circles $= \frac{1}{2} \rho v^2 \pi a^2$;

$$\therefore \text{ the resistance on the sphere} = \tfrac{1}{4} \pi \rho a^2 v^2.$$

And the mass of the sphere $= \frac{4}{3} \pi \rho' a^3$; hence

$$\text{the retarding force of resistance} = \frac{\frac{1}{4}\pi\rho a^2 v^2}{\frac{4}{3}\pi\rho' a^3} = \frac{3\rho}{16\rho' a} v^2$$

$$= kv^2, \text{ suppose.}$$

The force by which a body descends in a fluid is, neglecting the resistance, the excess of its weight above the weight of an equal bulk of the fluid.

The weight of the sphere $= \frac{4}{3} \pi \rho' g a^3$,

the weight of an equal bulk of the fluid $= \frac{4}{3} \pi \rho g a^3$.

Hence, subtracting and dividing by the mass of the sphere, this force

$$=\frac{\rho' - \rho}{\rho'}g = (1-r)g,$$

if r be the ratio of the density of the fluid to the density of the sphere.

The whole accelerating force on the sphere

$$=(1-r)g - kv^2.$$

Now generally, if f be the accelerating force, v the velocity, and s the space,

$$vdv = fds;$$

$$\therefore\ vdv = \{(1-r)g - kv^2\}\,ds,$$

$$\text{or}\quad d.v^2 + v^2.2kds = 2(1-r)gds,$$

which is a common linear equation, and will be rendered integrable by the factor ϵ^{2kds}. Multiplying and integrating, we have

$$v^2\epsilon^{2ks} = \frac{(1-r)g}{k}\epsilon^{2ks} + C \ldots\ldots\ldots(1).$$

To determine the arbitrary constant, let the sphere descend from rest, then v and s commence together;

$$\therefore\ 0 = \frac{(1-r)g}{k} + C,$$

subtracting and reducing,

$$v^2 = \frac{1-r}{k}g(1-\epsilon^{-2ks}).$$

When s is large, the second term may be omitted; the velocity then becomes constant, or is the terminal velocity. Let V be this value of v, then

$$V^2 = \frac{1-r}{k} g,$$

and the preceding becomes

$$v^2 = V^2(1 - \epsilon^{-2ks}) \ldots\ldots\ldots\ldots (2).$$

The constant $k = \frac{3r}{16a}$; $\therefore\ 2k = \frac{3r}{8a}$.

Let the sphere be double the density of the fluid, then $r = \frac{\rho}{\rho'} = \frac{1}{2}$; and let the sphere have descended through a space equal to 16 diameters.

Then $s = 32a$; $\therefore\ k = \epsilon^{-2ks} = \epsilon^{-6} = \frac{1}{400}$, nearly.

Hence $v^2 = V^2(1 - \frac{1}{400})$; $\therefore\ v = V(1 - \frac{1}{800})$, or, by the time that a sphere of twice the density of the fluid has descended through 16 diameters, which, when the particles are small in an insensible space, the velocity is within $\frac{1}{800}$th of the terminal velocity, and may after that be considered as moving uniformly.

The terminal velocity $V = \sqrt{\frac{(1-r)g}{k}}$, and substituting the preceding values, $V = \sqrt{\frac{16ag}{3}}$, which varies as the square root of the diameter of the sphere.

Hence it appears that the smaller the sphere the sooner it acquires its terminal velocity, and the less that velocity is when acquired. If then any small spherical bodies, as small dust, descend in water, or condensed vapour, as very small rain, descend in air, the velocity will be uniform and almost imperceptible.

159. PROP. *To determine the motion of an air-bubble ascending in a fluid.*

The air-bubble will increase in magnitude as it ascends; and let it be supposed to start from a depth in the fluid at which its density is very nearly that of the fluid; let b be its radius at this instant, and a its depth below the surface of the fluid, and ρ its density or that of the fluid.

After it has ascended through some distance, let y be its radius, x its depth, and ρ' its density. Then, since its magnitude is as the cube of the radius and inversely as its density, and the pressure, being proportional to the depth, is as the density, we have

$$y^3 : b^3 :: \frac{1}{x} : \frac{1}{a}; \quad \therefore y = \frac{a^{\frac{1}{3}}b}{x^{\frac{1}{3}}} \ldots\ldots\ldots (1).$$

$$\text{Also } \rho' : \rho :: x : a; \quad \therefore \rho' = \frac{\rho x}{a} \ldots\ldots\ldots (2).$$

The accelerating force upwards of the fluid displaced is $\frac{\rho - \rho'}{\rho'} g$, which

$$= \left(\frac{\rho}{\rho'} - 1\right) g = \left(\frac{a}{x} - 1\right) g, \text{ by (2).}$$

