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A HISTORY OF MATHEMATICS


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A HISTORY OF

MATHEMATICS

BY

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"I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history." — J. W. L. GLAISHER

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PREFACE.

An increased interest in the history of the exact sciences manifested in recent years by teachers everywhere, and the attention given to historical inquiry in the mathematical class-rooms and seminaries of our leading universities, cause me to believe that a brief general History of Mathematics will be found acceptable to teachers and students.

The pages treating — necessarily in a very condensed form — of the progress made during the present century, are put forth with great diffidence, although I have spent much time in the effort to render them accurate and reasonably complete. Many valuable suggestions and criticisms on the chapter on "Recent Times" have been made by Dr. E. W. Davis, of the University of Nebraska. The proof-sheets of this chapter have also been submitted to Dr. J. E. Davies and Professor C. A. Van Velzer, both of the University of Wisconsin; to Dr. G. B. Halsted, of the University of Texas; Professor L. M. Hoskins, of the Leland Stanford Jr. University; and Professor G. D. Olds, of Amherst College, — all of whom have afforded valuable assistance. I am specially indebted to Professor F. H. Loud, of Colorado College, who has read the proof-sheets throughout. To all the gentlemen above named, as well as to Dr. Carlo Veneziani

of Salt Lake City, who read the first part of my work in manuscript, I desire to express my hearty thanks. But in acknowledging their kindness, I trust that I shall not seem to lay upon them any share in the responsibility for errors which I may have introduced in subsequent revision of the text.

FLORIAN CAJORI.

COLORADO COLLEGE, December, 1893.

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A HISTORY OF MATHEMATICS.

INTRODUCTION.

THE contemplation of the various steps by which mankind has come into possession of the vast stock of mathematical knowledge can hardly fail to interest the mathematician. He takes pride in the fact that his science, more than any other, is an *exact* science, and that hardly anything ever done in mathematics has proved to be useless. The chemist smiles at the childish efforts of alchemists, but the mathematician finds the geometry of the Greeks and the arithmetic of the Hindoos as useful and admirable as any research of to-day. He is pleased to notice that though, in course of its development, mathematics has had periods of slow growth, yet in the main it has been pre-eminently a *progressive* science.

The history of mathematics may be instructive as well as agreeable; it may not only remind us of what we have, but may also teach us how to increase our store. Says De Morgan, "The early history of the mind of men with regard to mathematics leads us to point out our own errors; and in this respect it is well to pay attention to the history of mathematics." It warns us against hasty conclusions; it points out the importance of a good notation upon the progress of the science; it discourages excessive specialisation on the part of

investigators, by showing how apparently distinct branches have been found to possess unexpected connecting links; it saves the student from wasting time and energy upon problems which were, perhaps, solved long since; it discourages him from attacking an unsolved problem by the same method which has led other mathematicians to failure; it teaches that fortifications can be taken in other ways than by direct attack, that when repulsed from a direct assault it is well to reconnoitre and occupy the surrounding ground and to discover the secret paths by which the apparently unconquerable position can be taken.[1] The importance of this strategic rule may be emphasised by citing a case in which it has been violated. An untold amount of intellectual energy has been expended on the quadrature of the circle, yet no conquest has been made by direct assault. The circle-squarers have existed in crowds ever since the period of Archimedes. After innumerable failures to solve the problem at a time, even, when investigators possessed that most powerful tool, the differential calculus, persons versed in mathematics dropped the subject, while those who still persisted were completely ignorant of its history and generally misunderstood the conditions of the problem. "Our problem," says De Morgan, "is to square the circle with the *old allowance of means:* Euclid's postulates and nothing more. We cannot remember an instance in which a question to be solved by a *definite method* was tried by the best heads, and answered at last, *by that method*, after thousands of complete failures." But progress was made on this problem by approaching it from a different direction and by newly discovered paths. Lambert proved in 1761 that the ratio of the circumference of a circle to its diameter is incommensurable. Some years ago, Lindemann demonstrated that this ratio is also transcendental and that the quadrature of the circle, by means of the ruler and compass only, is *impos-*

sible. He thus showed by actual proof that which keen-minded mathematicians had long suspected; namely, that the great army of circle-squarers have, for two thousand years, been assaulting a fortification which is as indestructible as the firmament of heaven.

Another reason for the desirability of historical study is the value of historical knowledge to the teacher of mathematics. The interest which pupils take in their studies may be greatly increased if the solution of problems and the cold logic of geometrical demonstrations are interspersed with historical remarks and anecdotes. A class in arithmetic will be pleased to hear about the Hindoos and their invention of the "Arabic notation"; they will marvel at the thousands of years which elapsed before people had even thought of introducing into the numeral notation that Columbus-egg — the zero; they will find it astounding that it should have taken so long to *invent* a notation which they themselves can now *learn* in a month. After the pupils have learned how to bisect a given angle, surprise them by telling of the many futile attempts which have been made to solve, by elementary geometry, the apparently very simple problem of the trisection of an angle. When they know how to construct a square whose area is double the area of a given square, tell them about the duplication of the cube — how the wrath of Apollo could be appeased only by the construction of a cubical altar double the given altar, and how mathematicians long wrestled with this problem. After the class have exhausted their energies on the theorem of the right triangle, tell them the legend about its discoverer — how Pythagoras, jubilant over his great accomplishment, sacrificed a hecatomb to the Muses who inspired him. When the value of mathematical training is called in question, quote the inscription over the entrance into the academy of Plato, the philosopher: "Let no one who is

unacquainted with geometry enter here." Students in analytical geometry should know something of Descartes, and, after taking up the differential and integral calculus, they should become familiar with the parts that Newton, Leibniz, and Lagrange played in creating that science. In his historical talk it is possible for the teacher to make it plain to the student that mathematics is not a dead science, but a living one in which steady progress is made.[2]

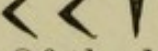
The history of mathematics is important also as a valuable contribution to the history of civilisation. Human progress is closely identified with scientific thought. Mathematical and physical researches are a reliable record of intellectual progress. The history of mathematics is one of the large windows through which the philosophic eye looks into past ages and traces the line of intellectual development.

ANTIQUITY.

THE BABYLONIANS.

The fertile valley of the Euphrates and Tigris was one of the primeval seats of human society. Authentic history of the peoples inhabiting this region begins only with the foundation, in Chaldæa and Babylonia, of a united kingdom out of the previously disunited tribes. Much light has been thrown on their history by the discovery of the art of reading the *cuneiform* or wedge-shaped system of writing.

In the study of Babylonian mathematics we begin with the notation of numbers. A vertical wedge  stood for 1, while the characters  and  signified 10 and 100 respectively. Grotefend believes the character for 10 originally to have been the picture of two hands, as held in prayer, the palms being pressed together, the fingers close to each other, but the thumbs thrust out. In the Babylonian notation two principles were employed — the additive and multiplicative. Numbers below 100 were expressed by symbols whose respective values had to be *added*. Thus,  stood for 2,  for 3,  for 4,  for 23,  for 30. Here the symbols of higher order appear always to the left of those of lower order. In writing the hundreds, on the other hand, a *smaller* symbol was placed to the left of the 100, and was, in that case, to be *multiplied* by 100. Thus,  signified

10 times 100, or 1000. But this symbol for 1000 was itself taken for a new unit, which could take smaller coefficients to its left. Thus,  denoted, not 20 times 100, but 10 times 1000. Of the largest numbers written in cuneiform symbols, which have hitherto been found, none go as high as a million.[3]

If, as is believed by most specialists, the early Sumerians were the inventors of the cuneiform writing, then they were, in all probability, also familiar with the notation of numbers. Most surprising, in this connection, is the fact that Sumerian inscriptions disclose the use, not only of the above *decimal* system, but also of a *sexagesimal* one. The latter was used chiefly in constructing tables for weights and measures. It is full of historical interest. Its consequential development, both for integers and fractions, reveals a high degree of mathematical insight. We possess two Babylonian tablets which exhibit its use. One of them, probably written between 2300 and 1600 B.C., contains a table of square numbers up to 60^2. The numbers 1, 4, 9, 16, 25, 36, 49, are given as the squares of the first seven integers respectively. We have next $1.4 = 8^2$, $1.21 = 9^2$, $1.40 = 10^2$, $2.1 = 11^2$, etc. This remains unintelligible, unless we assume the sexagesimal scale, which makes $1.4 = 60 + 4$, $1.21 = 60 + 21$, $2.1 = 2.60 + 1$. The second tablet records the magnitude of the illuminated portion of the moon's disc for every day from new to full moon, the whole disc being assumed to consist of 240 parts. The illuminated parts during the first five days are the series 5, 10, 20, 40, 1.20 (= 80), which is a geometrical progression. From here on the series becomes an arithmetical progression, the numbers from the fifth to the fifteenth day being respectively 1.20, 1.36, 1.52, 2.8, 2.24, 2.40, 2.56, 3.12, 3.28, 3.44, 4. This table not only exhibits the use of the sexagesimal system, but also indicates the acquaintance of the Babylonians with progressions.

Not to be overlooked is the fact that in the sexagesimal notation of integers the "principle of position" was employed. Thus, in 1.4 (= 64), the 1 is made to stand for 60, the unit of the second order, by virtue of its position with respect to the 4. The introduction of this principle at so early a date is the more remarkable, because in the decimal notation it was not introduced till about the fifth or sixth century after Christ. The principle of position, in its general and systematic application, requires a symbol for zero. We ask, Did the Babylonians possess one? Had they already taken the gigantic step of representing by a symbol the *absence* of units? Neither of the above tables answers this question, for they happen to contain no number in which there was occasion to use a zero. The sexagesimal system was used also in fractions. Thus, in the Babylonian inscriptions, $\frac{1}{2}$ and $\frac{1}{3}$ are designated by 30 and 20, the reader being expected, in his mind, to supply the word "sixtieths." The Greek geometer Hypsicles and the Alexandrian astronomer Ptolemæus borrowed the sexagesimal notation of fractions from the Babylonians and introduced it into Greece. From that time sexagesimal fractions held almost full sway in astronomical and mathematical calculations until the sixteenth century, when they finally yielded their place to the decimal fractions. It may be asked, What led to the invention of the sexagesimal system? Why was it that 60 parts were selected? To this we have no positive answer. *Ten* was chosen, in the decimal system, because it represents the number of fingers. But nothing of the human body could have suggested 60. Cantor offers the following theory: At first the Babylonians reckoned the year at 360 days. This led to the division of the circle into 360 degrees, each degree representing the daily amount of the supposed yearly revolution of the sun around the earth. Now they were, very probably, familiar with the

fact that the radius can be applied to its circumference as a chord 6 times, and that each of these chords subtends an arc measuring exactly 60 degrees. Fixing their attention upon these degrees, the division into 60 parts may have suggested itself to them. Thus, when greater precision necessitated a subdivision of the degree, it was partitioned into 60 minutes. In this way the sexagesimal notation may have originated. The division of the day into 24 hours, and of the hour into minutes and seconds on the scale of 60, is due to the Babylonians.

It appears that the people in the Tigro-Euphrates basin had made very creditable advance in arithmetic. Their knowledge of arithmetical and geometrical progressions has already been alluded to. Iamblichus attributes to them also a knowledge of proportion, and even the invention of the so-called *musical* proportion. Though we possess no conclusive proof, we have nevertheless reason to believe that in practical calculation they used the *abacus*. Among the races of middle Asia, even as far as China, the abacus is as old as fable. Now, Babylon was once a great commercial centre, — the metropolis of many nations, — and it is, therefore, not unreasonable to suppose that her merchants employed this most improved aid to calculation.

In geometry the Babylonians accomplished almost nothing. Besides the division of the circumference into 6 parts by its radius, and into 360 degrees, they had some knowledge of geometrical figures, such as the triangle and quadrangle, which they used in their auguries. Like the Hebrews (1 Kin. 7 : 23), they took $\pi = 3$. Of geometrical demonstrations there is, of course, no trace. "As a rule, in the Oriental mind the intuitive powers eclipse the severely rational and logical."

The astronomy of the Babylonians has attracted much attention. They worshipped the heavenly bodies from the earliest historic times. When Alexander the Great, after

the battle of Arbela (331 B.C.), took possession of Babylon, Callisthenes found there on burned brick astronomical records reaching back as far as 2234 B.C. Porphyrius says that these were sent to Aristotle. Ptolemy, the Alexandrian astronomer, possessed a Babylonian record of eclipses going back to 747 B.C. Recently Epping and Strassmaier[4] threw considerable light on Babylonian chronology and astronomy by explaining two calendars of the years 123 B.C. and 111 B.C., taken from cuneiform tablets coming, presumably, from an old observatory. These scholars have succeeded in giving an account of the Babylonian calculation of the new and full moon, and have identified by calculations the Babylonian names of the planets, and of the twelve zodiacal signs and twenty-eight normal stars which correspond to some extent with the twenty-eight *nakshatras* of the Hindoos. We append part of an Assyrian astronomical report, as translated by Oppert: —

"To the King, my lord, thy faithful servant, Mar-Istar."

". . . On the first day, as the new moon's day of the month Thammuz declined, the moon was again visible over the planet Mercury, as I had already predicted to my master the King. I erred not."

THE EGYPTIANS.

Though there is great difference of opinion regarding the antiquity of Egyptian civilisation, yet all authorities agree in the statement that, however far back they go, they find no uncivilised state of society. "Menes, the first king, changes the course of the Nile, makes a great reservoir, and builds the temple of Phthah at Memphis." The Egyptians built the pyramids at a very early period. Surely a people engaging in

enterprises of such magnitude must have known something of mathematics — at least of practical mathematics.

All Greek writers are unanimous in ascribing, without envy, to Egypt the priority of invention in the mathematical sciences. Plato in *Phædrus* says: "At the Egyptian city of Naucratis there was a famous old god whose name was Theuth; the bird which is called the Ibis was sacred to him, and he was the inventor of many arts, such as arithmetic and calculation and geometry and astronomy and draughts and dice, but his great discovery was the use of letters."

Aristotle says that mathematics had its birth in Egypt, because there the priestly class had the leisure needful for the study of it. Geometry, in particular, is said by Herodotus, Diodorus, Diogenes Laertius, Iamblichus, and other ancient writers to have originated in Egypt.[5] In Herodotus we find this (II. c. 109): "They said also that this king [Sesostris] divided the land among all Egyptians so as to give each one a quadrangle of equal size and to draw from each his revenues, by imposing a tax to be levied yearly. But every one from whose part the river tore away anything, had to go to him and notify what had happened; he then sent the overseers, who had to measure out by how much the land had become smaller, in order that the owner might pay on what was left, in proportion to the entire tax imposed. In this way, it appears to me, geometry originated, which passed thence to Hellas."

We abstain from introducing additional Greek opinion regarding Egyptian mathematics, or from indulging in wild conjectures. We rest our account on documentary evidence. A hieratic papyrus, included in the Rhind collection of the British Museum, was deciphered by Eisenlohr in 1877, and found to be a mathematical manual containing problems in arithmetic and geometry. It was written by **Ahmes** some

time before 1700 B.C., and was founded on an older work believed by Birch to date back as far as 3400 B.C.! This curious papyrus — the most ancient mathematical handbook known to us — puts us at once in contact with the mathematical thought in Egypt of three or five thousand years ago. It is entitled "Directions for obtaining the Knowledge of all Dark Things." We see from it that the Egyptians cared but little for theoretical results. Theorems are not found in it at all. It contains "hardly any general rules of procedure, but chiefly mere statements of results intended possibly to be explained by a teacher to his pupils."[6] In geometry the forte of the Egyptians lay in making constructions and determining areas. The area of an isosceles triangle, of which the sides measure 10 *ruths* and the base 4 *ruths*, was erroneously given as 20 square ruths, or half the product of the base by one side. The area of an isosceles trapezoid is found, similarly, by multiplying half the sum of the parallel sides by one of the non-parallel sides. The area of a circle is found by deducting from the diameter $\frac{1}{9}$ of its length and squaring the remainder. Here π is taken $= (\frac{16}{9})^2 = 3.1604\cdots$, a very fair approximation.[6] The papyrus explains also such problems as these, — To mark out in the field a right triangle whose sides are 10 and 4 units; or a trapezoid whose parallel sides are 6 and 4, and the non-parallel sides each 20 units.

Some problems in this papyrus seem to imply a rudimentary knowledge of proportion.

The base-lines of the pyramids run north and south, and east and west, but probably only the lines running north and south were determined by astronomical observations. This, coupled with the fact that the word *harpedonaptæ*, applied to Egyptian geometers, means "rope-stretchers," would point to the conclusion that the Egyptian, like the Indian and Chinese

geometers, constructed a right triangle upon a given line, by stretching around three pegs a rope consisting of three parts in the ratios 3 : 4 : 5, and thus forming a right triangle.[3] If this explanation is correct, then the Egyptians were familiar, 2000 years B.C., with the well-known property of the right triangle, for the special case at least when the sides are in the ratio 3 : 4 : 5.

On the walls of the celebrated temple of Horus at Edfu have been found hieroglyphics, written about 100 B.C., which enumerate the pieces of land owned by the priesthood, and give their areas. The area of any quadrilateral, however irregular, is there found by the formula $\frac{a+b}{2} \cdot \frac{c+d}{2}$. Thus, for a quadrangle whose opposite sides are 5 and 8, 20 and 15, is given the area $113\frac{1}{2}\frac{1}{4}$.[7] The incorrect formulæ of Ahmes of 3000 years B.C. yield generally closer approximations than those of the Edfu inscriptions, written 200 years after Euclid!

The fact that the geometry of the Egyptians consists chiefly of constructions, goes far to explain certain of its great defects. The Egyptians failed in two essential points without which a *science* of geometry, in the true sense of the word, cannot exist. In the first place, they failed to construct a rigorously logical system of geometry, resting upon a few axioms and postulates. A great many of their rules, especially those in solid geometry, had probably not been proved at all, but were known to be true merely from observation or as matters of fact. The second great defect was their inability to bring the numerous special cases under a more general view, and thereby to arrive at broader and more fundamental theorems. Some of the simplest geometrical truths were divided into numberless special cases of which each was supposed to require separate treatment.

Some particulars about Egyptian geometry can be mentioned more advantageously in connection with the early Greek mathematicians who came to the Egyptian priests for instruction.

An insight into Egyptian methods of numeration was obtained through the ingenious deciphering of the hieroglyphics by Champollion, Young, and their successors. The symbols used were the following:  for 1,  for 10,  for 100,  for 1000,  for 10,000,  for 100,000,  for 1,000,000,  for 10,000,000.[3] The symbol for 1 represents a vertical staff; that for 10,000 a pointing finger; that for 100,000 a burbot; that for 1,000,000, a man in astonishment. The significance of the remaining symbols is very doubtful. The writing of numbers with these hieroglyphics was very cumbrous. The unit symbol of each order was repeated as many times as there were units in that order. The principle employed was the *additive*. Thus, 23 was written 

Besides the hieroglyphics, Egypt possesses the *hieratic* and *demotic* writings, but for want of space we pass them by.

Herodotus makes an important statement concerning the mode of computing among the Egyptians. He says that they "*calculate with pebbles* by moving the hand from right to left, while the Hellenes move it from left to right." Herein we recognise again that *instrumental* method of figuring so extensively used by peoples of antiquity. The Egyptians used the decimal scale. Since, in figuring, they moved their hands horizontally, it seems probable that they used ciphering-boards with vertical columns. In each column there must have been not more than nine pebbles, for ten pebbles would be equal to one pebble in the column next to the left.

The *Ahmes papyrus* contains interesting information on the way in which the Egyptians employed fractions. Their

methods of operation were, of course, radically different from ours. Fractions were a subject of very great difficulty with the ancients. Simultaneous changes in both numerator and denominator were usually avoided. In manipulating fractions the Babylonians kept the denominators (60) constant. The Romans likewise kept them constant, but equal to 12. The Egyptians and Greeks, on the other hand, kept the numerators constant, and dealt with variable denominators. Ahmes used the term "fraction" in a restricted sense, for he applied it only to *unit-fractions*, or fractions having unity for the numerator. It was designated by writing the denominator and then placing over it a dot. Fractional values which could not be expressed by any one unit-fraction were expressed as the *sum* of two or more of them. Thus, he wrote $\frac{1}{3}\ \frac{1}{15}$ in place of $\frac{2}{5}$. The first important problem naturally arising was, how to represent any fractional value as the sum of unit-fractions. This was solved by aid of a table, given in the papyrus, in which all fractions of the form $\frac{2}{2n+1}$ (where n designates successively all the numbers up to 49) are reduced to the sum of unit-fractions. Thus, $\frac{2}{7} = \frac{1}{4}\ \frac{1}{28}$; $\frac{2}{99} = \frac{1}{66}\ \frac{1}{198}$. When, by whom, and how this table was calculated, we do not know. Probably it was compiled empirically at different times, by different persons. It will be seen that by repeated application of this table, a fraction whose numerator exceeds two can be expressed in the desired form, provided that there is a fraction in the table having the same denominator that *it* has. Take, for example, the problem, to divide 5 by 21. In the first place, $5 = 1 + 2 + 2$. From the table we get $\frac{2}{21} = \frac{1}{14}\ \frac{1}{42}$. Then $\frac{5}{21} = \frac{1}{21} + (\frac{1}{14}\ \frac{1}{42}) + (\frac{1}{14}\ \frac{1}{42}) = \frac{1}{21} + (\frac{2}{14}\ \frac{2}{42})$ $= \frac{1}{21}\ \frac{1}{7}\ \frac{1}{21} = \frac{1}{7}\ \frac{2}{21} = \frac{1}{7}\ \frac{1}{14}\ \frac{1}{42}$. The papyrus contains problems in which it is required that fractions be raised by addition or multiplication to given whole numbers or to other fractions.

For example, it is required to increase $\frac{1}{4}\ \frac{1}{8}\ \frac{1}{10}\ \frac{1}{30}\ \frac{1}{45}$ to 1. The common denominator taken appears to be 45, for the numbers are stated as $11\frac{1}{4}$, $5\frac{1}{2}\ \frac{1}{8}$, $4\frac{1}{2}$, $1\frac{1}{2}$, 1. The sum of these is $23\frac{1}{2}\ \frac{1}{4}\ \frac{1}{8}$ forty-fifths. Add to this $\frac{1}{9}\ \frac{1}{40}$, and the sum is $\frac{2}{3}$. Add $\frac{1}{3}$, and we have 1. Hence the quantity to be added to the given fraction is $\frac{1}{3}\ \frac{1}{9}\ \frac{1}{40}$.

Having finished the subject of fractions, Ahmes proceeds to the solution of equations of one unknown quantity. The unknown quantity is called 'hau' or heap. Thus the problem, "heap, its $\frac{1}{7}$, its whole, it makes 19," *i.e.* $\frac{x}{7} + x = 19$. In this case, the solution is as follows: $\frac{8x}{7} = 19$; $\frac{x}{7} = 2\frac{1}{4}\ \frac{1}{8}$; $x = 16\frac{1}{2}\ \frac{1}{8}$. But in other problems, the solutions are effected by various other methods. It thus appears that the beginnings of algebra are as ancient as those of geometry.

The principal defect of Egyptian arithmetic was the lack of a simple, comprehensive symbolism — a defect which not even the Greeks were able to remove.

The Ahmes papyrus doubtless represents the most advanced attainments of the Egyptians in arithmetic and geometry. It is remarkable that they should have reached so great proficiency in mathematics at so remote a period of antiquity. But strange, indeed, is the fact that, during the next two thousand years, they should have made no progress whatsoever in it. The conclusion forces itself upon us, that they resemble the Chinese in the *stationary character*, not only of their government, but also of their learning. All the knowledge of geometry which they possessed when Greek scholars visited them, six centuries B.C., was doubtless known to them two thousand years earlier, when they built those stupendous and gigantic structures — the pyramids. An explanation for this stagnation of learning has been sought in the fact that their early discoveries in mathematics and medicine had the misfortune of

being entered upon their sacred books and that, in after ages, it was considered heretical to augment or modify anything therein. Thus the books themselves closed the gates to progress.

THE GREEKS.

About the seventh century B.C. an active commercial intercourse sprang up between Greece and Egypt. Naturally there arose an interchange of ideas as well as of merchandise. Greeks, thirsting for knowledge, sought the Egyptian priests for instruction. Thales, Pythagoras, Œnopides, Plato, Democritus, Eudoxus, all visited the land of the pyramids. Egyptian ideas were thus transplanted across the sea and there stimulated Greek thought, directed it into new lines, and gave to it a basis to work upon. Greek culture, therefore, is not primitive. Not only in mathematics, but also in mythology and art, Hellas owes a debt to older countries. To Egypt Greece is indebted, among other things, for its elementary geometry. But this does not lessen our admiration for the Greek mind. From the moment that Hellenic philosophers applied themselves to the study of Egyptian geometry, this science assumed a radically different aspect. "Whatever we Greeks receive, we improve and perfect," says Plato. The Egyptians carried geometry no further than was absolutely necessary for their practical wants. The Greeks, on the other hand, had within them a strong speculative tendency. They felt a craving to discover the reasons for things. They found pleasure in the contemplation of *ideal* relations, and loved science *as* science.

Our sources of information on the history of Greek geometry before Euclid consist merely of scattered notices in ancient writers. The early mathematicians, Thales and Pythagoras,

left behind no written records of their discoveries. A full history of Greek geometry and astronomy during this period, written by Eudemus, a pupil of Aristotle, has been lost. It was well known to Proclus, who, in his commentaries on Euclid, gives a brief account of it. This abstract constitutes our most reliable information. We shall quote it frequently under the name of *Eudemian Summary*.

The Ionic School.

To **Thales** of Miletus (640–546 B.C.), one of the "seven wise men," and the founder of the Ionic school, falls the honour of having introduced the study of geometry into Greece. During middle life he engaged in commercial pursuits, which took him to Egypt. He is said to have resided there, and to have studied the physical sciences and mathematics with the Egyptian priests. Plutarch declares that Thales soon excelled his masters, and amazed King Amasis by measuring the heights of the pyramids from their shadows. According to Plutarch, this was done by considering that the shadow cast by a vertical staff of known length bears the same ratio to the shadow of the pyramid as the height of the staff bears to the height of the pyramid. This solution presupposes a knowledge of proportion, and the Ahmes papyrus actually shows that the rudiments of proportion were known to the Egyptians. According to Diogenes Laertius, the pyramids were measured by Thales in a different way; viz. by finding the length of the shadow of the pyramid at the moment when the shadow of a staff was equal to its own length.

The *Eudemian Summary* ascribes to Thales the invention of the theorems on the equality of vertical angles, the equality of the angles at the base of an isosceles triangle, the bisection of a circle by any diameter, and the congruence of two

triangles having a side and the two adjacent angles equal respectively. The last theorem he applied to the measurement of the distances of ships from the shore. Thus Thales was the first to apply theoretical geometry to practical uses. The theorem that all angles inscribed in a semicircle are right angles is attributed by some ancient writers to Thales, by others to Pythagoras. Thales was doubtless familiar with other theorems, not recorded by the ancients. It has been inferred that he knew the sum of the three angles of a triangle to be equal to two right angles, and the sides of equiangular triangles to be proportional.[8] The Egyptians must have made use of the above theorems on the straight line, in some of their constructions found in the Ahmes papyrus, but it was left for the Greek philosopher to give these truths, which others saw, but did not formulate into words, an explicit, abstract expression, and to put into scientific language and subject to proof that which others merely felt to be true. Thales may be said to have created the geometry of lines, essentially abstract in its character, while the Egyptians studied only the geometry of surfaces and the rudiments of solid geometry, empirical in their character.[8]

With Thales begins also the study of scientific astronomy. He acquired great celebrity by the prediction of a solar eclipse in 585 B.C. Whether he predicted the day of the occurrence, or simply the year, is not known. It is told of him that while contemplating the stars during an evening walk, he fell into a ditch. The good old woman attending him exclaimed, "How canst thou know what is doing in the heavens, when thou seest not what is at thy feet?"

The two most prominent pupils of Thales were **Anaximander** (b. 611 B.C.) and **Anaximenes** (b. 570 B.C.). They studied chiefly astronomy and physical philosophy. Of **Anaxagoras**, a pupil of Anaximenes, and the last philosopher of the Ionic

school, we know little, except that, while in prison, he passed his time attempting to square the circle. This is the first time, in the history of mathematics, that we find mention of the famous problem of the quadrature of the circle, that rock upon which so many reputations have been destroyed. It turns upon the determination of the exact value of π. Approximations to π had been made by the Chinese, Babylonians, Hebrews, and Egyptians. But the invention of a method to find its *exact* value, is the knotty problem which has engaged the attention of many minds from the time of Anaxagoras down to our own. Anaxagoras did not offer any solution of it, and seems to have luckily escaped paralogisms.

About the time of Anaxagoras, but isolated from the Ionic school, flourished **Œnopides** of Chios. Proclus ascribes to him the solution of the following problems: From a point without, to draw a perpendicular to a given line, and to draw an angle on a line equal to a given angle. That a man could gain a reputation by solving problems so elementary as these, indicates that geometry was still in its infancy, and that the Greeks had not yet gotten far beyond the Egyptian constructions.

The Ionic school lasted over one hundred years. The progress of mathematics during that period was slow, as compared with its growth in a later epoch of Greek history. A new impetus to its progress was given by Pythagoras.

The School of Pythagoras.

Pythagoras (580?–500? B.C.) was one of those figures which impressed the imagination of succeeding times to such an extent that their real histories have become difficult to be discerned through the mythical haze that envelops them. The following account of Pythagoras excludes the most doubtful

statements. He was a native of Samos, and was drawn by the fame of Pherecydes to the island of Syros. He then visited the ancient Thales, who incited him to study in Egypt. He sojourned in Egypt many years, and may have visited Babylon. On his return to Samos, he found it under the tyranny of Polycrates. Failing in an attempt to found a school there, he quitted home again and, following the current of civilisation, removed to Magna Græcia in South Italy. He settled at Croton, and founded the famous Pythagorean school. This was not merely an academy for the teaching of philosophy, mathematics, and natural science, but it was a brotherhood, the members of which were united for life. This brotherhood had observances approaching masonic peculiarity. They were forbidden to divulge the discoveries and doctrines of their school. Hence we are obliged to speak of the Pythagoreans as a body, and find it difficult to determine to whom each particular discovery is to be ascribed. The Pythagoreans themselves were in the habit of referring every discovery back to the great founder of the sect.

This school grew rapidly and gained considerable political ascendency. But the mystic and secret observances, introduced in imitation of Egyptian usages, and the aristocratic tendencies of the school, caused it to become an object of suspicion. The democratic party in Lower Italy revolted and destroyed the buildings of the Pythagorean school. Pythagoras fled to Tarentum and thence to Metapontum, where he was murdered.

Pythagoras has left behind no mathematical treatises, and our sources of information are rather scanty. Certain it is that, in the Pythagorean school, mathematics was the principal study. Pythagoras raised mathematics to the rank of a science. Arithmetic was courted by him as fervently as geometry. In fact, arithmetic is the foundation of his philosophic system.

The *Eudemian Summary* says that "Pythagoras changed the study of geometry into the form of a liberal education, for he examined its principles to the bottom, and investigated its theorems in an immaterial and intellectual manner." His geometry was connected closely with his arithmetic. He was especially fond of those geometrical relations which admitted of arithmetical expression.

Like Egyptian geometry, the geometry of the Pythagoreans is much concerned with areas. To Pythagoras is ascribed the important theorem that the square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides. He had probably learned from the Egyptians the truth of the theorem in the special case when the sides are 3, 4, 5, respectively. The story goes, that Pythagoras was so jubilant over this discovery that he sacrificed a hecatomb. Its authenticity is doubted, because the Pythagoreans believed in the transmigration of the soul and opposed, therefore, the shedding of blood. In the later traditions of the Neo-Pythagoreans this objection is removed by replacing this bloody sacrifice by that of "an ox made of flour"! The proof of the law of three squares, given in Euclid's *Elements*, I. 47, is due to Euclid himself, and not to the Pythagoreans. What the Pythagorean method of proof was has been a favourite topic for conjecture.

The theorem on the sum of the three angles of a triangle, presumably known to Thales, was proved by the Pythagoreans after the manner of Euclid. They demonstrated also that the plane about a point is completely filled by six equilateral triangles, four squares, or three regular hexagons, so that it is possible to divide up a plane into figures of either kind.

From the equilateral triangle and the square arise the solids, namely the tetraedron, octaedron, icosaedron, and the cube. These solids were, in all probability, known to the Egyptians,

excepting, perhaps, the icosaedron. In Pythagorean philosophy, they represent respectively the four elements of the physical world; namely, fire, air, water, and earth. Later another regular solid was discovered, namely the dodecaedron, which, in absence of a fifth element, was made to represent the universe itself. Iamblichus states that Hippasus, a Pythagorean, perished in the sea, because he boasted that he first divulged "the sphere with the twelve pentagons." The star-shaped pentagram was used as a symbol of recognition by the Pythagoreans, and was called by them Health.

Pythagoras called the sphere the most beautiful of all solids, and the circle the most beautiful of all plane figures. The treatment of the subjects of proportion and of irrational quantities by him and his school will be taken up under the head of arithmetic.

According to Eudemus, the Pythagoreans invented the problems concerning the application of areas, including the cases of defect and excess, as in Euclid, VI. 28, 29.

They were also familiar with the construction of a polygon equal in area to a given polygon and similar to another given polygon. This problem depends upon several important and somewhat advanced theorems, and testifies to the fact that the Pythagoreans made no mean progress in geometry.

Of the theorems generally ascribed to the Italian school, some cannot be attributed to Pythagoras himself, nor to his earliest successors. The progress from empirical to reasoned solutions must, of necessity, have been slow. It is worth noticing that on the circle no theorem of any importance was discovered by this school.

Though politics broke up the Pythagorean fraternity, yet the school continued to exist at least two centuries longer. Among the later Pythagoreans, Philolaus and Archytas are the most prominent. **Philolaus** wrote a book on the Pythago-

rean doctrines. By him were first given to the world the teachings of the Italian school, which had been kept secret for a whole century. The brilliant **Archytas** of Tarentum (428–347 B.C.), known as a great statesman and general, and universally admired for his virtues, was the only great geometer among the Greeks when Plato opened his school. Archytas was the first to apply geometry to mechanics and to treat the latter subject methodically. He also found a very ingenious mechanical solution to the problem of the duplication of the cube. His solution involves clear notions on the generation of cones and cylinders. This problem reduces itself to finding two mean proportionals between two given lines. These mean proportionals were obtained by Archytas from the section of a half-cylinder. The doctrine of proportion was advanced through him.

There is every reason to believe that the later Pythagoreans exercised a strong influence on the study and development of mathematics at Athens. The Sophists acquired geometry from Pythagorean sources. Plato bought the works of Philolaus, and had a warm friend in Archytas.

The Sophist School.

After the defeat of the Persians under Xerxes at the battle of Salamis, 480 B.C., a league was formed among the Greeks to preserve the freedom of the now liberated Greek cities on the islands and coast of the Ægæan Sea. Of this league Athens soon became leader and dictator. She caused the separate treasury of the league to be merged into that of Athens, and then spent the money of her allies for her own aggrandisement. Athens was also a great commercial centre. Thus she became the richest and most beautiful city of antiquity. All menial work was performed by slaves. The

citizen of Athens was well-to-do and enjoyed a large amount of leisure. The government being purely democratic, every citizen was a politician. To make his influence felt among his fellow-men he must, first of all, be educated. Thus there arose a demand for teachers. The supply came principally from Sicily, where Pythagorean doctrines had spread. These teachers were called *Sophists*, or "wise men." Unlike the Pythagoreans, they accepted pay for their teaching. Although rhetoric was the principal feature of their instruction, they also taught geometry, astronomy, and philosophy. Athens soon became the headquarters of Grecian men of letters, and of mathematicians in particular. The home of mathematics among the Greeks was first in the Ionian Islands, then in Lower Italy, and during the time now under consideration, at Athens.

The geometry of the circle, which had been entirely neglected by the Pythagoreans, was taken up by the Sophists. Nearly all their discoveries were made in connection with their innumerable attempts to solve the following three famous problems: —

(1) To trisect an arc or an angle.

(2) To "double the cube," *i.e.* to find a cube whose *volume* is double that of a given cube.

(3) To "square the circle," *i.e.* to find a square or some other rectilinear figure exactly equal in area to a given circle.

These problems have probably been the subject of more discussion and research than any other problems in mathematics. The bisection of an angle was one of the easiest problems in geometry. The trisection of an angle, on the other hand, presented unexpected difficulties. A right angle had been divided into three equal parts by the Pythagoreans. But the general problem, though easy in appearance, transcended the power of *elementary* geometry. Among the first

to wrestle with it was **Hippias of Elis**, a contemporary of Socrates, and born about 460 B.C. Like all the later geometers, he failed in effecting the trisection by means of a ruler and compass only. Proclus mentions a man, Hippias, presumably Hippias of Elis, as the inventor of a transcendental curve which served to divide an angle not only into three, but into any number of equal parts. This same curve was used later by Deinostratus and others for the quadrature of the circle. On this account it is called the *quadratrix*.

The Pythagoreans had shown that the diagonal of a square is the side of another square having double the area of the original one. This probably suggested the problem of the duplication of the cube, *i.e.* to find the edge of a cube having double the volume of a given cube. Eratosthenes ascribes to this problem a different origin. The Delians were once suffering from a pestilence and were ordered by the oracle to double a certain cubical altar. Thoughtless workmen simply constructed a cube with edges twice as long, but this did not pacify the gods. The error being discovered, Plato was consulted on the matter. He and his disciples searched eagerly for a solution to this "Delian Problem." **Hippocrates of Chios** (about 430 B.C.), a talented mathematician, but otherwise slow and stupid, was the first to show that the problem could be reduced to finding two mean proportionals between a given line and another twice as long. For, in the proportion $a : x = x : y = y : 2a$, since $x^2 = ay$ and $y^2 = 2ax$ and $x^4 = a^2y^2$, we have $x^4 = 2a^3x$ and $x^3 = 2a^3$. But he failed to find the two mean proportionals. His attempt to square the circle was also a failure; for though he made himself celebrated by squaring a lune, he committed an error in attempting to apply this result to the squaring of the circle.

In his study of the quadrature and duplication-problems, Hippocrates contributed much to the geometry of the circle.

The subject of similar figures was studied and partly developed by Hippocrates. This involved the theory of proportion. Proportion had, thus far, been used by the Greeks only in numbers. They never succeeded in uniting the notions of numbers and magnitudes. The term "number" was used by them in a restricted sense. What we call irrational numbers was not included under this notion. Not even rational fractions were called numbers. They used the word in the same sense as we use "integers." Hence numbers were conceived as *discontinuous*, while magnitudes were *continuous*. The two notions appeared, therefore, entirely distinct. The chasm between them is exposed to full view in the statement of Euclid that "incommensurable magnitudes do not have the same ratio as numbers." In Euclid's *Elements* we find the theory of proportion of magnitudes developed and treated independent of that of numbers. The transfer of the theory of proportion from numbers to magnitudes (and to lengths in particular) was a difficult and important step.

Hippocrates added to his fame by writing a geometrical text-book, called the *Elements*. This publication shows that the Pythagorean habit of secrecy was being abandoned; secrecy was contrary to the spirit of Athenian life.

The Sophist **Antiphon**, a contemporary of Hippocrates, introduced the *process* of exhaustion for the purpose of solving the problem of the quadrature. He did himself credit by remarking that by inscribing in a circle a square, and on its sides erecting isosceles triangles with their vertices in the circumference, and on the sides of these triangles erecting new triangles, etc., one could obtain a succession of regular polygons of 8, 16, 32, 64 sides, and so on, of which each approaches nearer to the circle than the previous one, until the circle is finally *exhausted*. Thus is obtained an inscribed

polygon whose sides coincide with the circumference. Since there can be found squares equal in area to any polygon, there also can be found a square equal to the last polygon inscribed, and therefore equal to the circle itself. **Bryson of Heraclea**, a contemporary of Antiphon, advanced the problem of the quadrature considerably by circumscribing polygons at the same time that he inscribed polygons. He erred, however, in assuming that the area of a circle was the arithmetical mean between circumscribed and inscribed polygons. Unlike Bryson and the rest of Greek geometers, Antiphon seems to have believed it possible, by continually doubling the sides of an inscribed polygon, to obtain a polygon coinciding with the circle. This question gave rise to lively disputes in Athens. If a polygon can coincide with the circle, then, says Simplicius, we must put aside the notion that magnitudes are divisible *ad infinitum*. Aristotle always supported the theory of the infinite divisibility, while Zeno, the Stoic, attempted to show its absurdity by proving that if magnitudes are infinitely divisible, motion is impossible. Zeno argues that Achilles could not overtake a tortoise; for while he hastened to the place where the tortoise had been when he started, the tortoise crept some distance ahead, and while Achilles reached that second spot, the tortoise again moved forward a little, and so on. Thus the tortoise was always in advance of Achilles. Such arguments greatly confounded Greek geometers. No wonder they were deterred by such paradoxes from introducing the idea of infinity into their geometry. It did not suit the rigour of their proofs.

The process of Antiphon and Bryson gave rise to the cumbrous but perfectly rigorous "method of exhaustion." In determining the ratio of the areas between two curvilinear plane figures, say two circles, geometers first inscribed or circumscribed similar polygons, and then by increasing indefi-

nitely the number of sides, nearly exhausted the spaces between the polygons and circumferences. From the theorem that similar polygons inscribed in circles are to each other as the squares on their diameters, geometers may have divined the theorem attributed to Hippocrates of Chios that the circles, which differ but little from the last drawn polygons, must be to each other as the squares on their diameters. But in order to exclude all vagueness and possibility of doubt, later Greek geometers applied reasoning like that in Euclid, XII. 2, as follows: Let C and c, D and d be respectively the circles and diameters in question. Then if the proportion $D^2 : d^2 = C : c$ is not true, suppose that $D^2 : d^2 = C : c'$. If $c' < c$, then a polygon p can be inscribed in the circle c which comes nearer to it in area than does c'. If P be the corresponding polygon in C, then $P : p = D^2 : d^2 = C : c'$, and $P : C = p : c'$. Since $p > c'$, we have $P > C$, which is absurd. Next they proved by this same method of *reductio ad absurdum* the falsity of the supposition that $c' > c$. Since c' can be neither larger nor smaller than c, it must be equal to it, Q.E.D. Hankel refers this Method of Exhaustion back to Hippocrates of Chios, but the reasons for assigning it to this early writer, rather than to Eudoxus, seem insufficient.

Though progress in geometry at this period is traceable only at Athens, yet Ionia, Sicily, Abdera in Thrace, and Cyrene produced mathematicians who made creditable contributions to the science. We can mention here only **Democritus of Abdera** (about 460–370 B.C.), a pupil of Anaxagoras, a friend of Philolaus, and an admirer of the Pythagoreans. He visited Egypt and perhaps even Persia. He was a successful geometer and wrote on incommensurable lines, on geometry, on numbers, and on perspective. None of these works are extant. He used to boast that in the construction of plane figures with proof no one had yet surpassed him, not even

the so-called harpedonaptæ ("rope-stretchers") of Egypt. By this assertion he pays a flattering compliment to the skill and ability of the Egyptians.

The Platonic School.

During the Peloponnesian War (431–404 B.C.) the progress of geometry was checked. After the war, Athens sank into the background as a minor political power, but advanced more and more to the front as the leader in philosophy, literature, and science. **Plato** was born at Athens in 429 B.C., the year of the great plague, and died in 348. He was a pupil and near friend of Socrates, but it was not from him that he acquired his taste for mathematics. After the death of Socrates, Plato travelled extensively. In Cyrene he studied mathematics under Theodorus. He went to Egypt, then to Lower Italy and Sicily, where he came in contact with the Pythagoreans. Archytas of Tarentum and Timæus of Locri became his intimate friends. On his return to Athens, about 389 B.C., he founded his school in the groves of the *Academia*, and devoted the remainder of his life to teaching and writing.

Plato's physical philosophy is partly based on that of the Pythagoreans. Like them, he sought in arithmetic and geometry the key to the universe. When questioned about the occupation of the Deity, Plato answered that "He geometrises continually." Accordingly, a knowledge of geometry is a necessary preparation for the study of philosophy. To show how great a value he put on mathematics and how necessary it is for higher speculation, Plato placed the inscription over his porch, "Let no one who is unacquainted with geometry enter here." Xenocrates, a successor of Plato as teacher in the Academy, followed in his master's footsteps, by declining to admit a pupil who had no mathematical training,

with the remark, "Depart, for thou hast not the grip of philosophy." Plato observed that geometry trained the mind for correct and vigorous thinking. Hence it was that the *Eudemian Summary* says, "He filled his writings with mathematical discoveries, and exhibited on every occasion the remarkable connection between mathematics and philosophy."

With Plato as the head-master, we need not wonder that the Platonic school produced so large a number of mathematicians. Plato did little real original work, but he made valuable improvements in the logic and methods employed in geometry. It is true that the Sophist geometers of the previous century were rigorous in their proofs, but as a rule they did not reflect on the inward nature of their methods. They used the axioms without giving them explicit expression, and the geometrical concepts, such as the point, line, surface, etc., without assigning to them formal definitions. The Pythagoreans called a point "unity in position," but this is a statement of a philosophical theory rather than a definition. Plato objected to calling a point a "geometrical fiction." He defined a point as the "beginning of a line" or as "an indivisible line," and a line as "length without breadth." He called the point, line, surface, the 'boundaries' of the line, surface, solid, respectively. Many of the definitions in Euclid are to be ascribed to the Platonic school. The same is probably true of Euclid's axioms. Aristotle refers to Plato the axiom that "equals subtracted from equals leave equals."

One of the greatest achievements of Plato and his school is the invention of *analysis* as a method of proof. To be sure, this method had been used unconsciously by Hippocrates and others; but Plato, like a true philosopher, turned the instinctive logic into a conscious, legitimate method.

The terms *synthesis* and *analysis* are used in mathematics in a more special sense than in logic. In ancient mathematics

they had a different meaning from what they now have. The oldest definition of mathematical analysis as opposed to synthesis is that given in Euclid, XIII. 5, which in all probability was framed by Eudoxus: "Analysis is the obtaining of the thing sought by assuming it and so reasoning up to an admitted truth; synthesis is the obtaining of the thing sought by reasoning up to the inference and proof of it." The analytic method is not conclusive, unless all operations involved in it are known to be reversible. To remove all doubt, the Greeks, as a rule, added to the analytic process a synthetic one, consisting of a reversion of all operations occurring in the analysis. Thus the aim of analysis was to aid in the discovery of synthetic proofs or solutions.

Plato is said to have solved the problem of the duplication of the cube. But the solution is open to the very same objection which he made to the solutions by Archytas, Eudoxus, and Menæchmus. He called their solutions not geometrical, but mechanical, for they required the use of other instruments than the ruler and compasses. He said that thereby "the good of geometry is set aside and destroyed, for we again reduce it to the world of sense, instead of elevating and imbuing it with the eternal and incorporeal images of thought, even as it is employed by God, for which reason He always is God." These objections indicate either that the solution is wrongly attributed to Plato or that he wished to show how easily non-geometric solutions of that character can be found. It is now generally admitted that the duplication problem, as well as the trisection and quadrature problems, cannot be solved by means of the ruler and compass only.

Plato gave a healthful stimulus to the study of stereometry, which until his time had been entirely neglected. The sphere and the regular solids had been studied to some extent, but the prism, pyramid, cylinder, and cone were hardly known to

exist. All these solids became the subjects of investigation by the Platonic school. One result of these inquiries was epoch-making. **Menæchmus**, an associate of Plato and pupil of Eudoxus, invented the conic sections, which, in course of only a century, raised geometry to the loftiest height which it was destined to reach during antiquity. Menæchmus cut three kinds of cones, the 'right-angled,' 'acute-angled,' and 'obtuse-angled,' by planes at right angles to a side of the cones, and thus obtained the three sections which we now call the parabola, ellipse, and hyperbola. Judging from the two very elegant solutions of the "Delian Problem" by means of intersections of these curves, Menæchmus must have succeeded well in investigating their properties.

Another great geometer was **Dinostratus**, the brother of Menæchmus and pupil of Plato. Celebrated is his mechanical solution of the quadrature of the circle, by means of the *quadratrix* of Hippias.

Perhaps the most brilliant mathematician of this period was **Eudoxus.** He was born at Cnidus about 408 B.C., studied under Archytas, and later, for two months, under Plato. He was imbued with a true spirit of scientific inquiry, and has been called the father of scientific astronomical observation. From the fragmentary notices of his astronomical researches, found in later writers, Ideler and Schiaparelli succeeded in reconstructing the system of Eudoxus with its celebrated representation of planetary motions by "concentric spheres." Eudoxus had a school at Cyzicus, went with his pupils to Athens, visiting Plato, and then returned to Cyzicus, where he died 355 B.C. The fame of the academy of Plato is to a large extent due to Eudoxus's pupils of the school at Cyzicus, among whom are Menæchmus, Dinostratus, Athenæus, and Helicon. Diogenes Laertius describes Eudoxus as astronomer, physician, legislator, as well as geometer. The *Eudemian Summary*

says that Eudoxus "first increased the number of general theorems, added to the three proportions three more, and raised to a considerable quantity the learning, begun by Plato, on the subject of the section, to which he applied the analytical method." By this 'section' is meant, no doubt, the "golden section" (*sectio aurea*), which cuts a line in extreme and mean ratio. The first five propositions in Euclid XIII. relate to lines cut by this section, and are generally attributed to Eudoxus. Eudoxus added much to the knowledge of solid geometry. He proved, says Archimedes, that a pyramid is exactly one-third of a prism, and a cone one-third of a cylinder, having equal base and altitude. The proof that spheres are to each other as the cubes of their radii is probably due to him. He made frequent and skilful use of the method of exhaustion, of which he was in all probability the inventor. A scholiast on Euclid, thought to be Proclus, says further that Eudoxus practically invented the whole of Euclid's fifth book. Eudoxus also found two mean proportionals between two given lines, but the method of solution is not known.

Plato has been called a maker of mathematicians. Besides the pupils already named, the *Eudemian Summary* mentions the following: **Theætetus** of Athens, a man of great natural gifts, to whom, no doubt, Euclid was greatly indebted in the composition of the 10th book,[8] treating of incommensurables; **Leodamas** of Thasos; **Neocleides** and his pupil **Leon**, who added much to the work of their predecessors, for Leon wrote an *Elements* carefully designed, both in number and utility of its proofs; **Theudius of Magnesia**, who composed a very good book of *Elements* and generalised propositions, which had been confined to particular cases; **Hermotimus of Colophon**, who discovered many propositions of the *Elements* and composed some on *loci;* and, finally, the names of **Amyclas of Heraclea**, **Cyzicenus of Athens**, and **Philippus of Mende.**

A skilful mathematician of whose life and works we have no details is **Aristæus**, the elder, probably a senior contemporary of Euclid. The fact that he wrote a work on conic sections tends to show that much progress had been made in their study during the time of Menæchmus. Aristæus wrote also on regular solids and cultivated the analytic method. His works contained probably a summary of the researches of the Platonic school.[8]

Aristotle (384–322 B.C.), the systematiser of deductive logic, though not a professed mathematician, promoted the science of geometry by improving some of the most difficult definitions. His *Physics* contains passages with suggestive hints of the principle of virtual velocities. About his time there appeared a work called *Mechanica*, of which he is regarded by some as the author. Mechanics was totally neglected by the Platonic school.

The First Alexandrian School.

In the previous pages we have seen the birth of geometry in Egypt, its transference to the Ionian Islands, thence to Lower Italy and to Athens. We have witnessed its growth in Greece from feeble childhood to vigorous manhood, and now we shall see it return to the land of its birth and there derive new vigour.

During her declining years, immediately following the Peloponnesian War, Athens produced the greatest scientists and philosophers of antiquity. It was the time of Plato and Aristotle. In 338 B.C., at the battle of Chæronea, Athens was beaten by Philip of Macedon, and her power was broken forever. Soon after, Alexander the Great, the son of Philip, started out to conquer the world. In eleven years he built up a great empire which broke to pieces in a day. Egypt

fell to the lot of Ptolemy Soter. Alexander had founded the seaport of Alexandria, which soon became "the noblest of all cities." Ptolemy made Alexandria the capital. The history of Egypt during the next three centuries is mainly the history of Alexandria. Literature, philosophy, and art were diligently cultivated. Ptolemy created the university of Alexandria. He founded the great Library and built laboratories, museums, a zoölogical garden, and promenades. Alexandria soon became the great centre of learning.

Demetrius Phalereus was invited from Athens to take charge of the Library, and it is probable, says Gow, that **Euclid** was invited with him to open the mathematical school. Euclid's greatest activity was during the time of the first Ptolemy, who reigned from 306 to 283 B.C. Of the life of Euclid, little is known, except what is added by Proclus to the *Eudemian Summary*. Euclid, says Proclus, was younger than Plato and older than Eratosthenes and Archimedes, the latter of whom mentions him. He was of the Platonic sect, and well read in its doctrines. He collected the *Elements*, put in order much that Eudoxus had prepared, completed many things of Theætetus, and was the first who reduced to unobjectionable demonstration the imperfect attempts of his predecessors. When Ptolemy once asked him if geometry could not be mastered by an easier process than by studying the *Elements*, Euclid returned the answer, "There is no royal road to geometry." Pappus states that Euclid was distinguished by the fairness and kindness of his disposition, particularly toward those who could do anything to advance the mathematical sciences. Pappus is evidently making a contrast to Apollonius, of whom he more than insinuates the opposite character.[9] A pretty little story is related by Stobæus:[6] "A youth who had begun to read geometry with Euclid, when he had learnt the first proposition, inquired,

'What do I get by learning these things?' So Euclid called his slave and said, 'Give him threepence, since he must make gain out of what he learns.'" These are about all the personal details preserved by Greek writers. Syrian and Arabian writers claim to know much more, but they are unreliable. At one time Euclid of Alexandria was universally confounded with Euclid of Megara, who lived a century earlier.

The fame of Euclid has at all times rested mainly upon his book on geometry, called the *Elements*. This book was so far superior to the *Elements* written by Hippocrates, Leon, and Theudius, that the latter works soon perished in the struggle for existence. The Greeks gave Euclid the special title of "the author of the *Elements*." It is a remarkable fact in the history of geometry, that the *Elements* of Euclid, written two thousand years ago, are still regarded by many as the best introduction to the mathematical sciences. In England they are used at the present time extensively as a text-book in schools. Some editors of Euclid have, however, been inclined to credit him with more than is his due. They would have us believe that a finished and unassailable system of geometry sprang at once from the brain of Euclid, "an armed Minerva from the head of Jupiter." They fail to mention the earlier eminent mathematicians from whom Euclid got his material. Comparatively few of the propositions and proofs in the *Elements* are his own discoveries. In fact, the proof of the "Theorem of Pythagoras" is the only one directly ascribed to him. Allman conjectures that the substance of Books I., II., IV. comes from the Pythagoreans, that the substance of Book VI. is due to the Pythagoreans and Eudoxus, the latter contributing the doctrine of proportion as applicable to incommensurables and also the Method of Exhaustions (Book XII.), that Theætetus contributed much toward Books X. and XIII.,

that the principal part of the original work of Euclid himself is to be found in Book X.[8] Euclid was the greatest systematiser of his time. By careful selection from the material before him, and by logical arrangement of the propositions selected, he built up, from a few definitions and axioms, a proud and lofty structure. It would be erroneous to believe that he incorporated into his *Elements* all the elementary theorems known at his time. Archimedes, Apollonius, and even he himself refer to theorems not included in his *Elements*, as being well-known truths.

The text of the *Elements* now commonly used is Theon's edition. Theon of Alexandria, the father of Hypatia, brought out an edition, about 700 years after Euclid, with some alterations in the text. As a consequence, later commentators, especially Robert Simson, who laboured under the idea that Euclid must be absolutely perfect, made Theon the scapegoat for all the defects which they thought they could discover in the text as they knew it. But among the manuscripts sent by Napoleon I. from the Vatican to Paris was found a copy of the *Elements* believed to be anterior to Theon's recension. Many variations from Theon's version were noticed therein, but they were not at all important, and showed that Theon generally made only verbal changes. The defects in the *Elements* for which Theon was blamed must, therefore, be due to Euclid himself. The *Elements* has been considered as offering models of scrupulously rigorous demonstrations. It is certainly true that in point of rigour it compares favourably with its modern rivals; but when examined in the light of strict mathematical logic, it has been pronounced by C. S. Peirce to be "riddled with fallacies." The results are correct only because the writer's experience keeps him on his guard.

At the beginning of our editions of the *Elements*, under the head of definitions, are given the assumptions of such

notions as the point, line, etc., and some verbal explanations. Then follow three postulates or demands, and twelve axioms. The term 'axiom' was used by Proclus, but not by Euclid. He speaks, instead, of 'common notions'—common either to all men or to all sciences. There has been much controversy among ancient and modern critics on the postulates and axioms. An immense preponderance of manuscripts and the testimony of Proclus place the 'axioms' about *right angles* and *parallels* (Axioms 11 and 12) among the postulates.[9, 10] This is indeed their proper place, for they are really *assumptions*, and not *common notions* or axioms. The postulate about *parallels* plays an important rôle in the history of non-Euclidean geometry. The only postulate which Euclid missed was the one of superposition, according to which figures can be moved about in space without any alteration in form or magnitude.

The *Elements* contains thirteen books by Euclid, and two, of which it is supposed that Hypsicles and Damascius are the authors. The first four books are on plane geometry. The fifth book treats of the theory of proportion as applied to magnitudes in general. The sixth book develops the geometry of similar figures. The seventh, eighth, ninth books are on the theory of numbers, or on arithmetic. In the ninth book is found the proof to the theorem that the number of primes is infinite. The tenth book treats of the theory of incommensurables. The next three books are on stereometry. The eleventh contains its more elementary theorems; the twelfth, the metrical relations of the pyramid, prism, cone, cylinder, and sphere. The thirteenth treats of the regular polygons, especially of the triangle and pentagon, and then uses them as faces of the five regular solids; namely, the tetraedron, octaedron, icosaedron, cube, and dodecaedron. The regular solids were studied so extensively by the Platonists that they

received the name of "Platonic figures." The statement of Proclus that the whole aim of Euclid in writing the *Elements* was to arrive at the construction of the regular solids, is obviously wrong. The fourteenth and fifteenth books, treating of solid geometry, are apocryphal.

A remarkable feature of Euclid's, and of all Greek geometry before Archimedes is that it eschews mensuration. Thus the theorem that the area of a triangle equals half the product of its base and its altitude is foreign to Euclid.

Another extant book of Euclid is the *Data*. It seems to have been written for those who, having completed the *Elements*, wish to acquire the power of solving new problems proposed to them. The *Data* is a course of practice in *analysis*. It contains little or nothing that an intelligent student could not pick up from the *Elements* itself. Hence it contributes little to the stock of scientific knowledge. The following are the other extant works generally attributed to Euclid: *Phœnomena*, a work on spherical geometry and astronomy; *Optics*, which develops the hypothesis that light proceeds from the eye, and not from the object seen; *Catoptrica*, containing propositions on reflections from mirrors; *De Divisionibus*, a treatise on the division of plane figures into parts having to one another a given ratio; *Sectio Canonis*, a work on musical intervals. His treatise on *Porisms* is lost; but much learning has been expended by Robert Simson and M. Chasles in restoring it from numerous notes found in the writings of Pappus. The term 'porism' is vague in meaning. The aim of a porism is not to state some property or truth, like a theorem, nor to effect a construction, like a problem, but to find and bring to view a thing which necessarily exists with given numbers or a given construction, as, to find the centre of a given circle, or to find the G.C.D. of two given numbers.[6] His other lost works are *Fallacies*, containing

exercises in detection of fallacies; *Conic Sections*, in four books, which are the foundation of a work on the same subject by Apollonius; and *Loci on a Surface*, the meaning of which title is not understood. Heiberg believes it to mean "loci which are surfaces."

The immediate successors of Euclid in the mathematical school at Alexandria were probably **Conon**, **Dositheus**, and **Zeuxippus**, but little is known of them.

Archimedes (287 ?–212 B.C.), the greatest mathematician of antiquity, was born in Syracuse. Plutarch calls him a relation of King Hieron; but more reliable is the statement of Cicero, who tells us he was of low birth. Diodorus says he visited Egypt, and, since he was a great friend of Conon and Eratosthenes, it is highly probable that he studied in Alexandria. This belief is strengthened by the fact that he had the most thorough acquaintance with all the work previously done in mathematics. He returned, however, to Syracuse, where he made himself useful to his admiring friend and patron, King Hieron, by applying his extraordinary inventive genius to the construction of various war-engines, by which he inflicted much loss on the Romans during the siege of Marcellus. The story that, by the use of mirrors reflecting the sun's rays, he set on fire the Roman ships, when they came within bow-shot of the walls, is probably a fiction. The city was taken at length by the Romans, and Archimedes perished in the indiscriminate slaughter which followed. According to tradition, he was, at the time, studying the diagram to some problem drawn in the sand. As a Roman soldier approached him, he called out, "Don't spoil my circles." The soldier, feeling insulted, rushed upon him and killed him. No blame attaches to the Roman general Marcellus, who admired his genius, and raised in his honour a tomb bearing the figure of a sphere inscribed in a cylinder. When

Cicero was in Syracuse, he found the tomb buried under rubbish.

Archimedes was admired by his fellow-citizens chiefly for his mechanical inventions; he himself prized far more highly his discoveries in pure science. He declared that "every kind of art which was connected with daily needs was ignoble and vulgar." Some of his works have been lost. The following are the extant books, arranged approximately in chronological order: 1. Two books on *Equiponderance of Planes* or *Centres of Plane Gravities*, between which is inserted his treatise on the *Quadrature of the Parabola;* 2. Two books on the *Sphere* and *Cylinder;* 3. The *Measurement of the Circle;* 4. *On Spirals;* 5. *Conoids* and *Spheroids;* 6. The *Sand-Counter;* 7. Two books on *Floating Bodies;* 8. Fifteen *Lemmas.*

In the book on the *Measurement of the Circle*, Archimedes proves first that the area of a circle is equal to that of a right triangle having the length of the circumference for its base, and the radius for its altitude. In this he assumes that there exists a straight line equal in length to the circumference —an assumption objected to by some ancient critics, on the ground that it is not evident that a straight line can equal a curved one. The finding of such a line was the next problem. He first finds an upper limit to the ratio of the circumference to the diameter, or π. To do this, he starts with an equilateral triangle of which the base is a tangent and the vertex is the centre of the circle. By successively bisecting the angle at the centre, by comparing ratios, and by taking the irrational square roots always a little too small, he finally arrived at the conclusion that $\pi < 3\frac{1}{7}$. Next he finds a lower limit by inscribing in the circle regular polygons of 6, 12, 24, 48, 96 sides, finding for each successive polygon its perimeter, which is, of course, always less than the circumference. Thus he finally concludes that "the circumference of a circle ex-

ceeds three times its diameter by a part which is less than $\frac{1}{7}$ but more than $\frac{10}{71}$ of the diameter." This approximation is exact enough for most purposes.

The *Quadrature of the Parabola* contains two solutions to the problem — one mechanical, the other geometrical. The method of exhaustion is used in both.

Archimedes studied also the ellipse and accomplished its quadrature, but to the hyperbola he seems to have paid less attention. It is believed that he wrote a book on conic sections.

Of all his discoveries Archimedes prized most highly those in his *Sphere and Cylinder*. In it are proved the new theorems, that the surface of a sphere is equal to four times a great circle; that the surface of a segment of a sphere is equal to a circle whose radius is the straight line drawn from the vertex of the segment to the circumference of its basal circle; that the volume and the surface of a sphere are $\frac{2}{3}$ of the volume and surface, respectively, of the cylinder circumscribed about the sphere. Archimedes desired that the figure to the last proposition be inscribed on his tomb. This was ordered done by Marcellus.

The spiral now called the "spiral of Archimedes," and described in the book *On Spirals*, was discovered by Archimedes, and not, as some believe, by his friend Conon.[3] His treatise thereon is, perhaps, the most wonderful of all his works. Nowadays, subjects of this kind are made easy by the use of the infinitesimal calculus. In its stead the ancients used the method of exhaustion. Nowhere is the fertility of his genius more grandly displayed than in his masterly use of this method. With Euclid and his predecessors the method of exhaustion was only the means of proving propositions which must have been seen and believed before they were proved. But in the hands of Archimedes it became an instrument of discovery.[9]

By the word 'conoid,' in his book on *Conoids and Spheroids*, is meant the solid produced by the revolution of a parabola or a hyperbola about its axis. Spheroids are produced by the revolution of an ellipse, and are long or flat, according as the ellipse revolves around the major or minor axis. The book leads up to the cubature of these solids.

We have now reviewed briefly all his extant works on geometry. His arithmetical treatise and problems will be considered later. We shall now notice his works on mechanics. Archimedes is the author of the first sound knowledge on this subject. Archytas, Aristotle, and others attempted to form the known mechanical truths into a science, but failed. Aristotle knew the property of the lever, but could not establish its true mathematical theory. The radical and fatal defect in the speculations of the Greeks, says Whewell, was "that though they had in their possession facts and ideas, *the ideas were not distinct and appropriate to the facts.*" For instance, Aristotle asserted that when a body at the end of a lever is moving, it may be considered as having two motions; one in the direction of the tangent and one in the direction of the radius; the former motion is, he says, *according to nature*, the latter *contrary to nature.* These inappropriate notions of 'natural' and 'unnatural' motions, together with the habits of thought which dictated these speculations, made the perception of the true grounds of mechanical properties impossible.[11] It seems strange that even after Archimedes had entered upon the right path, this science should have remained absolutely stationary till the time of Galileo — a period of nearly two thousand years.

The proof of the property of the lever, given in his *Equiponderance of Planes*, holds its place in text-books to this day. His estimate of the efficiency of the lever is expressed in the

saying attributed to him, "Give me a fulcrum on which to rest, and I will move the earth."

While the *Equiponderance* treats of solids, or the equilibrium of solids, the book on *Floating Bodies* treats of hydrostatics. His attention was first drawn to the subject of specific gravity when King Hieron asked him to test whether a crown, professed by the maker to be pure gold, was not alloyed with silver. The story goes that our philosopher was in a bath when the true method of solution flashed on his mind. He immediately ran home, naked, shouting, "I have found it!" To solve the problem, he took a piece of gold and a piece of silver, each weighing the same as the crown. According to one author, he determined the volume of water displaced by the gold, silver, and crown respectively, and calculated from that the amount of gold and silver in the crown. According to another writer, he weighed separately the gold, silver, and crown, while immersed in water, thereby determining their loss of weight in water. From these data he easily found the solution. It is possible that Archimedes solved the problem by both methods.

After examining the writings of Archimedes, one can well understand how, in ancient times, an 'Archimedean problem' came to mean a problem too deep for ordinary minds to solve, and how an 'Archimedean proof' came to be the synonym for unquestionable certainty. Archimedes wrote on a very wide range of subjects, and displayed great profundity in each. He is the Newton of antiquity.

Eratosthenes, eleven years younger than Archimedes, was a native of Cyrene. He was educated in Alexandria under Callimachus the poet, whom he succeeded as custodian of the Alexandrian Library. His many-sided activity may be inferred from his works. He wrote on *Good and Evil, Measurement of the Earth, Comedy, Geography, Chronology, Constel-*

lations, and the *Duplication of the Cube*. He was also a philologian and a poet. He measured the obliquity of the ecliptic and invented a device for finding prime numbers. Of his geometrical writings we possess only a letter to Ptolemy Euergetes, giving a history of the duplication problem and also the description of a very ingenious mechanical contrivance of his own to solve it. In his old age he lost his eyesight, and on that account is said to have committed suicide by voluntary starvation.

About forty years after Archimedes flourished **Apollonius of Perga**, whose genius nearly equalled that of his great predecessor. He incontestably occupies the second place in distinction among ancient mathematicians. Apollonius was born in the reign of Ptolemy Euergetes and died under Ptolemy Philopator, who reigned 222–205 B.C. He studied at Alexandria under the successors of Euclid, and for some time, also, at Pergamum, where he made the acquaintance of that Eudemus to whom he dedicated the first three books of his *Conic Sections*. The brilliancy of his great work brought him the title of the "Great Geometer." This is all that is known of his life.

