A

TREATISE

ON

DYNAMICS,

CONTAINING

A CONSIDERABLE COLLECTION

OF

Mechanical Problems.

BY

WILLIAM WHEWELL, M.A. F.R.S.
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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PREFACE.

THE following pages are intended to convey the mathematical doctrine of the Motion of Bodies, presented with the symmetry and generality which the science has now attained, satisfactorily established, and illustrated by its application to a variety of instances. Our language could scarcely be said to possess a scientific treatise on Dynamics, till Mr. Cresswell published his translation of Venturoli; and it is still presumed that the appearance of that elegant compendium of the subject may not have at all superseded the utility of a work like the present, in which the demonstrations are fully developed, and elucidated by a considerable collection of mechanical problems, selected from the works of the best mathematicians, and arranged with their solutions under the different divisions of the science.

From the time when Mechanics began to be cultivated as a branch of mathematics, but more particularly while the great Geometers of the last century were employed upon it, and before they had succeeded in compressing the whole science into a few short formulæ, the solution of particular mechanical problems, interesting from their nature or application, was a favorite employment of those who gave their attention to mathematical pursuits. An immense number of such investigations is to be found in the different Academical Collections of Europe, the work of various hands, but above all of Euler, whose love for the speculations of the science of quantity, and long life of uninterrupted application to them, led him in succession to almost every possible question of this kind. Even for a great number of years after his death, the transactions of the Imperial Academy of Petersburg were enriched by the materials of this description which he left accumulated for it. Among more recent mathematicians however, less attention has naturally been directed to this kind of researches. They became less attractive as the labours of those who went first gradually exhausted the problems which were obvious without being too difficult; and as the improvement of general methods, and the reduction of difficulties to classes, made the questions less interesting by making them less particular, and by diminishing the uncertainty of success. Perhaps it cannot now be considered as the most profitable or advisable employment of mathematical talent to exercise it upon particular problems, distinguished only by the simplicity of their conditions or the elegance of their results; especially while so many general methods remain to be cultivated and perfected, and so many practical questions to be subjected to calculation. Still, at a certain period of the mathematician's progress, this employment is instructive, and is generally found to be interesting. The object of the present work is therefore to present such a selection of problems as may sufficiently shew the application of the general formulæ, and may at the same time satisfy the curiosity which the student may feel with regard to the labours of those to whom the science owes so much. It seems no more than a proper respect for the Maclaurins, Bernoullis, Eulers, and Simpsons of the last age, to preserve in our treatises some specimens of the questions which occupied so much of their attention, and which probably, if they had not cleared the way for us, might have occupied much of ours at present. If a mathematician should now give up his mind to such subjects, without looking to what has been done by his predecessors, he might easily, in following a favourite path of research, spend too much of his time in inventing, solving and generalizing particular problems. Instead of this, a proper selection, such as is here attempted, may shew him what questions have been undertaken with success, and by sparing him all fruitless endeavours may leave him at liberty to employ his powers on those parts of mathematics where their exertion may really be of service. The investigations relating to the system of the universe, much as has been done, still offer an ample field for the display of the talents, however great, of this and future generations. Even without an extraordinary degree of the inventive faculty, much may be executed. The student who feels a proper admiration for the system of the *Principia*, ought to look forwards to the complete developement of it in the *Mecanique Celeste*, as the ulterior subject of his labours; and those who shall simplify the different parts of that work and reduce them to the level of ordinary readers, as far as they admit of it, will deserve to be considered as real benefactors to the commonwealth of science.

Perhaps it is not too much to hope that the succession of mathematical ability which the system of this University calls out, might, if directed to such objects, enable us to add much of what is most valuable in modern writers to the present subjects of our studies. It is undoubtedly desirable to give a sound knowledge of several branches of mathematical science, and of the Newtonian System, to a greater number of persons than we can expect to form into profound analysts; but if we could save the time which is now often lost in searching and selecting through a variety of books, and in unprofitable and unsystematic reading, we might make the extent of our *encyclopedia* less inconvenient than it is sometimes found to be at present. Instead of balancing the simplicity and evidence of the mathematics of a century ago against the generality and rapidity of modern analysis, it might be better to attempt to combine them; and if our University were provided with a course of elementary works written with this view, and if the higher branches of the science were simplified and made to correspond with these introductory steps, we might include in the circle of our studies a larger portion of the modern additions to mathematical knowledge than is now in most cases practicable.

The present Work is in some measure an attempt to facilitate some of the higher problems of the science, though the mathematician will find in it little except what he will consider as elementary. Its object is the mathematical developement of the doctrine of motion, beginning with the application of the differential calculus; and hence the fundamental laws and principles from which the science is derived are only briefly stated at the outset. Some proofs and illustrations of them are however added in Appendix (A), and it is presumed that the three laws of motion are there given in the form which is most distinct and simple, and which makes them a satisfactory foundation for the principles to be established on them. The common doctrine of projectiles and of bodies falling by gravity is omitted, as naturally belonging to a more elementary treatise. In the third Book, which treats of the motion of a solid body, a principle equivalent to D'Alembert's principle in its simplest form, viz. the equilibrium of impressed and effective forces, is made the foundation of each proposition. The examples given, will, it is hoped, sufficiently familiarize the student with the meaning and application of this important and extensive law. With respect to the evidence which is in general offered for the truth of the principle, it is in fact the extension of a statical law to a system in motion; and an attempt is here made to render this proof more clear and satisfactory.

After mature deliberation, the plan has been adopted of dividing the work into short Propositions, and enunciating each at the beginning of its Article. Though this method interrupts the succession of truths as they grow out of each other in course of our analytical reasoning, and so far may be considered as a blemish by the practised analyst, in an elementary work on such a subject its advantages seem to preponderate, and especially for the purposes of academical instruction. The propositions are a sort of perpetual recapitulation of what has been done; and the reader generally attends to the successive portions of the reasoning more clearly and steadily, from having each of them separated, circumscribed, and its object pointed out beforehand: besides which, if he meet with any difficulties, it is easily seen, in a work of such a form, exactly where and what they are. Thus, though it is not to be imagined that this work has any claim to be called *synthetical* from thus imitating the division of treatises of that description, it may perhaps, while its demonstrations are in general entirely analytical, possess some of the advantages of the synthetical form as a means of conveying information.

Among other matter, the work contains propositions corresponding to nearly all those in the two first books of the *Principia;* and references to these are given where the propositions occur. Partly also with the object of making our arrangement correspond with that of Newton, the motion of bodies in fluids is separated from their free motion, and forms a second book; an order which possesses some advantages, though it is different from that of most analytical treatises. The doctrine of central forces is deduced for all the cases from the same differential equation* which is employed by writers who treat of the perturbations in Physical Astronomy; and which, besides its claim to notice on that account,

* The equation $\frac{d^2u}{dv^2}+u-\frac{P}{h^2u^2}=0.$

gives all the required results with remarkable simplicity and uniformity; applying also immediately to the cases of orbits where the line of apsides is in motion, as in Newton's ninth section; and to the cases of orbits which have an asymptotic circle, first considered by Maclaurin. The problem of three bodies, as treated in Physical Astronomy, could not be introduced without leading to a mass of calculations which would of itself form a treatise; and the reader is therefore referred to works professedly on that subject. The problems of the motions of bodies on surfaces of revolution, of the motions of various systems of connected points, of curves of equal pressure, tractories, synchronous curves, &c. in Chapters V, VI, VII, will be found useful as specimens of the various problems which have been solved, and of the artifices which have been employed.

The third Book, on the motion of any system of rigid bodies, is the one where the work of simplification was at the same time most difficult and most necessary. The treatise of Atwood, unscientific and cumbrous in the highest degree, and in some respects erroneous, was till lately the principal English treatise on this subject. The more analytical modes of investigation presented to many readers great difficulties in this department in consequence of their conciseness and generality, and of the want of examples in which the results of the formulæ might be followed into detail. By breaking up the reasoning into distinct and short propositions, as is here done, it is hoped that the subject is rendered more easily accessible; and the examples in Chap. VII and VIII, besides illustrating the formulæ, are themselves very curious. Among other exemplifications are introduced those which are suggested by Captain Kater's method of determining the length of a seconds' pendulum, (Arts. 91, 92,) and the proof of the property established successively by Euler, Laplace, and Dr. Young, that in this method the time of vibration is independent of the bluntness of the axes, if their radii be the same. (Art. 131, Prob. VII.) The demonstration of the existence of three principal axes, (Art. 112,) differs from the one generally given, and is borrowed from Lagrange: it is recommended, like most of the processes of that great mathematician, by its remarkable analytical symmetry. The Appendix contains several investigations not absolutely essential to the treatise, but worth notice on various accounts, and most of them new to the English reader.

Besides the direct advantages of introducing into the following treatise many problems not usually found in elementary works, this collateral purpose is answered. Several of these investigations shew very remarkably the application and utility of some of those particular cases and branches of analysis, which might otherwise be considered as merely subjects of mathematical curiosity. Thus in finding the time of a body's descent to a centre of force varying inversely as the distance, we have to obtain the *definite integral* of $\epsilon^{-x^2} dx$. (p. 15). In considering the motion of a complex pendulum, (p. 136,) we employ the *simultaneous integration* of *n* differential equations, and are led to Daniel Bernoulli's important principle of *co-existent vibrations.* We are naturally introduced to the *calculus of variations* in solving the different cases of brachystochronous curves, (p. 161,) and in proving the principle of least action. (p. 401). In determining the perturbation of elliptical motion, arising from the resistance of a medium, (p. 201,) we use the method of the *variation of parameters.* In investigating the motion of a body about two centres of force, (p. 353,) we have occasion to refer to the criterion of the difference between *particular solutions* and *particular integrals.* In finding the attractions of spheres by Laplace's method, (p. 365,) we have to *differentiate under the sign of integration;* and in solving the problems of the vibrations of strings, and of elastic rods, (p. 378, 386,) we have to use and integrate *partial differential equations.* The student will be induced to consider the different branches of analysis more closely, when he thus sees instances of their use and necessity, and he may learn at the same time, that no portion of pure science is to be rejected as barren and useless; it is impossible to say what value and what results the extension of applied mathematics may sometime give to it.

From the nature of the work, it must be in a great measure borrowed from many preceding writers; and it is neither very easy nor very necessary to point out all who have been of service. Besides Euler, however, who is a never-failing guide and assistant, I may mention my obligations to the excellent Treatise on Mechanics of Mr. Poisson. A considerable portion of the Transactions of learned Societies has been examined, but without any pretensions to the merit of having made a complete abstract of the problems on this subject contained in the different Academical Collections.

In going through the detail of so many cases, properties and methods have occurred, as was to be expected, which so far as the author is aware, have been hitherto unnoticed by mathematicians. But in such cases there is always a strong probability that a more extended examination of what has been previously written, would overthrow most of the claims to originality which, in this late period of the history of science, can be advanced. In the present instance, the author would willingly abandon all such pretensions for the praise of having written a useful and perspicuous treatise on the subject on which he has laboured.

The Syndics of the University Press have, from the funds at their disposal, contributed liberally to the expense of the work; and the author gladly takes this opportunity of acknowledging his obligations to them.

CONTENTS.

Book II.

Book III.

ERRATA.

Page 48. l. 18 in denominator, *for* cot. β *read* 2 cos. β.

125. l. 6 from bottom, *for* pg read pq.

130. l. 7 *for* $-$ *read* $+$.

— l. 9 *for* greatest *read* least.

— l. 14 *for* least *read* greatest.

132. l. 3 from bottom, *for* ^1E sin. $^1\epsilon$ *read* $-\,^1E$ sin. $^1\epsilon$, and *for* ^2E sin.$^2\epsilon$ *read* $-\,^2E$ sin.$^2\epsilon$.

227. l. 1 in numerator, *for* $M.Cm$ *read* $M.Cn$.

— l. 9 from bottom, the $+$ should be in the denominator.

229. l. 11 *for* into its *read* into the square of its.

279. l. 10 *for* U cos. ψ *read* U cos. ϕ.

303. l. last but one, *for* $\int xy\,dm$ *read* $\Sigma\,xym$.

314. l. 5 of note, *for* Multiplying *read* Multiply.

321. l. 8 *dele* be.

— l. 3 from bottom, *for* oscillate *read* revolve.

330. ll. 5 and 8 *for* $\sqrt{\frac{g}{l}}$ *read* $\sqrt{\frac{l}{g}}$.

347. l. 12 *for* in the next *read* in the following pages.

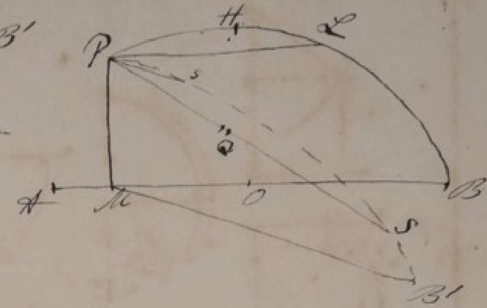
377. l. 9 in denominator, *for* m^2 *read* m.

387. l. 5 from bottom, *for* four *read* two.

393. l. 12 at the beginning, *insert* or.

To find the Att^{n} on any pt in surf. of a Spheroid.

Let P be the pt A the pole AB the axis major of meridn PAB
take PSB' any other Sn thro PM cutting — thro in MB'
put $PQ = \rho$ $PS = r$ $\angle BMB' = \phi$ inclin^{n} of
PS to MB' $= \theta$ & suppose PSB' to turn
thro a small $\angle\, d\phi$ $PM = \beta$ $OM = \alpha$



∴ Elem^{t} of solid at $Q = k.\rho\, d\theta . d\rho . \rho \cos\theta . d\phi$

∴ its att^{n} ∥ $MB' = k. \cos^2\theta . d\theta\, d\rho\, d\phi$

∴ att^{n} of PS & Ps ∥ to $MB' = k\iint \overline{r+r'}. \cos^2\theta\, d\theta\, d\phi$

∴ D^{o} ∥ $AB = k. \overline{r+r'}. \cos^2\theta \cos\phi\, d\theta\, d\phi$

∴ att^{n} of entire wedge $PSS' = k \cos\phi\, d\phi \int \overline{r+r'}. \cos^2\theta . d\theta$ between Ls

But $r+r'$ depends $r+r'$ depends only on α θ & m where $m = \frac{b^2}{a^2}$

∴ Att^{n} of entire wedge depends only on α ϕ & m and is ∴ the same as the att^{n} at the pt M in the surface of a sim^{r} concentric spheroid whose $\frac{1}{2}$ axis maj. is OM

∴ Entire att^{n} of on P ∥ AB is same as that on M in latter spheroid

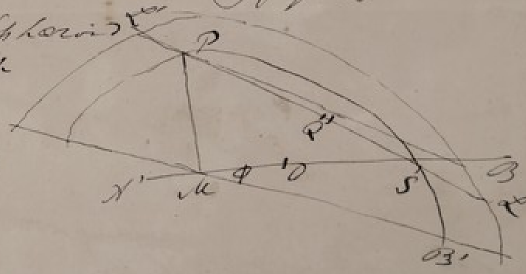
and is ∴ $= \frac{2\pi k\alpha\sqrt{1-e^2}}{e^2}\left\{\frac{\sin^{-1}e}{e} - \sqrt{1-e^2}\right\}$ by page (2)

Sim^{ly} att^{n} on P ∥ axis minor $= \frac{4\pi k\beta}{e^2}\left\{1 - \frac{\sqrt{1-e^2}}{e}\sin^{-1}e\right\}$ by page (1)

= To find the Att^{n} on any pt within a spheroid

Thro P the given pt take as a before a Sn PSB' and produce to cut exterior surf

Take in this a narrow pyramid LPL vertex P the segts of the PL PL are $= R$ & R'



as before Elem^{t} $Q = k\rho\, d\rho\, d\theta . \rho\cos\theta . d\phi$

∴ its att^{n} $= k\, d\rho\, d\theta \cos\theta\, d\phi$ (from $\rho = R'$ to R)

∴ Entire att^{n} of pyramid $= k . \overline{R - R'} . \cos\theta\, d\theta\, d\phi$

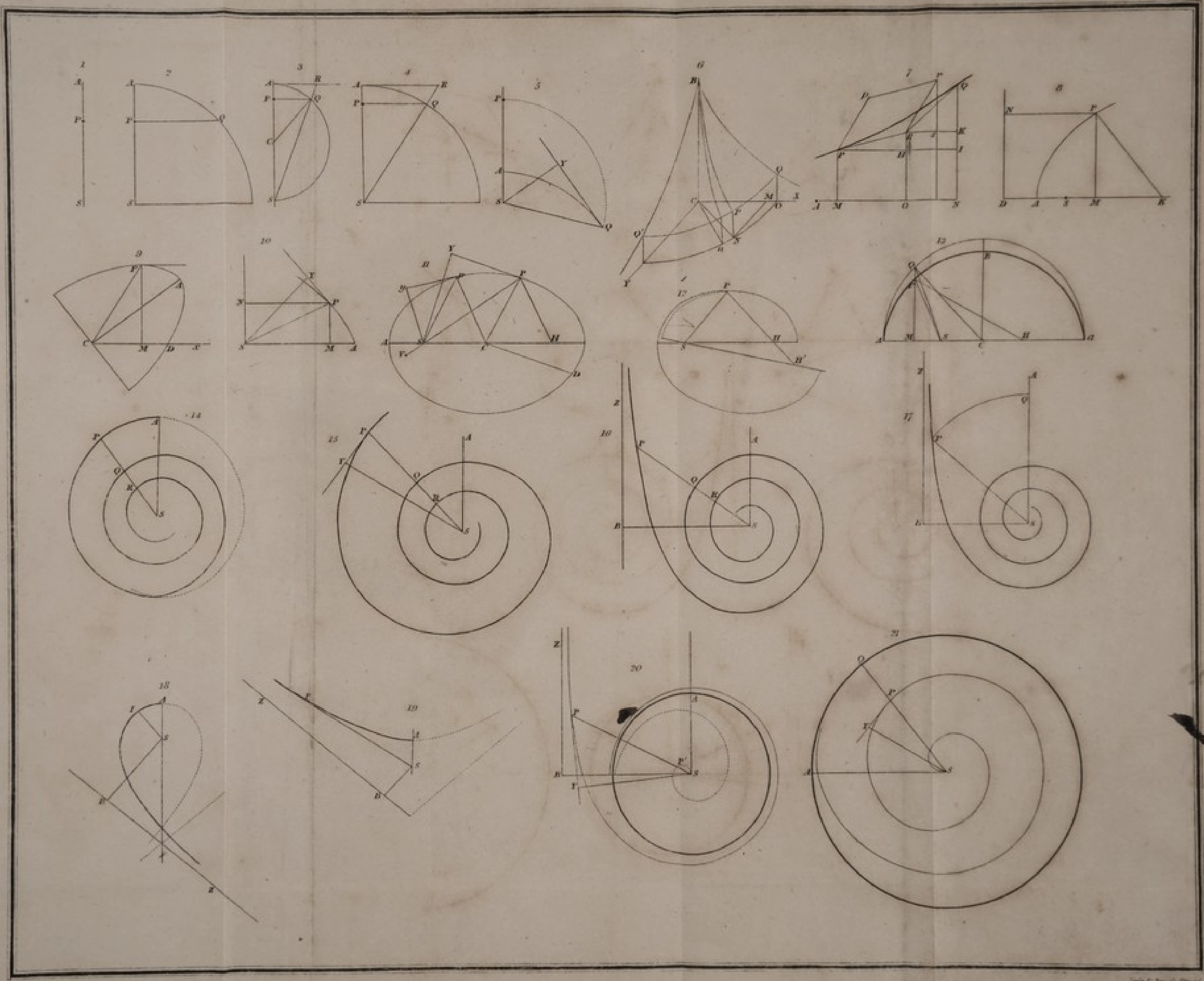
$= k . PS . \cos\theta\, d\theta\, d\phi$ = Att^{n} of Pyramid PS

∴ the only effective att^{n} is that of the interior concentric sim^{r} spheroid thro P

put $OM = \alpha$ $MP = \beta$ ∴ att^{n} on P ⊥^{r} axis of rotation $= \frac{2\pi k\alpha\sqrt{1-e^2}}{e^2}\left\{\frac{\sin^{-1}e}{e} - \sqrt{1-e^2}\right\}$

& D^{o} ⊥^{r} equator $= 4\pi k\beta\left\{1 - \frac{\sqrt{1-e^2}}{e}\sin^{-1}e\right\}$

PLATE I.



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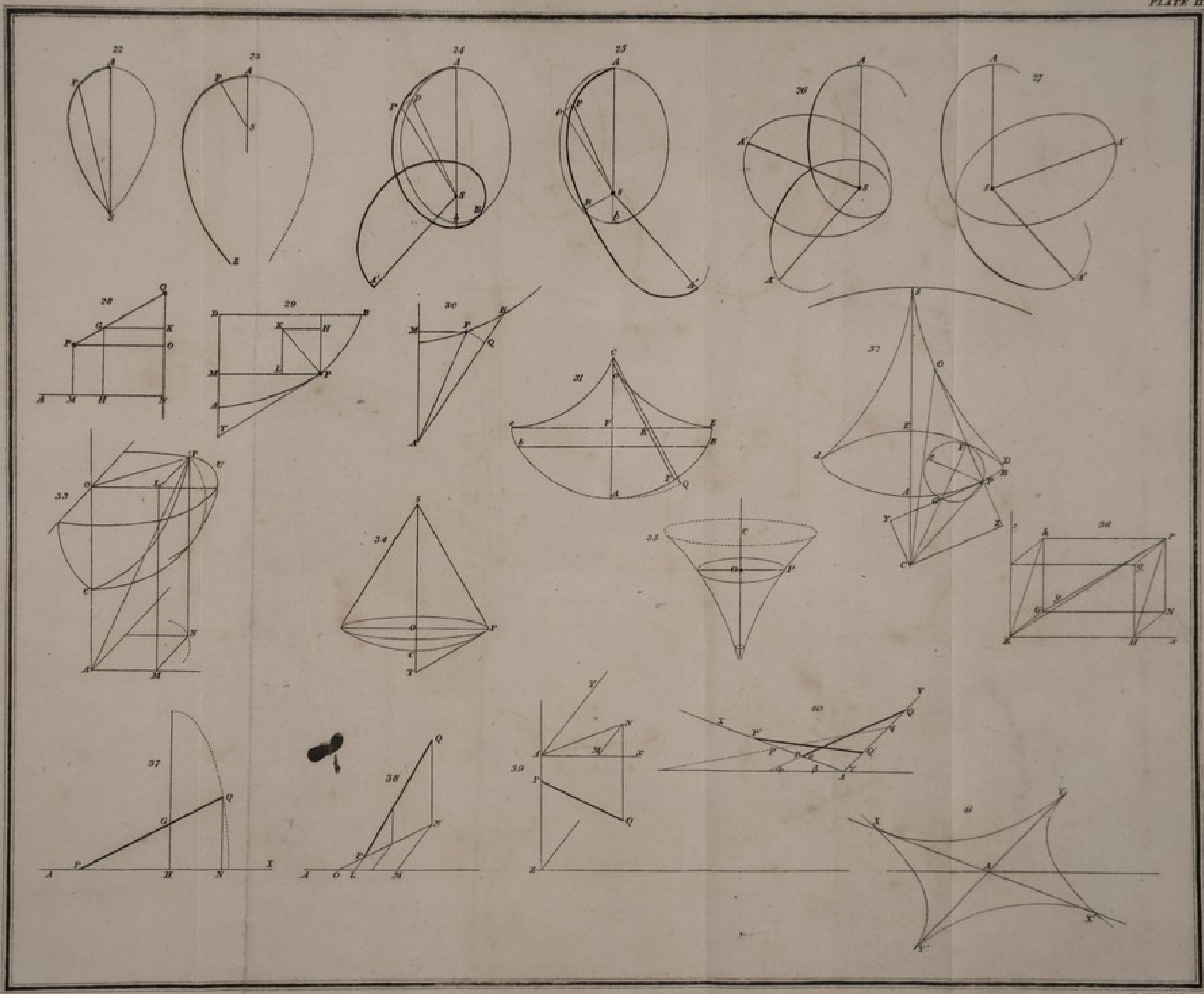

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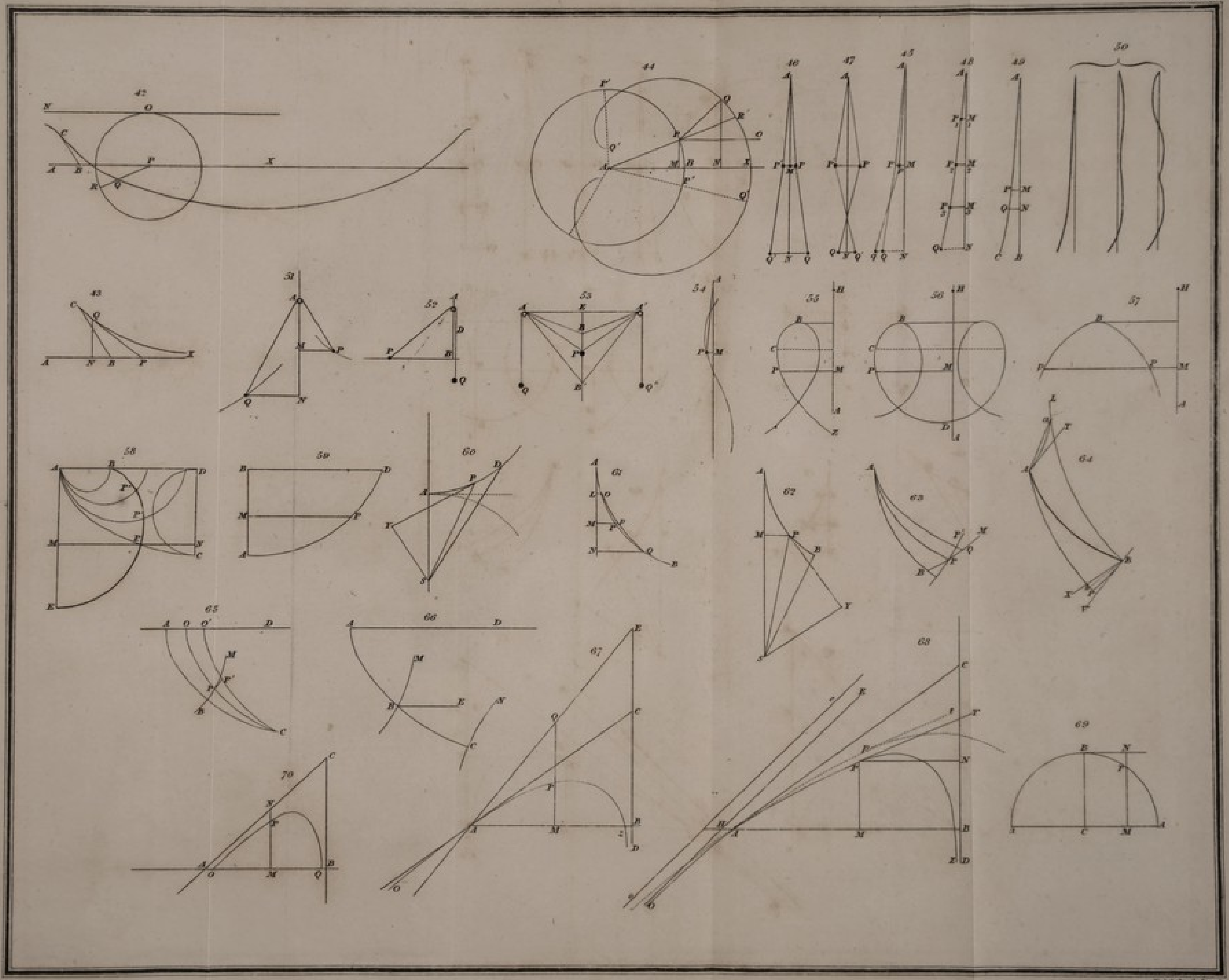

42
43
44
45
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47
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49
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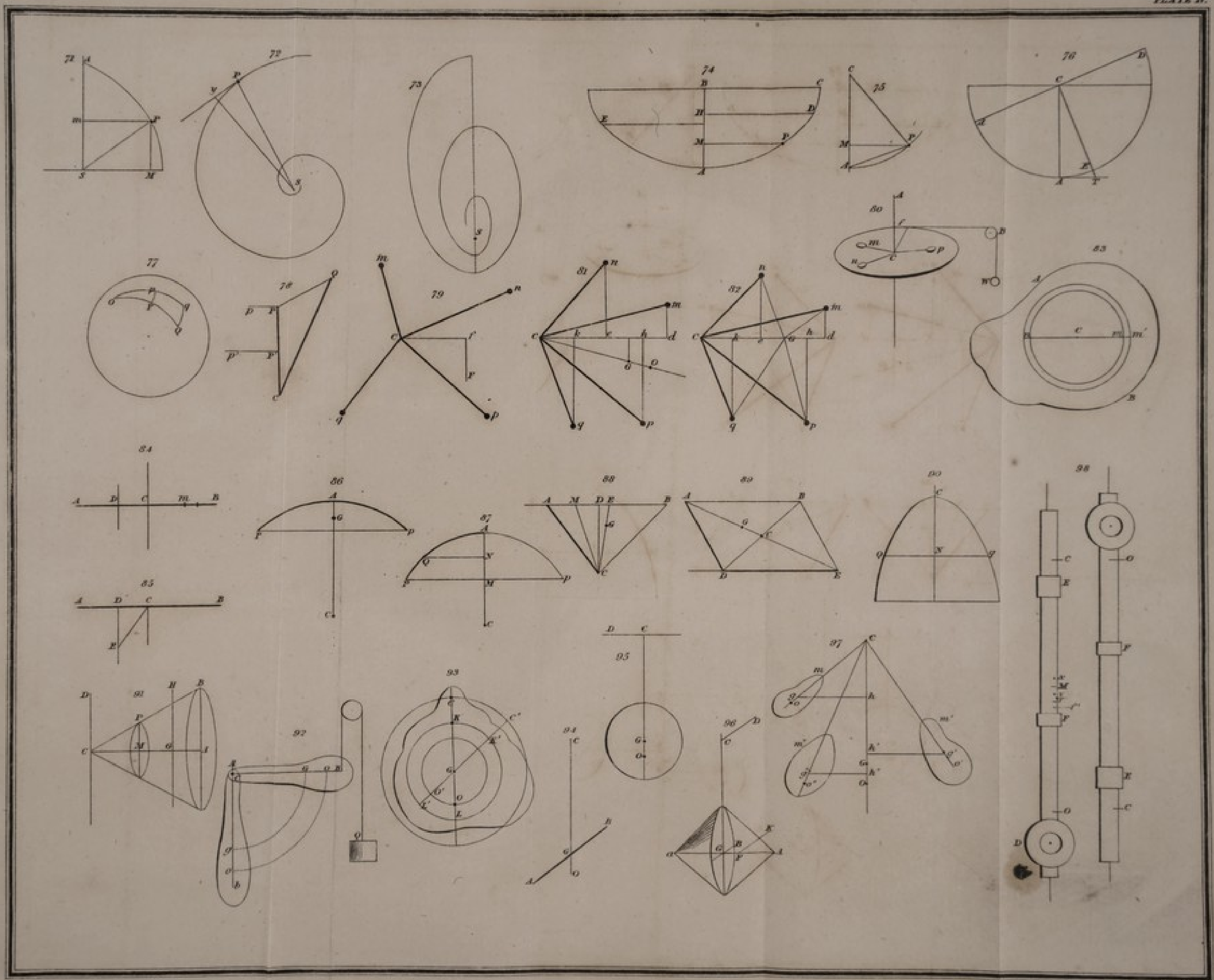
68
69
70

PLATE IV.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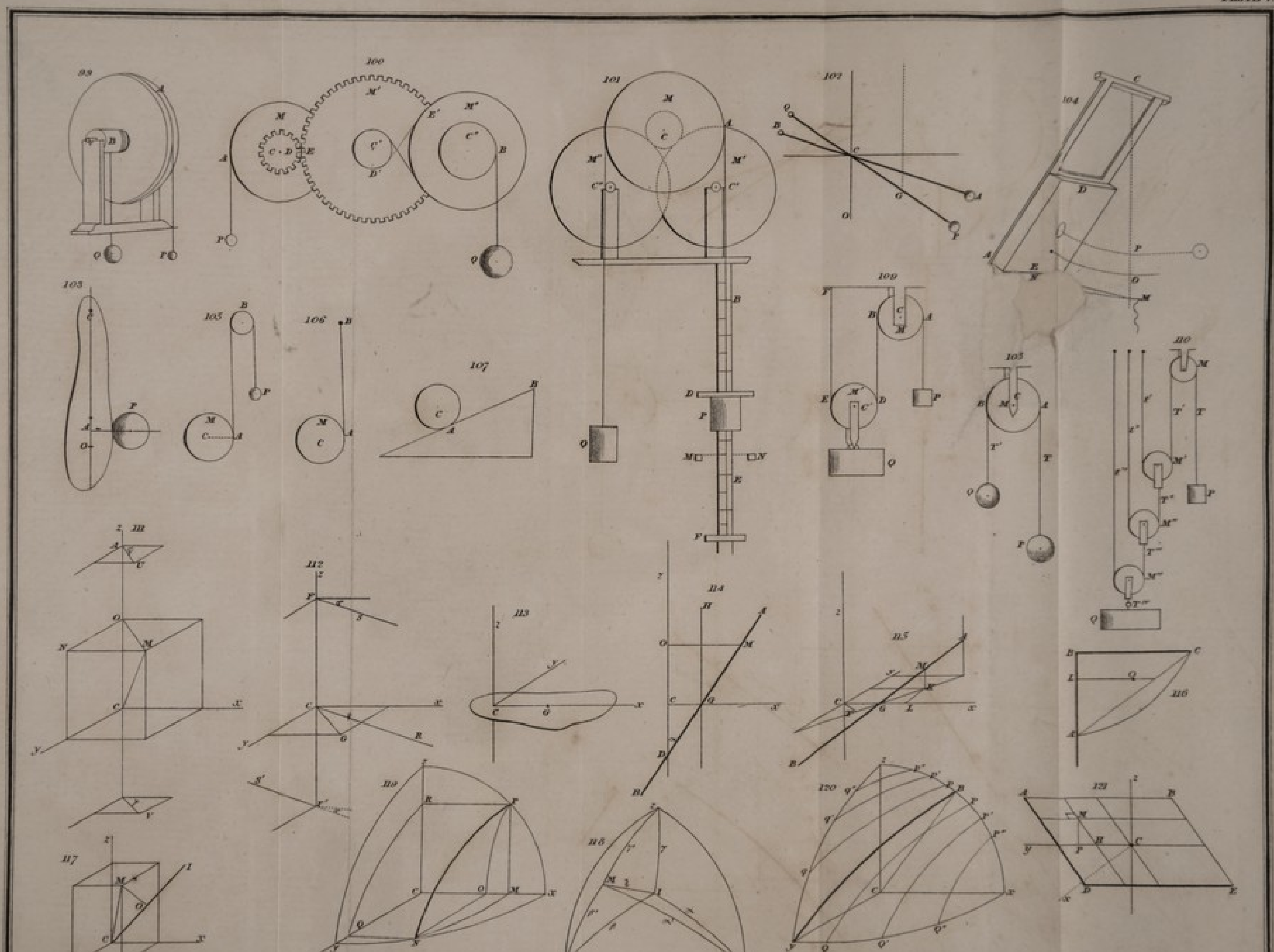
98


99
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107
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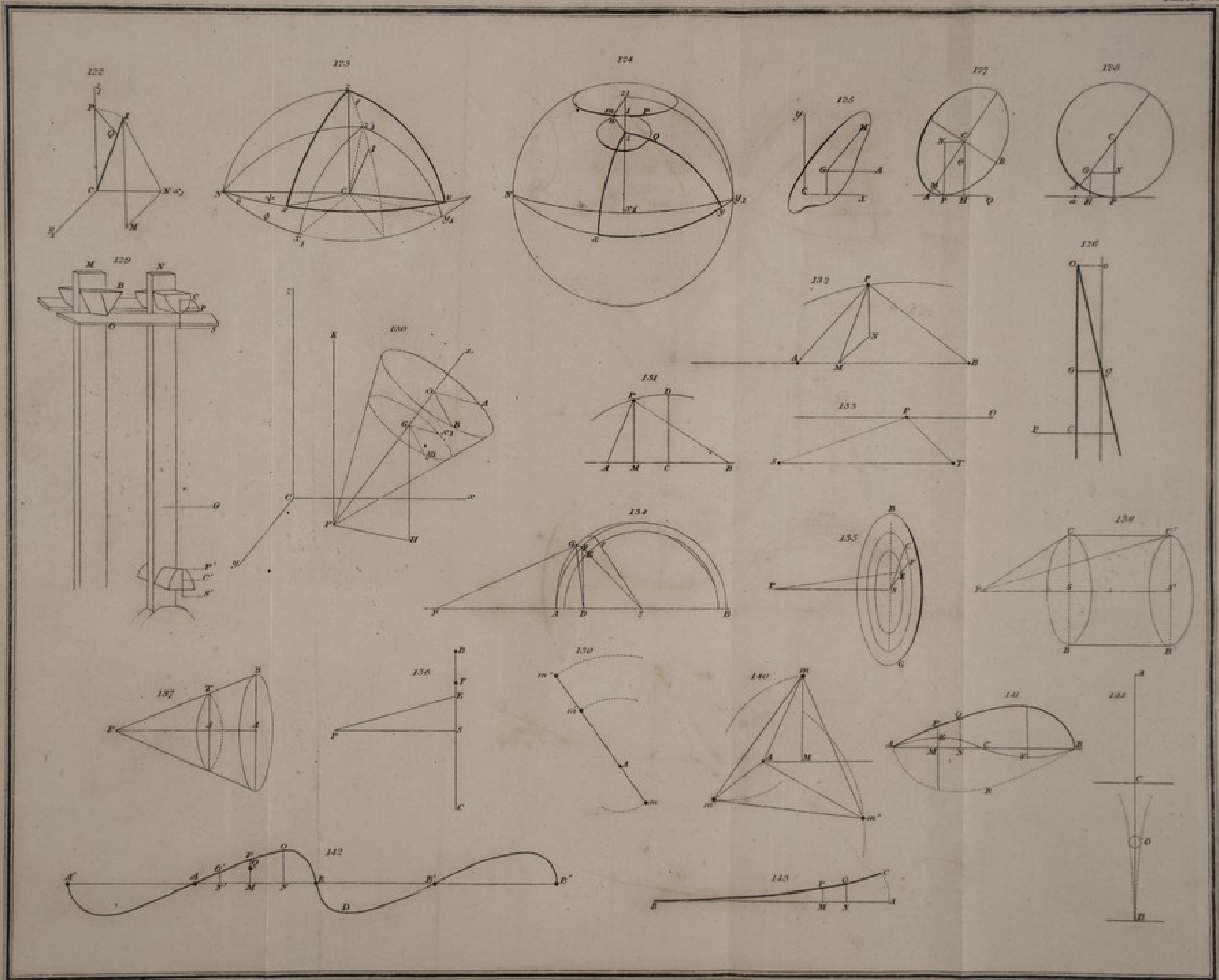
119
120
121



122
123
124
125
127
128
129
130
126
132
131
133
134
135
136
137
138
139
140
141
144
142
143

A

TREATISE

ON

DYNAMICS.

DIVISION OF THE WORK.

1. THE object of the present Volume is to determine, for any body or system of bodies, the motion which corresponds to any forces; and conversely, to determine the forces which correspond to any motion. That is, we have to investigate the relation of the time, space, velocity, and force, when bodies are in motion under given circumstances.

The subdivision of the Work will be as follows:

2. It will consist of THREE BOOKS; the two first treating of the motion of a *point*, and the third, of that of a *body* of finite magnitude. Thus, the former books apply to the motion of a particle of matter which is conceived to be indefinitely small, so that the path of every part of it may be considered as the same line: they include also the case of a finite body which moves so that all the points of it describe lines parallel and equal to each other. In these cases there is only a progressive motion, or *motion of translation*. The third book extends our reasonings to masses of determinate magnitude and figure, moving into various positions while their parts still remain at the same distance from each other in consequence of the rigidity of the bodies. In this case we may have, besides a motion of progression, *a motion of rotation* about an axis either fixed or variable; and we thus comprehend any motion which can exist in a body.

3. The first and second Books consider the motion of a point as it moves in a *non-resisting space*, or in a *resisting medium.* In the FIRST BOOK we conceive a body to move freely: but if, instead of this, we suppose it to be retarded by moving through a medium, as air or water, which does not allow it to pass without affecting its motion; the mathematical difficulties of the question will be somewhat modified: and this case will form the subject of the SECOND BOOK. In the THIRD BOOK, which treats of the motion of finite bodies, we shall not consider them as moving in a resisting medium.

4. Also, a point or body may either move freely in space, or may be constrained to move upon a given line or surface: and these different conditions will give rise to various problems relating to motions both in resisting and in non-resisting media.

Again, instead of considering a single point only, we may suppose that several points, connected either by invariable straight lines, or by laws of mutual attraction and repulsion, influence each other's motions: and in the first Book we shall, after investigating the motion of one point, proceed to the case of several.

The problems which we shall have to solve may differ also by the different suppositions which we make with respect to the force. They will vary, accordingly as we suppose the force to act in parallel lines, or in lines tending to a centre, &c. and likewise according to the power or other function of the distance from the centre, &c. which we suppose to express the force. We may also consider the motions of bodies when attracted to two or more centres of force.

The centres of force which we assume, are points from which attraction or repulsion emanates in every direction: that is, to or from which bodies tend to every side. The intensity of the force is supposed to be the same at the same distance from the centre, in every direction, and to vary according to some function of the distance. There are not, strictly speaking, such fixed centres of force actually existing in nature; because, though attractions and repulsions do appear, they take place to and from bodies, and not mathematical points; and these attractive and repulsive bodies are themselves attracted and repelled, and generally are themselves moving while they influence other motions. Still the introduction of such imaginary centres of force, besides being easily and distinctly conceived, and containing the most natural mathematical simplification of the conditions which actually exist, does, in most cases, offer an approximate solution of the problem in nature, and leads to others yet more exact.

The law of attraction which appears to prevail in nature, or rather the universal law of one of the kinds of attraction which particles of matter exert, is that the force increases in the proportion in which the square of the distance decreases. All the attempts to prove this general fact to be a necessary truth, seem to be completely unsatisfactory. At all events innumerable other laws are mathematically possible, and will be supposed in the problems which will occur; and more especially those laws in which the force varies according to some other *power* of the distance, direct or inverse.

DEFINITIONS AND PRINCIPLES.

5. *Velocity* is the degree in which a body moves quickly or slowly. When constant, it is *measured* by the space described in a unit of time. This is the same as measuring it by the ratio of the space to the time. When variable, it is measured by the *limit* of this ratio.

Force is that which produces or tends to produce motion.

Accelerating Force is force considered with respect only to the velocity produced, without regard to the magnitude of the body moved. It is measured, when constant, by the velocity generated in a unit of time; or by the ratio of the velocity generated, to the time. When variable, it is measured by the limit of this ratio.

If t be the time, s the space described, v the velocity, f the accelerating force acting in the direction of the motion,

$$v = \frac{ds}{dt}, f = \frac{dv}{dt} \ldots\ldots\ldots\ldots(a).$$

Moving Force is the product of the accelerating force into the quantity of matter moved.

Momentum is the product of the velocity into the quantity of matter of the moving body. Hence, moving force is, when constant, proportional to the momentum generated in a given time, as accelerating force is to the velocity generated in a given time.

6. The following are *the Laws of Motion.*

LAW 1. A body in motion, not acted upon by any force, will move on in a straight line and with a uniform velocity.

LAW 2. When any force acts upon a body in motion, the change of motion which it produces is the same, in magnitude and direction, as the effect of the force upon a body at rest.

LAW 3. When pressure communicates motion, the moving force is as the pressure.

Hence, the accelerating force is as the pressure directly and as the quantity of matter moved inversely.

The *Inertia* of a body is its resistance to the communication of motion; and since the velocity communicated by a given pressure is inversely as the quantity of matter; the inertia is directly as the quantity of matter.

When bodies in motion press each other, *Reaction* is equal and opposite to *Action;* that is, the pressures on each other are equal and in opposite directions.

Hence, the moving forces are equal, and the momenta communicated in opposite directions also equal.

Impact is a very short and violent pressure. And hence it appears, that in impact the momenta gained and lost are equal.

When a force acts to turn a body round an axis, its *Moment* is the product of the force into the perpendicular upon it from the axis.

When the moments of forces are equal, their effects are equal.

BOOK I.

THE MOTION OF A POINT IN A NON-RESISTING SPACE.

7. THE laws and equations of motion are here immediately applicable, putting for the force f, the value of the extraneous accelerating forces which act upon the point in motion. We shall first consider the motion in a straight line, and afterwards in a curve. For the first case we shall require no principles, except the nature of our ideas of force and velocity*: for the second, it is necessary to introduce also the second law of motion.

CHAP. I.

THE RECTILINEAR MOTION OF A POINT.

8. WHEN a point moves in a straight line, this will be the line in which the force acts, and we can immediately apply the equations,

$$v = \frac{ds}{dt}, \quad f = \frac{dv}{dt} \ldots\ldots\ldots\ldots (a).$$

Where t, s, v, are the time of motion, space described, and velocity of a body, which is acted on by an accelerating force f in the direction of its motion.

* It is here assumed that the forces and attractions in this case are independent of the mass of the body acted on.

These equations would enable us to obtain finite relations among the quantities in question, in several cases. For instance, if the velocity were given in terms of the time, we could find the space described by integrating the first equation, and if the force were known in terms of the time, we might in the same manner obtain the velocity from the second. In general, however, the force depends upon the position of a body, and in this case, therefore, is a function of s.

9. PROPOSITION. When the force is a function of the space, to find the velocity and time.

Take the equations

$$v = \frac{ds}{dt}, \quad f = \frac{dv}{dt};$$

multiply them crossways to eliminate dt, and we have

$$v \cdot \frac{dv}{dt} = f \cdot \frac{ds}{dt}, \text{ or } v\,dv = f\,ds \ldots\ldots\ldots (b).$$

If we put for f its value in terms of s we can integrate equation (b), and shall thus obtain $\frac{1}{2}v^2 = \int f\,ds$: whence v is known.

After this, t is determined by the equation

$$v = \frac{ds}{dt}, \text{ or } dt = \frac{ds}{v};$$

where a value of v being obtained in terms of s, we can integrate, and find $t = \int \frac{ds}{v}$, and thus the relation of the quantities in question is completely determined.

10. We may now proceed to the calculation for different suppositions of force.

One of the most common and useful suppositions is, that the force tends to a certain point or centre, and varies according to some direct or inverse power of the distance from this centre. Let S, fig. 1, be the centre of force; $SP = x$; and when a point is at P let it be urged towards S by a force $\frac{m}{x^n}$, or mx^n, m being a constant quantity. The quantity m depends upon the attractive power residing in S, and is called the *absolute force*. It is measured by the accelerating force at distance 1, for making $x = 1$, we have the force $= m$.

It is supposed that $\frac{m}{x^n}$ or $m x^n$ expresses the accelerating force on a particle P, whatever be the magnitude of the particle: the moving force or pressure produced by the attraction of S is greater as the mass acted on is greater.

PROBLEM I. *A body P falls from rest from a given point A, fig. 1, towards a centre of force S, varying as some power of the distance SP: to determine the motion**.

Let $SA = a$, $AP = s$; $\therefore x = a - s$; and take the force to be as some *inverse* power of the distance and $= \frac{m}{x^n}$.

1st, To find the velocity

$$v\,dv = f\,ds = \frac{m}{x^n} \times - dx;$$

$$\therefore 2v\,dv = -\frac{2m\,dx}{x^n},\ v^2 = \frac{2m}{(n-1)\,x^{n-1}} + C,$$

C being an arbitrary constant quantity, to be determined.

When $x = a$, $v = 0$; $\therefore v^2 = \frac{2m}{n-1}\left\{\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right\}$

$$v = \frac{(2m)^{\frac{1}{2}}}{(n-1)^{\frac{1}{2}}}\left(\frac{a^{n-1} - x^{n-1}}{a^{n-1}\,x^{n-1}}\right)^{\frac{1}{2}}$$

This gives the velocity when $n > 1$.

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* This is Prop. 39, Book I, of the *Principia*.

'To determine the motion,' "definire motum," implies the problems of obtaining the relation of the space, velocity, and time in finite terms, that is, freed from differentials.

It will be recollected that the bodies spoken of in this PART are always to be considered as physical points.

If $n = 1$ this integration fails, and recurring to the differential expression, we have $v^2 = -2m$ hyp. log. $x + C = 2m$ hyp. log. $\frac{a}{x}$.

If $n < 1$, $v^2 = \frac{2m}{1-n}(a^{1-n} - x^{1-n})$.

If the force vary as some *direct* power of the distance, let $f = m x^n$, and we have

$$v^2 = \frac{2m}{n+1}(a^{n+1} - x^{n+1}).$$

In cases where the force varies as some inverse power n, greater than 1, when $x = 0$, $v = inf.$ or the velocity of falling to the centre is infinite.

In the same case; when $a = inf.$ v remains finite, and

$$v^2 = \frac{2m}{(n-1)\, x^{n-1}};$$

or the velocity of falling from an infinite distance to the distance x is finite.

If the force vary inversely as the distance, both these velocities are infinite.

In all other cases, the velocity from an infinite distance to a finite one, is infinite; and the velocity from a finite distance to the centre, is finite.

If the body, instead of falling *from rest* at the distance a, be *projected* upwards or downwards with a velocity V, we have, when $x = a$, $v = V$, if the body be projected in the direction of the force, and $v = -V$, if it be projected in the opposite direction.

In this case, $v^2 = V^2 + \frac{2m}{n-1}\left(\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right)$.

2d, To find the time;

$$dt = \frac{ds}{v} = \frac{-dx}{\frac{(2m)^{\frac{1}{2}}}{(n-1)^{\frac{1}{2}}} \cdot \left(\frac{a^{n-1} - x^{n-1}}{a^{n-1}\, x^{n-1}}\right)^{\frac{1}{2}}}$$

$$= -\frac{(n-1)^{\frac{1}{2}}\cdot a^{\frac{n-1}{2}}}{(2m)^{\frac{1}{2}}}\;\frac{x^{\frac{n-1}{2}}\,dx}{(a^{n-1}-x^{n-1})^{\frac{1}{2}}},$$

which can be integrated only in particular cases: see *Lacroix*, Elem. Treat. Art. 169.

1st, We can integrate if $\frac{m'}{n'}$ be a whole number where $m'-1 = \frac{n-1}{2}$ and $n' = n-1$; that is, calling r a whole number, if

$$\frac{\frac{n-1}{2}+1}{n-1} = r, \text{ or } n+1 = 2rn - 2r; \;\therefore\; n = \frac{2r+1}{2r-1};$$

this comprehends the cases $n = -1,\ \frac{1}{3},\ \frac{3}{5},\ \frac{5}{7}$, &c.

also $n = \frac{3}{1},\ \frac{5}{3},\ \frac{7}{5}$, &c.

2d, We can integrate if $\frac{m'}{n'} - \frac{1}{2}$ be a whole number; suppose

$$\frac{n+1}{2(n-1)} - \frac{1}{2} = r; \;\therefore\; \frac{1}{n-1} = r; \;\therefore\; n = \frac{r+1}{r},$$

this comprehends the cases $n = 2,\ \frac{3}{2},\ \frac{4}{3},\ \frac{5}{4}$, &c.

also $n = 0,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5}$, &c.

Hence the only laws of force expressed by integral powers, for which we can find the time, are

force $\propto$ const., force $\propto$ dist., force $\propto \frac{1}{(\text{dist.})^2}$, force $\propto \frac{1}{(\text{dist.})^3}$.

The most simple fractional powers are

force $\propto \frac{1}{(\text{dist.})^{\frac{1}{2}}}$, force $\propto \frac{1}{(\text{dist.})^{\frac{3}{2}}}$, force $\propto \frac{1}{(\text{dist.})^{\frac{4}{3}}}$.

If the force be repulsive, the process of finding the velocity and time will be the same as above, except that the signs will be different.

In that case if force $= \frac{m}{x^n}$, $s = x - a$;

$$v\,dv = \frac{m\,dx}{x^n}, \quad dt = \frac{dx}{v}.$$

In many of the integrable cases, it is better to employ particular methods, than the general substitution for making the differential expressions rational.

Ex. 1. The force varies directly as the distance:

$$f = mx\,; \ \therefore\ v\,dv = mx\,ds = -mx\,dx,$$

$$v^2 = C - mx^2 = m(a^2 - x^2),$$

$$v = m^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{2}};$$

$$dt = \frac{ds}{v} = -\frac{dx}{m^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{2}}},$$

$$t = \frac{1}{m^{\frac{1}{2}}}.\text{arc}\left(\cos. = \frac{x}{a}\right) + C;$$

and if t begin when $x = a$,

$$t = \frac{1}{m^{\frac{1}{2}}}.\,\text{arc}\left(\cos. = \frac{x}{a}\right).$$

Cor. If with radius $SA = a$, fig. 2, a quadrant be described, and PQ be perpendicular to SA; and if $SP = x$:

$$AQ = \text{arc}\,(\cos. = x,\ \text{rad.} = a) = a\ \text{arc}\left(\cos. = \frac{x}{a},\ \text{rad.} = 1\right);$$

$$\therefore \text{ velocity at } P = m^{\frac{1}{2}}.PQ,$$

$$\text{time in } AP = \frac{\text{arc } AQ}{m^{\frac{1}{2}}\,a} = \frac{\text{arc } AQ}{\text{vel. at } S}*.$$

Ex. 2. The force is constant:

* *Principia*, Book I. Prop. 38.

Hence, time in $AP = \left(\frac{a}{2m}\right)^{\frac{1}{2}} (PQ + \text{arc } AQS - \text{arc } SQ)$

$$= \left(\frac{a}{2m}\right)^{\frac{1}{2}} (PQ + \text{arc } AQ).$$

Hence, also time $= \left(\frac{a}{2m}\right)^{\frac{1}{2}} \left(\frac{PQ . \frac{1}{2} SC + \text{arc } AQ . \frac{1}{2} AC}{\frac{1}{2} AC}\right)$

$$= \frac{2\sqrt{2}}{(am)^{\frac{1}{2}}} \text{ area } ASQ^*.$$

If SQ be produced to meet in R a tangent to the semi-circle at A;

$$AR = SA . \frac{PQ}{PS} = \frac{a\sqrt{(ax - x^2)}}{x} = a^{\frac{3}{2}} \left(\frac{1}{x} - \frac{1}{a}\right)^{\frac{1}{2}};$$

$$\therefore v = \frac{(2m)^{\frac{1}{2}}}{a^{\frac{3}{2}}} . AR.$$

Ex. 4. The force varies inversely as the cube of the distance;

$$v = \sqrt{m} . \frac{\sqrt{(a^2 - x^2)}}{ax}; \quad t = a . \frac{\sqrt{(a^2 - x^2)}}{\sqrt{m}}.$$

Cor. If with centre S and radius SA, fig. 4, we describe a circle, and make PQ, AR perpendicular to SA, and draw SQR;

$$v = \frac{\sqrt{m}}{a^2} . AR; \quad t = \frac{a}{\sqrt{m}} . PQ.$$

Ex. 5. The force varies inversely as the square of the distance, and is repulsive:

$$v = \sqrt{(2m)} \left\{\frac{1}{a} - \frac{1}{x}\right\}^{\frac{1}{2}}: \; dt = \left(\frac{a}{2m}\right)^{\frac{1}{2}} . \frac{x\,dx}{\sqrt{(x^2 - ax)}}.$$

Cor. If with focus S, fig. 5, and vertex A, the point from which the body begins to move, we describe a parabola, and take $SQ = SP$; SY being the perpendicular upon the tangent QY, we have

$$d . \text{arc } AQ = \frac{SQ}{QY} . d . SQ = \frac{SQ . d . SQ}{\sqrt{(SQ^2 - SY^2)}}$$

$$= \frac{SQ . d . SQ}{\sqrt{(SQ^2 - SQ . SA)}} = \frac{x\,dx}{\sqrt{(x^2 - ax)}};$$

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* *Principia*, Book I. Prop. 32.

$v\,dv = f\,ds$; $\therefore v^2 = 2fs$; the motion beginning when $s = 0$.

$$dt = \frac{ds}{\sqrt{(2fs)}}; \quad \therefore t = \sqrt{\frac{2s}{f}}; \ t \text{ being } 0 \text{ when } s \text{ is } 0.$$

If the constant force be gravity, represented by g,

$$v^2 = 2gs, \text{ and } t = \sqrt{\frac{2s}{g}}, \text{ or } s = \tfrac{1}{2}gt^2.$$

Ex. 3. The force varies inversely as the square of the distance,

$$f = \frac{m}{x^2}, \ v\,dv = -\frac{m\,dx}{x^2}, \ v^2 = 2m\left(\frac{1}{x} - \frac{1}{a}\right);$$

$$dt = -\frac{a^{\frac{1}{2}}x^{\frac{1}{2}}\,dx}{(2m)^{\frac{1}{2}}(a-x)^{\frac{1}{2}}} = -\left(\frac{a}{2m}\right)^{\frac{1}{2}}\frac{x\,dx}{(ax-x^2)^{\frac{1}{2}}}.$$

$$t = \left(\frac{a}{2m}\right)^{\frac{1}{2}}\left\{(ax-x^2)^{\frac{1}{2}} - \frac{a}{2}\text{ arc }\left(\text{ver. sin.} = \frac{2x}{a}\right)\right\} + C;$$

and, t being supposed to begin when $x = a$, since ver. sin. $\pi = 2$,

$$t = \left(\frac{a}{2m}\right)^{\frac{1}{2}}\left\{(ax-x^2)^{\frac{1}{2}} + \frac{a}{2}\left[\pi - \text{arc }\left(\text{ver. sin.} = \frac{2x}{a}\right)\right]\right\}^*.$$

Cor. On $AS = a$, fig. 3, let a semi-circle be described, with centre C; and let PQ be drawn perpendicular to AS meeting it,

$$PQ = \sqrt{(SP \,.\, PA)} = \sqrt{(ax - x^2)}. \quad AQS = \frac{\pi a}{2}.$$

$$\text{arc } SQ = SC \times \text{ang. } SCQ = SC \text{ ang. }\left(\text{ver. sin.} = \frac{SP}{SC}\right)$$

$$= \frac{a}{2}\text{ arc }\left(\text{ver. sin.} = \frac{x}{\frac{1}{2}a}\right).$$

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* When the force is as the distance, we have for the whole time of falling to the centre, making $x = 0$; $t = \dfrac{\pi}{2\,m^{\frac{1}{2}}}$.

When the force is inversely as the square of the distance we have for the whole time of falling to the centre, making $x = 0$, $t = \dfrac{\pi a^{\frac{3}{2}}}{2\,(2m)^{\frac{1}{2}}}$.

$$\therefore\ dt = \left(\frac{a}{2m}\right)^{\frac{1}{2}} d.\,\text{arc } AQ, \text{ and } t = \left(\frac{a}{2m}\right)^{\frac{1}{2}}.\,AQ.$$

Ex. 6. The force varies inversely as the square root of the distance:

$$v = 2m^{\frac{1}{2}}.\,\{a^{\frac{1}{2}} - x^{\frac{1}{2}}\}^{\frac{1}{2}};$$

$$t = \frac{2}{3m^{\frac{1}{2}}}.\,\{x^{\frac{1}{2}} + 2a^{\frac{1}{2}}\}.\,\{a^{\frac{1}{2}} - x^{\frac{1}{2}}\}^{\frac{1}{2}}.$$

Prob. II. *A body acted upon by a force varying as any power of the distance, falls to the centre from a given distance (a): to find the whole time of falling to the centre.*

By Prob. I, we have, if force $= \dfrac{m}{x^n}$,

$$dt = -\frac{(n-1)^{\frac{1}{2}}.\,a^{\frac{n-1}{2}}}{(2m)^{\frac{1}{2}}}.\,\frac{x^{\frac{n-1}{2}}\,dx}{\{a^{n-1} - x^{n-1}\}^{\frac{1}{2}}},$$

$$= -\frac{(n-1)^{\frac{1}{2}}\,x^{\frac{n-1}{2}}\,dx}{(2m)^{\frac{1}{2}}}.\left\{1 - \frac{x^{n-1}}{a^{n-1}}\right\}^{-\frac{1}{2}},$$

$$= -\frac{(n-1)^{\frac{1}{2}}\,x^{\frac{n-1}{2}}\,dx}{(2m)^{\frac{1}{2}}}.\left\{1 + \frac{1}{2}.\,\frac{x^{n-1}}{a^{n-1}} + \frac{1.3}{2.4}.\,\frac{x^{2n-2}}{a^{2n-2}} + \&c.\right\}.$$

Multiplying and integrating, we shall have

$$t = \frac{(n-1)^{\frac{1}{2}}}{(2m)^{\frac{1}{2}}}\left\{C - \frac{2}{n+1}.\,x^{\frac{n+1}{2}} - \frac{1}{2}.\,\frac{2}{3n-1}.\,\frac{x^{\frac{3n-1}{2}}}{a^{n-1}}\right.$$

$$\left. - \frac{1.3}{2.4}.\,\frac{2}{5n-3}.\,\frac{x^{\frac{5n-3}{2}}}{a^{2n-2}} - \&c.\right\}:$$

and this, taken from $x = a$, to $x = 0$, gives for the whole time

$$t = \frac{(n-1)^{\frac{1}{2}}}{(2m)^{\frac{1}{2}}}.\,2a^{\frac{n+1}{2}}\left\{\frac{1}{n+1} + \frac{1}{2}.\,\frac{1}{3n-1}\right.$$

$$\left. + \frac{1.3}{2.4}.\,\frac{1}{5n-3} + \&c.\right\}.$$

COR. 1. From different distances the times of falling to the same centre are as $a^{\frac{n+1}{2}}$.

Hence if force $\propto$ dist. time $\propto 1$, or is constant,

if force $\propto 1$, or is constant, time $\propto \sqrt{(\text{dist.})}$

if force $\propto \frac{1}{(\text{dist.})}$, time $\propto$ dist.

if force $\propto \frac{1}{(\text{dist.})^2}$, time $\propto (\text{dist.})^{\frac{3}{2}}$,

if force $\propto \frac{1}{(\text{dist.})^3}$, time $\propto (\text{dist.})^2$,

if force $\propto \frac{1}{(\text{dist.})^{\frac{1}{2}}}$, time $\propto (\text{dist.})^{\frac{3}{4}}$.

COR. 2. In all these cases the time is greater as the distance is greater; but if the force vary in a higher direct ratio than the simple power of the distance, the contrary will be the case.

Thus if force $\propto (\text{dist.})^2$, time $\propto \frac{1}{(\text{dist.})^{\frac{1}{2}}}$

if force $\propto (\text{dist.})^3$, time $\propto \frac{1}{(\text{dist.})}$.

COR. 3. The integration for finding the time when the force is inversely as the distance, is not properly included in this case, and is considered in the following problem.

PROB. III. *When the force varies inversely as the distance, to find the whole time of the descent to the centre.*

Let any distance SP, fig. 1, $= r$; hence,

$$f = \frac{m}{r}, \; v\,dv = -\frac{m\,dr}{r}, \; v^2 = 2m \text{ hyp. log. } \frac{a}{r},$$

$$dt = -\frac{dr}{v} = -\frac{1}{(2m)^{\frac{1}{2}}} \cdot \frac{dr}{\sqrt{\text{hyp. log. } \frac{a}{r}}}.$$

And our object must now be to integrate this expression from $r = a$, to $r = 0$.

Let $\frac{a}{r} = e^{x^2}$; $\therefore$ hyp. log. $\frac{a}{r} = x^2$, $r = \frac{a}{e^{x^2}}$, $dr = -\frac{2axdx}{e^{x^2}}$;

$\therefore dt = \frac{1}{(2m)^{\frac{1}{2}}} \cdot \frac{2adx}{e^{x^2}}$; and time to centre $= \frac{2a}{(2m)^{\frac{1}{2}}} \int e^{-x^2} dx$, from $x = 0$ to $x = \infty$.

Now let there be a curve BQ, fig. 6, of which the ordinates are $CO = u$, $OQ = z$: and let its equation be $z = e^{-u^2}$. Let this curve revolve round the axis CB, parallel to z, through a quadrant, so as to generate the surface BQQ'. We may find the solid content thus generated by supposing the plane $CBQO$ to revolve through an angle $d\theta = NCn$. If $CN = u$, we shall have $Nn = u\,d\theta$, and if we take a portion of the triangle whose breadth along CN is du, and conceive standing upon it a prism whose height is NP or z, the solid content of this prism will be $z \cdot u d\theta \cdot du$: and the solid content of the wedge $BCPn$ will be the integral of this from C to $N = \int z u du d\theta = \int e^{-u^2} u du . d\theta = d\theta \int e^{-u^2} u du$. And the solid content of the figure when the plane has revolved through a quadrant, will manifestly be $\frac{\pi}{2} \int e^{-u^2} u du = \frac{\pi}{4} . \{C - e^{-u^2}\}$, and if this be taken from C, when $u = 0$, integral $= 0$; $\therefore$ content $= \frac{\pi}{4} . \{1 - e^{-u^2}\}$. And if we suppose the solid to be extended to infinity, so as to comprehend the whole space between the planes BCX, BCY, and the curve surface, we must make u infinite, and we have the content $= \frac{\pi}{4}$.

But we may find this solid content in another manner, by referring the surface to their rectangular co-ordinates, $CM = x$, $MN = y$, $NP = z$: and it will then be equal to $\iint z dx dy$ (*Lacroix*, Elem. Treat. Art. 247.)

Now $u^2 = CN^2 = x^2 + y^2$, and $z = e^{-u^2} = e^{-(x^2 + y^2)}$.

Hence, content $=\iint e^{-x^2-y^2}dx\,dy$

$=\iint e^{-x^2}dx.\,e^{-y^2}dy$

$=\int e^{-x^2}dx.\int e^{-y^2}dy$; because in integrating with respect to y, x may be considered as constant.

And for the whole content we must take the integrals from $x=0$ to $x=\infty$, and from $y=0$ to $y=\infty$; and in this case, $\int e^{-x^2}dx$ and $\int e^{-y^2}dy$ will manifestly be equal. Hence, whole content $=(\int e^{-x^2}dx)^2$ from $x=0$ to $x=\infty$;

$$\therefore \frac{\pi}{4}=(\int e^{-x^2}dx)^2, \text{ from } x=0 \text{ to } x=\infty,$$

$$\frac{\sqrt{\pi}}{2}=\int e^{-x^2}dx, \text{ from } x=0 \text{ to } x=\infty.$$

And time to centre $=\dfrac{a\sqrt{\pi}}{\sqrt{2m}}$.

CHAP. II.

THE CURVILINEAR MOTION OF A POINT.

11. When a point in motion is acted on by a force which is not in the direction of its motion, it will be perpetually deflected from its path, so as to describe a curve line. The quantity of this deflection will be regulated by the second law of motion, in the manner which we shall explain. By that law it is asserted that if a point at P be moving with a velocity which would in a given time carry it through the space PR, fig. 7; and if, during its motion it be acted on by a constant force always parallel to itself, which would in the same time make it move through a space Pp

from rest, it will be found, at the end of that time, in a point r, determined by completing the parallelogram Rp.

If the force which acts upon the body, be variable in magnitude, or direction, or both, we can no longer in the same manner find the place of the body at the end of a *finite* time from P. The second law of motion is then applicable *ultimately* only; that is, to the motion of the body during an indefinitely small time*. This may be stated also thus. Let PR be the space which would be described in any time in consequence of the velocity; PQ the path which is actually described in the same time, in consequence of the action of a variable deflecting force; Pp the space through which the force, retaining the magnitude and direction which it has at P, would cause a body to move from rest in the same time; Rr equal and parallel to Pp: then will RQ be *ultimately* equal to Rr, and coincident with it in direction.

12. Prop. To find the equations of motion of a body, moving in a plane and acted upon by any forces in that plane. Fig. 7.

Let t be the time from a given epoch, till the body arrives at P, and $t+h$ till it arrives at Q, so that h is the time of motion in PQ. Also let AM, MP be rectangular co-ordinates to the point P, and be called x and y: similarly, let AN, NQ, and AO, OR, be co-ordinates, parallel to these: PI and RK parallel to AN. Let the force at P be called P, and the angle which it makes with x, be called α. Also let the velocity at P be called V, and the angle which it makes with x, be called θ.

We shall then have by supposition $PR = Vh$: and $Pp = \frac{1}{2}Ph^2$; because Pp is described by a constant force. (Ch. I. Ex. 2.)

Hence, $PH = Vh \cos. \theta$; $RH = Vh \sin. \theta$.

Also if Rs, sr be parallel to AM, MP,

$$Rs = \frac{1}{2} Ph^2 \cos. \alpha; \quad sr = \frac{1}{2} Ph^2 \sin. \alpha.$$

* The second law of motion is proved by experiment in the case of a constant force; and it is manifest that the effect of a variable finite force for an indefinitely small time may be considered to be the same as if it were constant.

But by Taylor's Theorem, considering x and y as functions of t, and dt constant,

$$AN = x + \frac{dx}{dt}.h + \frac{d^2x}{dt^2}.\frac{h^2}{1.2} + \&c.$$

$$NQ = y + \frac{dy}{dt}.h + \frac{d^2y}{dt^2}.\frac{h^2}{1.2} + \&c.$$

Hence, $RK = MN - PH = \frac{dx}{dt}.h + \frac{d^2x}{dt^2}.\frac{h^2}{1.2} + \&c. - Vh \cos.\theta.$

$$KQ = IQ - RH = \frac{dy}{dt}.h + \frac{d^2y}{dt^2}.\frac{h^2}{1.2} + \&c. - Vh \sin.\theta.$$

Now since Rr ultimately coincides with RQ, we have ultimately Rs, RK equal, and also sr, KQ. Hence, ultimately

$$\left(\frac{dx}{dt} - V\cos.\theta\right)h + \frac{d^2x}{dt^2}.\frac{h^2}{1.2} + \&c. = \tfrac{1}{2}Ph^2.\cos.\alpha.$$

$$\left(\frac{dy}{dt} - V\sin.\theta\right)h + \frac{d^2y}{dt^2}.\frac{h^2}{1.2} + \&c. = \tfrac{1}{2}Ph^2.\sin.\alpha.$$

Whence we must necessarily have, equating coefficients of h,

$$\frac{dx}{dt} - V\cos.\theta = 0, \quad \frac{dy}{dt} - V\sin.\theta = 0,$$

$$\frac{d^2x}{dt^2} = P\cos.\alpha, \quad \frac{d^2y}{dt^2} = P\sin.\alpha.$$

Hence, $\frac{dx}{dt}$ = velocity in x, $\frac{dy}{dt}$ = velocity in y,

$\frac{d^2x}{dt^2}$ = force in x, $\frac{d^2y}{dt^2}$ = force in y.

If we represent by X and Y, the whole forces which act on the point in the direction of x and of y, we have

$$\frac{d^2x}{dt^2} = X, \text{ and } \frac{d^2y}{dt^2} = Y; \ldots\ldots\ldots\ldots (c)$$

where dt is constant, and X and Y positive, when they tend to increase x and y.

Cor. 1. It is clear that if we had referred the path of the body to *three* rectangular co-ordinates, x, y, z, and if we had made X, Y, Z, represent the whole forces in the directions of these co-ordinates, we should have had, by reasoning exactly similar,

$$\frac{d^2 x}{dt^2} = X, \quad \frac{d^2 y}{dt^2} = Y, \quad \frac{d^2 z}{dt^2} = Z \ldots\ldots\ldots (c')$$

13. These equations enable us to solve various problems respecting the motions of bodies acted on by any forces. If the motion be known, we can, from them, find the forces in the directions of the co-ordinates, and by compounding these, the whole force which acts upon the body. If on the other hand, the force depends, in a known manner, on the position of the body, we can, by resolving it in the proper directions, find X, Y, Z, in terms of x, y, and z; and we shall then, by integrating the equations, have the motion of the body determined. If we can eliminate t, we obtain a relation among the co-ordinates which defines the curve described by the body. We shall have instances of these various applications in what follows.

Ex. 1. To find the forces which must act upon a point, so that it may describe the arc of a parabola with a uniform motion.

If x, y, s, represent the abscissa, ordinate, and curve of the parabola, we shall have, since the velocity is constant,

$$\frac{ds}{dt} = c, \text{ a constant quantity;}$$

$$\therefore \; c^2 = \frac{ds^2}{dt^2} = \frac{dx^2 + dy^2}{dt^2}.$$

Now if $4a$ be the principal parameter of the parabola, we have

$$y = 2\sqrt{ax}; \; \therefore \; dy = dx\sqrt{\frac{a}{x}};$$

Hence, $c^2 = \dfrac{dx^2}{dt^2}\left(1 + \dfrac{a}{x}\right)$, and $\dfrac{dx^2}{dt^2} = \dfrac{c^2}{1 + \dfrac{a}{x}}$;

Differentiating, $\frac{2\,dx\,d^2x}{dt^2} = \frac{c^2\frac{a\,dx}{x^2}}{\left(1+\frac{a}{x}\right)^2}$;

$$\therefore \frac{d^2x}{dt^2} = \frac{c^2 a}{2(a+x)^2},$$

which gives the force parallel to the abscissa.

Again, $x = \frac{y^2}{4a}$; $\therefore dx = \frac{y\,dy}{2a}$;

$$\therefore c^2 = \frac{dy^2}{dt^2}\left(\frac{y^2}{4a^2}+1\right);$$

$$\therefore \frac{dy^2}{dt^2} = \frac{4a^2c^2}{4a^2+y^2};$$ and differentiating,

$$\frac{2\,dy\,d^2y}{dt^2} = -\frac{8a^2c^2y\,dy}{(4a^2+y^2)^2};$$

$$\therefore \frac{d^2y}{dt^2} = -\frac{4a^2c^2y}{(4a^2+y^2)^2} = -\frac{4a^2c^2y}{(4a^2+4ax)^2}$$

$$= -\frac{c^2y}{4(a+x)^2},$$

which gives the force parallel to the ordinate, the negative sign shewing that it tends towards the axis.

If S, fig. 8, be the focus, and SA the axis of the parabola; $AD = AS = a$, and DN perpendicular to AD, so that DN is the directrix; we shall have DM or $NP = a + x$. Hence, the forces in NP and PM are respectively as

$$\frac{2AS}{PN^2}, \text{ and } \frac{PM}{NP^2}; \text{ or as } \frac{MK}{PN^2}, \text{ and } \frac{PM}{PN^2};$$

PK being the normal, and therefore $MK = 2AS$. Hence, the whole force on P, which is compounded of these two, is in the direction PK*, and proportional to $\frac{PK}{PN^2}$.

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* It may easily be shewn that if a body move uniformly in any curve, the force which retains it is perpendicular to the curve.

Ex. 2. A body is acted upon at every point of its path, by a force which is proportional to its distance from a given centre towards which it tends: to find its path.

Let the centre of force C, fig. 9. be made the origin of rectangular co-ordinates CM, MP: the force in the direction PC is every where proportional to PC. Resolve it in the directions PM, MC; and these lines will be proportional to the resolved parts. Hence, we shall have

force in direction of $x = -mx$, force in direction of $y = -my$: m being some constant quantity, and the negative signs indicating the direction of the forces.

$$\text{Hence, } \frac{d^2x}{dt^2} = -mx, \quad \frac{d^2y}{dt^2} = -my;$$

$$\therefore \frac{2\,dx\,d^2x}{dt^2} = -2mx\,dx, \quad \frac{2dy\,d^2y}{dt^2} = -2my\,dy;$$

$$\text{integrating, } \frac{dx^2}{dt^2} = C - mx^2; \quad \frac{dy^2}{dt^2} = D - my^2:$$

where C, D are arbitrary quantities depending on the velocity and direction of the body's motion at some given point. We may evidently, without restricting their values, put for C and D, mh^2 and mk^2; and thus we have

$$\frac{dx}{\sqrt{(C - mx^2)}} = dt = \frac{dy}{\sqrt{(D - my^2)}};$$

$$\text{or } \frac{dx}{\sqrt{(h^2 - x^2)}} = \frac{dy}{\sqrt{(k^2 - y^2)}}.$$

Integrating, we have $\alpha = \beta + \gamma$; where α is the arc whose sine is $\frac{x}{h}$, β is the arc whose sine is $\frac{y}{k}$, and γ an arbitrary arc. Therefore, we have

$$\sin.\ \alpha = \sin.\ \beta \cos.\ \gamma + \sin.\ \gamma \cos.\ \beta:$$

or if n be the cosine of γ, and consequently $\sqrt{(1 - n^2)}$ its sine,

$$\frac{x}{h} = \frac{ny}{k} + \sqrt{(1 - n^2)} \,.\, \sqrt{\left(1 - \frac{y^2}{k^2}\right)},$$

transposing and squaring, we get

$$\frac{x^2}{h^2}+\frac{n^2y^2}{k^2}-\frac{2nxy}{hk}=(1-n^2).\left(1-\frac{y^2}{k^2}\right)=1-n^2-\frac{y^2}{k^2}+\frac{n^2y^2}{k^2}.$$

Whence $\frac{x^2}{h^2}+\frac{y^2}{k^2}-\frac{2nxy}{hk}=1-n^2$:

which is the equation to an ellipse referred to rectangular co-ordinates measured from the centre. (*Wood's* Alg. Part IV.) Hence, the curve described is an ellipse, of which the centre is C.

The axes of the ellipse may be thus found: let the tangent at P, fig. 9. be parallel to Cx, whence CP and CD will be conjugate diameters, and hence

$$CP^2+CD^2=a^2+b^2,\ PM.CD=a.b,$$

where a and b are the semi-axes.

Now, since

$$\frac{dy}{dx}=\sqrt{\frac{k^2-y^2}{h^2-x^2}};$$

it is manifest that when the tangent is parallel to Cx, we have $y=k$: hence to find $x=CM$, we have

$$\frac{x^2}{h^2}+1-\frac{2nx}{h}=1-n^2;$$

$$\therefore\ \frac{x}{h}-n=0,\ x=nh=CM;\ \therefore\ CP^2=CM^2+MP^2=n^2h^2+k^2.$$

Also, to find CD, put $y=0$, and we have

$$\frac{x^2}{h^2}=1-n^2;\ \therefore\ x=h\sqrt{(1-n^2)}=CD.$$

Hence $a^2+b^2=h^2+k^2$, $ab=kh\sqrt{(1-n^2)}$, whence a, b are known.

To find the position of the major axis CA, or the angle $ACx=\theta$, we may proceed thus. Differentiating the equation to the curve, we find

$$\frac{dy}{dx}=-\frac{\frac{x}{h^2}-\frac{ny}{hk}}{\frac{y}{k^2}-\frac{nx}{hk}}.$$

Now at the point A, the curve is perpendicular to CA; and hence at that point the normal passes through the centre; therefore (*Lacroix*. Art. 65.)

$$\frac{y\,dy}{dx} = -x\,; \;\therefore\; \frac{\dfrac{x}{h^2} - \dfrac{n\,y}{h\,k}}{\dfrac{y}{k^2} - \dfrac{n\,x}{h\,k}} = \frac{x}{y}\,;$$

$$\therefore\; \frac{x\,y}{h^2} - \frac{n\,y^2}{h\,k} = \frac{x\,y}{k^2} - \frac{n\,x^2}{h\,k}\,;$$ whence we find

$$\frac{y^2}{x^2} + \frac{h^2 - k^2}{n\,h\,k}\,\frac{y}{x} - 1 = 0\,;$$ and hence

$$\frac{2\,n\,h\,k}{h^2 - k^2} = \frac{2\dfrac{y}{x}}{1 - \dfrac{y^2}{x^2}} = \frac{2\,\tan.\,\theta}{1 - \tan.^2\,\theta} = \tan.\,2\theta\,;$$

hence θ is known.

To find the time of describing any portion, we have

$$dt = \frac{d\,x}{\sqrt{(C - m x^2)}} = \frac{d\,x}{\sqrt{m}\,\sqrt{(h^2 - x^2)}}\,;$$

$$\therefore\, t = \frac{1}{\sqrt{m}}\,.\,\text{arc}\left(\sin. = \frac{x}{h}\right) + \text{const.}$$

the constant being determined by the place of the body at a given time.

For a whole revolution, we have time $= \dfrac{1}{\sqrt{m}}\,.\,2\,\pi$. Hence, it is independent in the size of the orbit.

CHAP. III.

CENTRAL FORCES.

14. **The** equations of the preceding Chapter would enable us to determine the motions of bodies acted on by any forces whatever, and of course, among the rest, in the case where the force is supposed always to tend to a centre, and to be represented by some function of the distance from that point, of which we had an instance in the last example. But problems respecting the action of *central forces* as these are called, are of such importance, and lead to such simplifications of our general formula, that it is convenient to consider separately this application of our reasonings.

15. Prop. A body acted upon by a central force will describe a curve lying in one plane.

If we consider a body, moving in any direction, to be acted on by a force tending to a given centre, it is clear that the body will be deflected from its rectilinear path, and will describe a curve. And this curve will be in one plane, namely, the plane passing through the centre of force and the original direction of its motion. For the body, by the combination of its original velocity with the action of the force, will, in a small time, describe a path by the second law of motion, (see p. 17.) which will be in the plane in which both the motion and the line drawn to the centre are. And at the end of this small time it will tend to move on in the same plane; but being deflected as before by the central force, which is still in the same plane, its actual motion will still be in the same plane. And similarly, after the lapse of any number of such intervals, that is, of any finite time, its motion will be still in the plane in which it originally was.

Hence, it is only necessary to consider the two equations, Art. 12, which belong to the motion of a body in a plane.

16. Prop. A body being acted on by a central force, the sectorial areas described are proportional to the times.

Let the centre of force S, fig. 10, be the centre of rectangular co-ordinates $SM=x$, $MP=y$; let $SP=r$; and let the whole force at P, in the direction PS, be called P, P being a function of r. Also let the angle $ASP=v$; then, P being resolved into its components X and Y parallel to SM and PM respectively, we have $X=-P\cos. v$, $Y=-P\sin. v$. Hence, equations (c) become

$$\frac{d^2x}{dt^2}=-\frac{Px}{r},\ \frac{d^2y}{dt^2}=-\frac{Py}{r};$$

multiply by y and x respectively, and subtract;

$$\text{hence}\quad \frac{xd^2y-yd^2x}{dt^2}=0;$$

$$\text{integrating,}\quad \frac{xdy-ydx}{dt}=h, \text{ a constant quantity.}$$

Now sector $ASP=ASNP-SNP$, MN, being a parallelogram;

$\therefore$ diff^1. of sector $ASP=$diff^1. of $ASNP-$diff^1. of SNP

$$=xdy-d.\frac{xy}{2}$$

$$=\frac{xdy-ydx}{2}$$

$$=\frac{hdt}{2}.$$

Hence sector $ASP=\frac{ht}{2}$, the time and the area beginning at A.

And any portion of this sector is proportional to the time of describing that portion*.

Cor. 1. Also since we have $ht=2$ area described in t, if we make $t=1$, (the unit of time, for instance $1''$,) we have $h=2$.area described in time 1.

Cor. 2. If $SP=r$, and angle $ASP=v$, we shall have diff^1. of sector $ASP=\frac{1}{2}r^2dv$; (*Lacroix*, Elem. Treat. Art. 111.);

hence $r^2dv=hdt$.

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* *Principia*, Book I, Prop. 1.

17. PROP. A body being acted on by a central force, to find the velocity at any distance from the centre.

Take the equations

$$\frac{d^2x}{dt^2} = -\frac{Px}{r}, \quad \frac{d^2y}{dt^2} = -\frac{Py}{r};$$

Multiply by $2dx$, $2dy$, and add;

$$\therefore \frac{2dx\,d^2x + 2dy\,d^2y}{dt^2} = -\frac{2P\,(x\,dx + y\,dy)}{r}$$

$$= -2P\,dr,$$

because $x^2 + y^2 = r^2$, and $x\,dx + y\,dy = r\,dr$:

$$\text{or } d\,.\frac{dx^2 + dy^2}{dt^2} = -2P\,dr.$$

If s be the length of the curve $\dfrac{dx^2 + dy^2}{dt^2} = \dfrac{ds^2}{dt^2} = (\text{velocity})^2$.

Hence the velocity will be known by integrating the expression

$$d\,.\frac{ds^2}{dt^2} = -2P\,dr,$$

which may be done when P is a function of r; and we shall have

$$\frac{ds^2}{dt^2} = C - 2\int P\,dr.$$

If C be the $(\text{velocity})^2$ at any distance a, the integral $\int P\,dr$ must be taken between the limits a and r.

COR. 1. If the velocity at the distance a be given, the velocity at the distance r will be the same, whatever be the path which the body describes. For the integral depends only on the force P, and not on the path.

COR. 2. If angle $ASP = v$, as before; $ds^2 = dr^2 + r^2dv^2$; (*Lacroix*, Art. 110.).

$$\therefore d \cdot \frac{dr^2 + r^2 dv^2}{dt^2} = -2P dr.$$

18. Prop. A body being acted on by a central force, to find the polar equation to the curve described.

By Cor. 2. to Art. 16, $dt = \frac{r^2 dv}{h}$;

$$\therefore \frac{dr^2 + r^2 dv^2}{dt^2} = \frac{h^2 dr^2}{r^4 dv^2} + \frac{h^2}{r^2};$$

$\therefore$ by Cor. 2. to Art. 17.

$$d \cdot \left\{ \frac{h^2 dr^2}{r^4 dv^2} + \frac{h^2}{r^2} \right\} = -2P dr.$$

Now make $\frac{1}{r} = u$, and, $\therefore \frac{dr}{r^2} = -du$, $dr = -\frac{du}{u^2}$;

$$\therefore d \left\{ \frac{h^2 du^2}{dv^2} + h^2 u^2 \right\} = \frac{2P du}{u^2},$$

or $\frac{2h^2 du d^2 u}{dv^2} + 2h^2 u du = \frac{2P du}{u^2}$, supposing dv constant,

or $\frac{d^2 u}{dv^2} + u - \frac{P}{h^2 u^2} = 0 : \ldots\ldots\ldots\ldots (d).$

which is the equation of which we shall make use most commonly in the consideration of orbits described about a centre. It may be employed either in determining the law of force, when we know the curve, and consequently the relation of u and v, which is called the direct problem of central forces; or if P be known in function of r, and therefore of u, we may, by integrating, find the relation of u and v, which gives us the nature of the orbit; this is called the inverse problem of central forces.

19. Prop. The orbit being given, to find the *time* of describing any part of it, and the *velocity* at any point.

The expression for the *time* is $dt = \frac{r^2 dv}{h} = \frac{dv}{hu^2}$:

and having expressed dv in terms of u, or u in terms of v, we can find t by integrating.

With respect to the *velocity*, we may thus obtain an expression for it. We have $ds^2 = dr^2 + r^2dv^2$, where ds is the differential of the curve;

$$\text{and } dt^2 = \frac{r^4dv^2}{h^2};$$

$$\therefore (\text{velocity})^2 = \frac{ds^2}{dt^2} = \frac{h^2}{r^4}\frac{dr^2}{dv^2} + \frac{h^2}{r^2}; \text{ or, since } \frac{dr}{r^2} = -du,$$

$$= h^2\left(\frac{du^2}{dv^2} + u^2\right).$$

Cor. 1. Multiply the equation

$$\frac{d^2u}{dv^2} + u - \frac{P}{h^2u^2} = 0,$$

by $2h^2du$, and we have

$$h^2.\frac{2dud^2u}{dv^2} + h^2.2udu - \frac{2Pdu}{u^2} = 0;$$

$$\therefore h^2\left(\frac{du^2}{dv^2} + u^2\right) - 2\int\frac{Pdu}{u^2} = C$$

$$h^2\left(\frac{du^2}{dv^2} + u^2\right) = C + 2\int\frac{Pdu}{u^2} = C - 2\int Pdr,$$

which agrees with Art. 17.

Cor. 2. If we draw SY a perpendicular upon the tangent, and suppose $p = SY$, it will be easy to see that we have

$$\frac{p}{r} = \frac{rdv}{ds}, \text{ or } p = \frac{r^2dv}{ds}.$$

Hence by what precedes, $p = \dfrac{hdt}{ds}$, and $\dfrac{ds}{dt} = \dfrac{h}{p}$;

therefore *the velocity is inversely as the perpendicular on the tangent**.

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* *Principia*, Book I. Prop. 1. Cor. 1. It has been usual among English Mathematicians to define a spiral by the equation between the radius vector r, and the perpendicular on the tangent p. This is virtually only a differential equation to the curve, but its use is sometimes convenient.

Prop.

20. Prop. When bodies revolve in circles, having the centre of force in the centre, to determine the periodic times†.

In this case r is constant, and $x = r \cos. v$. Also, since r is constant, by Art. 17, $\frac{r\,dv}{dt}$, the velocity, is constant; and therefore, dt being constant, $d^2 v = 0$.

Hence $x = r \cos. v$, $dx = -r \sin. v . dv$, $d^2 x = -r \cos. v . dv^2$; which substituted in the equation

$$\frac{d^2 x}{dt^2} = -\frac{Px}{r},$$

Prop. To obtain the central force in terms of r and p.

By Cor. 2, Art. 19, we have $p = \frac{h\,dt}{ds}$;

$$\therefore \frac{h^2}{p^2} = \frac{ds^2}{dt^2}; \text{ and differentiating,}$$

$$-\frac{2\,h^2\,dp}{p^3} = d . \frac{ds^2}{dt^2} = -2\,P\,dr, \text{ by Art. 19.};$$

$$\therefore P = \frac{h^2\,dp}{p^3\,dr}.$$

Cor. The velocity in the curve at any point is equal to that generated by the force P at that point, continued constant, and acting on a body while it moves through one-fourth the chord of curvature drawn at that point through the centre of force.

For $(\text{velocity})^2 = \frac{h^2}{p^2} = \frac{P . p\,dr}{dp} = 2\,P . \frac{1}{4}$ chord of curvature,

(for chord $= \frac{2\,p\,dr}{dp}$. *Lacroix*, Note H.)

$$= (\text{velocity})^2 \text{ by force } P \text{ through } \tfrac{1}{4} \text{ chord,}$$

because for constant forces, $(\text{velocity})^2 = 2fs$.

† *Principia*, Book. I. Prop. 6.

$$\text{give } \frac{r\,dv^2}{dt^2} = P, \text{ and } \frac{dv^2}{dt^2} = \frac{P}{r}.$$

The fraction $\frac{dv}{dt}$ is the angular velocity, and if 2π be four right angles, and T the time of a revolution,

$$\frac{dv}{dt} = \frac{2\pi}{T}; \therefore \frac{4\pi^2}{T^2} = \frac{P}{r},$$

$$T = \frac{2\pi\sqrt{r}}{\sqrt{P}},$$

$$\text{velocity} = \frac{r\,dv}{dt} = \sqrt{(Pr)}.$$

Cor. 1. If $P \propto r$, T is constant, velocity $\propto r$,

$P \propto 1$, $T \propto \sqrt{r}$, velocity $\propto \sqrt{r}$,

$P \propto \frac{1}{r^2}$, $T \propto r^{\frac{3}{2}}$, velocity $\propto \frac{1}{\sqrt{r}}$,

$P \propto \frac{1}{r^3}$, $T \propto r^2$, velocity $\propto \frac{1}{r}$.

Cor. 2. Similarly if the variation of T were given, that of P would be known.

21. We shall now proceed to determine the paths and motions of a point acted on by any central force whatever, beginning with the cases in which it is proportional to some power of the distance from the centre.

Prob. I. *Let the force be directly as the distance, or* $\mathrm{P} = \mathrm{m\,r} = \frac{\mathrm{m}}{\mathrm{u}}$, m *being a constant quantity.*

In this case the equation (d) becomes

$$\frac{d^2u}{dv^2} + u - \frac{m}{h^2u^3} = 0;$$

which might be integrated: but we have already solved this case by a different method, and found the orbit to be an ellipse, the centre of which is the centre of force*. See Chap. II. Ex. 2.

If the force be repulsive, and as the distance, the process is nearly the same, and a hyperbola is described, the centre of which is the centre of force.

PROB. II. *Let the force be inversely as the square of the distance, or* $P = \frac{m}{r^2} = mu^2$.

Hence, by (*d*), $\frac{d^2u}{dv^2} + u - \frac{m}{h^2} = 0.$

To integrate this, let $u - \frac{m}{h^2} = w$;

$$\therefore \frac{d^2w}{dv^2} + w = 0;$$

and if ϵ^{kv} represent a particular value of w, we have

$$k^2 + 1 = 0, \ k = \pm \sqrt{-1}.$$

(*Lacroix*, Art. 280, 281.)

Hence, the general value is

$$w = C\epsilon^{v\sqrt{-1}} + C'\epsilon^{-v\sqrt{-1}}$$

$$= \frac{(C+C')}{2}(\epsilon^{v\sqrt{-1}} + \epsilon^{-v\sqrt{-1}}) + \left(\frac{C-C'}{2}\right)(\epsilon^{v\sqrt{-1}} - \epsilon^{-v\sqrt{-1}})$$

$$\left(\text{making } C + C' = C_1,\ C - C' = \frac{C_2}{\sqrt{-1}},\right)$$

$$= C_1 \cos. v + C_2 \sin. v.$$

Hence, $u = \frac{1}{r} = C_1 \cos. v + C_2 \sin. v + \frac{m}{h^2}$,

and $\frac{du}{dv} = -C_1 \sin. v + C_2 \cos. v.$

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* *Principia*, Book I, Prop. 10.

$$\text{Now when } v = 0,\ \frac{du}{dv} = C_2,$$

$$\text{when } v = \pi,\ \frac{du}{dv} = -\ C_2;$$

hence, between $v=0$, and $v=\pi$, there must be a value of v which makes $\frac{du}{dv}=0$: let this value of v be a;

$$\therefore -\ C_1 \sin. a + C_2 \cos. a = 0;\ C_2 \cos. a = C_1 \sin. a;$$

$$\therefore \frac{1}{r} = \frac{C_1 \cos. a \cos. v + C_2 \cos. a \sin. v}{\cos. a} + \frac{m}{h^2}$$

$$= \frac{C_1}{\cos. a} (\cos. a \cos. v + \sin. a \sin. v) + \frac{m}{h^2}$$

$$= \frac{C_1 \cos. (v-a)}{\cos. a} + \frac{m}{h^2}.$$

Let r' and r'' be the values of r, for $v=a$, and $v=\pi+a$; both being supposed positive; hence,

$$\frac{1}{r'} = \frac{C_1}{\cos. a} + \frac{m}{h^2},$$

$$\frac{1}{r''} = -\ \frac{C_1}{\cos. a} + \frac{m}{h^2};\ \text{ and, adding and subtracting,}$$

$$\frac{m}{h^2} = \tfrac{1}{2}\left\{\frac{1}{r'} + \frac{1}{r''}\right\};\ \frac{C_1}{\cos. a} = \tfrac{1}{2}\left\{\frac{1}{r'} - \frac{1}{r''}\right\}.$$

Hence, the equation becomes

$$\frac{1}{r} = \tfrac{1}{2}\left\{\frac{1}{r'} - \frac{1}{r''}\right\} \cos. (v-a) + \tfrac{1}{2}\left\{\frac{1}{r'} + \frac{1}{r''}\right\};$$

$$\therefore r = \frac{2r'r''}{r' + r'' + (r'' - r') \cos. (v-a)}.$$

Now r', r'' are opposite parts of the same line: let $r''+r'=2a$, $r''-r'=2ae$; $\therefore r'r''=a^2-a^2e^2$, and

$$r = \frac{a(1-e^2)}{1+e\cos. (v-a)};$$

the equation to an ellipse, $v-a$ being measured from the vertex nearest to S*.

* *Principia*, Book I, Prop. 11.

If r'' be less than r', e will be negative, and the angle $v-a$ will be measured from the larger portion of the axis.

The curve may assume different forms by the alteration of the arbitrary quantities C_1, C_2. If u ever become 0, or negative, the form of the curve is no longer an ellipse.

Now the two values corresponding to $v=a$, and $v=\pi+a$, being those for which $\frac{du}{dv}=0$, are manifestly the greatest and least values of u. Hence, if any value of u be negative, one of these will be so. And hence the curve will no longer be an ellipse, if either

$$\frac{C_1}{\cos. a}+\frac{m}{h^2}, \text{ or } -\frac{C_1}{\cos. a}+\frac{m}{h^2}, \text{ be negative.}$$

If for instance, the latter be negative, we may suppose

$\frac{1}{r''}=\frac{C_1}{\cos. a}-\frac{m}{h^2}$; where r'' is positive, whence, as before,

$$r=\frac{2r''r'}{(r''+r')\cos.(v-a)+r''-r'};$$

and making $r''-r'=2a$, $r''+r'=2ae$, (supposing $r''>r'$) we have

$$r=\frac{a(e^2-1)}{e\cos.(v-a)+1}, \text{ the equation to an hyperbola.}$$

If we have $-\frac{C_1}{\cos. a}+\frac{m}{h^2}=0$, we get, putting $\frac{m}{h^2}$ for $\frac{C_1}{\cos. a}$,

$$\frac{1}{r}=\frac{m}{h^2}\{\cos.(v-a)+1\}; \text{ and if } \frac{h^2}{m}=2a,$$

$$r=\frac{2a}{1+\cos.(v-a)}; \text{ the equation to a parabola.}$$

And similarly if $\frac{C_1}{\cos. a}+\frac{m}{h^2}=0$, the curve will be a parabola, but in a different position.

Hence, in all cases the curve will be a conic section: and supposing C_1 positive, it will be

$$\text{an ellipse if } -\frac{C_1}{\cos. \alpha}+\frac{m}{h^2} \text{ be positive, or } \frac{m}{h^2} > \frac{C_1}{\cos. \alpha}:$$

$$\text{a parabola if } -\frac{C_1}{\cos. \alpha}+\frac{m}{h^2}=0, \text{ or } \frac{m}{h^2}=\frac{C_1}{\cos. \alpha}:$$

$$\text{an hyperbola if } -\frac{C_1}{\cos. \alpha}+\frac{m}{h^2} \text{ be negative, or } \frac{m}{h^2} < \frac{C_1}{\cos. \alpha}.$$

PROB. III. *A body being projected from a given point, with a given velocity, in a given direction; and acted on by a given force varying inversely as the square of the distance; to find the trajectory described* *.

This problem might be solved by the preceding formulæ, but more simply as follows.

In fig. 11, let a body be projected from a distance $SP=D$, in a direction PY making the angle $SPY=\delta$, and let the velocity at $P=V$; force $=\frac{m}{r^2}$; $Sp=r$.

By Art. 17, $\text{velocity}^2=C-2\int P\,dr=C-2\int\frac{m\,dr}{r^2}=C+\frac{2m}{r}$;

and when $r=D$, velocity $=V$; $\therefore \text{velocity}^2=V^2+2m\left\{\frac{1}{r}-\frac{1}{D}\right\}$.

But by Art. 19, Cor. 2, velocity $=\frac{h}{p}$, p being the perpendicular on the tangent.

Hence, at P, $\frac{h^2}{D^2\sin.^2\delta}=V^2$; because perp^r. $=D\sin.\delta$

$$\text{at } p,\ \frac{h^2}{p^2}=V^2+2m\left\{\frac{1}{r}-\frac{1}{D}\right\};$$

$$\therefore \text{dividing, } \frac{D^2\sin.^2\delta}{p^2}=1+\frac{2m}{V^2}\left\{\frac{1}{r}-\frac{1}{D}\right\};$$

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* *Principia*, Book I, Prop. 17.

$$\therefore \frac{1}{p^2} = \frac{1}{D^2 \sin.^2 \delta} - \frac{2m}{V^2 D^3 \sin.^2 \delta} + \frac{2m}{V^2 D^2 \sin.^2 \delta} \cdot \frac{1}{r};$$

now in the ellipse $\frac{1}{p^2} = -\frac{1}{b^2} + \frac{2a}{b^2} \cdot \frac{1}{r}$ *;

in the hyperbola $\frac{1}{p^2} = \frac{1}{b^2} + \frac{2a}{b^2} \cdot \frac{1}{r}$;

and the expression for $\frac{1}{p^2}$ in the trajectory may manifestly be made to agree with one or the other of these, as the part of it independent of r is positive or negative.