The retarding force of the resistance is $\frac{3\rho v^2}{16\rho' x}$, which

$$= \frac{3a^{\frac{2}{3}}}{16b}\frac{v^2}{x^{\frac{2}{3}}} = n\frac{v^2}{x^{\frac{2}{3}}},$$

by substitution from (1) and (2), if n be assumed $= \frac{3a^{\frac{2}{3}}}{16b}$; therefore the whole accelerating force upwards

$$= \left(\frac{a}{x} - 1\right) g - n\frac{v^2}{x^{\frac{2}{3}}}.$$

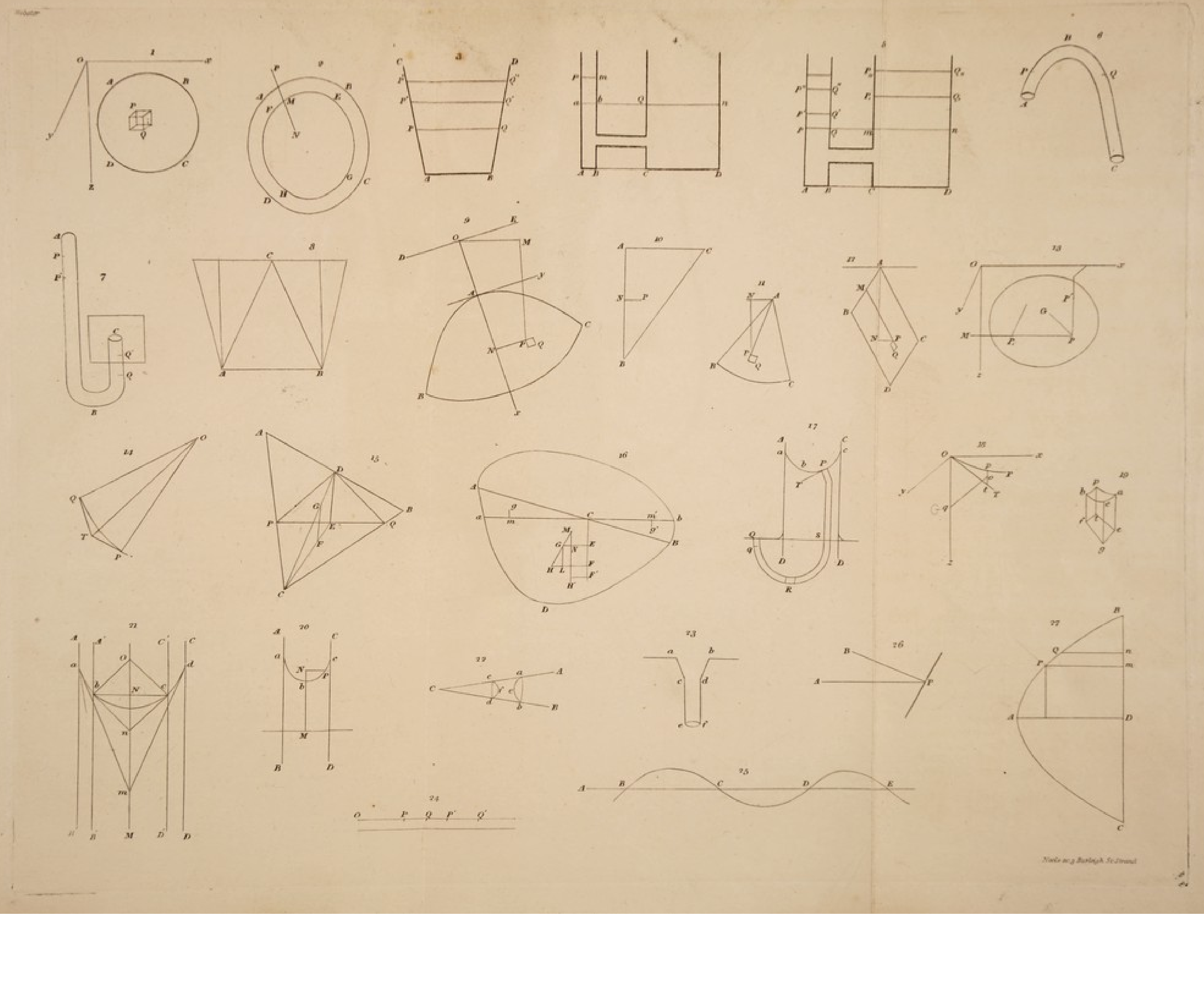
Then, since $v\,dv = f\,ds$, and the force diminishes x, we have

$$v\,dv = -\left\{\left(\frac{a}{x} - 1\right)g - n\frac{v^2}{x^{\frac{1}{3}}}\right\}dx,$$

whence

$$d\,.\,v^2 - v^2\,.\,\frac{2n}{x^{\frac{1}{3}}}\,dx = -\,2\left(\frac{a}{x} - 1\right)g\,dx,$$

a linear equation, whence the motion may be determined in particular cases.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