His *Conic Sections* were in eight books, of which the first four only have come down to us in the original Greek. The next three books were unknown in Europe till the middle of the seventeenth century, when an Arabic translation, made about 1250, was discovered. The eighth book has never been found. In 1710 Halley of Oxford published the Greek text of the first four books and a Latin translation of the remaining three, together with his conjectural restoration of the eighth book, founded on the introductory lemmas of Pappus. The first four books contain little more than the substance of what earlier geometers had done. Eutocius tells us that Heraclides, in his life of Archimedes, accused Apollonius of

having appropriated, in his *Conic Sections*, the unpublished discoveries of that great mathematician. It is difficult to believe that this charge rests upon good foundation. Eutocius quotes Geminus as replying that neither Archimedes nor Apollonius claimed to have invented the conic sections, but that Apollonius had introduced a real improvement. While the first three or four books were founded on the works of Menæchmus, Aristæus, Euclid, and Archimedes, the remaining ones consisted almost entirely of new matter. The first three books were sent to Eudemus at intervals, the other books (after Eudemus's death) to one Attalus. The preface of the second book is interesting as showing the mode in which Greek books were 'published' at this time. It reads thus: "I have sent my son Apollonius to bring you (Eudemus) the second book of my Conics. Read it carefully and communicate it to such others as are worthy of it. If Philonides, the geometer, whom I introduced to you at Ephesus, comes into the neighbourhood of Pergamum, give it to him also." [12]

The first book, says Apollonius in his preface to it, "contains the mode of producing the three sections and the conjugate hyperbolas and their principal characteristics, more fully and generally worked out than in the writings of other authors." We remember that Menæchmus, and all his successors down to Apollonius, considered only sections of *right* cones by a plane perpendicular to their sides, and that the three sections were obtained each from a different cone. Apollonius introduced an important generalisation. He produced all the sections from one and the same cone, whether right or scalene, and by sections which may or may not be perpendicular to its sides. The old names for the three curves were now no longer applicable. Instead of calling the three curves, sections of the 'acute-angled,' 'right-angled,' and 'obtuse-angled' cone, he called them *ellipse*, *parabola*, and

hyperbola, respectively. To be sure, we find the words 'parabola' and 'ellipse' in the works of Archimedes, but they are probably only interpolations. The word 'ellipse' was applied because $y^2 < px$, p being the parameter; the word 'parabola' was introduced because $y^2 = px$, and the term 'hyperbola' because $y^2 > px$.

The treatise of Apollonius rests on a unique property of conic sections, which is derived directly from the nature of the cone in which these sections are found. How this property forms the key to the system of the ancients is told in a masterly way by M. Chasles.[13] "Conceive," says he, "an oblique cone on a circular base; the straight line drawn from its summit to the centre of the circle forming its base is called the *axis* of the cone. The plane passing through the axis, perpendicular to its base, cuts the cone along two lines and determines in the circle a diameter; the triangle having this diameter for its base and the two lines for its sides, is called *the triangle through the axis.* In the formation of his conic sections, Apollonius supposed the cutting plane to be perpendicular to the plane of the triangle through the axis. The points in which this plane meets the two sides of this triangle are the *vertices* of the curve; and the straight line which joins these two points is a diameter of it. Apollonius called this diameter *latus transversum.* At one of the two vertices of the curve erect a perpendicular (*latus rectum*) to the plane of the triangle through the axis, of a certain length, to be determined as we shall specify later, and from the extremity of this perpendicular draw a straight line to the other vertex of the curve; now, through any point whatever of the diameter of the curve, draw at right angles an *ordinate:* the square of this ordinate, comprehended between the diameter and the curve, will be equal to the rectangle constructed on the portion of the ordinate comprised between the diameter and the straight

line, and the part of the diameter comprised between the first vertex and the foot of the ordinate. Such is the characteristic property which Apollonius recognises in his conic sections and which he uses for the purpose of inferring from it, by adroit transformations and deductions, nearly all the rest. It plays, as we shall see, in his hands, almost the same rôle as the equation of the second degree with two variables (abscissa and ordinate) in the system of analytic geometry of Descartes.

"It will be observed from this that the diameter of the curve and the perpendicular erected at one of its extremities suffice to construct the curve. These are the two elements which the ancients used, with which to establish their theory of conics. The perpendicular in question was called by them *latus erectum;* the moderns changed this name first to that of *latus rectum,* and afterwards to that of *parameter.*"

The first book of the *Conic Sections* of Apollonius is almost wholly devoted to the generation of the three principal conic sections.

The second book treats mainly of asymptotes, axes, and diameters.

The third book treats of the equality or proportionality of triangles, rectangles, or squares, of which the component parts are determined by portions of transversals, chords, asymptotes, or tangents, which are frequently subject to a great number of conditions. It also touches the subject of foci of the ellipse and hyperbola.

In the fourth book, Apollonius discusses the harmonic division of straight lines. He also examines a system of two conics, and shows that they cannot cut each other in more than four points. He investigates the various possible relative positions of two conics, as, for instance, when they have one or two points of contact with each other.

The fifth book reveals better than any other the giant

intellect of its author. Difficult questions of *maxima and minima*, of which few examples are found in earlier works, are here treated most exhaustively. The subject investigated is, to find the longest and shortest lines that can be drawn from a given point to a conic. Here are also found the germs of the subject of *evolutes* and *centres of osculation.*

The sixth book is on the similarity of conics.

The seventh book is on conjugate diameters.

The eighth book, as restored by Halley, continues the subject of conjugate diameters.

It is worthy of notice that Apollonius nowhere introduces the notion of *directrix* for a conic, and that, though he incidentally discovered the *focus* of an ellipse and hyperbola, he did not discover the focus of a parabola.[6] Conspicuous in his geometry is also the absence of technical terms and symbols, which renders the proofs long and cumbrous.

The discoveries of Archimedes and Apollonius, says M. Chasles,[13] marked the most brilliant epoch of ancient geometry. Two questions which have occupied geometers of all periods may be regarded as having originated with them. The first of these is the quadrature of curvilinear figures, which gave birth to the infinitesimal calculus. The second is the theory of conic sections, which was the prelude to the theory of geometrical curves of all degrees, and to that portion of geometry which considers only the forms and situations of figures, and uses only the intersection of lines and surfaces and the ratios of rectilineal distances. These two great divisions of geometry may be designated by the names of *Geometry of Measurements* and *Geometry of Forms and Situations*, or, Geometry of Archimedes and of Apollonius.

Besides the *Conic Sections*, Pappus ascribes to Apollonius the following works: *On Contacts, Plane Loci, Inclinations, Section of an Area, Determinate Section*, and gives lemmas

from which attempts have been made to restore the lost originals. Two books on *De Sectione Rationis* have been found in the Arabic. The book on *Contacts*, as restored by Vieta, contains the so-called "Apollonian Problem": Given three circles, to find a fourth which shall touch the three.

Euclid, Archimedes, and Apollonius brought geometry to as high a state of perfection as it perhaps could be brought without first introducing some more general and more powerful method than the old method of exhaustion. A briefer symbolism, a Cartesian geometry, an infinitesimal calculus, were needed. The Greek mind was not adapted to the invention of general methods. Instead of a climb to still loftier heights we observe, therefore, on the part of later Greek geometers, a descent, during which they paused here and there to look around for details which had been passed by in the hasty ascent.[3]

Among the earliest successors of Apollonius was **Nicomedes**. Nothing definite is known of him, except that he invented the *conchoid* ("mussel-like"). He devised a little machine by which the curve could be easily described. With aid of the conchoid he duplicated the cube. The curve can also be used for trisecting angles in a way much resembling that in the eighth lemma of Archimedes. Proclus ascribes this mode of trisection to Nicomedes, but Pappus, on the other hand, claims it as his own. The conchoid was used by Newton in constructing curves of the third degree.

About the time of Nicomedes, flourished also **Diocles**, the inventor of the *cissoid* ("ivy-like"). This curve he used for finding two mean proportionals between two given straight lines.

About the life of **Perseus** we know as little as about that of Nicomedes and Diocles. He lived some time between 200 and 100 B.C. From Heron and Geminus we learn that he wrote a

work on the *spire*, a sort of anchor-ring surface described by Heron as being produced by the revolution of a circle around one of its chords as an axis. The sections of this surface yield peculiar curves called *spiral sections*, which, according to Geminus, were thought out by Perseus. These curves appear to be the same as the *Hippopede* of Eudoxus.

Probably somewhat later than Perseus lived **Zenodorus**. He wrote an interesting treatise on a new subject; namely, *isoperimetrical figures*. Fourteen propositions are preserved by Pappus and Theon. Here are a few of them: Of isoperimetrical, regular polygons, the one having the largest number of angles has the greatest area; the circle has a greater area than any regular polygon of equal periphery; of all isoperimetrical polygons of n sides, the regular is the greatest; of all solids having surfaces equal in area, the sphere has the greatest volume.

Hypsicles (between 200 and 100 B.C.) was supposed to be the author of both the fourteenth and fifteenth books of Euclid, but recent critics are of opinion that the fifteenth book was written by an author who lived several centuries after Christ. The fourteenth book contains seven elegant theorems on *regular solids*. A treatise of Hypsicles on *Risings* is of interest because it is the first Greek work giving the division of the circumference into 360 degrees after the fashion of the Babylonians.

Hipparchus of Nicæa in Bithynia was the greatest astronomer of antiquity. He established inductively the famous theory of epicycles and eccentrics. As might be expected, he was interested in mathematics, not *per se*, but only as an aid to astronomical inquiry. No mathematical writings of his are extant, but Theon of Alexandria informs us that Hipparchus originated the science of *trigonometry*, and that he calculated a "table of chords" in twelve books. Such calculations

must have required a ready knowledge of arithmetical and algebraical operations.

About 100 B.C. flourished **Heron the Elder** of Alexandria. He was the pupil of Ctesibius, who was celebrated for his ingenious mechanical inventions, such as the hydraulic organ, the water-clock, and catapult. It is believed by some that Heron was a son of Ctesibius. He exhibited talent of the same order as did his master by the invention of the eolipile and a curious mechanism known as "Heron's fountain." Great uncertainty exists concerning his writings. Most authorities believe him to be the author of an important *Treatise on the Dioptra*, of which there exist three manuscript copies, quite dissimilar. But M. Marie[14] thinks that the *Dioptra* is the work of *Heron the Younger*, who lived in the seventh or eighth century after Christ, and that *Geodesy*, another book supposed to be by Heron, is only a corrupt and defective copy of the former work. *Dioptra* contains the important formula for finding the area of a triangle expressed in terms of its sides; its derivation is quite laborious and yet exceedingly ingenious. "It seems to me difficult to believe," says Chasles, "that so beautiful a theorem should be found in a work so ancient as that of Heron the Elder, without that some Greek geometer should have thought to cite it." Marie lays great stress on this silence of the ancient writers, and argues from it that the true author must be Heron the Younger or some writer much more recent than Heron the Elder. But no reliable evidence has been found that there actually existed a second mathematician by the name of Heron.

"Dioptra," says Venturi, were instruments which had great resemblance to our modern theodolites. The book *Dioptra* is a treatise on geodesy containing solutions, with aid of these instruments, of a large number of questions in geometry, such as to find the distance between two points, of which one only

is accessible, or between two points which are visible but both inaccessible; from a given point to draw a perpendicular to a line which cannot be approached; to find the difference of level between two points; to measure the area of a field without entering it.

Heron was a practical surveyor. This may account for the fact that his writings bear so little resemblance to those of the Greek authors, who considered it degrading the science to apply geometry to surveying. The character of his geometry is not Grecian, but decidedly Egyptian. This fact is the more surprising when we consider that Heron demonstrated his familiarity with Euclid by writing a commentary on the *Elements.*[21] Some of Heron's formulas point to an old Egyptian origin. Thus, besides the above exact formula for the area of a triangle in terms of its sides, Heron gives the formula $\frac{a_1 + a_2}{2} \times \frac{b}{2}$, which bears a striking likeness to the formula $\frac{a_1 + a_2}{2} \times \frac{b_1 + b_2}{2}$ for finding the area of a quadrangle, found in the Edfu inscriptions. There are, moreover, points of resemblance between Heron's writings and the ancient Ahmes papyrus. Thus Ahmes used unit-fractions exclusively; Heron uses them oftener than other fractions. Like Ahmes and the priests at Edfu, Heron divides complicated figures into simpler ones by drawing auxiliary lines; like them, he shows, throughout, a special fondness for the isosceles trapezoid.

The writings of Heron satisfied a practical want, and for that reason were borrowed extensively by other peoples. We find traces of them in Rome, in the Occident during the Middle Ages, and even in India.

Geminus of Rhodes (about 70 B.C.) published an astronomical work still extant. He wrote also a book, now lost, on the *Arrangement of Mathematics,* which contained many valuable

notices of the early history of Greek mathematics. Proclus and Eutocius quote it frequently. **Theodosius** of Tripolis is the author of a book of little merit on the geometry of the sphere. **Dionysodorus** of Amisus in Pontus applied the intersection of a parabola and hyperbola to the solution of a problem which Archimedes, in his *Sphere and Cylinder*, had left incomplete. The problem is "to cut a sphere so that its segments shall be in a given ratio."

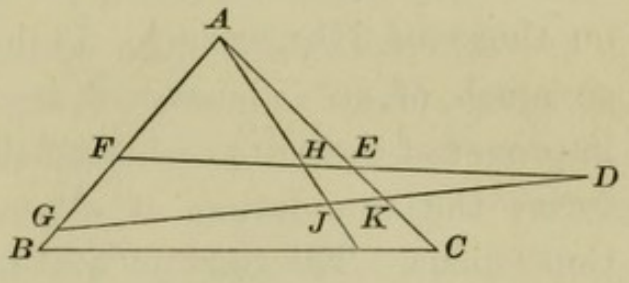
We have now sketched the progress of geometry down to the time of Christ. Unfortunately, very little is known of the history of geometry between the time of Apollonius and the beginning of the Christian era. The names of quite a number of geometers have been mentioned, but very few of their works are now extant. It is certain, however, that there were no mathematicians of real genius from Apollonius to Ptolemy, excepting Hipparchus and perhaps Heron.

The Second Alexandrian School.

The close of the dynasty of the Lagides which ruled Egypt from the time of Ptolemy Soter, the builder of Alexandria, for 300 years; the absorption of Egypt into the Roman Empire; the closer commercial relations between peoples of the East and of the West; the gradual decline of paganism and spread of Christianity, — these events were of far-reaching influence on the progress of the sciences, which then had their home in Alexandria. Alexandria became a commercial and intellectual emporium. Traders of all nations met in her busy streets, and in her magnificent Library, museums, lecture-halls, scholars from the East mingled with those of the West; Greeks began to study older literatures and to compare them with their own. In consequence of this interchange of ideas the Greek philosophy became fused with Oriental

philosophy. Neo-Pythagoreanism and Neo-Platonism were the names of the modified systems. These stood, for a time, in opposition to Christianity. The study of Platonism and Pythagorean mysticism led to the revival of the theory of numbers. Perhaps the dispersion of the Jews and their introduction to Greek learning helped in bringing about this revival. The theory of numbers became a favourite study. This new line of mathematical inquiry ushered in what we may call a new school. There is no doubt that even now geometry continued to be one of the most important studies in the Alexandrian course. This Second Alexandrian School may be said to begin with the Christian era. It was made famous by the names of Claudius Ptolemæus, Diophantus, Pappus, Theon of Smyrna, Theon of Alexandria, Iamblichus, Porphyrius, and others.

By the side of these we may place **Serenus** of Antissa, as having been connected more or less with this new school. He wrote on sections of the cone and cylinder, in two books, one of which treated only of the triangular section of the cone through the apex. He solved the problem, "given a cone (cylinder), to find a cylinder (cone), so that the section of both by the same plane gives similar ellipses." Of particular interest is the following theorem, which is the foundation of the modern theory of harmonics: If from D we draw DF, cutting the triangle ABC, and choose H on it, so that $DE : DF = EH : HF$, and if we draw the line AH, then every transversal through D, such as DG, will be divided by AH so that $DK : DG = KJ : JG$.



Menelaus of Alexandria (about 98 A.D.) was the author of *Sphærica*, a work extant in Hebrew and Arabic, but not

in Greek. In it he proves the theorems on the congruence of spherical triangles, and describes their properties in much the same way as Euclid treats plane triangles. In it are also found the theorems that the sum of the three sides of a spherical triangle is less than a great circle, and that the sum of the three angles exceeds two right angles. Celebrated are two theorems of his on plane and spherical triangles. The one on plane triangles is that, "if the three sides be cut by a straight line, the product of the three segments which have no common extremity is equal to the product of the other three." The illustrious Carnot makes this proposition, known as the 'lemma of Menelaus,' the base of his theory of transversals. The corresponding theorem for spherical triangles, the so-called 'regula sex quantitatum,' is obtained from the above by reading "chords of three segments doubled," in place of "three segments."

Claudius Ptolemæus, a celebrated astronomer, was a native of Egypt. Nothing is known of his personal history except that he flourished in Alexandria in 139 A.D. and that he made the earliest astronomical observations recorded in his works, in 125 A.D., the latest in 151 A.D. The chief of his works are the *Syntaxis Mathematica* (or the *Almagest*, as the Arabs call it) and the *Geographica*, both of which are extant. The former work is based partly on his own researches, but mainly on those of Hipparchus. Ptolemy seems to have been not so much of an independent investigator, as a corrector and improver of the work of his great predecessors. The *Almagest* forms the foundation of all astronomical science down to Copernicus. The fundamental idea of his system, the "Ptolemaic System," is that the earth is in the centre of the universe, and that the sun and planets revolve around the earth. Ptolemy did considerable for mathematics. He created, for astronomical use, a *trigonometry* remarkably perfect in form.

The foundation of this science was laid by the illustrious Hipparchus.

The *Almagest* is in 13 books. Chapter 9 of the first book shows how to calculate tables of chords. The circle is divided into 360 degrees, each of which is halved. The diameter is divided into 120 divisions; each of these into 60 parts, which are again subdivided into 60 smaller parts. In Latin, these parts were called *partes minutæ primæ* and *partes minutæ secundæ*. Hence our names, 'minutes' and 'seconds.'[3] The sexagesimal method of dividing the circle is of Babylonian origin, and was known to Geminus and Hipparchus. But Ptolemy's method of calculating chords seems original with him. He first proved the proposition, now appended to Euclid VI. (*D*), that "the rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to both the rectangles contained by its opposite sides." He then shows how to find from the chords of two arcs the chords of their sum and difference, and from the chord of any arc that of its half. These theorems he applied to the calculation of his tables of chords. The proofs of these theorems are very pretty.

Another chapter of the first book in the *Almagest* is devoted to *trigonometry*, and to *spherical* trigonometry in particular. Ptolemy proved the 'lemma of Menelaus,' and also the 'regula sex quantitatum.' Upon these propositions he built up his trigonometry. The fundamental theorem of plane trigonometry, that two sides of a triangle are to each other as the chords of double the arcs measuring the angles opposite the two sides, was not stated explicitly by him, but was contained implicitly in other theorems. More complete are the propositions in spherical trigonometry.

The fact that trigonometry was cultivated not for its own sake, but to aid astronomical inquiry, explains the rather

startling fact that spherical trigonometry came to exist in a developed state earlier than plane trigonometry.

The remaining books of the *Almagest* are on astronomy. Ptolemy has written other works which have little or no bearing on mathematics, except one on geometry. Extracts from this book, made by Proclus, indicate that Ptolemy did not regard the parallel-axiom of Euclid as self-evident, and that Ptolemy was the first of the long line of geometers from ancient time down to our own who toiled in the vain attempt to prove it.

Two prominent mathematicians of this time were Nicomachus and Theon of Smyrna. Their favourite study was theory of numbers. The investigations in this science culminated later in the algebra of Diophantus. But no important geometer appeared after Ptolemy for 150 years. The only occupant of this long gap was **Sextus Julius Africanus**, who wrote an unimportant work on geometry applied to the art of war, entitled *Cestes*.

Pappus, probably born about 340 A.D., in Alexandria, was the last great mathematician of the Alexandrian school. His genius was inferior to that of Archimedes, Apollonius, and Euclid, who flourished over 500 years earlier. But living, as he did, at a period when interest in geometry was declining, he towered above his contemporaries "like the peak of Teneriffa above the Atlantic." He is the author of a *Commentary on the Almagest*, a *Commentary on Euclid's Elements*, a *Commentary on the Analemma of Diodorus*, — a writer of whom nothing is known. All these works are lost. Proclus, probably quoting from the *Commentary on Euclid*, says that Pappus objected to the statement that an angle equal to a right angle is always itself a right angle.

The only work of Pappus still extant is his *Mathematical Collections*. This was originally in eight books, but the first

and portions of the second are now missing. The *Mathematical Collections* seems to have been written by Pappus to supply the geometers of his time with a succinct analysis of the most difficult mathematical works and to facilitate the study of them by explanatory lemmas. But these lemmas are selected very freely, and frequently have little or no connection with the subject on hand. However, he gives very accurate summaries of the works of which he treats. The *Mathematical Collections* is invaluable to us on account of the rich information it gives on various treatises by the foremost Greek mathematicians, which are now lost. Mathematicians of the last century considered it possible to restore lost works from the *résumé* by Pappus alone.

We shall now cite the more important of those theorems in the *Mathematical Collections* which are supposed to be original with Pappus. First of all ranks the elegant theorem re-discovered by *Guldin*, over 1000 years later, that the volume generated by the revolution of a plane curve which lies wholly on one side of the axis, equals the area of the curve multiplied by the circumference described by its centre of gravity. Pappus proved also that the centre of gravity of a triangle is that of another triangle whose vertices lie upon the sides of the first and divide its three sides in the same ratio. In the fourth book are new and brilliant propositions on the quadratrix which indicate an intimate acquaintance with curved surfaces. He generates the quadratrix as follows: Let a spiral line be drawn upon a right circular cylinder; then the perpendiculars to the axis of the cylinder drawn from each point of the spiral line form the surface of a screw. A plane passed through one of these perpendiculars, making any convenient angle with the base of the cylinder, cuts the screw-surface in a curve, the orthogonal projection of which upon the base is the *quadratrix*. A second mode of generation is

no less admirable: If we make the spiral of Archimedes the base of a right cylinder, and imagine a cone of revolution having for its axis the side of the cylinder passing through the initial point of the spiral, then this cone cuts the cylinder in a curve of double curvature. The perpendiculars to the axis drawn through every point in this curve form the surface of a screw which Pappus here calls the *plectoidal surface.* A plane passed through one of the perpendiculars at any convenient angle cuts that surface in a curve whose orthogonal projection upon the plane of the spiral is the required *quadratrix.* Pappus considers curves of double curvature still further. He produces a *spherical spiral* by a point moving uniformly along the circumference of a great circle of a sphere, while the great circle itself revolves uniformly around its diameter. He then finds the area of that portion of the surface of the sphere determined by the spherical spiral, "a complanation which claims the more lively admiration, if we consider that, although the entire surface of the sphere was known since Archimedes' time, to measure portions thereof, such as spherical triangles, was then and for a long time afterwards an unsolved problem."[3] A question which was brought into prominence by Descartes and Newton is the "problem of Pappus." Given several straight lines in a plane, to find the locus of a point such that when perpendiculars (or, more generally, straight lines at given angles) are drawn from it to the given lines, the product of certain ones of them shall be in a given ratio to the product of the remaining ones. It is worth noticing that it was Pappus who first found the focus of the parabola, suggested the use of the directrix, and propounded the theory of the involution of points. He solved the problem to draw through three points lying in the same straight line, three straight lines which shall form a triangle inscribed in a given circle.[3] From the *Mathematical Collections*

many more equally difficult theorems might be quoted which are original with Pappus as far as we know. It ought to be remarked, however, that he is known in three instances to have copied theorems without giving due credit, and that he may have done the same thing in other cases in which we have no data by which to ascertain the real discoverer.

About the time of Pappus lived **Theon** of Alexandria. He brought out an edition of Euclid's *Elements* with notes, which he probably used as a text-book in his classes. His commentary on the *Almagest* is valuable for the many historical notices, and especially for the specimens of Greek arithmetic which it contains. Theon's daughter **Hypatia**, a woman celebrated for her beauty and modesty, was the last Alexandrian teacher of reputation, and is said to have been an abler philosopher and mathematician than her father. Her notes on the works of Diophantus and Apollonius have been lost. Her tragic death in 415 A.D. is vividly described in Kingsley's *Hypatia*.

From now on, mathematics ceased to be cultivated in Alexandria. The leading subject of men's thoughts was Christian theology. Paganism disappeared, and with it pagan learning. The Neo-Platonic school at Athens struggled on a century longer. Proclus, Isidorus, and others kept up the "golden chain of Platonic succession." **Proclus**, the successor of Syrianus, at the Athenian school, wrote a commentary on Euclid's *Elements*. We possess only that on the first book, which is valuable for the information it contains on the history of geometry. **Damascius** of Damascus, the pupil of Isidorus, is now believed to be the author of the fifteenth book of Euclid. Another pupil of Isidorus was **Eutocius** of Ascalon, the commentator of Apollonius and Archimedes. **Simplicius** wrote a commentary on Aristotle's *De Cœlo*. In the year 529, Justinian, disapproving heathen learning, finally closed by imperial edict the schools at Athens.

As a rule, the geometers of the last 500 years showed a lack of creative power. They were commentators rather than discoverers.

The principal characteristics of ancient geometry are: —

(1) A wonderful clearness and definiteness of its concepts and an almost perfect logical rigour of its conclusions.

(2) A complete want of general principles and methods. Ancient geometry is decidedly *special.* Thus the Greeks possessed no general method of drawing tangents. "The determination of the tangents to the three conic sections did not furnish any rational assistance for drawing the tangent to any other new curve, such as the conchoid, the cissoid, etc." [15] In the demonstration of a theorem, there were, for the ancient geometers, as many different cases requiring separate proof as there were different positions for the lines. The greatest geometers considered it necessary to treat all possible cases independently of each other, and to prove each with equal fulness. To devise methods by which the various cases could all be disposed of by one stroke, was beyond the power of the ancients. "If we compare a mathematical problem with a huge rock, into the interior of which we desire to penetrate, then the work of the Greek mathematicians appears to us like that of a vigorous stonecutter who, with chisel and hammer, begins with indefatigable perseverance, from without, to crumble the rock slowly into fragments; the modern mathematician appears like an excellent miner, who first bores through the rock some few passages, from which he then bursts it into pieces with one powerful blast, and brings to light the treasures within." [16]

GREEK ARITHMETIC.

Greek mathematicians were in the habit of discriminating between the *science* of numbers and the *art* of calculation. The former they called *arithmetica*, the latter *logistica*. The drawing of this distinction between the two was very natural and proper. The difference between them is as marked as that between theory and practice. Among the Sophists the art of calculation was a favourite study. Plato, on the other hand, gave considerable attention to philosophical arithmetic, but pronounced calculation a vulgar and childish art.

In sketching the history of Greek calculation, we shall first give a brief account of the Greek mode of counting and of writing numbers. Like the Egyptians and Eastern nations, the earliest Greeks counted on their fingers or with pebbles. In case of large numbers, the pebbles were probably arranged in parallel vertical lines. Pebbles on the first line represented units, those on the second tens, those on the third hundreds, and so on. Later, frames came into use, in which strings or wires took the place of lines. According to tradition, Pythagoras, who travelled in Egypt and, perhaps, in India, first introduced this valuable instrument into Greece. The *abacus*, as it is called, existed among different peoples and at different times, in various stages of perfection. An abacus is still employed by the Chinese under the name of *Swan-pan*. We possess no specific information as to how the Greek abacus looked or how it was used. Boethius says that the Pythagoreans used with the abacus certain nine signs called *apices*, which resembled in form the nine "Arabic numerals." But the correctness of this assertion is subject to grave doubts.

The oldest Grecian numerical symbols were the so-called *Herodianic signs* (after Herodianus, a Byzantine grammarian of about 200 A.D., who describes them). These signs occur fre-

quently in Athenian inscriptions and are, on that account, now generally called *Attic*. For some unknown reason these symbols were afterwards replaced by the *alphabetic numerals*, in which the letters of the Greek alphabet were used, together with three strange and antique letters ϛ, ϟ, and Ϡ, and the symbol M. This change was decidedly for the worse, for the old Attic numerals were less burdensome on the memory, inasmuch as they contained fewer symbols and were better adapted to show forth analogies in numerical operations. The following table shows the Greek alphabetic numerals and their respective values: —

α	β	γ	δ	ε	ϛ	ζ	η	θ	ι	κ	λ	μ	ν	ξ	ο	π	ϟ
1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90

ρ	σ	τ	υ	ϕ	χ	ψ	ω	Ϡ	,α	,β	,γ	etc.
100	200	300	400	500	600	700	800	900	1000	2000	3000	

M	$\overset{\beta}{\mathrm{M}}$	$\overset{\gamma}{\mathrm{M}}$	etc.
10,000	20,000	30,000	

It will be noticed that at 1000, the alphabet is begun over again, but, to prevent confusion, a stroke is now placed before the letter and generally somewhat below it. A horizontal line drawn over a number served to distinguish it more readily from words. The coefficient for M was sometimes placed before or behind instead of over the M. Thus 43,678 was written $\overline{\delta \mathrm{M}, \gamma\chi o\eta}$. It is to be observed that the Greeks had no zero.

Fractions were denoted by first writing the numerator marked with an accent, then the denominator marked with two accents and written twice. Thus, $\iota\gamma'\kappa\theta''\kappa\theta'' = \frac{13}{29}$. In case of fractions having unity for the numerator, the α' was omitted and the denominator was written only once. Thus $\mu\delta'' = \frac{1}{44}$.

Greek writers seldom refer to calculation with alphabetic numerals. Addition, subtraction, and even multiplication were probably performed on the abacus. Expert mathematicians may have used the symbols. Thus Eutocius, a commentator of the sixth century after Christ, gives a great many multiplications of which the following is a specimen :[6]—

$\overline{\sigma\,\xi\,\epsilon}$	2 6 5
$\overline{\sigma\,\xi\,\epsilon}$	2 6 5
$\overset{\delta}{\mathrm{M}}\,\overset{\alpha}{\mathrm{M}}\,{}_{,}\beta\,{}_{,}\alpha$	40000, 12000, 1000
$\overset{\alpha}{\mathrm{M}}\,{}_{,}\beta\,{}_{,}\overline{\gamma\chi}\,\bar{\tau}$	12000, 3600, 300
${}_{,}\alpha\;\bar{\tau}\;\overline{\kappa\epsilon}$	1000, 300, 25
$\overset{\zeta}{\mathrm{M}}\;\overline{\sigma\,\kappa\,\epsilon}$	70225

The operation is explained sufficiently by the modern numerals appended. In case of mixed numbers, the process was still more clumsy. Divisions are found in Theon of Alexandria's commentary on the *Almagest*. As might be expected, the process is long and tedious.

We have seen in geometry that the more advanced mathematicians frequently had occasion to extract the square root. Thus Archimedes in his *Mensuration of the Circle* gives a large number of square roots. He states, for instance, that $\sqrt{3} < \frac{1351}{780}$ and $\sqrt{3} > \frac{265}{153}$, but he gives no clue to the method by which he obtained these approximations. It is not improbable that the earlier Greek mathematicians found the square root by trial only. Eutocius says that the method of extracting it was given by Heron, Pappus, Theon, and other commentators on the *Almagest*. Theon's is the only ancient method known to us. It is the same as the one used nowadays, except that sexagesimal fractions are employed in place of our decimals. What the mode of procedure actually was when sexagesimal fractions were not used, has been the subject of conjecture on the part of numerous modern writers.[17]

Of interest, in connection with arithmetical symbolism, is the *Sand-Counter* (Arenarius), an essay addressed by **Archi-**

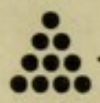
medes to Gelon, king of Syracuse. In it Archimedes shows that people are in error who think the sand cannot be counted, or that if it can be counted, the number cannot be expressed by arithmetical symbols. He shows that the number of grains in a heap of sand not only as large as the whole earth, but as large as the entire universe, can be arithmetically expressed. Assuming that 10,000 grains of sand suffice to make a little solid of the magnitude of a poppy-seed, and that the diameter of a poppy-seed be not smaller than $\frac{1}{40}$ part of a finger's breadth; assuming further, that the diameter of the universe (supposed to extend to the sun) be less than 10,000 diameters of the earth, and that the latter be less than 1,000,000 stadia, Archimedes finds a number which would exceed the number of grains of sand in the sphere of the universe. He goes on even further. Supposing the universe to reach out to the fixed stars, he finds that the sphere, having the distance from the earth's centre to the fixed stars for its radius, would contain a number of grains of sand less than 1000 myriads of the eighth octad. In our notation, this number would be 10^{63} or 1 with 63 ciphers after it. It can hardly be doubted that one object which Archimedes had in view in making this calculation was the improvement of the Greek symbolism. It is not known whether he invented some short notation by which to represent the above number or not.

We judge from fragments in the second book of Pappus that Apollonius proposed an improvement in the Greek method of writing numbers, but its nature we do not know. Thus we see that the Greeks never possessed the boon of a clear, comprehensive symbolism. The honour of giving such to the world, once for all, was reserved by the irony of fate for a nameless Indian of an unknown time, and we know not whom to thank for an invention of such importance to the general progress of intelligence.[6]

Passing from the subject of *logistica* to that of *arithmetica*, our attention is first drawn to the science of numbers of **Pythagoras**. Before founding his school, Pythagoras studied for many years under the Egyptian priests and familiarised himself with Egyptian mathematics and mysticism. If he ever was in Babylon, as some authorities claim, he may have learned the sexagesimal notation in use there; he may have picked up considerable knowledge on the theory of proportion, and may have found a large number of interesting astronomical observations. Saturated with that speculative spirit then pervading the Greek mind, he endeavoured to discover some principle of homogeneity in the universe. Before him, the philosophers of the Ionic school had sought it in the matter of things; Pythagoras looked for it in the structure of things. He observed various numerical relations or analogies between numbers and the phenomena of the universe. Being convinced that it was in numbers and their relations that he was to find the foundation to true philosophy, he proceeded to trace the origin of all things to numbers. Thus he observed that musical strings of equal length stretched by weights having the proportion of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, produced intervals which were an octave, a fifth, and a fourth. Harmony, therefore, depends on musical proportion; it is nothing but a mysterious numerical relation. Where harmony is, there are numbers. Hence the order and beauty of the universe have their origin in numbers. There are seven intervals in the musical scale, and also seven planets crossing the heavens. The same numerical relations which underlie the former must underlie the latter. But where numbers are, there is harmony. Hence his spiritual ear discerned in the planetary motions a wonderful 'harmony of the spheres.' The Pythagoreans invested particular numbers with extraordinary attributes. Thus *one* is the essence of things; it is an absolute number; hence the origin of all numbers and

so of all things. *Four* is the most perfect number, and was in some mystic way conceived to correspond to the human soul. Philolaus believed that 5 is the cause of color, 6 of cold, 7 of mind and health and light, 8 of love and friendship.[6] In Plato's works are evidences of a similar belief in religious relations of numbers. Even Aristotle referred the virtues to numbers.

Enough has been said about these mystic speculations to show what lively interest in mathematics they must have created and maintained. Avenues of mathematical inquiry were opened up by them which otherwise would probably have remained closed at that time.

The Pythagoreans classified numbers into odd and even. They observed that the sum of the series of odd numbers from 1 to $2n+1$ was always a complete square, and that by addition of the even numbers arises the series 2, 6, 12, 20, in which every number can be decomposed into two factors differing from each other by unity. Thus, $6 = 2 \cdot 3$, $12 = 3 \cdot 4$, etc. These latter numbers were considered of sufficient importance to receive the separate name of *heteromecic* (not equilateral).[7] Numbers of the form $\frac{n(n+1)}{2}$ were called *triangular*, because they could always be arranged thus,  . Numbers which were equal to the sum of all their possible factors, such as 6, 28, 496, were called *perfect;* those exceeding that sum, *excessive;* and those which were less, *defective*. *Amicable* numbers were those of which each was the sum of the factors in the other. Much attention was paid by the Pythagoreans to the subject of proportion. The quantities a, b, c, d were said to be in *arithmetical* proportion when $a - b = c - d$; in *geometrical* proportion, when $a : b = c : d$; in *harmonic* proportion, when $a - b : b - c = a : c$. It is probable that the Pythagoreans were also familiar with the *musical*

proportion $a : \frac{a+b}{2} = \frac{2ab}{a+b} : b$. Iamblichus says that Pythagoras introduced it from Babylon.

In connection with arithmetic, Pythagoras made extensive investigations into geometry. He believed that an arithmetical fact had its analogue in geometry, and *vice versa*. In connection with his theorem on the right triangle he devised a rule by which integral numbers could be found, such that the sum of the squares of two of them equalled the square of the third. Thus, take for one side an odd number $(2n+1)$; then $\frac{(2n+1)^2-1}{2} = 2n^2+2n =$ the other side, and $(2n^2+2n+1) =$ hypotenuse. If $2n+1=9$, then the other two numbers are 40 and 41. But this rule only applies to cases in which the hypotenuse differs from one of the sides by 1. In the study of the right triangle there doubtless arose questions of puzzling subtlety. Thus, given a number equal to the side of an isosceles right triangle, to find the number which the hypotenuse is equal to. The side may have been taken equal to 1, 2, $\frac{3}{2}$, $\frac{6}{5}$, or any other number, yet in every instance all efforts to find a number exactly equal to the hypotenuse must have remained fruitless. The problem may have been attacked again and again, until finally "some rare genius, to whom it is granted, during some happy moments, to soar with eagle's flight above the level of human thinking," grasped the happy thought that this problem cannot be solved. In some such manner probably arose the theory of *irrational quantities*, which is attributed by Eudemus to the Pythagoreans. It was indeed a thought of extraordinary boldness, to assume that straight lines could exist, differing from one another not only in length, — that is, in quantity, — but also in a quality, which, though real, was absolutely invisible.[7] Need we wonder that the Pythagoreans saw in

irrationals a deep mystery, a symbol of the unspeakable? We are told that the one who first divulged the theory of irrationals, which the Pythagoreans kept secret, perished in consequence in a shipwreck. Its discovery is ascribed to Pythagoras, but we must remember that all important Pythagorean discoveries were, according to Pythagorean custom, referred back to him. The first incommensurable ratio known seems to have been that of the side of a square to its diagonal, as $1 : \sqrt{2}$. **Theodorus of Cyrene** added to this the fact that the sides of squares represented in length by $\sqrt{3}$, $\sqrt{5}$, etc., up to $\sqrt{17}$, and Theætetus, that the sides of any square, represented by a surd, are incommensurable with the linear unit. **Euclid** (about 300 B.C.), in his *Elements*, X. 9, generalised still further: Two magnitudes whose squares are (or are not) to one another as a square number to a square number are commensurable (or incommensurable), and conversely. In the tenth book, he treats of incommensurable quantities at length. He investigates every possible variety of lines which can be represented by $\sqrt{\sqrt{a} \pm \sqrt{b}}$, a and b representing two commensurable lines, and obtains 25 species. Every individual of every species is incommensurable with all the individuals of every other species. "This book," says De Morgan, "has a completeness which none of the others (not even the fifth) can boast of; and we could almost suspect that Euclid, having arranged his materials in his own mind, and having completely elaborated the tenth book, wrote the preceding books after it, and did not live to revise them thoroughly."[9] The theory of incommensurables remained where Euclid left it, till the fifteenth century.

Euclid devotes the seventh, eighth, and ninth books of his *Elements* to arithmetic. Exactly how much contained in these books is Euclid's own invention, and how much is borrowed from his predecessors, we have no means of knowing.

Without doubt, much is original with Euclid. The *seventh book* begins with twenty-one definitions. All except that for 'prime' numbers are known to have been given by the Pythagoreans. Next follows a process for finding the G.C.D. of two or more numbers. The *eighth book* deals with numbers in continued proportion, and with the mutual relations of squares, cubes, and plane numbers. Thus, XXII., if three numbers are in continued proportion, and the first is a square, so is the third. In the *ninth book*, the same subject is continued. It contains the proposition that the number of primes is greater than any given number.

After the death of Euclid, the theory of numbers remained almost stationary for 400 years. Geometry monopolised the attention of all Greek mathematicians. Only two are known to have done work in arithmetic worthy of mention. **Eratosthenes** (275–194 B.C.) invented a 'sieve' for finding prime numbers. All composite numbers are 'sifted' out in the following manner: Write down the odd numbers from 3 up, in succession. By striking out every third number after the 3, we remove all multiples of 3. By striking out every fifth number after the 5, we remove all multiples of 5. In this way, by rejecting multiples of 7, 11, 13, etc., we have left prime numbers only. **Hypsicles** (between 200 and 100 B.C.) worked at the subjects of polygonal numbers and arithmetical progressions, which Euclid entirely neglected. In his work on 'risings of the stars,' he showed (1) that in an arithmetical series of $2n$ terms, the sum of the last n terms exceeds the sum of the first n by a multiple of n^2; (2) that in such a series of $2n+1$ terms, the sum of the series is the number of terms multiplied by the middle term; (3) that in such a series of $2n$ terms, the sum is half the number of terms multiplied by the two middle terms.[6]

For two centuries after the time of Hypsicles, arithmetic

disappears from history. It is brought to light again about 100 A.D. by **Nicomachus**, a Neo-Pythagorean, who inaugurated the final era of Greek mathematics. From now on, arithmetic was a favourite study, while geometry was neglected. Nicomachus wrote a work entitled *Introductio Arithmetica*, which was very famous in its day. The great number of commentators it has received vouch for its popularity. Boethius translated it into Latin. Lucian could pay no higher compliment to a calculator than this: "You reckon like Nicomachus of Gerasa." The *Introductio Arithmetica* was the first exhaustive work in which arithmetic was treated quite independently of geometry. Instead of drawing lines, like Euclid, he illustrates things by real numbers. To be sure, in his book the old geometrical nomenclature is retained, but the method is inductive instead of deductive. "Its sole business is classification, and all its classes are derived from, and exhibited by, actual numbers." The work contains few results that are really original. We mention one important proposition which is probably the author's own. He states that cubical numbers are always equal to the sum of successive odd numbers. Thus, $8 = 2^3 = 3 + 5$, $27 = 3^3 = 7 + 9 + 11$, $64 = 4^3 = 13 + 15 + 17 + 19$, and so on. This theorem was used later for finding the sum of the cubical numbers themselves. **Theon** of Smyrna is the author of a treatise on "the mathematical rules necessary for the study of Plato." The work is ill arranged and of little merit. Of interest is the theorem, that every square number, or that number minus 1, is divisible by 3 or 4 or both. A remarkable discovery is a proposition given by **Iamblichus** in his treatise on Pythagorean philosophy. It is founded on the observation that the Pythagoreans called 1, 10, 100, 1000, units of the first, second, third, fourth 'course' respectively. The theorem is this: If we add any three consecutive numbers, of which the highest

is divisible by 3, then add the digits of that sum, then, again, the digits of *that* sum, and so on, the final sum will be 6. Thus, $61 + 62 + 63 = 186$, $1 + 8 + 6 = 15$, $1 + 5 = 6$. This discovery was the more remarkable, because the ordinary Greek numerical symbolism was much less likely to suggest any such property of numbers than our "Arabic" notation would have been.

The works of Nicomachus, Theon of Smyrna, Thymaridas, and others contain at times investigations of subjects which are really algebraic in their nature. Thymaridas in one place uses the Greek word meaning "unknown quantity" in a way which would lead one to believe that algebra was not far distant. Of interest in tracing the invention of algebra are the arithmetical epigrams in the *Palatine Anthology*, which contain about fifty problems leading to linear equations. Before the introduction of algebra these problems were propounded as puzzles. A riddle attributed to Euclid and contained in the *Anthology* is to this effect: A mule and a donkey were walking along, laden with corn. The mule says to the donkey, "If you gave me one measure, I should carry twice as much as you. If I gave you one, we should both carry equal burdens. Tell me their burdens, O most learned master of geometry." [6]

It will be allowed, says Gow, that this problem, if authentic, was not beyond Euclid, and the appeal to geometry smacks of antiquity. A far more difficult puzzle was the famous 'cattle-problem,' which Archimedes propounded to the Alexandrian mathematicians. The problem is indeterminate, for from only seven equations, eight unknown quantities in integral numbers are to be found. It may be stated thus: The sun had a herd of bulls and cows, of different colours. (1) Of Bulls, the white (W) were, in number, $(\frac{1}{2} + \frac{1}{3})$ of the blue (B) and yellow (Y): the B were $(\frac{1}{4} + \frac{1}{5})$ of the Y and piebald (P): the

P were $(\frac{1}{6}+\frac{1}{7})$ of the W and Y. (2) Of Cows, which had the same colours (w, b, y, p),

$$w=(\tfrac{1}{3}+\tfrac{1}{4})(B+b) : b=(\tfrac{1}{4}+\tfrac{1}{5})(P+p) : p=(\tfrac{1}{5}+\tfrac{1}{6})(Y+y) : y=(\tfrac{1}{6}+\tfrac{1}{7})(W+w).$$

Find the number of bulls and cows.[6] Another problem in the *Anthology* is quite familiar to school-boys: "Of four pipes, one fills the cistern in one day, the next in two days, the third in three days, the fourth in four days: if all run together, how soon will they fill the cistern?" A great many of these problems, puzzling to an arithmetician, would have been solved easily by an algebraist. They became very popular about the time of Diophantus, and doubtless acted as a powerful stimulus on his mind.

Diophantus was one of the last and most fertile mathematicians of the second Alexandrian school. He died about 330 A.D. His age was eighty-four, as is known from an epitaph to this effect: Diophantus passed $\frac{1}{6}$ of his life in childhood, $\frac{1}{12}$ in youth, and $\frac{1}{7}$ more as a bachelor; five years after his marriage was born a son who died four years before his father, at half his father's age. The place of nativity and parentage of Diophantus are unknown. If his works were not written in Greek, no one would think for a moment that they were the product of Greek mind. There is nothing in his works that reminds us of the classic period of Greek mathematics. His were almost entirely new ideas on a new subject. In the circle of Greek mathematicians he stands alone in his specialty. Except for him, we should be constrained to say that among the Greeks *algebra* was always an unknown science.

Of his works we have lost the *Porisms*, but possess a fragment of *Polygonal Numbers*, and seven books of his great work on *Arithmetica*, said to have been written in 13 books.

If we except the Ahmes papyrus, which contains the first

suggestions of algebraic notation, and of the solution of equations, then his *Arithmetica* is the earliest treatise on algebra now extant. In this work is introduced the idea of an algebraic equation expressed in algebraic symbols. His treatment is purely analytical and completely divorced from geometrical methods. He is, as far as we know, the first to state that "a negative number multiplied by a negative number gives a positive number." This is applied to the multiplication of differences, such as $(x-1)(x-2)$. It must be remarked, however, that Diophantus had no notion whatever of negative numbers standing by themselves. All he knew were differences, such as $(2x-10)$, in which $2x$ could not be smaller than 10 without leading to an absurdity. He appears to be the first who could perform such operations as $(x-1) \times (x-2)$ without reference to geometry. Such identities as $(a+b)^2 = a^2 + 2ab + b^2$, which with Euclid appear in the elevated rank of geometric theorems, are with Diophantus the simplest consequences of the algebraic laws of operation. His sign for subtraction was $\pitchfork$, for equality ι. For unknown quantities he had only one symbol, ς. He had no sign for addition except juxtaposition. Diophantus used but few symbols, and sometimes ignored even these by describing an operation in words when the symbol would have answered just as well.

In the solution of simultaneous equations Diophantus adroitly managed with only one symbol for the unknown quantities and arrived at answers, most commonly, by the method of *tentative assumption*, which consists in assigning to some of the unknown quantities preliminary values, that satisfy only one or two of the conditions. These values lead to expressions palpably wrong, but which generally suggest some stratagem by which values can be secured satisfying all the conditions of the problem.

Diophantus also solved determinate equations of the second degree. We are ignorant of his method, for he nowhere goes through with the whole process of solution, but merely states the result. Thus, "$84x^2 + 7x = 7$, whence x is found $= \frac{1}{4}$." Notice he gives only one root. His failure to observe that a quadratic equation has two roots, even when both roots are positive, rather surprises us. It must be remembered, however, that this same inability to perceive more than one out of the several solutions to which a problem may point is common to all Greek mathematicians. Another point to be observed is that he never accepts as an answer a quantity which is negative or irrational.

Diophantus devotes only the first book of his *Arithmetica* to the solution of determinate equations. The remaining books extant treat mainly of *indeterminate quadratic equations* of the form $Ax^2+Bx+C=y^2$, or of two simultaneous equations of the same form. He considers several but not all the possible cases which may arise in these equations. The opinion of Nesselmann on the method of Diophantus, as stated by Gow, is as follows: "(1) Indeterminate equations of the second degree are treated completely only when the quadratic or the absolute term is wanting: his solution of the equations $Ax^2+C=y^2$ and $Ax^2+Bx+C=y^2$ is in many respects cramped. (2) For the 'double equation' of the second degree he has a definite rule only when the quadratic term is wanting in both expressions: even then his solution is not general. More complicated expressions occur only under specially favourable circumstances." Thus, he solves $Bx + C^2 = y^2$, $B_1x + C_1^2 = y_1^2$.

The extraordinary ability of Diophantus lies rather in another direction, namely, in his wonderful ingenuity to reduce all sorts of equations to particular forms which he knows how to solve. Very great is the variety of problems considered. The 130 problems found in the great work of Diophantus con-

tain over 50 different classes of problems, which are strung together without any attempt at classification. But still more multifarious than the problems are the solutions. General methods are unknown to Diophantus. Each problem has its own distinct method, which is often useless for the most closely related problems. "It is, therefore, difficult for a modern, after studying 100 Diophantine solutions, to solve the 101st." [7]

That which robs his work of much of its scientific value is the fact that he always feels satisfied with one solution, though his equation may admit of an indefinite number of values. Another great defect is the absence of general methods. Modern mathematicians, such as Euler, La Grange, Gauss, had to begin the study of indeterminate analysis anew and received no direct aid from Diophantus in the formulation of methods. In spite of these defects we cannot fail to admire the work for the wonderful ingenuity exhibited therein in the solution of particular equations.

It is still an open question and one of great difficulty whether Diophantus derived portions of his algebra from Hindoo sources or not.

THE ROMANS.

Nowhere is the contrast between the Greek and Roman mind shown forth more distinctly than in their attitude toward the mathematical science. The sway of the Greek was a flowering time for mathematics, but that of the Roman a period of sterility. In philosophy, poetry, and art the Roman was an imitator. But in mathematics he did not even rise to the desire for imitation. The mathematical fruits of Greek genius lay before him untasted. In him a science which had

no direct bearing on practical life could awake no interest. As a consequence, not only the higher geometry of Archimedes and Apollonius, but even the *Elements* of Euclid, were entirely neglected. What little mathematics the Romans possessed did not come from the Greeks, but from more ancient sources. Exactly where and how it originated is a matter of doubt. It seems most probable that the "Roman notation," as well as the practical geometry of the Romans, came from the old Etruscans, who, at the earliest period to which our knowledge of them extends, inhabited the district between the Arno and Tiber.

Livy tells us that the Etruscans were in the habit of representing the number of years elapsed, by driving yearly a nail into the sanctuary of Minerva, and that the Romans continued this practice. A less primitive mode of designating numbers, presumably of Etruscan origin, was a notation resembling the present "Roman notation." This system is noteworthy from the fact that a principle is involved in it which is not met with in any other; namely, the principle of subtraction. If a letter be placed before another of greater value, its value is not to be added to, but subtracted from, that of the greater. In the designation of large numbers a horizontal bar placed over a letter was made to increase its value one thousand fold. In fractions the Romans used the duodecimal system.

Of arithmetical calculations, the Romans employed three different kinds: Reckoning on the fingers, upon the abacus, and by tables prepared for the purpose.[3] Finger-symbolism was known as early as the time of King Numa, for he had erected, says Pliny, a statue of the double-faced Janus, of which the fingers indicated 365 (355?), the number of days in a year. Many other passages from Roman authors point out the use of the fingers as aids to calculation. In fact, a finger-symbolism of practically the same form was in use not only in

Rome, but also in Greece and throughout the East, certainly as early as the beginning of the Christian era, and continued to be used in Europe during the Middle Ages. We possess no knowledge as to where or when it was invented. The second mode of calculation, by the abacus, was a subject of elementary instruction in Rome. Passages in Roman writers indicate that the kind of abacus most commonly used was covered with dust and then divided into columns by drawing straight lines. Each column was supplied with pebbles (calculi, whence 'calculare' and 'calculate') which served for calculation. Additions and subtractions could be performed on the abacus quite easily, but in multiplication the abacus could be used only for adding the particular products, and in division for performing the subtractions occurring in the process. Doubtless at this point recourse was made to mental operations and to the multiplication table. Possibly finger-multiplication may also have been used. But the multiplication of large numbers must, by either method, have been beyond the power of the ordinary arithmetician. To obviate this difficulty, the arithmetical tables mentioned above were used, from which the desired products could be copied at once. Tables of this kind were prepared by *Victorius* of Aquitania. His tables contain a peculiar notation for fractions, which continued in use throughout the Middle Ages. Victorius is best known for his *canon paschalis*, a rule for finding the correct date for Easter, which he published in 457 A.D.

Payments of interest and problems in interest were very old among the Romans. The Roman laws of inheritance gave rise to numerous arithmetical examples. Especially unique is the following: A dying man wills that, if his wife, being with child, gives birth to a son, the son shall receive $\frac{2}{3}$ and she $\frac{1}{3}$ of his estates; but if a daughter is born, she shall receive $\frac{1}{3}$ and his wife $\frac{2}{3}$. It happens that twins are born, a boy and a girl.

How shall the estates be divided so as to satisfy the will? The celebrated Roman jurist, Salvianus Julianus, decided that the estates shall be divided into seven equal parts, of which the son receives four, the wife two, the daughter one.

We next consider Roman geometry. He who expects to find in Rome a science of geometry, with definitions, axioms, theorems, and proofs arranged in logical order, will be disappointed. The only geometry known was a *practical* geometry, which, like the old Egyptian, consisted only of empirical rules. This practical geometry was employed in surveying. Treatises thereon have come down to us, compiled by the Roman surveyors, called *agrimensores* or *gromatici*. One would naturally expect rules to be clearly formulated. But no; they are left to be abstracted by the reader from a mass of numerical examples. "The total impression is as though the Roman gromatic were thousands of years older than Greek geometry, and as though a deluge were lying between the two." Some of their rules were probably inherited from the Etruscans, but others are identical with those of Heron. Among the latter is that for finding the area of a triangle from its sides and the approximate formula, $\frac{13}{30}a^2$, for the area of equilateral triangles (a being one of the sides). But the latter area was also calculated by the formulas $\frac{1}{2}(a^2+a)$ and $\frac{1}{2}a^2$, the first of which was unknown to Heron. Probably the expression $\frac{1}{2}a^2$ was derived from the Egyptian formula $\frac{a+b}{2}\cdot\frac{c+d}{2}$ for the determination of the surface of a quadrilateral. This Egyptian formula was used by the Romans for finding the area, not only of rectangles, but of any quadrilaterals whatever. Indeed, the gromatici considered it even sufficiently accurate to determine the areas of cities, laid out irregularly, simply by measuring their circumferences.[7] Whatever Egyptian geometry the Romans possessed was transplanted across the Mediterranean at the

time of *Julius Cæsar*, who ordered a survey of the whole empire to secure an equitable mode of taxation. Cæsar also reformed the calendar, and, for that purpose, drew from Egyptian learning. He secured the services of the Alexandrian astronomer, *Sosigenes*.

In the fifth century, the Western Roman Empire was fast falling to pieces. Three great branches — Spain, Gaul, and the province of Africa — broke off from the decaying trunk. In 476, the Western Empire passed away, and the Visigothic chief, Odoacer, became king. Soon after, Italy was conquered by the Ostrogoths under Theodoric. It is remarkable that this very period of political humiliation should be the one during which Greek science was studied in Italy most zealously. School-books began to be compiled from the elements of Greek authors. These compilations are very deficient, but are of absorbing interest, from the fact that, down to the twelfth century, they were the only sources of mathematical knowledge in the Occident. Foremost among these writers is **Boethius** (died 524). At first he was a great favourite of King Theodoric, but later, being charged by envious courtiers with treason, he was imprisoned, and at last decapitated. While in prison he wrote *On the Consolations of Philosophy*. As a mathematician, Boethius was a Brobdingnagian among Roman scholars, but a Liliputian by the side of Greek masters. He wrote an *Institutis Arithmetica*, which is essentially a translation of the arithmetic of Nicomachus, and a *Geometry* in several books. Some of the most beautiful results of Nicomachus are omitted in Boethius' arithmetic. The first book on geometry is an extract from Euclid's *Elements*, which contains, in addition to definitions, postulates, and axioms, the theorems in the first three books, without proofs. How can this omission of proofs be accounted for? It has been argued by some that Boethius possessed an incomplete Greek copy of

the *Elements;* by others, that he had Theon's edition before him, and believed that only the theorems came from Euclid, while the proofs were supplied by Theon. The second book, as also other books on geometry attributed to Boethius, teaches, from numerical examples, the mensuration of plane figures after the fashion of the agrimensores.

A celebrated portion in the geometry of Boethius is that pertaining to an abacus, which he attributes to the Pythagoreans. A considerable improvement on the old abacus is there introduced. Pebbles are discarded, and *apices* (probably small cones) are used. Upon each of these apices is drawn a numeral giving it some value below 10. The names of these numerals are pure Arabic, or nearly so, but are added, apparently, by a later hand. These figures are obviously the parents of our modern "Arabic" numerals. The 0 is not mentioned by Boethius in the text. These numerals bear striking resemblance to the Gubar-numerals of the West-Arabs, which are admittedly of Indian origin. These facts have given rise to an endless controversy. Some contended that Pythagoras was in India, and from there brought the nine numerals to Greece, where the Pythagoreans used them secretly. This hypothesis has been generally abandoned, for it is not certain that Pythagoras or any disciple of his ever was in India, nor is there any evidence in any Greek author, that the apices were known to the Greeks, or that numeral signs of any sort were used by them with the abacus. It is improbable, moreover, that the Indian signs, from which the apices are derived, are so old as the time of Pythagoras. A second theory is that the *Geometry* attributed to Boethius is a forgery; that it is not older than the tenth, or possibly the ninth, century, and that the apices are derived from the Arabs. This theory is based on contradictions between passages in the *Arithmetica* and others in the *Geometry.* But

there is an Encyclopædia written by *Cassiodorius* (died about 570) in which both the arithmetic and geometry of Boethius are mentioned. There appears to be no good reason for doubting the trustworthiness of this passage in the Encyclopædia. A third theory (Woepcke's) is that the Alexandrians either directly or indirectly obtained the nine numerals from the Hindoos, about the second century A.D., and gave them to the Romans on the one hand, and to the Western Arabs on the other. This explanation is the most plausible.

MIDDLE AGES.

THE HINDOOS.

The first people who distinguished themselves in mathematical research, after the time of the ancient Greeks, belonged, like them, to the Aryan race. It was, however, not a European, but an Asiatic nation, and had its seat in far-off India.

Unlike the Greek, Indian society was fixed into castes. The only castes enjoying the privilege and leisure for advanced study and thinking were the *Brahmins*, whose prime business was religion and philosophy, and the *Kshatriyas*, who attended to war and government.

Of the development of Hindoo mathematics we know but little. A few manuscripts bear testimony that the Indians had climbed to a lofty height, but their path of ascent is no longer traceable. It would seem that Greek mathematics grew up under more favourable conditions than the Hindoo, for in Greece it attained an independent existence, and was studied for its own sake, while Hindoo mathematics always remained merely a servant to astronomy. Furthermore, in Greece mathematics was a science of the people, free to be cultivated by all who had a liking for it; in India, as in Egypt, it was in the hands chiefly of the priests. Again, the Indians were in the habit of putting into verse all mathematical results they obtained, and of clothing them in obscure and mystic language,

which, though well adapted to aid the memory of him who already understood the subject, was often unintelligible to the uninitiated. Although the great Hindoo mathematicians doubtless reasoned out most or all of their discoveries, yet they were not in the habit of preserving the proofs, so that the naked theorems and processes of operation are all that have come down to our time. Very different in these respects were the Greeks. Obscurity of language was generally avoided, and proofs belonged to the stock of knowledge quite as much as the theorems themselves. Very striking was the difference in the bent of mind of the Hindoo and Greek; for, while the Greek mind was pre-eminently *geometrical*, the Indian was first of all *arithmetical*. The Hindoo dealt with number, the Greek with form. Numerical symbolism, the science of numbers, and algebra attained in India far greater perfection than they had previously reached in Greece. On the other hand, we believe that there was little or no geometry in India of which the source may not be traced back to Greece. Hindoo trigonometry might possibly be mentioned as an exception, but it rested on arithmetic more than on geometry.

An interesting but difficult task is the tracing of the relation between Hindoo and Greek mathematics. It is well known that more or less trade was carried on between Greece and India from early times. After Egypt had become a Roman province, a more lively commercial intercourse sprang up between Rome and India, by way of Alexandria. *A priori*, it does not seem improbable, that with the traffic of merchandise there should also be an interchange of ideas. That communications of thought from the Hindoos to the Alexandrians actually did take place, is evident from the fact that certain philosophic and theologic teachings of the Manicheans, Neo-Platonists, Gnostics, show unmistakable likeness to

Indian tenets. Scientific facts passed also from Alexandria to India. This is shown plainly by the Greek origin of some of the technical terms used by the Hindoos. Hindoo astronomy was influenced by Greek astronomy. Most of the geometrical knowledge which they possessed is traceable to Alexandria, and to the writings of Heron in particular. In algebra there was, probably, a mutual giving and receiving. We suspect that Diophantus got the first glimpses of algebraic knowledge from India. On the other hand, evidences have been found of Greek algebra among the Brahmins. The earliest knowledge of algebra in India may possibly have been of Babylonian origin. When we consider that Hindoo scientists looked upon arithmetic and algebra merely as tools useful in astronomical research, there appears deep irony in the fact that these secondary branches were after all the only ones in which they won real distinction, while in their pet science of astronomy they displayed an inaptitude to observe, to collect facts, and to make inductive investigations.

We shall now proceed to enumerate the names of the leading Hindoo mathematicians, and then to review briefly Indian mathematics. We shall consider the science only in its complete state, for our data are not sufficient to trace the history of the development of methods. Of the great Indian mathematicians, or rather, astronomers, — for India had no mathematicians proper, — **Aryabhatta** is the earliest. He was born 476 A.D., at Pataliputra, on the upper Ganges. His celebrity rests on a work entitled *Aryabhattiyam*, of which the third chapter is devoted to mathematics. About one hundred years later, mathematics in India reached the highest mark. At that time flourished **Brahmagupta** (born 598). In 628 he wrote his *Brahma-sphuta-siddhanta* ("The Revised System of Brahma"), of which the twelfth and eighteenth chapters belong to mathematics. To the fourth or fifth century belongs

an anonymous astronomical work, called *Surya-siddhanta* ("Knowledge from the Sun"), which by native authorities was ranked second only to the *Brahma-siddhanta*, but is of interest to us merely as furnishing evidence that Greek science influenced Indian science even before the time of Aryabhatta. The following centuries produced only two names of importance; namely, **Cridhara**, who wrote a *Ganita-sara* ("Quintessence of Calculation"), and **Padmanabha**, the author of an algebra. The science seems to have made but little progress at this time; for a work entitled *Siddhantaciromani* ("Diadem of an Astronomical System"), written by **Bhaskara Acarya** in 1150, stands little higher than that of Brahmagupta, written over 500 years earlier. The two most important mathematical chapters in this work are the *Lilavati* (= "the beautiful," *i.e.* the noble science) and *Viga-ganita* (= "root-extraction"), devoted to arithmetic and algebra. From now on, the Hindoos in the Brahmin schools seemed to content themselves with studying the masterpieces of their predecessors. Scientific intelligence decreases continually, and in modern times a very deficient Arabic work of the sixteenth century has been held in great authority.[7]

The mathematical chapters of the *Brahma-siddhanta* and *Siddhantaciromani* were translated into English by H. T. Colebrooke, London, 1817. The *Surya-siddhanta* was translated by E. Burgess, and annotated by W. D. Whitney, New Haven, Conn., 1860.

The grandest achievement of the Hindoos and the one which, of all mathematical inventions, has contributed most to the general progress of intelligence, is the invention of the principle of position in writing numbers. Generally we speak of our notation as the "Arabic" notation, but it should be called the "Hindoo" notation, for the Arabs borrowed it from the Hindoos. That the invention of this notation was

not so easy as we might suppose at first thought, may be inferred from the fact that, of other nations, not even the keen-minded Greeks possessed one like it. We inquire, who invented this ideal symbolism, and when? But we know neither the inventor nor the time of invention. That our system of notation is of Indian origin is the only point of which we are certain. From the evolution of ideas in general we may safely infer that our notation did not spring into existence a completely armed Minerva from the head of Jupiter. The nine figures for writing the units are supposed to have been introduced earliest, and the sign of zero and the principle of position to be of later origin. This view receives support from the fact that on the island of Ceylon a notation resembling the Hindoo, but without the zero has been preserved. We know that Buddhism and Indian culture were transplanted to Ceylon about the third century after Christ, and that this culture remained stationary there, while it made progress on the continent. It seems highly probable, then, that the numerals of Ceylon are the old, imperfect numerals of India. In Ceylon, nine figures were used for the units, nine others for the tens, one for 100, and also one for 1000. These 20 characters enabled them to write all the numbers up to 9999. Thus, 8725 would have been written with six signs, representing the following numbers: 8, 1000, 7, 100, 20, 5. These Singhalesian signs, like the old Hindoo numerals, are supposed originally to have been the initial letters of the corresponding numeral adjectives. There is a marked resemblance between the notation of Ceylon and the one used by Aryabhatta in the first chapter of his work, and there only. Although the zero and the principle of position were unknown to the scholars of Ceylon, they were probably known to Aryabhatta; for, in the second chapter, he gives directions for extracting the square and cube roots, which seem to indicate a knowledge of them.

It would appear that the zero and the accompanying principle of position were introduced about the time of Aryabhatta. These are the inventions which give the Hindoo system its great superiority, its admirable perfection.

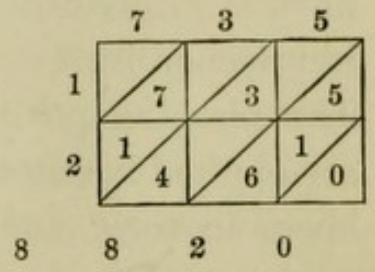
There appear to have been several notations in use in different parts of India, which differed, not in principle, but merely in the forms of the signs employed. Of interest is also a *symbolical system of position*, in which the figures generally were not expressed by numerical adjectives, but by objects suggesting the particular numbers in question. Thus, for 1 were used the words *moon*, *Brahma*, *Creator*, or *form;* for 4, the words *Veda*, (because it is divided into four parts) or *ocean*, etc. The following example, taken from the *Surya-siddhanta*, illustrates the idea. The number 1,577,917,828 is expressed from right to left as follows: Vasu (a class of 8 gods) + two + eight + mountains (the 7 mountain-chains) + form + digits (the 9 digits) + seven + mountains + lunar days (half of which equal 15). The use of such notations made it possible to represent a number in several different ways. This greatly facilitated the framing of verses containing arithmetical rules or scientific constants, which could thus be more easily remembered.

At an early period the Hindoos exhibited great skill in calculating, even with large numbers. Thus, they tell us of an examination to which Buddha, the reformer of the Indian religion, had to submit, when a youth, in order to win the maiden he loved. In arithmetic, after having astonished his examiners by naming all the periods of numbers up to the 53d, he was asked whether he could determine the number of primary atoms which, when placed one against the other, would form a line one mile in length. Buddha found the required answer in this way: 7 primary atoms make a very minute grain of dust, 7 of these make a minute grain of dust,

7 of *these* a grain of dust whirled up by the wind, and so on. Thus he proceeded, step by step, until he finally reached the length of a mile. The multiplication of all the factors gave for the multitude of primary atoms in a mile a number consisting of 15 digits. This problem reminds one of the 'Sand-Counter' of Archimedes.

After the numerical symbolism had been perfected, figuring was made much easier. Many of the Indian modes of operation differ from ours. The Hindoos were generally inclined to follow the motion from left to right, as in writing. Thus, they *added* the left-hand columns first, and made the necessary corrections as they proceeded. For instance, they would have added 254 and 663 thus: $2+6=8$, $5+6=11$, which changes 8 into 9, $4+3=7$. Hence the sum 917. In *subtraction* they had two methods. Thus in $821-348$ they would say, 8 from $11=3$, 4 from $11=7$, 3 from $7=4$. Or they would say, 8 from $11=3$, 5 from $12=7$, 4 from $8=4$. In *multiplication* of a number by another of only one digit, say 569 by 5, they generally said, $5\cdot5=25$, $5\cdot6=30$, which changes 25 into 28, $5\cdot9=45$, hence the 0 must be increased by 4. The product is 2845. In the multiplication with each other of many-figured numbers, they first multiplied, in the manner just indicated, with the left-hand digit of the multiplier, which was written above the multiplicand, and placed the product above the multiplier. On multiplying with the next digit of the multiplier, the product was not placed in a new row, as with us, but the first product obtained was corrected, as the process continued, by erasing, whenever necessary, the old digits, and replacing them by new ones, until finally the whole product was obtained. We who possess the modern luxuries of pencil and paper, would not be likely to fall in love with this Hindoo method. But the Indians wrote "with a cane-pen upon a small blackboard with a white,

thinly liquid paint which made marks that could be easily erased, or upon a white tablet, less than a foot square, strewn with red flour, on which they wrote the figures with a small stick, so that the figures appeared white on a red ground."[7] Since the digits had to be quite large to be distinctly legible, and since the boards were small, it was desirable to have a method which would not require much space. Such a one was the above method of multiplication. Figures could be easily erased and replaced by others without sacrificing neatness. But the Hindoos had also other ways of multiplying, of which we mention the following: The tablet was divided into squares like a chess-board. Diagonals were also drawn, as seen in the figure. The multiplication of $12 \times 735 = 8820$ is exhibited in the adjoining diagram.[3] The manuscripts extant give no information of how *divisions* were executed. The correctness of their additions, subtractions, and multiplications was tested "by excess of 9's." In writing fractions, the numerator was placed above the denominator, but no line was drawn between them.