Hence, the trajectory will be an ellipse

if $\frac{1}{D^2 \sin.^2 \delta} - \frac{2m}{V^2 D^3 \sin.^2 \delta}$ be negative;

that is, if $1 < \frac{2m}{V^2 D}$; or if $V^2 < \frac{2m}{D}$.

Similarly, if $V^2 > \frac{2m}{D}$, the curve will be an hyperbola.

If $V^2 = \frac{2m}{D}$, the curve will be a parabola.

In the case of the ellipse, we must have

$$\frac{1}{b^2} = \frac{2m}{V^2 D^3 \sin.^2 \delta} - \frac{1}{D^2 \sin.^2 \delta} = \frac{2m}{V^2 D^2 \sin.^2 \delta} \left\{ \frac{1}{D} - \frac{V^2}{2m} \right\};$$

$$\frac{2a}{b^2} = \frac{2m}{V^2 D^2 \sin.^2 \delta};$$

$$\therefore \frac{1}{2a} = \frac{1}{D} - \frac{V^2}{2m}.$$

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* For $p^2 = \frac{b^2 r}{2a - r}$ in the ellipse,

and $p^2 = \frac{b^2 r}{2a + r}$ in the hyperbola, by Conic Sections.

Hence, a and b, the semi-axes of the ellipse, are known: and hence, $e^2 = \frac{a^2 - b^2}{a^2}$ is known. To find the position of the major axis we have*

$$D = \frac{a(1-e^2)}{1+e \cos.(v-\alpha)},$$

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* We may find the position of the major axis in the following manner also.

If PV, fig. 11, be the chord of curvature at P, by Note, p. 29,

$$V^2 = \frac{2m}{D^2} \cdot \frac{1}{4} PV; \therefore PV = \frac{2D^2V^2}{m} \text{ is known.}$$

Also, by the property of the ellipse, $PV = \frac{2CD^2}{AC} = \frac{4SP.PH}{SP+PH}$;

$\therefore PH = \frac{SP.PV}{4SP-VP}$; whence PH is known: and it must make with PY an angle equal to SPY, and hence the point H is known. Hence C is known, and $AC = \frac{1}{2}(SP+PH)$: from which data the dimensions and position of the ellipse are easily found.

From the same principles we may solve other problems where the velocity in the conic section is concerned.

Prop. To compare the velocity in the ellipse with the velocity in a circle about the same centre of force and at the same distance.

Let PV be the chord of curvature at P.

In ellipse, $\text{velocity}^2 = \frac{2m}{r^2} \cdot \frac{PV}{4} = \frac{m.PV}{2r^2}$,

$$PV = \frac{2CD^2}{AC} = \frac{2SP.PH}{AC} = \frac{2r(2a-r)}{a};$$

$$\therefore \text{velocity}^2 = \frac{m(2a-r)}{ar}.$$

And in the circle $\text{velocity}^2 = \frac{m}{r^2}.r = \frac{m}{r}$, by Art. 20.

Hence, velocity^2 in ellipse : velocity^2 in circle :: $\frac{2a-r}{ar} : \frac{1}{r} .\ 2a-r : a$.

This agrees with Newton Book I, Prop. 16, Cor. 3, 4, 6, 7.

And

where v is the angle which determines the position of P, and is therefore known. Hence, cos. $(v-\alpha)$ is known, and hence α, which determines the position of the major axis.

Cor. 1. It appears by Art. 10, that the velocity² from an infinite distance $=\frac{2m}{D}$; hence, the trajectory will be an ellipse, a

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And in the same manner may the velocities in the other conic sections be compared.

Prop. A body is revolving in an ellipse, and the force is suddenly altered in the ratio $n : 1$; to find the alteration which takes place in the orbit, fig. 12.

If m be the force before the alteration, nm will be the force after. Let a be the semi-axis major before, and a' after; and let r be the radius vector where the alteration takes place.

Then since the velocity at this point may be considered as belonging to both the first and the last orbit, we have

$$\frac{m(2a-r)}{ar}=\text{velocity}^2=\frac{nm(2a'-r)}{a'r};$$

$$\therefore\ 2aa'-ra'=n(2aa'-ra);$$

$$\therefore\ a'=\frac{nar}{2(n-1)a+r};$$

$$\text{Hence, } 2a'-r=\frac{(2a-r)r}{2(n-1)a+r}.$$

This is the value of PH', S and H' being the foci of the new orbit, and PH' will be in the same line with PH; hence the new orbit is known.

If PH' be infinite, the new orbit will be a parabola with its axis parallel to PH. This will be the case if

$$2(n-1)a+r=0;$$

$$\text{if } n=\frac{2a-r}{2a}=\frac{PH}{SP+PH}.$$

If n be less than this, the new orbit will be an hyperbola, and PH' must be measured in the opposite direction.

In nearly the same manner we may find the alteration produced in the orbit, if the velocity be suddenly altered in any ratio.

parabola, or an hyperbola, as the velocity is less than, equal to, or greater than that acquired from infinity.

Cor. 2. In the first case

$$\frac{1}{2a} = \frac{1}{D} - \frac{V^2}{2m}; \therefore V^2 = 2m\left\{\frac{1}{D} - \frac{1}{2a}\right\}:$$

therefore V is the velocity acquired by falling from a distance $2a$ to a distance D, (see p. 11.). Hence $2a$ is the distance from the centre at which a body must begin to fall, so that when it reaches the curve, it may have the velocity of the body in the curve; and this distance is the same for every point of the curve.

Cor. 3. Let the velocity $= n$ times the velocity from infinity, or

$$V^2 = n^2 \cdot \frac{2m}{D}; \therefore \text{ by last Cor. } \frac{1}{2a} = \frac{(1-n^2)}{D};$$

$$2a = \frac{D}{1-n^2}, \frac{1}{b^2} = \frac{1-n^2}{n^2 D^2 \sin.^2 \delta}, b^2 = \frac{n^2 D^2 \sin.^2 \delta}{1-n^2}.$$

Hence $e^2 = 1 - \frac{b^2}{a^2} = 1 - 4n^2(1-n^2)\sin.^2 \delta.$

By means of these formulæ, we may, under given circumstances, find the magnitude and position of the trajectory.

Cor. 4. It appears from the preceding investigation that the major axis is independent of the direction of projection. And that, if n be given, the excentricity is independent of the distance of projection.

Example. A body is projected at an angle of 30° with the distance, and with a velocity which is to the velocity from infinity as 4 to 5: to determine the ellipse described.

In this case $n = \frac{4}{5}$, $\sin. \delta = \frac{1}{2}$;

$$\therefore 2a = \frac{D}{1-n^2} = \frac{25D}{9},$$

$$b^2 = \frac{n^2 D^2 \sin.^2 \delta}{1-n^2} = \frac{4D^2}{9};$$

$$\therefore \frac{b}{a} = \frac{12}{25} = .48 ; \therefore e^2 = 1 - \frac{b^2}{a^2} = .7696$$

$$e = .877.$$

And at the point of projection

$$D = \frac{a(1-e^2)}{1 + e.\cos.(v-\alpha)} = \frac{\frac{25\,D}{18} \times .2304}{1 + .877 \cos.(v-\alpha)};$$

$$\therefore \cos.(v-\alpha) = \frac{\frac{25}{18} \times .2304 - 1}{.877} = -\frac{680}{877}$$

$$= -.7753, \&c. = -\cos. 39^0. 10'.$$

Hence, $v - \alpha = 140^\circ\, 50' = ASP$, fig. 11.

Prob. IV. *To find the time of describing any portion of the elliptical orbit**.

We have the equation $dt = \frac{r^2 dv}{h}$, (Art. 19.); $\therefore t = \frac{\int r^2 dv}{h}$;

$$\text{and since } \frac{m}{h^2} = \tfrac{1}{2}\left\{\frac{1}{r'} + \frac{1}{r''}\right\} = \tfrac{1}{2} \cdot \frac{2\,a}{a^2 - a^2 e^2};$$

$$\therefore h = \sqrt{(am)} \cdot \sqrt{(1-e^2)}.$$

Instead of substituting for r, its value in terms of v, which would produce for dt an expression not readily integrable; we shall express the time in terms of another angle u, as follows.

On the major axis of the ellipse Aa, fig. 13, let a semi-circle be described, and MPQ drawn perpendicular to the axis, and SQ, CQ joined: and let $ACQ = u$.

The expression $\int r^2 dv$, beginning from A, is twice the area ASP. Now it is easily seen that

$$\frac{\text{area } ASP}{\text{area } ASQ} = \frac{MP}{MQ} = \frac{BC}{AC}.$$

* *Principia*, Book I, Prop. 31.

$$\therefore \text{ area } ASP = \frac{b}{a} \cdot \text{area } ASQ = \frac{b}{a} \text{ (area } ACQ - SCQ)$$

$$= \frac{b}{a} (\tfrac{1}{2} AC \cdot AQ - \tfrac{1}{2} SC \cdot MQ)$$

$$= \frac{b}{2a} (a \cdot au - ae \cdot a \sin. u);$$

$$\therefore \int r^2 dv = 2 \text{ area } ASP = ab\,(u - e \sin. u),$$

and putting for b, $a\sqrt{(1 - e^2)}$, and for h its value,

$$t = \frac{\int r^2 dv}{h} = \frac{a^{\frac{3}{2}}}{m^{\frac{1}{2}}} (u - e \sin. u) \ldots\ldots\ldots\ldots (1),$$

the time being supposed to begin from A.

We have now to find the relation between u and v;

$$HP^2 - HM^2 = PM^2 = SP^2 - SM^2;$$

$$\therefore HP^2 - SP^2 = HM^2 - SM^2;$$

$$(HP + SP)(HP - SP) = (HM + SM)(HM - SM);$$

$$2AC \cdot (2AC - 2SP) = 2CM \cdot 2CS;$$

or, dividing by $2 \cdot 2$: and putting $a \cos. u$ for CM,

$$a(a - r) = a \cos. u \cdot ae;$$

$$\therefore SP = r = a - ae \cos. u,$$

$$\text{and } \cos. v = \frac{SM}{SP} = \frac{CM - CS}{SP} = \frac{a \cos. u - ae}{a - ae \cos. u};$$

$$\therefore \cos. v = \frac{\cos. u - e}{1 - e \cos. u}.$$

$$\text{Hence } \frac{1 - \cos. v}{1 + \cos. v} = \frac{1 - e \cos. u + e - \cos. u}{1 - e \cos. u - e + \cos. u}$$

$$= \frac{(1 + e)(1 - \cos. u)}{(1 - e)(1 + \cos. u)};$$

$$\therefore \tan.^2 \frac{v}{2} = \frac{1 + e}{1 - e} \cdot \tan.^2 \frac{u}{2};$$

$$\text{and } \tan. \frac{v}{2} = \sqrt{\frac{1 + e}{1 - e}} \cdot \tan. \frac{u}{2} \ldots\ldots\ldots\ldots (2).$$

Hence, we can find u in terms of v by (2), and then t in terms of u by (1); and conversely. The angle u, or ACQ, is called in Astronomy the *eccentric anomaly*, ASP being called the *true anomaly**.

Cor. To find the time of a half revolution from A to a, it is evident that we must take u from 0 to π: which will give

$$t = \frac{a^{\frac{3}{2}}}{m^{\frac{1}{2}}} \cdot \pi.$$

Hence, the time of a revolution is $\dfrac{2\, a^{\frac{3}{2}} \pi}{m^{\frac{1}{2}}}$.

If the trajectory be an hyperbola, the calculations will be nearly the same as in the case of the ellipse.

If the force be repulsive, an hyperbola will be described having the centre of force in its exterior focus: and its properties will be analogous to those in the other cases.

Prob. V. *In the case when the orbit is a parabola; to find the expression for the time*†.

As before, $dt = \dfrac{r^2 dv}{h}$.

And $r = \dfrac{2\, a}{1 + \cos. v}$; measuring angles from the vertex, so that $\alpha = 0$.

Also $h = \sqrt{(2\, m a)}$; see p. 33.

$$\text{Hence,} \quad dt = \frac{4\, a^2 dv}{\sqrt{(2\, m a)} . (1 + \cos. v)^2}$$

$$= \frac{a^{\frac{3}{2}}}{(2\, m)^{\frac{1}{2}}} \cdot \frac{dv}{\cos.^4 \frac{v}{2}};$$

* See Woodhouse's *Astron.* p. 190, edit. 1.

For expansions of v and r, see Laplace, *Mec. Cel.* Part. I, Liv. 2. p. 180, &c.

† *Principia*, Book I, Prop. 30.

$$= \frac{a^{\frac{3}{2}}}{(2\,m)^{\frac{1}{2}}} \,.\, dv \left\{ \frac{\cos.^2 \frac{v}{2} + \sin.^2 \frac{v}{2}}{\cos.^4 \frac{v}{2}} \right\}$$

$$= \frac{a^{\frac{3}{2}}}{(2\,m)^{\frac{1}{2}}} \cdot \frac{dv}{\cos.^2 \frac{v}{2}} \left\{ 1 + \tan.^2 \frac{v}{2} \right\},$$

and since $d\,.\tan.\frac{v}{2} = \frac{dv}{\cos.^2\frac{v}{2}}$, we can integrate; and find

$$t = \frac{a^{\frac{3}{2}}}{2\,m^{\frac{1}{2}}} \left\{ \tan.\frac{v}{2} + \frac{1}{3}\tan.^3\frac{v}{2} \right\};$$

t being supposed to begin at the vertex, where $v = 0$.

Prob. VI. *Let the force be inversely as the cube of the distance, or* $\mathrm{P} = \frac{\mathrm{m}}{\mathrm{r}^3} = \mathrm{m\,u}^3$*.

The equation in this case becomes

$$\frac{d^2u}{dv^2} + u - \frac{mu}{h^2} = 0.$$

To integrate this equation, let $u = \epsilon^{kv}$ be a particular solution. (See *Lacroix*, Elem. Treat. Art. 280.)

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* Newton considered the curves described when the force is inversely as the cube of the distance, and besides the logarithmic spiral, noticed the curves, Species I, V, and VI; but omitted the examination of the others, by supposing the body to move from an apse. *Principia*, Book. I, Prop. 9, and Prop. 41, Cor. 3. The complete analysis of this case was given by Cotes in his *Logometria;* Phil. Trans. 1715; from which circumstance these curves are sometimes called Cotes's Spirals. From certain analogies observed by Newton, Species I, and V, are called the Hyperbolic and Elliptic Spiral, respectively. It may be remarked, however, that the Reciprocal Spiral is sometimes, by foreign writers, called the Hyperbolic Spiral.

Thence, $k^2 + 1 - \frac{m}{h^2} = 0$;

and if γ, $-\gamma$ be the two values of k in this equation, the general integral will be

$$u = C\epsilon^{\gamma v} + C'\epsilon^{-\gamma' v};$$

C, C' being any two arbitrary constants.

The curves described will be different, as the values of $\pm\gamma$ are possible or impossible, and as the arbitrary constants are positive or negative. We shall consider the different species thus produced.

Species I. Let $\frac{m}{h^2} > 1$, and C, C' both the same sign.

Hence, $k = \pm\sqrt{\left(\frac{m}{h^2} - 1\right)} = \pm\gamma$.

Suppose therefore $u = C\epsilon^{\gamma v} + C'\epsilon^{-\gamma v}$;

hence, $\frac{du}{dv} = \gamma\,\{C\epsilon^{\gamma v} - C'\epsilon^{-\gamma v}\}$.

Now when $\frac{du}{dv} = 0$, $C\epsilon^{\gamma v} = C'\epsilon^{-\gamma v}$, or $\epsilon^{2\gamma v} = \frac{C'}{C}$; which can always be fulfilled by a possible value of v: let this value be a, so that

$$C\epsilon^{\gamma a} = C'\epsilon^{-\gamma a} = c; \therefore C = c\epsilon^{-\gamma a}, C' = c\epsilon^{\gamma a}.$$

Hence, $u = c\,\{\epsilon^{\gamma(v-a)} + \epsilon^{-\gamma(v-a)}\}$.

When $v = a$, $u = 2c$; and since at that point $\frac{du}{dv} = 0$, the curve is perpendicular to the radius, or there is an apse*. As v increases, u increases, and therefore r diminishes, and when v becomes infinite, r becomes 0. Hence, the curve is such as is represented in fig. 14.

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* An apse is a point where the curve is perpendicular to the radius vector, and where, consequently, in general, the radius vector will be either a maximum or minimum.

If C, C' be both negative, the curve will be the same. The sign can only indicate that the angle v is to be measured in the opposite direction.

SPECIES II. Let $\frac{m}{h^2} > 1$; and $C' = 0$.

Therefore $u = C\epsilon^{\gamma v}$; $\gamma v = \text{hyp. log.} \frac{u}{C} = \text{hyp. log.} \frac{a}{r}$, if $a = \frac{1}{C}$.

Hence, the curve is the *logarithmic spiral*, fig. 15.

Differentiating, $\gamma dv = -\frac{dr}{r}$; $-\frac{dr}{r.dv} = \gamma = \sqrt{\left(\frac{m}{h^2} - 1\right)}$:

hence, $\sqrt{\left(\frac{m}{h^2} - 1\right)}$ is the co-tangent of the constant angle SPY, which the tangent makes with the radius vector: and therefore $\frac{\sqrt{m}}{h}$ is the co-secant, and $\frac{h}{\sqrt{m}}$ the sine of SPY.

$$\text{Let } \frac{h}{\sqrt{m}} = \sin.\beta; \therefore h = \sin.\beta\sqrt{m}.$$

If $C = 0$, the curve will be the same.

SPECIES III. Let $\frac{m}{h^2} > 1$, and C' negative.

Therefore $u = C\epsilon^{\gamma v} - C'\epsilon^{-\gamma v}$.

Now when $u = 0$, $C\epsilon^{\gamma v} = C'\epsilon^{-\gamma v}$ and $\epsilon^{2\gamma v} = \frac{C'}{C}$, for which there is always a possible value of v: let this value be α; and let $C\epsilon^{\gamma\alpha} = C'\epsilon^{-\gamma\alpha} = c$,

$\therefore C = c\epsilon^{-\gamma\alpha}$; $C' = c\epsilon^{\gamma\alpha}$: and hence $u = c\left\{\epsilon^{\gamma(v-\alpha)} - \epsilon^{-\gamma(v-\alpha)}\right\}$.

When $v = \alpha$, since $u = 0$, r is infinite. As v increases, u increases, and r decreases; and when v is infinite, u is also infinite, and r vanishes. Hence, the form of the curve is that in fig. 16, $v - \alpha$ being the angle ASP.

If p be the perpendicular from S upon the tangent, we have, (p. 28,)

$$\frac{1}{p^2} = \frac{ds^2}{r^4 dv^2} = \frac{r^2 dv^2 + dr^2}{r^4 dv^2} = u^2 + \frac{du^2}{dv^2}$$

$$= c^2 \{\epsilon^{\gamma(v-\alpha)} - \epsilon^{-\gamma(v-\alpha)}\}^2 + c^2\gamma^2 \{\epsilon^{\gamma(v-\alpha)} + \epsilon^{-\gamma(v-\alpha)}\}^2$$

$$= c^2(1+\gamma^2)\{\epsilon^{2\gamma(v-\alpha)} + \epsilon^{-2\gamma(v-\alpha)}\} + 2c^2(\gamma^2 - 1),$$

$$\text{and when } v = \alpha, \ \frac{1}{p^2} = c^2(1+\gamma^2).2 + 2c^2(\gamma^2 - 1)$$

$$= 4c^2\gamma^2; \text{ and } p = \frac{1}{2c\gamma}.$$

And, hence there is an asymptote BZ to the curve, parallel to SA, at a distance $SB = \dfrac{1}{2c\gamma}$.

Similarly, if C' be positive and C negative.

Species IV. Let $\dfrac{m}{h^2} = 1$.

In this case we must return to the original equation, which here gives us

$$\frac{d^2u}{dv^2} = 0; \ \therefore \frac{du}{dv} = C, \ u = C(v-\alpha): \ r = \frac{1}{C(v-\alpha)} = \frac{a}{v-\alpha};$$

it being supposed that when $u = 0$, $v = \alpha$.

For this position r is infinite: any other value of r, as SP, is reciprocally as the angle $v - \alpha$, or ASP. Hence, the curve in this case is the *Reciprocal Spiral*, fig. 17.

If a circular arc PQ be described with centre S, $PQ = r(v-\alpha) = a$; and hence, is at every point the same.

It is manifest that the curve will have an asymptote BZ, such, that $SB = a$.

Species V. Let $\dfrac{m}{h^2} < 1$.

In this case the values of k in the equation $k^2 + 1 - \dfrac{m}{h^2} = 0$, are impossible. Let them be $\pm\gamma\sqrt{-1}$. Therefore for the general integral of the equation we have

$$u = C\epsilon^{\gamma v\sqrt{-1}} + C'\epsilon^{-\gamma v\sqrt{-1}}$$

$$= \tfrac{1}{2}(C+C')(\epsilon^{\gamma v\sqrt{-1}} + \epsilon^{-\gamma v\sqrt{-1}})$$

$$+ \tfrac{1}{2}(C-C')(\epsilon^{\gamma v\sqrt{-1}} - \epsilon^{-\gamma v\sqrt{-1}})$$

$$= C_1 \cos.\gamma v + C_2 \sin.\gamma v,$$

making $C_1 = C + C'$, and $C_2 = \sqrt{-1}\,(C-C')$.

Hence, $\dfrac{du}{dv} = -\gamma C_1 \sin.\gamma v + \gamma C_2 \cos.\gamma v$;

and when $\dfrac{du}{dv} = 0$, $\tan.\gamma v = \dfrac{C_2}{C_1}$: for which there is always a value of v, whether C_1 and C_2 be of the same or of different signs. Let a be this value;

$$\therefore C_2 = C_1 \frac{\sin.\gamma a}{\cos.\gamma a};$$

$$\therefore u = \frac{C_1}{\cos.\gamma a}\{\cos.\gamma v \cos.\gamma a + \sin.\gamma v \sin.\gamma a\}$$

$$= \frac{C_1}{\cos.\gamma a}\cos.\gamma(v-a),$$

or, making $\dfrac{C_1}{\cos.\gamma a} = \dfrac{1}{a}$, $u = \dfrac{\cos.\gamma(v-a)}{a}$, and $r = \dfrac{a}{\cos.\gamma(v-a)}$; when $v = a$, $r = a$, and there is an apse. When $\gamma(v-a) = \dfrac{\pi}{2}$, r is infinite, and, therefore, may be parallel to an asymptote; to find the position of the asymptote, we have

$$\frac{1}{p^2} = u^2 + \frac{du^2}{dv^2} = \frac{\cos.^2\gamma(v-a)}{a^2} + \frac{\gamma^2\sin.^2\gamma(v-a)}{a^2};$$

and when $\gamma(v-a) = \dfrac{\pi}{2}$, $\dfrac{1}{p^2} = \dfrac{\gamma^2}{a^2}$, $p^2 = \dfrac{a^2}{1-\dfrac{m}{h^2}} = SB^2$.

The form of the curve is given in fig. 18.

Prob. VII. *To determine in what cases each of these curves will be described.*

We may observe, that in the case where the body describes a circle, and consequently where $\frac{d^2u}{dv^2} = 0$, we have, in a circle,

$$u - \frac{mu}{h^2} = 0, \text{ and } \frac{m}{h^2} = 1, \text{ or } h = \sqrt{m}, \text{ the area in time 1.}$$

Now the species varies as $\frac{m}{h^2}$ in the curve is greater than, equal to, or less than 1: that is, as h, the area in time 1 in the curve, is less than, equal to, or greater than $\sqrt{m}$, its value in the circle. So that if the area in a given time be *less* than that in a circle with the same force, we shall have Species I, II, or III; if the areas be *equal*, we have Species IV; if the area in the curve be *greater*, we have Species V.

In these two latter cases it is clear, that since the area is not less than it is in the circle when the radii vectors are the same, the velocity will be greater than it is in a circle. In the three first cases we may thus compare those velocities.

In the circle whose radius is r, since $r^2 \cdot \text{velocity}^2 = h^2 = m$, we have $\text{velocity}^2 = \frac{m}{r^2} = mu^2$*.

In the curve, $\text{velocity}^2 = h^2\left(u^2 + \frac{du^2}{dv^2}\right)$.

$$\text{But } u = C\epsilon^{\gamma v} + C'\epsilon^{-\gamma v},$$

$$\left(\frac{du}{dv}\right)^2 = \{\gamma C\epsilon^{\gamma v} - \gamma C'\epsilon^{-\gamma v}\}^2$$

$$= \gamma^2 u^2 - 4\gamma^2 CC'.$$

$$\text{Hence, velocity}^2 = h^2\left((1+\gamma^2)u^2 - 4\gamma^2 CC'\right)$$

$$= mu^2 - 4h^2\gamma^2 CC'; \text{ since } h^2(1+\gamma^2) = m.$$

If the velocity be *less* than that in a circle, we have CC' negative,

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* This is also the velocity from an infinite distance.

and therefore the curve is Species I. If the velocity be *equal* to that in a circle, we have Species II. If the velocity in the curve be *greater*, we have Species III.

If the force be repulsive, the equation will resemble the one for Species V, and the curve, which we may call Species VI, will be as in fig. 19.

PROB. VIII. *To find the time of describing any portion of the curves in Prob. VI.*

In Species I, if we suppose the angle v to be measured from the apse, and consequently $\alpha = 0$, we shall have

$$hdt = r^2 dv = \frac{dv}{u^2} = \frac{1}{c^2} \frac{dv}{\{\epsilon^{\gamma v} + \epsilon^{-\gamma v}\}^2}$$

$$= \frac{1}{c^2} \cdot \frac{\epsilon^{2\gamma v} dv}{\{\epsilon^{2\gamma v} + 1\}^2};$$

$$\therefore ht = C - \frac{1}{2c^2\gamma} \cdot \frac{1}{\epsilon^{2\gamma v} + 1}.$$

We may suppose the time to begin when $v = 0$: on this supposition we have

$$t = \frac{1}{2c^2\gamma h}\left\{\frac{1}{2} - \frac{1}{\epsilon^{2\gamma v} + 1}\right\} = \frac{a^2}{\gamma h} \cdot \frac{\epsilon^{2\gamma v} - 1}{\epsilon^{2\gamma v} + 1}; \text{ if } a = SA = \frac{1}{2c}.$$

Similarly, we should find

in Species II, $t = \dfrac{a^2}{2\gamma h}\{1 - \epsilon^{-2\gamma v}\} = \dfrac{a^2}{\text{cot.}\,\beta\sqrt{m}}\{1 - \epsilon^{-2\gamma v}\}$,

v and t being measured from the point where the radius vector $= a$.

In Species III, $t = \dfrac{2a^2}{\gamma h} \dfrac{1}{\epsilon^{2\gamma v} - 1}$, if $a = \dfrac{1}{2c}$; $v = ASP$, being measured from SA, and t being the time from P to the centre.

In Species IV, $t = \dfrac{a^2}{hv} = \dfrac{a^2}{\sqrt{m} \cdot v}$; v being measured from SA, and t being the time to the centre.

In Species V, $t = \frac{a^2}{\gamma h}$ tan. γv; v and t being measured from the apse.

Cor. 1. In order to find the time τ of describing a given angle δ, we must take the value of t between the values v and $v+\delta$; we shall thus have in Species I,

$$h\tau = \frac{1}{2c^2\gamma}\left\{\frac{1}{\epsilon^{2\gamma v}+1} - \frac{1}{\epsilon^{2\gamma(v+\delta)}+1}\right\}$$

$$= \frac{1}{2c^2\gamma} \cdot \frac{\epsilon^{2\gamma(v+\delta)} - \epsilon^{2\gamma v}}{(\epsilon^{2\gamma v}+1)(\epsilon^{2\gamma(v+\delta)}+1)}$$

$$= \frac{1}{2c^2\gamma} \frac{\epsilon^{\gamma\delta} - \epsilon^{-\gamma\delta}}{\{\epsilon^{\gamma v}+\epsilon^{-\gamma v}\}\{\epsilon^{\gamma(v+\delta)}+\epsilon^{-\gamma(v+\delta)}\}}$$

$$= \frac{\epsilon^{\gamma\delta} - \epsilon^{-\gamma\delta}}{2\gamma} \cdot r_1 r_2;$$

r_1 and r_2 being the radii at the beginning and end of the given angle.

Similarly,

in Species II, $h\tau = \frac{(\epsilon^{\gamma\delta} - \epsilon^{-\gamma\delta})}{2\gamma} r_1 r_2$,

in Species III, $h\tau = \frac{(\epsilon^{\gamma\delta} - \epsilon^{-\gamma\delta})}{2\gamma} r_1 r_2$,

in Species IV, $h\tau = \delta \cdot r_1 r_2$,

in Species V, $h\tau = \frac{\sin.\ \gamma\delta}{\gamma} r_1 r_2$.

Cor. 2. In all the cases, the times of successive revolutions in the same spiral are as the extreme radii.

Let a straight line $SRQP$, fig. 14, drawn from S, cut the spiral successively in R, Q, P: thence, since in this case $\delta = 2\pi$ is constant, we have

time from P to Q : time from Q to R :: $SP.SQ$: $SQ.SR$
:: SP : SR.

PROB. IX. *Let the force be inversely as the 5th power of the distance, or* $P = \frac{m}{r^5} = m u^5$.

Therefore, by (*d*), $\frac{d^2 u}{dv^2} + u - \frac{m u^3}{h^2} = 0$;

and multiplying by $2du$ and integrating,

$$\frac{du^2}{dv^2} + u^2 - \frac{m u^4}{2h^2} = C;$$

C being an arbitrary constant; hence,

$$\frac{du^2}{dv^2} = C - u^2 + \frac{m u^4}{2h^2}.$$

This equation cannot be integrated generally by the common methods.

When the right hand member is a square, it becomes simple; that is, if 4 times the product of the extreme terms be equal to the square of the middle term;

$$\text{if } \frac{2Cm}{h^2} = 1; \text{ if } C = \frac{h^2}{2m}.$$

On this supposition,

$$\frac{du}{dv} = \pm \frac{1}{\sqrt{2}} \cdot \left\{\frac{h}{\sqrt{m}} - \frac{u^2 \sqrt{m}}{h}\right\} = \pm \frac{\sqrt{m}}{h\sqrt{2}} \left\{\frac{h^2}{m} - u^2\right\};$$

and we shall have two different Species as we take the + or the − sign.

In the first case, $\dfrac{\frac{2h\,du}{\sqrt{m}}}{\frac{h^2}{m} - u^2} = dv\sqrt{2}$;

$$\therefore \text{hyp. log. } \frac{\frac{h}{\sqrt{m}} + u}{\frac{h}{\sqrt{m}} - u} = \sqrt{2} \cdot (v - a);$$

a being the value of *v*, when $u = 0$.

When $v=a$, r is infinite; as u increases and r decreases, v increases, and when $u=\frac{h}{\sqrt{m}}$, or $r=\frac{\sqrt{m}}{h}$, v is infinite. Hence, the curve, fig. 20, has what may be called an *asymptotic circle* with radius $SA = \frac{\sqrt{m}}{h}$, to which circle it perpetually approximates, but which it never actually reaches.

We have $\frac{1}{p^2}=u^2+\frac{du^2}{dv^2}=u^2+\frac{h^2}{2m}-u^2+\frac{mu^4}{2h^2}=\frac{h^2}{2m}+\frac{mu^4}{2h^2}$;

and when r is infinite, or $u=0$, $p=\frac{\sqrt{(2m)}}{h}$; which is SB, the distance of the asymptote BZ from SA.

In the second case, we have

$$\frac{\frac{2h\,du}{\sqrt{m}}}{u^2-\frac{h^2}{m}}=dv\sqrt{2};$$

$$\therefore \text{hyp. log.}\ \frac{u-\frac{h}{\sqrt{m}}}{u+\frac{h}{\sqrt{m}}}=\sqrt{2}\,.\,(v-a),$$

a being the value of v when u is infinite.

$$\text{Hence, hyp. log.}\ \frac{u+\frac{h}{\sqrt{m}}}{u-\frac{h}{\sqrt{m}}}=\sqrt{2}\,.\,(a-v).$$

When $v=a$, u is infinite, and $r=0$; as u decreases, or r increases, $a-v$ also increases; and when $u=\frac{h}{\sqrt{m}}$, or $r=\frac{\sqrt{m}}{h}$, $a-v$ is infinite. Hence, the curve has, in this case also, an asymptotic circle, and is situated within it, as it was before without it. See fig. 21. SQ comes to SA when $v=0$; and $ASP=a-v$.

Cor. 1. We shall now compare the velocity with that in a circle.

In a circle with radius $= r$, velocity^2 $= Pr$, (see Art. 20.) $= \frac{m}{r^4} = mu^4$.

In the curve, velocity^2 $= h^2 \left(u^2 + \frac{du^2}{dv^2}\right)$

$$= \frac{h^4}{2m} + \frac{mu^4}{2}.$$

Now when $r = SA$, or $u^2 = \frac{h^2}{m}$, velocity^2 in curve $= \frac{mu^4}{2} + \frac{mu^4}{2} = mu^4 =$ velocity^2 in circle; which it manifestly should be, because as the radius approximates to SA, the motion approximates to circular motion.

In the first case u is always less than $\frac{h}{\sqrt{m}}$, and hence the velocity is always less than that in a circle.

In the second case u is always greater than this value, and the velocity is greater than that in a circle at the same distance.

Cor. 2. To find the velocity, so that one of these curves may be described.

Let, at any point P, the angle $SPY = \beta$, SY being a perpendicular on the tangent. Therefore $h^2 =$ velocity^2. $SY^2 =$ velocity^2 $r^2 \sin.^2 \beta$.

Now let the velocity be ϵ times that in a circle at the distance SP: that is, velocity^2 $= \epsilon^2 mu^4$: hence,

$$\epsilon^2 mu^4 = \frac{h^4}{2m} + \frac{mu^4}{2}; \ \therefore (2\epsilon^2 - 1)\, m^2 u^4 = h^4;$$

$$\text{but } h^2 = \epsilon^2 mu^4 \,.\, \frac{\sin.^2 \beta}{u^2}; \ \therefore \sin.^2 \beta = \frac{h^2}{\epsilon^2 mu^2} = \frac{\sqrt{(2\epsilon^2 - 1)}}{\epsilon^2}.$$

Hence, if ϵ be given, we can find $\sin.^2 \beta$; and hence, the direction in which the body must be projected to describe the curve. It will belong to the first or second Species, as ϵ is less or greater than 1.

Also $\epsilon^2 \sin.^2\beta = \sqrt{(2\epsilon^2-1)}$; $\epsilon^4 \sin.^4\beta - 2\epsilon^2 + \frac{1}{\sin.^4\beta} = \frac{1}{\sin.^4\beta} - 1$;

$$\therefore \epsilon^2 \sin.^2\beta - \frac{1}{\sin.^2\beta} = \pm \frac{\sqrt{(1-\sin.^4\beta)}}{\sin.^2\beta};\ \epsilon^2 = \frac{1 \pm \sqrt{(1-\sin.^4\beta)}}{\sin.^4\beta}:$$

and the first or second curve will be described, as we take the lower or the upper sign.

COR. 3. By equating the values of ASP in the two species, we should find for the same angle ASP, fig. 20, $SP'.SP = SA^2$.

PROB. X. *Let the force vary inversely as any power of the distance, or* $P = mu^n$.

$$\text{Therefore } \frac{d^2u}{dv^2} + u - \frac{mu^{n-2}}{h^2} = 0;$$

multiplying by $2du$, and integrating,

$$\frac{du^2}{dv^2} + u^2 - \frac{2mu^{n-1}}{(n-1)h^2} = C;$$

$$\text{whence, } \frac{du}{\sqrt{\left\{C - u^2 + \frac{2mu^{n-1}}{(n-1)h^2}\right\}}} = dv;$$

and if the expression on the first side be integrable, we can find the relation between u and v.

To find the time, we have

$$dt = \frac{dv}{hu^2} = \frac{du}{hu^2\sqrt{\left\{C - u^2 + \frac{2mu^{n-1}}{(n-1)h^2}\right\}}}.$$

The quantity C will depend upon the velocity, and will be known, if we know the velocity for a given point; which may be called *the velocity of projection,* if we consider this point as the beginning of the motion. For we have

$$\text{velocity}^2 = h^2\left(u^2 + \frac{du^2}{dv^2}\right) = \frac{2mu^{n-1}}{(n-1)} + h^2C,$$

and if, when $u = a$, we have velocity $= V$,

$$V^2 = \frac{2ma^{n-1}}{(n-1)} + h^2C; \text{ whence } C \text{ is known.}$$

It may be convenient to compare the velocity with that acquired by falling from an infinite distance. Let q be the velocity acquired by falling through any space towards the centre. Therefore

$$qdq = -\frac{mdr}{r^n},$$

$$q^2 = \frac{2m}{(n-1)\,r^{n-1}} + \text{const.};$$

and if q be the velocity acquired from infinity,

$$\text{const.} = 0,\ q^2 = \frac{2m}{(n-1)\,r^{n-1}} = \frac{2mu^{n-1}}{n-1}.$$

Hence, if at the point of projection, when $u = a$, the velocity be ϵ times that from infinity, we have

$$\epsilon^2 . \frac{2ma^{n-1}}{n-1} = \frac{2ma^{n-1}}{n-1} + h^2 C;$$

$$\therefore\ h^2 C = (\epsilon^2 - 1) . \frac{2ma^{n-1}}{n-1}.$$

Cor. At the apsides we have $\dfrac{du}{dv} = 0$;

$$\therefore C - u^2 + \frac{2mu^{n-1}}{(n-1)\,h^2} = 0,$$

or, putting for C its value,

$$(\epsilon^2 - 1)\,a^{n-1} - \frac{(n-1)\,h^2}{2m} . u^2 + u^{n-1} = 0.$$

This may have four roots possible, {for instance, if $n = 5$, and $\dfrac{(n-1)^2 h^4}{4m^2} > 4(\epsilon^2 - 1)\,a^4$,} but only two give apsidal distances; in fact the other two are always negative.

Prob. XI. *In the particular case where the velocity is equal to that from infinity*, to find the curves.*

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* If the velocity be at one point that from infinity, it will be so at all points. For, by Art. 17, Cor. 2, if the velocity at the distance a be that from infinity, it will, at the distance r, be the same as if the body had continued to descend in a straight line.

Here in the last Prob. $C=0$, and we can integrate. For we have

$$dv = \frac{du}{\sqrt{\left\{\frac{2\,m u^{n-1}}{(n-1)h^2} - u^2\right\}}} = \frac{du}{u\sqrt{\left\{\frac{2\,m u^{n-3}}{(n-1)h^2} - 1\right\}}}.$$

Let $\frac{2\,mu^{n-3}}{(n-1)h^2} = y^2$; $\therefore \frac{(n-3)\,du}{u} = \frac{2\,dy}{y}$;

$$dv = \frac{2}{n-3} \cdot \frac{dy}{y\sqrt{(y^2-1)}};$$

$$(v-a) = \frac{2}{n-3} \text{ arc (sec.} = y);$$

$$\therefore \cos. \frac{n-3}{2}(v-a) = \frac{1}{y} = \frac{h\sqrt{(n-1)}}{\sqrt{(2m)}\,.\,u^{\frac{n-3}{2}}}.$$

Hence, if $n > 3$; $\frac{h\sqrt{(n-1)}}{\sqrt{(2m)}}.r^{\frac{n-3}{2}} = \cos. \frac{n-3}{2}(v-a)$.

Similarly, if $n < 3$; $\frac{h\sqrt{(n-1)}}{\sqrt{(2\,m)}}\,\frac{1}{r^{\frac{3-n}{2}}} = \cos. \frac{3-n}{2}(a-v)$.

In the first case, it is manifest that when the first side is $= 1$, or $r^{\frac{n-3}{2}} = \frac{\sqrt{(2\,m)}}{h\,\sqrt{(n-1)}}$, the figure has an apse. It is symmetrical on the two sides of this apse, and r diminishes as $v-a = ASP$, fig. 22, increases. When $\frac{n-3}{2}(v-a) = \frac{\pi}{2}$, or $v-a = \frac{\pi}{n-3}$, we have $r=0$, and the curve passes through the centre, as in fig. 22.

In the second case, r increases as $a-v = ASP$, fig. 23, increases, when $\frac{3-n}{2}(a-v) = \frac{\pi}{2}$, or $a-v = \frac{\pi}{3-n}$, r is infinite, and the curve is parallel to it. To find the nature of the infinite branch we have $\frac{1}{p^2} = u^2 + \frac{du^2}{dv^2} = \frac{2\,m\,u^{n-1}}{(n-1)\,h^2}$; and when r is infinite $u=0$, and p is infinite; hence, the branch AZ has no asymptote.

Prob. XII. *Let the force vary inversely as any power of the distance; it is required to find the conditions requisite that the orbit may have an asymptotic circle.* See Prob. IX.

$P = mu^n$, and as before in Prob. X.

$$dv = \frac{du}{\sqrt{\left\{C - u^2 + \frac{2mu^{n-1}}{(n-1)h^2}\right\}}},$$

$$v = \int \frac{du}{\sqrt{\left\{C - u^2 + \frac{2mu^{n-1}}{(n-1)h^2}\right\}}}.$$

Now if the orbit have an asymptotic circle, of which the radius is $\frac{1}{c}$, it is manifest that the value of v, taken from any value of u up to $u = c$, will be infinite. That is, the integral on the right hand side must be infinite when $u = c$. Also $u - c$ is necessarily a factor of the denominator, because when $u = c$, $\frac{du}{dv} = 0$, and therefore $C - u^2 + \frac{2mu^{n-1}}{(n-1)h^2} = 0$. But if the denominator has *two* factors $u - c$, the integral will be infinite for $u = c$. For in that case

$$v = \int \frac{du}{\sqrt{\{(u-c)^2 \cdot Q\}}},$$

Q involving u^{n-3}, and inferior powers of u. And if we put $u = c + z$, it is manifest that Q will become $A + Bz +$ &c. and

$$v = \int \frac{dz}{\sqrt{\{z^2 \cdot (A + Bz + \&c.)\}}}$$

$$= \int \frac{dz}{z\sqrt{A}} \left\{1 - \frac{Bz}{2A} + \&c.\right\}$$

$$= \frac{\text{hyp. log. } z}{\sqrt{A}} - \frac{Bz}{2A\sqrt{A}} + \&c.;$$

the other terms involving direct powers of z. Hence, when $u = c$, $z = 0$, v becomes infinite.

We shall therefore have an asymptotic circle if there be the factor $u - c$ twice in the denominator of dv; that is, if the equation

$$\frac{2\,mu^{n-1}}{(n-1)h^2} - u^2 + C = 0$$

have two roots c, c.

But in this case the equation

$\dfrac{2\,mu^{n-2}}{h^2} - 2\,u = 0$, has one of these roots; therefore

$$\frac{2\,mc^{n-1}}{(n-1)h^2} - c^2 + C = 0, \text{ or } C = c^2 - \frac{2\,mc^{n-1}}{(n-1)h^2},$$

and $\dfrac{2\,mc^{n-2}}{h^2} - 2\,c = 0$, or $h^2 = m\,c^{n-3}$.

Now, in the curve, velocity$^2 = h^2 \left(\dfrac{du^2}{dv^2} + u^2\right)$

$$= h^2 \left(\frac{2\,m\,u^{n-1}}{(n-1)h^2} + C\right);$$

(putting for C its value,) $= \dfrac{2\,mu^{n-1}}{n-1} + c^2 h^2 - \dfrac{2\,mc^{n-1}}{(n-1)}$;

(putting for h its value,) $= \dfrac{2\,mu^{n-1}}{n-1} + mc^{n-1} - \dfrac{2\,mc^{n-1}}{n-1}$*;

$$= \frac{m}{n-1}\left\{2\,u^{n-1} + (n-3)\,c^{n-1}\right\}.$$

Let at any point the value of u be b, and the angle SPY, which the tangent makes with the radius vector, β; and suppose that at this point the velocity is ϵ times that in a circle. Now, in a circle velocity^2 = force $\times$ radius (Art. 20.) $= m\,u^n \cdot \dfrac{1}{u} = m\,u^{n-1}$. And

* Hence, when $u = c$, velocity$^2 = m\,c^{n-1}$ = velocity^2 in a circle; as it manifestly should be.

when $u=b$, velocity^2 in circle $= mb^{n-1}$, and velocity^2 in curve $= \epsilon^2 mb^{n-1}$.

$$\text{Hence, } \epsilon^2 mb^{n-1} = \frac{m}{n-1}\{2b^{n-1} + (n-3)c^{n-1}\};$$

$$\text{and } \{(n-1)\epsilon^2 - 2\}b^{n-1} = (n-3)c^{n-1};$$

$$\therefore\ c = b\left\{\frac{(n-1)\epsilon^2 - 2}{n-3}\right\}^{\frac{1}{n-1}}.$$

$$\text{Also } h^2 = \frac{1}{b^2}.\text{velocity}^2 \sin.^2\beta = \frac{1}{b^2}.\epsilon^2 mb^{n-1}.\sin.^2\beta$$

$$= \epsilon^2.mb^{n-3}\sin.^2\beta.$$

But $h^2 = mc^{n-3}$;

$$\therefore\ c^{n-3} = \epsilon^2 b^{n-3}.\sin.^2\beta;\ \text{and } c = b.(\epsilon.\sin.\beta)^{\frac{2}{n-3}}.$$

$$\text{Therefore } (\epsilon \sin.\beta)^{\frac{2}{n-3}} = \left\{\frac{(n-1)\epsilon^2 - 2}{n-3}\right\}^{\frac{1}{n-1}}.$$

$$\text{And } \sin.\beta = \frac{1}{\epsilon}.\left\{\frac{(n-1)\epsilon^2 - 2}{n-3}\right\}^{\frac{n-3}{2n-2}},$$

which gives the relation between the velocity and the direction of projection, in order that the curve may have an asymptotic circle.

The radius $\left(=\frac{1}{c}\right)$, of the circle, is easily found by the preceding formulæ. If $\frac{1}{b}$ is greater than $\frac{1}{c}$, the circle is an interior one as in fig. 20: that is,

$$\text{if } \frac{(n-1)\epsilon^2 - 2}{n-3} > 1,$$

$$\text{if } (n-1)\epsilon^2 - 2 > n-3,$$

$$\text{if } \epsilon^2 > 1, \text{ or if } \epsilon > 1.$$

If on the contrary ϵ be less than 1, the circle is exterior to the curve, as in fig. 21.

It is clear that we must have $n > 3$.

In nearly the same way we may find the conditions requisite for the description of an orbit, with an asymptotic circle, when the force is represented by any function whatever of u.

Prob. XIII. *Let the force consist of two parts, one of which varies inversely as the square, and the other inversely as the cube, of the distance.*

$$P = mu^2 + m'u^3;$$

$$\therefore \frac{d^2u}{dv^2} + u - \frac{m}{h^2} - \frac{m'u}{h^2} = 0;$$

$$\text{or } \frac{d^2u}{dv^2} + \left(1 - \frac{m'}{h^2}\right) u - \frac{m}{h^2} = 0.$$

To integrate, let $\left(1 - \frac{m'}{h^2}\right) u - \frac{m}{h^2} = \left(1 - \frac{m'}{h^2}\right) w,$

$$\text{or } u = w + \frac{m}{h^2 - m'};$$

$$\therefore \frac{d^2w}{dv^2} + \left(1 - \frac{m'}{h^2}\right) w = 0; \text{ or if } 1 - \frac{m'}{h^2} = \gamma^2,$$

$$\frac{d^2w}{dv^2} + \gamma^2 w = 0:$$

of which, by nearly the same process as in Prob. II, we shall find the integral to be

$$w = C_1 \cos. \gamma v + C_2 \sin. \gamma v;$$

$$\therefore u = C_1 \cos. \gamma v + C_2 \sin. \gamma v + \frac{m}{h^2 - m'}.$$

This may be transformed in exactly the same manner as in Prob. II; that is, let α be the value of v, which makes $\frac{du}{dv} = 0$, then the value which gives $\gamma v = \pi + \gamma\alpha$ will also make $\frac{du}{dv} = 0$; and if $\frac{1}{r'}$, $\frac{1}{r''}$, be the values of u, corresponding to these values of v, we shall have

$$u = \frac{1}{r} = \tfrac{1}{2}\left\{\frac{1}{r'} - \frac{1}{r''}\right\}\cos.\gamma(v-\alpha) + \tfrac{1}{2}\left\{\frac{1}{r'} + \frac{1}{r''}\right\};$$

which is the equation to the curve described, if r' and r'' be positive.

This manifestly agrees with the equation to an ellipse, of which the focus is in the centre of forces, except in having $\gamma(v-\alpha)$ instead of $v-\alpha$. Hence, the curve may be thus described: if, in fig. 24, and 25, Ap be an ellipse of which the focus is S; and Sp being any radius, if we take $ASP = \frac{ASp}{\gamma}$; so that ASP being $v-\alpha$ we may have $ASp = \gamma(v-\alpha)$; then $SP = Sp$ may be r, and the equation just found for r will be satisfied; therefore the curve APB thus described will be the path.

APB will be without the ellipse Ap, fig. 24, if γ be less than unity; that is, if $1 - \frac{m'}{h^2} < 1$, or if m' be positive. If m' be negative, or the force be $P = mu^2 - m'u^3$, the path described will be within the ellipse, as in fig. 25.

In both cases we shall have an apse B, corresponding to an apse b in the ellipse; at which point $ASb = \gamma . ASB$, and $SB = Sb$. Hence, since $ASb = \pi$, we have $ASB = \frac{ASb}{\gamma} = \frac{\pi}{\sqrt{\left(1 \pm \frac{m'}{h^2}\right)}}$.

ASB is *the angle between the apsides.*

After describing an angle $BSA' = ASB$, the body will come again to an apse at A', and so go on perpetually revolving about S; and approaching to it, and receding from it alternately.

The *line of apsides* SA retains always the same position, when a body describes an ellipse as in Prob. II. In the case of the present problem, this line, which is at first in the position SA, fig. 26, 27, would after one revolution come into the position SA', after a second, into the position SA'', and so on; the angles ASA', $A'SA''$, &c. being equal. Hence, this line is said to *revolve* round S. If it revolve in the direction of the body's motion, as in fig. 26, it is said to move *in consequentia*, or to *progress*; if it move in the opposite direction, as in fig. 27, it is said to move in *antecedentia*, or to *regress*. It appears by what has preceded, that the first or the second of these cases will occur, as the part of the force $m'u^3$ which varies inversely as the cube of the distance, is additive, or subtractive*.

If $P = mu^2 + m'u^3$ so that we have $ASB = \dfrac{\pi}{\sqrt{\left(1 - \frac{m'}{h^2}\right)}}$

it is manifest that we must have $\dfrac{m'}{h^2} < 1$; and therefore $h^2 > m'$.

When $h^2 < m'$, the body will fall into the centre without coming to a second apse, as might be shewn by integrating the equation

$$\frac{d^2u}{dv^2} - \left(\frac{m'}{h^2} - 1\right)u - \frac{m}{h^2} = 0.$$

When $h^2 = m'$, $\dfrac{d^2u}{dv^2} - \dfrac{m}{h^2} = 0$; $\dfrac{du}{dv} = \dfrac{m}{h^2}(v - a)$;

$$u = \frac{m}{h^2} \cdot \frac{(v-a)^2}{2} + c;$$ supposing that $u = c$ when $v = a$.

In this case the body approaches the centre by an indefinite number of revolutions.

Prob. XIV. *Let the force be represented by any function of the distance; it is required to find what value the angle between the apsides approximates to, when the orbit becomes very nearly a circle*†.

It is manifest, that if we project a body perpendicularly to the radius vector, with a velocity very little greater or less than the velocity in a circle for the same distance and force, the path of the body will not differ much from a circle. With many laws of force, the body will revolve perpetually between its greatest and least apsidal distances, as in last Prob. fig. 26, 27: and the angle between the

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* This corresponds with *Principia*, Book I, Prop. 43, 44.

† *Principia*, Book I, Prop. 45.

apsides will depend both upon the velocity and the law of force. As, however, the velocity approaches more nearly to that in a circle, the angle between the apsides will tend nearer and nearer to a certain limit. This limit it can never reach, because when the velocity becomes accurately that in a circle, the apsidal distances are equal, a circle is the curve described, and there is no longer, properly speaking, an angle between the apsides, as every point is an apse. But if we find this limiting angle, it may serve to indicate what is the angle between the apsides, when the difference of the higher and lower apsidal distances is small, but finite.

Let $P = u^2\phi u$; when ϕu is a function of u; so that P may be any function whatever of u;

$$\therefore \text{ by } (d), \ \frac{d^2 u}{dv^2} + u - \frac{\phi u}{h^2} = 0.$$

Now at the point where the body is projected perpendicularly to the radius, let $u = c$; and for any other point let $u = c + z$, z being small. Then $\phi u = \phi c + \phi' c . z + \phi'' c . \frac{z^2}{1.2} + \&c.$

Also if $1 : 1 + \delta$ were the ratio of the velocity^2 to the velocity^2 in a circle at the point of projection, we should have

in the circle, velocity^2 = force × radius (Art. 20) $= c^2 \phi c . \frac{1}{c} = c\phi c.$

$\therefore$ in the curve, velocity$^2 = \frac{c\phi c}{1+\delta}$;

$$\therefore \text{ in curve } h^2 = \frac{\text{velocity}^2}{c^2} = \frac{\phi c}{c(1+\delta)};$$

therefore, substituting in the original equation, we have

$$\frac{d^2 z}{dv^2} + c + z - \frac{(1+\delta)c}{\phi c}\left\{\phi c + \phi' c . z + \phi'' c . \frac{z^2}{1.2} + \&c.\right\} = 0;$$

$$\text{or } \left.\begin{aligned} &\frac{d^2 z}{dv^2} + \left(1 - \frac{c\phi' c}{\phi c}\right) z - \frac{c\phi'' c}{\phi c} . \frac{z^2}{1.2} - \&c. \\ &\qquad - c\delta - \frac{c\phi' c}{\phi c} z\delta - \&c. \end{aligned}\right\} = 0.$$

And when the orbit becomes indefinitely near a circle, δ becomes indefinitely small, as does z; and hence, z^2, $z\delta$, &c. may be omitted in comparison of z:

$$\text{hence, } \frac{d^2 z}{dv^2} + \left(1 - \frac{c\phi' c}{\phi c}\right) z - c\delta = 0.$$

If we make $1 - \dfrac{c\phi' c}{\phi c} = \gamma^2$, we shall have, as in Prob. XIII, for the integral of this equation,

$$z = C_1 \cos. \gamma v + C_2 \sin. \gamma v + \frac{c\delta}{\gamma^2};$$

and $u = c + z = C_1 \cos. \gamma v + C_2 \sin. \gamma v + c + \dfrac{c\delta}{\gamma^2}$:

which indicates the same kind of orbit as is described in the last problem. And here, as there, we shall have

$$A = \text{the angle between the apsides} = \frac{\pi}{\gamma} = \frac{\pi}{\sqrt{\left\{1 - \dfrac{c\phi' c}{\phi c}\right\}}}.$$

Ex. 1. Let the force vary as any power of the distance,

$$P = mu^n = u^2 . mu^{n-2}; \therefore \phi u = mu^{n-2}; \ \phi' u = (n-2)\, mu^{n-3};$$

$$\therefore \gamma^2 = 1 - \frac{(n-2)\, mc^{n-2}}{mc^{n-2}} = 3 - n;$$

$$\therefore \text{the angle between the apsides} = \frac{\pi}{\sqrt{(3-n)}}.$$

When $n = -1$, $A = \dfrac{\pi}{2}$, which agrees with Prob. I, of this Chapter;

when $n = 0$, $\quad A = \dfrac{\pi}{\sqrt{3}}$;

when $n = 1$, $\quad A = \dfrac{\pi}{\sqrt{2}}$;

when $n = 2$, $\quad A = \pi$, which agrees with Prob. II;

when $n = 3$, $\quad A$ is infinite;

and when $n > 3$, the expression is impossible. In fact, in this case, if the body leave one apse, it will never reach another, but will go off to infinity, if the velocity be greater than that in a circle, and fall to the centre if the velocity be less *.

If n be a little greater than 2, the apsides progress slowly: thus the apse will advance about 3^0 in one revolution, or $1\frac{1}{2}^0$ in a semi-revolution, if $n = 2\frac{4}{243}$.

Ex. 2. Let the force consist of two parts, each varying as any power of the distance;

$$P = mu^n + m'u^{n'} = u^2(mu^{n-2} + m'u^{n'-2});$$

$$\therefore \phi u = mu^{n-2} + m'u^{n'-2},\ \phi' u = (n-2)\,mu^{n-3} + (n'-2)\,m'u^{n'-3};$$

$$\therefore \gamma^2 = 1 - \frac{(n-2)\,mc^{n-2} + (n'-2)\,m'c^{n'-2}}{mc^{n-2} + m'c^{n'-2}}$$

$$= \frac{(3-n)\,mc^{n-2} + (3-n')\,m'c^{n'-2}}{mc^{n-2} + m'c^{n'-2}};$$

whence $A = \frac{\pi}{\sqrt{\gamma}}$, is known.

Ex. 3. Let the force vary as the sine of the distance from the centre: the distance being considered as an arc.

Let q be the distance, which, in this variation is considered as a quadrant; and m the force at that distance: then,

$$\sin. q : \sin. r :: m : m.\frac{\sin. r}{\sin. q} = \text{force at distance } r\text{: the}$$

sines being taken to such a radius that q is a quadrant.

But, if the sines of the corresponding angles be taken to radius

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* This is also true if the velocity be not nearly equal to that in a circle, as might be shewn.

The Student will find an investigation of the angle between the apsides, in some cases, when the orbit is not nearly circular, in the Transactions of the Cambridge Philosophical Society, Vol. I, Part I, p. 179.

1, they will be in the same ratio: and $q : r :: \frac{\pi}{2} : \frac{\pi r}{2q}$, the angle corresponding to r;

$$\therefore \text{ force} = P = m \,.\, \sin. \frac{\pi r}{2q} = m \,.\, \sin. \frac{\pi}{2qu}$$

$$= u^2 \,.\, \frac{m}{u^2} \,.\, \sin. \frac{\pi}{2qu};$$

$$\therefore \phi u = \frac{m}{u^2} \,.\, \sin. \frac{\pi}{2qu},$$

$$\phi' u = -\frac{2m}{u^3} \,.\, \sin. \frac{\pi}{2qu} - \frac{m}{u^2} \,.\, \frac{\pi}{2qu^2} \cos. \frac{\pi}{2qu}.$$

Hence, $\gamma^2 = 1 - \frac{c\phi' c}{\phi c} = 3 + \frac{\pi}{2qc} \text{ cotan. } \frac{\pi}{2qc}$;

where $\frac{1}{c}$ is the radius of the circle to which the orbit approximates.

If we make $\frac{1}{c} = a$, we have

$$\gamma^2 = 3 + \frac{\pi a}{2q} \text{ cotan. } \frac{\pi a}{2q}.$$

If $a = 0, \quad \gamma^2 = 4, \quad \gamma = 2.$

If $a = \frac{1}{2} q, \gamma^2 = 3 + \frac{\pi}{4}.$

If $a = q, \quad \gamma^2 = 3.$

The angle between the apsides varies from $\frac{\pi}{2}$ to $\frac{\pi}{\sqrt{3}}$ according to the different magnitudes of the circle described.

CHAP. IV.

THE MOTION OF SEVERAL POINTS.

22. In the last chapter we have supposed a single body to be acted on by forces tending to fixed mathematical points, and on that supposition have calculated its motion. But we may suppose those points, from which the force emanates, to be themselves moveable bodies, acted upon by their mutual forces, or by any others, and we may then have to calculate the motions of each of these bodies. This is in fact the problem which occurs in nature; for we do not there find forces tending to mathematical points, but residing in physical bodies, and connected with their material properties; and, with respect to those forces which we have most frequently to consider, depending entirely on the quantity of matter. By considering the motions of the planets, and other bodies of which the universe is composed, it was discovered by Newton, that each of them exerts upon all the others a force which is at different distances inversely as the square of the distance; and that at the same distance from each, the forces with which a point would be impelled towards them, are directly as their quantities of matter. It further appeared, that this force or attraction exerted by each mass, is the result of an attraction exerted by each particle of which it is composed; so that we may conceive every physical point in the universe to exert a force varying inversely as the square of the distance.

We shall consider more particularly bodies exerting forces of this kind; but it is manifest, that any other hypothesis of the variation of the force is equally possible, speaking mathematically, and may occasionally be introduced in our problems.

23. The forces which are exerted by these bodies, are of the kind which we have called *accelerating forces;* that is, they are measured solely by the velocity produced in a given time, and are entirely independent of the mass moved. Thus, if a body M exert upon a particle P, a certain accelerating force, which is represented by f, it will, under the same circumstances, exert upon $2P$ or $3P$ the same accelerating force; though it is manifest, that for this purpose the pressure or moving force exerted, or the weight produced in the particle $2P$ or $3P$, must be two or three times as great, respectively, as it was in P. At a given distance f is proportional to M.

By the third law of motion, the accelerating force f is proportional to the pressure exerted on P directly, and to the mass of P inversely. Hence, by what has been said, we have

$$M \propto f \propto \frac{\text{pressure on } P}{P};$$

and hence, pressure on $P \propto MP$, and $= \mu MP$, suppose: μ being the same for all bodies.

Hence, if a body M act upon any particle at a distance r, the accelerating force which it exerts may be represented by $\frac{m}{r^2}$, where m is proportional to the body itself. By properly assuming the unit, m may be considered *equal* to the body. And to this force acting upon the particle, estimated in the proper direction, we may apply the equations of motion in the same manner as if it tended to a fixed point. We shall now proceed to the different cases of the problems to which we are led by these considerations.

We have taken both the bodies M and P as points. If they be spheres of finite magnitude, the effect will (in the case in which the force varies inversely as the square of the distance) be the same as if they were collected at the centre; if they be of any other form, an irregularity will be introduced into the variation of the force: this will be shewn in treating of the attractions of bodies.

Sect. I. *Problem of two Bodies.*

24. Prop. When two bodies are acted on by no forces except their mutual attractions, their centre of gravity will either remain at rest, or move uniformly in a straight line. The motion will evidently be in the same plane.

Let P, Q, fig. 28, be the two bodies, referred to rectangular co-ordinates x', y' for P, and x'', y'' for Q; and let their masses be m', m'', respectively, and their distance r. Also let $\phi(r)$ represent the function of r, according to which their force varies; so that $m'\phi(r)$ is P's action on Q, and $m''\phi(r)$ is Q's action on P. Resolving these forces parallel to x' and y', respectively, we have by equations (c),

$$\frac{d^2x'}{dt^2} = m''\phi(r)\frac{x''-x'}{r}, \quad \frac{d^2y'}{dt^2} = m''\phi(r)\frac{y''-y'}{r} \ldots\ldots (1),$$

$$\frac{d^2x''}{dt^2} = -m'\phi(r)\frac{x''-x'}{r}, \quad \frac{d^2y''}{dt^2} = -m'\phi(r)\frac{y''-y'}{r} \ldots\ldots (2).$$

Now, multiplying equations (1) by m', and equations (2) by m'', and adding those which stand under each other, we have

$$\frac{m'd^2x' + m''d^2x''}{dt^2} = 0, \quad \frac{m'd^2y' + m''d^2y''}{dt^2} = 0;$$

integrating these, we have

$$m'\frac{dx'}{dt} + m''\frac{dx''}{dt} = A, \quad m'\frac{dy'}{dt} + m''\frac{dy''}{dt} = B.$$

If $\bar{x}$ and $\bar{y}$ be the co-ordinates of the centre of gravity of m', m'', we have by the formula for the centre of gravity

$$\bar{x} = \frac{m'x' + m''x''}{m' + m''}, \quad \bar{y} = \frac{m'y' + m''y''}{m' + m''}.$$

$$\text{Hence, } \frac{d\bar{x}}{dt} = \frac{A}{m'+m''}, \quad \frac{d\bar{y}}{dt} = \frac{B}{m'+m''}.$$

But $\frac{d\bar{x}}{dt}$, $\frac{d\bar{y}}{dt}$, manifestly represent the resolved parts of the velocity of the centre of gravity, in directions parallel to the co-ordinates x and y, respectively; and it here appears, that these parts are constant. Hence, the motion of the centre of gravity is uniform in these directions, and consequently, uniform and rectilinear in its own.

If $A=0$, and $B=0$, the centre is at rest.

This proposition was first stated by Newton; and, as we shall see afterwards, is true of any number of bodies*.

25. Prop. The motion of each body about the centre of gravity, is the same as if that point was a centre of force, the law of which was the same as the law of the attraction of the bodies†.

Let us suppose

$$x' = \bar{x} + x_1 \quad y' = \bar{y} + y_1,$$
$$x'' = \bar{x} + x_2 \quad y'' = \bar{y} + y_2;$$

so that x_1, y_1, x_2, y_2, indicate the co-ordinates of the points P, Q, measured from the centre of gravity.

Then since, as appears above,

$$\frac{d^2\bar{x}}{dt^2} = 0, \quad \frac{d^2\bar{y}}{dt^2} = 0,$$

we have $\dfrac{d^2x'}{dt^2} = \dfrac{d^2x_1}{dt^2}$, &c.;

and equations (2) become

$$\frac{d^2x_2}{dt^2} = -m\phi(r)\frac{x_2 - x_1}{r}, \quad \frac{d^2y_2}{dt^2} = -m\phi(r)\frac{y_2 - y_1}{r};$$

but it follows from the property of the centre of gravity, that if we make $GQ = r_2$, we have $r = \dfrac{m' + m''}{m'} r_2$.

Also, $\dfrac{x_2 - x_1}{r} = \dfrac{PO}{PQ} = \dfrac{GK}{GQ} = \dfrac{x_2}{r_2}$.

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* *Principia*, Cor. 4. to the Laws of Motion.

† Ibid. Book I, Prop. 61.

Hence, our equations become, for the body Q,

$$\frac{d^2x_2}{dt^2} = -\, m'\phi\left(\frac{m'+m''}{m'}\cdot r_2\right)\cdot\frac{x_2}{r_2},$$

$$\frac{d^2y_2}{dt^2} = -\, m'\phi\left(\frac{m'+m''}{m'}\cdot r_2\right)\cdot\frac{y_2}{r_2}.$$

Now these are the equations we shall have, if we suppose the centre of gravity a fixed point, to which a force tends, represented by $m'\phi\left(\frac{m'+m''}{m'}\cdot r_2\right)$; therefore the motion of the body Q *about the centre of gravity*, will be the same as if such a force resided in that point.

It might in the same manner be shewn, that the body P will move about G as if there were in G a force $= m''\phi\left(\frac{m'+m''}{m''} r_1\right)$.

And it is evident, that if $\phi(r)$ represent any power of r, $\phi\left(\frac{m'+m''}{m''} r_1\right)$ will vary according to the same power of r_1: Hence, the *law* of the force about G will be the same as that of the attraction to P or Q supposed fixed.

Cor. 1. The angular velocities of P and Q round G will always be equal, and PG, QG will always be in a given ratio. Hence, the figures described by P, Q are every way similar*.

Cor. 2. The velocities of P and Q relatively to G, will always be parallel, in opposite directions, and proportional to GP and GQ.

Prob. I. *Let the force vary inversely as the square of the distance, and* P, Q *have no angular motion originally: it is required to determine their motions*†.

It is manifest, that since P has no angular motion round G, it will descend in a straight line to G. Similarly, Q will descend to G in a straight line. And QG, PG, will be described in equal

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* *Principia*, Book I, Prop. 57.

† Ibid. Book. I, Prop. 62.

times, so that the bodies will meet in G. For, since the accelerating forces on P and Q are inversely as P and Q, that is, directly as GP and GQ, the velocities will be in the same proportion, and corresponding portions of GP, GQ, will be described in equal times, so that the whole will be described in the same time. Hence, also, after these bodies meet, they will go on together with the same velocity and direction with which the centre of gravity moved before they met.

By last Article, since $\phi(r)$ is here r^{-2}, the body P will move towards G as if there were in G a force $= m'' \left(\frac{m' + m''}{m''} . r_1\right)^{-2} = \frac{m''^3}{(m' + m'')^2} \cdot \frac{1}{r_1^{\,2}}$.

Hence, if a be the original distance of P from G, and P be supposed to have no original velocity towards G, we have, by Chap. I, Ex. 3, time of P falling to G

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\dfrac{2m''^3}{(m'+m'')^2}}} \cdot \frac{\pi}{2} = \frac{(m'+m'')\, a^{\frac{3}{2}}}{m''^{\frac{3}{2}}} \cdot \frac{\pi}{2\sqrt{2}} \cdot$$

Similarly, if b be the original distance of Q from G,

$$\text{time of } Q \text{ falling to } G = \frac{(m'+m'')\, b^{\frac{3}{2}}}{m'^{\frac{3}{2}}} \cdot \frac{\pi}{2\sqrt{2}} \cdot$$

And since $\frac{a}{m''} = \frac{b}{m'}$, these are equal, agreeably to what has just been said.

In the same manner, by Chap. I, Ex. 3, we might find the velocity at any point, and the time of falling through any portion of the distance.

Prob. II. *Let the force vary inversely as the square of the distance, and let the bodies* P, Q *have any velocities whatever originally; it is required to determine their motions**.

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* *Principia*, Book I, Prop. 63.

It has already been proved, (Art. 24.) that the centre of gravity G will move uniformly in a right line; and that (Art. 25.) P and Q will describe about G similar figures; P moving as if it were acted on by a force $\frac{m''^3}{(m'+m'')^2} \cdot \frac{1}{r_1^{\,2}}$, and Q as if it were acted on by a force $\frac{m'^3}{(m'+m'')^2} \cdot \frac{1}{r_2^{\,2}}$, placed in G. Hence, the curves described about G by P and Q will be similar ellipses, with G in the focus; and if we knew the original velocity of P and Q *about* G, we might determine the ellipse, as in Chap. III, Prob. III.

The velocities of P and Q at any moment, and consequently at the beginning, will be compounded of two velocities; viz., that which the whole system has in consequence of the motion of the centre of gravity, and the velocity of each point P and Q about this centre. Now these last velocities are, by what has already been said, (Art. 25, Cor. 2.) parallel to each other, in opposite directions, and proportional to GP, GQ, or to m'' and m'. The whole original velocities being known, we may thus find the separate parts of them.

Let P and Q have the original velocities p and q, making with QP angles $= \alpha, \beta$, respectively. Let the velocity *of the centre* G resulting from them be c, and γ the angle which this velocity makes with GP: let also v' and v'' be the velocities of P and Q *about* G, and let ϕ be the angle which this motion makes with GP or SQ; which will be the same for both bodies.

Now the component of the velocity of P resolved parallel to PQ is $p \cos. \alpha$: but since p is compounded of the motion c of the centre of gravity, and v' about the centre of gravity, its component will be $c \cos. \gamma + v' \cos. \phi$. And thus equating the expressions for the components of the velocities of P and Q, parallel and perpendicular to PQ, we have

$$p \cos. \alpha = c \cos. \gamma + v' \cos. \phi; \quad q \cos. \beta = c \cos. \gamma - v'' \cos. \phi;$$

$$p \sin. \alpha = c \sin. \gamma + v' \sin. \phi; \quad q \sin. \beta = c \sin. \gamma - v'' \sin. \phi.$$

Multiply the two upper equations by m' and m'', and add them, observing that $m'v' = m''v''$;

$$\therefore (m'p \cos. \alpha + m''q \cos. \beta) = (m' + m'') \, c \,.\, \cos. \gamma.$$

In the same way, the two lower equations give us

$$m'p \sin. \alpha + m''q \sin. \beta = (m' + m'')\, c . \sin. \gamma.$$

By adding the squares of these two equations, we have

$$m'^2p^2 + m''^2q^2 + 2m'm''pq \cos. (\alpha - \beta) = (m' + m'')^2c^2;$$

$$\text{whence, } c = \frac{\sqrt{\{m'^2p^2 + m''^2q^2 + 2m'm''pq \cos. (\alpha - \beta)\}}}{m' + m''}.$$

By taking the quotient of the same two equations, we have

$$\tan. \gamma = \frac{m'p \sin. \alpha + m''q \sin. \beta}{m'p \cos. \alpha + m''q \cos. \beta}.$$

Again, subtracting the upper equations, and also the lower, we get

$$p \cos. \alpha - q \cos. \beta = (v' + v'') \cos. \phi = v' \frac{m' + m''}{m''} \cos. \phi;$$

$$p \sin. \alpha - q \sin. \beta = (v' + v'') \sin. \phi = v' \frac{m' + m''}{m''} \sin. \phi.$$

Adding the squares of these equations, we have

$$p^2 + q^2 - 2pq \cos. (\alpha - \beta) = v'^2 \left(\frac{m' + m''}{m''}\right)^2 :$$

$$\text{whence } v' = \frac{m''}{m' + m''} \sqrt{\{p^2 + q^2 - 2pq \cos. (\alpha - \beta)\}};$$

$$v'' = \frac{m'}{m' + m''} \sqrt{\{p^2 + q^2 - 2pq \cos. (\alpha - \beta)\}};$$

and dividing the same two equations

$$\tan. \phi = \frac{p \sin. \alpha - q \sin. \beta}{p \cos. \alpha - q \cos. \beta}.$$

Hence, we know the velocity and direction of projection of P round G, and we can therefore, by Chap. III, Prob. 3, find the conic section described. And, combining the motion in this, with the motion of the centre of gravity, which we have also found, we have the motion of P.

Cor. 1. By Art. 25, Cor. 1, it appears that the curve described by P relatively to Q, will be similar to the curve which P describes about G. If a_1 be the semi-axis of the ellipse which P describes round G, and a the semi-axis of the ellipse which P describes relatively to Q, which is also in motion; we shall have

$$a_1 : a :: m'' : m' + m''.$$

Cor. 2. Since an ellipse with semi-axis $= a_1$, is described by a force $= \frac{m''^3}{(m'+m'')^2} \cdot \frac{1}{r_1^{\ 2}}$, we shall have the periodic time T by Chap. III, Prob. 4, Cor.; putting a_1 for a, and $\frac{m''^3}{(m'+m'')^2}$ for m;

$$\therefore T = \frac{2\, a_1^{\frac{3}{2}} \pi}{\sqrt{\frac{m''^3}{(m'+m'')^2}}} = \frac{2\pi a_1^{\frac{3}{2}} (m'+m'')}{m''^{\frac{3}{2}}} = \frac{2\pi a^{\frac{3}{2}}}{(m'+m'')^{\frac{1}{2}}}$$

by last corollary.

Cor. 3. If the body Q were *at rest*, and P were to revolve about it, at the same distance from it as in last Cor.; the ellipse would have its semi-axis major $= a$, and we should have for the periodic time T',

$$T' = \frac{2\pi a^{\frac{3}{2}}}{m''^{\frac{1}{2}}}.$$

Hence, $T : T' :: m''^{\frac{1}{2}} : (m'+m'')^{\frac{1}{2}}$ *.

Cor. 4. If P were to revolve round Q at rest, in an ellipse of which the semi-axis major was A, we should have for the periodic time T'',

$$T'' = \frac{2\pi A^{\frac{3}{2}}}{m''^{\frac{1}{2}}}.$$

And we may find A, such that T'', about Q at rest, may be equal to T, about Q in motion. For this purpose,

$$\frac{2\pi A^{\frac{3}{2}}}{m''^{\frac{1}{2}}} = \frac{2\pi a^{\frac{3}{2}}}{(m'+m'')^{\frac{1}{2}}};$$

$$\therefore a : A :: (m'+m'')^{\frac{1}{3}} : m''^{\frac{1}{3}} \dagger.$$

Sect. II. *Problem of three or more Bodies.*

26. If we suppose three bodies to act upon each other, we shall no longer be able generally to find the paths described as in the former case. Any two of them would describe regular orbits as in the preceding Section, but these will be changed by the action

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* *Principia*, Book I, Prop. 59. † *Principia*, Book I, Prop. 60.

of the third; and, in consequence of this change, an orbit will be described completely different in kind from the former one. In particular cases, however, the third body will only slightly alter the regular orbit described by the two others, and in these instances, this slight deviation from the regular orbit may be approximated to by particular methods. As this is the problem which nature actually presents to us in the case of the Earth, Moon, and Sun; the great importance, joined to the great difficulties of it, have made the *problem of three bodies* very celebrated; and since it was first suggested, it has employed a large portion of the attention of the best mathematicians, down to the present day. It does not, however, fall within the plan of the present Treatise: the student will find the different steps of the solution in Professor Woodhouse's *Physical Astronomy*.

27. Prop. If any number of bodies be acted on only by their mutual attraction, their centre of gravity will either be at rest, or will move uniformly in a straight line.

Let P, Q, R, &c. be any number of bodies distributed in space, of which the masses are m', m'', m''', &c. And let them be referred to co-ordinates parallel to three rectangular axes, viz.

$$x', y', z'; \; x'', y'', z''; \; x''', y''', z''', \text{ \&c.}$$

Also, let $r_{1,2}$ be the distance of m' and m'',
$r_{1,3}$ ....of m' and m''',
$r_{2,3}$ ....of m'' and m''',
&c.

And let the law of attraction between P and Q be $\phi(r_{1,2})$, between P and R, $\phi(r_{1,3})$; between Q and R, $\phi(r_{2,3})$, &c.

Hence, since each body attracts all the rest, by resolving the forces, and applying equations (c'), we have

$$\frac{d^2x'}{dt^2} = m''\phi(r_{1,2})\frac{x'-x''}{r_{1,2}} + m'''\phi(r_{1,3})\frac{x'-x'''}{r_{1,3}} + \text{\&c.}$$

$$\frac{d^2x''}{dt^2} = -m'\phi(r_{1,2})\frac{x'-x''}{r_{1,2}} + m'''\phi(r_{2,3})\frac{x''-x'''}{r_{2,3}} + \text{\&c.}$$

$$\frac{d^2x'''}{dt^2} = -m'\phi(r_{1,3})\frac{x'-x'''}{r_{1,3}} - m''\phi(r_{2,3})\frac{x''-x'''}{r_{2,3}} + \text{\&c.}$$

$$\text{\&c.} = \text{\&c.}$$

Multiply the first equation by m', the second by m'', the third by m''', &c. and add; and we have

$$\frac{m'd^2x' + m''d^2x'' + m'''d^2x''' + \&c.}{dt^2} = 0.$$

Similarly, we find

$$\frac{m'd^2y' + m''d^2y'' + m'''d^2y''' + \&c.}{dt^2} = 0;$$

$$\frac{m'd^2z' + m''d^2z'' + m'''d^2z''' + \&c.}{dt^2} = 0.$$

Integrating, we have

$$m'\frac{dx'}{dt} + m''\frac{dx''}{dt} + m'''\frac{dx'''}{dt} + \&c. = A,$$

$$m'\frac{dy'}{dt} + m''\frac{dy''}{dt} + m'''\frac{dy'''}{dt} + \&c. = B,$$

$$m'\frac{dz'}{dt} + m''\frac{dz''}{dt} + m'''\frac{dz'''}{dt} + \&c. = C.$$

Now, if $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of the centre of gravity, we have

$$\bar{x} = \frac{m'x' + m''x'' + m'''x''' + \&c.}{m' + m'' + m''' + \&c.},$$

$$\bar{y} = \frac{m'y' + m''y'' + m'''y''' + \&c.}{m' + m'' + m''' + \&c.},$$

$$\bar{z} = \frac{m'z' + m''z'' + m'''z''' + \&c.}{m' + m'' + m''' + \&c.}.$$

Hence, if for the sake of abbreviation we make $m' + m'' + m''' + \&c. = M$, we shall have

$$\frac{d\bar{x}}{dt} = \frac{A}{M}, \quad \frac{d\bar{y}}{dt} = \frac{B}{M}, \quad \frac{d\bar{z}}{dt} = \frac{C}{M};$$

and therefore the resolved parts of the velocity of the centre of gravity are uniform; and hence, this centre moves with a uniform velocity, and in a straight line.

28. Prop. Let any number of bodies, whose magnitudes are m', m'', m''' &c., act upon each other with forces which are directly as the distances: it is required to determine their motions*.

The distance of any two bodies m', m'', being r, the force of m'' on m' may be represented by $m''r$, and the part of it parallel to x, will be $m''r\,\frac{x'-x''}{r} = m''(x'-x'')$; similarly, the force of m''' on m', parallel to x, will be $m'''(x'-x''')$; and similarly for the other bodies, and also for the other co-ordinates y, z. Hence, by (c'),

$$\frac{d^2x'}{dt^2} = m''(x'-x'') + m'''(x'-x''') + \&c.$$

$$\frac{d^2x''}{dt^2} = -m'(x'-x'') + m'''(x''-x''') + \&c.$$

$$\frac{d^2x'''}{dt^2} = -m'(x'-x''') - m''(x''-x''') + \&c.$$

$$\&c. = \&c.$$

which may be put in this form,

$$\frac{d^2x'}{dt^2} = (m'+m''+m'''+\&c.)\,x' - m'x' - m''x'' - m'''x''', \ \&c.$$

$$\frac{d^2x''}{dt^2} = (m'+m''+m'''+\&c.)\,x'' - m'x' - m''x'' - m'''x''', \ \&c.$$

$$\frac{d^2x'''}{dt^2} = (m'+m''+m'''+\&c.)\,x''' - m'x' - m''x'' - m'''x''', \ \&c.$$

$$\&c. = \&c.$$

or, (observing that if we make $m'+m''+m'''+\&c. = M$ and $\bar{x}, \bar{y}, \bar{z}$, the co-ordinates of the centre of gravity, we have

$$m'x' + m''x'' + m'''x''' + \&c. = M\bar{x},)$$

$$\frac{d^2x'}{dt^2} = M(x'-\bar{x});$$

$$\frac{d^2x''}{dt^2} = M(x''-\bar{x});$$

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* *Principia*, Book I, Prop. 64.

$$\frac{d^2 x'''}{dt^2} = M\,(x''' - \bar{x}),$$

$$\&c. = \&c.$$

Similarly, we should have

$$\frac{d^2 y'}{dt^2} = M\,(y' - \bar{y}), \quad \&c.$$

$$\frac{d^2 z'}{dt^2} = M\,(z' - \bar{z}), \quad \&c.$$

Now $x' - \bar{x}$, &c. are the co-ordinates of m', &c. measured from the centre of gravity; and it has already been seen that

$$\frac{d^2\,(x' - \bar{x})}{dt^2} = \frac{d^2 x'}{dt^2}, \quad \&c.:$$

hence it appears by comparing these equations with Chap. II, Ex. 2, that the motion about the centre of gravity is the same as if there were no force but one residing in the centre of gravity, and equal to $M \times$ distance. Hence, the bodies will all describe ellipses about the centre of gravity, as a centre. And the periodic times in these ellipses will all be the same. Their magnitude, eccentricity, the positions of the planes of the orbits, and of the major axes, may differ in any manner.

Also, the motion of any one body relative to any other, will be governed by the same laws as the motion of a body relative to a centre of force, varying as the distance; for if we take the equations

$$\frac{d^2 x'}{dt^2} = M'\,(x' - \bar{x}), \qquad \frac{d^2 x''}{dt^2} = M'\,(x'' - \bar{x}),$$

and subtract them, we have

$$\frac{d^2\,(x' - x'')}{dt^2} = M\,(x' - x'');$$

and similarly for the ys and zs, from which it appears that the motion of m' about m'' is of the kind described.

CHAP. V.

THE CONSTRAINED MOTION OF A POINT ON A GIVEN LINE OR SURFACE.

29. If we suppose a body to slide along a surface, as the inside of a bowl which is perfectly smooth, it is evident that we cannot apply immediately the reasonings of the preceding Chapters; for by the impenetrability of the surface, the body is perpetually deflected from the path in which it would move in consequence of the action of the forces alone. Since the surface is supposed perfectly smooth, the force which it exerts upon the body must be, at each point, perpendicular to it. For there is no reason why the direction of the action of the surface should be inclined on one side rather than the other, to this perpendicular; if any lateral force do exist, it is attributed to the defect from absolute smoothness, and called *friction*.

This perpendicular force exerted by the surface, varies perpetually during the motion of the body: it is always such as exactly to keep the body in the given surface, that is, to resist the tendency to move *through* the surface. Now, if we were to suppose the body to move freely, and a force of the same magnitude and direction as this reaction, but not arising from the contact of the body with a surface, always to act upon it, its motion would be exactly the same. But in this case, we may calculate the motion by the formulæ of the preceding Chapters, introducing among the forces this new one, and making it such as always to keep the body in the surface.

30. Prop. The reaction of the surface does not increase or diminish the velocity of the body.

This will be seen when we come to deduce the formulæ for the motion. But it appears to be nearly evident from this consideration;

that the force of reaction is entirely employed in deflecting the body. If it were not perpendicular to the direction of the motion, we might resolve it into two, one perpendicular to this direction which deflects the body from its path; and one in the direction of the motion, which alone would affect the velocity.

We may, as the simplest case, consider the body to move on a curve *line*, lying in one plane. This will happen when the original motion and the forces are all in one plane, and that plane also every where perpendicular to the given surface. For then the reaction will be in the same plane, and there will be nothing to deflect the body from it. It will therefore be sufficient to consider the motion as on a plane curve. The next simplest case will be when the surface is one of revolution, and all the forces act in planes passing through the axis. We shall afterwards consider any surface whatever, with any forces.

Instead of supposing the body retained by the reaction of an impenetrable surface, we may suppose any other means. For instance, if a body be fastened by an inextensible string to a given point, it can move in the surface of a sphere: and the conditions of its motion will be the same as if it moved on a smooth spherical surface: the tension of the string supplying the place of the reaction of the surface. And by supposing the string during its motion to wrap round other curves and surfaces, the surface to which the body is confined may become any whatever.

We may also, instead of supposing the body to move *on* a curve, conceive it to move *in* a curvilinear *tube*, indefinitely narrow, the body being considered as a point. The difference between this case and the former one, will be, that in this, the reaction can operate in any direction, whereas before it could only act on one side of the body. So that in the tube, if the reaction were to become first nothing and then negative, (or opposite to its former side) the body would still be retained; while, in the other case, it would, on this supposition, fly off from the curve and describe a path in free space.

In this manner also we may suppose a body to be compelled to move in *a curve of double curvature*.

SECT. I. *The Motion of a Point on a plane Curve.*

31. PROP. When a body moves on a curve, acted on by given forces, to determine its velocity.

Let a body move on the curve PA, fig. 29, referred to the co-ordinates AM, MP, which are represented by x, y. And let the forces which act upon it be resolved into X, Y, parallel, respectively, to these co-ordinates. Besides these forces we have to consider the reaction of the curve, which is in the direction PK, perpendicular to the curve, and which being represented by PK, may be resolved into PL, PH. If we call the reaction R, we shall have the resolved parts in PL, and in PH parallel to AM,

$$R.\frac{PL}{PK} \text{ and } R.\frac{PH}{PK}, \text{ respectively;}$$

or, (since the triangles PLK and TMP are manifestly similar, PT being a tangent),

$$R.\frac{MT}{PT}, \text{ and } R.\frac{MP}{PT}; \text{ that is, } R\frac{dx}{ds}, \text{ and } R\frac{dy}{ds}:$$

supposing $ds = \sqrt{(dx^2 + dy^2)}$ = the differential of the curve AP.

Hence, collecting the whole forces in the directions PH and MP, we have, by equations (c),

$$\frac{d^2x}{dt^2} = X + R\frac{dy}{ds},$$

$$\frac{d^2y}{dt^2} = Y - R\frac{dx}{ds}.$$

Now, to eliminate R, multiply by $2\,dx$ and $2\,dy$, respectively, and add; and we have

$$\frac{2\,dx\,d^2x + 2\,dy\,d^2y}{dt^2} = 2\,Xdx + 2\,Ydy,$$

$$\text{and } \frac{dx^2 + dy^2}{dt^2} = 2\int(X\,dx + Y\,dy).$$

This expression is the same as when the body moves freely. Hence, it appears that when a body, acted on by given forces, moves from one given point to another, as from B to P, the velocity is the same, whatever be the path it takes, and whether it moves freely or be constrained to move along a given curve.

If we suppose the body to be acted on by a force in parallel lines, we may suppose the axis of x to be parallel to these lines; and we have then $Y=0$.

If the force be also supposed to be constant, as for instance, gravity, and x to be measured upwards, we have $X=-g$; and $2\int X\,dx = C - 2gx$. Or, if we put $C=2gh$, h being an arbitrary quantity, we have

$$\text{velocity}^2 = \frac{dx^2+dy^2}{dt^2} = 2g(h-x).$$

Here, when $x=h$, the velocity is $=0$; therefore h is the height from which the body begins to fall. Also since the velocity depends on x alone, it appears that it is the same, whether the body fall down the perpendicular DM, or down any curve BP of the same vertical height.

If we suppose the force to tend to a centre, and to vary as some function of the distance from it; the centre of force may be made the origin of co-ordinates A, fig. 30. Let $AP=r$, and the force in $PA=P$, a function of r. Hence, we have

$$X=-P\frac{x}{r},\quad Y=-P\frac{y}{r},$$

$$\int(X\,dx+Y\,dy) = -\int P\frac{x\,dx+y\,dy}{r} = -\int P\,dr;$$

$$\therefore\ \frac{dx^2+dy^2}{dt^2} = C-2\int P\,dr.$$

Or if $PAM=v$,

$$\text{velocity}^2 = \frac{dr^2+r^2dv^2}{dt^2} = C-2\int P\,dr.$$

Since the velocity depends on r alone, it is the same whether the body fall down the curve BP, or down the line BQ (making $AQ=AP$) acted on by the same force. And if the velocity in the curve and in the straight line AB, be equal at any corresponding equal distances from the centre, they will be equal at any other equal distances*.

32. Having found the velocity in terms of the co-ordinates, or of the radius vector: and knowing moreover the nature of the curve, we can find the time of the motion, as will be seen in the following examples.

Since $dt = \frac{ds}{v}$, when the force acts parallel to x, we have

$$dt = \frac{ds}{\sqrt{\{2g(h-x)\}}}.$$

And when the force acts to a centre, $dt = \frac{ds}{\sqrt{(C - 2\int P dr)}}$.

Prob. I. *Let* AP, *fig.* 29, *be a cycloid with its axis vertical: to determine the motion upon it when the body is acted on by gravity.*

We have here $dy = dx\sqrt{\frac{2a-x}{x}}$; (*Lacroix*, Art. 102.), where a is the radius of the generating circle;

$$\therefore\ ds = dx\sqrt{\frac{2a}{x}};$$

$$dt = -\frac{ds}{\sqrt{2g}\ \sqrt{(h-x)}} = -\sqrt{\frac{a}{g}}\cdot\frac{dx}{\sqrt{(hx-x^2)}};$$

$$\therefore\ t = C - \sqrt{\frac{a}{g}}\ .\ \text{arc}\left(\text{ver. sin.} = \frac{2x}{h}\right)$$

$$= \sqrt{\frac{a}{g}}\cdot\left\{\pi - \text{arc}\left(\text{ver. sin.} = \frac{2x}{h}\right)\right\};$$

for when $x = h$, $t = 0$, and arc $= \pi$.

And for the whole time of descent to A, $t = \pi\sqrt{\frac{a}{g}}$.

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* *Principia*, Book I, Prop. 40.

Cor. 1. Hence, the time of descent is the same whatever be the arc BA.

Cor. 2. If the cycloid be completed, fig. 31, the body, after descending from B to A, will ascend by the velocity acquired, to b, in the same horizontal line with B: for the height up which the body must ascend to lose the velocity, will be the same as that down which it descended to gain it. And the time up Ab, will equal the time down BA.

When the body comes to b, it will have lost all its velocity: it will then descend to A, and rise again to B, and so go on oscillating for ever on the supposition that the surface is perfectly smooth.

Cor. 3. By *Lacroix*, Elem. Treat. Art. 103, the evolute to the cycloid $EPAe$, consists of two semi-cycloids, CE, Ce. Hence, if a body be suspended by a string of proper length, which wraps round the curves CE, Ce, and oscillates, the conditions of the motion of the point P will be the same as those of a body upon a cycloidal surface, just investigated.

If l be the length of the pendulum = curve EOC;

$$l = 4a, \text{ and time in } BA = \frac{\pi}{2}\sqrt{\frac{l}{g}};$$

$$\therefore \text{ time of an oscillation from } B \text{ to } b = \pi\sqrt{\frac{l}{g}}.$$

Cor. 4. The time which a body would employ in falling down the vertical length l is $\sqrt{\frac{l}{g}}$;

$$\therefore \text{ time of oscillation : time down pendulum :: } \pi : 1.$$

The semi-cubical parabola, with its axis vertical, is another case in which we can integrate the expression for the time.

Prob. II. *Let AQ, fig. 31, be a circle whose radius is c: the body being acted on by gravity.*

$$y = \sqrt{(2cx - x^2)}; \; ds = \frac{c\,dx}{\sqrt{(2cx - x^2)}};$$

$$dt = -\frac{ds}{2g.\sqrt{(h\sqrt{} - x)}} = -\frac{c}{\sqrt{2g}}\cdot\frac{dx}{\sqrt{\{(h-x)(2cx-x^2)\}}};$$

$$= -\frac{c}{\sqrt{2g}} \cdot \frac{dx}{\sqrt{\{(hx-x^2)(2c-x)\}}}.$$

We might integrate by expanding $(2c-x)^{-\frac{1}{2}}$. But we may do it better thus.

$$\text{Let } \theta = \text{arc}\left(\sin. = \sqrt{\frac{x}{h}}\right); \therefore d\theta = \frac{dx}{2\sqrt{(hx-x^2)}}.$$

Also since $\sqrt{\frac{x}{h}} = \sin.\theta$, $x = h\sin.^2\theta$, $2c - x = 2c - h\sin.^2\theta$

$= 2c(1-\delta^2\sin.^2\theta)$: putting $\delta^2 = \frac{h}{2c}$. Hence,

$$dt = -\frac{c}{\sqrt{(4gc)}} \cdot \frac{2\,d\theta}{\sqrt{(1-\delta^2\sin.^2\theta)}}$$

$$= -\sqrt{\frac{c}{g}} \cdot \frac{d\theta}{\sqrt{(1-\delta^2\sin.^2\theta)}}.$$

The integral of this is an elliptic transcendent, and may be found by the methods of approximation, or the tables, given by Legendre for such functions. The integral must be taken from $x=h$, that is, from $\theta = \frac{\pi}{2}$: and for the whole arc AP, it must be taken to $x=0$, and therefore $\theta = 0$.

If the oscillations be small, h is small; and therefore δ. In this case, we may approximate by expanding. We have thus

$$\int \frac{d\theta}{\sqrt{(1-\delta^2\sin.^2\theta)}} = \int d\theta\,\{1 + \frac{\delta^2}{2}\sin.^2\theta + \frac{1.3}{2.4}\delta^4\sin.^4\theta + \&c.\}.$$

Now, to find $\int \sin.^m\theta\, d\theta$, where m is an even number, we have (*Lacroix*, Elem. Treat. 205.)

$$\int d\theta\sin.^m\theta = -\frac{1}{m}\cos.\theta\sin.^{m-1}\theta + \frac{m-1}{m}\int d\theta\sin.^{m-2}\theta;$$

and taking the integral from $\theta = 0$ to $\theta = \frac{\pi}{2}$, the first term vanishes;

and

$$\int d\theta \sin.^m \theta = \frac{m-1}{m}\int d\theta \sin.^{m-2}\theta \begin{Bmatrix}\theta = 0\\ \theta = \frac{\pi}{2}\end{Bmatrix}.$$

Similarly, $\int d\theta \sin.^{m-2}\theta = \frac{m-3}{m-2}\int d\theta \sin.^{m-4}\theta \begin{Bmatrix}\theta = 0\\ \theta = \frac{\pi}{2}\end{Bmatrix}$;

and so on to $\int d\theta \sin.^2 \theta = \frac{1}{2}\int d\theta = \frac{1}{2}.\frac{\pi}{2}$.

Hence, $\int d\theta \sin.^m\theta = \frac{(m-1)(m-3)\ldots\ldots 1}{m.(m-2)\ldots\ldots\ldots 2}.\frac{\pi}{2}$,

and $\int \frac{d\theta}{\sqrt{(1-\delta^2 \sin.^2\theta)}}$, from $\theta = 0$ to $\theta = \frac{\pi}{2}$, is the same as $\int -\frac{d\theta}{\sqrt{(1-\delta^2\sin.^2\theta)}}$, from $\theta = \frac{\pi}{2}$ to $\theta = 0$. Hence, we have

$$t = \sqrt{\frac{c}{g}}\left\{1 + \left(\frac{\delta}{2}\right)^2 + \left(\frac{1.3.\delta^2}{2.4}\right)^2 + \left(\frac{1.3.5.\delta^3}{2.4.6}\right)^2 + \&c.\right\}.\frac{\pi}{2}.$$

If we neglect all the terms after the first, we have $t = \pi\sqrt{\frac{c}{4g}}$: which is the time of descent down a cycloidal arc of which the radius is $\frac{c}{4}$. Hence, if $AF = \frac{1}{2}\,AC$ be the axis of a cycloid AP, the times in PA, and QA are equal. In fact, C is the centre of curvature of both, and they may, for small arcs, be supposed to coincide.

If we take two terms, we have, since $\delta^2 = \frac{h}{2c}$,

$$t = \frac{\pi}{2}\sqrt{\frac{c}{g}}\left\{1 + \frac{\delta^2}{4}\right\} = \frac{\pi}{2}\sqrt{\frac{c}{g}}.\left\{1 + \frac{h}{8c}\right\}.$$

But if AB, the chord of the whole descent $= k$, we have $h = \frac{k^2}{2c}$;

$$\therefore\ t = \frac{\pi}{2}\sqrt{\frac{c}{g}}\left\{1 + \frac{k^2}{16c^2}\right\}.$$

Hence, if a pendulum oscillate on each side of the vertical through an arc, the chord of which is $\frac{1}{10}$ of the length, $\frac{k}{c} = \frac{1}{10}$, and the oscillation is longer by $\frac{1}{1600}$ of the whole.

Prob. III. *Let the curve be the hypocycloid, and the force tend to the centre of the globe, and be as the distance.*

A hypocycloid is a curve, *APD*, fig. 32, generated by a point *P*, in the circumference of a circle *GPe*, which rolls along the inside of the circumference *EFD* of another circle. The circle *EFD* is called a circle of *the globe*, and the rolling circle *FPG* the *wheel* or generating circle.

Suppose, that when the describing point *P* was at *A*, the diameter *Pe* of the generating circle was in the position *EA*, coinciding with *CE*. And suppose that when *A* comes to *P*, *E* comes to *e*, so that $FE = Fe$. Now when the point is at *P*, the circle is turning on the point *F*; hence, the motion of *P* at that point will be perpendicular to *FP*; and hence, a tangent at *P* will be at right angles to *FP*, and will therefore pass through the point *G*. Let *CY* be a perpendicular on *PG*. And let $CF = a$, $OF = b$; $CP = r$, $CY = p$.

Now by similar triangles, $\frac{PY^2}{CF^2} = \frac{GY^2}{CG^2}$;

$$\text{or } \frac{r^2 - p^2}{a^2} = \frac{(a - 2b)^2 - p^2}{(a - 2b)^2} = \frac{e^2 - p^2}{e^2};$$

putting $e = a - 2b = CA$.

$$\text{Hence, } p^2 = \frac{e^2 (a^2 - r^2)}{a^2 - e^2}.$$

(See Mr. Peacock's Collection of Examples, p. 195.)

$$\text{Also, } r^2 - p^2 = \frac{a^2 (r^2 - e^2)}{a^2 - e^2}.$$

Now, we have in spirals, $ds = \dfrac{r\,dr}{\sqrt{(r^2 - p^2)}}$

$$= \frac{\sqrt{(a^2 - e^2)}\; r\,dr}{a\sqrt{(r^2 - e^2)}} \text{ in this case.}$$

And, $P = mr$; $\int P\,dr = \dfrac{mr^2}{2}$; $C - 2\int P dr = m(h^2 - r^2)$,

making $C = mh^2$; where h is the height CB, when the body begins to move.