We shall now proceed to the consideration of some arithmetical problems and the Indian modes of solution. A favourite method was that of *inversion*. With laconic brevity, Aryabhatta describes it thus: "Multiplication becomes division, division becomes multiplication; what was gain becomes loss, what loss, gain; inversion." Quite different from this quotation in style is the following problem from Aryabhatta, which illustrates the method:[3] "Beautiful maiden with beaming eyes, tell me, as thou understandst the right method of inversion, which is the number which multiplied by 3, then increased by $\frac{3}{4}$ of the product, divided by 7, diminished by $\frac{1}{3}$ of the quotient, multiplied by itself, diminished by 52, the square

root extracted, addition of 8, and division by 10, gives the number 2?" The process consists in beginning with 2 and working backwards. Thus, $(2 \cdot 10 - 8)^2 + 52 = 196$, $\sqrt{196} = 14$, and $14 \cdot \frac{3}{2} \cdot 7 \cdot \frac{4}{7} \div 3 = 28$, the answer.

Here is another example taken from *Lilavati*, a chapter in Bhaskara's great work: "The square root of half the number of bees in a swarm has flown out upon a jessamine-bush, $\frac{8}{9}$ of the whole swarm has remained behind; one female bee flies about a male that is buzzing within a lotus-flower into which he was allured in the night by its sweet odour, but is now imprisoned in it. Tell me the number of bees." Answer, 72. The pleasing poetic garb in which all arithmetical problems are clothed is due to the Indian practice of writing all school-books in verse, and especially to the fact that these problems, propounded as puzzles, were a favourite social amusement. Says Brahmagupta: "These problems are proposed simply for pleasure; the wise man can invent a thousand others, or he can solve the problems of others by the rules given here. As the sun eclipses the stars by his brilliancy, so the man of knowledge will eclipse the fame of others in assemblies of the people if he proposes algebraic problems, and still more if he solves them."

The Hindoos solved problems in interest, discount, partnership, alligation, summation of arithmetical and geometric series, devised rules for determining the numbers of combinations and permutations, and invented magic squares. It may here be added that chess, the profoundest of all games, had its origin in India.

The Hindoos made frequent use of the "rule of three," and also of the method of "falsa positio," which is almost identical with that of the "tentative assumption" of Diophantus. These and other rules were applied to a large number of problems.

Passing now to *algebra*, we shall first take up the symbols of operation. Addition was indicated simply by juxtaposition as in Diophantine algebra; subtraction, by placing a dot over the subtrahend; multiplication, by putting after the factors, *bha*, the abbreviation of the word *bhavita*, "the product"; division, by placing the divisor beneath the dividend; square-root, by writing *ka*, from the word *karana* (irrational), before the quantity. The unknown quantity was called by Brahmagupta *yâvattâvat* (*quantum tantum*). When several unknown quantities occurred, he gave, unlike Diophantus, to each a distinct name and symbol. The first unknown was designated by the general term "unknown quantity." The rest were distinguished by names of colours, as the black, blue, yellow, red, or green unknown. The initial syllable of each word constituted the symbol for the respective unknown quantity. Thus *yâ* meant x; *kâ* (from *kâlaka* = black) meant y; *yâ kâ bha*, "x times y"; *ka* 15 *ka* 10, "$\sqrt{15} - \sqrt{10}$."

The Indians were the first to recognise the existence of absolutely negative quantities. They brought out the difference between positive and negative quantities by attaching to the one the idea of 'possession,' to the other that of 'debts.' The conception also of opposite directions on a line, as an interpretation of + and − quantities, was not foreign to them. They advanced beyond Diophantus in observing that a quadratic has always two roots. Thus Bhaskara gives $x = 50$ and $x = -5$ for the roots of $x^2 - 45x = 250$. "But," says he, "the second value is in this case not to be taken, for it is inadequate; people do not approve of negative roots." Commentators speak of this as if negative roots were seen, but not admitted.

Another important generalisation, says Hankel, was this, that the Hindoos never confined their arithmetical operations to rational numbers. For instance, Bhaskara showed how,

by the formula $\sqrt{a+\sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} + \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$ the square root of the sum of rational and irrational numbers could be found. The Hindoos never discerned the dividing line between numbers and magnitudes, set up by the Greeks, which, though the product of a scientific spirit, greatly retarded the progress of mathematics. They passed from magnitudes to numbers and from numbers to magnitudes without anticipating that gap which to a sharply discriminating mind exists between the continuous and discontinuous. Yet by doing so the Indians greatly aided the general progress of mathematics. "Indeed, if one understands by algebra the application of arithmetical operations to complex magnitudes of all sorts, whether rational or irrational numbers or space-magnitudes, then the learned Brahmins of Hindostan are the real inventors of algebra." [7]

Let us now examine more closely the Indian algebra. In extracting the square and cube roots they used the formulas $(a+b)^2 = a^2 + 2ab + b^2$ and $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. In this connection Aryabhatta speaks of dividing a number into periods of two and three digits. From this we infer that the principle of position and the zero in the numeral notation were already known to him. In figuring with zeros, a statement of Bhaskara is interesting. A fraction whose denominator is zero, says he, admits of no alteration, though much be added or subtracted. Indeed, in the same way, no change takes place in the infinite and immutable Deity when worlds are destroyed or created, even though numerous orders of beings be taken up or brought forth. Though in this he apparently evinces clear mathematical notions, yet in other places he makes a complete failure in figuring with fractions of zero denominator.

In the Hindoo solutions of determinate equations, Cantor

thinks he can see traces of Diophantine methods. Some technical terms betray their Greek origin. Even if it be true that the Indians borrowed from the Greeks, they deserve great credit for improving and generalising the solutions of linear and quadratic equations. Bhaskara advances far beyond the Greeks and even beyond Brahmagupta when he says that "the square of a positive, as also of a negative number, is positive; that the square root of a positive number is twofold, positive and negative. There is no square root of a negative number, for it is not a square." Of equations of higher degrees, the Indians succeeded in solving only some special cases in which both sides of the equation could be made perfect powers by the addition of certain terms to each.

Incomparably greater progress than in the solution of determinate equations was made by the Hindoos in the treatment of *indeterminate equations*. Indeterminate analysis was a subject to which the Hindoo mind showed a happy adaptation. We have seen that this very subject was a favourite with Diophantus, and that his ingenuity was almost inexhaustible in devising solutions for particular cases. But the glory of having invented *general* methods in this most subtle branch of mathematics belongs to the Indians. The Hindoo indeterminate analysis differs from the Greek not only in method, but also in aim. The object of the former was to find all possible integral solutions. Greek analysis, on the other hand, demanded not necessarily integral, but simply rational answers. Diophantus was content with a single solution; the Hindoos endeavoured to find all solutions possible. Aryabhatta gives solutions in integers to linear equations of the form $ax \pm by = c$, where a, b, c are integers. The rule employed is called the *pulveriser*. For this, as for most other rules, the Indians give no proof. Their solution is essentially the same as the one of

Euler. Euler's process of reducing $\frac{a}{b}$ to a continued fraction amounts to the same as the Hindoo process of finding the greatest common divisor of a and b by division. This is frequently called the Diophantine method. Hankel protests against this name, on the ground that Diophantus not only never knew the method, but did not even aim at solutions purely integral.[7] These equations probably grew out of problems in astronomy. They were applied, for instance, to determine the time when a certain constellation of the planets would occur in the heavens.

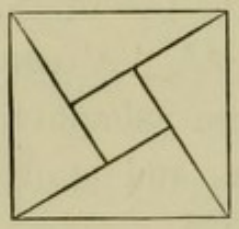
Passing by the subject of linear equations with more than two unknown quantities, we come to indeterminate quadratic equations. In the solution of $xy = ax + by + c$, they applied the method re-invented later by Euler, of decomposing $(ab + c)$ into the product of two integers $m \cdot n$ and of placing $x = m + b$ and $y = n + a$.

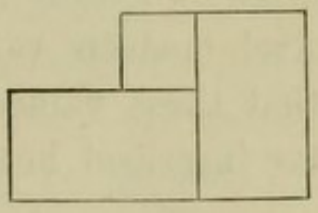
Remarkable is the Hindoo solution of the quadratic equation $cy^2 = ax^2 + b$. With great keenness of intellect they recognised in the special case $y^2 = ax^2 + 1$ a fundamental problem in indeterminate quadratics. They solved it by the *cyclic method*. "It consists," says De Morgan, "in a rule for finding an indefinite number of solutions of $y^2 = ax^2 + 1$ (a being an integer which is not a square), by means of one solution given or found, and of feeling for one solution by making a solution of $y^2 = ax^2 + b$ give a solution of $y^2 = ax^2 + b^2$. It amounts to the following theorem: If p and q be one set of values of x and y in $y^2 = ax^2 + b$ and p' and q' the same or another set, then $qp + pq'$ and $app' + qq'$ are values of x and y in $y^2 = ax^2 + b^2$. From this it is obvious that one solution of $y^2 = ax^2 + 1$ may be made to give any number, and that if, taking b at pleasure, $y^2 = ax^2 + b^2$ can be solved so that x and y are divisible by b, then one preliminary solution of $y^2 = ax^2 + 1$

can be found. Another mode of trying for solutions is a combination of the preceding with the *cuttaca* (pulveriser)." These calculations were used in astronomy.

Doubtless this "cyclic method" constitutes the greatest invention in the theory of numbers before the time of Lagrange. The perversity of fate has willed it, that the equation $y^2 = ax^2 + 1$ should now be called *Pell's* problem, while in recognition of Brahmin scholarship it ought to be called the "Hindoo problem." It is a problem that has exercised the highest faculties of some of our greatest modern analysts. By them the work of the Hindoos was done over again; for, unfortunately, the Arabs transmitted to Europe only a small part of Indian algebra and the original Hindoo manuscripts, which we now possess, were unknown in the Occident.

Hindoo *geometry* is far inferior to the Greek. In it are found no definitions, no postulates, no axioms, no logical chain of reasoning or rigid form of demonstration, as with Euclid. Each theorem stands by itself as an independent truth. Like the early Egyptian, it is empirical. Thus, in the proof of the theorem of the right triangle, Bhaskara draws the right triangle four times in the square of the hypotenuse, so that in the middle there remains a square whose side equals the difference between the two sides of the right triangle. Arranging this square and the four triangles in a different way, they are seen, together, to make up the sum of the square of the two sides. "Behold!" says Bhaskara, without adding another word of explanation. Bretschneider conjectures that the Pythagorean proof was substantially the same as this. In another place, Bhaskara gives a second demonstration of this theorem by drawing from the vertex of





the right angle a perpendicular to the hypotenuse, and comparing the two triangles thus obtained with the given triangle to which they are similar. This proof was unknown in Europe till Wallis rediscovered it. The Brahmins never inquired into the properties of figures. They considered only metrical relations applicable in practical life. In the Greek sense, the Brahmins never had a science of geometry. Of interest is the formula given by Brahmagupta for the area of a triangle in terms of its sides. In the great work attributed to Heron the Elder this formula is first found. Whether the Indians themselves invented it, or whether they borrowed it from Heron, is a disputed question. Several theorems are given by Brahmagupta on quadrilaterals which are true only of those which can be inscribed on a circle — a limitation which he omits to state. Among these is the proposition of Ptolemæus, that the product of the diagonals is equal to the sum of the products of the opposite sides. The Hindoos were familiar with the calculation of the areas of circles and their segments, of the length of chords and perimeters of regular inscribed polygons. An old Indian tradition makes $\pi = 3$, also $= \sqrt{10}$; but Aryabhatta gives the value $\frac{31416}{10000}$. Bhaskara gives two values, — the 'accurate,' $\frac{3927}{1250}$, and the 'inaccurate,' Archimedean value, $\frac{22}{7}$. A commentator on *Lilavati* says that these values were calculated by beginning with a regular inscribed hexagon, and applying repeatedly the formula $AD = \sqrt{2 - \sqrt{4 - \overline{AB}^2}}$, wherein AB is the side of the given polygon, and AD that of one with double the number of sides. In this way were obtained the perimeters of the inscribed polygons of 12, 24, 48, 96, 192, 384 sides. Taking the radius = 100, the perimeter of the last one gives the value which Aryabhatta used for π.

Greater taste than for geometry was shown by the Hindoos for *trigonometry*. Like the Babylonians and Greeks, they

divided the circle into quadrants, each quadrant into 90 degrees and 5400 minutes. The whole circle was therefore made up of 21,600 equal parts. From Bhaskara's 'accurate' value for π it was found that the radius contained 3438 of these circular parts. This last step was not Grecian. The Greeks might have had scruples about taking a part of a *curve* as the measure of a *straight line.* Each quadrant was divided into 24 equal parts, so that each part embraced 225 units of the whole circumference, and corresponds to $3\frac{3}{4}$ degrees. Notable is the fact that the Indians never reckoned, like the Greeks, with the whole chord of double the arc, but always with the *sine* (*joa*) and *versed sine.* Their mode of calculating tables was theoretically very simple. The sine of 90° was equal to the radius, or 3438; the sine of 30° was evidently half that, or 1719. Applying the formula $\sin^2 a + \cos^2 a = r^2$, they obtained $\sin 45° = \sqrt{\frac{r^2}{2}} = 2431$. Substituting for $\cos a$ its equal $\sin (90 - a)$, and making $a = 60°$, they obtained $\sin 60° = \frac{\sqrt{3\,r^2}}{2} = 2978$. With the sines of 90, 60, 45, and 30 as starting-points, they reckoned the sines of half the angles by the formula $\text{ver}\sin 2a = 2\sin^2 a$, thus obtaining the sines of 22° 30′, 11° 15′, 7° 30′, 3° 45′. They now figured out the sines of the complements of these angles, namely, the sines of 86° 15′, 82° 30′, 78° 45′, 75°, 67° 30′; then they calculated the sines of half these angles; then of their complements; then, again, of half their complements; and so on. By this very simple process they got the sines of angles at intervals of 3° 45′. In this table they discovered the unique law that if a, b, c be three successive arcs such that $a - b = b - c = 3° 45'$, then $\sin a - \sin b = (\sin b - \sin c) - \frac{\sin b}{225}$. This formula was afterwards used whenever a re-calculation of tables had to be made. No Indian trigonometrical treatise on the triangle

is extant. In astronomy they solved plane and spherical right triangles.[18]

It is remarkable to what extent Indian mathematics enters into the science of our time. Both the form and the spirit of the arithmetic and algebra of modern times are essentially Indian and not Grecian. Think of that most perfect of mathematical symbolisms — the Hindoo notation, think of the Indian arithmetical operations nearly as perfect as our own, think of their elegant algebraical methods, and then judge whether the Brahmins on the banks of the Ganges are not entitled to some credit. Unfortunately, some of the most brilliant of Hindoo discoveries in indeterminate analysis reached Europe too late to exert the influence they would have exerted, had they come two or three centuries earlier.

THE ARABS.

After the flight of Mohammed from Mecca to Medina in 622 A.D., an obscure people of Semitic race began to play an important part in the drama of history. Before the lapse of ten years, the scattered tribes of the Arabian peninsula were fused by the furnace blast of religious enthusiasm into a powerful nation. With sword in hand the united Arabs subdued Syria and Mesopotamia. Distant Persia and the lands beyond, even unto India, were added to the dominions of the Saracens. They conquered Northern Africa, and nearly the whole Spanish peninsula, but were finally checked from further progress in Western Europe by the firm hand of Charles Martel (732 A.D.). The Moslem dominion extended now from India to Spain; but a war of succession to the caliphate ensued, and in 755 the Mohammedan empire was divided, — one caliph reigning at Bagdad, the other at Cordova

in Spain. Astounding as was the grand march of conquest by the Arabs, still more so was the ease with which they put aside their former nomadic life, adopted a higher civilisation, and assumed the sovereignty over cultivated peoples. Arabic was made the written language throughout the conquered lands. With the rule of the Abbasides in the East began a new period in the history of learning. The capital, Bagdad, situated on the Euphrates, lay half-way between two old centres of scientific thought, — India in the East, and Greece in the West. The Arabs were destined to be the custodians of the torch of Greek and Indian science, to keep it ablaze during the period of confusion and chaos in the Occident, and afterwards to pass it over to the Europeans. Thus science passed from Aryan to Semitic races, and then back again to the Aryan. The Mohammedans have added but little to the knowledge in mathematics which they received. They now and then explored a small region to which the path had been previously pointed out, but they were quite incapable of discovering new fields. Even the more elevated regions in which the Hellenes and Hindoos delighted to wander — namely, the Greek conic sections and the Indian indeterminate analysis — were seldom entered upon by the Arabs. They were less of a speculative, and more of a practical turn of mind.

The Abbasides at Bagdad encouraged the introduction of the sciences by inviting able specialists to their court, irrespective of nationality or religious belief. Medicine and astronomy were their favourite sciences. Thus Haroun-al-Raschid, the most distinguished Saracen ruler, drew Indian physicians to Bagdad. In the year 772 there came to the court of Caliph Almansur a Hindoo astronomer with astronomical tables which were ordered to be translated into Arabic. These tables, known by the Arabs as the *Sindhind*, and

probably taken from the *Brahma-sphuta-siddhanta* of Brahmagupta, stood in great authority. They contained the important Hindoo table of sines.

Doubtless at this time, and along with these astronomical tables, the Hindoo numerals, with the zero and the principle of position, were introduced among the Saracens. Before the time of Mohammed the Arabs had no numerals. Numbers were written out in words. Later, the numerous computations connected with the financial administration over the conquered lands made a short symbolism indispensable. In some localities, the numerals of the more civilised conquered nations were used for a time. Thus in Syria, the Greek notation was retained; in Egypt, the Coptic. In some cases, the numeral adjectives may have been abbreviated in writing. The *Diwani-numerals*, found in an Arabic-Persian dictionary, are supposed to be such abbreviations. Gradually it became the practice to employ the 28 Arabic letters of the alphabet for numerals, in analogy to the Greek system. This notation was in turn superseded by the Hindoo notation, which quite early was adopted by merchants, and also by writers on arithmetic. Its superiority was so universally recognised, that it had no rival, except in astronomy, where the alphabetic notation continued to be used. Here the alphabetic notation offered no great disadvantage, since in the sexagesimal arithmetic, taken from the *Almagest*, numbers of generally only one or two places had to be written.[7]

As regards the form of the so-called Arabic numerals, the statement of the Arabic writer *Albiruni* (died 1039), who spent many years in India, is of interest. He says that the shape of the numerals, as also of the letters in India, differed in different localities, and that the Arabs selected from the various forms the most suitable. An Arabian astronomer says there was among people much difference in the use of

symbols, especially of those for 5, 6, 7, and 8. The symbols used by the Arabs can be traced back to the tenth century. We find material differences between those used by the Saracens in the East and those used in the West. But most surprising is the fact that the symbols of both the East and of the West Arabs deviate so extraordinarily from the Hindoo *Devanagari* numerals (= divine numerals) of to-day, and that they resemble much more closely the apices of the Roman writer Boethius. This strange similarity on the one hand, and dissimilarity on the other, is difficult to explain. The most plausible theory is the one of Woepcke: (1) that about the second century after Christ, before the zero had been invented, the Indian numerals were brought to Alexandria, whence they spread to Rome and also to West Africa; (2) that in the eighth century, after the notation in India had been already much modified and perfected by the invention of the zero, the Arabs at Bagdad got it from the Hindoos; (3) that the Arabs of the West borrowed the Columbus-egg, the zero, from those in the East, but retained the old forms of the nine numerals, if for no other reason, simply to be contrary to their political enemies of the East; (4) that the old forms were remembered by the West-Arabs to be of Indian origin, and were hence called *Gubar-numerals* (= dust-numerals, in memory of the Brahmin practice of reckoning on tablets strewn with dust or sand; (5) that, since the eighth century, the numerals in India underwent further changes, and assumed the greatly modified forms of the modern Devanagari-numerals.[3] This is rather a bold theory, but, whether true or not, it explains better than any other yet propounded, the relations between the apices, the Gubar, the East-Arabic, and Devanagari numerals.

It has been mentioned that in 772 the Indian *Siddhanta* was brought to Bagdad and there translated into Arabic. There

is no evidence that any intercourse existed between Arabic and Indian astronomers either before or after this time, excepting the travels of Albiruni. But we should be very slow to deny the probability that more extended communications actually did take place.

Better informed are we regarding the way in which Greek science, in successive waves, dashed upon and penetrated Arabic soil. In Syria the sciences, especially philosophy and medicine, were cultivated by Greek Christians. Celebrated were the schools at Antioch and Emesa, and, first of all, the flourishing Nestorian school at Edessa. From Syria, Greek physicians and scholars were called to Bagdad. Translations of works from the Greek began to be made. A large number of Greek manuscripts were secured by Caliph *Al Mamun* (813–833) from the emperor in Constantinople and were turned over to Syria. The successors of Al Mamun continued the work so auspiciously begun, until, at the beginning of the tenth century, the more important philosophic, medical, mathematical, and astronomical works of the Greeks could all be read in the Arabic tongue. The translations of mathematical works must have been very deficient at first, as it was evidently difficult to secure translators who were masters of both the Greek and Arabic and at the same time proficient in mathematics. The translations had to be revised again and again before they were satisfactory. The first Greek authors made to speak in Arabic were Euclid and Ptolemæus. This was accomplished during the reign of the famous Haroun-al-Raschid. A revised translation of Euclid's *Elements* was ordered by Al Mamun. As this revision still contained numerous errors, a new translation was made, either by the learned Honein ben Ishak, or by his son, Ishak ben Honein. To the thirteen books of the *Elements* were added the fourteenth, written by Hypsicles, and the fifteenth by Damascius. But it remained for Tabit

ben Korra to bring forth an Arabic Euclid satisfying every need. Still greater difficulty was experienced in securing an intelligible translation of the *Almagest*. Among other important translations into Arabic were the works of Apollonius, Archimedes, Heron, and Diophantus. Thus we see that in the course of one century the Arabs gained access to the vast treasures of Greek science. Having been little accustomed to abstract thought, we need not marvel if, during the ninth century, all their energy was exhausted merely in appropriating the foreign material. No attempts were made at original work in mathematics until the next century.

In astronomy, on the other hand, great activity in original research existed as early as the ninth century. The religious observances demanded by Mohammedanism presented to astronomers several practical problems. The Moslem dominions being of such enormous extent, it remained in some localities for the astronomer to determine which way the "Believer" must turn during prayer that he may be facing Mecca. The prayers and ablutions had to take place at definite hours during the day and night. This led to more accurate determinations of time. To fix the exact date for the Mohammedan feasts it became necessary to observe more closely the motions of the moon. In addition to all this, the old Oriental superstition that extraordinary occurrences in the heavens in some mysterious way affect the progress of human affairs added increased interest to the prediction of eclipses.[7]

For these reasons considerable progress was made. Astronomical tables and instruments were perfected, observatories erected, and a connected series of observations instituted. This intense love for astronomy and astrology continued during the whole Arabic scientific period. As in India, so here, we hardly ever find a man exclusively devoted to pure mathematics. Most of the so-called mathematicians were first of all astronomers.

The first notable author of mathematical books was **Mohammed ben Musa Al Hovarezmi,** who lived during the reign of Caliph Al Mamun (813–833). He was engaged by the caliph in making extracts from the *Sindhind,* in revising the tablets of Ptolemæus, in taking observations at Bagdad and Damascus, and in measuring a degree of the earth's meridian. Important to us is his work on algebra and arithmetic. The portion on arithmetic is not extant in the original, and it was not till 1857 that a Latin translation of it was found. It begins thus: "Spoken has Algoritmi. Let us give deserved praise to God, our leader and defender." Here the name of the author, *Al Hovarezmi,* has passed into *Algoritmi,* from which comes our modern word *algorithm,* signifying the art of computing in any particular way. The arithmetic of Hovarezmi, being based on the principle of position and the Hindoo method of calculation, "excels," says an Arabic writer, "all others in brevity and easiness, and exhibits the Hindoo intellect and sagacity in the grandest inventions." This book was followed by a large number of arithmetics by later authors, which differed from the earlier ones chiefly in the greater variety of methods. Arabian arithmetics generally contained the four operations with integers and fractions, modelled after the Indian processes. They explained the operation of *casting out the 9's,* which was sometimes called the "Hindoo proof." They contained also the *regula falsa* and the *regula duorum falsorum,* by which algebraical examples could be solved without algebra. Both these methods were known to the Indians. The *regula falsa* or *falsa positio* was the assigning of an assumed value to the unknown quantity, which value, if wrong, was corrected by some process like the "rule of three." Diophantus used a method almost identical with this. The *regula duorum falsorum* was as follows:[7] To solve an equation $f(x) = V$, assume, for the moment, two values for x; namely, $x = a$ and $x = b$.

Then form $f(a) = A$ and $f(b) = B$, and determine the errors $V - A = E_a$ and $V - B = E_b$; then the required $x = \frac{bE_a - aE_b}{E_a - E_b}$ is generally a close approximation, but is absolutely accurate whenever $f(x)$ is a linear function of x.

We now return to Hovarezmi, and consider the other part of his work, — the *algebra*. This is the first book known to contain this word itself as title. Really the title consists of two words, *aldshebr walmukabala*, the nearest English translation of which is "restoration" and "reduction." By "restoration" was meant the transposing of negative terms to the other side of the equation; by "reduction," the uniting of similar terms. Thus, $x^2 - 2x = 5x + 6$ passes by aldshebr into $x^2 = 5x + 2x + 6$; and this, by walmukabala, into $x^2 = 7x + 6$. The work on algebra, like the arithmetic, by the same author, contains nothing original. It explains the elementary operations and the solutions of linear and quadratic equations. From whom did the author borrow his knowledge of algebra? That it came entirely from Indian sources is impossible, for the Hindoos had no such rules like the "restoration" and "reduction." They were, for instance, never in the habit of making all terms in an equation positive, as is done by the process of "restoration." Diophantus gives two rules which resemble somewhat those of our Arabic author, but the probability that the Arab got all his algebra from Diophantus is lessened by the considerations that he recognised both roots of a quadratic, while Diophantus noticed only one; and that the Greek algebraist, unlike the Arab, habitually rejected irrational solutions. It would seem, therefore, that the algebra of Hovarezmi was neither purely Indian nor purely Greek, but was a hybrid of the two, with the Greek element predominating.

The algebra of Hovarezmi contains also a few meagre

fragments on *geometry*. He gives the theorem of the right triangle, but proves it after Hindoo fashion and only for the simplest case, when the right triangle is isosceles. He then calculates the areas of the triangle, parallelogram, and circle. For π he uses the value $3\frac{1}{7}$, and also the two Indian, $\pi = \sqrt{10}$ and $\pi = \frac{62832}{20000}$. Strange to say, the last value was afterwards forgotten by the Arabs, and replaced by others less accurate. This bit of geometry doubtless came from India. Later Arabic writers got their geometry almost entirely from Greece.

Next to be noticed are the three sons of **Musa ben Sakir,** who lived in Bagdad at the court of the Caliph Al Mamun. They wrote several works, of which we mention a geometry in which is also contained the well-known formula for the area of a triangle expressed in terms of its sides. We are told that one of the sons travelled to Greece, probably to collect astronomical and mathematical manuscripts, and that on his way back he made acquaintance with Tabit ben Korra. Recognising in him a talented and learned astronomer, Mohammed procured for him a place among the astronomers at the court in Bagdad. **Tabit ben Korra** (836–901) was born at Harran in Mesopotamia. He was proficient not only in astronomy and mathematics, but also in the Greek, Arabic, and Syrian languages. His translations of Apollonius, Archimedes, Euclid, Ptolemy, Theodosius, rank among the best. His dissertation on *amicable numbers* (of which each is the sum of the factors of the other) is the first known specimen of original work in mathematics on Arabic soil. It shows that he was familiar with the Pythagorean theory of numbers. Tabit invented the following rule for finding amicable numbers: If $p = 3 \cdot 2^n - 1$, $q = 3 \cdot 2^{n-1} - 1$, $r = 9 \cdot 2^{2n-1} - 1$ (n being a whole number) are three primes, then $a = 2^n pq$, $b = 2^n r$ are a pair of amicable numbers. Thus, if $n = 2$, then $p = 11$,

$q = 5$, $r = 71$, and $a = 220$, $b = 284$. Tabit also trisected an angle.

Foremost among the astronomers of the ninth century ranked **Al Battani**, called *Albategnius* by the Latins. Battan in Syria was his birthplace. His observations were celebrated for great precision. His work, *De scientia stellarum*, was translated into Latin by Plato Tiburtinus, in the twelfth century. Out of this translation sprang the word 'sinus,' as the name of a trigonometric function. The Arabic word for "sine," *dschiba*, was derived from the Sanscrit *jiva*, and resembled the Arabic word *dschaib*, meaning an indentation or gulf. Hence the Latin "sinus."[3] Al Battani was a close student of Ptolemy, but did not follow him altogether. He took an important step for the better, when he introduced the Indian "sine" or *half* the chord, in place of the *whole* chord of Ptolemy. Another improvement on Greek trigonometry made by the Arabs points likewise to Indian influences. Propositions and operations which were treated by the Greeks geometrically are expressed by the Arabs algebraically. Thus, *Al Battani* at once gets from an equation $\frac{\sin\theta}{\cos\theta} = D$, the value of θ by means of $\sin\theta = \frac{D}{\sqrt{1+D^2}}$, — a process unknown to the ancients. He knows, of course, all the formulas for spherical triangles given in the *Almagest*, but goes further, and adds an important one of his own for oblique-angled triangles; namely, $\cos a = \cos b \cos c + \sin b \sin c \cos A$.

At the beginning of the tenth century political troubles arose in the East, and as a result the house of the Abbasides lost power. One province after another was taken, till, in 945, all possessions were wrested from them. Fortunately, the new rulers at Bagdad, the Persian Buyides, were as much interested in astronomy as their predecessors. The progress of the sciences was not only unchecked, but the conditions

for it became even more favourable. The Emir *Adud-ed-daula* (978–983) gloried in having studied astronomy himself. His son *Saraf-ed-daula* erected an observatory in the garden of his palace, and called thither a whole group of scholars.[7] Among them were *Abul Wefa*, *Al Kuhi*, *Al Sagani*.

Abul Wefa (940–998) was born at Buzshan in Chorassan, a region among the Persian mountains, which has brought forth many Arabic astronomers. He forms an important exception to the unprogressive spirit of Arabian scientists by his brilliant discovery of the *variation* of the moon, an inequality usually supposed to have been first discovered by Tycho Brahe.[11] Abul Wefa translated Diophantus. He is one of the last Arabic translators and commentators of Greek authors. The fact that he esteemed the algebra of Mohammed ben Musa Hovarezmi worthy of his commentary indicates that thus far algebra had made little or no progress on Arabic soil. Abul Wefa invented a method for computing tables of sines which gives the sine of half a degree correct to nine decimal places. He did himself credit by introducing the *tangent* into trigonometry and by calculating a table of tangents. The first step toward this had been taken by Al Battani. Unfortunately, this innovation and the discovery of the moon's variation excited apparently no notice among his contemporaries and followers. "We can hardly help looking upon this circumstance as an evidence of a servility of intellect belonging to the Arabian period." A treatise by Abul Wefa on "geometric constructions" indicates that efforts were being made at that time to improve draughting. It contains a neat construction of the corners of the regular polyedrons on the circumscribed sphere. Here, for the first time, appears the condition which afterwards became very famous in the Occident, that the construction be effected with a single opening of the compass.

Al Kuhi, the second astronomer at the observatory of the emir at Bagdad, was a close student of Archimedes and Apollonius. He solved the problem, to construct a segment of a sphere equal in volume to a given segment and having a curved surface equal in area to that of another given segment. He, **Al Sagani,** and **Al Biruni** made a study of the trisection of angles. **Abul Gud,** an able geometer, solved the problem by the intersection of a parabola with an equilateral hyperbola.

The Arabs had already discovered the theorem that the sum of two cubes can never be a cube. **Abu Mohammed Al Hogendi** of Chorassan thought he had proved this, but we are told that the demonstration was defective. Creditable work in theory of numbers and algebra was done by **Al Karhi** of Bagdad, who lived at the beginning of the eleventh century. His treatise on algebra is the greatest algebraic work of the Arabs. In it he appears as a disciple of Diophantus. He was the first to operate with higher roots and to solve equations of the form $x^{2n} + ax^n = b$. For the solution of quadratic equations he gives both arithmetical and geometric proofs. He was the first Arabic author to give and prove the theorems on the summation of the series: —

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = (1 + 2 + \cdots + n)^{\frac{2n+1}{3}},$$
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

Al Karhi also busied himself with indeterminate analysis. He showed skill in handling the methods of Diophantus, but added nothing whatever to the stock of knowledge already on hand. As a subject for original research, indeterminate analysis was too subtle for even the most gifted of Arabian minds. Rather surprising is the fact that Al Karhi's algebra shows no traces whatever of Hindoo indeterminate analysis.

But most astonishing it is, that an arithmetic by the same author completely excludes the Hindoo numerals. It is constructed wholly after Greek pattern. Abul Wefa also, in the second half of the tenth century, wrote an arithmetic in which Hindoo numerals find no place. This practice is the very opposite to that of other Arabian authors. The question, why the Hindoo numerals were ignored by so eminent authors, is certainly a puzzle. Cantor suggests that at one time there may have been rival schools, of which one followed almost exclusively Greek mathematics, the other Indian.

The Arabs were familiar with geometric solutions of quadratic equations. Attempts were now made to solve cubic equations geometrically. They were led to such solutions by the study of questions like the Archimedean problem, demanding the section of a sphere by a plane so that the two segments shall be in a prescribed ratio. The first to state this problem in form of a cubic equation was **Al Mahani** of Bagdad, while **Abu Gafar Al Hazin** was the first Arab to solve the equation by conic sections. Solutions were given also by Al Kuhi, Al Hasan ben Al Haitam, and others.[20] Another difficult problem, to determine the side of a regular heptagon, required the construction of the side from the equation $x^3 - x^2 - 2x + 1 = 0$. It was attempted by many and at last solved by Abul Gud.

The one who did most to elevate to a *method* the solution of algebraic equations by intersecting conics, was **Omar al Hayyami** of Chorassan, about 1079 A.D. He divides cubics into two classes, the trinomial and quadrinomial, and each class into families and species. Each species is treated separately but according to a general plan. He believed that cubics could not be solved by calculation, nor bi-quadratics by geometry. He rejected negative roots and often failed to discover all the positive ones. Attempts at bi-quadratic equations

were made by Abul Wefa,[30] who solved geometrically $x^4 = a$ and $x^4 + ax^3 = b$.

The solution of cubic equations by intersecting conics was the greatest achievement of the Arabs in algebra. The foundation to this work had been laid by the Greeks, for it was Menæchmus who first constructed the roots of $x^3 - a = 0$ or $x^3 - 2a^3 = 0$. It was not his aim to find the number corresponding to x, but simply to determine the side x of a cube double another cube of side a. The Arabs, on the other hand, had another object in view: to find the roots of given numerical equations. In the Occident, the Arabic solutions of cubics remained unknown until quite recently. Descartes and Thomas Baker invented these constructions anew. The works of Al Hayyami, Al Karhi, Abul Gud, show how the Arabs departed further and further from the Indian methods, and placed themselves more immediately under Greek influences. In this way they barred the road of progress against themselves. The Greeks had advanced to a point where material progress became difficult with their methods; but the Hindoos furnished new ideas, many of which the Arabs now rejected.

With Al Karhi and Omar Al Hayyami, mathematics among the Arabs of the East reached flood-mark, and now it begins to ebb. Between 1100 and 1300 A.D. come the crusades with war and bloodshed, during which European Christians profited much by their contact with Arabian culture, then far superior to their own; but the Arabs got no science from the Christians in return. The crusaders were not the only adversaries of the Arabs. During the first half of the thirteenth century, they had to encounter the wild Mongolian hordes, and, in 1256, were conquered by them under the leadership of *Hulagu*. The caliphate at Bagdad now ceased to exist. At the close of the fourteenth century still another empire was formed by Timur

or *Tamerlane*, the Tartar. During such sweeping turmoil, it is not surprising that science declined. Indeed, it is a marvel that it existed at all. During the supremacy of Hulagu, lived **Nasir Eddin** (1201–1274), a man of broad culture and an able astronomer. He persuaded Hulagu to build him and his associates a large observatory at Maraga. Treatises on algebra, geometry, arithmetic, and a translation of Euclid's *Elements*, were prepared by him. Even at the court of Tamerlane in Samarkand, the sciences were by no means neglected. A group of astronomers was drawn to this court. **Ulug Beg** (1393–1449), a grandson of Tamerlane, was himself an astronomer. Most prominent at this time was **Al Kaschi**, the author of an arithmetic. Thus, during intervals of peace, science continued to be cultivated in the East for several centuries. The last Oriental writer was *Beha Eddin* (1547–1622). His *Essence of Arithmetic* stands on about the same level as the work of Mohammed ben Musa Hovarezmi, written nearly 800 years before.

"Wonderful is the expansive power of Oriental peoples, with which upon the wings of the wind they conquer half the world, but more wonderful the energy with which, in less than two generations, they raise themselves from the lowest stages of cultivation to scientific efforts." During all these centuries, astronomy and mathematics in the Orient greatly excel these sciences in the Occident.

Thus far we have spoken only of the Arabs in the East. Between the Arabs of the East and of the West, which were under separate governments, there generally existed considerable political animosity. In consequence of this, and of the enormous distance between the two great centres of learning, Bagdad and Cordova, there was less scientific intercourse among them than might be expected to exist between peoples having the same religion and written language. Thus the

course of science in Spain was quite independent of that in Persia. While wending our way westward to Cordova, we must stop in Egypt long enough to observe that there, too, scientific activity was rekindled. Not Alexandria, but Cairo with its library and observatory, was now the home of learning. Foremost among her scientists ranked **Ben Junus** (died 1008), a contemporary of Abul Wefa. He solved some difficult problems in spherical trigonometry. Another Egyptian astronomer was **Ibn Al Haitam** (died 1038), who wrote on geometric loci. Travelling westward, we meet in Morocco **Abul Hasan Ali**, whose treatise 'on astronomical instruments' discloses a thorough knowledge of the *Conics* of Apollonius. Arriving finally in Spain at the capital, Cordova, we are struck by the magnificent splendour of her architecture. At this renowned seat of learning, schools and libraries were founded during the tenth century.

Little is known of the progress of mathematics in Spain. The earliest name that has come down to us is **Al Madshriti** (died 1007), the author of a mystic paper on 'amicable numbers.' His pupils founded schools at Cordova, Dania, and Granada. But the only great astronomer among the Saracens in Spain is **Gabir ben Aflah** of Sevilla, frequently called *Geber*. He lived in the second half of the eleventh century. It was formerly believed that he was the inventor of algebra, and that the word *algebra* came from 'Gabir' or 'Geber.' He ranks among the most eminent astronomers of this time, but, like so many of his contemporaries, his writings contain a great deal of mysticism. His chief work is an astronomy in nine books, of which the first is devoted to trigonometry. In his treatment of spherical trigonometry, he exercises great independence of thought. He makes war against the time-honoured procedure adopted by Ptolemy of applying "the rule of six quantities," and gives a new way of his own, based on the 'rule of four

quantities.' This is: If PP_1 and QQ_1 be two arcs of great circles intersecting in A, and if PQ and P_1Q_1 be arcs of great circles drawn perpendicular to QQ_1, then we have the proportion

$$\sin AP : \sin PQ = \sin AP_1 : \sin P_1Q_1.$$

From this he derives the formulas for spherical right triangles. To the four fundamental formulas already given by Ptolemy, he added a fifth, discovered by himself. If a, b, c, be the sides, and A, B, C, the angles of a spherical triangle, right-angled at A, then $\cos B = \cos b \sin C$. This is frequently called "Geber's Theorem." Radical and bold as were his innovations in spherical trigonometry, in plane trigonometry he followed slavishly the old beaten path of the Greeks. Not even did he adopt the Indian 'sine' and 'cosine,' but still used the Greek 'chord of double the angle.' So painful was the departure from old ideas, even to an independent Arab! After the time of Gabir ben Aflah there was no mathematician among the Spanish Saracens of any reputation. In the year in which Columbus discovered America, the Moors lost their last foothold on Spanish soil.

We have witnessed a laudable intellectual activity among the Arabs. They had the good fortune to possess rulers who, by their munificence, furthered scientific research. At the courts of the caliphs, scientists were supplied with libraries and observatories. A large number of astronomical and mathematical works were written by Arabic authors. Yet we fail to find a single important principle in mathematics brought forth by the Arabic mind. Whatever discoveries they made, were in fields previously traversed by the Greeks or the Indians, and consisted of objects which the latter had overlooked in their rapid march. The Arabic mind did not possess that penetrative insight and invention by which mathematicians in Europe afterwards revolutionised the science.

The Arabs were learned, but not original. Their chief service to science consists in this, that they adopted the learning of Greece and India, and kept what they received with scrupulous care. When the love for science began to grow in the Occident, they transmitted to the Europeans the valuable treasures of antiquity. Thus a Semitic race was, during the Dark Ages, the custodian of the Aryan intellectual possessions.

EUROPE DURING THE MIDDLE AGES.

With the third century after Christ begins an era of migration of nations in Europe. The powerful Goths quit their swamps and forests in the North and sweep onward in steady southwestern current, dislodging the Vandals, Sueves, and Burgundians, crossing the Roman territory, and stopping and recoiling only when reaching the shores of the Mediterranean. From the Ural Mountains wild hordes sweep down on the Danube. The Roman Empire falls to pieces, and the Dark Ages begin. But dark though they seem, they are the germinating season of the institutions and nations of modern Europe. The Teutonic element, partly pure, partly intermixed with the Celtic and Latin, produces that strong and luxuriant growth, the modern civilisation of Europe. Almost all the various nations of Europe belong to the Aryan stock. As the Greeks and the Hindoos — both Aryan races — were the great thinkers of antiquity, so the nations north of the Alps became the great intellectual leaders of modern times.

Introduction of Roman Mathematics.

We shall now consider how these as yet barbaric nations of the North gradually came in possession of the intellectual

treasures of antiquity. With the spread of Christianity the Latin language was introduced not only in ecclesiastical but also in scientific and all important worldly transactions. Naturally the science of the Middle Ages was drawn largely from Latin sources. In fact, during the earlier of these ages Roman authors were the only ones read in the Occident. Though Greek was not wholly unknown, yet before the thirteenth century not a single Greek scientific work had been read or translated into Latin. Meagre indeed was the science which could be gotten from Roman writers, and we must wait several centuries before any substantial progress is made in mathematics.

After the time of Boethius and Cassiodorius mathematical activity in Italy died out. The first slender blossom of science among tribes that came from the North was an encyclopædia entitled *Origines*, written by **Isidorus** (died 636 as bishop of Seville). This work is modelled after the Roman encyclopædias of Martianus Capella of Carthage and of Cassiodorius. Part of it is devoted to the quadrivium, arithmetic, music, geometry, and astronomy. He gives definitions and grammatical explications of technical terms, but does not describe the modes of computation then in vogue. After Isidorus there follows a century of darkness which is at last dissipated by the appearance of **Bede the Venerable** (672–735), the most learned man of his time. He was a native of Ireland, then the home of learning in the Occident. His works contain treatises on the *Computus*, or the computation of Easter-time, and on finger-reckoning. It appears that a finger-symbolism was then widely used for calculation. The correct determination of the time of Easter was a problem which in those days greatly agitated the Church. It became desirable to have at least one monk at each monastery who could determine the day of religious festivals and could compute the calendar.

Such determinations required some knowledge of arithmetic. Hence we find that the art of calculating always found some little corner in the curriculum for the education of monks.

The year in which Bede died is also the year in which **Alcuin** (735–804) was born. Alcuin was educated in Ireland, and was called to the court of Charlemagne to direct the progress of education in the great Frankish Empire. Charlemagne was a great patron of learning and of learned men. In the great sees and monasteries he founded schools in which were taught the psalms, writing, singing, computation (*computus*), and grammar. By *computus* was here meant, probably, not merely the determination of Easter-time, but the art of computation in general. Exactly what modes of reckoning were then employed we have no means of knowing. It is not likely that Alcuin was familiar with the apices of Boethius or with the Roman method of reckoning on the abacus. He belongs to that long list of scholars who dragged the theory of numbers into theology. Thus the number of beings created by God, who created all things well, is 6, because 6 is a perfect number (the sum of its divisors being $1 + 2 + 3 = 6$); 8, on the other hand, is an imperfect number ($1 + 2 + 4 < 8$); hence the second origin of mankind emanated from the number 8, which is the number of souls said to have been in Noah's ark.

There is a collection of "Problems for Quickening the Mind" (*propositiones ad acuendos iuvenes*), which are certainly as old as 1000 A.D. and possibly older. Cantor is of the opinion that they were written much earlier and by Alcuin. The following is a specimen of these "Problems": A dog chasing a rabbit, which has a start of 150 feet, jumps 9 feet every time the rabbit jumps 7. In order to determine in how many leaps the dog overtakes the rabbit, 150 is to be divided by 2. In this collection of problems, the areas of triangular and quadrangular pieces of land are found by the same formulas of

approximation as those used by the Egyptians and given by Boethius in his geometry. An old problem is the "cistern-problem" (given the time in which several pipes can fill a cistern singly, to find the time in which they fill it jointly), which has been found previously in Heron, in the Greek *Anthology*, and in Hindoo works. Many of the problems show that the collection was compiled chiefly from Roman sources. The problem which, on account of its uniqueness, gives the most positive testimony regarding the Roman origin is that on the interpretation of a will in a case where twins are born. The problem is identical with the Roman, except that different ratios are chosen. Of the exercises for recreation, we mention the one of the wolf, goat, and cabbage, to be rowed across a river in a boat holding only one besides the ferry-man. Query: How must he carry them across so that the goat shall not eat the cabbage, nor the wolf the goat? The solutions of the "problems for quickening the mind" require no further knowledge than the recollection of some few formulas used in surveying, the ability to solve linear equations and to perform the four fundamental operations with integers. Extraction of roots was nowhere demanded; fractions hardly ever occur.[3]

The great empire of Charlemagne tottered and fell almost immediately after his death. War and confusion ensued. Scientific pursuits were abandoned, not to be resumed until the close of the tenth century, when under Saxon rule in Germany and Capetian in France, more peaceful times began. The thick gloom of ignorance commenced to disappear. The zeal with which the study of mathematics was now taken up by the monks is due principally to the energy and influence of one man, — **Gerbert.** He was born in Aurillac in Auvergne. After receiving a monastic education, he engaged in study, chiefly of mathematics, in Spain. On his return he taught

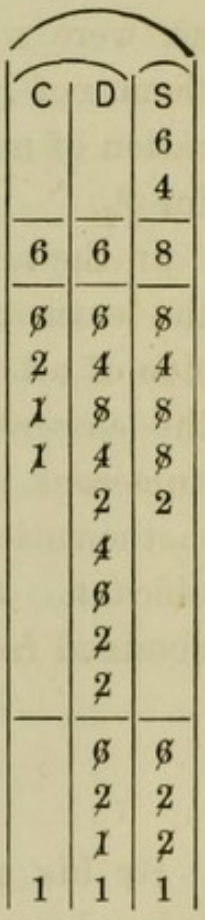
school at Rheims for ten years and became distinguished for his profound scholarship. By King Otto I. and his successors Gerbert was held in highest esteem. He was elected bishop of Rheims, then of Ravenna, and finally was made Pope under the name of Sylvester II. by his former pupil Emperor Otho III. He died in 1003, after a life intricately involved in many political and ecclesiastical quarrels. Such was the career of the greatest mathematician of the tenth century in Europe. By his contemporaries his mathematical knowledge was considered wonderful. Many even accused him of criminal intercourse with evil spirits.

Gerbert enlarged the stock of his knowledge by procuring copies of rare books. Thus in Mantua he found the geometry of Boethius. Though this is of small scientific value, yet it is of great importance in history. It was at that time the only book from which European scholars could learn the elements of geometry. Gerbert studied it with zeal, and is generally believed himself to be the author of a geometry. H. Weissenborn denies his authorship, and claims that the book in question consists of three parts which cannot come from one and the same author.[21] This geometry contains nothing more than the one of Boethius, but the fact that occasional errors in the latter are herein corrected shows that the author had mastered the subject. "The first mathematical paper of the Middle Ages which deserves this name," says Hankel, "is a letter of Gerbert to Adalbold, bishop of Utrecht," in which is explained the reason why the area of a triangle, obtained "geometrically" by taking the product of the base by half its altitude, differs from the area calculated "arithmetically," according to the formula $\frac{1}{2}a\,(a+1)$, used by surveyors, where a stands for a side of an equilateral triangle. He gives the correct explanation that in the latter formula all the small squares, in which the triangle is sup-

posed to be divided, are counted in wholly, even though parts of them project beyond it.

Gerbert made a careful study of the arithmetical works of Boethius. He himself published two works, — *Rule of Computation on the Abacus*, and *A Small Book on the Division of Numbers*. They give an insight into the methods of calculation practised in Europe before the introduction of the Hindoo numerals. Gerbert used the abacus, which was probably unknown to Alcuin. **Bernelinus**, a pupil of Gerbert, describes it as consisting of a smooth board upon which geometricians were accustomed to strew blue sand, and then to draw their diagrams. For arithmetical purposes the board was divided into 30 columns, of which 3 were reserved for fractions, while the remaining 27 were divided into groups with 3 columns in each. In every group the columns were marked respectively by the letters C (*centum*), D (*decem*), and S (*singularis*) or M (*monas*). Bernelinus gives the nine numerals used, which are the apices of Boethius, and then remarks that the Greek letters may be used in their place.[3] By the use of these columns any number can be written without introducing a zero, and all operations in arithmetic can be performed in the same way as we execute ours without the columns, but with the symbol for zero. Indeed, the methods of adding, subtracting, and multiplying in vogue among the abacists agree substantially with those of to-day. But in a division there is very great difference. The early rules for division appear to have been framed to satisfy the following three conditions: (1) The use of the multiplication table shall be restricted as far as possible; at least, it shall never be required to multiply mentally a figure of two digits by another of one digit. (2) Subtractions shall be avoided as much as possible and replaced by additions. (3) The operation shall proceed in a purely mechanical way, without requiring trials.[7]

That it should be necessary to make such conditions seems strange to us; but it must be remembered that the monks of the Middle Ages did not attend school during childhood and learn the multiplication table while the memory was fresh. Gerbert's rules for division are the oldest extant. They are so brief as to be very obscure to the uninitiated. They were probably intended simply to aid the memory by calling to mind the successive steps in the work. In later manuscripts they are stated more fully. In dividing any number by another of one digit, say 668 by 6, the divisor was first increased to 10 by adding 4. The process is exhibited in the adjoining figure.[3] As it continues, we must imagine the digits which are crossed out, to be erased and then replaced by the ones beneath. It is as follows: $600 \div 10 = 60$, but, to rectify the error, 4×60, or 240, must be added; $200 \div 10 = 20$, but 4×20, or 80, must be added. We now write for $60 + 40 + 80$, its sum 180, and continue thus: $100 \div 10 = 10$; the correction necessary is 4×10, or 40, which, added to 80, gives 120. Now $100 \div 10 = 10$, and the correction 4×10, together with the 20, gives 60. Proceeding as before, $60 \div 10 = 6$; the correction is $4 \times 6 = 24$. Now $20 \div 10 = 2$, the correction being $4 \times 2 = 8$. In the column of units we have now $8 + 4 + 8$, or 20. As before, $20 \div 10 = 2$; the correction is $2 \times 4 = 8$, which is not divisible by 10, but only by 6, giving the quotient 1 and the remainder 2. All the partial quotients taken together give $60 + 20 + 10 + 10 + 6 + 2 + 2 + 1 = 111$, and the remainder 2.



Similar but more complicated, is the process when the divisor contains two or more digits. Were the divisor 27,

then the next higher multiple of 10, or 30, would be taken for the divisor, but corrections would be required for the 3. He who has the patience to carry such a division through to the end, will understand why it has been said of Gerbert that "Regulas dedit, quæ a sudantibus abacistis vix intelliguntur." He will also perceive why the Arabic method of division, when first introduced, was called the *divisio aurea*, but the one on the abacus, the *divisio ferrea*.

In his book on the abacus, Bernelinus devotes a chapter to fractions. These are, of course, the *duodecimals*, first used by the Romans. For want of a suitable notation, calculation with them was exceedingly difficult. It would be so even to us, were we accustomed, like the early abacists, to express them, not by a numerator or denominator, but by the application of names, such as *uncia* for $\frac{1}{12}$, *quincunx* for $\frac{5}{12}$, *dodrans* for $\frac{9}{12}$.

In the tenth century, Gerbert was the central figure among the learned. In his time the Occident came into secure possession of all mathematical knowledge of the Romans. During the eleventh century it was studied assiduously. Though numerous works were written on arithmetic and geometry, mathematical knowledge in the Occident was still very insignificant. Scanty indeed were the mathematical treasures obtained from Roman sources.

Translation of Arabic Manuscripts.

By his great erudition and phenomenal activity, Gerbert infused new life into the study not only of mathematics, but also of philosophy. Pupils from France, Germany, and Italy gathered at Rheims to enjoy his instruction. When they themselves became teachers, they taught of course not only the use of the abacus and geometry, but also what they had

learned of the philosophy of Aristotle. His philosophy was known, at first, only through the writings of Boethius. But the growing enthusiasm for it created a demand for his complete works. Greek texts were wanting. But the Latins heard that the Arabs, too, were great admirers of Peripatetism, and that they possessed translations of Aristotle's works and commentaries thereon. This led them finally to search for and translate Arabic manuscripts. During this search, mathematical works also came to their notice, and were translated into Latin. Though some few unimportant works may have been translated earlier, yet the period of greatest activity began about 1100. The zeal displayed in acquiring the Mohammedan treasures of knowledge excelled even that of the Arabs themselves, when, in the eighth century, they plundered the rich coffers of Greek and Hindoo science.

Among the earliest scholars engaged in translating manuscripts into Latin was **Athelard of Bath.** The period of his activity is the first quarter of the twelfth century. He travelled extensively in Asia Minor, Egypt, and Spain, and braved a thousand perils, that he might acquire the language and science of the Mohammedans. He made the earliest translations, from the Arabic, of Euclid's *Elements* and of the astronomical tables of Mohammed ben Musa Hovarezmi. In 1857, a manuscript was found in the library at Cambridge, which proved to be the arithmetic by Mohammed ben Musa in Latin. This translation also is very probably due to Athelard.

At about the same time flourished *Plato of Tivoli* or *Plato Tiburtinus.* He effected a translation of the astronomy of Al Battani and of the *Sphærica* of Theodosius. Through the former, the term *sinus* was introduced into trigonometry.

About the middle of the twelfth century there was a group of Christian scholars busily at work at Toledo, under the

leadership of Raymond, then archbishop of Toledo. Among those who worked under his direction, **John of Seville** was most prominent. He translated works chiefly on Aristotelian philosophy. Of importance to us is a *liber algorismi*, compiled by him from Arabic authors. On comparing works like this with those of the abacists, we notice at once the most striking difference, which shows that the two parties drew from independent sources. It is argued by some that Gerbert got his apices and his arithmetical knowledge, not from Boethius, but from the Arabs in Spain, and that part or the whole of the geometry of Boethius is a forgery, dating from the time of Gerbert. If this were the case, then the writings of Gerbert would betray Arabic sources, as do those of John of Seville. But no points of resemblance are found. Gerbert could not have learned from the Arabs the use of the abacus, because all evidence we have goes to show that they did not employ it. Nor is it probable that he borrowed from the Arabs the apices, because they were never used in Europe except on the abacus. In illustrating an example in division, mathematicians of the tenth and eleventh centuries state an example in Roman numerals, then draw an abacus and insert in it the necessary numbers with the apices. Hence it seems probable that the abacus and apices were borrowed from the same source. The contrast between authors like John of Seville, drawing from Arabic works, and the abacists, consists in this, that, unlike the latter, the former mention the Hindoos, use the term *algorism*, calculate with the zero, and do not employ the abacus. The former teach the extraction of roots, the abacists do not; they teach the sexagesimal fractions used by the Arabs, while the abacists employ the duodecimals of the Romans.[3]

A little later than John of Seville flourished **Gerard of Cremona** in Lombardy. Being desirous to gain possession of

the *Almagest*, he went to Toledo, and there, in 1175, translated this great work of Ptolemy. Inspired by the richness of Mohammedan literature, he gave himself up to its study. He translated into Latin over 70 Arabic works. Of mathematical treatises, there were among these, besides the *Almagest*, the 15 books of Euclid, the *Sphærica* of Theodosius, a work of Menelaus, the algebra of Mohammed ben Musa Hovarezmi, the astronomy of Dshabir ben Aflah, and others less important.

In the thirteenth century, the zeal for the acquisition of Arabic learning continued. Foremost among the patrons of science at this time ranked Emperor Frederick II. of Hohenstaufen (died 1250). Through frequent contact with Mohammedan scholars, he became familiar with Arabic science. He employed a number of scholars in translating Arabic manuscripts, and it was through him that we came in possession of a new translation of the *Almagest*. Another royal head deserving mention as a zealous promoter of Arabic science was Alfonso X. of Castile (died 1284). He gathered around him a number of Jewish and Christian scholars, who translated and compiled astronomical works from Arabic sources. **Rabbi Zag** and **Iehuda ben Mose Cohen** were the most prominent among them. Astronomical tables prepared by these two Jews spread rapidly in the Occident, and constituted the basis of all astronomical calculation till the sixteenth century.[7] The number of scholars who aided in transplanting Arabic science upon Christian soil was large. But we mention only one more. **Giovanni Campano** of Novara (about 1260) brought out a new translation of Euclid, which drove the earlier ones from the field, and which formed the basis of the printed editions.[7]

At the close of the twelfth century, the Occident was in possession of the so-called Arabic notation. The Hindoo methods of calculation began to supersede the cumbrous meth-

ods inherited from Rome. Algebra, with its rules for solving linear and quadratic equations, had been made accessible to the Latins. The geometry of Euclid, the *Sphærica* of Theodosius, the astronomy of Ptolemy, and other works were now accessible in the Latin tongue. Thus a great amount of new scientific material had come into the hands of the Christians. The talent necessary to digest this heterogeneous mass of knowledge was not wanting. The figure of Leonardo of Pisa adorns the vestibule of the thirteenth century.

It is important to notice that no work either on mathematics or astronomy was translated directly from the Greek previous to the fifteenth century.

The First Awakening and its Sequel.

Thus far, France and the British Isles have been the headquarters of mathematics in Christian Europe. But at the beginning of the thirteenth century the talent and activity of one man was sufficient to assign the mathematical science a new home in Italy. This man was not a monk, like Bede, Alcuin, or Gerbert, but a merchant, who in the midst of business pursuits found time for scientific study. **Leonardo of Pisa** is the man to whom we owe the first renaissance of mathematics on Christian soil. He is also called *Fibonacci*, *i.e.* son of Bonaccio. His father was secretary at one of the numerous factories erected on the south and east coast of the Mediterranean by the enterprising merchants of Pisa. He made Leonardo, when a boy, learn the use of the abacus. The boy acquired a strong taste for mathematics, and, in later years, during his extensive business travels in Egypt, Syria, Greece, and Sicily, collected from the various peoples all the knowledge he could get on this subject. Of all the methods of calculation, he found the Hindoo to be unquestionably the

best. Returning to Pisa, he published, in 1202, his great work, the *Liber Abaci*. A revised edition of this appeared in 1228. This work contains about all the knowledge the Arabs possessed in arithmetic and algebra, and treats the subject in a free and independent way. This, together with the other books of Leonardo, shows that he was not merely a compiler, or, like other writers of the Middle Ages, a slavish imitator of the form in which the subject had been previously presented, but that he was an original worker of exceptional power.

He was the first great mathematician to advocate the adoption of the "Arabic notation." The calculation with the zero was the portion of Arabic mathematics earliest adopted by the Christians. The minds of men had been prepared for the reception of this by the use of the abacus and the apices. The reckoning with columns was gradually abandoned, and the very word *abacus* changed its meaning and became a synonym for *algorism*. For the zero, the Latins adopted the name *zephirum*, from the Arabic *sifr* (*sifra* = empty); hence our English word *cipher*. The new notation was accepted readily by the enlightened masses, but, at first, rejected by the learned circles. The merchants of Italy used it as early as the thirteenth century, while the monks in the monasteries adhered to the old forms. In 1299, nearly 100 years after the publication of Leonardo's *Liber Abaci*, the Florentine merchants were forbidden the use of the Arabic numerals in book-keeping, and ordered either to employ the Roman numerals or to write the numeral adjectives out in full. In the fifteenth century the abacus with its counters ceased to be used in Spain and Italy. In France it was used later, and it did not disappear in England and Germany before the middle of the seventeenth century.[22] Thus, in the *Winter's Tale* (iv. 3), Shakespeare lets the clown be embarrassed by

a problem which he could not do without counters. Iago (in *Othello*, i. 1) expresses his contempt for Michael Cassio, "forsooth a great mathematician," by calling him a "counter-caster." So general, indeed, says Peacock, appears to have been the practice of this species of arithmetic, that its rules and principles form an essential part of the arithmetical treatises of that day. The real fact seems to be that the old methods were used long after the Hindoo numerals were in common and general use. With such dogged persistency does man cling to the old!

The *Liber Abaci* was, for centuries, the storehouse from which authors got material for works on arithmetic and algebra. In it are set forth the most perfect methods of calculation with integers and fractions, known at that time; the square and cube root are explained; equations of the first and second degree leading to problems, either determinate or indeterminate, are solved by the methods of 'single' or 'double position,' and also by real algebra. The book contains a large number of problems. The following was proposed to Leonardo of Pisa by a magister in Constantinople, as a difficult problem: If A gets from B 7 denare, then A's sum is five-fold B's; if B gets from A 5 denare, then B's sum is seven-fold A's. How much has each? The *Liber Abaci* contains another problem, which is of historical interest, because it was given with some variations by Ahmes, 3000 years earlier: 7 old women go to Rome; each woman has 7 mules, each mule carries 7 sacks, each sack contains 7 loaves, with each loaf are 7 knives, each knife is put up in 7 sheaths. What is the sum total of all named? *Ans.* 137,256.[3]

In 1220, Leonardo of Pisa published his *Practica Geometriæ*, which contains all the knowledge of geometry and trigonometry transmitted to him. The writings of Euclid and of some other Greek masters were known to him, either from Arabic

manuscripts directly or from the translations made by his countrymen, Gerard of Cremona and Plato of Tivoli. Leonardo's *Geometry* contains an elegant geometrical demonstration of Heron's formula for the area of a triangle, as a function of its three sides. Leonardo treats the rich material before him with skill and Euclidean rigour.

Of still greater interest than the preceding works are those containing Fibonacci's original investigations. We must here preface that after the publication of the *Liber Abaci*, Leonardo was presented by the astronomer Dominicus to Emperor Frederick II. of Hohenstaufen. On that occasion, John of Palermo, an imperial notary, proposed several problems, which Leonardo solved promptly. The first problem was to find a number x, such that $x^2 + 5$ and $x^2 - 5$ are each square numbers. The answer is $x = 3\frac{5}{12}$; for $(3\frac{5}{12})^2 + 5 = (4\frac{1}{12})^2$, $(3\frac{5}{12})^2 - 5 = (2\frac{7}{12})^2$. His masterly solution of this is given in his *liber quadratorum*, a copy of which work was sent by him to Frederick II. The problem was not original with John of Palermo, since the Arabs had already solved similar ones. Some parts of Leonardo's solution may have been borrowed from the Arabs, but the method which he employed of building squares by the summation of odd numbers is original with him.

The second problem proposed to Leonardo at the famous scientific tournament which accompanied the presentation of this celebrated algebraist to that great patron of learning, Emperor Frederick II., was the solving of the equation $x^3 + 2x^2 + 10x = 20$. As yet cubic equations had not been solved algebraically. Instead of brooding stubbornly over this knotty problem, and after many failures still entertaining new hopes of success, he changed his method of inquiry and showed by clear and rigorous demonstration that the roots of this equation could not be represented by the Euclidean irrational quantities, or, in other words, that they could not be

constructed with the ruler and compass only. He contented himself with finding a very close approximation to the required root. His work on this cubic is found in the *Flos*, together with the solution of the following third problem given him by John of Palermo: Three men possess in common an unknown sum of money t; the share of the first is $\frac{t}{2}$; that of the second, $\frac{t}{3}$; that of the third, $\frac{t}{6}$. Desirous of depositing the sum at a safer place, each takes at hazard a certain amount; the first takes x, but deposits only $\frac{x}{2}$; the second carries y, but deposits only $\frac{y}{3}$; the third takes z, and deposits $\frac{z}{6}$. Of the amount deposited each one must receive exactly $\frac{1}{3}$, in order to possess his share of the whole sum. Find x, y, z. Leonardo shows the problem to be indeterminate. Assuming 7 for the sum drawn by each from the deposit, he finds $t = 47$, $x = 33$, $y = 13$, $z = 1$.

One would have thought that after so brilliant a beginning, the sciences transplanted from Mohammedan to Christian soil would have enjoyed a steady and vigorous development. But this was not the case. During the fourteenth and fifteenth centuries, the mathematical science was almost stationary. Long wars absorbed the energies of the people and thereby kept back the growth of the sciences. The death of Frederick II. in 1254 was followed by a period of confusion in Germany. The German emperors and the popes were continually quarrelling, and Italy was inevitably drawn into the struggles between the Guelphs and the Ghibellines. France and England were engaged in the Hundred Years' War (1338–1453). Then followed in England the Wars of the Roses. The growth of science was retarded not only by war, but also by the injurious influence of scholastic philosophy. The intellectual leaders of those times quarrelled over subtle subjects in meta-

physics and theology. Frivolous questions, such as "How many angels can stand on the point of a needle?" were discussed with great interest. Indistinctness and confusion of ideas characterised the reasoning during this period. Among the mathematical productions of the Middle Ages, the works of Leonardo of Pisa appear to us like jewels among quarry-rubbish. The writers on mathematics during this period were not few in number, but their scientific efforts were vitiated by the method of scholastic thinking. Though they possessed the *Elements* of Euclid, yet the true nature of a mathematical proof was so little understood, that Hankel believes it no exaggeration to say that "since Fibonacci, not a single proof, not borrowed from Euclid, can be found in the whole literature of these ages, which fulfils all necessary conditions."

The only noticeable advance is a simplification of numerical operations and a more extended application of them. Among the Italians are evidences of an early maturity of arithmetic. Peacock[22] says: The Tuscans generally, and the Florentines in particular, whose city was the cradle of the literature and arts of the thirteenth and fourteenth centuries, were celebrated for their knowledge of arithmetic and book-keeping, which were so necessary for their extensive commerce; the Italians were in familiar possession of commercial arithmetic long before the other nations of Europe; to them we are indebted for the formal introduction into books of arithmetic, under distinct heads, of questions in the single and double rule of three, loss and gain, fellowship, exchange, simple and compound interest, discount, and so on.

There was also a slow improvement in the algebraic notation. The Hindoo algebra possessed a tolerable symbolic notation, which was, however, completely ignored by the Mohammedans. In this respect, Arabic algebra approached much more closely to that of Diophantus, which can scarcely

be said to employ symbols in a systematic way. Leonardo of Pisa possessed no algebraic symbolism. Like the Arabs, he expressed the relations of magnitudes to each other by lines or in words. But in the mathematical writings of the monk *Luca Pacioli* (also called Lucas de Burgo sepulchri) symbols began to appear. They consisted merely in abbreviations of Italian words, such as *p* for *piu* (more), *m* for *meno* (less), *co* for *cosa* (the thing or unknown quantity). "Our present notation has arisen by almost insensible degrees as convenience suggested different marks of abbreviation to different authors; and that perfect symbolic language which addresses itself solely to the eye, and enables us to take in at a glance the most complicated relations of quantity, is the result of a large series of small improvements."[23]

We shall now mention a few authors who lived during the thirteenth and fourteenth and the first half of the fifteenth centuries. About the time of Leonardo of Pisa (1200 A.D.), lived the German monk **Jordanus Nemorarius,** who wrote a once famous work on the properties of numbers (1496), modelled after the arithmetic of Boethius. The most trifling numeral properties are treated with nauseating pedantry and prolixity. A practical arithmetic based on the Hindoo notation was also written by him. **John Halifax** (Sacro Bosco, died 1256) taught in Paris and made an extract from the *Almagest* containing only the most elementary parts of that work. This extract was for nearly 400 years a work of great popularity and standard authority. Other prominent writers are **Albertus Magnus** and **George Purbach** in Germany, and **Roger Bacon** in England. It appears that here and there some of our modern ideas were anticipated by writers of the Middle Ages. Thus, **Nicole Oresme,** a bishop in Normandy (died 1382), first conceived a notation of fractional powers, afterwards re-discovered by Stevinus, and gave rules for operating with them.

His notation was totally different from ours. **Thomas Bradwardine,** archbishop of Canterbury, studied star-polygons, — a subject which has recently received renewed attention. The first appearance of such polygons was with Pythagoras and his school. We next meet with such polygons in the geometry of Boethius and also in the translation of Euclid from the Arabic by Athelard of Bath. Bradwardine's philosophic writings contain discussions on the infinite and the infinitesimal — subjects never since lost sight of. To England falls the honour of having produced the earliest European writers on trigonometry. The writings of Bradwardine, of Richard of Wallingford, and John Maudith, both professors at Oxford, and of Simon Bredon of Winchecombe, contain trigonometry drawn from Arabic sources.