Hence, $dt = -\dfrac{\sqrt{(a^2 - e^2)}}{a\sqrt{m}} \cdot \dfrac{r\,dr}{\sqrt{\{(r^2 - e^2)(h^2 - r^2)\}}}$,

to integrate this, let $\sqrt{(r^2 - e^2)} = u$; $\therefore \dfrac{r\,dr}{\sqrt{(r^2 - e^2)}} = du$;

$$\text{and} \quad h^2 - r^2 = h^2 - e^2 - u^2,$$

$$dt = -\frac{\sqrt{(a^2 - e^2)}}{a\sqrt{m}} \cdot \frac{du}{\sqrt{(h^2 - e^2 - u^2)}},$$

$$t = \frac{\sqrt{(a^2 - e^2)}}{a\sqrt{m}} \text{ arc } \left(\cos. = \frac{u}{\sqrt{(h^2 - e^2)}} = \frac{\sqrt{(r^2 - e^2)}}{\sqrt{(h^2 - e^2)}}\right),$$

which is $= 0$, when $r = h$, as it should be.

Hence, the time of falling to A, found by making $r = e$, is

$$t = \frac{\sqrt{(a^2 - e^2)}}{a\sqrt{m}} \, \frac{\pi}{2}.$$

Cor. 1. Since h does not appear in this result, the time of descending to A is the same, whatever be the arc; that is, the descents are *isochronous**.

Since $e = a - 2b$, $a^2 - e^2 = 4ab - 4b^2$;

$$\therefore t = \pi \sqrt{\left(\frac{b}{am} - \frac{b^2}{a^2 m}\right)}.$$

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* *Principia*, Book I, Prop. 51.

Cor. 2. If am = force at E, be put = g, and if we suppose a large compared with b, so that $\frac{b}{a}$ may be rejected, we have

$$t = \pi\sqrt{\frac{b}{g}},$$

which coincides with the time in a common cycloid (see Prob. I.), as it manifestly should; for if a be very large, ED will be nearly a straight line; and the force in DA constant and parallel to EC.

Cor. 3. Hence, also the *oscillations* in such a curve are isochronous, and the time is found as above. The time of an oscillation $= \frac{\pi\sqrt{(a^2-e^2)}}{a\sqrt{m}}$.

If we suppose the Earth spherical and homogeneous, the force, in proceeding from the surface to the centre, varies as the distance from the centre. Hence, if we were to suppose a body to move on a hypocycloid, generated by the rolling of a circle on the interior of the circumference of a circle within the Earth, and concentric with it, the descents and oscillations in such a curve would be isochronous.

Cor. 4. Instead of supposing the body to move on a curve, we may suppose it to be suspended by a string, which, during its oscillations, wraps round the curves SD, Sd, the evolutes to the portions AD, Ad of the hypocycloid. The motions will then be the same as before, and the oscillations of such a pendulum will therefore be isochronous.

*The evolutes SD, Sd are also hypocycloids**.

Let PO be the radius of curvature, CZ a perpendicular upon it, $CO = r'$, $CZ = p'$,

$$PO = \frac{r\,dr}{dp}. \quad (\textit{Lacroix}, \text{ Elem. Treatise, Note H, p. 668.})$$

$$p = \frac{e\sqrt{(a^2-r^2)}}{\sqrt{(a^2-e^2)}}; \therefore dp = -\frac{e r\,dr}{\sqrt{\{(a^2-e^2)(a^2-r^2)\}}},$$

* *Principia*, Book I, Prop. 51.

$$PO = \frac{\sqrt{\{(a^2 - e^2)(a^2 - r^2)\}}}{e}.$$

Also, $OZ = OP + CY = \dfrac{a^2\sqrt{(a^2 - r^2)}}{e\sqrt{(a^2 - e^2)}}$,

$$CZ = \sqrt{(r^2 - p^2)} = \frac{a\sqrt{(r^2 - e^2)}}{\sqrt{(a^2 - e^2)}}.$$

Hence, squaring, $\dfrac{a^4(a^2 - r^2)}{e^2(a^2 - e^2)} = r'^2 - p'^2$,

$$\frac{a^2(r^2 - e^2)}{(a^2 - e^2)} = p'^2.$$

Multiply the first by e^2, and the second by a^2, and add, and we have

$$a^4 = e^2 r'^2 - e^2 p'^2 + a^2 p'^2;$$

$$\therefore p'^2 = \frac{a^4 - e^2 r'^2}{a^2 - e^2}.$$

Let $\dfrac{a^2}{e} = a'$ so that $\dfrac{a^2}{e^2} = \dfrac{a'^2}{a^2}$,

$$p'^2 = \frac{a'^2 - r'^2}{\dfrac{a'^2}{a^2} - 1} = \frac{a^2(a'^2 - r'^2)}{a'^2 - a^2};$$

which is the same as the equation which we had to the hypocycloid, only putting a for e, and a' for a.

Hence, take $CS = \dfrac{a^2}{e}$, a third proportional to CA and CE; and describing a circle with radius CE, suppose a wheel, whose diameter is ES, to roll on this circle, so as to describe the hypocycloids SD, Sd; these will be the evolutes of AD, Ad. And a pendulum oscillating between them will always have its extremity in the hypocycloid DAd.

The length of the hypocycloid is thus found*,

$$ds = \frac{\sqrt{(a^2 - e^2)}\,.\,r\,dr}{a\sqrt{(r^2 - e^2)}};$$

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* *Principia*, Book I, Props. 48, 49.

$$\therefore s = \frac{\sqrt{\{(a^2-e^2)(r^2-e^2)\}}}{a}, \text{ measuring from } A.$$

And for the whole AD, $s = \dfrac{a^2-e^2}{a}$, for $r=a$.

Similarly, $SD = Sd = \dfrac{a'^2-a^2}{a'} = \dfrac{a^4-a^2e^2}{a^2e} = \dfrac{a^2-e^2}{e}$.

Cor. 5. If the length of the pendulum $=l$, $\dfrac{a^2-e^2}{e}=l$,

$$\text{whence, } a^2-e^2 = le = \frac{la^2}{a'}.$$

And time of oscillation $= \pi\,\dfrac{\sqrt{(a^2-e^2)}}{a\sqrt{m}} = \dfrac{\pi\sqrt{l}}{\sqrt{(ma')}}$.

Cor. 6. By Chap. I, Ex. 1, the time of falling to the centre C will be $\dfrac{\pi}{2\sqrt{m}} = \dfrac{\pi\sqrt{a'}}{2\sqrt{(ma')}}$. Hence, $\frac{1}{2}$ time of an oscillation : time to centre :: $\sqrt{l}$: $\sqrt{a'}$ :: $\sqrt{SA}$: $\sqrt{SC}$*.

If the curve be an *Epicycloid*, and the body be acted upon by a repulsive force, varying as the distance from the centre of the globe, we shall obtain similar results: and the oscillations will be isochronous†.

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* *Principia*, Book I, Prop. 52.

† If we resolve the force in these isochronous cases, so as to obtain the portion which acts along the curve, we shall find, that this portion is as the length of the curve from the lowest point; consequently, the force which accelerates the body, is as the space to be described; and, therefore, the time to the lowest point is independent of the original distance by Chap. I, p. 14.

Conversely, in any curve the descents to a given point will be isochronous, if the force be such, that the resolved part of it along the curve is as the arc from that point. If the force be P, and act parallel to x, $P\dfrac{dx}{ds}$ will be the resolved part.

Ex.

33. Prop. When a body moves upon a curve, to find the magnitude of the reaction R, which is also equal to the pressure upon the curve, we resume the equations

$$\frac{d^2x}{dt^2} = X + R\frac{dy}{ds};$$

$$\frac{d^2y}{dt^2} = Y - R\frac{dx}{ds}.$$

Multiplying the first by dy, and the second by dx, and subtracting, we have

$$\frac{dy\,d^2x - dx\,d^2y}{dt^2} = Xdy - Ydx + Rds.$$

$$\text{Hence, } R = -\frac{Xdy - Ydx}{ds} + \frac{dy\,d^2x - dx\,d^2y}{dt^2 ds}.$$

If ρ be the radius of curvature, we have $\rho = \dfrac{ds^3}{dy\,d^2x - dx\,d^2y}$;

$$\text{hence, } R = \frac{Ydx - Xdy}{ds} + \frac{ds^2}{\rho\,dt^2}.$$

Of the two parts of which this expression consists, the first is what we obtain by supposing X and Y resolved perpendicular to a tangent to the curve. The second, $\dfrac{ds^2}{\rho\,dt^2}$, or $\dfrac{\text{velocity}^2}{\rho}$, is the force which would retain the body in the curve if it were moving in a circle whose radius is ρ. This latter is called the *centrifugal force*, and arises entirely from the tendency of the body to go in a

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Ex. To find the force which must act in parallel lines, that the descents in a circle may be isochronous.

If c be the radius, and θ the angle which the radius CQ, fig. 31, makes with the vertical, x will be $= c$ ver. sin. θ, and $s = c\theta$; $\therefore \dfrac{dx}{ds} = \sin.\ \theta$.

Hence, P sin. θ must vary as $c\theta$, and P must vary as $\dfrac{\theta}{\sin.\ \theta}$. in order that the circle may be isochronous.

This problem is in the *Principia*, Book I, Prop. 53.

straight line, instead of the curvilinear path in which it is compelled to move: it is greater as the curvature is greater.

If the body be acted on by no force, $R = \frac{ds^2}{\rho dt^2} = \frac{\text{velocity}^2}{\rho}$.

In the case where the body is acted on by gravity only,

$$R = \frac{g\,dy}{ds} + \frac{ds^2}{\rho dt^2}.$$

In the case where the force P tends to a centre at the origin of co-ordinates,

$$Ydx - Xdy = P\,\frac{xdy - ydx}{r} = Prdv;$$

for $xdy - ydx = r^2 dv$, (see Art. 16.)

$$R = \frac{Prdv}{ds} + \frac{ds^2}{\rho dt^2}.$$

PROB. IV. *A body oscillates in a cycloid acted on by gravity: to find the tension of the string.* Fig. 31.

$$R = g\,.\,\frac{dy}{ds} + \frac{ds^2}{\rho dt^2},$$

we have $dy = dx\sqrt{\frac{2a-x}{x}}$, $ds = dx\sqrt{\frac{2a}{x}}$, $\frac{dy}{ds} = \sqrt{\frac{2a-x}{2a}}$,

$\rho = 2\sqrt{\{2a(2a-x)\}}$, (*Lacroix*, Art. 103), $\frac{ds^2}{dt^2} = 2g\,(h-x)$.

Hence, $R = g\sqrt{\frac{2a-x}{2a}} + \frac{g\,(h-x)}{\sqrt{\{2a\,.\,(2a-x)\}}}$

$$= g\,.\,\frac{2a+h-2x}{\sqrt{(4a^2-2ax)}}.$$

When $x = h$, $R = g\frac{2a-h}{\sqrt{(4a^2-2ah)}} = g\,\frac{\sqrt{(2a-h)}}{\sqrt{(2a)}}$.

When $x = 0$, $R = g\,.\,\frac{2a+h}{2a} = g\left(1+\frac{h}{2a}\right)$;

the part g arises from gravity, the other part $\frac{gh}{2a}$ from centrifugal force.

If $h = 2a$, or the body fall from the highest point; pressure at $A = 2g$.

Prob. V. *A body oscillates in a circular arc acted on by gravity: to find the tension of the string.*

$$dy = \frac{(c-x)\,dx}{\sqrt{(2cx-x^2)}},\ ds = \frac{c\,dx}{\sqrt{(2cx-x^2)}},\ \frac{dy}{ds} = \frac{c-x}{c},$$

$$\rho = c,\ \frac{ds^2}{dt^2} = 2g\,(h-x);$$

$$\therefore R = g.\frac{c-x}{c} + \frac{2g\,(h-x)}{c} = g.\frac{c+2h-3x}{c}.$$

When $x = 0$, this gives $R = g\,\frac{c+2h}{c}$. If $h = c$, or the body fall through the whole quadrant, $R = 3g$; and the tension at the lowest point is three times the weight.

If the body fall through the whole semi-circle from the highest point of the circle, $h = 2c$, $R = 5g$, and the tension at the lowest point is five times the weight.

This tension, by the increase of x, may become 0, and afterwards negative, or opposite to its former direction: the body will then tend to approach the centre of suspension.

It will $= 0$, when $c + 2h - 3x = 0$, or $x = \frac{c+2h}{3}$.

If a body move on the upper *convex surface* of a circle, it will only remain upon it while its pressure is towards the centre. It will fly off and describe a parabola, when the pressure becomes $= 0$; that is, when $x = \frac{c+2h}{3}$; x and h being measured from the lowest point of the circle.

The parabola which the body describes, will be one which has a common tangent with the circle at the point where the body leaves it, and the velocity at that point $= \sqrt{\{2g.(h-x)\}} = \sqrt{\left\{2g.\frac{h-c}{3}\right\}}$.

Since x is less than $2c$, we have $c + 2h < 6c$, or $h < \frac{5}{2}c$. If h be greater than this, the body will not move along the circle at all, but will leave it at the highest point.

By a similar application of our formulæ we might easily find the tension, when a body oscillates in a *hypocycloid*.

Sect. II. *The Motion of a Body on a Surface of Revolution.*

34. Prop. To find the motion of a body upon a surface of revolution, and acted on by forces in a plane passing through the axis.

Let CP, fig. 33, be the curve by the revolution of which the surface is described, AC its axis, and let AMN, be a fixed plane perpendicular to this axis, AM, MN, NP three rectangular coordinates to the point P; represented by x, y, z, respectively. Also let OLP be a plane parallel to AMN; and let $OP = r$. Then since $AO = NP = z$, and $OP = r$, knowing the nature of the curve CP, we know the relation of r and z.

The forces which act upon the body when at P are the reaction and the extraneous force: let this latter, which is by supposition in the plane POA, be resolved into two, P in the direction PO, and Z in the direction PN. Also the reaction is manifestly perpendicular to the curve CP, and in the plane AOP; and if we resolve it, as in the case of a plane curve, we shall have its component

$$\text{in direction } PO = R\,\frac{dz}{ds},$$

$$\text{in direction } NP = R.\frac{dr}{ds};$$

supposing $ds = \sqrt{(dz^2 + dr^2)}$.

Hence, the whole force in $PO = P + R\,\frac{dz}{ds}$, and if we resolve this parallel to x and y, we shall have, since OL, LP, OP are x, y, r, respectively,

$$\text{component in } x = -\left(P + R\frac{dz}{ds}\right).\frac{x}{r},$$

$$\text{in } y = -\left(P + R\frac{dz}{ds}\right).\frac{y}{r}.$$

$$\text{Also force in } z = - \quad Z + R\frac{dr}{ds}.$$

Hence, by the equations (c') we have

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -\left(P + R\frac{dz}{ds}\right).\frac{x}{r}, \\ \frac{d^2y}{dt^2} &= -\left(P + R\frac{dz}{ds}\right).\frac{y}{r}, \\ \frac{d^2z}{dt^2} &= - Z + R\frac{dr}{ds}. \end{aligned}\right\}\ldots\ldots\ldots(1).$$

Hence, we find

$$\frac{xd^2y - yd^2x}{dt} = 0; \text{ or } d\left(\frac{xdy - ydx}{dt}\right) = 0\ldots.(2),$$

$$\frac{dx\,d^2x + dy\,d^2y + dz\,d^2z}{dt^2} = -P\frac{xdx + ydy}{r} - Zdz$$

$$- R.\left\{\frac{dz}{ds}.\frac{xdx + ydy}{r} - \frac{dr\,dz}{ds}\right\};$$

which since $\dfrac{xdx + ydy}{r} = dr$, becomes, multiplying by 2,

$$d.\left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) = -2Pdr - 2Zdz\ldots.(3).$$

Now $\dfrac{dz^2}{dt^2} = \dfrac{dz^2}{dr^2}.\dfrac{dr^2}{dt^2}$; and from the nature of the curve CP, $\dfrac{dz}{dr}$ is known in terms of r. Let $\dfrac{dz}{dr} = p$, p representing the tangent of the angle which the surface at P makes with the horizon. Therefore $\dfrac{dz^2}{dt^2} = p^2.\dfrac{dr^2}{dt^2}$.

Also let the angle $LOP = v$, and we shall have, (as in Art. 16, 17,) $xdy - ydx = r^2 dv$;

$$dx^2 + dy^2 = dr^2 + r^2 dv^2.$$

Hence, the equations (2) and (3) become

$$d \cdot \frac{r^2 dv}{dt} = 0,$$

$$d \cdot \left\{\frac{dr^2}{dt^2} + \frac{r^2 dv^2}{dt^2} + p^2 \cdot \frac{dr^2}{dt^2}\right\} = -2 P dr - 2 Z dz.$$

If we integrate the first of these, we obtain

$r^2 dv = h dt$; h being a constant quantity,

or $dt = \dfrac{r^2 dv}{h}$.

The second of the two equations just found, might be integrated, if we could integrate the right-hand side $- 2 P dr - 2 Z dz$. If we put for P, Z, and z, their values in terms of r, this expression will become a function of r, and its integral will be a function of r. Let $\int (P dr + Z dz) = Q$: hence

$$\frac{dr^2}{dt^2} + \frac{r^2 dv^2}{dt^2} + p^2 \cdot \frac{dr^2}{dt^2} = C - 2 Q.$$

To find the path of the body, put for dt its value $\dfrac{r^2 dv}{h}$, and we have

$$(1 + p^2) \cdot \frac{h^2 dr^2}{r^4 dv^2} + \frac{h^2}{r^2} = C - 2 Q;$$

$$\therefore dv = \frac{\sqrt{(1 + p^2)} \cdot h dr}{r \sqrt{\{(C - 2 Q) r^2 - h^2\}}} \ldots\ldots (e).$$

Hence, $dt = \dfrac{\sqrt{(1 + p^2)} \cdot r dr}{\sqrt{\{(C - 2 Q) r^2 - h^2\}}} \ldots\ldots\ldots (f)$.

If equation (e) be integrable, we have v in terms of r, whence the locus of N is known. And r being known, we know z; for $dz = p dr$.

If the force be always parallel to the axis, we have $P = 0$; and if also Z be a constant force, as gravity, and $= g$, $Q = \int Z dz = gz$, and z is to be expressed in terms of r.

If the force tend to a fixed point in the axis, we may make this the origin A. Let $AP = r'$, and the force in $PA = P'$; therefore $P = P'\frac{r}{r'}$, $Z = P'\frac{z}{r'}$; and $Q = \int(P\,dr + Z\,dz)$

$$= \int P'\frac{r\,dr + z\,dz}{r'} = \int P'dr';$$

because $r^2 + z^2 = r'^2$. And if P' be a function of r' we find Q by integrating.

There will be apsides when $\frac{dr}{dv} = 0$, and therefore when

$$(C - 2Q)\,r^2 - h^2 = 0.$$

35. Prop. To find under what circumstances a body will describe a circle on a surface of revolution.

For this purpose it must always move in a plane perpendicular to the axis of revolution; r and z will be constant; and as in Art. 20, $r\cos. v = x$; therefore $\frac{d^2x}{dt^2} = -\frac{r\cos. v\,.\,dv^2}{dt^2}$;

and if V be the velocity, $V = \frac{rdv}{dt}$; $\therefore \frac{d^2x}{dt^2} = -\frac{V^2\cos. v}{r}$.

Hence, the first and third of equations (1) in last Article become

$$\frac{V^2}{r} = P + R\frac{dz}{ds},$$

$$0 = -Z + R\frac{dr}{ds};$$

$$\therefore \frac{V^2}{r} = P + Z\frac{dz}{dr}\ldots\ldots(g).$$

If the force be gravity acting in the direction of the axis OA, so that $P = 0$, $Z = g$; $\frac{V^2}{r} = g\,.\frac{dz}{dr}$.

Since $h = 2$ area in time $1 = Vr$, $h^2 = gr^3\frac{dz}{dr} = gr^3p$.

Prob. VI. *To determine the time of a pendulum performing a conical revolution.*

A body suspended by a string SP from a point S, fig. 34, will be retained in a spherical surface whose centre is S, and may, by properly adjusting the velocity, revolve in a circle, SP describing a conical surface, which is the motion here spoken of.

In a spherical surface of which the radius is c,

$$z = AS - \sqrt{(c^2 - r^2)}; \quad \frac{dz}{dr} = \frac{r}{\sqrt{(c^2 - r^2)}},$$

$$V^2 = \frac{g r^2}{\sqrt{(c^2 - r^2)}}.$$

If we draw PT a tangent at P, $OT = \dfrac{r^2}{\sqrt{(c^2 - r^2)}}$; $V^2 = 2g . \frac{1}{2} OT$; hence, the velocity of P is that acquired by falling down half OT.

The circle described by P has its circumference $= 2\pi r$; hence, we have the time of a revolution $= \dfrac{2\pi r}{V} = \dfrac{2\pi (c^2 - r^2)^{\frac{1}{4}}}{\sqrt{g}} = \dfrac{2\pi \sqrt{SO}}{\sqrt{g}}$.

Cor. 1. By Prob. I, Cor. 3, the time of a double oscillation of a pendulum whose length is l, would be $2\pi \sqrt{\dfrac{l}{g}}$. Hence, in order that this oscillation may employ the same time as the revolution, we must have $l = SO$.

Cor. 2. The last corollary is true for any surface, PS being a normal.

Cor. 3. If a body were to revolve in a parabolical surface, the times of all circular revolutions would be the same. For in this case, SO the subnormal is constant.

Prob. VII. *A body moves in a spherical surface acted on by gravity, and so as not to describe a circle, to determine its motion.*

Let c be the radius of the sphere. We have $Q = gz$, and $C - 2Q = 2g(k - z)$, k being an arbitrary quantity.

Also $r^2 = 2cz - z^2$, z being measured from the surface:

$$r dr = (c - z) dz; \quad 1 + p^2 = 1 + \frac{r^2}{(c - z)^2} = \frac{c^2}{(c - z)^2}.$$

Hence, by equation (f) $dt = \frac{\sqrt{(1+p^2)}\,r\,dr}{\sqrt{\{(C-2Q)r^2-h^2\}}}$

$$= \frac{c\,dz}{\sqrt{\{2g(k-z)(2cz-z^2)-h^2\}}}.$$

In order that $\frac{dz}{dt}$ may $= 0$, the denominator of the right-hand member of this equation must$=0$; that is

$$2g(k-z)(2cz-z^2)-h^2=0;$$

$$\text{or } z^3-(k+2c)z^2+2kcz-\frac{h^2}{2g}=0;$$

which equation will necessarily have two possible roots; because, as the body moves, it will necessarily reach one highest and one lowest point, and, therefore two places when $\frac{dz}{dt}=0$. Hence, the equation has also a third possible root. Suppose it to be identical with

$$(z-\alpha)(z-\beta)(z-\gamma)=0:$$

where α is the greatest value of z, and β the least, which occur during the body's motion.

Hence, $dt = \frac{c\,dz}{\sqrt{(2g)}\cdot\sqrt{\{(\alpha-z)(z-\beta)(\gamma-z)\}}}$.

To integrate this, let $\theta = \text{arc}\left(\sin.=\sqrt{\frac{z-\beta}{\alpha-\beta}}\right)$;

$$\therefore\ d\theta = \frac{dz}{2\sqrt{\{(z-\beta)(\alpha-\beta)\}}\cdot\sqrt{\left\{1-\frac{z-\beta}{\alpha-\beta}\right\}}}$$

$$= \frac{dz}{2\sqrt{\{(\alpha-z)(z-\beta)\}}}.$$

Also $\sin^2\theta = \frac{z-\beta}{\alpha-\beta}$; $\therefore\ z=\beta+(\alpha-\beta)\sin^2\theta$.

And $\gamma-z=\gamma-\{\beta+(\alpha-\beta)\sin.^2\theta\}=(\gamma-\beta)\{1-\delta^2\sin.^2\theta\}$; if $\delta=\sqrt{\frac{\alpha-\beta}{\gamma-\beta}}$;

$$\therefore\ dt = \frac{2c\,d\theta}{\sqrt{\{2g(\gamma-\beta)\}}\cdot\sqrt{\{1-\delta^2\sin.^2\theta\}}},$$

to be integrated from $z=\beta$ to $z=\alpha$; that is, from $\theta=0$, to $\theta=\frac{\pi}{2}$: this expanded in the same manner as in Prob. II, gives

$$t = \frac{2c}{\sqrt{2g(\gamma-\beta)}}\left\{1+\left(\frac{\delta}{2}\right)^2+\left(\frac{1.3.\delta^2}{2.4}\right)^2 + \left(\frac{1.3.5\delta^3}{2.4.6}\right)^2 + \&c.\right\}.\frac{\pi}{2};$$

which is the time of a whole oscillation from the least to the greatest distance.

Also $dv = \frac{h\,dt}{r^2} = \frac{h\,dt}{2cz-z^2}$; and is hence known in terms of z.

36. Prop. A body acted on by gravity moves on a surface of revolution, whose axis is vertical: when its path is nearly circular, it is required to find the angle between the apsides of N's path, fig. 33.

In this case, $\int Z\,dz = gz = Q$.

And if at an apse $r=a$, $z=k$, we have

$$(C-2gk)\,a^2-h^2=0;\ \therefore\ C=\frac{h^2}{a^2}+2gk.$$

Hence, equation (e) becomes

$$dv = \frac{\sqrt{(1+p^2)}\,.\,h\,dr}{r\sqrt{\left\{2g(k-z)r^2-\frac{h^2.(a^2-r^2)}{a^2}\right\}}}$$

$$= \frac{\sqrt{(1+p^2)}\,.\,\frac{h\,dr}{r^2}}{\sqrt{\left\{2g(k-z)-h^2\left(\frac{1}{r^2}-\frac{1}{a^2}\right)\right\}}}.$$

Let $\frac{1}{r}=\frac{1}{a}+w$; $\therefore\ -\frac{dr}{r^2}=dw$,

$$dv = -\frac{\sqrt{(1+p^2)}\,.\,h\,dw}{\sqrt{\left\{2g(k-z)-h^2\left(\frac{1}{r^2}-\frac{1}{a^2}\right)\right\}}};$$

$$\therefore\ \frac{dw^2}{dv^2} = \frac{2g(k-z)-h^2\left(\frac{1}{r^2}-\frac{1}{a^2}\right)}{h^2(1+p^2)}.$$

It is requisite to express the right-hand side of this equation in terms of w.

Now since at an apse we have $w=0,\ z=k,\ r=a$,

we have generally $z = k + \frac{dz}{dw}.w + \frac{d^2z}{dw^2}.\frac{w^2}{1.2} + \&c.$

the values of the differential coefficients being taken for $w=0$.

And $dz = p\,dr = -pr^2dw$,

$$d^2z = -2p\,r\,dr\,dw - r^2dw\,dp;\ \text{or, making } dp = q\,dr$$
$$= -(2p+qr)\,r\,dr\,dw = (2p+qr)\,r^3\,dw^2.$$

And if p_1 and q_1 be the values which p and q assume when $w=0$, and $r=a$, we have for that case, $\frac{d^2z}{dw^2} = (2p_1+q_1a)a^3$,

$$z = k - p_1a^2w + (2p_1+q_1a)a^3.\frac{w^2}{2} - \&c.$$

Also $\frac{1}{r^2} = \left(\frac{1}{a}+w\right)^2 = \frac{1}{a^2} + \frac{2w}{a} + w^2$.

Hence, $2g(k-z)-h^2\left(\frac{1}{r^2}-\frac{1}{a^2}\right)$ becomes

$$2g\left(p_1a^2w-(2p_1+q_1a)a^3.\frac{w^2}{2}+\&c.\right)-h^2\left(\frac{2w}{a}+w^2\right).$$

But when a body moves in a circle of radius $=a$, we have $h^2 = gr^3p = ga^3p_1$ in this case, (Art. 35.). And when the body moves *nearly* in a circle, h^2 will have nearly this value. If we put $h^2=(1+\delta)ga^3p_1$, we shall finally have to put $\delta=0$, in order to get the ultimate angle when the orbit becomes indefinitely near a circle. Hence, we may put $h^2 = g\,a^3 p_1$, and

$$2g(k-z) - h^2\left(\frac{1}{r^2} - \frac{1}{a^2}\right) \text{ becomes } -\{3ga^3p_1 + ga^4q_1\}\,w^2 + \&c.$$

in which the higher powers of w may be neglected in comparison of w^2;

$$\therefore\ \frac{dw^2}{dv^2} = -\frac{g\,a^3\{3p_1 + q_1 a\}\,w^2}{h^2(1+p^2)} = -\frac{(3p_1 + q_1 a)\,w^2}{p_1(1+p^2)}$$

$$= -\frac{(3p_1 + q_1 a)\,w^2}{p_1(1+p_1^2)},$$

again omitting powers above w^2: for $p = p_1 + Aw + \&c.$

Differentiate and divide by $2\,dw$, and we have

$$\frac{d^2 w}{dv^2} = -\frac{3p_1 + q_1 a}{p_1(1+p_1^2)}\,.\,w = -Nw, \text{ suppose.}$$

Of which the integral, taken so that $v=0$ when $w=0$, is

$$w = C \sin.\, v\sqrt{N}.$$

And w passes from 0 to its greatest value, and consequently r passes from the value a, to another maximum or minimum, while the arc $v\sqrt{N}$ passes from 0 to π. Hence, for the angle A between the apsides we have

$$A\sqrt{N} = \pi;\ A = \frac{\pi}{\sqrt{N}};$$

$$\therefore \text{ where } N = \frac{3p_1 + q_1 a}{p_1(1+p_1^2)}.$$

Prob. VIII. *Let the surface be a sphere; and let the path described be nearly a circle: to find the horizontal angle between the apsides.*

Supposing the origin to be at the lowest point of the surface, we have

$$z = c - \sqrt{(c^2 - r^2)};\ p = \frac{dz}{dr} = \frac{r}{\sqrt{(c^2 - r^2)}};\ q = \frac{dp}{dr} = \frac{c^2}{(c^2 - r^2)^{\frac{3}{2}}};$$

$$\therefore p_1 = \frac{a}{\sqrt{(c^2-a^2)}}; \quad q_1 = \frac{c^2}{(c^2-a^2)^{\frac{3}{2}}}; \quad 1+p_1^2 = \frac{c^2}{c^2-a^2};$$

$$\therefore N = \frac{4c^2-3a^2}{c^2}.$$

Hence, the angle between the apsides $= \dfrac{\pi c}{\sqrt{(4c^2-3a^2)}}$.

The motion of a point on a spherical surface, is manifestly the same as the motion of a simple pendulum or heavy body, suspended by an inextensible string from a fixed point; the body being considered as a point, and the string without weight. If the pendulum begin to move in a vertical plane, it will go on oscillating in the same plane, in the manner already considered in Sect. I, of this Chapter. But, if the pendulum have any lateral motion, it will go on revolving about the lowest point, and generally alternately approaching to it, and receding from it. By a proper adjustment of the velocity and direction, it may describe, by Art. 35, a circle; and if the velocity, when it is moving parallel to the horizon, be *nearly* equal to the velocity in a circle, it will describe a curve little differing from a circle. In this case, we can find the angle between the greatest and least distances, by the formula just deduced.

$$\text{Since} \quad A = \frac{\pi c}{\sqrt{(4c^2-3a^2)}};$$

If $a=0$, $A=\frac{\pi}{2}$; the apsides are 90^0 from each other, which also appears by observing, that when the amplitude of the pendulum's revolutions is very small, the force is nearly as the distance; and the body describes ellipses nearly; of which the lowest point is the centre.

If $a=c$, $A=\pi=180^0$; this is when the pendulum string is horizontal; and requires an infinite velocity.

If $a=\frac{c}{2}$; so that the string is inclined 30^0 to the vertical;

$$A = \frac{2\pi}{\sqrt{13}} = 99^0\ 50'.$$

If $a^2 = \frac{c^2}{2}$; so that the string is inclined 45° to the vertical;

$$A = \pi\sqrt{\frac{2}{5}} = 113^\circ.\ 56'.$$

If $a^2 = \frac{3c^2}{4}$ so that the string is inclined 60° to the vertical;

$$A = \frac{2\pi}{\sqrt{7}} = 136^\circ, \text{ nearly.}$$

PROB. IX. *Let the surface be an inverted cone, with its axis vertical: to find the horizontal angle between the apsides when the orbit is nearly a circle.*

Let a be the radius of the circle, and γ the angle which the slant side makes with the horizon.

Hence, $z = r \tan.\ \gamma$, $p = \tan.\ \gamma$, $q = 0$;

$$\therefore\ N = \frac{3 \tan.\ \gamma}{\tan.\ \gamma \,.\, \sec.^2 \gamma} = 3 \cos.^2 \gamma:$$

and the angle between the apsides $= \frac{\pi}{\cos.\ \gamma \sqrt{3}}$.

If $\gamma = 60^\circ$, angle $= \frac{2\pi}{3} = 120^\circ$.

PROB. X. *Let the surface be an inverted paraboloid whose parameter is* c.

$$r^2 = cz;\ \therefore p = \frac{dz}{dr} = \frac{2r}{c};\ q = \frac{2}{c};$$

$$\therefore\ N = \frac{\frac{6a}{c} + \frac{2a}{c}}{\frac{2a}{c}\left(1 + \frac{4a^2}{c^2}\right)} = \frac{4c^2}{c^2 + 4a^2}.$$

If $a = \frac{c}{2}$, or the body revolve at the extremity of the focal ordinate,

$$N = 2;\ \text{angle between apsides} = \frac{\pi}{\sqrt{2}}.$$

Prob. XI. *Let in fig. 35, OP vary inversely as CO^4.*

Let $AC = h$, $h - z = \frac{c^4}{r^3}$; $\therefore p = \frac{dz}{dr} = \frac{3c^4}{r^4}$; $q = -\frac{12c^4}{r^5}$;

$$\therefore N = \frac{\frac{9c^4}{a^4} - \frac{12c^4}{a^4}}{\frac{3c^4}{a^4}\left(1 + \frac{9c^8}{a^8}\right)} = -\frac{a^8}{a^8 + 9c^8}.$$

This is negative, and hence the angle is impossible, and the body will never come to a second apse. If the velocity be at all less than that in a circle, the body will go on descending continually down the funnel PA.

37. **Prop.** When a body moves on a conical surface, acted on by a force tending to the vertex; its motion in the surface will be the same, as if the surface were unwrapped, and made plane, the force remaining at the vertex.

Let A, fig. 33, be at the vertex, $AP = r'$, force $= P'$, and the angle which the slant side makes with the base $= \gamma$; $\therefore z = r \tan. \gamma$, $p = \tan. \gamma$, $1 + p^2 = \sec.^2 \gamma$.

Also $Q = \int (P dr + Z dz) = \int P' dr'$.

Hence, by equation (*e*), Art. 34,

$$dv = \frac{\sec. \gamma . h dr}{r \sqrt{\{(C - 2\int P' dr') r^2 - h^2\}}},$$

or putting $h' \cos. \gamma$ for h, $dv' . \sec. \gamma$ for dv, and for r its equal $r' \cos \gamma$,

$$dv' = \frac{h' dr'}{r' \sqrt{\{(C - 2\int P' dr') r'^2 - h'^2\}}}.$$

Now dv' is the differential of the angle described *along the conical surface*, and it appears that the relation between v' and r' will be the same as in a plane, where a body is acted upon by a central force P'. For we have in that case, (see page 27,)

$$d\left\{\frac{h'^2 dr'^2}{r'^4 dv'^2} + \frac{h'^2}{r'^2}\right\} = -2P' dr';$$

and integrating,

$$\frac{h'^2 dr'^2}{r'^4 dv^2} + \frac{h'^2}{r'^2} = C - 2\int P' dr',$$

which agrees with the equation just found.

38. Prop. When a body moves on a surface of revolution, to find the reaction R.

Take the three original equations, (1) p. 96, and multiply them respectively by xdz, ydz, and rdr; and the two first become

$$\frac{xd^2x\,dz}{dt^2} = -\frac{Px^2dz}{r} - R\frac{dz^2}{ds}.\frac{x^2}{r},$$

$$\frac{yd^2y\,dz}{dt^2} = -\frac{Py^2dz}{r} - R\frac{dz^2}{ds}.\frac{y^2}{r}.$$

Add these, observing that $x^2 + y^2 = r^2$, and we have

$$\frac{(xd^2x + yd^2y)\,dz}{dt^2} = -Prdz - Rr\frac{dz^2}{ds};$$

and the third is $\dfrac{rdrd^2z}{dt^2} = -Zrdr + Rr\dfrac{dr^2}{ds}.$

Subtract this, observing that $dz^2 + dr^2 = ds^2$, and we have

$$\frac{(xd^2x + yd^2y)\,dz - rdrd^2z}{dt^2} = r(Zdr - Pdz) - Rrds.$$

But $x^2 + y^2 = r^2$, $xdx + ydy = rdr$,

$xd^2x + yd^2y + dx^2 + dy^2 = rd^2r + dr^2$.

Hence the equation becomes

$$\frac{(dr^2 - dx^2 - dy^2)\,dz}{dt^2} + \frac{rdzd^2r - rdrd^2z}{dt^2} = r(Zdr - Pdz) - Rrds,$$

and $dr^2 = ds^2 - dz^2$.

$$\text{Hence, } R = \frac{Zdr - Pdz}{ds} + \frac{drd^2z - dzd^2r}{dt^2ds}$$

$$+ \frac{(dx^2 + dy^2 + dz^2 - ds^2)\,dz}{rdt^2ds}.$$

Now if ρ be the radius of curvature of CP at P,

$$\rho = \frac{ds^3}{drd^2z - dzd^2r} \text{ (\textit{Lacroix}, Traité du Cal. Diff. Art. 351.)}$$

and $dx^2 + dy^2 + dz^2 = d\sigma^2$, σ being the arc described,

$$\text{therefore } R = \frac{Zdr - Pdz}{ds} + \frac{ds^2}{\rho dt^2} + \frac{(d\sigma^2 - ds^2)}{rdt^2} \cdot \frac{dz}{ds};$$

Here it is manifest that $\frac{ds^2}{dt^2}$ is the square of the velocity resolved in the curve PC, fig. 33, and that $\frac{d\sigma^2 - ds^2}{dt^2}$ is the square of the velocity resolved perpendicular to OP in the plane OLP. The two last terms, which involve these quantities, together form that part of the resistance which is due to the centrifugal force; the first term is that which arises from the resolved part of the forces.

From this expression we know the value of R; for we have, as before,

$$\frac{d\sigma^2}{dt^2} = C - 2\int(Pdr + Zdz);$$

also $\frac{d\sigma^2 - ds^2}{dt^2} = \text{vel.}^2 \text{ in } PU = \frac{r^2dv^2}{dt^2} = \frac{h^2}{r^2}$.

$$\text{Hence, } \frac{ds^2}{dt^2} = C - 2\int(Pdr + Zdz) - \frac{h^2}{r^2}.$$

Prob. XI. *To find the tension of a pendulum moving in a spherical surface.*

Retaining the denominations of the Prop. $\rho = 0$;

$$C - 2(\int Pdr + Zdz) = 2g(k - z),$$

$$r = \sqrt{(2cz - z^2)}; \quad \frac{dr}{dz} = \frac{c - z}{\sqrt{(2cz - z^2)}}; \quad \frac{ds}{dr} = \frac{c}{c - z};$$

$$\frac{ds}{dz} = \frac{c}{\sqrt{(2cz - z^2)}} = \frac{c}{r}.$$

$$\text{Hence, } R = \frac{g(c - z)}{c} + \frac{2g(k - z) - \frac{h^2}{r^2}}{c} + \frac{h^2}{r^3} \cdot \frac{r}{c}$$

$$= \frac{g \cdot (c + 2k - 3z)}{c}$$

and hence it is the same as that of the pendulm oscillating in a vertical plane with the same velocity at the same distances: see Prob. V.

SECT. III. *The Motion of a Point upon any Surface.*

39. PROP. To find the velocity, reaction, and motion of a body upon any surface whatever.

Let R be the reaction of the surface, which is of course in the direction of a normal to it at each point. And let ϵ, ϵ', ϵ'', be the angles which this normal makes with the axes of x, y, and z respectively; we shall then have, considering the resolved parts of R among the forces which act on the point, by the formulæ (c'),

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= X + R \cos.\ \epsilon, \\ \frac{d^2y}{dt^2} &= Y + R \cos.\ \epsilon', \\ \frac{d^2z}{dt^2} &= Z + R \cos.\ \epsilon''. \end{aligned}\right\} \cdots\cdots\cdots(1)$$

Now the nature of the surface is expressed by an equation between x, y, z: and if we suppose that we have, deduced from this equation,

$$dz = pdx + qdy, \text{ so that } p = \frac{dz}{dx},\ q = \frac{dz}{dy},$$

p, q, being the partial differential coefficients of z; (*Lacroix*, 125); we have for the equations to the normal of the point whose co-ordinates are x, y, z, (*Lacroix*, Elem. Treat. No. 143); x', y', z', being co-ordinates to any point in the normal;

$$x' - x + p(z' - z) = 0,$$
$$y' - y + q(z' - z) = 0.$$

Hence, it appears that if PK, fig. 36, be the normal, PG, PH its projections on planes parallel to xz and yz respectively; the equation of PG is $x' - x + p(z' - z) = 0$; and hence $GN + p.PN = 0$, and $GN = -p.PN$, similarly the equation of PH is $y' - y + q(z' - z) = 0$, whence $HN + q.PN = 0$, and $HN = -q.PN$.

And hence, $\cos. \epsilon = \cos. KPh = \frac{Ph}{PK} = \frac{GN}{\sqrt{(PN^2 + NG^2 + HN^2)}}$

$$= - \frac{p}{\sqrt{(1+p^2+q^2)}}$$

$$\cos. \epsilon' = \cos. KPg = \frac{Pg}{PK} = - \frac{HN}{\sqrt{(PN^2+NG^2+HN^2)}}$$

$$= - \frac{q}{\sqrt{(1+p^2+q^2)}}.$$

Whence, $\cos. \epsilon'' = \sqrt{(1 - \cos.^2 \epsilon - \cos.^2 \epsilon')} = \frac{1}{\sqrt{(1+p^2+q^2)}}.$

Substituting these values; multiplying by dx, dy, and dz *, respectively, in the three equations; and observing that $dz - pdx - qdy = 0$, we have

$$\frac{dx d^2x + dy d^2y + dz d^2z}{dt^2} = Xdx + Ydy + Zdz.$$

Hence, $\frac{dx^2 + dy^2 + dz^2}{dt^2} = 2\int(X dx + Y dy + Z dz)$,

and if this can be integrated, we have the velocity.

If we take the three original equations (1), and multiply them respectively by $-p$, $-q$, and 1, and then add, we obtain

$$-p \cdot \frac{d^2x}{dt^2} - q \cdot \frac{d^2y}{dt^2} + \frac{d^2z}{dt^2} = -pX - qY + Z + R\sqrt{(1+p^2+q^2)}.$$

But $dz = pdx + qdy$;

$$\text{hence, } \frac{d^2z}{dt^2} = p\frac{d^2x}{dt^2} + q\frac{d^2y}{dt^2} + \frac{dp\,dx + dq\,dy}{dt^2};$$

substituting this on the first side of the above equation, and taking the value of R, we find

$$R = \frac{pX + qY - Z}{\sqrt{(1+p^2+q^2)}} + \frac{dp dx + dq dy}{dt^2\sqrt{(1+p^2+q^2)}}.$$

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* Here dz indicates the differential of z considered as a function of *both* the quantities x and y. It is what *Lacroix*, Elem. Treat. 126, indicates by $d(z)$.

If in the three original equations we eliminate R, we find two second differential equations, involving the known forces X, Y, Z, and p, q, which are also known when the surface is known, combining with these the equation to the surface, by which z is known in x and y, we have equations from which we can find the relation between the time and the three co-ordinates.

40. **Prop.** To find the path which a body will describe upon a given surface, when acted on by no force.

In this case, we must make X, Y, Z each$=0$. Then, if we multiply the three equations (1) of last Art. respectively by $-(q\,dz+dy)$, $p\,dz+dx$, $q\,dx-p\,dy$, and add them, we find,

$$-(q\,dz+dy)\,d^2x+(p\,dz+dx)\,d^2y+(q\,dx-p\,dy)\,d^2z$$

$$=R\,dt^2\left\{\begin{array}{l}-(q\,dz+dy)\cos.\epsilon\\+(p\,dz+dx)\cos.\epsilon'\\+(q\,dx-p\,dy)\cos.\epsilon''\end{array}\right\}$$

or, putting for $\cos.\epsilon$, $\cos.\epsilon'$, $\cos.\epsilon''$, their values,

$$=\frac{R\,dt^2}{\sqrt{(1+p^2+q^2)}}\{p(q\,dz+dy)-q(p\,dz+dx)+q\,dx-p\,dy\}=0.$$

Hence, for the curve described in this case, we have

$$(p\,dz+dx)\,d^2y=(p\,dy-q\,dx)\,d^2z+(q\,dz+dy)\,d^2x\,*.$$

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* If we make *ds constant*, in this we have $dx\,d^2x+dy\,d^2y+dz\,d^2z=0$, and eliminating d^2x we obtain

$$\{(p\,dz+dx)\,dx+(q\,dz+dy)\,dy\}\,d^2y=\{(p\,dy-q\,dx)\,dx$$
$$-(q\,dz+dy)\,dz\}\,d^2z.$$

But the first side$=d^2y\,\{(p\,dx+q\,dy)\,dz+dx^2+dy^2\}$
$$=d^2y\,\{dz^2+dx^2+dy^2\}=ds^2\,d^2y.$$

And the second side$=d^2z\,\{(p\,dx-dz)\,dy-q\,(dx^2+dz^2)\}$
$$=d^2z\,\{-q\,dy^2-q\,dx^2-q\,dz^2\}=-q\,ds^2\,d^2z.$$

Hence, the equation becomes, omitting ds^2,

$$d^2y+q\,d^2z=0.$$

This equation expresses a relation among the differentials of x, y, z, without any regard to the time. Hence, we may suppose x the independent variable, and $d^2 x = 0$; whence we have

$$(p\,dz + dx)\,d^2 y = (p\,dy - q\,dx)\,d^2 z.$$

This equation, combined with $dz = p\,dx + q\,dy$, gives the curve described, where the body is left to itself, and moves along the surface.

The curve thus described is the shortest line which can be drawn from one of its points to another, upon the surface.

The velocity is constant, as appears from the equation

$$\text{velocity}^2 = 2\int(X\,dx + Y\,dy + Z\,dz).$$

By methods somewhat similar we might determine the motion of a point upon a given curve of double curvature when acted upon by given forces.

CHAP. VI.

THE CONSTRAINED MOTION OF SEVERAL POINTS.

41. In the present Chapter it is intended to consider some problems in which several material points are connected with each other by means of cords or rods, not possessing inertia or weight, and generally inflexible and inextensible, while one or more of the points are compelled to move on given lines or surfaces.

Some of these problems will approach in their nature to those in which we consider the motion of a finite body. For two material points connected by a rigid rod may be considered as a finite body, and accordingly, the formulæ will be the same as those for a rigid body in some cases. But by considering the tension of the rod or string, which connects two points, as one of the forces which acts upon them, we can reduce these problems under the formulæ for the motion of points. And as this tension will always be the same for both the bodies which the rod or cord connects, we shall be able to eliminate it, and to obtain the motions of the points.

The *tensions* here spoken of are of the nature of those which we have called *moving forces*, or *pressures*. Consequently, the accelerating forces which they produce upon the bodies are, by the third law of motion, as the pressures or tensions directly, and as the mass of the bodies inversely. If the tension of a string which acts upon a body m be equal to the weight of a mass p, and if g be the *accelerating force* of gravity, we have

$$\frac{pg}{m} = \text{accelerating force produced by tension.}$$

SECT. I. *The Motion of a Rod on Planes**.

PROB. I. Fig. 37. P, Q, *are two material points, connected by an inflexible rod* PQ; PQ *falls by gravity in a vertical plane, while* P *moves along a horizontal plane: to define the motion of* Q.

N.B. If PQ have at first no motion, except in a vertical plane, it will continue to move in a vertical plane, because all the forces are in the same plane.

Let APN be the horizontal line, G the centre of gravity of P, Q; GH, QN vertical; let A be a fixed point, $AP = x$, $AN = x'$, $NQ = y'$. And let p = the tension of the rod PQ; (that is, the force with which it presses P in the direction QP, and Q in the direction PQ;) also, let θ be the angle NPQ; then the resolved parts of the tension will be $p \,.\, \cos.\,\theta$ parallel to AN, and $p \,.\, \sin.\,\theta$ parallel to NQ. And, if m, m', be the masses of P, Q, respectively, the accelerating forces will be

$$-\frac{pg\cos.\,\theta}{m} \text{ on } P;\ \frac{pg\cos.\,\theta}{m'},\ \text{and } \frac{pg\sin.\,\theta}{m'} \text{ on } Q.$$

* The motion of a rod without weight connecting two heavy bodies, will be the same as the motion of a heavy rod: this will appear hereafter.

Hence, we have, by equations (c),

$$\left.\begin{array}{l}\dfrac{d^2 x}{dt^2} = -\dfrac{pg\cos.\theta}{m}, \\ \dfrac{d^2 x'}{dt^2} = \dfrac{pg\cos.\theta}{m'},\ \dfrac{d^2 y'}{dt^2} = \dfrac{pg\sin.\theta}{m'} - g\end{array}\right\}\ldots\ldots(1);$$

$$\therefore \frac{md^2 x + m'd^2 x'}{dt^2} = 0.$$

Let $\bar{x} = AH = \dfrac{mx + m'x'}{m + m'}$; $\therefore d^2\bar{x} = 0$.

Hence, the point H will either remain at rest, or move uniformly along AN.

If we make $PG = b$, $QG = c$,

$$x = AP = AH - PH = \bar{x} - b\cos.\theta;\ \therefore d^2 x = -bd^2\cos.\theta,$$
$$x' = AN = AH + HN = \bar{x} + c\cos.\theta;\ \therefore d^2 x' = \ \ cd^2\cos.\theta.$$

$$\text{Also, } y' = QN = \frac{QP}{PG}.GH = \frac{m + m'}{m'}\, b\sin.\theta.$$

Hence, the first and last of equations (1) give, substituting these values, and multiplying by m, m',

$$mb\frac{d^2\cos.\theta}{dt^2} = pg\cos.\theta;$$

$$(m + m')b.\frac{d^2\sin.\theta}{dt^2} = pg\sin.\theta - gm'.$$

Multiply the first of these equations by $2d.\cos.\theta = -2\sin.\theta d\theta$, and the second by $2d.\sin.\theta = 2\cos.\theta d\theta$, and add them, and we have

$$mb\frac{2d\cos.\theta . d^2\cos.\theta}{dt^2} + (m + m')b\frac{2d\sin.\theta\, d^2\sin.\theta}{dt^2}$$
$$= -2gm'\cos.\theta d\theta.$$

This integrated gives

$$mb.\left(\frac{d.\cos.\theta}{dt}\right)^2 + (m+m')\,b\left(\frac{d\sin.\theta}{dt}\right)^2 = C - 2gm'\sin.\theta:$$

$$\text{or, } \{m\sin.^2\theta + (m+m')\cos.^2\theta\}\, b.\frac{d\theta^2}{dt^2} = C - 2gm'\sin.\theta:$$

which reduced, gives

$$b.\frac{d\theta^2}{dt^2} = \frac{C - 2gm'\sin.\theta}{m + m'\cos.^2\theta}.$$

To determine C, suppose that when PQ reaches the line AN, or when $\theta = 0$, the angular velocity $\frac{d\theta}{dt}$ is a: hence,

$$ba^2 = \frac{C}{m+m'};$$

$$\therefore \frac{d\theta^2}{dt^2} = \frac{ba^2(m+m') - 2gm'\sin.\theta}{b(m+m'\cos.^2\theta)} \ldots\ldots\ldots(2).$$

There may be a position when the angular velocity is 0; suppose that there θ is θ_1; then, we shall have

$$ba^2(m+m') - 2gm\sin.\theta_1 = 0\ldots\ldots\ldots(3);$$

$$\frac{d\theta^2}{dt^2} = \frac{2gm}{b}\cdot\frac{\sin.\theta_1 - \sin.\theta}{m+m'\cos.^2\theta}.$$

If the rod fall from rest in a given position, θ_1 is known: and hence, the angular velocity at any point. To determine the position at any time, we must obtain t in terms of θ, by integrating again; and hence θ in terms of t.

If the line fall from a vertical position $\theta_1 = \frac{\pi}{2}$; hence, to find the angular velocity a, acquired when it becomes horizontal, we have, by (3),

$$a^2 = \frac{2gm}{b(m+m')};\quad \therefore b^2a^2 = \frac{2bgm}{m+m'}.$$

Now ba is the linear velocity of the point G at H, and it appears from this, that the space through which a body must fall freely to acquire this velocity, is $\frac{bm'}{m+m'}$, a third proportional to QP and GP.

If it were supposed that P were constrained to move in a horizontal groove AP, while PQ were so connected with it, that PQ could descend below the horizontal position, so as to revolve entirely round P; the mechanical conditions would be the same, and the expression obtained above for the angular velocity is true in this case also.

$$\text{Since } \frac{d\theta}{dt} = \sqrt{\frac{ba^2(m+m') - 2m'g \sin.\theta}{b(m+m' \cos.^2\theta)}},$$

the motion is possible, so long as the numerator of the fraction is positive. The negative part is greatest, when $\theta = \frac{\pi}{2}$; hence, the numerator is always positive, if

$$ba^2(m+m') > 2mg, \text{ or if } a^2 > \frac{2mg}{(m+m')b}.$$

In this case, if H be at rest at first, the line PQ will go on revolving about the point H; the centre of gravity G ascending and descending in a vertical line.

If H be at rest, the path described by Q will be an ellipse, whose semi-axes are GQ and PQ,

$$\text{For } \left(\frac{GO}{GQ}\right)^2 + \left(\frac{QN}{PQ}\right)^2 = 1, \text{ or } \frac{x^2}{c^2} + \frac{y^2}{(b+c)^2} = 1:$$

the equation to an ellipse, of which the semi-axes are $b+c$ and c.

$$\text{The velocity of } P = \frac{bd.\cos.\theta}{dt} = -b \sin.\theta.\frac{d\theta}{dt}$$

$$= -b \sin.\theta \sqrt{\frac{ba^2(m+m') - 2m'g \sin.\theta}{b(m+m' \cos.^2\theta)}} \ldots\ldots(4).$$

If H be in motion, and move uniformly so as to carry the whole line PQ in the direction AX, the relative motion of the points P, Q will not be altered. The system will have the motion of rotation already investigated, combined with the uniform motion of translation in the direction AX; and the point Q by the mixture of its elliptical motion with this rectilinear one, will describe a kind of trochoidal curve.

Suppose Q *to hang vertically from* P, *and a given velocity* V *to be communicated to* P: *to determine the motion of* P *and* Q.

The velocity of G parallel to AX is constant, because $\frac{d^2\bar{x}}{dt^2}=0$, and $\frac{d\bar{x}}{dt}=$ constant. Hence, G will retain the velocity parallel to AX, which it has at first. But at first, when Q is at rest, and P moves with the velocity V, the velocity of G parallel to AX will be $\frac{V.QG}{QP}$ $=\frac{Vc}{b+c}$. And at first the angular velocity of Q round P will be $\frac{V}{b+c}$; hence, by (2), putting $\frac{\pi}{2}$ for θ,

$$\frac{V^2}{(b+c)^2}=a^2.\frac{m+m'}{m}-\frac{2gm'}{mb};$$

whence, a is known; and hence the motion of the system is completely determined.

The quantity g which represents the vertical force may be of any value. If we make it $=0$, we have the motion of a rod moving freely, for instance, on a horizontal plane, while one end moves in a rectilinear groove. In this case

$$\frac{d\theta^2}{dt^2}=\frac{a^2(m+m')}{m+m'\cos.^2\theta}.$$

This velocity is a, when the rod coincides with the groove, and increases to $a\sqrt{\frac{m+m'}{m}}$ when it is perpendicular to it.

If we make $m=0$, we have, by (2),

$$\frac{d\theta^2}{dt^2}=\frac{ba^2-2g\sin.^2\theta}{b\cos.\theta}.$$

Here H coincides with N; we might find θ by integrating. It is easily seen that the curve described is a parabola.

PROB. II. *The same things being supposed, except that the plane* AX *is inclined to the horizon; to determine the motion.*

The rod is still supposed to move in a vertical plane.

If we resolve the forces parallel and perpendicular to the plane AX, we shall find that the point H will descend down the inclined plane in the same manner as a heavy point would do: that is, acted upon by a constant force g sin. ι, ι being the inclination. And that the angular motion will be obtained in the same manner as in the last problem, except that instead of g we shall have g cos. ι, the force perpendicular to the plane.

If the line AX be vertical, the motion of PQ, with respect to AX, will be the same as in the last problem when we made $g=0$.

PROB. III. *A rod* PQ, *connecting two material points* P, Q, *acted on by gravity, moves so that one of them slides along a horizontal plane: to define the motion.*

This differs from Prob. I, in not supposing the motion to be in a vertical plane.

Let AL, LP, fig. 38, be rectangular co-ordinates of P in the horizontal plane, AM, MN, NQ co-ordinates of Q,

$$AL=x,\ LP=y,\ AM=x',\ MP=y',\ NQ=z',\ PQ=b.$$

Also let $NOM=\phi$, and $QPN=\theta$; hence,

$$z'=b \sin.\theta,\ x'-x=b\cos.\theta\cos.\phi,\ y'-y=b\cos.\theta\sin.\phi.$$

Let p be the tension of PQ, m, m' the masses of P, Q.

And resolving the forces, we have

$$\left.\begin{aligned} &\frac{d^2x}{dt^2}=-\frac{pg}{m}\cos.\theta\cos.\phi,\ \frac{d^2y}{dt^2}=-\frac{pg}{m}\cos.\theta\sin.\phi,\\ &\frac{d^2x'}{dt^2}=\frac{pg}{m}\cos.\theta\cos.\phi,\ \frac{d^2y'}{dt^2}=\frac{pg}{m'}\cos.\theta\sin.\phi,\\ &\qquad\frac{d^2z'}{dt^2}=\frac{pg}{m'}\sin.\theta-g.\end{aligned}\right\}\ldots.(1).$$

Hence, we have $\frac{m d^2 x + m' d^2 x'}{d t^2} = 0$; $\frac{m d^2 y + m' d^2 y'}{d t^2} = 0$.

From which it appears that the projection of the centre of gravity of P, Q, on the plane of xy moves in a straight line with a uniform velocity.

$$\left.\begin{aligned} \text{Also } \frac{d^2 x' - d^2 x}{d t^2} &= \left(\frac{1}{m} + \frac{1}{m'}\right) p g \cos.\, \theta \cos.\, \phi, \\ \frac{d^2 y' - d^2 y}{d t^2} &= \left(\frac{1}{m} + \frac{1}{m'}\right) p g \cos.\, \theta \sin.\, \phi \end{aligned}\right\} \ldots (2).$$

And $y' - y = b \cos.\, \theta \sin.\, \phi$, $x' - x = b \cos.\, \theta \cos.\, \phi$;

$$\therefore \frac{(x' - x)(d^2 y' - d^2 y) - (y' - y)(d^2 x' - d^2 x)}{d t^2} = 0.$$

Whence it appears that N describes about P areas proportional to the times; (see Art. 16,) and therefore

$b^2 \cos.^2 \theta \, d\phi = h\, d t$: h being a constant quantity....(3).

Again, equations (2) may be put in this form

$$\frac{d^2 . \cos.\, \theta \cos.\, \phi}{d t^2} = \left(\frac{1}{m} + \frac{1}{m'}\right) \frac{pg}{b} \cos.\, \theta \cos.\, \phi;$$

$$\frac{d^2 . \cos.\, \theta \sin.\, \phi}{d t^2} = \left(\frac{1}{m} + \frac{1}{m'}\right) \frac{pg}{b} \cos.\, \theta \sin.\, \phi.$$

And multiplying by $\cos.\, \phi$, $\sin.\, \phi$, and adding

$$\cos.\, \phi\, d^2 . \cos.\, \theta \cos.\, \phi + \sin.\, \phi\, d^2 \cos.\, \theta \sin.\, \phi$$

$$= \left(\frac{1}{m} + \frac{1}{m'}\right) \frac{pg}{b} \cos.\, \theta\, d t^2.$$

$$\begin{aligned} \text{But } d . \cos.\, \theta \cos.\, \phi &= \cos.\, \phi\, d . \cos.\, \theta - \cos.\, \theta \sin.\, \phi\, d\phi, \\ d^2 . \cos.\, \theta \cos.\, \phi &= \cos.\, \phi\, d^2 \cos.\, \theta - 2 \sin.\, \phi\, d\phi\, d . \cos.\, \theta \\ &\quad - \cos.\, \theta \cos.\, \phi\, d\phi^2 - \cos.\, \theta \sin.\, \phi\, d^2\phi, \\ d . \cos.\, \theta \sin.\, \phi &= \sin.\, \phi\, d . \cos.\, \theta + \cos.\, \theta \cos.\, \phi\, d\phi, \\ d^2 . \cos.\, \theta \sin.\, \phi &= \sin.\, \phi\, d^2 \cos.\, \theta + 2 \cos.\, \phi\, d\phi . d . \cos.\, \theta \\ &\quad - \cos.\, \theta \sin.\, \phi\, d\phi^2 + \cos.\, \theta \cos.\, \phi\, d^2\phi. \end{aligned}$$

Hence, $\cos.\phi d^2 . \cos.\theta\cos.\phi + \sin.\phi d^2\cos.\theta\sin.\phi$

becomes $d^2\cos.\theta - \cos.\theta d\phi^2$;

or since, by (3), $d\phi^2 = \dfrac{h^2dt^2}{b^4\cos.^4\theta}$, it becomes

$$d^2\cos.\theta - \frac{h^2dt^2}{b^4\cos.^3\theta}.$$

And the above equation is equivalent to

$$d^2\cos.\theta - \frac{h^2dt^2}{b^4\cos.^3\theta} = \left(\frac{1}{m} + \frac{1}{m'}\right)\frac{pg}{b}\cos.\theta dt^2 \dots\dots (4).$$

Also the equation in z' may be written thus,

$$d^2\sin.\theta = \left(\frac{pg}{bm'}.\sin.\theta - \frac{g}{b}\right)dt^2 \dots\dots\dots\dots\dots\dots (5).$$

Eliminate p by multiplying these by $m\sin.\theta$, and $(m+m')\cos.\theta$, and subtracting

$$(m+m')\cos.\theta\, d^2\sin.\theta - m\sin.\theta d^2\cos.\theta - \frac{mh^2\sin.\theta dt^2}{b^4\cos.^3\theta}$$

$$= -(m+m')\frac{g}{b}\cos.\theta dt^2.$$

Multiply by $2d\theta$, and we may put the equation in this form,

$$2(m+m')d\sin.\theta d^2\sin.\theta + 2md\cos.\theta d^2\cos.\theta$$

$$+ \frac{2mh^2d\cos.\theta dt^2}{b^4\cos.^3\theta} = -\frac{2(m+m')g}{b}d.\sin.\theta dt^2.$$

Integrating, $(m+m')(d\sin.\theta)^2 + m(d\cos.\theta)^2 - \dfrac{mh^2dt^2}{b^4\cos.^2\theta}$

$$= Cdt^2 - \frac{2(m+m')g}{b}\sin.\theta.dt^2;$$

$$\therefore \frac{d\theta^2}{dt^2}\{(m+m')\cos.^2\theta + m\sin.^2\theta\}$$

$$= \left\{C - \frac{2(m+m')g\sin.\theta}{b} + \frac{mh^2}{b^4\cos.^2\theta}\right\},$$

$$\frac{d\theta^2}{dt^2}\{m+m'\cos.^2\theta\}=C-\frac{2(m+m').g\sin.\theta}{b}+\frac{mh^2}{b^4\cos.^2\theta}\ldots(6).$$

When $\theta=0$, $\dfrac{d\theta^2}{dt^2}(m+m')=C+\dfrac{mh^2}{b^4}$.

We can never have $\theta=\dfrac{\pi}{2}$; for then $\dfrac{d\theta}{dt}$ is infinite.

To find when $\dfrac{d\theta}{dt}=0$, we have a cubic equation. But it is manifest that if θ be a value which satisfies this equation, $\pi-\theta$ will also satisfy it.

Having found θ in terms of t by (6), we find ϕ by equation (3). If in equation (3) we eliminate dt by (6), we have $d\phi=d\theta\times$ a function of θ. And if we put $b\cos.\theta=PN=r$, $d\theta=-\dfrac{dr}{\sqrt{(b^2-r^2)}}$, whence $d\phi=dr\times$ a function of r, which is the equation to the orbit of N about P.

If $m=0$, we have

$$\frac{d\theta^2}{dt^2}.m'.\cos.^2\theta=C-\frac{2m'g}{b}\sin.\theta.$$

And when $\theta=0$,

$$\frac{d\theta^2}{dt^2}=\frac{C}{m'}.$$

In this case we might integrate. The body Q will describe a parabola.

PROB. IV. *Two points* P, Q, *fig.* 39, *are connected by a rod, and one of them* P, *slides along a vertical line* AZ: *to define the motion.*

Let $AP=z$; and let $AM=x'$, $MN=y'$, $NQ=z'$, three rectangular co-ordinates of the point Q from a fixed point A.

Let $QPZ=\theta$, $MAN=\phi$, $PQ=b$;

$\therefore x'=b\sin.\theta.\cos.\phi$, $y=b\sin.\theta\sin.\phi$, $z'-z=b\cos.\theta$.

And if p be the tension of PQ, the equations of motion will be

$$\left.\begin{aligned}\frac{d^2x'}{dt^2} = \frac{pg}{m'}\ \text{sin.}\ \theta\ \text{cos.}\ \phi,\ \frac{d^2y'}{dt^2} = \frac{pg}{m'}\ \text{sin.}\ \theta\ \text{sin.}\ \phi \\ \frac{d^2z'}{dt^2} = g + \frac{pg}{m'}\ \text{cos.}\ \theta,\ \frac{d^2z}{dt^2} = g - \frac{pg}{m}\ \text{cos.}\ \theta\end{aligned}\right\}\cdot\cdot(1).$$

Hence, as in last problem, we find

$$\frac{x'd^2y' - y'd^2x'}{dt^2} = 0, \text{ and } b^2\ \text{sin.}^2\ \theta d\phi = h dt\ \ldots\ldots\ldots (2).$$

$$\text{Also, cos. } \phi \frac{d^2x'}{dt^2} + \text{sin. } \phi \frac{d^2y'}{dt^2} = \frac{pg}{m'}\ \text{sin. } \theta,$$

which, transformed as in last problem, gives

$$d^2\ \text{sin.}\ \theta - \text{sin.}\ \theta d\phi^2 = \frac{pg}{m'b}.\ \text{sin.}\ \theta dt^2;$$

$$\therefore \text{ by (2), } d^2\ \text{sin.}\ \theta - \frac{h^2dt^2}{b^4\ \text{sin.}^3\ \theta} = \frac{pg}{m'b}\ \text{sin.}\ \theta dt^2\ \ldots\ldots (3).$$

And $b d^2$ cos. $\theta = d^2z' - d^2z$; hence, by equations (1),

$$d^2\ \text{cos.}\ \theta = \left(\frac{1}{m} + \frac{1}{m'}\right)\frac{pg}{b}\ \text{cos.}\ \theta\ldots\ldots\ldots(4).$$

Eliminating p, as in last problem, we obtain

$$2.(m+m')\,d.\ \text{sin.}\ \theta d^2\ \text{sin.}\ \theta + 2.md.\ \text{cos.}\ \theta d^2\ \text{cos.}\ \theta$$
$$- \frac{2h^2\,(m+m')\,dt^2.\,d\ \text{sin.}\ \theta}{b^4\ \text{sin.}^3\ \theta} = 0.$$

Integrating, $(m+m')\,(d\ \text{sin.}\ \theta)^2 + (d.\ \text{cos.}\ \theta)^2$

$$+ \frac{h^2\,(m+m')\,dt^2}{b^4\ \text{sin.}^2\ \theta} = Cdt^2,$$

$$\frac{d\theta^2}{dt^2}.(m+m'\ \text{cos.}^2\ \theta) = C - \frac{h^2\,(m+m')}{b^4\ \text{sin.}^2\ \theta}\ \ldots\ldots (5).$$

Hence, the angular velocity of PQ in a vertical direction is known. And hence, by (2), we may, as in last problem, find the differential equation to the orbit described by N round A. AN describes areas proportional to the times by (2).

The relation between t and θ may be expressed by means of an elliptical arc. Equation (5) may be put in this form,

$$dt^2 = \text{sin.}^2\ \theta\, d\theta^2.\ \frac{m+m'\ \text{cos.}^2\ \theta}{C - \dfrac{h^2\,(m+m')}{b^4} - C\ \text{cos.}^2\ \theta}$$

$$= \sin.^2 \theta \, d\theta^2 \, . \frac{m + m' \cos.^2 \theta}{D - C \cos.^2 \theta} .$$

Making $D = C - \dfrac{h^2 (m + m')}{b^4}$.

Now, if 1 and a be the semi-axis minor and major of an ellipse, σ the arc measured from the extremity of the semi-axis, and ξ its abscissa from the centre along the axis-minor; we shall have

$$d\sigma^2 = d\xi^2 \, . \frac{1 + (a^2 - 1)\, \xi^2}{1 - \xi^2} .$$

If, in the value of dt^2 we make $C \cos.^2 \theta = D \xi^2$, we have

$$\cos. \theta = \sqrt{\frac{D}{C}} \, . \xi ; \quad - \sin. \theta d\theta = \sqrt{\frac{D}{C}} \, . d\xi ; \text{ and hence,}$$

$$dt^2 = d\xi^2 \, . \frac{D}{C} \, . \frac{m + \dfrac{Dm'}{C} \xi^2}{D - D\xi^2} = \frac{m}{C} . \, d\xi^2 \, . \frac{1 + \dfrac{Dm'}{Cm} \xi^2}{1 - \xi^2} ;$$

which agrees with $\dfrac{m}{C} \, . \, d\sigma^2$, if we make

$$a^2 - 1 = \frac{Dm'}{Cm} , \text{ whence } a \text{ is known.}$$

$$t + C' = \sqrt{\frac{m}{C}} \times \text{ elliptical arc } \left(\text{abscissa} = \cos. \theta \sqrt{\frac{C}{D}} \right) ;$$

$$\therefore \cos. \theta \sqrt{\frac{C}{D}} = \text{abscissa of elliptical arc } (t + C') . \sqrt{\frac{C}{m}} .$$

When $\theta = \dfrac{\pi}{2}$, $\xi = 0$, and the arc $= 0$, and if at that time t is t_1, $C = - t_1$.

Also, when $\theta = \dfrac{3\pi}{2}$, $\xi = 0$, and the arc $=$ a semi-ellipse; hence,

$\tau \sqrt{\dfrac{C}{m}} =$ semi-ellipse, gives the interval τ between two horizontal positions of the rod.

When $\theta = 0$, the angular velocity is infinite; hence, the rod will never coincide with the vertical line.

This is true only if the body have some angular motion horizontally, for if it move at first in a vertical plane, it will continue to do so, and its motion may be calculated by Prob. III.

PROB. V. *Two points* P, Q, *connected by a rod, slide along two given inclined planes, acted on by gravity; to determine the motion.*

The motion is supposed to take place in a vertical plane.

Let AX, AY, fig. 40, be the two planes, making with the horizon angles β, γ. Let PQ make an angle θ with AX, and η with AY. $AP=x$, $AQ=y$, $PQ=b$; tension of $PQ=p$, masses of P, $Q=m, m'$.

Resolving the forces in the lines AX, AY, we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=\frac{pg}{m}\cos.\theta-g\sin.\beta\\ \frac{d^2y}{dt^2}&=\frac{pg}{m'}\cos.\eta-g\sin.\gamma\end{aligned}\right\}\ldots.(1).$$

$$\text{Hence, }\frac{m'\cos.\eta d^2x-m'\cos.\theta d^2y}{dt^2}$$

$$=g\,(m'\sin.\gamma\cos.\theta-m\sin.\beta\cos.\eta)\ldots.(2).$$

$$\text{Now } x=\frac{b\sin.\eta}{\sin.(\beta+\gamma)},\ y=\frac{b\sin.\theta}{\sin.(\beta+\gamma)}.$$

Substituting these values in (2), multiplying some of the terms by $2d\theta$, and others by $-2d\eta$, (since $d\theta=-d\eta$) and integrating, we have

$$\frac{b}{\sin.(\beta+\gamma)}\left\{m\left(\frac{d.\sin.\eta}{dt}\right)^2+m'\left(\frac{d.\sin.\theta}{dt}\right)^2\right\}$$

$$=C-2g\,(m'\sin.\gamma\sin.\theta+m\sin.\beta\sin.\eta);$$

$$\therefore\frac{b}{\sin.(\beta+\gamma)}\cdot\frac{d\theta^2}{dt^2}.(m\cos.^2\eta+m'\cos.^2\theta)$$

$$=C-2g\,(m'\sin.\gamma\sin\theta+m\sin.\beta\sin.\eta)\ldots.(3).$$

Hence, the angular velocity is known.

This may be reduced as follows. Let the angle which PQ

makes with the horizon be ψ. Then, $\theta=\beta+\psi$, $\eta=\gamma-\psi$. Also, let $C=2gn$; then the right-hand side of equation (3) becomes

$$2g\,\{n-m' \text{ sin. } \gamma \,.\, \text{sin. } (\beta+\psi)-m \text{ sin. } \beta \,.\, \text{sin. } (\gamma-\psi)\}:$$

expanding, the part in brackets becomes

$$n-(m'+m) \text{ sin. } \beta \text{ sin. } \gamma \text{ cos. } \psi$$
$$-(m' \text{ sin. } \gamma \text{ cos. } \beta+m \text{ sin. } \beta \text{ cos. } \gamma) \text{ sin. } \psi.$$

$$\left.\begin{array}{r}\text{Suppose } (m+m') \text{ sin. } \beta \text{ sin. } \gamma=r \text{ cos. } \delta \\ m' \text{ sin. } \gamma \text{ cos. } \beta-m \text{ sin. } \beta \text{ cos. } \gamma=r \text{ sin. } \delta\end{array}\right\}\ldots\ldots\ldots(4)$$

which always give possible values of r and δ; and our expression becomes

$$n-r \text{ cos. } \delta \text{ cos. } \psi-r \text{ sin. } \delta \text{ sin. } \psi, \text{ or } n-r \text{ cos. } (\psi-\delta).$$

Hence, equation (3) gives

$$\frac{d\theta^2}{dt^2}=\frac{2g \text{ sin. } (\beta+\gamma)}{b}\cdot\frac{n-r \text{ cos. } (\psi-\delta)}{m \text{ cos.}^2 (\gamma-\psi)+m' \text{ cos.}^2 (\beta+\psi)}.$$

By equations (4), $\tan.\ \delta=\dfrac{m' \text{ cot. } \beta-m \text{ cot. } \gamma}{m+m'}$;

and hence, by Statics, (Chap. VII, Prob. 13.) δ is the value of ψ in the position of equilibrium of PQ. When $\psi=\delta$, the angular velocity is greatest. About this position PQ oscillates both ways; (except its velocity be too great). We shall find the limit of the oscillations by making

$$\frac{d\theta}{dt}=0;\ \therefore\ n-r \text{ cos. } (\psi-\delta)=0;\ \text{cos. } (\psi-\delta)=\frac{n}{r}.$$

And $\dfrac{n}{r}$ is the cosine of a positive arc, and of a negative arc of equal magnitude. Hence, the limiting positions PQ, $P'Q'$, make equal angles with pg the position of equilibrium.

If we know the velocity at any point, we know C, and consequently n, by equation (3); r is known by equation (4).

If the velocity be so great that n exceeds r, there will no longer be a limit to the angular motion of the rod. If we suppose XA, YA to be grooves in which P, Q, slide, instead of planes; and to be produced beyond A, which will not alter the mathematical conditions; the rod PQ will perform complete revolutions, and the limits to its positions will be curves $XYX'Y'$, fig. 41. The end P will oscillate backwards and forwards through XX', and Q through YY'.

If we suppose P, Q to move in two grooves, not affected by gravity, we must in equation (3), make $g=0$. Hence,

$$\frac{b}{\sin.(\beta+\gamma)}\cdot\frac{d\theta^2}{dt^2}(m\cos.^2\eta+m'\cos.^2\theta)=C.$$

If the grooves cross each other at right angles, and $m=m'$, PQ will revolve with a uniform angular velocity.

PROP. VI. *The rod* PQ *descends, one end sliding along a horizontal and another along a vertical plane: to determine its motion.*

The rod is supposed to move in a vertical plane.

This is a particular case of last problem. It might be simplified by beginning with the equations (1) properly modified; but it will be sufficient to apply our equation (3). Making, in that, $\beta=0$, $\gamma=$ right angle, $\eta=$ comp. θ; we have

$$b\cdot\frac{d\theta^2}{dt^2}(m\sin.^2\theta+m'\cos.^2\theta)=C-2gm'\sin.\theta.$$

If we suppose a to be the angular velocity when the rod is horizontal, or when $\theta=0$;

$$ba^2m'=C.$$

If we suppose that when the angular velocity is 0, θ is θ_1,

$$0=C-2gm'\sin.\theta_1;\ \therefore ba^2=2g\sin.\theta_1.$$

Hence, the angular velocity a acquired by falling from rest through an angle θ, to a horizontal position, is independent of m, m'.

When Q comes to the horizontal plane

$$\text{velocity}^2 \text{ of } Q=b^2a^2=2gb\sin.\theta_1;$$

hence, the velocity is the same as if Q had fallen freely through the space $b\sin.\theta_1$, which is its actual descent.

If $ba^2>2g$, there is no position where the velocity $=0$, and the rod will *revolve* in the manner described in the last problem.

SECT. II. *Tractories.*

42. When one end of a string or rod is drawn with a given velocity along a given straight line or curve, which is called the *Directrix*, so that a body fastened to the other end is dragged along by it, and made to describe a certain path, this path is called a *Tractory*. The motion of the body at any instant will depend upon its preceding motion, and upon the tension of the string. The body is considered as a material point.

PROB. VII. *A point* P, *fig.* 42, *moves uniformly along a straight line* AX, *drawing along with it, in the same plane, a body* Q, *by means of a string* PQ: *to find the tractory described by* Q*.

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* This problem is sometimes solved without taking account of the tendency which the angular motion, generated in PQ, has to continue itself. This mode of considering it would be true, if we were to consider Q to move upon a plane, and the friction to be such, as instantly to destroy any motion communicated to the body. In that case, Q, fig. 43, would always move in the direction QP, in which the string draws it; QP would be a tangent; and the curve would be determined by the condition, that the tangent QP, intercepted by the abscissa, is a constant quantity. Hence, if $AN=x$, $NQ=y$, rectangular co-ordinates; $QP=a$; we have

$$NP=-\frac{y\,dx}{dy};\quad \therefore\ QP=a=y\sqrt{\left(1+\frac{dx^2}{dy^2}\right)};$$

$$\therefore\ dx=dy\,\frac{\sqrt{(a^2-y^2)}}{y};$$

and by integrating we get the relation of x and y. (See Mr. Peacock's Examples, p. 174, for the properties of this curve.)

This curve has an asymptote, as represented in fig. 43. We may reserve for it the name of *Tractrix*, by which it is frequently designated; giving to the curves, which really solve the problem in the text, the name of *Tractory*, which is analogous to the names of other curves which occur in Mechanics.

If the directrix be a circle, and the friction be such as immediately to destroy the motion of the body, it will move in a curve, the tangent of which, intercepted by a circle, is a constant quantity. This curve is sometimes called the *Complicated Tractrix*. See Cotes, *Harm. Mens.*

For

We might solve this by the equations of motion: but we may find the curve more simply by the following reasoning.

If, when any system is in motion, we suppose the space in which it is, to move uniformly, so as to carry all the parts of the system in parallel directions, and with equal velocities; the relative motion of the parts, and their action upon one another will remain the same as before, by the second law of motion.

While P is moving with a uniform velocity along the line AX, let this line and the space containing PQ move in the opposite direction XA, with an equal uniform velocity. Therefore, by what has been said, the angular motion of PQ will remain the same as before. And P, having two equal and opposite velocities, will be at rest. Now, if a body fastened to a string, revolve round a fixed point P, it will revolve uniformly. Hence, the angular motion of PQ round P when fixed, will be uniform; and therefore it will be uniform when P moves uniformly along AX.

Hence, the motion of Q arises from a uniform angular motion round P while P moves uniformly in a straight line: and hence its path will be a cycloid, or a trochoid, within or without the cycloid, (see Mr. Peacock's Examples, p. 186.). Take a point R in the radius PQ, produced if necessary, such that the velocity of R round P at rest, may be the same as the velocity of P along AX. Let a circle be described with radius PR, and let NO, parallel to AX, be a tangent to it. Then, if the circle RO roll uniformly along the line NO, the point Q will trace out a trochoid, which will be the tractory in question. For, in this case it is manifest that P the centre of the circle moves uniformly along the line AX, and that PQ revolves uniformly round P; which are the conditions requisite.

For the tractory, when the directrix is any given curve, see Euler, in the *Nova Acta Acad. Petrop.* 1784. He has there also considered the problem, taking a finite friction into their account: and likewise, what he calls *Compound Tractories*, where there are more points than one attached to the string.

The curve will be a cycloid if R coincide with Q; a trochoid or *prolate* cycloid if R be without Q; and a *curtate* cycloid if R be within Q.

If BC be the original position of PQ, when Q is at rest, and if the point P begin to move along AX, BC will be a tangent to the tractory. The curve will be a trochoid of the first kind, and C will be its point of contrary flexure. If BC be perpendicular to AB, the curve will be a cycloid.

PROB. VII. *Supposing the same things, and that* PQ *does not move in the plane* APQ; *to find the tractory.*

By reasoning similar to that of last problem, it will appear that the motion of Q will be determined by supposing P to move uniformly along the directrix, and PQ to revolve uniformly round P: its motion relatively to P, being always parallel to a certain fixed plane. Hence, the path of Q will be an oblique helix, which may be supposed to be described on the surface of an elliptical cylinder, of which the axis is AX.

PROB. VIII. *The point* P, *fig.* 44, *moves uniformly in the circumference of a circle* BP, *drawing the point* Q: *to find the tractory described by* Q.

The motion is supposed to be in the plane of the circle BP.

Let AX be any fixed line; AM, $MP = x, y$, and AN, $NQ = x'$, y', rectangular co-ordinates to P and Q. The tension of $PQ = p$, and the mass of $Q = m$. Also, let t be the time; and BP, proportional to the time, $= nt$, $QPO = \phi$, PO being parallel to AX, $AP = a$, $PQ = b$. Hence, by resolving the forces,

$$\frac{d^2x'}{dt^2} = -\frac{pg}{m}\cos.\phi, \quad \frac{d^2y'}{dt^2} = -\frac{pg}{m}.\sin.\phi,$$

$$x = a\cos.nt, \quad y = a\sin.nt;$$

$$\therefore \frac{d^2x}{dt^2} = -an^2\cos.nt, \quad \frac{d^2y}{dt^2} = -an^2\sin.nt;$$

$$\therefore \frac{b d^2 \cos.\phi}{dt^2} = \frac{d^2x' - d^2x}{dt^2} = -\frac{pg}{m}\cos.\phi + an^2\cos.nt,$$

$$\frac{b d^2 \sin.\phi}{dt^2} = \frac{d^2y' - d^2y}{dt^2} = -\frac{pg}{m}\sin.\phi + an^2\sin.nt;$$

$$\therefore \frac{b\sin.\phi\, d^2\cos.\phi - b\cos.\phi\, d^2\sin.\phi}{dt^2} = an^2(\sin.\phi\cos.nt - \cos.\phi\sin.nt);$$

$$\therefore \frac{b d^2\phi}{dt^2} = -an^2\sin.(\phi - nt).$$

Let $(\phi - nt) = \psi$; then $d^2\phi = d^2\psi$; and substituting and multiplying by $2d\psi$, and integrating, we have

$$\frac{d\psi^2}{dt^2} = \frac{2an^2}{b}(C - \cos.\psi).$$

The angle $\phi - nt$ is $QPO - RPO = QPR$. The angular velocity of PQ will be greatest when $\phi - nt = \pi$, or when PQ coincides with PA in direction, as at $P'\,Q'$. If C be less than unity, PQ will oscillate about PA, while P moves uniformly in the circle. If C be greater than unity, Q will revolve about P with a variable velocity, while P revolves about A uniformly. The least velocity will be when $\phi - nt = 0$, or when AP, PQ are in a straight line, as at Q''.

Prob. IX. *Let* P *move uniformly along a given straight line, while* Q *is drawn along, and also acted on by gravity: to find the motion of* Q.

By reasoning, as in Prob. VI, it will appear that the motion of Q with respect to P, will be the same as if P were fixed. Consequently, if PQ move in the same plane, it will be the motion of a circular pendulum, and if not, it will be the *turbinatory* motion of a point in a sphere. This, combined with the rectilinear motion of the point P, will give the actual motion of Q.

Sect. III. *Complex Pendulums.*

Prob. X. P, Q, *fig.* 45, *are two bodies, of which the first hangs from a fixed point, and the second from the first, by means of inextensible strings* AP, PQ: *it is required to determine the small oscillations.*

Let $AM=x$, $MP=y$, $AN=x_1$, $NQ=y_1$, $AP=a$, $PQ=a_1$. Mass of $P=m$, of $Q=m_1$; tension of $AP=p$, of $PQ=p_1$.

Hence, resolving the forces p, p_1, we have

$$\left.\begin{aligned}\frac{d^2y}{dt^2}&=\frac{p_1g}{m}\cdot\frac{y_1-y}{a_1}-\frac{pg}{m}\cdot\frac{y}{a}\\ \frac{d^2y_1}{dt^2}&=-\frac{p_1g}{m_1}\cdot\frac{y_1-y}{a_1}\end{aligned}\right\}\ldots\ldots\ldots(1).$$

By combining these with the equations in x, x_1, and with the two

$$x^2+y^2=a^2,\quad (x_1-x)^2+(y_1-y)^2=a_1^2;$$

we should, by eliminating p, p^1, find the motion. But when the oscillations are small, we may approximate in a more simple manner.

Let β, β_1 be the initial values of y, y_1. Then manifestly, p, p_1 will depend on the initial position of the bodies, and on their position at the time t: and hence we may suppose

$p=M+P\beta+Q\beta_1+Ry+Sy_1+$&c.: and similarly for p_1.

Now, in the equations of motion above, p, p_1, are multiplied by y, y_1-y, which, since the oscillations are very small, are also very small quantities (viz. of the order β). Hence, their products with β will be of the order β^2, and may be neglected, and we may suppose p reduced to its first term M.

M is the tension of AP, when β, β_1, &c. are all $=0$. Hence, it is the tension when P, Q hang at rest from A, and consequently, $M=m+m_1$; similarly, the first term of p_1, which may be put for it, is m_1. Substituting these values, and dividing by g, equations (1) become

$$\left.\begin{aligned}\frac{d^2y}{gdt^2}&=-\left(\frac{m_1}{ma_1}+\frac{m+m_1}{ma}\right)y+\frac{m_1}{ma_1}\cdot y_1\\ \frac{d^2y_1}{gdt^2}&=\qquad\qquad\frac{y}{a_1}-\frac{y_1}{a_1}\end{aligned}\right\}\ldots\ldots(2).$$

Multiply the second of these equations by λ, and add it to the first, and we have

$$\frac{d^2y+\lambda d^2y_1}{g\,dt^2}=-\left(\frac{m_1}{ma_1}+\frac{m+m_1}{ma}-\frac{\lambda}{a_1}\right)y-\left(\frac{\lambda}{a_1}-\frac{m_1}{ma_1}\right)y_1:$$

and manifestly this can be solved, if the second side can be put in the form $-k(y+\lambda y_1)$; that is, if

$$k=\frac{m_1}{ma_1}+\frac{m+m_1}{ma}-\frac{\lambda}{a_1};$$

$$k\lambda=\frac{\lambda}{a_1}-\frac{m_1}{ma_1};$$

$$\left.\begin{aligned}\text{or } a_1k&=\frac{m_1}{m}+\frac{a_1}{a}+\frac{m_1a_1}{ma}-\lambda\\ -\frac{m_1}{m}&=(a_1k-1)\lambda\end{aligned}\right\}\ldots\ldots\ldots(3).$$

Eliminating λ, we have

$$(a_1k-1)a_1k-\frac{m_1}{m}=(a_1k-1)\left(\frac{m_1}{m}+\frac{a_1}{a}+\frac{m_1a_1}{ma}\right).$$

Hence, $(a_1k)^2-\left(1+\frac{m_1}{m}\right)\left(1+\frac{a_1}{a}\right)a_1k=-\frac{a_1}{a}-\frac{m_1a_1}{ma}$. (4).

From this equation we obtain two values of k. Let these be denoted by ${}^1k$, ${}^2k$; and let the corresponding values of λ be ${}^1\lambda$, ${}^2\lambda$. Hence, we have these equations

$$\frac{d^2y+{}^1\lambda d^2y_1}{g\,dt^2}=-{}^1k(y+{}^1\lambda y_1),$$

$$\frac{d^2y+{}^2\lambda d^2y_1}{g\,dt^2}=-{}^2k(y+{}^2\lambda y_1).$$

And it is easily seen, as in Prob. II, Chap. III, that the integrals of these equations are

$$y+{}^1\lambda y_1={}^1C\cos.t\sqrt{({}^1kg)}+{}^1D\sin.t\sqrt{({}^1kg)},$$
$$y+{}^2\lambda y_1={}^2C\cos.t\sqrt{({}^2kg)}+{}^2D\sin.t\sqrt{({}^2kg)}.$$

${}^1C$, ${}^1D$, ${}^2C$, ${}^2D$ being arbitrary constants. But we may suppose ${}^1C={}^1E\cos.{}^1\epsilon$, ${}^1D={}^1E\sin.{}^1\epsilon$, ${}^2C={}^2E\cos.{}^2\epsilon$, ${}^2D={}^2E\sin.{}^2\epsilon$, where ${}^1E$, ${}^2E$, ${}^1\epsilon$, ${}^2\epsilon$, are other arbitrary constants. By introducing these values, we find

$$\left.\begin{aligned} y + {}^1\lambda y_1 &= {}^1E \cos. \{t \sqrt{({}^1kg)} + {}^1\epsilon\} \\ y + {}^2\lambda y_1 &= {}^2E \cos. \{t \sqrt{({}^2kg)} + {}^2\epsilon\} \end{aligned}\right\} \ldots\ldots(5).$$

From these we easily find

$$\left.\begin{aligned} y &= \frac{{}^2\lambda\, {}^1E}{{}^2\lambda - {}^1\lambda} . \cos. \{t \sqrt{({}^1kg)} + {}^1\epsilon\} + \frac{{}^1\lambda\, {}^2E}{{}^2\lambda - {}^1\lambda} \cos. \{t \sqrt{({}^2kg)} + {}^2\epsilon\} \\ y_1 &= \frac{{}^1E}{{}^1\lambda - {}^2\lambda} . \cos. \{t \sqrt{({}^1kg)} + {}^1\epsilon\} + \frac{{}^2E}{{}^1\lambda - {}^2\lambda} \cos. \{t \sqrt{({}^2kg)} + {}^2\epsilon\} \end{aligned}\right\} (6).$$

The arbitrary quantities ${}^1E$, ${}^1\epsilon$, &c. depend on the initial position and velocity of the points. If the velocities of P, $Q = 0$, when $t = 0$, we shall have ${}^1\epsilon$, ${}^2\epsilon$ each $= 0$, as appears by taking the differentials of y, y_1.

If either of the two ${}^1E$, ${}^2E$, be $= 0$, we shall have, (supposing the latter case, and omitting ${}^1\epsilon$)

$$y = \frac{{}^2\lambda\, {}^1E}{{}^2\lambda - {}^1\lambda} \cos. t \sqrt{{}^1kg},$$

$$y_1 = \frac{{}^1E}{{}^1\lambda - {}^2\lambda} \cos. t \sqrt{{}^1kg}.$$

Hence, it appears that the oscillations in this case are *symmetrical:* that is, the bodies P, Q come to the vertical line at the same time, have similar and equal motions on the two sides of it, and reach their greatest distances from it at the same time. It is easy to see that in this case, the motion has the same law of time and velocity as in a cycloidal pendulum; and the time of an oscillation, in this case, extends from when $t = 0$ to when $t \sqrt{({}^1kg)} = \pi$, or $t = \frac{\pi}{\sqrt{({}^1kg)}}$. Also if β, β_1 be the greatest horizontal deviation of P, Q, we shall have

$$y = \beta \cos. t \sqrt{({}^1kg)}, \; y_1 = \beta_1 \cos. t \sqrt{({}^1kg)}.$$

In order to find the original relation of β, β_1 *, that the oscil-

* The oscillations will be symmetrical if the forces which urge P and Q to the vertical, be as PM, QN, as is easily seen. Hence, the conditions for symmetrical oscillation might be determined by finding the position of P, Q, that this might originally be the relation of the forces.

lations may be of this kind (the original velocities being 0), we must have, by equation (5), since ${}^2E=0$,

$$\beta+{}^2\lambda\beta_1=0.$$

Similarly, if we had $\beta+{}^1\lambda\beta_1=0$, we should have ${}^1E=0$, and the oscillations would be symmetrical, and would employ a time

$$\frac{\pi}{\sqrt{({}^2kg)}}.$$

When neither of these relations obtains, the oscillations may be considered as compounded of two, in the following manner. Suppose that we put

$$y=H\cos.t\sqrt{({}^1kg)}+K\cos.t\sqrt{({}^2kg)}\ldots.(7),$$

omitting ${}^1\epsilon$, ${}^2\epsilon$, and altering the constants in equations (6); and suppose that in fig. 45, we take $Mp=H\cos.t\sqrt{({}^1kg)}$. Then p will oscillate about M, according to the law of a cycloidal pendulum (neglecting the vertical motion). Also pP will $=K\cos.t\sqrt{({}^2kg)}$. Hence, P oscillates about p according to a similar law, while p oscillates about M. And in the same way, we may have a point q so moved, that Q shall oscillate about q in a time $\frac{\pi}{\sqrt{({}^2kg)}}$, while q oscillates about N in a time $\frac{\pi}{\sqrt{({}^1kg)}}$. And hence, the motion of the pendulum APQ is compounded of the motion of Apq oscillating symmetrically about the vertical line, and of APQ oscillating symmetrically about Apq, as if that were a fixed vertical line.

When a pendulum oscillates in this manner, it will never return exactly to its original position, if $\sqrt{{}^1k}$ and $\sqrt{{}^2k}$ are incommensurable. If $\sqrt{{}^1k}$ and $\sqrt{{}^2k}$ are commensurable, so that we have $m\sqrt{{}^1k}=n\sqrt{{}^2k}$, m and n being whole numbers, the pendulum will at certain intervals, return to its original position. For let

$$t\sqrt{({}^1kg)}=2n\pi;\text{ then }t\sqrt{({}^2kg)}\text{ will}=2m\pi;\text{ and by (7),}$$

$$y=H\cos.2n\pi+K\cos.2m\pi=H+K,$$

which is the same as when $t=0$. And similarly, after an interval such that $t\sqrt{({}^1kg)}=4n\pi$, $6n\pi$, &c. the pendulum will return to its original position, having described in the intermediate times, similar cycles of oscillations.

Ex. Let $m_1=m$, and $a_1=a$, to determine the oscillations.

Here equation (4) becomes,

$$a^2k^2-4ak=-2,\ ak=2\pm\sqrt{2}.$$

Also, by equation (3),

$$ak=3-\lambda;\ \therefore\ {}^1\lambda=1+\sqrt{2};\ {}^2\lambda=1-\sqrt{2}.$$

Hence, in order that the oscillations may be symmetrical, we must either have

$$\beta+(1+\sqrt{2})\,\beta_1=0;\ \text{whence}\ \beta_1=-(\sqrt{2}-1)\,\beta:$$
$$\text{or}\ \beta-(\sqrt{2}-1)\,\beta_1=0;\ \text{whence}\ \beta_1=(\sqrt{2}+1)\,\beta.$$

The two arrangements indicated by these equations are represented, fig. 46, and fig. 47. The first corresponds to $\beta_1=(\sqrt{2}+1)\,\beta$, or $QN=(\sqrt{2}+1)\,PM$. In this case, the pendulum will oscillate into the position $AP'Q'$, similarly situated on the other side of the line; and the time of this complete oscillation will be

$$\frac{\pi}{\sqrt{\left\{\frac{g}{a}(2-\sqrt{2})\right\}}}=\frac{\pi}{\sqrt{(2-\sqrt{2})}}\sqrt{\frac{a}{g}}.$$

In the other case, corresponding to $\beta_1=-(\sqrt{2}-1)\,\beta$, Q is on the other side of the vertical line, and $QN=(\sqrt{2}-1)\,PM$. The pendulum oscillates into the position $AP'Q'$, the point O remaining always in the vertical line; and the time of an oscillation is

$$\frac{\pi}{\sqrt{(2+\sqrt{2})}}\sqrt{\frac{a}{g}}.$$

The lengths of simple pendulums which would oscillate respectively in these times, would be

$$\frac{a}{2-\sqrt{2}},\ \text{and}\ \frac{a}{2+\sqrt{2}},\ \text{or}\ 1.707\,a\ \text{and}\ .293\,a.$$

If neither of these arrangements exist originally, let β, β_1, be the original values of y, y_1, when t is 0. Then making $t=0$ in equations (5), we have

$$^1E=\beta+(\sqrt{2}+1)\beta_1,\ ^2E=\beta-(\sqrt{2}-1)\beta_1.$$

And these being known, we have the motion by equations (6).

PROB. XI. *Any number of material points* $P_1\ P_2\ P_3 \ldots Q$, *fig.* 48, *hang, by means of a string without weight, from a point* A: *it is required to determine their small oscillations in a vertical plane.*

Let AN be a vertical abscissa, and P_1M_1, P_2M_2, &c. horizontal ordinates; so that

$$AM_1=x_1,\ AM_2=x_2,\ AM_3=x_3, \text{ \&c.}$$
$$P_1M_1=y_1,\ P_2M_2=y_2,\ P_3M_3=y_3, \text{ \&c.}$$
$$AP_1=a_1,\ P_1P_2=a_2,\ P_2P_3=a_3, \text{ \&c.}$$
$$\text{tension of } AP_1=p_1, \text{ of } P_1P_2=p_2, \text{ of } P_2P_3=p_3, \text{ \&c.}$$
$$\text{mass of } P_1=m_1, \text{ of } P_2=m_2, \text{ of } P_3=m_3, \text{ \&c.}$$

Hence, we have these equations, by resolving the forces parallel to the horizon,

$$\left.\begin{aligned}
\frac{d^2y_1}{dt^2}&=-\frac{p_1g}{m_1}\cdot\frac{y_1}{a_1}+\frac{p_2g}{m_1}\cdot\frac{y_2-y_1}{a_2}\\
\frac{d^2y_2}{dt^2}&=-\frac{p_2g}{m_2}\cdot\frac{y_2-y_1}{a_2}+\frac{p_3g}{m_2}\cdot\frac{y_3-y_2}{a_3}\\
\frac{d^2y_3}{dt^2}&=-\frac{p_3g}{m_3}\cdot\frac{y_3-y_2}{a_3}+\frac{p_4g}{m_3}\cdot\frac{y_4-y_3}{a_4}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{d^2y_n}{dt^2}&=-\frac{p_ng}{m_n}\cdot\frac{y_n-y_{n-1}}{a_n}
\end{aligned}\right\}\cdots\cdots\cdots(1).$$

And, as in the last problem, it will appear that p_1, p_2, &c. may, for these small oscillations, be considered as constant, and the same as in the state of rest. Hence, if $m_1+m_2+m_3\ldots+m_n=M$,

$$p_1=M,\ p_2=M-m_1,\ p_3=M-m_1-m_2, \text{ \&c.}$$

Also dividing by g, and arranging the above equations, may be put in this form

$$\left.\begin{aligned}
\frac{d^2 y_1}{g\,dt^2} &= -\left(\frac{p_1}{m_1 a_1} + \frac{p_2}{m_1 a_2}\right) y_1 + \frac{p_2 y_2}{m_1 a_2} \\
\frac{d^2 y_2}{g\,dt^2} &= \frac{p_2 y_1}{m_2 a_2} - \left(\frac{p_2}{m_2 a_2} + \frac{p_3}{m_2 a_3}\right) \cdot y_2 + \frac{p_3 y_3}{m_2 a_3} \\
\frac{d^2 y_3}{g\,dt^2} &= \frac{p_3 y_2}{m_3 a_3} - \left(\frac{p_3}{m_3 a_3} + \frac{p_4}{m_3 a^4}\right) y_3 + \frac{p_4\, y_4}{m_3 a_4}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{d^2 y_n}{g\,dt^2} &= \frac{p_n y_{n-1}}{m_n a_n} - \frac{p_n y_n}{m_n a_n}.
\end{aligned}\right\} \cdots\cdots(2).$$

The first and last equations become symmetrical with the rest if we observe that $y_0 = 0$, and $p_{n+1} = 0$.