The works of the Greek monk **Maximus Planudes,** who lived in the first half of the fourteenth century, are of interest only as showing that the Hindoo numerals were then known in Greece. A writer belonging, like Planudes, to the Byzantine school, was **Moschopulus,** who lived in Constantinople in the early part of the fifteenth century. To him appears to be due the introduction into Europe of magic squares. He wrote a treatise on this subject. Magic squares were known to the Arabs, and perhaps to the Hindoos. Mediæval astrologers and physicians believed them to possess mystical properties and to be a charm against plague, when engraved on silver plate.

In 1494 was *printed* the *Summa de Arithmetica, Geometria, Proportione et Proportionalita,* written by the Tuscan monk **Lucas Pacioli,** who, as we remarked, first introduced symbols in algebra. This contains all the knowledge of his day on arithmetic, algebra, and trigonometry, and is the first comprehensive work which appeared after the *Liber Abaci* of Fibonacci. It contains little of importance which cannot be

found in Fibonacci's great work, published three centuries earlier.[1]

Perhaps the greatest result of the influx of Arabic learning was the establishment of universities. What was their attitude toward mathematics? The *University of Paris*, so famous at the beginning of the twelfth century under the teachings of Abelard, paid but little attention to this science during the Middle Ages. Geometry was neglected, and Aristotle's logic was the favourite study. In 1336, a rule was introduced that no student should take a degree without attending lectures on mathematics, and from a commentary on the first six books of Euclid, dated 1536, it appears that candidates for the degree of A.M. had to give an oath that they had attended lectures on these books.[7] Examinations, when held at all, probably did not extend beyond the first book, as is shown by the nickname "magister matheseos," applied to the Theorem of Pythagoras, the last in the first book. More attention was paid to mathematics at the *University of Prague*, founded 1384. For the Baccalaureate degree, students were required to take lectures on Sacro Bosco's famous work on astronomy. Of candidates for the A.M. were required not only the six books of Euclid, but an additional knowledge of applied mathematics. Lectures were given on the *Almagest*. At the *University of Leipzig*, the daughter of Prague, and at *Cologne*, less work was required, and, as late as the sixteenth century, the same requirements were made at these as at Prague in the fourteenth. The universities of Bologna, Padua, Pisa, occupied similar positions to the ones in Germany, only that purely astrological lectures were given in place of lectures on the *Almagest*. At Oxford, in the middle of the fifteenth century, the first two books of Euclid were read.[6]

Thus it will be seen that the study of mathematics was

maintained at the universities only in a half-hearted manner. No great mathematician and teacher appeared, to inspire the students. The best energies of the schoolmen were expended upon the stupid subtleties of their philosophy. The genius of Leonardo of Pisa left no permanent impress upon the age, and another Renaissance of mathematics was wanted.

MODERN EUROPE.

We find it convenient to choose the time of the capture of Constantinople by the Turks as the date at which the Middle Ages ended and Modern Times began. In 1453, the Turks battered the walls of this celebrated metropolis with cannon, and finally captured the city; the Byzantine Empire fell, to rise no more. Calamitous as was this event to the East, it acted favourably upon the progress of learning in the West. A great number of learned Greeks fled into Italy, bringing with them precious manuscripts of Greek literature. This contributed vastly to the reviving of classic learning. Up to this time, Greek masters were known only through the often very corrupt Arabic manuscripts, but now they began to be studied from original sources and in their own language. The first English translation of Euclid was made in 1570 from the Greek by *Sir Henry Billingsley*, assisted by *John Dee*.[29] About the middle of the fifteenth century, printing was invented; books became cheap and plentiful; the printing-press transformed Europe into an audience-room. Near the close of the fifteenth century, America was discovered, and, soon after, the earth was circumnavigated. The pulse and pace of the world began to quicken. Men's minds became less servile; they became clearer and stronger. The indistinctness of thought, which was the characteristic feature of mediæval learning, began to be remedied chiefly by the steady cultiva-

tion of Pure Mathematics and Astronomy. Dogmatism was attacked; there arose a long struggle with the authority of the Church and the established schools of philosophy. The Copernican System was set up in opposition to the time-honoured Ptolemaic System. The long and eager contest between the two culminated in a crisis at the time of Galileo, and resulted in the victory of the new system. Thus, by slow degrees, the minds of men were cut adrift from their old scholastic moorings and sent forth on the wide sea of scientific inquiry, to discover new islands and continents of truth.

THE RENAISSANCE.

With the sixteenth century began a period of increased intellectual activity. The human mind made a vast effort to achieve its freedom. Attempts at its emancipation from Church authority had been made before, but they were stifled and rendered abortive. The first great and successful revolt against ecclesiastical authority was made in Germany. The new desire for judging freely and independently in matters of religion was preceded and accompanied by a growing spirit of scientific inquiry. Thus it was that, for a time, Germany led the van in science. She produced *Regiomontanus*, *Copernicus*, *Rhœticus*, *Kepler*, and *Tycho Brahe*, at a period when France and England had, as yet, brought forth hardly any great scientific thinkers. This remarkable scientific productiveness was no doubt due, to a great extent, to the commercial prosperity of Germany. Material prosperity is an essential condition for the progress of knowledge. As long as every individual is obliged to collect the necessaries for his subsistence, there can be no leisure for higher pursuits. At this time, Germany had accumulated considerable wealth. The

Hanseatic League commanded the trade of the North. Close commercial relations existed between Germany and Italy. Italy, too, excelled in commercial activity and enterprise. We need only mention Venice, whose glory began with the crusades, and Florence, with her bankers and her manufacturers of silk and wool. These two cities became great intellectual centres. Thus, Italy, too, produced men in art, literature, and science, who shone forth in fullest splendour. In fact, Italy was the fatherland of what is termed the Renaissance.

For the first great contributions to the mathematical sciences we must, therefore, look to Italy and Germany. In Italy brilliant accessions were made to algebra, in Germany to astronomy and trigonometry.

On the threshold of this new era we meet in Germany with the figure of John Mueller, more generally called **Regiomontanus** (1436–1476). Chiefly to him we owe the revival of trigonometry. He studied astronomy and trigonometry at Vienna under the celebrated George Purbach. The latter perceived that the existing Latin translations of the *Almagest* were full of errors, and that Arabic authors had not remained true to the Greek original. Purbach therefore began to make a translation directly from the Greek. But he did not live to finish it. His work was continued by Regiomontanus, who went beyond his master. Regiomontanus learned the Greek language from Cardinal Bessarion, whom he followed to Italy, where he remained eight years collecting manuscripts from Greeks who had fled thither from the Turks. In addition to the translation of and the commentary on the *Almagest*, he prepared translations of the *Conics* of Apollonius, of Archimedes, and of the mechanical works of Heron. Regiomontanus and Purbach adopted the Hindoo *sine* in place of the Greek *chord of double the arc*. The Greeks and afterwards the Arabs divided the radius into 60 equal parts, and each of these again

into 60 smaller ones. The Hindoos expressed the length of the radius by parts of the circumference, saying that of the 21,600 equal divisions of the latter, it took 3438 to measure the radius. Regiomontanus, to secure greater precision, constructed one table of sines on a radius divided into 600,000 parts, and another on a radius divided decimally into 10,000,000 divisions. He emphasised the use of the *tangent* in trigonometry. Following out some ideas of his master, he calculated a table of tangents. German mathematicians were not the first Europeans to use this function. In England it was known a century earlier to Bradwardine, who speaks of tangent (*umbra recta*) and cotangent (*umbra versa*), and to John Maudith. Regiomontanus was the author of an arithmetic and also of a complete treatise on trigonometry, containing solutions of both plane and spherical triangles. The form which he gave to trigonometry has been retained, in its main features, to the present day.

Regiomontanus ranks among the greatest men that Germany has ever produced. His complete mastery of astronomy and mathematics, and his enthusiasm for them, were of far-reaching influence throughout Germany. So great was his reputation, that Pope Sixtus IV. called him to Italy to improve the calendar. Regiomontanus left his beloved city of Nürnberg for Rome, where he died in the following year.

After the time of Purbach and Regiomontanus, trigonometry and especially the calculation of tables continued to occupy German scholars. More refined astronomical instruments were made, which gave observations of greater precision; but these would have been useless without trigonometrical tables of corresponding accuracy. Of the several tables calculated, that by *Georg Joachim* of Feldkirch in Tyrol, generally called **Rhæticus**, deserves special mention. He calculated a table of sines with the radius = 10,000,000,000 and from 10″ to 10″;

and, later on, another with the radius = 1,000,000,000,000,000, and proceeding from 10″ to 10″. He began also the construction of tables of tangents and secants, to be carried to the same degree of accuracy; but he died before finishing them. For twelve years he had had in continual employment several calculators. The work was completed by his pupil, **Valentine Otho**, in 1596. This was indeed a gigantic work, — a monument of German diligence and indefatigable perseverance. The tables were republished in 1613 by **Pitiscus**, who spared no pains to free them of errors. Astronomical tables of so great a degree of accuracy had never been dreamed of by the Greeks, Hindoos, or Arabs. That Rhæticus was not a ready calculator only, is indicated by his views on trignometrical lines. Up to his time, the trigonometric functions had been considered always with relation to the arc; he was the first to construct the right triangle and to make them depend directly upon its angles. It was from the right triangle that Rhæticus got his idea of calculating the hypotenuse; *i.e.* he was the first to plan a table of secants. Good work in trigonometry was done also by Vieta and Romanus.

We shall now leave the subject of trigonometry to witness the progress in the solution of algebraical equations. To do so, we must quit Germany for Italy. The first comprehensive algebra printed was that of Lucas Pacioli. He closes his book by saying that the solution of the equations $x^3 + mx = n$, $x^3 + n = mx$ is as impossible at the present state of science as the quadrature of the circle. This remark doubtless stimulated thought. The first step in the algebraic solution of cubics was taken by **Scipio Ferro** (died 1526), a professor of mathematics at Bologna, who solved the equation $x^3 + mx = n$. Nothing more is known of his discovery than that he imparted it to his pupil, *Floridas*, in 1505. It was the practice in those days and for two centuries afterwards to keep discoveries

secret, in order to secure by that means an advantage over rivals by proposing problems beyond their reach. This practice gave rise to numberless disputes regarding the priority of inventions. A second solution of cubics was given by *Nicolo* of Brescia (1506(?)–1557). When a boy of six, Nicolo was so badly cut by a French soldier that he never again gained the free use of his tongue. Hence he was called **Tartaglia**, *i.e.* the stammerer. His widowed mother being too poor to pay his tuition in school, he learned to read and picked up a knowledge of Latin, Greek, and mathematics by himself. Possessing a mind of extraordinary power, he was able to appear as teacher of mathematics at an early age. In 1530, one Colla proposed him several problems, one leading to the equation $x^3 + px^2 = q$. Tartaglia found an imperfect method for solving this, but kept it secret. He spoke about his secret in public and thus caused Ferro's pupil, Floridas, to proclaim his own knowledge of the form $x^3 + mx = n$. Tartaglia, believing him to be a mediocrist and braggart, challenged him to a public discussion, to take place on the 22d of February, 1535. Hearing, meanwhile, that his rival had gotten the method from a deceased master, and fearing that he would be beaten in the contest, Tartaglia put in all the zeal, industry, and skill to find the rule for the equations, and he succeeded in it ten days before the appointed date, as he himself modestly says.[7] The most difficult step was, no doubt, the passing from quadratic irrationals, used in operating from time of old, to cubic irrationals. Placing $x = \sqrt[3]{t} - \sqrt[3]{u}$, Tartaglia perceived that the irrationals disappeared from the equation $x^3 + mx = n$, making $n = t - u$. But this last equality, together with $(\frac{1}{3}m)^3 = tu$, gives at once

$$t = \sqrt{\left(\frac{n}{2}\right)^3 + \left(\frac{m}{3}\right)^3} + \frac{n}{2}, \quad u = \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{2}\right)^3} - \frac{n}{2}.$$

This is Tartaglia's solution of $x^3 + mx = n$. On the 13th of February, he found a similar solution for $x^3 = mx + n$. The contest began on the 22d. Each contestant proposed thirty problems. The one who could solve the greatest number within fifty days should be the victor. Tartaglia solved the thirty problems proposed by Floridas in two hours; Floridas could not solve any of Tartaglia's. From now on, Tartaglia studied cubic equations with a will. In 1541 he discovered a general solution for the cubic $x^3 \pm px^2 = \pm q$, by transforming it into the form $x^3 \pm mx = \pm n$. The news of Tartaglia's victory spread all over Italy. Tartaglia was entreated to make known his method, but he declined to do so, saying that after his completion of the translation from the Greek of Euclid and Archimedes, he would publish a large algebra containing his method. But a scholar from Milan, named **Hieronimo Cardano** (1501–1576), after many solicitations, and after giving the most solemn and sacred promises of secrecy, succeeded in obtaining from Tartaglia a knowledge of his rules.

At this time Cardan was writing his *Ars Magna*, and he knew no better way to crown his work than by inserting the much sought for rules for solving cubics. Thus Cardan broke his most solemn vows, and published in 1545 in his *Ars Magna* Tartaglia's solution of cubics. Tartaglia became desperate. His most cherished hope, of giving to the world an immortal work which should be the monument of his deep learning and power for original research, was suddenly destroyed; for the crown intended for his work had been snatched away. His first step was to write a history of his invention; but, to completely annihilate his enemies, he challenged Cardan and his pupil Lodovico Ferrari to a contest: each party should propose thirty-one questions to be solved by the other within fifteen days. Tartaglia solved most questions in seven days, but the other party did not send in their solution before the expiration

of the fifth month; moreover, all their solutions except one were wrong. A replication and a rejoinder followed. Endless were the problems proposed and solved on both sides. The dispute produced much chagrin and heart-burnings to the parties, and to Tartaglia especially, who met with many other disappointments. After having recovered himself again, Tartaglia began, in 1556, the publication of the work which he had had in his mind for so long; but he died before he reached the consideration of cubic equations. Thus the fondest wish of his life remained unfulfilled; the man to whom we owe the greatest contribution to algebra made in the sixteenth century was forgotten, and his method came to be regarded as the discovery of Cardan and to be called Cardan's solution.

Remarkable is the great interest that the solution of cubics excited throughout Italy. It is but natural that after this great conquest mathematicians should attack bi-quadratic equations. As in the case of cubics, so here, the first impulse was given by Colla, who, in 1540, proposed for solution the equation $x^4 + 6x^2 + 36 = 60x$. To be sure, Cardan had studied particular cases as early as 1539. Thus he solved the equation $13x^2 = x^4 + 2x^3 + 2x + 1$ by a process similar to that employed by Diophantus and the Hindoos; namely, by adding to both sides $3x^2$ and thereby rendering both numbers complete squares. But Cardan failed to find a general solution; it remained for his pupil **Ferrari** to prop the reputation of his master by the brilliant discovery of the general solution of bi-quadratic equations. Ferrari reduced Colla's equation to the form $(x^2 + 6)^2 = 60x + 6x^2$. In order to give also the right member the form of a complete square he added to both members the expression $2(x^2 + 6)y + y^2$, containing a new unknown quantity y. This gave him $(x^2 + 6 + y)^2 = (6 + 2y)x^2 + 60x + (12y + y^2)$. The condition that the right member be a complete square is expressed by the

cubic equation $(2y + 6)(12y + y^2) = 900$. Extracting the square root of the bi-quadratic, he got $x^2 + 6 + y = x\sqrt{2y + 6} + \frac{900}{\sqrt{2y + 6}}$. Solving the cubic for y and substituting, it remained only to determine x from the resulting quadratic. Ferrari pursued a similar method with other numerical bi-quadratic equations.[7] Cardan had the pleasure of publishing this discovery in his *Ars Magna* in 1545. Ferrari's solution is sometimes ascribed to *Bombelli*, but he is no more the discoverer of it than Cardan is of the solution called by his name.

To Cardan algebra is much indebted. In his *Ars Magna* he takes notice of negative roots of an equation, calling them *fictitious*, while the positive roots are called *real*. Imaginary roots he does not consider; cases where they appear he calls impossible. Cardan also observed the difficulty in the irreducible case in the cubics, which, like the quadrature of the circle, has since "so much tormented the perverse ingenuity of mathematicians." But he did not understand its nature. It remained for **Raphael Bombelli** of Bologna, who published in 1572 an algebra of great merit, to point out the reality of the apparently imaginary expression which the root assumes, and thus to lay the foundation of a more intimate knowledge of imaginary quantities.

After this brilliant success in solving equations of the third and fourth degrees, there was probably no one who doubted, that with aid of irrationals of higher degrees, the solution of equations of any degree whatever could be found. But all attempts at the algebraic solution of the quintic were fruitless, and, finally, Abel demonstrated that all hopes of finding algebraic solutions to equations of higher than the fourth degree were purely Utopian.

Since no solution by radicals of equations of higher degrees

could be found, there remained nothing else to be done than the devising of rules by which at least the numerical values of the roots could be ascertained. Cardan applied the Hindoo rule of "false position" (called by him *regula aurea*) to the cubic, but this mode of approximating was exceedingly rough. An incomparably better method was invented by **Franciscus Vieta**, a French mathematician, whose transcendent genius enriched mathematics with several important innovations. Taking the equation $f(x) = Q$, wherein $f(x)$ is a polynomial containing different powers of x, with numerical coefficients, and Q is a given number, Vieta first substitutes in $f(x)$ a known approximate value of the root, and then shows that another figure of the root can be obtained by division. A repetition of the same process gives the next figure of the root, and so on. Thus, in $x^2+14x = 7929$, taking 80 for the approximate root, and placing $x = 80 + b$, we get

$$(80 + b)^2 + 14(80 + b) = 7929,$$

or

$$174b + b^2 = 409.$$

Since $174b$ is much greater than b^2, we place $174b = 409$, and obtain thereby $b = 2$. Hence the second approximation is 82. Put $x = 82 + c$, then $(82 + c)^2+14(82 + c) = 7929$, or $178c+c^2 = 57$. As before, place $178c = 57$, then $c = .3$, and the third approximation gives 82.3. Assuming $x = 82.3 + d$, and substituting, gives $178.6d + d^2 = 3.51$, and $178.6d = 3.51$, $\therefore d = .01$; giving for the fourth approximation 82.31. In the same way, $e = .009$, and the value for the root of the given equation is 82.319... For this process, Vieta was greatly admired by his contemporaries. It was employed by Harriot, Oughtred, Pell, and others. Its principle is identical with the main principle involved in the methods of approximation of Newton and Horner. The only change lies in the arrangement of the

work. This alteration was made to afford facility and security in the process of evolution of the root.

We pause a moment to sketch the life of Vieta, the most eminent French mathematician of the sixteenth century. He was born in Poitou in 1540, and died in 1603 at Paris. He was employed throughout life in the service of the state, under Henry III. and Henry IV. He was, therefore, not a mathematician by profession, but his love for the science was so great that he remained in his chamber studying, sometimes several days in succession, without eating and sleeping more than was necessary to sustain himself. So great devotion to abstract science is the more remarkable, because he lived at a time of incessant political and religious turmoil. During the war against Spain, Vieta rendered service to Henry IV. by deciphering intercepted letters written in a species of cipher, and addressed by the Spanish Court to their governor of Netherlands. The Spaniards attributed the discovery of the key to magic.

An ambassador from Netherlands once told Henry IV. that France did not possess a single geometer capable of solving a problem propounded to geometers by a Belgian mathematician, Adrianus Romanus. It was the solution of the equation of the forty-fifth degree : —

$$45y - 3795y^3 + 95634y^5 - \cdots + 945y^{41} - 45y^{43} + y^{45} = C.$$

Henry IV. called Vieta, who, having already pursued similar investigations, saw at once that this awe-inspiring problem was simply the equation by which $C = 2 \sin \phi$ was expressed in terms of $y = 2 \sin \frac{1}{45} \phi$; that, since $45 = 3 \cdot 3 \cdot 5$, it was necessary only to divide an angle once into 5 equal parts, and then twice into 3, — a division which could be effected by corresponding equations of the fifth and third degrees. Brilliant was the discovery by Vieta of 23 roots to this equation, instead

of only one. The reason why he did not find 45 solutions, is that the remaining ones involve negative sines, which were unintelligible to him. Detailed investigations on the famous old problem of the section of an angle into an odd number of equal parts, led Vieta to the discovery of a trigonometrical solution of Cardan's irreducible case in cubics. He applied the equation $(2\cos\frac{1}{3}\phi)^3 - 3(2\cos\frac{1}{3}\phi) = 2\cos\phi$ to the solution of $x^3 - 3a^2x = a^2b$, when $a > \frac{1}{2}b$, by placing $x = 2a\cos\frac{1}{3}\phi$, and determining ϕ from $b = 2a\cos\phi$.

The main principle employed by him in the solution of equations is that of *reduction*. He solves the quadratic by making a suitable substitution which will remove the term containing x to the first degree. Like Cardan, he reduces the general expression of the cubic to the form $x^3 + mx + n = 0$; then, assuming $x = (\frac{1}{3}a - z^2) \div z$ and substituting, he gets $z^6 - bz^3 - \frac{1}{27}a^3 = 0$. Putting $z^3 = y$, he has a quadratic. In the solution of bi-quadratics, Vieta still remains true to his principle of reduction. This gives him the well-known cubic resolvent. He thus adheres throughout to his favourite principle, and thereby introduces into algebra a uniformity of method which claims our lively admiration. In Vieta's algebra we discover a partial knowledge of the relations existing between the coefficients and the roots of an equation. He shows that if the coefficient of the second term in an equation of the second degree is minus the sum of two numbers whose product is the third term, then the two numbers are roots of the equation. Vieta rejected all except positive roots; hence it was impossible for him to fully perceive the relations in question.

The most epoch-making innovation in algebra due to Vieta is the denoting of general or indefinite quantities by letters of the alphabet. To be sure, Regiomontanus and Stifel in Germany, and Cardan in Italy, used letters before him, but

Vieta extended the idea and first made it an essential part of algebra. The new algebra was called by him *logistica speciosa* in distinction to the old *logistica numerosa*. Vieta's formalism differed considerably from that of to-day. The equation $a^3 + 3\,a^2b + 3\,ab^2 + b^3 = (a+b)^3$ was written by him "a cubus $+ b$ in a quadr. $3 + a$ in b quadr. $3 + b$ cubo æqualia $\overline{a+b}$ cubo." In numerical equations the unknown quantity was denoted by N, its square by Q, and its cube by C. Thus the equation $x^3 - 8\,x^2 + 16\,x = 40$ was written $1\,C - 8\,Q + 16\,N$ *æqual.* 40. Observe that exponents and our symbol $(=)$ for equality were not yet in use; but that Vieta employed the Maltese cross $(+)$ as the short-hand symbol for addition, and the $(-)$ for subtraction. These two characters had not been in general use before the time of Vieta. "It is very singular," says Hallam, "that discoveries of the greatest convenience, and, apparently, not above the ingenuity of a village schoolmaster, should have been overlooked by men of extraordinary acuteness like Tartaglia, Cardan, and Ferrari; and hardly less so that, by dint of that acuteness, they dispensed with the aid of these contrivances in which we suppose that so much of the utility of algebraic expression consists." Even after improvements in notation were once proposed, it was with extreme slowness that they were admitted into general use. They were made oftener by accident than design, and their authors had little notion of the effect of the change which they were making. The introduction of the $+$ and $-$ symbols seems to be due to the Germans, who, although they did not enrich algebra during the Renaissance with great inventions, as did the Italians, still cultivated it with great zeal. The arithmetic of **John Widmann,** printed A.D. 1489 in Leipzig, is the earliest book in which the $+$ and $-$ symbols have been found. There are indications leading us to surmise that they were in use first among merchants. They occur again in the arith-

metic of Grammateus, a teacher at the University of Vienna. His pupil, Christoff Rudolff, the writer of the first text-book on algebra in the German language (printed in 1525), employs these symbols also. So did Stifel, who brought out a second edition of Rudolff's *Coss* in 1553. Thus, by slow degrees, their adoption became universal. There is another short-hand symbol of which we owe the origin to the Germans. In a manuscript published sometime in the fifteenth century, a dot placed before a number is made to signify the extraction of a root of that number. This dot is the embryo of our present symbol for the square root. Christoff Rudolff, in his algebra, remarks that "the radix quadrata is, for brevity, designated in his algorithm with the character $\surd$, as $\surd 4$." Here the dot has grown into a symbol much like our own. This same symbol was used by *Michael Stifel.* Our sign of equality is due to **Robert Recorde** (1510–1558), the author of *The Whetstone of Witte* (1557), which is the first English treatise on algebra. He selected this symbol because no two things could be more equal than two parallel lines $=$. The sign $\div$ for division was first used by *Johann Heinrich Rahn*, a Swiss, in 1659, and was introduced in England by *John Pell* in 1668.

Michael Stifel (1486 ?–1567), the greatest German algebraist of the sixteenth century, was born in Esslingen, and died in Jena. He was educated in the monastery of his native place, and afterwards became Protestant minister. The study of the significance of mystic numbers in Revelation and in Daniel drew him to mathematics. He studied German and Italian works, and published in 1544, in Latin, a book entitled *Arithmetica integra.* Melanchthon wrote a preface to it. Its three parts treat respectively of rational numbers, irrational numbers, and algebra. Stifel gives a table containing the numerical values of the binomial coefficients for powers below the 18th. He observes an advantage in letting a geometric progres-

sion correspond to an arithmetical progression, and arrives at the designation of integral powers by numbers. Here are the germs of the theory of exponents. In 1545 Stifel published an arithmetic in German. His edition of Rudolff's *Coss* contains rules for solving cubic equations, derived from the the writings of Cardan.

We remarked above that Vieta discarded negative roots of equations. Indeed, we find few algebraists before and during the Renaissance who understood the significance even of negative quantities. Fibonacci seldom uses them. Pacioli states the rule that "minus times minus gives plus," but applies it really only to the development of the product of $(a-b)\ (c-d)$; purely negative quantities do not appear in his work. The great German "Cossist" (algebraist), *Michael Stifel*, speaks as early as 1544 of numbers which are "absurd" or "fictitious below zero," and which arise when "real numbers above zero" are subtracted from zero. Cardan, at last, speaks of a "pure minus"; "but these ideas," says Hankel, "remained sparsely, and until the beginning of the seventeenth century, mathematicians dealt exclusively with absolute positive quantities." The first algebraist who occasionally places a purely negative quantity by itself on one side of an equation, is *Harriot* in England. As regards the recognition of negative roots, Cardan and Bombelli were far in advance of all writers of the Renaissance, including Vieta. Yet even they mentioned these so-called false or fictitious roots only in passing, and without grasping their real significance and importance. On this subject Cardan and Bombelli had advanced to about the same point as had the Hindoo Bhaskara, who saw negative roots, but did not approve of them. The generalisation of the conception of quantity so as to include the negative, was an exceedingly slow and difficult process in the development of algebra.

We shall now consider the history of geometry during the Renaissance. Unlike algebra, it made hardly any progress. The greatest gain was a more intimate knowledge of Greek geometry. No essential progress was made before the time of Descartes. Regiomontanus, Xylander of Augsburg, Tartaglia, Commandinus of Urbino in Italy, Maurolycus, and others, made translations of geometrical works from the Greek. **John Werner** of Nürnberg published in 1522 the first work on conics which appeared in Christian Europe. Unlike the geometers of old, he studied the sections in relation with the cone, and derived their properties directly from it. This mode of studying the conics was followed by **Maurolycus** of Messina (1494–1575). The latter is, doubtless, the greatest geometer of the sixteenth century. From the notes of Pappus, he attempted to restore the missing fifth book of Apollonius on *maxima* and *minima*. His chief work is his masterly and original treatment of the conic sections, wherein he discusses tangents and asymptotes more fully than Apollonius had done, and applies them to various physical and astronomical problems.

The foremost geometrician of Portugal was **Nonius**; of France, before Vieta, was **Peter Ramus,** who perished in the massacre of St. Bartholomew. *Vieta* possessed great familiarity with ancient geometry. The new form which he gave to algebra, by representing general quantities by letters, enabled him to point out more easily how the construction of the roots of cubics depended upon the celebrated ancient problems of the duplication of the cube and the trisection of an angle. He reached the interesting conclusion that the former problem includes the solutions of all cubics in which the radical in Tartaglia's formula is real, but that the latter problem includes only those leading to the irreducible case.

The problem of the quadrature of the circle was revived in

this age, and was zealously studied even by men of eminence and mathematical ability. The army of circle-squarers became most formidable during the seventeenth century. Among the first to revive this problem was the German Cardinal **Nicolaus Cusanus** (died 1464), who had the reputation of being a great logician. His fallacies were exposed to full view by Regiomontanus. As in this case, so in others, every quadrator of note raised up an opposing mathematician: Orontius was met by Buteo and Nonius; Joseph Scaliger by Vieta, Adrianus Romanus, and Clavius; A. Quercu by Peter Metius. Two mathematicians of Netherlands, **Adrianus Romanus** and **Ludolph van Ceulen,** occupied themselves with approximating to the ratio between the circumference and the diameter. The former carried the value π to 15, the latter to 35, places. The value of π is therefore often named "Ludolph's number." His performance was considered so extraordinary, that the numbers were cut on his tomb-stone in St. Peter's church-yard, at Leyden. Romanus was the one who propounded for solution that equation of the forty-fifth degree solved by Vieta. On receiving Vieta's solution, he at once departed for Paris, to make his acquaintance with so great a master. Vieta proposed to him the Apollonian problem, to draw a circle touching three given circles. "Adrianus Romanus solved the problem by the intersection of two hyperbolas; but this solution did not possess the rigour of the ancient geometry. Vieta caused him to see this, and then, in his turn, presented a solution which had all the rigour desirable."[25] Romanus did much toward simplifying spherical trigonometry by reducing, by means of certain projections, the 28 cases in triangles then considered to only six.

Mention must here be made of the improvements of the Julian calendar. The yearly determination of the movable feasts had for a long time been connected with an untold

amount of confusion. The rapid progress of astronomy led to the consideration of this subject, and many new calendars were proposed. Pope Gregory XIII. convoked a large number of mathematicians, astronomers, and prelates, who decided upon the adoption of the calendar proposed by the Jesuit **Lilius Clavius.** To rectify the errors of the Julian calendar it was agreed to write in the new calendar the 15th of October immediately after the 4th of October of the year 1582. The Gregorian calendar met with a great deal of opposition both among scientists and among Protestants. Clavius, who ranked high as a geometer, met the objections of the former most ably and effectively; the prejudices of the latter passed away with time.

The passion for the study of mystical properties of numbers descended from the ancients to the moderns. Much was written on numerical mysticism even by such eminent men as Pacioli and Stifel. The *Numerorum Mysteria* of Peter Bungus covered 700 quarto pages. He worked with great industry and satisfaction on 666, which is the number of the beast in Revelation (xiii. 18), the symbol of Antichrist. He reduced the name of the 'impious' Martin Luther to a form which may express this formidable number. Placing $a = 1$, $b = 2$, etc., $k = 10$, $l = 20$, etc., he finds, after misspelling the name, that $M_{(30)}\,A_{(1)}\,R_{(80)}\,T_{(100)}\,I_{(9)}\,N_{(40)}\,L_{(20)}\,V_{(200)}\,T_{(100)}\,E_{(5)}\,R_{(80)}\,A_{(1)}$ constitutes the number required. These attacks on the great reformer were not unprovoked, for his friend, Michael Stifel, the most acute and original of the early mathematicians of Germany, exercised an equal ingenuity in showing that the above number referred to Pope Leo X., — a demonstration which gave Stifel unspeakable comfort.[22]

Astrology also was still a favourite study. It is well known that Cardan, Maurolycus, Regiomontanus, and many other eminent scientists who lived at a period even later than

this, engaged in deep astrological study; but it is not so generally known that besides the occult sciences already named, men engaged in the mystic study of star-polygons and magic squares. "The pentagramma gives you pain," says Faust to Mephistopheles. It is of deep psychological interest to see scientists, like the great Kepler, demonstrate on one page a theorem on star-polygons, with strict geometric rigour, while on the next page, perhaps, he explains their use as amulets or in conjurations.[1] Playfair, speaking of Cardan as an astrologer, calls him "a melancholy proof that there is no folly or weakness too great to be united to high intellectual attainments."[26] Let our judgment not be too harsh. The period under consideration is too near the Middle Ages to admit of complete emancipation from mysticism even among scientists. Scholars like Kepler, Napier, Albrecht Duerer, while in the van of progress and planting one foot upon the firm ground of truly scientific inquiry, were still resting with the other foot upon the scholastic ideas of preceding ages.

VIETA TO DESCARTES.

The ecclesiastical power, which in the ignorant ages was an unmixed benefit, in more enlightened ages became a serious evil. Thus, in France, during the reigns preceding that of Henry IV., the theological spirit predominated. This is painfully shown by the massacres of Vassy and of St. Bartholomew. Being engaged in religious disputes, people had no leisure for science and for secular literature. Hence, down to the time of Henry IV., the French "had not put forth a single work, the destruction of which would now be a loss to Europe." In England, on the other hand, no religious wars were waged. The people were comparatively indifferent about

religious strifes; they concentrated their ability upon secular matters, and acquired, in the sixteenth century, a literature which is immortalised by the genius of Shakespeare and Spenser. This great literary age in England was followed by a great scientific age. At the close of the sixteenth century, the shackles of ecclesiastical authority were thrown off by France. The ascension of Henry IV. to the throne was followed in 1598 by the Edict of Nantes, granting freedom of worship to the Huguenots, and thereby terminating religious wars. The genius of the French nation now began to blossom. Cardinal Richelieu, during the reign of Louis XIII., pursued the broad policy of not favouring the opinions of any sect, but of promoting the interests of the nation. His age was remarkable for the progress of knowledge. It produced that great secular literature, the counterpart of which was found in England in the sixteenth century. The seventeenth century was made illustrious also by the great French mathematicians, Roberval, Descartes, Desargues, Fermat, and Pascal.

More gloomy is the picture in Germany. The great changes which revolutionised the world in the sixteenth century, and which led England to national greatness, led Germany to degradation. The first effects of the Reformation there were salutary. At the close of the fifteenth and during the sixteenth century, Germany had been conspicuous for her scientific pursuits. She had been the leader in astronomy and trigonometry. Algebra also, excepting for the discoveries in cubic equations, was, before the time of Vieta, in a more advanced state there than elsewhere. But at the beginning of the seventeenth century, when the sun of science began to rise in France, it set in Germany. Theologic disputes and religious strife ensued. The Thirty Years' War (1618–1648) proved ruinous. The German empire was shattered, and became a mere lax confederation of petty despotisms. Com-

merce was destroyed; national feeling died out. Art disappeared, and in literature there was only a slavish imitation of French artificiality. Nor did Germany recover from this low state for 200 years; for in 1756 began another struggle, the Seven Years' War, which turned Prussia into a wasted land. Thus it followed that at the beginning of the seventeenth century, the great Kepler was the only German mathematician of eminence, and that in the interval of 200 years between Kepler and Gauss, there arose no great mathematician in Germany excepting Leibniz.

Up to the seventeenth century, mathematics was cultivated but little in Great Britain. During the sixteenth century, she brought forth no mathematician comparable with Vieta, Stifel, or Tartaglia. But with the time of Recorde, the English became conspicuous for numerical skill. The first important arithmetical work of English authorship was published in Latin in 1522 by **Cuthbert Tonstall** (1474–1559). He had studied at Oxford, Cambridge, and Padua, and drew freely from the works of Pacioli and Regiomontanus. Reprints of his arithmetic appeared in England and France. After Recorde the higher branches of mathematics began to be studied. Later, Scotland brought forth Napier, the inventor of logarithms. The instantaneous appreciation of their value is doubtless the result of superiority in calculation. In Italy, and especially in France, geometry, which for a long time had been an almost stationary science, began to be studied with success. Galileo, Torricelli, Roberval, Fermat, Desargues, Pascal, Descartes, and the English Wallis are the great revolutioners of this science. Theoretical mechanics began to be studied. The foundations were laid by Fermat and Pascal for the theory of numbers and the theory of probability.

We shall first consider the improvements made in the art of calculating. The nations of antiquity experimented thou-

sands of years upon numeral notations before they happened to strike upon the so-called "Arabic notation." In the simple expedient of the cipher, which was introduced by the Hindoos about the fifth or sixth century after Christ, mathematics received one of the most powerful impulses. It would seem that after the "Arabic notation" was once thoroughly understood, decimal fractions would occur at once as an obvious extension of it. But "it is curious to think how much science had attempted in physical research and how deeply numbers had been pondered, before it was perceived that the all-powerful simplicity of the 'Arabic notation' was as valuable and as manageable in an infinitely descending as in an infinitely ascending progression."[28] Simple as decimal fractions appear to us, the invention of them is not the result of one mind or even of one age. They came into use by almost imperceptible degrees. The first mathematicians identified with their history did not perceive their true nature and importance, and failed to invent a suitable notation. The idea of decimal fractions makes its first appearance in methods for approximating to the square roots of numbers. Thus John of Seville, presumably in imitation of Hindoo rules, adds $2n$ ciphers to the number, then finds the square root, and takes this as the numerator of a fraction whose denominator is 1 followed by n ciphers. The same method was followed by *Cardan*, but it failed to be generally adopted even by his Italian contemporaries; for otherwise it would certainly have been at least mentioned by *Cataldi* (died 1626) in a work devoted exclusively to the extraction of roots. Cataldi finds the square root by means of continued fractions — a method ingenious and novel, but for practical purposes inferior to Cardan's. **Orontius Finaeus** (died 1555) in France, and **William Buckley** (died about 1550) in England extracted the square root in the same way as Cardan and John of Seville.

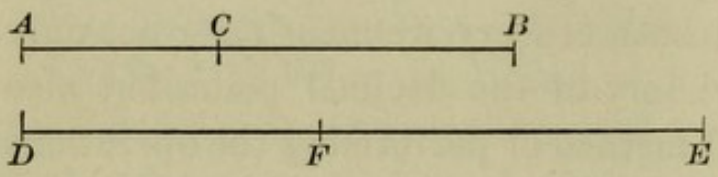
The invention of decimals is frequently attributed to Regiomontanus, on the ground that instead of placing the sinus totus, in trigonometry, equal to a multiple of 60, like the Greeks, he put it = 100,000. But here the trigonometrical lines were expressed in *integers*, and not in fractions. Though he adopted a decimal division of the radius, he and his successors did not apply the idea outside of trigonometry and, indeed, had no notion whatever of decimal *fractions*. To **Simon Stevin** of Bruges in Belgium (1548–1620), a man who did a great deal of work in most diverse fields of science, we owe the first systematic treatment of decimal fractions. In his *La Disme* (1585) he describes in very express terms the advantages, not only of decimal fractions, but also of the decimal division in systems of weights and measures. Stevin applied the new fractions "to all the operations of ordinary arithmetic."[25] What he lacked was a suitable notation. In place of our decimal point, he used a cipher; to each place in the fraction was attached the corresponding index. Thus, in his notation, the number 5.912 would be $5^{0}9^{1}1^{2}2^{3}$ or 5⓪9①1②2③. These indices, though cumbrous in practice, are of interest, because they are the germ of an important innovation. To Stevin belongs the honour of inventing our present mode of designating powers and also of introducing fractional exponents into algebra. Strictly speaking, this had been done much earlier by *Oresme*, but it remained wholly unnoticed. Not even Stevin's innovations were immediately appreciated or at once accepted, but, unlike Oresme's, they remained a secure possession. No improvement was made in the notation of decimals till the beginning of the seventeenth century. After Stevin, decimals were used by **Joost Bürgi**, a Swiss by birth, who prepared a manuscript on arithmetic soon after 1592, and by **Johann Hartmann Beyer**, who assumes the invention as his own. In 1603, he published at Frankfurt on the Main a *Logistica*

Decimalis. With Bürgi, a zero placed underneath the digit in unit's place answers as sign of separation. Beyer's notation resembles Stevin's. The decimal point, says Peacock, is due to Napier, who in 1617 published his *Rabdologia*, containing a treatise on decimals, wherein the decimal point is used in one or two instances. In the English translation of Napier's *Mirifici logarithmorum canonis descriptio*, executed by Edward Wright in 1616, and corrected by the author, the decimal point occurs in the tables. There is no mention of decimals in English arithmetics between 1619 and 1631. *Oughtred* in 1631 designates the fraction .56 thus, $0|\underline{56}$. *Albert Girard*, a pupil of Stevin, in 1629 uses the point on one occasion. John Wallis in 1657 writes $12|\underline{345}$, but afterwards in his algebra adopts the usual point. De Morgan says that "to the first quarter of the eighteenth century we must refer not only the complete and final victory of the decimal point, but also that of the now universal method of performing the operations of division and extraction of the square root."[27] We have dwelt at some length on the progress of the decimal notation, because "the history of language . . . is of the highest order of interest, as well as utility: its suggestions are the best lesson for the future which a reflecting mind can have."[27]

The miraculous powers of modern calculation are due to three inventions: the Arabic Notation, Decimal Fractions, and Logarithms. The invention of logarithms in the first quarter of the seventeenth century was admirably timed, for Kepler was then examining planetary orbits, and Galileo had just turned the telescope to the stars. During the Renaissance German mathematicians had constructed trigonometrical tables of great accuracy, but this greater precision enormously increased the work of the calculator. It is no exaggeration to say that the invention of logarithms "by shortening the labours doubled the life of the astronomer." Logarithms were

invented by **John Napier**, Baron of Merchiston, in Scotland (1550–1617). It is one of the greatest curiosities of the history of science that Napier constructed logarithms before exponents were used. To be sure, Stifel and Stevin made some attempts to denote powers by indices, but this notation was not generally known, — not even to *Harriot*, whose algebra appeared long after Napier's death. That logarithms flow naturally from the exponential symbol was not observed until much later. It was Euler who first considered logarithms as being indices of powers. What, then, was Napier's line of thought?

Let AB be a definite line, DE a line extending from D indefinitely. Imagine two points starting at the same moment; the one moving from A toward B, the other from D toward E. Let the velocity during the first moment be the same for both: let that of the point on line DE be uniform; but the velocity of the point on AB decreasing in such a way that when it arrives at any point C, its velocity is proportional to the remaining distance BC. While the first point moves over a distance AC, the second one moves over a distance DF. Napier calls DF the logarithm of BC.



Napier's process is so unique and so different from all other modes of presenting the subject that there cannot be the shadow of a doubt that this invention is entirely his own; it is the result of unaided, isolated speculation. He first sought the logarithms only of sines; the line AB was the sine of 90° and was taken $= 10^7$; BC was the sine of the arc, and DF its logarithm. We notice that as the motion proceeds, BC decreases in geometrical progression, while DF increases in arithmetical progression. Let $AB = a = 10^7$, let $x = DF$,

$y = BC$, then $AC = a - y$. The velocity of the point C is $\frac{d(a-y)}{dt} = y$; this gives $-$ nat. log $y = t + c$. When $t = 0$, then $y = a$ and $c = -$ nat. log a. Again, let $\frac{dx}{dt} = a$ be the velocity of the point F, then $x = at$. Substituting for t and c their values and remembering that $a = 10^7$ and that by definition $x =$ Nap. log y, we get

$$\text{Nap. log } y = 10^7 \text{ nat. log } \frac{10^7}{y}.$$

It is evident from this formula that Napier's logarithms are not the same as the natural logarithms. Napier's logarithms increase as the number itself decreases. He took the logarithm of $\sin 90 = 0$; *i.e.* the logarithm of $10^7 = 0$. The logarithm of $\sin \alpha$ increased from zero as α decreased from 90°. Napier's genesis of logarithms from the conception of two flowing points reminds us of Newton's doctrine of fluxions. The relation between geometric and arithmetical progressions, so skilfully utilised by Napier, had been observed by Archimedes, Stifel, and others. Napier did not determine the base to his system of logarithms. The notion of a "base" in fact never suggested itself to him. The one demanded by his reasoning is the reciprocal of that of the natural system, but such a base would not reproduce accurately all of Napier's figures, owing to slight inaccuracies in the calculation of the tables. Napier's great invention was given to the world in 1614 in a work entitled *Mirifici logarithmorum canonis descriptio*. In it he explained the nature of his logarithms, and gave a logarithmic table of the natural sines of a quadrant from minute to minute.

Henry Briggs (1556–1631), in Napier's time professor of geometry at Gresham College, London, and afterwards professor at Oxford, was so struck with admiration of Napier's book, that he left his studies in London to do

homage to the Scottish philosopher. Briggs was delayed in his journey, and Napier complained to a common friend, "Ah, John, Mr. Briggs will not come." At that very moment knocks were heard at the gate, and Briggs was brought into the lord's chamber. Almost one-quarter of an hour was spent, each beholding the other without speaking a word. At last Briggs began: "My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy, viz. the logarithms; but, my lord, being by you found out, I wonder nobody found it out before, when now known it is so easy."[28] Briggs suggested to Napier the advantage that would result from retaining zero for the logarithm of the whole sine, but choosing 10,000,000,000 for the logarithm of the 10th part of that same sine, *i.e.* of 5° 44′ 22″. Napier said that he had already thought of the change, and he pointed out a slight improvement on Briggs' idea; viz. that zero should be the logarithm of 1, and 10,000,000,000 that of the whole sine, thereby making the characteristic of numbers greater than unity positive and not negative, as suggested by Briggs. Briggs admitted this to be more convenient. The invention of "Briggian logarithms" occurred, therefore, to Briggs and Napier independently. The great practical advantage of the new system was that its fundamental progression was accommodated to the base, 10, of our numerical scale. Briggs devoted all his energies to the construction of tables upon the new plan. Napier died in 1617, with the satisfaction of having found in Briggs an able friend to bring to completion his unfinished plans. In 1624 Briggs published his *Arithmetica logarithmica*, containing the logarithms to 14 places of numbers, from 1 to 20,000 and from 90,000 to 100,000. The gap from 20,000 to 90,000 was filled up by that illustrious successor of

Napier and Briggs, **Adrian Vlacq** of Gouda in Holland. He published in 1628 a table of logarithms from 1 to 100,000, of which 70,000 were calculated by himself. The first publication of Briggian logarithms of trigonometric functions was made in 1620 by **Gunter**, a colleague of Briggs, who found the logarithmic sines and tangents for every minute to seven places. Gunter was the inventor of the words *cosine* and *cotangent*. Briggs devoted the last years of his life to calculating more extensive Briggian logarithms of trigonometric functions, but he died in 1631, leaving his work unfinished. It was carried on by the English **Henry Gellibrand**, and then published by Vlacq at his own expense. Briggs divided a degree into 100 parts, but owing to the publication by Vlacq of trigonometrical tables constructed on the old sexagesimal division, Briggs' innovation remained unrecognised. Briggs and Vlacq published four fundamental works, the results of which "have never been superseded by any subsequent calculations."

The first logarithms upon the *natural* base e were published by **John Speidell** in his *New Logarithmes* (London, 1619), which contains the natural logarithms of sines, tangents, and secants.

The only possible rival of John Napier in the invention of logarithms was the Swiss **Justus Byrgius** (Joost Bürgi). He published a rude table of logarithms six years after the appearance of the *Canon Mirificus*, but it appears that he conceived the idea and constructed that table as early, if not earlier, than Napier did his. But he neglected to have the results published until Napier's logarithms were known and admired throughout Europe.

Among the various inventions of Napier to assist the memory of the student or calculator, is "Napier's rule of circular parts" for the solution of spherical right triangles. It is, perhaps, "the happiest example of artificial memory that is known."

The most brilliant conquest in algebra during the sixteenth century had been the solution of cubic and bi-quadratic equations. All attempts at solving algebraically equations of higher degrees remaining fruitless, a new line of inquiry — the properties of equations and their roots — was gradually opened up. We have seen that Vieta had attained a partial knowledge of the relations between roots and coefficients. **Peletarius**, a Frenchman, had observed as early as 1558, that the root of an equation is a divisor of the last term. One who extended the theory of equations somewhat further than Vieta, was **Albert Girard** (1590–1634), a Flemish mathematician. Like Vieta, this ingenious author applied algebra to geometry, and was the first who understood the use of negative roots in the solution of geometric problems. He spoke of imaginary quantities; inferred by induction that every equation has as many roots as there are units in the number expressing its degree; and first showed how to express the sums of their powers in terms of the coefficients. Another algebraist of considerable power was the English **Thomas Harriot** (1560–1621). He accompanied the first colony sent out by Sir Walter Raleigh to Virginia. After having surveyed that country he returned to England. As a mathematician, he was the boast of his country. He brought the theory of equations under one comprehensive point of view by grasping that truth in its full extent to which Vieta and Girard only approximated; viz. that in an equation in its simplest form, the coefficient of the second term with its sign changed is equal to the sum of the roots; the coefficient of the third is equal to the sum of the products of every two of the roots; etc. He was the first to decompose equations into their simple factors; but, since he failed to recognise imaginary and even negative roots, he failed also to prove that every equation could be thus decomposed. Harriot made some changes in algebraic nota-

tion, adopting small letters of the alphabet in place of the capitals used by Vieta. The symbols of inequality $>$ and $<$ were introduced by him. Harriot's work, *Artis Analyticæ praxis*, was published in 1631, ten years after his death. **William Oughtred** (1574–1660) contributed vastly to the propagation of mathematical knowledge in England by his treatises, which were long used in the universities. He introduced $\times$ as symbol of multiplication, and :: as that of proportion. By him ratio was expressed by only one dot. In the eighteenth century *Christian Wolf* secured the general adoption of the dot as a symbol of multiplication, and the sign for ratio was thereupon changed to two dots. Oughtred's ministerial duties left him but little time for the pursuit of mathematics during daytime, and evenings his economical wife denied him the use of a light.

Algebra was now in a state of sufficient perfection to enable Descartes to take that important step which forms one of the grand epochs in the history of mathematics, — the application of algebraic analysis to define the nature and investigate the properties of algebraic curves.

In geometry, the determination of the areas of curvilinear figures was diligently studied at this period. **Paul Guldin** (1577–1643), a Swiss mathematician of considerable note, rediscovered the following theorem, published in his *Centrobaryca*, which has been named after him, though first found in the *Mathematical Collections* of Pappus: The volume of a solid of revolution is equal to the area of the generating figure, multiplied by the circumference described by the centre of gravity. We shall see that this method excels that of Kepler and Cavalieri in following a more exact and natural course; but it has the disadvantage of necessitating the determination of the centre of gravity, which in itself may be a more difficult problem than the original one of finding the

volume. Guldin made some attempts to prove his theorem, but Cavalieri pointed out the weakness of his demonstration.

Johannes Kepler (1571–1630) was a native of Würtemberg and imbibed Copernican principles while at the University of Tübingen. His pursuit of science was repeatedly interrupted by war, religious persecution, pecuniary embarrassments, frequent changes of residence, and family troubles. In 1600 he became for one year assistant to the Danish astronomer, Tycho Brahe, in the observatory near Prague. The relation between the two great astronomers was not always of an agreeable character. Kepler's publications are voluminous. His first attempt to explain the solar system was made in 1596, when he thought he had discovered a curious relation between the five regular solids and the number and distance of the planets. The publication of this pseudo-discovery brought him much fame. Maturer reflection and intercourse with Tycho Brahe and Galileo led him to investigations and results more worthy of his genius — "Kepler's laws." He enriched pure mathematics as well as astronomy. It is not strange that he was interested in the mathematical science which had done him so much service; for "if the Greeks had not cultivated conic sections, Kepler could not have superseded Ptolemy."[11] The Greeks never dreamed that these curves would ever be of practical use; Aristæus and Apollonius studied them merely to satisfy their intellectual cravings after the ideal; yet the conic sections assisted Kepler in tracing the march of the planets in their elliptic orbits. Kepler made also extended use of logarithms and decimal fractions, and was enthusiastic in diffusing a knowledge of them. At one time, while purchasing wine, he was struck by the inaccuracy of the ordinary modes of determining the contents of kegs. This led him to the study of the volumes of solids of revolution and to the publication of the *Stereometria Doliorum* in 1615. In it he deals first with the

solids known to Archimedes and then takes up others. Kepler introduced a new idea into geometry; namely, that of infinitely great and infinitely small quantities. Greek mathematicians always shunned this notion, but with it modern mathematicians have completely revolutionised the science. In comparing rectilinear figures, the method of superposition was employed by the ancients, but in comparing rectilinear and curvilinear figures with each other, this method failed because no addition or subtraction of rectilinear figures could ever produce curvilinear ones. To meet this case, they devised the Method of Exhaustion, which was long and difficult; it was purely synthetical, and in general required that the conclusion should be known at the outset. The new notion of infinity led gradually to the invention of methods immeasurably more powerful. Kepler conceived the circle to be composed of an infinite number of triangles having their common vertices at the centre, and their bases in the circumference; and the sphere to consist of an infinite number of pyramids. He applied conceptions of this kind to the determination of the areas and volumes of figures generated by curves revolving about any line as axis, but succeeded in solving only a few of the simplest out of the 84 problems which he proposed for investigation in his *Stereometria.*

Other points of mathematical interest in Kepler's works are (1) the statement of the earliest problem of inverse tangents; (2) an investigation which amounts to the evaluation of the definite integral $\int_0^{\phi} \sin \phi d\phi = 1 - \cos \phi$; (3) the assertion that the circumference of an ellipse, whose axes are $2a$ and $2b$, is nearly $\pi (a + b)$; (4) a passage from which it has been inferred that Kepler knew the variation of a function near its maximum value to disappear; (5) the assumption of the principle of continuity (which differentiates modern from ancient geometry), when he shows that a parabola has a focus at

infinity, that lines radiating from this "cæcus focus" are parallel and have no other point at infinity.

The *Stereometria* led Cavalieri, an Italian Jesuit, to the consideration of infinitely small quantities. **Bonaventura Cavalieri** (1598–1647), a pupil of Galileo and professor at Bologna, is celebrated for his *Geometria indivisibilibus continuorum nova quadam ratione promota*, 1635. This work expounds his method of Indivisibles, which occupies an intermediate place between the method of exhaustion of the Greeks and the methods of Newton and Leibniz. He considers lines as composed of an infinite number of points, surfaces as composed of an infinite number of lines, and solids of an infinite number of planes. The relative magnitude of two solids or surfaces could then be found simply by the summation of series of planes or lines. For example, he finds the sum of the squares of all lines making up a triangle equal to one-third the sum of the squares of all lines of a parallelogram of equal base and altitude; for if in a triangle, the first line at the apex be 1, then the second is 2, the third is 3, and so on; and the sum of their squares is

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = n\,(n+1)\,(2n+1) \div 6.$$

In the parallelogram, each of the lines is n and their number is n; hence the total sum of their squares is n^3. The ratio between the two sums is therefore

$$n\,(n+1)\,(2n+1) \div 6\,n^3 = \tfrac{1}{3},$$

since n is infinite. From this he concludes that the pyramid or cone is respectively $\frac{1}{3}$ of a prism or cylinder of equal base and altitude, since the polygons or circles composing the former decrease from the base to the apex in the same way as the squares of the lines parallel to the base in a triangle decrease from base to apex. By the Method of Indivisibles, Cavalieri

solved the majority of the problems proposed by Kepler. Though expeditious and yielding correct results, Cavalieri's method lacks a scientific foundation. If a line has absolutely no width, then no number, however great, of lines can ever make up an area; if a plane has no thickness whatever, then even an infinite number of planes cannot form a solid. The reason why this method led to correct conclusions is that one area is to another area in the same ratio as the sum of the series of lines in the one is to the sum of the series of lines in the other. Though unscientific, Cavalieri's method was used for fifty years as a sort of integral calculus. It yielded solutions to some difficult problems. Guldin made a severe attack on Cavalieri and his method. The latter published in 1647, after the death of Guldin, a treatise entitled *Exercitationes geometricæ sex*, in which he replied to the objections of his opponent and attempted to give a clearer explanation of his method. Guldin had never been able to demonstrate the theorem named after him, except by metaphysical reasoning, but Cavalieri proved it by the method of indivisibles. A revised edition of the *Geometry of Indivisibles* appeared in 1653.

There is an important curve, not known to the ancients, which now began to be studied with great zeal. Roberval gave it the name of "trochoid," Pascal the name of "roulette," Galileo the name of "cycloid." The invention of this curve seems to be due to Galileo, who valued it for the graceful form it would give to arches in architecture. He ascertained its area by weighing paper figures of the cycloid against that of the generating circle, and found thereby the first area to be nearly but not exactly thrice the latter. A mathematical determination was made by his pupil, **Evangelista Torricelli** (1608–1647), who is more widely known as a physicist than as a mathematician.

By the Method of Indivisibles he demonstrated its area to be triple that of the revolving circle, and published his solution. This same quadrature had been effected a few years earlier by Roberval in France, but his solution was not known to the Italians. Roberval, being a man of irritable and violent disposition, unjustly accused the mild and amiable Torricelli of stealing the proof. This accusation of plagiarism created so much chagrin with Torricelli that it is considered to have been the cause of his early death. **Vincenzo Viviani**, another prominent pupil of Galileo, determined the tangent to the cycloid. This was accomplished in France by Descartes and Fermat.

In France, where geometry began to be cultivated with greatest success, Roberval, Fermat, Pascal, employed the Method of Indivisibles and made new improvements in it. **Giles Persone de Roberval** (1602–1675), for forty years professor of mathematics at the College of France in Paris, claimed for himself the invention of the Method of Indivisibles. Since his complete works were not published until after his death, it is difficult to settle questions of priority. Montucla and Chasles are of the opinion that he invented the method independent of and earlier than the Italian geometer, though the work of the latter was published much earlier than Roberval's. Marie finds it difficult to believe that the Frenchman borrowed nothing whatever from the Italian, for both could not have hit independently upon the word *Indivisibles,* which is applicable to infinitely small quantities, as conceived by Cavalieri, but not as conceived by Roberval. Roberval and Pascal improved the rational basis of the Method of Indivisibles, by considering an area as made up of an indefinite number of rectangles instead of lines, and a solid as composed of indefinitely small solids instead of surfaces. Roberval applied the method to the finding of

areas, volumes, and centres of gravity. He effected the quadrature of a parabola of any degree $y^m = a^{m-1}x$, and also of a parabola $y^m = a^{m-n}x^n$. We have already mentioned his quadrature of the cycloid. Roberval is best known for his method of drawing tangents. He was the first to apply motion to the resolution of this important problem. His method is allied to Newton's principle of fluxions. Archimedes conceived his spiral to be generated by a double motion. This idea Roberval extended to all curves. Plane curves, as for instance the conic sections, may be generated by a point acted upon by two forces, and are the resultant of two motions. If at any point of the curve the resultant be resolved into its components, then the diagonal of the parallelogram determined by them is the tangent to the curve at that point. The greatest difficulty connected with this ingenious method consisted in resolving the resultant into components having the proper lengths and directions. Roberval did not always succeed in doing this, yet his new idea was a great step in advance. He broke off from the ancient definition of a tangent as a straight line having only one point in common with a curve, — a definition not valid for curves of higher degrees, nor apt even in curves of the second degree to bring out the properties of tangents and the parts they may be made to play in the generation of the curves. The subject of tangents received special attention also from Fermat, Descartes, and Barrow, and reached its highest development after the invention of the differential calculus. Fermat and Descartes defined tangents as secants whose two points of intersection with the curve coincide; Barrow considered a curve a polygon, and called one of its sides produced a tangent.

A profound scholar in all branches of learning and a mathematician of exceptional powers was **Pierre de Fermat** (1601–1665). He studied law at Toulouse, and in 1631 was made

councillor for the parliament of Toulouse. His leisure time was mostly devoted to mathematics, which he studied with irresistible passion. Unlike Descartes and Pascal, he led a quiet and unaggressive life. Fermat has left the impress of his genius upon all branches of mathematics then known. A great contribution to geometry was his *De maximis et minimis.* About twenty years earlier, Kepler had first observed that the increment of a variable, as, for instance, the ordinate of a curve, is evanescent for values very near a maximum or a minimum value of the variable. Developing this idea, Fermat obtained his rule for maxima and minima. He substituted $x + e$ for x in the given function of x and then equated to each other the two consecutive values of the function and divided the equation by e. If e be taken 0, then the roots of this equation are the values of x, making the function a maximum or a minimum. Fermat was in possession of this rule in 1629. The main difference between it and the rule of the differential calculus is that it introduces the indefinite quantity e instead of the infinitely small dx. Fermat made it the basis for his method of drawing tangents.

Owing to a want of explicitness in statement, Fermat's method of maxima and minima, and of tangents, was severely attacked by his great contemporary, Descartes, who could never be brought to render due justice to his merit. In the ensuing dispute, Fermat found two zealous defenders in Roberval and Pascal, the father; while Mydorge, Desargues, and Hardy supported Descartes.

Since Fermat introduced the conception of infinitely small differences between consecutive values of a function and arrived at the principle for finding the maxima and minima, it was maintained by Lagrange, Laplace, and Fourier, that Fermat may be regarded as the first inventor of the differential calculus. This point is not well taken, as will be seen

from the words of Poisson, himself a Frenchman, who rightly says that the differential calculus "consists in a system of rules proper for finding the differentials of all functions, rather than in the use which may be made of these infinitely small variations in the solution of one or two isolated problems."

A contemporary mathematician, whose genius excelled even that of the great Fermat, was **Blaise Pascal** (1623–1662). He was born at Clermont in Auvergne. In 1626 his father retired to Paris, where he devoted himself to teaching his son, for he would not trust his education to others. Blaise Pascal's genius for geometry showed itself when he was but twelve years old. His father was well skilled in mathematics, but did not wish his son to study it until he was perfectly acquainted with Latin and Greek. All mathematical books were hidden out of his sight. The boy once asked his father what mathematics treated of, and was answered, in general, "that it was the method of making figures with exactness, and of finding out what proportions they relatively had to one another." He was at the same time forbidden to talk any more about it, or ever to think of it. But his genius could not submit to be confined within these bounds. Starting with the bare fact that mathematics taught the means of making figures infallibly exact, he employed his thoughts about it and with a piece of charcoal drew figures upon the tiles of the pavement, trying the methods of drawing, for example, an exact circle or equilateral triangle. He gave names of his own to these figures and then formed axioms, and, in short, came to make perfect demonstrations. In this way he arrived unaided at the theorem that the sum of the three angles of a triangle is equal to two right angles. His father caught him in the act of studying this theorem, and was so astonished at the sublimity and force of his genius as to weep for joy. The father now gave

him Euclid's *Elements*, which he, without assistance, mastered easily. His regular studies being languages, the boy employed only his hours of amusement on the study of geometry, yet he had so ready and lively a penetration that, at the age of sixteen, he wrote a treatise upon conics, which passed for such a surprising effort of genius, that it was said nothing equal to it in strength had been produced since the time of Archimedes. Descartes refused to believe that it was written by one so young as Pascal. This treatise was never published, and is now lost. Leibniz saw it in Paris and reported on a portion of its contents. The precocious youth made vast progress in all the sciences, but the constant application at so tender an age greatly impaired his health. Yet he continued working, and at nineteen invented his famous machine for performing arithmetical operations mechanically. This continued strain from overwork resulted in a permanent indisposition, and he would sometimes say that from the time he was eighteen, he never passed a day free from pain. At the age of twenty-four he resolved to lay aside the study of the human sciences and to consecrate his talents to religion. His Provincial Letters against the Jesuits are celebrated. But at times he returned to the favourite study of his youth. Being kept awake one night by a toothache, some thoughts undesignedly came into his head concerning the roulette or cycloid; one idea followed another; and he thus discovered properties of this curve even to demonstration. A correspondence between him and Fermat on certain problems was the beginning of the theory of probability. Pascal's illness increased, and he died at Paris at the early age of thirty-nine years.[30] By him the answer to the objection to Cavalieri's Method of Indivisibles was put in the clearest form. Like Roberval, he explained "the sum of right lines" to mean "the sum of infinitely small rectangles." Pascal greatly advanced

the knowledge of the cycloid. He determined the area of a section produced by any line parallel to the base; the volume generated by it revolving around its base or around the axis; and, finally, the centres of gravity of these volumes, and also of half these volumes cut by planes of symmetry. Before publishing his results, he sent, in 1658, to all mathematicians that famous challenge offering prizes for the first two solutions of these problems. Only Wallis and A. La Louère competed for them. The latter was quite unequal to the task; the former, being pressed for time, made numerous mistakes: neither got a prize. Pascal then published his own solutions, which produced a great sensation among scientific men. Wallis, too, published his, with the errors corrected. Though not competing for the prizes, Huygens, Wren, and Fermat solved some of the questions. The chief discoveries of **Christopher Wren** (1632–1723), the celebrated architect of St. Paul's Cathedral in London, were the rectification of a cycloidal arc and the determination of its centre of gravity. Fermat found the area generated by an arc of the cycloid. Huygens invented the cycloidal pendulum.

The beginning of the seventeenth century witnessed also a revival of synthetic geometry. One who treated conics still by ancient methods, but who succeeded in greatly simplifying many prolix proofs of Apollonius, was **Claude Mydorge** in Paris (1585–1647), a friend of Descartes. But it remained for **Girard Desargues** (1593–1662) of Lyons, and for Pascal, to leave the beaten track and cut out fresh paths. They introduced the important method of Perspective. All conics on a cone with circular base appear circular to an eye at the apex. Hence Desargues and Pascal conceived the treatment of the conic sections as projections of circles. Two important and beautiful theorems were given by Desargues: The one is on the "involution of the six points," in which a transversal

meets a conic and an inscribed quadrangle; the other is that, if the vertices of two triangles, situated either in space or in a plane, lie on three lines meeting in a point, then their sides meet in three points lying on a line; and conversely. This last theorem has been employed in recent times by Brianchon, Sturm, Gergonne, and Poncelet. Poncelet made it the basis of his beautiful theory of homoligical figures. We owe to Desargues the theory of involution and of transversals; also the beautiful conception that the two extremities of a straight line may be considered as meeting at infinity, and that parallels differ from other pairs of lines only in having their points of intersection at infinity. Pascal greatly admired Desargues' results, saying (in his *Essais pour les Coniques*), "I wish to acknowledge that I owe the little that I have discovered on this subject, to his writings." Pascal's and Desargues' writings contained the fundamental ideas of modern synthetic geometry. In Pascal's wonderful work on conics, written at the age of sixteen and now lost, were given the theorem on the anharmonic ratio, first found in Pappus, and also that celebrated proposition on the mystic hexagon, known as "Pascal's theorem," viz. that the opposite sides of a hexagon inscribed in a conic intersect in three points which are collinear. This theorem formed the keystone to his theory. He himself said that from this alone he deduced over 400 corollaries, embracing the conics of Apollonius and many other results. Thus the genius of Desargues and Pascal uncovered several of the rich treasures of modern synthetic geometry; but owing to the absorbing interest taken in the analytical geometry of Descartes and later in the differential calculus, the subject was almost entirely neglected until the present century.

In the theory of numbers no new results of scientific value had been reached for over 1000 years, extending from the

times of Diophantus and the Hindoos until the beginning of the seventeenth century. But the illustrious period we are now considering produced men who rescued this science from the realm of mysticism and superstition, in which it had been so long imprisoned; the properties of numbers began again to be studied scientifically. Not being in possession of the Hindoo indeterminate analysis, many beautiful results of the Brahmins had to be re-discovered by the Europeans. Thus a solution in integers of linear indeterminate equations was re-discovered by the Frenchman **Bachet de Méziriac** (1581–1638), who was the earliest noteworthy European Diophantist. In 1612 he published *Problèmes plaisants et délectables qui se font par les nombres*, and in 1621 a Greek edition of *Diophantus* with notes. The father of the modern theory of numbers is **Fermat**. He was so uncommunicative in disposition, that he generally concealed his methods and made known his results only. In some cases later analysts have been greatly puzzled in the attempt of supplying the proofs. Fermat owned a copy of Bachet's *Diophantus*, in which he entered numerous marginal notes. In 1670 these notes were incorporated in a new edition of *Diophantus*, brought out by his son. Other theorems on numbers, due to Fermat, were published in his *Opera varia* (edited by his son) and in Wallis's *Commercium epistolicum* of 1658. Of the following theorems, the first seven are found in the marginal notes: —

(1) $x^n + y^n = z^n$ is impossible for integral values of x, y, and z, when $n > 2$. Remark: "I have found for this a truly wonderful proof, but the margin is too small to hold it." Repeatedly was this theorem made the prize question of learned societies. It has given rise to investigations of great interest and difficulty on the part of Euler, Lagrange, Dirichlet, and Kummer.

(2) A prime of the form $4n+1$ is only once the hypothenuse

of a right triangle; its square is twice; its cube is three times, etc. Example: $5^2 = 3^2 + 4^2$; $25^2 = 15^2 + 20^2 = 7^2 + 24^2$; $125^2 = 75^2 + 100^2 = 35^2 + 120^2 = 44^2 + 117^2$.

(3) A prime of the form $4n + 1$ can be expressed once, and only once, as the sum of two squares. Proved by Euler.

(4) A number composed of two cubes can be resolved into two other cubes in an infinite multiplicity of ways.

(5) Every number is either a triangular number or the sum of two or three triangular numbers; either a square or the sum of two, three, or four squares; either a pentagonal number or the sum of two, three, four, or five pentagonal numbers; similarly for polygonal numbers in general. The proof of this and other theorems is promised by Fermat in a future work which never appeared. This theorem is also given, with others, in a letter of 1637(?) addressed to *Pater Mersenne*.

(6) As many numbers as you please may be found, such that the square of each remains a square on the addition to or subtraction from it of the sum of all the numbers.

(7) $x^4 + y^4 = z^2$ is impossible.