Now, if we multiply these equations respectively by $1, \lambda, \lambda', \lambda''$, &c. and add them, we have $\dfrac{d^2 y_1 + \lambda d^2 y_2 + \lambda' d^2 y_3 + \&c.}{g\,dt^2}$

$$= \left\{ -\frac{p_1}{m_1 a_1} - \frac{p_2}{m_1 a_2} + \frac{\lambda p_2}{m_2 a_2} \right\} y_1$$

$$+ \left\{ \frac{p_2}{m_1 a_2} - \lambda \left(\frac{p_2}{m_2 a_2} + \frac{p_3}{m_2 a_3} \right) + \frac{\lambda' p_3}{m_3 a_3} \right\} y_2$$

$$+ \left\{ \frac{\lambda p_3}{m_2 a_3} - \lambda' \left(\frac{p_3}{m_3 a_3} + \frac{p_4}{m_3 a_4} \right) + \frac{\lambda'' p_4}{m_4 a_4} \right\} y_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \left\{ \frac{\lambda'^{(n-3)} p_n}{m_{n-1} a_n} - \frac{\lambda'^{(n-2)} p_n}{m_n a_n} \right\} y_n.$$

And this will be integrable, if the right hand side of the equation be reducible to this form

$$-k\,(y_1 + \lambda y_2 + \lambda' y_3 + \&c.).$$

That is, if

$$\left.\begin{aligned} k &= \frac{p_1}{m_1 a_1} + \frac{p_2}{m_1 a_2} - \frac{\lambda p_2}{m_2 a_2} \\ k\lambda &= -\frac{p_2}{m_1 a_2} + \lambda\left(\frac{p_2}{m_2 a_2} + \frac{p_3}{m_2 a_3}\right) - \frac{\lambda' p_3}{m_3 a_3} \\ k\lambda' &= -\frac{\lambda p_3}{m_2 a_3} + \lambda'\left(\frac{p_3}{m_3 a_3} + \frac{p_4}{m_3 a_4}\right) + \frac{\lambda'' p_4}{m_4 a_4} \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ k\lambda'^{(n-2)} &= -\frac{\lambda'^{(n-3)} p_n}{m_{n-1} a_n} + \frac{\lambda'^{(n-2)} p_n}{m_n a_n} \end{aligned}\right\} \ldots\ldots(3).$$

If we now eliminate λ, λ', λ'', &c. from these n equations, it is easily seen that we shall have an equation of n dimensions in k. Let ${}^1k$, ${}^2k$, ${}^3k$. . ${}^nk$ be the n values of k; then, for each of these there is a value of λ', λ'', λ''' easily deduced from equations (3), which we may represent by ${}^1\lambda$, ${}^1\lambda'$, ${}^1\lambda''$, ${}^1\lambda'''$, &c., ${}^2\lambda'$, ${}^2\lambda''$, ${}^2\lambda'''$, &c. Hence, we have these equations, by taking corresponding values of λ and k,

$$\frac{d^2y_2 + {}^1\lambda d^2y_2 + {}^1\lambda' d^2y_3 + \&c.}{g\,dt^2} = -{}^1k.(y_1 + {}^1\lambda y_2 + {}^1\lambda' y_3 + \&c.)$$

$$\frac{d^2y_1 + {}^2\lambda d^2y_2 + {}^2\lambda'' d^2y_3 + \&c.}{g\,dt^2} = -{}^2k\,(y_2 + {}^2\lambda y_2 + {}^2\lambda' y_3 + \&c.)$$

and so on, making n equations.

Integrating each of these equations we get, as in last problem

$$\left.\begin{aligned} y_1 + {}^1\lambda y_2 + {}^1\lambda' y_3 + \&c. &= {}^1E \cos. \{t\sqrt{({}^1kg)} + {}^1\epsilon\} \\ y_1 + {}^2\lambda y_2 + {}^2\lambda' y_3 + \&c. &= {}^2E \cos. \{t\sqrt{({}^2kg)} + {}^2\epsilon\} \\ \&c. &= \&c. \end{aligned}\right\} \ldots\ldots(5).$$

${}^1E$, ${}^2E$, &c. ${}^1\epsilon$, ${}^2\epsilon$, &c. being arbitrary constants.

From these n simple equations, we can, without difficulty, obtain the n quantities y_1, y_2, &c. And it is manifest that the results will be of this form

$$\left.\begin{aligned} y_1 &= {}^1H_1 \cos. \{t\sqrt{({}^1kg)} + {}^1\epsilon\} + {}^2H_1 \cos. \{t\sqrt{({}^2kg)} + {}^2\epsilon\} + \&c. \\ y_2 &= {}^1H_2 \cos. \{t\sqrt{({}^1kg)} + {}^1\epsilon\} + {}^2H_2 \cos. \{t\sqrt{({}^2kg)} + {}^2\epsilon\} + \&c. \\ \&c. &= \&c. \end{aligned}\right\} \ (6$$

where ${}^1H_1$, ${}^1H_2$, &c. must be deduced from β_1, β_2, &c. the original values of y_1, y_2, &c.

If the points have no initial velocities, (i. e. when $t=0$) we shall have ${}^1\epsilon=0$, ${}^2\epsilon=0$, &c.

We may have symmetrical oscillations in the following manner. If, of the quantities ${}^1E$, ${}^2E$, ${}^3E$, &c. all vanish except one, for instance, ${}^nE$; we have

$$\left.\begin{array}{l} y_1+{}^1\lambda y_2+{}^1\lambda' y_3+\&c.=0, \\ y_1+{}^2\lambda y_2+{}^2\lambda' y_3+\&c.=0, \\ y_1+{}^3\lambda y_2+{}^3\lambda' y_3+\&c.=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ y_1+{}^n\lambda y_2+{}^n\lambda' y_3+\&c.={}^nE\cos. t\sqrt{({}^nkg)},\text{ omitting }{}^n\epsilon \end{array}\right\} (8).$$

From the first $n-1$ of these equations, it appears that y_2, y_3, &c. are in a given ratio to y_1; and hence, $y_1+{}^n\lambda y_2+{}^n\lambda' y_3+$&c. is a given multiple of y_1 and $=my_1$ suppose. Hence, we have

$$my_1={}^nE\cos. t\sqrt{({}^nkg)};$$

or, omitting the index n, which is now unnecessary,

$$my_1=E\cos. t\sqrt{(kg)}. \quad \text{Also, if } y_2=e_2y_1,$$

$$my_2=Ee_2\cos. t\sqrt{(kg)}, \text{ and similarly for } y_3, \&c.$$

Hence, it appears that in this case the oscillations are *symmetrical*. All the points come into the vertical line at the same time, and move similarly and contemporaneously on the two sides of it. The relation among the original ordinates β_1, β_2, β_3, &c. which must subsist in order that the oscillations may be of this kind, is given by the $n-1$ equations (8),

$$\begin{array}{l} \beta_1+{}^1\lambda\beta_2+{}^1\lambda'\beta_3+\&c.=0, \\ \beta_1+{}^2\lambda\beta_2+{}^2\lambda'\beta_3+\&c.=0, \\ \beta_1+{}^3\lambda\beta_2+{}^3\lambda'\beta_3+\&c.=0, \\ \qquad\qquad\qquad\qquad \&c.=\&c. \end{array}$$

These give the *proportion* of β_1, β_2, &c.; the arbitrary constant ${}^nE$, in the remaining equation, gives the *actual quantity* of the original displacement.

Also, we may take any one of the quantities ${}^1E$, ${}^2E$, ${}^3E$, &c. for that which does not vanish; and hence obtain, in a different way, such a system of $n-1$ equations as has just been described. Hence, there are n different relations among β_1, β_2, &c., or n different modes of arrangement, in which the points may be placed, so as to oscillate symmetrically*.

The time of oscillation in each of these arrangements is easily known; the equation

$$my_1 = {}^nE \cos. t \sqrt{({}^nkg)},$$

shews that an oscillation employs a time $t = \frac{\pi}{\sqrt{({}^nkg)}}$. And hence, if all the roots ${}^1k$, ${}^2k$, ${}^3k$, &c. be different, the time is different for each different arrangement.

If the initial arrangement of the points be different from all those thus obtained, the oscillations of the pendulum may always be considered as compounded of n symmetrical oscillations. That is, if an imaginary pendulum oscillate symmetrically about the vertical line in a time $\frac{\pi}{\sqrt{({}^1kg)}}$; and a second imaginary pendulum oscillate about the place of the first, considered as a fixed line, in the time $\frac{\pi}{\sqrt{({}^2kg)}}$; and a third about the second, in the same manner, in the time $\frac{\pi}{\sqrt{({}^3kg)}}$; and so on; the nth pendulum may always be made to coincide perpetually with the real pendulum, by properly adjusting the amplitudes of the imaginary oscillations. This appears by considering the equations (6),

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* We might here also find these positions, which give symmetrical oscillations, by requiring the force in each of the ordinates P_1M_1, P_2M_2, to be as the distance; in which case the points P_1, P_2, &c. would all come to the vertical at the same time.

If the quantities $\sqrt{{}^1k}$, $\sqrt{{}^2k}$, &c. have one common measure, there will be a time after which the pendulum will come into its original position. And it will describe similar successive cycles of vibrations. If these quantities be not commensurable, no portion of its motion will be similar to any preceding portion.

$$y_1 = {}^1H_1 \cos. t\sqrt{({}^1kg)} + {}^2H_1 \cos. t\sqrt{({}^2kg)} + \&c.$$
$$\&c. = \&c.$$

This principle of the *co-existence of vibrations* is applicable in all cases where the vibrations are indefinitely small. In all such cases each set of symmetrical vibrations takes place, and affects the system as if that were the only motion which it experienced.

A familiar instance of this principle is seen in the manner in which the circular vibrations, produced by dropping stones into still water, spread from their respective centres, and cross without disfiguring each other.

If the oscillations be not all made in one vertical plane, we may take a horizontal ordinate z perpendicular to y. The oscillations in the direction of y will be the same as before, and there will be similar results obtained with respect to the oscillations in the direction of z.

We have supposed that the motion in the direction of x, the vertical axis, may be neglected, which is true when the oscillations are very small.

Ex. Let there be three bodies all equal; (each $= m$,) and also their distances a_1, a_2, a_3, all equal; (each $= a$).

Here $p_1 = 3m$, $p_2 = 2m$, $p_3 = m$, and equations (3) become

$$ak = 5 - 2\lambda,$$
$$ak\lambda = -2 + 3\lambda - \lambda',$$
$$ak\lambda' = -\lambda + \lambda'.$$

Eliminating k, we have

$$5\lambda - 2\lambda^2 = -2 + 3\lambda - \lambda',$$
$$5\lambda' - 2\lambda\lambda' = -\lambda + \lambda',$$
$$\text{or, } \lambda' = 2\lambda^2 - 2\lambda - 2,$$
$$4\lambda' - 2\lambda\lambda' = -\lambda,\ \lambda' = \frac{\lambda}{2\lambda - 4};$$
$$\therefore (2\lambda^2 - 2\lambda - 2)(2\lambda - 4) = \lambda,$$
$$\text{or, } \lambda^3 - 3\lambda^2 + \tfrac{3}{4}\lambda + 2 = 0,$$

which may be solved by Trigonometrical Tables. We shall find three values of λ.

Hence, we have a value of λ' corresponding to each value of λ; and then by equations (8),

$$\left.\begin{array}{l}\beta_1 + {}^1\lambda\beta_2 + {}^1\lambda'\beta_3 = 0\\ \beta_1 + {}^2\lambda\beta_2 + {}^2\lambda'\beta_3 = 0\end{array}\right\}\ldots\ldots(8').$$

Whence we find β_2 and β_3 in terms of β_1.

We shall thus find* $\beta_2 = 2.295\ \beta_1$,

or $\beta_2 = 1.348\ \beta_1$,

or $\beta_2 = -.643\ \beta_1$,

according as we take the different values of λ.

And the times of oscillation in each case will be found by taking the value of $ak = 5 - 2\lambda$; that value of λ being taken which is not used in equation (8′). For the time of oscillation will be given by making $t\sqrt{(kg)} = \pi$.

If the values of β_1, β_2, β_3 have not this initial relation, the oscillations will be compounded in a manner similar to that described, p. 135, in the Example for two bodies only.

PROB. XII. *A flexible chain, of uniform thickness, hangs from a fixed point: to find its initial form, that its small oscillations may be symmetrical.*

Let, in fig. 49, AM, the vertical abscissa $= x$; MP, the horizontal ordinate $= y$; $AP = s$, and the whole length $AC = a$; $\therefore AP = a - s$. And, in the same way as in Prob. XI, the tension at P will, when the oscillations are small, be the weight of PC, and may be represented by $a - s$. This tension will act in the direction of a tangent at P, and hence, the part of it in the direction PM will be *tension* $\times \frac{dy}{ds}$, or $(a-s)\frac{dy}{ds}$.

Now, if we take any portion $PQ = h$, we shall find the horizontal force at Q in the same manner. For the point Q, supposing ds constant,

$$\frac{dy}{ds} \text{ becomes } \frac{dy}{ds} + \frac{d^2y}{ds^2}\cdot\frac{h}{1} + \frac{d^3y}{ds^3}\cdot\frac{h^2}{1.2} + \&c.$$

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* Euler, *Com. Petrop.* tom. VIII, p. 37.

Also, the tension will be $(a-s-h)$. Hence, the horizontal force in direction NQ, is

$$(a-s-h)\left(\frac{dy}{ds}+\frac{d^2y}{ds^2}\cdot\frac{h}{1}+\frac{d^3y}{ds^3}\,\frac{h^2}{1\,.\,2},\ \&c.\right)$$

Subtracting from this the force in PM, we have the force on PQ horizontally

$$=(a-s)\left(\frac{d^2y}{ds^2}\cdot\frac{h}{1}+\frac{d^3y}{ds^3}\cdot\frac{h^2}{1\,.\,2}+\&c.\right)$$
$$-h\left(\frac{dy}{ds}+\frac{d^2y}{ds^2}\cdot\frac{h}{1}+\&c.\right).$$

And the mass of PQ being represented by h, the accelerating force $\left(=\frac{\text{pressure}\,.\,g}{\text{mass}}\right)$ is found. But since the different points of PQ move with different velocities, this expression is only applicable when h is indefinitely small. Hence, supposing Q to approach to, and coincide with P, we have, when h vanishes,

$$\text{accelerating force on } P=(a-s)\frac{d^2y}{ds^2}-\frac{dy}{ds}.$$

But, since the oscillations are indefinitely small, x coincides with s, and we have

$$\text{accelerating force on } P=(a-x)\frac{d^2y}{dx^2}-\frac{dy}{dx}.$$

Now, in order that the oscillations may be symmetrical, this force must be in the direction PM, and proportional to PM, in which case all the points of AC will come to the vertical AB at once. Hence, we must have

$$(a-x)\frac{d^2y}{dx^2}-\frac{dy}{dx}=-\,ky\,\ldots\ldots..(1),$$

k being some constant quantity to be determined.

This equation cannot be integrated in finite terms. To obtain a series, let

$$y = A + B(a-x) + C(a-x)^2 + D(a-x)^3 + \&c.$$

$$\frac{dy}{dx} = - B - 2C(a-x) - 3D(a-x)^2 + \&c.$$

$$\frac{d^2y}{dx^2} = 1.2C + 2.3D(a-x) + \&c.$$

Hence, $0 = (a-x)\frac{d^2y}{dx^2} - \frac{dy}{dx} + ky$; gives

$$\begin{aligned} 0 = {} & 1.2C(a-x) + 2.3D(a-x)^2 + \&c. \\ & + B + 2C(a-x) + 3D(a-x)^2 + \&c. \\ & + kA + kB(a-x) + kC(a-x)^2 + \&c. \end{aligned}$$

Equating coefficients, we have

$$B = -kA,\ 2^2.C = -kB,\ 3^2D = -kC, \&c.$$

$$\therefore B = -kA, \quad C = \frac{k^2A}{2^2}, \quad D = -\frac{k^3A}{2^2.3^2}, \&c.$$

and $y = A\left\{1 - k(a-x) + \frac{k^2}{2^2}(a-x)^2 - \frac{k^3}{2^2 3^2}(a-x)^3 + \&c.\right\}$..(2).

Here A is BC, the value of y when $x = a$.

When $x = 0, y = 0$; $\therefore 1 - ka + \frac{k^2a^2}{2^2} - \frac{k^3a^3}{2^2 3^2} + \&c. = 0$..(3).

From this equation k is to be found. The equation has an infinite number of dimensions, and hence k will have an infinite number of values, which we may call ${}^1k, {}^2k, {}^3k \ldots {}^nk, \ldots$; and these give an infinite number of initial forms, for which the chain may perform symmetrical oscillations.

The time of oscillation for each of these forms will be found thus. At the distance y, the force is kgy; hence, by Chap. I, Ex. 1.

time to the vertical $= \frac{\pi}{2\sqrt{(kg)}}$; and time of oscillation $= \frac{\pi}{\sqrt{(kg)}}$*.

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* The greatest value of ka is about 1.44, (Euler, *Com. Acad. Petrop.* tom. VIII, p. 43.). And the time of oscillation for this value, is the same as that of a simple pendulum, whose length is $\frac{2}{3}a$, nearly.

The points where the curve cuts the axis will be found by putting $y=0$. Hence, taking the value ${}^n k$ of k, we have

$$0 = 1 - {}^n k\,(a-x) + \frac{{}^n k^2\,(a-x)^2}{2^2} - \frac{{}^n k^3\,(a-x)^2}{2^2 \cdot 3^2} + \text{\&c.}$$

which will manifestly be verified, if

$${}^n k\,(a-x) = {}^1 k a, \text{ or } {}^n k\,(a-x) = {}^2 k a, \text{ or } {}^n k\,(a-x) = {}^3 k a, \text{ \&c.}$$

because ${}^1 ka$, ${}^2 ka$, &c. are roots of equation (3).

That is, if

$$x = a\left(1 - \frac{{}^1 k}{{}^n k}\right), \text{ or} = a\left(1 - \frac{{}^2 k}{{}^n k}\right), \text{ or} = a\left(1 - \frac{{}^3 k}{{}^n k}\right), \text{ \&c.}$$

Suppose ${}^1 k$, ${}^2 k$, ${}^3 k$, &c. to be the roots in the order of their magnitude, ${}^1 k$ being the least.

Then, if for ${}^n k$ we take ${}^1 k$, all these values of x will be negative, and the curve will never cut the vertical axis below A.

If for ${}^n k$ we take ${}^2 k$, all the values of x will be negative except the first; therefore, the curve will cut AB in one point. If we take ${}^3 k$, all the values will be negative except the two first, and the curve cuts AB in two points; and so on.

Hence, the forms for which the oscillations will be symmetrical, are of the kind represented in fig. 50. And there are an infinite number of them, each cutting the axis in a different number of points.

If we represent equation (2) in this manner, $y = A\phi\,(k, x)$; it is evident that $y = {}^1 A\,\phi\,({}^1 k, x)$, $y = {}^2 A\,\phi\,({}^2 k, x)$, &c. will each satisfy equation (1). Hence, as in Prob. XI, if we put

$$y = {}^1 A\phi\,({}^1 k, x) + {}^2 A\phi\,({}^2 k, x) + \text{\&c.}$$

and if ${}^1 A$, ${}^2 A$, &c. can be so assumed, that this shall represent a given initial form of the chain, its oscillations will be compounded of as many co-existing symmetrical ones, as there are terms ${}^1 A$, ${}^2 A$, &c.

SECT. IV. *The Motion of Bodies connected by Strings.*

PROB. XIII. *Two bodies, connected by a string passing over a fixed pully, move on two given curves: to determine their motions.*

The motions are supposed to take place in a vertical plane.

Let A, fig. 51, be the pully, AM a vertical line, BP, CQ, the given curves, and MP, NQ horizontal ordinates to the places of the bodies.

$AM=x$, $MP=y$, $AN=x'$, $NQ=y'$; $AP=r$, $AQ=r'$, curve $BP=s$, $CQ=s'$: mass of $P=m$, of $Q=m'$; re-action of surface $BP=n$, of $CQ=n'$; tension of the string $PAQ=p$.

Then, by resolving the forces which act upon each of the bodies, we shall easily find the equations

$$\left.\begin{array}{l} \frac{d^2x}{dt^2}=g-\frac{pg}{m}.\frac{x}{r}-\frac{ng}{m}.\frac{dy}{ds};\ \frac{d^2y}{dt^2}=-\frac{pg}{m}.\frac{y}{r}+\frac{ng}{m}.\frac{dx}{ds} \\ \frac{d^2x'}{dt^2}=g-\frac{p'g}{m'}.\frac{x'}{r'}-\frac{n'g}{m'}.\frac{dy'}{ds'};\ \frac{d^2y'}{dt^2}=-\frac{p'g}{m'}.\frac{y'}{r'}+\frac{n'g}{m'}.\frac{dx'}{ds'} \end{array}\right\}\ldots.(1).$$

Multiply by $2\,m\,dx$, $2\,m\,dy$, $2\,m'\,dx'$, $2\,m'dy'$, and add the two upper ones, and the two lower ones; and we have

$$\frac{2\,m\,(dx\,d^2x+dy\,d^2y)}{dt^2}=2\,mg\,dx-2\,pg\,.\,\frac{x\,dx+y\,dy}{r},$$

$$\frac{2\,m'\,(dx'd^2x'+dy'd^2y')}{dt^2}=2\,m'g\,dx'-2\,pg\,\frac{x'dx'+y'dy'}{r'}.$$

Add these equations, observing that $r^2=x^2+y^2$, whence

$$dr=\frac{xdx+ydy}{r};\ \text{similarly,}\ dr'=\frac{x'dx'+y'dy'}{r'};$$

and $r+r'=$ a constant length APC; $\therefore\ dr+dr'=0$. Thus we have

$$2m.\frac{dxd^2x+dyd^2y}{dt^2}+2m'.\frac{dx'd^2x'+dy'd^2y'}{dt^2}=2g\,(mdx+m'dx').$$

Integrating, we have

$$m\frac{dx^2+dy^2}{dt^2}+m'\frac{dx'^2+dy'^2}{dt^2}=2g(mx+m'x')+C.$$

If we suppose a, a', to be the values of x, x', when the velocities are 0, we shall have

$$m\frac{ds^2}{dt^2}+m'\frac{ds'^2}{dt^2}=m.2g(x-a)-m'.2g(a'-x')\ldots.(2).$$

The quantities $x-a$, $a'-x'$ are the spaces through which P has descended and Q ascended: $2g(x-a)$, $2g(a'-x')$ are the squares of the velocities which would have been generated in falling freely through these spaces.

The product of the mass and square of the velocity is sometimes called the *vis viva* of a body: hence the equation just found, shews that in our problem.

The sum of the vis viva of the bodies in constrained motion, is the sum or difference of the vis viva which they would have had, if they had descended freely through the same vertical spaces *.

The sum if both descend : the difference if one ascend.

By introducing into equation (2), the relations among x, x', s, s', given by the nature of the curve, and by the condition $r+r'=$ a constant quantity, we have the equations which determine the motion.

Ex. Let Q, hanging freely, draw P along a horizontal plane, fig. 52. Let the original position of Q, when the bodies begin to move, be D; $AD=a$, $AB=c$; length of string $PAQ=l$: $AP=r$, $AQ=l-r$.

$$\text{Velocity of } Q=-\frac{dr}{dt}; \text{ and } BP=\sqrt{(r^2-c^2)};$$

$$\therefore \text{ velocity of } P=\frac{r}{\sqrt{(r^2-c^2)}}\cdot\frac{dr}{dt};$$

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* This is *the principle of the conservation of vis viva*, which applies not only to this problem but to all cases whatever of mechanical action.

P neither ascends nor descends: hence, the equation (2) gives

$$m \cdot \frac{r^2}{r^2 - c^2} \cdot \frac{dr^2}{dt^2} + m' \cdot \frac{dr^2}{dt^2} = 2\,m'g\,(l - r - a),$$

$$\frac{dr^2}{dt^2} = \frac{2\,m'g\,(l - a - r)\,(r^2 - c^2)}{(m + m')\,r^2 - m'c^2};$$

which gives the relation between r and t.

PROB. XIV. *A body P, fig. 53, is fastened to two equal weights Q, Q′, by strings passing over pullies A, A′, equidistant from it, and in the same horizontal line: to determine its motion.*

The vertical line PE will bisect AA', and the body being acted upon by equal forces on the two sides of the vertical line, will not be drawn from it, and we shall only have the vertical motion to consider.

Let $AE = EA' = a$, $EP = x$; mass of $P = m$, of $Q = Q' = m'$: tension of PA, or $PA' = p$; $AP = AP' = r$: hence, the accelerating force which each string exerts vertically upon P will be $\frac{pg}{m} \cdot \frac{x}{r}$; therefore

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= g - \frac{2pg}{m} \cdot \frac{x}{r}, \text{ from the motion of } P \\ -\frac{d^2r}{dt^2} &= g - \frac{pg}{m'}, \text{ from the motion of } Q \end{aligned}\right\} \ldots\ldots (1);$$

$$\therefore \frac{2\,m\,dx\,d^2x + 4\,m'\,dr\,d^2r}{dt^2} = 2\,mg\,dx - 4\,m'g\,dr:$$

(for since $r^2 - x^2 = a^2$, $r\,dr - x\,dx = 0$.)

Integrating,

$$m\frac{dx^2}{dt^2} + 2\,m'\frac{dr^2}{dt^2} = 2\,mg\,(x - b) - 4\,m'g\,(r - c) \ldots\ldots (2).$$

Supposing that when the velocities are 0, $x = b = EB$, $r = c = AB$. Hence, $c^2 = a^2 + b^2$.

But, since $r = \sqrt{(a^2+x^2)}$, $dr = \frac{x\,dx}{\sqrt{(a^2+x^2)}}$; this becomes

$$\left\{m + \frac{2m'x^2}{a^2+x^2}\right\} \frac{dx^2}{dt^2} = 2mg(x-b) - 4m'g(r-c),$$

$$\frac{dx^2}{dt^2} = \frac{a^2+x^2}{ma^2+(m+2m')x^2} \cdot 2g\,\{m(x-b) - 2m'(r-c)\}.$$

To find when the velocity again becomes 0, we must have

$$m(x-b) - 2m'(r-c) = 0;$$

or, putting for r its value $\sqrt{(a^2+x^2)}$, transposing and squaring,

$$(4m'^2 - m^2)x^2 - 2m(2m'c - mb)x + 4m'^2a^2 - (2m'c - mb)^2 = 0,$$

the two values of x in this equation give the points where the velocity is 0; hence, b is one of them; and if b' be the other, we shall have

$$b + b' = \frac{2m(2m'c - mb)}{4m'^2 - m^2},$$

$$b' = \frac{4mm'c - (4m'^2+m^2)b}{4m'^2 - m^2}.$$

If $2m' > m$, make $EB' = b'$, and P will go on oscillating between B and B'. If $2m' < m$, b' will be negative, and P will never come to a second point where its velocity is 0;

The velocity of P will be greatest when it is in the position in which there would be an equilibrium: for then the force is 0,

and therefore $\frac{d^2x}{dt^2} = 0$.

That is, when $x = \frac{ma}{\sqrt{(4m'^2 - m^2)}}$, see Statics, Chap. IV, Prob. II.

Prob. XV. *A weight* Q *draws a weight* P *over a fixed pully* A, *fig.* 54, P *in the mean time making small oscillations: to determine the motion.*

When the oscillations are very small, we may here, as in Prob. XI, suppose the tension to be the same as if P did not oscillate: and it will thus be found.

Let m, m' be the masses of P, Q; then $m'-m$ is the mass employed in producing motion; and $m'+m$ the mass moved: hence, the accelerating force on Q is $\frac{m'-m}{m'+m}g$.

But the force on Q downwards is manifestly the excess of its weight downwards, above the tension which acts upwards. Hence, if p be the tension,

$$g-\frac{pg}{m'}=\frac{m'-m}{m'+m}g; \therefore p=\frac{2m'm}{m'+m}.$$

Let AM vertical $=x$, MP horizontal $=y$, $AP=r$, $AQ=l-r$;

$$\therefore \frac{d^2y}{dt^2}=-\frac{pg}{m}\cdot\frac{y}{r}=-\frac{2m'g}{m'+m}\cdot\frac{y}{r}\ \ldots\ldots\ldots(1).$$

CASE 1. When $m'=m$, accelerating force on $Q=0$; $\therefore \frac{dr}{dt}$ is constant.

Let $\frac{dr}{dt}=-b$, P being supposed to ascend; $\therefore dt=-\frac{dr}{b}$, and equation (1) becomes

$$\frac{b^2d^2y}{dr^2}=-g\cdot\frac{y}{r}, \text{ or } \frac{d^2y}{dr^2}=-\frac{ky}{r}, \ k \text{ being constant.}$$

Let the integral of this be $y=Ar+Br^2+Cr^3+\&c.$

$$\left.\begin{aligned}\therefore \frac{d^2y}{dr^2}&=1.2.B+2.3\,Cr+\&c.\\ +\frac{ky}{r}&=kA+kBr+\&c.\end{aligned}\right\}=0;$$

whence $B=-\frac{kA}{2}$, $C=\frac{k^2A}{2^2.3}$, $D=-\frac{k^3A}{2^2.3^2.4}$, &c.

$$\therefore y=A\left\{r-\frac{kr^2}{2}+\frac{k^2r^3}{2^2\,3}-\frac{k^3r^4}{2^2\,3^2\,4}+\&c.\right\}.$$

By making $y=0$, we have an equation of an infinite number of dimensions; shewing that the curve described by P cuts the axis in an infinite number of points.

In order to determine the points where y is a maximum, we must have

$$\frac{dy}{dr}=0; \text{ or } 1-kr+\frac{k^2r^2}{2^2}-\frac{k^3r^3}{2^2 3^2}+\&c.=0,$$

(agreeing with equation (3), Prob. XII,) which gives the points of extreme deviation of the pendulum from the vertical. The times of successive oscillations will be as the differences of the successive values of r, because r diminishes uniformly *.

CASE 2. When m, m' are unequal.

Here $-\frac{d^2r}{dt^2}=\frac{m'-m}{m'+m}\cdot g$; $\frac{dr}{dt}=-\frac{m'-m}{m'+m}gt$; velocity being 0 when t is 0,

$$r=a-\frac{m'-m}{m'+m}\cdot\frac{gt^2}{2}=a-\tfrac{1}{2}ngt^2; \text{ making } \frac{m'-m}{m'+m}=n;$$

$$\text{whence } \frac{2m'}{m'+m}=1+n.$$

And equation (1) becomes

$$\frac{d^2y}{dt^2}=-\frac{(1+n)gy}{a-\frac{1}{2}ngt^2};$$

whence y may be found by series in terms of t.

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* This problem has been differently and erroneously solved by some authors. No solution but an approximate one is attainable. Euler, *Com. Petrop.* tom. VIII, p. 137, &c. obtains an equation not integrable, and then observes, "Ita ut determinationem hujus motus oscillatorii, quo corpora A et B cientur, dum filum super trochleam *uniformiter* promovetur, pro casu desparato declarare simus coacti."

The equation in Case 2, is integrable for some particular values of P and Q: for instance, if $Q=3P$.

CHAP. VII.

INVERSE PROBLEMS RESPECTING THE MOTION OF POINTS ON CURVES.

In the fifth Chapter we supposed a body to move on a curve, the curve being given, and the motion being the thing to be determined. In the present one we shall collect several questions which have occupied the attention of mathematicians, in which some property or consequence of the body's motion is given, and the curve is required.

Sect. I. *Curve of equal Pressure.*

43. Prop. To find the curve on which a body, descending by the force of gravity, presses equally at all points.

Let AM, fig. 55, be the vertical abscissa $= x$, MP the horizontal ordinate $= y$; the arc of the curve s, the time t, and the radius of curvature at $P = \rho$, ρ being positive when the curve is concave to the axis; then, R being the re-action at P, we have by Art. 33, page 93,

$$R = \frac{gdy}{ds} + \frac{ds^2}{\rho dt^2} \dots\dots\dots\dots (1).$$

But if HM be the height due to the velocity at P, $AH = h$, we have, by Art. 31,

$$\frac{ds^2}{dt^2} = 2g(h - x).$$

Also, if we suppose ds constant, we have $\rho = -\dfrac{dsdx}{d^2y}$; and if the constant value of R be k, equation (1) becomes

$$k = \frac{g\,dy}{ds} - \frac{2g\,(h-x)\,d^2y}{ds\,dx};$$

$$\therefore -\frac{k}{g}.\frac{dx}{2\sqrt{(h-x)}} = \sqrt{(h-x)}\,.\frac{d^2y}{ds} - \frac{dy}{ds}.\frac{dx}{2\sqrt{(h-x)}}.$$

The right hand side is obviously the differential of $\sqrt{(h-x)}\,.\frac{dy}{ds}$; hence, integrating,

$$\frac{k}{g}.\sqrt{(h-x)} = \sqrt{(h-x)}\,.\frac{dy}{ds} + C,$$

$$\frac{dy}{ds} = \frac{k}{g} - \frac{C}{\sqrt{(h-x)}}\,\ldots\ldots\ldots\ldots(2).$$

If $C = 0$, the curve becomes a straight line inclined to the horizon, which obviously answers the condition. The sine of inclination is $\frac{k}{g}$.

In other cases the curve is found by equation (2), putting $\sqrt{(dx^2+dy^2)}$ for ds, and integrating.

If we differentiate equation (2), ds being constant, we have

$$\frac{d^2y}{ds} = -\frac{C\,dx}{2\,(h-x)^{\frac{3}{2}}};\ \rho = -\frac{ds\,dx}{d^2y} = \frac{2\,(h-x)^{\frac{3}{2}}}{C}\,\ldots\ldots(3).$$

And if C be positive, ρ is positive, and the curve is concave to the axis.

We have the curve parallel to the axis, as at C, when $\frac{dy}{ds} = 0$, that is, when $\frac{k}{g} = \frac{C}{\sqrt{(h-x)}}$; when $x = h - \frac{C^2g^2}{k^2}$.

When x increases beyond this, the curve approaches the axis, and $\frac{dy}{ds}$ is negative; it can never become < -1; hence, the limit of x as B, is found by making $\frac{k}{g} - \frac{C}{\sqrt{(h-x)}} = -1$;

$$\text{or, } x = h - \frac{C^2g^2}{(k+g)^2}.$$

If $k < g$, as the curve descends towards Z, it approximates perpetually to the inclination, the sine of which is $\frac{k}{g}$. If $k > g$, fig. 56, there will be a point when the curve becomes horizontal as at D, after which it will ascend in a form similar to the descending branch.

C is known from equation (2) or (3), if we know the pressure or the radius of curvature at a given point.

If C be negative, the curve is convex to the axis. In this case the part of the pressure arising from centrifugal force diminishes the part arising from gravity, and k must be less than g, fig, 57.

Sect. II. *Synchronous Curves.*

44. In fig. 58, let AP, AP', AP'', &c. be curves of the same kind, referred to a common base AD, and differing only in their *parameters**: a curve P, P', P'', &c. cutting them, so that the arcs AP, AP', AP'', &c. may all be described in the same time, by a body descending from A by gravity, is said to make them *synchronous.*

Prop. To find the curve which cuts a given assemblage of curves, so as to make them synchronous.

Let AM vertical $= x$, MP horizontal $= y$; y and x being connected by an equation involving a. The time down AP is $\int \frac{dx}{\sqrt{(2gx)}}$, the integral being taken from $x = 0$ to $x = AM$; and this must be the same for all curves whatever be a. Hence, we may put

$$\int \frac{ds}{\sqrt{(2gx)}} = k \dots\dots (1),$$

k being a constant quantity, and in differentiating, we must suppose

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* Any constant line is called a *parameter*, which occurs in the equation to a curve, and by its different values gives different magnitudes to corresponding portions of the curve. Thus the radius of a circle, and the semi-axis of a cycloid are parameters.

a variable as well as x and s. Let $ds = pdx$, p being a function of x and a, which will be of 0 dimensions, because dx and ds are quantities of the same dimensions. Hence, $\int \frac{pdx}{\sqrt{(2gx)}} = k$, and differentiating

$$\frac{pdx}{\sqrt{(2gx)}} + qda = 0 \ldots\ldots\ldots (2),$$

q being the differential coefficient of $\int \frac{pdx}{\sqrt{(2gx)}}$, with respect to a. Now, since p is of 0 dimensions in x and a, it is easily seen that $\int \frac{pdx}{\sqrt{(2gx)}}$ is a function whose dimensions in x and a are $\frac{1}{2}$, because the dimensions of an expression are increased by 1 in integrating. Hence, by a known property of homogeneous functions, (see *Lacroix*, Elem. Treat. Art. 266,) we have

$$\frac{p}{\sqrt{(2gx)}} \cdot x + q \cdot a = \tfrac{1}{2}k;$$

$$\therefore q = \frac{k}{2a} - \frac{p\sqrt{x}}{a\sqrt{(2g)}},$$

substituting this in equation (2), it becomes

$$\frac{pdx}{\sqrt{(2gx)}} + \frac{kda}{2a} - \frac{pda\sqrt{x}}{a\sqrt{(2g)}} = 0 \ldots\ldots (3),$$

in which, if we put for a its value in x and y, we have an equation to the curve $PP'P''$.

If the given time k be the time of falling down a vertical height h, we have $k = \sqrt{\frac{2h}{g}}$, and hence, equation (3) becomes

$$p\,(adx - xda) + da\sqrt{(hx)} = 0 \ldots\ldots\ldots (4).$$

Ex. Let the curves AP, AP', AP'' be all cycloids of which the bases coincide with AD.

Let CD be the axis of any one of these cycloids $= 2a$, a being the radius of the generating circle. If $CN = x'$, we shall have as before

$$-ds = dx'\sqrt{\frac{2a}{x'}}; \text{ and since } x' = 2a - x,$$

$$ds = dx\sqrt{\frac{2a}{2a-x}}.$$

Hence, $p = \sqrt{\frac{2a}{2a-x}}$; and equation (4) becomes

$$\frac{\sqrt{(2a)}\,.\,(adx - xda)}{\sqrt{(2a-x)}} + da\sqrt{(hx)} = 0 \ldots\ldots\ldots(5).$$

Let $\frac{x}{a} = u$, so that $adx - xda = a^2du$, $x = au$; and substituting,

$$\frac{a^2du\sqrt{2}}{\sqrt{(2-u)}} + da\sqrt{(hau)} = 0;$$

$$\therefore \frac{du\sqrt{2}}{\sqrt{(2u-u^2)}} + \frac{da\sqrt{h}}{a^{\frac{3}{2}}} = 0;$$

$$\sqrt{2}\,.\,\text{arc (ver. sin.} = u) - \frac{2\sqrt{h}}{\sqrt{a}} = C.$$

When a is infinite, the portion AP of the cycloid becomes a vertical line, and $x = h$; $\therefore u = 0$; $\therefore C = 0$.

$$\text{Hence, } \frac{x}{a} = \text{ver. sin.} \sqrt{\frac{2h}{a}} \ldots\ldots\ldots\ldots(7).$$

From this equation a should be eliminated by the equation to the cycloid, which is

$$y = a\,.\,\text{arc.}\left(\text{ver. sin.} = \frac{x}{a}\right) - \sqrt{(2ax - x^2)} \ldots\ldots\ldots(8),$$

and we should have the equation to the curve required.

Substituting in (8) from (7), we have

$$y = \sqrt{(2ah)} - \sqrt{(2ax - x^2)},$$

$$dy = \frac{da\sqrt{h}}{\sqrt{(2a)}} - \frac{xda + adx - xdx}{\sqrt{(2ax - x^2)}};$$

and eliminating da by (5),

$$\frac{dy}{dx} = -\frac{2a-x}{\sqrt{(2ax-x^2)}} = -\frac{\sqrt{(2a-x)}}{\sqrt{x}}.$$

But differentiating (8) supposing a constant, we have in the cycloid

$$\frac{dy}{dx} = \frac{\sqrt{x}}{\sqrt{(2a-x)}}.$$

And hence the curve P, P', P'', cuts all the cycloids at right angles *.

The curve $PP'P''$ will meet AD in the point B, such that the given time is that of describing the whole cycloid AB. It will meet the vertical line in E, so that the body falls through AE in the given time.

COR. If instead of supposing all the cycloids to meet in the point A, we suppose them all to pass through any point C, fig. 65, their bases still being in the same line AD; a curve PP' drawn so that the times down PC, $P'C$, &c. are all equal, will cut all the cycloids at right angles. This may easily be collected from the preceding reasoning.

SECT. III. *Tautochronous Curves.*

45. If a body move upon a curve, the curve is said to be *tautochronous*, if the time of descent *to* a given point be the same, *from* whatever point the body begin to descend. We shall consider the body as descending to the lowest point.

PROP. When a body is acted upon by a constant force in parallel lines, to find the tautochronous curve.

Let A, fig. 59, be the lowest point, D the point from which the body falls, AB vertical, BD, MP horizontal. $AM = x$, $AP = s$, $AB = h$, the constant force $= g$.

Hence, the velocity at $P = \sqrt{\{2g(h-x)\}}$,

$$dt = -\frac{ds}{\sqrt{\{2g(h-x)\}}};$$

* For the subnormal of the former coincides with the subtangent of the latter, each being $\frac{y\sqrt{(2a-x)}}{\sqrt{x}}$.

and the whole time of descent will be found by integrating this from $x=h$, to $x=0$.

Now, since the time is to be the same, from whatever point D the body falls, that is, whatever be h, the integral just mentioned, taken between the limits, must be independent of h. That is, if we take the integral so as to vanish when $x=0$, and then put h for x, h will disappear altogether from the result. This must manifestly arise from its being possible to put the result in a form involving only $\frac{x}{h}$, and functions of $\frac{x}{h}$, as $\frac{x^2}{h^2}$, &c.; that is, from its being of 0 dimensions in x and h.

Let $ds=pdx$, where p depends only upon the curve, and does not involve h. Then, we have

$$t=-\int\frac{pdx}{\sqrt{\{2g(h-x)\}}}$$

$$=-\frac{1}{\sqrt{(2g)}}\int\left\{\frac{pdx}{h^{\frac{1}{2}}}+\frac{1}{2}\cdot\frac{pxdx}{h^{\frac{3}{2}}}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{px^2dx}{h^{\frac{5}{2}}}+\text{\&c.}\right\};$$

and from what has been said, it is evident, that each of the quantities $\int\frac{pdx}{h^{\frac{1}{2}}}$, $\int\frac{pxdx}{h^{\frac{3}{2}}}$, and generally $\int\frac{px^n}{h^{\frac{2n+1}{2}}}$, must be of the form $\frac{cx^{\frac{2n+1}{2}}}{h^{\frac{2n+1}{2}}}$; that is, $\int px^n dx$ must $=cx^{\frac{2n+1}{2}}$; hence, $px^n dx$ $=\frac{2n+1}{2}cx^{\frac{2n-1}{2}}dx$; $p=\frac{2n+1}{2}\frac{c}{x^{\frac{1}{2}}}$; or, if $\frac{2n+1}{2}c=a^{\frac{1}{2}}$, $p=\sqrt{\frac{a}{x}}$, $ds=dx\sqrt{\frac{a}{x}}$, which is the property of a cycloid*.

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* Without expanding, we may reason thus. If p be a function of m dimensions in x, $\frac{p}{\sqrt{(h-x)}}$ is of $m-\frac{1}{2}$ dimensions; and as the dimensions of an expression are increased by 1 in integrating, $\int\frac{p\,dx}{\sqrt{(h-x)}}$ is of $m+\frac{1}{2}$ dimensions in x, and when h is put for x, of $m+\frac{1}{2}$ dimensions in h. But it ought to be independent of h, or of 0 dimensions. Hence, $m+\frac{1}{2}=0$, $m=-\frac{1}{2}$. Therefore $p=a^{\frac{1}{2}}x^{-\frac{1}{2}}$ as before.

46. Prop. When the body is acted upon by a force tending to a centre, and varying as any function of the distance, to find the tautochronous curve.

Let S, fig. 60, be the centre of force, A the point to which the body must descend; D the point from which it descends. Let $SA=e$, $SD=f$, $SP=r$, P being any point, $AP=s$.

Now we have velocity$^2 = C - 2\int Pdr$, (Art. 31. p. 82.), or, if $2\int Pdr = \phi(r)$, velocity$^2 = \phi(f) - \phi(r)$; because the velocity$^2 = 0$, when $r=f$. Hence, the time of describing DA is $t = \int -\frac{ds}{\sqrt{\{\phi(f)-\phi(r)\}}}$, taken from $r=f$, to $r=e$. And since the time must be the same whatever is D, the integral so taken must be independent of f. Let $\phi(r)-\phi(e)=z$, $\phi(f)-\phi(e)=h$, $ds=pdz$, p depending only on the nature of the curve, and not involving f. Then

$$t = -\int \frac{pdz}{\sqrt{(h-z)}}, \text{ taken from } z=h, \text{ to } z=0$$

$$= \int \frac{pdz}{\sqrt{(h-z)}}, \text{ from } z=0, \text{ to } z=h.$$

And this must be independent of f, and therefore of $\phi(f)$, and of h. Hence, after taking the integral, the result must be 0, when $z=0$, and when h is put for z, must be independent of h. Therefore it must be of 0 dimensions in z and h. But if p be of n dimensions in z, or if $p=cz^n$, $\frac{p}{\sqrt{(h-z)}}$ will be of $n-\frac{1}{2}$ dimensions, and $\int \frac{pdz}{\sqrt{(h-z)}}$ of $n+\frac{1}{2}$. Hence, $n+\frac{1}{2}=0$, $n=-\frac{1}{2}$, and $p=\sqrt{\frac{c}{z}}$. Therefore $ds = dz\sqrt{\frac{c}{z}} = \phi' r dr \sqrt{\frac{c}{\phi(r)-\phi(e)}}$; whence the curve is known.

If v be the angle ASP, we have

$$ds^2 = dr^2 + r^2dv^2,\ dv^2 = \frac{ds^2-dr^2}{r^2},$$

whence we may find a polar equation to the curve.

Ex. 1. Let the force vary as the distance, and be attractive.

Here $P = mr$, $\phi(r) = mr^2$;

$$z = \phi(r) - \phi(e) = m(r^2 - e^2);\ dz = 2mrdr,$$

$$ds = dz\sqrt{\frac{c}{z}} = 2mrdr\sqrt{\frac{c}{m(r^2-e^2)}} = rdr\sqrt{\frac{4cm}{r^2-e^2}}.$$

When $r = e$, $\frac{ds}{dr}$ is infinite, or the curve is perpendicular to SA at A.

If SY, perpendicular upon the tangent PY, be called p, we have

$$\frac{p^2}{r^2} = \frac{ds^2 - dr^2}{ds^2} = 1 - \frac{dr^2}{ds^2} = 1 - \frac{r^2 - e^2}{4cmr^2},$$

$$p^2 = \frac{e^2 - (1 - 4cm)r^2}{4cm}.$$

If $e = 0$, or the body descend to the centre, this gives the logarithmic spiral.

In other cases let $1 - 4cm = \frac{e^2}{a^2}$; $\therefore\ 4cm = \frac{a^2 - e^2}{a^2}$;

$p^2 = \frac{e^2(a^2 - r^2)}{a^2 - e^2}$; the equation to a hypocycloid, see p. 87.

If $4cm = 1$, the curve becomes a straight line, to which SA is perpendicular at A.

If $4cm > 1$, the curve will be concave to the centre, and will go off to infinity.

Ex. 2. Let the force vary inversely as the square of the distance.

$P = \frac{m}{r^2}$; and as before, we shall find

$$p^2 = r^2 - \frac{r^5(r-e)}{2mce}.$$

Sect. IV. *Brachystochronous Curves.*

47. The *brachystochron* is the curve, down which a body must descend from one point to another, so that the time of descent may be less than that down any other curve, under the same circumstances.

Prop. A body being acted upon by a force in parallel lines, in its descent from one point to another; to find the brachystochron.

Let A, B, fig. 61, be the given points, and $AOPQB$ the required curve. Since the time down $AOPQB$ is less than down any other curve, if we take another curve $AOpQB$, which coincides with the former, except for the arc OPQ, we shall have

time down AO + time down OPQ + time down QB, less than
time down AO + time down OpQ + time down QB:

and if the times down QB be the same on the two suppositions, we shall have

time down OPQ less than time down any other arc OpQ.

The times down QB will be the same in the two cases, if the velocity at Q be the same. But it has been seen, (Art. 31,) that the velocity acquired at Q is the same, whether the body descend down $AOPQ$, or $AOpQ$*. Hence, it appears that if the time down $AOPQB$ be a minimum, the time down any portion OPQ is also a minimum.

Let a vertical line of abscissas be taken in the direction of the force, and perpendicular ordinates OL, PM, QN be drawn, it being supposed that $LM=MN$. Then, if LM, MN be taken indefinitely small, we may consider them as representing the differential of x: on this supposition, OP, PQ, will represent the differentials of the curve, and the velocity may be supposed constant in OP, and in PQ. Let $AL=x$, $LO=y$, $AO=s$; and let dx, dy, ds represent the differentials of the abscissa, ordinate and curve at Q, and v the velocity there; and dx', dy', ds', v', be the corres-

* This is true, whenever the body descends in a non-resisting space, or when the forces are necessarily the same in the same points.

ponding quantities at P. Hence, the time of describing OPQ will be

$$\frac{ds}{v} + \frac{ds'}{v'};$$

which is a minimum; and consequently, its differential is equal 0. This differential is that which arises from supposing P to assume any position, as p, out of the curve OPQ; and, as the differentials indicated by d arise from supposing P to vary its position along the curve OPQ, we shall use δ to indicate the differentiation, on hypothesis of passing from one curve to another, or the *variations* of the quantities to which it is prefixed. We shall also suppose p to be in the line MP, so that dx is not supposed to vary. These considerations being introduced, we may proceed thus,

$$\delta \left\{\frac{ds}{v} + \frac{ds'}{v'}\right\} = 0 \ldots\ldots\ldots (1).$$

And v, v' are the same whether we take OPQ or OpQ; for the velocity at $p =$ velocity at P. Hence, $\delta v = 0$, $\delta v' = 0$: and

$$\frac{\delta ds}{v} + \frac{\delta ds'}{v'} = 0.$$

Now, $ds^2 = dx^2 + dy^2$; $\therefore ds . \delta ds = dy . \delta dy$, because $\delta dx = 0$.

Similarly, $ds' . \delta ds' = dy' . \delta dy'$.

Substituting the values of δds, $\delta ds'$ which these equations give, we have

$$\frac{dy . \delta dy}{ds . v} + \frac{dy' . \delta dy'}{ds' . v'} = 0.$$

And since the points O, Q, remain fixed during the variation of P's position, we have

$$dy + dy' = \text{const.} \quad \delta dy' = - \delta dy.$$

Substituting, and omitting δdy,

$$\frac{dy}{ds.v} - \frac{dy'}{ds'.v'} = 0.$$

Or, since the two terms belong to the successive points O, P, their difference will be the differential indicated by d; hence,

$$d \cdot \frac{dy}{ds \cdot v} = 0; \quad \frac{dy}{ds \cdot v} = \text{constant} \ldots\ldots..(2),$$

which is the property of the curve; and v being known in terms of x, we may determine its nature.

Cor. 1. Let the force be gravity: then $v = \sqrt{(2gx)}$;

$$\frac{dy}{ds\sqrt{(2gx)}} = \text{constant}, \quad \frac{dy}{ds\sqrt{x}} = \frac{1}{\sqrt{a}};$$

$\sqrt{a}$ being a constant quantity;

$$\therefore \frac{dy}{ds} = \sqrt{\frac{x}{a}},$$

which is a property of the cycloid, of which the axis is parallel to x, and of which the base passes through the point from which the body falls.

Cor. 2. If the body fall from a given point to another given point, setting off with the velocity acquired down a given height: the curve of quickest descent is a cycloid, of which the base coincides with the horizontal line, from which the body acquires its velocity.

48. Prop. If a body be acted on by gravity, the curve of its quickest descent from a given point to a given curve, cuts the latter at right angles.

Let A, fig. 63, be the given point, and BM the given curve; AB the curve of quickest descent cuts BM at right angles.

It is manifest the curve AB must be a cycloid, for otherwise a cycloid might be drawn from A to B, in which the descent would be shorter. If possible, let AQ be the cycloid of quickest descent, the angle AQB being acute. Draw another cycloid AP, and let PP' be the curve which cuts AP, AQ, so as to make the arcs AP, AP' *synchronous*. Then, by Art. 44, PP' is perpendicular to AQ, and therefore manifestly P' is between A and Q, and the time down AP is less than the time down AQ; therefore, this latter is not the curve of quickest descent. Hence, if AQ be not perpendicular to BM, it is not the curve of quickest descent*.

* The cycloid which is perpendicular to BM may be the cycloid of *longest* descent from A to BM.

49. Prop. If a body be acted on by gravity, and if AB, fig. 64, be the curve of quickest descent from the curve AL to the point B; AT, the tangent of AL at A, is parallel to BV, a perpendicular to the curve AB at B.

If BV be not parallel to AT, draw BX parallel to AT, and falling between BV and A. In the curve AL take a point a near to A. Let aB be the cycloid of quickest descent from the point a to the point B; and Bb being taken equal and parallel to aA, let Ab be a cycloid equal and similar to aB. Since ABV is a right angle, the curve BP, which cuts off AP synchronous to AB, has BV for a tangent, (Art. 44.). Also, ultimately Aa coincides with AT, and therefore Bb with BX. Hence, b is between A and P. Hence, the time down Ab is less than the time down AP, and therefore, than that down AB. And hence the time down aB (which is the same as that down Ab,) is less than that down AB. Hence, if BV be not parallel to AT, AB is not the line of quickest descent from AL to B.

50. Prop. If a body acted on by gravity descend to a given point C, fig. 65, setting off from a curve BM, with a velocity acquired in falling from a given horizontal line AD, the curve of quickest descent cuts the curve BM at right angles.

As before, BC the curve of quickest descent, will be a cycloid, by Cor. 2 to Art. 47.

If possible, let QC be the cycloid of quickest descent, making CQB an acute angle. By Cor. 2 to Art. 47, the base of this cycloid will be in the horizontal line AD. Let OPC be another cycloid, of which the base is in AD. And by Cor. to Art. 44, if PP' cut off synchronous arcs PC, $P'C$, PP' will be perpendicular to the curves PC, QC. Hence, P' will fall between Q and C, and the time down PC, being equal to that down $P'C$, will be less than that down QC. Hence, if QC be not perpendicular to QB, it cannot be the curve of shortest descent.

From this it appears in what manner a cycloid must be drawn, so that it may be the curve of quickest descent from one given curve to another.

If the body descend from rest, from the curve *BM*, fig. 66, to the curve *CN*, by the action of gravity, the curve of quickest descent will be a cycloid, of which the base is the horizontal line *BE*, which cuts *CN* at right angles, and which is so situated, that the tangents to *BM* at *B*, and *CN* at *C*, are parallel.

If the body descend from the curve *BM* to the curve *CN*, the velocity being that acquired in falling from the horizontal line *AD*, the curve of quickest descent will be a cycloid, of which the base is the horizontal line *AD*, and which cuts both the curves *BM*, *CN*, at right angles.

For the brachystochron, when the length of the curve is given, see Mr. Woodhouse's *Isoperimetrical Problems*, p. 122.

51. Prop. Supposing a body to be acted on by any forces whatever, to determine the brachystochron.

Making the same notations and suppositions as before, *AL*, *LO*, fig. 61, being any rectangular co-ordinates; since as before, the time down *OPQ* is a minimum; we have, by equation (1) of last Article,

$$\delta\left\{\frac{ds}{v}+\frac{ds'}{v'}\right\}=0\ldots\ldots\ldots\ldots(1),$$

$$\frac{\delta ds}{v}+\frac{\delta ds'}{v'}-\frac{ds\,\delta v}{v^2}-\frac{ds'\,\delta v'}{v'^2}=0.$$

Now we have as before $\delta ds=\dfrac{dy\,.\,\delta dy}{ds}$, supposing $\delta dx=0$,

$$\delta ds'=\frac{dy'\,.\,\delta dy'}{ds'}=-\frac{dy'\,.\,\delta dy}{ds'}.$$

$\delta v=0$, for v is the velocity at O, and does not vary by altering the curve.

$$v'=v+dv;\ \therefore\ \delta v'=\delta v+\delta dv=\delta dv.$$

Hence, $\dfrac{dy\,.\,\delta dy}{ds\,.\,v}-\dfrac{dy'\,.\,\delta dy}{ds'\,.\,v'}-\dfrac{ds'\,.\,\delta dv}{v'^2}=0.$

Also $\dfrac{1}{v'}=\dfrac{1}{v+dv}=\dfrac{1}{v}-\dfrac{dv}{v^2}$; for dv^2, &c. must be omitted.

Substituting this in the second term, we have

$$\frac{dy\,.\,\delta dy}{ds\,.\,v} - \frac{dy'\,\delta\,dy}{ds'\,.\,v} + \frac{dy'\,.\,dv\,.\,\delta dy}{ds'\,.\,v^2} - \frac{ds'\,.\,\delta\,dv}{v'^2} = 0,$$

$$\text{or } -\left(\frac{dy'}{ds'} - \frac{dy}{ds}\right).\frac{1}{v} + \frac{dy'\,.\,dv}{ds'\,.\,v^2} - \frac{ds'}{v'^2}.\frac{\delta\,dv}{\delta\,dy} = 0.$$

Now as before $\frac{dy'}{ds'} - \frac{dy}{ds}$ is $d\,.\frac{dy}{ds}$. And in the other terms we may, since O, P, are indefinitely near, put ds, dy, v, for ds', dy', v': if we do this, and multiply by $-v$, we have

$$d\,.\frac{dy}{ds} - \frac{dy\,.\,dv}{ds\,.\,v} + \frac{ds}{v}.\frac{\delta\,dv}{\delta dy} = 0 \cdots\cdots\cdots(2),$$

which will give the nature of the curve.

If the forces which act on the body at O, be equivalent to X in the direction of x, and Y in the direction of y, we have

$$vdv = Xdx + Ydy, \text{ (Art. 31)};$$

$$\therefore\ dv = \frac{Xdx + Ydy}{v},$$

$$\delta\,dv = \frac{Y\delta dy}{v},$$

because $\delta v = 0$, $\delta dx = 0$; also X and Y are functions of AL and LO, and therefore not affected by δ.

Substituting these values in the equation to the curve, we have

$$d\,.\frac{dy}{ds} - \frac{dy}{ds}.\frac{Xdx + Ydy}{v^2} + \frac{ds}{v}.\frac{Y}{v} = 0;$$

$$\text{or } d\,.\frac{dy}{ds} - \frac{dx}{ds}.\frac{Xdy - Ydx}{v^2} = 0,$$

which will give the nature of the curve.

Cor. 1. If ρ be the radius of curvature, and ds constant, we have $\rho = \frac{ds\,dx}{d^2y}$: ρ being positive when the curve is convex to AM;

$$d\,.\frac{dy}{ds} = \frac{dx}{\rho}; \text{ and hence,}$$

$$\frac{v^2}{\rho} = \frac{Xdy - Ydx}{ds}.$$

The quantity $\frac{v^2}{\rho}$ is the centrifugal force, and therefore that part of the pressure which arises from it. And $\frac{Xdy - Ydx}{ds}$ is the pressure which arises from resolving the forces perpendicular to the axis. Hence, it appears then in the brachystochron for any given forces, the parts of the pressure which arise from the given forces, and from the centrifugal force, must be equal.

Cor. 2. If we suppose the force to tend to a centre S, fig. 62, which may be assumed to be in the line AM, and P to be the whole force; also $SA = a$, $SP = r$, SY, perpendicular on the tangent $PY = p$; we have

$$\frac{Xdy - Ydx}{ds} = \text{force in } PS \text{, resolved parallel to } YS = P\,.\frac{p}{r},$$

$$v^2 = C - 2\int P\,dr, \text{ (Art. 31)};$$

$$\therefore \frac{C - 2\int Pdr}{\rho} = \frac{Pp}{r}.$$

$$\text{Also } \rho = -\frac{rdr}{dp};$$

$$\therefore C - 2\int Pdr = -\frac{Ppdr}{dp};$$

$$\therefore \frac{2dp}{p} = \frac{-2Pdr}{C - 2\int Pdr}, \text{ and integrating,}$$

$p^2 = C'\{C - 2\int Pdr\}$; whence the relation of p and r is known.

If the body begin to descend from A, $C - 2\int Pdr$ must $= 0$ when $r = a$.

Ex. 1. Let the force vary directly as the distance.

$P = mr$, $C - 2\int P\,dr = \text{velocity}^2 = m\,(a^2 - r^2)$, $p^2 = C'\,m\,(a^2 - r^2)$, which agrees with the equation to the hypocycloid, p. 87.

Ex. 2. Let the force vary inversely as the square of the distance,

$$P = \frac{m}{r^2},\ C - 2\int P\,dr = \frac{2m}{r} - \frac{2m}{a};$$

$$p^2 = \frac{2mC'}{a} \cdot \frac{a-r}{r} = c^2 \cdot \frac{a-r}{r},\ \text{putting } c^2 = \frac{2mC'}{a};$$

$$r^2 - p^2 = \frac{r^3 + c^2 r - c^2 a}{r},$$

$$d\theta = \frac{p\,dr}{r\sqrt{(r^2 - p^2)}} = \frac{c\sqrt{(a-r)} \cdot dr}{r\sqrt{(r^3 + c^2 r - c^2 a)}} = \frac{c\,dr}{r\sqrt{\left\{\frac{r^3}{a-r} - c^2\right\}}}.$$

When $r = a$, $d\theta = 0$: when $r^3 + c^2 r - c^2 a = 0$, $d\theta$ is infinite, and the curve is perpendicular to the radius as at B. This equation has only one root.

If we have $c = \frac{a}{2}$, $SB = \frac{a}{2}$, B being an apse,

$$\text{if } c = \frac{a}{10},\ SB = \frac{a}{5},$$

$$\text{if } c = \frac{a}{30},\ SB = \frac{a}{10},$$

$$\text{if } c = \frac{a}{n^3 + n},\ SB = \frac{a}{n^2 + 1}.$$

52. Prop. When a body moves on a given surface, to determine the brachystochron.

Let x, y, z be rectangular co-ordinates, x being vertical; and as before, let ds, ds' be two successive elements of the curve: and let dx, dy, dz; dx', dy', dz' be the corresponding elements of x, y, z; then, since the minimum property will be true of the indefinitely small portion of the curve, we have, as before, supposing v, v' the velocities,

$$\frac{ds}{v} + \frac{ds'}{v'} = \text{min.}$$

$$\delta \left\{ \frac{ds}{v} + \frac{ds'}{v'} \right\} = 0 \cdots\cdots\cdots (1).$$

The variations indicated by δ are those which arise, supposing dx, dx' to be equal and constant, and dy, dz, dy' and dz' to vary. Now,

$$ds^2 = dx^2 + dy^2 + dz^2 \,;\ \therefore ds\,\delta ds = dy\,\delta dy + dz\,\delta dz,$$

$$\text{similarly,} \qquad ds'\delta ds' = dy'\delta dy' + dz'\delta dz'.$$

Also, the extremities of the arc $ds + ds'$ being fixed, we have

$$dy + dy' = \text{const.} \qquad dz + dz' = \text{const.}$$

$$\therefore \delta dy + \delta dy' = 0; \qquad \delta dz + \delta dz' = 0.$$

$$\left.\begin{aligned} \text{Hence,}\quad \delta ds &= \frac{dy}{ds}\,\delta dy + \frac{dz}{ds}\,\delta dz \\ \delta ds' &= -\frac{dy'}{ds'}\,\delta dy - \frac{dz'}{ds'}\,\delta dz \end{aligned}\right\} \cdots\cdots\cdots (2).$$

And the surface is defined by an equation between x, y, z, which we may call $L = 0$. Let this, differentiated, give

$$dz = p\,dx + q\,dy \cdots\cdots\cdots\cdots\cdots\cdots\cdots (3).$$

Hence, since dx, p, q are not affected by δ,

$$\delta dz = q\,\delta dy \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (4).$$

For the sake of simplicity, we will suppose the body to be acted on only by a force in the direction of x, so that v, v' will depend on x alone, and will not be affected by the variation of dy, dz. Hence, we have by (1),

$\dfrac{\delta ds}{v} + \dfrac{\delta ds'}{v'} = 0$; which, by substituting from (2), becomes,

$$\left\{ \frac{dy'}{v'ds'} - \frac{dy}{vds} \right\} \delta dy + \left\{ \frac{dz'}{v'ds'} - \frac{dz}{vds} \right\} \delta dz = 0.$$

Therefore we shall have, as before,

$$d\,.\,\frac{dy}{vds}\,.\,\delta dy + d\,.\,\frac{dz}{vds}\,\delta dz = 0\,;$$

and by equation (4), this becomes

$$d \cdot \frac{dy}{vds} + q \cdot d\,\frac{dz}{vds} = 0 \cdots\cdots\cdots\cdots (5),$$

whence the equation to the curve is known.

If we suppose the body not to be acted on by any force, v will be constant, (Art. 39;) and the path described will manifestly be the shortest line which can be drawn on the given surface, and will be determined by the equation

$$d \cdot \frac{dy}{ds} + qd \cdot \frac{dz}{ds} = 0 \cdots\cdots\cdots\cdots (6).$$

If we suppose, as in Note, p. 111, ds to be constant, we have

$$d^2 y + q d^2 z = 0;$$

which agrees with the equation there deduced, for the path when the body is acted on by no forces. Hence, it appears, that when a body moves along a surface undisturbed, it will describe the shortest line which can be drawn on that surface, between any points of its path.

For a more general investigation of the nature of the Brachystochron, see Poisson, *Traite de Mec.* No. 288, &c.

BOOK II.

THE MOTION OF A POINT IN A RESISTING MEDIUM.

53. The preceding reasonings go upon the supposition, that a body in motion, and left to itself, would move on for ever with a uniform velocity. This would be true, if the body moved in a perfect vacuum; but when the motion takes place in any fluid or *medium*, whose density is finite; then will be a retardation, arising from the *resistance* which the medium offers to the motion of the body: and if the body be acted on by any extraneous forces, the effects of the resistance will be combined with those of the forces, and the curves, &c., described by the body, as calculated in the preceding Book, will be modified in a manner which we shall consider in this.

The resistance arises, in part, from the friction and tenacity of the fluid, but principally from its inertia, that is, from the force which the body, moving through the medium, necessarily exerts in putting the fluid particles in motion. Hence, it will be in a direction opposite to the body's motion: and if, *cæteris paribus*, the velocity be increased, the resistance will also be increased; for the body will strike more particles, and with greater violence. The *law*, according to which the resistance varies with respect to the velocity, is to be deduced from experiment: in most fluids, and for a moderate velocity, it appears to be nearly, but not accurately, as the square of the velocity. In the following problems we shall suppose various laws of resistance.

The resistance is of the nature of a *pressure*, or *moving* force, and may be represented by a weight. Hence, its effect on the body, that is, the *accelerating* force (or, as it may here be called, the *retarding force*,) is to be found by dividing this resisting force by the mass of the body.

The quantity of the resistance, (considered as accelerating or retarding force,) will of course depend upon the law, and upon a constant coefficient. Thus, if the resistance vary as the square of the velocity (v), it may be supposed equal to kv^2. The quantity k will vary with the density, and other circumstances of the fluid, and also with the mass, magnitude, and form of the body.

We may also represent the absolute quantity of the resistance thus. Since it assumes different values as the velocity varies, we may suppose a velocity such, that with it, the resistance shall be equal to gravity (g). Suppose V to be this velocity, and let the resistance vary as v^n, v being any velocity. We have then

$$V^n : v^n :: g : \frac{gv^n}{V^n} = \text{resistance to velocity } v.$$

V is such, that if a body were moving downwards with that velocity, it would move on for ever uniformly, for the action of gravity downwards, and of the resistance upwards, would exactly counteract each other; and the motion would be the same as if the body were acted on by no force at all.

If we suppose the resistance $= kv^n$, we have $k = \frac{g}{V^n}$.

CHAP. I.

RECTILINEAR MOTION OF A POINT IN A RESISTING MEDIUM.

54. THE formulæ $dv = fdt$, $ds = vdt$, $vdv = fds$, are applicable to all cases of this kind, (v being the velocity, f the force in the direction of the motion, and s the space described,) provided we put for f the whole force arising from the attractions, &c., and diminished by that arising from the resistance. The forces in this Chapter are supposed to act in the line of the bodies' motion.

Sect. I. *No Forces but the Resistance.*

As an example of this case, we may suppose a body to move on a horizontal plane perfectly smooth.

Prop. The resistance varying as any power of the velocity, to determine the motion.

Let the resistance $= kv^n$: hence, in this case $f = -kv^n$, where the negative sign is used, because the force tends to diminish the velocity.

First, for the space,

$$vdv = -kv^n ds,$$

$$kds = -\frac{dv}{v^{n-1}},$$

$$ks = \frac{1}{(n-2)v^{n-2}} + C.$$

And if v_1 be the velocity of the body when $s = 0$,

$$(n-2)ks = \frac{1}{v^{n-2}} - \frac{1}{v_1^{\ n-2}}, \text{ if } n > 2,$$

$$(2-n)ks = v_1^{\ 2-n} - v^{2-n}, \text{ if } n < 2.$$

If $n = 2$, this integral fails, and we must return to the original equation.

Second, for the time,

$$dv = fdt = -kv^n dt,$$

$$kdt = -\frac{dv}{v^n},$$

$$kt = \frac{1}{(n-1)v^{n-1}} + C;$$

and if $t = 0$, when $v = v_1$,

$$(n-1)kt = \frac{1}{v^{n-1}} - \frac{1}{v_1^{\ n-1}} \text{ if } n > 1;$$

$$(1-n)kt = v_1^{\ 1-n} - v^{1-n}, \text{ if } n < 1.$$

If $n=1$, the integral fails, and must be obtained differently.

Ex. 1. Let $n=1$; our expression will be

$$ks = v_1 - v;$$

or, the velocity lost is as the space*.

$$\text{And in this case, } kdt = -\frac{dv}{v};$$

$$\therefore\ kt = \text{hyp. log. } \frac{v_1}{v};$$

therefore, if times increase uniformly, velocities decrease in geometrical progression†.

$$\text{When } v=0,\ s=\frac{v_1}{k},\quad t=\infty,$$

that is, in losing its whole velocity, a body will employ an infinite term, but will describe only a finite space.

Ex. 2. Let $n=2$. In this case,

$$kds = -\frac{dv}{v};$$

$$\therefore\ ks = \text{hyp. log. } \frac{v_1}{v}.$$

$$\text{Again,}\quad kt = \frac{1}{v} - \frac{1}{v_1};$$

$$\therefore\ 1 + kv_1 t = \frac{v_1}{v},\quad \text{or, if } T = \frac{1}{kv_1},$$

$$k(T+t) = \frac{1}{v}.$$

Hence, if the times $(T+t)$ increase in geometrical progression, the reciprocals of the velocities increase, or the velocities decrease, in the same progression‡.

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* *Principia*, Book II, Prop. 1. † Ibid. Prop. 2.

‡ Ibid. Prop. 5.

If $t \propto \frac{1}{v_1}$, s is constant, and $v_1 - v \propto \frac{1}{v}$. Ibid. Prop. 6.

$$\text{Also } \frac{v_1}{v} = \epsilon^{ks};$$

$$\therefore\ kt = \frac{1}{v_1}\{\epsilon^{ks} - 1\},$$

$$1 + kv_1 t = \epsilon^{ks}, \text{ or } kv_1(T+t) = \epsilon^{ks}.$$

Hence, if the times $(T+t)$ increase in geometrical progression, the spaces increase in arithmetical progression, and the spaces in the successive intervals are all equal.

When $v = 0$, we have s and t both infinite, or the body will describe an infinite space, and move for an infinite time before it loses all its velocity.

Ex. 3. Let $n = 4$. In this case,

$$2ks = \frac{1}{v^2} - \frac{1}{v_1^{\,2}},$$

$$3kt = \frac{1}{v^3} - \frac{1}{v_1^{\,3}}.$$

Scholium. It is manifest, that if we suppose the body to move till all its velocity be destroyed,

if $n < 1$, both space and time are finite,
if $n = 1$, or > 1 and < 2, space is finite, and time infinite,
if $n = 2$, or > 2, both space and time are infinite.

55. Prop. The resistance being $= hv + kv^2$, to determine the motion*.

$$\text{Here } ds = -\frac{vdv}{f} = -\frac{dv}{h+kv};$$

$$\therefore\ s = \frac{1}{k}\text{ hyp. log. }\frac{h+kv_1}{h+kv},$$

$$dt = \frac{dv}{f} = -\frac{dv}{hv+kv^2}$$

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* *Principia*, Book II, Prop. 11, 12.

$$= \frac{1}{h}\left\{\frac{dv}{v} - \frac{k\,dv}{h+kv}\right\}$$

$$t = \frac{1}{h}\left\{\text{hyp. log.}\,\frac{v}{v_1} + \text{hyp. log.}\,\frac{h+kv_1}{h+kv}\right\}$$

$$= \frac{1}{h}\,\text{hyp. log.}\,\frac{v\,(h+kv_1)}{v_1\,(h+kv)}.$$

When $v=0$, $s=\frac{1}{k}\,\text{hyp. log.}\left(1+\frac{kv_1}{h}\right)$, $t=\text{inf.}$

SECT. II. *The Body acted on by a constant Force besides Resistance.*

56. The constant force may be supposed to be gravity ($=g$), and the body to move in a vertical line, upwards or downwards. In the former case, the force which acts upon it is the *sum* of gravity and the resistance; in the latter, it is the *difference*. Hence, the motions in ascent and descent are not similar, and cannot be obtained from the same equation, as in the case of a vacuum. They may however easily be obtained separately, as in the following examples.

PROP. A body is acted on by gravity, and also by a resistance varying as the velocity; to determine the motion*.

For the descent, $f=g-kv$;

$$ds = \frac{v\,dv}{f} = \frac{v\,dv}{g-kv}$$

$$= \frac{1}{k}\left\{\frac{g\,dv}{g-kv} - dv\right\};$$

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* *Principia*, Book II, Prop. 3.

It may be observed, that the constant force of gravity, urging a body to descend in a resisting medium, is not the same as the whole force of gravity g in a vacuum. It is only the *relative gravity* of the body, with respect to the fluid; that is, its weight, diminished by the weight of an equal bulk of the fluid.

$\therefore\ s = \frac{g}{k^2}$ hyp. log. $\frac{g - kv_1}{g - kv} - \frac{v - v_1}{k}$ supposing $g - kv$ positive, and v_1 being the velocity when $s = 0$.

It is manifest that the greatest value which v can assume, is that which makes $g - kv = 0$, or $v = \frac{g}{k} = V$ suppose; so that $k = \frac{g}{V}$.

This velocity V is called the *terminal* velocity; it is the limit to which the velocity perpetually approximates as the body descends, but which it never actually attains in a finite time.

If we put $\frac{g}{V}$ for k, we have

$$s = \frac{V^2}{g} \text{ hyp. log. } \frac{V - v_1}{V - v} - \frac{V}{g}(v - v_1).$$

$$\text{Again, } dt = \frac{dv}{f} = \frac{dv}{g - kv};$$

$$\therefore\ t = \frac{1}{k} \text{ hyp. log. } \frac{g - kv_1}{g - kv}$$

$$= \frac{V}{g} \text{ hyp. log. } \frac{V - v_1}{V - v}.$$

From this we should easily obtain the relation between s and t.

Supposing $kv - g$ positive, we have

$$s = \frac{g}{k^2} \text{ hyp. log. } \frac{kv_1 - g}{kv - g} + \frac{v_1 - v}{k} = \frac{V^2}{g} \text{ hyp. log. } \frac{v_1 - V}{v - V} + \frac{V}{g}(v_1 - v).$$

Hence, if v be at one point greater than V, it will always continue so. This belongs to the case when the body is projected downwards with a velocity *greater than the terminal velocity*. The velocity will, in this case, *decrease* and approach to the terminal velocity as its limit.

The *ascent* may be obtained in nearly the same manner by making $f = -(g + kv)$.

57. Prop. A body is acted upon by gravity, and by a resistance varying as the square of the velocity: to determine the ascent and descent*.

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* *Principia*, Book II. Prop. 8, 9.

For the ascent: $f = -(g + kv^2)$,

$$ds = \frac{v\,dv}{f} = -\frac{v\,dv}{g + kv^2};$$

$$s = \frac{1}{2k}\text{ hyp. log. }\frac{g + kv_1^2}{g + kv^2}.$$

When $v = 0$, $s = \frac{1}{2k}\text{ hyp. log. }\left(1 + \frac{kv_1^2}{g}\right)$, which gives the whole height ascended.

When k is small, or the density small,

whole height $= \frac{1}{2k}\left\{\frac{kv_1^2}{g} - \frac{1}{2}\cdot\frac{k^2v_1^4}{g^2}\right\}$, neglecting terms beyond k

$$= \frac{v_1^2}{2g} - \frac{kv_1^4}{4g^2}.$$

The first term is the ascent in a vacuum, (Book I. Chap. I. Ex. 3.) and therefore the second is the *defect* of height produced by a medium of small density. It appears from the expression that this defect is as the square of the height ascended.

Again, $dt = \frac{dv}{f} = -\frac{dv}{g + kv^2}$,

$$t = \frac{1}{\sqrt{(kg)}}\cdot\left\{\text{arc}\left(\tan. = v_1\sqrt{\frac{k}{g}}\right) - \text{arc}\left(\tan. = v\sqrt{\frac{k}{g}}\right)\right\}.$$

When $v = 0$, time of whole ascent $= \frac{1}{\sqrt{(kg)}}\cdot\text{arc}\left(\tan. = v_1\sqrt{\frac{k}{g}}\right)$.

If v_1 be infinite, $t = \frac{\pi}{2\sqrt{(kg)}}$, and is therefore finite, but the height ascended is infinite.

When k is small, expanding the arc by the formula,

$$\text{arc} = \tan. - \frac{\tan.^3}{3} + \&c.$$

we have time $= \frac{1}{\sqrt{(kg)}}\left\{v_1\sqrt{\frac{k}{g}} - \frac{v_1^3}{3}\cdot\frac{k^{\frac{3}{2}}}{g^{\frac{3}{2}}}\right\}$ neglecting higher powers

$$= \frac{v_1}{g} - \frac{kv_1^3}{3g^2}.$$

of which expression the first term is the whole time of ascent in a vacuum, and the second the *diminution* of the time by the resistance.

For the descent: $f = g - kv^2$,

$$ds = \frac{vdv}{f} = \frac{vdv}{g - kv^2}; \text{ and if } g - kv^2 \text{ be positive,}$$

$$s = \frac{1}{2k} \text{ hyp. log. } \frac{g - kv_1^2}{g - kv^2}; \text{ or if } k = \frac{g}{V^2}, V = \sqrt{\frac{g}{k}},$$

$$s = \frac{V^2}{2g} \text{ hyp. log. } \frac{V^2 - v_1^2}{V^2 - v^2}.$$

But if $g - kv^2$ be negative,

$$s = \frac{1}{2k} \text{ hyp. log. } \frac{kv_1^2 - g}{kv^2 - g}$$

$$= \frac{V^2}{2g} \text{ hyp. log. } \frac{v_1^2 - V^2}{v^2 - V^2}.$$

V is the terminal velocity as before; and the former expression is for the case when the body is projected with a less, and the latter, when with a greater velocity than V.

$$dt = \frac{dv}{f} = \frac{dv}{g - kv^2} = \frac{1}{2\sqrt{(kg)}} \cdot \left\{ \frac{dv\sqrt{k}}{\sqrt{g} + v\sqrt{k}} + \frac{dv\sqrt{k}}{\sqrt{g} - v\sqrt{k}} \right\}$$

$$t = \frac{1}{2\sqrt{(kg)}} \left\{ \text{hyp. log. } \frac{\sqrt{g} + v\sqrt{k}}{\sqrt{g} + v_1\sqrt{k}} \right.$$

$$\left. + \text{hyp. log. } \frac{\sqrt{g} - v_1\sqrt{k}}{\sqrt{g} - v\sqrt{k}} \right\} (v_1 < V)$$

$$= \frac{V}{2g} \left\{ \text{hyp. log. } \frac{V + v}{V + v_1} + \text{hyp. log. } \frac{V - v_1}{V - v} \right\}$$

$$= \frac{V}{2g} \text{ hyp. log. } \left\{ \frac{V - v_1}{V + v_1} \cdot \frac{V + v}{V - v} \right\}.$$

When $v_1 > V$, the expression is similarly integrated.

If the body fall from rest, $v_1 = 0$,

$$s = \frac{V^2}{2g} \text{ hyp. log. } \frac{V^2}{V^2 - v^2}, \quad t = \frac{V}{2g} \text{ hyp. log. } \frac{V + v}{V - v}.$$

The last equation gives

$$\therefore \frac{V-v}{V+v} = \epsilon^{-\frac{2gt}{V}}.$$

When t becomes considerable, the right hand side of this equation is very small, and v very nearly $= V$. Hence, though the body never acquires the terminal velocity, after the lapse of a certain finite time, it comes very near it, and the velocity afterwards may be considered constant.

58. Prop. The resistance varying partly as the velocity, and partly as the square of the velocity: to determine the ascent of a body by gravity*.

$$\text{Resistance} = hv + kv^2;\ f = -g - hv - kv^2;$$

$$ds = \frac{vdv}{f} = -\frac{vdv}{g + hv + kv^2}.$$

$$\text{Let } v + \frac{h}{2k} = u,\ v^2 + \frac{hv}{k} + \frac{h^2}{4k^2} = u^2,$$

$$g + hv + kv^2 = g - \frac{h^2}{4k} + ku^2.$$

and we shall have two different cases, as $\frac{h^2}{4k}$ is less or greater than g.

In the former case,

$$ds = -\frac{udu - \frac{h}{2k}.du}{g - \frac{h^2}{4k} + ku^2}$$

$$= -\frac{4kudu}{4kg - h^2 + 4k^2u^2} + \frac{2hdu}{4kg - h^2 + 4k^2u^2},$$

$$s = -\frac{1}{2k}\text{ hyp. log. } (4kg - h^2 + 4k^2u^2)$$

$$+ \frac{h}{k}\text{ arc. } \left(\tan. = \frac{2ku}{\sqrt{(4kg - h^2)}}\right) + C$$

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* *Principia*, Book II. Prop. 13, 14.

$$= -\frac{1}{2k}\text{ hyp. log. } 4k\,(g+hv+kv^2)$$

$$+\frac{h}{k}\text{. arc. }\left(\text{tan.} = \frac{2kv+h}{\sqrt{(4kg-h^2)}}\right) + C$$

$$= \frac{1}{2k}\text{ hyp. log. }\frac{g+hv_1+kv_1^{\,2}}{g+hv+kv^2}$$

$$+\frac{h}{k}\cdot\left\{\text{arc. tan.} = \frac{2kv+h}{\sqrt{(4kg-h^2)}} - \text{arc. tan.} = \frac{2kv_1+h}{\sqrt{(4kg-h^2)}}\right\}.$$

In the latter case, when $\frac{h^2}{4k}$ is greater than g, we have

$$ds = -\frac{4kudu}{4k^2u^2-(h^2-4kg)} + \frac{2hdu}{4k^2u^2-(h^2-4kg)},$$

and both the terms may be integrated by logarithms.

$$\text{Also } dt = \frac{dv}{f} = -\frac{dv}{g+hv+kv^2},$$

which may be integrated in the same manner.

In this case the terminal velocity V is found by solving the equation $g-hV-kV^2=0$. One of the roots of this will be negative, and the other gives the velocity.

Sect. III. *The Body being acted on by a variable Force.*

59. In this case, to make the problem more general, we may suppose that the density of the medium, or its power of producing resistance, is different in different places. On this supposition k will be variable.

Prop. A body is acted on by a force P in the line of its motion, and by a resistance which is as the square of the velocity: to determine its motion.

Let s be the body's distance from a given point towards which the force tends.

In the descent towards the given point $f = -P+kv^2$,

$$vdv = -Pds + kv^2ds,$$

$$\text{or, } vdv - kv^2ds = -Pds\ldots\ldots(1).$$

Now $d.v^2\epsilon^{-\int 2kds} = 2vdv\epsilon^{-\int 2kds} - 2kv^2ds\epsilon^{-\int 2kds}$;

hence, the first side of equation (1), multiplied by $2\epsilon^{-\int 2kds}$, becomes integrable.

Multiplying and integrating, we have

$$v^2\epsilon^{-\int 2kds} = -2\int Pds\epsilon^{-\int 2kds} + C,$$

$$v^2 = 2\epsilon^{2\int kds}\int Pds\epsilon^{-2\int kds} + C\epsilon^{2\int kds}\ldots\ldots(2).$$

If the integrals on the right hand side of equation (2) can be taken, the velocity is found. And this being known, the time may be found.

Ex. Let the force be inversely as s^2, and the density inversely as s.

Here $k = \frac{h}{s}$, h being constant. $\int kds = h\int\frac{ds}{s} = h$ hyp. log. s.

$\epsilon^{-2\int kds} = s^{-2h}$; and the force being $= \frac{m}{s^2}$,

$$\int Pds\epsilon^{-2\int kds} = \int\frac{mds}{s^{2h+2}} = -\frac{2m}{(2h+1)\,s^{2h+1}};$$

$$\therefore v^2 = Cs^{2h} - \frac{2m}{(h+1)\,s}.$$

And if $v = 0$, when $s = a$, $C = \frac{2m}{(2h+1)\,a^{2h+1}}$;

$$\therefore v^2 = \frac{2m}{(2h+1)\,a}\left\{\frac{s^{2h}}{a^{2h}} - \frac{a}{s}\right\}.$$

For the time, $dt = \frac{ds}{v}$; and putting for v its value, the possibility of the integration will depend on the value of h.

In the same manner, if the force vary as any power of the distance, the velocity may be found.

If the density vary inversely as the square of the distance, and the force as any inverse power of it, the integration is possible.

If the density vary inversely as the cube of the distance, and the force as any inverse odd power, the integration is possible.

And generally, if the density vary as any inverse power r, of the distance, the integration is possible, if the force vary as any inverse power of the distance, whose index is contained in the series, r, $2r-1$, $3r-2$, &c.

CHAP. II.

THE FREE CURVILINEAR MOTION OF A BODY IN A RESISTING MEDIUM.

60. The second law of motion, and the results deduced from it, are true when the body moves in a resisting medium, provided we comprehend the resistance among the forces which operate to change the motion. Hence, we may apply to this case also, the equations (c), Art. 12, (the motion being in the same plane,) viz.

$$\frac{d^2x}{dt^2} = X, \frac{d^2y}{dt^2} = Y,$$

taking care to include in X, Y the resolved parts of the resistance in the directions of x, y.

Let R be the resistance in the direction of the curve, and opposite to the body's motion. Then it is clear, that the resolved parts of this in the direction of x and y are

$$-R\frac{dx}{ds}, -R\frac{dy}{ds};$$

where the negative sign is used, because when x and y are increasing, the resistance tends to diminish them. Adding these to the other forces which act on the body, we have the whole values of X and Y, and may proceed to determine the motion nearly in the same manner as in the former book.

If the motion of the body be not all in the same plane, we shall have, besides the two equations already mentioned, a third $\frac{d^2 z}{dt^2} = Z$; and Z will involve a term $- R \cdot \frac{dz}{ds}$, arising from the resistance.

Sect. I. *The Force acting in parallel Lines and constant.*

61. Prop. Let a body, acted on by a constant force, as gravity, $(=g)$, be projected in a uniform medium, of which the resistance is as the velocity: it is required to find the curve*, fig. 67.

Let x, measured from the point of projection A, be horizontal, y vertical, s the curve, t the time. In this case $R = k \cdot \frac{ds}{dt}$; hence, our equations become

$$\frac{d^2 x}{dt^2} = -k \cdot \frac{dx}{dt}, \quad \frac{d^2 y}{dt^2} = -g - k \frac{dy}{dt} \ldots\ldots\ldots (1).$$

Integrating,

$$\text{hyp. log.} \frac{dx}{dt} = C - kt; \quad \text{hyp. log.} \left(g + k\frac{dy}{dt}\right) = C' - kt,$$

and when $t = 0$, if a be the velocity of the body, and α the angle which its direction makes with the horizon, we have at that time

$$\frac{dx}{dt} = a \cos. \alpha, \quad \frac{dy}{dt} = a \sin. \alpha.$$

Hence, obtaining the values of C, C', and reducing,

$$\frac{dx}{dt} = a \cos. \alpha \epsilon^{-kt}, \quad g + k \cdot \frac{dy}{dt} = (g + ka \sin. \alpha)\, \epsilon^{-kt} \ldots (2).$$

Now integrate the first of these, and substitute in the second;

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* *Principia*, Book II, Prop. 4.

$$x = -\frac{a \cos. a \epsilon^{-kt}}{k} + \text{constant}$$

$$= \frac{a \cos. a\ (1-\epsilon^{-kt})}{k}, \text{ for } x=0, \text{ when } t=0;$$

$$\therefore\ 1-\epsilon^{-kt} = \frac{kx}{a \cos. a};\ \epsilon^{-kt} = 1 - \frac{kx}{a \cos. a}.$$

$$-kt = \text{hyp. log.} \left(1 - \frac{kx}{a \cos. a}\right);\ \therefore\ dt = \frac{dx}{a \cos. a - kx}.$$

Putting the values of dt and of ϵ^{-kt} in the second of equations (2),

$$g+k\ (a \cos. a-kx) \cdot \frac{dy}{dx} = (g+ka \sin. a) \left(\frac{a \cos. a-kx}{a \cos. a}\right);$$

$$\frac{dy}{dx} = \frac{g+ka \sin. a}{ka \cos. a} - \frac{g}{k} \cdot \frac{1}{a \cos. a-kx} \ldots\ldots (3).$$

Multiplying by dx and integrating, y being $=0$, when $x=0$,

$$y = \left(\tan. a + \frac{g}{ka \cos. a}\right) x - \frac{g}{k^2} \cdot \text{hyp. log.} \frac{a \cos. a}{a \cos. a - kx} \ldots (4),$$

the equation to the path of the projectile.

COR. 1. Expanding the logarithm, this becomes

$$y = \tan. a.x + \frac{gx}{ka\cos.a} - \frac{gx}{ka\cos.a} - \frac{1}{2} \cdot \frac{gx^2}{a^2\cos.^2a} - \frac{1}{3} \cdot \frac{gkx^3}{a^3\cos.^3a} + \text{\&c.}$$

where, omitting the terms which destroy each other, the two first terms represent a parabola, the curve described in a vacuum; and the remaining ones the alteration introduced by resistance.

COR. 2. It is manifest, that x cannot become greater than $\frac{a \cos. a}{k}$, for if it were so, $a \cos. a-kx$ would be negative, and its logarithm impossible. Hence, if we take $AB = \frac{a \cos. a}{k}$, a vertical line BD will be an asymptote to the descending branch.

COR. 3. If we suppose the curve to be continued backwards beyond A, it will not have an asymptote to the branch AO, but it will perpetually approximate to a certain angle of inclination with the horizon. To find this angle, make, in equation (3), x infinite and negative: we have thus,

$$\frac{dy}{dx} = \tan. a + \frac{g}{ka \cos. a},$$

for the ultimate position of the tangent at 0.

Cor. 4. If we draw AE parallel to this tangent, we have, since $AB = \frac{a \cos. a}{k}$;

$$BE = AB.\frac{dy}{dx} = \frac{a \sin. a}{k} + \frac{g}{k^2}.$$

Also $BC = AB. \tan. a = \frac{a \sin. a}{k}$. Hence, $CE = \frac{g}{k^2}$.

Also $AC = AB \sec. a = \frac{a}{k}$.

Hence, $AC : CE :: ka : g ::$ initial resistance : gravity*.

Cor. 5. To find when the body is highest, we must have $\frac{dy}{dx} = 0$; hence, by (3),

$$x = \frac{a^2 \sin. a \cos. a}{g + ka \sin. a}.$$

62. Prop. Let a body, acted on by gravity, be projected in a uniform medium, in which the resistance is as the square of the velocity: it is required to find the curve, fig. 68.

The co-ordinates are measured from the point of projection A, as before.

We have here $R = k . \frac{ds^2}{dt^2}$, and our equations become

* We shall have $PQ = CE$ hyp. log. $\frac{AB}{MB}$.

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -k.\frac{dsdx}{dt^2},\\ \frac{d^2y}{dt^2} &= -g-k\frac{dsdy}{dt^2}\end{aligned}\right\}\ldots\ldots(1).$$

The first gives

$$\frac{d^2x}{dt} = -kds.\frac{dx}{dt},$$

$$\frac{dx}{dt} = C.\epsilon^{-ks}\ \ldots\ldots\ldots\ldots\ldots\ (2).$$

C being a constant quantity, which is to be so taken that when $s=0$, $\frac{dx}{dt}$ may be the velocity of projection in a horizontal direction. Hence, if a be the velocity, and α the angle of projection BAC, we have

$$a\cos.\alpha = C;$$

$$\therefore \frac{dx}{dt} = a\cos.\alpha\epsilon^{-ks},\quad dt = \frac{\epsilon^{ks}dx}{a\cos.\alpha}.$$

Let at any point $dy=pdx$, whence p will be the trigonometrical tangent of the angle TPN, which the curve at P, or its tangent PT, makes with the horizontal line PN. Hence, we have

$$d^2y = dpdx + pd^2x,$$

which values, substituted in the second of our equations (1), give

$$\frac{dpdx}{dt^2} + p\frac{d^2x}{dt^2} = -g-k.\frac{pdsdx}{dt^2},$$

and by the first,

$$p\frac{d^2x}{dt^2} = -k\frac{pdsdx}{dt^2};$$

$$\therefore\ dpdx = -gdt^2\ldots\ldots\ldots(3),$$

and putting for dt its value already found, and omitting dx,

$$dp = -\frac{g\epsilon^{2ks}dx}{a^2.\cos.^2\alpha}\ldots\ldots\ldots\ldots\ldots(4).$$

This equation expresses the nature of the curve. It may be made integrable, by multiplying it by the equation

$$dx\sqrt{(1+p^2)} = ds, \text{ which gives}$$

$$dp\sqrt{(1+p^2)} = -\frac{g\epsilon^{2ks}ds}{a^2\cos.^2\alpha}\ldots\ldots\ldots(5).$$

Integrating this equation, we have

$$p\sqrt{(1+p^2)} + \text{hyp. log. } \{p+\sqrt{(1+p^2)}\} = C - \frac{g\epsilon^{2ks}}{ka^2\cos.^2\alpha}\ldots.(6).$$

When C is to be such, that when $s=0$, $p=\tan.\alpha = \frac{\sin.\alpha}{\cos.\alpha}$.

Hence,

$$\frac{\sin.\alpha}{\cos.^2\alpha} + \text{hyp. log. } \frac{\sin.\alpha+1}{\cos.\alpha} = C - \frac{g}{ka^2\cos.^2\alpha}.$$

We may eliminate ϵ^{2ks} between equations (4) and (6), and thus obtain dx in terms of p and dp; and, by integrating, x in terms of p: we have also $dy=p\,dx$, and hence, y in terms of p. Finally, we have $dt^2 = -\frac{dp\,dx}{g}$, and for dx we may substitute its value, ana thus by integrating, obtain t in p. If these integrations could be performed, and p eliminated, we should have a complete solution of the problem.

Cor. 1. If we make $k=0$ in the value of $dp\sqrt{(1+p^2)}$, we have an equation which belongs to a projectile in a non-resisting medium; the original velocity and direction of projection being the same as in this problem.

Let AP be the curve in a resisting, Ap in a non-resisting medium; and let these be taken, so that the tangents PT, pt are always parallel; therefore the values of p are the same in the two cases; $AP=s$, $Ap=S$. Then by (5),

$$-\frac{gdS}{a^2\cos.^2\alpha} = dp\sqrt{(1+p^2)}, \text{ making } k=0;$$

And $-\frac{g\epsilon^{2ks}ds}{a^2\cos.^2\alpha} = dp\sqrt{(1+p^2)}$. Hence, p being the same in both,

$$2\,k\,dS = \epsilon^{2ks}.2\,k\,ds;\ 2\,kS = \epsilon^{2ks} - 1,$$

supposing S and s to begin together.

And hence,

$$2\,ks = \text{hyp. log. } (1 + 2\,k\,S).$$

Cor. 2. The quantity p will be positive, and x will increase, to a point which is found by making $p = 0$. After that, p will be negative, and will become numerically greater and greater without limit, as the curve becomes more nearly vertical.

Cor. 3. The curve has a vertical asymptote BD, to the descending branch Pz.

To prove this, eliminate ϵ^{2ks} in (6) by means of (4), and we have

$$p\sqrt{(1+p^2)} + \text{hyp. log. } \{p + \sqrt{(1+p^2)}\} = C + \frac{dp}{k\,dx}.$$

Dividing by p^2,

$$\sqrt{\left(\frac{1}{p^2} + 1\right)} + \frac{\text{hyp. log. } \{p + \sqrt{(1+p^2)}\}}{p^2} = \frac{C}{p^2} + \frac{dp}{kp^2dx}.$$

Now, when p becomes very large and negative, all the terms on the first side become very small, except $\sqrt{1}$ *. Hence, this must ultimately be equal to the term on the other side; and we have, taking the $-$ value of $\sqrt{1}$, because p decreases as x increases,

$$-1 = \frac{dp}{kp^2dx}, \text{ and } dx = -\frac{dp}{kp^2}.$$

This, which is nearly true when p is very large, may be used in finding the value of x, in the interval between $p =$ a large quantity, and $p =$ infinity. And, as the value of x corresponding to the former quantity is not infinite, if the value in this interval be finite, the whole value of the abscissa will be finite, and the curve will have a vertical asymptote. Now by integrating, we have

* Hyp. log. $\{\sqrt{(1+p^2)} - p\}$ is infinitely small compared with p, when p is infinite; and *a fortiori* compared with p^2.

$$x = \text{const.} + \frac{1}{kp};$$

the constant being known by giving p and x, known finite values. And when p becomes infinite and negative, we have $x =$ constant, which is finite, and there is therefore a vertical asymptote.

Cor. 4. The curve has an oblique asymptote EH, to AO the ascending branch continued backwards.

Put, in the equation $2ks =$ hyp. log. $(1 + 2kS)$, $S = -\frac{1}{2k}$, and we shall have s infinite. Hence, if we take, in the parabola described in a non-resisting medium, an arc $Ao = \frac{1}{2k}$, and draw a tangent oe, the curve AO becomes ultimately parallel to oe.

To shew that there is an asymptote parallel to this line, we must prove that AH is always finite, H being the intersection of the tangent with the axis.

$$AH = \frac{y\,dx}{dy} - x; \quad d \,.\, AH = yd \,.\, \frac{dx}{dy} = -\frac{y\,dp}{p^2}.$$

Also, if the ultimate value of p be called n, we shall have ultimately, the abscissa being $-x$,

$$y = -nx, \quad s = -x\sqrt{(1+n^2)}; \text{ and by (4),}$$

$$dp = -\frac{g\epsilon^{2ks}dx}{a^2 \cos.^2 \alpha};$$

$\therefore\ AH = -\int \frac{y\,dp}{p^2}$, will be finite, if the integral be finite for the portion, when p becomes n, and when x and y become infinite.