(8) In a letter of 1640 he gives the celebrated theorem generally known as "Fermat's theorem," which we state in Gauss's notation: If p is prime, and a is prime to p, then $a^{p-1} \equiv 1 \pmod{p}$. It was proved by Euler.

(9) Fermat died with the belief that he had found a long-sought-for law of prime numbers in the formula $2^{2^n} + 1 =$ a prime, but he admitted that he was unable to prove it rigorously. The law is not true, as was pointed out by Euler in the example $2^{2^5} + 1 = 4{,}294{,}967{,}297 = 6{,}700{,}417$ times 641. The American lightning calculator *Zerah Colburn*, when a boy, readily found the factors, but was unable to explain the method by which he made his marvellous mental computation.

(10) An odd prime number can be expressed as the difference of two squares in one, and only one, way. This theorem,

given in the *Relation*, was used by Fermat for the decomposition of large numbers into prime factors.

(11) If the integers a, b, c represent the sides of a right triangle, then its area cannot be a square number. This was proved by Lagrange.

(12) Fermat's solution of $ax^2 + 1 = y^2$, where a is integral but not a square, has come down in only the broadest outline, as given in the *Relation*. He proposed the problem to the Frenchman, *Bernhard Frenicle de Bessy*, and in 1657 to all living mathematicians. In England, Wallis and Lord Brounker conjointly found a laborious solution, which was published in 1658, and also in 1668, in an algebraical work brought out by *John Pell*. Though Pell had no other connection with the problem, it went by the name of "Pell's problem." The first solution was given by the Hindoos.

We are not sure that Fermat subjected all his theorems to rigorous proof. His methods of proof were entirely lost until 1879, when a document was found buried among the manuscripts of Huygens in the library of Leyden, entitled *Relation des découvertes en la science des nombres*. It appears from it that he used an inductive method, called by him *la descente infinie ou indefinie*. He says that this was particularly applicable in proving the impossibility of certain relations, as, for instance, Theorem 11, given above, but that he succeeded in using the method also in proving affirmative statements. Thus he proved Theorem 3 by showing that if we suppose there be a prime $4n + 1$ which does not possess this property, then there will be a smaller prime of the form $4n + 1$ not possessing it; and a third one smaller than the second, not possessing it; and so on. Thus descending indefinitely, he arrives at the number 5, which is the smallest prime factor of the form $4n + 1$. From the above supposition it would follow that 5 is not the sum of two squares — a conclusion

contrary to fact. Hence the supposition is false, and the theorem is established. Fermat applied this method of descent with success in a large number of theorems. By this method Euler, Legendre, Dirichlet, proved several of his enunciations and many other numerical propositions.

A correspondence between *Pascal* and *Fermat* relating to a certain game of chance was the germ of the theory of probabilities, which has since attained a vast growth. Chevalier de Méré proposed to Pascal the fundamental problem, to determine the probability which each player has, at any given stage of the game, of winning the game. Pascal and Fermat supposed that the players have equal chances of winning a single point.

The former communicated this problem to Fermat, who studied it with lively interest and solved it by the theory of combinations, a theory which was diligently studied both by him and Pascal. The calculus of probabilities engaged the attention also of Huygens. The most important theorem reached by him was that, if A has p chances of winning a sum a, and q chances of winning a sum b, then he may expect to win the sum $\frac{ap+bq}{p+q}$. The next great work on the theory of probability was the *Ars conjectandi* of Jakob Bernoulli.

Among the ancients, Archimedes was the only one who attained clear and correct notions on theoretical statics. He had acquired firm possession of the idea of pressure, which lies at the root of mechanical science. But his ideas slept nearly twenty centuries, until the time of **Stevin** and **Galileo.** Stevin determined accurately the force necessary to sustain a body on a plane inclined at any angle to the horizon. He was in possession of a complete doctrine of equilibrium. While Stevin investigated statics, Galileo pursued principally dynamics. Galileo was the first to abandon the Aristotelian idea that bodies descend more quickly in proportion as they are

heavier; he established the first law of motion; determined the laws of falling bodies; and, having obtained a clear notion of acceleration and of the independence of different motions, was able to prove that projectiles move in parabolic curves. Up to his time it was believed that a cannon-ball moved forward at first in a straight line and then suddenly fell vertically to the ground. Galileo had an understanding of *centrifugal forces*, and gave a correct definition of *momentum*. Though he formulated the fundamental principle of statics, known as the *parallelogram of forces*, yet he did not fully recognise its scope. The principle of virtual velocities was partly conceived by **Guido Ubaldo** (died 1607), and afterwards more fully by Galileo.

Galileo is the founder of the science of dynamics. Among his contemporaries it was chiefly the novelties he detected in the sky that made him celebrated, but Lagrange claims that his astronomical discoveries required only a telescope and perseverance, while it took an extraordinary genius to discover laws from phenomena, which we see constantly and of which the true explanation escaped all earlier philosophers. The first contributor to the science of mechanics after Galileo was Descartes.

DESCARTES TO NEWTON.

Among the earliest thinkers of the seventeenth and eighteenth centuries, who employed their mental powers toward the destruction of old ideas and the up-building of new ones, ranks **René Descartes** (1596–1650). Though he professed orthodoxy in faith all his life, yet in science he was a profound sceptic. He found that the world's brightest thinkers had been long exercised in metaphysics, yet they had discovered nothing

certain; nay, had even flatly contradicted each other. This led him to the gigantic resolution of taking nothing whatever on authority, but of subjecting everything to scrutinous examination, according to new methods of inquiry. The certainty of the conclusions in geometry and arithmetic brought out in his mind the contrast between the true and false ways of seeking the truth. He thereupon attempted to apply mathematical reasoning to all sciences. "Comparing the mysteries of nature with the laws of mathematics, he dared to hope that the secrets of both could be unlocked with the same key." Thus he built up a system of philosophy called Cartesianism.

Great as was Descartes' celebrity as a metaphysician, it may be fairly questioned whether his claim to be remembered by posterity as a mathematician is not greater. His philosophy has long since been superseded by other systems, but the analytical geometry of Descartes will remain a valuable possession forever. At the age of twenty-one, Descartes enlisted in the army of Prince Maurice of Orange. His years of soldiering were years of leisure, in which he had time to pursue his studies. At that time mathematics was his favourite science. But in 1625 he ceased to devote himself to pure mathematics. Sir William Hamilton is in error when he states that Descartes considered mathematical studies absolutely pernicious as a means of internal culture. In a letter to Mersenne, Descartes says: "M. Desargues puts me under obligations on account of the pains that it has pleased him to have in me, in that he shows that he is sorry that I do not wish to study more in geometry, but I have resolved to quit only abstract geometry, that is to say, the consideration of questions which *serve only to exercise the mind*, and this, in order to study another kind of geometry, which has for its object the explanation of the phenomena of nature. . . . You know that all my physics is nothing else than geometry." The years between 1629 and

1649 were passed by him in Holland in the study, principally, of physics and metaphysics. His residence in Holland was during the most brilliant days of the Dutch state. In 1637 he published his *Discours de la Méthode*, containing among others an essay of 106 pages on geometry. His *Geometry* is not easy reading. An edition appeared subsequently with notes by his friend *De Beaune*, which were intended to remove the difficulties.

It is frequently stated that Descartes was the first to apply algebra to geometry. This statement is inaccurate, for Vieta and others had done this before him. Even the Arabs sometimes used algebra in connection with geometry. The new step that Descartes did take was the introduction into geometry of an analytical method based on the notion of variables and constants, which enabled him to represent curves by algebraic equations. In the Greek geometry, the idea of motion was wanting, but with Descartes it became a very fruitful conception. By him a point on a plane was determined in position by its distances from two fixed right lines or axes. These distances varied with every change of position in the point. This geometric idea of *co-ordinate representation*, together with the algebraic idea of *two variables in one equation* having an indefinite number of simultaneous values, furnished a method for the study of loci, which is admirable for the generality of its solutions. Thus the entire conic sections of Apollonius is wrapped up and contained in a single equation of the second degree.

The Latin term for "ordinate," used by Descartes comes from the expression *lineæ ordinatæ*, employed by Roman surveyors for parallel lines. The term *abscissa* occurs for the first time in a Latin work of 1659, written by *Stefano degli Angeli* (1623–1697), a professor of mathematics in Rome.[3] Descartes' geometry was called "analytical geometry," partly

because, unlike the synthetic geometry of the ancients, it is actually *analytical*, in the sense that the word is used in logic; and partly because the practice had then already arisen, of designating by the term *analysis* the calculus with general quantities.

The first important example solved by Descartes in his geometry is the "problem of Pappus"; viz. "Given several straight lines in a plane, to find the locus of a point such that the perpendiculars, or more generally, straight lines at given angles, drawn from the point to the given lines, shall satisfy the condition that the product of certain of them shall be in a given ratio to the product of the rest." Of this celebrated problem, the Greeks solved only the special case when the number of given lines is four, in which case the locus of the point turns out to be a conic section. By Descartes it was solved completely, and it afforded an excellent example of the use which can be made of his analytical method in the study of loci. Another solution was given later by Newton in the *Principia*.

The methods of drawing tangents invented by Roberval and Fermat were noticed earlier. Descartes gave a third method. Of all the problems which he solved by his geometry, none gave him as great pleasure as his mode of constructing tangents. It is profound but operose, and, on that account, inferior to Fermat's. His solution rests on the method of *Indeterminate Coefficients*, of which he bears the honour of invention. Indeterminate coefficients were employed by him also in solving bi-quadratic equations.

The essays of Descartes on dioptrics and geometry were sharply criticised by Fermat, who wrote objections to the former, and sent his own treatise on "maxima and minima" to show that there were omissions in the geometry. Descartes thereupon made an attack on Fermat's method of tangents.

Descartes was in the wrong in this attack, yet he continued the controversy with obstinacy. He had a controversy also with Roberval on the cycloid. This curve has been called the "Helen of geometers," on account of its beautiful properties and the controversies which their discovery occasioned. Its quadrature by Roberval was generally considered a brilliant achievement, but Descartes commented on it by saying that any one moderately well versed in geometry might have done this. He then sent a short demonstration of his own. On Roberval's intimating that he had been assisted by a knowledge of the solution, Descartes constructed the tangent to the curve, and challenged Roberval and Fermat to do the same. Fermat accomplished it, but Roberval never succeeded in solving this problem, which had cost the genius of Descartes but a moderate degree of attention.

He studied some new curves, now called "ovals of Descartes," which were intended by him to serve in the construction of converging lenses, but which yielded no results of practical value.

The application of algebra to the doctrine of curved lines reacted favourably upon algebra. As an abstract science, Descartes improved it by the systematic use of exponents and by the full interpretation and construction of negative quantities. Descartes also established some theorems on the theory of equations. Celebrated is his "rule of signs" for determining the number of positive and negative roots; viz. an equation may have as many + roots as there are variations of signs, and as many − roots as there are permanencies of signs. Descartes was charged by Wallis with availing himself, without acknowledgment, of Harriot's theory of equations, particularly his mode of generating equations; but there seems to be no good ground for the charge. Wallis also claimed that Descartes failed to observe that the above rule of signs is not true whenever the

equation has imaginary roots; but Descartes does not say that the equation *always has*, but that it *may have* so many roots. It is true that Descartes does not consider the case of imaginaries directly, but further on in his *Geometry* he gives incontestable evidence of being able to handle this case also.

In mechanics, Descartes can hardly be said to have advanced beyond Galileo. The latter had overthrown the ideas of Aristotle on this subject, and Descartes simply "threw himself upon the enemy" that had already been "put to the rout." His statement of the first and second laws of motion was an improvement in form, but his third law is false in substance. The motions of bodies in their direct impact was imperfectly understood by Galileo, erroneously given by Descartes, and first correctly stated by Wren, Wallis, and Huygens.

One of the most devoted pupils of Descartes was the learned *Princess Elizabeth*, daughter of Frederick V. She applied the new analytical geometry to the solution of the "Apollonian problem." His second royal follower was *Queen Christina*, the daughter of Gustavus Adolphus. She urged upon Descartes to come to the Swedish court. After much hesitation he accepted the invitation in 1649. He died at Stockholm one year later. His life had been one long warfare against the prejudices of men.

It is most remarkable that the mathematics and philosophy of Descartes should at first have been appreciated less by his countrymen than by foreigners. The indiscreet temper of Descartes alienated the great contemporary French mathematicians, Roberval, Fermat, Pascal. They continued in investigations of their own, and on some points strongly opposed Descartes. The universities of France were under strict ecclesiastical control and did nothing to introduce his mathematics and philosophy. It was in the youthful universities of

Holland that the effect of Cartesian teachings was most immediate and strongest.

The only prominent Frenchman who immediately followed in the footsteps of the great master was **De Beaune** (1601–1652). He was one of the first to point out that the properties of a curve can be deduced from the properties of its tangent. This mode of inquiry has been called the *inverse method of tangents.* He contributed to the theory of equations by considering for the first time the upper and lower limits of the roots of numerical equations.

In the Netherlands a large number of distinguished mathematicians were at once struck with admiration for the Cartesian geometry. Foremost among these are *van Schooten, John de Witt, van Heuraet, Sluze,* and *Hudde.* **Van Schooten** (died 1660), professor of mathematics at Leyden, brought out an edition of Descartes' geometry, together with the notes thereon by De Beaune. His chief work is his *Exercitationes Mathematicæ,* in which he applies the analytical geometry to the solution of many interesting and difficult problems. The noble-hearted **Johann de Witt,** grand-pensioner of Holland, celebrated as a statesman and for his tragical end, was an ardent geometrician. He conceived a new and ingenious way of generating conics, which is essentially the same as that by projective pencils of rays in modern synthetic geometry. He treated the subject not synthetically, but with aid of the Cartesian analysis. **René François de Sluze** (1622–1685) and **Johann Hudde** (1633–1704) made some improvements on Descartes' and Fermat's methods of drawing tangents, and on the theory of maxima and minima. With Hudde, we find the first use of three variables in analytical geometry. He is the author of an ingenious rule for finding equal roots. We illustrate it by the equation $x^3 - x^2 - 8x + 12 = 0$. Taking an arithmetical progression 3, 2, 1, 0, of which the highest term is equal to the degree of

the equation, and multiplying each term of the equation respectively by the corresponding term of the progression, we get $3x^3 - 2x^2 - 8x = 0$, or $3x^2 - 2x - 8 = 0$. This last equation is by one degree lower than the original one. Find the G.C.D. of the two equations. This is $x - 2$; hence 2 is one of the two equal roots. Had there been no common divisor, then the original equation would not have possessed equal roots. Hudde gave a demonstration for this rule.[24]

Heinrich van Heuraet must be mentioned as one of the earliest geometers who occupied themselves with success in the rectification of curves. He observed in a general way that the two problems of quadrature and of rectification are really identical, and that the one can be reduced to the other. Thus he carried the rectification of the hyperbola back to the quadrature of the hyperbola. The semi-cubical parabola $y^3 = ax^2$ was the first curve that was ever rectified absolutely. This appears to have been accomplished independently by Van Heuraet in Holland and by **William Neil** (1637–1670) in England. According to Wallis the priority belongs to Neil. Soon after, the cycloid was rectified by Wren and Fermat.

The prince of philosophers in Holland, and one of the greatest scientists of the seventeenth century, was **Christian Huygens** (1629–1695), a native of the Hague. Eminent as a physicist and astronomer, as well as mathematician, he was a worthy predecessor of Sir Isaac Newton. He studied at Leyden under the younger *Van Schooten*. The perusal of some of his earliest theorems led Descartes to predict his future greatness. In 1651 Huygens wrote a treatise in which he pointed out the fallacies of Gregory St. Vincent (1584–1667) on the subject of quadratures. He himself gave a remarkably close and convenient approximation to the length of a circular arc. In 1660 and 1663 he went to Paris and to London. In 1666 he was appointed by Louis XIV. member of the French

Academy of Sciences. He was induced to remain in Paris from that time until 1681, when he returned to his native city, partly for consideration of his health and partly on account of the revocation of the Edict of Nantes.

The majority of his profound discoveries were made with aid of the ancient geometry, though at times he used the geometry of Descartes or of Cavalieri and Fermat. Thus, like his illustrious friend, Sir Isaac Newton, he always showed partiality for the Greek geometry. Newton and Huygens were kindred minds, and had the greatest admiration for each other. Newton always speaks of him as the "Summus Hugenius."

To the two curves (cubical parabola and cycloid) previously rectified he added a third, — the cissoid. He solved the problem of the catenary, determined the surface of the parabolic and hyperbolic conoid, and discovered the properties of the logarithmic curve and the solids generated by it. Huygens' *De horologio oscillatorio* (Paris, 1673) is a work that ranks second only to the *Principia* of Newton and constitutes historically a necessary introduction to it.[13] The book opens with a description of pendulum clocks, of which Huygens is the inventor. Then follows a treatment of accelerated motion of bodies falling free, or sliding on inclined planes, or on given curves, — culminating in the brilliant discovery that the cycloid is the tautochronous curve. To the theory of curves he added the important theory of "evolutes." After explaining that the tangent of the evolute is normal to the involute, he applied the theory to the cycloid, and showed by simple reasoning that the evolute of this curve is an equal cycloid. Then comes the complete general discussion of the centre of oscillation. This subject had been proposed for investigation by Mersenne and discussed by Descartes and Roberval. In Huygens' assumption that the common centre

of gravity of a group of bodies, oscillating about a horizontal axis, rises to its original height, but no higher, is expressed for the first time one of the most beautiful principles of dynamics, afterwards called the principle of the conservation of *vis viva*.[32] The thirteen theorems at the close of the work relate to the theory of centrifugal force in circular motion. This theory aided Newton in discovering the law of gravitation.

Huygens wrote the first formal treatise on probability. He proposed the wave-theory of light and with great skill applied geometry to its development. This theory was long neglected, but was revived and successfully worked out by Young and Fresnel a century later. Huygens and his brother improved the telescope by devising a better way of grinding and polishing lenses. With more efficient instruments he determined the nature of Saturn's appendage and solved other astronomical questions. Huygens' *Opuscula posthuma* appeared in 1703.

Passing now from Holland to England, we meet there one of the most original mathematicians of his day — **John Wallis** (1616–1703). He was educated for the Church at Cambridge and entered Holy Orders. But his genius was employed chiefly in the study of mathematics. In 1649 he was appointed Savilian professor of geometry at Oxford. He was one of the original members of the Royal Society, which was founded in 1663. Wallis thoroughly grasped the mathematical methods both of Cavalieri and Descartes. His *Conic Sections* is the earliest work in which these curves are no longer considered as sections of a cone, but as curves of the second degree, and are treated analytically by the Cartesian method of co-ordinates. In this work Wallis speaks of Descartes in the highest terms, but in his *Algebra* he, without good reason, accuses Descartes of plagiarising from Harriot. We have

already mentioned elsewhere Wallis's solution of the prize questions on the cycloid, which were proposed by Pascal.

The *Arithmetic of Infinites*, published in 1655, is his greatest work. By the application of analysis to the Method of Indivisibles, he greatly increased the power of this instrument for effecting quadratures. He advanced beyond Kepler by making more extended use of the "law of continuity" and placing full reliance in it. By this law he was led to regard the denominators of fractions as powers with negative exponents. Thus, the descending geometrical progression x^3, x^2, x^1, x^0, if continued, gives x^{-1}, x^{-2}, x^{-3}, etc.; which is the same thing as $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$. The exponents of this geometric series are in continued arithmetical progression, 3, 2, 1, 0, -1, -2, -3. He also used fractional exponents, which, like the negative, had been invented long before, but had failed to be generally introduced. The symbol ∞ for infinity is due to him.

Cavalieri and the French geometers had ascertained the formula for squaring the parabola of any degree, $y = x^m$, m being a positive integer. By the summation of the powers of the terms of infinite arithmetical series, it was found that the curve $y = x^m$ is to the area of the parallelogram having the same base and altitude as 1 is to $m + 1$. Aided by the law of continuity, Wallis arrived at the result that this formula holds true not only when m is positive and integral, but also when it is fractional or negative. Thus, in the parabola $y = \sqrt{px}$, $m = \frac{1}{2}$; hence the area of the parabolic segment is to that of the circumscribed rectangle as $1 : 1\frac{1}{2}$, or as $2 : 3$. Again, suppose that in $y = x^m$, $m = -\frac{1}{2}$; then the curve is a kind of hyperbola referred to its asymptotes, and the hyperbolic space between the curve and its asymptotes is to the corresponding parallelogram as $1 : \frac{1}{2}$. If $m = -1$, as in the common equilateral hyperbola $y = x^{-1}$ or $xy = 1$, then this ratio is $1 : -1 + 1$, or $1 : 0$, showing that its asymptotic space

is infinite. But in the case when m is greater than unity and negative, Wallis was unable to interpret correctly his results. For example, if $m = -3$, then the ratio becomes $1 : -2$, or as unity to a negative number. What is the meaning of this? Wallis reasoned thus: If the denominator is only zero, then the area is already infinite; but if it is less than zero, then the area must be more than infinite. It was pointed out later by Varignon, that this space, supposed to exceed infinity, is really finite, but taken negatively; that is, measured in a contrary direction.[31] The method of Wallis was easily extended to cases such as $y = ax^{\frac{m}{n}} + bx^{\frac{p}{q}}$ by performing the quadrature for each term separately, and then adding the results.

The manner in which Wallis studied the quadrature of the circle and arrived at his expression for the value of π is extraordinary. He found that the areas comprised between the axes, the ordinate corresponding to x, and the curves represented by the equations $y = (1-x^2)^0$, $y = (1-x^2)^1$, $y = (1-x^2)^2$, $y = (1-x^2)^3$, etc., are expressed in functions of the circumscribed rectangles having x and y for their sides, by the quantities forming the series

$$x,$$
$$x - \tfrac{1}{3}x^3,$$
$$x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5,$$
$$x - \tfrac{3}{3}x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7, \text{ etc.}$$

When $x = 1$, these values become respectively 1, $\frac{2}{3}$, $\frac{8}{15}$, $\frac{48}{105}$, etc. Now since the ordinate of the circle is $y = (1-x^2)^{\frac{1}{2}}$, the exponent of which is $\frac{1}{2}$ or the mean value between 0 and 1, the question of this quadrature reduced itself to this: If 0, 1, 2, 3, etc., operated upon by a certain law, give 1, $\frac{2}{3}$, $\frac{8}{15}$, $\frac{48}{105}$, what will $\frac{1}{2}$ give, when operated upon by the same law? He attempted to solve this by *interpolation*, a method first brought into prominence by him, and arrived by a highly complicated

and difficult analysis at the following very remarkable expression:

$$\frac{\pi}{2} = \frac{2.2 \cdot 4.4 \cdot 6.6 \cdot 8.8 \cdots}{1 \cdot 3.3 \cdot 5.5 \cdot 7.7 \cdot 9 \cdots}.$$

He did not succeed in making the interpolation itself, because he did not employ literal or general exponents, and could not conceive a series with more than one term and less than two, which it seemed to him the interpolated series must have. The consideration of this difficulty led Newton to the discovery of the Binomial Theorem. This is the best place to speak of that discovery. Newton virtually assumed that the same conditions which underlie the general expressions for the areas given above must also hold for the expression to be interpolated. In the first place, he observed that in each expression the first term is x, that x increases in odd powers, that the signs alternate $+$ and $-$, and that the second terms $\frac{0}{3}x^3$, $\frac{1}{3}x^3$, $\frac{2}{3}x^3$, $\frac{3}{3}x^3$, are in arithmetical progression. Hence the first two terms of the interpolated series must be $x - \frac{\frac{1}{2}x^3}{3}$. He next considered that the denominators 1, 3, 5, 7, etc., are in arithmetical progression, and that the coefficients in the numerators in each expression are the digits of some power of the number 11; namely, for the first expression, 11^0 or 1; for the second, 11^1 or 1, 1; for the third, 11^2 or 1, 2, 1; for the fourth, 11^3 or 1, 3, 3, 1; etc. He then discovered that, having given the second digit (call it m), the remaining digits can be found by continual multiplication of the terms of the series $\frac{m-0}{1} \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \cdot$ etc. Thus, if $m = 4$, then $4 \cdot \frac{m-1}{2}$ gives 6; $6 \cdot \frac{m-2}{3}$ gives 4; $4 \cdot \frac{m-3}{4}$ gives 1. Applying this rule to the required series, since the second term is $\frac{\frac{1}{2}x^3}{3}$, we have $m = \frac{1}{2}$, and then get for the succeeding co-

efficients in the numerators respectively $-\frac{1}{8}$, $+\frac{1}{16}$, $-\frac{5}{128}$, etc.; hence the required area for the circular segment is $x-\frac{\frac{1}{2}x^3}{3}-\frac{\frac{1}{8}x^5}{5}-\frac{\frac{1}{16}x^7}{7}-$ etc. Thus he found the interpolated expression to be an infinite series, instead of one having more than one term and less than two, as Wallis believed it must be. This interpolation suggested to Newton a mode of expanding $(1-x^2)^{\frac{1}{2}}$, or, more generally, $(1-x^2)^m$, into a series. He observed that he had only to omit from the expression just found the denominators 1, 3, 5, 7, etc., and to lower each power of x by unity, and he had the desired expression. In a letter to Oldenburg (June 13, 1676), Newton states the theorem as follows: The extraction of roots is much shortened by the theorem

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\text{etc.},$$

where A means the first term, $P^{\frac{m}{n}}$, B the second term, C the third term, etc. He verified it by actual multiplication, but gave no regular proof of it. He gave it for any exponent whatever, but made no distinction between the case when the exponent is positive and integral, and the others.

It should here be mentioned that very rude beginnings of the binomial theorem are found very early. The Hindoos and Arabs used the expansions of $(a+b)^2$ and $(a+b)^3$ for extracting roots; Vieta knew the expansion of $(a+b)^4$; but these were the results of simple multiplication without the discovery of any law. The binomial coefficients for positive whole exponents were known to some Arabic and European mathematicians. Pascal derived the coefficients from the method of what is called the "arithmetical triangle." Lucas de Burgo, Stifel, Stevinus, Briggs, and others, all possessed something from which one would think the binomial theorem could have

been gotten with a little attention, "if we did not know that such simple relations were difficult to discover."

Though Wallis had obtained an entirely new expression for π, he was not satisfied with it; for instead of a finite number of terms yielding an absolute value, it contained merely an infinite number, approaching nearer and nearer to that value. He therefore induced his friend, **Lord Brouncker** (1620 ?–1684), the first president of the Royal Society, to investigate this subject. Of course Lord Brouncker did not find what they were after, but he obtained the following beautiful equality: —

$$\pi = \cfrac{4}{1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\text{etc.}}}}}}$$

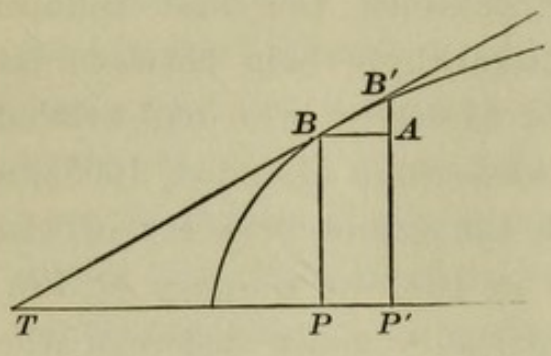
Continued fractions, both ascending and descending, appear to have been known already to the Greeks and Hindoos, though not in our present notation. Brouncker's expression gave birth to the theory of continued fractions.

Wallis' method of quadratures was diligently studied by his disciples. Lord Brouncker obtained the first infinite series for the area of an equilateral hyperbola between its asymptotes. **Nicolaus Mercator** of Holstein, who had settled in England, gave, in his *Logarithmotechnia* (London, 1668), a similar series. He started with the grand property of the equilateral hyperbola, discovered in 1647 by *Gregory St. Vincent*, which connected the hyperbolic space between the asymptotes with the natural logarithms and led to these logarithms being called hyperbolic. By it Mercator arrived at the logarithmic series, which Wallis had attempted but failed to obtain. He showed how the construction of logarith-

mic tables could be reduced to the quadrature of hyperbolic spaces. Following up some suggestions of Wallis, *William Neil* succeeded in rectifying the cubical parabola, and *Wren* in rectifying any cycloidal arc.

A prominent English mathematician and contemporary of Wallis was **Isaac Barrow** (1630–1677). He was professor of mathematics in London, and then in Cambridge, but in 1669 he resigned his chair to his illustrious pupil, Isaac Newton, and renounced the study of mathematics for that of divinity. As a mathematician, he is most celebrated for his method of tangents. He simplified the method of Fermat by introducing two infinitesimals instead of one, and approximated to the course of reasoning afterwards followed by Newton in his doctrine on Ultimate Ratios.

He considered the infinitesimal right triangle ABB' having for its sides the difference between two successive ordinates, the distance between them, and the portion of the curve intercepted by them. This triangle is similar to BPT, formed by the ordinate, the tangent, and the sub-tangent. Hence, if we know the ratio of $B'A$ to BA, then we know the ratio of the ordinate and the sub-tangent, and the tangent can be constructed at once. For any curve, say $y^2 = px$, the ratio of $B'A$ to BA is determined from its equation as follows: If x receives an infinitesimal increment $PP' = e$, then y receives an increment $B'A = a$, and the equation for the ordinate $B'P'$ becomes $y^2 + 2ay + a^2 = px + pe$.



Since $y^2 = px$, we get $2ay + a^2 = pe$; neglecting higher powers of the infinitesimals, we have $2ay = pe$, which gives

$$a : e = p : 2y = p : 2\sqrt{px}.$$

But $a : e =$ the ordinate : the sub-tangent ; hence

$$p : 2\sqrt{px} = \sqrt{px} : \text{sub-tangent},$$

giving $2x$ for the value of the sub-tangent. This method differs from that of the differential calculus only in notation.[31]

NEWTON TO EULER.

It has been seen that in France prodigious scientific progress was made during the beginning and middle of the seventeenth century. The toleration which marked the reign of Henry IV. and Louis XIII. was accompanied by intense intellectual activity. Extraordinary confidence came to be placed in the power of the human mind. The bold intellectual conquests of Descartes, Fermat, and Pascal enriched mathematics with imperishable treasures. During the early part of the reign of Louis XIV. we behold the sunset splendour of this glorious period. Then followed a night of mental effeminacy. This lack of great scientific thinkers during the reign of Louis XIV. may be due to the simple fact that no great minds were born ; but, according to Buckle, it was due to the paternalism, to the spirit of dependence and subordination, and to the lack of toleration, which marked the policy of Louis XIV.

In the absence of great French thinkers, Louis XIV. surrounded himself by eminent foreigners. Römer from Denmark, Huygens from Holland, Dominic Cassini from Italy, were the mathematicians and astronomers adorning his court. They were in possession of a brilliant reputation before going to Paris. Simply because they performed scientific work in Paris, that work belongs no more to France than the discoveries of Descartes belong to Holland, or those of Lagrange to Germany, or those of Euler and Poncelet to Russia. We

must look to other countries than France for the great scientific men of the latter part of the seventeenth century.

About the time when Louis XIV. assumed the direction of the French government Charles II. became king of England. At this time England was extending her commerce and navigation, and advancing considerably in material prosperity. A strong intellectual movement took place, which was unwittingly supported by the king. The age of poetry was soon followed by an age of science and philosophy. In two successive centuries England produced Shakespeare and Newton!

Germany still continued in a state of national degradation. The Thirty Years' War had dismembered the empire and brutalised the people. Yet this darkest period of Germany's history produced Leibniz, one of the greatest geniuses of modern times.

There are certain focal points in history toward which the lines of past progress converge, and from which radiate the advances of the future. Such was the age of Newton and Leibniz in the history of mathematics. During fifty years preceding this era several of the brightest and acutest mathematicians bent the force of their genius in a direction which finally led to the discovery of the infinitesimal calculus by Newton and Leibniz. Cavalieri, Roberval, Fermat, Descartes, Wallis, and others had each contributed to the new geometry. So great was the advance made, and so near was their approach toward the invention of the infinitesimal analysis, that both Lagrange and Laplace pronounced their countryman, Fermat, to be the true inventor of it. The differential calculus, therefore, was not so much an individual discovery as the grand result of a succession of discoveries by different minds. Indeed, no great discovery ever flashed upon the mind at once, and though those of Newton will

influence mankind to the end of the world, yet it must be admitted that Pope's lines are only a "poetic fancy": —

> "Nature and Nature's laws lay hid in night;
> God said, 'Let Newton be,' and all was light."

Isaac Newton (1642–1727) was born at Woolsthorpe, in Lincolnshire, the same year in which Galileo died. At his birth he was so small and weak that his life was despaired of. His mother sent him at an early age to a village school, and in his twelfth year to the public school at Grantham. At first he seems to have been very inattentive to his studies and very low in the school; but when, one day, the little Isaac received a severe kick upon his stomach from a boy who was above him, he laboured hard till he ranked higher in school than his antagonist. From that time he continued to rise until he was the head boy.[33] At Grantham, Isaac showed a decided taste for mechanical inventions. He constructed a water-clock, a wind-mill, a carriage moved by the person who sat in it, and other toys. When he had attained his fifteenth year his mother took him home to assist her in the management of the farm, but his great dislike for farm-work and his irresistible passion for study, induced her to send him back to Grantham, where he remained till his eighteenth year, when he entered Trinity College, Cambridge (1660). Cambridge was the real birthplace of Newton's genius. Some idea of his strong intuitive powers may be drawn from the fact that he regarded the theorems of ancient geometry as self-evident truths, and that, without any preliminary study, he made himself master of Descartes' *Geometry*. He afterwards regarded this neglect of elementary geometry a mistake in his mathematical studies, and he expressed to Dr. Pemberton his regret that "he had applied himself to the works of Descartes and other algebraic writers before he had

considered the *Elements* of Euclid with that attention which so excellent a writer deserves." Besides Descartes' *Geometry*, he studied Oughtred's *Clavis*, Kepler's *Optics*, the works of Vieta, Schooten's *Miscellanies*, Barrow's *Lectures*, and the works of Wallis. He was particularly delighted with Wallis' *Arithmetic of Infinites*, a treatise fraught with rich and varied suggestions. Newton had the good fortune of having for a teacher and fast friend the celebrated Dr. Barrow, who had been elected professor of Greek in 1660, and was made Lucasian professor of mathematics in 1663. The mathematics of Barrow and of Wallis were the starting-points from which Newton, with a higher power than his masters', moved onward into wider fields. Wallis had effected the quadrature of curves whose ordinates are expressed by any integral and positive power of $(1-x^2)$. We have seen how Wallis attempted but failed to interpolate between the areas thus calculated, the areas of other curves, such as that of the circle; how Newton attacked the problem, effected the interpolation, and discovered the Binomial Theorem, which afforded a much easier and direct access to the quadrature of curves than did the method of interpolation; for even though the binomial expression for the ordinate be raised to a fractional or negative power, the binomial could at once be expanded into a series, and the quadrature of each separate term of that series could be effected by the method of Wallis. Newton introduced the system of literal indices.

Newton's study of quadratures soon led him to another and most profound invention. He himself says that in 1665 and 1666 he conceived the method of fluxions and applied them to the quadrature of curves. Newton did not communicate the invention to any of his friends till 1669, when he placed in the hands of Barrow a tract, entitled *De Analysi per Æquationes Numero Terminorum Infinitas*, which was sent

by Barrow to Collins, who greatly admired it. In this treatise the principle of fluxions, though distinctly pointed out, is only partially developed and explained. Supposing the abscissa to increase uniformly in proportion to the time, he looked upon the area of a curve as a nascent quantity increasing by continued fluxion in the proportion of the length of the ordinate. The expression which was obtained for the fluxion he expanded into a finite or infinite series of monomial terms, to which Wallis' rule was applicable. Barrow urged Newton to publish this treatise; "but the modesty of the author, of which the excess, if not culpable, was certainly in the present instance very unfortunate, prevented his compliance."[26] Had this tract been published then, instead of forty-two years later, there would probably have been no occasion for that long and deplorable controversy between Newton and Leibniz.

For a long time Newton's method remained unknown, except to his friends and their correspondents. In a letter to Collins, dated December 10th, 1672, Newton states the fact of his invention with one example, and then says: "This is one particular, or rather corollary, of a general method, which extends itself, without any troublesome calculation, not only to the drawing of tangents to any curve lines, whether geometrical or mechanical, or anyhow respecting right lines or other curves, but also to the resolving other abstruser kinds of problems about the crookedness, areas, lengths, centres of gravity of curves, etc.; nor is it (as Hudden's method of Maximis and Minimis) limited to equations which are free from surd quantities. This method I have interwoven with that other of working in equations, by reducing them to infinite series."

These last words relate to a treatise he composed in the year 1671, entitled *Method of Fluxions*, in which he aimed to represent his method as an independent calculus and as

a complete system. This tract was intended as an introduction to an edition of Kinckhuysen's *Algebra*, which he had undertaken to publish. "But the fear of being involved in disputes about this new discovery, or perhaps the wish to render it more complete, or to have the sole advantage of employing it in his physical researches, induced him to abandon this design." [33]

Excepting two papers on optics, all of his works appear to have been published only after the most pressing solicitations of his friends and against his own wishes.[34] His researches on light were severely criticised, and he wrote in 1675: "I was so persecuted with discussions arising out of my theory of light that I blamed my own imprudence for parting with so substantial a blessing as my quiet to run after a shadow."

The *Method of Fluxions*, translated by J. Colson from Newton's Latin, was first published in 1736, or sixty-five years after it was written. In it he explains first the expansion into series of fractional and irrational quantities, — a subject which, in his first years of study, received the most careful attention. He then proceeds to the solution of the two following mechanical problems, which constitute the pillars, so to speak, of the abstract calculus: —

"I. The length of the space described being continually (*i.e.* at all times) given; to find the velocity of the motion at any time proposed.

"II. The velocity of the motion being continually given; to find the length of the space described at any time proposed."

Preparatory to the solution, Newton says: "Thus, in the equation $y = x^2$, if y represents the length of the space at any time described, which (time) another space x, by increasing with an uniform celerity $\dot{x}$, measures and exhibits as described: then $2x\dot{x}$ will represent the celerity by which the space y,

at the same moment of time, proceeds to be described; and contrarywise."

"But whereas we need not consider the time here, any farther than it is expounded and measured by an equable local motion; and besides, whereas only quantities of the same kind can be compared together, and also their velocities of increase and decrease; therefore, in what follows I shall have no regard to time formally considered, but I shall suppose some one of the quantities proposed, being of the same kind, to be increased by an equable fluxion, to which the rest may be referred, as it were to time; and, therefore, by way of analogy, it may not improperly receive the name of time." In this statement of Newton there is contained a satisfactory answer to the objection which has been raised against his method, that it introduces into analysis the foreign idea of motion. A quantity thus increasing by uniform fluxion, is what we now call an independent variable.

Newton continues: "Now those quantities which I consider as gradually and indefinitely increasing, I shall hereafter call *fluents*, or *flowing quantities*, and shall represent them by the final letters of the alphabet, v, x, y, and z; . . . and the velocities by which every fluent is increased by its generating motion (which I may call *fluxions*, or simply velocities, or celerities), I shall represent by the same letters pointed, thus, $\dot{v}$, $\dot{x}$, $\dot{y}$, $\dot{z}$. That is, for the celerity of the quantity v I shall put $\dot{v}$, and so for the celerities of the other quantities x, y, and z, I shall put $\dot{x}$, $\dot{y}$, and $\dot{z}$, respectively." It must here be observed that Newton does not take the fluxions themselves infinitely small. The "moments of fluxions," a term introduced further on, are infinitely small quantities. These "moments," as defined and used in the *Method of Fluxions*, are substantially the differentials of Leibniz. De Morgan points out that no small amount of confusion has arisen from the use of the word *fluxion* and the

notation $\dot{x}$ by all the English writers previous to 1704, excepting Newton and Cheyne, in the sense of an infinitely small increment.[35] Strange to say, even in the *Commercium Epistolicum* the words *moment* and *fluxion* appear to be used as synonymous.

After showing by examples how to solve the first problem, Newton proceeds to the demonstration of his solution : —

"The moments of flowing quantities (that is, their indefinitely small parts, by the accession of which, in infinitely small portions of time, they are continually increased) are as the velocities of their flowing or increasing.

"Wherefore, if the moment of any one (as x) be represented by the product of its celerity $\dot{x}$ into an infinitely small quantity 0 (*i.e.* by $\dot{x}0$), the moments of the others, v, y, z, will be represented by $\dot{v}0$, $\dot{y}0$, $\dot{z}0$; because $\dot{v}0$, $\dot{x}0$, $\dot{y}0$, and $\dot{z}0$ are to each other as $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$.

"Now since the moments, as $\dot{x}0$ and $\dot{y}0$, are the indefinitely little accessions of the flowing quantities x and y, by which those quantities are increased through the several indefinitely little intervals of time, it follows that those quantities, x and y, after any indefinitely small interval of time, become $x + \dot{x}0$ and $y + \dot{y}0$, and therefore the equation, which at all times indifferently expresses the relation of the flowing quantities, will as well express the relation between $x + \dot{x}0$ and $y + \dot{y}0$, as between x and y; so that $x + \dot{x}0$ and $y + \dot{y}0$ may be substituted in the same equation for those quantities, instead of x and y. Thus let any equation $x^3 - ax^2 + axy - y^3 = 0$ be given, and substitute $x + \dot{x}0$ for x, and $y + \dot{y}0$ for y, and there will arise

$$\left.\begin{array}{llll} & x^3 & +3x^2\dot{x}0 & +3x\dot{x}0\dot{x}0 + \dot{x}^3 0^3 \\ - & ax^2 & -2ax\dot{x}0 & -a\dot{x}0\dot{x}0 \\ + & axy & +ay\dot{x}0 & +a\dot{x}0\dot{y}0 \\ & & +ax\dot{y}0 & \\ - & y^3 & -3y^2\dot{y}0 & -3y\dot{y}0\dot{y}0 - \dot{y}^3 0^3 \end{array}\right\} = 0.$$

"Now, by supposition, $x^3 - ax^2 + axy - y^3 = 0$, which therefore, being expunged and the remaining terms being divided by 0, there will remain

$$3x^2\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} + 3x\dot{x}\dot{x}0 - a\dot{x}\dot{x}0 + a\dot{x}\dot{y}0 - 3y\dot{y}\dot{y}0 + \dot{x}^3 00 - \dot{y}^3 00 = 0.$$

But whereas zero is supposed to be infinitely little, that it may represent the moments of quantities, the terms that are multiplied by it will be nothing in respect of the rest (*termini in eam ducti pro nihilo possunt haberi cum aliis collati*); therefore I reject them, and there remains

$$3x^2\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0,$$

as above in Example I." Newton here uses infinitesimals.

Much greater than in the first problem were the difficulties encountered in the solution of the second problem, involving, as it does, inverse operations which have been taxing the skill of the best analysts since his time. Newton gives first a special solution to the second problem in which he resorts to a rule for which he has given no proof.

In the general solution of his second problem, Newton assumed homogeneity with respect to the fluxions and then considered three cases: (1) when the equation contains two fluxions of quantities and but one of the fluents; (2) when the equation involves both the fluents as well as both the fluxions; (3) when the equation contains the fluents and the fluxions of three or more quantities. The first case is the easiest since it requires simply the integration of $\frac{dy}{dx} = f(x)$, to which his "special solution" is applicable. The second case demanded nothing less than the general solution of a differential equation of the first order. Those who know what efforts were afterwards needed for the complete exploration of this field in analysis, will not depreciate Newton's work

even though he resorted to solutions in form of infinite series. Newton's third case comes now under the solution of partial differential equations. He took the equation $2\dot{x} - \dot{z} + x\dot{y} = 0$ and succeeded in finding a particular integral of it.

The rest of the treatise is devoted to the determination of maxima and minima, the radius of curvature of curves, and other geometrical applications of his fluxionary calculus. All this was done previous to the year 1672.

It must be observed that in the *Method of Fluxions* (as well as in his *De Analysi* and all earlier papers) the method employed by Newton is strictly infinitesimal, and in substance like that of Leibniz. Thus, the original conception of the calculus in England, as well as on the Continent, was based on infinitesimals. The fundamental principles of the fluxionary calculus were first given to the world in the *Principia;* but its peculiar notation did not appear until published in the second volume of Wallis' *Algebra* in 1693. The exposition given in the *Algebra* was substantially a contribution of Newton; it rests on infinitesimals. In the first edition of the *Principia* (1687) the description of fluxions is likewise founded on infinitesimals, but in the second (1713) the foundation is somewhat altered. In Book II. Lemma II. of the first edition we read: "Cave tamen intellexeris particulas finitas. *Momenta quam primum finitæ sunt magnitudinis, desinunt esse momenta. Finiri enim repugnat aliquatenus perpetuo eorum incremento vel decremento.* Intelligenda sunt principia jamjam nascentia finitorum magnitudinum." In the second edition the two sentences which we print in italics are replaced by the following: "Particulæ finitæ non sunt momenta sed quantitates ipsæ ex momentis genitæ." Through the difficulty of the phrases in both extracts, this much distinctly appears, that in the first, moments are infinitely small quantities. What else they are in the second is not clear.[35] In the

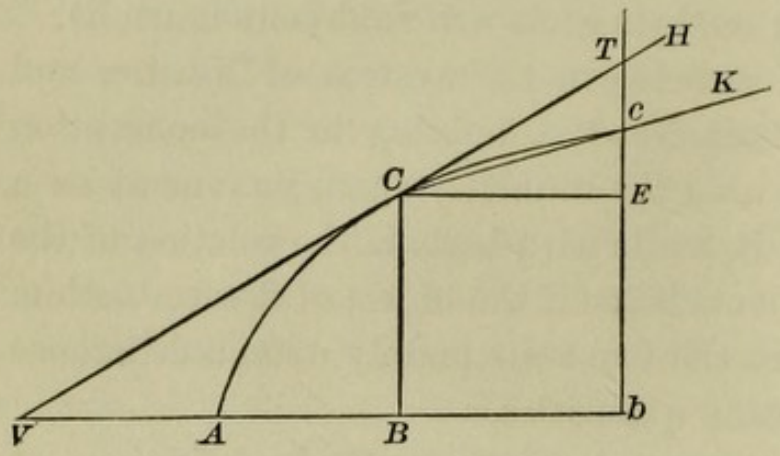
Quadrature of Curves of 1704, the infinitely small quantity is completely abandoned. It has been shown that in the *Method of Fluxions* Newton rejected terms involving the quantity 0, because they are infinitely small compared with other terms. This reasoning is evidently erroneous; for as long as 0 is a quantity, though ever so small, this rejection cannot be made without affecting the result. Newton seems to have felt this, for in the *Quadrature of Curves* he remarked that "in mathematics the minutest errors are not to be neglected" (errores quam minimi in rebus mathematicis non sunt contemnendi).

The early distinction between the system of Newton and Leibniz lies in this, that Newton, holding to the conception of velocity or fluxion, used the infinitely small increment as a means of determining it, while with Leibniz the relation of the infinitely small increments is itself the object of determination. The difference between the two rests mainly upon a difference in the mode of generating quantities.[35]

We give Newton's statement of the method of fluxions or rates, as given in the introduction to his *Quadrature of Curves.* "I consider mathematical quantities in this place not as consisting of very small parts, but as described by a continued motion. Lines are described, and thereby generated, not by the apposition of parts, but by the continued motion of points; superficies by the motion of lines; solids by the motion of superficies; angles by the rotation of the sides; portions of time by continual flux: and so on in other quantities. These geneses really take place in the nature of things, and are daily seen in the motion of bodies. . . .

"Fluxions are, as near as we please (*quam proxime*), as the increments of fluents generated in times, equal and as small as possible, and to speak accurately, they are in the prime ratio of nascent increments; yet they can be expressed by any lines whatever, which are proportional to them."

Newton exemplifies this last assertion by the problem of tangency: Let *AB* be the abscissa, *BC* the ordinate, *VCH* the tangent, *Ec* the increment of the ordinate, which produced meets *VH* at *T*, and *Cc* the increment of the curve. The right line *Cc* being produced to *K*, there are formed three small triangles, the rectilinear *CEc*, the mixtilinear *CEc*, and the rectilinear *CET*. Of these, the first is evidently the smallest, and the last the greatest. Now suppose the ordinate *bc* to move into the place *BC*, so that the



point *c* exactly coincides with the point *C*; *CK*, and therefore the curve *Cc*, is coincident with the tangent *CH*, *Ec* is absolutely equal to *ET*, and the mixtilinear evanescent triangle *CEc* is, in the last form, similar to the triangle *CET*; and its evanescent sides *CE*, *Ec*, *Cc*, will be proportional to *CE*, *ET*, and *CT*, the sides of the triangle *CET*. Hence it follows that the fluxions of the lines *AB*, *BC*, *AC*, being in the last ratio of their evanescent increments, are proportional to the sides of the triangle *CET*, or, which is all one, of the triangle *VBC* similar thereunto. As long as the points *C* and *c* are distant from each other by an interval, however small, the line *CK* will stand apart by a small angle from the tangent *CH*. But when *CK* coincides with *CH*, and the lines *CE*, *Ec*, *cC* reach their ultimate ratios, then the points *C* and *c* accurately coincide and are one and the same. Newton then adds that "in mathematics the minutest errors are not to be neglected." This is plainly a rejection of the postulates of Leibniz. The doctrine of infinitely small quan-

tities is here renounced in a manner which would lead one to suppose that Newton had never held it himself. Thus it appears that Newton's doctrine was different in different periods. Though, in the above reasoning, the Charybdis of infinitesimals is safely avoided, the dangers of a Scylla stare us in the face. We are required to believe that a point may be considered a triangle, or that a triangle can be inscribed in a point; nay, that three dissimilar triangles become similar and equal when they have reached their ultimate form in one and the same point.

In the introduction to the *Quadrature of Curves* the fluxion of x^n is determined as follows: —

"In the same time that x, by flowing, becomes $x+0$, the power x^n becomes $(x+0)^n$, *i.e.* by the method of infinite series

$$x^n + n0\,x^{n-1} + \frac{n^2-n}{2}0^2\,x^{n-2} + \text{etc.},$$

and the increments

$$0 \text{ and } n0\,x^{n-1} + \frac{n^2-n}{2}0^2\,x^{n-2} + \text{etc.},$$

are to one another as

$$1 \text{ to } nx^{n-1} + \frac{n^2-n}{2}0\;x^{n-2} + \text{etc.}$$

"Let now the increments vanish, and their last proportion will be 1 to nx^{n-1}: hence the fluxion of the quantity x is to the fluxion of the quantity x^n as $1 : nx^{n-1}$.

"The fluxion of lines, straight or curved, in all cases whatever, as also the fluxions of superficies, angles, and other quantities, can be obtained in the same manner by the method of prime and ultimate ratios. But to establish in this way the analysis of infinite quantities, and to investigate prime and ultimate ratios of finite quantities, nascent or evanescent, is in harmony with the geometry of the ancients; and I have endeavoured to show that, in the method of fluxions, it is not

necessary to introduce into geometry infinitely small quantities." This mode of differentiating does not remove all the difficulties connected with the subject. When 0 becomes nothing, then we get the ratio $\frac{0}{0} = nx^{n-1}$, which needs further elucidation. Indeed, the method of Newton, as delivered by himself, is encumbered with difficulties and objections. Among the ablest admirers of Newton, there have been obstinate disputes respecting his explanation of his method of "prime and ultimate ratios."

The so-called "method of limits" is frequently attributed to Newton, but the pure method of limits was never adopted by him as his method of constructing the calculus. All he did was to establish in his *Principia* certain principles which are applicable to that method, but which he used for a different purpose. The first lemma of the first book has been made the foundation of the method of limits: —

"Quantities and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal."

In this, as well as in the lemmas following this, there are obscurities and difficulties. Newton appears to teach that a variable quantity and its limit will ultimately coincide and be equal. But it is now generally agreed that in the clearest statements which have been made of the theory of limits, the variable does not actually reach its limit, though the variable may approach it as near as we please.

The full title of Newton's *Principia* is *Philosophiæ Naturalis Principia Mathematica*. It was printed in 1687 under the direction, and at the expense, of Dr. Edmund Halley. A second edition was brought out in 1713 with many alterations and improvements, and accompanied by a preface from

Mr. Cotes. It was sold out in a few months, but a pirated edition published in Amsterdam supplied the demand.[34] The third and last edition which appeared in England during Newton's lifetime was published in 1726 by Henry Pemberton. The *Principia* consists of three books, of which the first two, constituting the great bulk of the work, treat of the mathematical principles of natural philosophy, namely, the laws and conditions of motions and forces. In the third book is drawn up the constitution of the universe as deduced from the foregoing principles. The great principle underlying this memorable work is that of universal gravitation. The first book was completed on April 28, 1686. After the remarkably short period of three months, the second book was finished. The third book is the result of the next nine or ten months' labours. It is only a sketch of a much more extended elaboration of the subject which he had planned, but which was never brought to completion.

The law of gravitation is enunciated in the first book. Its discovery envelops the name of Newton in a halo of perpetual glory. The current version of the discovery is as follows: it was conjectured by Hooke, Huygens, Halley, Wren, Newton, and others, that, if Kepler's third law was true (its absolute accuracy was doubted at that time), then the attraction between the earth and other members of the solar system varied inversely as the square of the distance. But the proof of the truth or falsity of the guess was wanting. In 1666 Newton reasoned, in substance, that if g represent the acceleration of gravity on the surface of the earth, r be the earth's radius, R the distance of the moon from the earth, T the time of lunar revolution, and a a degree at the equator, then, if the law is true,

$$g\frac{r^2}{R^2} = 4\pi^2\frac{R}{T^2}, \text{ or } g = \frac{4\pi}{T^2}\left(\frac{R}{r}\right)^3 \cdot 180\,a.$$

The data at Newton's command gave $R = 60.4\,r$, $T = 2{,}360{,}628$ seconds, but a only 60 instead of $69\frac{1}{2}$ English miles. This wrong value of a rendered the calculated value of g smaller than its true value, as known from actual measurement. It looked as though the law of inverse squares were not the true law, and Newton laid the calculation aside. In 1684 he casually ascertained at a meeting of the Royal Society that Jean Picard had measured an arc of the meridian, and obtained a more accurate value for the earth's radius. Taking the corrected value for a, he found a figure for g which corresponded to the known value. Thus the law of inverse squares was verified. In a scholium in the *Principia*, Newton acknowledged his indebtedness to Huygens for the laws on centrifugal force employed in his calculation.

The perusal by the astronomer Adams of a great mass of unpublished letters and manuscripts of Newton forming the Portsmouth collection (which remained private property until 1872, when its owner placed it in the hands of the University of Cambridge) seems to indicate that the difficulties encountered by Newton in the above calculation were of a different nature. According to Adams, Newton's numerical verification was fairly complete in 1666, but Newton had not been able to determine what the attraction of a spherical shell upon an external point would be. His letters to Halley show that he did not suppose the earth to attract as though all its mass were concentrated into a point at the centre. He could not have asserted, therefore, that the assumed law of gravity was verified by the figures, though for long distances he might have claimed that it yielded close approximations. When Halley visited Newton in 1684, he requested Newton to determine what the orbit of a planet would be if the law of attraction were that of inverse squares. Newton had solved a similar problem for Hooke in 1679, and replied at once that it

was an ellipse. After Halley's visit, Newton, with Picard's new value for the earth's radius, reviewed his early calculation, and was able to show that if the distances between the bodies in the solar system were so great that the bodies might be considered as points, then their motions were in accordance with the assumed law of gravitation. In 1685 he completed his discovery by showing that a sphere whose density at any point depends only on the distance from the centre attracts an external point as though its whole mass were concentrated at the centre.[34]

Newton's unpublished manuscripts in the Portsmouth collection show that he had worked out, by means of fluxions and fluents, his lunar calculations to a higher degree of approximation than that given in the *Principia*, but that he was unable to interpret his results geometrically. The papers in that collection throw light upon the mode by which Newton arrived at some of the results in the *Principia*, as, for instance, the famous construction in Book II., Prop. 25, which is unproved in the *Principia*, but is demonstrated by him twice in a draft of a letter to David Gregory, of Oxford.[34]

It is chiefly upon the *Principia* that the fame of Newton rests. Brewster calls it "the brightest page in the records of human reason." Let us listen, for a moment, to the comments of Laplace, the foremost among those followers of Newton who grappled with the subtle problems of the motions of planets under the influence of gravitation: "Newton has well established the existence of the principle which he had the merit of discovering, but the development of its consequences and advantages has been the work of the successors of this great mathematician. The imperfection of the infinitesimal calculus, when first discovered, did not allow him completely to resolve the difficult problems which the theory of the universe offers; and he was oftentimes forced to give mere hints, which

were always uncertain till confirmed by rigorous analysis. Notwithstanding these unavoidable defects, the importance and the generality of his discoveries respecting the system of the universe, and the most interesting points of natural philosophy, the great number of profound and original views, which have been the origin of the most brilliant discoveries of the mathematicians of the last century, which were all presented with much elegance, will insure to the *Principia* a lasting pre-eminence over all other productions of the human mind."

Newton's *Arithmetica Universalis*, consisting of algebraical lectures delivered by him during the first nine years he was professor at Cambridge, were published in 1707, or more than thirty years after they were written. This work was published by Mr. Whiston. We are not accurately informed how Mr. Whiston came in possession of it, but according to some authorities its publication was a breach of confidence on his part.

The *Arithmetica Universalis* contains new and important results on the theory of equations. His theorem on the sums of powers of roots is well known. Newton showed that in equations with real coefficients, imaginary roots always occur in pairs. His inventive genius is grandly displayed in his rule for determining the inferior limit of the number of imaginary roots, and the superior limits for the number of positive and negative roots. Though less expeditious than Descartes', Newton's rule always gives as close, and generally closer, limits to the number of positive and negative roots. Newton did not prove his rule. It awaited demonstration for a century and a half, until, at last, Sylvester established a remarkable general theorem which includes Newton's rule as a special case.

The treatise on *Method of Fluxions* contains Newton's method

of approximating to the roots of numerical equations. This is simply the method of Vieta improved. The same treatise contains "Newton's parallelogram," which enabled him, in an equation, $f(x, y) = 0$, to find a series in powers of x equal to the variable y. The great utility of this rule lay in its determining the *form* of the series; for, as soon as the law was known by which the exponents in the series vary, then the expansion could be effected by the method of indeterminate coefficients. The rule is still used in determining the infinite branches to curves, or their figure at multiple points. Newton gave no proof for it, nor any clue as to how he discovered it. The proof was supplied half a century later, by Kaestner and Cramer, independently.[37]

In 1704 was published, as an appendix to the *Opticks*, the *Enumeratio linearum tertii ordinis*, which contains theorems on the theory of curves. Newton divides cubics into seventy-two species, arranged in larger groups, for which his commentators have supplied the names "genera" and "classes," recognising fourteen of the former and seven (or four) of the latter. He overlooked six species demanded by his principles of classification, and afterwards added by Stirling, Murdoch, and Cramer. He enunciates the remarkable theorem that the five species which he names "divergent parabolas" give by their projection every cubic curve whatever. As a rule, the tract contains no proofs. It has been the subject of frequent conjecture how Newton deduced his results. Recently we have gotten at the facts, since much of the analysis used by Newton and a few additional theorems have been discovered among the Portsmouth papers. An account of the four holograph manuscripts on this subject has been published by W. W. Rouse Ball, in the *Transactions of the London Mathematical Society* (vol. xx., pp. 104–143). It is interesting to observe how Newton begins his research on the classification of cubic

curves by the algebraic method, but, finding it laborious, attacks the problem geometrically, and afterwards returns again to analysis.[36]

Space does not permit us to do more than merely mention Newton's prolonged researches in other departments of science. He conducted a long series of experiments in optics and is the author of the corpuscular theory of light. The last of a number of papers on optics, which he contributed to the Royal Society, 1687, elaborates the theory of "fits." He explained the decomposition of light and the theory of the rainbow. By him were invented the reflecting telescope and the sextant (afterwards re-discovered by Thomas Godfrey of Philadelphia[2] and by John Hadley). He deduced a theoretical expression for the velocity of sound in air, engaged in experiments on chemistry, elasticity, magnetism, and the law of cooling, and entered upon geological speculations.

During the two years following the close of 1692, Newton suffered from insomnia and nervous irritability. Some thought that he laboured under temporary mental aberration. Though he recovered his tranquillity and strength of mind, the time of great discoveries was over; he would study out questions propounded to him, but no longer did he by his own accord enter upon new fields of research. The most noted investigation after his sickness was the testing of his lunar theory by the observations of Flamsteed, the astronomer royal. In 1695 he was appointed warden, and in 1699 master, of the mint, which office he held until his death. His body was interred in Westminster Abbey, where in 1731 a magnificent monument was erected, bearing an inscription ending with, "Sibi gratulentur mortales tale tantumque exstitisse humani generis decus." It is not true that the Binomial Theorem is also engraved on it.

We pass to Leibniz, the second and independent inventor

of the calculus. **Gottfried Wilhelm Leibniz** (1646–1716) was born in Leipzig. No period in the history of any civilised nation could have been less favourable for literary and scientific pursuits than the middle of the seventeenth century in Germany. Yet circumstances seem to have happily combined to bestow on the youthful genius an education hardly otherwise obtainable during this darkest period of German history. He was brought early in contact with the best of the culture then existing. In his fifteenth year he entered the University of Leipzig. Though law was his principal study, he applied himself with great diligence to every branch of knowledge. Instruction in German universities was then very low. The higher mathematics was not taught at all. We are told that a certain John Kuhn lectured on Euclid's *Elements*, but that his lectures were so obscure that none except Leibniz could understand them. Later on, Leibniz attended, for a half-year, at Jena, the lectures of Erhard Weigel, a philosopher and mathematician of local reputation. In 1666 Leibniz published a treatise, *De Arte Combinatoria*, in which he does not pass beyond the rudiments of mathematics. Other theses written by him at this time were metaphysical and juristical in character. A fortunate circumstance led Leibniz abroad. In 1672 he was sent by Baron Boineburg on a political mission to Paris. He there formed the acquaintance of the most distinguished men of the age. Among these was Huygens, who presented a copy of his work on the oscillation of the pendulum to Leibniz, and first led the gifted young German to the study of higher mathematics. In 1673 Leibniz went to London, and remained there from January till March. He there became incidentally acquainted with the mathematician Pell, to whom he explained a method he had found on the summation of series of numbers by their differences. Pell told him that a similar formula had been published by Mouton

as early as 1670, and then called his attention to Mercator's work on the rectification of the parabola. While in London, Leibniz exhibited to the Royal Society his arithmetical machine, which was similar to Pascal's, but more efficient and perfect. After his return to Paris, he had the leisure to study mathematics more systematically. With indomitable energy he set about removing his ignorance of higher mathematics. Huygens was his principal master. He studied the geometric works of Descartes, Honorarius Fabri, Gregory St. Vincent, and Pascal. A careful study of infinite series led him to the discovery of the following expression for the ratio of the circumference to the diameter of the circle, previously discovered by James Gregory : —

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc.}$$

This elegant series was found in the same way as Mercator's on the hyperbola. Huygens was highly pleased with it and urged him on to new investigations. Leibniz entered into a detailed study of the quadrature of curves and thereby became intimately acquainted with the higher mathematics. Among the papers of Leibniz is still found a manuscript on quadratures, written before he left Paris in 1676, but which was never printed by him. The more important parts of it were embodied in articles published later in the *Acta Eruditorum*.

In the study of Cartesian geometry the attention of Leibniz was drawn early to the direct and inverse problems of tangents. The direct problem had been solved by Descartes for the simplest curves only; while the inverse had completely transcended the power of his analysis. Leibniz investigated both problems for any curve; he constructed what he called the *triangulum characteristicum* — an infinitely small triangle between the infinitely small part of the curve coinciding with the tangent, and the differences of the ordinates and abscissas.

A curve is here considered to be a polygon. The *triangulum characteristicum* is similar to the triangle formed by the tangent, the ordinate of the point of contact, and the sub-tangent, as well as to that between the ordinate, normal, and sub-normal. It was first employed by Barrow in England, but appears to have been reinvented by Leibniz. From it Leibniz observed the connection existing between the direct and inverse problems of tangents. He saw also that the latter could be carried back to the quadrature of curves. All these results are contained in a manuscript of Leibniz, written in 1673. One mode used by him in effecting quadratures was as follows: The rectangle formed by a sub-tangent p and an element a (*i.e.* infinitely small part of the abscissa) is equal to the rectangle formed by the ordinate y and the element l of that ordinate; or in symbols, $pa = yl$. But the summation of these rectangles from zero on gives a right triangle equal to half the square of the ordinate. Thus, using Cavalieri's notation, he gets

$$\text{omn. } pa = \text{omn. } yl = \frac{y^2}{2} \quad (\textit{omn.} \text{ meaning } \textit{omnia}, \text{ all}).$$

But $y = \text{omn. } l$; hence

$$\overline{\text{omn. omn. } l}\,\frac{l}{a} = \frac{\overline{\text{omn. } l^2}}{2a}.$$

This equation is especially interesting, since it is here that Leibniz first introduces a new notation. He says: "It will be useful to write $\int$ for *omn.*, as $\int l$ for *omn.* l, that is, the sum of the l's"; he then writes the equation thus: —

$$\frac{\int \overline{l^2}}{2a} = \int \overline{\int l \frac{l}{a}}.$$

From this he deduced the simplest integrals, such as

$$\int x = \frac{x^2}{2}, \quad \int (x+y) = \int x + \int y.$$

Since the symbol of summation $\int$ raises the dimensions, he concluded that the opposite calculus, or that of differences d, would lower them. Thus, if $\int l = ya$, then $l = \frac{ya}{d}$. The symbol d was at first placed by Leibniz in the denominator, because the lowering of the power of a term was brought about in ordinary calculation by division. The manuscript giving the above is dated October 29th, 1675.[39] This, then, was the memorable day on which the notation of the new calculus came to be, — a notation which contributed enormously to the rapid growth and perfect development of the calculus.

Leibniz proceeded to apply his new calculus to the solution of certain problems then grouped together under the name of the Inverse Problems of Tangents. He found the cubical parabola to be the solution to the following: To find the curve in which the sub-normal is reciprocally proportional to the ordinate. The correctness of his solution was tested by him by applying to the result Sluze's method of tangents and reasoning backwards to the original supposition. In the solution of the third problem he changes his notation from $\frac{x}{d}$ to the now usual notation dx. It is worthy of remark that in these investigations, Leibniz nowhere explains the significance of dx and dy, except at one place in a marginal note: "Idem est dx et $\frac{x}{d}$, id est, differentia inter duas x proximas." Nor does he use the term *differential*, but always *difference*. Not till ten years later, in the *Acta Eruditorum*, did he give further explanations of these symbols. What he aimed at principally was to determine the change an expression undergoes when the symbol $\int$ or d is placed before it. It may be a consolation to students wrestling with the elements of the differential calculus to know that it required Leibniz considerable thought and atten-

tion[39] to determine whether $dx\,dy$ is the same as $d(xy)$, and $\frac{dx}{dy}$ the same as $d\frac{x}{y}$. After considering these questions at the close of one of his manuscripts, he concluded that the expressions were not the same, though he could not give the true value for each. Ten days later, in a manuscript dated November 21, 1675, he found the equation $y\bar{dx} = \bar{dxy} - x\bar{dy}$, giving an expression for $d(xy)$, which he observed to be true for all curves. He succeeded also in eliminating dx from a differential equation, so that it contained only dy, and thereby led to the solution of the problem under consideration. "Behold, a most elegant way by which the problems of the inverse methods of tangents are solved, or at least are reduced to quadratures!" Thus he saw clearly that the inverse problems of tangents could be solved by quadratures, or, in other words, by the integral calculus. In course of a half-year he discovered that the direct problem of tangents, too, yielded to the power of his new calculus, and that thereby a more general solution than that of Descartes could be obtained. He succeeded in solving all the special problems of this kind, which had been left unsolved by Descartes. Of these we mention only the celebrated problem proposed to Descartes by De Beaune, viz. to find the curve whose ordinate is to its sub-tangent as a given line is to that part of the ordinate which lies between the curve and a line drawn from the vertex of the curve at a given inclination to the axis.

Such was, in brief, the progress in the evolution of the new calculus made by Leibniz during his stay in Paris. Before his departure, in October, 1676, he found himself in possession of the most elementary rules and formulæ of the infinitesimal calculus.

From Paris, Leibniz returned to Hanover by way of London and Amsterdam. In London he met Collins, who showed him

a part of his scientific correspondence. Of this we shall speak later. In Amsterdam he discussed mathematics with Sluze, and became satisfied that his own method of constructing tangents not only accomplished all that Sluze's did, but even more, since it could be extended to three variables, by which tangent planes to surfaces could be found; and especially, since neither irrationals nor fractions prevented the immediate application of his method.

In a paper of July 11, 1677, Leibniz gave correct rules for the differentiation of sums, products, quotients, powers, and roots. He had given the differentials of a few negative and fractional powers, as early as November, 1676, but had made some mistakes. For $d\sqrt{x}$ he had given the erroneous value $\frac{1}{\sqrt{x}}$, and in another place the value $-\frac{1}{2}x^{-\frac{1}{2}}$; for $d\frac{1}{x^2}$ occurs in one place the wrong value, $-\frac{2}{x^2}$, while a few lines lower is given $-\frac{3}{x^4}$, its correct value.

In 1682 was founded in Berlin the *Acta Eruditorum*, a journal usually known by the name of *Leipzig Acts*. It was a partial imitation of the French *Journal des Savans* (founded in 1665), and the literary and scientific review published in Germany. Leibniz was a frequent contributor. Tschirnhaus, who had studied mathematics in Paris with Leibniz, and who was familiar with the new analysis of Leibniz, published in the *Acta Eroditorum* a paper on quadratures, which consists principally of subject-matter communicated by Leibniz to Tschirnhaus during a controversy which they had had on this subject. Fearing that Tschirnhaus might claim as his own and publish the notation and rules of the differential calculus, Leibniz decided, at last, to make public the fruits of his inventions. In 1684, or nine years after the new calculus first dawned upon the mind of Leibniz, and nineteen years after Newton first worked at fluxions,

and three years before the publication of Newton's *Principia*, Leibniz published, in the *Leipzig Acts*, his first paper on the differential calculus. He was unwilling to give to the world all his treasures, but chose those parts of his work which were most abstruse and least perspicuous. This epoch-making paper of only six pages bears the title : "Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis calculi genus." The rules of calculation are briefly stated without proof, and the meaning of dx and dy is not made clear. It has been inferred from this that Leibniz himself had no definite and settled ideas on this subject. Are dy and dx finite or infinitesimal quantities? At first they appear, indeed, to have been taken as finite, when he says: "We now call any line selected at random dx, then we designate the line which is to dx as y is to the sub-tangent, by dy, which is the difference of y." Leibniz then ascertains, by his calculus, in what way a ray of light passing through two differently refracting media, can travel easiest from one point to another; and then closes his article by giving his solution, in a few words, of De Beaune's problem. Two years later (1686) Leibniz published in the *Acta Eruditorum* a paper containing the rudiments of the integral calculus. The quantities dx and dy are there treated as infinitely small. He showed that by the use of his notation, the properties of curves could be fully expressed by equations. Thus the equation

$$y = \sqrt{2x - x^2} + \int \frac{dx}{\sqrt{2x - x^2}}$$

characterises the cycloid.[38]

The great invention of Leibniz, now made public by his articles in the *Leipzig Acts*, made little impression upon the mass of mathematicians. In Germany no one comprehended

the new calculus except Tschirnhaus, who remained indifferent to it. The author's statements were too short and succinct to make the calculus generally understood. The first to recognise its importance and to take up the study of it were two foreigners, — the Scotchman *John Craig*, and the Swiss *James Bernoulli*. The latter wrote Leibniz a letter in 1687, wishing to be initiated into the mysteries of the new analysis. Leibniz was then travelling abroad, so that this letter remained unanswered till 1790. James Bernoulli succeeded, meanwhile, by close application, in uncovering the secrets of the differential calculus without assistance. He and his brother John proved to be mathematicians of exceptional power. They applied themselves to the new science with a success and to an extent which made Leibniz declare that it was as much theirs as his. Leibniz carried on an extensive correspondence with them, as well as with other mathematicians. In a letter to John Bernoulli he suggests, among other things, that the integral calculus be improved by reducing integrals back to certain fundamental irreducible forms. The integration of logarithmic expressions was then studied. The writings of Leibniz contain many innovations, and anticipations of since prominent methods. Thus he made use of variable parameters, laid the foundation of *analysis in situ*, introduced the first notion of determinants in his effort to simplify the expression arising in the elimination of the unknown quantities from a set of linear equations. He resorted to the device of breaking up certain fractions into the sum of other fractions for the purpose of easier integration; he explicitly assumed the principle of continuity; he gave the first instance of a "singular solution," and laid the foundation to the theory of envelopes in two papers, one of which contains for the first time the terms *co-ordinate* and *axes of co-ordinates*. He wrote on osculating curves, but his paper contained the

error (pointed out by John Bernoulli, but not admitted by him) that an osculating circle will necessarily cut a curve in four consecutive points. Well known is his theorem on the nth differential coefficient of the product of two functions of a variable. Of his many papers on mechanics, some are valuable, while others contain grave errors.

Before tracing the further development of the calculus we shall sketch the history of that long and bitter controversy between English and Continental mathematicians on the invention of the calculus. The question was, did Leibniz invent it independently of Newton, or was he a plagiarist?

We must begin with the early correspondence between the parties appearing in this dispute. Newton had begun using his notation of fluxions in 1666.[41] In 1669 Barrow sent Collins Newton's tract, *De Analysi per Equationes*, etc.

The first visit of Leibniz to London extended from the 11th of January until March, 1673. He was in the habit of committing to writing important scientific communications received from others. In 1890 Gerhardt discovered in the royal library at Hanover a sheet of manuscript with notes taken by Leibniz during this journey.[40] They are headed "Observata Philosophica in itinere Anglicano sub initium anni 1673." The sheet is divided by horizontal lines into sections. The sections given to Chymica, Mechanica, Magnetica, Botanica, Anatomica, Medica, Miscellanea, contain extensive memoranda, while those devoted to mathematics have very few notes. Under Geometrica he says only this: "Tangentes omnium figurarum. Figurarum geometricarum explicatio per motum puncti in moto lati." We suspect from this that Leibniz had read Barrow's lectures. Newton is referred to only under Optica. Evidently Leibniz did not obtain a knowledge of fluxions during this visit to London, nor is it claimed that he did by his opponents.

Various letters of Newton, Collins, and others, up to the beginning of 1676, state that Newton invented a method by which tangents could be drawn without the necessity of freeing their equations from irrational terms. Leibniz announced in 1674 to Oldenburg, then secretary of the Royal Society, that he possessed very general analytical methods, by which he had found theorems of great importance on the quadrature of the circle by means of series. In answer, Oldenburg stated Newton and James Gregory had also discovered methods of quadratures, which extended to the circle. Leibniz desired to have these methods communicated to him; and Newton, at the request of Oldenburg and Collins, wrote to the former the celebrated letters of June 13 and October 24, 1676. The first contained the Binomial Theorem and a variety of other matters relating to infinite series and quadratures; but nothing directly on the method of fluxions. Leibniz in reply speaks in the highest terms of what Newton had done, and requests further explanation. Newton in his second letter just mentioned explains the way in which he found the Binomial Theorem, and also communicates his method of fluxions and fluents in form of an anagram in which all the letters in the sentence communicated were placed in alphabetical order. Thus Newton says that his method of drawing tangents was

$$6accdæ\ 13eff\ 7i\ 3l\ 9n\ 4o\ 4qrr\ 4s\ 9t\ 12vx.$$

The sentence was, "Data æquatione quotcunque fluentes quantitates involvente fluxiones invenire, et vice versa." ("Having any given equation involving never so many flowing quantities, to find the fluxions, and vice versa.") Surely this anagram afforded no hint. Leibniz wrote a reply to Collins, in which, without any desire of concealment, he explained the principle, notation, and the use of the differential calculus.

The death of Oldenburg brought this correspondence to a

close. Nothing material happened till 1684, when Leibniz published his first paper on the differential calculus in the *Leipzig Acts*, so that while Newton's claim to the priority of invention must be admitted by all, it must also be granted that Leibniz was the first to give the full benefit of the calculus to the world. Thus, while Newton's invention remained a secret, communicated only to a few friends, the calculus of Leibniz was spreading over the Continent. No rivalry or hostility existed, as yet, between the illustrious scientists. Newton expressed a very favourable opinion of Leibniz's inventions, known to him through the above correspondence with Oldenburg, in the following celebrated scholium (*Principia*, first edition, 1687, Book II., Prop. 7, scholium): —

"In letters which went between me and that most excellent geometer, G. G. Leibniz, ten years ago, when I signified that I was in the knowledge of a method of determining maxima and minima, of drawing tangents, and the like, and when I concealed it in transposed letters involving this sentence (Data æquatione, etc., above cited), that most distinguished man wrote back that he had also fallen upon a method of the same kind, and communicated his method, which hardly differed from mine, except in his forms of words and symbols."

As regards this passage, we shall see that Newton was afterwards weak enough, as De Morgan says: "First, to deny the plain and obvious meaning, and secondly, to omit it entirely from the third edition of the *Principia*." On the Continent, great progress was made in the calculus by Leibniz and his coadjutors, the brothers James and John Bernoulli, and Marquis de l'Hospital. In 1695 Wallis informed Newton by letter that "he had heard that his notions of fluxions passed in Holland with great applause by the name of 'Leibniz's Calculus Differentialis.'" Accordingly Wallis stated in the preface to a volume of his works that the calculus differen-

tialis was Newton's method of fluxions which had been communicated to Leibniz in the Oldenburg letters. A review of Wallis' works, in the *Leipzig Acts* for 1696, reminded the reader of Newton's own admission in the scholium above cited.

For fifteen years Leibniz had enjoyed unchallenged the honour of being the inventor of his calculus. But in 1699 Fato de Duillier, a Swiss, who had settled in England, stated in a mathematical paper, presented to the Royal Society, his conviction that Newton was the first inventor; adding that, whether Leibniz, the second inventor, had borrowed anything from the other, he would leave to the judgment of those who had seen the letters and manuscripts of Newton. This was the first distinct insinuation of plagiarism. It would seem that the English mathematicians had for some time been cherishing suspicions unfavourable to Leibniz. A feeling had doubtless long prevailed that Leibniz, during his second visit to London in 1676, had or might have seen among the papers of Collins, Newton's *Analysis per æquationes*, etc., which contained applications of the fluxionary method, but no systematic development or explanation of it. Leibniz certainly did see at least part of this tract. During the week spent in London, he took note of whatever interested him among the letters and papers of Collins. His memoranda discovered by Gerhardt in 1849 in the Hanover library fill two sheets.[40] The one bearing on our question is headed "Excerpta ex tractatu Newtoni Msc. de Analysi per æquationes numero terminorum infinitas." The notes are very brief, excepting those *De Resolutione æquationum affectarum*, of which there is an almost complete copy. This part was evidently new to him. If he examined Newton's entire tract, the other parts did not particularly impress him. From it he seems to have gained nothing pertaining to the infinitesimal calculus. By the previous intro-

duction of his own algorithm he had made greater progress than by what came to his knowledge in London. Nothing mathematical that he had received engaged his thoughts in the immediate future, for on his way back to Holland he composed a lengthy dialogue on mechanical subjects.

Duillier's insinuations lighted up a flame of discord which a whole century was hardly sufficient to extinguish. Leibniz, who had never contested the priority of Newton's discovery, and who appeared to be quite satisfied with Newton's admission in his scholium, now appears for the first time in the controversy. He made an animated reply in the *Leipzig Acts*, and complained to the Royal Society of the injustice done him.

Here the affair rested for some time. In the *Quadrature of Curves*, published 1704, for the first time, a formal exposition of the method and notation of fluxions was made public. In 1705 appeared an unfavourable review of this in the *Leipzig Acts*, stating that Newton uses and always has used fluxions for the differences of Leibniz. This was considered by Newton's friends an imputation of plagiarism on the part of their chief, but this interpretation was always strenuously resisted by Leibniz. Keill, professor of astronomy at Oxford, undertook with more zeal than judgment the defence of Newton. In a paper inserted in the *Philosophical Transactions* of 1708, he claimed that Newton was the first inventor of fluxions and "that the same calculus was afterward published by Leibniz, the name and the mode of notation being changed." Leibniz complained to the secretary of the Royal Society of bad treatment and requested the interference of that body to induce Keill to disavow the intention of imputing fraud. Keill was not made to retract his accusation; on the contrary, was authorised by Newton and the Royal Society to explain and defend his statement. This he did in a long letter. Leibniz thereupon complained that the charge was now more open than

before, and appealed for justice to the Royal Society and to Newton himself. The Royal Society, thus appealed to as a judge, appointed a committee which collected and reported upon a large mass of documents — mostly letters from and to Newton, Leibniz, Wallis, Collins, etc. This report, called the *Commercium Epistolicum*, appeared in the year 1712 and again in 1725, with a Recensio prefixed, and additional notes by Keill. The final conclusion in the *Commercium Epistolicum* was that Newton was the first inventor. But this was not to the point. The question was not whether Newton was the first inventor, but whether Leibniz had stolen the method. The committee had not formally ventured to assert their belief that Leibniz was a plagiarist. Yet there runs throughout the document a desire of proving Leibniz guilty of more than they meant positively to affirm. Leibniz protested only in private letters against the proceeding of the Royal Society, declaring that he would not answer an argument so weak. John Bernoulli, in a letter to Leibniz, which was published later in an anonymous tract, is as decidedly unfair towards Newton as the friends of the latter had been towards Leibniz. Keill replied, and then Newton and Leibniz appear as mutual accusers in several letters addressed to third parties. In a letter to Conti, April 9, 1716, Leibniz again reminded Newton of the admission he had made in the scholium, which he was now desirous of disavowing; Leibniz also states that he always believed Newton, but that, seeing him connive at accusations which he must have known to be false, it was natural that he (Leibniz) should begin to doubt. Newton did not reply to this letter, but circulated some remarks among his friends which he published immediately after hearing of the death of Leibniz, November 14, 1716. This paper of Newton gives the following explanation pertaining to the scholium in question: "He [Leibniz] pretends that in my

book of principles I allowed him the invention of the calculus differentialis, independently of my own; and that to attribute this invention to myself is contrary to my knowledge there avowed. But in the paragraph there referred unto I do not find one word to this purpose." In the third edition of the *Principia*, 1726, Newton omitted the scholium and substituted in its place another, in which the name of Leibniz does not appear.

National pride and party feeling long prevented the adoption of impartial opinions in England, but now it is generally admitted by nearly all familiar with the matter, that Leibniz really was an independent inventor. Perhaps the most telling evidence to show that Leibniz was an independent inventor is found in the study of his mathematical papers (collected and edited by C. I. Gerhardt, in six volumes, Berlin, 1849–1860), which point out a gradual and natural evolution of the rules of the calculus in his own mind. "There was throughout the whole dispute," says De Morgan, "a confusion between the knowledge of fluxions or differentials and that of a *calculus* of fluxions or differentials; that is, a digested method with general rules."

This controversy is to be regretted on account of the long and bitter alienation which it produced between English and Continental mathematicians. It stopped almost completely all interchange of ideas on scientific subjects. The English adhered closely to Newton's methods and, until about 1820, remained, in most cases, ignorant of the brilliant mathematical discoveries that were being made on the Continent. The loss in point of scientific advantage was almost entirely on the side of Britain. The only way in which this dispute may be said, in a small measure, to have furthered the progress of mathematics, is through the challenge problems by which each side attempted to annoy its adversaries.

The recurring practice of issuing challenge problems was inaugurated at this time by Leibniz. They were, at first, not intended as defiances, but merely as exercises in the new calculus. Such was the problem of the isochronous curve (to find the curve along which a body falls with uniform velocity), proposed by him to the Cartesians in 1687, and solved by James Bernoulli, himself, and John Bernoulli. James Bernoulli proposed in the Leipzig *Journal* the question to find the curve (the catenary) formed by a chain of uniform weight suspended freely from its ends. It was resolved by Huygens, Leibniz, and himself. In 1697 John Bernoulli challenged the best mathematicians in Europe to solve the difficult problem, to find the curve (the cycloid) along which a body falls from one point to another in the shortest possible time. Leibniz solved it the day he received it. Newton, de l'Hospital, and the two Bernoullis gave solutions. Newton's appeared anonymously in the *Philosophical Transactions*, but John Bernoulli recognised in it his powerful mind, "tanquam," he says, "ex ungue leonem." The problem of orthogonal trajectories (a system of curves described by a known law being given, to describe a curve which shall cut them all at right angles) had been long proposed in the *Acta Eruditorum*, but failed at first to receive much attention. It was again proposed in 1716 by Leibniz, to feel the pulse of the English mathematicians.

This may be considered as the first defiance problem professedly aimed at the English. Newton solved it the same evening on which it was delivered to him, although he was much fatigued by the day's work at the mint. His solution, as published, was a general plan of an investigation rather than an actual solution, and was, on that account, criticised by Bernoulli as being of no value. Brook Taylor undertook the defence of it, but ended by using very reprehensible language.

Bernoulli was not to be outdone in incivility, and made a bitter reply. Not long afterwards Taylor sent an open defiance to Continental mathematicians of a problem on the integration of a fluxion of complicated form which was known to very few geometers in England and supposed to be beyond the power of their adversaries. The selection was injudicious, for Bernoulli had long before explained the method of this and similar integrations. It served only to display the skill and augment the triumph of the followers of Leibniz. The last and most unskilful challenge was by John Keill. The problem was to find the path of a projectile in a medium which resists proportionally to the square of the velocity. Without first making sure that he himself could solve it, Keill boldly challenged Bernoulli to produce a solution. The latter resolved the question in very short time, not only for a resistance proportional to the square, but to any power of the velocity. Suspecting the weakness of the adversary, he repeatedly offered to send his solution to a confidential person in London, provided Keill would do the same. Keill never made a reply, and Bernoulli abused him and cruelly exulted over him.[26]

The explanations of the fundamental principles of the calculus, as given by Newton and Leibniz, lacked clearness and rigour. For that reason it met with opposition from several quarters. In 1694 Bernard Nieuwentyt of Holland denied the existence of differentials of higher orders and objected to the practice of neglecting infinitely small quantities. These objections Leibniz was not able to meet satisfactorily. In his reply he said the value of $\frac{dy}{dx}$ in geometry could be expressed as the ratio of finite quantities. In the interpretation of dx and dy Leibniz vacillated. At one time they appear in his writings as finite lines; then they are called infinitely small

quantities, and again, *quantitates inassignabiles*, which spring from *quantitates assignabiles* by the law of continuity. In this last presentation Leibniz approached nearest to Newton.

In England the principles of fluxions were boldly attacked by Bishop Berkeley, the eminent metaphysician, who argued with great acuteness, contending, among other things, that the fundamental idea of supposing a finite ratio to exist between terms absolutely evanescent — "the ghosts of departed quantities," as he called them — was absurd and unintelligible. The reply made by Jurin failed to remove all the objections. Berkeley was the first to point out what was again shown later by Lazare Carnot, that correct answers were reached by a "compensation of errors." Berkeley's attack was not devoid of good results, for it was the immediate cause of the work on fluxions by Maclaurin. In France *Michel Rolle* rejected the differential calculus and had a controversy with *Varignon* on the subject.

Among the most vigorous promoters of the calculus on the Continent were the Bernoullis. They and Euler made Basel in Switzerland famous as the cradle of great mathematicians. The family of Bernoullis furnished in course of a century eight members who distinguished themselves in mathematics. We subjoin the following genealogical table: —

Nicolaus Bernoulli, the Father

Jacob, 1654–1705	*Nicolaus*	**Johann**, 1667–1748
	|	|
	Nicolaus, 1687–1759	*Nicolaus*, 1695–1726
		Daniel, 1700–1782
		Johann, 1710–1790

Daniel	*Johann*, 1744–1807	*Jacob*, 1758–1789

Most celebrated were the two brothers Jacob (James) and Johann (John), and Daniel, the son of John. James and

John were staunch friends of Leibniz and worked hand in hand with him. **James Bernoulli** (1654–1705) was born in Basel. Becoming interested in the calculus, he mastered it without aid from a teacher. From 1687 until his death he occupied the mathematical chair at the University of Basel. He was the first to give a solution to Leibniz's problem of the isochronous curve. In his solution, published in the *Acta Eruditorum*, 1690, we meet for the first time with the word *integral*. Leibniz had called the integral calculus *calculus summatorius*, but in 1696 the term *calculus integralis* was agreed upon between Leibniz and John Bernoulli. James proposed the problem of the catenary, then proved the correctness of Leibniz's construction of this curve, and solved the more complicated problems, supposing the string to be (1) of variable density, (2) extensible, (3) acted upon at each point by a force directed to a fixed centre. Of these problems he published answers without explanations, while his brother John gave in addition their theory. He determined the shape of the "elastic curve" formed by an elastic plate or rod fixed at one end and bent by a weight applied to the other end; of the "lintearia," a flexible rectangular plate with two sides fixed horizontally at the same height, filled with a liquid; of the "volaria," a rectangular sail filled with wind. He studied the loxodromic and logarithmic spirals, in the last of which he took particular delight from its remarkable property of reproducing itself under a variety of conditions. Following the example of Archimedes, he willed that the curve be engraved upon his tombstone with the inscription "*eadem mutata resurgo.*" In 1696 he proposed the famous problem of isoperimetrical figures, and in 1701 published his own solution. He wrote a work on *Ars Conjectandi*, which is a development of the calculus of probabilities and contains the investigation now called "Bernoulli's theorem" and the so-called "numbers

of Bernoulli," which are in fact (though not so considered by him) the coefficients of $\frac{x^n}{n!}$ in the expansion of $(e^x-1)^{-1}$. Of his collected works, in three volumes, one was printed in 1713, the other two in 1744.

John Bernoulli (1667–1748) was initiated into mathematics by his brother. He afterwards visited France, where he met Malebranche, Cassini, De Lahire, Varignon, and de l'Hospital. For ten years he occupied the mathematical chair at Gröningen and then succeeded his brother at Basel. He was one of the most enthusiastic teachers and most successful original investigators of his time. He was a member of almost every learned society in Europe. His controversies were almost as numerous as his discoveries. He was ardent in his friendships, but unfair, mean, and violent toward all who incurred his dislike —even his own brother and son. He had a bitter dispute with James on the isoperimetrical problem. James convicted him of several paralogisms. After his brother's death he attempted to substitute a disguised solution of the former for an incorrect one of his own. John admired the merits of Leibniz and Euler, but was blind to those of Newton. He immensely enriched the integral calculus by his labours. Among his discoveries are the exponential calculus, the line of swiftest descent, and its beautiful relation to the path described by a ray passing through strata of variable density. He treated trigonometry by the analytical method, studied caustic curves and trajectories. Several times he was given prizes by the Academy of Science in Paris.

Of his sons, **Nicholas** and **Daniel** were appointed professors of mathematics at the same time in the Academy of St. Petersburg. The former soon died in the prime of life; the latter returned to Basel in 1733, where he assumed the chair of experimental philosophy. His first mathematical publi-

cation was the solution of a differential equation proposed by Riccati. He wrote a work on hydrodynamics. His investigations on probability are remarkable for their boldness and originality. He proposed the theory of *moral expectation*, which he thought would give results more in accordance with our ordinary notions than the theory of *mathematical probability*. His "moral expectation" has become classic, but no one ever makes use of it. He applies the theory of probability to insurance; to determine the mortality caused by small-pox at various stages of life; to determine the number of survivors at a given age from a given number of births; to determine how much inoculation lengthens the average duration of life. He showed how the differential calculus could be used in the theory of probability. He and Euler enjoyed the honour of having gained or shared no less than ten prizes from the Academy of Sciences in Paris.

Johann Bernoulli (born 1710) succeeded his father in the professorship of mathematics at Basel. He captured three prizes (on the capstan, the propagation of light, and the magnet) from the Academy of Sciences at Paris. **Nicolaus Bernoulli** (born 1687) held for a time the mathematical chair at Padua which Galileo had once filled. **Johann Bernoulli** (born 1744) at the age of nineteen was appointed astronomer royal at Berlin, and afterwards director of the mathematical department of the Academy. His brother *Jacob* took upon himself the duties of the chair of experimental physics at Basel, previously performed by his uncle Jacob, and later was appointed mathematical professor in the Academy at St. Petersburg.

Brief mention will now be made of some other mathematicians belonging to the period of Newton, Leibniz, and the elder Bernoullis.

Guillaume François Antoine l'Hospital (1661–1704), a pupil

of John Bernoulli, has already been mentioned as taking part in the challenges issued by Leibniz and the Bernoullis. He helped powerfully in making the calculus of Leibniz better known to the mass of mathematicians by the publication of a treatise thereon in 1696. This contains for the first time the method of finding the limiting value of a fraction whose two terms tend toward zero at the same time.

Another zealous French advocate of the calculus was **Pierre Varignon** (1654–1722). **Joseph Saurin** (1659–1737) solved the delicate problem of how to determine the tangents at the multiple points of algebraic curves. **François Nicole** (1683–1758) in 1717 issued the first systematic treatise on finite differences, in which he finds the sums of a considerable number of interesting series. He wrote also on roulettes, particularly spherical epicycloids, and their rectification. Also interested in finite differences was **Pierre Raymond de Montmort** (1678–1719). His chief writings, on the theory of probability, served to stimulate his more distinguished successor, De Moivre. **Jean Paul de Gua** (1713–1785) gave the demonstration of Descartes' rule of signs, now given in books. This skilful geometer wrote in 1740 a work on analytical geometry, the object of which was to show that most investigations on curves could be carried on with the analysis of Descartes quite as easily as with the calculus. He shows how to find the tangents, asymptotes, and various singular points of curves of all degrees, and proved by perspective that several of these points can be at infinity. A mathematician who clung to the methods of the ancients was **Philippe de Lahire** (1640–1718), a pupil of Desargues. His work on conic sections is purely synthetic, but differs from ancient treatises in deducing the properties of conics from those of the circle in the same manner as did Desargues and Pascal. His innovations stand in close relation with modern synthetic geometry. He wrote on roulettes, on

graphical methods, epicycloids, conchoids, and on magic squares. **Michel Rolle** (1652–1719) is the author of a theorem named after him.

Of Italian mathematicians, Riccati and Fagnano must not remain unmentioned. **Jacopo Francesco, Count Riccati** (1676–1754) is best known in connection with his problem, called Riccati's equation, published in the *Acta Eruditorum* in 1724. He succeeded in integrating this differential equation for some special cases. A geometrician of remarkable power was **Giulio Carlo, Count de Fagnano** (1682–1766). He discovered the following formula, $\pi = 2i \log \frac{1-i}{1+i}$, in which he anticipated Euler in the use of imaginary exponents and logarithms. His studies on the rectification of the ellipse and hyperbola are the starting-points of the theory of elliptic functions. He showed, for instance, that two arcs of an ellipse can be found in an indefinite number of ways, whose difference is expressible by a right line.

In Germany the only noted contemporary of Leibniz is **Ehrenfried Walter Tschirnhausen** (1651–1708), who discovered the caustic of reflection, experimented on metallic reflectors and large burning-glasses, and gave us a method of transforming equations named after him. Believing that the most simple methods (like those of the ancients) are the most correct, he concluded that in the researches relating to the properties of curves the calculus might as well be dispensed with.

After the death of Leibniz there was in Germany not a single mathematician of note. **Christian Wolf** (1679–1754), professor at Halle, was ambitious to figure as successor of Leibniz, but he "forced the ingenious ideas of Leibniz into a pedantic scholasticism, and had the unenviable reputation of having presented the elements of the arithmetic, algebra, and

analysis developed since the time of the Renaissance in the form of Euclid, — of course only in outward form, for into the spirit of them he was quite unable to penetrate." [16]

The contemporaries and immediate successors of Newton in Great Britain were men of no mean merit. We have reference to Cotes, Taylor, Maclaurin, and De Moivre. We are told that at the death of **Roger Cotes** (1682–1716), Newton exclaimed, "If Cotes had lived, we might have known something." It was at the request of Dr. Bentley that Cotes undertook the publication of the second edition of Newton's *Principia*. His mathematical papers were published after his death by Robert Smith, his successor in the Plumbian professorship at Trinity College. The title of the work, *Harmonia Mensurarum*, was suggested by the following theorem contained in it: If on each radius vector, through a fixed point O, there be taken a point R, such that the reciprocal of OR be the arithmetic mean of the reciprocals of OR_1, OR_2, $\cdots$ OR_n, then the locus of R will be a straight line. In this work progress was made in the application of logarithms and the properties of the circle to the calculus of fluents. To Cotes we owe a theorem in trigonometry which depends on the forming of factors of $x^n - 1$. Chief among the admirers of Newton were Taylor and Maclaurin. The quarrel between English and Continental mathematicians caused them to work quite independently of their great contemporaries across the Channel.

Brook Taylor (1685–1731) was interested in many branches of learning, and in the latter part of his life engaged mainly in religious and philosophic speculations. His principal work, *Methodus incrementorum directa et inversa*, London, 1715–1717, added a new branch to mathematics, now called "finite differences." He made many important applications of it, particularly to the study of the form of movement of vibrating

strings, first reduced to mechanical principles by him. This work contains also "Taylor's theorem," the importance of which was not recognised by analysts for over fifty years, until Lagrange pointed out its power. His proof of it does not consider the question of convergency, and is quite worthless. The first rigorous proof was given a century later by Cauchy. Taylor's work contains the first correct explanation of astronomical refraction. He wrote also a work on linear perspective, a treatise which, like his other writings, suffers for want of fulness and clearness of expression. At the age of twenty-three he gave a remarkable solution of the problem of the centre of oscillation, published in 1714. His claim to priority was unjustly disputed by John Bernoulli.

Colin Maclaurin (1698–1746) was elected professor of mathematics at Aberdeen at the age of nineteen by competitive examination, and in 1725 succeeded James Gregory at the University of Edinburgh. He enjoyed the friendship of Newton, and, inspired by Newton's discoveries, he published in 1719 his *Geometria Organica*, containing a new and remarkable mode of generating conics, known by his name. A second tract, *De Linearum geometricarum Proprietatibus*, 1720, is remarkable for the elegance of its demonstrations. It is based upon two theorems: the first is the theorem of Cotes; the second is Maclaurin's: If through any point O a line be drawn meeting the curve in n points, and at these points tangents be drawn, and if any other line through O cut the curve in R_1, R_2, etc., and the system of n tangents in r_1, r_2, etc., then $\Sigma \frac{1}{OR} = \Sigma \frac{1}{Or}$.

This and Cotes' theorem are generalisations of theorems of Newton. Maclaurin uses these in his treatment of curves of the second and third degree, culminating in the remarkable theorem that if a quadrangle has its vertices and the two points of intersection of its opposite sides upon a curve of the

third degree, then the tangents drawn at two opposite vertices cut each other on the curve. He deduced independently Pascal's theorem on the hexagram. The following is his extension of this theorem (*Phil. Trans.*, 1735): If a polygon move so that each of its sides passes through a fixed point, and if all its summits except one describe curves of the degrees m, n, p, etc., respectively, then the free summit moves on a curve of the degree $2\,mnp\cdots$, which reduces to $mnp\cdots$ when the fixed points all lie on a straight line. Maclaurin wrote on pedal curves. He is the author of an *Algebra*. The object of his treatise on *Fluxions* was to found the doctrine of fluxions on geometric demonstrations after the manner of the ancients, and thus, by rigorous exposition, answer such attacks as Berkeley's that the doctrine rested on false reasoning. The *Fluxions* contained for the first time the correct way of distinguishing between maxima and minima, and explained their use in the theory of multiple points. "Maclaurin's theorem" was previously given by James Stirling, and is but a particular case of "Taylor's theorem." Appended to the treatise on *Fluxions* is the solution of a number of beautiful geometric, mechanical, and astronomical problems, in which he employs ancient methods with such consummate skill as to induce Clairaut to abandon analytic methods and to attack the problem of the figure of the earth by pure geometry. His solutions commanded the liveliest admiration of Lagrange. Maclaurin investigated the attraction of the ellipsoid of revolution, and showed that a homogeneous liquid mass revolving uniformly around an axis under the action of gravity must assume the form of an ellipsoid of revolution. Newton had given this theorem without proof. Notwithstanding the genius of Maclaurin, his influence on the progress of mathematics in Great Britain was unfortunate; for, by his example, he induced his countrymen to neglect analysis and to be indifferent to the

wonderful progress in the higher analysis made on the Continent.

It remains for us to speak of **Abraham de Moivre** (1667–1754), who was of French descent, but was compelled to leave France at the age of eighteen, on the Revocation of the Edict of Nantes. He settled in London, where he gave lessons in mathematics. He lived to the advanced age of eighty-seven and sank into a state of almost total lethargy. His subsistence was latterly dependent on the solution of questions on games of chance and problems on probabilities, which he was in the habit of giving at a tavern in St. Martin's Lane. Shortly before his death he declared that it was necessary for him to sleep ten or twenty minutes longer every day. The day after he had reached the total of over twenty-three hours, he slept exactly twenty-four hours and then passed away in his sleep. De Moivre enjoyed the friendship of Newton and Halley. His power as a mathematician lay in analytic rather than geometric investigation. He revolutionised higher trigonometry by the discovery of the theorem known by his name and by extending the theorems on the multiplication and division of sectors from the circle to the hyperbola. His work on the theory of probability surpasses anything done by any other mathematician except Laplace. His principal contributions are his investigations respecting the Duration of Play, his Theory of Recurring Series, and his extension of the value of Bernoulli's theorem by the aid of Stirling's theorem.[42] His chief works are the *Doctrine of Chances*, 1716, the *Miscellanea Analytica*, 1730, and his papers in the *Philosophical Transactions*.

EULER, LAGRANGE, AND LAPLACE.

During the epoch of ninety years from 1730 to 1820 the French and Swiss cultivated mathematics with most brilliant success. No previous period had shown such an array of illustrious names. At this time Switzerland had her Euler; France, her Lagrange, Laplace, Legendre, and Monge. The mediocrity of French mathematics which marked the time of Louis XIV. was now followed by one of the very brightest periods of all history. England and Germany, on the other hand, which during the unproductive period in France had their Newton and Leibniz, could now boast of no great mathematician. France now waved the mathematical sceptre. Mathematical studies among the English and German people had sunk to the lowest ebb. Among them the direction of original research was ill-chosen. The former adhered with excessive partiality to ancient geometrical methods; the latter produced the combinatorial school, which brought forth nothing of value.

The labours of Euler, Lagrange, and Laplace lay in higher analysis, and this they developed to a wonderful degree. By them analysis came to be completely severed from geometry. During the preceding period the effort of mathematicians not only in England, but, to some extent, even on the continent, had been directed toward the solution of problems clothed in geometric garb, and the results of calculation were usually reduced to geometric form. A change now took place. Euler brought about an emancipation of the analytical calculus from geometry and established it as an independent science. Lagrange and Laplace scrupulously adhered to this separation. Building on the broad foundation laid for higher analysis and mechanics by Newton and Leibniz, Euler, with matchless fertility of mind, erected

an elaborate structure. There are few great ideas pursued by succeeding analysts which were not suggested by Euler, or of which he did not share the honour of invention. With, perhaps, less exuberance of invention, but with more comprehensive genius and profounder reasoning, Lagrange developed the infinitesimal calculus and put analytical mechanics into the form in which we now know it. Laplace applied the calculus and mechanics to the elaboration of the theory of universal gravitation, and thus, largely extending and supplementing the labours of Newton, gave a full analytical discussion of the solar system. He also wrote an epoch-marking work on Probability. Among the analytical branches created during this period are the calculus of Variations by Euler and Lagrange, Spherical Harmonics by Laplace and Legendre, and Elliptic Integrals by Legendre.

Comparing the growth of analysis at this time with the growth during the time of Gauss, Cauchy, and recent mathematicians, we observe an important difference. During the former period we witness mainly a development with reference to *form*. Placing almost implicit confidence in results of calculation, mathematicians did not always pause to discover rigorous proofs, and were thus led to general propositions, some of which have since been found to be true in only special cases. The Combinatorial School in Germany carried this tendency to the greatest extreme; they worshipped formalism and paid no attention to the actual contents of formulæ. But in recent times there has been added to the dexterity in the formal treatment of problems, a much-needed rigour of demonstration. A good example of this increased rigour is seen in the present use of infinite series as compared to that of Euler, and of Lagrange in his earlier works.

The ostracism of geometry, brought about by the master-minds of this period, could not last permanently. Indeed, a

new geometric school sprang into existence in France before the close of this period. Lagrange would not permit a single diagram to appear in his *Mécanique analytique*, but thirteen years before his death, Monge published his epoch-making *Géometrie descriptive.*

Leonhard Euler (1707–1783) was born in Basel. His father, a minister, gave him his first instruction in mathematics and then sent him to the University of Basel, where he became a favourite pupil of John Bernoulli. In his nineteenth year he composed a dissertation on the masting of ships, which received the second prize from the French Academy of Sciences. When John Bernoulli's two sons, Daniel and Nicolaus, went to Russia, they induced Catharine I., in 1727, to invite their friend Euler to St. Petersburg, where Daniel, in 1733, was assigned to the chair of mathematics. In 1735 the solving of an astronomical problem, proposed by the Academy, for which several eminent mathematicians had demanded some months' time, was achieved in three days by Euler with aid of improved methods of his own. But the effort threw him into a fever and deprived him of the use of his right eye. With still superior methods this same problem was solved later by the illustrious Gauss in one hour![47] The despotism of Anne I. caused the gentle Euler to shrink from public affairs and to devote all his time to science. After his call to Berlin by Frederick the Great in 1747, the queen of Prussia, who received him kindly, wondered how so distinguished a scholar should be so timid and reticent. Euler naïvely replied, "Madam, it is because I come from a country where, when one speaks, one is hanged." In 1766 he with difficulty obtained permission to depart from Berlin to accept a call by Catharine II. to St. Petersburg. Soon after his return to Russia he became blind, but this did not stop his wonderful literary productiveness, which continued for seventeen years, until the

day of his death.[45] He dictated to his servant his *Anleitung zur Algebra*, 1770, which, though purely elementary, is meritorious as one of the earliest attempts to put the fundamental processes on a sound basis.

Euler wrote an immense number of works, chief of which are the following: *Introductio in analysin infinitorum*, 1748, a work that caused a revolution in analytical mathematics, a subject which had hitherto never been presented in so general and systematic manner; *Institutiones calculi differentialis*, 1755, and *Institutiones calculi integralis*, 1768–1770, which were the most complete and accurate works on the calculus of that time, and contained not only a full summary of everything then known on this subject, but also the Beta and Gamma Functions and other original investigations; *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*, 1744, which, displaying an amount of mathematical genius seldom rivalled, contained his researches on the calculus of variations (a subject afterwards improved by Lagrange), to the invention of which Euler was led by the study of isoperimetrical curves, the brachistochrone in a resisting medium, and the theory of geodesics (subjects which had previously engaged the attention of the elder Bernoullis and others); the *Theoria motuum planetarum et cometarum*, 1744, *Theoria motus lunæ*, 1753, *Theoria motuum lunæ*, 1772, are his chief works on astronomy; *Ses lettres à une princesse d'Allemagne sur quelques sujets de Physique et de Philosophie*, 1770, was a work which enjoyed great popularity.

We proceed to mention the principal innovations and inventions of Euler. He treated trigonometry as a branch of analysis, introduced (simultaneously with Thomas Simpson in England) the now current abbreviations for trigonometric functions, and simplified formulæ by the simple expedient of designating the angles of a triangle by A, B, C, and the

opposite sides by a, b, c, respectively. He pointed out the relation between trigonometric and exponential functions. In a paper of 1737 we first meet the symbol π to denote 3.14159....[21] Euler laid down the rules for the transformation of co-ordinates in space, gave a methodic analytic treatment of plane curves and of surfaces of the second order. He was the first to discuss the equation of the second degree in three variables, and to classify the surfaces represented by it. By criteria analogous to those used in the classification of conics he obtained five species. He devised a method of solving biquadratic equations by assuming $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$, with the hope that it would lead him to a general solution of algebraic equations. The method of elimination by solving a series of linear equations (invented independently by Bezout) and the method of elimination by symmetric functions, are due to him.[20] Far reaching are Euler's researches on logarithms. Leibniz and John Bernoulli once argued the question whether a negative number has a logarithm. Bernoulli claimed that since $(-a)^2 = (+a)^2$, we have $\log(-a)^2 = \log(+a)^2$ and $2\log(-a) = 2\log(+a)$, and finally $\log(-a) = \log(+a)$. Euler proved that a has really an infinite number of logarithms, all of which are imaginary when a is negative, and all except one when a is positive. He then explained how $\log(-a)^2$ might equal $\log(+a)^2$, and yet $\log(-a)$ not equal $\log(+a)$.

The subject of infinite series received new life from him. To his researches on series we owe the creation of the theory of definite integrals by the development of the so-called *Eulerian integrals*. He warns his readers occasionally against the use of divergent series, but is nevertheless very careless himself. The rigid treatment to which infinite series are subjected now was then undreamed of. No clear notions existed as to what constitutes a convergent series. Neither Leibniz nor Jacob

and John Bernoulli had entertained any serious doubt of the correctness of the expression $\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$. Guido Grandi went so far as to conclude from this that $\frac{1}{2} = 0 + 0 + 0 + \cdots$. In the treatment of series Leibniz advanced a metaphysical method of proof which held sway over the minds of the elder Bernoullis, and even of Euler.[46] The tendency of that reasoning was to justify results which seem to us now highly absurd. The looseness of treatment can best be seen from examples. The very paper in which Euler cautions against divergent series contains the proof that

$$\cdots \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + \cdots = 0 \text{ as follows:}$$

$$n + n^2 + \cdots = \frac{n}{1-n}, \quad 1 + \frac{1}{n} + \frac{1}{n^2} + \cdots = \frac{n}{n-1};$$

these added give zero. Euler has no hesitation to write $1 - 3 + 5 - 7 + \cdots = 0$, and no one objected to such results excepting Nicolaus Bernoulli, the nephew of John and Jacob. Strange to say, Euler finally succeeded in converting Nicolaus Bernoulli to his own erroneous views. At the present time it is difficult to believe that Euler should have confidently written $\sin\phi - 2\sin 2\phi + 3\sin 3\phi - 4\sin 4\phi + \cdots = 0$, but such examples afford striking illustrations of the want of scientific basis of certain parts of analysis at that time. Euler's proof of the binomial formula for negative and fractional exponents, which has been reproduced in elementary text-books of even recent years, is faulty. A remarkable development, due to Euler, is what he named the hypergeometric series, the summation of which he observed to be dependent upon the integration of a linear differential equation of the second order, but it remained for Gauss to point out that for special values of its letters, this series represented nearly all functions then known.

Euler developed the calculus of finite differences in the first

chapters of his *Institutiones calculi differentialis*, and then deduced the differential calculus from it. He established a theorem on homogeneous functions, known by his name, and contributed largely to the theory of differential equations, a subject which had received the attention of Newton, Leibniz, and the Bernoullis, but was still undeveloped. Clairaut, Fontaine, and Euler about the same time observed criteria of integrability, but Euler in addition showed how to employ them to determine integrating factors. The principles on which the criteria rested involved some degree of obscurity. The celebrated addition-theorem for elliptic integrals was first established by Euler. He invented a new algorithm for continued fractions, which he employed in the solution of the indeterminate equation $ax + by = c$. We now know that substantially the same solution of this equation was given 1000 years earlier, by the Hindoos. By giving the factors of the number $2^{2^n} + 1$ when $n = 5$, he pointed out that this expression did not always represent primes, as was supposed by Fermat. He first supplied the proof to "Fermat's theorem," and to a second theorem of Fermat, which states that every prime of the form $4n + 1$ is expressible as the sum of two squares in one and only one way. A third theorem of Fermat, that $x^n + y^n = z^n$, has no integral solution for values of n greater than 2, was proved by Euler to be correct when $n = 3$. Euler discovered four theorems which taken together make out the great law of quadratic reciprocity, a law independently discovered by Legendre.[48] Euler enunciated and proved a well-known theorem, giving the relation between the number of vertices, faces, and edges of certain polyhedra, which, however, appears to have been known to Descartes. The powers of Euler were directed also towards the fascinating subject of the theory of probability, in which he solved some difficult problems.

Of no little importance are Euler's labours in analytical mechanics. Says Whewell: "The person who did most to give to analysis the generality and symmetry which are now its pride, was also the person who made mechanics analytical; I mean Euler."[11] He worked out the theory of the rotation of a body around a fixed point, established the general equations of motion of a free body, and the general equation of hydrodynamics. He solved an immense number and variety of mechanical problems, which arose in his mind on all occasions. Thus, on reading Virgil's lines, "The anchor drops, the rushing keel is staid," he could not help inquiring what would be the ship's motion in such a case. About the same time as Daniel Bernoulli he published the *Principle of the Conservation of Areas* and defended the principle of "least action," advanced by Maupertius. He wrote also on tides and on sound.

Astronomy owes to Euler the method of the variation of arbitrary constants. By it he attacked the problem of perturbations, explaining, in case of two planets, the secular variations of eccentricities, nodes, etc. He was one of the first to take up with success the theory of the moon's motion by giving approximate solutions to the "problem of three bodies." He laid a sound basis for the calculation of tables of the moon. These researches on the moon's motion, which captured two prizes, were carried on while he was blind, with the assistance of his sons and two of his pupils.

Most of his memoirs are contained in the transactions of the Academy of Sciences at St. Petersburg, and in those of the Academy at Berlin. From 1728 to 1783 a large portion of the Petropolitan transactions were filled by his writings. He had engaged to furnish the Petersburg Academy with memoirs in sufficient number to enrich its acts for twenty years — a promise more than fulfilled, for down to 1818 the volumes usually contained one or more papers of his. It has

been said that an edition of Euler's complete works would fill 16,000 quarto pages. His mode of working was, first to concentrate his powers upon a special problem, then to solve separately all problems growing out of the first. No one excelled him in dexterity of accommodating methods to special problems. It is easy to see that mathematicians could not long continue in Euler's habit of writing and publishing. The material would soon grow to such enormous proportions as to be unmanageable. We are not surprised to see almost the opposite in Lagrange, his great successor. The great Frenchman delighted in the general and abstract, rather than, like Euler, in the special and concrete. His writings are condensed and give in a nutshell what Euler narrates at great length.

Jean-le-Rond D'Alembert (1717–1783) was exposed, when an infant, by his mother in a market by the church of St. Jean-le-Rond, near the Nôtre-Dame in Paris, from which he derived his Christian name. He was brought up by the wife of a poor glazier. It is said that when he began to show signs of great talent, his mother sent for him, but received the reply, "You are only my step-mother; the glazier's wife is my mother." His father provided him with a yearly income. D'Alembert entered upon the study of law, but such was his love for mathematics, that law was soon abandoned. At the age of twenty-four his reputation as a mathematician secured for him admission to the Academy of Sciences. In 1743 appeared his *Traité de dynamique*, founded upon the important general principle bearing his name: The impressed forces are equivalent to the effective forces. D'Alembert's principle seems to have been recognised before him by Fontaine, and in some measure by John Bernoulli and Newton. D'Alembert gave it a clear mathematical form and made numerous applications of it. It enabled the laws of motion and the reason-

ings depending on them to be represented in the most general form, in analytical language. D'Alembert applied it in 1744 in a treatise on the equilibrium and motion of fluids, in 1746 to a treatise on the general causes of winds, which obtained a prize from the Berlin Academy. In both these treatises, as also in one of 1747, discussing the famous problem of vibrating chords, he was led to partial differential equations. He was a leader among the pioneers in the study of such equations. To the equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$, arising in the problem of vibrating chords, he gave as the general solution,

$$y = f(x + at) + \phi(x - at),$$

and showed that there is only one arbitrary function, if y be supposed to vanish for $x = 0$ and $x = l$. Daniel Bernoulli, starting with a particular integral given by Brook Taylor, showed that this differential equation is satisfied by the trigonometric series

$$y = \alpha \sin\frac{\pi x}{l} \cdot \cos\frac{\pi t}{l} + \beta \sin\frac{2\pi x}{l} \cdot \cos\frac{2\pi t}{l} + \cdots,$$

and claimed this expression to be the most general solution. Euler denied its generality, on the ground that, if true, the doubtful conclusion would follow that the above series represents any arbitrary function of a variable. These doubts were dispelled by Fourier. Lagrange proceeded to find the sum of the above series, but D'Alembert rightly objected to his process, on the ground that it involved divergent series.[46]

A most beautiful result reached by D'Alembert, with aid of his principle, was the complete solution of the problem of the precession of the equinoxes, which had baffled the talents of the best minds. He sent to the French Academy in 1747, on the same day with Clairaut, a solution of the problem of three bodies. This had become a question of universal inter-

est to mathematicians, in which each vied to outdo all others. The problem of two bodies, requiring the determination of their motion when they attract each other with forces inversely proportional to the square of the distance between them, had been completely solved by Newton. The "problem of three bodies" asks for the motion of three bodies attracting each other according to the law of gravitation. Thus far, the complete solution of this has transcended the power of analysis. The general differential equations of motion were stated by Laplace, but the difficulty arises in their integration. The "solutions" hitherto given are merely convenient methods of approximation in special cases when one body is the sun, disturbing the motion of the moon around the earth, or where a planet moves under the influence of the sun and another planet.

In the discussion of the meaning of negative quantities, of the fundamental processes of the calculus, and of the theory of probability, D'Alembert paid some attention to the philosophy of mathematics. His criticisms were not always happy. In 1754 he was made permanent secretary of the French Academy. During the last years of his life he was mainly occupied with the great French encyclopædia, which was begun by Diderot and himself. D'Alembert declined, in 1762, an invitation of Catharine II. to undertake the education of her son. Frederick the Great pressed him to go to Berlin. He made a visit, but declined a permanent residence there.

Alexis Claude Clairaut (1713–1765) was a youthful prodigy. He read l'Hospital's works on the infinitesimal calculus and on conic sections at the age of ten. In 1731 was published his *Recherches sur les courbes à double courbure*, which he had ready for the press when he was sixteen. It was a work of remarkable elegance and secured his admission to the Academy of Sciences when still under legal age. In 1731 he gave a proof of

the theorem enunciated by Newton, that every cubic is a projection of one of five divergent parabolas. Clairaut formed the acquaintance of Maupertius, whom he accompanied on an expedition to Lapland to measure the length of a degree of the meridian. At that time the shape of the earth was a subject of serious disagreement. Newton and Huygens had concluded from theory that the earth was flattened at the poles. About 1713 Dominico Cassini measured an arc extending from Dunkirk to Perpignan and arrived at the startling result that the earth is elongated at the poles. To decide between the conflicting opinions, measurements were renewed. Maupertius earned by his work in Lapland the title of "earth flattener" by disproving the Cassinian tenet that the earth was elongated at the poles, and showing that Newton was right. On his return, in 1743, Clairaut published a work, *Théorie de la figure de la Terre,* which was based on the results of Maclaurin on homogeneous ellipsoids. It contains a remarkable theorem, named after Clairaut, that the sum of the fractions expressing the ellipticity and the increase of gravity at the pole is equal to $2\frac{1}{2}$ times the fraction expressing the centrifugal force at the equator, the unit of force being represented by the force of gravity at the equator. This theorem is independent of any hypothesis with respect to the law of densities of the successive strata of the earth. It embodies most of Clairaut's researches. Todhunter says that "in the figure of the earth no other person has accomplished so much as Clairaut, and the subject remains at present substantially as he left it, though the form is different. The splendid analysis which Laplace supplied, adorned but did not really alter the theory which started from the creative hands of Clairaut."

In 1752 he gained a prize of the St. Petersburg Academy for his paper on *Théorie de la Lune,* in which for the first time modern analysis is applied to lunar motion. This contained

the explanation of the motion of the lunar apsides. This motion, left unexplained by Newton, seemed to him at first inexplicable by Newton's law, and he was on the point of advancing a new hypothesis regarding gravitation, when, taking the precaution to carry his calculation to a higher degree of approximation, he reached results agreeing with observation. The motion of the moon was studied about the same time by Euler and D'Alembert. Clairaut predicted that "Halley's Comet," then expected to return, would arrive at its nearest point to the sun on April 13, 1759, a date which turned out to be one month too late. He was the first to detect singular solutions in differential equations of the first order but of higher degree than the first.

In their scientific labours there was between Clairaut and D'Alembert great rivalry, often far from friendly. The growing ambition of Clairaut to shine in society, where he was a great favourite, hindered his scientific work in the latter part of his life.

Johann Heinrich Lambert (1728–1777), born at Mühlhausen in Alsace, was the son of a poor tailor. While working at his father's trade, he acquired through his own unaided efforts a knowledge of elementary mathematics. At the age of thirty he became tutor in a Swiss family and secured leisure to continue his studies. In his travels with his pupils through Europe he became acquainted with the leading mathematicians. In 1764 he settled in Berlin, where he became member of the Academy, and enjoyed the society of Euler and Lagrange. He received a small pension, and later became editor of the Berlin *Ephemeris*. His many-sided scholarship reminds one of Leibniz. In his *Cosmological Letters* he made some remarkable prophecies regarding the stellar system. In mathematics he made several discoveries which were extended and overshadowed by his great contemporaries. His first research on pure mathe-

matics developed in an infinite series the root x of the equation $x^m + px = q$. Since each equation of the form $ax^r + bx^s = d$ can be reduced to $x^m + px = q$ in two ways, one or the other of the two resulting series was always found to be convergent, and to give a value of x. Lambert's results stimulated Euler, who extended the method to an equation of four terms, and particularly Lagrange, who found that a function of a root of $\alpha - x + \phi(x) = 0$ can be expressed by the series bearing his name. In 1761 Lambert communicated to the Berlin Academy a memoir, in which he proves that π is irrational. This proof is given in Note IV. of Legendre's *Géometrie*, where it is extended to π^2. To the genius of Lambert we owe the introduction into trigonometry of hyperbolic functions, which he designated by *sinh x*, *cosh x*, etc. His *Freye Perspective*, 1759 and 1773, contains researches on descriptive geometry, and entitle him to the honour of being the forerunner of Monge. In his effort to simplify the calculation of cometary orbits, he was led geometrically to some remarkable theorems on conics, for instance this: "If in two ellipses having a common major axis we take two such arcs that their chords are equal, and that also the sums of the radii vectores, drawn respectively from the foci to the extremities of these arcs, are equal to each other, then the sectors formed in each ellipse by the arc and the two radii vectores are to each other as the square roots of the parameters of the ellipses." [13]

John Landen (1719–1790) was an English mathematician whose writings served as the starting-point of investigations by Euler, Lagrange, and Legendre. Landen's capital discovery, contained in a memoir of 1755, was that every arc of the hyperbola is immediately rectified by means of two arcs of an ellipse. In his "residual analysis" he attempted to obviate the metaphysical difficulties of fluxions by adopting a purely algebraic method. Lagrange's *Calcul des Fonctions* is based

upon this idea. Landen showed how the algebraic expression for the roots of a cubic equation could be derived by application of the differential and integral calculus. Most of the time of this suggestive writer was spent in the pursuits of active life.

Étienne Bézout (1730–1783) was a French writer of popular mathematical school-books. In his *Théorie générale des Équations Algébriques*, 1779, he gave the method of elimination by linear equations (invented also by Euler). This method was first published by him in a memoir of 1764, in which he uses determinants, without, however, entering upon their theory. A beautiful theorem as to the degree of the resultant goes by his name.

Louis Arbogaste (1759–1803) of Alsace was professor of mathematics at Strasburg. His chief work, the *Calcul des Dérivations*, 1800, gives the method known by his name, by which the successive coefficients of a development are derived from one another when the expression is complicated. De Morgan has pointed out that the true nature of derivation is differentiation accompanied by integration. In this book for the first time are the symbols of operation separated from those of quantity. The notation $D_x y$ for $\frac{dy}{dx}$ is due to him.

Maria Gaetana Agnesi (1718–1799) of Milan, distinguished as a linguist, mathematician, and philosopher, filled the mathematical chair at the University of Bologna during her father's sickness. In 1748 she published her *Instituzioni Analitiche*, which was translated into English in 1801. The "witch of Agnesi" or "versiera" is a plane curve containing a straight line, $x = 0$, and a cubic $\left(\frac{y}{c}\right)^2 + 1 = \frac{c}{x}$.

Joseph Louis Lagrange (1736–1813), one of the greatest mathematicians of all times, was born at Turin and died at Paris. He was of French extraction. His father, who had

charge of the Sardinian military chest, was once wealthy, but lost all he had in speculation. Lagrange considered this loss his good fortune, for otherwise he might not have made mathematics the pursuit of his life. While at the college in Turin his genius did not at once take its true bent. Cicero and Virgil at first attracted him more than Archimedes and Newton. He soon came to admire the geometry of the ancients, but the perusal of a tract of Halley roused his enthusiasm for the analytical method, in the development of which he was destined to reap undying glory. He now applied himself to mathematics, and in his seventeenth year he became professor of mathematics in the royal military academy at Turin. Without assistance or guidance he entered upon a course of study which in two years placed him on a level with the greatest of his contemporaries. With aid of his pupils he established a society which subsequently developed into the Turin Academy. In the first five volumes of its transactions appear most of his earlier papers. At the age of nineteen he communicated to Euler a general method of dealing with "isoperimetrical problems," known now as the Calculus of Variations. This commanded Euler's lively admiration, and he courteously withheld for a time from publication some researches of his own on this subject, so that the youthful Lagrange might complete *his* investigations and claim the invention. Lagrange did quite as much as Euler towards the creation of the Calculus of Variations. As it came from Euler it lacked an analytic foundation, and this Lagrange supplied. He separated the principles of this calculus from geometric considerations by which his predecessor had derived them. Euler had assumed as fixed the limits of the integral, *i.e.* the extremities of the curve to be determined, but Lagrange removed this restriction and allowed all co-ordinates of the curve to vary at the same time. Euler introduced in 1766 the

name "calculus of variations," and did much to improve this science along the lines marked out by Lagrange.

Another subject engaging the attention of Lagrange at Turin was the propagation of sound. In his papers on this subject in the *Miscellanea Taurinensia*, the young mathematician appears as the critic of Newton, and the arbiter between Euler and D'Alembert. By considering only the particles which are in a straight line, he reduced the problem to the same partial differential equation that represents the motions of vibrating strings. The general integral of this was found by D'Alembert to contain two arbitrary functions, and the question now came to be discussed whether an arbitrary function may be discontinuous. D'Alembert maintained the negative against Euler, Daniel Bernoulli, and finally Lagrange, — arguing that in order to determine the position of a point of the chord at a time t, the initial position of the chord must be continuous. Lagrange settled the question in the affirmative.

By constant application during nine years, Lagrange, at the age of twenty-six, stood at the summit of European fame. But his intense studies had seriously weakened a constitution never robust, and though his physicians induced him to take rest and exercise, his nervous system never fully recovered its tone, and he was thenceforth subject to fits of melancholy.

In 1764 the French Academy proposed as the subject of a prize the theory of the libration of the moon. It demanded an explanation, on the principle of universal gravitation, why the moon always turns, with but slight variations, the same phase to the earth. Lagrange secured the prize. This success encouraged the Academy to propose as a prize the theory of the four satellites of Jupiter, — a problem of six bodies, more difficult than the one of three bodies previously solved by Clairaut, D'Alembert, and Euler. Lagrange overcame the difficulties, but the shortness of time did not permit him to

exhaust the subject. Twenty-four years afterwards it was completed by Laplace. Later astronomical investigations of Lagrange are on cometary perturbations (1778 and 1783), on Kepler's problem, and on a new method of solving the problem of three bodies.

Being anxious to make the personal acquaintance of leading mathematicians, Lagrange visited Paris, where he enjoyed the stimulating delight of conversing with Clairaut, D'Alembert, Condorcet, the Abbé Marie, and others. He had planned a visit to London, but he fell dangerously ill after a dinner in Paris, and was compelled to return to Turin. In 1766 Euler left Berlin for St. Petersburg, and he pointed out Lagrange as the only man capable of filling the place. D'Alembert recommended him at the same time. Frederick the Great thereupon sent a message to Turin, expressing the wish of "the greatest king of Europe" to have "the greatest mathematician" at his court. Lagrange went to Berlin, and staid there twenty years. Finding all his colleagues married, and being assured by their wives that the marital state alone is happy, he married. The union was not a happy one. His wife soon died. Frederick the Great held him in high esteem, and frequently conversed with him on the advantages of perfect regularity of life. This led Lagrange to cultivate regular habits. He worked no longer each day than experience taught him he could without breaking down. His papers were carefully thought out before he began writing, and when he wrote he did so without a single correction.

During the twenty years in Berlin he crowded the transactions of the Berlin Academy with memoirs, and wrote also the epoch-making work called the *Mécanique Analytique*. He enriched algebra by researches on the solution of equations. There are two methods of solving directly algebraic equations, — that of substitution and that of combination. The

former method was developed by Ferrari, Vieta, Tchirnhausen, Euler, Bézout, and Lagrange; the latter by Vandermonde and Lagrange.[20] In the method of substitution the original forms are so transformed that the determination of the roots is made to depend upon simpler functions (resolvents). In the method of combination auxiliary quantities are substituted for certain simple combinations ("types") of the unknown roots of the equation, and auxiliary equations (resolvents) are obtained for these quantities with aid of the coefficients of the given equation. Lagrange traced all known algebraic solutions of equations to the uniform principle consisting in the formation and solution of equations of lower degree whose roots are linear functions of the required roots, and of the roots of unity. He showed that the quintic cannot be reduced in this way, its resolvent being of the sixth degree. His researches on the theory of equations were continued after he left Berlin. In the *Résolution des équations numériques* (1798) he gave a method of approximating to the real roots of numerical equations by continued fractions. Among other things, it contains also a proof that every equation must have a root, — a theorem which appears before this to have been considered self-evident. Other proofs of this were given by Argand, Gauss, and Cauchy. In a note to the above work Lagrange uses Fermat's theorem and certain suggestions of Gauss in effecting a complete algebraic solution of any binomial equation.

While in Berlin Lagrange published several papers on the theory of numbers. In 1769 he gave a solution in integers of indeterminate equations of the second degree, which resembles the Hindoo cyclic method; he was the first to prove, in 1771, "Wilson's theorem," enunciated by an Englishman, John Wilson, and first published by Waring in his *Meditationes Algebraicæ;* he investigated in 1775 under what conditions ± 2 and ± 5 (-1 and ± 3 having been discussed by Euler)

are quadratic residues, or non-residues of odd prime numbers, q; he proved in 1770 Méziriac's theorem that every integer is equal to the sum of four, or a less number, of squares. He proved Fermat's theorem on $x^n + y^n = z^n$, for the case $n = 4$, also Fermat's theorem that, if $a^2 + b^2 = c^2$, then ab is not a square.

In his memoir on Pyramids, 1773, Lagrange made considerable use of determinants of the third order, and demonstrated that the square of a determinant is itself a determinant. He never, however, dealt explicitly and directly with determinants; he simply obtained accidentally identities which are now recognised as relations between determinants.

Lagrange wrote much on differential equations. Though the subject of contemplation by the greatest mathematicians (Euler, D'Alembert, Clairaut, Lagrange, Laplace), yet more than other branches of mathematics did they resist the systematic application of fixed methods and principles. Lagrange established criteria for singular solutions (*Calcul des Fonctions*, Lessons 14–17), which are, however, erroneous. He was the first to point out the geometrical significance of such solutions. He generalised Euler's researches on total differential equations of two variables, and of the ninth order; he gave a solution of partial differential equations of the first order (*Berlin Memoirs*, 1772 and 1774), and spoke of their singular solutions, extending their solution in *Memoirs* of 1779 and 1785 to equations of any number of variables. The discussion on partial differential equations of the second order, carried on by D'Alembert, Euler, and Lagrange, has already been referred to in our account of D'Alembert.

While in Berlin, Lagrange wrote the "*Méchanique Analytique*," the greatest of his works (Paris, 1788). From the principle of virtual velocities he deduced, with aid of the calculus of variations, the whole system of mechanics so elegantly and

harmoniously that it may fitly be called, in Sir William Rowan Hamilton's words, "a kind of scientific poem." It is a most consummate example of analytic generality. Geometrical figures are nowhere allowed. "On ne trouvera point de figures dans cet ouvrage" (Preface). The two divisions of mechanics — statics and dynamics — are in the first four sections of each carried out analogously, and each is prefaced by a historic sketch of principles. Lagrange formulated the principle of least action. In their original form, the equations of motion involve the co-ordinates x, y, z, of the different particles m or dm of the system. But x, y, z, are in general not independent, and Lagrange introduced in place of them any variables ξ, ψ, ϕ, whatever, determining the position of the point at the time. These may be taken to be independent. The equations of motion may now assume the form

$$\frac{d}{dt}\frac{dT}{d\xi'} - \frac{dT}{d\xi} + \Xi = 0;$$

or when Ξ, ψ, ϕ, . . . are the partial differential coefficients with respect to ξ, ψ, ϕ, . . . of one and the same function V, then the form

$$\frac{d}{dt}\frac{dT}{d\xi'} - \frac{dT}{d\xi} + \frac{dV}{d\xi} = 0.$$

The latter is *par excellence* the Lagrangian form of the equations of motion. With Lagrange originated the remark that mechanics may be regarded as a geometry of four dimensions. To him falls the honour of the introduction of the potential into dynamics.[49] Lagrange was anxious to have his *Mécanique Analytique* published in Paris. The work was ready for print in 1786, but not till 1788 could he find a publisher, and then only with the condition that after a few years he would purchase all the unsold copies. The work was edited by Legendre.

After the death of Frederick the Great, men of science were no longer respected in Germany, and Lagrange accepted an invitation of Louis XVI. to migrate to Paris. The French queen treated him with regard, and lodging was procured for him in the Louvre. But he was seized with a long attack of melancholy which destroyed his taste for mathematics. For two years his printed copy of the *Mécanique*, fresh from the press, — the work of a quarter of a century, — lay unopened on his desk. Through Lavoisier he became interested in chemistry, which he found "as easy as algebra." The disastrous crisis of the French Revolution aroused him again to activity. About this time the young and accomplished daughter of the astronomer Lemonnier took compassion on the sad, lonely Lagrange, and insisted upon marrying him. Her devotion to him constituted the one tie to life which at the approach of death he found it hard to break.

He was made one of the commissioners to establish weights and measures having units founded on nature. Lagrange strongly favoured the decimal subdivision, the general idea of which was obtained from a work of Thomas Williams, London, 1788. Such was the moderation of Lagrange's character, and such the universal respect for him, that he was retained as president of the commission on weights and measures even after it had been *purified* by the Jacobins by striking out the names of Lavoisier, Laplace, and others. Lagrange took alarm at the fate of Lavoisier, and planned to return to Berlin, but at the establishment of the *École Normale* in 1795 in Paris, he was induced to accept a professorship. Scarcely had he time to elucidate the foundations of arithmetic and algebra to young pupils, when the school was closed. His additions to the algebra of Euler were prepared at this time. In 1797 the *École Polytechnique* was founded, with Lagrange as one of the professors. The earliest triumph of this institution was

the restoration of Lagrange to analysis. His mathematical activity burst out anew. He brought forth the *Théorie des fonctions analytiques* (1797), *Leçons sur le calcul des fonctions*, a treatise on the same lines as the preceding (1801), and the *Résolution des équations numeriques* (1798). In 1810 he began a thorough revision of his *Mécanique analytique*, but he died before its completion.

The *Théorie des fonctions*, the germ of which is found in a memoir of his of 1772, aimed to place the principles of the calculus upon a sound foundation by relieving the mind of the difficult conception of a limit or infinitesimal. John Landen's residual calculus, professing a similar object, was unknown to him. Lagrange attempted to prove Taylor's theorem (the power of which he was the first to point out) by simple algebra, and then to develop the entire calculus from that theorem. The principles of the calculus were in his day involved in philosophic difficulties of a serious nature. The infinitesimals of Leibniz had no satisfactory metaphysical basis. In the differential calculus of Euler they were treated as absolute zeros. In Newton's limiting ratio, the magnitudes of which it is the ratio cannot be found, for at the moment when they should be caught and equated, there is neither arc nor chord. The chord and arc were not taken by Newton as equal before vanishing, nor after vanishing, but *when* they vanish. "That method," said Lagrange, "has the great inconvenience of considering quantities in the state in which they cease, so to speak, to be quantities; for though we can always well conceive the ratios of two quantities, as long as they remain finite, that ratio offers to the mind no clear and precise idea, as soon as its terms become both nothing at the same time." D'Alembert's method of limits was much the same as the method of prime and ultimate ratios. D'Alembert taught that a variable actually reached its limit. When Lagrange

endeavoured to free the calculus of its metaphysical difficulties, by resorting to common algebra, he avoided the whirlpool of Charybdis only to suffer wreck against the rocks of Scylla. The algebra of his day, as handed down to him by Euler, was founded on a false view of infinity. No correct theory of infinite series had then been established. Lagrange proposed to define the differential coefficient of $f(x)$ with respect to x as the coefficient of h in the expansion of $f(x+h)$ by Taylor's theorem, and thus to avoid all reference to limits. But he used infinite series without ascertaining that they were convergent, and his proof that $f(x+h)$ can always be expanded in a series of ascending powers of h, labours under serious defects. Though Lagrange's method of developing the calculus was at first greatly applauded, its defects were fatal, and to-day his "method of derivatives," as it was called, has been generally abandoned. He introduced a notation of his own, but it was inconvenient, and was abandoned by him in the second edition of his *Mécanique*, in which he used infinitesimals. The primary object of the *Théorie des fonctions* was not attained, but its secondary results were far-reaching. It was a purely abstract mode of regarding functions, apart from geometrical or mechanical considerations. In the further development of higher analysis a function became the leading idea, and Lagrange's work may be regarded as the starting-point of the theory of functions as developed by Cauchy, Riemann, Weierstrass, and others.

In the treatment of infinite series Lagrange displayed in his earlier writings that laxity common to all mathematicians of his time, excepting Nicolaus Bernoulli II. and D'Alembert. But his later articles mark the beginning of a period of greater rigour. Thus, in the *Calcul de fonctions* he gives his theorem on the limits of Taylor's theorem. Lagrange's mathematical researches extended to subjects which have not been men-

tioned here — such as probabilities, finite differences, ascending continued fractions, elliptic integrals. Everywhere his wonderful powers of generalisation and abstraction are made manifest. In that respect he stood without a peer, but his great contemporary, Laplace, surpassed him in practical sagacity. Lagrange was content to leave the application of his general results to others, and some of the most important researches of Laplace (particularly those on the velocity of sound and on the secular acceleration of the moon) are implicitly contained in Lagrange's works.