That is, if $$\int \frac{gx\,dx}{a^2 \cos.^2 a \,.\, n} \,.\, \epsilon^{-2kx\sqrt{(1+n^2)}}$$

be finite when x is infinite; which it will be seen to be by integrating.

Cor. 5. Hence, it appears that the ascending and descending branches are not similar. The body rises more obliquely, and descends more vertically than it would do in a vacuum *.

Sect. II. *Any Force acting in parallel Lines.*

63. Prop. Let the force act parallel to the ordinate y, and be equal to P; to find the equations of motion †.

We shall here have

$$\frac{d^2x}{dt^2} = -R\frac{dx}{ds},\ \frac{d^2y}{dt^2} = -P - R\frac{dy}{ds}\ldots\ldots(1).$$

Hence, $$\frac{2dxd^2x + 2dyd^2y}{dt^2} = -2Pdy - 2Rds;$$

$$\text{or, } d\,.\frac{ds^2}{dt^2} = -2Pdy - 2Rds\ldots\ldots\ldots\ldots(2).$$

Also, by (1),

$$\frac{dxd^2y - dyd^2x}{dt^2} = -Pdx;$$

$$\text{or, } \frac{dx^2}{dt^2}d\,.\frac{dy}{dx} = -Pdx.$$

We may in this suppose dx constant; and differentiating on this supposition, we have

$$\frac{d^2y}{dt^2} = -P\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

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* For the determination of other circumstances in the curve, and for methods of constructing the curve approximately, see *Journal de l'Ec. Polyt.* tom. IV, p. 204. This part of Mechanics has been called *Ballistics.*

When the angle of elevation is small, we may modify our formulæ and obtain an approximation for that case.

For the path of the projectile when the resistance varies as $(\text{velocity})^{2n}$, see *John Bernoulli's* works, tom. II, p. 293. This is the problem which Keil proposed to foreign Mathematicians as a challenge, and of which he was unable to produce a solution when Bernoulli called upon him to compare it with his own.

† *Principia*, Book II. Prop. 10. This problem, which was erroneously solved in the first Edition of the Principia, gave rise to some of the most angry disputes in the controversy between the English and Foreign Mathematicians of that period.

Putting in (2) the value of dt^2 from (3), we have

$$d\,.\frac{Pds^2}{d^2y} = 2Pdy + 2Rds \cdots\cdots\cdots\cdots (4).$$

This is an equation to the curve, if the resistance be known, and conversely.

Cor. 1. If P be constant and $=g$, since $d\,.\,ds^2 = 2dyd^2y$, we have

$$2gdy - \frac{gds^2d^3y}{(d^2y)^2} = 2gdy + 2Rds;$$

$$\therefore R = -\frac{gdsd^3y}{2(d^2y)^2} \cdots\cdots\cdots\cdots (A).$$

Cor. 2. In this case, if R be as the square of the velocity,

$$\text{let } R = Q\frac{ds^2}{dt^2} = -Q\,.\,\frac{gds^2}{d^2y} \text{ by (3);}$$

$$\therefore Q = \frac{d^3y}{2\,ds\,d^2y} \cdots\cdots\cdots\cdots (B),$$

which gives the density.

Ex. 1. A body moves in a semi-circle, acted on by a constant force in parallel lines, the resistance varying as the density and square of the velocity; it is required to find the variation of the resistance and density, fig. 69.

If we measure x from C the centre, and call the radius a, we have

$$y = \sqrt{(a^2 - x^2)},$$

$$dy = -\frac{x\,dx}{\sqrt{(a^2-x^2)}};\ ds = \frac{a\,dx}{\sqrt{(a^2-x^2)}},$$

$$d^2y = -\frac{a^2\,dx^2}{(a^2-x^2)^{\frac{3}{2}}},$$

$$d^3y = -\frac{3\,a^2\,x\,dx^3}{(a^2-x^2)^{\frac{5}{2}}}.$$

Hence, by (A), $R = -\dfrac{gdsd^3y}{2(d^2y)^2} = \dfrac{3ga^3x}{2\,a^4} = \dfrac{3\,x}{2\,a}\cdot g.$

Also velocity$^2 = -\dfrac{g\,ds^2}{d^2y} = \dfrac{g(a^2-x^2)^{\frac{3}{2}}}{a^2}$.

Hence, density $= \dfrac{R}{\text{velocity}^2} = \dfrac{3ax}{2(a^2-x^2)^{\frac{3}{2}}}$.

Hence, the density $=0$ at B, when $x=0$; and is infinite at A. Between B and a the expression will be negative; and in order that the body may describe the arc aB, it must be *propelled* by the medium, which physically speaking, is absurd.

Ex. 2. Let the body, acted on by a constant and parallel force, move in a parabola of any order; it is required to find the resistance and density.

Let BN, fig. 69, be a tangent at the highest point; and let NP vary as any power of BN. Then if $CM=x$, $MP=y$, $CB=b$, we shall have $y=b-cx^n$; and differentiating

$$dy=-ncx^{n-1}dx,\ ds=dx\sqrt{(1+n^2c^2x^{2n-2})},$$
$$d^2y=-n(n-1)cx^{n-2}dx^2,\ d^3y=-n(n-1)(n-2)cx^{n-3}dx^3.$$

Then by (A), $R = \dfrac{n(n-1)(n-2)cx^{n-3}\sqrt{(1+n^2c^2x^{2n-2})}g}{2n^2(n-1)^2c^2x^{2n-2}}$

$$= \frac{(n-2)\sqrt{(1+n^2c^2x^{2n-2})}g}{2n(n-1)cx^{n+1}}.$$

If n be greater than 1, this value of R will manifestly become infinite, when $x=0$, and negative, when x is negative. Also if NP do not generally vary as some power of BN, yet if the curve be symmetrical with respect to a line BC, it is manifest that ultimately, when BN is small, NP may be considered as proportional to some power of it. Hence, any symmetrical curve will require an infinite resistance at the vertex B; and hence, conversely, the curve described in a medium, the density of which is every where finite, cannot be symmetrical with respect to its ascending and descending branches.

Let us now take an example of a curve not symmetrical.

Ex. 3. Let the curve OPQ, fig. 70, be an hyperbola of any order, of which one of the asymptotes CB is vertical; it is required to find the resistance and density at any point*.

* *Principia*, Book II, Prop. 10. Ex. 3.

Let $AB = a$, $BC = b$, $AM = x$, $MP = y$. And suppose NP to be as $\frac{1}{CN^n}$.

Also let $MB = z$, so that $a - z = x$.

$$MN = \frac{bx}{a} = b - \frac{bz}{a};$$

NP which varies as $\frac{1}{CN^n}$ varies as $\frac{1}{MB^n}$.

$$\text{Let } NP = \frac{c}{z^n};$$

$$\therefore\ y = b - \frac{bz}{a} - \frac{c}{z^n},$$

$$dy = -\left(\frac{b}{a} - \frac{nc}{z^{n+1}}\right) dz,$$

$$ds = -dz\sqrt{\left\{1 + \left(\frac{b}{a} - \frac{nc}{z^{n+1}}\right)^2\right\}}$$

$$= -\frac{dz}{z}\sqrt{\left\{z^2 + \left(\frac{bz}{a} + \frac{nc}{z^n}\right)^2\right\}},$$

$$d^2y = -\frac{n.(n+1)c\,dz^2}{z^{n+2}},$$

$$d^3y = \frac{n(n+1)(n+2)c\,dz^3}{z^{n+3}}.$$

Making these substitutions in the expression for the density, we have by (B)

$$Q = \frac{d^3y}{2ds\,d^2y} = \frac{n(n+1)(n+2)z^{n+2}}{2n(n+1)z^{n+3}} \cdot \frac{z}{\sqrt{\left\{z^2 + \left(\frac{bz}{a} - \frac{nc}{z^n}\right)^2\right\}}}$$

$$= \frac{n+2}{2\sqrt{\left\{z^2 + \left(\frac{bz}{a} - \frac{nc}{z^n}\right)^2\right\}}}.$$

Now this does not vanish for any value of z, except $z = 0$.

Hence, if the density be every where finite, the curve will approach more nearly to this form than to the symmetrical curves before-mentioned.

SECT. III. *Central Forces.*

64. PROP. A body moves in any resisting medium acted upon by any force tending to a centre; it is required to find the curve described.

Let, in fig. 71, $SM=x$, $MP=y$, be rectangular co-ordinates to the place P of the body. $SP=r$; and P the force, any function of r, and tending to the point S. Also let R be the resistance which acts in the direction of the curve, and depends upon the velocity, and on the density of the medium. And let s be the curve described.

The resolved parts of the force P are $P\frac{x}{r}$ and $P\frac{y}{r}$ in directions Pm parallel to MS, and PM. Also if the force R be resolved, the components in the same directions are $R\frac{dx}{ds}$, and $R\frac{dy}{ds}$; hence, we have (the body moving in the direction AP),

$$\frac{d^2x}{dt^2}=-\frac{Px}{r}-\frac{Rdx}{ds},\ \frac{d^2y}{dt^2}=-\frac{Py}{r}-\frac{Rdy}{ds}\ldots\ldots\ldots(1);$$

$$\therefore\ \frac{2dxd^2x+2dyd^2y}{dt^2}=-2P.\frac{xdx+ydy}{r}-2R.\frac{dx^2+dy^2}{ds};$$

or since $dx^2+dy^2=ds^2$, $2dxd^2x+2dyd^2y=d.ds^2$,

$$x^2+y^2=r^2,\ xdx+ydy=rdr,$$

$$d.\frac{ds^2}{dt^2}=-2Pdr-2Rds\ldots\ldots\ldots\ldots(2).$$

Again, the original equations give us

$$\frac{yd^2x-xd^2y}{dt^2}=-R.\frac{ydx-xdy}{ds}.$$

But if we call $ASP = v$, we have

$$y\,dx - x\,dy = r^2\,dv,\quad y\,d^2x - x\,d^2y = d\,.\,r^2\,dv;$$

$$\therefore\ \frac{d\,.\,r^2\,dv}{dt^2} = -\,R\,.\,\frac{r^2\,dv}{ds};\ \ldots\ldots.(3).$$

R is a function of $\frac{ds}{dt}$ the velocity, and of r, if the density vary with r. Hence, putting for R its value, we may eliminate $\frac{ds}{dt}$ between the equations (2), (3), and obtain an equation in r and v, which by integration will give the curve.

65. Prop. Under the same circumstances, it is required to find the relation between the radius vector (r), and the perpendicular upon the tangent (p),

$r^2\,dv = p\,ds$, and equation (3) gives

$$d\,.\,p\,ds = -\,R\,p\,dt^2,$$

$$\text{or } p\,d^2s + dp\,ds = -\,R\,p\,dt^2.$$

And equation (2) gives

$$ds\,d^2s = -\,P\,dr\,dt^2 - R\,ds\,dt^2.$$

Eliminating R, we have

$$ds^2\,dp = P\,p\,dr\,dt^2;\quad \frac{ds^2}{dt^2} = \frac{P\,p\,dr}{dp}\ldots\ldots\ldots.(C).$$

Let q be one-fourth the chord of curvature through S: then

$$\frac{p\,dr}{dp} = 2q,$$

$$\frac{ds^2}{dt^2} = 2Pq^*.$$

* Hence, the velocity in the curve is equal to that acquired by falling down one-fourth the chord of curvature by the action of the constant force P; the same property which obtains in a vacuum.

And substituting in equation (2) of last Article,

$$d \,.\, Pq = -\, Pdr - Rds;$$

$$\therefore\ R = -\,\frac{d \,.\, Pq + Pdr}{ds} {}^{*} \ldots\ldots\ldots\ldots (D).$$

When the relation between p and r is given, this formula enables us to find the resistance, and conversely.

Cor. 1. If $R = Q\,\frac{ds^2}{dt^2} = Q \,.\, 2\,Pq$; we shall have

$$Q = -\,\frac{d \,.\, Pq + Pdr}{2\,Pq\,ds} {}^{*} \ldots\ldots\ldots\ldots (E).$$

Ex. 1. To find the resistance in the circle, the centre of force being in the circumference, and the force as any power o the distance. The body moving towards the centre.

Here $p = \frac{r^2}{a}$, $P = \frac{m}{r^n}$,

$$\frac{dp}{p} = \frac{2\,dr}{r};\ q = \frac{p\,dr}{2\,dp} = \frac{r}{4};\quad ds = -\,\frac{dr}{\sqrt{\left(1 - \frac{r^2}{a^2}\right)}};$$

$$Pq = \frac{m}{4\,r^{n-1}};\ dPq + Pdr = -\,\frac{m\,(n-1)\,dr}{4\,r^n} + \frac{m\,dr}{r^n}$$

$$= -\,\frac{m\,(n-5)}{4\,r^n};$$

$$\therefore \text{ by } (D),\ R = \frac{m\,(5-n)}{4\,r^n}\sqrt{\left(1 - \frac{r^2}{a^2}\right)} = \frac{m\,(5-n)\sin.\theta}{4\,r^n};$$

θ being the angle which r makes with a line through the centre.

Also $\frac{ds^2}{dt^2} = 2\,Pq = \frac{m}{2\,r^{n-1}}$; $\therefore$ if $R = Q\frac{ds^2}{dt^2}$, $Q = \frac{(5-n)\sin.\theta}{2\,r}$.

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* *Principia*, Book II, Prop. 17, 18.

Ex. 2. To find the resistance in the logarithmic spiral; the force as any power.

$$\text{Here, } p = ar; \quad \therefore q = \frac{p\,dr}{2dp} = \frac{r}{2},$$

$$\frac{ds^2}{dt^2} = 2Pq = Pr;$$

$$\therefore R = -\frac{\frac{1}{2}d.Pr + Pdr}{ds}; \text{ or since } ds = -\frac{dr}{\sqrt{(1-a^2)}},$$

the negative sign indicating that the body is moving *towards* the centre;

$$R = \left(\frac{d.Pr}{2\,dr} + P\right).\sqrt{(1-a^2)}.$$

$$\text{And if } P = \frac{m}{r^n}; \quad Pr = \frac{m}{r^{n-1}};$$

$$R = \left\{-\frac{n-1}{2}.\frac{m}{r^n} + \frac{m}{r^n}\right\}\sqrt{(1-a^2)} = \frac{(3-n)}{2}.\frac{m}{r^n}\sqrt{(1-a^2)};$$

$$R = \frac{(3-n)}{2}.\frac{m}{r^n}\sqrt{(1-a^2)},$$

$$\text{and, if } R = Q.\text{velocity}^2 = Q.Pr = \frac{Qm}{r^{n-1}},$$

$$Q = \frac{(3-n)\sqrt{(1-a^2)}}{2r}.$$

Hence, the density is inversely as the distance from the centre*.

If $n > 3$, R and Q will become negative, and it will no longer be possible for a body to move in this spiral towards the centre; but if the motion be *from* the centre, we shall find in the same manner

$$Q = \frac{(n-3)\sqrt{(1-a^2)}}{2r}.$$

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* *Principia*, Book II, Prop. 15, 16.

From the expression for R, it appears that (fig. 72,)

$$R : \text{force to centre} :: \frac{3-n}{2} Py : Sy.$$

If $n=2$, this gives

$$R : \text{force to centre} :: Py : 2\,Sy.$$

Hence, when the force varies inversely as the square of the distance, and the density inversely as the distance, the resistance being small, a body may, as in fig. 72, approach the centre in a logarithmic spiral not much differing from a circle.

This motion requires a particular relation of the velocity and direction. If the velocity be not that which the direction requires, the path described will no longer be a logarithmic spiral. But if the resistance be small, the curve will still be a spiral, approaching the centre in successive oblong revolutions, as in fig. 73. And if the density, instead of being inversely as the distance, follow any other law, the orbit will still be of a similar form. We shall in a future Article consider the effect of resistance in altering an elliptical orbit.

66. Prop. The resistance varying as the square of the velocity and the density and the force being given; it is required to find the polar equation to the trajectory.

Taking the equations of Art. 64.

$$d\,.\frac{ds^2}{dt^2} = -\,2\,Pdr - 2\,Rds \cdots\cdots\cdots\cdots (2),$$

$$d\,.\frac{r^2dv}{dt^2} = -\,R\,\frac{r^2dv}{ds} \cdots\cdots\cdots\cdots\cdots\cdots (3).$$

Let $R = Q\dfrac{ds^2}{dt^2}$; $\therefore$ dt being constant, (3) becomes

$$d\,.r^2dv = -\,Q\,ds\,.r^2dv;$$

integrating, $r^2dv = hdt\epsilon^{-\int Qds}$,.. ($F$), h being a constant quantity.

$$\therefore\ \frac{ds}{dt} = \frac{hds\epsilon^{-\int Qds}}{r^2dv};\quad \frac{ds^2}{dt^2} = \frac{h^2ds^2\epsilon^{-2\int Qds}}{r^4dv^2};$$

$$d.\frac{ds^2}{dt^2} = -2\,Qds.\frac{h^2 ds^2 \epsilon^{-2\int Qds}}{r^4 dv^2} + h^2 \epsilon^{-2\int Qds}.d.\frac{ds^2}{r^4 dv^2}.$$

And by (2),

$$d.\frac{ds^2}{dt^2} = -2\,P dr - 2\,Qds.\frac{h^2 ds^2 \epsilon^{-2\int Qds}}{r^4 dv^2};$$

putting for R, $Q\frac{ds^2}{dt^2}$, and for $\frac{ds^2}{dt^2}$ its value.

Hence, equating,

$$h^2 \epsilon^{-2\int Qds} d.\frac{ds^2}{r^4 dv^2} = -2\,P dr,$$

$$d.\frac{ds^2}{r^4 dv^2} + \frac{2\,P dr \epsilon^{2\int Qds}}{h^2} = 0.$$

But if we make $\frac{1}{r} = u$; whence $\frac{dr}{r^2} = -du$,

$$\frac{ds^2}{r^4 dv^2} = \frac{dr^2 + r^2 dv^2}{r^4 dv^2} = \frac{du^2}{dv^2} + u^2;$$

and substituting, differentiating, and reducing,

$$\frac{d^2 u}{dv^2} + u - \frac{P\epsilon^{2\int Qds}}{h^2 u^2} = 0 \ldots\ldots\ldots (G).$$

This differs from the equation (d), Art. 18. for motion in a vacuum, only in having the last term multiplied by $\epsilon^{2\int Qds}$.

If Q be small, the equation (G) will enable us to approximate to the effect of resistance in altering the orbit, as will appear in the following Article.

67. Prop. A body acted upon by a central force, varying inversely as the square of the distance, moves in a medium of small density (Q); to find what alteration will be produced in the eccentricity, and in the place of the major axis.

By (F), $\quad r^2 dv = h dt \epsilon^{-\int Qds}$;

or, if h' be the area in time 1 at the end of time t,

$$r^2 dv = h' dt, \quad h' = h\epsilon^{-\int Qds} \ldots\ldots\ldots (1).$$

Also by (G), since $P = mu^2$,

$$\frac{d^2u}{dv^2} + u - \frac{m\epsilon^{2\int Q ds}}{h^2} = 0 \ldots\ldots\ldots (2);$$

$$\text{or} \quad \frac{d^2u}{dv^2} + u - \frac{m}{h'^2} = 0.$$

If h' were constant, we should find the integral as in Book I, Chap. III, Prob. 2, and it would be

$$u = C \cos. v + C' \sin. v + \frac{m}{h'^2} \ldots\ldots\ldots (3);$$

but since h' varies a little, u will differ a little from this form. We may suppose u to be still expressed by the preceding formula, C and C' being now taken variable, and determined so as to satisfy equation (2).

Differentiating (3), we have

$$\frac{du}{dv} = -C \sin. v + C' \cos. v + \cos. v \frac{dC}{dv} + \sin. v \frac{dC'}{dv} + \frac{d \cdot \frac{m}{h'^2}}{dv}.$$

But since we have yet only one condition to determine C, C', we may assume another: let therefore

$$\cos. v \frac{dC}{dv} + \sin. v \frac{dC'}{dv} + \frac{d \cdot \frac{m}{h'^2}}{dv} = 0 \ldots\ldots (4).$$

Hence, $\frac{du}{dv}$ will be reduced to its two first terms; and differentiating again,

$$\frac{d^2u}{dv^2} = -C \cos. v - C' \sin. v - \sin. v \frac{dC}{dv} + \cos. v \frac{dC'}{dv};$$

substituting in (2), it becomes

$$-\sin. v \frac{dC}{dv} + \cos. v \frac{dC'}{dC} = 0 \ldots\ldots\ldots\ldots (5).$$

Hence, by (4) and (5), we have

$$dC = -\cos. v . d . \frac{m}{h'^2}, \quad dC' = -\sin. v . d . \frac{m}{h'^2}.$$

Since by (1) $\frac{m}{h'^2} = \frac{m\epsilon^{2\int Q ds}}{h^2}$,

$$d\frac{m}{h'^2} = \frac{m}{h^2} . 2Q ds \epsilon^{2\int Q ds};$$

and if Q and $\int Q ds$ be small, so that $\epsilon^{2\int Q ds}$ may be considered equal to unity; $d\frac{m}{h'^2} = \frac{m}{h^2} . 2Q ds$.

Hence, we have

$$dC = -\frac{m}{h^2} . 2Q \cos. v . ds, \quad dC' = -\frac{m}{h^2} . 2Q \sin. v . ds \ldots . (6).$$

Now, if in equation (3), C, C', and h be supposed to belong to the beginning of a revolution, we may make $C = c \cos. a$, $C' = c \sin. a$, and we shall have

$$u = \frac{m}{h^2} + c \cos. (v - a) = \frac{m}{h^2} \left\{1 + \frac{ch^2}{m} \cos. (v - a)\right\},$$

which agrees with the equation to an ellipse

$$u = \frac{a}{b^2} \{1 + e \cos (v - a)\}, \text{ if } \frac{m}{h^2} = \frac{a}{b^2}, \text{ and } \frac{ch^2}{m} = e.$$

And we shall find the variation of a and of e in the course of a revolution in the following manner.

Make $C = c' \cos. a'$, $C' = c' \sin. a'$, supposing c', a', variable; and we have

$$\cos. a' dC' - \sin. a' dC = c' da',$$
$$\sin. a' dC' + \cos. a' dC = dc'.$$

Hence, putting for dC, dC' the values from (6),

$$\left.\begin{aligned} c' da' &= -\frac{m}{h^2} 2Q \sin. (v - a') ds \\ dc' &= -\frac{m}{h^2} 2Q \cos. (v - a') . ds \end{aligned}\right\} \ldots\ldots\ldots (7).$$

For a first approximation we may suppose the expression for r and for ds to be the same as in an ellipse; and Q being any function of the distance, it will easily be seen, that, supposing e small, and neglecting powers of it, we have

$$Qds = \{A + eB \cos.(v - \alpha')\}\, dv,$$

A, B being constant quantities. Hence, by (7),

$$\left.\begin{aligned} c'd\alpha' &= -\frac{m}{h^2}\{2A \sin.(v-\alpha') + eB \sin. 2(v-\alpha')\}\, dv \\ dc' &= -\frac{m}{h^2}\{2A \cos.(v-\alpha') + eB\,[1 + \cos. 2(v-\alpha')]\}\, dv \end{aligned}\right\} \ldots (8).$$

In order to find the effect in a whole revolution we must integrate these expressions through 2π. And the values when $v = \beta$ being α, c, e, $\frac{m}{h^2}$, let them when $v = 2\pi + \beta$ be $\alpha + \Delta\alpha$, $c + \Delta c$, $e + \Delta e$, $\frac{m}{h^2} + \Delta\frac{m}{h^2}$, then we shall have from the first of equations (8) making c' and α' constant in the integration on the right hand side,

$$c'\Delta\alpha = 0.$$

Hence, $\Delta\alpha = 0$, and the position of the major axis is not altered after a whole revolution.

From the second of equations (8), making α' constant in the integration,

$$\Delta c = -\frac{m}{h^2} \cdot 2\pi e B.$$

But since $c = \frac{me}{h^2}$, we have $\Delta c = \frac{m\Delta e}{h^2} + e\Delta\frac{m}{h^2}$, neglecting powers of Δe, &c.;

$$\text{also, } \frac{m}{h'^2} = \frac{m}{h^2}\, e^{2\int Qds} = \frac{m}{h^2} \cdot 2\int Qds$$

$$= \frac{2m}{h^2}\int \{A + eB \cos.(v - \alpha')\}\, dv.$$

Hence, the integral being taken through 2π, we have

$$\Delta\frac{m}{h^2} = \frac{2m}{h^2} \cdot 2\pi A\,;$$

$$\therefore \frac{m}{h^2}\Delta e + e\frac{2m}{h^2} \cdot 2\pi A = -\frac{m}{h^2} 2\pi e B,$$

$$\Delta e = -\,2\pi e\,(2A + B).$$

Hence, it appears that by the effect of the resistance of a medium of small density the eccentricity diminishes perpetually, while the major axis contains the same position, see fig. 73.

Also, if we consider the remaining terms in Qds, which will be of the form, E cos. $2\,(v-\alpha')$, F cos. $3\,(v-\alpha')$, &c. the expressions in (8) arising from these, will, when integrated through 2π, become $=0$; hence, the truth of our result does not depend on supposing e small.

CHAP. III.

THE CONSTRAINED MOTION OF A POINT ON A GIVEN CURVE IN A RESISTING MEDIUM.

68. THIS case might be treated in the same manner as those in the last Chapter, by considering in addition to other forces, the re-action of the curve in the direction of its normal, and resolving this in the direction of the co-ordinates. It will, however, generally be more simple to consider only the forces along the curve. It has already been seen, that the alteration of velocity in the curve is entirely due to the resolved parts of the forces in the direction of the curve ; and the same is true in a resisting medium, if we con-

sider the resistance, which is wholly in the same direction. Hence, we may apply the formulæ for rectilinear motion to the motion along the curve, as will be seen in the following examples.

PROB. I. *A body moves on an inverted cycloid, with a vertical axis, in a medium, the resistance of which is as the velocity; to determine the motion, fig.* 74.

Let s be the arc AP from the lowest point A. Then the force of gravity in the direction of the curve is as AP; and if $AC=l$, force at $P=\frac{gs}{l}$, g being gravity; call this fs. Also, v being the velocity, let the resistance be $2kv$, when $2k$ is used rather than k, to avoid fractions, as will be seen. Then, by the formulæ, $dv=\text{force}\,.\,dt$, we shall have, when the body *descends*,

$$dv=(fs-2kv)\,dt;$$

and the same expression is true for the *ascent*: for s becomes negative, and the term which arises from gravity changes its sign as the force changes its direction. Now we have $v=-\frac{ds}{dt}$; for s decreases as t increases. Hence, making dt constant, we have

$$-\frac{d^2s}{dt}=\left(fs+2k\frac{ds}{dt}\right)dt;$$

$$\therefore \frac{d^2s}{dt^2}+2k.\frac{ds}{dt}+fs=0.$$

We shall obtain a particular integral of this by making $s=\epsilon^{mt}$, m being constant;

$$\therefore m^2+2km+f=0,$$

$$m=-k\pm\sqrt{(-1)}\,.\,\sqrt{(f-k^2)};\text{ or if } f-k^2=h^2,$$

$$m=-k\pm h\sqrt{(-1)}.$$

Hence, particular integrals are

$$s=\epsilon^{-[k-h\sqrt{(-1)}]t},\quad s=\epsilon^{-[k+h\sqrt{(-1)}]t};$$

and the complete integral

$$s = C_1 \epsilon^{-[k-h\sqrt{(-1)}]t} + C_2 \epsilon^{-[k+h\sqrt{(-1)}]t}$$

$$= \epsilon^{-kt} \{C_1 \epsilon^{ht\sqrt{(-1)}} + C_2 \epsilon^{-ht\sqrt{(-1)}}\}$$

$$= \epsilon^{-kt} \{C_1[\cos. ht + \sqrt{(-1)}.\sin. ht] + C_2[\cos. ht - \sqrt{(-1)}\sin. ht]\}.$$

And if $C_1 + C_2 = C$, $(C_1 - C_2)\sqrt{(-1)} = C'$,

$$s = \epsilon^{-kt}(C \cos. ht + C' \sin. ht).$$

C and C' must be determined by considering, that at the beginning of the motion, when we shall suppose $t = 0$, we have $s = a =$ the arc AD; and the velocity $\frac{ds}{dt} = 0$.

$$\text{Now } \frac{ds}{dt} = \epsilon^{-kt} \{(C'h - Ck) \cos. ht - (Ch + C'k) \sin. ht\}.$$

Hence, when $t = 0$,

$$a = C,$$

$$0 = C'h - Ck; \therefore C' = \frac{kC}{h} = \frac{ka}{h};$$

$$\therefore s = \frac{a\epsilon^{-kt}}{h}.(h \cos. ht + k \sin. ht),$$

$$\frac{ds}{dt} = -\frac{a(h^2 + k^2)}{h}\epsilon^{-kt} \sin. ht = -\frac{af}{h}.\epsilon^{-kt} \sin. ht.$$

If the body descend, and then ascend till its whole velocity is destroyed, it will have performed one oscillation. This will be when $\frac{ds}{dt} = 0$, or $\sin. ht = 0$, which will first be the case when $ht = \pi$. Hence, if T be the time of an oscillation

$$T = \frac{\pi}{h} = \frac{\pi}{\sqrt{(f - k^2)}} = \frac{\pi}{\sqrt{\left(\frac{g}{l} - k^2\right)}}.$$

This is independent of a, and hence the oscillations occupy the same time whatever be the length of the arc. The *cycloid*, which is the *tautochronous* curve *in vacuo*, also possesses that property in a medium whose resistance is proportional to the velocity*.

Cor. 1. *To find where the velocity is the greatest.* This will be the case when the accelerating force is 0; that is, when $fs + 2k \cdot \frac{ds}{dt} = 0$.

$$\text{Or, } f(h \cos. ht + k \sin. ht) - 2kf \sin. ht = 0;$$

$$\therefore \tan. ht = \frac{h}{k}.$$

Cor. 2. *To find the decrement of the arc*, that is, the difference of the arcs of descent and ascent. At the extremity of the arc of ascent we shall have $\frac{ds}{dt} = 0$, and $\therefore ht = \pi$. If we suppose the arc of ascent to be numerically represented by b, we shall have, putting $-b$ for s, and $\frac{\pi}{h}$ for t,

$$b = a\epsilon^{-\frac{k\pi}{h}};$$

$$\therefore (a-b) = a\left\{1 - \epsilon^{-\frac{k\pi}{h}}\right\},$$

and if k be small, neglecting its powers

$$a - b = a \cdot \frac{k\pi}{h} = \frac{\pi k a}{\sqrt{\left(\frac{g}{l} - k^2\right)}}.$$

Cor. 3. The body, after reaching the extremity of the arc of ascent, will again descend and ascend on the first side. Let the arc through which it ascends be a_1; we have then

$$a_1 = b\epsilon^{-\frac{k\pi}{h}} = a\epsilon^{-\frac{2k\pi}{h}}.$$

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* *Principia*, Book II. Prop. 26.

If the body make $2(n+1)$ oscillations, and return to distance a_n on the first side, we shall have

$$a_n = a_{n-1} \cdot \epsilon^{-\frac{2k\pi}{h}} = a_{n-2}\epsilon^{-\frac{4k\pi}{h}} = a\epsilon^{-\frac{2nk\pi}{h}}.$$

Hence, if a and a_n be observed, we may find $\frac{k}{h}$, and thence k.

PROB. II. *A body moves in a cycloid as before, the resistance being as the square of the velocity; to determine the motion*.*

Let the force of gravity in the curve $= \frac{gs}{l} = fs$; resistance $= kv^2$; and in descent, by the formulæ $vdv = -$ force . ds,

$$vdv = -(fs - kv^2)\,ds,$$

which is also true in the ascent as in last Problem.

$$\text{Let } v^2 = 2z; \therefore vdv = dz;$$

$$dz = -(fs - 2kz)\,ds;$$

$$dz - z\,.\,2kds = -fsds. \quad \text{Multiply by } \epsilon^{-2ks};$$

$$\therefore dz\epsilon^{-2ks} - z\epsilon^{-2ks}\,.\,2kds = -f\epsilon^{-2ks}sds.$$

Integrating both sides

$$z\epsilon^{-2ks} = C - f\int \epsilon^{-2ks}sds$$

$$= C + \frac{fs\epsilon^{-2ks}}{2k} - \frac{f}{2k}\int \epsilon^{-2ks}ds$$

$$= C + \frac{fs\epsilon^{-2ks}}{2k} + \frac{f\epsilon^{-2ks}}{4k^2},$$

$$z = C\epsilon^{2ks} + \frac{f}{4k^2}(2ks+1).$$

Now at the beginning of the motion, let $s = a$, and $v = 0$, and $\therefore z = 0$;

$$\therefore 0 = C\epsilon^{2ka} + \frac{f}{4k^2}(2ka+1);$$

* *Principia*, Book II. Prop. 29.

$$\therefore C = -\frac{f\epsilon^{-2ka}}{4k^2}\{2ka+1\},$$

$$z = \frac{f}{4k^2}\{2ks+1-(2ka+1)\epsilon^{-2k(a-s)}\}.$$

Let the body perform a whole oscillation, and let b be the numerical value of the arc of ascent. Therefore putting $-b$ for s, and 0 for v, and therefore for z;

$$0 = -2kb+1-(2ka+1)\epsilon^{-2k(a+b)},$$

$$\text{whence } (1-2kb)\epsilon^{2kb} = (1+2ka)\epsilon^{-2ka};$$

from which b may be found from a.

Cor. 1. *To find the decrement of the arc of ascent.*

The equation last found may be put in this form,

$$\frac{1-2kb}{1+2ka} = \epsilon^{-2k(a+b)}; \text{ and expanding,}$$

$$\left.\begin{array}{l} 1-2ka+4k^2a^2-8k^3a^3+\&c. \\ -2kb+4k^2ab-8k^3a^2b+\&c. \end{array}\right\} =$$

$$1-2k(a+b)+\frac{4k^2(a+b)^2}{1\,.\,2}-\frac{8k^3(a+b)^3}{1\,.\,2\,.\,3}+\&c.$$

which gives, omitting the identical terms, and dividing by $4k^2$,

$$a(a+b)-2ka^2(a+b)+\&c. = \frac{(a+b)^2}{1\,.\,2}-\frac{2k(a+b)^3}{1\,.\,2\,.\,3}+\&c.$$

Dividing by $(a+b)$, &c. we have, omitting k^2, &c.

$$b = a-\frac{2k}{3}(5a^2-2ab-b^2).$$

Hence, the decrement of the arc $= \frac{2k}{3}(5a^2-2ab-b^2)$; or,

$$(\text{since } b=a \text{ nearly,}) = \frac{4ka^2}{3}, \text{ nearly.}$$

Cor. 2. *To find where the velocity is the greatest.*

We must have $\frac{dv}{ds}=0$, and therefore $\frac{dz}{ds}=0$;

$$\therefore 2k-2k(2ka+1)\epsilon^{-2k(a-s)}=0;$$

$$\therefore \epsilon^{2k(a-s)} = 1+2ka,\quad 2ka-2ks = \text{hyp. log. }(1+2ka).$$

If we expand, and neglect powers of k, we shall find

$$s = k\,a^2.$$

Prob. III. *A body oscillates in a circle, the resistance being as the square of the velocity; to find the velocity at any point.*

If, in fig. 75, the radius $CP = l$, arc $AP = s$, $ACP = \theta$, we shall have the force of gravity along the curve $= g \sin.\theta$, and the resistance

$$= k\,.\frac{ds^2}{dt^2}.$$

Hence, since $\frac{d^2 s}{dt^2} =$ force in direction of the curve, we have

$$\frac{d^2 s}{dt^2} = - g \sin.\theta + k\frac{ds^2}{dt^2}.$$

Or because $s = l\theta$,

$$l\frac{d^2\theta}{dt^2} = - g \sin.\theta + k l^2 . \frac{d\theta^2}{dt^2}.$$

Which is true both for ascent and descent.

This may be thus integrated. Multiply by $\frac{2d\theta}{l}$ and integrate, and we have

$$\frac{d\theta^2}{dt^2} = \frac{2g\cos.\theta}{l} + 2kl\int\frac{d\theta^2}{dt^2}\,d\theta.$$

Let $\int\frac{d\theta^2}{dt^2}\,d\theta = z$; $\therefore \frac{d\theta^2}{dt^2} = \frac{dz}{d\theta}$,

$$\frac{dz}{d\theta} = \frac{2g\cos.\theta}{l} + 2klz.$$

Which is a linear equation of the first order, and may be integrated. Transpose, and multiply by $d\theta\epsilon^{-2kl\theta}$, and we have

$$dz\epsilon^{-2kl\theta} - z\epsilon^{-2kl\theta}2kld\theta = \frac{2g}{l}.\epsilon^{-2kl\theta}\cos.\theta\,.d\theta.$$

Where both sides are integrable. Therefore

$$z\epsilon^{-2kl\theta} = \frac{2g}{l}\int\epsilon^{-2kl\theta}\cos.\theta\,d\theta + C.$$

$$\text{Now} \int \epsilon^{-m\theta} \cos.\theta d\theta = -\frac{\epsilon^{-m\theta}\cos.\theta}{m} - \frac{1}{m}\int \epsilon^{-m\theta}.\sin.\theta d\theta$$

$$= -\frac{\epsilon^{-m\theta}\cos.\theta}{m} - \frac{1}{m}\left\{-\frac{\epsilon^{-m\theta}\sin.\theta}{m} + \frac{1}{m}\int \epsilon^{-m\theta}\cos.\theta d\theta\right\};$$

$$\therefore \left(1 + \frac{1}{m^2}\right)\int \epsilon^{-m\theta}\cos.\theta\, d\theta = \frac{\epsilon^{-m\theta}}{m^2}\{\sin.\theta - m\cos.\theta\};$$

$$\therefore \int \epsilon^{-2kl\theta}\cos.\theta\, d\theta = \frac{\epsilon^{-2kl\theta}}{1+4k^2l^2}\{\sin.\theta - 2kl\cos.\theta\}.$$

And hence our equation becomes

$$z = \frac{2g}{l}.\frac{\sin.\theta - 2kl\cos.\theta}{1+4k^2l^2} + C\epsilon^{2kl\theta}.$$

$$\text{Hence, } \frac{dz}{d\theta} = \frac{2g}{l}\frac{\cos.\theta + 2kl\sin.\theta}{1+4k^2l^2} + 2klC\epsilon^{2kl\theta} = \frac{d\theta^2}{dt^2}.$$

And hence, the angular velocity is known.

If a be the value of θ at the beginning of the motion, we have

$$\frac{2g}{l}.\frac{\cos.a + 2kl\sin.a}{1+4k^2l^2} + 2klC\epsilon^{2kla} = 0;$$

$$\therefore \frac{d\theta^2}{dt^2} = \frac{2g}{l}\left\{\frac{\cos.\theta + 2kl\sin.\theta - (\cos.a + 2kl\sin.a)\epsilon^{-2kl(a-\theta)}}{1+4k^2l^2}\right\}.$$

COR. 1. If we make $2kl = \tan.\beta$, we shall have

$$\frac{d\theta^2}{dt^2} = \frac{2g\cos.\beta}{l}.\{\cos.(\theta-\beta) - \cos.(a-\beta)\epsilon^{-(a-\theta)\tan.\beta}\}.$$

$$\text{Here } \tan.\beta = \frac{k.2gl}{g} = \frac{\text{resistance to velocity acquired down } l}{\text{gravity}}.$$

COR. 2. If we make $a = \frac{\pi}{2} + \beta$, this becomes

$$\frac{d\theta^2}{dt^2} = \frac{2g\cos.\beta}{l}\cos.(\theta-\beta).$$

Hence, we have this curious property. If in fig. 76, we take AT horizontal, such that $AC : AT ::$ gravity : resistance to velocity acquired down AC, and make DE a quadrant, the body, setting off from D, will move as if it were acted on only by a uniform force $g\cos.\beta$ in a direction parallel to AT.

In this case, the angular velocity is greatest at E, when $\theta = \beta$, and vanishes again when $\theta - \beta = -\frac{\pi}{2}$, or $\theta = -\left(\frac{\pi}{2} - \beta\right)$.

The whole arc described $DA\,d$ is a semi-circle, and the decrement of the arc of ascent is 2β.

Cor. 3. Generally, to find when the angular velocity is greatest; the expression for $\frac{d\theta^2}{dt^2}$ must be a maximum. Therefore

$$-\sin.(\theta - \beta) - \cos.(a - \beta)\tan.\beta\,\epsilon^{-(a-\theta)\tan.\beta} = 0.$$

And if the resistance, and therefore tan. β, be very small,

$$\sin.(\theta - \beta) = -\cos.(a - \beta)\tan.\beta.$$

Or, putting $\theta - \beta$, and β, for $\sin.(\theta - \beta)$, and tan. β,

$$\theta = \beta\,\{1 - \cos.(a - \beta)\} = 2\beta\,.\sin.^2\frac{a - \beta}{2}$$

$$= 2\beta\,.\sin.^2\frac{a}{2}, \text{ nearly.}$$

Prob. IV. *To find the time of oscillation in a cycloidal arc, in a medium, the resistance of which is as the square of the velocity.*

We have, (see Prob. II,)

$$v^2 = 2z = \frac{f}{2k^2}\,\{2ks + 1 - (2ka + 1)\epsilon^{-2k(a-s)}\}.$$

Put n for $2k$, and y for $a - s$, whence, $s = a - y$, and

$$v^2 = \frac{2f}{n^2}\,\{1 + ns - (1 + na)\,\epsilon^{-ny}\}.$$

Expanding the exponential, we have

$$v^2 = \frac{2f}{n^2}\left\{\frac{n^2}{1\,.\,2}(2ay - y^2) - \frac{n^3}{1\,.\,2\,.\,3}(3ay^2 - y^3)\right.$$

$$\left. + \frac{n^4}{1\,.\,2\,.\,3\,.\,4}(4ay^3 - y^4) - \&c.\right\}$$

$$= f\,\{2ay - y^2 - \frac{n}{3}(3ay^2 - y^3) + \frac{n^2}{3\,.\,4}(4ay^3 - y^4) - \&c.\}$$

$$= f\,\{A - Bn + Cn^2 - \&c.\} \text{ suppose;}$$

$$\therefore \frac{1}{v} = \frac{1}{\sqrt{f}}\{A - Bn + Cn^2 - \&c.\}^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{f}}\left\{\frac{1}{A^{\frac{1}{2}}} + n \cdot \frac{B}{2A^{\frac{3}{2}}} + n^2\left(\frac{1 \cdot 3 \cdot B^2}{2 \cdot 4 \cdot A^{\frac{5}{2}}} - \frac{C}{2A^{\frac{3}{2}}}\right) + \&c.\right\}.$$

And $dt = -\frac{ds}{v} = \frac{dy}{v}$; whence, t will be found by integrating each term of the above series multiplied by dy.

To find the time of descending to the lowest point, (down DA, fig. 74), we must take the integrals from $y=0$, to $y=a$.

Now $\int \frac{dy}{A^{\frac{1}{2}}} = \int \frac{dy}{\sqrt{(2ay-y^2)}} = \text{arc}\left(\text{ver. sin.} = \frac{y}{a}\right)$.

And to the lowest point $= \frac{\pi}{2}$.

Also $\int \frac{B\,dy}{2A^{\frac{3}{2}}} = \int \frac{(3ay^2-y^3)dy}{6(2ay-y^2)^{\frac{3}{2}}} = \frac{y^2}{6(2ay-y^2)^{\frac{1}{2}}}$,

and to the lowest point $= \frac{a}{6}$.

Again, $\int\left\{\frac{1 \cdot 3\,B^2}{2.4.A^{\frac{5}{2}}} - \frac{C}{2A^{\frac{3}{2}}}\right\}dy = \int \frac{3B^2 - 4AC}{8A^{\frac{5}{2}}} \cdot dy$

$$= \int \frac{a^2y^4\,dy}{24(2ay-y^2)^{\frac{5}{2}}} = \frac{a^2}{18}\,\frac{2y^3 - 3ay^2}{(2ay-y^2)^{\frac{3}{2}}}$$

$$+ \frac{a^2}{24}\,\text{arc}\left(\text{ver. sin.} = \frac{y}{a}\right),$$

and, to the lowest point $= \frac{a^2}{24} \cdot \frac{\pi}{2} - \frac{a^2}{18}$.

Hence, omitting terms involving n^3, &c., we have

$$\text{time down } DA = \frac{1}{\sqrt{f}}\left\{\frac{\pi}{2} + \frac{na}{6} + \frac{n^2a^2}{24}\left(\frac{\pi}{2} - \frac{4}{3}\right)\right\}.$$

The expression for the time up AD would be the same, except that the resistance acts in the opposite direction, and consequently it will be had by making n negative. Hence, if AE be b, we shall have

$$\text{time up } AE = \frac{1}{\sqrt{f}}\left\{\frac{\pi}{2} - \frac{nb}{6} + \frac{n^2 b^2}{24}\left(\frac{\pi}{2} - \frac{4}{3}\right)\right\}.$$

And the time of an oscillation through DAE

$$= \frac{1}{\sqrt{f}}\left\{\pi + \frac{n(a-b)}{6} + \frac{n^2(a^2+b^2)}{24}\left(\frac{\pi}{2} - \frac{4}{3}\right)\right\}.$$

And it has already been seen that

$$b = a - \frac{4ka^2}{3} = a - \frac{2na^2}{3};$$

hence, the time of an oscillation

$$= \frac{1}{\sqrt{f}}\left\{\pi + \frac{n^2 a^2}{9} + \frac{n^2 a^2}{12}\left(\frac{\pi}{2} - \frac{4}{3}\right)\right\}$$

$$= \frac{\pi}{\sqrt{f}}\left\{1 + \frac{n^2 a^2}{24}\right\} = \frac{\pi}{\sqrt{f}}\left\{1 + \frac{k^2 a^2}{6}\right\}.$$

The quantity f is $\frac{g}{l}$, l being the length of the pendulum; and $\frac{\pi}{\sqrt{f}}$ is the time of an oscillation in a vacuum. Hence it appears, that when the arc described (a) is small, the time of oscillation is the same as in a vacuum very nearly.

Cor. 1. The arcs described become smaller and smaller, in consequence of the resistance, and the times of the smaller oscillations are somewhat shorter.

The excess of the time in the medium above that in the vacuum, is as the square of the arc described *.

Cor. 2. The same things are true for small circular oscillations †.

* Hence, Cor. 2, to Prop. 27, Book II, of the *Principia*, is erroneous. It is there asserted, that the excesses of the times above those in a vacuum, are as the arcs themselves.

† For the direct proof of this Proposition in the case of circular arcs, see Poisson, *Traité de Mec.* Art. 273.

PROB. V. *A body oscillates in a cycloidal arc: the resistance being small, and varying as any power of the velocity; it is required to find the decrement of the arc of ascent.*

It will be shewn, that if the resistance vary as the n^{th} power of the velocity, the decremental arc will be as the n^{th} power of the arc of descent*.

Let resistance $= kv^n$; the other denominations as before;

$$\therefore\ vdv = -(fs - kv^n)\,ds,$$

$$\frac{v^2}{2} = f(a^2 - s^2) + k\int v^n\,ds.$$

And $v^2 = 2f(a^2 - s^2)$ nearly, since k is small;

$$\therefore \int v^n ds = (2f)^{\frac{n}{2}} \int (a^2 - s^2)^{\frac{n}{2}} ds, \text{ nearly.}$$

$$= (2f)^{\frac{n}{2}} \int \left\{a^n - \frac{n}{2} a^{n-2} s^2 + \frac{n(n-2)}{2.4} a^{n-4} s^4 - \&c.\right\} ds$$

$$= (2f)^{\frac{n}{2}} \left\{a^n s - \frac{na^{n-2} s^3}{2.3} + \frac{n(n-2)\,a^{n-4} s^5}{2.4.5} - \&c.\right\} + C;$$

$$\therefore\ v^2 = 2f(a^2 - s^2) + 2k(2f)^{\frac{n}{2}} \left\{a^n s - \frac{na^{n-2} s^3}{2.3} + \&c.\right\} + C.$$

Now $v = 0$, when $s = a$. And when v becomes 0 again, suppose $s = -a + \delta$, and neglecting powers of δ, and products $k\delta$, we have

$$0 = 2f.2a\delta + 4k(2f)^{\frac{n}{2}} \left\{- a^{n+1} + \frac{na^{n+1}}{2.3} - \&c.\right\};$$

$$\therefore\ \delta = 2ka^n(2f)^{\frac{n}{2}-1} \left\{1 - \frac{n}{2.3} + \frac{n(n-2)}{2.4.5} - \&c.\right\}.$$

And hence, δ is as a^n.

COR. If δ be known, k may be found†.

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* *Principia*, Book II, Prop. 31. Cor. † Ibid. Prop. 30. Cor.

CHAP. IV.

INVERSE PROBLEMS RESPECTING THE MOTION OF POINTS ON CURVES IN RESISTING MEDIA.

69. **Here**, as in the corresponding Chapter in the first Book, the curve is to be determined from some property of it which is given. Most of the problems, however, which occur, present considerable difficulties, and lead to very complicated calculations. We shall therefore take only one of the most remarkable and celebrated of the questions of this kind. The solution is nearly that given by Laplace, *Mecanique Celeste*, Prem. Par. Liv. I, No. 12.

Prop. To find the tautochronous curve in a medium, of which the resistance varies partly as the velocity, and partly as the square of the velocity; the body being acted on by gravity.

In fig. 74, let the vertical abscissa from the lowest point $AM = x$, $AP = s$; the velocity $= v$, and the resistance $= 2hv + kv^2$. The resolved part of gravity in the curve will be $g \cdot \frac{dx}{ds}$, and hence, making dt constant, for the descent

$$\frac{d^2 s}{dt^2} = -g \cdot \frac{dx}{ds} + (2hv + kv^2);$$

or since $v = -\frac{ds}{dt}$,

$$\frac{d^2 s}{dt^2} + g \cdot \frac{dx}{ds} + 2h \cdot \frac{ds}{dt} - k\frac{ds^2}{dt^2} = 0 \ldots\ldots\ldots (1).$$

Similarly, in ascent,

$$\frac{d^2 s}{dt^2} + g\,\frac{dx}{ds} + 2h\,\frac{ds}{dt} + k\,\frac{ds^2}{dt^2} = 0 \;\cdots\cdots\cdot(2),$$

s being positive.

To integrate equation (1). Let $s = -\frac{1}{k}$ hyp. log. $(1-z)$;

$$\therefore\ \frac{ds}{dt} = \frac{1}{k}\cdot\frac{1}{1-z}\cdot\frac{dz}{dt},$$

$$\frac{d^2 s}{dt^2} = \frac{1}{k}\cdot\frac{1}{(1-z)^2}\cdot\frac{dz^2}{dt^2} + \frac{1}{k}\cdot\frac{1}{1-z}\cdot\frac{d^2 z}{dt^2}.$$

Substituting these values, we have

$$\frac{1}{k}\cdot\frac{1}{1-z}\,\frac{d^2 z}{dt^2} + gk\,(1-z)\,\frac{dx}{dz} + \frac{2h}{k}\,\frac{1}{1-z}\cdot\frac{dz}{dt} = 0;$$

$$\therefore\ \frac{d^2 z}{dt^2} + 2h\,\frac{dz}{dt} + k^2 g\,(1-z)^2\cdot\frac{dx}{dz} = 0 \cdots\cdots\cdots(3).$$

Now, suppose the last term of this equation to be expanded in a series of powers of z. It will not involve any power of z below z^1; for at A, where $s = 0$, and therefore $z = 0$, we must have

$$x \propto s^2, \text{ and } \therefore\ x \propto z^2;\ \therefore\ \frac{dx}{dz} \propto z.$$

Let therefore

$$k^2 g\,(1-z)^2\,\frac{dx}{dz} = Az + Bz^{\beta} + \&c. \text{ where } \beta > 1 \cdots\cdots\cdots(4);$$

$$\therefore\ \frac{d^2 z}{dt^2} + 2h\,\frac{dz}{dt} + Az + Bz^{\beta} + \&c. = 0.$$

Let T be the time of descending to the lowest point, and put $T - t = t'$. Then our equation becomes

$$\frac{d^2 z}{dt'^2} - 2h\,\frac{dz}{dt'} + Az + Bz^{\beta} + \&c. = 0 \cdots\cdots\cdots\cdots(5).$$

If we omit the terms after Az, we can integrate this equation, and find a factor which will make the first part of the expression immediately integrable. This factor is

$$\epsilon^{-t'(h-\gamma\sqrt{-1})} \text{ where } \gamma = \sqrt{(A-h^2)},$$

for if we differentiate,

$$\epsilon^{-t'(h-\gamma\sqrt{-1})}\left\{\frac{dz}{dt'} - (h+\gamma\sqrt{-1})z\right\},$$

we find

$$\epsilon^{-t'(h-\gamma\sqrt{-1})}\left\{\frac{d^2z}{dt'^2} - 2h\frac{dz}{dt'} + (h^2+\gamma^2)z\right\} dt'.$$

Hence, multiplying equation (5), by $\epsilon^{-t'(h-\gamma\sqrt{-1})}$, and integrating, we have

$$\epsilon^{-t'(h-\gamma\sqrt{-1})}\left\{\frac{dz}{dt'} - (h+\gamma\sqrt{-1})z\right\}$$
$$+ B\int z^{\beta} dt' \epsilon^{-t'(h-\gamma\sqrt{-1})} + \&c. + C = 0.$$

Putting for $\epsilon^{t'\gamma\sqrt{-1}}$ its value $\cos. t'\gamma + \sqrt{(-1)}.\sin. t'\gamma$, and taking the impossible parts, we find

$$\epsilon^{-t'h}\left\{\left(\frac{dz}{dt'} - hz\right)\sin. t'\gamma - \gamma z \cos. t'\gamma\right\}$$
$$+ C' = - B\int z^{\beta} dt' \sin. t'\gamma \epsilon^{-t'h} + \&c.$$

But when $t'=0$, $t=T$, $s=0$; $\therefore$ $z=0$; and $\therefore$ $C'=0$, the integral on the right-hand side being taken from $t'=0$. Also when $t'=T$,

$$\frac{ds}{dt'} = 0, \text{ and } \frac{dz}{dt} = 0.$$

Let at that point $z=Z$; and we have

$$\epsilon^{-Th}\{h \sin. \gamma T + \gamma \cos. \gamma T\} Z = B\int z^{\beta} dt' \sin. \gamma t' \epsilon^{-ht'} + \&c.$$

the integral being taken from $t'=0$, to $t'=T$.

Now, when the oscillation is indefinitely small, and therefore

z indefinitely small, the second side vanishes (for $\beta > 1$). Hence, we have in this case,

$$h \sin. \gamma T + \gamma \cos. \gamma T = 0, \ \tan. \gamma T = -\frac{\gamma}{h} \ldots\ldots\ldots (6).$$

But the time T is, by the supposition of tautochronism, independent of the arc, and we must therefore have the same equation for it, when z is not indefinitely small. Hence, we must have

$$0 = B \int z^{\beta} dt' \sin. \gamma t' \epsilon^{-ht'} + \&c.$$

between the limits $t' = 0$, and $t' = T$. But this cannot be the case except $B = 0$. For the factor $\sin. \gamma t' \epsilon^{-ht'}$ is always positive between the limits; and when z is small, the first term may represent the whole expression. Hence, the integral cannot be $= 0$; and therefore we must have $B = 0$.

Hence, we have from (4),

$$k^2 g (1 - z)^2 \frac{dx}{dz} = A z;$$

or, since

$$1 - z = \epsilon^{-ks}, \text{ and } \frac{dz}{(1-z)^2} = k \epsilon^{ks}. ds,$$

$$k g \epsilon^{-ks}. \frac{dx}{ds} = A (1 - \epsilon^{-ks}),$$

$$k g dx = A ds (\epsilon^{ks} - 1).$$

Cor. 1. If we expand ϵ^{ks}, divide by k, and then make $k = 0$, we have

$$g dx = A s ds,$$

which is the expression for the tautochron in a non-resisting medium.

Cor. 2. The expression for T in (6), does not involve k, the coefficient of the square of the velocity. Also, if the resistance involved terms $mv^3 + nv^4 + \&c.$ this expression would be the same.

Cor. 3. For the arc of ascent we may proceed in the same manner, making

$$s = \frac{1}{k} \text{ hyp. log. } (1+z); \text{ and we shall have}$$

$$kg\,dx = A\,ds\,(1 - \epsilon^{-ks}).$$

For the brachystochron in a resisting medium, see Woodhouse's *Isoperimetrical Problems,* p. 141.

BOOK III.

THE MOTION OF A RIGID BODY OR SYSTEM.

70. The conclusions of the preceding part of the Work are true of the motion of a *point*: some of them are also true of a body of finite magnitude, if we suppose it, during its motion, always to retain the same position, so that any line drawn in the body may continue parallel to itself. But they are no longer necessarily true if the body have any *rotatory* or *angular motion*; that is, a motion such that a line drawn in the body, and retaining its position in it, revolves successively into different directions. In these cases new principles and new formulæ are to be applied.

CHAP. I.

DEFINITIONS AND PRINCIPLES.

71. When a solid body has any rotatory motion, its parts act upon one another, and thus modify the effects of the other forces by which they are acted on. Every body, under such circumstances, may be considered as a machine; and by means of its rigidity or other properties, the forces which are applied to one part propagate their effect to another. So that each particle both presses with its own force, and serves to form levers and rods by which

the pressures of other particles are communicated. The laws according to which this connexion of different particles modifies the effect of the forces which move them, are to be the subject of our consideration in the present division of the Work.

When a body revolves continually about the same fixed line, this is called a *permanent axis*. This axis may be merely a mathematical or imaginary line: the points where it meets the surface of the body are called *poles*.

A body, while revolving, may change its axis of rotation, so as to revolve first about one line, and then about another. And we may suppose, that this change of the axis to be perpetually and gradually taking place ; so that the body revolves about the same axis only for an indivisible instant, and the next moment proceeds to revolve about an axis immediately contiguous to the former, and so on continually ; the poles of the axis moving perpetually upon the surface of the body. In this case, the axis at any instant is called the *momentary axis*.

A *motion of translation* is distinguished from a motion of rotation : by the latter, the body merely changes its position about some point ; in it (as for instance, its centre of gravity ;) by the former, this point changes its place, and is *transferred* to some new point of space.

By combining a motion of translation with a rotation round a variable axis, we may produce any motion whatever ; conversely, the motion of a body in any manner whatever, may, at any instant, be resolved into a motion of translation of the centre of gravity, and a motion of rotation about an axis passing through that point*.

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* This may be thus proved. If the centre of gravity be in motion, let an equal velocity be communicated to the body in the opposite direction, so that that centre may be at rest. Then let the centre of gravity of the body be made the centre of a sphere, fig. 77, and let the points of the body be referred to the sphere's surface by lines drawn through them from the centre. Let P be one of these referred points, and in an indefinitely small time let P move to p. Draw a great circle PQ perpendicular to Pp, and in this small time

72. The principles upon which our reasonings must depend are the laws of motion, which have been already applied to points. The most important principles which will be requisite in applying them to the cases in question, are the two following.

Principle I. Instead of the forces which act upon any body *in motion*, we may substitute those which are equivalent to them according to the principles of Statics.

Thus a force P (Pp, fig. 78.) acting at a certain distance (CP) to turn a body round an axis (C), exerts the same effort as a force $2P$ ($P'p'$) acting similarly at half the radius (CP').

Thus any force which acts to turn a body round an axis, acts effectively upon all the particles; the body itself transmitting the action after the manner, and according to the laws, of a lever.

Principle II. If the particles of a system when *unconnected*, would move so as always to have the same relative situation, we may suppose them to be connected, and their motions will remain unaltered.

Thus if the particles P and Q, fig. 78, which are moveable separately in the same plane about the centre C, be acted upon by such forces that their angular velocities round C would always be equal, we may suppose P, Q connected by a rigid rod PQ, and they will move in the same manner as before. For the bodies have no tendency to increase or diminish their distance from each other, and therefore exert no force in PQ, and cannot disturb each other's motions by means of the rod PQ.

This principle is an extension of the one in Statics, that when a system is in equilibrium any two of its particles may be supposed to be connected.

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time suppose an arc PQ to come into the position pq; therefore $pq = PQ$, and Qq must be perpendicular to PQ. Let the arcs QP, qp produced meet in O; then O will have been at rest during this small time. And if we draw a line from the centre of gravity to O, all the points of the body in this line will have been at rest, and therefore the body during this instant has been revolving round this line as an axis, (Euler, *Theor. Corp. Sol.* Cap. II. Th. 9.)

These principles may be applied immediately to the motions of any bodies. We may also deduce from them a general theorem, equivalent to what is called D'Alembert's Principle, and reduce our mechanical conditions into equations by means of it.

The forces which really act upon a system are called the *impressed forces*. The forces which must act upon each of the points of the system, (supposing them unconnected,) in order to produce the effect which really takes place, are called the *effective forces*.

73. PROP. If any forces whatever act upon any points of a body or system, the *impressed forces*, and the *effective forces* on all the points of the system, would produce an equilibrium by acting on the same system at rest; (according to its statical properties;) the latter forces being supposed to act in a direction opposite to that in which the forces are impressed.

Let accelerating forces P, Q, R, &c. act upon a system; and let p, q, r, &c. be the masses of the particles on which they respectively act; also let m, m', m'', &c. be the other particles of the system.

Let the effect produced in an indefinitely small time be that which arises from compounding velocities a, β, γ, ω, ω', ω'', &c. with the original velocities of the points p, q, r, m, m', m'', &c. And let P', Q', R', M, M', M'', &c. be the accelerating forces which would produce this effect.

Then Pp, Qq, Rr, &c. are the moving forces *impressed*; and $P'p$, $Q'q$, $R'r$, Mm, $M'm$, &c. the *effective* moving forces; and it is to be shewn, that the latter are statically *equivalent* to the former.

If the forces impressed are not equivalent to $P'p$, $Q'q$, $R'r$, Mm, $M'm'$, &c. let them be equivalent to $kP'p$, $kQ'q$, $kR'r$, kMm, $kM'm'$, &c.*. And since $P'p$, &c. would produce in an indefinitely

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* This is always possible. For, from the nature of the system, the virtual velocities of the parts are given. Hence, in the equation of equilibrium given by the principle of virtual velocities, kP', kQ', &c. will alone be unknown. Also, since P', Q', &c. are known, k will be the only unknown quantity in the equation, and may be assumed so as to satisfy the equation.

small time velocities a, &c.; $kP'p$, &c. acting on the particles, *supposing them unconnected,* will produce velocities ka, &c. And since the system can have the velocities a, β, &c. communicated to its parts without destroying their connexion, the velocities ka, $k\beta$, &c. can be communicated without affecting that connexion. Hence, by Principle II, the forces $kP'p$, &c. acting upon the system, *supposing it rigid,* will produce the same velocities ka, &c. And therefore by Principle I, the impressed forces Pp, &c. which are supposed equivalent to $kP'p$, &c. will produce in the system the velocities ka, &c. But they produce the velocities a, &c. Therefore $k=1$, and Pp, Qq, Rr, are equivalent to $P'p$, $Q'q$, $R'r$, Mm, $M'm$, &c. Therefore Pp, Qq, Rr, along with $P'p$, $Q'q$, $R'r$, Mm, $M'm'$, &c. acting oppositely, will produce an equilibrium Q. E. D.

Cor. Each force may be considered as equivalent to the effective force and the *force lost*. Thus, if the force Pp be equivalent to $P'p$ and Ap; Qq to $Q'q$ and Bq; Rr to $R'r$ and Cr, &c.; the forces Ap, Bq, Cr, &c. are the forces lost. Also Mm, $M'm'$, &c. are the forces gained*.

This being premised, the following proposition is true.

When any forces act upon a system, and produce motion, the forces lost and gained would balance each other, acting upon the system in opposite directions.

It has already been shewn, that

$$-P'p, \ -Q'q, \ -R'r, \ -Mm, \ -M'm', \text{ \&c.}$$

would balance Pp, Qq, Rr. That is, they would balance

$$P'p, \ Ap, \ Q'q, \ Bq, \ R'r, \ Cr.$$

Therefore $-Mm$, $-M'm'$, &c. would balance Ap, Bq, Cr. Or Ap, Bq, Cr would balance Mm, $M'm'$, &c. acting in the opposite direction.

It is in this form that the principle was enunciated by D' Alembert.

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* It is evident that Mm, $M'm'$, &c. may be classed with the forces Ap, Bq, &c. for the force which acts on m is $0.m$; which is equivalent to Mm, the effective force, and $-Mm$, the force lost. Hence, Mm may be considered as the force gained.

CHAP. II.

ROTATION ABOUT A FIXED AXIS.

74. We proceed to determine the angular motion produced when forces act upon a body moveable about a fixed axis. We consider the effect in producing motion only. The other effects of producing pressure upon the axis, and affecting its motion when it is moveable, will be investigated afterwards.

Prop. In a system consisting of any number of points m, n, p, q, &c. fig. 79, in the same plane, moveable about an axis C, perpendicular to that plane, a force F acts to turn the system; to find the effective accelerating force on any point.

Let F be a moving force which acts perpendicularly at an arm Cf. And let M, N, P, &c. be the effective accelerating forces on m, n, p, &c. Therefore Mm, Nn, Pp, &c. are the effective moving forces; and they are perpendicular to CM, CN, CP, &c. because the motion is so.

Hence, we have

Impressed force .... F acting perpendicularly at an arm CF,

Effective forces .... Mm, Nn, Pp, &c. acting perpendicularly at arms Cm, Cn, Cp, &c.

Hence, by Art. 73, and by the general proposition of the lever,

$$F.Cf - M.m.Cm - N.n.Cn - P.p.Cp - \&c. = 0.$$

But since m, n, p, &c. must all move with the same angular velocity round C, their linear velocities must always be as Cm, Cn, Cp, &c. and therefore all alterations of the velocities must be in this ratio, and the accelerating forces which produce them in the same ratio. Hence, we have $M : N :: Cm : Cn$; therefore

$N = \frac{M . Cm}{Cm}$; similarly $P = \frac{M . Cp}{Cm}$, &c. And substituting these values in the equation

$$F . Cf - M . m . Cm - M . n . \frac{Cn^2}{Cm} - M . p . \frac{Cp^2}{Cm} - \&c. = 0.$$

$$\text{Whence } M = \frac{F . Cf . Cm}{m . Cm^2 + n . Cn^2 + p . Cp^2 + \&c.}\text{*}.$$

$$\text{Similarly } N = \frac{F . Cf . Cn}{m . Cm^2 + n . Cn^2 + p . Cp^2 + \&c.},$$

$$P = \frac{F . Cf . Cp}{m . Cm^2 + n . Cn^2 + p . Cp^2 + \&c.},$$

and so on for any other point.

$$\text{And the effective force on } f = \frac{F . Cf^2}{m . Cm^2 + n . Cn^2 + p . Cp^2 + \&c.}.$$

The effective force at a distance 1 from the axis

$$= \frac{F . Cf}{m . Cm^2 + n . Cn^2 + p . Cp^2 + \&c.}.$$

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* As this Proposition is the foundation of the whole doctrine of rotatory motion, we shall shew how it may be deduced from elementary laws, independently of D'Alembert's Principle.

Let F be statically equivalent to a number of accelerating forces M at m, N at n, &c. which act perpendicularly to Cm, Cn, &c., and are as Cm, Cn, &c. Therefore

$$F . Cf - M . m . Cm - N . n . Cn, \&c. = 0; \text{ and } N = M . \frac{Cn}{Cm}, \&c.$$

$$\therefore F.Cf - M . m.Cm - M.n.\frac{Cn^2}{Cm}, \&c. = 0; \text{ and } M = \frac{F . Cf . Cm}{m.Cm^2 + n.Cn^2} + \&c.$$

Now, by Principle I, Art. 72. M, N, &c. will produce the same effect as F. Also, if Cm, Cn be supposed unconnected and moveable independently about C, and if M, N, &c. act on m, n, the accelerations of m, n, will be proportional to Cm, Cn, and therefore the angular velocities of Cm, Cn will be equally increased, and they will retain the same position with respect to each other. Hence, by Principle II, Art. 72. we may suppose Cm, Cn, &c. to be connected, and the system to become rigid, and the effect will still be the same. Hence, when F acts, the effective force in m is M as before.

If instead of the force F we have any forces acting in any manner, we must substitute instead of $F \cdot Cf$ the *moment* of these forces about the axis C; that is, the sum of each into the perpendicular upon it from C; those being taken negative which tend to turn it in the opposite direction.

Cor. 1. Since the effective accelerating force on f

$$= \frac{F}{m \cdot \frac{Cm^2}{Cf^2} + n \cdot \frac{Cn^2}{Cf^2} + p \cdot \frac{Cp^2}{Cf^2}}.$$

It appears, that the resistance which m opposes to the communication of motion, is the same as that of a mass $m \cdot \frac{Cm^2}{Cf^2}$ placed at f, and acted upon immediately; and similarly of the other particles.

Cor. 2. It appears by the demonstration, that the effective forces on different points are as their distances from the axis C.

Cor. 3. If the force F, and the radius Cf be constant, the effective force on each point will be constant; the motions will be uniformly accelerated, and the formula for such motions may be applied. If F be variable, the formulæ for variable motions may be applied.

Cor. 4. If the force which acts be the *weight* of any body, this body must be included among the bodies m, n, p, &c. in the denominator.

Thus if a system of material points in horizontal planes, m, n, p, fig. 80, be moved about a vertical axis AC, by a weight W acting perpendicularly at the radius Cf, by means of a string passing over a pully B; W moves with the same velocity as a body at the extremity of the area Cf; and therefore the same effective force is employed in moving W, as if it were at f. Hence, we have

$$\text{effective force on } f = \frac{\text{weight of } W \cdot Cf^2}{W \cdot Cf^2 + m \cdot Cm^2 + n \cdot Cn^2 + \&c.},$$

and the effective force on W is the same.

Cor. 5. The quantities W, m, n, &c. in the denominator in the last Corollary, are the *masses* of the bodies; the weight of W in the numerator is a *moving force*. If g represent the accelerating force of gravity, the weight of W is Wg.

Cor. 6. If the lines m, n, p, &c. be not in the same plane perpendicular to the axis, if Cm, $C'n$, $C''p$, &c. be their perpendicular distances from the axis, the same formula will be true, putting these lines for Cm, Cn, Cp, &c.

Or, if we take a plane Cmn, perpendicular to the axis, and refer the points of the system to this plane, by lines parallel to the axis; if m, n, p, &c. be the points thus referred, the same formulæ will be true.

The denominator of the fractions which express the effective forces in the preceding formulæ, is *the sum of each particle multiplied into its distance from the axis.* This sum is called the *Moment of Inertia* with respect to this axis *. It occurs perpetually in considering the subject of rotation.

If the system, instead of consisting of distinct material points, be a continuous body of finite magnitude, the momentum of inertia will be the sum of *each* point into the square of its distance from the axis, and will consist of an indefinite number of terms. The sum of these terms may be found by the integral calculus, as will be shewn in the following Chapter.

If the points be m, m_1, m_2, m_3, &c. and their distances from the axis, Cm, Cm_1, Cm_2, Cm_3, &c. the moment of inertia may be represented by $\Sigma(m \,.\, Cm^2)$. And if F be a moving force which

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* *The inertia* of a body is its effect in resisting the communication of motion: in a single point, it is as the mass simply; but in a body revolving about an axis, the effect of a particle in resisting motion depends on the distance from the axis, like the effect of the force acting on a lever. The effect on a lever is as the product of the force and distance, and this product is called *the moment;* the effect of the inertia of the mass in resisting rotatory motion, appears from the above investigation to be as the product of the mass and square of the distance, and hence, this product is called *the moment of inertia.* And the sum of these products is called the moment of inertia of the system.

acts perpendicularly at a distance Cf, we shall have the accelerating force at the point where the force acts $= \frac{F.Cf^2}{\Sigma(m.Cm^2)}$.

If forces act upon every point of the system, the effect may be calculated by the same principle as before, as will be seen in the next Problem.

75. Prop. A system of material points, moveable about a horizontal axis, has all its parts acted on by gravity; it is required to determine the accelerating force.

Let C, fig. 81, be the axis, and m, n, p, the points. Draw a horizontal line through C, meeting vertical lines through m, n, p, in d, e, h. Then the moving forces impressed are the weights of m, n, p. Let M be the effective accelerating force upon m; therefore in the same way as befere, the effective accelerating forces on n, p, are

$$\frac{M.Cn}{Cm}, \quad \frac{M.Cp}{Cm}.$$

And the effective moving forces are

$$M.m, \quad \frac{M.n.Cn}{Cm}, \quad \frac{M.p.Cp}{Cm}.$$

Now Cd, Ce, Ch are perpendicular to the directions of the former, and Cm, Cn, Cp to those of the latter; also, if m be the mass of one of the bodies, its weight or moving force is mg, and so for the rest. Hence, by the equilibrium between the impressed and the effective forces, we have, by Art. 73.

$$m.g.Cd+n.g.Ce+p.g.Ch$$

$$= M.m.Cm + \frac{M.n.Cn^2}{Cm} + \frac{M.p.Cp^2}{Cm};$$

$$\therefore M = \frac{(m.Cd+n.Ce+p.Ch)\,Cm.g}{m.Cm^2+n.Cn^2+p.Cp^2}.$$

Cor. 1. If we had supposed more bodies, we should have had a corresponding number of terms, both in the numerator and denominator.

Cor. 2. There will be negative terms in the numerator, when any of the bodies are on the other side of the vertical through C; the terms in the denominator will always be positive, because the bodies all move in the same direction round C; and therefore the effective accelerating forces are always in the same direction.

Cor. 3. The effective accelerating force on any other point of the system, as n, will be

$$N = \frac{(m \cdot Cd + n \cdot Ce + p \cdot Ch)\, Cn \cdot g}{m \cdot Cm^2 + n \cdot Cn^2 + p \cdot Cp^2}.$$

Cor. 4. If G be the centre of gravity of the system, and if a perpendicular from G meet Ch in H, we have

$$(m \cdot Cd + n \cdot Ce + p \cdot Ch) = (m + n + p) \cdot CH.$$

And if θ be the angle which CG makes with the vertical,

$$CH = CG \cdot \sin. \theta.$$

Hence,

$$M = \frac{(m + n + p)\, CG \cdot \sin. \theta \cdot Cm \cdot g}{m \cdot Cm^2 + n \cdot Cn^2 + p \cdot Cp^2};$$

or, denoting $m + n + p$ by Σm, and the denominator by $\Sigma m \cdot Cm^2$, whatever be the number of bodies,

$$M = \frac{Cm . CG . g \sin. \theta . \Sigma m}{\Sigma (m . Cm^2)}.$$

76. Prop. To find a point of the system which shall be accelerated exactly as much as a single point in the same position.

If O be any point in CG, we shall have

$$\text{accelerating force on } O = \frac{CO . CG . g . \sin. \theta . \Sigma m}{\Sigma (m . Cm^2)}.$$

Now, if a single particle were placed in O, and all the rest removed, we should have

$$\text{accelerating force on particle in } O = g \sin. \theta.$$

And we have to find O, so that these accelerating forces may be equal. For this purpose, we must have

$$CO \cdot CG \cdot \Sigma m = \Sigma (m \cdot Cm^2);$$

$$\therefore\ CO = \frac{\Sigma \cdot (m \cdot Cm^2)}{CG \cdot \Sigma m} \ldots . (a).$$

The point O is called the *Centre of Oscillation;* a single point placed in O, would, in any position of CG, be acted on by the same accelerating force as when O is a point in the system; and therefore, the oscillations of CO and of the system would be exactly the same as if we had but one particle O.

Cor. 1. The time of oscillation of the system, is the same as that of a simple pendulum, whose length is CO. Hence, if we make $CO = l$, we shall have the time of one of the small oscillations $= \pi \sqrt{\frac{l}{g}}$.

Cor. 2. When we know the moment of inertia, and the place of the centre of gravity, the centre of oscillation with respect to the axis C is found by the formula

$$CO = \frac{\Sigma (m \cdot Cm^2)}{CG \cdot \Sigma m}.$$

And this is applicable, whether the system consist of distinct points, or of finite bodies.

CHAP. III.

MOMENT OF INERTIA.

77. IN the present Chapter we shall find the moment of inertia of a variety of different bodies, and with respect to any axis. From this it will be easy to deduce the position of the centre of oscillation. We shall in the first place, prove a property of the moment of inertia, by means of which, knowing this moment with respect to any axis passing through the centre of gravity, we can find it with respect to any other axis parallel to the former.

SECT. I. *General Properties.*

PROP. The moment of inertia of any system, with respect to any given axis, is equal to the moment about an axis parallel to this, passing through the centre of gravity, *plus* the moment of the whole body, (collected in its centre of gravity,) about the given axis.

Let fig. 82 represent any system, moveable about an axis C; and let m, n, p, q, be the particles of it, referred to a plane perpendicular to the axis. Let G be the centre of gravity of m, n, p, q. Draw md perpendicular on CG.

$$\text{Now,} \quad Cm^2 = CG^2 + Gm^2 + 2\,CG\,.\,Gd.$$

Similarly, if ne, ph, qk, be perpendicular on CG,

$$Cn^2 = CG^2 + Gn^2 - 2\,CG\,.\,Ge,$$

$$\&c. = \&c.\,;$$

$$\therefore m \cdot Cm^2 + n \cdot Cn^2 + p \cdot Cp^2 + \&c.$$
$$= m \cdot CG^2 + n \cdot CG^2 + p \cdot CG^2 + \&c.$$
$$+ m \cdot Gm^2 + n \cdot Gn^2 + p \cdot Cp^2 + \&c.$$
$$+ 2\,CG\,(m \cdot Gd - n \cdot Ge + p \cdot Gh - \&c.)$$

And, since by the property of the centre of gravity,

$$m \cdot Gd - n \cdot Ge + p \cdot Gh - \&c. = 0;$$

we have

$$m \cdot Cm^2 + n \cdot Cn^2 + p \cdot Cp^2 + \&c. = (m + n + p + \&c.)\,CG^2$$
$$+ m \cdot Gm^2 + n \cdot Gn^2 + p \cdot Gp^2 + \&c.$$

Or, moment of inertia round C = moment of $(m+n+p+\&c.)$ at G round C + moment round G.

Cor. 1. We may represent this theorem thus, whatever be the number of bodies;

$$\Sigma\,(m \cdot Cm^2) = CG^2 \Sigma m + \Sigma\,(m \cdot Gm^2).$$

Cor. 2. Knowing the moment of inertia round G, we may, from this expression, find the moment round C.

Cor. 3. The moment round G, the centre of gravity, is less than that round any other axis C, parallel to G.

78. Since the expression for the moment of inertia of a system, consisting of a finite number of points, is $\Sigma\,(m \cdot Cm^2)$, m being the portion of the mass which is at the distance Cm; when the number of points becomes indefinite, the expression will evidently become $\int dM \cdot Cm^2$, where dM is the differential of the mass at the distance Cm. For we approach the true value by dividing the mass into portions smaller and smaller indefinitely, and taking the sum of each into the square of its distance; but by the nature of the differential calculus, we thus approach the differential of the mass at each point, multiplied into the square of its distance, and the integral of this. Hence, if r be the distance of any point from the axis, and dM the differential of the mass corresponding to dr,

$$\text{moment of inertia} = \int r^2 dM\,^*.$$

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* This may be proved more distinctly thus.

Let AB, fig. 83, be any body, and $Cm = r$, $Cm' = r'$, be two distances from

This moment will be the same as that of the whole mass, collected at a certain distance. If we call this distance k, we have

$$k^2 M = \int r^2 dM\,*.$$

We proceed to find $k^2 M$ in different bodies. We may consider bodies as being either physical lines, surfaces, or solids, and apply our formulæ to each. The moment itself will depend upon the thickness of the lines and surfaces, and the density of the substance; but if the mass be homogeneous, the line k will depend only upon the geometrical form of the system. The other quantities will enter as multipliers on both sides of the equation

$$k^2 M = \int r^2 d M.$$

Hence, in finding k we may suppose the thickness of lines and surfaces, and the density of the mass, each to be unity.

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from the axis: M the mass included in the distance r from the axis, M' that included in the distance r'. And let the moment of inertia of the former portion $= R$, and of the latter $= R'$. Therefore, the moment of the portion $m\,n\,m'$, included between the two distances, will be $R' - R$. And the portion itself will be $M' - M$.

Now $R' - R$ is evidently greater than $(M' - M)\, r^2$,

and less than $(M' - M)\, r'^2$;

$$\therefore \frac{R' - R}{M' - M} \text{ is } > r^2, \text{ and } < r'^2.$$

But, as M' and M become ultimately equal, r^2 and r'^2 become ultimately equal, and therefore $\frac{R' - R}{M' - M}$ is ultimately equal to either of them. Also, ultimately

$$\frac{R' - R}{M' - M} \text{ is } \frac{d R}{d M};$$

therefore, we have

$$\frac{d R}{d M} = r^2, \quad d R = r^2 d M, \quad R = \int r^2 d M.$$

* The *centre of gyration* is defined to be 'the point at which the whole mass must be collected, that the rotatory motion communicated by a given force may be the same as before.' It appears by the last Chapter, that *any point*, whose distance from the axis is k, possesses this property. We may call k the *radius of gyration*.

SECT. II. *Moment of Inertia of a Line, revolving in its own Plane.*

79. If any distance from the axis be r, and ds the differential of the length of the line, corresponding to ds, it appears by what has just been said, that we may suppose the thickness and density of the line each $= 1$, and $dM = ds$. Then $k^2M = \int r^2 ds$.

We may take, for the centre of revolution, any point with respect to which the relation of ds and r is most simple; and then by Art. 77 find the moment of inertia about the centre of gravity, and from that, about any other point.

Ex. 1. To find k for a straight line revolving about an axis perpendicular to it in its middle point, fig. 84.

Call any distance $Cm = r$, $CA = CB = a$, $M = 2a$,

$$2ak^2 = \int r^2 dr = \frac{r^3}{3} + \text{constant}.$$

and, the integral being taken from $r = -a$ to $r = a$,

$$2ak^2 = \frac{2a^3}{3}; \therefore k^2 = \frac{a^2}{3}.$$

Hence, momentum of inertia $= k^2M = \dfrac{a^2M}{3}$, when M is the mass of the line, and its thickness and density may be any whatever. (The thickness must necessarily be small, that it may be considered as a line).

Ex. 2. A straight line about an axis perpendicular to it through any point of it; (D, fig. 84.).

Let mass $= M$, $AB = 2a$, $CD = b$.

By Art. 77, mom. ab. $D =$ mom. ab. $C +$ mom. of M at dist. CD,

$$k^2M = \frac{a^2M}{3} + b^2M,$$

$$k^2 = \frac{a^2}{3} + b^2.$$

Ex. 3. A straight line about any axis perpendicular to the plane in which it is.

AB fig. 85, about E,

$$AB = 2a,\ CD = b,\ DE = c;\ EC^2 = b^2 + c^2.$$

As before, $k^2M =$ mom. round $C +$ mom. of M at dist. EC.

$$= \frac{a^2M}{3} + (b^2 + c^2)\,M;$$

$$\therefore k^2 = \frac{a^2}{3} + b^2 + c^2.$$

Ex. 4. A circular arc about the centre of the circle.

If a be the radius, r always $= a$, and mom. of inertia $= a^2M$; $\therefore k^2 = a^2$.

Ex. 5. A circular arc about an axis perpendicular to its plane through its centre of gravity, fig. 86. Let C be the centre of the arc PAp, G its centre of gravity. Then by the rules for the centre of gravity, (*Statics*, Chap. VI, Ex. 29.)

$$CG = \frac{CA\ .\ \text{chord}\ Pp}{\text{arc}\ Pp} = \frac{aq}{p};$$

a being the radius, p the arc, q its chord.

Then, by Art. 77, mom. round $C =$ mom. round $G + CG^2\,.\,M$;

$$\therefore \text{mom. round } G = \text{mom. round } C - CG^2.\,M,$$

$$k^2M = a^2M - \frac{a^2q^2}{p^2}M,$$

$$k^2 = a^2\left(1 - \frac{q^2}{p^2}\right).$$

If PAp be a semi-circle, $q = 2a$, $p = \pi a$,

$$k^2 = a^2\left(1 - \frac{4}{\pi^2}\right).$$

Ex. 6. A circular arc about an axis perpendicular to its plane through its vertex A, fig. 86.

By the two last examples we shall here easily find

$$k^2 = 2a^2\left(1 - \frac{q}{p}\right).$$

If the arc be a whole circumference, $q=0$, $k^2 = 2a^2$.

Sect. III. *Moment of Inertia of a Line, revolving perpendicularly to its own Plane.*

80. In such cases the figure revolves about a line in its own plane, or parallel to it. If this line divide the figure symmetrically, as AC, fig. 87, we shall have $dM = 2$ differential of arc AQ, of which $NQ = r$ is the ordinate: and if $AQ = s$, $k^2 M = 2\int r^2 ds$.

Ex. 7. A circular arc about a radius through its vertex A, fig. 37.

Let $CA = a$, $ds = \dfrac{a\,dr}{\sqrt{(a^2 - r^2)}}$,

$$k^2 \,.\, \text{arc } PAp = \int \frac{2ar^2 dr}{\sqrt{(a^2 - r^2)}}$$

$$= C - ar\sqrt{(a^2 - r^2)} + a^3 \text{ arc}\left(\sin = \frac{r}{a}\right).$$

And this, being taken to begin when $r=0$, gives, putting p for the arc PAp, and c for PM,

$$k^2 p = \frac{a^2}{2}p - ac\sqrt{(a^2 - c^2)},$$

$$k^2 = \frac{a^2}{2} - \frac{ac\sqrt{(a^2 - c^2)}}{p}.$$

If the arc be a semi-circle, $c = a$, $k^2 = \dfrac{a^2}{2}$.

The same is true if the arc be a circle, and $\therefore c = 0$.

Sect. IV. *Moment of Inertia of a Surface, revolving in its own Plane.*

81. If the mass be a surface revolving about an axis perpendicular to it, we shall have $dM = \int dr \,.\, r d\theta$, taken between proper limits of θ; $k^2 M = \int\int r^3 dr d\theta$.

The integrations may be performed in any order. If we integrate first for r we shall have

$$k^2 M = \int \frac{r^4 d\theta}{4};$$

the value of r being taken which belongs to the boundary of the figure. The same expression might also be obtained by conceiving the figure divided into triangles of indefinitely small width by lines drawn in it from the axis.

Ex. 8. To find the moment of inertia in any triangle, about an axis perpendicular to its plane, and through one of its angles.

Let ACB, fig. 88, be the triangle, and C its axis; CD perpendicular on $AB = h$. $CB = a$, $CA = b$, $AB = c$; $DCM = \theta$, $CM = r$,

$$\theta = \text{angle}\left(\cos. = \frac{h}{r}\right); \; d\theta = \frac{h dr}{r \sqrt{(r^2 - h^2)}},$$

$$k^2 M = \int \frac{h r^3 dr}{4 \sqrt{(r^2 - h^2)}} = \frac{h r^2 + 2h^3}{12} \sqrt{(r^2 - h^2)} + \text{constant},$$

and for the triangle ADC, we must take the integral from $r = h$ to $r = b$, which gives $\frac{b^2 + 2h^2}{12} h \sqrt{(b^2 - h^2)}$. Similarly for BDC we have $\frac{a^2 + 2h^2}{12} h \sqrt{(a^2 - h^2)}$.

Hence, for ACB we have

$$k^2 M = \frac{a^2 + 2h^2}{12} h \sqrt{(a^2 - h^2)} + \frac{b^2 + 2h^2}{12} h \sqrt{(b^2 - h^2)}.$$

Cor. 1. If the triangle be isosceles, $k^2 M = \frac{a^2 + 2h^2}{6} \,.\, M.$

Cor. 2. Let AB be bisected in E, and $DE=q$;

$$\therefore BD=\sqrt{(a^2-h^2)}=\frac{c}{2}+q,\ AD=\sqrt{(b^2-h^2)}=\frac{c}{2}-q.$$

And $BC^2-BD^2=AC^2-AD^2$,

$$\text{or } a^2-\left(\frac{c}{2}+q\right)^2=b^2-\left(\frac{c}{2}-q\right)^2;$$

$$\therefore a^2-b^2=2cq.$$

Also $a^2+2h^2=a^2+2\left\{a^2-\left(\frac{c}{2}+q\right)^2\right\}=3a^2-2\left(\frac{c}{2}+q\right)^2$,

$$b^2+2h^2=b^2+2\left\{b^2-\left(\frac{c}{2}-q\right)^2\right\}=3b^2-2\left(\frac{c}{2}+q\right)^2.$$

Hence,

$$k^2M=\frac{h}{12}\left\{3a^2\left(\frac{c}{2}+q\right)-2\left(\frac{c}{2}+q\right)^3+3b^2\left(\frac{c}{2}-q\right)-2\left(\frac{c}{2}-q\right)^3\right\}$$

$$=\frac{h}{12}\left\{\frac{3}{2}(a^2+b^2)c+3(a^2-b^2)q-\frac{c^3}{2}-6cq^2\right\},$$

or, since $3(a^2-b^2)q=6cq^2$,

$$k^2M=\frac{h}{12}\left\{\frac{3}{2}(a^2+b^2)c-\frac{c^3}{2}\right\}$$

$$=\frac{ch}{24}\{3(a^2+b^2)-c^2\}$$

$$=\frac{M}{12}\{3(a^2+b^2)-c^2\};$$

$$\therefore k^2=\frac{3(a^2+b^2)-c^2}{12}.$$

Ex. 9. A triangle about an axis through its centre of gravity.

By the property of the centre of gravity, (*Statics*, Chap. VI, Ex. 3.)

$$CG^2=\frac{2(a^2+b^2)-c^2}{9};$$

$$\therefore \frac{M}{12}\{3(a^2+b^2)-c^2\} = M \cdot \frac{2(a^2+b^2)-c^2}{9} + k^2M,$$

$$k^2M = M\left\{\frac{a^2+b^2+c^2}{36}\right\} = \frac{M(h^2+k^2+l^2)}{12};$$

h, k, l being the distances from G to the angles A, B, C.

If the triangle be equilateral,

$$k^2M = M \cdot \frac{a^2}{12} = M \cdot \frac{l^2}{4}.$$

Ex. 10. A parallelogram about an axis perpendicular to it through its centre of gravity.

$ABDE$, fig. 89, the parallelogram, $AB = 2a$, $AD = 2b$, $BD = 2c$.

Let G be the centre of gravity of ABD; $\therefore$ by *Statics*,

$$GC^2 = \frac{2(4a^2+4b^2)-4c^2}{36} = \frac{2(a^2+b^2)-c^2}{9}.$$

And mom. of ABD round C = mom. round $G + ABD \cdot GC^2$

$$= ABD\left\{\frac{4a^2+4b^2+4c^2}{36} + \frac{2(a^2+b^2)-c^2}{9}\right\}$$

$$= ABD\left\{\frac{a^2+b^2}{3}\right\},$$

and doubling both sides, since $2ABD = M$,

$$k^2M = M \cdot \frac{a^2+b^2}{3}, \text{ and } k^2 = \frac{a^2+b^2}{3}.$$

Hence, it is independent of the angles of the parallelogram.

Ex. 11. Any regular polygon about an axis perpendicular to it through its centre.

The polygon may be divided into isosceles triangles, and for each of these, mom. $= \frac{a^2+2c^2}{6}A$, a and c being the radius of the circumscribed and inscribed circle, and A one of the triangles;

$$\therefore k^2M = \frac{a^2+2c^2}{6}nA = \frac{a^2+2c^2}{6}M = \frac{a^2}{6}\left(1+2\cos.^2\frac{\pi}{n}\right),$$

$$M = \frac{a^2}{6}\left(2 + \cos.\frac{2\pi}{n}\right) M,$$

or, if l be a side, $$k^2 M = \frac{l^2}{24}\left\{\frac{1 + 2\cos.^2\frac{\pi}{n}}{\sin.^2\frac{\pi}{n}}\right\} M.$$

Ex. 12. A circle revolving about an axis perpendicular to it through its centre.

Here r is constant, and $=a$, the radius;

$$k^2 M = \int \frac{a^4\, d\theta}{4} = \frac{2\pi a^4}{4}, \text{ integrating for } \theta;$$

$$= \frac{\pi a^4}{2} = \frac{a^2 M}{2},\ k^2 = \frac{a^2}{2}.$$

Ex. 13. An annulus whose external and internal radii are a, b,

$$k^2 M = \frac{\pi(a^4 - b^4)}{2} = \frac{a^2 + b^2}{2}\,.\,(\pi a^2 - \pi b^2) = \frac{a^2 + b^2}{2} M,$$

$$k^2 = \frac{a^2 + b^2}{2}.$$

Ex. 14. An ellipse revolving about the centre.

If θ be measured from the extremity of the major axis,

$$r^2 = \frac{b^2}{1 - e^2\cos.^2\theta};$$

$$\therefore\ k^2 M = b^4 \int \frac{d\theta}{(2 - 2e^2\cos.^2\theta)^2}$$

$$= b^4 \int \frac{d\theta}{(2 - e^2 - e^2\cos.2\theta)^2}$$

$$= \frac{b^4}{(2 - e^2)^2}\int \frac{d\theta}{(1 - n\cos.2\theta)^2},$$

putting n for $\frac{e^2}{2 - e^2}$.

To integrate, let $P = \frac{\sin. 2\theta}{1 - n\cos. 2\theta}$, and the integral $= S$,

$$dP = \frac{2\cos. 2\theta\, d\theta}{1 - n\cos. 2\theta} - \frac{2n\sin.^2 2\theta d\theta}{(1 - n\cos. 2\theta)^2}$$

$$= \frac{2\cos. 2\theta d\theta - 2n d\theta}{(1 - n\cos. 2\theta)^2}$$

$$= \frac{2}{n} \frac{(1 - n^2) d\theta - (1 - n\cos. 2\theta) d\theta}{(1 - n\cos. 2\theta)^2}$$

$$= \frac{2(1 - n^2)}{n} \cdot \frac{d\theta}{(1 - n\cos. 2\theta)^2} - \frac{1}{n} \frac{2 d\theta}{1 - n\cos. 2\theta};$$

$$\therefore P = \frac{2(1 - n^2)}{n} \cdot S - \frac{1}{n} \int \frac{d\theta}{1 - n\cos. 2\theta}.$$

And this latter integral is (*Lacroix*, Elem. Treat. Note K),

$$\frac{1}{\surd(1 - n^2)} \text{arc} \left(\cos. = \frac{\cos. 2\theta - n}{1 - n\cos. 2\theta}\right) = \frac{1}{\surd(1 - n^2)} A\text{: suppose,}$$

$$\therefore S = \frac{1}{2(1 - n^2)} \cdot P + \frac{1}{2(1 - n^2)^{\frac{3}{2}}} A.$$

If we take the integrals from $\theta = 0$, to $\theta = \pi$, and then double them, in order to obtain the whole value of S, we have

$$S = \frac{1}{(1 - n^2)^{\frac{3}{2}}} \text{arc} (\cos. = 1)$$

$$= \frac{2\pi}{(1 - n^2)^{\frac{3}{2}}};$$

$$\therefore k^2 M = \frac{2\pi b^4}{(2 - e^2)^2 (1 - n^2)^{\frac{3}{2}}};$$

and putting $a\surd(1 - e^2)$ for b, and $\frac{e}{\surd(2 - e^2)}$ for n,

$$k^2 M = \frac{\pi a^4}{4} (2 - e^2) \surd(1 - e^2).$$

But the area $M = \pi a^2 \surd(1 - e^2)$;

$$\therefore k^2 = \frac{a^2 (2 - e^2)}{4} = \frac{a^2 + a^2 (1 - e^2)}{4} = \frac{a^2 + b^2}{4}.$$

When $e=0$, $k^2 = \frac{a^2}{2}$, as in a circle.

When $e=1$, $b=0$, $k^2 = \frac{a^2}{4}$.

82. If the surface be bounded by rectangular co-ordinates, we may thus find its moment when revolving in its own plane.

Let the centre of motion C, fig. 90, be the origin of co-ordinates $CN=x$, $NQ=y$. And we have

moment of Qq about $C=$ moment of Qq about $N+CN^2 . Qq$, (by Ex. 1, Art. 79.),

$$= \frac{y^2}{3} . 2y + x^2 . 2y.$$

And multiplying by dx, and integrating,

$$k^2 M = 2\int \left(\frac{y^3}{3} + x^2 y\right) dx.$$

Ex. 15. A parabola revolving about its vertex. Here $y=a^{\frac{1}{2}}x^{\frac{1}{2}}$,

$$k^2 M = 2\int \left(\frac{a^{\frac{3}{2}} x^{\frac{3}{2}}}{3} + a^{\frac{1}{2}} x^{\frac{5}{2}}\right) dx = 2\left(\frac{2a^{\frac{3}{2}} x^{\frac{5}{2}}}{15} + \frac{2a^{\frac{1}{2}} x^{\frac{7}{2}}}{7}\right),$$

$$\text{and } M = \frac{4}{3} a^{\frac{1}{2}} x^{\frac{3}{2}}; \ \therefore\ k^2 = \frac{3x^2}{7} + \frac{y^2}{5},$$

x and y being the extreme ordinates.

Sect. V. *Moment of Inertia of a Plane revolving about an Axis in or parallel to the Plane.*

83. When any plane revolves about a line *in it*, we may call r the distance of any point from the axis; and if dz be the differential of the axis, $\int dz\,dr$ integrated with respect to z, will be the portion at distance r; and $\int\int r^2 dz\,dr$ the moment of inertia.

The integral $\int\int r^2 dz\,dr$ is $= \int \frac{r^3 dz}{3}$; and r is given in terms of z.

Ex. 16. A circle revolving about a diameter.

Let radius $=a$, z measured from the vertex;

$$\therefore\ a-z=\sqrt{(a^2-r^2)},$$

$$dz=\frac{r\,dr}{\sqrt{(a^2-r^2)}};\ k^2M=\frac{2}{3}\int\frac{r^4\,dr}{\sqrt{(a^2-r^2)}},$$

which, integrated from $z=0$, to $z=a$, gives for a semi-circle,

$$k^2M=\frac{\pi a^4}{8}.$$

And since $M=\frac{\pi a^2}{2}$, $k^2=\frac{a^2}{4}$.

The same expression is true for a whole circle.

Also for an ellipse revolving about either principal axis, $2a$ being the other.

Ex. 17. A circle revolving about a line parallel to its plane, at a distance c from its centre; radius $=a$.

A diameter being drawn parallel to the axis of rotation, we have, by last Example,

$$\text{moment round diameter}=\frac{a^2}{4}M.$$

And therefore by Art. 77,

$$k^2M=\frac{a^2}{4}M+c^2M,$$

$$k^2=\frac{a^2}{4}+c^2.$$

Ex. 18. An isosceles triangle about its perpendicular.

Perpendicular $=a$, base $=2b$; z measured from vertex,

$$r=\frac{bz}{a},\ k^2M=\frac{2b^3}{3a^3}\int z^3dz=\frac{b^3z^4}{6a^3}=\frac{b^3a}{6}.$$

But $M=ba$; $\therefore\ k^2=\frac{b^2}{6}$.

Sect. VI. *Moment of Inertia of a symmetrical Solid about its Axis.*

84. When we have a solid of which all the sections perpendicular to the axis are similar, it may, by planes perpendicular to this axis, be divided into indefinitely thin slices, and the moment of inertia of the whole, will be the sum of the moments of these parts; and ultimately, it will be the integral of a portion of the moment which corresponds to dz the differential of the axis. This portion will be found by taking the moment of the plane, which is the section of the solid, (found by last Article), and multiplying it by dz.

Ex. 19. A cone revolving about its axis.

The momentum of a circle, radius $= r$, is $\frac{\pi r^4}{2}$. And in the cone r is as z: let $r = nz$. The momentum of a differential slice

$$= \frac{\pi r^4 dz}{2} = \frac{\pi n^4 z^4 dz}{2},$$

$$k^2 M = \frac{\pi n^4 z^5}{10}, \text{ and taken to } z = a,$$

$$= \frac{\pi n^4 a^5}{10} = \frac{\pi b^4 a}{10},$$

if $na = b$, the radius of the base.