Lagrange was an extremely modest man, eager to avoid controversy, and even timid in conversation. He spoke in tones of doubt, and his first words generally were, "Je ne sais pas." He would never allow his portrait to be taken, and the only ones that were secured were sketched without his knowledge by persons attending the meetings of the Institute.

Pierre Simon Laplace (1749–1827) was born at Beaumont-en-Auge in Normandy. Very little is known of his early life. When at the height of his fame he was loath to speak of his boyhood, spent in poverty. His father was a small farmer. Some rich neighbours who recognised the boy's talent assisted him in securing an education. As an extern he attended the military school in Beaumont, where at an early age he became teacher of mathematics. At eighteen he went to Paris, armed with letters of recommendation to D'Alembert, who was then at the height of his fame. The letters remained unnoticed, but young Laplace, undaunted, wrote the great geometer a letter on the principles of mechanics, which brought the following enthusiastic response: "You needed no introduction; you have recommended yourself; my support is your due." D'Alembert secured him a position at the *École Militaire* of Paris as professor of mathe-

matics. His future was now assured, and he entered upon those profound researches which brought him the title of "the Newton of France." With wonderful mastery of analysis, Laplace attacked the pending problems in the application of the law of gravitation to celestial motions. During the succeeding fifteen years appeared most of his original contributions to astronomy. His career was one of almost uninterrupted prosperity. In 1784 he succeeded Bézout as examiner to the royal artillery, and the following year he became member of the Academy of Sciences. He was made president of the Bureau of Longitude; he aided in the introduction of the decimal system, and taught, with Lagrange, mathematics in the *École Normale*. When, during the Revolution, there arose a cry for the reform of everything, even of the calendar, Laplace suggested the adoption of an era beginning with the year 1250, when, according to his calculation, the major axis of the earth's orbit had been perpendicular to the equinoctial line. The year was to begin with the vernal equinox, and the zero meridian was to be located east of Paris by 185.30 degrees of the centesimal division of the quadrant, for by this meridian the beginning of his proposed era fell at midnight. But the revolutionists rejected this scheme, and made the start of the new era coincide with the beginning of the glorious French Republic.[50]

Laplace was justly admired throughout Europe as a most sagacious and profound scientist, but, unhappily for his reputation, he strove not only after greatness in science, but also after political honours. The political career of this eminent scientist was stained by servility and suppleness. After the 18th of Brumaire, the day when Napoleon was made emperor, Laplace's ardour for republican principles suddenly gave way to a great devotion to the emperor. Napoleon rewarded this devotion by giving him the post of minister of the interior,

but dismissed him after six months for incapacity. Said Napoleon, "Laplace ne saisissait aucune question sous son véritable point de vue; il cherchait des subtilités partout, n'avait que des idées problematiques, et portait enfin l'esprit des infiniment petits jusque dans l'administration." Desirous to retain his allegiance, Napoleon elevated him to the Senate and bestowed various other honours upon him. Nevertheless, he cheerfully gave his voice in 1814 to the dethronement of his patron and hastened to tender his services to the Bourbons, thereby earning the title of marquis. This pettiness of his character is seen in his writings. The first edition of the *Système du monde* was dedicated to the Council of Five Hundred. To the third volume of the *Mécanique Céleste* is prefixed a note that of all the truths contained in the book, that most precious to the author was the declaration he thus made of gratitude and devotion to the peace-maker of Europe. After this outburst of affection, we are surprised to find in the editions of the *Théorie analytique des probabilités*, which appeared after the Restoration, that the original dedication to the emperor is suppressed.

Though supple and servile in politics, it must be said that in religion and science Laplace never misrepresented or concealed his own convictions however distasteful they might be to others. In mathematics and astronomy his genius shines with a lustre excelled by few. Three great works did he give to the scientific world, — the *Mécanique Céleste*, the *Exposition du système du monde*, and the *Théorie analytique des probabilities*. Besides these he contributed important memoirs to the French Academy.

We first pass in brief review his astronomical researches. In 1773 he brought out a paper in which he proved that the mean motions or mean distances of planets are invariable or merely subject to small periodic changes. This was the first

and most important step in establishing the stability of the solar system.[51] To Newton and also to Euler it had seemed doubtful whether forces so numerous, so variable in position, so different in intensity, as those in the solar system, could be capable of maintaining permanently a condition of equilibrium. Newton was of the opinion that a powerful hand must intervene from time to time to repair the derangements occasioned by the mutual action of the different bodies. This paper was the beginning of a series of profound researches by Lagrange and Laplace on the limits of variation of the various elements of planetary orbits, in which the two great mathematicians alternately surpassed and supplemented each other. Laplace's first paper really grew out of researches on the theory of Jupiter and Saturn. The behaviour of these planets had been studied by Euler and Lagrange without receiving satisfactory explanation. Observation revealed the existence of a steady acceleration of the mean motions of our moon and of Jupiter and an equally strange diminution of the mean motion of Saturn. It looked as though Saturn might eventually leave the planetary system, while Jupiter would fall into the sun, and the moon upon the earth. Laplace finally succeeded in showing, in a paper of 1784–1786, that these variations (called the "great inequality") belonged to the class of ordinary periodic perturbations, depending upon the law of attraction. The cause of so influential a perturbation was found in the commensurability of the mean motion of the two planets.

In the study of the Jovian system, Laplace was enabled to determine the masses of the moons. He also discovered certain very remarkable, simple relations between the movements of those bodies, known as "Laws of Laplace." His theory of these bodies was completed in papers of 1788 and 1789. These, as well as the other papers here mentioned, were published in the *Mémoirs présentés par divers savans.* The year

1787 was made memorable by Laplace's announcement that the lunar acceleration depended upon the secular changes in the eccentricity of the earth's orbit. This removed all doubt then existing as to the stability of the solar system. The universal validity of the law of gravitation to explain all motion in the solar system was established. That system, as then known, was at last found to be a complete machine.

In 1796 Laplace published his *Exposition du système du monde*, a non-mathematical popular treatise on astronomy, ending with a sketch of the history of the science. In this work he enunciates for the first time his celebrated nebular hypothesis. A similar theory had been previously proposed by Kant in 1755, and by Swedenborg; but Laplace does not appear to have been aware of this.

Laplace conceived the idea of writing a work which should contain a complete analytical solution of the mechanical problem presented by the solar system, without deriving from observation any but indispensable data. The result was the *Mécanique Céleste*, which is a systematic presentation embracing all the discoveries of Newton, Clairaut, D'Alembert, Euler, Lagrange, and of Laplace himself, on celestial mechanics. The first and second volumes of this work were published in 1799; the third appeared in 1802, the fourth in 1805. Of the fifth volume, Books XI. and XII. were published in 1823; Books XIII., XIV., XV. in 1824, and Book XVI. in 1825. The first two volumes contain the general theory of the motions and figure of celestial bodies. The third and fourth volumes give special theories of celestial motions, — treating particularly of motions of comets, of our moon, and of other satellites. The fifth volume opens with a brief history of celestial mechanics, and then gives in appendices the results of the author's later researches. The *Mécanique Céleste* was such a master-piece, and so complete, that Laplace's successors have

been able to add comparatively little. The general part of the work was translated into German by Joh. Karl Burkhardt, and appeared in Berlin, 1800–1802. Nathaniel Bowditch brought out an edition in English, with an extensive commentary, in Boston, 1829–1839. The *Mécanique Céleste* is not easy reading. The difficulties lie, as a rule, not so much in the subject itself as in the want of verbal explanation. A complicated chain of reasoning receives often no explanation whatever. Biot, who assisted Laplace in revising the work for the press, tells that he once asked Laplace some explanation of a passage in the book which had been written not long before, and that Laplace spent an hour endeavouring to recover the reasoning which had been carelessly suppressed with the remark, "Il est facile de voir." Notwithstanding the important researches in the work, which are due to Laplace himself, it naturally contains a great deal that is drawn from his predecessors. It is, in fact, the organised result of a century of patient toil. But Laplace frequently neglects to properly acknowledge the source from which he draws, and lets the reader infer that theorems and formulæ due to a predecessor are really his own.

We are told that when Laplace presented Napoleon with a copy of the *Mécanique Céleste*, the latter made the remark, "M. Laplace, they tell me you have written this large book on the system of the universe, and have never even mentioned its Creator." Laplace is said to have replied bluntly, "Je n'avais pas besoin de cette hypothèse-la." This assertion, taken literally, is impious, but may it not have been intended to convey a meaning somewhat different from its literal one? Newton was not able to explain by his law of gravitation all questions arising in the mechanics of the heavens. Thus, being unable to show that the solar system was stable, and suspecting in fact that it was unstable, Newton expressed the

opinion that the special intervention, from time to time, of a powerful hand was necessary to preserve order. Now Laplace was able to prove by the law of gravitation that the solar system is stable, and in that sense may be said to have felt no necessity for reference to the Almighty.

We now proceed to researches which belong more properly to pure mathematics. Of these the most conspicuous are on the theory of probability. Laplace has done more towards advancing this subject than any one other investigator. He published a series of papers, the main results of which were collected in his *Théorie analytique des probabilités*, 1812. The third edition (1820) consists of an introduction and two books. The introduction was published separately under the title, *Essai philosophique sur les probabilités*, and is an admirable and masterly exposition without the aid of analytical formulæ of the principles and applications of the science. The first book contains the theory of generating functions, which are applied, in the second book, to the theory of probability. Laplace gives in his work on probability his method of approximation to the values of definite integrals. The solution of linear differential equations was reduced by him to definite integrals. One of the most important parts of the work is the application of probability to the method of least squares, which is shown to give the most probable as well as the most convenient results.

The first printed statement of the principle of least squares was made in 1806 by Legendre, without demonstration. Gauss had used it still earlier, but did not publish it until 1809. The first deduction of the law of probability of error that appeared in print was given in 1808 by Robert Adrain in the *Analyst*, a journal published by himself in Philadelphia.[2] Proofs of this law have since been given by Gauss, Ivory, Herschel, Hagen, and others; but all proofs contain some

point of difficulty. Laplace's proof is perhaps the most satisfactory.

Laplace's work on probability is very difficult reading, particularly the part on the method of least squares. The analytical processes are by no means clearly established or free from error. "No one was more sure of giving the result of analytical processes correctly, and no one ever took so little care to point out the various small considerations on which correctness depends" (De Morgan).

Of Laplace's papers on the attraction of ellipsoids, the most important is the one published in 1785, and to a great extent reprinted in the third volume of the *Mécanique Céleste*. It gives an exhaustive treatment of the general problem of attraction of any ellipsoid upon a particle situated outside or upon its surface. Spherical harmonics, or the so-called "Laplace's coefficients," constitute a powerful analytic engine in the theory of attraction, in electricity, and magnetism. The theory of spherical harmonics for two dimensions had been previously given by Legendre. Laplace failed to make due acknowledgment of this, and there existed, in consequence, between the two great men, "a feeling more than coldness." The potential function, V, is much used by Laplace, and is shown by him to satisfy the partial differential equation $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$. This is known as Laplace's equation, and was first given by him in the more complicated form which it assumes in polar co-ordinates. The notion of potential was, however, not introduced into analysis by Laplace. The honour of that achievement belongs to Lagrange.[49]

Among the minor discoveries of Laplace are his method of solving equations of the second, third, and fourth degrees, his memoir on singular solutions of differential equations, his

researches in finite differences and in determinants, the establishment of the expansion theorem in determinants which had been previously given by Vandermonde for a special case, the determination of the complete integral of the linear differential equation of the second order. In the *Mécanique Céleste* he made a generalisation of Lagrange's theorem on the development of functions in series known as Laplace's theorem.

Laplace's investigations in physics were quite extensive. We mention here his correction of Newton's formula on the velocity of sound in gases by taking into account the changes of elasticity due to the heat of compression and cold of rarefaction; his researches on the theory of tides; his mathematical theory of capillarity; his explanation of astronomical refraction; his formulæ for measuring heights by the barometer.

Laplace's writings stand out in bold contrast to those of Lagrange in their lack of elegance and symmetry. Laplace looked upon mathematics as the tool for the solution of physical problems. The true result being once reached, he spent little time in explaining the various steps of his analysis, or in polishing his work. The last years of his life were spent mostly at Arcueil in peaceful retirement on a country-place, where he pursued his studies with his usual vigour until his death. He was a great admirer of Euler, and would often say, "Lisez Euler, lisez Euler, c'est notre maître à tous."

Abnit-Théophile Vandermonde (1735–1796) studied music during his youth in Paris and advocated the theory that all art rested upon one general law, through which any one could become a composer with the aid of mathematics. He was the first to give a connected and logical exposition of the theory of determinants, and may, therefore, almost be regarded as the founder of that theory. He and Lagrange originated the method of combinations in solving equations.[20]

Adrien Marie Legendre (1752–1833) was educated at the

Collège Mazarin in Paris, where he began the study of mathematics under Abbé Marie. His mathematical genius secured for him the position of professor of mathematics at the military school of Paris. While there he prepared an essay on the curve described by projectiles thrown into resisting media (ballistic curve), which captured a prize offered by the Royal Academy of Berlin. In 1780 he resigned his position in order to reserve more time for the study of higher mathematics. He was then made member of several public commissions. In 1795 he was elected professor at the Normal School and later was appointed to some minor government positions. Owing to his timidity and to Laplace's unfriendliness toward him, but few important public offices commensurate with his ability were tendered to him.

As an analyst, second only to Laplace and Lagrange, Legendre enriched mathematics by important contributions, mainly on elliptic integrals, theory of numbers, attraction of ellipsoids, and least squares. The most important of Legendre's works is his *Fonctions elliptiques,* issued in two volumes in 1825 and 1826. He took up the subject where Euler, Landen, and Lagrange had left it, and for forty years was the only one to cultivate this new branch of analysis, until at last Jacobi and Abel stepped in with admirable new discoveries.[52] Legendre imparted to the subject that connection and arrangement which belongs to an independent science. Starting with an integral depending upon the square root of a polynomial of the fourth degree in x, he showed that such integrals can be brought back to three canonical forms, designated by $F(\phi)$, $E(\phi)$, and $\Pi(\phi)$, the radical being expressed in the form $\Delta(\phi) = \sqrt{1 - k^2 \sin^2 \phi}$. He also undertook the prodigious task of calculating tables of arcs of the ellipse for different degrees of amplitude and eccentricity, which supply the means of integrating a large number of differentials.

An earlier publication which contained part of his researches on elliptic functions was his *Calcul intégral* in three volumes (1811, 1816, 1817), in which he treats also at length of the two classes of definite integrals named by him *Eulerian*. He tabulated the values of $\log \Gamma(p)$ for values of p between 1 and 2.

One of the earliest subjects of research was the attraction of spheroids, which suggested to Legendre the function P_n, named after him. His memoir was presented to the Academy of Sciences in 1783. The researches of Maclaurin and Lagrange suppose the point attracted by a spheroid to be at the surface or within the spheroid, but Legendre showed that in order to determine the attraction of a spheroid on any external point it suffices to cause the surface of another spheroid described upon the same foci to pass through that point. Other memoirs on ellipsoids appeared later.

The two household gods to which Legendre sacrificed with ever-renewed pleasure in the silence of his closet were the elliptic functions and the theory of numbers. His researches on the latter subject, together with the numerous scattered fragments on the theory of numbers due to his predecessors in this line, were arranged as far as possible into a systematic whole, and published in two large quarto volumes, entitled *Théorie des nombres*, 1830. Before the publication of this work Legendre had issued at divers times preliminary articles. Its crowning pinnacle is the theorem of quadratic reciprocity, previously indistinctly given by Euler without proof, but for the first time clearly enunciated and partly proved by Legendre.[48]

While acting as one of the commissioners to connect Greenwich and Paris geodetically, Legendre calculated all the triangles in France. This furnished the occasion of establishing formulæ and theorems on geodesics, on the treatment of the spherical triangle as if it were a plane triangle, by applying

certain corrections to the angles, and on the method of least squares, published for the first time by him without demonstration in 1806.

Legendre wrote an *Éléments de Géométrie*, 1794, which enjoyed great popularity, being generally adopted on the Continent and in the United States as a substitute for Euclid. This great modern rival of Euclid passed through numerous editions; the later ones containing the elements of trigonometry and a proof of the irrationality of π and π^2. Much attention was given by Legendre to the subject of parallel lines. In the earlier editions of the *Éléments*, he made direct appeal to the senses for the correctness of the "parallel-axiom." He then attempted to demonstrate that "axiom," but his proofs did not satisfy even himself. In Vol. XII. of the Memoirs of the Institute is a paper by Legendre, containing his last attempt at a solution of the problem. Assuming space to be infinite, he proved satisfactorily that it is impossible for the sum of the three angles of a triangle to exceed two right angles; and that if there be any triangle the sum of whose angles is two right angles, then the same must be true of all triangles. But in the next step, to show that this sum cannot be less than two right angles, his demonstration necessarily failed. If it could be granted that the sum of the three angles is always equal to two right angles, then the theory of parallels could be strictly deduced.

Joseph Fourier (1768–1830) was born at Auxerre, in central France. He became an orphan in his eighth year. Through the influence of friends he was admitted into the military school in his native place, then conducted by the Benedictines of the Convent of St. Mark. He there prosecuted his studies, particularly mathematics, with surprising success. He wished to enter the artillery, but, being of low birth (the son of a tailor), his application was answered thus: "Fourier, not

being noble, could not enter the artillery, although he were a second Newton."[53] He was soon appointed to the mathematical chair in the military school. At the age of twenty-one he went to Paris to read before the Academy of Sciences a memoir on the resolution of numerical equations, which was an improvement on Newton's method of approximation. This investigation of his early youth he never lost sight of. He lectured upon it in the Polytechnic School; he developed it on the banks of the Nile; it constituted a part of a work entitled *Analyse des equationes determines* (1831), which was in press when death overtook him. This work contained "Fourier's theorem" on the number of real roots between two chosen limits. Budan had published this result as early as 1807, but there is evidence to show that Fourier had established it before Budan's publication. These brilliant results were eclipsed by the theorem of Sturm, published in 1835.

Fourier took a prominent part at his home in promoting the Revolution. Under the French Revolution the arts and sciences seemed for a time to flourish. The reformation of the weights and measures was planned with grandeur of conception. The Normal School was created in 1795, of which Fourier became at first pupil, then lecturer. His brilliant success secured him a chair in the Polytechnic School, the duties of which he afterwards quitted, along with Monge and Berthollet, to accompany Napoleon on his campaign to Egypt. Napoleon founded the Institute of Egypt, of which Fourier became secretary. In Egypt he engaged not only in scientific work, but discharged important political functions. After his return to France he held for fourteen years the prefecture of Grenoble. During this period he carried on his elaborate investigations on the propagation of heat in solid bodies, published in 1822 in his work entitled *La Theorie Analytique de la Chaleur*. This work marks an epoch in the history of

mathematical physics. "Fourier's series" constitutes its gem. By this research a long controversy was brought to a close, and the fact established that any arbitrary function can be represented by a trigonometric series. The first announcement of this great discovery was made by Fourier in 1807, before the French Academy. The trigonometric series $\sum_{n=0}^{n=\infty}(a_n \sin nx + b_n \cos nx)$ represents the function $\phi(x)$ for every value of x, if the coefficients $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\phi(x)\sin nx dx$, and b_n be equal to a similar integral. The weak point in Fourier's analysis lies in his failure to prove generally that the trigonometric series actually converges to the value of the function. In 1827 Fourier succeeded Laplace as president of the council of the Polytechnic School.

Before proceeding to the origin of modern geometry we shall speak briefly of the introduction of higher analysis into Great Britain. This took place during the first quarter of this century. The British began to deplore the very small progress that science was making in England as compared with its racing progress on the Continent. In 1813 the "Analytical Society" was formed at Cambridge. This was a small club established by George Peacock, John Herschel, Charles Babbage, and a few other Cambridge students, to promote, as it was humorously expressed, the principles of pure "*D*-ism," that is, the Leibnizian notation in the calculus against those of "dot-age," or of the Newtonian notation. This struggle ended in the introduction into Cambridge of the notation $\frac{dy}{dx}$, to the exclusion of the fluxional notation $\dot{y}$. This was a great step in advance, not on account of any great superiority of the Leibnizian over the Newtonian notation, but because the adoption of the former opened up to English students the vast storehouses of continental discoveries. Sir William Thomson, Tait, and some other modern writers find

it frequently convenient to use both notations. Herschel, Peacock, and Babbage translated, in 1816, from the French, Lacroix's treatise on the differential and integral calculus, and added in 1820 two volumes of examples. Lacroix's was one of the best and most extensive works on the calculus of that time. Of the three founders of the "Analytical Society," Peacock afterwards did most work in pure mathematics. Babbage became famous for his invention of a calculating engine superior to Pascal's. It was never finished, owing to a misunderstanding with the government, and a consequent failure to secure funds. John Herschel, the eminent astronomer, displayed his mastery over higher analysis in memoirs communicated to the Royal Society on new applications of mathematical analysis, and in articles contributed to cyclopædias on light, on meteorology, and on the history of mathematics.

George Peacock (1791–1858) was educated at Trinity College, Cambridge, became Lowndean professor there, and later, dean of Ely. His chief publications are his *Algebra*, 1830 and 1842, and his *Report on Recent Progress in Analysis*, which was the first of several valuable summaries of scientific progress printed in the volumes of the British Association. He was one of the first to study seriously the fundamental principles of algebra, and to fully recognise its purely symbolic character. He advances, though somewhat imperfectly, the "principle of the permanence of equivalent forms." It assumes that the rules applying to the symbols of arithmetical algebra apply also in symbolical algebra. About this time D. F. Gregory wrote a paper "on the real nature of symbolical algebra," which brought out clearly the commutative and distributive laws. These laws had been noticed years before by the inventors of symbolic methods in the calculus. It was Servois who introduced the names *commutative* and *distributive* in 1813.

Peacock's investigations on the foundation of algebra were considerably advanced by De Morgan and Hankel.

James Ivory (1765–1842) was a Scotch mathematician who for twelve years, beginning in 1804, held the mathematical chair in the Royal Military College at Marlow (now at Sandhurst). He was essentially a self-trained mathematician, and almost the only one in Great Britain previous to the organisation of the Analytical Society who was well versed in continental mathematics. Of importance is his memoir (*Phil. Trans.*, 1809) in which the problem of the attraction of a homogeneous ellipsoid upon an external point is reduced to the simpler problem of the attraction of a related ellipsoid upon a corresponding point interior to it. This is known as "Ivory's theorem." He criticised with undue severity Laplace's solution of the method of least squares, and gave three proofs of the principle without recourse to probability; but they are far from being satisfactory.

The Origin of Modern Geometry.

By the researches of Descartes and the invention of the calculus, the analytical treatment of geometry was brought into great prominence for over a century. Notwithstanding the efforts to revive synthetic methods made by Desargues, Pascal, De Lahire, Newton, and Maclaurin, the analytical method retained almost undisputed supremacy. It was reserved for the genius of Monge to bring synthetic geometry in the foreground, and to open up new avenues of progress. His *Géométrie descriptive* marks the beginning of a wonderful development of modern geometry.

Of the two leading problems of descriptive geometry, the one — to represent by drawings geometrical magnitudes — was brought to a high degree of perfection before the time of

Monge; the other — to solve problems on figures in space by constructions in a plane — had received considerable attention before his time. His most noteworthy predecessor in descriptive geometry was the Frenchman Frézier (1682–1773). But it remained for Monge to create descriptive geometry as a *distinct* branch of science by imparting to it geometric generality and elegance. All problems previously treated in a special and uncertain manner were referred back to a few general principles. He introduced the line of intersection of the horizontal and the vertical plane as the axis of projection. By revolving one plane into the other around this axis or ground-line, many advantages were gained.[54]

Gaspard Monge (1746–1818) was born at Beaune. The construction of a plan of his native town brought the boy under the notice of a colonel of engineers, who procured for him an appointment in the college of engineers at Mézières. Being of low birth, he could not receive a commission in the army, but he was permitted to enter the annex of the school, where surveying and drawing were taught. Observing that all the operations connected with the construction of plans of fortification were conducted by long arithmetical processes, he substituted a geometrical method, which the commandant at first refused even to look at, so short was the time in which it could be practised; when once examined, it was received with avidity. Monge developed these methods further and thus created his descriptive geometry. Owing to the rivalry between the French military schools of that time, he was not permitted to divulge his new methods to any one outside of this institution. In 1768 he was made professor of mathematics at Mézières. In 1780, when conversing with two of his pupils, S. F. Lacroix and Gayvernon in Paris, he was obliged to say, "All that I have here done by calculation, I could have

done with the ruler and compass, but I am not allowed to reveal these secrets to you." But Lacroix set himself to examine what the secret could be, discovered the processes, and published them in 1795. The method was published by Monge himself in the same year, first in the form in which the short-hand writers took down his lessons given at the Normal School, where he had been elected professor, and then again, in revised form, in the *Journal des écoles normales*. The next edition occurred in 1798–1799. After an ephemeral existence of only four months the Normal School was closed in 1795. In the same year the Polytechnic School was opened, in the establishing of which Monge took active part. He taught there descriptive geometry until his departure from France to accompany Napoleon on the Egyptian campaign. He was the first president of the Institute of Egypt. Monge was a zealous partisan of Napoleon and was, for that reason, deprived of all his honours by Louis XVIII. This and the destruction of the Polytechnic School preyed heavily upon his mind. He did not long survive this insult.

Monge's numerous papers were by no means confined to descriptive geometry. His analytical discoveries are hardly less remarkable. He introduced into analytic geometry the methodic use of the equation of a line. He made important contributions to surfaces of the second degree (previously studied by Wren and Euler) and discovered between the theory of surfaces and the integration of partial differential equations, a hidden relation which threw new light upon both subjects. He gave the differential of curves of curvature, established a general theory of curvature, and applied it to the ellipsoid. He found that the validity of solutions was not impaired when imaginaries are involved among subsidiary quantities. Monge published the following books: *Statics*, 1786; *Applications de l'algèbre à la géométrie*, 1805; *Applica-*

tion de l'analyse à la géométrie. The last two contain most of his miscellaneous papers.

Monge was an inspiring teacher, and he gathered around him a large circle of pupils, among which were Dupin, Servois, Brianchon, Hachette, Biot, and Poncelet.

Charles Dupin (1784–1873), for many years professor of mechanics in the Conservatoire des Arts et Métiers in Paris, published in 1813 an important work on *Développements de géométrie*, in which is introduced the conception of conjugate tangents of a point of a surface, and of the indicatrix.[55] It contains also the theorem known as "Dupin's theorem." Surfaces of the second degree and descriptive geometry were successfully studied by *Jean Nicolas Pierre Hachette* (1769–1834), who became professor of descriptive geometry at the Polytechnic School after the departure of Monge for Rome and Egypt. In 1822 he published his *Traité de géométrie descriptive.*

Descriptive geometry, which arose, as we have seen, in technical schools in France, was transferred to Germany at the foundation of technical schools there. G. Schreiber, professor in Karlsruhe, was the first to spread Monge's geometry in Germany by the publication of a work thereon in 1828–1829.[54] In the United States descriptive geometry was introduced in 1816 at the Military Academy in West Point by Claude Crozet, once a pupil at the Polytechnic School in Paris. Crozet wrote the first English work on the subject.[2]

Lazare Nicholas Marguerite Carnot (1753–1823) was born at Nolay in Burgundy, and educated in his native province. He entered the army, but continued his mathematical studies, and wrote in 1784 a work on machines, containing the earliest proof that kinetic energy is lost in collisions of bodies. With the advent of the Revolution he threw himself into politics, and when coalesced Europe, in 1793, launched against France a million soldiers, the gigantic task of organising fourteen

armies to meet the enemy was achieved by him. He was banished in 1796 for opposing Napoleon's *coup d'état.* The refugee went to Geneva, where he issued, in 1797, a work still frequently quoted, entitled, *Réflexions sur la Métaphysique du Calcul Infinitésimal.* He declared himself as an "irreconcilable enemy of kings." After the Russian campaign he offered to fight for France, though not for the empire. On the restoration he was exiled. He died in Magdeburg. His *Géométrie de position,* 1803, and his *Essay on Transversals,* 1806, are important contributions to modern geometry. While Monge revelled mainly in three-dimensional geometry, Carnot confined himself to that of two. By his effort to explain the meaning of the negative sign in geometry he established a "geometry of position," which, however, is different from the "Geometrie der Lage" of to-day. He invented a class of general theorems on projective properties of figures, which have since been pushed to great extent by Poncelet, Chasles, and others.

Jean Victor Poncelet (1788–1867), a native of Metz, took part in the Russian campaign, was abandoned as dead on the bloody field of Krasnoi, and taken prisoner to Saratoff. Deprived there of all books, and reduced to the remembrance of what he had learned at the Lyceum at Metz and the Polytechnic School, where he had studied with predilection the works of Monge, Carnot, and Brianchon, he began to study mathematics from its elements. He entered upon original researches which afterwards made him illustrious. While in prison he did for mathematics what Bunyan did for literature, — produced a much-read work, which has remained of great value down to the present time. He returned to France in 1814, and in 1822 published the work in question, entitled, *Traité des Propriétés projectives des figures.* In it he investigated the properties of figures which remain un-

altered by projection of the figures. The projection is not effected here by parallel rays of prescribed direction, as with Monge, but by central projection. Thus perspective projection, used before him by Desargues, Pascal, Newton, and Lambert, was elevated by him into a fruitful geometric method. In the same way he elaborated some ideas of De Lahire, Servois, and Gergonne into a regular method — the method of "reciprocal polars." To him we owe the Law of Duality as a consequence of reciprocal polars. As an independent principle it is due to Gergonne. Poncelet wrote much on applied mechanics. In 1838 the Faculty of Sciences was enlarged by his election to the chair of mechanics.

While in France the school of Monge was creating modern geometry, efforts were made in England to revive Greek geometry by **Robert Simson** (1687–1768) and **Matthew Stewart** (1717–1785). Stewart was a pupil of Simson and Maclaurin, and succeeded the latter in the chair at Edinburgh. During the eighteenth century he and Maclaurin were the only prominent mathematicians in Great Britain. His genius was ill-directed by the fashion then prevalent in England to ignore higher analysis. In his *Four Tracts, Physical and Mathematical,* 1761, he applied geometry to the solution of difficult astronomical problems, which on the Continent were approached analytically with greater success. He published, in 1746, *General Theorems*, and in 1763, his *Propositiones geometricæ more veterum demonstratæ.* The former work contains sixty-nine theorems, of which only five are accompanied by demonstrations. It gives many interesting new results on the circle and the straight line. Stewart extended some theorems on transversals due to Giovanni Ceva (1648–1737), an Italian, who published in 1678 at Mediolani a work containing the theorem now known by his name.

RECENT TIMES.

NEVER more zealously and successfully has mathematics been cultivated than in this century. Nor has progress, as in previous periods, been confined to one or two countries. While the French and Swiss, who alone during the preceding epoch carried the torch of progress, have continued to develop mathematics with great success, from other countries whole armies of enthusiastic workers have wheeled into the front rank. Germany awoke from her lethargy by bringing forward Gauss, Jacobi, Dirichlet, and hosts of more recent men; Great Britain produced her De Morgan, Boole, Hamilton, besides champions who are still living; Russia entered the arena with her Lobatchewsky; Norway with Abel; Italy with Cremona; Hungary with her two Bolyais; the United States with Benjamin Peirce.

The productiveness of modern writers has been enormous. "It is difficult," says Professor Cayley,[56] "to give an idea of the vast extent of modern mathematics. This word 'extent' is not the right one: I mean extent crowded with beautiful detail, — not an extent of mere uniformity such as an objectless plain, but of a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower." It is pleasant to the mathematician to think that in his, as in no other science, the achievements of

every age remain possessions forever; new discoveries seldom disprove older tenets; seldom is anything lost or wasted.

If it be asked wherein the utility of some modern extensions of mathematics lies, it must be acknowledged that it is at present difficult to see how they are ever to become applicable to questions of common life or physical science. But our inability to do this should not be urged as an argument against the pursuit of such studies. In the first place, we know neither the day nor the hour when these abstract developments will find application in the mechanic arts, in physical science, or in other branches of mathematics. For example, the whole subject of graphical statics, so useful to the practical engineer, was made to rest upon von Staudt's *Geometrie der Lage;* Hamilton's "principle of varying action" has its use in astronomy; complex quantities, general integrals, and general theorems in integration offer advantages in the study of electricity and magnetism. "The utility of such researches," says Spottiswoode,[57] "can in no case be discounted, or even imagined beforehand. Who, for instance, would have supposed that the calculus of forms or the theory of substitutions would have thrown much light upon ordinary equations; or that Abelian functions and hyperelliptic transcendents would have told us anything about the properties of curves; or that the calculus of operations would have helped us in any way towards the figure of the earth?" A second reason in favour of the pursuit of advanced mathematics, even when there is no promise of practical application, is this, that mathematics, like poetry and music, deserves cultivation for its own sake.

The great characteristic of modern mathematics is its generalising tendency. Nowadays little weight is given to isolated theorems, "except as affording hints of an unsuspected new sphere of thought, like meteorites detached from some

undiscovered planetary orb of speculation." In mathematics, as in all true sciences, no subject is considered in itself alone, but always as related to, or an outgrowth of, other things. The development of the notion of continuity plays a leading part in modern research. In geometry the principle of continuity, the idea of correspondence, and the theory of projection constitute the fundamental modern notions. Continuity asserts itself in a most striking way in relation to the circular points at infinity in a plane. In algebra the modern idea finds expression in the theory of linear transformations and invariants, and in the recognition of the value of homogeneity and symmetry.

SYNTHETIC GEOMETRY.

The conflict between geometry and analysis which arose near the close of the last century and the beginning of the present has now come to an end. Neither side has come out victorious. The greatest strength is found to lie, not in the suppression of either, but in the friendly rivalry between the two, and in the stimulating influence of the one upon the other. Lagrange prided himself that in his *Mecanique Analytique* he had succeeded in avoiding all figures; but since his time mechanics has received much help from geometry.

Modern synthetic geometry was created by several investigators about the same time. It seemed to be the outgrowth of a desire for general methods which should serve as threads of Ariadne to guide the student through the labyrinth of theorems, corollaries, porisms, and problems. Synthetic geometry was first cultivated by Monge, Carnot, and Poncelet in France; it then bore rich fruits at the hands of Möbius and Steiner in Germany and Switzerland, and was finally developed to still

higher perfection by Chasles in France, von Staudt in Germany, and Cremona in Italy.

Augustus Ferdinand Möbius (1790–1868) was a native of Schulpforta in Prussia. He studied at Göttingen under Gauss, also at Leipzig and Halle. In Leipzig he became, in 1815, privat-docent, the next year extraordinary professor of astronomy, and in 1844 ordinary professor. This position he held till his death. The most important of his researches are on geometry. They appeared in *Crelle's Journal*, and in his celebrated work entitled *Der Barycentrische Calcul*, Leipzig, 1827. As the name indicates, this calculus is based upon properties of the centre of gravity.[58] Thus, that the point S is the centre of gravity of weights a, b, c, d placed at the points A, B, C, D respectively, is expressed by the equation

$$(a+b+c+d)S = aA + bB + cC + dD.$$

His calculus is the beginning of a quadruple algebra, and contains the germs of Grassmann's marvellous system. In designating segments of lines we find throughout this work for the first time consistency in the distinction of positive and negative by the order of letters AB, BA. Similarly for triangles and tetrahedra. The remark that it is always possible to give three points A, B, C such weights α, β, γ that any fourth point M in their plane will become a centre of mass, led Möbius to a new system of co-ordinates in which the position of a point was indicated by an equation, and that of a line by co-ordinates. By this algorithm he found by algebra many geometric theorems expressing mainly invariantal properties, — for example, the theorems on the anharmonic relation. Möbius wrote also on statics and astronomy. He generalised spherical trigonometry by letting the sides or angles of triangles exceed 180°.

Jacob Steiner (1796–1863), "the greatest geometrician since the time of Euclid," was born in Utzendorf in the Canton of Bern. He did not learn to write till he was fourteen. At eighteen he became a pupil of Pestalozzi. Later he studied at Heidelberg and Berlin. When Crelle started, in 1826, the celebrated mathematical journal bearing his name, Steiner and Abel became leading contributors. In 1832 Steiner published his *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander*, "in which is uncovered the organism by which the most diverse phenomena (*Erscheinungen*) in the world of space are united to each other." Through the influence of Jacobi and others, the chair of geometry was founded for him at Berlin in 1834. This position he occupied until his death, which occurred after years of bad health. In his *Systematische Entwickelungen*, for the first time, is the principle of duality introduced at the outset. This book and von Staudt's lay the foundation on which synthetic geometry in its present form rests. Not only did he fairly complete the theory of curves and surfaces of the second degree, but he made great advances in the theory of those of higher degrees. In his hands synthetic geometry made prodigious progress. New discoveries followed each other so rapidly that he often did not take time to record their demonstrations. In an article in *Crelle's Journal* on *Allgemeine Eigenschaften Algebraischer Curven* he gives without proof theorems which were declared by Hesse to be "like Fermat's theorems, riddles to the present and future generations." Analytical proofs of some of them have been given since by others, but Cremona finally proved them all by a synthetic method. Steiner discovered synthetically the two prominent properties of a surface of the third order; viz. that it contains twenty-seven straight lines and a pentahedron which has the double points for its vertices and the lines of the Hessian of the given sur-

face for its edges.[55] The first property was discovered analytically somewhat earlier in England by Cayley and Salmon, and the second by Sylvester. Steiner's work on this subject was the starting-point of important researches by H. Schröter, F. August, L. Cremona, and R. Sturm. Steiner made investigations by synthetic methods on maxima and minima, and arrived at the solution of problems which at that time altogether surpassed the analytic power of the calculus of variations. He generalised the *hexagrammum mysticum* and also Malfatti's problem.[59] Malfatti, in 1803, proposed the problem, to cut three cylindrical holes out of a three-sided prism in such a way that the cylinders and the prism have the same altitude and that the volume of the cylinders be a maximum. This problem was reduced to another, now generally known as Malfatti's problem: to inscribe three circles in a triangle that each circle will be tangent to two sides of a triangle and to the other two circles. Malfatti gave an analytical solution, but Steiner gave without proof a construction, remarked that there were thirty-two solutions, generalised the problem by replacing the three lines by three circles, and solved the analogous problem for three dimensions. This general prob lem was solved analytically by C. H. Schellbach (1809–1892) and Cayley, and by Clebsch with the aid of the addition theorem of elliptic functions.[60]

Steiner's researches are confined to synthetic geometry. He hated analysis as thoroughly as Lagrange disliked geometry. Steiner's *Gesammelte Werke* were published in Berlin in 1881 and 1882.

Michel Chasles (1793–1880) was born at Epernon, entered the Polytechnic School of Paris in 1812, engaged afterwards in business, which he later gave up that he might devote all his time to scientific pursuits. In 1841 he became professor of geodesy and mechanics at the Polytechnic School; later,

"Professeur de Géométrie supérieure à la Faculté des Sciences de Paris." He was a voluminous writer on geometrical subjects. In 1837 he published his admirable *Aperçu historique sur l'origine et le développement des méthodes en géométrie,* containing a history of geometry and, as an appendix, a treatise "sur deux principes généraux de la Science." The *Aperçu historique* is still a standard historical work; the appendix contains the general theory of Homography (Collineation) and of duality (Reciprocity). The name *duality* is due to Joseph Diaz Gergonne (1771–1859). Chasles introduced the term *anharmonic ratio,* corresponding to the German *Doppelverhältniss* and to Clifford's *cross-ratio.* Chasles and Steiner elaborated independently the modern synthetic or projective geometry. Numerous original memoirs of Chasles were published later in the *Journal de l'École Polytechnique.* He gave a reduction of cubics, different from Newton's in this, that the five curves from which all others can be projected are symmetrical with respect to a centre. In 1864 he began the publication, in the *Comptes rendus,* of articles in which he solves by his "method of characteristics" and the "principle of correspondence" an immense number of problems. He determined, for instance, the number of intersections of two curves in a plane. The method of characteristics contains the basis of enumerative geometry. The application of the principle of correspondence was extended by Cayley, A. Brill, H. G. Zeuthen, H. A. Schwarz, G. H. Halphen (1844–1889), and others. The full value of these principles of Chasles was not brought out until the appearance, in 1879, of the *Kalkül der Abzählenden Geometrie* by Hermann Schubert of Hamburg. This work contains a masterly discussion of the problem of enumerative geometry, viz. to determine how many geometric figures of given definition satisfy a sufficient number of conditions. Schubert extended his enumerative geometry to n-dimensional space.[55]

To Chasles we owe the introduction into projective geometry of non-projective properties of figures by means of the infinitely distant imaginary sphero-circle.[61] Remarkable is his complete solution, in 1846, by synthetic geometry, of the difficult question of the attraction of an ellipsoid on an external point. This was accomplished analytically by Poisson in 1835. The labours of Chasles and Steiner raised synthetic geometry to an honoured and respected position by the side of analysis.

Karl Georg Christian von Staudt (1798–1867) was born in Rothenburg on the Tauber, and, at his death, was professor in Erlangen. His great works are the *Geometrie der Lage*, Nürnberg, 1847, and his *Beiträge zur Geometrie der Lage*, 1856–1860. The author cut loose from algebraic formulæ and from metrical relations, particularly the anharmonic ratio of Steiner and Chasles, and then created a geometry of position, which is a complete science in itself, independent of all measurements. He shows that projective properties of figures have no dependence whatever on measurements, and can be established without any mention of them. In his theory of what he calls "Würfe," he even gives a geometrical definition of a number in its relation to geometry as determining the position of a point. The *Beiträge* contains the first complete and general theory of imaginary points, lines, and planes in projective geometry. Representation of an imaginary point is sought in the combination of an involution with a determinate direction, both on the real line through the point. While purely projective, von Staudt's method is intimately related to the problem of representing by actual points and lines the imaginaries of analytical geometry. This was systematically undertaken by C. F. Maximilien Marie, who worked, however, on entirely different lines. An independent attempt has been made recently (1893) by F. H. Loud of Colorado

College. Von Staudt's geometry of position was for a long time disregarded, mainly, no doubt, because his book is extremely condensed. An impulse to the study of this subject was given by Culmann, who rests his graphical statics upon the work of von Staudt. An interpreter of von Staudt was at last found in Theodor Reye of Strassburg, who wrote a *Geometrie der Lage* in 1868.

Synthetic geometry has been studied with much success by **Luigi Cremona,** professor in the University of Rome. In his *Introduzione ad una teoria geometrica delle curve piane* he developed by a uniform method many new results and proved synthetically all important results reached before that time by analysis. His writings have been translated into German by M. Curtze, professor at the gymnasium in Thorn. The theory of the transformation of curves and of the correspondence of points on curves was extended by him to three dimensions. Ruled surfaces, surfaces of the second order, space-curves of the third order, and the general theory of surfaces have received much attention at his hands.

Karl Culmann, professor at the Polytechnicum in Zürich, published an epoch-making work on *Die graphische Statik,* Zürich, 1864, which has rendered graphical statics a great rival of analytical statics. Before Culmann, *B. E. Cousinery* had turned his attention to the graphical calculus, but he made use of perspective, and not of modern geometry.[62] Culmann is the first to undertake to present the graphical calculus as a symmetrical whole, holding the same relation to the new geometry that analytical mechanics does to higher analysis. He makes use of the polar theory of reciprocal figures as expressing the relation between the force and the funicular polygons. He deduces this relation without leaving the plane of the two figures. But if the polygons be regarded as projections of lines in space, these lines may be treated as recipro-

cal elements of a "Nullsystem." This was done by *Clerk Maxwell* in 1864, and elaborated further by *Cremona.*[63] The graphical calculus has been applied by *O. Mohr* of Dresden to the elastic line for continuous spans. *Henry T. Eddy*, of the Rose Polytechnic Institute, gives graphical solutions of problems on the maximum stresses in bridges under concentrated loads, with aid of what he calls "reaction polygons." A standard work, *La Statique graphique*, 1874, was issued by Maurice Levy of Paris.

Descriptive geometry (reduced to a science by Monge in France, and elaborated further by his successors, *Hachette, Dupin, Olivier, J. de la Gournerie*) was soon studied also in other countries. The French directed their attention mainly to the theory of surfaces and their curvature; the Germans and Swiss, through Schreiber, Pohlke, Schlessinger, and particularly Fiedler, interwove projective and descriptive geometry. Bellavitis in Italy worked along the same line. The theory of shades and shadows was first investigated by the French writers just quoted, and in Germany treated most exhaustively by Burmester.[62]

During the present century very remarkable generalisations have been made, which reach to the very root of two of the oldest branches of mathematics, — elementary algebra and geometry. In algebra the laws of operation have been extended; in geometry the axioms have been searched to the bottom, and the conclusion has been reached that the space defined by Euclid's axioms is not the only possible non-contradictory space. Euclid proved (I. 27) that "if a straight line falling on two other straight lines make the alternate angles equal to one another, the two straight lines shall be parallel to one another." Being unable to prove that in every other case the two lines are not parallel, he *assumed* this to be true in what is generally called the 12th "axiom," by some

the 11th "axiom." But this so-called axiom is far from axiomatic. After centuries of desperate but fruitless attempts to prove Euclid's assumption, the bold idea dawned upon the minds of several mathematicians that a geometry might be built up without assuming the parallel-axiom. While Legendre still endeavoured to establish the axiom by rigid proof, Lobatchewsky brought out a publication which assumed the contradictory of that axiom, and which was the first of a series of articles destined to clear up obscurities in the fundamental concepts, and to greatly extend the field of geometry.

Nicholaus Ivanovitch Lobatchewsky (1793–1856) was born at Makarief, in Nischni-Nowgorod, Russia, studied at Kasan, and from 1827 to 1846 was professor and rector of the University of Kasan. His views on the foundation of geometry were first made public in a discourse before the physical and mathematical faculty at Kasan, and first printed in the Kasan *Messenger* for 1829, and then in the *Gelehrte Schriften der Universität Kasan*, 1836–1838, under the title, "New Elements of Geometry, with a complete theory of Parallels." Being in the Russian language, the work remained unknown to foreigners, but even at home it attracted no notice. In 1840 he published a brief statement of his researches in Berlin. Lobatchewsky constructed an "imaginary geometry," as he called it, which has been described by Clifford as "quite simple, merely Euclid without the vicious assumption." A remarkable part of this geometry is this, that through a point an indefinite number of lines can be drawn in a plane, none of which cut a given line in the same plane. A similar system of geometry was deduced independently by the Bolyais in Hungary, who called it "absolute geometry."

Wolfgang Bolyai de Bolya (1775–1856) was born in Szekler-Land, Transylvania. After studying at Jena, he went to

Göttingen, where he became intimate with Gauss, then nineteen years old. Gauss used to say that Bolyai was the only man who fully understood his views on the metaphysics of mathematics. Bolyai became professor at the Reformed College of Maros-Vásárhely, where for forty-seven years he had for his pupils most of the present professors of Transylvania. The first publications of this remarkable genius were dramas and poetry. Clad in old-time planter's garb, he was truly original in his private life as well as in his mode of thinking. He was extremely modest. No monument, said he, should stand over his grave, only an apple-tree, in memory of the three apples; the two of Eve and Paris, which made hell out of earth, and that of Newton, which elevated the earth again into the circle of heavenly bodies.[64] His son, **Johann Bolyai** (1802–1860), was educated for the army, and distinguished himself as a profound mathematician, an impassioned violin-player, and an expert fencer. He once accepted the challenge of thirteen officers on condition that after each duel he might play a piece on his violin, and he vanquished them all.

The chief mathematical work of Wolfgang Bolyai appeared in two volumes, 1832–1833, entitled *Tentamen juventutem studiosam in elementa matheseos purœ . . . introducendi.* It is followed by an appendix composed by his son Johann on *The Science Absolute of Space.* Its twenty-six pages make the name of Johann Bolyai immortal. He published nothing else, but he left behind one thousand pages of manuscript which have never been read by a competent mathematician! His father seems to have been the only person in Hungary who really appreciated the merits of his son's work. For thirty-five years this appendix, as also Lobatchewsky's researches, remained in almost entire oblivion. Finally Richard Baltzer of the University of Giessen, in 1867, called attention to the wonderful researches. Johann Bolyai's *Science Absolute of*

Space and Lobatchewsky's *Geometrical Researches on the Theory of Parallels* (1840) were rendered easily accessible to American readers by translations into English made in 1891 by George Bruce Halsted of the University of Texas.

The Russian and Hungarian mathematicians were not the only ones to whom pangeometry suggested itself. A copy of the *Tentamen* reached Gauss, the elder Bolyai's former roommate at Göttingen, and this Nestor of German mathematicians was surprised to discover in it worked out what he himself had begun long before, only to leave it after him in his papers. As early as 1792 he had started on researches of that character. His letters show that in 1799 he was trying to prove *a priori* the reality of Euclid's system; but some time within the next thirty years he arrived at the conclusion reached by Lobatchewsky and Bolyai. In 1829 he wrote to Bessel, stating that his "conviction that we cannot found geometry completely *a priori* has become, if possible, still firmer," and that "if number is merely a product of our mind, space has also a *reality beyond* our mind of which we cannot fully foreordain the laws *a priori*." The term *non-Euclidean geometry* is due to Gauss. It has recently been brought to notice that *Geronimo Saccheri*, a Jesuit father of Milan, in 1733 anticipated Lobatchewsky's doctrine of the parallel angle. Moreover, G. B. Halsted has pointed out that in 1766 Lambert wrote a paper "Zur Theorie der Parallellinien," published in the *Leipziger Magazin für reine und angewandte Mathematik*, 1786, in which: (1) The failure of the parallel-axiom in surface-spherics gives a geometry with angle-sum > 2 right angles; (2) In order to make intuitive a geometry with angle-sum < 2 right angles we need the aid of an "imaginary sphere" (pseudo-sphere); (3) In a space with the angle-sum differing from 2 right angles, there is an absolute measure (Bolyai's natural unit for length).

In 1854, nearly twenty years later, Gauss heard from his pupil, *Riemann*, a marvellous dissertation carrying the discussion one step further by developing the notion of n-ply extended magnitude, and the measure-relations of which a manifoldness of n dimensions is capable, on the assumption that every line may be measured by every other. Riemann applied his ideas to space. He taught us to distinguish between "unboundedness" and "infinite extent." According to him we have in our mind a more general notion of space, *i.e.* a notion of non-Euclidean space; but we learn *by experience* that our physical space is, if not exactly, at least to high degree of approximation, Euclidean space. Riemann's profound dissertation was not published until 1867, when it appeared in the *Göttingen Abhandlungen*. Before this the idea of n dimensions had suggested itself under various aspects to Lagrange, Plücker, and H. Grassmann. About the same time with Riemann's paper, others were published from the pens of *Helmholtz* and *Beltrami*. These contributed powerfully to the victory of logic over excessive empiricism. This period marks the beginning of lively discussions upon this subject. Some writers — Bellavitis, for example — were able to see in non-Euclidean geometry and n-dimensional space nothing but huge caricatures, or diseased outgrowths of mathematics. Helmholtz's article was entitled *Thatsachen, welche der Geometrie zu Grunde liegen*, 1868, and contained many of the ideas of Riemann. Helmholtz popularised the subject in lectures, and in articles for various magazines.

Eugenio Beltrami, born at Cremona, Italy, in 1835, and now professor at Rome, wrote the classical paper *Saggio di interpretazione della geometria non-euclidea* (*Giorn. di Matem.*, 6), which is analytical (and, like several other papers, should be mentioned elsewhere were we to adhere to a strict separation between synthesis and analysis). He reached the brilliant

and surprising conclusion that the theorems of non-Euclidean geometry find their realisation upon surfaces of constant negative curvature. He studied, also, surfaces of constant positive curvature, and ended with the interesting theorem that the space of constant positive curvature is contained in the space of constant negative curvature. These researches of Beltrami, Helmholtz, and Riemann culminated in the conclusion that on surfaces of constant curvature we may have three geometries, — the non-Euclidean on a surface of constant negative curvature, the spherical on a surface of constant positive curvature, and the Euclidean geometry on a surface of zero curvature. The three geometries do not contradict each other, but are members of a system, — a geometrical trinity. The ideas of hyper-space were brilliantly expounded and popularised in England by Clifford.

William Kingdon Clifford (1845–1879) was born at Exeter, educated at Trinity College, Cambridge, and from 1871 until his death professor of applied mathematics in University College, London. His premature death left incomplete several brilliant researches which he had entered upon. Among these are his paper *On Classification of Loci* and his *Theory of Graphs.* He wrote articles *On the Canonical Form and Dissection of a Riemann's Surface,* on *Biquaternions,* and an incomplete work on the *Elements of Dynamic.* The theory of polars of curves and surfaces was generalised by him and by Reye. His classification of loci, 1878, being a general study of curves, was an introduction to the study of n-dimensional space in a direction mainly projective. This study has been continued since chiefly by G. Veronese of Padua, C. Segre of Turin, E. Bertini, F. Aschieri, P. Del Pezzo of Naples.

Beltrami's researches on non-Euclidean geometry were followed, in 1871, by important investigations of Felix Klein,

resting upon Cayley's *Sixth Memoir on Quantics*, 1859. The question whether it is not possible to so express the metrical properties of figures that they will not vary by projection (or linear transformation) had been solved for special projections by Chasles, Poncelet, and E. Laguerre (1834–1886) of Paris, but it remained for Cayley to give a general solution by defining the distance between two points as an arbitrary constant multiplied by the logarithm of the anharmonic ratio in which the line joining the two points is divided by the fundamental quadric. Enlarging upon this notion, Klein showed the independence of projective geometry from the parallel-axiom, and by properly choosing the law of the measurement of distance deduced from projective geometry the spherical, Euclidean, and pseudospherical geometries, named by him respectively the elliptic, parabolic, and hyperbolic geometries. This suggestive investigation was followed up by numerous writers, particularly by G. Battaglini of Naples, E. d' Ovidio of Turin, R. de Paolis of Pisa, F. Aschieri, A. Cayley, F. Lindemann of Munich, E. Schering of Göttingen, W. Story of Clark University, H. Stahl of Tübingen, A. Voss of Würzburg, Homersham Cox, A. Buchheim.[55] The geometry of n dimensions was studied along a line mainly metrical by a host of writers, among whom may be mentioned Simon Newcomb of the Johns Hopkins University, L. Schläfli of Bern, W. I. Stringham of the University of California, W. Killing of Münster, T. Craig of the Johns Hopkins, R. Lipschitz of Bonn. R. S. Heath and Killing investigated the kinematics and mechanics of such a space. Regular solids in n-dimensional space were studied by Stringham, Ellery W. Davis of the University of Nebraska, R. Hoppe of Berlin, and others. Stringham gave pictures of projections upon our space of regular solids in four dimensions, and Schlegel at Hagen constructed models of such projections. These are

among the most curious of a series of models published by L. Brill in Darmstadt. It has been pointed out that if a fourth dimension existed, certain motions could take place which we hold to be impossible. Thus Newcomb showed the possibility of turning a closed material shell inside out by simple flexure without either stretching or tearing; Klein pointed out that knots could not be tied; Veronese showed that a body could be removed from a closed room without breaking the walls; C. S. Peirce proved that a body in four-fold space either rotates about two axes at once, or cannot rotate without losing one of its dimensions.

ANALYTIC GEOMETRY.

In the preceding chapter we endeavoured to give a flash-light view of the rapid advance of synthetic geometry. In connection with hyperspace we also mentioned analytical treatises. Modern synthetic and modern analytical geometry have much in common, and may be grouped together under the common name "projective geometry." Each has advantages over the other. The continual direct viewing of figures as existing in space adds exceptional charm to the study of the former, but the latter has the advantage in this, that a well-established routine in a certain degree may outrun thought itself, and thereby aid original research. While in Germany Steiner and von Staudt developed synthetic geometry, Plücker laid the foundation of modern analytic geometry.

Julius Plücker (1801–1868) was born at Elberfeld, in Prussia. After studying at Bonn, Berlin, and Heidelberg, he spent a short time in Paris attending lectures of Monge and his pupils. Between 1826 and 1836 he held positions successively at Bonn, Berlin, and Halle. He then became professor of

physics at Bonn. Until 1846 his original researches were on geometry. In 1828 and in 1831 he published his *Analytisch-Geometrische Entwicklungen* in two volumes. Therein he adopted the abbreviated notation (used before him in a more restricted way by Bobillier), and avoided the tedious process of algebraic elimination by a geometric consideration. In the second volume the principle of duality is formulated analytically. With him duality and homogeneity found expression already in his system of co-ordinates. The homogenous or tri-linear system used by him is much the same as the co-ordinates of Möbius. In the identity of analytical operation and geometric construction Plücker looked for the source of his proofs. The *System der Analytischen Geometrie*, 1835, contains a complete classification of plane curves of the third order, based on the nature of the points at infinity. The *Theorie der Algebraischen Curven*, 1839, contains, besides an enumeration of curves of the fourth order, the analytic relations between the ordinary singularities of plane curves known as "Plücker's equations," by which he was able to explain "Poncelet's paradox." The discovery of these relations is, says Cayley, "the most important one beyond all comparison in the entire subject of modern geometry." But in Germany Plücker's researches met with no favour. His method was declared to be unproductive as compared with the synthetic method of Steiner and Poncelet! His relations with Jacobi were not altogether friendly. Steiner once declared that he would stop writing for *Crelle's Journal* if Plücker continued to contribute to it.[66] The result was that many of Plücker's researches were published in foreign journals, and that his work came to be better known in France and England than in his native country. The charge was also brought against Plücker that, though occupying the chair of physics, he was no physicist. This induced him to relin-

quish mathematics, and for nearly twenty years to devote his energies to physics. Important discoveries on Fresnel's wave-surface, magnetism, spectrum-analysis were made by him. But towards the close of his life he returned to his first love, — mathematics, — and enriched it with new discoveries. By considering space as made up of lines he created a "new geometry of space." Regarding a right line as a curve involving four arbitrary parameters, one has the whole system of lines in space. By connecting them by a single relation, he got a "complex" of lines; by connecting them with a twofold relation, he got a "congruency" of lines. His first researches on this subject were laid before the Royal Society in 1865. His further investigations thereon appeared in 1868 in a posthumous work entitled *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement*, edited by Felix Klein. Plücker's analysis lacks the elegance found in Lagrange, Jacobi, Hesse, and Clebsch. For many years he had not kept up with the progress of geometry, so that many investigations in his last work had already received more general treatment on the part of others. The work contained, nevertheless, much that was fresh and original. The theory of complexes of the second degree, left unfinished by Plücker, was continued by Felix Klein, who greatly extended and supplemented the ideas of his master.

Ludwig Otto Hesse (1811–1874) was born at Königsberg, and studied at the university of his native place under Bessel, Jacobi, Richelot, and F. Neumann. Having taken the doctor's degree in 1840, he became docent at Königsberg, and in 1845 extraordinary professor there. Among his pupils at that time were Durège, Carl Neumann, Clebsch, Kirchhoff. The Königsberg period was one of great activity for Hesse. Every new discovery increased his zeal for still greater achievement. His earliest researches were on surfaces of the second order,

and were partly synthetic. He solved the problem to construct any tenth point of such a surface when nine points are given. The analogous problem for a conic had been solved by Pascal by means of the hexagram. A difficult problem confronting mathematicians of this time was that of elimination. Plücker had seen that the main advantage of his special method in analytic geometry lay in the avoidance of algebraic elimination. Hesse, however, showed how by determinants to make algebraic elimination easy. In his earlier results he was anticipated by Sylvester, who published his dialytic method of elimination in 1840. These advances in algebra Hesse applied to the analytic study of curves of the third order. By linear substitutions, he reduced a form of the third degree in three variables to one of only four terms, and was led to an important determinant involving the second differential coefficient of a form of the third degree, called the "Hessian." The "Hessian" plays a leading part in the theory of invariants, a subject first studied by Cayley. Hesse showed that his determinant gives for every curve another curve, such that the double points of the first are points on the second, or "Hessian." Similarly for surfaces (Crelle, 1844). Many of the most important theorems on curves of the third order are due to Hesse. He determined the curve of the 14th order, which passes through the 56 points of contact of the 28 bitangents of a curve of the fourth order. His great memoir on this subject (Crelle, 1855) was published at the same time as was a paper by Steiner treating of the same subject.

Hesse's income at Königsberg had not kept pace with his growing reputation. Hardly was he able to support himself and family. In 1855 he accepted a more lucrative position at Halle, and in 1856 one at Heidelberg. Here he remained until 1868, when he accepted a position at a technic school in Munich.[67] At Heidelberg he revised and enlarged upon his

previous researches, and published in 1861 his *Vorlesungen über die Analytische Geometrie des Raumes, insbesondere über Flächen 2. Ordnung.* More elementary works soon followed. While in Heidelberg he elaborated a principle, his "Uebertragungsprincip." According to this, there corresponds to every point in a plane a pair of points in a line, and the projective geometry of the plane can be carried back to the geometry of points in a line.

The researches of Plücker and Hesse were continued in England by Cayley, Salmon, and Sylvester. It may be premised here that among the early writers on analytical geometry in England was **James Booth** (1806–1878), whose chief results are embodied in his *Treatise on Some New Geometrical Methods;* and **James MacCullagh** (1809–1846), who was professor of natural philosophy at Dublin, and made some valuable discoveries on the theory of quadrics. The influence of these men on the progress of geometry was insignificant, for the interchange of scientific results between different nations was not so complete at that time as might have been desired. In further illustration of this, we mention that Chasles in France elaborated subjects which had previously been disposed of by Steiner in Germany, and Steiner published researches which had been given by Cayley, Sylvester, and Salmon nearly five years earlier. Cayley and Salmon in 1849 determined the straight lines in a cubic surface, and studied its principal properties, while Sylvester in 1851 discovered the pentahedron of such a surface. Cayley extended Plücker's equations to curves of higher singularities. Cayley's own investigations, and those of M. Nöther of Erlangen, G. H. Halphen (1844–1889) of the Polytechnic School in Paris, De La Gournérie of Paris, A. Brill of Tübingen, lead to the conclusion that each higher singularity of a curve is equivalent to a certain number of simple singularities, — the node, the ordinary cusp, the double tangent,

and the inflection. Sylvester studied the "twisted Cartesian," a curve of the fourth order. Salmon helped powerfully towards the spreading of a knowledge of the new algebraic and geometric methods by the publication of an excellent series of text-books (*Conic Sections, Modern Higher Algebra, Higher Plane Curves, Geometry of Three Dimensions*), which have been placed within easy reach of German readers by a free translation, with additions, made by Wilhelm Fiedler of the Polytechnicum in Zürich. The next great worker in the field of analytic geometry was Clebsch.

Rudolf Friedrich Alfred Clebsch (1833–1872) was born at Königsberg in Prussia, studied at the university of that place under Hesse, Richelot, F. Neumann. From 1858 to 1863 he held the chair of theoretical mechanics at the Polytechnicum in Carlsruhe. The study of Salmon's works led him into algebra and geometry. In 1863 he accepted a position at the University of Giesen, where he worked in conjunction with Paul Gordan (now of Erlangen). In 1868 Clebsch went to Göttingen, and remained there until his death. He worked successively at the following subjects: Mathematical physics, the calculus of variations and partial differential equations of the first order, the general theory of curves and surfaces, Abelian functions and their use in geometry, the theory of invariants, and "Flächenabbildung."[68] He proved theorems on the pentahedron enunciated by Sylvester and Steiner; he made systematic use of "deficiency" (*Geschlecht*) as a fundamental principle in the classification of algebraic curves. The notion of deficiency was known before him to Abel and Riemann. At the beginning of his career, Clebsch had shown how elliptic functions could be advantageously applied to Malfatti's problem. The idea involved therein, viz. the use of higher transcendentals in the study of geometry, led him to his greatest discoveries. Not only did he apply Abelian

functions to geometry, but conversely, he drew geometry into the service of Abelian functions.

Clebsch made liberal use of determinants. His study of curves and surfaces began with the determination of the points of contact of lines which meet a surface in four consecutive points. Salmon had proved that these points lie on the intersection of the surface with a derived surface of the degree $11n - 24$, but his solution was given in inconvenient form. Clebsch's investigation thereon is a most beautiful piece of analysis.

The representation of one surface upon another (*Flächenabbildung*), so that they have a (1, 1) correspondence, was thoroughly studied for the first time by Clebsch. The representation of a sphere on a plane is an old problem which drew the attention of Ptolemæus, Gerard Mercator, Lambert, Gauss, Lagrange. Its importance in the construction of maps is obvious. Gauss was the first to represent a surface upon another with a view of more easily arriving at its properties. Plücker, Chasles, Cayley, thus represented on a plane the geometry of quadric surfaces; Clebsch and Cremona, that of cubic surfaces. Other surfaces have been studied in the same way by recent writers, particularly M. Nöther of Erlangen, Armenante, Felix Klein, Korndörfer, Caporali, H. G. Zeuthen of Copenhagen. A fundamental question which has as yet received only a partial answer is this: What surfaces can be represented by a (1, 1) correspondence upon a given surface? This and the analogous question for curves was studied by Clebsch. Higher correspondences between surfaces have been investigated by Cayley and Nöther. The theory of surfaces has been studied also by **Joseph Alfred Serret** (1819–1885), professor at the Sorbonne in Paris, **Jean Gaston Darboux** of Paris, *John Casey* of Dublin (died 1891), *W. R. W. Roberts* of Dublin, *H. Schröter* (1829–1892) of Breslau. Surfaces of the

fourth order were investigated by Kummer, and Fresnel's wave-surface, studied by Hamilton, is a particular case of Kummer's quartic surface, with sixteen canonical points and sixteen singular tangent planes.[56]

The infinitesimal calculus was first applied to the determination of the measure of curvature of surfaces by Lagrange, Euler, and Meunier (1754–1793) of Paris. Then followed the researches of Monge and Dupin, but they were eclipsed by the work of Gauss, who disposed of this difficult subject in a way that opened new vistas to geometricians. His treatment is embodied in the *Disquisitiones generales circa superficies curvas* (1827) and *Untersuchungen über gegenstände der höheren Geodäsie* of 1843 and 1846. He defined the measure of curvature at a point to be the reciprocal of the product of the two principal radii of curvature at that point. From this flows the theorem of *Johann August Grunert* (1797–1872; professor in Greifswald), that the arithmetical mean of the radii of curvature of all normal sections through a point is the radius of a sphere which has the same measure of curvature as has the surface at that point. Gauss's deduction of the formula of curvature was simplified through the use of determinants by *Heinrich Richard Baltzer* (1818–1887) of Giessen.[69] Gauss obtained an interesting theorem that if one surface be developed (*abgewickelt*) upon another, the measure of curvature remains unaltered at each point. The question whether two surfaces having the same curvature in corresponding points can be unwound, one upon the other, was answered by F. Minding in the affirmative only when the curvature is constant. The case of variable curvature is difficult, and was studied by Minding, J. Liouville (1806–1882) of the Polytechnic School in Paris, Ossian Bonnet of Paris (died 1892). Gauss's measure of curvature, expressed as a function of curvilinear co-ordinates, gave an impetus to the study of differ-

ential-invariants, or differential-parameters, which have been investigated by Jacobi, C. Neumann, Sir James Cockle, Halphen, and elaborated into a general theory by Beltrami, S. Lie, and others. Beltrami showed also the connection between the measure of curvature and the geometric axioms.

Various researches have been brought under the head of "analysis situs." The subject was first investigated by Leibniz, and was later treated by Gauss, whose theory of knots (*Verschlingungen*) has been employed recently by J. B. Listing, O. Simony, F. Dingeldey, and others in their "topologic studies." Tait was led to the study of knots by Sir William Thomson's theory of vortex atoms. In the hands of Riemann the analysis situs had for its object the determination of what remains unchanged under transformations brought about by a combination of infinitesimal distortions. In continuation of his work, Walter Dyck of Munich wrote on the analysis situs of three-dimensional spaces.

Of geometrical text-books not yet mentioned, reference should be made to **Alfred Clebsch's** *Vorlesungen über Geometrie*, edited by Ferdinand Lindemann, now of Munich; **Frost's** *Solid Geometry;* **Durège's** *Ebene Curven dritter Ordnung.*

ALGEBRA.

The progress of algebra in recent times may be considered under three principal heads: the study of fundamental laws and the birth of new algebras, the growth of the theory of equations, and the development of what is called modern higher algebra.

We have already spoken of George Peacock and D. F. Gregory in connection with the fundamental laws of algebra. Much was done in this line by De Morgan.

Augustus De Morgan (1806–1871) was born at Madura (Madras), and educated at Trinity College, Cambridge. His scruples about the doctrines of the established church prevented him from proceeding to the M.A. degree, and from sitting for a fellowship. In 1828 he became professor at the newly established University of London, and taught there until 1867, except for five years, from 1831–1835. De Morgan was a unique, manly character, and pre-eminent as a teacher. The value of his original work lies not so much in increasing our stock of mathematical knowledge as in putting it all upon a thoroughly logical basis. He felt keenly the lack of close reasoning in mathematics as he received it. He said once: "We know that mathematicians care no more for logic than logicians for mathematics. The two eyes of exact science are mathematics and logic: the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye; each believing that it can see better with one eye than with two." De Morgan saw with both eyes. He analysed logic mathematically, and studied the logical analysis of the laws, symbols, and operations of mathematics; he wrote a *Formal Logic* as well as a *Double Algebra*, and corresponded both with Sir William Hamilton, the metaphysician, and Sir William Rowan Hamilton, the mathematician. Few contemporaries were as profoundly read in the history of mathematics as was De Morgan. No subject was too insignificant to receive his attention. The authorship of "Cocker's Arithmetic" and the work of circle-squarers was investigated as minutely as was the history of the invention of the calculus. Numerous articles of his lie scattered in the volumes of the *Penny* and *English Cyclopædias*. His *Differential Calculus*, 1842, is still a standard work, and contains much that is original with the author. For the *Encyclopædia Metropolitana* he wrote on the calculus of functions (giving principles of symbolic reasoning)

and on the theory of probability. Celebrated is his *Budget of Paradoxes*, 1872. He published memoirs "On the Foundation of Algebra" (*Trans. of Cam. Phil. Soc.*, 1841, 1842, 1844, and 1847).