And $M = \frac{\pi b^2 a}{3}$; $\therefore\ k^2 = \frac{3 b^2}{10}$.

Ex. 20. A sphere about a diameter,

$$k^2 = \frac{2 a^2}{5}.$$

Ex. 21. A hollow sphere, of which the external and internal radii are a, b,

$$k^2 = \frac{2}{5} \cdot \frac{a^5 - b^5}{a^3 - b^3}.$$

Ex. 22. An ellipsoid about its axis: the semi-axes of the largest section perpendicular to the axis of rotation being a, b.

$$k^2 = \frac{a^2 + b^2}{5}.$$

Ex. 23. A parallelepiped, $k^2 = \frac{a^2 + b^2}{3}$.

Ex. 24. A cylinder, $k^2 = \frac{a^2}{2}$.

Ex. 25. A hollow cylinder, $k^2 = \frac{a^2 + b^2}{2}$.

Sect. VII. *Moment of Inertia of a Solid not symmetrical.*

85. When the solid comes under this description, different methods may be used, as in the following Examples.

Ex. 26. A cylinder about an axis perpendicular to its own, through its middle.

Let its radius $= b$, its length $= 2a$.

A circle perpendicular to the axis of the cylinder, and at a distance x from the axis of rotation, has its moment, (Ex. 17.)

$$= \pi b^2 \left(x^2 + \frac{b^2}{4}\right),$$

$$\therefore k^2 M = \int \pi b^2 \left\{x^2 + \frac{b^2}{4}\right\} dx = \pi b^2 \left\{\frac{x^3}{3} + \frac{b^2 x}{4}\right\}$$

$$= \pi b^2 \left\{\frac{2a^3}{3} + \frac{b^2 a}{2}\right\} = 2\pi b^2 a \left\{\frac{a^2}{3} + \frac{b^2}{4}\right\},$$

and since $M = 2\pi b^2 a$, $k^2 = \frac{a^2}{3} + \frac{b^2}{4}$.

Ex. 27. A cone about an axis perpendicular to its axis at its vertex.

Fig. 91. AB the base, CD the axis of rotation, MP any section parallel to the base,

$$CM = x,\ MP = nx,\ CA = a,\ AB = b = na.$$

By Ex. 17, moment of circle MP round CD

$$= \text{circle} \cdot \left(x^2 + \frac{n^2 x^2}{4}\right)$$

$$= \pi n^2 \left(1 + \frac{n^2}{4}\right) x^4;$$

$$\therefore k^2 M = \pi n^2 \left(1 + \frac{n^2}{4}\right) \int x^4 dx = \frac{\pi n^2 x^5}{5} \left(1 + \frac{n^2}{4}\right)$$

$$= \frac{\pi n^2 a^5}{5} \left(1 + \frac{n^2}{4}\right),$$

and $M = \frac{\pi n^2 a^2 . a}{3}$; $\therefore k^2 = \frac{3a^2}{5}\left(1 + \frac{n^2}{4}\right) = \frac{3}{5}\left(a^2 + \frac{b^2}{4}\right)$.

Ex. 28. A cone about an axis perpendicular to its axis through its centre of gravity.

The distance of the centre of gravity from the vertex is $\frac{3a}{4}$; hence, by Art. 77,

$$\text{moment round } CD = \text{moment round } GH + CG^2 . M,$$

$$\frac{3}{5}\left(a^2 + \frac{b^2}{4}\right) M = k^2 M + \frac{9a^2}{16} M,$$

$$k^2 = \frac{3(a^2 + 4b^2)}{80}.$$

And similarly for other figures of revolution*.

Sect. VIII. *Centre of Oscillation.*

86. Prop. To find the momentum of inertia of a given body, by a practical method.

Let the body AB, fig. 92, be suspended from an horizontal axis C; and let a vertical line be drawn from the axis, which will give

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* The expressions in this Chapter may also be found by using the following formulæ in rectangular co-ordinates. The body revolves about the axis of z.

For planes, $\iint (x^2 + y^2)\, dx\, dy$.

Where, after the first integration, in x for instance, limits are to be put in terms of y.

For solids, $\iiint (x^2 + y^2)\, dx\, dy\, dz = \iint (x^2 + y^2) z\, dx\, dy$.

Where z must be put in terms of x and y, and then the integrations performed, and the limits introduced as before. See Poisson, *Traité de Mec.* No. 348, 349.

the plane Cb, in which the centre of gravity is. Let the body be moved, so that Cb may be exactly horizontal (CB), and let the weight Q be ascertained, which, acting vertically at B, will support in this position. If W be the weight of the mass, we shall have

$$W = \frac{Q \cdot CB}{CG}.$$

Now let the body hang from C, and make small oscillations, and let it make n oscillations in a time t. Then the time of one oscillation is, by Cor. 1, Art. 76.

$$\frac{t}{n} = \pi \sqrt{\frac{CO}{g}};$$

O being the centre of oscillation;

$$\therefore CO = \frac{g t^2}{\pi^2 n^2}. \quad \text{And by Art. 76.}$$

$$\Sigma(m \cdot Cm^2) = CG \cdot CO \cdot \Sigma m = \frac{Q}{W} \cdot \frac{CB \cdot g \cdot t^2}{\pi^2 n^2} \cdot M,$$

M being the mass of the system;

$$\therefore k^2 = \frac{Q}{W} \cdot \frac{t^2}{\pi^2 n^2} \cdot CB \cdot g.$$

37. Prop. The centres of suspension and oscillation are reciprocal.

That is, if the centre of oscillation be made the point of suspension, the former point of suspension will become the centre of oscillation.

$$\text{In fig. 81,} \quad CO = \frac{\Sigma(m \cdot Cm^2)}{CG \cdot \Sigma m}.$$

But by Art. 77.

$$\Sigma(m \cdot Cm^2) = \Sigma(m \cdot Gm^2) + CG^2 \cdot \Sigma m;$$

or, putting M for Σm the mass, and $k^2 M$ for $\Sigma(m \cdot Gm^2)$,

$$CO = \frac{k^2}{CG} + CG, \text{ and } GO = \frac{k^2}{CG}, \text{ and } CG = \frac{k^2}{GO}.$$

If therefore, we suspend the body from O, C will be its centre of oscillation.

Cor. 1. CO depends on CG alone, and will be the same, so long as CG is the same. Hence, if with centre G, (fig. 93,) and radius GC, we describe a circle; (in the plane of oscillation, that is, a plane perpendicular to the axis of suspension;) CO is the same from whatever point of this circumference we suspend the body. And therefore, the time of oscillation is the same.

Cor. 2. Also, if we describe a circle with radius GO, the time of oscillation will be the same, whether the body is suspended from any point in the circumference OO', or CC'.

88. Prop. To find from what axis a body must be suspended, that it may oscillate in the least time possible.

That is, amongst all the axes which are parallel to one another.

The time of oscillation is that of a simple pendulum, whose length is CO, fig. 93, C being the axis of suspension, O the centre of oscillation. Therefore, the time will be least, when CO is least.

$$\therefore \text{ by last Art.} \quad CG + \frac{k^2}{CG} = \text{min.}$$

and taking the differential coefficient,

$$1 - \frac{k^2}{CG^2} = 0; \quad \therefore k = CG.$$

Hence, the time will be the least when $CG = k$.

In that case, $GO = \dfrac{k^2}{CG} = k$ also.

Hence, the least time of oscillation will take place when the body is suspended from K, so that L being the centre of oscillation, $KG = GL$.

Cor. 1. The least time of oscillation is equal to that of a simple pendulum, whose length is $2k$.

Cor. 2. If we describe a circle with radius GK, the time will be the same when the body is suspended from any point in this circumference KK'.

Cor. 3. It has been seen in last Art., that the times of oscillation (t), are the same when the point of suspension is in any point of the circumferences CC', OO'; for a point between these circumferences, the time is less than t; and least for a point in the circumference KK'. For points within OO', or without CC', the time is greater than t, and becomes infinite as the point of suspension approaches the centre, or goes off to an infinite distance.

89. Prop. To find the centre of oscillation in given figures.

We have, (Art. 87.)

$$CO = \frac{\text{moment of inertia about axis}}{\text{mass} \,.\, CG} \ldots\ldots\ldots (a).$$

$$\text{Also, } GO = \frac{k^2}{CG} \ldots\ldots\ldots\ldots (a');$$

$k^2 M$ being the moment of inertia about an axis through the centre of gravity, and parallel to the axis of suspension. And either of these formulæ, by the assistance of the preceding part of this Chapter, will enable us to find O.

Ex. 1. To find the centre of oscillation of a straight line, suspended from its extremity.

By Ex. 1, Art. 79, $k^2 = \frac{a^2}{3}$, $2a$ being the length of the line;

$\therefore$ by (a'), $GO = \frac{a}{3}$ for $CG = a$. And $CO = a + \frac{a}{3} = \frac{2}{3} \cdot 2a$.

Ex. 2. A line AB, fig. 94, oscillating in its plane.

G the middle point, $CG = l$, $AB = 2a$;

$$GO = \frac{k^2}{l} = \frac{a^2}{3l};$$

$\therefore$ it is independent of the angle COA.

Ex. 3. A circle AB, fig. 95, oscillating in its own plane.

$CG = l$, radius $= a$; and by Ex. 12, Art. 81, $k^2 = \frac{a^2}{2}$;

$$\therefore\ GO = \frac{k^2}{l} = \frac{a^2}{2l}.$$

If C be in the circumference,

$$GO = \frac{a}{2},\quad CO = \frac{3a}{2}.$$

The same is true for a cylinder, about a line parallel to its axis.

Ex. 4. A circle oscillating about an axis in its own plane; (about CD, fig. 95.)

By Ex. 16, Art. 83,

$$k^2 = \frac{a^2}{4};\ GO = \frac{a^2}{4l}.$$

If the axis be a tangent to the circumference,

$$l = a,\ GO = \frac{a}{4},\ CO = \frac{5a}{4}.$$

Ex. 5. A globe about any axis.

Distance to the centre of the globe $= l$, radius $= a$.

By Ex. 20, Art. 84,

$$k^2 = \frac{2a^2}{5},\ GO = \frac{2a^2}{5l},\ CO = l + \frac{2a^2}{5l}.$$

Ex. 6. A cone about its vertex.

Axis $= a$, radius of base $= b$.

By Ex. 27, Art. 85,

$$\text{moment of inertia} = \frac{3}{5}\left(a^2 + \frac{b^2}{4}\right) M. \quad \text{Also } CG = \frac{3a}{4};$$

$$\therefore\ CO = \frac{4}{5}\left(a + \frac{b^2}{4a}\right) = \frac{4a}{5} + \frac{b^2}{5a}.$$

That the centre of oscillation may be in the centre of the base, we must have $CO = a$; whence $b = a$, and the cone is a right angled one.

Ex. 7. Let a solid, composed of two equal cones set base to base, oscillate in the direction of its axis.

Let G, fig. 96, be the common centre of the bases, GB a horizontal radius, and therefore parallel to CD the axis of rotation.

We must find the moment of inertia of the figure about GB. Now for the cone GA, we have, if F be its centre of gravity, and FK parallel to GB,

$$\text{moment about } GB = \text{moment about } FK + GF^2 \,.\, \text{cone},$$

$$\text{by Ex. 28,} \quad = \frac{3(a^2 + 4b^2)}{80} \,.\, \text{cone} + \frac{a^2}{16} \,.\, \text{cone}$$

$$= \frac{2a^2 + 3b^2}{20} \,.\, \text{cone}.$$

And hence the whole moment, or

$$k^2 M = \frac{2a^2 + 3b^2}{20} \,.\, 2 \,.\, \text{cone};$$

$$\therefore k^2 = \frac{2a^2 + 3b^2}{20}. \quad \text{And if } CG = l,$$

$$CO = l + \frac{k^2}{l} = l + \frac{2a^2 + 3b^2}{20\,l}.$$

90. Prop. Having a system composed of several separate bodies, whose centres of gravity and oscillation are known; to find the centre of oscillation of the whole.

In fig. 97, let g, o, be the centres of gravity and oscillation of m; g', o', of m'; g'', o'', of m''; and so on. And let G be the centre of gravity, and O that of oscillation for the whole system. Also let gh, $g'h'$, $g''h''$, &c. be perpendiculars on CG.

$$\text{moment of inertia of } m \text{ about } C = m \,.\, Cg \,.\, Co,$$

$$\text{of } m' \ldots\ldots = m' \,.\, Cg' \,.\, Co',$$

$$\&c. \ldots\ldots = \&c.$$

$$\therefore \text{ whole moment} = m \,.\, Cg \,.\, Co + m \,.\, Cg' \,.\, Co' + \&c.\,;$$

$$\therefore CO = \frac{m \,.\, Cg \,.\, Co + m' \,.\, Cg' \,.\, Co' + \&c.}{(m + m' + \&c.)\, CG}$$

$$= \frac{m \,.\, Cg \,.\, Co + m' \,.\, Cg' \,.\, Co' + \&c.}{m \,.\, Ch + m' \,.\, Ch' + \&c.}.$$

91. Prop. To find practically the length of a pendulum, which oscillates seconds.

If we know the exact length of a simple pendulum which makes a given number of small oscillations in 24 hours, we can find the length of a pendulum which shall oscillate in any given time as 1 second. But it is impossible to form a pendulum which may, with sufficient regard for accuracy, be considered as a simple pendulum; that is, as a single point suspended by a string without weight. It is necessary, therefore, in our experiment, to find the distance between the centre of suspension, and of oscillation, of the oscillating body. And the difficulty of the case is to determine accurately this latter point; for the unavoidable irregularities of figure and density make its geometrical determination include errors which the delicacy of the inquiry renders important.

To avoid these sources of inaccuracy, Captain Kater has ingeniously employed the property of a compound pendulum, proved in Art. 87; viz., that the centres of oscillation and suspension are reciprocal. It follows from that property, that if a pendulum have two centres of suspension, and oscillate on them, first with one end uppermost, and then with the other; so that the times of oscillation in the two cases may be exactly equal; the distance of these two centres will be the length of the equivalent simple pendulum, whatever be the irregularities of form or composition in the instrument. The manner in which this effect was produced, was as follows:

A brass pendulum *CD*, fig. 98, was furnished with two axes, from which it could be suspended; one passing through *C*, and the other through *O*. Besides the principal weight *D*, it was provided with a smaller sliding weight *F*, which could be moved along the stem *CD*; and this weight was to be moved till the number of oscillations in a given time, (as 24 hours,) was the same, whether the pendulum was suspended from *C* or from *O*.

F was placed in such a position, that by moving it from *O*, as to *f*, the number of oscillations about *C* in twenty-four hours was increased; and by the same change, the number of oscillations about *O* in the same time was still more increased. We shall afterwards consider this position mathematically. The adjustment was thus made.

Let the weight be at *F*, and let the number of oscillations in 10^m about *C* be 606, and about *O* be 601. Now let *F* be moved to *f*; and let the oscillations in 10^m be 607 about *C*, and 609 about *O*, (because the latter are more affected than the former). Then, the proper position of the slider is somewhere between *F* and *f*. Let it be placed at f', bisecting Ff; and let the oscillations in this case be $606\frac{1}{2}$, and 606; then, the proper position is between *f* and f'; and so on. Observing always, that if the number of vibrations about *C* be the greater, the slider must move toward *C*; and if the contrary, it must move towards *O*. By this means, continually halving the distance last moved, we may make the oscillations about *C* and *O* approach within any required degree of exactness. The distance of *C* and *O* being then measured, will give the length of a pendulum which makes a known number of oscillations in 10 minutes*.

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* There were, in Captain Kater's experiments, a number of contrivances which it would detain us too long to describe. Besides the slider *F*, he had another moveable weight *E*; and he made the numbers of oscillations nearly equal by means of this, before he attempted a more accurate adjustment by the slider.

The axes through *C* and *O*, were made with *knife edges*, which resting on planes of agate, turned as nearly as possible on a mathematical line. It is however true, as has been proved by Laplace, and will be shewn hereafter, that, if they had been cylinders, their distance would still have given exactly the length of the simple pendulum. The method of determining the number of vibrations in 24 hours was elegant; it was done by placing the pendulum in front of a clock pendulum, oscillating nearly in the same time; and observing the intervals at which the two pendulums coincided. Corrections were also to be made, for the magnitude of the arc vibrated; for the bouyancy of the atmosphere; for the temperature, &c. The reader will find Captain Kater's account of his method, and its results, in the *Phil. Trans.* for 1818, p. 33.

92. Prop. To determine the effect produced by the change of position of the moveable weight in last Article.

We shall consider any pendulum, and a small weight moveable along the line passing through the centres of suspension and gravity.

Let m be the mass of the pendulum, independent of the moveable weight; h the distance of its centre of gravity from the centre of suspension C, l the distance of its centre of oscillation. And let μ be the moveable weight, λ the distance from C of its centre of gravity, or of oscillation, supposing them to coincide because it is small. And let L be the length of the corresponding simple pendulum; then, by Art. 90,

$$L = \frac{mhl + \mu\lambda^2}{mh + \mu\lambda}; \text{ and } L \text{ and } \lambda \text{ being variable,}$$

$$dL = \frac{\mu^2\lambda^2 + 2m\mu h\lambda - m\mu hl}{(mh + \mu\lambda)^2} \cdot d\lambda$$

$$= \frac{\mu^2(\lambda - \alpha)(\lambda + \beta)}{(mh + \mu\lambda)^2} d\lambda;$$

where $\mu\alpha = \sqrt{(m^2h^2 + m\mu hl)} - mh$;

$$\mu\beta = \sqrt{(m^2h^2 + m\mu hl)} + mh.$$

Let $Cx = \alpha$; then, if F be below x, dL will be negative, when $d\lambda$ is negative; and hence, if F be moved towards C, L will be shortened, and the number of oscillations about C in a given time will be increased. And this increase will be the slower, as F comes nearer to x. At x a small change in F's position would produce no effect, and, when F is between C and x, the effects would be contrary to the former ones.

Let the distance CO of the two points of suspension be $2a$; and M being the middle point of CO, let F be the position of the moveable weight, when the oscillations about C and about O are performed in the same time; therefore in this case $L = CO = 2a$. Let $MF = \delta$; therefore, when $L = 2a$, $\lambda = a + \delta$. Hence,

$$2a = \frac{mhl + \mu(a + \delta)^2}{mh + \mu(a + \delta)};$$

$$\therefore l = 2a + \frac{\mu(a^2 - \delta^2)}{mh}.$$

$$\text{Hence, } \mu a = mh\left\{\sqrt{\left(1+\frac{2\mu a}{mh}+\frac{\mu^2(a^2-\delta^2)}{m^2h^2}\right)}-1\right\}.$$

Expanding, and neglecting higher powers of $\frac{\delta\mu}{mh}$.

$$a = a - \frac{\mu\delta^2}{2mh}.$$

Hence, x is between M and C, and near to M.

Similarly, if y be the point where the weight F does not affect the oscillations about O, y is between M and O, and near to M.

If F be between x and y, we shall, by sliding F towards O, lengthen the oscillations about C, and shorten those about O, and *vice versâ*. But, if F be in Cx, or in Oy, we shall, by sliding it towards the middle, shorten both sets of oscillations; and by sliding it from the middle lengthen both, though in different degrees.

CHAP. IV.

MOTION OF MACHINES.

93. We shall in the present Chapter apply the preceding principles to determine the motion of the mechanical powers, and other simple combinations.

Sect. I. *Motion about a fixed Axis.*

Prop. To determine the motion of weights on a wheel and axle, fig. 99.

Let P draw up Q by means of strings wrapping round two cylinders A, B, which have a common horizontal axis. Let a, b, be the radii of the cylinders respectively; and Mk^2 the moment of inertia of the machine AB, about its axis. We shall then have

impressed forces, Pg at distance a, $-Qg$ at distance b;

of which the moment is $Pga - Qgb$.

Hence, by Cor. 4. to Art. 74, we have

$$\text{accelerating force on } P = \frac{(Pa - Qb)\,ga}{Pa^2 + Qb^2 + Mk^2} \text{ downwards;}$$

$$\text{accelerating force on } Q = \frac{(Pa - Qb)\,gb}{Pa^2 + Qb^2 + Mk^2} \text{ upwards.}$$

And these being constant, the motion may be found by the formulæ for constant forces.

Cor. 1. If $Qb > Pa$, the force will be in the opposite direction, and Q will descend.

Cor. 2. If Tg be the tension of the string by which P hangs, P is impelled downwards by its weight, and upwards by the tension. Hence, the moving force on P is $Pg - Tg$, and the accelerating force $\frac{(P-T)\,g}{P}$;

$$\therefore \frac{(P-T)}{P} g = \frac{(Pa - Qb)\,ga}{Pa^2 + Qb^2 + Mk^2},$$

$$T = \frac{P\,(Qb^2 + Qab + Mk^2)}{Pa^2 + Qb^2 + Mk^2}.$$

Similarly, if T' be the tension of the string by which Q hangs,

$$T' = \frac{Q\,(Pa^2 + Pab + Mk^2)}{Pa^2 + Qb^2 + Mk^2}.$$

Cor. 3. The pressure on the centre of motion arising from P, Q, will be the sum of these tensions, (see next Section)

$$\therefore \text{pressure on the centre} = \frac{PQ\,(a+b)^2 + (P+Q)\,Mk^2 g}{Pa^2 + Qb^2 + Mk^2}.$$

94. Prop. To determine the motion of weights acting on a combination of wheels and axles, fig. 100.

The wheels and axles may act on each other, either by means of teeth as at D, or by strings passing round both, and turning them by friction as at D'; or in other ways: the mechanical conditions of the problem are the same in all these cases.

Let a, b be the radii of the first wheel, and of its axle; a', b' of the second, a'', b'' of the third, and so on. Then the impressed moving forces are Pg acting at A, and Qg in the opposite direction at B. By Statics, the latter would be counterbalanced by a force at P equal to $Qg \cdot \frac{bb'b''}{aa'a''}$. Hence, the moving force impressed is equivalent to

$$Pg - Qg\frac{bb'b''}{aa'a''} \text{ at } A, \text{ at distance } a \text{ from } C.$$

Let Mk^2, $M'k'^2$, Mk''^2 be the moments of inertia of the respective wheels M, M', M'' about their centres, (including the axles).

And let x be the effective accelerating force on P or on A. Then, since the accelerating forces are as the velocities, the accelerating force at D or E will be $\frac{bx}{a}$; that at D' or E' will be $\frac{bb'x}{aa'}$; and that at B or Q, $\frac{bb'b''x}{aa'a''}$.

Now, since the effective accelerating force at P is x, that at any distance r from C, in the wheel M, is $\frac{rx}{a}$; and if m be a particle at that distance, the effective moving force is $\frac{mrx}{a}$. And this is equivalent in its moment round C, to a force $\frac{mr^2x}{a^2}$, acting at A. And hence, the whole effective moving forces in M are equivalent to a force $\frac{x\Sigma mr^2}{a^2}$ acting at A; that is, they are equivalent to $\frac{x \cdot Mk^2}{a^2}$ at A.

Similarly, the effective moving forces in M' are equivalent to

$\frac{bx}{a} \cdot \frac{M'k'^2}{a'^2}$ at E; which is, by the property of the machine, equivalent in its effect to turn the system round C, to a force $\frac{b^2x}{a^2} \cdot \frac{M'k'^2}{a'^2}$ at A.

And the effective moving forces of M'' are equivalent to $\frac{bb'x}{aa'} \cdot \frac{M'k''^2}{a''^2}$ at E'; which is, with respect to C, equivalent to $\frac{b^2b'^2x}{a^2a'^2} \cdot \frac{M'k''^2}{a''^2}$ at A.

The effective accelerating force on Q is $\frac{bb'b''x}{aa'a''}$; which gives a moving force $\frac{bb'b''x}{aa'a''}Q$, equivalent to $\frac{b^2b'^2b''^2x}{a^2a'^2a''^2}$ at A.

Hence, we have the moment of the impressed forces about C

$$= ga\left(P - Q\frac{bb'b''}{aa'a''}\right).$$

And the moment of all the effective forces about C

$$= Pax + Qax\frac{b^2b'^2b''^2}{a^2a'^2a''^2}$$

$$+ Max.\frac{k^2}{a^2} + M'ax.\frac{b^2}{a^2}.\frac{k'^2}{a'^2} + M''ax.\frac{b^2b'^2}{a^2a'^2}.\frac{k''^2}{a''^2}.$$

Equating these (by Art. 73,) and putting n, n', n'' for $\frac{b}{a}$, $\frac{b'}{a'}$, $\frac{b''}{a''}$, we have

$$x = \frac{(P - Qnn'n'')g}{P + Qn^2n'^2n''^2 + M\frac{k^2}{a^2} + M'n\frac{k'^2}{a'^2} + M''nn'\frac{k''^2}{a''^2}}.$$

The accelerating force on $Q = nn'n''x$. These forces are constant.

95. A machine was constructed by Atwood, to measure the spaces and velocities of bodies descending by gravity, in order to

compare them with theory. It is represented in fig. 101. Two equal weights P, Q, hang by a fine string over a fixed pully M. One of them is made to descend, by placing upon it a small weight D, and the times and spaces of the motion are observed. The weights at P and Q are inclosed in equal and cylindrical boxes; so that the effect of the resistance of the air will be the same upon both, and will not affect the motion. And the effect of friction is nearly removed by making the axis of M very slender, and causing each end of it as C to rest upon *Friction Wheels*, as M, M'*. The times are observed by means of a pendulum, and the spaces by a scale of inches BF, down which P descends. To determine the velocity, P is made to pass through an opening MN, in a stage fastened to the scale BF; and the weight D, which is too large to pass, is left resting on M, N. Therefore, after passing the point E, P will move uniformly with the velocity acquired. When it has passed through a given space EF, is stopped by striking the stage F, which is there fixed to the scale.

The body P being allowed to descend from rest at a given point B, descends till D is heard to strike the stage M, N, and the time is noted; it then descends till P is heard to strike the stage F, and the time is noted: the space EF, divided by the interval of the

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* To shew that these wheels will diminish the effect of friction, we may consider friction as a force acting in a tangent to the axle. If the axle C rested on immoveable surfaces, and the friction were F, its effect at A would be $\frac{Fb}{a}$. But if the axle C rest upon friction wheels, their circumferences will turn with the circumference of the axle, and between them there will be no friction. The friction will take place at the axles C, C'; and if we suppose it to be F at the angle C', this will be equivalent to $F\frac{b'}{a'}$ at the circumference of the axle C, and to $F\frac{bb'}{aa'}$ at A. And as there are four ends of the axles C' for one C, the friction with friction wheels is $4F.\frac{bb'}{aa'}$. Hence, by means of such a contrivance, it is diminished in the ratio of $a' : 4b'$, supposing F to be the same in both.

latter times gives the velocity ; and the space BF, and the time of describing it, being known, we can compare our results with theory. The velocities are small, both because D is small, and because the wheels F, M', M'' are to be moved, and their moment of inertia enters the denominator of the accelerating force.

Observing, that besides the friction wheels M', M'', there are two others at the other end of the axis A ; calling the moment of each of these $M'k'^2$, and of M, Mk^2, and the radii of the wheels and axles a, b, a', b', we have

$$\text{accelerating force on } P = \frac{Dg}{2P+D+M\frac{k^2}{a^2}+4M'.\frac{b}{a}\frac{k'^2}{a'^2}}.$$

The effect of the inertia of the wheels is the same as if a mass $M\frac{k^2}{a^2}+4M'.\frac{bk^2}{aa'^2}$ were collected at the circumference of M.

The reader will find in Atwood's *Treatise on Rectilinear and Rotatory Motion*, Sect. 7, an account of experiments made with this machine. They all agreed accurately with the formulæ for constant forces.

96. Prop. To determine the motion of weights on a lever, fig. 102.

Let P, Q, be attached to the extremities of a lever whose arms are a, b; and let M be the mass of the lever, and h the distance of its centre of gravity. Let PQ be any position in which it makes an angle θ with the vertical. Then $a \cos. \theta$, $b \cos. \theta$, $h \cos. \theta$ are the perpendiculars from the centre upon the vertical directions of the forces of P, Q, M. And the moment of the forces is $(Pa+Mh-Qb)\, g \cos. \theta$, to make P descend. Hence, if Mk^2 be the moment of inertia of the lever itself, we have

$$\text{accelerating force on } P = \frac{(Pa+Mh-Qb)\, ga \cos. \theta}{Pa^2+Qb^2+Mk^2}$$

acting perpendicular to CP.

The space described by P in dt is $-ad\theta$; hence, by the formula $vdv=fds$, we have, v representing P's velocity,

$$vdv = -\frac{(Pa+Mh-Qb)\,ga\cos.\,\theta d\theta}{Pa^2+Qb^2+Mk^2};$$ and integrating,

$$v^2 = C - \frac{2\,(Pa+Mh-Qb)\,ga\sin.\,\theta}{Pa^2+Qb^2+Mk^2}.$$

If the lever descend from rest from a position AB, let AB make an angle θ_1 with the vertical, and we have

$$\frac{2\,(Pa+Mh-Qb)\,ga}{Pa^2+Qb^2+Mk^2}(\sin.\,\theta_1 - \sin.\,\theta) = v^2 = a^2\frac{d\theta^2}{dt^2}.$$

And hence, by integrating, we should find the relation of θ and t.

If P descend through a quadrant, we have, at the lowest point O,

$$v^2 = \frac{2\,(Pa+Mh-Qb)\,ga}{Pa^2+Qb^2+Mk^2}.$$

97. PROP. A body moveable about an axis C is *struck* at a given point by a given mass with a given velocity; to determine its motion, fig. 103.

Impact is, properly speaking, a violent pressure continued for a short time. Now if any force act at a distance a from the axis of a body whose moment of inertia is Mk^2, the effect produced at any instant will be the same as if a mass $\frac{Mk^2}{a^2}$ were collected at the distance a. Hence, the whole effect produced will be the same as if such a mass were substituted for the body, whatever be the time which the charge employs. And hence, the effect of perpendicular impact at a distance a will be the same as if it took place upon a mass $\frac{Mk^2}{a^2}$ placed there.

In fig. 103, let a mass P impinge directly on a system CA, with a velocity V; and let CA be a perpendicular on P's direction. If $CA = a$, the effect will be the same as if P impinged on $\frac{Mk^2}{a^2}$. Let the substances be supposed inelastic; and the bodies will both move with the same velocity after the impact; and since, by the third law of motion, the mass multiplied into the velocity will be the same before and after the blow, we shall have, if x be the velocity of A after the stroke,

$$x\left(P + \frac{Mk^2}{a^2}\right) = PV,$$

$$x = \frac{PVa^2}{Pa^2 + Mk^2}.$$

If the body be acted upon by no force after the impact, it will revolve uniformly. If it move about a horizontal axis, and be acted on by gravity, it will ascend till all the velocity be destroyed, and then descend, and so oscillate.

If the bodies be elastic, we must apply the rules for impact in that case. On this supposition, P and M will separate after the impact. And if the impact be not direct, we must, supposing the bodies perfectly smooth, take that part of it which is perpendicular to the surfaces at the point of contact.

98. An instrument depending upon these principles was constructed by Robins for the purpose of measuring the velocities of musquet and cannon bullets, and called the Ballistic Pendulum. It consisted in an iron plate CA, fig. 104, suspended from a horizontal axis at C, and faced with a thick board DE. When this was at rest, a bullet was fired into it as at P, which caused it to move through an arc MN. The chord of this arc was known by means of a ribbon fastened to the pendulum, as at N, and sliding through a slit at M, so that when drawn to the length MN it did not return. The ball stuck in the wood, and was prevented from going through by the iron.

Let O be the centre of oscillation of the pendulum, including the bullet. Then the motion of the pendulum will be the same when left to itself, as if all the matter were collected in O. And hence, the arc through which O will move will be that down which it would acquire the velocity which it has at the lowest point. If θ be this angle, the velocity acquired in describing it would be that acquired down the versed sine of θ; or down a perpendicular height CO. ver. sin. θ. Let $CO = l$; ∴ velocity² of O at lowest point $= \sqrt{(2gl \text{ ver. sin. } \theta)} = 2 \sin. \frac{\theta}{2} \sqrt{(gl)}$.

But since the velocity of P at the lowest point is by last Article

$$\frac{P V a^2}{P a^2 + M k^2},$$

the velocity of O, which is to this as CO to CP, will be

$$\frac{P V a l}{P a^2 + M k^2} = 2 \sin. \frac{\theta}{2} \sqrt{(g l)} \text{ by what has been said.}$$

If h be the distance from C of the centre of gravity of the pendulum, including the ball,

$$l = \frac{P a^2 + M k^2}{(P + M) h}; \; P a^2 + M k^2 = (P + M) h l;$$

$$\therefore \; P V a l = 2 \sin. \frac{\theta}{2} (P + M) h l \sqrt{(g l)},$$

$$V = 2 \sin. \frac{\theta}{2} \cdot \frac{P + M}{P} \frac{h}{a} \sqrt{(g l)}.$$

If the pendulum after being struck by the ball, makes n oscillations in a minute, we take

$$\text{time of oscillation} = \frac{60}{n} = \pi \sqrt{\frac{l}{g}}; \;\; \therefore \; \sqrt{(g l)} = \frac{60 g}{\pi n}.$$

$$\text{And, } V = 2 \sin. \frac{\theta}{2} . \frac{P + M}{P} . \frac{60 g h}{\pi n a}.$$

This agrees with Dr. Hutton's formula. We have $2 \sin. \frac{\theta}{2}$ by dividing the chord MN by the radius CN.

Dr. Hutton himself however, in his own experiments, found the velocity of balls, by suspending the cannon which he used, and observing the arc through which it was driven by the recoil. The same formula is still applicable, M now representing the weight of the cannon and its appendages without the ball. For the effect will be the same, whether a velocity be communicated to the pendulous body by the impact of the ball, or its reaction. And the momentum communicated at the axis of the cannon will be PV, because the *momentum* communicated to the ball in one direction, and to the pendulum in the other, must be equal.

It is found by experiments of this kind, that the velocity of musquet and cannon bullets varies from 1600 to 2000 feet per second.

SECT. II. *Motion of Bodies unrolling.*

99. PROP. A cylindrical body unrolls itself from a vertical string, the other end of which passes over a fixed pully, and supports a weight; to determine the motion, fig. 105.

One end of the string is supposed to be fastened to the surface of the body M, so that it cannot slide, but can only descend by unrolling.

Let the tension of the string ABP be Tg; (if we neglect the inertia of B, it will be the same throughout). Let the moment of inertia of M be Mk^2, and $CA = a$: and the effective accelerating force on C downwards $= x$.

The effective moving force on P downwards is $(P - T)g$, and the accelerating force is $\left(1 - \frac{T}{P}\right)g$. And this will be the effective accelerating force upwards on any point of the string BA, as A. Now since C descends by an accelerating force x, and A ascends by an accelerating force $\left(1 - \frac{T}{P}\right)g$, the relative accelerating force of A round C is $x + \left(1 - \frac{T}{P}\right)g = y$ suppose. And hence, the effective accelerating force round C, of a point at a distance r from C is $\frac{ry}{a}$: and the moment of all the effective accelerating forces round C is $\frac{y\Sigma mr^2}{a}$, or $\frac{Mk^2 y}{a}$. The impressed forces on M are the weight Mg acting downwards at C, and Tg acting upwards at A. Hence, to establish an equilibrium between the impressed and effective forces, according to Art. 73, we must have the forces equal,

$$\text{or } Mx = Mg - Tg;$$

and their moments with respect to C equal,

$$\therefore \frac{Mk^2 y}{a} \text{ or } \frac{Mk^2}{a}\left\{x + \left(1 - \frac{T}{P}\right)g\right\} = Tga.$$

Eliminating T, we have

$$Mk^2\left\{x + g - \frac{M}{P}(g - x)\right\} = Ma^2(g - x),$$

$$x = \frac{Pa^2 + (M - P)k^2}{Pa^2 + (M + P)k^2} \cdot g.$$

Cor. 1. Since $Tg = M(g - x)$, we have

$$T = \frac{2MPk^2}{Pa^2 + (M + P)k^2}.$$

Cor. 2. The accelerating force on P, is

$$\left(1 - \frac{T}{P}\right)g = \frac{Pa^2 - (M - P)k^2}{Pa^2 + (M + P)k^2} \cdot g.$$

Cor. 3. It is not necessary that the whole body should be cylindrical, but only that the part of it from which the string unrolls should be a cylinder, of which the axis passes through the centre of gravity. The vertical plane of the string must be perpendicular to the axis of the cylinder, and pass through the centre of gravity.

Cor. 4. If the figure be a cylindrical shell of small thickness, $k = a$,

$$\text{accelerating force on } C = \frac{Mg}{2P + M},$$

$$\text{accelerating force on } P = \frac{2P - M}{2P + M} \cdot g,$$

$$\text{tension} = \frac{2MP}{2P + M}.$$

Cor. 5. If the figure be a solid homogeneous cylinder, $k^2 = \frac{a^2}{2}$,

$$\text{accelerating force on } C = \frac{P + M}{3P + M} \cdot g,$$

$$\text{accelerating force on } P = \frac{3P - M}{3P + M} \cdot g,$$

$$\text{tension} = \frac{2MP}{3P + M}.$$

100. Prop. A cylindrical body unrolls itself from a vertical string, the other end of which is fixed; to determine the motion, fig. 106.

If we assume P, in last Article, to be such that it shall neither ascend nor descend, we may suppose the string AB to be fixed at the point B, and the motion will be the same as before. We must therefore in this case, have the accelerating force on $P = 0$;

$$\text{or, } Pa^2 - (M - P)k^2 = 0, \text{ whence, } P = \frac{Mk^2}{a^2 + k^2}.$$

$$\text{Hence also, } T = \frac{Mk^2}{a^2 + k^2}.$$

$$\text{And accelerating force on } C = \frac{(M - T)g}{M} = \frac{a^2 g}{a^2 + k^2}.$$

Cor. 1. If the figure be a cylindrical shell, $k = a$;

$$\text{accelerating force on } C = \frac{g}{2}, \; T = \frac{M}{2}.$$

Cor. 2. If the figure be a solid cylinder, $k^2 = \frac{a^2}{2}$;

$$\text{accelerating force on } C = \frac{2g}{3}, \; T = \frac{M}{3}.$$

Cor. 3. If the figure be a globe, $k^2 = \frac{2a^2}{5}$;

$$\text{accelerating force on } C = \frac{5g}{7}, \; T = \frac{2M}{7}.$$

Cor. 4. If the string, instead of being vertical, be laid along an inclined plane as BA, fig. 107, the same conclusions are manifestly true; putting for g the force of gravity down the plane, which is g . sin. inclination. The tension will also be Tg . sin. inclination.

Cor. 5. If M, instead of rolling by means of a string, roll down the plane in consequence of the friction entirely preventing its sliding, the results will be the same. The tension of the string is now replaced by the effort which the friction exercises to prevent sliding. Hence, when a body rolls down an inclined plane, the accelerating force is $\frac{1}{2}$ if it be a hollow cylinder, $\frac{2}{3}$ if it be a solid cylinder, and $\frac{5}{7}$ if it be a globe, of the force with which a body would slide down the plane, if friction were removed.

SECT. III. *Motion of Pullies.*

101. PROP. One body draws another over a single fixed pully; to determine the motion, fig. 108.

Let Mk^2 be the moment of inertia of the pully, a its radius. And let x be the effective accelerating force on P downwards; which is therefore the accelerating force on the circumference of the pully M, and on Q upwards. Let Tg be the tension of the string AP, and $T'g$ of BQ. Hence, the force impressed at the circumference of the pully is $Tg - Tg'$, and therefore,

$$x = \frac{(T - T')\,ga^2}{Mk^2} \dots\dots\dots (1).$$

But the accelerating force on $P = x = \frac{(P - T)\,g}{P}$;

and the accelerating force on $Q = x = \frac{(T' - Q)\,g}{Q}$;

$$\therefore\ Px = (P - T)\,g, \quad Qx = (T' - Q)\,g,$$

$$(P + Q)\,x = (T' - T)\,g + (P - Q)\,g \dots\dots (2).$$

Multiply (1) by Mk^2, and (2) by a^2, and add;

$$\therefore\ Mk^2x + (P + Q)\,a^2x = (P - Q)\,ga^2,$$

$$x = \frac{(P - Q)\,ga^2}{Mk^2 + (P + Q)\,a^2}.$$

COR. 1. The tensions of AP, and BQ, are respectively

$$\frac{(Mk^2 + 2\,Qa^2)\,Pg}{Mk^2 + (P + Q)\,a^2}, \quad \frac{(Mk^2 + 2\,Pa^2)\,Qg}{Mk^2 + (P + Q)\,a^2}.$$

COR. 2. Hence, when strings are in motion about pullies, the tension of each string is no longer the same throughout its length.

A part of the tension of PA is employed in turning M; and it is only the remainder which is continued along the cord, so as to act in BQ.

The same results might have been obtained from Art. 93, by making the radii of the wheel and axle equal.

102. Prop. In the single moveable pully with the strings parallel; to determine the motion, fig. 109.

Let P, Q, be the weights; Mk^2, $M'k'^2$, the moments of the pullies; a, a' their diameters. And let the tension of $AP = Tg$, of $BD = T'g$, of $EF = t'g$. Then, if x be the accelerating force on P, $\frac{x}{2}$ will be the accelerating force on Q, because it moves with half the velocity. Also, the accelerating force at the circumference of M will be x: and since, while E remains fixed, the centre of M' rises with half P's velocity, the relative motion of E *round the centre*, is half P's velocity, and therefore, the effective accelerating force at the circumference of M' round C' is $\frac{x}{2}$.

And if we consider the forces which act upon M', we have

Impressed forces, $T'g$, $t'g$ upwards, Qg downwards;

Q including the weight of M'.

Effective forces, $\frac{x}{2}$ upwards for Q, and $\frac{x}{2} \frac{M'k'}{a'^2}$ at the circumference, turning M' round C'.

Hence, by Art. 73, we must have

$$(T' + t')g - Qg = Q\frac{x}{2};$$

and, considering the moments with respect to C',

$$(T' - t')a'g = \frac{M'k'^2}{a'} \cdot \frac{x}{2};$$

$$\therefore 2T'g = Qg + Q\frac{x}{2} + \frac{M'k'^2}{a'^2} \cdot \frac{x}{2}.$$

Also, we have, as before,

$$x = \frac{P-T}{P}g, \text{ or } (P-T)g = Px,$$

$$x = \frac{(T-T')ga^2}{Mk^2}, \text{ or } (T-T')g = \frac{Mk^2}{a^2}x;$$

add these, and the former one

$$T'g = \frac{Qg}{2} + \frac{Qx}{2^2} + \frac{M'k'^2}{a'^2} \cdot \frac{x}{2^2},$$

and we have

$$Pg = \frac{Qg}{2} + \left(P + \frac{Q}{2^2} + \frac{Mk^2}{a^2} + \frac{M'k'^2}{2^2a'^2}\right)x;$$

$$\therefore\ x = \frac{\left(P - \frac{Q}{2}\right)g}{P + \frac{Q}{2^2} + \frac{Mk^2}{a^2} + \frac{M'k'^2}{2^2a'^2}};$$

from this also the tensions might be found.

103. Prop. In the system of moveable pullies, where each hangs by a separate string; to determine the motion, fig. 110.

The strings are supposed parallel.

M, M', M'', M''' the pullies, Mk^2, $M'k'^2$, $M''k''^2$, &c. their moments; a, a', a'', &c. their radii. Let x be the effective accelerating force on P; then $\frac{x}{2}$ will be the accelerating force on M'; $\frac{x}{2^2}$ on M''; $\frac{x}{2^3}$ on M'''; and these will also be the effective accelerating forces producing rotation at the circumferences of M, M', M'', &c. Then, by reasoning with respect to each pully, as we have done for M' in last Article, we have

$$(P-T)g = Px, \quad (T-T')g = \frac{Mk^2}{a^2}x.$$

$$(T'+t')g - M'g - T''g = M'\frac{x}{2},\ (T'-t')g = \frac{M'k'^2}{a'^2}\cdot\frac{x}{2},$$

$$(T''+t'')g - M''g - T'''g = M''\frac{x}{2^2},\ (T''-t'')g = \frac{M''k''^2}{a''^2}\cdot\frac{x}{2^2};$$

and so on.

Eliminating t', t'', &c. from each successive pair, we have

$$Pg = Px + Tg,$$

$$Tg = \frac{Mk^2}{a^2}x + T'g,$$

$$T'g = \frac{M'k'^2}{a'^2}\cdot\frac{x}{2^2} + \frac{M'x}{2^2} + \frac{M'g}{2} + \frac{T''g}{2},$$

$$T''g = \frac{M''k''^2}{a''^2}\cdot\frac{x}{2^3} + \frac{M''x}{2^3} + \frac{M''g}{2} + \frac{T'''g}{2},$$

and so on. Therefore,

$$Pg = Px + \frac{Mk^2}{a^2}x + \frac{M'k'^2}{a'^2}\cdot\frac{x}{2^2} + \frac{M''k''^2}{a''^2}\cdot\frac{x}{2^4} + \&c.$$

$$+\frac{M'x}{2^2} + \frac{M''x}{2^4} + \&c. + \frac{M'g}{2} + \frac{M''g}{2^2} + \&c. + \frac{T^{iv}g}{2^3}.$$

The law of continuation is manifest. And the last tension (T^{iv} in the figure) is that which immediately raises Q. Hence, we have the effective accelerating force on Q,

$$Q = \frac{x}{2^3} = \frac{(T^{iv}-Q)g}{Q};$$

$$\therefore\ T^{iv}g = Qg + \frac{x}{2^3}.$$

Substituting this, and obtaining the value of x, we have

$$x = \frac{\left(P - \frac{M'}{2} - \frac{M''}{2^2} - \&c. - \frac{Q}{2^3}\right)}{P + \frac{Mk^2}{a^2} + \frac{M'}{2^2} + \frac{M'k'^2}{2^2a'^2} + \frac{M''}{2^4} + \frac{M''k''^2}{2^4a''^4} + \&c.};$$

and similarly for any number of pullies.

By similar reasoning, we shall have the accelerating force in the system of pullies, when each is attached to the weight. But more immediately in all such cases by the following Proposition.

104. PROP. To find the accelerating force on any machine whatever.

Let P be one of the bodies of the machine, and let P' be the mass, which, placed at P, would preserve the equilibrium. Then the weight $(P-P')\,g$ is the *impressed force*, which produces the motion.

Let u be the velocity of P, and v, v', &c. the velocities of any other bodies m, m', &c. in the system. Then, if x be the effective accelerating force on P, $\frac{vx}{u}$ will be that on m, and $\frac{mvx}{u}$ the effective moving force. Therefore, the forces which must balance each other by Art. 73, are $(P-P')\,g$ in one direction, and $\frac{mvx}{u}$, $\frac{m'v'x}{u}$, &c. in the opposite.

Now u, v, v', &c. may be considered as the *virtual velocities* of the points where these forces are applied. Hence, by the principle of virtual velocities,

$$(P-P')\,g\,.\,u - Px\,.\,u - \frac{mvx}{u}\,.\,v - \frac{m'v'x}{u}\,.\,v' - \text{\&c.} = 0;$$

$$\therefore\ x = \frac{(P-P')\,g}{P + \frac{mv^2}{u^2} + \frac{m'v'^2}{u^2} + \text{\&c.}} = \frac{(P-P')\,g}{P + \frac{\Sigma mv^2}{u^2}}.$$

Let the motion of any mass, as M, be considered as for a moment taking place about a fixed axis. This is always possible, (see Note, p. 222). Let a be the distance of the centre of gravity from this axis, and α its velocity; then, if m be a particle at the distance r, m's velocity $= \frac{r\alpha}{a}$. And for the whole of M,

$$\Sigma mv^2 = \Sigma \frac{mr^2\alpha^2}{a^2} = \frac{\alpha^2}{a^2}\,\Sigma mr^2 = \frac{\alpha^2}{a^2}\,M\,(a^2+k^2),$$

using the same notation as before,

$$= Ma^2 + M \cdot \frac{k^2 a^2}{a^2}.$$

Hence, in the denominator of the accelerating force x, we shall have, for each mass M, two terms in the denominator, such as we have just found. It may be observed, that $\frac{a}{a}$ is the angular velocity of M about its centre.

It will be seen by comparison, that this includes all the preceding propositions of this Chapter.

SECT. IV. *Maximum effect of Machines.*

105. PROB. I. *A weight* P, *acting at a wheel, produces rotation in a mass which moves about an axis passing through the centre of gravity; it is required to determine the distance at which* P *must act, that the angular velocity, generated in a given time, may be the greatest possible.*

Here the accelerating force on P is

$$f = \frac{Pa^2 g}{Pa^2 + Mk^2},$$

P acting at a radius a. And the velocity generated in time t in the circumference at which P acts, is ft. And hence,

$$\text{angular velocity} = \frac{ft}{a}; \; \therefore \frac{f}{a} = \text{max.},$$

$$\frac{a}{Pa^2 + Mk^2} = \text{max.} \quad \frac{Pa^2 + Mk^3}{a} = \text{min.},$$

$$Pa + \frac{Mk^2}{a} = \text{min. whence } P - \frac{Mk^2}{a^2} = 0,$$

$$a = k\sqrt{\frac{M}{P}}.$$

PROB. II. P *raises* Q *by means of a wheel and axle, as in Art.* 93; *the axle being given, to find the wheel, that the time of* Q*'s ascending through a given space, may be the least possible.*

The accelerating force on Q is

$$f = \frac{(Pa - Qb)gb}{Pa^2 + Qb^2 + Mk^2}.$$

And, as this is constant, we have $t = \sqrt{\frac{2s}{f}}$, which will manifestly be least when f is greatest. Therefore, we must have

$$\frac{Pa - Qb}{Pa^2 + Qb^2 + Mk^2} = \text{max.}$$

If we suppose a to vary, k will also vary in a manner depending on the form of the wheel; but if we suppose M to be small, we have, neglecting it,

$$\frac{Pa - Qb}{Pa^2 + Qb^2} = \text{max.}$$

and differentiating, supposing a variable,

$$P(Pa^2 + Qb^2) - 2Pa(Pa - Qb) = 0;$$

$$Pa^2 - 2Qab - Qb^2 = 0;$$

$$\therefore a = \frac{Qb}{P}\left\{1 + \sqrt{\left(1 + \frac{P}{Q}\right)}\right\}.$$

If P be small compared with Q, this will give nearly

$$a = \frac{2Qb}{P} + \frac{b}{2}.$$

The weight P must act at a little more than twice the distance at which it would balance Q.

Prob. III. *The wheel and axle, and the weight* P, *being given; to find* Q, *so that the momentum communicated to it in a given time, may be the greatest possible.*

The accelerating force f on Q, is the same as before, and the velocity acquired by it in a given time, is $v = ft$. And hence, we must have Qft a maximum, or Qf a maximum;

$$\therefore \frac{(Pa - Qb)\,Q}{Pa^2 + Qb^2 + Mk^2} = \text{max.};$$

$$\therefore (Pa - 2\,Qb)(Pa^2 + Qb^2 + Mk^2) - b^2\,(PQa - Q^2 b) = 0.$$

And hence, Q is found by the solution of a quadratic equation.

If we neglect M, we have

$$Q = \frac{Pa^2}{b^2}\left\{\sqrt{\left(1 + \frac{b}{a}\right)} - 1\right\}:$$

and if we suppose b small, compared with a,

$$Q = \frac{Pa}{2b} - \frac{P}{8};$$

Q is nearly half the weight which P would support.

PROB. IV. *A body revolving round an axis, strikes another given body* P *in a direction perpendicular to the radius; to find the distance at which the impact must take place that the velocity communicated may be the greatest possible.*

Let r be the distance of the striking point from the axis, and Mk^2 the moment of inertia with respect to the axis. Any pressure acting at the distance r, will produce the same effect as if there were collected at that distance a mass $\frac{Mk^2}{r^2}$. Hence, the impact, which is only a short and violent pressure, will produce the same effect, as if such a mass were to impinge on the given body P. Let α be the angular velocity, then $r\alpha$ is the velocity at the point of impact. And when a body A impinges with a velocity a on B at rest, the velocity after impact is $\frac{Aa}{A+B}$; supposing the bodies inelastic, (*Elem. Treat. on Mechanics*, Art. 166.). Hence, the velocity communicated to P is

$$\frac{\frac{Mk^2\alpha}{r}}{\frac{Mk^2}{r^2} + P} = \frac{Mk^2\alpha r}{Mk^2 + Pr^2},$$

which is to be a maximum. Whence we find

$$r = k\sqrt{\frac{M}{P}}.$$

If the body be in any degree elastic, the result will be the same, for the velocity of B in that case, is $\frac{(1+e)\,Aa}{A+B}$, (*Elem. Treat. on Mechanics*, Art. 167.).

CHAP. V.

PRESSURE ON A FIXED AXIS.

106. When a body revolves about a fixed axis, the axis in general suffers some pressure, depending upon the form and motion of the body, and on the forces which act upon it. The different parts of the system influence each other's motions by the intervention of this axis; and it supplies, as it were, the difference of the *forces impressed*, and the *effective forces*, so that they may balance each other. D'Alembert's Principle, Art. 73, being general for *all* the forces which act upon a system, will enable us to find the pressures in question.

If a body, not acted upon by any forces, have a rotatory motion, it will retain it for ever, (neglecting friction, &c.) and go on with a uniform velocity. If it be acted upon by external forces, its velocity will be variable. We shall consider successively these two cases.

Sect. I. *A Body revolving, acted on by no Forces.*

Prop. A system acted on by no forces, revolves about a fixed axis with *uniform* velocity; to find the pressure on the axis.

Let zC, fig. 111, be the axis of rotation, xCy a plane perpendicular to it, in which Cx and Cy are at right angles. The system may be referred to three rectangular co-ordinates, parallel to Cx, Cy, Cz, which we shall call x, y, z. And let M be any particle of the body, and MO perpendicular to the axis. M describes a circle about O, and for this purpose, it must be retained in a circle by a force in the direction MO, the magnitude of which force is known from Art. 20. If ω be the angular velocity, and $OM = r$, *the effective accelerating force in* $MO = r\omega^2$. Hence, if m be the mass of the particle at M, $mr\omega^2$ will be the *effective moving force* in MO.

The force in MO may be resolved in the directions MN, NO, parallel to Cx and Cy. And we shall have for the particle m,

$$\text{moving force parallel to } x = mr\omega^2 \frac{MN}{MO} = mx\omega^2;$$

$$\text{to } y = mr\omega^2 \frac{NO}{MO} = my\omega^2.$$

And *effective forces*, analogous to these, act on each of the points of the system.

Also, the moments of these forces will be

about the axis Cx, $\quad 0$, and $myz\omega^2$,
about the axis Cy, $\quad mxz\omega^2$, and 0.

And if Σ represent the sum of all the products corresponding to different points analogous to m, we shall have for the *whole effective forces*, $\omega^2\Sigma mx$, $\omega^2\Sigma my$, and for their moments about Cx, and Cy, $\omega^2\Sigma myz$, $\omega^2\Sigma mxz$: these forces acting to diminish x and y.

The *impressed forces* are none except the reaction of the axis, which is the pressures upon it inverted. We may reduce these pressures to two*, acting at given points.

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* They cannot always be reduced to one, as will be seen.

When forces are reducible to two equal ones, acting on a line in opposite directions, at different points, the effect produced is of a peculiar kind, and may be called *Torsion*. There is no tendency to a change of *place*, but only of position. The axis about which this torsion takes place, passes through the centre of gravity, as will be shewn.

If the axis be supported at two points A and B, we may suppose the forces to act at these points. Let U and V be the forces which act at those points, both being in planes perpendicular to the axis; for it is evident that there will be no pressure in the direction of the axis. Let U make an angle ϕ with the plane xz, and V an angle ψ with the same plane. And let $CA = a$, $CB = b$. Then the pressures on the axis will be

$$\text{parallel to } x, \quad U \cos. \phi \text{ at } A, \quad V \cos. \psi \text{ at } B,$$
$$\text{to } y, \quad U \sin. \phi \text{ at } A, \quad V \sin. \psi \text{ at } B.$$

$$\text{Hence, } U \cos. \psi + V \cos. \psi = \omega^2 \Sigma m x,$$
$$U \sin. \phi + V \sin. \psi = \omega^2 \Sigma m y;$$

because the forces must be equal.

$$\text{And } Ua \cos. \phi - Vb \cos. \psi = \omega^2 \Sigma m x z,$$
$$Ua \sin. \phi - Vb \sin. \psi = \omega^2 \Sigma m y z;$$

because the moments of the forces, with respect to Cx and Cy must be equal.

From these four equations, a and b being known, we may find the four quantities U, V, ϕ, ψ, (or rather, $U \cos. \phi$, $V \cos. \psi$, $U \sin. \phi$, $V \sin. \psi$.)

107. We may suppose C to be taken so that the plane xCy, fig. 112, passes through the centre of gravity G. And we shall be able to reduce all the forces to

(1) A force CR in the plane xCy.

(2) Two equal forces FS, $F'S'$ at equal distances CF, CF', and acting in opposite directions.

These latter forces will produce no effect, except a *torsion* round C. The tendency to a motion of the centre of gravity will entirely result from the force CR.

Prop. It is required to find the force which acts at C, and the forces which tend to turn the system round C.

Let the former force be R, and make with the plane xz an angle ρ; let the forces at F and F' be each $\frac{1}{2}S$, and make with xz an angle σ; and let $CF = CF' = f$. We shall then have

impressed forces parallel to x,

$$R \cos. \rho \text{ at } C, \ \frac{S}{2} \cos. \sigma \text{ at } F, \ -\frac{S}{2} \cos. \sigma \text{ at } F',$$

$$\text{to } y, \ R \sin. \rho \text{ at } C, \ \frac{S}{2} \sin. \sigma \text{ at } F, \ -\frac{S}{2} \sin. \sigma \text{ at } F'.$$

And considering the moments of the forces with respect to the areas Cx and Cy, we have, as before, by the conditions of equilibrium with the effective forces,

$$R \cos. \rho = \omega^2 \Sigma mx, \ R \sin. \rho = \omega^2 \Sigma my,$$

$$Sf \cos. \sigma = \omega^2 \Sigma mxz, \ Sf \sin. \sigma = \omega^2 \Sigma myz.$$

The two former equations give

$$\tan. \rho = \frac{\Sigma my}{\Sigma mx}; \ R = \omega^2 \surd \{(\Sigma mx)^2 + (\Sigma my)^2\}.$$

If $\bar{x}$ and $\bar{y}$ be the co-ordinates x and y for the centre of gravity, h its distance from the axis, and M the whole mass, we have

$$\tan. \rho = \frac{\bar{y}}{\bar{x}}, \ R = \omega^2 M \surd (\bar{x}^2 + \bar{y}^2) = \omega^2 Mh.$$

Hence, the force R passes through the axis and the centre of gravity; and its magnitude is the same as if the whole mass were collected in the centre of gravity.

If the axis pass through the centre of gravity, the force R is 0.

The two latter equations give

$$\tan. \sigma = \frac{\Sigma myz}{\Sigma mxz}; \ Sf = \omega^2 \surd \{(\Sigma mxz)^2 + (\Sigma myz)^2\}.$$

It appears by this, that we cannot determine the force S, or the distance f, but only their product, or the moment of the forces in SF and $S'F'$ to turn the axis round C.

Cor. 1. Hence, we may assign to S any value, and find the corresponding distance f, and *vice versâ*. As we suppose f larger and larger, S becomes smaller and smaller, and when f is indefinitely large, S is indefinitely small. Hence, in this case the force in SF would not affect the value of R, even if it were not neutralized by that in $S'F'$; and we may suppose the two portions $\frac{1}{2}S$ and $\frac{1}{2}S$ to act at the same point F.

Thus it appears, that forces which tend at the same time to give a rotatory motion, and a motion of translation, to a system, may be resolved into two ; a finite force, producing the motion of translation, and an indefinitely small force, acting at an indefinitely great distance, to produce the rotation. This latter force, being indefinitely small, would not affect the motion of translation, if it were transferred to the centre of rotation. This resolution is frequently used by Euler.

Cor. 2. The directions of the forces R, S, change as the position of the system changes, and in the case we are considering, these directions revolve uniformly round the axis. So that to keep the axis at rest during a whole revolution, it must oppose the tendency to motion in *every* direction with sufficient force.

Cor. 3. If the system, instead of consisting of separate particles, be a continuous body, we must put the differential of the mass instead of m, and the integral $\int$, instead of the sum Σ: we shall then have

$$\tan.\,\sigma = \frac{\int yzdM}{\int xzdM}; \quad Sf = \omega^2 \sqrt{\{(\int xzdM)^2 + (\int yzdM)^2\}}.$$

Cor. 4. S cannot $=0$, except $\int xzdM = 0$, $\int yzdM = 0$.

Prob. I. *A plane revolves about an axis perpendicular to it; to find the pressure on this axis, fig.* 113.

We may suppose C to be placed at the intersection of the plane and the axis; we shall then have $z=0$, and consequently $S=0$.

If G be the centre of gravity, and M represent the mass of the plane, the force R acts in CG, and is $= \omega^2 Mh$; CG being h. The same is true for any body which is symmetrical with respect to the plane xCy.

Prob. II. *A uniform straight line revolves about an axis meeting it at any angle; to find the pressure on the axis; fig.* 114.

Let AB be the line, G its centre of gravity; M any point, Dz the axis, GC, MO perpendiculars upon the axis. $CO=z$, $OM=x$, $CG=h$, and if the angle $ODM=\alpha$, $GM=s$, we shall have $x=h+s.\sin.\alpha$, $z=s.\cos.\alpha$, $y=0$, $dM=ds$. And let $GA=GB=a$. Then

$$\int xzdM=\int(h+s.\sin.\alpha).s.\cos.\alpha.ds$$
$$=\cos.\alpha\left(\frac{hs^2}{2}+\frac{s^3\sin.\alpha}{3}\right),$$

and the integral taken from $s=-a$ to $s=a$, gives

$$\cos.\alpha.\sin.\alpha.\frac{2a^3}{3}=\frac{Ma^2}{3}\sin.\alpha.\cos.\alpha.$$

$$\text{Also, }\int yzdM=0;\ \therefore\ Sf=\frac{Ma^2\omega^2}{3}\sin.\alpha.\cos.\alpha.$$

And the effort to turn AB round G is the same as if we had a mass $\frac{M}{3}$ placed at A, and revolving round an axis GH parallel to CZ.

Besides this, we have $R=Mh\omega^2$ acting at G as before.

Prob. III. *A straight line revolves in any position not meeting the axis; to find the pressures; fig.* 115.

G the centre of gravity, GC a perpendicular on the axis. Let the axis of x be parallel to CG. And let the position of the line be such, that it makes an angle θ with the plane of xy, and that its projection GK in that plane makes an angle η with x. Also let $GM=s$, $CG=h$, $GA=GB=a$. We shall then have

$$s\sin.\theta=MK=z;$$
$$s\cos.\theta.\sin.\eta=KL;\ s\cos.\theta\cos.\eta=GL;$$
$$\therefore\ s\cos.\theta\sin.\eta=y,\ h+s\cos.\theta\cos.\eta=x,$$
$$\int xzdM=\int(h+s\cos.\theta\cos.\eta)s\sin.\theta.ds$$
$$=\frac{hs^2}{2}\sin.\theta+\frac{s^3}{3}\cos.\theta\sin.\theta\cos.\eta+\text{const.}$$

which, taken from $s=-a$, to $s=a$, gives

$$\frac{2a^3}{3}\cos.\theta\sin.\theta\cos.\eta,$$

$$\int yz\,dM=\int s\cos.\theta\sin.\eta.s.\sin.\theta.ds$$
$$=\frac{2a^3}{3}\cos.\theta.\sin.\theta.\sin.\eta,$$

$$\tan.\sigma=\tan.\eta;$$

Hence, $$Sf=\frac{2a^3\omega^2}{3}.\sin.\theta.\cos.\theta=\frac{Ma^2\omega^2}{3}.\sin.\theta\cos.\theta.$$

If from C a perpendicular CT be drawn on KG; the axis will be drawn in the direction CG by the force R, and at the same time will have an effort to turn round the axis CT by the force S.

SECT. II. *A Body revolving acted on by any Forces.*

108. PROP. Let any forces act upon a system moving round a fixed axis; to find the pressure on the axis.

Let the forces at each point be resolved in the directions of the co-ordinates; $x_1, y_1, z_1, x_2, y_2, z_2$, &c. the co-ordinates of their points of application; and suppose

X_1, X_2, &c. the parts parallel to x,
Y_1, Y_2, &c. .... to y,
Z_1, Z_2, &c. .... to z.

The pressures, and the forces which support the axis, may still be reduced to two, acting at different points of the axis, but not in planes perpendicular to it. For the sake of simplicity, let one of these act at the origin C. We may resolve each of these forces in the directions of the axis, and perpendicular to it. Let U, V be the forces perpendicular to the axis, and T the sum of the forces in the axis. Let U pass through the origin, and make an angle ϕ with x; let V act at a distance b from the origin along the axis, and let it make an angle ψ with the line of x; and let the symbol Σ be used to represent the sum of all the products with respect to X_1, X_2, &c.

We shall then have for the whole *impressed forces, U, V, T,* acting to diminish $x, y, z,$

$$\Sigma X - U \cos.\ \phi - V \cos.\ \psi, \text{ parallel to } x,$$
$$\Sigma Y - U \sin.\ \phi - V \sin.\ \psi, \ldots\ldots \text{ to } y,$$
$$\Sigma Z - T \ldots\ldots\ldots\ldots\ldots\ldots \text{ to } z.$$

And their moments

$$Vb \sin.\ \psi + \Sigma\,(Zy - Yz) \text{ with respect to } Cx,$$
$$Vb \cos.\ \psi + \Sigma\,(Zx - Xz) \text{ with respect to } Cy,$$
$$\Sigma\,(Yx - Xy) \text{ with respect to } Cz:$$

these forces acting to turn the system in directions yz, xz, xy.

The effective forces are the same as in the former section, adding the effective forces which accelerate the motion round the axis. These act upon each particle, and are as the distance from the axis; if F be the force at distance 1, in direction xy, Fr will be the force at a distance r; Fmr will be the moving force, and its resolved parts parallel to Cx and Cy will be $-Fmy$ and Fmx. And the moments about Cx, Cy, and Cz arising from this force will be $-Fmxz$, $Fmyz$, Fmr^2.

Hence we have the whole *effective forces*

$$-\omega^2\Sigma mx - F\Sigma my \text{ parallel to } x,$$
$$-\omega^2\Sigma my + F\Sigma mx \ldots\ldots \text{ to } y,$$
$$0 \ldots\ldots\ldots\ldots\ldots\ldots \text{ to } z;$$

and their moments in directions xy, xz, yz,

$$\omega^2\Sigma myz - F\Sigma mxz \text{ about } Cx,$$
$$\omega^2\Sigma mxz + F\Sigma myz \ldots\ldots Cy,$$
$$F\Sigma mr^2 \ldots\ldots\ldots\ldots\ldots z.C$$

These effective forces inverted (that is, with their signs changed) must produce an equilibrium with the effective forces: (Art. 73.) hence, we have these six equations,

$$\Sigma X - U \cos.\phi - V \cos.\psi + \omega^2 \Sigma mx + F\Sigma my = 0,$$
$$\Sigma Y - U \sin.\phi - V \sin.\psi + \omega^2 \Sigma my - F\Sigma mx = 0,$$
$$Vb \sin.\psi + \Sigma(Zy - Yz) - \omega^2 \Sigma myz + F\Sigma mxz = 0,$$
$$Vb \cos.\psi + \Sigma(Zx - Xz) - \omega^2 \Sigma mxz - F\Sigma myz = 0.$$
$$\Sigma Z - T = 0,$$
$$\Sigma(Yx - Xy) - F\Sigma mr^2 = 0.$$

From the last, F is determined; and then the four first equations serve to determine U, ϕ, ψ, and Vb.

Prob. IV. *Let a body which is symmetrical with respect to a plane passing through the centre of gravity, revolve about a horizontal axis, so that this plane may be vertical; the body being acted on by gravity only, to find the pressure on the axis.*

Let C be taken at the point where the plane of symmetry meets the axis. In this case Σmxz, Σmyz each $=0$, because every positive value of zm will have a corresponding equal negative value. Also if $\bar{x}$ and $\bar{y}$ be the co-ordinates x and y of the centre of gravity, and M the whole mass; $\Sigma mx = M\bar{x}$, $\Sigma my = M\bar{y}$. Let x be vertical, and y horizontal; then $\Sigma Y = 0$, $\Sigma Z = 0$, and $\Sigma X = Mg$. Also $\Sigma Xy = Mg\bar{y}$. Hence, our equations give

$$-Mg\bar{y} - F\Sigma mr^2 = 0,$$

or, putting Mk^2 for Σmr^2, and $h \sin.\theta$ for $\bar{y}$, h being the distance of the centre of gravity from the axis, and θ the angle which h makes with the vertical,

$$-gh \sin.\theta = Fk^2, \quad F = -\frac{gh \sin.\theta}{k^2}.$$

And the four equations become

$$Mg - U \cos.\phi - V \cos.\psi + \omega^2 M\bar{x} + FM\bar{y} = 0,$$
$$-U \sin.\phi - V \sin.\psi + \omega^2 M\bar{y} - FM\bar{x} = 0,$$
$$Vb \sin.\psi = 0,$$
$$Vb \cos.\psi = 0.$$

Hence, $Vb = 0$, and we may make $V = 0$, b remaining indeterminate. This is also seen by considering that the whole pressure will obviously pass through the origin. Hence, putting for F its value, and for $\bar{x}$, $\bar{y}$, h cos. θ; h sin. θ, we have

$$Mg - U \cos. \phi = -M\left(\omega^2 h \cos. \theta - \frac{h^2}{k^2} g \sin.^2 \theta\right),$$

$$\text{or} \quad U \cos. \phi = M\left(\omega^2 h \cos. \theta - \frac{h^2}{k^2} g \sin.^2 \theta + g\right),$$

$$\text{and } U \sin. \phi = M\left(\omega^2 h \sin. \theta + \frac{h^2}{k^2} g \sin. \theta \cos. \theta\right).$$

Let the force U at the axis be resolved into two; R in the direction of the line passing through the centre of gravity, and S perpendicular to this line. Then taking the forces in the direction of x and y, we have

$$R \cos. \theta + S \sin. \theta = M\left(\omega^2 h \cos. \theta - \frac{h^2}{k^2} g \sin.^2 \theta + g\right),$$

$$R \sin. \theta - S \cos. \theta = M\left(\omega^2 h \sin. \theta + \frac{h^2}{k^2} g \sin. \theta \cos. \theta\right).$$

Multiply by cos. θ, sin. θ, and add, and we have

$$R = M\omega^2 h + Mg \cos. \theta.$$

Multiply by sin. θ, cos. θ, and subtract, and we have

$$S = Mg \sin. \theta \left(1 - \frac{h^2}{k^2}\right).$$

In the value of R, the first term, $M\omega^2 h$, arises from the centrifugal force, and is the same as if the mass were collected at the centre of gravity. The second, Mg cos. θ, is the resolved portion of the weight in the direction of a line passing through the centre of gravity. The forces U and V, which support the axis, are supposed to act opposite to the pressures on it. R and S in this case will act upwards, if θ be less than a right angle.

If the body were a line, revolving in a vertical plane, about one of its extremities, its length being a, we should have $h = \frac{a}{2}$,

$k^2 = \frac{a^2}{3}$. Hence, in this case, $S = \frac{1}{4} Mg \sin.\theta$. And when the line is horizontal, $S = \frac{1}{4} Mg$, or the pressure downwards is one fourth the weight of the body.

Prob. V. *A uniform straight line revolves about a horizontal axis meeting it in any position; to determine the pressure upon the axis.*

In fig. 114, let CG be considered as the direction of gravity; notation as in Prob. II.

The forces will be more distinctly conceived by separating them in the following manner.

We have the forces arising from the rotation, calculated in Prob. II, of this Chapter. They are

(1) A force $Mh\omega^2$ acting in CG.

(2) A force whose moment is $\frac{1}{3} Ma^2\omega^2$, $\sin.\alpha.\cos.\alpha$ to turn zC about C in the direction zA.

We have also the forces arising from gravity. The force of gravity on each point may be resolved parallel to CG, and perpendicular to the plane zCG; and if θ be the angle which this plane makes with a vertical plane, we shall have, arising from the forces parallel to CG,

(3) A force $Mg \cos.\theta$ acting in CG; and we have to find

(4) The pressure arising from the forces perpendicular to the plane zCG.

These pressures will be perpendicular to the plane zCG, and may be represented by Q and S acting in the axis at distances e and f from C. Hence, the impressed forces are $-Q$, $-S$, and $Mg \sin.\theta$, and the moment is $Qe + Sf$ about CG.

Now if F be the effective force at distance 1, acting to diminish θ, we have $F = \frac{gh \sin.\theta}{k^2}$; and at the point M, effective force $= \frac{gh \sin.\theta}{k^2} x\,ds$, and its moment round $CG = \frac{gh \sin.\theta}{k^2} xz\,ds$.

Observing that $x = h + s \sin. a$, $z = s \cos. a$, we have

$$\int x\,ds = hs + \frac{s^2}{2} \sin. a = 2\,ha = Mh,$$

$$\int xz\,ds = h \cos. a \frac{s^2}{2} + \sin. a \,.\, \cos. a \frac{s^3}{3}$$

$$= \frac{2\,a^3}{3} \sin. a \,.\cos. a = M \,.\frac{a^2}{3} \sin. a \cos. a.$$

Hence, the whole effective force is $Mg \,.\dfrac{h^2 \sin. \theta}{k^2}$,

and its moment about CG is $Mg \,.\, \dfrac{ha^2 \sin. \theta \sin. a \cos. a}{3\,k^2}$;

therefore,

$$-Q - S + Mg \sin. \theta = Mg \,.\frac{h^2 \sin. \theta}{k^2},$$

$$Qe + Sf = Mg \,.\frac{h\,a^2 \sin. \theta \,.\sin. a \,.\cos. a}{3\,k^2}.$$

Whence the forces must be determined.

If we suppose Q to be the force acting at the centre C, and Sf to be the moment of the force of torsion, represented by an infinitely small force at an infinite distance, we have $S = 0$, $e = 0$; whence

$$Q = Mg \sin. \theta \left(1 - \frac{h^2}{k^2}\right),$$

$$Sf = Mg \sin. \theta \,.\frac{h\,a^2 \sin. a \cos. a}{3\,k^2}.$$

These two forces, with the three before-mentioned, give the whole pressure on the axis. The last tends to turn zC about CG as an axis. Hence, on the whole, besides the pressure at C, there will be a tendency to turn Cz about a line inclined to the plane zCG.

We may find k by the Chapter III.

For Sf we may substitute two equal forces acting at equal finite distances from C in opposite directions. The forces will be $\frac{S}{2}$, $\frac{S}{2}$, and the distances f, Sf being of the proper value as above.

109. Prop. When a body revolving about an axis is acted on by a single force in a plane perpendicular to the axis, to find the pressure on the axis.

This is a particular case of the general proposition, and might be so considered; but as it leads immediately to the centre of percussion, we shall consider it separately.

Let P be the force, and let it act parallel to the axis of x, at a point of which the co-ordinates are y', z'. Let the body be supposed to be at rest. And let the reaction of the axis be resolved into a force S parallel to x at an ordinate z'', and a force T parallel to y at ordinate z'''. Let as before F be the effective force which produces rotation about the axis z, at distance 1. Then Frm is the effective moving force of m at distance r, $-Fym$, and Fxm, the parts parallel to x and y. And $-Fxzm$ and $Fyzm$ are the moments of these forces about the axes x and y respectively. Also, Fr^2m is the moment about z. Hence, we have

the *whole effective forces*

$$-F\Sigma ym, \quad F\Sigma xm, \quad 0; \text{ and their moments}$$

$$-F\Sigma xzm, \quad F\Sigma yzm, \quad F\Sigma r^2m.$$

The impressed forces are P, $-S$ and $-T$; and therefore we have

the whole impressed forces

parallel to x, $P-S$, to y, $-T$; to z, 0; and their moments about x, Tz'''; about y, $-Pz'+Sz''$; about z, $-Py'$.

$$\text{Hence, } P-S=-F\Sigma ym, \quad T=-F\Sigma xm,$$

$$-Pz'+Sz''=F\Sigma yzm; \quad Tz'''=-F\Sigma xzm,$$

$$-Py'=F\Sigma r^2m,$$

$$\text{Hence, } P-S=\frac{P\Sigma ym}{\Sigma r^2m}.y'; \qquad S=P\left(1-\frac{y'\Sigma ym}{\Sigma r^2m}\right),$$

$$Sz''=P\left(z'-\frac{y'\Sigma yzm}{\Sigma r^2m}\right).$$

If the body, instead of being at rest, have any angular velocity, this will produce a centrifugal force in the parts, the result of which will, by the first section of this Chapter, be a pressure on the axis in the line passing through the centre of gravity. Hence, if we do not consider this force, S will be the same as before.

Cor. 1. If the centre of gravity be in the plane yz, we shall have $\Sigma xm = 0$, and $T = 0$. And if the figure be symmetrical with respect to the plane yz, we shall have $\Sigma xzm = 0$, and $Tz''' = 0$. Σxzm will also $= 0$ in many other cases.

Cor. 2. To find at what point P must be applied, in order that the effect of S upon the axis may be 0. We must have $S = 0$, and $Sz'' = 0$*;

$$\therefore 0 = P\left(1 - \frac{y'\Sigma ym}{\Sigma r^2 m}\right); \quad 0 = P\left(z' - \frac{y'\Sigma yzm}{\Sigma r^2 m}\right).$$

$$\text{Hence, } y' = \frac{\Sigma r^2 m}{\Sigma ym}; \quad z' = \frac{y'\Sigma yzm}{\Sigma r^2 m} = \frac{\Sigma yzm}{\Sigma ym}.$$

110. Prop. To find the *Centre of Percussion* of any system.

If a body be *struck*, in a direction perpendicular to the plane passing through the axis and the centre of gravity, the *centre of percussion* is the point at which the impact must take place, that it may produce no pressure upon the axis.

P, in the last Article, represents any pressure, and therefore, the conclusions will be true of a *sudden* and *violent* pressure, which is an impact.

The impact is perpendicular on the plane yz, in which the centre of gravity is.

And we have, by Cor. 2, to last Article,

$$y' = \frac{\Sigma r^2 m}{\Sigma ym} = \frac{\Sigma r^2 m}{Mh},$$

if h be the distance of the centre of gravity from the axis. Hence,

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* It does not follow, that if $S = 0$, $Sz'' = 0$; for Sz'' expresses the force of torsion exerted by S, which may be represented by an indefinitely small force acting at an indefinitely great distance. The same is true of T and Tz'''. See p. 281.

the distance of the centre of percussion from the axis is the same as that of the centre of oscillation

$$z' = \frac{\Sigma yzm}{\Sigma ym} = \frac{\Sigma yzm}{Mh}.$$

If the system be symmetrical with respect to a plane perpendicular to the axis, and through the centre of gravity, and if c be the value of z corresponding to this plane, $\Sigma yzm = Mhc$, and $z' = h$. Therefore, the centre of percussion will be in the same plane; and of course, in the line passing through the centre of gravity.

In other cases, we must find Σxzm by integrating.

If Σxzm do not vanish, the whole force on the axis will not vanish. There will be a tendency to twist the system round the axis x.

Ex. 1. A semi-parabola ABC, fig. 116, is moveable about its axis AB: to find the point O, at which it must be struck perpendicularly to its plane, that there may be no stress on the axis.

If z be measured along the axis, and y perpendicular to it, we have

$$\Sigma r^2 m = \iint y^2 dz dy = \int \frac{y^3 dz}{3};$$

and y is to be taken from 0 to $\sqrt{(az)}$, a being the parameter;

$$\therefore \Sigma r^2 m = \tfrac{1}{3}\int a^{\frac{3}{2}} z^{\frac{3}{2}} dz = \frac{2a^{\frac{3}{2}} z^{\frac{5}{2}}}{15}.$$

And if b be the whole length AB, this $= \dfrac{2a^{\frac{3}{2}} b^{\frac{5}{2}}}{15}$.

$$\Sigma yzm = \iint yz dy dz = \int \frac{y^2}{2} z dz = \tfrac{1}{2}\int az^2 dz = \frac{az^3}{6} = \frac{ab^3}{6},$$

$$\Sigma ym = \iint y dy dz = \int \frac{y^2}{2} dz = \tfrac{1}{2}\int az dz = \frac{az^2}{4} = \frac{ab^2}{4},$$

$$y' = \frac{\Sigma r^2 m}{\Sigma y m} = \frac{2a^{\frac{3}{2}} b^{\frac{5}{2}}}{15} \cdot \frac{4}{ab^2} = \frac{8}{15} a^{\frac{1}{2}} b^{\frac{1}{2}},$$

$$z' = \frac{\Sigma y z m}{\Sigma y m} = \frac{ab^3}{6} \cdot \frac{4}{ab^2} = \frac{2}{3} b.$$

Hence, we find the point O by making

$$AL = \frac{2}{3} AB, \ LO = \frac{8}{15} BC.$$

Ex. 2. Let ABC be a right-angled triangle,

$$AL = \tfrac{3}{4} AB, \ \ LO = \tfrac{1}{2} BC.$$

CHAP. VI.

THE THREE PRINCIPAL AXES OF ROTATION.

111. There are in every body or system, three lines so situated, that if the body revolve about any one of them, the pressure upon the axis is 0. The same lines also have this property; that the moment of inertia about one of them is greater, and about another less, than it is about any other axis. These three lines are called the *principal axes:* they are all three at right angles to each other. The properties, &c., of these axes will be the subject of the present Chapter.

Prop. To find the moment of inertia of a system about an axis, passing through the origin of co-ordinates, and making with the axes of x, y, z, any angles α, β, γ *.

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* The equation $\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma = 1$, connects α, β, γ.

Let Cx, Cy, Cz, fig. 117, be the axes of co-ordinates, and CI the axis of rotation. Let M be any point, and $MO = r$ a perpendicular on CI. Also, let the co-ordinates of the point M be x, y, z; and $CM = D$.

If the angles which CM makes with Cx, Cy, Cz respectively, be α', β', γ': and δ the angle which CM makes with CI, we shall have

$$\cos.\ \delta = \cos.\ \alpha\ .\ \cos.\ \alpha' + \cos.\ \beta\ .\ \cos.\ \beta' + \cos.\ \gamma\ .\ \cos.\ \gamma'\ *.$$

But, $\cos.\ \alpha' = \dfrac{x}{D}$, &c.;

$$\therefore\ \cos.\ \delta = \frac{x}{D}\cos.\ \alpha + \frac{y}{D}\cos.\ \beta + \frac{z}{D}\cos.\ \gamma.$$

Also, $r^2 = D^2 \sin.^2 \delta = D^2 - D^2 \cos.^2 \delta$; and $D^2 = x^2 + y^2 + z^2$;

$$\begin{aligned}\therefore\ r^2 = {} & x^2 + y^2 + z^2 \\ & - x^2 \cos.^2 \alpha - y^2 \cos.^2 \beta - z^2 \cos.^2 \gamma \\ & - 2yz \cos.\ \beta\ \cos.\ \gamma - 2xz \cos.\ \alpha\ .\ \cos.\ \gamma - 2xy \cos.\ \alpha\ \cos.\ \beta.\end{aligned}$$

Putting $x^2 \sin.^2 \alpha$ for $x^2 - x^2 \cos.^2 \alpha$, &c., multiplying by m, the particle at M, and taking the sum indicated by Σ;

$$\begin{aligned}\Sigma r^2 m = {} & \sin.^2 \alpha\, \Sigma x^2 m + \sin.^2 \beta\, \Sigma y^2 m + \sin.^2 g\, \Sigma z^2 m \\ & - 2 \cos.\beta \cos.\gamma\, \Sigma yzm - 2 \cos.\alpha \cos.\gamma\, \Sigma xzm - 2 \cos.\alpha \cos.\beta\, \Sigma xym.\end{aligned}$$

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* Let a sphere, fig. 118, described with centre C, meet Cx, Cy, Cz, CM, CI, in x, y, z, M, I. Then, arcs Ix, Iy, Iz, will be α, β, γ; Mx, My, Mz, will be α', β', γ'; MD will be δ. And we shall have, since xzy is a right angle, and xz a quadrant

$$\cos.\ Izx = \frac{\cos.\ \alpha}{\sin.\ \gamma};\quad \sin.\ Izx = \cos.\ Izy = \frac{\cos.\ \beta}{\sin.\ \gamma};$$

$$\cos.\ Mzx = \frac{\cos.\ \alpha'}{\sin.\ \gamma'};\quad \sin.\ Mzx = \cos.\ Mzy = \frac{\cos.\ \beta'}{\sin.\ \gamma'};$$

$$\therefore\ \cos.\ IzM = \frac{\cos.\ \alpha\ .\ \cos.\ \alpha' + \cos.\ \beta\ .\ \cos.\ \beta'}{\sin.\ \alpha\ .\ \sin.\ \alpha'}.$$

$$\begin{aligned}\text{And, } \cos.\ \delta & = \cos.\ IzM . \sin.\ \alpha\ . \sin.\ \alpha' + \cos.\ \alpha\ .\ \cos.\ \alpha' \\ & = \cos.\ \alpha\ . \cos.\ \alpha' + \cos.\ \beta\ , \cos.\ \beta' + \cos.\ \gamma\ . \cos.\ \gamma'.\end{aligned}$$

If we put

$$\Sigma x^2 m = f,\ \Sigma y^2 m = g,\ \Sigma z^2 m = h,$$
$$\Sigma yzm = F,\ \Sigma xzm = G,\ \Sigma xym = H,$$
$$\Sigma r^2 m = f \sin.^2 \alpha + g \sin.^2 \beta + h \sin.^2 \gamma$$
$$- 2F \cos.\beta \cos.\gamma - 2G \cos.\alpha \cos.\gamma - 2H \cos.\alpha \cos.\beta.$$

If the three quantities F, G, H, be each equal to 0, the expression will be simplified. We shall shew hereafter, that it is always possible to place the axes of x, y, z, in such a position, that F, G, H, shall $= 0$. In that case,

$$\Sigma r^2 m = f \sin.^2 \alpha + g \sin.^2 \beta + h \sin.^2 \gamma.$$

112. Prop. To find the position of three axes of rectangular co-ordinates x', y', z', such that

$$\Sigma y'z'm = 0,\ \Sigma x'z'm = 0,\ \Sigma x'y'm = 0.$$

Let x, y, z, be rectangular co-ordinates, having the same origin as x', y' z', and a known position with respect to the body.

Let x' make with x, y, z, angles, whose cosines are a, b, c,
y' .... a', b', c',
z' .... a'', b'', c''.

Hence, by note, p. 293, the cosine of the angle which x and y make, is $aa' + bb' + cc'$; but this angle is a right angle;

$$\left.\begin{array}{rl} \therefore & aa' + bb' + cc' = 0 \\ \text{Similarly,} & aa'' + bb'' + cc'' = 0 \\ & a'a'' + b'b'' + c'c'' = 0. \\ \text{Also,} & a^2 + b^2 + c^2 = 1 \\ & a'^2 + b'^2 + c'^2 = 1 \\ & a''^2 + b''^2 + c''^2 = 1 \end{array}\right\} \cdots\cdots\cdots\cdots (\beta)^*,$$

by Note, p. 292.

* We have also
$$\left.\begin{array}{l} ab + a'b' + a''b'' = 0 \\ ac + a'c' + a''c'' = 0 \\ bc + b'c' + b''c'' = 0 \\ a^2 + a'^2 + a''^2 = 1 \\ b^2 + b'^2 + b''^2 = 1 \\ c^2 + c'^2 + c''^2 = 1 \end{array}\right\} \cdots\cdots\cdots (\gamma).$$

These are six equations of condition.

Now let the moment of inertia be found for any axis, making angles α, β, γ, with x, y, z. It will be

$$f \sin.^2 \alpha + g \sin.^2 \beta + h \sin.^2 \gamma$$
$$-2F\cos.\beta\cos.\gamma - 2G\cos.\alpha.\cos.\gamma - 2H\cos.\alpha.\cos.\beta.$$

But, if this axis make angles α', β', γ', with x', y', z', and if $\Sigma x'^2 m = X$, $\Sigma y'^2 m = Y$, $\Sigma z'^2 m = Z$; since by supposition, $\Sigma y'z'm = 0$, &c., the moment of inertia will be

$$X \sin.^2 \alpha' + Y \sin.^2 \beta' + Z \sin.^2 \gamma';$$

and these two expressions must be equal.

The latter is equal to

$$X + Y + Z - X\cos.^2\alpha' - Y\cos.^2\beta' - Z\cos.^2\gamma',$$

and $X + Y + Z = \Sigma(x'^2 + y'^2 + z'^2)m = \Sigma D^2 m$, if D be the distance of m from the origin; and this is

$$= \Sigma(x^2 + y^2 + z^2)m = f + g + h.$$

Also, $f\sin.^2\alpha = f - f\cos.^2\alpha$, &c.: hence, substituting, the expressions to be made equal, are

$$f\cos.^2\alpha + \&c. + 2F\cos.\beta\cos.\gamma + \&c.$$
$$\text{and } X\cos.^2\alpha' + \&c.$$

By the Note, p. 293, we have

$$\cos.\alpha' = a\cos.\alpha + b\cos.\beta + c.\cos.\gamma,$$
$$\cos.\beta' = a'\cos.\alpha + b'\cos.\beta + c'\cos.\gamma,$$
$$\cos.\gamma' = a''\cos.\alpha + b''\cos.\beta + c''\cos.\gamma.$$

Hence, $X\cos.^2\alpha' + Y\cos.^2\beta' + Z\cos.^2\gamma'$ becomes

$$X(a^2\cos.^2\alpha + b^2\cos.^2\beta + c^2\cos.^2\gamma$$
$$+ 2bc\cos.\beta\cos.\gamma + 2ac\cos.\alpha\cos.\gamma + 2ab\cos.\alpha\cos.\beta)$$
$$+ Y(a'^2\cos.^2\alpha + b'^2\cos.^2\beta + c'^2\cos.^2\gamma$$
$$+ 2b'c'\cos.\beta\cos.\gamma + 2a'c'\cos.\alpha\cos.\gamma + 2a'b'\cos.\alpha\cos.\beta)$$
$$+ Z(a''^2\cos.^2\alpha + b''^2\cos.^2\beta + c''^2\cos.^2\gamma$$
$$+ 2''bc''\cos.\beta\cos.\gamma + 2a''c''\cos.\alpha\cos.\gamma + 2a''b''\cos.\alpha\cos.\beta);$$

which must be identical with

$$f \cos.^2 \alpha + g \cos.^2 \beta + h \cos.^2 \gamma$$
$$+ 2 F \cos. \beta \cos. \gamma + 2 G \cos. \alpha . \cos. \gamma + 2 H \cos. \alpha . \cos. \beta.$$

Equating coefficients of $\cos.^2 \alpha$, &c., we have

$$X a^2 + Y a'^2 + Z a''^2 = f. \dots\dots\dots (1),$$
$$X b^2 + Y b'^2 + Z' b''^2 = g \dots\dots\dots (2),$$
$$X c^2 + Y c'^2 + Z c''^2 = h. \dots\dots\dots (3),$$
$$X b c + Y b' c' + Z b'' c'' = F. \dots\dots (4),$$
$$X a c + Y a' c' + Z a'' c'' = G. \dots\dots (5),$$
$$X a b + Y a' b' + Z a'' b'' = H. \dots\dots (6),$$

(1) $a +$ (6) $b +$ (5) c, gives

$$X a (a^2 + b^2 + c^2) + Y a' (a a' + b b' + c c') + Z a'' (a a'' + b b'' + c c'')$$
$$= f a + H b + G c;$$

or, by the equations of condition (β),

$$\left.\begin{aligned} X a &= f a + H b + G c \\ \text{Similarly, (2)} b + \text{(4)} c + \text{(6)} a \text{ gives } X b &= g b + F c + H a \\ \text{(3)} c + \text{(5)} a + \text{(4)} b \dots\dots X c &= h c + G a + F b \end{aligned}\right\} \dots\dots (7).$$

From these three equations, and $a^2 + b^2 + c^2 = 1$, we must find X, a, b, c.

By the two first, we get

$$(X - f) a - H b - G c = 0,$$
$$(X - g) b - F c - H a = 0.$$

Eliminate c,

$$\therefore \{(X - f) F + G H\} a - \{(X - g) G + F H\} b = 0;$$
$$\therefore b = \frac{(X - f) F + G H}{(X - g) G + F H} . a.$$

Eliminate b,

$$\therefore \{(X - f)(X - g) - H^2\} a - \{(X - g) G + F H\} c = 0;$$
$$\therefore c = \frac{(X - f)(X - g) - H^2}{(X - g) G + F H} a.$$

Substitute these values in the third of equations (7), viz.,

$$(X-h)\,c-Ga-Fb=0, \text{ and we have}$$

$$(X-h)\,.\,\frac{(X-f)(X-g)-H^2}{(X-g)\,G+FH}-G-F\,.\,\frac{(X-f)\,F+GH}{(X-g)\,G+FH}=0;$$

$$\text{or, } (X-f)(X-g)(X-h)$$

$$-\{(X-f)\,F^2+(X-g)\,G^2+(X-h)H^2\}-2\,FGH=0;$$

a cubic equation, from which X may be determined.

X being known, we can find a, b, c, by the equations

$$b=\frac{(X-f)\,F+GH}{(X-g)\,G+FH}\,.\,a, \quad c=\frac{(X-f)(X-g)-H^2}{(X-g)\,G+FH}\,a,$$

$$a^2+b^2+c^2=1.$$

And hence the position of the axis x' is determined.

If we make the same combinations as before, of equations (1), (2), (3), (4), (5), (6); only using a', b', c', instead of a, b, c, we shall have the same final result, with the difference of Y instead of X. If we use a'', b'', c'', instead of a, b, c, we shall have Z instead of X in the result. Hence, the same cubic equation will give X, Y, Z; and therefore, these must be its three roots. And hence, there is only one system of three axes, possessing the property required; for the first root of the cubic giving one axis, the second and third roots give the two other axes of the same system.

This cubic has necessarily one root possible; and it may be shewn that the other roots are also possible. If they be impossible, suppose them to be of the form $m+n\sqrt{(-1)}$, and $m-n\sqrt{(-1)}$. The quantities a, b, c, are possible when X is so; and for one of the impossible roots, the corresponding quantities a', b', c', will be of the form $p+q\sqrt{(-1)}$, $p'+q'\sqrt{(-1)}$, $p''+q''\sqrt{(-1)}$; and for the other root, a'', b'', c'', they will be of the form $p+q\sqrt{(-1)}$, $p'-q'\sqrt{(-1)}$, $p''-q''\sqrt{(-1)}$. Now, $a'a''+b'b''+c'c''=0$;

$$\therefore\ p^2+p'^2+p''^2+q^2+q'^2+q''^2=0,$$

which cannot be, if p, q, &c., are possible. Therefore, the roots are not of the form supposed.

113. Prop. Given one principal axis, to find the other two.

Let the given principal axis coincide with the axis of x. Therefore, $a=1$, $b=0$, $c=0$. And putting these values in the four equations in X, a, b, c, we find

$$X=f, \quad 0=H, \quad 0=G;$$

the first equation is equivalent to $X-f=0$: substituting the two latter values in the cubic, dividing out the factor $X-f$, and putting Y for X, we have

$$(Y-g)(Y-h)-F^2=0.$$

and the value of Y, obtained from this, must be substituted in the equations

$$Yb'=gb'+Fc'+Ha',$$
$$a'^2+b'^2+c'^2=1.$$

But $H=0$, and $a'=0$, since y' must make a right angle with x;

$$\therefore (Y-g)b'=Fc'; \quad b'^2+c'^2=1.$$

If θ be the angle which y' makes with y, $b'=\cos.\theta$, $c'=\sin.\theta$;

$$\therefore \tan.\theta=\frac{c'}{b'}=\frac{Y-g}{F};$$

$$\therefore \tan.2\theta=\frac{2F(Y-g)}{F^2-(Y-g)^2}.$$

$$\text{But} \quad Y^2-(g+h)Y+gh-F^2=0.$$

$$\text{Add} \quad (h-g)Y-(h-g)g=(Y-g)(h-g);$$

$$\therefore (Y-g)^2-F^2=(Y-g)(h-g);$$

$$\therefore \tan.2\theta=\frac{2F}{g-h} \;\ldots\ldots\ldots (\delta);$$

this gives two values of θ, differing by a right angle, which determines the required positions of y' and z'.

114. Prop. To find the moment of inertia about any axis, in terms of the moments about the principal axes.

Let A be the moment about the axis of x: then, if r be the distance of a particle m from that axis,

$$r^2 = x^2 + y^2,\ A = \Sigma r^2 m = \Sigma (x^2 + y^2) m = \Sigma x^2 m + \Sigma y^2 m = g + h.$$

Similarly, if B be the moment about the axis of y, and C about the axis of z, we shall have

$$B = f + h,\ C = f + g.$$

Now, it is shewn at the end of Art. 111, that if f, g, h, belong to axes for which $\Sigma yzm = 0$, &c., we shall have for the moment of inertia (μ), about any other axis,

$$\mu = f \sin.^2 \alpha + g \sin.^2 \beta + h \sin.^2 \gamma.$$

But, since

$$\cos.^2 \alpha + \cos.^2 \beta + \cos.^2 \gamma = 1,\ \sin.^2 \alpha = \cos.^2 \beta + \cos.^2 \gamma;$$

similarly,

$$\sin.^2 \beta = \cos.^2 \alpha + \cos.^2 \gamma, \text{ and } \sin.^2 \gamma = \cos.^2 \alpha + \cos.^2 \beta;$$

and substituting these values,

$$\begin{aligned}\mu &= (g+h) \cos.^2 \alpha + (f+h) \cos.^2 \beta + (f+g) \cos.^2 \gamma \\ &= A \cos.^2 \alpha + B \cos.^2 \beta + C \cos.^2 \gamma.\end{aligned}$$

115. Prop. Of the moments A, B, C about the principal arcs, one is greater and another less than the moments about any other axes.

Let A be the greatest of the three, and C the least, and μ the moment about any axis, and, putting $1 - \cos.^2 \beta - \cos.^2 \gamma$ for $\cos.^2 \alpha$,

$$\begin{aligned}\mu &= A (1 - \cos.^2 \beta - \cos.^2 \gamma) + B \cos.^2 \beta + C \cos.^2 \gamma \\ &= A - (A - B) \cos.^2 \beta - (A - C) \cos.^2 \gamma,\end{aligned}$$

and $A - B$, $A - C$ are positive; $\therefore$ μ is less than A.

Similarly,
$$\begin{aligned}\mu &= A \cos.^2 \alpha + B \cos.^2 \beta + C (1 - \cos.^2 \alpha - \cos.^2 \beta) \\ &= C + (A - C) \cos.^2 \alpha + (B - C) \cos.^2 \beta;\end{aligned}$$

$\therefore$ μ is greater than C.

If two of the moments as B, C, be equal

$$\begin{aligned}\mu &= A \cos.^2 \alpha + B (\cos.^2 \beta + \cos.^2 \gamma) \\ &= A \cos.^2 \alpha + B \sin.^2 \alpha.\end{aligned}$$

If all the three A, B, C are equal, $\mu = A$ for every axis.

116. Prop. To find all the axes for which the moments are equal.

If μ be the moment, all these axes are connected by the equation

$$A \cos.^2 \alpha + B \cos.^2 \beta + C \cos.^2 \gamma = \mu, \text{ a constant quantity,}$$

$$(A - C) \cos.^2 \alpha + (B - C) \cos.^2 \beta = \mu - C.$$

If we suppose a sphere whose radius is 1, to be described with its centre at the origin; and suppose the axis to meet it in a point, which we may call the *pole* of rotation; the co-ordinates of this pole parallel to Cx and Cy are $\cos. \alpha$ and $\cos. \beta$, and these determine the projection of the pole on the plane xy. Hence, the above equation is the equation to the projection, on the plane of xy, of the *locus* of all the poles for which μ is the same, or, as we may call them, the *equi-momental* poles. And it appears by that equation, that this projection is an ellipse with its centre C, and its semi-axes in the direction of x and y, equal to

$$\sqrt{\frac{\mu - C}{A - C}}, \text{ and } \sqrt{\frac{\mu - C}{B - C}},$$

and xy is the plane perpendicular to the axis of least moment.

Similarly, $(A - B) \cos.^2 \beta + (A - C) \cos.^2 \gamma = A - \mu$,

and hence, the projections of the locus of the equi-momental poles on a plane yz perpendicular to the axis of the greatest moment are ellipses.

Again, $(A - B) \cos.^2 \alpha - (B - C) \cos.^2 \gamma = \mu - B$.

Hence, the projection on the plane perpendicular to the plane xz of the mean moment is a hyperbola.

Thus, in fig. 119, PQ is the locus of equi-momental poles on the surface of a sphere. And the projection of PQ on the plane xy is the ellipse MN; on the plane yz it is the ellipse RQ, and on the plane xz it is the hyperbola PO.

In fig. 120, are represented the loci of equi-momental poles, on the surface of a sphere concentric with the body. If we make $\mu = B$, we have the locus a quadrant of a great circle yB, and its projection BC on xz a straight line. When $\mu > B$, we have curves PQ, $P'Q'$, &c. approaching nearer to x as μ is larger. When $\mu < B$, we have curves pq, $p'q'$, &c., approaching to C as μ is smaller.

117. We shall now find the principal axes in given figures.

PROB. *To find the principal axes of a given parallelogram.*

Let, in fig. 121, $AB=2a$, $AD=2b$, angle $A=\zeta$. And let the axis of x be perpendicular to the plane in its centre C; y parallel to BA, z perpendicular to BA. $CP=y$, $PM=z$. And let $CH=u$, $HM=v$.

It is manifest, that the axis of x is a principal axis, and we have to find the other two by Art. 113. The figure being supposed to be divided into particles by lines parallel to AB and AD; one of these at M will be $du\,dv \sin.\zeta$. Also, we shall have

$$y=v\cos.\zeta+u,\quad z=v\sin.\zeta,$$

$$g=\int y^2dM=\iint(v\cos.\zeta+u)^2\,du\,dv\sin.\zeta.$$

And, integrating for u, $=\int\frac{1}{3}(u+v\cos.\zeta)^3.\,dv\sin.\zeta$, which taken from $u=-a$ to $u=a$, gives

$$g=\int\frac{2a^3+6av^2\cos.^2\zeta}{3}\,dv\sin.\zeta$$

$$=\frac{2a^3v\sin.\zeta}{3}+\frac{2av^3\sin.\zeta\cos.^2\zeta}{3}.$$

This taken from $v=-b$ to $v=b$ gives,

$$g=4ab\sin.\zeta.\frac{a^2+b^2\cos.^2\zeta}{3}.$$

Also $h=\int z^2dM=\iint v^2\sin.^3\zeta\,du\,dv=2a\sin.^3\zeta\int v^2dv$

$$=\frac{4ab^3\sin.^3\zeta}{3}=4ab\sin.\zeta.\frac{b^2\sin.^2\zeta}{3}.$$

And $F=\int yz\,dM=\iint(u+v\cos\zeta)\,v\sin.^2\zeta\,du\,dv$,

and integrating for u, $=\int\frac{1}{2}(v\cos.\zeta+u)^2.\,v\,dv\sin.^2\zeta$;

which taken from $u=-a$ to $u=a$, gives

$$F=\int 2av^2dv\cos.\zeta\sin.^2\zeta$$

$$=2a\cos.\zeta\sin.^2\zeta\int v^2dv=\frac{4ab^3\cos.\zeta.\sin.^2\zeta}{3}$$

$$=4ab\sin.\zeta.\frac{b^2\sin.\zeta\cos.\zeta}{3}.$$

Hence, $\tan 2\theta = \frac{2F}{g-h} = \frac{2b^2 \sin.\zeta \cos.\zeta}{a^2 + b^2 \cos.^2\zeta - b^2 \sin.^2\zeta}$

$$= \frac{b^2 \sin. 2\zeta}{a^2 + b^2 \cos. 2\zeta},$$

which gives two values of θ, corresponding to the two principal axes in the plane of the parallelogram.

To find the momentum with respect to one of these principal axes, we have by the formula, Art. 111, observing that $\int x^2 dM = 0$, $\alpha = \frac{\pi}{2}$, $\beta = \theta$, $\gamma = \frac{\pi}{2} - \theta$,

$$\int r^2 dM = g \sin.^2\theta + h \cos.^2\theta - 2F \sin.\theta \cos.\theta$$

$$= 4ab \sin\zeta \left\{\frac{a^2 + b^2 \cos.^2\zeta}{3} \sin.^2\theta + \frac{b^2 \sin.^2\zeta}{3} \cos.^2\theta - \frac{2b^2 \sin.\zeta . \cos.\zeta}{3} \sin.\theta \cos.\theta\right\}$$

$$= \frac{4ab \sin.\zeta}{3} \{a^2 \sin.^2\theta + b^2 (\cos.\zeta \sin.\theta - \cos.\theta \sin.\zeta)^2\}$$

$$= \frac{4ab \sin.\zeta}{3} \{a^2 \sin.^2\theta + b^2 \sin.^2 (\zeta - \theta)\}$$

$$= \frac{2ab \sin.\zeta}{3} \{a^2 (1 - \cos. 2\theta) + b^2 [1 - \cos. 2(\zeta - \theta)]\}.$$

Now, $a^2 \cos. 2\theta + b^2 \cos. 2(\zeta - \theta)$

$$= a^2 \cos. 2\theta + b^2 \cos. 2\zeta \cos. 2\theta + b^2 \sin. 2\theta . \sin. 2\zeta$$

$$= \cos. 2\theta \{a^2 + b^2 \cos. 2\zeta + b^2 \sin. 2\zeta \tan. 2\theta\}$$

$$= \cos. 2\theta \left\{a^2 + b^2 \cos. 2\zeta + \frac{b^4 \sin.^2 2\zeta}{a^2 + b^2 \cos. 2\zeta}\right\}$$

$$= \cos. 2\theta \left\{\frac{a^4 + 2a^2b^2 \cos. 2\zeta + b^4}{a^2 + b^2 \cos. 2\zeta}\right\}.$$

Also, $\cos.^2 2\theta = \frac{1}{1 + \tan.^2 2\theta} = \frac{(a^2 + b^2 \cos. 2\zeta)^2}{a^4 + 2a^2b^2 \cos. 2\zeta + b^4}$;

$$\therefore \int r^2 dM = \frac{2ab \sin.\zeta}{3} \{a^2 + b^2 \pm \sqrt{(a^4 + 2a^2b^2 \cos. 2\zeta + b^4)}\},$$

and the two values give B and C.

Cor. The two moments B and C together are equal to $4ab \sin. \zeta, \frac{a^2+b^2}{3}$ the moment A about the axis x.

This proposition is general for plane figures.

In any symmetrical plane figure, the principal axes are, the axes of symmetry, and the axis perpendicular to the plane.

In a sphere, all the axes are principal axes.

In a cube, the same is true.

In a parallelepiped, the lines perpendicular to the surfaces are principal axes, and the moments are as $b^2 + c^2$, $a^2 + c^2$, $a^2 + b^2$.

In any figure of revolution, the axis of revolution is a principal axis, and any other in a plane perpendicular to this through the centre of gravity, is a principal axis.

118. Prop. The principal axes are axes about which the body can revolve permanently.

By Art. 107, Cor. 4, if the body revolve about an axis z, such that $\Sigma xzm = 0$, $\Sigma yzm = 0$; the effort to *turn* the axis round the centre of gravity of the body will be 0; and therefore if the point of the axis, where the perpendicular from the centre of gravity meets it, be fixed, the axis will be fixed. And if, besides this condition, the axis of rotation pass through the centre of gravity, the pressure on this axis will $= 0$. Hence, if the axis be not fixed, but the body left to itself, it will still revolve about this axis; the axis z for which $\Sigma xzm = 0$, $\Sigma yzm = 0$, is therefore a *permanent axis of free rotation*. Similarly, if $\int xy\,dm = 0$, y and x are also permanent axes of free rotation.

CHAP. VII.

MOTION OF ANY RIGID BODY ABOUT ITS CENTRE OF GRAVITY.

119. **Our** object at present is to obtain the equations for the motion of rotation of any body about its centre of gravity, supposing that point to be fixed. It has already been shewn, that when forces act upon any body, the centre of gravity is the centre about which the rotation, when separated from the motion of translation, does take place, and it will be seen hereafter, that the motion of the centre of gravity will be the same as if the same forces acted immediately on it.

Let three rectangular axes, *fixed in space*, pass through the centre of gravity C, fig. 122, and let x, y, z, be co-ordinates parallel to these axes. Let three rectangular axes, fixed in the body, and moveable with it, likewise pass through the centre, and let x_1, y_1, z_1, be co-ordinates parallel to these. We shall also for the sake of simplicity suppose these latter axes to be the principal axes of the body.

Let, as in Art. 112,

x make with x_1, y_1, z_1, angles whose cosines a, b, c,
y . . . . a', b', c',
z . . . . a'', b'', c'',

we shall then have the same equations of condition (β), (γ), as in p. 294. Also, if D be the distance from the origin C, of any point, by p. 293,

$$\frac{x}{D} = a\frac{x_1}{D} + b\frac{y_1}{D} + c\frac{z_1}{D},$$

$$\left.\begin{aligned} \text{or,}\quad x &= a\,x_1 + b\,y_1 + c\,z_1 \\ \text{similarly,}\quad y &= a'\,x_1 + b'\,y_1 + c'z_1 \\ z &= a''x_1 + b''y_1 + c''z_1 \end{aligned}\right\} \ldots\ldots (\epsilon);$$

$$\text{Also, } \frac{x_1}{D} = a\frac{x}{D} + a'\frac{x_1}{D} + a''\frac{z}{D},$$

$$\text{or } \left.\begin{aligned} x_1 &= ax + a'y + a''z \\ y_1 &= bx + b'y + b''z \\ z_1 &= cx + c'y + c''z \end{aligned}\right\} \ldots\ldots (\zeta).$$

120. Prop. Whatever be the motions of the different parts of the body, it may at any instant be considered as moving round some axis *.

This axis is called *the axis of instantaneous rotation.*

By differentiating equations (ϵ) with respect to t, observing that for a given point, x_1, y_1, z_1, are constant, we have

$$\left.\begin{aligned} \frac{dx}{dt} &= x_1\frac{da}{dt} + y_1\frac{db}{dt} + z_1\frac{dc}{dt} \\ \frac{dy}{dt} &= x_1\frac{da'}{dt} + y_1\frac{db'}{dt} + z_1\frac{dc'}{dt} \\ \frac{dz}{dt} &= x_1\frac{da''}{dt} + y_1\frac{db''}{dt} + z_1\frac{dc''}{dt} \end{aligned}\right\} \ldots\ldots\ldots (\eta).$$

Which give the motions of any point in consequence of the change of the angles whose cosines are a, b, c, a', &c.

If we put $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$, $\frac{dz}{dt} = 0$, we find the points of the system for which the velocity $= 0$; that is,

$$x_1\,da + y_1\,db + z_1\,dc = 0 \ldots\ldots\ldots\ldots (1),$$
$$x_1\,da' + y_1\,db' + z_1\,dc' = 0 \ldots\ldots\ldots\ldots (2),$$
$$x_1\,da'' + y_1\,db'' - z_1\,dc'' = 0 \ldots\ldots\ldots\ldots (3).$$

And $(1)\,c + (2)\,c' + (3)\,c''$ gives

$$x_1(c\,da + c'\,da' + c''\,da'') + y_1(c\,db + c'\,db' + c''\,db'') + z_1(c\,dc + c'\,dc' + c''\,dc'') = 0.$$

Also $c^2 + c'^2 + c''^2 = 1$; $\therefore\ c\,dc + c'\,dc' + c''\,dc'' = 0.$

And we shall have similar equations by taking

$$(1)\,b + (2)\,b' + (3)\,b'', \text{ and } (1)\,a + (2)\,a' + (3)\,a''.$$

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* See proof also in Chap. I. of this Book, Note, p. 222.

The differentials da, &c. are taken with respect to t: we will suppose*

$$c\,db + c'db' + c''db'' = p\,dt,$$

and since $c\,b + c'b' + c''b'' = 0$,

$$b\,dc + b'dc' + b''dc'' = -c\,db - c'db' - c''db'' = -p\,dt.$$

Similarly, we will suppose $a\,dc + a'dc' + a''dc'' = q\,dt$;

whence, $c\,da + c'da' + c''da'' = -q\,dt$,

and $b\,da + b'da' + b''da'' = r\,dt$;

whence, $a\,db + a'db' + a''db'' = -r\,dt$.

And our equations become

$$py_1 - qx_1 = 0,\ r_1x_1 - pz_1 = 0,\ qz_1 - ry_1 = 0.$$

And hence, the points for which the velocity is 0, lie in a straight line, passing through the origin. Therefore, this line is, for an instant, immoveable, and the body revolves round it; and these are the equations to the *axis of instantaneous rotation*.

121. PROP. To find the angles, which the axis of instantaneous rotation makes with x_1, y_1 z_1. If IC, fig. 122, be the axis of instantaneous rotation, and if CN, NM, MI, be x_1, y_1, z_1, IN will be perpendicular to Cx_1, and

$$\cos.\ ICx_1 = \frac{CN}{CI} = \frac{x_1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}}$$

$$= \frac{px_1}{\sqrt{(p^2x_1^2 + p^2y_1^2 + p^2z_1^2)}}$$

$$= \frac{px_1}{\sqrt{(p^2x_1^2 + q^2x_1^2 + r^2x_1^2)}},$$

by the equations just found.

$$\cos.\ ICx_1 = \frac{p}{\sqrt{(p^2 + q^2 + r^2)}}.$$

Similarly, $\cos.\ ICy_1 = \dfrac{q}{\sqrt{(p^2 + q^2 + r^2)}}$,

$$\cos.\ ICz_1 = \frac{r}{\sqrt{(p^2 + q^2 + r^2)}}.$$

* The quantities p, q, r, are the angular velocity resolved respectively parallel to the planes yz, xz, and xy.

122. PROP. The angular velocity about the axis IC is

$$\sqrt{(p^2 + q^2 + r^2)}.$$

Take a point P in the axis Cz_1 at a distance 1 from C; so that for it $x_1 = 0$, $y_1 = 0$, $z_1 = 1$.

Therefore, by (e), $x = c$, $y = c'$, $z = c''$.

$$\text{And velocity}^2 \text{ of } P = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = \frac{dc^2 + dc'^2 + dc''^2}{dt^2}.$$

$$\text{Now } -p\,dt = b\,dc + b'dc' + b''dc'', \quad q\,dt = a\,dc + a'dc' + a''dc'';$$

$$0 = c\,dc + c'dc' + c''dc''.$$

Adding together the squares of these three equations, and taking account of the equations (β), we have

$$(p^2 + q^2)\,dt^2 = dc^2 + dc'^2 + dc''^2;$$

$$\therefore \text{ velocity of } P = \sqrt{(p^2 + q^2)}.$$

Also if we draw PQ perpendicular on CI,

$$PQ = \sin. ICz_1 = \sqrt{(1 - \cos.^2 ICz_1)}$$

$$= \frac{\sqrt{(p^2 + q^2)}}{\sqrt{(p^2 + q^2 + r^2)}};$$

$$\therefore \text{ angular velocity of } P = \frac{\text{velocity of } P}{PQ} = \sqrt{(p^2 + q^2 + r^2)}.$$

And the angular velocity is necessarily the same for all the points, in consequence of the rigidity of the system.

123. PROP. To find the velocities parallel to x_1, y_1, z_1, in terms of p, q, r.

The position of the axes parallel to x_1, y_1, z_1, is perpetually varying, but taking their position at any moment, we may resolve the velocities and the forces in their directions. And we may transform the expressions for such quantities in the directions x, y, z, into corresponding expressions for the same quantities (velocities or forces), in the directions x_1, y_1, z_1, by the formulæ for the transformation of co-ordinates; for a co-ordinate x_1 will have the same relation to x, a co-ordinate to the same point, as a velocity or force in the direction of x_1 has to its resolved part in the direction of x.

Now $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, are the velocities in the directions of x, y, z.

Hence, we have by (ζ),

$$\text{velocity parallel to } x_1 = a.\frac{dx}{dt} + a'\frac{dy}{dt} + a''\frac{dz}{dt},$$

$$\text{to } y_1 = b.\frac{dx}{dt} + b'\frac{dy}{dt} + b''\frac{dz}{dt},$$

$$\text{to } z_1 = c.\frac{dx}{dt} + c'\frac{dy}{dt} + c''\frac{dz}{dt}.$$

But from equations (η) by the same reductions as in Art. 120.

$$a\frac{dx}{dt} + a'.\frac{dy}{dt} + a''\frac{dz}{dt} = qz_1 - ry_1,$$

$$b\frac{dx}{dt} + b'.\frac{dy}{dt} + b''\frac{dz}{dt} = rx_1 - pz_1,$$

$$c\frac{dx}{dt} + c'.\frac{dy}{dt} + c''\frac{dz}{dt} = py_1 - qx_1.$$

Hence, it appears that the quantities $qz_1 - ry_1$, $rx_1 - pz_1$, $py_1 - qx_1$, which are 0 where the velocity is 0, do at other points represent the velocities in the directions parallel to x_1 to y_1 and to z_1.

124. Prop. To find the effective forces parallel to x_1, y_1, z_1, in terms of p, q, r.

For the sake of abbreviation, let

$$qz_1 - ry_1 = \pi,\ rx_1 - pz_1 = \chi,\ py_1 - qx_1 = \rho.$$

And by the last

$$a\frac{dx}{dt} + a'\frac{dy}{dt} + a''\frac{dz}{dt} = \pi,$$

$$b\frac{dx}{dt} + b'\frac{dy}{dt} + b''\frac{dz}{dt} = \chi,$$

$$c\frac{dx}{dt} + c'\frac{dy}{dt} + c''\frac{dz}{dt} = \rho.$$

Hence, eliminating, and taking account of equations (β),

$$\frac{dx}{dt} = a\pi + b\chi + c\rho,$$

$$\frac{dy}{dt} = a'\pi + b'\chi + c'\rho,$$

$$\frac{dz}{dt} = a''\pi + b''\chi + c''\rho;$$

$$\therefore \frac{d^2x}{dt} = ad\pi + bd\chi + cd\rho + \pi da + \chi db + \rho dc,$$

$$\frac{d^2y}{dt^2} = a'd\pi + b'd\chi + c'd\rho + \pi'da' + \chi db' + \rho dc',$$

$$\frac{d^2z}{dt} = a''d\pi + b''d\chi + c''d\rho + \pi da'' + \chi db'' + \rho dc''.$$

And $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$, are the components of the effective accelerating forces in directions x, y, z; hence, if π', χ', ρ', be the effective accelerating forces in directions x_1, y_1, z_1, we shall have

$$\pi' = a\frac{d^2x}{dt^2} + a'.\frac{d^2y}{dt^2} + a''.\frac{d^2z}{dt^2},$$

and by the values just found, this becomes, observing the equations of condition,

$$\pi'dt = d\pi - r\chi dt + q\rho dt;$$

similarly,

$$\chi'dt = d\chi + r\pi dt - p\rho dt,$$

$$\rho'dt = d\rho - q\pi dt + p\chi dt.$$

Or restoring the values of π, χ, ρ,

$$\pi'dt = z_1 dq - y_1 dr + q(py_1 - qx_1)dt - r(rx_1 - pz_1)dt,$$

$$\chi'dt = x_1 dr - z_1 dp + r(qz_1 - ry_1)dt - p(py_1 - qx_1)dt,$$

$$\rho'dt = y_1 dp - x_1 dq + p(rx_1 - pz_1)dt - q(ry_1 - qz_1)dt.$$

124. Prop. A body being acted upon by given forces; to find the equations of its motion in p, q, r.

By Dalembert's principle, Art. 73, the forces impressed must be equivalent to the effective forces. In expressing this equivalence by the equations of equilibrium, we may refer them to any axes; we shall refer them to the axes x_1, y_1, z_1, at any moment; for though these axes are not fixed, the statical properties of the system being true for any axes, are true for the position which the axes x_1, y_1, z_1, have at any moment. The effective forces π', χ', ρ', on a particle m, whose co-ordinates are x_1, y_1, z_1, have been found in the last Article: their moment with respect to the axis Oz_1 is $(x_1\chi' - y_1\pi')m$, and the whole moment with respect to this axis is

$$\Sigma(x_1\chi' - y_1\pi')m,$$

and putting for π', χ', their values from the last, this becomes

$$\Sigma\left\{\begin{array}{l} x_1^2\dfrac{dr}{dt} - x_1 z_1\dfrac{dp}{dt} + x_1 r(q z_1 - r y_1) - x_1 p(p y_1 - q x_1) \\ -y_1 z_1\dfrac{dq}{dt} + y_1^2\dfrac{dr}{dt} - y_1 q(p y_1 - q x_1) + y_1 r(r x_1 - p z_1) \end{array}\right\} . m.$$

Now, $\Sigma x_1 y_1 m = 0$, $\Sigma x_1 z_1 m = 0$, $\Sigma y_1 z_1 m = 0$; and p, q, r, are constant in the integrals expressed by Σ; whence this becomes

$$\Sigma(x_1^2 + y_1^2)m . \frac{dr}{dt} + \Sigma(x_1^2 - y_1^2) . m . p q.$$

$$\text{But } \Sigma(x_1^2 + y_1^2)m = C;$$

$$\Sigma(x_1^2 - y_1^2)m = \Sigma\{(x_1^2 + z_1^2) - (y_1^2 + z_1^2)\} m = B - A;$$

whence the moment of the effective forces with respect to the axis of z_1, becomes

$$C\frac{dr}{dt} + (B - A)pq.$$

Similarly, $$B\frac{dq}{dt} + (A - C)pr;$$

and $$A\frac{dp}{dt} + (C - B)qr,$$

are the moments of the *effective forces* with respect to the axes of y_1 and z_1 respectively.

Now, let the moments of the *forces impressed* with respect to the axes x_1, y_1, z_1, be respectively N, N', N'': hence, we shall have, by Dalembert's principle,

$$\left.\begin{array}{l} N = C\dfrac{dr}{dt} + (B - A)pq \\ N' = B\dfrac{dq}{dt} + (A - C)pr \\ N'' = A . \dfrac{dp}{dt} + (C - B)qr \end{array}\right\} \ldots\ldots\ldots (\theta);$$

which are the equations of motion.

If the forces which act on any particle m, be resolved into X_1, Y_1, Z_1, in the directions of x_1, y_1, z_1, respectively, we shall have their moment with respect to the axis $Cz_1 = (x_1 Y_1 - y_1 X_1)m$; and the whole moment for this axis will be

$$\Sigma (x_1 Y_1 - y_1 X_1) m = N;$$

similarly, $$\Sigma (z_1 X_1 - x_1 Z_1) m = N',$$

$$\Sigma (y_1 Z_1 - z_1 Y_1) m = N''.$$

COR. If the motion take place parallel to a fixed plane, the expressions may be simplified. (The plane will be one of the principal planes.)

Let the plane be that of xy, and of x_1, y_1; then, z and z_1 are always in the same direction. Therefore, $c''=1$, $c'=0$, $c=0$, $a''=0$, $b''=0$,

$$a^2+a'^2=1,\ ab+a'b'=0,$$

$$b^2+b'^2=1,$$

$$p=0,\ q=0,\quad r\,dt=b\,da+b'da'.$$

If ϕ be the angle which x_1 makes with x,

$$a=\cos.\phi,\ a'=\sin.\phi;\ b=-\sin.\phi,\ b'=\cos.\phi;$$

$$b\,da+b'\,da'=\sin.^2\phi\,d\phi+\cos.^2\phi\,d\phi;\ r=\frac{d\phi}{dt};$$

and the first of the equations (θ) becomes

$$N=C.\frac{d^2\phi}{dt^2},$$

which may be deduced independently.

125. PROP. The quantities p, q, r, being known, to determine the position of the body.

Let the plane xCy, fig. 123, cut x_1Cy_1 in the line NC, and let the angle $x_1CN=\phi$, $NCx=\psi$; and let the angle which the planes xCy, x_1Cy_1 make, be θ; this angle will be the same as z_1Cz.

Let a sphere be described about the point C with radius 1, and let the points x, y, z; x_1, y_1, z_1; N; be upon its surface; we shall then have

$$x_1N=\phi,\ xN=\psi,\ z_1z=\theta;\ \text{also angle } N=\theta.$$

And, Nz_1 Nz_1, quadrants.

And joining every two of the points x, y, z, x_1, y_1, z_1, by arcs of great circles, we shall have the following equations by the properties of spherical triangles,

$a=\cos.x_1x=\cos.\theta\sin.\phi\sin.\psi+\cos.\phi\cos.\psi$ by triangle x_1Nx,
$b=\cos.y_1x=\cos.\theta\cos.\phi\sin.\psi-\sin.\phi\cos.\psi$ . . . . y_1Nx,

$c = \cos. z_1 x = \sin. \theta . \sin. \psi$; for in triangle $z_1 N x$, the angle $z_1 N x$ is the complement of θ;

$a' = \cos. x_1 y = \cos. \theta \sin. \phi \cos. \psi - \cos. \phi \sin. \psi$ by triangle $x_1 N y$,
$b' = \cos. y_1 y = \cos. \theta \cos. \phi \cos. \psi + \sin. \phi \sin. \psi . \ldots\ldots . y_1 N y$,
$c' = \cos. z_1 y = \sin. \theta \cos. \psi$; for in triangle $z_1 N y$, the angle $z_1 N y$ is the complement of θ;

$a'' = \cos. x_1 z = - \sin. \theta . \sin. \phi$ by triangle $x_1 N z$ where $x_1 N z$ is $\frac{\pi}{2} + \theta$,
$b'' = \cos. y_1 z = - \sin. \theta \cos. \phi$ by triangle $y_1 N z$,
$c'' = \cos. z_1 z = \cos. \theta$.

Hence,
$$\begin{aligned} -p\,dt &= b\,dc + b'dc' + b''dc'' \\ &= \{\cos. \theta \cos. \phi \sin. \psi - \sin. \phi \cos. \psi\} \\ &\quad . \{\cos. \theta \sin. \psi d\theta + \sin. \theta \cos. \psi d\psi\} \\ &\quad + \{\cos. \theta \cos. \phi \cos. \psi + \sin. \phi \sin. \psi\} \\ &\quad . \{\cos. \theta \cos. \psi d\theta - \sin. \theta \sin. \psi d\psi\} \\ &\quad + \sin. \theta \cos. \phi . \sin. \theta . d\theta \\ &= \cos.^2 \theta . \cos. \phi . d\theta - \sin. \theta . \sin. \phi d\psi + \sin.^2 \theta \cos. \phi d\theta \\ &= \cos. \phi d\theta - \sin. \theta \sin. \phi d\psi; \end{aligned}$$

$\therefore\ p\,dt = \sin. \theta \sin. \phi d\psi - \cos. \phi d\theta.$

Similarly,

$$q\,dt = a\,dc + a'dc' + a''dc'' = \sin. \theta \cos. \phi d\psi + \sin. \phi . d\theta.$$

And,

$$r\,dt = b\,da + b'da' + b''da''.$$

$$\begin{aligned} \text{Now,}\quad da &= -\sin. \theta \sin. \phi \sin. \psi d\theta \\ &\quad + \{\cos. \theta \cos. \phi \sin. \psi - \sin. \phi \cos. \psi\}\, d\phi \\ &\quad + \{\cos. \theta \sin. \phi \cos. \psi - \cos. \phi \sin. \psi\}\, d\psi; \end{aligned}$$

$$\begin{aligned} \therefore\ b\,da = &- \{\cos. \theta \cos. \phi \sin. \psi - \sin. \phi \cos. \psi\} . \\ &\quad \sin. \theta . \sin. \phi . \sin. \psi d\theta \\ &+ \{\cos. \theta \cos. \phi \sin. \psi - \sin. \phi \cos. \psi\}^2 d\phi \\ &+ \{\cos. \theta \sin. \phi \cos. \psi - \cos. \phi \sin. \psi\} \\ &\quad . \{\cos. \theta \cos. \phi \sin. \psi - \sin. \phi \cos. \psi\}\, d\psi, \end{aligned}$$

and we shall have $b'da'$ by putting $\psi + \frac{\pi}{2}$ for ψ in this expression.

Hence, $bda + b'da'$

$$= -\sin.\theta\cos.\theta\sin.\phi\cos.\phi\, d\theta + \{\cos.^2\theta\cos.^2\phi + \sin.^2\phi\}\, d\phi$$
$$- \{\cos.\theta\cos.^2\phi + \cos.\theta\sin.^2\phi\}\, d\psi.$$

Also, $b''da''$

$$= \sin.\theta\cos.\phi\,.\,\{\cos.\theta.\sin.\phi.d\theta + \sin.\theta\cos.\phi d\phi\}.$$

Hence, $rdt = bda + b'da' + b''da'' = d\phi - \cos.\theta d\psi.$

Hence, having found $p, q, r,$ we must determine $\phi, \psi, \theta,$ by means of the equations

$$\left.\begin{aligned} pdt &= \sin.\phi\sin.\theta d\psi - \cos.\phi d\theta \\ qdt &= \cos.\phi\sin.\theta d\psi + \sin.\phi d\theta \\ rdt &= d\phi - \cos.\theta d\psi \end{aligned}\right\}\ldots\ldots\ldots(\iota).$$

And ϕ, ψ, θ being known, the position of the body is completely determined.

N, N', N'', may be functions of $\phi, \psi. \theta$. Hence, the six equations (θ) and (ι), will determine the quantities $\phi, \psi, \theta, p, q, r$.

126. Prop. A body revolves about its centre of gravity acted upon by no forces; it is required to integrate the equations already found.

Take the equations (θ),

$$\left.\begin{aligned} Cdr + (B-A)\,pqdt &= 0 \\ Bdq + (A-C)\,prdt &= 0 \\ Adp + (C-B)\,qrdt &= 0 \end{aligned}\right\}\ldots\ldots\ldots(\kappa).$$

Multiply by r, q, p respectively, and add, and we have

$$Crdr + Bqdq + Apdp = 0;$$

$$\therefore\ Cr^2 + Bq^2 + Ap^2 = h^2\ldots\ldots\ldots(\lambda),$$

h being a constant quantity.

Again, multiply equations (κ), by Cr, Bq, and Ap respectively, and we have

$$C^2rdr + B^2qdq + A^2pdp = 0;$$

$$\therefore\ C^2r^2 + B^2q^2 + A^2p^2 = k^2, \text{ a constant quantity}\ldots\ldots(\mu).$$

Again, multiply by c, b, a, and add

$$C\{cdr+(qa-pb)rdt\}+$$
$$B\{bdq+(pc-ra)qdt\}+A\{adp+(rb-qc)pdt\}=0.$$

But it may be proved*, that

$$(qa-pb)dt=dc,\quad (pc-ra)dt=db,\quad (rb-qc)dt=da;$$
$$\therefore\ C.d.rc+Bd.qb+A.d.pa=0,$$

Similarly, $$\left.\begin{array}{l} Crc+Bqb+Apa=l.\\ Crc'+Bqb'+Apa'=l'\\ Crc''+Bqb''+Apa''=l''\end{array}\right\}\dots\dots\dots(\nu).$$

If we add then three, we get

$$C^2r^2+B^2q^2+A^2p^2=l^2+l'^2+l''^2=k^2.$$

127. Prop. When a body revolves, acted on by no forces, there exists a plane, to which it may be referred, which plane is invariable in position.

If we draw a line mC, making with x, y, z, angles of which the cosines are

$$-\frac{Apa+Bqb+Crc}{\sqrt{(A^2p^2+B^2q^2+C^2r^2)}}\ \&\text{c.}:$$

since these quantities are constant, this line will have the same position during the whole motion of the body, and a plane perpendicular to it will be fixed.

---

* Take the three equations

$$adc+a'dc'+a''dc''=qdt,$$
$$bdc+b'dc'+b''dc''=-pdt,$$
$$cdc+c'dc'+c''dc''=0.$$

Multiplying the first by a, the second by b, the third by c, and add them; and we find, taking account of the equations of condition,

$$dc=(aq-bp)dt.$$

And in the same manner we shall find

$$db=(cp-ar)dt,\quad da=(br-cp)dt.$$

And so for the other similar quantities.

This plane is called the *Principal Plane of Moments* *.

Cor. 1. We may thus find the angles which the line mC makes with x_1, y_1, z_1.

$$\cos mCx_1 = a . \cos. mCx + a' \cos. mCy + a''. \cos. mCz$$

$$\cos. mCx_1 = -\frac{Ap}{k}, \quad \text{similarly,} \quad \cos. mCy_1 = -\frac{Bq}{k},$$

$$\cos. mCz_1 = -\frac{Cr}{k}.$$

Cor. 2. If we take the plane perpendicular to mC for the plane of xy, we have

$$a'' = -\frac{Ap}{k}, \; b'' = -\frac{Bq}{k}, \; C'' = -\frac{Cr}{k};$$

or $\sin. \theta \sin. \phi = \frac{Ap}{k}$, $\sin. \theta \cos. \phi = \frac{Bq}{k}$, $\cos. \theta = -\frac{Cr}{k}$.

Prob. I. *All the three moments* A, B, C *being equal, and the body not being acted upon by any forces; to determine its motion.*

In this case the equations (κ) become

$$\frac{dr}{dt} = 0, \; \frac{dq}{dt} = 0, \; \frac{dp}{dt} = 0.$$

Hence, p, q, r are constant quantities.

Hence, the position of the axis of instantaneous rotation is fixed with respect to the body, and the motion of the poles of instantaneous rotation is therefore nothing, and hence this axis is fixed in space.

All the axes of the body possess the properties of *principal axes:* therefore let the axis of instantaneous rotation coincide with

* If we consider at any instant the momentum of each particle of the system; (that is, the product of its velocity and mass;) and if we resolve this momentum parallel to a given plane passing through C, and multiply the resolved part by the perpendicular from C, so as to get its *moment*, we may obtain the sum of the moments of the particles referred to this plane. And the plane for which this sum is the greatest, is the plane found above, and hence denominated the Principal Plane of Moments.

Cz_1; hence, $p=0$, $q=0$, $r=$ constant; and equations (ι) become

$$\sin.\ \phi \sin.\ \theta d\psi - \cos.\ \phi d\theta = 0,$$
$$\cos.\ \phi \sin.\ \theta d\psi + \sin.\ \phi d\theta = 0,$$
$$d\phi - \cos.\ \theta d\psi = r dt.$$

the two first give $d\theta=0$, and $\sin.\ \theta d\psi = 0$, whence $d\psi=0$; $\therefore$ θ and ψ are constant.

Hence, $d\phi = r dt$, and $\frac{d\phi}{dt} = r$, a constant quantity.

Therefore the body will revolve about a fixed axis and with a constant velocity.

If the motion have been produced by a single force, acting at a point of the system, (impact or pressure,) the axis of rotation will be perpendicular to the plane which passes through the direction of the force and the fixed centre of gravity.

These conclusions are applicable to a sphere, a cube, &c.

PROB. II. *If two of the moments,* A *and* B, *be equal, and the body be not acted upon by any forces; to determine the motion.*

(This will be the case for all figures of revolution, the axis of revolution coinciding with the axis Cz_1).

The equations (κ) here become

$$Cdr = 0,$$
$$A\frac{dq}{dt} + (A-C)\,pr = 0,$$
$$A\frac{dp}{dt} - (A-C)\,qr = 0.$$

From the first of these it appears that r is a constant quantity, and the third being differentiated and divided by dt, gives

$$A\frac{d^2p}{dt^2} = (A-C)\,r\,.\frac{dq}{dt},$$

and substituting from this the value of $\frac{dq}{dt}$ in the second, we have

$$\frac{d^2p}{dt^2} + \frac{(A-C)^2r^2}{A^2}\,.p = 0,$$

whence we easily obtain

$$p = a \,.\, \sin.\ (nt + \gamma)\,; \text{ putting } n \text{ for } \frac{(A - C)\, r}{A},$$

and supposing $A > C$; and we have from this

$$q = a \cos.\ (nt + \gamma)\,;$$

where a and γ are two arbitrary quantities, to be determined from the circumstances of the motion at a given time.

Since r is a constant quantity, and also $p^2 + q^2$ a constant quantity, viz. a^2, it appears that if IC be the axis of instantaneous rotation, ICz_1 is a constant angle. Hence, the axis of instantaneous rotation describes, with respect to the body, a conical surface about Cz_1.

By substituting these values in equations (ι), we obtain

$$a\,dt \,.\, \sin.\ (nt + \gamma) = \sin.\ \phi \,.\, \sin.\ \theta\, d\psi - \cos.\ \phi\, d\theta,$$

$$a\,dt \,.\, \cos.\ (nt + \gamma) = \cos.\ \phi \,.\, \sin.\ \theta\, d\psi + \sin.\ \phi\, d\theta,$$

$$r\,dt = d\phi - \cos.\ \theta\, d\psi.$$

Take the plane of xy to be the principal plane of moments; we then have, by Cor. 2 to Art. 127,

$$\sin.\ \theta \sin.\ \phi = \frac{Ap}{k} = \frac{Aa}{k} \sin.\ (nt + \gamma),$$

$$\sin.\ \theta \cos.\ \phi = \frac{Bq}{k} = \frac{Aa}{k} \cos.\ (nt + \gamma),$$

$$\cos.\ \theta = -\frac{Cr}{k}\,;$$

$$\therefore \tan.\ \phi = \tan.\ (nt + \gamma)\,;$$

$$\phi = nt + \gamma\,;\ \frac{d\phi}{dt} = n.$$

$$\sin.\ \theta = \frac{Aa}{k}\,;\ \cos.\ \theta = -\frac{Cr}{k}\,;\ \tan.\ \theta = -\frac{Aa}{Cr},$$

and θ is constant. And by the third of equations (ι)

$$\frac{d\psi}{dt} = \frac{1}{\cos.\ \theta}\left(\frac{d\phi}{dt} - r\right) = \frac{k\,(r - n)}{Cr} = \frac{k}{A}.$$

Hence, the body revolves uniformly about Cz_1, while CN, the *line of nodes*, moves uniformly round C in the plane xy.

Suppose, that at the first instant, when $t=0$, the instantaneous axis CI is in the plane z_1Cx_1* and makes an angle δ with Cz_1; and that the angular velocity about it is ϵ. Then, by the formulæ of Art. 121, since IC is perpendicular to Cy_1 we have

$$\cos. ICz_1 = \cos. \delta = \frac{r}{\sqrt{(p^2+q^2+r^2)}},$$

$$\cos. ICx_1 = \sin. \delta = \frac{p}{\sqrt{(p^2+q^2+r^2)}},$$

$$\cos. ICy_1 = 0 = \frac{q}{\sqrt{(p^2+q^2+r^2)}}; \therefore q=0.$$

$$\text{Also, } \epsilon = \sqrt{(p^2+q^2+r^2)}.$$

Hence we have, since $t=0$,

$$\cos. \delta = \frac{r}{\epsilon}, \ \sin. \delta = \frac{a \sin. \gamma}{\epsilon}, \ 0 = \frac{a \cos. \gamma}{\epsilon}; \ \therefore \gamma = \frac{\pi}{2};$$

$$\therefore r = \epsilon \cos. \delta, \ a = \epsilon \sin. \delta, \ \tan. \theta = -\frac{A}{C} \tan. \delta.$$

$$\text{And } n = \frac{A-C}{A} r = \frac{A-C}{A} . \epsilon \cos. \delta.$$

Hence, (see Art. 126,) $k^2 = A^2a^2 + C^2r^2 = A^2\epsilon^2 \sin.^2 \delta + C^2\epsilon^2 \cos.^2 \delta$

$$= \epsilon^2 \{A^2 - (A^2 - C^2) \cos.^2 \delta\}.$$

$$\text{Hence, } \frac{d\psi}{dt} = \epsilon \sqrt{\left(1 - \frac{A^2-C^2}{A^2} \cos.^2 \delta\right)}.$$

If $C=A$, nearly, n is small; and since $\frac{d\phi}{dt} = n$, and $\frac{d\psi}{dt} = \epsilon$, nearly, the motion of the body round the axis Cz_1 will be slow, and the motion of Cz_1 about Cz, comparatively quick.

When ϵ and δ are positive, I is within the quadrantal space $x_1 y_1 z_1$ fig. 123; and it appears that in this case θ is negative,

* Since any line in the plane x_1Cy_1 is a principal axis, we may take for Cx_1 the intersection of x_1Cy_1 and z_1CI.

and $Nx_1\,y_1$ which was in the preceding Articles supposed below Nxy, is here above it.

We have, when $t=0$, $\phi=\gamma=\frac{\pi}{2}$. Hence, N is the pole of z_1x_1, and z, which is a quadrant distant from N, is in z_1x_1. Therefore in this case I is in the arc z_1z. And any moment may be considered as that when $t=0$. Therefore I, the pole of instantaneous rotation is always in z_1z. And since r and p^2+q^2 are constant, CI always makes the same angle with Cz_1.

$$\text{Since } zz_1=\theta,\ z_1I=\delta,$$

we have $zI=\theta+\delta$, $\sin. zI=\sin.\theta\cos.\delta+\cos.\theta\sin.\delta$,

$$\frac{\sin. zI}{\sin. z_1I}=\cos.\theta\left(\frac{\tan.\theta}{\tan.\delta}+1\right)=-\frac{Cr}{k}\left(-\frac{A}{C}+1\right)$$

$$=\frac{(A-C)r}{k}$$

$$\text{Also, } \frac{d\phi}{d\psi}=\frac{nA}{k}=\frac{(A-C)r}{k};$$

$$\therefore \sin. z_1I.d\phi=\sin. zI.d\psi.$$

If with centres z, z_1, we describe circles in fig. 124, on the surface of the sphere, since $d\phi$ is the angle described in dt by the body about z_1 and $\sin. z_1I$ the radius; $\sin. z_1I.d\phi$ will be Im, the arc of the circle which a point would describe in dt. Similarly, $d\psi$ is the angle which the surface $z_1x_1y_1$, describes about z; $\sin. zI$ is the radius: therefore $\sin. zI.d\psi$ is the arc In described by a point about z in dt. And $Im=In$.

Hence, if the circle Qn be fixed, and Pm carried by the pole I of the body, *roll* on Qn; the angular motion will be exactly the same as when the body is left to itself; and this supposition represents the motion of the body in the Problem. The reasonings will be the same if $A < C$. In this case, we shall have $n=\frac{C-A}{A}r$. The point z will be between z_1 and I, and the circle described about z_1 will roll with its interior circumference on the outside of the circle described about z.

Prob. III. *When all the moments are unequal, and the body is acted upon by no forces; to determine the motion.*

Equations (κ) become in this case

$$C\frac{dr}{dt} + (B-A)\,pq = 0,$$

$$B\frac{dq}{dt} + (A-C)\,pr = 0,$$

$$A\frac{dp}{dt} + (C-B)\,qr = 0.$$

Let $pqr\,.\,dt = d\phi$, and we have

$$Crdr + (B-A)\,d\phi = 0,$$

$$Bqdq + (A-C)\,d\phi = 0,$$

$$Apdp + (C-B)\,d\phi = 0;$$

$$\therefore\ r^2 = \frac{2(A-B)}{C}\phi + c^2,$$

$$q^2 = \frac{2(C-A)}{B}\phi + b^2,$$

$$p^2 = \frac{2(B-C)}{A}\phi + a^2,$$

a, b, c being the values of p, q, r, when ϕ is 0;

$$\therefore\ dt = \frac{d\phi}{pqr}$$

$$= \frac{d\phi}{\sqrt{\left\{\left(2\phi\frac{B-C}{A} + a^2\right)\left(2\phi\frac{C-A}{B} + b^2\right)\left(2\phi\frac{A-B}{C} + c^2\right)\right\}}}.$$

And integrating, we have t in terms of ϕ, and hence ϕ in terms of t. And hence, we have p, q, r; and by Art. 125, the position of the body.

Cor. By Art. 114, we have the moment of inertia at any time

$$\Sigma r'^2 m = A\cos.^2\alpha + B\cos.^2\beta + C\cos^2.\gamma$$

$$=\frac{Ap^2+Bq^2+Cr^2}{p^2+q^2+r^2}=\frac{h^2}{p^2+q^2+r^2}, \text{ by Art. 126.}$$

$$\therefore (p^2+q^2+r^2)\,\Sigma r'^2 m=h^2.$$

Hence, the sum of each particle into the square of its velocity, continues constant during the motion.

PROB. IV. *A solid body revolves about its centre of gravity, so that its axis of rotation coincides nearly with one of its principal axes; to find the motion.*

If IC be the instantaneous axis, always nearly coincide with Cz_1,

$$\sin. ICz_1=\frac{\sqrt{(p^2+q^2)}}{\sqrt{(p^2+q^2+r^2)}};$$

$\therefore p$ and q must both be small, $Cdr=0$, nearly, and r is nearly constant, and = the velocity of rotation $=n$;

$$\therefore Bdq+(A-C)\,npdt=0,\ Adp+(C-B)\,nqdt=0;$$

$$\therefore \text{ as before, } p=\alpha \sin.(n't+\gamma),\ q=\beta\cos.(n't+\gamma),$$

$$\text{where } n'=n\sqrt{\frac{(A-C).(B-C)}{AB}},\ \beta=\alpha\sqrt{\frac{(A-C).A}{(B-C).B}},$$

and knowing p and q, we might find ϕ, ψ, θ.

Now if IC and z_1C coincide, $\alpha=0$, and $\beta=0$. Hence, α and β are small when ICz_1 is small, and if n be real, p and q will always be small, but if n' be imaginary, p and q become exponentials, and increase beyond small values; and the solution is not applicable. In the first case, the axis will oscillate about Cz_1. In the second case, the axis IC will leave Cz_1, and oscillate about another of the principal axes.

The first case will happen, if $(A-C).(B-C)$ be positive; $\therefore$ if C be the *greatest* or *least* of the moments A, B, C.

The second case, if C be the *mean* moment.

Hence, if in the second case the body at first oscillate *accurately* about Cz_1, and if that axis be disturbed ever so little, the axis will entirely leave its position.

Cor. 1. We may prove that p and q cannot increase beyond a certain limit.

$$A p^2 + B q^2 + C r^2 = h^2 \dots (\lambda), \quad A^2 p^2 + h^2 q^2 + C r^2 = k^2 \dots (\mu),$$

$$(\mu) - (\lambda)\, C \text{ gives } A(A - C)\, p^2 + B(B - C)\, q^2 = k^2 - C h^2.$$

Hence, if p and q are small at first, and $A - C$ and $B - C$ of the same sign, p and q will always remain small.

Cor. 2. The limits of p^2 and q^2 are

$$p^2 < \frac{k^2 - C h^2}{A(A - C)}, \quad q^2 < \frac{k^2 - C h^2}{B(B - C)}.$$

CHAP. VIII.

MOTION OF A RIGID BODY ACTED ON BY ANY FORCES.

128. We shall consider, in the present Chapter, the motion of a body, whether it be in free space, or constrained to move upon a given plane. The same principles are applicable in both cases, if we include, among the forces which act upon the body, the reaction of the plane which it touches, and then eliminate this reaction.

In a large class of problems of this kind, one of the principal axes moves parallel to itself, and consequently, all the particles move in planes perpendicular to it; for instance, if a body bounded by a cylindrical surface of any form roll upon a plane. We shall take this more simple case separately.

Sect. 1. *When the Motions of all the Particles are in Parallel Planes.*

Prop. The body being acted upon by any forces, the motion of the centre of gravity will be the same as if all those forces acted at the centre.

By Dalembert's principle, Art. 73, the impressed and effective forces must be equivalent, and their moments about any point also equivalent; the former consideration will determine the motion of translation of the centre of gravity, and the latter the motion of rotation about it.

If x', y' be the co-ordinates of any particle, the velocity in the direction of x' is $\frac{dx'}{dt}$, and therefore the effective accelerating force is $\frac{d^2x'}{dt^2}$, and the effective moving force $m\frac{d^2x'}{dt^2}$. Thus we have

$$\text{whole effective force parallel to } x' = \frac{\Sigma m d^2x'}{dt^2}; \text{ to } y' = \frac{\Sigma m d^2y'}{dt^2}.$$

Also, if X and Y be the impressed accelerating forces on any particle m,

whole impressed forces are $\Sigma m X$, $\Sigma m Y$.

Hence we have

$$\Sigma \frac{md^2x'}{dt^2} = \Sigma m X, \; \Sigma \frac{md^2y'}{dt^2} = \Sigma m Y.$$

Let $\overline{x}$, $\overline{y}$ be the co-ordinates of the centre of gravity; and x, y the co-ordinates of m from the centre of gravity, so that $x' = \overline{x} + x$, $y' = \overline{y} + y$;

$$\therefore \Sigma m x = 0, \; \Sigma m y = 0; \; \therefore \Sigma m d^2x = 0, \; \Sigma m d^2y = 0;$$

$$\therefore \Sigma \frac{m\,d^2x'}{dt^2} = \Sigma \frac{m\,d^2\overline{x}}{dt^2} = M \cdot \frac{d^2\overline{x}}{dt^2};$$

(M being Σm the whole mass;) because $\overline{x}$ is the same for all particles. Similarly, $\Sigma \frac{m\,d^2y'}{dt^2} = M\frac{d^2\overline{y}}{dt^2}$;

$$\therefore \frac{d^2\overline{x}}{dt^2} = \frac{\Sigma m X}{M}, \; \frac{d^2\overline{y}}{dt^2} = \frac{\Sigma m Y}{M} \; \ldots\ldots\ldots (\xi),$$

Which are the equations that would result if all the forces were applied to the centre of gravity.

Cor. The forces mX, mY, are here understood to be pressures; and as impact is only a short pressure, the results are true of impact. The forces mX, mY, are measured by the momenta generated in a time $1''$; the force of impact may be measured by the *whole* momentum which it would generate.

129. Prop. The body being acted upon by any forces, the motion of rotation will be affected as if the centre of gravity were fixed, and the same forces were applied.

The moment of the effective force with respect to C is

$$\Sigma m \left(\frac{x' d^2 y' - y' d^2 x'}{dt^2}\right).$$

And putting for x', $\bar{x} + x$, and for y', $\bar{y} + y$, it is

$$\Sigma . m \left\{ \begin{array}{l} \frac{\bar{x} d^2 \bar{y}}{dt^2} + \frac{x d^2 \bar{y}}{dt^2} + \frac{\bar{x} d^2 y}{dt^2} + \frac{x d^2 y}{dt^2} \\ - \frac{\bar{y} d^2 \bar{x}}{dt^2} - \frac{\bar{y} d^2 x}{dt^2} - \frac{y d^2 \bar{x}}{dt^2} - \frac{y d^2 x}{dt^2} \end{array} \right\}$$

And observing that $\Sigma m = M$, $\Sigma m x = 0$, $\Sigma m y = 0$, $\Sigma m d^2 x = 0$, $\Sigma m d^2 y = 0$; and that $\bar{x}$ and $\bar{y}$ are not affected by Σ; it becomes

$$M . \frac{\bar{x} d^2 \bar{y} - \bar{y} d^2 \bar{x}}{dt^2} + \Sigma . m \frac{x d^2 y - y d^2 x}{dt^2}.$$

Also the moment of the effective forces is

$$\Sigma . m (Y x' - X y') = \Sigma . m (Y \bar{x} - X \bar{y} + Y x - X y)$$
$$= M (Y \bar{x} - X \bar{y}) + \Sigma . m (Y x - X y).$$

Equating, we have, observing that the terms multiplied by M are equal by last Article,

$$\Sigma . m \frac{x d^2 y - y d^2 x}{dt^2} = \Sigma . m (Y x - X y) \ldots\ldots\ldots (\pi).$$

Which is the equation that would result if the centre of gravity were fixed.

Cor. If in fig. 125, GA be a line always parallel to a line fixed in space, GM a line in the plane of xy, fixed with respect to the body, M any particle, of which the co-ordinates from G are x, y: it will be seen as in Art. 16, that $\frac{1}{2}(x\,d^2y - y\,d^2x)$ is the second differential of the sector AGM; and hence, since GM is constant, if $GM = r$, $AGM = \theta$, we have $x\,d^2y - y\,d^2x = r^2\,d^2\theta$. Therefore

$$\Sigma \,.\, mr^2 \,.\, \frac{d^2\theta}{dt^2} = \Sigma \,.\, m\,(Yx - Xy).$$

Or if as before, $\Sigma \,.\, m\,r^2 = M\,k^2$,

$$\frac{d^2\theta}{dt^2} = \frac{\Sigma \,.\, m\,(Yx - Xy)}{M\,k^2} \dots\dots\dots\dots (\rho)^*.$$

130. Prop. To find the centre of spontaneous rotation.

The centre of spontaneous rotation is the point which remains at rest for an instant when the body is put in motion by any force.

Thus, if a body GC, fig. 126, be acted upon by a force PC, it will, for the first instant, revolve round some point O, which may be thus determined.

Let the force in PC be perpendicular to GC, and $= P$; and the mass of the body being M, the space Gg described by G in a small time t, will be $\frac{P}{M} \cdot \frac{t^2}{2}$, the force being supposed constant for the time t. And by this motion of translation, any point, as O will be transferred to o, Oo being equal to Gg. Let $GO = l$, $CG = h$; and by Art. 75, the accelerating force causing O to revolve round C is $\frac{Phl}{Mk^2}$, and the space generated by it in the time t is $\frac{Phl}{Mk^2} \cdot \frac{t^2}{2}$. And if this be equal to Oo, the point O will be

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* These formulæ might be used in solving many of the problems in Chap. VI. So long as Mk^2 is the same, the motion will be the same whatever be the form of the body. Thus if, in Prob. V, fig. 40, instead of a straight line PQ, we had a body of any form, of which one point slides along AX, and the other along AY, the motion would be the same as is investigated, p. 124.

carried backwards by the rotation, just as much as it is carried forwards by the translation, and will be absolutely at rest: that is, if

$$\frac{P}{M}\cdot\frac{t^2}{2}=\frac{Phl}{Mk^2}\cdot\frac{t^2}{2},$$

$$\text{or } l=\frac{k^2}{h}.$$

Hence, by (a) C and O, are so situated that if one be the centre of suspension, the other is the centre of oscillation, or percussion.

Cor. If the force which acts at C be a force of impact, the point O will be the same, Gg and Oo being in this case described by the uniform velocities which the impact generates.

Prob. I. *To find at what distance from the Earth's centre a force must have acted, to generate at the same time its progressive and rotatory motion.*

This problem stated generally, is, having given the velocity of the centre of gravity and the angular velocity in a body revolving about a permanent axis; to find at what distance from the centre of gravity a single force would produce them.

Let P be the force acting at a distance h; then $\frac{P}{M}$ and $\frac{Ph}{Mk^2}$ are the forces which accelerate the centre of gravity, and a point at distance 1 about the centre of gravity; and hence, the whole velocities generated in any times will be as these forces. If v be the velocity of the centre, and a the angular velocity, measured by the space described by a particle at distance 1, we have

$$\frac{P}{M}:\frac{Ph}{Mk^2}::v:a;\ \therefore\ h=\frac{k^2a}{v}.$$

In the case of the Earth, let its radius be taken $=1$, and let the unit of time be 1 day: therefore, $a=2\pi$. And the radius of the Earth's orbit being r, the space described in 366.24 sidereal days is $2\pi r$, and therefore $v=\frac{2\pi r}{366.24}$. And since the Earth is a sphere,

$k^2 = \frac{2}{5}$. Hence, $h = \frac{2}{5} \cdot \frac{366.24}{r}$, and $r = 24266$, nearly;

$$\therefore h = \frac{1}{165.6}, \text{ nearly.}$$

Hence, we may find the centre of spontaneous rotation by the formula $l = \frac{k^2}{h} = 66.25$, which is a little greater than the Moon's distance from the Earth.

In the same way for Mars we should have $h = \frac{1}{464}$ of his radius, nearly. And for Jupiter, $h = \frac{1}{3}$ of his radius, nearly. In the latter planet, the centre of spontaneous rotation is only $\frac{6}{5}$ of the radius from the centre *.

131. The following problems refer to such motions as the *rocking* of a cradle, the oscillation, or as it has been called, *titubation* of a body on a curved base, and the rolling of a body not spherical. We shall solve some of them, first neglecting, and afterwards considering, friction.

PROB. II. *An ellipse with its plane vertical, rolls upon an horizontal plane which is perfectly smooth; to determine the motion, fig.* 127.

Let P be the point of contact with the horizontal plane, $CM = x$, $MP = y$, $CA = a$, $CB = b$, the semi-axes; PN vertical, CN horizontal; and let the angle ACH be θ,

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)};\ \frac{dy}{dx} = -\frac{b}{a}\frac{x\,dx}{\sqrt{(a^2 - x^2)}};$$

$$\text{but at } P,\ -\frac{dx}{dy} = \tan.\theta;\ \therefore \frac{\sin.\theta}{\cos.\theta} = \frac{a\sqrt{(a^2 - x^2)}}{bx};$$

$$\text{hence, } x = \frac{a^2\cos.\theta}{\sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)}};\ \text{and } y = \frac{b^2\sin.\theta}{\sqrt{(a^2\cos.^2\theta + b^2\sin^2\theta)}}.$$

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* John Bernoulli's Works, Vol. IV. p. 284.

Hence, if $\bar{y}$ be CH or NP,

$$\bar{y} = x \cos.\theta + y \sin.\theta = \sqrt{(a^2 \cos.^2\theta + b^2 \sin.^2\theta)}.$$

Also $CN = x \sin.\theta - y \cos.\theta = \dfrac{(a^2 - b^2)\sin.\theta\cos.\theta}{\sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)}}$.

Now let R be the reaction upwards at the point P, and $R.CN$ its moment; M the mass of the body, Mg its weight, Mk^2 its moment of inertia; the centre of gravity being supposed to be at C. Then by Art. 128, and 129,

$$\frac{d^2\bar{y}}{dt^2} = \frac{R}{M} - g,$$

$$\frac{d^2\theta}{dt^2} = \frac{R(a^2 - b^2)\sin.\theta\cos.\theta}{Mk^2\sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)}}.$$

But since $\bar{y} = \sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)}$;

$$d\bar{y} = -\frac{(a^2 - b^2)\sin.\theta\cos.\theta\, d\theta}{\sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)}}.$$

And the second equation becomes, multiplying by $d\theta$,

$\dfrac{d\theta d^2\theta}{dt^2} = -\dfrac{R}{Mk^2}\cdot d\bar{y}$; and eliminating R by the first,

$$\frac{d\theta\, d^2\theta}{dt^2} + \frac{d\bar{y}\, d^2\bar{y}}{k^2 dt^2} + \frac{g\, d\bar{y}}{k^2} = 0.$$

Multiply by $2k^2$ and integrate,

$$\frac{k^2 d\theta^2}{dt^2} + \frac{d\bar{y}^2}{dt^2} + 2g\bar{y} = C.$$

Put for $\bar{y}$ and for $d\bar{y}$ their values, and this becomes

$$\left\{k^2 + \frac{a^4 e^4 \sin.^2\theta\cos.^2\theta}{a^2\cos.^2\theta + b^2\sin.^2\theta}\right\}\frac{d\theta^2}{dt^2}$$
$$+ 2g\sqrt{(a^2\cos.^2\theta + b^2\sin.^2\theta)} = C.$$

Hence the angular velocity $\dfrac{d\theta}{dt}$ is known. C is to be determined by knowing the value of this velocity for a given value of θ.

If when CA is vertical, the angular velocity be α, we have $C = k^2 \alpha^2 + 2ga$. And when CA becomes horizontal,

$$k^2 \frac{d\theta^2}{dt^2} + 2gb = k^2 \alpha^2 + 2ga;$$

$$\therefore \frac{d\theta^2}{dt^2} = \alpha^2 + \frac{2g(a-b)}{k^2};$$

therefore, a point at the distance k from C acquires as much velocity as if it had fallen vertically through the difference of the axes CA, CB.

Since there is no lateral force, the centre of gravity C will ascend and descend in a vertical line. The surface will slide along the horizontal line PQ. If there be so much friction as to prevent this sliding, it will roll; the centre of gravity will have a lateral motion, and the problem will no longer be the same.

The figure may be an elliptical cylinder with a horizontal axis. In that case, $k^2 = \frac{1}{4}(a^2 + b^2)$. Or it may be an elliptical spheroid, APB being one of its principal sections. In this case, $k^2 = \frac{1}{5}(a^2 + b^2)$. It may be any figure in which the vertical ellipse is the part which touches the horizontal line, and the centre of gravity is at C.

If the plane were inclined at an angle ϵ, the rotatory motion would be the same, putting $g \cos. \epsilon$ for g. And the body would move along the plane, so that the motion parallel to the plane should be that of a point sliding down an inclined plane. For the part of gravity parallel to the plane could not affect the rotatory motion, since its result would pass through the centre of gravity.

PROB. III. *When the major axis of the ellipse, in last Problem, is nearly horizontal; to determine the small oscillations.*

Let ϕ be the angle which CB makes with the vertical; $\therefore \phi = \frac{\pi}{2} - \theta$. And when ϕ is very small, $\sin.^2\theta = \cos.^2\phi = 1 - \phi^2$, $\cos.^2\theta = \sin.^2\phi = \phi^2$. And, by last Problem, neglecting ϕ^4,

$$\left(k^2 + \frac{a^4 e^4 \phi^2}{b^2}\right)\frac{d\phi^2}{dt^2} = C - 2g\sqrt{(b^2 + a^2 e^2 \phi^2)}$$

$$= C - 2g\left(b + \frac{a^2 e^2 \phi^2}{2b}\right).$$

$$\left(\text{And if } \phi = \beta \text{ when } \frac{d\phi}{dt} = 0\right) = \frac{a^2 e^2 g}{b} (\beta^2 - \phi^2).$$

And neglecting $\beta^2 \phi^2$,

$$\frac{d\phi^2}{dt^2} = \frac{a^2 e^2 g}{k^2 b} (\beta^2 - \phi^2);$$

$$\therefore \text{ if } \frac{k^2 b}{a^2 e^2} = l,$$

$$\sqrt{\frac{g}{l}} \cdot \frac{d\phi}{\sqrt{(\beta^2 - \phi^2)}} = - dt; \; t = - \sqrt{\frac{g}{l}} \cdot \text{arc} \left(\text{sin.} = \frac{\phi}{\beta}\right);$$

and for a whole oscillation, this must be taken from $\phi = \beta$ to $\phi = -\beta$. Hence, if time of an oscillation $= T$,

$$T = \pi \sqrt{\frac{g}{l}}.$$

It appears from this, that the length of the isochronous pendulum is $l = \frac{k^2 b}{a^2 e^2}$.

Prob. IV. *Let the body roll on a horizontal plane, the part which comes in contact with the plane being a portion of a common cylinder with its axis horizontal; to determine the motion, fig* 128.

Let a line be drawn through the centre of the cylinder, and through the centre of gravity; let the latter point be at a distance c from the former; let θ be the angle which this line makes with the vertical; then it will be found by proceeding as in Prob. II, that

$$\frac{d\theta^2}{dt^2} = \frac{2\,cg\,(\text{cos.}\ \theta - \text{cos.}\ \beta)}{k^2 + c^2 \text{ sin.}^2 \theta};$$

where Mk^2 is the moment of inertia about the centre of gravity.

If the body perform small oscillations, we shall have for the length of the isochronous pendulum, $l = \frac{k^2}{c}$.

In these cases, the centre of gravity ascends and descends vertically, and goes twice up and down while the body goes once back and forwards.

PROB. V. *A cylindrical body is supported on a horizontal plane and oscillates; the friction being such as to prevent all sliding; to determine its motion, fig.* 128.

Let C be the centre of the circle AP, G the centre of gravity; P the point of contact with the horizontal plane; PC will be vertical; let GN be horizontal. Let a be the point in which A comes in contact with the plane: then, $AP = aP$.

Let $CA = a$, $CG = c$, $aH = x$, $HG = y$; $ACP = \theta$,

$$x = AP - PH = a\theta - c \sin.\theta,$$
$$y = CP - CN = a - c \cos.\theta.$$

Now, the forces which act upon the body are the force of gravity, and the force which arises from the pressure and the friction at P. Let these forces at P be composed of a horizontal force Q, and a vertical force R. And, by Art. 128, the effect on the motion of the centre of gravity will be the same as if all the forces were applied immediately at that point. Hence,

$$\frac{d^2x}{dt^2} = \frac{Q}{M}, \quad \frac{d^2y}{dt^2} = \frac{R}{M} - g.$$

And, by putting for x and y their values,

$$\frac{d\,(a\,d\theta - c\cos.\theta\,d\theta)}{dt^2} = \frac{Q}{M},$$

$$\frac{d\,(c\sin.\theta\,d\theta)}{dt^2} + g = \frac{R}{M}.$$

Also, by Art. 129, the effect of the forces in producing rotation about the centre of gravity, is the same as if it were fixed, and the forces Q, R, acted on the body; hence,

$$\frac{d^2\theta}{dt^2} = -\frac{Q\,.\,GH + R\,.\,GN}{Mk^2} = -\frac{Q\,(a - c\cos.\theta) + Rc\sin.\theta}{Mk^2}.$$

Substituting for Q and R, &c. we find

$$k^2 . \frac{2d\theta d^2\theta}{dt^2} + \frac{2(ad\theta - c\cos.\theta d\theta)\, d(ad\theta - c\cos.\theta d\theta)}{dt^2}$$

$$+ \frac{2c . \sin.\theta d\theta\, d(c\sin.d\theta)}{dt^2} + 2gc\sin.\theta d\theta = 0.$$

Integrating,

$$k^2\frac{d\theta^2}{dt^2} + (a - c\cos.\theta)^2 . \frac{d\theta^2}{dt^2} + c^2\sin.^2\theta\frac{d\theta^2}{dt^2} - 2gc\cos.\theta = C;$$

$$\text{or, } (k^2 + a^2 + c^2 - 2ac\cos.\theta)\frac{d\theta^2}{dt^2} = 2gc(\cos.\theta - \cos.\beta),$$

β being the value of θ when the velocity is 0.

If we make $k^2 + (a-c)^2 = b^2$, we have

$$\left(b^2 + 4ac . \sin.^2\frac{\theta}{2}\right)\frac{d\theta^2}{dt^2} = 4gc\left(\sin.^2\frac{\beta}{2} - \sin.^2\frac{\theta}{2}\right).$$

If we suppose β and θ small, and neglect θ^4, $\beta^2\theta^2$, &c. we have

$$\frac{d\theta^2}{dt^2} = \frac{gc}{b^2}(\beta^2 - \theta^2),$$

whence it appears, as in Prob. III, that the length of the isochronous pendulum is

$$= \frac{b^2}{c} = \frac{(a-c)^2 + k^2}{c} = c - 2a + \frac{a^2 + k^2}{c}.$$

If the curve be not a circle, the results will still be true for small oscillations, if we take C the centre of curvature.

G may be at any distance from C, and hence it may be beyond A, as when a body hangs by means of an axis passing through it, and supported on a plane PQ, fig. 129.

Prob. VI. *To find the correction due to the length of a pendulum, for the thickness of its axis.*

When it is requisite that a pendulum should oscillate very accurately about a horizontal line, it has an axis as B in the pendulum

M, fig. 129, of which the section is triangular; and this, with the edge O downwards, being placed with each end upon a hard plane, the pendulum will turn round the axis O.

But, if the axis have a curvilinear section as CP, in the pendulum N, fig. 129; or, if the edge be blunt, the pendulum no longer oscillates about a mathematical line. In this case it is required to find from what *point* the pendulum must be suspended, that it may oscillate in the same time.

Let the body oscillate about a fixed point S, C being the centre of curvature, and CS being $=\delta$, $CG=c$, and $CP=a$. Then, the length of the pendulum

$$=SG+\frac{k^2}{SG}=c-\delta+\frac{k^2}{c-\delta}=c-\delta+\frac{k^2}{c}+\frac{k^2\delta}{c^2};$$

omitting powers of δ, because CP, and therefore CS is small.

Hence, that the time of this may be the same as the time of the pendulum with the axis CP, we must have, by last Problem,

$$c-2a+\frac{a^2+k^2}{c}=c-\delta+\frac{k^2}{c}+\frac{k^2\delta}{c^2}.$$

Hence, omitting $\frac{a^2}{c}$, which is small, we have

$$\delta=\frac{2ac^2}{c^2-k^2}.$$

If $c>k$, the point S will lie towards G, if $c<k$, it will be beyond C. CS is always greater than CP.

Prob. VII. *The pendulum having two axes which are isochronous to each other, (as in Art.* 91.), *supposing them cylindrical, to find the corresponding length of the simple pendulum.*

In Captain Kater's experiments, mentioned Art. 91, the pendulum was supposed to turn about a mathematical line; if this supposition be not true, we shall find the consequence of the alteration.

Let c, c', be the distances of the centres of curvature of the two blunt axes from the centre of gravity; a, a' their radii. Then, since the pendulum oscillates in the same time about both, the length of the isochronous pendulum must be the same for both. Hence, by Prob. V,

$$\frac{(c'-a')^2+k^2}{c'}=\frac{(c-a)^2+k^2}{c};\ \therefore\ k^2=\frac{c'(c-a)^2-c(c'-a')^2}{c-c'};$$

$$\therefore\ l=\frac{(c-c')(c-a)^2+(c-c')k^2}{(c-c')c}=\frac{(c-a)^2-(c'-a')^2}{(c-c')}$$

$$=c+c'-\frac{2(ca-c'a')}{c-c'}+\frac{a^2-a'^2}{c-c'}.$$

If we suppose a and a' to be equal, we have

$$l=c+c'-2a,$$

the exact distance between the surfaces of the two axes.

Hence, the method of finding the length of the pendulum is equally correct, whether the edges be sharp or not.

If S, S', be the points from which the body would oscillate in the same time as about the surfaces P, P', we have $SS'=PP'$.

Sect. II. *When the Body moves in any manner whatever.*

132. Prop. The body being acted upon by any forces, the motion of the centre of gravity will be the same as if all those forces acted at the centre.

If x', y', z', be the co-ordinates of a particle m; X, Y, Z, the forces on it; we shall have, as in last Section,

effective forces, $\Sigma\dfrac{m\,d^2x'}{dt^2}$, $\Sigma\dfrac{m\,d^2y'}{dt^2}$, $\Sigma\dfrac{m\,d^2z'}{dt^2}$;

impressed forces, ΣmX, ΣmY, ΣmZ.

And as before, if $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of the centre of gravity, $x'=\bar{x}+x$, &c. we have

$$\Sigma\frac{m\,d^2x'}{dt^2}=\Sigma\frac{m\,d^2\bar{x}}{dt^2}=M.\frac{d^2\bar{x}}{dt^2},\ \text{\&c.};$$

$$\therefore\ \frac{d^2\bar{x}}{dt^2}=\frac{\Sigma mX}{M};\ \frac{d^2\bar{y}}{dt^2}=\frac{\Sigma mY}{M};\ \frac{d^2\bar{z}}{dt^2}=\frac{\Sigma mZ}{M};$$

whence the proposition is true.

133. Prop. The body being acted on by any forces, the motion of rotation will be affected as if the centre of gravity were fixed, and the same forces were applied.

This might be proved as in the corresponding proposition of last Section; but it appears also thus.

Let the centre of gravity have a velocity V, and let the effect produced on it by the forces, be the same as if a single force P acted there. Let there be communicated to each point of the system a velocity equal and opposite to V. Then, the centre of gravity will be at rest; and the forces which communicated these velocities will not affect the rotation about that centre, because their resultant will pass through that point*. Let there also be communicated to each point of the system a force equal and opposite to P. Then, the centre of gravity will have no tendency to move, and for the same reason as before, the rotation about that centre will not be affected.

Hence, if we suppose the centre of gravity to be fixed, and the same forces as before to act upon the body; the effect on the motion of rotation will not be altered.

Cor. 1. If the system be at rest when a force acts upon it, the instantaneous axis, about which it begins to revolve, must be perpendicular to the plane in which the line of force and the centre of gravity are. For otherwise the effective forces could not be equivalent to the force impressed, since the former would all be in planes parallel to each other, and oblique to the latter.

Cor. 2. In this case, the velocity communicated in the first instant is the same as if this instantaneous axis were fixed, and may be found by Art. 74.

Cor. 3. To find the circumstances of the motion of a body of any form, acted on by any forces, we must find the motion of the

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* If equal and parallel forces be communicated to each point of a system, their resultant will pass through the centre of gravity; for the force of gravity acts in this manner, and the centre of gravity is the point through which the resultant passes in that case.

centre of gravity by the formulæ of Book I, and determine the motion of rotation by equations (θ), Art. 124.

PROB. I. *A solid of revolution terminated by a point,* (PAB, *fig.* 130.) *moves so that the point* (P) *is always upon a given horizontal plane; to determine its motion.*

Let PO be the axis of revolution; G the centre of gravity; Gz_1 in the direction of PO, and Gx_1, Gy_1, perpendicular to it, moveable rectangular co-ordinates. And let Cx, Cy in the horizontal plane, and Cz perpendicular to it, be fixed rectangular co-ordinates.

The forces which act on the body, are gravity (g), downwards at G, and the reaction of the plane (R), upwards at the point P. Let a'', and b'', as in Art. 119, be the cosines of the angles which the vertical line PK, or Cz, makes with Gx_1 and Gy_1; then, the components of R at P, parallel respectively to Gx_1 and Gy_1, will be Ra'' and Rb''; and if $GP=l$, the moments of this force, with respect to the axes Gy_1 and Gx_1 will be $-Rla''$ and Rlb'' respectively. The moment, with respect to Gz_1, will manifestly be 0.

If c'' be the angle which PK or Cz makes with Pz_1, as in Art. 119, we have $GH=lc''$; and for the motion of the centre of gravity, M being the mass,

$$\frac{d^2.lc''}{dt^2}=\frac{R}{M}-g;$$

$$\text{or } R=Mg+Ml\frac{d^2c''}{dt^2}\ldots\ldots\ldots(1).$$

And by equations (θ) for the motion of rotation, observing that $B=A$,

$$Cdr=0\ldots\ldots\ldots\ldots\ldots(2),$$

$$Adq+(A-C)\,rpdt=-Rla''dt\ldots\ldots\ldots(3),$$

$$Adp-(A-C)\,qrdt=Rlb''dt\ldots\ldots\ldots\ldots(4).$$

From (2) we have $r=0$, a constant quantity.

Also, taking (3) b'' + (4) a'', we have

$$A\ \{a''dp + (b''r - c''q)\,p\,dt + b''dq + (c''p - a''r)\,q\,dt\}$$
$$+ C\,(a''q - b''p)\,r\,dt = 0.$$

Whence, reducing as in Note, p. 134, and integrating,

$$A\,(a''p + b''q) + Crc'' = k\ \ldots\ldots(5).$$

Again, (3) q + (4) p gives

$$A\,(p\,dp + q\,dq) = Rl\,(pb'' - qa'')\,dt.$$

Or, putting $-dc''$ for $(pb'' - qa'')\,dt$, and for R its value from (1),

$$A\,(pdp + qdq) + Ml^2\frac{dc''d^2c''}{dt^2} + Mlgdc'' = 0.$$

Multiply by 2, and integrate;

$$A\,(p^2 + q^2) + Ml\left(\frac{dc''}{dt}\right)^2 + 2Mlgc'' = h\ldots\ldots(6).$$

Now let θ be the angle GPK; then $c'' = \cos.\,\theta$, $dc'' = -\sin.\,\theta\,d\theta$. And by Art. 125, we find

$$a'' = -\sin.\,\theta\sin.\,\phi,\ \ b'' = -\sin.\,\theta\cos.\,\phi;$$

and hence, by equations (2),

$$pa'' + qb'' = -\sin.^2\theta\frac{d\psi}{dt};\ \ p^2 + q^2 = \sin.^2\theta\left(\frac{d\psi}{dt}\right)^2 + \left(\frac{d\theta}{dt}\right)^2.$$

Hence, equations (5) and (6) become

$$\left.\begin{aligned} -A\sin.^2\theta\frac{d\psi}{dt} + Cr\cos.\,\theta = k \\ (A + Ml^2\sin.^2\theta)\frac{d\theta^2}{dt^2} + A\sin.^2\theta\frac{d\psi^2}{dt^2} + 2Mgl\cos.\,\theta = h \end{aligned}\right\}\ldots(7).$$

Eliminating $\frac{d\psi}{dt}$, we find

$$(A^2\sin.^2\theta + AMl^2\sin.^4\theta)\frac{d\theta^2}{dt^2} = A\sin.^2\theta\,(h - 2Mgl\cos.\,\theta)$$
$$-(k - Cr\cos.\,\theta)^2\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

If we deduce from this the value of dt, we shall find $dt = F\theta \, . \, d\theta$; $F\theta$ being a function of θ; and by integrating this, we have t in terms of θ, and θ in terms of t. And hence, by (7), we have ψ in terms of t; and hence, by equations (ι) we have ϕ. These integrations cannot be performed in finite terms.

The quantities h and k are to be determined from the given initial circumstances of the motion, by equations (7).

PROB. II. *The body having at first no motion except a rotation about its axis* PO; *to determine its motion afterwards.*

Let the original velocity about $PO = \epsilon$, and the original inclination $= \theta_1$. And since at first $\frac{d\psi}{dt} = 0$, $\frac{d\theta}{dt} = 0$, we have

$$r = \frac{d\phi}{dt} = \epsilon \text{; and by equations (7),}$$

$$C\epsilon \cos. \theta_1 = k, \quad 2\,Mgl \cos. \theta_1 = h.$$

Hence, equation (8) becomes

$$(A^2 \sin.^2 \theta + AMl^2 \sin.^4 \theta) \frac{d\theta^2}{dt^2}$$

$$= 2\,AMgl \sin.^2 \theta (\cos. \theta_1 - \cos. \theta) - C^2\epsilon^2 (\cos. \theta_1 - \cos. \theta)^2.$$

The expression on the right hand side consists of two factors; $\cos. \theta_1 - \cos. \theta$, and (putting $1 - \cos.^2 \theta$ for $\sin.^2 \theta$),

$$2\,AMgl - C^2\epsilon^2 \cos. \theta_1 + C^2\epsilon^2 \cos. \theta - 2\,AMgl \cos.^2 \theta.$$

If we put $\frac{C^2}{4\,AMgl} = m$, and

$$\alpha = m\epsilon^2 - \sqrt{(1 - 2\,m\epsilon^2 \cos. \theta_1 + m^2\epsilon^4)},$$

$$\beta = m\epsilon^2 + \sqrt{(1 - 2\,m\epsilon^2 \cos. \theta_1 + m^2\epsilon^4)}.$$

the above expression becomes

$$2\,AMgl (\cos. \theta - \alpha) (\beta - \cos. \theta).$$

Hence, we shall have

$$\frac{d\theta^2}{dt^2} = \frac{2\,Mgl (\cos. \theta_1 - \cos. \theta) (\cos. \theta - \alpha) (\beta - \cos. \theta)}{\sin.^2 \theta (A + Ml^2 \sin.^2 \theta)}.$$

It is easily seen that β is greater than $\cos. \theta_1$, and that α is less than 1; let $\alpha = \cos.\ \theta_2$, and θ_2 and θ_1, will be the greatest and least values of θ. The axis will oscillate perpetually between these inclinations, and the rotation will continue for ever.

If PO be inclined beyond a certain limit, the sides of the body POB will touch the horizontal plane, and the rotation can no longer continue in the same manner. If we suppose that PO is susceptible of all positions above a horizontal one, the rotation will not be stopped if θ_2 be less than a right angle. That is, if

$$\cos.\ \theta_2 > 0, \text{ or if } m\epsilon^2 - \surd(1 - 2m\epsilon^2 \cos.\ \theta_1 + m^2\epsilon^4) > 0,$$

$$\text{if } 2m\epsilon^2 \cos.\ \theta_1 > 1\,; \text{ if } \epsilon^2 > \frac{1}{2m\cos.\ \theta_1}, \text{ or } \epsilon^2 > \frac{2AMgl}{C^2 \cos.\ \theta_1}.$$

If ϵ, the original velocity, be less than this, the body will fall.

If $\theta_1 = \theta_2$, the axis will always retain the same inclination. This supposes

$$\cos.\ \theta_1 = m\epsilon^2 - \surd(1 - 2m\epsilon^2 \cos.\ \theta_1 + m^2\epsilon^4),$$

and cannot be except ϵ, the velocity of rotation, be infinite.

Prob. III. *The body and the axis having velocities communicated to them; to determine the conditions that the axis* PO *may always retain the same inclination.*

In this case θ, and therefore c'', is constant; and by equation (1) of Prob. I, $R = Mg$. Also, by (7), $\frac{d\psi}{dt}$ will be constant; let it $= \delta$.

$$\text{Then, } p = \sin.\ \theta \sin.\ \phi \frac{d\psi}{dt} = -a''\delta\,;$$

$$\therefore\ a'' = -\frac{p}{\delta}\,; \text{ similarly, } b'' = -\frac{q}{\delta}\,;$$

and equations (3), (4) become

$$Adq + \left(Ar - Cr + \frac{Mgl}{\delta}\right) pdt = 0,$$

$$Adp - \left(Ar - Cr + \frac{Mgl}{\delta}\right) qdt = 0.$$

Hence, we find as in p. 316, making $\frac{(A-C)r\delta+Mgl}{A\delta}=\epsilon$,

$$p=a \sin. (\epsilon t+\gamma),\ q=a \cos. (\epsilon t+\gamma).$$

Substituting in the first and second of equations (ι), and dividing, we have

$$\tan. (\epsilon t+\gamma)=\tan. \phi;\ \epsilon t+\gamma=\phi,\ \frac{d\phi}{dt}=\epsilon.$$

And by the third of equations (ι), $r=\epsilon-\cos. \theta . \delta$.

$$\text{But } r=\frac{A\delta\epsilon-Mgl}{(A-C)\delta}.$$

$$\text{Hence, } C\delta\epsilon+(A-C)\cos. \theta . \delta^2=Mgl.$$

Hence, δ and θ being known, ϵ is known.

If $\delta=0$, we have ϵ infinite, agreeing with last Problem.

The results in the preceding Problems are applicable to the motion of a *spinning top*, considering it as upon a perfectly smooth plane, and supported on a mathematical point. It has appeared, that under these circumstances the top will, if a sufficient velocity be communicated to it, go on revolving for ever; but, in consequence of the absence of friction, the motion will be a good deal different from that which we observe actually to take place in such cases. The centre of gravity in our Problem either remains at rest, or moves up and down in a vertical line, and cannot have any curvilinear motion. The axis can never become more vertical than it was at the beginning of the motion, though it will at intervals return to the inclination which it then had. But in the experiment, the top, if inclined at first, will approach to a vertical position, which it will, as near as the senses can judge, attain and preserve for some time; and the centre of gravity will frequently describe a curve approaching to a circle, while the foot of the instrument remains stationary. These differences of theory and practice appear to be attributable to the effects of friction*.

* Euler thus explains the effect of friction in causing a top to raise itself into a vertical position. "The friction will perpetually retard the motion of the

134. A Problem of great consequence, depending on the principles of this Chapter, is that of the *Precession of the Equinoxes*, or motion of the nodes of the Earth's equator on the ecliptic, and similar motions in the heavenly bodies. This motion in the Earth arises from the attraction of the Sun on the Earth, which, in consequence of the spheroidal form of the latter, produces a force tending to turn its poles, one towards, and one from, the Sun. And this combined with the Earth's rotation, produces a motion like that described in p. 319. If, in fig. 124, xy be the fixed plane of the ecliptic, $x_1\,y_1$ the Earth's equator, and z_1 its north pole, N, the intersection of xy and x_1y_1, moves along xz in a direction opposite to the diurnal rotation. The investigation of this subject belongs to Physical Astronomy; but, in order to shew the nature of the action exercised, we shall take the inverse Problem, the motion being given to find the forces.

PROB. IV. *A figure of revolution, turning uniformly on its axis, retains the same inclination, while the nodes of its equator move uniformly on a fixed plane; to find the forces by which it is acted on.*

The notation remaining as in Art. 133, fig. 124, let

$$\frac{d\phi}{dt} = \epsilon, \text{ the velocity of rotation;}$$

$$\frac{d\psi}{dt} = \delta, \text{ the velocity of the node; } \theta \text{ constant.}$$

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the *point* P of the instrument, and at last reduce it to rest. If this happen before the top fall, it must then be spinning in such a position, that the point can remain stationary. But this cannot be if it be inclined. Hence, it must have a tendency to erect itself into a vertical position." *Theor. Mot. Corp. Solid et Rigid*, Suppl. Cap. 4.

This property has been applied to obtain an artificial horizon. Since the axis tends to become vertical, a plane perpendicular to the axis tends to become horizontal; and for a considerable length of time may be considered as accurately so. See *Phil Trans.* 1752.

Mr. Landen's solution of this Problem appears to contain several mistakes. See his *Math. Lucubr.*

Hence, by equations (ι),

$$p=\delta \sin.\phi \sin.\theta,\ q=\delta \cos.\phi \sin.\theta,\ r=\epsilon-\delta \cos.\theta,$$

$$\text{and } B=A.$$

Hence, since δ, ϵ are constant, we have, by equations (θ),

$$N=0,$$

$$N'=-A\delta\epsilon \sin.\phi \sin.\theta$$
$$+(A-C)\delta\epsilon \sin.\phi \sin.\theta-(A-C)\delta^2 \sin.\phi \sin.\theta \cos.\theta$$
$$=-\{C\epsilon+(A-C)\delta \cos.\theta\}\delta \sin.\theta \sin.\phi,$$

$$N''=A\delta\epsilon \cos.\phi \sin.\theta$$
$$+(C-A)\delta\epsilon \cos.\phi \sin.\theta-(C-A)\delta^2 \cos.\phi \sin.\theta \cos.\theta$$
$$=\{C\epsilon+(A-C)\delta \cos.\theta\}\delta \sin.\theta \cos.\phi.$$

Hence, $N'=-F\sin.\phi$, $N''=F\cos.\phi$; F being constant.

These forces N', N'' will be supplied, if we suppose a force F to act any where in the circle zz_1, fig. 124, urging z_1 towards z.

APPENDIX.

APPENDIX (A) to the INTRODUCTION, p. 4.

On the Definitions and Principles.

THE equations $v = \frac{ds}{dt}$ and $f = \frac{dv}{dt}$ may be considered as the mathematical *definitions* of velocity and force. They express that the velocity is the limit of the ratio of the increment of the space, to the time in which it is described; and the force, the limit of the ratio of the increment of the velocity, to the time in which it is described. And though these definitions are perhaps not the simplest descriptions of the vague and popular meanings of the words *velocity* and *force*; they may be shewn to agree with those significations as far as they go; and to be the limitations to which we are naturally led in making those notions exact and measurable. The quantities are greater or less according to our definitions, when they are so according to the common ideas; and the definitions are capable of being applied to any portions of time however small, which is requisite for the purpose of considering velocities and forces perpetually variable.

In fact, we may look upon space and time as the two variables, whose relations we have to investigate, and consider the general Problem of Dynamics to be this, "to find the place of a body at the end of a given time." The space being thus a function of the time t, it becomes convenient to give a name to the first differential coefficient $\frac{ds}{dt}$; and to the second, $\frac{d\left(\frac{ds}{dt}\right)}{dt}$; we call the first, *velocity*, and the second, *force*.

We shall here briefly state the proofs of the laws of motion.

Law 1. A body in motion, not acted on by any force, will move on in a straight line, and with a uniform velocity.

First, it will move in a straight line. For if it do not, it must move in some curve, and it must depend upon external circumstances, towards which side the convexity must lie, and how great the curvature must be. But, when a body's motion is influenced by external bodies, those bodies are said to exert force upon it, which is contrary to the supposition. Hence, a body influenced by no force, cannot describe any path but a straight line.

Next, it will move with a uniform velocity. As we remove the known causes which retard a body's motion, we find that we remove the retardation, and this without limit; so that it is evident, that if we could entirely remove the external causes of retardation, the body would not be retarded at all; there is no internal principle which tends to diminish the velocity.

The common causes by which motions are retarded, and finally stopped, are friction and the resistance of the air. If a wheel turn on a very smooth axle, it will revolve for a long time; and the longer, as we remove more of the friction by making the axle smoother; and if we also diminish the resistance of the air by making the wheel revolve in an exhausted receiver, the motion will continue still longer. We can never quite remove the friction or the resistance; and it is on that account, that the rotation cannot be made to continue for ever without diminution.

Law 2. When any force acts upon a body in motion, the change of motion which it produces is the same, in magnitude and direction, as the effect of the force upon a body at rest.

Both the original motion, and the change of motion communicated, are retained in their own directions. Thus, in fig. 7, if the body be in motion with a velocity which would carry it through *PR*, and be acted on by a force which would carry it through *Pp* in the same time, it will at the end of that time be found at the point *r*, *PRrp* being a parallelogram.

The proofs of this law are of the following nature.

A body let fall from the top of the mast of a vessel in motion, will fall down the mast, (if vertical;) thus retaining the horizontal motion of the ship, as well as the motion communicated by gravity.

A body thrown across the deck by a person on board, will in the same manner proceed in the direction in which it is thrown relatively to the vessel; thus both retaining the motion of the vessel, and obeying the force by which it is projected.

The motions impressed on bodies by the same agent, are the same, whatever be their direction with respect to the direction of the Earth's motion. Thus a pendulum oscillates in the same time east and west, or north and south.

The motions impressed on bodies in different parts of the Earth, are the same, relatively to the Earth, if the forces be the same; thus shewing, that besides the motions impressed, they retain the motions of the parts of the Earth where they are, which vary infinitely in velocity.

Law 3. When pressure communicates motion, the moving force is as the pressure.

This is proved from experiment. The pressure is the weight which produces motion, and the moving force is measured by the momentum generated in a given time. Thus, in fig. 107, when two bodies P, Q are suspended over a pully, if P be the heavier, $P-Q$ is the mass whose weight is the pressure producing motion; and if we neglect the pully, $P+Q$ is the mass moved; and this, multiplied into the velocity generated in a given time, is proportional to $P-Q$. This is found to agree with experiments. Atwood's machine, fig. 101, is the one with which the greatest number of experiments were made. In this, the mass of the wheels over which the string passes must be allowed for. See Art. 95, and Atwood on *Rect. and Rotatory Motion*, Sect. 7. Also Mr. Smeaton's Experiments, *Phil. Trans.* Vol. LXVI.

Action and Reaction signify the mutual pressures of two bodies which influence each other's motions. Action and Reaction are sometimes defined to be the momenta gained and lost; and in that case, in order to prove the equality of action and reaction, it is necessary to shew, that these momenta are as the pressures which produce them.

This third law of motion is also proved by shewing, that in the case of impact, the momenta gained and lost by the mutual collision are equal. Newton, *Scholium to the Laws of Motion, Principia*, Book I. Impact is in some respects the simplest case of pressure, because in it the consideration of time does not enter. But properly speaking, we cannot consider the proof of the third law of motion in this case, as sufficient to establish it in all others.

The *Inertia* or resistance of different bodies to motion is, at the same place, proportional to the weight, but does not vary, as the weight does, according to the different action of gravity, &c. The *Inertia of Rotation*, or resistance to the communication of rotatory motion, does not depend on the mass only, but also on its distance from the axis, as is seen in Book III.

The third Book supposes the general property of the lever to be established; namely, that the effects of forces to turn a body round an axis, will be the same, when the sum of their *moments*, or products by their perpendiculars from the axis, is the same. See *Statics*, Art. 26. It supposes also the composition of forces to be demonstrated; which is, however, included in the general property of the lever.

APPENDIX (B) to CHAP. III. BOOK I.

On the Motion of a Body about two Centres of Force.

THIS Problem is remarkable both for the elegance of the results to which our investigations lead us, and for being almost the only addition which has been made since Newton's time to the *exact* solution of inverse problems of central forces. This acquisition we owe in the first place to Euler, who in the Transactions of the Academy of Petersburg, (*Nov. Com. Petrop.* 1764,) published about 1766, examined the question of the motion of a body acted on by two centres of force, when it is supposed to move always in one plane. His analysis of this case was complete, but he informs us, that in his first attempts he was led into a mistake by the method which he employed. This he detected, by the absurdities which resulted from supposing one of the forces to vanish, which of course reduced it to a known problem; and in investigating the origin of the inconsistencies in his solution, he was led to the complete solution. So that he attributes his success to this "most fortunate error," as he calls it. Nearly at the same time he published in the Berlin Memoirs, (*Hist. de l'Acad. Royale des Sciences*, Berlin 1760, published 1767;) an examination of the cases in which the curve described is an algebraical curve.

At the end of his first paper, Euler had promised a solution of the problem, when the motion is not in the same plane. This he performs in the *Nov. Com. Petrop.* for 1765, (published 1767,): where he gives a new method of obtaining the differential equations: and it is this which, with some modifications, is adopted in the next.

Before the publication of this last paper of Euler, Lagrange had taken up the problem, and had written a Memoir, in which, with very great elegance and simplicity, he solves it for the motion in a curve of double curvature. This Memoir appears in the *Melanges de la Soc. de Turin* for 1766—1769, (published some years afterwards). In the same Volume there is also another Memoir of Lagrange, containing a discussion of the laws of force for which the integration is possible; and an observation founded upon this, that it cannot be applied to the motion of the Moon, which the Student will find explained at the end of the problem as here given. Legendre has more recently examined the cases of this Problem, in his *Exercices du Calc. Integ.* (tom. II, p. 366.).

The body will not necessarily perform its motion always in the same plane, but will in general describe a curve of double curvature. It is manifest, however, that if we suppose the direction to be originally in the plane passing through the two centres, there will be nothing to deflect the body from this plane, and hence it will continue in it during its whole motion. We shall first, therefore, consider this, the simple problem.

Prob. I. *A body, acted upon by forces tending to two fixed centres, and varying inversely as the square of the distances from their respective centres; moving so as to be always in the same plane; to determine its motion, fig.* 131.

Let the motion be in the plane, passing through A, B, the centres of force. Let P be any point, $AP = r$, $BP = r'$: the forces which A and B exert upon P, $\frac{m}{r^2}$, $\frac{m'}{r'^2}$, respectively. $AB = 2c$; PM perpendicular to AB; and $AM = x$, $BM = x'$; $MP = y$; hence, $x + x' = 2c$. Also,

$$r^2 = x^2 + y^2;\ r'^2 = x'^2 + y^2.$$

Now, if we resolve the force $\frac{m}{r^2}$, in direction PA, into two forces parallel respectively to y, and to x, as in Art. 12, we shall have the resolved parts $\frac{m}{r^2} \cdot \frac{x}{r}$, $\frac{m}{r^2} \cdot \frac{y}{r}$: similarly, the resolved parts of B's force will be $\frac{m'}{r'^2} \cdot \frac{x'}{r'}$, $\frac{m'}{r'^2} \cdot \frac{y}{r'}$. Hence, observing the directions of the forces, we have

$$\text{force in } x = -\frac{mx}{r^3} + \frac{m'x'}{r'^3};$$

$$\text{force in } y = -\frac{my}{r^3} - \frac{m'y}{r'^3}.$$

And hence, the equations of motion (c), Art. 12, are

$$\frac{d^2x}{dt^2} = -\left\{\frac{mx}{r^3} - \frac{m'x'}{r'^3}\right\} \dots\dots\dots\dots (1),$$

$$\frac{d^2y}{dt^2} = -\left\{\frac{my}{r^3} + \frac{m'y}{r'^3}\right\} \dots\dots\dots\dots (2).$$

We shall indicate these equations by the figures which stand opposite to them, and operations performed upon these equations by the usual algebraical symbols; a mode of notation which will be easily understood.

Thus, $(1) \times dx + (2) \times dy$, gives

$$\frac{dx\,d^2x + dy\,d^2y}{dt^2} = -\left\{\frac{mr\,dr}{r^3} + \frac{m'r'\,dr'}{r'^3}\right\};$$

the right hand side being reduced by observing, that

$$x\,dx + y\,dy = r\,dr;\quad -x'\,dx + y\,dy = x'\,dx' + y\,dy = r'\,dr'.$$

Now this equation integrated and multiplied by 2, gives

$$\frac{dx^2+dy^2}{dt^2}=2\left\{\frac{m}{r}+\frac{m'}{r'}+\frac{C}{c}\right\}\dots\dots\dots\dots(3).$$

Here C is a constant quantity which depends upon the velocity at a given point; and the right hand side of the equation gives the velocity at any point.

We shall now make another integrable combination of equations (1) and (2); observing that $d^2x=-d^2x'$,

$(2)\times x'+(1)\times y$ gives

$$\frac{x'd^2y-yd^2x'}{dt^2}=-\frac{2mcy}{r^3}, \text{ since } x+x'=2c;$$

$(2)\times x-(1)\times y$ gives

$$\frac{xd^2y-yd^2x}{dt^2}=-\frac{2m'cy}{r'^3}.$$

Multiplying by $xdy-ydx$, and by $x'dy-ydx'$, respectively, we have

$$\frac{\{x'd^2y-yd^2x'\}\{xdy-ydx\}}{dt^2}=-\frac{2mcy(xdy-ydx)}{r^3},$$

$$\frac{\{xd^2y-yd^2x\}\{x'dy-ydx'\}}{dt^2}=-\frac{2m'cy(x'dy-ydx')}{r'^3}.$$

Add these equations together, and the sum becomes integrable; for the numerator of the first side will be the differential of

$\{x'dy-ydx'\}\{xdy-ydx\}$. Also

$$d.\frac{x}{r}=\frac{rdx-xdr}{r^2}=\frac{r^2dx-xrdr}{r^3}=\frac{(x^2+y^2)dx-x(xdx+ydy)}{r^3}$$

$$=\frac{y^2dx-xydy}{r^3};$$ whence one of the terms on the second side.

Similarly, $$d.\frac{x'}{r'}=\frac{y^2dx'-x'ydy}{r'^3}.$$

Hence, the integral of the sum just mentioned is

$$\frac{\{x'dy - ydx'\}\,\{xdy - ydx\}}{dt^2} = 2mc \cdot \frac{x}{r} + 2m'c\frac{x'}{r'} + 2\,Dc \ldots (4),$$

where D is a constant quantity, which depends upon the direction and velocity of the motion at a given point. If we eliminate t, we have the equation to the curve.

We shall transform equations (3), and (4), by the following assumptions.

Let $r = c\,(u + v)$, $r' = c\,(u - v)$.

Now, by the triangle APM,

$$x = \frac{4c^2 + r^2 - r'^2}{4c} = \frac{4c^2 + 4c^2uv}{4c};$$

$$x = c\,(1 + uv); \text{ hence also, } x' = c\,(1 - uv);$$

and $y^2 = r^2 - x^2 = c^2\,(u^2 + v^2 - 1 - u^2v^2) = c^2\,(u^2 - 1)(1 - v^2)$;

therefore, $dx = c\,(udv + vdu)$, $dx' = -c\,(udv + vdu)$;

$$y = c\,\sqrt{(u^2 - 1)}\,\sqrt{(1 - v^2)},\ dy = cudu\sqrt{\frac{1 - v^2}{u^2 - 1}} - cvdv\sqrt{\frac{u^2 - 1}{1 - v^2}}.$$

Hence, we obtain

$$dx^2 = c^2\,\{v^2du^2 + 2\,uvdudv + u^2dv^2\},$$

$$dy^2 = c^2\left\{u^2du^2 \cdot \frac{1 - v^2}{u^2 - 1} - 2\,uvdudv + v^2dv^2 \cdot \frac{u^2 - 1}{1 - v^2}\right\};$$

$$\therefore\ dx^2 + dy^2 = c^2\left\{du^2\,\frac{u^2 - v^2}{u^2 - 1} + dv^2\,\frac{u^2 - v^2}{1 - v^2}\right\}.$$

Also, $xdy - ydx$

$$= c^2\left\{udu\,(1 + uv)\sqrt{\frac{1 - v^2}{u^2 - 1}} - vdv\,(1 + uv)\sqrt{\frac{u^2 - 1}{1 - v^2}}\right\}$$

$$- c^2\,\{vdu\,\sqrt{(u^2 - 1)}\,\sqrt{(1 - v^2)} + udv\,\sqrt{(u^2 - 1)}\,\sqrt{(1 - v^2)}\}$$

$$= c^2\left\{du\,(u + v)\sqrt{\frac{1 - v^2}{u^2 - 1}} - dv\,(u + v)\sqrt{\frac{u^2 - 1}{1 - v^2}}\right\}.$$

Similarly, $x'dy - y\,dx'$

$$= c^2 \left\{du\,(u-v)\sqrt{\frac{1-v^2}{u^2-1}} + dv\,(u-v)\sqrt{\frac{u^2-1}{1-v^2}}\right\};$$

hence, $(x\,dy - y\,dx)\,(x'dy - y\,dx')$

$$= c^4\,(u^2-v^2)\left\{du^2\,\frac{1-v^2}{u^2-1} - dv^2\,\frac{u^2-1}{1-v^2}\right\}.$$

Substituting these values in equations (3) and (4) they become, (dividing by c^2 and c^4),

$$(u^2-v^2)\left\{\frac{du^2}{dt^2}\cdot\frac{1}{u^2-1} + \frac{dv^2}{dt^2}\cdot\frac{1}{1-v^2}\right\}$$

$$= \frac{2}{c^3}\left\{\frac{m}{u+v} + \frac{m'}{u-v} + C\right\}\ldots\ldots\ldots\ldots(5),$$

$$(u^2-v^2)\left\{\frac{du^2}{dt^2}\cdot\frac{1-v^2}{u^2-1} - \frac{dv^2}{dt^2}\;\frac{u^2-1}{1-v^2}\right\}$$

$$= \frac{2}{c^3}\left\{m\,.\frac{1+uv}{u+v} + m'\,.\frac{1-uv}{u-v} + D\right\}..(6).$$

Now to eliminate dv, we take $(5)\times(u^2-1)+(6)$; which gives

$$\frac{(u^2-v^2)^2}{u^2-1}\;\frac{du^2}{dt^2} = \frac{2}{c^3}\,\{(m+m')\,u + C\,(u^2-1) + D\}\ldots\ldots(7).$$

Similarly, to eliminate du, $(5)\times(1-v^2)-(6)$ gives

$$\frac{(u^2-v^2)^2}{u-v^2}\cdot\frac{dv^2}{dt^2} = \frac{2}{c^3}\,\{-(m-m')\,v + C\,(1-v^2) - D\}..(8).$$

If we now divide (7) by (8), we obtain

$$\frac{1-v^2}{u^2-1}\cdot\frac{du^2}{dv^2} = \frac{(m+m')\,u + C\,(u^2-1) + D}{(m-m')\,v + C\,(1-v^2) - D}.$$

If we make $D - C = E$, we have

$$\frac{du}{dv} = \sqrt{\frac{(u^2-1)\,\{E+(m+m')\,u + Cu^2\}}{(v^2-1)\,\{E+(m-m')\,v + Cv^2\}}}\ldots\ldots\ldots(9),$$

which gives us immediately an equation in which the variables are separated; viz.

$$\frac{du}{\sqrt{\{(u^2-1)[E+(m+m')u+Cu^2]\}}}$$
$$=\frac{dv}{\sqrt{\{(v^2-1)[E+(m-m')v+Cv^2]\}}} \cdots\cdot (10).$$

If we could integrate equation (10), we should have the equation to the curve described. The variable quantities are separated in this equation, but the integrals are what are called *Elliptic Transcendants*, and cannot be obtained in finite terms. There are, however, an infinite number of cases, in which (10) belongs to a remarkable class of equations, to which the researches of Euler and others have shewn how to find algebraical integrals; namely, when the quantities C, E, m, m' have such relations, that, making $u=a\phi$, $v=b\psi$, it can be reduced to the form

$$\frac{\mu d\phi}{\sqrt{(\alpha+\beta\phi+\gamma\phi^2+\delta\phi^3+\epsilon\phi^4)}}$$
$$=\frac{\nu d\psi}{\sqrt{(\alpha+\beta\psi+\gamma\psi^2+\delta\psi^3+\epsilon\psi^4)}},$$

μ, ν, being any whole numbers whatever. See Lacroix, *Traité du Calc. Diff. et du Calc. Integ.* tom. II, No. 702.

But the expressions thus obtained are of so complicated a form, that we shall not attempt to examine the solutions which they offer.

There are two remarkable cases, in which we can more simply find the curve described. If we have $u = \alpha$, a constant quantity, such that

$$(\alpha^2-1)\{E+(m+m')\alpha+C\alpha^2\}=0,$$

it is manifest, that both sides of equation (9) vanish, and hence, $u=\alpha$, satisfies the equation. Similarly, if we invert the equation, and suppose

$$(\beta^2-1)\{E+(m-m')\beta+C\beta^2\}=0,$$

it is clear, that $v=\beta$ will satisfy it.

But though $u=\alpha$, or $v=\beta$ satisfy equation (9), it does not necessarily follow, that they give the curve described in a particular case. They will not do this, except they be integrals derived from the *complete integral* by giving particular values to the constants. If, instead of being such *particular integrals*, they be *particular solutions*, (*Lacroix*, Art. 293.) of the differential equation, they no longer solve the problem.

Now we have this rule, by which we can determine whether an equation which satisfies a given differential equation in u, v, is a particular solution or a particular integral. (*Lacroix*, Art. 297, 298). *Put* $\frac{du}{dv} = p$, *and take the value of* $\frac{dp}{du}$; *every particular solution will make* $\frac{dp}{du}$ *infinite.*

In this case, if we call $\sqrt{U}$ and $\sqrt{V}$ the numerator and denominator of the fraction in (10), we have

$$\frac{du}{dv} = p = \frac{\sqrt{U}}{\sqrt{V}};$$

and if U', V' represent the differential coefficients of U and V;

$$\therefore \frac{dp}{du} = \frac{U'}{2\sqrt{U}.\sqrt{V}} - \frac{V'\sqrt{U}}{2V\sqrt{V}}.\frac{dv}{du}$$

$$= \frac{U'}{2\sqrt{U}\sqrt{V}} - \frac{V'}{2V}, \text{ putting for } \frac{dv}{du} \text{ its value;}$$

$$= \frac{U'\sqrt{V} - V'\sqrt{U}}{2V\sqrt{U}},$$

which will be infinite, when $U=0$, except $U'=0$ at the same time,

and when $V=0$, except $V'=0$ at the same time.

Hence, if $U=0$, and $U'=0$, we have not a particular solution, but a particular integral, and consequently a solution of the problem. Similarly, if $V=0$, and $V'=0$.

Let us consider the equations $u=a$, $U=0$, $U'=0$, or

$$u = a,$$

$$\left.\begin{array}{r}(u^2-1)\{E+(m+m')u+Cu^2\}=0\\ 2u\{E+(m+m')u+Cu^2\}+(u^2-1)\{(m+m')+2Cu\}=0\end{array}\right\}(11).$$

Now the first equation $u=a$, whence $r+r'=2cu=2ca$, shews that the curve is an ellipse: the other equations determine the values of C and E.

The second equation gives either $E+(m+m')\,u+Cu^2=0$, or $u^2-1=0$; but it is clear, that the latter cannot be an answer to the problem, since it gives $y=0$ for every point. Hence, the problem requires that

$$E+(m+m')\,a+Ca^2=0,$$
$$(m+m')+2Ca=0.$$

From the latter,

$$C=-\frac{m+m'}{2a};$$

$$\therefore E=-\frac{(m+m')}{2}a, \text{ by the former.}$$

$$\text{And therefore } D=E+C=-\frac{m+m'}{2}\,\frac{a^2+1}{a}.$$

If we call $2a$ the major axis of the ellipse, we have

$$2a=r+r'=2ca;\ \therefore a=\frac{a}{c}=\frac{1}{e}, \text{ } e \text{ being the eccentricity.}$$

$$\text{Hence, } C=-\frac{m+m'}{2}\cdot\frac{c}{a};$$

and substituting in equation (3), we have

$$\text{velocity}=2\left\{\frac{m}{r}+\frac{m'}{r}-\frac{m+m'}{2a}\right\}\ldots\ldots\ldots(12).$$

at the extremity D of the axis minor, $r=r'=a$; and here

$$\text{velocity}^2=\frac{m+m'}{a}.$$

By Prob. II. of Chap. III. it appears, that this is the velocity which the body would have in the ellipse at D, if there were at A or at B a single force equal to the sum of the forces at A and at B. Hence, conversely, if the body at the point D, equidistant from A and B, be projected parallel to AB with this velocity, it will describe an ellipse about the centres of force A, B, as foci.

If the body were projected in any other manner, if we know the velocity, we find a by equation (12); and hence, successively, e, α, C, D, and E; and knowing C, there we can find the direction of projection by equations (6), (7), or (8).

Similarly, if we make $v=\beta$, β being subject to equations analogous to α in (11), we shall have

$$r - r' = 2C\beta = 2a \text{ suppose};$$

therefore the curve is a hyperbola, B being its interior focus.

The eccentricity $e = \frac{C}{a} = \frac{1}{\beta}$.

$$\text{Also, } E + (m - m')\,\beta + C\beta^2 = 0,$$

$$(m - m') + 2C\beta = 0;$$

$$\therefore C = -\frac{m - m'}{2\beta} = -\frac{(m - m')}{2}\cdot\frac{c}{a};\ E = -\frac{(m - m')\,a}{2},$$

$$\text{velocity}^2 = 2\left\{\frac{m}{r} + \frac{m'}{r'} - \frac{m - m'}{2a}\right\}.$$

If we had originally put

$$x = c\,(u - v)\ r' = c\,(u + v),$$

the expressions would have been every where the same, excepting that we should have $-v$, for v at each step. Hence, we should have in this case

$$v = \beta,$$

$$E - (m - m')\,\beta + C\beta^2 = 0,$$

$$-(m - m') + 2C\beta = 0;$$

$$\text{therefore } r' - r = u\beta = 2a,$$

and the curve is a hyperbola, of which A is the interior focus.

$$C = \frac{m - m'}{2\beta} = \frac{m - m'}{2}\ \frac{c}{a},$$

$$E = \frac{(m - m')\,\beta}{2};\ D = E + C = \frac{m - m'}{2}\cdot\frac{\beta^2 + 1}{\beta},$$

$$\text{velocity}^2 = 2\left\{\frac{m}{r} + \frac{m'}{r'} + \frac{m - m'}{2a}\right\}.$$

In order to complete the determination of the motion, we ought to have the time. Now this is easily obtained from equations (7) and (8). Retaining the notation of p. 353, we have by the equations just mentioned,

$$\frac{(u^2-v^2)\,dv}{\sqrt{V}}=\frac{\sqrt{2}dt}{c^{\frac{3}{2}}},$$

$$\frac{u^2\,dv}{\sqrt{V}}-\frac{v^2\,dv}{\sqrt{V}}=\frac{u^2\,du}{\sqrt{U}}-\frac{v^2\,dv}{\sqrt{V}}=\frac{\sqrt{2}dt}{c^{\frac{3}{2}}},$$

which reduces it to the integration of expressions of one variable.

In the particular case which we have considered, (when $u=a$) equation (8) gives

$$\frac{dt\,\sqrt{2}}{c^{\frac{3}{2}}}=\frac{(a^2-v^2)\,dv}{\sqrt{\{(v^2-1)[E+(m-m')\,v+Cv^2]\}}}$$

$$=\frac{(a^2-v^2)\,dv\,\sqrt{(2a)}}{\sqrt{\{(1-v^2)\,[(m+m')\,a^2+2\,(m-m')\,av+(m+m')\,v^2]\}}}.$$

Prob. II. *When the body does not move in the same plane; to determine the motion, fig.* 123.

Let the path be referred to three rectangular co-ordinates x, y, z; the plane of xy passing through AB, so that $AM=x$, $MN=y$, $NP=z$; the plane PMN will be perpendicular to AM. Let also $AB=2c$, $BM=x'$, $AP=r$, $BP=r'$. Hence,

$$x+x'=2c,\ r^2=x^2+y^2+z^2,\ r'^2=x'^2+y^2+z^2,$$

and we shall then have the forces $\frac{m}{r^2}$ and $\frac{m'}{r'^2}$ in the directions PA and PB; and the resolved parts of the first parallel to MA, to NM, and to PN respectively, will be $\frac{m}{r^2}\cdot\frac{x}{r}$, $\frac{m}{r^2}\cdot\frac{y}{r}$, $\frac{m}{r^2}\cdot\frac{z}{r}$; similarly of the other force; and hence, the equations of motion become

$$\frac{d^2x}{dt^2}=-\left\{\frac{mx}{r^3}-\frac{m'x'}{r'^3}\right\}\ldots\ldots\ldots(1),$$

$$\frac{d^2y}{dt^2} = -\left\{\frac{my}{r^3} + \frac{m'y'}{r'^3}\right\} \ldots\ldots\ldots (2),$$

$$\frac{d^2z}{dt^2} = -\left\{\frac{mz}{r^3} + \frac{m'z'}{r'^3}\right\} \ldots\ldots\ldots (3).$$

Then (1) $2dx + (2) . 2dy + (3)\, 2dz$ gives, (observing that

$$rdr = xdx + ydy + zdz,$$

$$r'dr' = x'dx' + ydy + zdz = -x'dx + ydy + zdz),$$

$$\frac{2dxd^2x + 2dyd^2x + 2dzd^2z}{dt^2} = -2\left\{\frac{mdr}{r^2} + \frac{m'dr'}{r'^2}\right\};$$

and integrating,

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = 2\left\{\frac{m}{r} + \frac{m'}{r'} + \frac{C}{c}\right\};\ C \text{ a constant quantity. . (4).}$$

Also, $(3)y - (2)z$ gives

$$\frac{yd^2z - zd^2y}{dt^2} = 0;$$

$$\therefore \frac{ydz - zdy}{dt} = h, \text{ a constant quantity..(5).}$$

It is necessary to get another integrable combination of our equations; which we may thus perform; (2) $x - (1)\, y$ gives

$$\frac{d\,(xdy - ydx)}{dt^2} = -\frac{m'y\,(x + x')}{r'^3} = -\frac{2m'cy}{r'^3},$$

(2) $x' + (1)\, y$ gives, since $-d^2x' = d^2x$,

$$\frac{d\,(x'dy - ydx')}{dt^2} = -\frac{my\,(x + x')}{r^3} = -\frac{2mcy}{r^3}.$$

Multiplying these by $(x'dy - y'dx)$, and $(xdy - ydx)$ respectively, and adding, we have

$$\frac{d.(xdy - ydx)(x'dy - ydx')}{dt^2}$$

$$= -\frac{2m'cy\,(x'dy - ydx')}{r'^3} - \frac{2mcy(xdy - ydx)}{r^3}.$$

Similarly, by using (3) for (2), and z for y, we shall get

$$\frac{d\cdot(xdz-zdx)(x'dz-zdx')}{dt^2}$$

$$=-\frac{2m'cz(x'dz-zdx')}{r'^3}-\frac{2mcz(xdz-zdx)}{r^3}.$$

Now add these, observing that

$$d\frac{x}{r}=\frac{rdx-xdr}{r^2}=\frac{(x^2+y^2+z^2)dx-x(xdx+ydy+zdz)}{r^3}$$

$$=-\frac{y(xdy-ydx)-z(xdz-zdx)}{r^3},$$

and

$$d\frac{x'}{r'}=-\frac{y(x'dy-ydx')-z(x'dz-zdx')}{r'^3};$$

and we have

$$\frac{d.(xdy-ydx)(x'dy-ydx')+d.(xdz-zdx)(x'dz-zdx')}{dt^2}$$

$$=2cmd\frac{x}{r}+2cm'd\frac{x'}{r'}.$$

Hence, integrating,

$$\frac{(xdy-ydx)(x'dy-ydx')+(xdz-zdx)(x'dz-zdx')}{dt^2}$$

$$=2cm\frac{x}{r}+2cm'\frac{x'}{r'}+2cD\ldots\ldots(6).$$

Equations (4), (5), and (6), contain the complete determination of the motion. We shall transform them by the following substitutions.

Let $MP=\rho$, and the angle $NMP=\phi$: hence, as in Arts. 16, 17, we have

$$y^2+z^2=\rho^2;\ dy^2+dz^2=d\rho^2+\rho^2d\phi^2,\ ydz-zdy=\rho^2d\phi.$$

So that the equations (4) and (5) become, removing the denominators,

$$dx^2+d\rho^2+\rho^2d\phi^2=2dt^2\left\{\frac{m}{r}+\frac{m'}{r'}+\frac{C}{c}\right\}\ldots\ldots(7),$$

$$\rho^2 d\phi = h dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

To transform equation (6), we may observe that

$$(x dy - y dx)(x' dy - y dx')$$
$$= xx' dy^2 - x dx' . y dy - x' dx . y dy + y^2 dx dx',$$
$$(x dz - z dx)(x' dz - z dx')$$
$$= xx' dz^2 - x dx' . z dz - x' dx . z dz + z^2 dx dx'.$$

And the sum

$$= xx' (d\rho^2 + \rho^2 d\phi^2) - (x dx' + x' dx) \rho d\rho + \rho^2 dx dx'$$
$$= (x d\rho - \rho dx)(x' d\rho - \rho dx') + xx' \rho^2 d\phi^2.$$

Substituting in equation (6), and multiplying by dt^2,

$$(x d\rho - \rho dx)(x' d\rho - \rho dx') + xx' \rho^2 d\phi^2$$
$$= 2 c dt^2 \left\{ \frac{m x}{r} + \frac{m' x'}{r'} + D \right\} \ldots\ldots\ldots(9).$$

We shall now make the same substitution as at p. 350, in the former case, viz.,

$$r = c(u + v), \quad r' = c(u - v).$$

And we shall have the same values of x, x', &c., as we then had, ρ corresponding to y. Hence,

$$dx^2 + d\rho^2 = c^2 (u^2 - v^2) \left\{ \frac{du^2}{u^2 - 1} + \frac{dv^2}{1 - v^2} \right\},$$

$$(x d\rho - \rho dx)(x' d\rho - \rho dx') = c^4 (u^2 - v^2) \left\{ du^2 \frac{1 - v^2}{u^2 - 1} - dv^2 \frac{u^2 - 1}{1 - v^2} \right\}.$$

Substituting these in (7), and (9), observing also that

$$xx' = c^2 (1 - u^2 v^2);$$

and, dividing the latter by c^2, we shall have results analogous to (5) and (6), of the former case, viz.,

$$c^2 (u^2 - v^2) \left\{ \frac{du^2}{u^2 - 1} + \frac{dv^2}{1 - v^2} \right\} + \rho^2 d\phi^2$$
$$= \frac{2 dt^2}{c} \left\{ \frac{m}{u + v} + \frac{m'}{u - v} + C \right\} \ldots(10),$$

$$c^2(u^2-v^2)\left\{du^2.\frac{1-v^2}{u^2-1}-dv^2.\frac{u^2-1}{1-v^2}\right\}+(1-u^2v^2)\rho^2d\phi^2$$

$$=\frac{2dt^2}{c}\left\{\frac{m(1+uv)}{u+v}+\frac{m'(1-uv)}{u-v}+D\right\}\ldots\ldots(11).$$

Now, (10) $(u^2-1)+(11)$ gives

$$c^2(u^2-v^2).du^2.\frac{u^2-v^2}{u^2-1}+u^2(1-v^2)\rho^2d\phi^2$$

$$=\frac{2dt^2}{c}\{(m+m')u+C(u^2-1)+D\}.$$

Put for dt^2 its value from (8); multiply by (u^2-1), observing that $(u^2-1)(1-v^2)=\frac{\rho^2}{c^2}$; and transpose; and we have

$$c^2(u^2-v^2)^2du^2=$$

$$\frac{2\rho^4d\phi^2}{ch^2}(u^2-1)\{(m+m')u+C(u^2-1)+D\}-\frac{u^2\rho^4d\phi^2}{c^2}\ldots(12).$$

Similarly, (10) $(1-v^2)-(11)$, gives

$$c^2(u^2-v^2)dv^2\frac{u^2-v^2}{1-v^2}+v^2(u^2-1)\rho^2d\phi^2$$

$$=\frac{2dt^2}{c}\{-(m-m')v+C(1-v^2)-D\}.$$

And, reducing this in the same way as the other,

$$c^2(u^2-v^2)^2dv^2=$$

$$\frac{2\rho^4d\phi^2}{ch^2}(1-v^2)\{-(m-m')v+C(1-v^2)-D\}-\frac{v^2\rho^4d\phi^2}{c^2}\ldots(13).$$

If we now divide (12) by (13), we have

$$\frac{du^2}{dv^2}=\frac{\frac{2c}{h^2}(u^2-1)\{(m+m')u+C(u^2-1)+D\}-u^2}{\frac{2c}{h^2}(1-v^2)\{-(m-m')v+C(1-v^2)-D\}-v^2}.$$

If we call $\frac{c}{h^2} = n$, and $D - C = E$, we have

$$\frac{du^2}{dv^2} = \frac{2n(u^2 - 1)\{E + (m + m')u + Cu^2\} - u^2}{2n(v^2 - 1)\{E + (m - m')v + Cv^2\} - v^2} \ldots (14)$$

$$= \frac{U}{V} \text{ suppose;}$$

and $\frac{du}{\sqrt{U}} = \frac{dv}{\sqrt{V}}$, by integrating which equation, we have the curve described: the variables separated, but the expressions transcendant, as before.

Knowing the relation between u and v, we have, from equation (12) or (13), the value of $d\phi$;

$$\text{for } c^2(u^2 - v^2)^2 du^2 = \frac{\rho^4}{c^2} U d\phi^2$$

$$= c^2 . (u^2 - 1)^2 (1 - v^2)^2 U d\phi^2;$$

$$\therefore d\phi = \frac{(u^2 - v^2) du}{(u^2 - 1)(1 - v^2)\sqrt{U}}$$

$$= \frac{du}{(u^2 - 1)\sqrt{U}} + \frac{du}{(1 - v^2)\sqrt{U}}$$

$$= \frac{du}{(u^2 - 1)\sqrt{U}} + \frac{dv}{(1 - v^2)\sqrt{V}};$$

and for the time we have, since $h = \sqrt{\frac{c}{n}}$,

$$dt = \frac{\rho^2 d\phi}{h} = c^{\frac{3}{2}} \sqrt{n} \left\{\frac{(1 - v^2) dv}{\sqrt{V}} + \frac{(u^2 - 1) du}{\sqrt{U}}\right\}$$

$$= c^{\frac{3}{2}} \sqrt{n} \left\{\frac{u^2 du}{\sqrt{U}} - \frac{v^2 dv}{\sqrt{V}}\right\}.$$

It is manifest, as in the former case, that the equation (14) is satisfied by supposing $u = a$, where a is such a value of u, as to make the numerator on the right hand side vanish. But it may be shewn as before, that a solution to the problem is not given by $U = 0$, except also $U' = 0$; that is, we have the path in a particular case, if

$$\left.\begin{array}{c} u = a \\ 2n(a^2-1)\{E+(m+m')a+Ca^2\} - a^2 = 0 \\ 2na\{E+(m+m')a+Ca^2\} \\ +n(a^2-1)\{(m+m')+2Ca\} - a = 0 \end{array}\right\} \ldots\ldots (15).$$

Hence, $r+r' = 2ca = 2a$ suppose, and P is always situated in an ellipse which has for its foci A, B, and which revolves about its major axis AB with an angular velocity $\frac{d\phi}{dt}$.

$$\text{eccentricity} = e = \frac{c}{a} = \frac{1}{a},$$

$$a^2 - 1 = \frac{a^2 - c^2}{c^2} = \frac{b^2}{c^2}.$$

To find C, multiply the second of equations (15) by a, and the third by $a^2 - 1$, and subtract; we have thus,

$$n(a^2-1)^2\{m+m'+2Ca\}+a=0,$$

$$C = -\frac{m+m'}{2a} - \frac{1}{2n(a^2-1)^2}$$

$$= -\frac{m+m'}{2}\cdot\frac{c}{a} - \frac{c^4}{2nb^4};$$

and substituting this, we find

$$E = -\frac{(m+m')a}{2} + \frac{a^4}{2n(a^2-1)^2};$$

and $D = C + E = -\frac{m+m'}{2}\cdot\frac{a^2+1}{a} + \frac{1}{2n}\cdot\frac{a^2+1}{a^2-1}$.

Hence, by (4),

$$\text{velocity}^2 = 2\left\{\frac{m}{r} + \frac{m'}{r'} - \frac{m+m'}{2a} - \frac{1}{2n}\cdot\frac{c^3}{b^4}\right\}.$$

And we have $d\phi$ and dt, by equations (13) and (7).

If we make $v = \beta$, we may find a solution in the same manner, which will give for the place of P a hyperboloid, of which A, B, are the foci.

If we had supposed the force to vary according to any other law, the process of determining the motion would, to a certain extent, have been the same. There are, however, only a few cases, besides that which we have considered, in which we can integrate and eliminate, and in these, the methods are of considerable complexity. The conditions of integrability are very completely discussed by Lagrange in the *Turin Memoirs* for 1766—1769: p. 216, &c., and it there appears, that if P and Q represent the forces acting towards A and B respectively, we shall be able to reduce the problem to differential equations of the first order, in the following cases,

$$1^0 \text{ if } P = \alpha r + \frac{\beta}{r^2}, \qquad Q = \alpha r' + \frac{\beta'}{r'^2};$$

$$2^0 \text{ if } P = \alpha r + \beta r^3, \qquad Q = \alpha' r' + \beta' r'^3;$$

$$3^0 \text{ if } P = \alpha r + \beta r^3 + \gamma r^5, \quad Q = \alpha r' + \beta r'^3 + \gamma r'^5.$$

The first of these three cases is somewhat remarkable. The parts αr, and $\alpha r'$, or, $\alpha . PA$, and $\alpha . PB$ of the forces are equivalent to a force $2\,\alpha . PC$, C being the bisection of AB, (by Statics). Hence, P is acted on by three forces; one to C varying as the distance, and one to each of the points A, B varying inversely as the square. Now, by each of these three forces, the body might describe an ellipse, of which the foci are A and B, and it is a curious circumstance, that it may be made to describe the same ellipse when acted on by all the three forces*. The proof of this is contained in the Memoir just referred to.

It may be worth while to notice how far this problem bears upon those which the solar system offers to us. The Moon is acted upon by two forces, tending to the Earth and to the Sun, and each varying inversely as the square of the distance. If therefore we could suppose the Sun and the Earth to be fixed points, the path of the Moon would be that determined in the preceding problems. But this is not the case, for the Sun also acts upon the Earth, and though the Earth's distance from it remains nearly constant, and consequently the Sun's force nearly constant, the centrifugal force

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* It may also be observed, that $\alpha r + \frac{\beta}{r^2}$ expresses the law of attraction, which particles must exert, that the attraction to a sphere may follow the same law as it does to one of the particles. *Appendix* (C).

by which the Sun's attraction is counteracted, operates also upon the Moon. We may, however, reduce this case to our problem by this consideration. If we suppose equal and parallel forces to act upon the Earth and the Moon, the motion of the Moon relatively to the Earth, will remain unaltered. Let therefore such forces act, and let them be equal and opposite to that which the Sun exerts upon the Earth. Then, the Earth will be acted upon by forces which destroy each other, and will therefore be at rest: and the Moon (*P*, fig. 133,) will be acted on by the forces which tend to the Earth and the Sun, in the directions *PS*, *PT*, $\left(\text{suppose } \frac{\beta}{r^2}, \frac{\beta'}{r'^2}\right)$, and also by a force in *PO* parallel to *ST*, which may be considered as constant. Let *ST* represent this force; *ST* is equivalent to *SP*, *PT*; that is, if the force $= \alpha . ST$, ar, ar', are the parts in *SP*, *PT*. Hence, *P* is acted on by a force $-ar + \frac{\beta}{r^2}$ to *S*, and a force $ar' + \frac{\beta'}{r'^2}$ to *T*. This does not come under the integrable cases of the problem; and hence, we cannot apply the method to determine the motions of the Moon. The proper mode of considering the question of the motions of these bodies *S*, *P*, *T*, acted on by their mutual attractions, belongs to the subject of the succeeding Chapter.

APPENDIX (C) to CHAP. IV. p. 66.

On the Attractions of Bodies.

IT is observed in Chapter IV, that the attractions of bodies are the collective effect of the attractions of all their particles. We shall here shew how, from the law of the attractive power of the elementary parts, we may find the attractions of finite bodies *.

PROP. I. A spherical shell of indefinitely small thickness, being composed of particles attracting according to a given law; to find the attraction on any point.

* *Newton*, Book I, Sect. 12. Laplace *Mec. Cel.* Liv. II. Art. 11.

Let S, fig. 134, be the centre of the spherical shell, SE, its radius $= a$; EF, its thickness $= \alpha$; P the point attracted; $PS = r$, $PF = f$, F being any particle. And let $PSE = \theta$.

Suppose two planes FSP, GSP, passing through SP, to make with each other an indefinitely small angle $FDG = \delta$, FDG being a plane perpendicular to PD. Then, $FG = DF \,.\, \delta = a \sin.\, \theta \,.\, \delta$. And if we suppose $ESe = d\theta$, $Ee = a d\theta$, and the solid content of the small portion of the shell $EFGe$ will be $\delta \alpha d\theta a^2 \sin.\, \theta$.

Now since this portion is indefinitely small, its attraction on P may be considered as that of a single particle at the distance f. Let $\phi(f)$ be the function of f expressing the law of attraction; then the attraction of the elementary solid Ge on P will be $\delta \alpha d\theta \,.\, a^2 \sin.\, \theta \, \phi(f)$. To reduce this to the direction PS, we must multiply it by $\frac{PD}{PF}$, or $\frac{r - a \cos.\, \theta}{f}$; hence, the attraction by the slice AEB, towards S, is

$$\int \delta \alpha d\theta \,.\, a^2 \sin.\, \theta \, \phi(f) \,.\, \frac{r - a \cos.\, \theta}{f};$$

the integral being taken from A to B.

The attraction, by varying the angle δ, manifestly varies in the same ratio; hence, for the whole shell we must put 2π for δ, and we have for the whole attraction

$$2\pi a^2 \alpha \int d\theta \sin.\, \theta \, \phi(f) \frac{r - a \cos.\, \theta}{f} = A, \text{ suppose.}$$

Since $f^2 = r^2 - 2ra \cos.\, \theta + a^2$, $\left(\frac{df}{dr}\right) = \frac{r - a \cos.\, \theta}{f}$; $\left(\frac{df}{dr}\right)$ indicating the differential coefficient where r alone is supposed to vary.

$$\text{Hence, } A = 2\pi a^2 \alpha \int d\theta \sin.\, \theta \phi(f) \left(\frac{df}{dr}\right).$$

Now let $\int df \phi(f) = \phi_1(f)$, and we have, since $\phi(f)$ is the differential coefficient of $\phi_1(f)$,

$$\left(\frac{d\phi_1(f)}{dr}\right) = \phi(f) \cdot \left(\frac{df}{dr}\right).$$

Hence, if we take $B = 2\pi a^2 \alpha \int d\theta \sin.\ \theta \phi_1(f)$, we shall have

$$\frac{dB}{dr} = 2\pi a^2 \alpha \int d\theta \sin.\ \theta \left(\frac{d\phi_1(f)}{dr}\right)$$

$$= 2\pi a^2 \alpha \int d\theta \sin.\ \theta \phi(f) \cdot \left(\frac{df}{dr}\right) = A,$$

for since the variations of θ and of r are independent, it makes no difference, whether we perform the differentiations before or after integration*.

Now, since $f^2 = r^2 - 2ra \cos.\ \theta + a^2$, we have

$$fdf = ra \sin.\ \theta d\theta;$$

$$\text{and } \sin.\ \theta d\theta = \frac{fdf}{ra};$$

$$\text{hence, } B = \frac{2\pi a \alpha}{r} \int f df \cdot \phi_1(f).$$

The integral is to be taken from $\theta = 0$, to $\theta = \pi$; that is, from $f = r - a$, to $f = r + a$. If $\int f df \cdot \phi_1(f) = \psi(f)$, we have for the whole figure,

$$B = \frac{2\pi a \alpha}{r} \{\psi(r+a) - \psi(r-a)\}.$$

And the attraction $= A = \frac{dB}{dr}$ is thus known.

For a point within the shell the process will be the same, except that the integral must be taken between the limits $a + r$, and $a - r$.

Ex. 1. Let the force of each particle vary inversely as the square of the distance.

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* If F be a function of r and θ, and $B = \int F d\theta$,

$$\frac{dB}{dr} = \frac{d\int F d\theta}{dr}.$$

$$\text{But } \frac{dB}{d\theta} = F,\ \frac{dF}{dr} = \frac{d^2B}{d\theta\, dr}.$$

Hence, (Lacroix, *Elem. Treat.* 122.)

$$\frac{d^2B}{dr\, d\theta} = \frac{dF}{dr},\ \text{and } \frac{d^2B}{dr} = \frac{dF}{dr} \cdot d\theta;\ \therefore \frac{dB}{dr} = \int \frac{dF}{dr} \cdot d\theta.$$

Here $\phi(f)=\frac{1}{f^2}$; $\phi_1(f)=\int df.\phi(f)=-\frac{1}{f}$;

$$\psi f=\int f df \phi_1(f)=-f,$$

$$B=\frac{2\pi a \alpha}{r}\{(r-a)-(r+a)\}=-\frac{4\pi a^2\alpha}{r},$$

$$A=\frac{dB}{dr}=\frac{4\pi a^2\alpha}{r^2}.$$

The surface of the shell is $4\pi a^2$, and hence, its mass is $4\pi a^2\alpha$, and the attraction is the same as if it were collected at the centre of the sphere.

Ex. 2. Let the force of each particle vary as any power of the distance.

Let $\phi(f)=f^n$, $\phi_1(f)=\frac{f^{n+1}}{n+1}$, $\psi(f)=\frac{f^{n+3}}{(n+1)(n+3)}$,

$$B=\frac{2\pi a\alpha}{(n+1)(n+3)r}\{(r+a)^{n+3}-(r-a)^{n+3}\}$$

$$=\frac{4\pi a^2\alpha}{n+1}\Big\{r^{n+1}+\frac{(n+2)(n+1)}{2.3}r^{n-1}a^2$$

$$+\frac{(n+2)(n+1)n.(n-1)}{2.3.4.5}r^{n-3}a^4+\&c.\Big\}.$$

And $A=\frac{dB}{dr}=4\pi a^2\alpha\Big\{r^n+\frac{(n+2)(n-1)}{2.3}r^{n-2}a^2$

$$+\frac{(n+2)n(n-1)(n-3)}{2.3.4.5}r^{n-4}a^4+\&c.\Big\}.$$

This series terminates, if n be a whole positive number.

If $n=1$, or $n=-2$, that is, if the attraction varies directly as the distance, or inversely as the square of the distance, the terms after the first vanish, and the attraction is the same, as if the mass were collected at the centre of the sphere.

Hence, if the particles exert a force which is as $mr+\frac{m'}{r^2}$, the whole force will be the same as if the mass were so collected; for we may suppose the shell to consist of particles which attract with forces mr, and of an equal number of others which attract with forces $\frac{m'}{r^2}$.

If $n=-1$, or $n=-3$, the integrations for $\psi(f)$ fail, and we must employ other methods.

Ex. 3. Let the force vary inversely as the cube of the distance.

$$\phi(f)=\frac{1}{f^3},\ \phi_1(f)=-\frac{1}{2f^2},\ \psi(f)=-\tfrac{1}{2}\text{ hyp. log. } f.$$

$$B=\frac{\pi a a}{r}\text{ hyp. log. }\frac{r-a}{r+a},$$

$$A=\frac{dB}{dr}=\frac{\pi a a}{r^2}\left\{\frac{2ar}{r^2-a^2}-\text{hyp. log. }\frac{r-a}{r+a}\right\}.$$

Ex. 4. Let the force vary inversely as the distance.

$$A=\pi a a\left\{\frac{2a}{r}+\left(1-\frac{a^2}{r^2}\right)\text{ hyp. log. }\frac{r+a}{r-a}\right\}.$$

Ex. 5. The force varying as any power of the distance; to find the attraction on a point within the shell.

As in Ex. 2, $\psi(f)=\dfrac{f^{n+3}}{(n+1)(n+3)}$,

$$B=\frac{2\pi a a}{(n+1)(n+3)r}\{(a+r)^{n+3}-(a-r)^{n+3}\}$$

$$=\frac{4\pi a}{n+1}\left\{a^{n+3}+\frac{(n+2)(n+1)}{2.3}a^{n+1}r^2\right.$$

$$\left.+\frac{(n+2)(n+1)n(n-1)}{2.3.4.5}a^{n-1}r^4+\&c.\right\},$$

$$A=\frac{dB}{dr}=4\pi a\left\{\frac{n+2}{3}a^{n+1}r\right.$$

$$\left.+\frac{2(n+2)n(n-1)}{3.4.5}a^{n-1}r^3+\&c.\right\}.$$

If $n=-2$, or the force be inversely as the square of the distance, we have $A=0$; the attractions in different directions counterbalance each other.

Prop. II. To find the attraction of a sphere composed of particles attracting according to a given law.

If in the last proposition we put u for a, and du for a, the thickness of the shell, and integrate from $u=0$, to $u=a$, we shall have the attraction of a solid sphere of radius a.

By this means, from the expression for A in Ex. 2, we find for the attraction of a sphere

$$\frac{4\pi a^3}{3}\left\{r^n + \frac{(n+2)(n-1)}{2.3}.\frac{3r^{n-2}a^2}{5} + \&c.\right\}.$$

In the cases where the attraction of a shell is the same as if the matter were collected at the centre, the attraction of a sphere will also follow the same law. For the sphere may be supposed to be composed of concentric shells, each of which attracts as if it were collected at the centre, and therefore the whole will attract as if all its parts were there collected.

PROP. III. To find the attraction of a circle on a point in a line perpendicular to its plane, and passing through its centre.

Let BC, fig. 135, be the circle, P the attracted point, $SP=r$, $SE=f$, SE any radius $=u$, and SF a radius indefinitely nearly equal to this, so that $EF=du$. Let a small angle $FSG=\delta$, then the quadrilateral $EG=u.\delta.du$. And, if the law of the attraction be represented by $\phi(f)$, the attraction of EG is $\delta.udu.\phi(f)$; which resolved in the direction PS becomes $\delta.udu.\phi(f).\frac{r}{f}$. And for the whole annulus, whose breadth is EF, we must put 2π for δ; whence it becomes $2\pi udu.\phi(f).\frac{r}{f}$. Hence, the attraction of the whole circle

$$=2\pi\int udu\phi(f)\frac{r}{f};\ \text{where } f=\sqrt{(r^2+u^2)},$$

the integral being taken from $u=0$, to $u=a$, the radius of the circle.

If $\phi(f)=f^n$,

$$\text{attraction} = 2\pi r\int udu\,(r^2+u^2)^{\frac{n-1}{2}}$$

$$=\frac{4\pi r}{n+1}(r^2+u^2)^{\frac{n+1}{2}} + \text{constant}$$

$$=\frac{4\pi r}{n+1}\{(r^2+a^2)^{\frac{n+1}{2}} - r^{n+1}\}.$$

Ex. 1. Let $n=-2$, or the force vary inversely as the square of the distance.

$$\text{Here, attraction} = 4\pi\left\{1 - \frac{r}{\sqrt{(r^2+a^2)}}\right\}.$$

Ex. 2. Let the circle be infinite, and $n < -1$.

In this case $(r^2+a^2)^{\frac{n+1}{2}}$ becomes 0, and we have, putting $-m$ for n,

$$\text{attraction} = \frac{4\pi}{(m-1)\,r^{m-2}}.$$

If $m=2$, attraction $=4\pi$, and is the same at all distances.

Prop. IV. To find the attraction of a solid of revolution on a point in the axis.

We must here multiply the attraction of the circle, found in the last Proposition, by the thickness dr, for the attraction of a differential slice; and if we then put for a its value in terms of r, and integrate, we have the attraction of the whole solid.

Ex. 1. The attraction of a cylinder on a point in its axis; fig. 136.

$$\text{attraction} = \frac{4\pi}{n+1}\int\left\{r\,dr\,(r^2+a^2)^{\frac{n+1}{2}} - r^{n+2}dr\right\}$$

$$= \frac{4\pi}{n+1}\left\{\frac{(r^2+a^2)^{\frac{n+3}{2}}}{n+3} - \frac{r^{n+3}}{n+3} + \text{constant}\right\}.$$

If BSC and $B'S'C'$ be the two ends of the cylinder, and if $PS=b$, $PS'=b'$, $PC=c$, $PC'=c'$, we have

$$\text{attraction} = \frac{4\pi}{(n+1)(n+3)}\left\{c'^{n+3} - c^{n+3} - (b'^{n+3} - b'^{n+3})\right\}.$$

If the force vary inversely as the square of the distance, $n=-2$,

$$\text{attraction} = 4\pi\left\{b' - b - (c'-c)\right\}.$$

Ex. 2. The attraction of an infinite solid bounded by planes.

In last Prop. Ex. 2, multiply by dr, and we have

$$\text{attraction} = \frac{4\pi}{m-1}\int\frac{dr}{r^{m-2}} = \frac{4\pi}{(m-1)(m-3)}\left\{\frac{1}{b^{m-3}} - \frac{1}{r^{m-3}}\right\},$$

where b is the distance of the attracted point from the surface of the solid.

If $m = 2$, attraction $= 4\pi (r - b)$.

If $m = 3$, attraction $= 2\pi$ hyp. log. $\frac{r}{b}$.

If $m > 3$, the attraction is finite, when r is infinite, and we have

$$\text{attraction} = \frac{4\pi}{(m-1)(m-3)\, b^{m-3}}.$$

Ex. 3. The attraction of a cone on a point at the vertex.

In fig. 137, let $PS = r$, $ST = kr$, and putting kr for a in last Prop.

$$\text{attraction} = \frac{4\pi}{n+1} \int r^{n+2} dr \left\{ (1+k^2)^{\frac{n+1}{2}} - 1 \right\}$$

$$= \frac{4\pi r^{n+3}}{(n+1)(n+3)} \left\{ \sqrt{(1+k^2)} - 1 \right\}.$$

Where r is to be made $= PA$ the axis.

Prop. V. To find the attraction of a straight line upon a point at any distance from it.

Let BC, fig. 138, be the attracting line, P the point attracted, PS perpendicular on $BC = r$, $SE = u$, $PE = f$, and let the force of each particle be as $\phi(f)$. We may suppose EF an indefinitely small portion, to be du; and its attraction on P will be $\phi(f) du$, and part resolved *perpendicular* to BC will be $\frac{r}{f} \phi(f)\, du$, where $f = \sqrt{(r^2 + u^2)}$. This is to be integrated from $u = 0$, to $u = a = SB$, for the attraction of SB; and the attraction of SC is to be found in the same manner, and added to the former.

The attraction of du *parallel* to SB, will be $\frac{u}{f} \phi(f)\, du$; which is to be integrated in the same manner as before; and the difference taken of the parts belonging to SB and to SC.

Ex. Let the force vary inversely as the square of the distance. Here $\phi(f) = \frac{1}{f^2}$;

$$\text{attraction in } PS = \int \frac{r\,du}{(r^2+u^2)^{\frac{3}{2}}} = \frac{u}{r\sqrt{(r^2+u^2)}},$$

$$\text{attraction perpendicular to } PS = \int \frac{u\,du}{(r^2+u^2)^{\frac{3}{2}}} = \frac{1}{r} - \frac{1}{\sqrt{(r^2+u^2)}}.$$

And this is to be taken for SB and for SC, and combined.

APPENDIX (D) to CHAP. IV. p. 78.

On some particular Cases of the Motions of three Bodies.

THERE are one or two other particular cases in which the motions of bodies acting upon each other can be accurately obtained. These cases may be deduced from the following principles.

PROP. If any number of bodies be acted upon every where by forces which are to each other as their distances from a given point, to which they tend, the bodies may be made to describe similar curves round this point in equal times.

For if they be projected at equal angles to their distances, and with velocities proportional to them, they will manifestly in the first instant describe similar curves. And at the end of this first instant, the forces will still be proportional to their distances, their directions will make equal angles with the distances, and their velocities will be proportional. Hence, they will in the second instant describe similar curves. And so on for any number of instants. And hence, the curves will be similar, when they are described continuously.

Hence, if any number of bodies acting upon each other, be so placed, that the resultant of all the actions on each tends to the centre of gravity of the whole, and is proportional to its distance from that point, they will, being projected with proportional velocities in similar directions, describe similar figures round this centre in the same times. We shall take two examples*.

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* Laplace, *Mec. Cel.* Liv. X. Chap. VI. No. 17.

Prob. I. *Three bodies in a straight line,* m, m′, m″, fig. 139, *attract each other with forces varying inversely as the* n*th power of the distance; it is required to find their positions, that they may describe similar figures round* A *the centre of gravity.*

We must have the whole force on m : whole force on m' :: Am : Am', and whole force on m : whole force on m'' :: Am ; Am''. Hence, if $Am=r$, $Am'=r'$, $Am''=r''$, we must have

force on $m=kr$, force on $m'=kr'$, force on $m''=kr''$;

k being any quantity. Hence,

$$\left.\begin{aligned} \frac{m'}{(r+r')^n}+\frac{m''}{(r+r'')^n}&=kr \\ \frac{m}{(r+r')^n}-\frac{m''}{(r''-r')^n}&=kr' \\ \frac{m}{(r+r'')^n}+\frac{m'}{(r''-r')^n}&=kr'' \end{aligned}\right\}\ldots\ldots\ldots(1).$$

Eliminating k in the two first, we have

$$\frac{m'r'}{(r+r')^n}+\frac{m''r'}{(r+r'')^n}-\frac{mr}{(r+r')^n}+\frac{m''r}{(r''-r')^n}=0,$$

$$\text{or } m'r'+m''r'\left(\frac{r+r'}{r+r''}\right)^n-mr+m''r\left(\frac{r+r'}{r''-r'}\right)^n=0\ldots.(2).$$

$$\text{Let } \frac{r''-r'}{r+r'}=z\,;\ \therefore\ 1+z=\frac{r''+r}{r+r'}.$$

Also, $mr-m'r'-m''r''=0$, because A is the centre of gravity.

$$\text{Hence, } z=\frac{r''-r'}{r+r'}=\frac{m''r''-m''r'}{m''(r+r')}=\frac{mr-m'r'-m''r'}{m''(r+r')};$$

$$\therefore\ m''rz+m''r'z=mr-(m'+m'')r',$$

$$\text{and } r'=\frac{(m-m''z)r}{m'+m''(1+z)}.$$

Substituting these values in equation (2), it becomes

$$\frac{(m-m''z)}{m'+m''(1+z)}\left\{m'+\frac{m''}{(1+z)^n}\right\}-m+\frac{m''}{z^n}=0\ldots\ldots(3);$$

whence z may be determined, and thence r'; and thence

$$r'' = \frac{mr - m'r'}{m''}.$$

If we multiply the first and second of equations (1) by m, m', respectively, and subtract, observing that $mr - m'r' = m''r''$, we have the third. Hence, the values of r', r'' so found, will answer the conditions. And if, the bodies being at these distances, they be projected in similar directions with proportional velocities, they will describe similar figures.

If the force vary inversely as the square of the distance, equation (3) becomes

$$z^2(m - m''z)\{m'(1+z)^2 + m''\} - (mz^2 - m'')(1+z)^2\{m' + m''(1+z)\} = 0,$$

$$\therefore \left.\begin{array}{l} \{mm'(1+z)^2 + mm'' - m'm''z(1+z)^2 - m''^2 z\}\, z^2 \\ - \{mm'z^2 + mm''(1+z)z^2 - m'm'' - m''^2(1+z)\}\,(1+z)^2 \end{array}\right\} = 0;$$

or, dividing by m'', and changing the signs,

$$mz^2\{(1+z)^3 - 1\} - m'(1+z)^2(1-z^3) - m''\{(1+z)^3 - z^3\} = 0.$$

If m be the Sun, m' the Earth, and m'' the Moon, m is large compared with m' and m''. Hence, it appears that z will be small, and that mz^3 is of the order of m' and m''. Neglecting therefore higher powers of z, we have

$$3mz^3 - m' - m'' = 0; \quad z = \sqrt[3]{\frac{m' + m''}{3m}};$$

which gives $z = \frac{1}{100}$, nearly. Hence, in fig. 139, $m'm'' = \frac{1}{100}\, mm'$. If therefore, the Earth and Moon had been placed in the same straight line, at distances respectively from the Sun, proportional to 1 and $1 + \frac{1}{100}$, and if they had had velocities parallel to each other, and proportional to those distances, they would have moved about the Sun, the Moon being perpetually in opposition. Also at this distance, which is about four times the Moon's actual distance, she would have been beyond the Earth's shadow. But then, instead of occupying an angle of half a degree, she would only have subtended an angle of about 8′, and, presenting only $\frac{1}{16}$ of the present full Moon's surface, would have answered, in a very imperfect degree, the purposes of replacing the light of the Sun.

PROB. II. *Three bodies,* m, m′, m″, *fig.* 140, *attract each other according to any function,* ϕ (s), *of the distance* s; *to find their positions, not in a straight line, that they may describe similar figures.*

Let the bodies be referred to rectangular co-ordinates in the plane of the three, and measured from the centre of gravity A. Let these co-ordinates be x, y, for m; x' y' for m'; x'', y'', for m''. Also, let the distance mm' be s, mm'' be s', $m'm''$ be s''. Hence, we shall have, for the forces on m,

$$\text{parallel to } x,\ m'\phi(s).\frac{x-x'}{s}+m''\phi(s')\frac{x-x''}{s};$$

$$\text{to } y,\ m'\phi(s)\frac{y-y'}{s}+m''\phi(s')\frac{y-y''}{s};$$

and similarly for the other bodies m', m''.

Now, in order that the three bodies may describe similar figures, the forces upon them must be proportional to their distances Am, &c. from the centre of gravity. Hence, the resolved part of the force on m, in direction of AM, will be proportional to the co-ordinate AM, or x; and similarly for the others; so that K being a constant quantity, we shall have

$$\left.\begin{aligned} m'\frac{\phi(s)}{s}(x-x')+m''\frac{\phi(s')}{s'}(x-x'')&=Kx\\ m\frac{\phi(s)}{s}(x'-x)+m''\frac{\phi(s'')}{s''}(x'-x'')&=Kx'\\ m\frac{\phi(s')}{s'}(x''-x)+m'\frac{\phi(s'')}{s''}(x''-x')&=Kx'' \end{aligned}\right\}\ldots\ldots(1);$$

where, as before, the third equation is deducible from the other two, observing that

$$mx+m'x'+m''x''=0.$$

If we combine this equation with the first of equations (1), we find

$$x\left\{m'\frac{\phi(s)}{s}+(m+m'')\frac{\phi(s')}{s'}\right\}+m'x'\left\{\frac{\phi(s')}{s'}-\frac{\phi(s)}{s}\right\}=Kx.$$

If we suppose $s'=s$, this gives

$$K=(m+m'+m'')\frac{\phi(s)}{s}.$$

If we also suppose $s''=s$, the two last of equations (1) will give the same value of K, and all the three equations will be satisfied. In the same manner the three equations in $y, y' y''$, will be satisfied by the same supposition. In this case, (that is, when $s=s'=s''$,) it is obvious that the forces tending to the centre of gravity are $K\sqrt{(x^2+y^2)}$, &c.: or, if we call r, r', r'', the distances from the centre, the forces towards that point are Kr, Kr', Kr''. Hence, if the bodies be projected in similar directions, with velocities proportional to r, r', r'', they will describe similar curves.

We shall now find the force tending to the centre of gravity. On this supposition, we have

$$mx=-m'x'-m''x'';$$

$$\therefore (m+m'+m'')x=m'(x-x')+m''(x-x'').$$

Similarly, $\quad (m+m'+m'')y=m'(y-y')+m''(y-y'').$

Squaring and adding,

$$\left.\begin{aligned}&(m+m'+m'')^2(x^2+y^2)\\ =m'^2(x-x')^2&+2m'm''(x-x')(x-x'')+m''^2(x-x'')^2\\ +m'^2(y-y')^2&+2m'm''(y-y')(y-y'')+m''^2(y-y'')^2\end{aligned}\right\}\ldots\ (2).$$

But, $x^2+y^2=r^2$; also $(x-x')^2+(y-y'^2)^2=s^2$

$$(x-x'')^2+(y-y'')^2=s^2.$$

Again, since $s'=s''$,

$$(x-x'')^2-(x'-x'')^2+(y-y'')^2-(y'-y'')^2=0.$$

And $(x-x')^2+(y-y')^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots =s^2.$

Adding

$$2x^2-2xx'-2xx''+2x'x''+2y^2-2yy'-2yy''+2y'y''=s^2;$$

$$\text{or, } 2(x-x')(x-x'')+2(y-y')(y-y'')=s^2.$$

Hence, equation (2) becomes

$$(m+m'+m'')^2 r^2 = (m'^2+m'm''+m''^2)\, s^2,$$

$$s = \frac{(m+m'+m'')\, r}{\sqrt{(m'^2+m'm''+m''^2)}}.$$

And putting this for s in Kr, where

$$K = (m+m'+m'')\,\frac{\phi(s)}{s},$$

we have the force to A.

If the force vary inversely as the square of the distance, we have $\phi s = \frac{1}{s^2}$. Hence, K is

$$(m+m'+m'')\,.\,\frac{(m'^2+m'm''+m''^2)^{\frac{3}{2}}}{(m^2+m'+m'')^3 r^3};$$

$$\text{and the force} = \frac{(m'^2+m'm''+m''^2)^{\frac{3}{2}}}{(m+m'+m'')^3 r^2}.$$

If the bodies be properly projected, they will move about the centre, so as to describe similar paths, (namely, conic sections,) and always forming an equilateral triangle by the lines that join them.

If we do not suppose $s = s'$, we shall find that the only way in which equations (1) can be satisfied, requires that

$$x : y :: x' : y';$$

whence the bodies are in a straight line. And therefore this case is reduced to the last problem.

APPENDIX (E) to CHAP. VI. p. 142.

On the Vibrations of Strings.

THE vibrations of strings stretched between two points, such, for instance, as musical strings, were the subjects of long and various investigations and discussions among the mathematicians of the last century. We shall here as briefly as possible, give the statement and solution of the Problem, in its simplest form.

PROB. I. *A uniform string being stretched between two given points; to find the time of its small vibrations.*

Let a be the length of the string between the fixed points; Wg its weight, and Fg the force by which it is stretched. It is supposed, that the vibrations are exceedingly small, so that the length, and consequently the tension, may remain the same in all the forms which the curve assumes. Let APB, fig. 141, be the form at a time t, and let AM, MP, an abscissa and ordinate, be x and y, and $AP=s$. Let AN, NQ be another abscissa and ordinate, and $PQ=h$. The tension will be the same throughout, and $=Fg$; hence, at P the part of it resolved in the direction PM is $Fg\,.\,\frac{dy}{ds}$.

At Q, $\frac{dy}{ds}$ becomes $\frac{dy}{ds}+\frac{d^2y}{ds^2}\,.\,h+\frac{d^3y}{ds^3}\,.\,\frac{h^2}{1\,.\,2}+$ &c., supposing ds constant. Hence, the difference of the forces at P, Q, or the moving force by which PQ is drawn in the direction MP, is

$$Fg\,.\left\{\frac{d^2y}{ds^2}h+\frac{d^3y}{ds^3}\,.\,\frac{h^2}{1\,.\,2}+\text{\&c.}\right\}.$$

And since the weight of a length a is Wg, the weight of a length h is $\frac{Wgh}{a}$; hence, the mass of PQ is $\frac{Wh}{a}$, and the accelerating force on PQ is

$$\frac{Fag}{W}\left\{\frac{d^2y}{ds^2}+\frac{d^3y}{ds^3}\,.\,\frac{h}{2}+\text{\&c.}\right\}.$$

And when PQ is taken very small, every part of it moves with the same velocity, and this is the true expression for the accelerating force. That is, accelerating force on P in $MP=\frac{Fag}{W}\,.\,\frac{d^2y}{ds^2}$.

Since the vibrations are very small, we may suppose P to move always perpendicular to AM in MP; and hence, the accelerating force on $P = \frac{d^2y}{dt^2}$. Also, for the same reason, we may suppose s to be equal to x. Hence, we have

$$\left(\frac{d^2y}{dt^2}\right) = \frac{Fag}{W}\left(\frac{d^2y}{dx^2}\right) \ldots\ldots (1).$$

The quantities are put in brackets to indicate that they are *partial* differential coefficients; in the one y is differentiated, supposing t only to vary, and in the other, supposing x only to vary. The former differentiation refers to a change in the position of a given point of the curve; the latter, to a passage from one point of the curve to another at a given moment of time.

The ordinate y will be a function of x and t, which is to be determined by integrating the partial differential equation (1). Its integration is given, Lacroix, *Elem. Treat.* Art. 319; and if $\frac{Fag}{W} = b^2$, it is

$$y = \phi\,(x + bt) + \psi\,(x - bt) \ldots\ldots\ldots (2).$$

It is evident from trial, that this satisfies equation (1), for if we differentiate twice with respect to x, we find

$$\left(\frac{d^2y}{dx^2}\right) = \phi''\,(x + bt) + \psi''\,(x - bt),$$

where $\phi''z$ indicates $\frac{d^2\phi z}{dz^2}$. Also, differentiating for t, we have

$$\left(\frac{d^2y}{dt^2}\right) = b^2\phi''\,(x + bt) + b^2\psi''\,(x - bt).$$

And it is manifest, that these values verify equation (1). And the form of the functions ϕ and ψ is to be determined from the initial circumstances of the string.

Differentiating equation (2), we have

$$\frac{dy}{dt} = b\,\{\phi'\,(x + bt) - \psi'\,(x - bt)\}.$$

And if we suppose that the curve was at rest in any position when $t = 0$, we must have $\phi'\,(x) - \psi'\,(x) = 0$. Hence, $\psi'\,(x) = \phi'\,(x)$, and $\psi(x) = \phi(x)$. Let $\phi(x) = \frac{1}{2} f(x)$, and equation (2) becomes

$$y = \frac{1}{2}\{f(x+bt) + f(x-bt)\} \ldots .(3),$$

$$\frac{dy}{dt} = \frac{b}{2}\{f'(x+bt) - f'(x-bt)\} \ldots .(4).$$

Equation (3) gives the position of the curve at any time, and equation (4) the velocity of any point; b is equal to $\sqrt{\frac{Fag}{W}}$.

If we make $t=0$, we have $y=f(x)$, which is the equation to the original form of the string, and hence, the form of f is known. The original form may be any whatever, and f any function whatever, subject to the conditions that it must $= 0$, when $x = 0$, and when $x = a$. It appears from the theory of partial differential equations, that the function f, introduced in integration, may be *discontinuous*; that is, the initial form of the string APB, may be composed of different lines, not expressed by the same equation all the way from A to B.

But, from the nature of the reasoning, we obtain other properties of the function f. The points A, B, are fixed; hence, we must have $y=0$, when $x=0$, whatever be t. Therefore by equation (3),

$$0 = f(bt) + f(-bt).$$

Or, if we make $bt = u$, $f(-u) = -f(u)$ .... (5).

Also, if $x=a$, $y=0$, whatever be t;

$\therefore 0 = f(a+u) + f(a-u)$, and $f(a+u) = -f(a-u)$ ....(6).

From equation (5) it appears, that the curve represented by $y=f(x)$, is continued with similar forms on each side of A, the curve being on one side above, and on the other below the axis.

Also, from equation (6) it appears, that the curve is continued with similar forms on each side of B, above the axis on one side, and below it on the other. Hence, the curve is continued indefinitely on both directions, in the manner represented in fig. 142, and it is the ordinate of this curve which is to be considered as $f(x)$ in finding y from equation (3). To find the position of a point P at the end of any time t, we must take $MN = MN' = bt$, and $MQ = \frac{1}{2}(NQ + N'O')$ will give Q, the position of P after a time t. And as the curve is continued indefinitely, the time may be supposed to increase indefinitely, and the same construction will always determine the position.

The string will perform oscillations, assume a position similar to its original one on the opposite side of the string, return to its original position, and so on perpetually; as may thus be shewn:

By (6), $f(a+u) = -f(a-u)$. Let $u - a = u'$; $\therefore u = a + u'$,

$$f(2a+u') = -f(-u') = f(u') \text{ by (5), or } f(2a+u) = f(u):$$

sim^{ly}. $f(4a+u) = f(u)$, and generally,

$$f(2ma+u) = f(u), \ m \text{ being any whole number.}$$

Hence, the ordinate for an abscissa $2ma + x$ is the same as for x, as also appears from fig. 142.

If, therefore, we assume $bt = 2a$, $bt = 4a, \ldots . \, bt = 2ma$, at the times corresponding to each of these values, we shall have $y = \frac{1}{2}\{f(x) + f(x)\} = f(x)$, and every point of the curve is in its original position. The intervals between these times are the lengths of complete vibrations of the string, and we have for these intervals,

$$bt = 2a, \ t = \frac{2a}{b} = 2\sqrt{\frac{Wa}{Fg}}.$$

If the thickness and material of the string be given, W is as a. Hence, *for a given string, the time of vibration is as the length directly, and as the square root of the tension inversely.*

If we make $bt = a$, or $t = \frac{b}{a}$, we have

$$y = \frac{1}{2}\{f(x+a) + f(x-a)\}.$$

And $f(x-a) = -f(a-x)$ by (5), $= f(a+x)$ by (6);

$$\therefore y = f(x+a).$$

And hence, after this time, the form of the curve is ARB, in fig. 141, identical with BDB' in fig. 142, and exactly similar to the original curve inverted; the greatest ordinate being now at the same distance from B, as it was from A at the beginning of the motion. Similarly, at the end of times $\frac{3b}{a}$, $\frac{5b}{a}$, &c., the figure will be the same as ARB.

If we make $bt = \frac{a}{2}$, we have the figure in the middle of the time between the positions APB and ARB.

$$\text{Hence } y = \tfrac{1}{2}\left\{f\left(x + \frac{a}{2}\right) + f\left(x - \frac{a}{2}\right)\right\}.$$

$$\text{But } f\left(x - \frac{a}{2}\right) = -f\left(\frac{a}{2} - x\right) \text{ by (5);}$$

$$\therefore y = \tfrac{1}{2}\left\{f\left(\frac{a}{2} + x\right) - f\left(\frac{a}{2} - x\right)\right\},$$

$\frac{a}{2} + x$, and $\frac{a}{2} - x$ correspond to points at equal distances from the middle point C of AB. If in the original form of the curve, the ordinates for the portion CB be greater than for AB, the position of the portion AEC at the middle of the time of oscillation will be above the axis AC. In the same manner it may be shewn, that the position of the portion CFB at that time will be below the axis CB, and similar to the portion AEC inverted. The string never becomes a straight line.

If the curve, instead of being at first in the position APB, have an original position $APBDB'$, fig. 142, A and B' being the extreme points, and the curve consisting of two equal and similar portions APB, $B'DB$, it will oscillate so that the point B will remain fixed, and each half APB, BDB' will oscillate as if the string were fixed, at A, B, and at B, B', and the vibrations will employ half the time. Similarly, if the original form be $APBDB'B''$, the points B and B' will remain fixed, and the oscillations will employ one-third of the time, and so on. The points B, B' which remain fixed, are sometimes called *nodes*.

The musical tone, or *note*, produced by a vibrating string, depends upon the rapidity of the vibrations. If a string a vibrate twice as fast as another A, the note produced by a is said to be an *octave* above that produced by A. If a string e vibrate three times, while a vibrates twice, the note of e is a *fifth* above that of a; and so on. And notes where numbers of vibrations are to each other in ratios so simple as these, are found when combined, to be agreeable to the ear.

It appears from what has been said, that a string may vibrate so that no point of it is at rest, except its two extremities, and thus give the *fundamental note;* or it may vibrate so that its middle point is at rest, in which case it will produce the octave to the fundamental note. Or it may vibrate so as to have two points at rest, dividing it into three equal parts, and it will then give the fifth above the octave; and so on.

It appears by experiment, that a wire which performs 240 vibrations in a second, sounds the note called C in the middle of the scale of a harpsichord. It has been proposed however to consider 256, or 2^8 as the number of vibrations determining this note, in which case every other note C would also have a power of 2 for the number of its vibrations. Hence, we may find the note produced by any string; by finding the number of its vibrations, and comparing them with the number just mentioned.

Prob. II. *To find the form of a string, that it may oscillate symmetrically, that is, that all its points may come to the axis at the same time.*

In this case, the force which accelerates each particle, must be as its distance from the axis; that is, as y. Hence, taking the expression for the force obtained in p. 378, we have

$$\frac{Fag}{W} \cdot \frac{d^2y}{dx^2} = -my, \text{ } m \text{ being a constant quantity.}$$

$$\text{or } \frac{d^2y}{dx^2} + \frac{Wm}{Fag} \cdot y = 0.$$

Of this the integral is, if $\frac{Wm}{Fag} = k^2$,

$$y = C \sin. kx + C' \cos. kx;$$

$$\therefore \frac{dy}{dx} = kC \cos. kx - kC' \sin. kx.$$

$$\text{And when } x = 0, \frac{dy}{dx} = 0; \therefore C' = 0,$$

$$y = C \sin. kx.$$

The quantity C is arbitrary, and determines the magnitude of the original ordinate. It must necessarily be small, because the vibrations are supposed to be small.

The time of a point coming to the axis will be $\frac{\pi}{2\sqrt{m}}$, and hence, the time of an oscillation to the opposite side, is $\frac{\pi}{\sqrt{m}}$, and the time of a complete vibration, the string returning into the original position, is $\frac{2\pi}{\sqrt{m}}$.

The curve must meet the axis again when $x = a$, hence,

$$0 = C \cdot \sin. ka; \therefore ka = \pi, \text{ and } k = \frac{\pi}{a};$$

$$\therefore \sqrt{\frac{Wm}{Fag}} = \frac{\pi}{a}, \text{ and } \sqrt{m} = \pi \sqrt{\frac{Fg}{Wa}}.$$

Hence, the time of a vibration $= \frac{2\pi}{\sqrt{m}} = 2\sqrt{\frac{Wa}{Fg}}$, the same as before.

APPENDIX (F) to CHAP. VI. p. 142.

On the Vibrations of Springs.

As another case of small oscillations, we shall obtain the equations for the motion of an elastic rod.

PROP. I. *A uniform elastic rod* BC, fig. 143, *firmly fixed at* B, *and naturally straight, vibrates by its elasticity; to find the equations of its motion in small vibrations.*

We shall find first the forces which must act at every point to keep the rod *at rest* in the position BPC. Let AB be its position when left to itself, AN, NQ, an abscissa and ordinate; $CQ = s$. And suppose that a force F, acting at every point Q, would keep the rod $BPQC$ in equilibrium. Since the whole is in equilibrium the part PQC must be so, and hence, the forces which act on PC must balance the tendency which the rod has to straight itself at P by its elasticity. Now the moment of the elasticity is directly as the curvature, and therefore inversely as the radius of the curvature at P. Let this radius be ρ; and let the effect of the elasticity to turn PC round P be that of a force K acting at any arm c. Hence,

$$Kc \propto \frac{1}{\rho}. \quad \text{Let } Kc = \frac{E}{\rho}.$$

And, if $AM = x'$, $MP = y'$, $AN = x$, $NQ = y$, the moment of any force F about P is $F(x' - x)$; and, as the forces F act at every point of PC, the moment of the forces on ds is $F(x' - x)\,ds$, or $F(x' - x)dx$, when the flexure is very small, so that dx and ds may be conceived to coincide. Hence, $\int F(x' - x)\,dx$ is the whole moment of the forces on PC, the integral being taken from $x = 0$ to $x = x'$. Therefore,

$$\frac{E}{\rho} = \int F(x' - x)\,dx.$$

$$\text{Now,} \quad \rho = -\frac{ds'^3}{dx'd^2y'} = -\frac{dx'^2}{d^2y'}.$$

$$\text{And} \int F(x' - x)\,dx = x'\int F dx - \int F x dx.$$

If we suppose $\int F dx$ taken from $x = 0$, to be $= F_1$, we shall have

$$\int F x dx = \int x\,dF_1 = F_1 x' - \int F_1 dx.$$

Hence, $\int F(x' - x)\,dx$ becomes $\int F_1 dx$. And we have

$$-\frac{Ed^2y'}{dx'^2} = \int F_1\,dx$$

$$-\frac{Ed^3y'}{dx'^3} = F_1.$$

for it is clear that $\int F_1 dx$, from $x=0$ to $x=x'$, will be a function of x', and that its differential with respect to x' will be F_1

$$-\frac{E d^4 y'}{dx'^4} = \frac{d F_1}{d x} = F,$$

on the same account. Hence, F is known at every point.

Now, if when the spring is kept at rest by these forces F acting at every point, we suppose at each point a force equal to F in the opposite direction, these forces will counterbalance each other, and the rod will be left to its own elasticity. But, in this case the points are manifestly each urged *from rest* by a force F. Hence, we shall have $\frac{d^2 y}{d t^2} = F$; or, using the same notation as before,

$$-E\left(\frac{d^4 y}{dx^4}\right) = \left(\frac{d^2 y}{dt^2}\right) \ldots\ldots\ldots (1),$$

which is the equation to the motion. This cannot be integrated in finite terms, and though we might obtain integrals in series, we shall confine ourselves to the consideration of symmetrical oscillations.

Prob. II. *To find the form of an elastic rod, that it may vibrate symmetrically.*

For this purpose the force which urges each point towards the axis must be as the distance from the axis. Hence, we must have

$$E\frac{d^4 y}{dx^4} = my, \text{ or } \frac{d^4 y}{dx^4} = \frac{my}{E} = k^4 y \text{ suppose} \ldots\ldots (2).$$

This equation may be integrated, and gives

$$y = A \cos. kx + B \sin. kx + C\epsilon^{kx} + D\epsilon^{-kx} \ldots\ldots (3).$$

A, B, C, D, being arbitrary quantities. To determine them, we may observe, that when $x=0$, we must have $\frac{d^3 y}{dx^3} = 0$, because $\frac{d^3 y}{dx^3} = F_1$ in last Prop. For the same reason, $\frac{d^2 y}{dx^2} = 0$, when $x=0$. Hence,

$$0 = -B + C - D, \quad 0 = -A + C + D.$$

$$\therefore A = C + D, \qquad B = C - D.$$

Again, since the extremity B is fixed, we have, when $x=a$, the whole length, $y=0$. And, since that extremity is also fixed in direction, we have $\frac{dy}{dx}=0$, when $x=a$. Hence, we find

$$0=(C+D)\cos. ka+(C-D)\sin. ka+C\epsilon^{ka}+D\epsilon^{-ka},$$

$$0=-(C+D)\sin. ka+(C-D)\cos. ka+C\epsilon^{ka}-D\epsilon^{-ka};$$

whence

$$\frac{C}{D}=\frac{\sin. ka-\cos. ka-\epsilon^{-ka}}{\sin. ka+\cos. ka+\epsilon^{ka}}=\frac{\cos. ka+\sin. ka+\epsilon^{-ka}}{\cos. ka-\sin. ka+\epsilon^{ka}}.$$

Hence, we find $2+\cos. ka\,(\epsilon^{ka}+\epsilon^{-ka})=0 \ldots\ldots (4).$

From this equation k is to be determined: k will have an infinite number of different values; and to each of these will correspond a different form of the curve, determined by equation (3), in which the rod will oscillate symmetrically.

The least positive value of ka is 1.8751, nearly. The other values of ka are very nearly $\frac{3\pi}{2}$, $\frac{5\pi}{2}$, $\frac{7\pi}{2}$, &c.

The time of a vibration will be $\frac{2\pi}{\sqrt{m}}$. Now $\frac{m}{E}=k^4$, whence $m=Ek^4$. And hence, the time of vibration is

$$\frac{2\pi}{k^2\sqrt{E}}=\frac{2\pi a^2}{k^2a^2\sqrt{E}}.$$

For the same value of ka, that is, for the same form, in rods of different lengths, it appears that *the time of the vibration is as the square of the length directly, and as the square root of the elasticity inversely.*

Hence, if a rod A' were twice as long as A, and of the same elasticity, the note produced by A' would be four octaves below that produced by A.

The different values of ka in equation (4), taken for the same rod, give a series of different forms, resembling those in fig. 50, and differing like them in the number of their nodes. The times of vibration will be successively as the values of $\frac{1}{k^2 a^2}$; and the numbers of vibrations in a given time, and consequently, the note, will be as $k^2 a^2$. Hence, the note produced by the successive figures will be as

1, 6.2673, 17.549, 34.386, &c.

Hence, the notes which an elastic rod can produce, besides the fundamental note, are, the fifth above the double octave, somewhat too sharp: the half note above the fourth octave nearly: the half note above the fifth octave nearly: and so on.

If the rod be not fixed into a support at B, but be subject to other conditions, for instance, if both ends were fastened, or both free, we shall have to make some other supposition instead of putting $y=0$, and $\frac{dy}{dx}=0$, when $x=a$. The rest of the process will be nearly the same as before.

We may find E by the equation $E=Kc\rho$, where a force K, acting at an arm c, produces a radius of curvature ρ. See *Statics*.

For an examination of the different cases of this Problem, see *Com. Acad. Petrop.* for 1741—1743, p. 105, and *Novi Com. Acad. Petrop.* for 1772, p. 449. Dr. Young, *Elem. of Nat. Phil.* Art. 398. For experiments, see Chladni, *Traité d'Aconstique*, p. 92, Biot, *Traité de Physique*, tom. II, p. 74.

APPENDIX (G) to BOOK II. CHAP. I. p. 180.

On the Descent of small Bodies in Fluids.—On the Ascent of an Air-Bubble.

THESE problems do not belong immediately to our subject, but they depend upon little in addition to the principles employed in the text: and as the latter problem has been imperfectly solved by Atwood in treating of this subject, and the solution copied into other works, we shall introduce them here.

The resistance on a plane moving perpendicularly to itself in a fluid, is the weight of a column of fluid, whose altitude is the height due to the velocity.

The base of the column is understood to be the surface resisted.

It appears probable in the first place, that the resistance should be as the square of the velocity: for the momentum lost by a body moving in a fluid is, by the equality of action and reaction, the same as the momentum communicated to the fluid. Now, the momentum communicated will be as the product of the quantity of fluid-matter moved, and the velocity communicated to it. But the quantity moved will be nearly as the space through which the velocity moves in a given time; and the velocity communicated will be nearly that of the body itself. Hence, the momentum, which is in the compound ratio of these, will be as the square of the velocity nearly. And this is confirmed by experiment.

The height fallen through to acquire the velocity, is as the square of the velocity; hence, the resistance is as that height. Also, it is *cæteris paribus* as the surface resisted, and as the density of the fluid. Hence, it is as the surface, the height due to the velocity, and the density; and therefore it is *as* the column of fluid, whose base is the plane resisted, and its height that due to the velocity.

Though this reasoning is far from being exact and conclusive, experiment confirms the result of it; and shews that the resistance or moving force retarding the body, is *equal* to the weight of a column of the fluid, whose base is the surface resisted, and its height the height due to the velocity.

Let A be the area of the plane which moves perpendicularly to itself, V its velocity, D the density of the fluid; then the mass of the column spoken of is $A \cdot \frac{V^2}{2g} \cdot D$; and the moving force of resistance, which is equal to the weight of this column, is $\frac{1}{2} A V^2 D$.

If the moving body be a cylinder, in which b is the radius of the base, a the length, and Δ the density; we shall have $A = \pi h^2$, and $\frac{1}{2} \pi b^2 V^2 D =$ the moving force of resistance. Also, the mass of the body is $\pi b^2 a \Delta$: and hence, the *accelerating* force, (or properly the retarding force) of resistance, is $\frac{DV^2}{2\Delta a}$; and the quantity which corresponds to k in the text, is $\frac{D}{2\Delta a}$.

The moving force of the resistance upon a globe is $\frac{1}{2}$ the resistance upon one of its great circles, moving perpendicularly to itself. This appears by resolving the force on each particle of the spherical surface, in a direction perpendicular to the surface, then resolving this part again in the direction of the motion, and taking the sum of all the forces so resolved. It appears by experiment, that the resistance on the globe is a little greater, in proportion to the resistance on a circle, than this theory gives it.

Taking for granted the truth of the theory, the moving force of resistance on a globe of radius b, is $\frac{1}{4}\pi b^2 V^2 D$. And the mass of the globe is $\frac{4}{3}\pi b^3 \Delta$. Hence, the accelerating force is $\frac{3DV^2}{16\Delta b}$: and $\frac{3D}{16\Delta b}$ corresponds to k in the text.

The force by which a body descends in a fluid, is the excess of its weight above the weight of an equal bulk of fluid. This appears by the laws of Hydrostatics. If the body be a sphere, $\frac{4}{3}\pi b^3 \Delta g$ is its weight, and $\frac{4}{3}\pi b^3 D g$ the weight of an equal bulk of the fluid, and hence, the body descends with a moving force $\frac{4}{3}\pi b^3 (\Delta - D) g$, or an accelerating force $\frac{\Delta - D}{\Delta} g$. And if D be greater than Δ, the accelerating force with which the body ascends will be $\frac{D - \Delta}{\Delta} g$.

PROP. I. *To find the descent of very small bodies in fluids.*

In p. 179, we have the equation

$$s = \frac{V^2}{2g} \text{ hyp. log. } \frac{V^2}{V^2 - v^2};$$

where $V^2 = \frac{g}{k}$, V being the terminal velocity.

$$\therefore\ 2ks = -\text{hyp. log.}\left(1 - \frac{v^2}{V^2}\right),$$

$$\frac{v^2}{V^2} = 1 - \epsilon^{-2ks} = 1 - \epsilon^{-\frac{3\,Ds}{8\,\Delta\,b}}.$$

Now, if b be small, and s considerable, D not being very small compared with Δ, the quantity $\epsilon^{-\frac{3\,Ds}{16\,\Delta\,b}}$ will be very small, and $\frac{v^2}{V^2} = 1$, very nearly. Hence, after the body has descended through a certain space, it will have acquired very nearly the terminal velocity, and may after that be considered as going on with that uniform velocity.

Thus, let $\Delta = 2\,D$, $s = 32\,b$, and we have

$$\frac{v^2}{V^2} = 1 - \epsilon^{-6}.$$

Now $\epsilon^{-6} = \frac{1}{403}$, nearly. Hence, v^2 is within $\frac{1}{400}$ of V^2, and v within $\frac{1}{800}$ of V. Therefore while a sphere of twice the density of the fluid, descends through 16 diameters, it acquires a velocity within $\frac{1}{800}$ of the terminal velocity, and may after that be considered as moving uniformly.

The terminal velocity

$$V = \sqrt{\frac{g}{k}} = \sqrt{\frac{16\,\Delta\,bg}{3\,D}}.$$

And the height due to it is $\frac{8\,\Delta\,b}{3\,D}$.

Hence, if small spherical particles be dispersed in a fluid, they will, as to sense, descend with uniform velocities, for the space at first during which a sensible acceleration takes place, may be neglected. And if the particles be very small, the velocities will be very small. Thus, if they be $\frac{1}{1000}$ of an inch in diameter, and $\Delta = 2D$, we shall have $b = \frac{1}{6000}$ of a foot: $V = .24$ feet per second.

Prob. II. *To determine the ascent of an air-bubble.*

In this case the body will ascend by a force $\frac{D-\Delta}{\Delta} g$ where Δ is exceedingly small, being the density of atmospheric air. Also, in consequence of the elasticity of the air, the spherule will vary its dimensions as it ascends, and as the pressure upon it diminishes. The density of air is directly as the pressure. Let AC, fig. 144, be the depth of water which is equivalent to the pressure of the atmosphere on the surface of the fluid, CO the depth of the spherule below the surface of the water; then, the pressure on the spherule is equivalent to a column of fluid of the height AO. The density of the spherule is as AO, and therefore, as the quantity of air in it is given, its magnitude is inversely as AO. Let B be the original place of the spherule, and its radius at $B = b$; its radius at $O = y$, $AB = a$. Then, since the magnitude is as the cube of the radius,

$$y^3 : b^3 :: a : x, \quad y = \frac{b a^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

Also, the density is as the pressure, and therefore, if c be the density at B, $\Delta = \frac{cx}{a}$.

Now, the force upwards is

$$\frac{D-\Delta}{\Delta} g - \frac{3Dv^2}{16\Delta y}$$

$$= \left(\frac{Da}{cx} - 1\right) g - \frac{3Da^{\frac{2}{3}}}{16bcx^{\frac{2}{3}}} . v^2.$$

Let $\frac{c}{D} = m$, a very small quantity; and $\frac{3a^{\frac{2}{3}}}{16b} = n$; then, we shall have

$$v\,dv = -\left(\frac{a}{mx} - 1\right) g\,dx + \frac{nv^2}{mx^{\frac{2}{3}}}.\,dx;$$

$$2\,v\,dv - \frac{2\,nv^2\,dx}{m\,x^{\frac{2}{3}}} = -2\left(\frac{a}{mx} - 1\right) g\,dx:$$

of which the first side becomes integrable, when multiplied by $\epsilon^{-\frac{6nx^{\frac{1}{3}}}{m}}$. Integrating, we have

$$v^2\,\epsilon^{-\frac{6nx^{\frac{1}{3}}}{m}} = -2g\int \epsilon^{-\frac{6nx^{\frac{1}{3}}}{m}} \left(\frac{a}{mx} - 1\right) dx.$$

If we make $x^{\frac{1}{3}} = z$, the integral on the right hand side becomes

$$\frac{3\,a}{m}\int \frac{dz.\,\epsilon^{-\frac{6nz}{m}}}{z} - 3\int dz.\,z^2\,\epsilon^{-\frac{6nz}{m}}.$$

The latter integral will evidently have each term multiplied by m, and may be rejected, because m is very small. The former term gives

$$-\frac{3\,a}{m}\left\{\frac{m}{6\,n}.\frac{\epsilon^{-\frac{6nz}{m}}}{z} - \frac{m}{6\,n}\int \frac{dz\,\epsilon^{-\frac{6nz}{m}}}{z^2}\right\}$$

$$-\frac{a}{2n}\,\frac{\epsilon^{-\frac{6nz}{m}}}{z} + \frac{a}{2\,n}\int \frac{dz\,\epsilon^{-\frac{6nz}{m}}}{x^2}.$$

And all the following terms in the integral will be multiplied by m, and may be rejected. Hence

$$v^2\,\epsilon^{-\frac{6nx^{\frac{1}{3}}}{m}} = C + \frac{g\,a}{n}\,\frac{\epsilon^{-\frac{6nx^{\frac{1}{3}}}{m}}}{x^{\frac{1}{3}}}$$

$$v^2 = \frac{g\,a}{nx^{\frac{1}{3}}} + C\,\epsilon^{-\frac{6nx^{\frac{1}{3}}}{m}}$$

$$= \frac{16\,b\,g\,a^{\frac{1}{3}}}{3\,x^{\frac{1}{3}}} + C\,\epsilon^{\frac{ga^{\frac{2}{3}}x^{\frac{1}{3}}}{8bm}},$$

When $x = a$, $v = 0$. Hence, $0 = \frac{16bg}{3} + C\epsilon^{\frac{ga}{8bm}}$,

$$v^2 = \frac{16bg}{3}\left\{\frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}} - \epsilon^{-\frac{ga^{\frac{2}{3}}}{8bm}(a^{\frac{1}{3}} - x^{\frac{1}{3}})}\right\}.$$

When x is very nearly equal to a, that is, when the bubble is near to the point B, the latter term is considerable. But when $a^{\frac{1}{3}} - x^{\frac{1}{3}}$ is not very small, this term becomes inconsiderable, because m is very small, and therefore the index a large negative quantity. Hence, after the early part of the motion, we have

$$v^2 = \frac{16\,b\,g}{3} \cdot \frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}},\quad v = \frac{a^{\frac{1}{6}}}{x^{\frac{1}{6}}} \cdot \sqrt{\frac{16\,b\,g}{3}}.$$

Appendix (H).

General Mechanical Principles.

There are several properties of the motion of a system of bodies, some of which Authors have attempted to establish by peculiar reasonings, and to found upon them different branches of Mechanics. Properly speaking, however, they are consequences of the elementary laws of motion; and as they are remarkable either in themselves, or for the facilities they offer in the solution of problems, we shall here give the most important of such properties.

I. *Principle of the conservation of the motion of the centre of gravity.*

The centre of gravity of a system of bodies moving freely, moves in the same manner as if all the masses were collected in the centre of gravity, and all the forces which act upon the system were transferred to that point, retaining their magnitude and direction.

This is proved in the text, Art. 172, for the reasoning there will be the same, whether the particles m, &c. be connected or not.

Hence, if the particles of a system be acted upon only by their mutual forces, the centre of gravity will either remain at rest, or move uniformly in a straight line.

For if m act on m', m' re-acts upon m with an equal force; and these two forces transferred to the centre of gravity, would destroy each other. Therefore the centre of gravity, acted upon only by such pairs of forces, would be affected as if it were not acted upon by any forces. Hence, it will either be at rest, or move uniformly in a straight line.

It follows from this, that the *momentum* estimated in a given direction, is always the same in a system acted on only by the mutual forces of the parts. For if x, x', x'', &c. be the distances of the particles m, m', m'', &c. measured from a fixed plane in the given direction, $\bar{x}$ the distance of the centre of gravity, we have

$$\bar{x} = \frac{mx + m'x' + m''x'' + \&c.}{m + m' + m'' + \&c.},$$

$$(m + m' + m'' + \&c.)\,\frac{d\bar{x}}{dt} = m\,\frac{dx}{dt} + m'\,\frac{dx'}{dt} + m''\,\frac{dx''}{dt} + \&c.$$

And $\dfrac{dx}{dt}$, &c. are the velocities. Hence, the sum of all the momenta estimated in the given direction is equal to the momentum of the whole system collected in the centre of gravity. And the latter momentum is constant; therefore the former sum is so.

This may be called the *Principle of the conservation of momentum*. What is said of mutual actions, refers either to pressures, attractions, or impacts; and the latter, either supposing the bodies elastic or inelastic.

II. *Principle of the conservation of areas.*

If a system of bodies moving freely is acted upon only by their mutual forces, the sum of each particle multiplied into the projections of the areas described about a fixed point is proportional to the time.

Let x, y be the co-ordinates of a particle m, from the fixed point; and since the mutual action of the bodies will, in all cases, produce equal and opposite moving forces, the moment of the impressed forces will be 0. And taking the moment of the effective forces as in Art. 129, we shall have

$$\Sigma . m . \frac{x d^2 y - y d^2 x}{dt^2} = 0; \text{ and similarly,}$$

$$\Sigma . m . \frac{z d^2 x - x d^2 z}{dt^2} = 0,$$

$$\Sigma . m . \frac{y d^2 z - z d^2 y}{dt^2} = 0.$$

If α be the projection of the area described by m on the plane of xy, we have

$$\frac{d\alpha}{dt} = \frac{x dy - y dx}{2dt}; \quad \frac{d^2\alpha}{dt^2} = \frac{x d^2 y - y d^2 x}{2dt^2}.$$

And if α', α'', &c. be the same quantity for m', m'', &c., we have

$$\Sigma m \frac{d^2\alpha}{dt^2} = 0, \quad \Sigma m \frac{d\alpha}{dt} = c, \text{ a constant quantity};$$

$$\therefore \Sigma m\alpha = ct; \text{ or } m\alpha + m'\alpha' + m''\alpha'' + \text{\&c.} = ct.$$

Similarly, if β, β', β'', &c. be the projections of the areas described by m, m', m'', &c., on the planes of xz; $\gamma, \gamma', \gamma''$, &c. the projections of the same areas on the plane of yz,

$$m\beta + m'\beta' + m''\beta'' + \text{\&c.} = c't,$$

$$m\gamma + m'\gamma' + m''\gamma'' + \text{\&c.} = c''t.$$

If, besides the mutual actions of the bodies, they be acted on by forces tending to the origin of co-ordinates, the same proposition will manifestly be true.

In the same manner it would appear, that if instead of the fixed point, we take the centre of gravity, the same properties will be true.

Let a line be drawn, making with the axis of z, of y, and of x, angles of which the cosines are

$$\frac{c}{\sqrt{(c^2 + c'^2 + c''^2)}}, \quad \frac{c'}{\sqrt{(c^2 + c'^2 + c''^2)}}, \quad \frac{c''}{\sqrt{(c^2 + c'^2 + c''^2)}},$$

the position of this line will be the same at all points of time*; and a plane perpendicular to it is called the *Invariable Plane*. It may always be found, when the motions of the particles are given; for we have

$$m\frac{d\alpha}{dt} + m'\frac{d\alpha'}{dt} + \&c. = c,$$

$$m\frac{d\beta}{dt} + m'\frac{d\beta'}{dt} + \&c. = c',$$

$$m\frac{d\gamma}{dt} + m'\frac{d\gamma'}{dt} + \&c. = c''.$$

In the same manner we may find a plane of invariable position passing through the centre of gravity of the system. Thus a plane passing through the centre of gravity of the solar system, and determined by these formulæ, will, during all the motions which the different bodies undergo, retain the same position.

III. *Principle of the conservation of vis viva.*

The *vis viva* of a body is the product of its mass into the square of its velocity †. The sum of the vires vivæ in any system is

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* This plane will be that on which the sum of the projections of the areas multiplied respectively into the particles which describe them, is the greatest possible. Upon planes perpendicular to it, this sum is 0. Also, if A, A', A'', &c. be these projections for m, m', m'', &c., and $mA + m'A' + m''A'' + \&c. = \Sigma . mA$, we have

$$(\Sigma . mA)^2 = (\Sigma . m\alpha)^2 + (\Sigma . m\beta)^2 + (\Sigma . m\gamma)^2.$$

See Poisson, *Traité de Mec.* No. 84. Laplace, *Mec. Cel.* Liv. I. No. 21.

It is the same plane which is called the Principal Plane of Moments in Art. 127.

† The *force* or *quantity of motion* of a body is generally understood to mean the product of the mass into the velocity, and is the same as the *momentum*. The conservation of the quantity of force thus measured has been proved in proving the conservation of the motion of the centre of gravity. But if the force of a body in motion be measured by the whole effect which it will produce before the velocity is destroyed, or by the whole effort which

has

the same as if its particles, being separate had been acted upon by the same forces through the same spaces.

Let P be the force which acts on the particle m, p its distance from a fixed point in the direction of the force; therefore $\frac{dp}{dt}$ is m's velocity in the direction of the force. Also, if q be the distance of the body from a fixed point in the direction of its motion, $\frac{dq}{dt}$ is its velocity, and $\frac{d^2q}{dt^2}$, the effective accelerating force. And if we make similar suppositions with respect to m', m'', &c. we may consider the velocities of m, m', m'', &c. as virtual velocities, since they are consistent with the connexion of the system; and we have

impressed forces, Pm, $P'm'$, $P''m''$, &c.

with virtual velocities $\frac{dp}{dt}$, $\frac{dp'}{dt}$, $\frac{dp''}{dt}$, &c.

effective forces $\frac{m\,d^2q}{dt^2}$, $\frac{m'\,d^2q'}{dt^2}$, $\frac{m''d^2q''}{dt^2}$, &c.

with virtual velocities $\frac{dq}{dt}$, $\frac{dq'}{dt}$, $\frac{dq''}{dt}$, &c.

Hence, since these forces must balance each other, we have by the principle of virtual velocities, (see *Statics*, Art. 105.),

$$mP\,.\frac{dp}{dt} + m'P'.\frac{dp'}{dt} + \text{\&c.} = m\,.\,\frac{dq\,d^2q}{dt^3} + m'\frac{dq'd^2q'}{dt^3} + \text{\&c.}$$

---

has been exercised in generating it, without regard to the time, it must be measured by the mass into the square of the velocity. Thus balls of the same size projected into a resisting substance, as a bed of clay, will go to the same depth so long as their weights into the squares of their velocities are the same. Force thus measured is called *vis viva*, in opposition to force measured by momentum, which is proportional to the pressure, or *dead pull*, producing it. And it appears from the text, that forces will always produce a certain quantity of *vis viva* by acting through a given space, whatever be the manner in which the bodies are constrained to move.

$$\text{or, } \Sigma m \cdot \frac{dq\,d^2q}{dt^2} = \Sigma m P\,dp \ldots\ldots\ldots\ldots (1).$$

And multiplying by 2 and integrating, and putting v for $\frac{dq}{dt}$,

$$\Sigma \cdot mv^2 = C + 2\Sigma \cdot m \int P\,dp \ldots\ldots\ldots (2).$$

Now $2\int P\,dp$ is the square of the velocity which the force P would have generated in a point separated from the rest; hence, the integral being taken between the same limits, the vis viva is the same as it would have been in that case.

Let x, y, z, be the co-ordinates of m, and we shall have the square of the velocity, or

$$v^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2}$$

Also, if X, Y, Z, be the resolved parts of P, parallel respectively to x, y, z, and if α, β, γ, be the co-ordinates of a fixed point O, in the direction in which this force acts, so that $Om = p$, we shall have

$$X = P \cdot \frac{x-\alpha}{p}, \quad Y = P \cdot \frac{y-\beta}{p}, \quad Z = P \cdot \frac{z-\gamma}{p}.$$

$$\text{And } p^2 = (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2;$$

$$\therefore p\,dp = (x-\alpha)\,dx + (y-\beta)\,dy + (z-\gamma)\,dx.$$

Hence,

$$Xdx + Ydy + Zdz = P \cdot \frac{(x-\alpha)\,dx + (y-\beta)\,dy + (z-\gamma)\,dz}{p}$$

$$= P\,dp.$$

By substituting these values, the equation before obtained becomes

$$\Sigma \cdot m \cdot \frac{dx^2 + dy^2 + dz^2}{dt^2} = C + 2\Sigma \cdot m \int (Xdx + Ydy + Zdz).$$

If the system be acted upon by no forces, we shall have

$$\Sigma \cdot mv^2 = C.$$

the sum of each particle multiplied into the square of its velocity, will always be equal to a constant quantity.

If the system be acted on by gravity only, let x be vertical, and we shall have

$$\Sigma mv^2 = C - 2\Sigma \,.\, mgx = 2\Sigma \,.\, mg\,(h - x),$$

h being the height from which m must have fallen to acquire its velocity at any point.

This Proposition may be employed for the solution of a variety of Problems respecting bodies acted on by gravity. For instance, the greater part of the Examples in Chap. VI. Book I, might have been readily solved by means of it.

The mutual action of the parts of a system may increase or diminish its vis viva in any degree. Their attraction or repulsion may augment the velocities, and consequently, the vis viva. Their collision will generally diminish the vis viva, except they be perfectly elastic, in which case, after the impact, the vis viva will be the same as before.

As the vis viva varies, it may become a *maximum* or *minimum*. This happens when the system passes through a position of equilibrium.

When the vis viva is a *maximum*, the body passes a position of *stable* equilibrium, when it is a *minimum*, it passes a position of *unstable* equilibrium. See Poisson, *Traité de Mec.* No. 472, &c.

IV. *Principle of least action.*

The *action* of a particle is here measured by the product of the momentum, and of the space through which the body moves. If the velocity be perpetually varying, the action will be the integral of this product taken for a differential of the space. If m be the mass, v the velocity, and s the space, the whole action of the body is $\int mvds$, where the integral is to be taken between the beginning and end of the motion. And the principle to be proved is this:

When a system moves in any manner, the actual motion is such, that the sum of the *actions* of each particle for the whole motion is less than if the particles had taken any other paths between the same points.

That is, $\int \Sigma mvds$ is a minimum.

To prove this, we must, according to the Calculus of Variations, prove that the *variation* of this integral, between the proper limits, is 0. See Lacroix, *Elem. Treat.* Art. 331.

In the preceding proof of the principle of *vis viva*, we have taken the actual velocities $\frac{dp}{dt}$, &c., $\frac{dq}{dt}$, &c. for the virtual velocities. But if δp, &c., δq, &c., be any possible corresponding variations of p and q, they will be as the virtual velocities ; and we shall have, instead of equation (1), this equation,

$$\Sigma m \frac{d^2 q}{dt^2} \delta q = \Sigma m P \delta p \ldots\ldots\ldots (3).$$

Also, as in p. 399, it will appear, that

$$P\delta p = X\delta x + Y\delta y + Z\delta z.$$

And since $dq^2 = dx^2 + dy^2 + dz^2$, differentiating, &c.

$$\frac{d^2 q}{dt^2} \cdot dq = \frac{d^2 x}{dt^2} dx + \frac{d^2 y}{dt^2} dy + \frac{d^2 z}{dt^2} dz.$$

Hence, considering the first side as the differential of a function of q, and the second as the differential of a function of x, y, z, we shall have the variations in the same manner, or

$$\frac{d^2 q}{dt^2} \delta q = \frac{d^2 x}{dt^2} \delta x + \frac{d^2 y}{dt^2} \delta y + \frac{d^2 z}{dt^2} \delta z.$$

Hence, equation (3) becomes

$$\Sigma m \left\{ \frac{d^2 x}{dt^2} \delta x + \frac{d^2 y}{dt^2} \delta y + \frac{d^2 z}{dt^2} dz \right\} = \Sigma m P \delta p.$$

$$\text{Now } \delta \int \Sigma mvds = \int \Sigma m \delta vds ;$$

$$\text{and } \Sigma m \delta vds = \Sigma mv\delta ds + \Sigma mds\delta v.$$

$$\text{And } ds = vdt ; \ \therefore\ ds\delta v = dt \cdot v\delta v = \frac{dt}{2} \cdot \delta \cdot v^2 ;$$

$$\therefore \Sigma mds\delta v = \frac{dt}{2} \Sigma m \delta \cdot v^2.$$

$$\text{But } \Sigma m v^2 = C + 2\Sigma . m \int P\, dp;$$

$$\therefore \Sigma m \delta . v^2 = 2\Sigma . m P \delta p$$

$$= 2\Sigma . m \left\{ \frac{d^2 x}{dt^2} \delta x + \frac{d^2 y}{dt^2} \delta y + \frac{d^2 z}{dt^2} \delta z \right\}.$$

$$\text{Also } ds^2 = dx^2 + dy^2 + dz^2;$$

$$\therefore \frac{ds}{dt} \delta ds = \frac{dx}{dt} \delta dx + \frac{dy}{dt} \delta dy + \frac{dz}{dt} \delta dz,$$

$$\text{or } v \delta ds = \frac{dx}{dt} \delta dx + \frac{dy}{dt} \delta dy + \frac{dz}{dt} \delta dz.$$

Hence, transposing δd, and adding; $\Sigma m ds \delta v + \Sigma m v \delta ds$, or

$$\delta \Sigma m v ds = \Sigma . m \begin{cases} \dfrac{dx}{dt} d\delta x + \dfrac{dy}{dt} d\delta y + \dfrac{dz}{dt} d\delta z \\ \dfrac{d^2 x}{dt} \delta x + \dfrac{d^2 y}{dt} \delta y + \dfrac{d^2 z}{dt} \delta z \end{cases}$$

$$= \Sigma . m d \left\{ \frac{dx}{dt} \delta x + \frac{dy}{dt} \delta y + \frac{dz}{dt} \delta z \right\}$$

$$= d \Sigma . m \left\{ \frac{dx}{dt} \delta x + \frac{dy}{dt} \delta y + \frac{dz}{dt} \delta z \right\};$$

$$\therefore \delta \int \Sigma m v ds = \int \delta \Sigma m v ds$$

$$= \Sigma . m \left\{ \frac{dx}{dt} \delta x + \frac{dy}{dt} \delta y + \frac{dz}{dt} \delta z \right\}.$$

Now the right hand side is to be taken for the limits of the motion of the points m, m', &c.; and these limits being fixed, δx, δy, δz, are each 0.

Hence, $\delta \int \Sigma m v ds = 0$, and $\int \Sigma m v ds$ is a *minimum.*

It follows from this, that the sum of the *vis viva* of a system in passing from one position to another, is a minimum. For $\int \Sigma m v ds = \int \Sigma m v^2 dt$. Hence, this sum of the *vis viva* with respect to the time is less than if the system had moved in any other manner to the same position.

When the particles are not acted upon by any accelerating force, the *vis viva* is constant. Hence, the sum of the *vis viva* for any time is proportional to the time. And hence, in passing from one position to another, the time actually employed is the least possible.

The principle of least action was announced by Maupertuis as a fundamental law of Mechanics, and attempted to be proved *a priori*. It appears from what has been said, that it is a consequence of the elementary laws of motion.