In Germany symbolical algebra was studied by Martin Ohm, who wrote a *System der Mathematik* in 1822. The ideas of Peacock and De Morgan recognise the possibility of algebras which differ from ordinary algebra. Such algebras were indeed not slow in forthcoming, but, like non-Euclidean geometry, some of them were slow in finding recognition. This is true of Grassmann's, Bellavitis's, and Peirce's discoveries, but Hamilton's quaternions met with immediate appreciation in England. These algebras offer a geometrical interpretation of imaginaries. During the times of Descartes, Newton, and Euler, we have seen the negative and the imaginary, $\sqrt{-1}$, accepted as numbers, but the latter was still regarded as an algebraic fiction. The first to give it a geometric picture, analogous to the geometric interpretation of the negative, was *H. Kühn*, a teacher in Danzig, in a publication of 1750–1751. He represented $a\sqrt{-1}$ by a line perpendicular to the line a, and equal to a in length, and construed $\sqrt{-1}$ as the mean proportional between $+1$ and -1. This same idea was developed further, so as to give a geometric interpretation of $a+\sqrt{-b}$, by *Jean-Robert Argand* (1768–?) of Geneva, in a remarkable *Essai* (1806).[70] The writings of Kühn and Argand were little noticed, and it remained for Gauss to break down the last opposition to the imaginary. He introduced i as an independent unit co-ordinate to 1, and $a+ib$ as a "complex number." The connection between complex numbers and points on a plane, though artificial, constituted a powerful aid in the further study of symbolic algebra. The mind required a visual representation to aid it. The notion of what we now call vectors was growing upon mathematicians,

and the geometric addition of vectors in space was discovered independently by Hamilton, Grassmann, and others, about the same time.

William Rowan Hamilton (1805–1865) was born of Scotch parents in Dublin. His early education, carried on at home, was mainly in languages. At the age of thirteen he is said to have been familiar with as many languages as he had lived years. About this time he came across a copy of Newton's *Universal Arithmetic.* After reading that, he took up successively analytical geometry, the calculus, Newton's *Principia*, Laplace's *Mécanique Céleste.* At the age of eighteen he published a paper correcting a mistake in Laplace's work. In 1824 he entered Trinity College, Dublin, and in 1827, while he was still an undergraduate, he was appointed to the chair of astronomy. His early papers were on optics. In 1832 he predicted conical refraction, a discovery by aid of mathematics which ranks with the discovery of Neptune by Le Verrier and Adams. Then followed papers on the *Principle of Varying Action* (1827) and a general method of dynamics (1834–1835). He wrote also on the solution of equations of the fifth degree, the hodograph, fluctuating functions, the numerical solution of differential equations.

The capital discovery of Hamilton is his quaternions, in which his study of algebra culminated. In 1835 he published in the *Transactions of the Royal Irish Academy* his Theory of Algebraic Couples. He regarded algebra "as being no mere art, nor language, nor primarily a science of quantity, but rather as the science of order of progression." Time appeared to him as the picture of such a progression. Hence his definition of algebra as "the science of pure time." It was the subject of years' meditation for him to determine what he should regard as the product of each pair of a system of perpendicular directed lines. At last, on the 16th of October,

1843, while walking with his wife one evening, along the Royal Canal in Dublin, the discovery of quaternions flashed upon him, and he then engraved with his knife on a stone in Brougham Bridge the fundamental formula $i^2 = j^2 = k^2 = ijk = -1$. At the general meeting of the Irish Academy, a month later, he made the first communication on quaternions. An account of the discovery was given the following year in the *Philosophical Magazine*. Hamilton displayed wonderful fertility in their development. His *Lectures on Quaternions*, delivered in Dublin, were printed in 1852. His *Elements of Quaternions* appeared in 1866. Quaternions were greatly admired in England from the start, but on the Continent they received less attenttion. P. G. Tait's *Elementary Treatise* helped powerfully to spread a knowledge of them in England. Cayley, Clifford, and Tait advanced the subject somewhat by original contributions. But there has been little progress in recent years, except that made by Sylvester in the solution of quaternion equations, nor has the application of quaternions to physics been as extended as was predicted. The change in notation made in France by Hoüel and by Laisant has been considered in England as a wrong step, but the true cause for the lack of progress is perhaps more deep-seated. There is indeed great doubt as to whether the quaternionic product can claim a necessary and fundamental place in a system of vector analysis. Physicists claim that there is a loss of naturalness in taking the square of a vector to be negative. In order to meet more adequately their wants, *J. W. Gibbs* of Yale University and *A. Macfarlane* of the University of Texas, have each suggested an algebra of vectors with a new notation. Each gives a definition of his own for the product of two vectors, but in such a way that the square of a vector is positive. A third system of vector analysis has been used by *Oliver Heaviside* in his electrical researches.

Hermann Grassmann (1809–1877) was born at Stettin, attended a gymnasium at his native place (where his father was teacher of mathematics and physics), and studied theology in Berlin for three years. In 1834 he succeeded Steiner as teacher of mathematics in an industrial school in Berlin, but returned to Stettin in 1836 to assume the duties of teacher of mathematics, the sciences, and of religion in a school there.[71] Up to this time his knowledge of mathematics was pretty much confined to what he had learned from his father, who had written two books on "Raumlehre" and "Grössenlehre." But now he made his acquaintance with the works of Lacroix, Lagrange, and Laplace. He noticed that Laplace's results could be reached in a shorter way by some new ideas advanced in his father's books, and he proceeded to elaborate this abridged method, and to apply it in the study of tides. He was thus led to a new geometric analysis. In 1840 he had made considerable progress in its development, but a new book of Schleiermacher drew him again to theology. In 1842 he resumed mathematical research, and becoming thoroughly convinced of the importance of his new analysis, decided to devote himself to it. It now became his ambition to secure a mathematical chair at a university, but in this he never succeeded. In 1844 appeared his great classical work, the *Lineale Ausdehnungslehre*, which was full of new and strange matter, and so general, abstract, and out of fashion in its mode of exposition, that it could hardly have had less influence on European mathematics during its first twenty years, had it been published in China. Gauss, Grunert, and Möbius glanced over it, praised it, but complained of the strange terminology and its "philosophische Allgemeinheit." Eight years afterwards, Bretschneider of Gotha was said to be the only man who had read it through. An article in *Crelle's Journal*, in which Grassmann eclipsed the geometers of that

time by constructing, with aid of his method, geometrically any algebraic curve, remained again unnoticed. Need we marvel if Grassmann turned his attention to other subjects, — to Schleiermacher's philosophy, to politics, to philology? Still, articles by him continued to appear in *Crelle's Journal*, and in 1862 came out the second part of his *Ausdehnungslehre*. It was intended to show better than the first part the broad scope of the Ausdehnungslehre, by considering not only geometric applications, but by treating also of algebraic functions, infinite series, and the differential and integral calculus. But the second part was no more appreciated than the first. At the age of fifty-three, this wonderful man, with heavy heart, gave up mathematics, and directed his energies to the study of Sanskrit, achieving in philology results which were better appreciated, and which vie in splendour with those in mathematics.

Common to the Ausdehnungslehre and to quaternions are geometric addition, the function of two vectors represented in quaternions by $S\alpha\beta$ and $V\alpha\beta$, and the linear vector functions. The quaternion is peculiar to Hamilton, while with Grassmann we find in addition to the algebra of vectors a geometrical algebra of wide application, and resembling Möbius's *Barycentrische Calcul*, in which the point is the fundamental element. Grassmann developed the idea of the "external product," the "internal product," and the "open product." The last we now call a matrix. His Ausdehnungslehre has very great extension, having no limitation to any particular number of dimensions. Only in recent years has the wonderful richness of his discoveries begun to be appreciated. A second edition of the *Ausdehnungslehre* of 1844 was printed in 1877. C. S. Peirce gave a representation of Grassmann's system in the logical notation, and E. W. Hyde of the University of Cincinnati wrote the first text-book on Grassmann's calculus in the English language.

Discoveries of less value, which in part covered those of Grassmann and Hamilton, were made by *Saint-Venant* (1797–1886), who described the multiplication of vectors, and the addition of vectors and oriented areas; by *Cauchy*, whose "clefs algébriques" were units subject to combinatorial multiplication, and were applied by the author to the theory of elimination in the same way as had been done earlier by Grassmann; by **Justus Bellavitis** (1803–1880), who published in 1835 and 1837 in the *Annali delle Scienze* his calculus of æquipollences. Bellavitis, for many years professor at Padua, was a self-taught mathematician of much power, who in his thirty-eighth year laid down a city office in his native place, Bassano, that he might give his time to science.[65]

The first impression of Grassmann's ideas is marked in the writings of **Hermann Hankel** (1839–1873), who published in 1867 his *Vorlesungen über die Complexen Zahlen.* Hankel, then docent in Leipzig, had been in correspondence with Grassmann. The "alternate numbers" of Hankel are subject to his law of combinatorial multiplication. In considering the foundations of algebra Hankel affirms the principle of the permanence of formal laws previously enunciated incompletely by Peacock. Hankel was a close student of mathematical history, and left behind an unfinished work thereon. Before his death he was professor at Tübingen. His *Complexe Zahlen* was at first little read, and we must turn to **Victor Schlegel** of Hagen as the successful interpreter of Grassmann. Schlegel was at one time a young colleague of Grassmann at the Marienstifts-Gymnasium in Stettin. Encouraged by Clebsch, Schlegel wrote a *System der Raumlehre* which explained the essential conceptions and operations of the Ausdehnungslehre.

Multiple algebra was powerfully advanced by Peirce, whose theory is not geometrical, as are those of Hamilton and Grass-

mann. **Benjamin Peirce** (1809–1880) was born at Salem, Mass., and graduated at Harvard College, having as undergraduate carried the study of mathematics far beyond the limits of the college course.[2] When Bowditch was preparing his translation and commentary of the *Mécanique Céleste*, young Peirce helped in reading the proof-sheets. He was made professor at Harvard in 1833, a position which he retained until his death. For some years he was in charge of the *Nautical Almanac* and superintendent of the United States Coast Survey. He published a series of college text-books on mathematics, an *Analytical Mechanics*, 1855, and calculated, together with Sears C. Walker of Washington, the orbit of Neptune. Profound are his researches on *Linear Associative Algebra*. The first of several papers thereon was read at the first meeting of the American Association for the Advancement of Science in 1864. Lithographed copies of a memoir were distributed among friends in 1870, but so small seemed to be the interest taken in this subject that the memoir was not printed until 1881 (*Am. Jour. Math.*, Vol. IV., No. 2). Peirce works out the multiplication tables, first of *single* algebras, then of *double* algebras, and so on up to sextuple, making in all 162 algebras, which he shows to be possible on the consideration of symbols A, B, etc., which are linear functions of a determinate number of letters or units i, j, k, l, etc., with coefficients which are ordinary analytical magnitudes, real or imaginary, — the letters i, j, etc., being such that every binary combination i^2, ij, ji, etc., is equal to a linear function of the letters, but under the restriction of satisfying the associative law.[56] *Charles S. Peirce*, a son of Benjamin Peirce, and one of the foremost writers on mathematical logic, showed that these algebras were all defective forms of quadrate algebras which he had previously discovered by logical analysis, and for which he had devised a simple notation. Of these quadrate algebras quaternions is a simple

example; nonions is another. C. S. Peirce showed that of all linear associative algebras there are only three in which division is unambiguous. These are ordinary single algebra, ordinary double algebra, and quaternions, from which the imaginary scalar is excluded. He showed that his father's algebras are operational and matricular. Lectures on multiple algebra were delivered by J. J. Sylvester at the Johns Hopkins University, and published in various journals. They treat largely of the algebra of matrices. The theory of matrices was developed as early as 1858 by Cayley in an important memoir which, in the opinion of Sylvester, ushered in the reign of Algebra the Second. Clifford, Sylvester, H. Taber, C. H. Chapman, carried the investigations much further. The originator of matrices is really Hamilton, but his theory, published in his *Lectures on Quaternions*, is less general than that of Cayley. The latter makes no reference to Hamilton.

The theory of determinants[73] was studied by Hoëné Wronski in Italy and J. Binet in France; but they were forestalled by the great master of this subject, Cauchy. In a paper (*Jour. de l'ecole Polyt.*, IX., 16) Cauchy developed several general theorems. He introduced the name *determinant*, a term previously used by Gauss in the functions considered by him. In 1826 Jacobi began using this calculus, and he gave brilliant proof of its power. In 1841 he wrote extended memoirs on determinants in *Crelle's Journal*, which rendered the theory easily accessible. In England the study of linear transformations of quantics gave a powerful impulse. Cayley developed skew-determinants and Pfaffians, and introduced the use of determinant brackets, or the familiar pair of upright lines. More recent researches on determinants appertain to special forms. "Continuants" are due to Sylvester; "alternants," originated by Cauchy, have been developed by Jacobi, N. Trudi, H. Nägelbach, and G. Garbieri; "axisymmetric determinants,"

first used by Jacobi, have been studied by V. A. Lebesgue, Sylvester, and Hesse; "circulants" are due to E. Catalan of Liège, W. Spottiswoode (1825–1883), J. W. L. Glaisher, and R. F. Scott; for "centro-symmetric determinants" we are indebted to G. Zehfuss. E. B. Christoffel of Strassburg and G. Frobenius discovered the properties of "Wronskians," first used by Wronski. V. Nachreiner and S. Günther, both of Munich, pointed out relations between determinants and continued fractions; Scott uses Hankel's alternate numbers in his treatise. Text-books on determinants were written by Spottiswoode (1851), Brioschi (1854), Baltzer (1857), Günther (1875), Dostor (1877), Scott (1880), Muir (1882), Hanus (1886).

Modern higher algebra is especially occupied with the theory of linear transformations. Its development is mainly the work of Cayley and Sylvester.

Arthur Cayley, born at Richmond, in Surrey, in 1821, was educated at Trinity College, Cambridge.[74] He came out Senior Wrangler in 1842. He then devoted some years to the study and practice of law. On the foundation of the Sadlerian professorship at Cambridge, he accepted the offer of that chair, thus giving up a profession promising wealth for a very modest provision, but which would enable him to give all his time to mathematics. Cayley began his mathematical publications in the *Cambridge Mathematical Journal* while he was still an undergraduate. Some of his most brilliant discoveries were made during the time of his legal practice. There is hardly any subject in pure mathematics which the genius of Cayley has not enriched, but most important is his creation of a new branch of analysis by his theory of invariants. Germs of the principle of invariants are found in the writings of Lagrange, Gauss, and particularly of Boole, who showed, in 1841, that invariance is a property of discrimi-

nants generally, and who applied it to the theory of orthogonal substitution. Cayley set himself the problem to determine *a priori* what functions of the coefficients of a given equation possess this property of invariance, and found, to begin with, in 1845, that the so-called "hyper-determinants" possessed it. Boole made a number of additional discoveries. Then Sylvester began his papers in the *Cambridge and Dublin Mathematical Journal* on the Calculus of Forms. After this, discoveries followed in rapid succession. At that time Cayley and Sylvester were both residents of London, and they stimulated each other by frequent oral communications. It has often been difficult to determine how much really belongs to each.

James Joseph Sylvester was born in London in 1814, and educated at St. Johns College, Cambridge. He came out Second Wrangler in 1837. His Jewish origin incapacitated him from taking a degree. In 1846 he became a student at the Inner Temple, and was called to the bar in 1850. He became professor of natural philosophy at University College, London; then, successively, professor of mathematics at the University of Virginia, at the Royal Military Academy in Woolwich, at the Johns Hopkins University in Baltimore, and is, since 1883, professor of geometry at Oxford. His first printed paper was on Fresnel's optic theory, 1837. Then followed his researches on invariants, the theory of equations, theory of partitions, multiple algebra, the theory of numbers, and other subjects mentioned elsewhere. About 1874 he took part in the development of the geometrical theory of linkwork movements, originated by the beautiful discovery of A. Peaucellier, Capitaine du Génie à Nice (published in *Nouvelles Annales*, 1864 and 1873), and made the subject of close study by A. B. Kempe. To Sylvester is ascribed the general statement of the theory of contravariants, the dis-

covery of the partial differential equations satisfied by the invariants and covariants of binary quantics, and the subject of mixed concomitants. In the *American Journal of Mathematics* are memoirs on binary and ternary quantics, elaborated partly with aid of *F. Franklin*, now professor at the Johns Hopkins University. At Oxford, Sylvester has opened up a new subject, the theory of reciprocants, treating of the functions of a dependent variable y and the functions of its differential coefficients in regard to x, which remain unaltered by the interchange of x and y. This theory is more general than one on differential invariants by Halphen (1878), and has been developed further by J. Hammond of Oxford, McMahon of Woolwich, A. R. Forsyth of Cambridge, and others. Sylvester playfully lays claim to the appellation of the Mathematical Adam, for the many names he has introduced into mathematics. Thus the terms *invariant*, *discriminant*, *Hessian*, *Jacobian*, are his.

The great theory of invariants, developed in England mainly by Cayley and Sylvester, came to be studied earnestly in Germany, France, and Italy. One of the earliest in the field was **Siegfried Heinrich Aronhold** (1819–1884), who demonstrated the existence of invariants, S and T, of the ternary cubic. Hermite discovered evectants and the theorem of reciprocity named after him. Paul Gordan showed, with the aid of symbolic methods, that the number of distinct forms for a binary quantic is finite. Clebsch proved this to be true for quantics with any number of variables. A very much simpler proof of this was given in 1891, by David Hilbert of Königsberg. In Italy, F. Brioschi of Milan and *Faà de Bruno* (1825–1888) contributed to the theory of invariants, the latter writing a text-book on binary forms, which ranks by the side of Salmon's treatise and those of Clebsch and Gordan. Among other writers on invariants are E. B. Chris-

toffel, Wilhelm Fiedler, P. A. McMahon, J. W. L. Glaisher of Cambridge, Emory McClintock of New York. McMahon discovered that the theory of semi-invariants is a part of that of symmetric functions. The modern higher algebra has reached out and indissolubly connected itself with several other branches of mathematics — geometry, calculus of variations, mechanics. Clebsch extended the theory of binary forms to ternary, and applied the results to geometry. Clebsch, Klein, Weierstrass, Burckhardt, and Bianchi have used the theory of invariants in hyperelliptic and Abelian functions.

In the theory of equations Lagrange, Argand, and Gauss furnished proof to the important theorem that every algebraic equation has a real or a complex root. Abel proved rigorously that the general algebraic equation of the fifth or of higher degrees cannot be solved by radicals (*Crelle*, I., 1826). A modification of Abel's proof was given by Wantzel. Before Abel, an Italian physician, *Paolo Ruffini* (1765–1822), had printed proofs of the insolvability, which were criticised by his countryman Malfatti. Though inconclusive, Ruffini's papers are remarkable as containing anticipations of Cauchy's theory of groups.[76] A transcendental solution of the quintic involving elliptic integrals was given by Hermite (*Compt. Rend.*, 1858, 1865, 1866). After Hermite's first publication, Kronecker, in 1858, in a letter to Hermite, gave a second solution in which was obtained a simple resolvent of the sixth degree. *Jerrard*, in his *Mathematical Researches* (1832–1835), reduced the quintic to the trinomial form by an extension of the method of Tschirnhausen. This important reduction had been effected as early as 1786 by *E. S. Bring*, a Swede, and brought out in a publication of the University of Lund. Jerrard, like Tschirnhausen, believed that his method furnished a general algebraic solution of equations of any degree. In 1836 William R. Hamilton made a report on the validity of Jerrard's

method, and showed that by his process the quintic could be transformed to any one of the four trinomial forms. Hamilton defined the limits of its applicability to higher equations. Sylvester investigated this question, What is the lowest degree an equation can have in order that it may admit of being deprived of i consecutive terms by aid of equations not higher than ith degree. He carried the investigation as far as $i = 8$, and was led to a series of numbers which he named "Hamilton's numbers." A transformation of equal importance to Jerrard's is that of Sylvester, who expressed the quintic as the sum of three fifth-powers. The covariants and invariants of higher equations have been studied much in recent years.

Abel's proof that higher equations cannot always be solved algebraically led to the inquiry as to what equations of a given degree can be solved by radicals. Such equations are the ones discussed by Gauss in considering the division of the circle. Abel advanced one step further by proving that an irreducible equation can always be solved in radicals, if, of two of its roots, the one can be expressed rationally in terms of the other, provided that the degree of the equation is prime; if it is not prime, then the solution depends upon that of equations of lower degree. Through geometrical considerations, Hesse came upon algebraically solvable equations of the ninth degree, not included in the previous groups. The subject was powerfully advanced in Paris by the youthful *Evariste Galois* (born, 1811; killed in a duel, 1832), who introduced the notion of a group of substitutions. To him are due also some valuable results in relation to another set of equations, presenting themselves in the theory of elliptic functions, viz. the modular equations. Galois's labours gave birth to the important theory of substitutions, which has been greatly advanced by *C. Jordan* of Paris, *J. A. Serret* (1819–1885) of the Sorbonne in Paris, **L. Kronecker** (1823–1891) of

Berlin, Klein of Göttingen, M. Nöther of Erlangen, C. Hermite of Paris, A. Capelli of Naples, L. Sylow of Friedrichshald, E. Netto of Giessen. Netto's book, the *Substitutionstheorie*, has been translated into English by F. N. Cole of the University of Michigan, who contributed to the theory. A simple group of 504 substitutions of nine letters, discovered by Cole, has been shown by E. H. Moore of the University of Chicago to belong to a doubly-infinite system of simple groups. The theory of substitutions has important applications in the theory of differential equations. Kronecker published, in 1882, his *Grundzüge einer Arithmetischen Theorie der Algebraischen Grössen.*

Since Fourier and Budan, the solution of numerical equations has been advanced by *W. G. Horner* of Bath, who gave an improved method of approximation (*Philosophical Transactions*, 1819). **Jacques Charles François Sturm** (1803–1855), a native of Geneva, Switzerland, and the successor of Poisson in the chair of mechanics at the Sorbonne, published in 1829 his celebrated theorem determining the number and situation of roots of an equation comprised between given limits. Sturm tells us that his theorem stared him in the face in the midst of some mechanical investigations connected with the motion of a compound pendulum.[77] This theorem, and Horner's method, offer together sure and ready means of finding the real roots of a numerical equation.

The symmetric functions of the sums of powers of the roots of an equation, studied by Newton and Waring, was considered more recently by Gauss, Cayley, Sylvester, Brioschi. Cayley gives rules for the "weight" and "order" of symmetric functions.

The theory of elimination was greatly advanced by Sylvester, Cayley, Salmon, Jacobi, Hesse, Cauchy, Brioschi, and Gordan. Sylvester gave the dialytic method (*Philosophical*

Magazine, 1840), and in 1852 established a theorem relating to the expression of an eliminant as a determinant. Cayley made a new statement of Bézout's method of elimination and established a general theory of elimination (1852).

ANALYSIS.

Under this head we find it convenient to consider the subjects of the differential and integral calculus, the calculus of variations, infinite series, probability, and differential equations. Prominent in the development of these subjects was Cauchy.

Augustin-Louis Cauchy[78] (1789–1857) was born in Paris, and received his early education from his father. Lagrange and Laplace, with whom the father came in frequent contact, foretold the future greatness of the young boy. At the École Centrale du Panthéon he excelled in ancient classical studies. In 1805 he entered the Polytechnic School, and two years later the École des Ponts et Chaussées. Cauchy left for Cherbourg in 1810, in the capacity of engineer. Laplace's *Mécanique Céleste* and Lagrange's *Fonctions Analytiques* were among his book companions there. Considerations of health induced him to return to Paris after three years. Yielding to the persuasions of Lagrange and Laplace, he renounced engineering in favour of pure science. We find him next holding a professorship at the Polytechnic School. On the expulsion of Charles X., and the accession to the throne of Louis Philippe in 1830, Cauchy, being exceedingly conscientious, found himself unable to take the oath demanded of him. Being, in consequence, deprived of his positions, he went into voluntary exile. At Fribourg in Switzerland, Cauchy resumed his studies, and in 1831 was induced by the king of Piedmont to

accept the chair of mathematical physics, especially created for him at the university of Turin. In 1833 he obeyed the call of his exiled king, Charles X., to undertake the education of a grandson, the Duke of Bordeaux. This gave Cauchy an opportunity to visit various parts of Europe, and to learn how extensively his works were being read. Charles X. bestowed upon him the title of Baron. On his return to Paris in 1838, a chair in the College de France was offered to him, but the oath demanded of him prevented his acceptance. He was nominated member of the Bureau of Longitude, but declared ineligible by the ruling power. During the political events of 1848 the oath was suspended, and Cauchy at last became professor at the Polytechnic School. On the establishment of the second empire, the oath was re-instated, but Cauchy and Arago were exempt from it. Cauchy was a man of great piety, and in two of his publications staunchly defended the Jesuits.

Cauchy was a prolific and profound mathematician. By a prompt publication of his results, and the preparation of standard text-books, he exercised a more immediate and beneficial influence upon the great mass of mathematicians than any contemporary writer. He was one of the leaders in infusing rigour into analysis. His researches extended over the field of series, of imaginaries, theory of numbers, differential equations, theory of substitutions, theory of functions, determinants, mathematical astronomy, light, elasticity, etc., — covering pretty much the whole realm of mathematics, pure and applied.

Encouraged by Laplace and Poisson, Cauchy published in 1821 his *Cours d'Analyse de l'École Royale Polytechnique*, a work of great merit. Had it been studied more diligently by writers of text-books in England and the United States, many a lax and loose method of analysis hardly as yet eradicated

from elementary text-books would have been discarded over half a century ago. Cauchy was the first to publish a rigorous proof of Taylor's theorem. He greatly improved the exposition of fundamental principles of the differential calculus by his mode of considering limits and his new theory on the continuity of functions. The method of Cauchy and Duhamel was accepted with favour by Hoüel and others. In England special attention to the clear exposition of fundamental principles was given by De Morgan. Recent American treatises on the calculus introduce time as an independent variable, and the allied notions of velocity and acceleration — thus virtually returning to the method of fluxions.

Cauchy made some researches on the calculus of variations. This subject is now in its essential principles the same as when it came from the hands of Lagrange. Recent studies pertain to the variation of a double integral when the limits are also variable, and to variations of multiple integrals in general. Memoirs were published by Gauss in 1829, Poisson in 1831, and Ostrogradsky of St. Petersburg in 1834, without, however, determining in a general manner the number and form of the equations which must subsist at the limits in case of a double or triple integral. In 1837 Jacobi published a memoir, showing that the difficult integrations demanded by the discussion of the second variation, by which the existence of a maximum or minimum can be ascertained, are included in the integrations of the first variation, and thus are superfluous. This important theorem, presented with great brevity by Jacobi, was elucidated and extended by V. A. Lebesgue, C. E. Delaunay, Eisenlohr, S. Spitzer, Hesse, and Clebsch. An important memoir by Sarrus on the question of determining the limiting equations which must be combined with the indefinite equations in order to determine completely the maxima and minima of multiple integrals, was awarded a

prize by the French Academy in 1845, honourable mention being made of a paper by Delaunay. Sarrus's method was simplified by Cauchy. In 1852 G. Mainardi attempted to exhibit a new method of discriminating maxima and minima, and extended Jacobi's theorem to double integrals. Mainardi and F. Brioschi showed the value of determinants in exhibiting the terms of the second variation. In 1861 *Isaac Todhunter* (1820–1884) of St. John's College, Cambridge, published his valuable work on the *History of the Progress of the Calculus of Variations*, which contains researches of his own. In 1866 he published a most important research, developing the theory of discontinuous solutions (discussed in particular cases by Legendre), and doing for this subject what Sarrus had done for multiple integrals.

The following are the more important authors of systematic treatises on the calculus of variations, and the dates of publication: Robert Woodhouse, Fellow of Caius College, Cambridge, 1810; Richard Abbatt in London, 1837; John Hewitt Jellett (1817–1888), once Provost of Trinity College, Dublin, 1850; G. W. Strauch in Zurich, 1849; Moigno and Lindelöf, 1861; Lewis Buffett Carll of Flushing in New York, 1881.

The lectures on definite integrals, delivered by Dirichlet in 1858, have been elaborated into a standard work by G. F. Meyer. The subject has been treated most exhaustively by D. Bierens de Haan of Leiden in his *Exposé de la théorie des intégrals définies*, Amsterdam, 1862.

The history of infinite series illustrates vividly the salient feature of the new era which analysis entered upon during the first quarter of this century. Newton and Leibniz felt the necessity of inquiring into the convergence of infinite series, but they had no proper criteria, excepting the test advanced by Leibniz for alternating series. By Euler and his contemporaries the *formal* treatment of series was greatly extended,

while the necessity for determining the convergence was generally lost sight of. Euler reached some very pretty results on infinite series, now well known, and also some very absurd results, now quite forgotten. The faults of his time found their culmination in the Combinatorial School in Germany, which has now passed into deserved oblivion. At the beginning of the period now under consideration, the doubtful, or plainly absurd, results obtained from infinite series stimulated profounder inquiries into the validity of operations with them. Their *actual contents* came to be the primary, *form* a secondary, consideration. The first important and strictly rigorous investigation of series was made by Gauss in connection with the hypergeometric series. The criterion developed by him settles the question of convergence in every case which it is intended to cover, and thus bears the stamp of generality so characteristic of Gauss's writings. Owing to the strangeness of treatment and unusual rigour, Gauss's paper excited little interest among the mathematicians of that time.

More fortunate in reaching the public was Cauchy, whose *Analyse Algébrique* of 1821 contains a rigorous treatment of series. All series whose sum does not approach a fixed limit as the number of terms increases indefinitely are called divergent. Like Gauss, he institutes comparisons with geometric series, and finds that series with positive terms are convergent or not, according as the nth root of the nth term, or the ratio of the $(n+1)$th term and the nth term, is ultimately less or greater than unity. To reach some of the cases where these expressions become ultimately unity and fail, Cauchy established two other tests. He showed that series with negative terms converge when the absolute values of the terms converge, and then deduces Leibniz's test for alternating series. The product of two convergent series was not found to be necessarily convergent. Cauchy's theorem that the

product of two absolutely convergent series converges to the product of the sums of the two series was shown half a century later by F. Mertens of Graz to be still true if, of the two convergent series to be multiplied together, only one is absolutely convergent.

The most outspoken critic of the old methods in series was Abel. His letter to his friend Holmboe (1826) contains severe criticisms. It is very interesting reading, even to modern students. In his demonstration of the binomial theorem he established the theorem that if two series and their product series are all convergent, then the product series will converge towards the product of the sums of the two given series. This remarkable result would dispose of the whole problem of multiplication of series if we had a universal practical criterion of convergency for semi-convergent series. Since we do not possess such a criterion, theorems have been recently established by A. Pringsheim of Munich and A. Voss of Würzburg which remove in certain cases the necessity of applying tests of convergency to the product series by the application of tests to easier related expressions. Pringsheim reaches the following interesting conclusions: The product of two semi-convergent series can never converge absolutely, but a semi-convergent series, or even a divergent series, multiplied by an absolutely convergent series, *may* yield an absolutely convergent product.

The researches of Abel and Cauchy caused a considerable stir. We are told that after a scientific meeting in which Cauchy had presented his first researches on series, Laplace hastened home and remained there in seclusion until he had examined the series in his *Mécanique Céleste*. Luckily, every one was found to be convergent! We must not conclude, however, that the new ideas at once displaced the old. On the contrary, the new views were generally accepted only after a

severe and long struggle. As late as 1844 De Morgan began a paper on "divergent series" in this style: "I believe it will be generally admitted that the heading of this paper describes the only subject yet remaining, of an elementary character, on which a serious schism exists among mathematicians as to the absolute correctness or incorrectness of results."

First in time in the evolution of more delicate criteria of convergence and divergence come the researches of Josef Ludwig Raabe (*Crelle*, Vol. IX.); then follow those of De Morgan as given in his calculus. De Morgan established the logarithmic criteria which were discovered in part independently by J. Bertrand. The forms of these criteria, as given by Bertrand and by Ossian Bonnet, are more convenient than De Morgan's. It appears from Abel's posthumous papers that he had anticipated the above-named writers in establishing logarithmic criteria. It was the opinion of Bonnet that the logarithmic criteria never fail; but Du Bois-Reymond and Pringsheim have each discovered series demonstrably convergent in which these criteria fail to determine the convergence. The criteria thus far alluded to have been called by Pringsheim *special* criteria, because they all depend upon a comparison of the nth term of the series with special functions a^n, n^z, $n(\log n)^z$, etc. Among the first to suggest *general* criteria, and to consider the subject from a still wider point of view, culminating in a regular mathematical theory, was Kummer. He established a theorem yielding a test consisting of two parts, the first part of which was afterwards found to be superfluous. The study of general criteria was continued by U. Dini of Pisa, Paul Du Bois-Reymond, G. Kohn of Minden, and Pringsheim. Du Bois-Reymond divides criteria into two classes: criteria of the *first kind* and criteria of the *second kind*, according as the general nth term, or the ratio of the $(n+1)$th term and

the nth term, is made the basis of research. Kummer's is a criterion of the second kind. A criterion of the first kind, analogous to this, was invented by Pringsheim. From the general criteria established by Du Bois-Reymond and Pringsheim respectively, all the special criteria can be derived. The theory of Pringsheim is very complete, and offers, in addition to the criteria of the first kind and second kind, entirely new criteria of a *third kind*, and also generalised criteria of the second kind, which apply, however, only to series with never increasing terms. Those of the third kind rest mainly on the consideration of the limit of the difference either of consecutive terms or of their reciprocals. In the generalised criteria of the second kind he does not consider the ratio of two consecutive terms, but the ratio of any two terms however far apart, and deduces, among others, two criteria previously given by Kohn and Ermakoff respectively.

Difficult questions arose in the study of Fourier's series.[79] Cauchy was the first who felt the necessity of inquiring into its convergence. But his mode of proceeding was found by Dirichlet to be unsatisfactory. Dirichlet made the first thorough researches on this subject (*Crelle*, Vol. IV.). They culminate in the result that whenever the function does not become infinite, does not have an infinite number of discontinuities, and does not possess an infinite number of maxima and minima, then Fourier's series converges toward the value of that function at all places, except points of discontinuity, and there it converges toward the mean of the two boundary values. Schläfli of Bern and Du Bois-Reymond expressed doubts as to the correctness of the mean value, which were, however, not well founded. Dirichlet's conditions are sufficient, but not necessary. Lipschitz, of Bonn, proved that Fourier's series still represents the function when the number of discontinuities is infinite, and

established a condition on which it represents a function having an infinite number of maxima and minima. Dirichlet's belief that all continuous functions can be represented by Fourier's series at all points was shared by Riemann and H. Hankel, but was proved to be false by Du Bois-Reymond and H. A. Schwarz.

Riemann inquired what properties a function must have, so that there may be a trigonometric series which, whenever it is convergent, converges toward the value of the function. He found necessary and sufficient conditions for this. They do not decide, however, whether such a series actually represents the function or not. Riemann rejected Cauchy's definition of a definite integral on account of its arbitrariness, gave a new definition, and then inquired when a function has an integral. His researches brought to light the fact that continuous functions need not always have a differential coefficient. But this property, which was shown by Weierstrass to belong to large classes of functions, was not found necessarily to exclude them from being represented by Fourier's series. Doubts on some of the conclusions about Fourier's series were thrown by the observation, made by Weierstrass, that the integral of an infinite series can be shown to be equal to the sum of the integrals of the separate terms only when the series converges *uniformly* within the region in question. The subject of uniform convergence was investigated by Philipp Ludwig Seidel (1848) and G. G. Stokes (1847), and has assumed great importance in Weierstrass' theory of functions. It became necessary to prove that a trigonometric series representing a continuous function converges uniformly. This was done by Heinrich Eduard Heine (1821–1881), of Halle. Later researches on Fourier's series were made by G. Cantor and Du Bois-Reymond.

As compared with the vast development of other mathe-

matical branches, the theory of probability has made very insignificant progress since the time of Laplace. Improvements and simplications in the mode of exposition have been made by A. De Morgan, G. Boole, A. Meyer (edited by E. Czuber), J. Bertrand. Cournot's and Westergaard's treatment of insurance and the theory of life-tables are classical. Applications of the calculus to statistics have been made by L. A. J. Quetelet (1796–1874), director of the observatory at Brussels; by Lexis; Harald Westergaard, of Copenhagen; and Düsing.

Worthy of note is the rejection of inverse probability by the best authorities of our time. This branch of probability had been worked out by Thomas Bayes (died 1761) and by Laplace (Bk. II., Ch. VI. of his *Théorie Analytique*). By it some logicians have explained induction. For example, if a man, who has never heard of the tides, were to go to the shore of the Atlantic Ocean and witness on m successive days the rise of the sea, then, says Quetelet, he would be entitled to conclude that there was a probability equal to $\frac{m+1}{m+2}$ that the sea would rise next day. Putting $m = 0$, it is seen that this view rests upon the unwarrantable assumption that the probability of a totally unknown event is $\frac{1}{2}$, or that of all theories proposed for investigation one-half are true. W. S. Jevons in his *Principles of Science* founds induction upon the theory of inverse probability, and F. Y. Edgeworth also accepts it in his *Mathematical Psychics.*

The only noteworthy recent addition to probability is the subject of "local probability," developed by several English and a few American and French mathematicians. The earliest problem on this subject dates back to the time of Buffon, the naturalist, who proposed the problem, solved by himself and Laplace, to determine the probability that a short needle, thrown at random upon a floor ruled with equidistant parallel

lines, will fall on one of the lines. Then came Sylvester's four-point problem: to find the probability that four points, taken at random within a given boundary, shall form a re-entrant quadrilateral. Local probability has been studied in England by A. R. Clarke, H. McColl, S. Watson, J. Wolstenholme, but with greatest success by *M. W. Crofton* of the military school at Woolwich. It was pursued in America by E. B. Seitz; in France by C. Jordan, E. Lemoine, E. Barbier, and others. Through considerations of local probability, Crofton was led to the evaluation of certain definite integrals.

The first full scientific treatment of differential equations was given by Lagrange and Laplace. This remark is especially true of partial differential equations. The latter were investigated in more recent time by Monge, Pfaff, Jacobi, Émile Bour (1831–1866) of Paris, A. Weiler, Clebsch, A. N. Korkine of St. Petersburg, G. Boole, A. Meyer, Cauchy, Serret, Sophus Lie, and others. In 1873 their reseaches, on partial differential equations of the first order, were presented in text-book form by Paul Mansion, of the University of Gand. The keen researches of **Johann Friedrich Pfaff** (1795–1825) marked a decided advance. He was an intimate friend of young Gauss at Göttingen. Afterwards he was with the astronomer Bode. Later he became professor at Helmstädt, then at Halle. By a peculiar method, Pfaff found the general integration of partial differential equations of the first order for any number of variables. Starting from the theory of ordinary differential equations of the first order in n variables, he gives first their general integration, and then considers the integration of the partial differential equations as a particular case of the former, assuming, however, as known, the general integration of differential equations of any order between two variables. His researches led Jacobi to introduce the name "Pfaffian problem." From the connection, observed by Hamilton, between

a system of ordinary differential equations (in analytical mechanics) and a partial differential equation, Jacobi drew the conclusion that, of the series of systems whose successive integration Pfaff's method demanded, all but the first system were entirely superfluous. Clebsch considered Pfaff's problem from a new point of view, and reduced it to systems of simultaneous linear partial differential equations, which can be established independently of each other without any integration. Jacobi materially advanced the theory of differential equations of the first order. The problem to determine unknown functions in such a way that an integral containing these functions and their differential coefficients, in a prescribed manner, shall reach a maximum or minimum value, demands, in the first place, the vanishing of the first variation of the integral. This condition leads to differential equations, the integration of which determines the functions. To ascertain whether the value is a maximum or a minimum, the second variation must be examined. This leads to new and difficult differential equations, the integration of which, for the simpler cases, was ingeniously deduced by Jacobi from the integration of the differential equations of the first variation. Jacobi's solution was perfected by Hesse, while Clebsch extended to the general case Jacobi's results on the second variation. Cauchy gave a method of solving partial differential equations of the first order having any number of variables, which was corrected and extended by Serret, J. Bertrand, O. Bonnet in France, and Imschenetzky in Russia. Fundamental is the proposition of Cauchy that every ordinary differential equation admits in the vicinity of any non-singular point of an integral, which is synectic within a certain circle of convergence, and is developable by Taylor's theorem. Allied to the point of view indicated by this theorem is that of Riemann, who regards a function of a single variable as

defined by the position and nature of its singularities, and who has applied this conception to that linear differential equation of the second order, which is satisfied by the hypergeometric series. This equation was studied also by Gauss and Kummer. Its general theory, when no restriction is imposed upon the value of the variable, has been considered by J. Tannery, of Paris, who employed Fuchs' method of linear differential equations and found all of Kummer's twenty-four integrals of this equation. This study has been continued by Édouard Goursat of Paris.

A standard text-book on *Differential Equations*, including original matter on integrating factors, singular solutions, and especially on symbolical methods, was prepared in 1859 by **George Boole** (1815–1864), at one time professor in Queen's University, Cork, Ireland. He was a native of Lincoln, and a self-educated mathematician of great power. His treatise on *Finite Differences* (1860) and his *Laws of Thought* (1854) are works of high merit.

The fertility of the conceptions of Cauchy and Riemann with regard to differential equations is attested by the researches to which they have given rise on the part of **Lazarus Fuchs** of Berlin (born 1835), **Felix Klein** of Göttingen (born 1849), **Henri Poincaré** of Paris (born 1854), and others. The study of linear differential equations entered a new period with the publication of Fuchs' memoirs of 1866 and 1868. Before this, linear equations with *constant* coefficients were almost the only ones for which general methods of integration were known. While the general theory of these equations has recently been presented in a new light by Hermite, Darboux, and Jordan, Fuchs began the study from the more general standpoint of the linear differential equations whose coefficients are not constant. He directed his attention mainly to those whose integrals are all regular.

If the variable be made to describe all possible paths enclosing one or more of the critical points of the equation, we have a certain substitution corresponding to each of the paths; the aggregate of all these substitutions being called a *group*. The forms of integrals of such equations were examined by Fuchs and by G. Frobenius by independent methods. Logarithms generally appear in the integrals of a group, and Fuchs and Frobenius investigated the conditions under which no logarithms shall appear. Through the study of groups the reducibility or irreducibility of linear differential equations has been examined by Frobenius and Leo Königsberger. The subject of linear differential equations, not all of whose integrals are regular, has been attacked by G. Frobenius of Berlin, W. Thomé of Greifswald (born 1841), and Poincaré, but the resulting theory of irregular integrals is as yet in very incomplete form.

The theory of invariants associated with linear differential equations has been developed by Halphen and by A. R. Forsyth.

The researches above referred to are closely connected with the theory of functions and of groups. Endeavours have thus been made to determine the nature of the function defined by a differential equation from the differential equation itself, and not from any analytical expression of the function, obtained first by solving the differential equation. Instead of studying the properties of the integrals of a differential equation for all the values of the variable, investigators at first contented themselves with the study of the properties in the vicinity of a given point. The nature of the integrals at singular points and at ordinary points is entirely different. *Albert Briot* (1817–1882) and *Jean Claude Bouquet* (1819–1885), both of Paris, studied the case when, near a singular point, the differential equations take the form $(x - x_0)\frac{dy}{dx} = \int(xy)$. Fuchs

gave the development in series of the integrals for the particular case of linear equations. Poincaré did the same for the case when the equations are not linear, as also for partial differential equations of the first order. The developments for ordinary points were given by Cauchy and Madame Kowalevsky.

The attempt to express the integrals by developments that are always convergent and not limited to particular points in a plane necessitates the introduction of new transcendents, for the old functions permit the integration of only a small number of differential equations. Poincaré tried this plan with linear equations, which were then the best known, having been studied in the vicinity of given points by Fuchs, Thomé, Frobenius, Schwarz, Klein, and Halphen. Confining himself to those with rational algebraical coefficients, Poincaré was able to integrate them by the use of functions named by him *Fuchsians*.[81] He divided these equations into "families." If the integral of such an equation be subjected to a certain transformation, the result will be the integral of an equation belonging to the same family. The new transcendents have a great analogy to elliptic functions; while the region of the latter may be divided into parallelograms, each representing a group, the former may be divided into curvilinear polygons, so that the knowledge of the function inside of one polygon carries with it the knowledge of it inside the others. Thus Poincaré arrives at what he calls *Fuchsian groups*. He found, moreover, that Fuchsian functions can be expressed as the ratio of two transcendents (theta-fuchsians) in the same way that elliptic functions can be. If, instead of linear substitutions with real coefficients, as employed in the above groups, imaginary coefficients be used, then discontinuous groups are obtained, which he called *Kleinians*. The extension to non-linear equations of the method thus applied to linear equations has been begun by Fuchs and Poincaré.

We have seen that among the earliest of the several kinds of "groups" are the finite discontinuous groups (groups in the theory of substitution), which since the time of Galois have become the leading concept in the theory of algebraic equations; that since 1876 Felix Klein, H. Poincaré, and others have applied the theory of finite and infinite discontinuous groups to the theory of functions and of differential equations. The finite continuous groups were first made the subject of general research in 1873 by Sophus Lie, now of Leipzig, and applied by him to the integration of ordinary linear partial differential equations.

Much interest attaches to the determination of those linear differential equations which can be integrated by simpler functions, such as algebraic, elliptic, or Abelian. This has been studied by C. Jordan, P. Appel of Paris (born 1858), and Poincaré.

The mode of integration above referred to, which makes known the properties of equations from the standpoint of the theory of functions, does not suffice in the application of differential equations to questions of mechanics. If we consider the function as defining a plane curve, then the general form of the curve does not appear from the above mode of investigation. It is, however, often desirable to construct the curves defined by differential equations. Studies having this end in view have been carried on by Briot and Bouquet, and by Poincaré.[81]

The subject of singular solutions of differential equations has been materially advanced since the time of Boole by G. Darboux and Cayley. The papers prepared by these mathematicians point out a difficulty as yet unsurmounted: whereas a singular solution, from the point of view of the integrated equation, ought to be a phenomenon of universal, or at least of general occurrence, it is, on the other hand, a very special and

exceptional phenomenon from the point of view of the differential equation.[89] A geometrical theory of singular solutions resembling the one used by Cayley was previously employed by W. W. Johnson of Annapolis.

An advanced *Treatise on Linear Differential Equations* (1889) was brought out by Thomas Craig of the Johns Hopkins University. He chose the algebraic method of presentation followed by Hermite and Poincaré, instead of the geometric method preferred by Klein and Schwarz. A notable work, the *Traité d'Analyse*, is now being published by Émile Picard of Paris, the interest of which is made to centre in the subject of differential equations.

THEORY OF FUNCTIONS.

We begin our sketch of the vast progress in the theory of functions by considering the special class called elliptic functions. These were richly developed by Abel and Jacobi.

Niels Henrick Abel (1802–1829) was born at Findoë in Norway, and was prepared for the university at the cathedral school in Christiania. He exhibited no interest in mathematics until 1818, when B. Holmboe became lecturer there, and aroused Abel's interest by assigning original problems to the class. Like Jacobi and many other young men who became eminent mathematicians, Abel found the first exercise of his talent in the attempt to solve by algebra the general equation of the fifth degree. In 1821 he entered the University in Christiania. The works of Euler, Lagrange, and Legendre were closely studied by him. The idea of the inversion of elliptic functions dates back to this time. His extraordinary success in mathematical study led to the offer of a stipend by the government, that he might continue his studies

in Germany and France. Leaving Norway in 1825, Abel visited the astronomer, Schumacher, in Hamburg, and spent six months in Berlin, where he became intimate with **August Leopold Crelle** (1780–1855), and met Steiner. Encouraged by Abel and Steiner, Crelle started his journal in 1826. Abel began to put some of his work in shape for print. His proof of the impossibility of solving the general equation of the fifth degree by radicals, — first printed in 1824 in a very concise form, and difficult of apprehension, — was elaborated in greater detail, and published in the first volume. He entered also upon the subject of infinite series (particularly the binomial theorem, of which he gave in *Crelle's Journal* a rigid general investigation), the study of functions, and of the integral calculus. The obscurities everywhere encountered by him owing to the prevailing loose methods of analysis he endeavoured to clear up. For a short time he left Berlin for Freiberg, where he had fewer interruptions to work, and it was there that he made researches on hyperelliptic and Abelian functions. In July, 1826, Abel left Germany for Paris without having met Gauss! Abel had sent to Gauss his proof of 1824 of the impossibility of solving equations of the fifth degree, to which Gauss never paid any attention. This slight, and a haughtiness of spirit which he associated with Gauss, prevented the genial Abel from going to Göttingen. A similar feeling was entertained by him later against Cauchy. Abel remained ten months in Paris. He met there Dirichlet, Legendre, Cauchy, and others; but was little appreciated. He had already published several important memoirs in *Crelle's Journal*, but by the French this new periodical was as yet hardly known to exist, and Abel was too modest to speak of his own work. Pecuniary embarrassments induced him to return home after a second short stay in Berlin. At Christiania he for some time gave private lessons, and served

as docent. Crelle secured at last an appointment for him at Berlin; but the news of it did not reach Norway until after the death of Abel at Froland.[82]

At nearly the same time with Abel, Jacobi published articles on elliptic functions. Legendre's favourite subject, so long neglected, was at last to be enriched by some extraordinary discoveries. The advantage to be derived by inverting the elliptic integral of the first kind and treating it as a function of its amplitude (now called elliptic function) was recognised by Abel, and a few months later also by Jacobi. A second fruitful idea, also arrived at independently by both, is the introduction of imaginaries leading to the observation that the new functions simulated at once trigonometric and exponential functions. For it was shown that while trigonometric functions had only a real period, and exponential only an imaginary, elliptic functions had both sorts of periods. These two discoveries were the foundations upon which Abel and Jacobi, each in his own way, erected beautiful new structures. Abel developed the curious expressions representing elliptic functions by infinite series or quotients of infinite products. Great as were the achievements of Abel in elliptic functions, they were eclipsed by his researches on what are now called Abelian functions. Abel's theorem on these functions was given by him in several forms, the most general of these being that in his *Mémoire sur une propriété générale d'une classe très-étendue de fonctions transcendentes* (1826). The history of this memoir is interesting. A few months after his arrival in Paris, Abel submitted it to the French Academy. Cauchy and Legendre were appointed to examine it; but said nothing about it until after Abel's death. In a brief statement of the discoveries in question, published by Abel in *Crelle's Journal*, 1829, reference is made to that memoir. This led Jacobi to inquire of Legendre what had become of it. Le-

gendre says that the manuscript was so badly written as to be illegible, and that Abel was asked to hand in a better copy, which he neglected to do. The memoir remained in Cauchy's hands. It was not published until 1841. By a singular mishap, the manuscript was lost before the proof-sheets were read.

In its form, the contents of the memoir belongs to the integral calculus. Abelian integrals depend upon an irrational function y which is connected with x by an algebraic equation $F(x, y) = 0$. Abel's theorem asserts that a sum of such integrals can be expressed by a definite number p of similar integrals, where p depends merely on the properties of the equation $F(x, y) = 0$. It was shown later that p is the deficiency of the curve $F(x, y) = 0$. The addition theorems of elliptic integrals are deducible from Abel's theorem. The hyperelliptic integrals introduced by Abel, and proved by him to possess multiple periodicity, are special cases of Abelian integrals whenever $p =$ or > 3. The reduction of Abelian to elliptic integrals has been studied mainly by Jacobi, Hermite, Königsberger, Brioschi, Goursat, E. Picard, and O. Bolza of the University of Chicago.

Two editions of Abel's works have been published: the first by Holmboe in 1839, and the second by Sylow and Lie in 1881.

Abel's theorem was pronounced by Jacobi the greatest discovery of our century on the integral calculus. The aged Legendre, who greatly admired Abel's genius, called it "*monumentum aere perennius.*" During the few years of work allotted to the young Norwegian, he penetrated new fields of research, the development of which has kept mathematicians busy for over half a century.

Some of the discoveries of Abel and Jacobi were anticipated by Gauss. In the *Disquisitiones Arithmeticæ* he observed

that the principles which he used in the division of the circle were applicable to many other functions, besides the circular, and particularly to the transcendents dependent on the integral $\int \frac{dx}{\sqrt{1-x^4}}$. From this Jacobi [83] concluded that Gauss had thirty years earlier considered the nature and properties of elliptic functions and had discovered their double periodicity. The papers in the collected works of Gauss confirm this conclusion.

Carl Gustav Jacob Jacobi [84] (1804–1851) was born of Jewish parents at Potsdam. Like many other mathematicians he was initiated into mathematics by reading Euler. At the University of Berlin, where he pursued his mathematical studies independently of the lecture courses, he took the degree of Ph.D. in 1825. After giving lectures in Berlin for two years, he was elected extraordinary professor at Königsberg, and two years later to the ordinary professorship there. After the publication of his *Fundamenta Nova* he spent some time in travel, meeting Gauss in Göttingen, and Legendre, Fourier, Poisson, in Paris. In 1842 he and his colleague, Bessel, attended the meetings of the British Association, where they made the acquaintance of English mathematicians.

His early researches were on Gauss' approximation to the value of definite integrals, partial differential equations, Legendre's coefficients, and cubic residues. He read Legendre's *Exercises*, which give an account of elliptic integrals. When he returned the book to the library, he was depressed in spirits and said that important books generally excited in him new ideas, but that this time he had not been led to a single original thought. Though slow at first, his ideas flowed all the richer afterwards. Many of his discoveries in elliptic functions were made independently by Abel. Jacobi communicated his first researches to *Crelle's Journal*. In 1829, at the age

of twenty-five, he published his *Fundamenta Nova Theoriæ Functionum Ellipticarum*, which contains in condensed form the main results in elliptic functions. This work at once secured for him a wide reputation. He then made a closer study of theta-functions and lectured to his pupils on a new theory of elliptic functions based on the theta-functions. He developed a theory of transformation which led him to a multitude of formulæ containing q, a transcendental function of the modulus, defined by the equation $q = e^{-\pi k'/k}$. He was also led by it to consider the two new functions H and Θ, which taken each separately with two different arguments are the four (single) theta-functions designated by the Θ_1, Θ_2, Θ_3, Θ_4.[56] In a short but very important memoir of 1832, he shows that for the hyperelliptic integral of any class the direct functions to which Abel's theorem has reference are not functions of a single variable, such as the elliptic *sn*, *cn*, *dn*, but functions of p variables.[56] Thus in the case $p = 2$, which Jacobi especially considers, it is shown that Abel's theorem has reference to two functions $\lambda(u, v)$, $\lambda_1(u, v)$, each of two variables, and gives in effect an addition-theorem for the expression of the functions $\lambda(u + u', v + v')$, $\lambda_1(u + u', v + v')$ algebraically in terms of the functions $\lambda(u, v)$, $\lambda_1(u, v)$, $\lambda(u', v')$, $\lambda_1(u', v')$. By the memoirs of Abel and Jacobi it may be considered that the notion of the Abelian function of p variables was established and the addition-theorem for these functions given. Recent studies touching Abelian functions have been made by Weierstrass, E. Picard, Madame Kowalevski, and Poincaré. Jacobi's work on differential equations, determinants, dynamics, and the theory of numbers is mentioned elsewhere.

In 1842 Jacobi visited Italy for a few months to recuperate his health. At this time the Prussian government gave him a pension, and he moved to Berlin, where the last years of his life were spent.

The researches on functions mentioned thus far have been greatly extended. In 1858 **Charles Hermite** of Paris (born 1822), introduced in place of the variable q of Jacobi a new variable ω connected with it by the equation $q = e^{i\pi\omega}$, so that $\omega = ik'/k$, and was led to consider the functions $\phi(\omega)$, $\psi(\omega)$, $\chi(\omega)$.[56] Henry Smith regarded a theta-function with the argument equal to zero, as a function of ω. This he called an omega-function, while the three functions $\phi(\omega)$, $\psi(\omega)$, $\chi(\omega)$, are his modular functions. Researches on theta-functions with respect to real and imaginary arguments have been made by Meissel of Kiel, J. Thomae of Jena, Alfred Enneper of Göttingen (1830–1885). A general formula for the product of two theta-functions was given in 1854 by H. Schröter of Breslau (1829–1892). These functions have been studied also by Cauchy, Königsberger of Heidelberg (born 1837), F. S. Richelot of Königsberg (1808–1875), Johann Georg Rosenhain of Königsberg (1816–1887), L. Schläfli of Bern (born 1818).[85]

Legendre's method of reducing an elliptic differential to its normal form has called forth many investigations, most important of which are those of Richelot and of Weierstrass of Berlin.

The algebraic transformations of elliptic functions involve a relation between the old modulus and the new one which Jacobi expressed by a differential equation of the third order, and also by an algebraic equation, called by him "modular equation." The notion of modular equations was familiar to Abel, but the development of this subject devolved upon later investigators. These equations have become of importance in the theory of algebraic equations, and have been studied by Sohnke, E. Mathieu, L. Königsberger, E. Betti of Pisa (died 1892), C. Hermite of Paris, Joubert of Angers, Francesco Brioschi of Milan. Schläfli, H. Schröter, M. Gudermann of Cleve, Gützlaff.

Felix Klein of Göttingen has made an extensive study of modular functions, dealing with a type of operations lying between the two extreme types, known as the theory of substitutions and the theory of invariants and covariants. Klein's theory has been presented in book-form by his pupil, Robert Fricke. The bolder features of it were first published in his *Ikosaeder*, 1884. His researches embrace the theory of modular functions as a specific class of elliptic functions, the statement of a more general problem as based on the doctrine of groups of operations, and the further development of the subject in connection with a class of Riemann's surfaces.

The elliptic functions were expressed by Abel as quotients of doubly infinite products. He did not, however, inquire rigorously into the convergency of the products. In 1845 Cayley studied these products, and found for them a complete theory, based in part upon geometrical interpretation, which he made the basis of the whole theory of elliptic functions. Eisenstein discussed by purely analytical methods the general doubly infinite product, and arrived at results which have been greatly simplified in form by the theory of primary factors, due to Weierstrass. A certain function involving a doubly infinite product has been called by Weierstrass the sigma-function, and is the basis of his beautiful theory of elliptic functions. The first systematic presentation of Weierstrass' theory of elliptic functions was published in 1886 by G. H. Halphen in his *Théorie des fonctions elliptiques et des leurs applications.* Applications of these functions have been given also by A. G. Greenhill. Generalisations analogous to those of Weierstrass on elliptic functions have been made by Felix Klein on hyperelliptic functions.

Standard works on elliptic functions have been published by *Briot and Bouquet* (1859), by *Königsberger, Cayley, Heinrich Durège* of Prague (1821–1893), and others.

Jacobi's work on Abelian and theta-functions was greatly extended by **Adolph Göpel** (1812–1847), professor in a gymnasium near Potsdam, and **Johann Georg Rosenhain** of Königsberg (1816–1887). Göpel in his *Theoriæ transcendentium primi ordinis adumbratio levis* (*Crelle*, 35, 1847) and Rosenhain in several memoirs established each independently, on the analogy of the single theta-functions, the functions of two variables, called double theta-functions, and worked out in connection with them the theory of the Abelian functions of two variables. The theta-relations established by Göpel and Rosenhain received for thirty years no further development, notwithstanding the fact that the double theta series came to be of increasing importance in analytical, geometrical, and mechanical problems, and that Hermite and Königsberger had considered the subject of transformation. Finally, the investigations of C. W. Borchardt of Berlin (1817–1880), treating of the representation of Kummer's surface by Göpel's biquadratic relation between four theta-functions of two variables, and researches of H. H. Weber of Marburg, F. Prym of Würzburg, Adolf Krazer, and Martin Krause of Dresden led to broader views. Researches on double theta-functions, made by Cayley, were extended to quadruple theta-functions by Thomas Craig of the Johns Hopkins University.

Starting with the integrals of the most general form and considering the inverse functions corresponding to these integrals (the Abelian functions of p variables), *Riemann* defined the theta-functions of p variables as the sum of a p-tuply infinite series of exponentials, the general term depending on p variables. Riemann shows that the Abelian functions are algebraically connected with theta-functions of the proper arguments, and presents the theory in the broadest form.[56] He rests the theory of the multiple theta-functions upon the general principles of the theory of functions of a complex variable.

Through the researches of A. Brill of Tübingen, M. Nöther of Erlangen, and Ferdinand Lindemann of Munich, made in connection with Riemann-Roch's theorem and the theory of residuation, there has grown out of the theory of Abelian functions a theory of algebraic functions and point-groups on algebraic curves.

Before proceeding to the general theory of functions, we make mention of the "calculus of functions," studied chiefly by C. Babbage, J. F. W. Herschel, and De Morgan, which was not so much a theory of functions as a theory of the solution of functional equations by means of known functions or symbols.

The history of the general theory of functions begins with the adoption of new definitions of a function. With the Bernoullis and Leibniz, y was called a function of x, if there existed an equation between these variables which made it possible to calculate y for any given value of x lying anywhere between $-\infty$ and $+\infty$. The study of Fourier's theory of heat led Dirichlet to a new definition: y is called a function of x, if y possess one or more definite values for each of certain values that x is assumed to take in an interval x_0 to x_1. In functions thus defined, there need be no analytical connection between y and x, and it becomes necessary to look for possible discontinuities. A great revolution in the ideas of a function was brought about by Cauchy when, in a function as defined by Dirichlet, he gave the variables imaginary values, and when he extended the notion of a definite integral by letting the variable pass from one limit to the other by a succession of imaginary values along arbitrary paths. Cauchy established several fundamental theorems, and gave the first great impulse to the study of the general theory of functions. His researches were continued in France by Puiseux and Liouville. But more profound investigations were made in Germany by Riemann.

Georg Friedrich Bernhard Riemann (1826–1866) was born at Breselenz in Hanover. His father wished him to study theology, and he accordingly entered upon philological and theological studies at Göttingen. He attended also some lectures on mathematics. Such was his predilection for this science that he abandoned theology. After studying for a time under Gauss and Stern, he was drawn, in 1847, to Berlin by a galaxy of mathematicians, in which shone Dirichlet, Jacobi, Steiner, and Eisenstein. Returning to Göttingen in 1850, he studied physics under Weber, and obtained the doctorate the following year. The thesis presented on that occasion, *Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen complexen Grösse*, excited the admiration of Gauss to a very unusual degree, as did also Riemann's trial lecture, *Ueber die Hypothesen welche der Geometrie zu Grunde liegen.* Riemann's Habilitationsschrift was on the Representation of a Function by means of a Trigonometric Series, in which he advanced materially beyond the position of Dirichlet. Our hearts are drawn to this extraordinarily gifted but shy genius when we read of the timidity and nervousness displayed when he began to lecture at Göttingen, and of his jubilation over the unexpectedly large audience of eight students at his first lecture on differential equations.

Later he lectured on Abelian functions to a class of three only, — Schering, Bjerknes, and Dedekind. Gauss died in 1855, and was succeeded by Dirichlet. On the death of the latter, in 1859, Riemann was made ordinary professor. In 1860 he visited Paris, where he made the acquaintance of French mathematicians. The delicate state of his health induced him to go to Italy three times. He died on his last trip at Selasca, and was buried at Biganzolo.

Like all of Riemann's researches, those on functions were profound and far-reaching. He laid the foundation for a

general theory of functions of a complex variable. The theory of potential, which up to that time had been used only in mathematical physics, was applied by him in pure mathematics. He accordingly based his theory of functions on the partial differential equation, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u = 0$, which must hold for the analytical function $w = u + iv$ of $z = x + iy$. It had been proved by Dirichlet that (for a plane) there is always one, and only one, function of x and y, which satisfies $\Delta u = 0$, and which, together with its differential quotients of the first two orders, is for all values of x and y within a given area one-valued and continuous, and which has for points on the boundary of the area arbitrarily given values.[86] Riemann called this "Dirichlet's principle," but the same theorem was stated by Green and proved analytically by Sir William Thomson. It follows then that w is uniquely determined for all points within a closed surface, if u is arbitrarily given for all points on the curve, whilst v is given for one point within the curve. In order to treat the more complicated case where w has n values for one value of z, and to observe the conditions about continuity, Riemann invented the celebrated surfaces, known as "Riemann's surfaces," consisting of n coincident planes or sheets, such that the passage from one sheet to another is made at the branch-points, and that the n sheets form together a multiply-connected surface, which can be dissected by cross-cuts into a singly-connected surface. The n-valued function w becomes thus a one-valued function. Aided by researches of J. Lüroth of Freiburg and of Clebsch, W. K. Clifford brought Riemann's surface for algebraic functions to a canonical form, in which only the two last of the n leaves are multiply-connected, and then transformed the surface into the surface of a solid with p holes. A. Hurwitz of Zürich discussed the question, how far a Riemann's surface is deter-

minate by the assignment of its number of sheets, its branch-points and branch-lines.[62]

Riemann's theory ascertains the criteria which will determine an analytical function by aid of its discontinuities and boundary conditions, and thus defines a function independently of a mathematical expression. In order to show that two different expressions are identical, it is not necessary to transform one into the other, but it is sufficient to prove the agreement to a far less extent, merely in certain critical points.

Riemann's theory, as based on Dirichlet's principle (Thomson's theorem), is not free from objections. It has become evident that the existence of a derived function is not a consequence of continuity, and that a function may be integrable without being differentiable. It is not known how far the methods of the infinitesimal calculus and the calculus of variations (by which Dirichlet's principle is established) can be applied to an unknown analytical function in its generality. Hence the use of these methods will endow the functions with properties which themselves require proof. Objections of this kind to Riemann's theory have been raised by Kronecker, Weierstrass, and others, and it has become doubtful whether his most important theorems are actually proved. In consequence of this, attempts have been made to graft Riemann's speculations on the more strongly rooted methods of Weierstrass. The latter developed a theory of functions by starting, not with the theory of potential, but with analytical expressions and operations. Both applied their theories to Abelian functions, but there Riemann's work is more general.[86]

The theory of functions of one complex variable has been studied since Riemann's time mainly by **Karl Weierstrass** of Berlin (born 1815), **Gustaf Mittag-Leffler** of Stockholm (born 1846), and Poincaré of Paris. Of the three classes of such

functions (viz. functions uniform throughout, functions uniform only in lacunary spaces, and non-uniform functions) Weierstrass showed that those functions of the first class which can be developed according to ascending powers of x into converging series, can be decomposed into a product of an infinite number of primary factors. A primary factor of the species n is the product $\left(1-\frac{x}{a}\right)e^{P_{(x)}}$, $P_{(x)}$ being an entire polynomial of the nth degree. A function of the species n is one, all the primary factors of which are of species n. This classification gave rise to many interesting problems studied also by Poincaré.

The first of the three classes of functions of a complex variable embraces, among others, functions having an infinite number of singular points, but no singular lines, and at the same time no isolated singular points. These are Fuchsian functions, existing throughout the whole extent. Poincaré first gave an example of such a function.

Uniform functions of two variables, unaltered by certain linear substitutions, called hyperfuchsian functions, have been studied by E. Picard of Paris, and by Poincaré.[81]

Functions of the second class, uniform only in lacunary spaces, were first pointed out by Weierstrass. The Fuchsian and the Kleinian functions do not generally exist, except in the interior of a circle or of a domain otherwise bounded, and are therefore examples of functions of the second class. Poincaré has shown how to generate functions of this class, and has studied them along the lines marked out by Weierstrass. Important is his proof that there is no way of generalising them so as to get rid of the lacunæ.

Non-uniform functions are much less developed than the preceding classes, even though their properties in the vicinity of a given point have been diligently studied, and though

much light has been thrown on them by the use of Riemann's surfaces. With the view of reducing their study to that of uniform transcendents, Poincaré proved that if y is any analytical non-uniform function of x, one can always find a variable z, such that x and y are uniform functions of z.

Weierstrass and Darboux have each given examples of continuous functions having no derivatives. Formerly it had been generally assumed that every function had a derivative. Ampère was the first who attempted to prove analytically (1806) the existence of a derivative, but the demonstration is not valid. In treating of discontinuous functions, Darboux established rigorously the necessary and sufficient condition that a continuous or discontinuous function be susceptible of integration. He gave fresh evidence of the care that must be exercised in the use of series by giving an example of a series always convergent and continuous, such that the series formed by the integrals of the terms is always convergent, and yet does not represent the integral of the first series.[87]

The general theory of functions of two variables has been investigated to some extent by Weierstrass and Poincaré.

H. A. Schwarz of Berlin (born 1845), a pupil of Weierstrass, has given the conform representation (*Abbildung*) of various surfaces on a circle. In transforming by aid of certain substitutions a polygon bounded by circular arcs into another also bounded by circular arcs, he was led to a remarkable differential equation $\psi(u', t) = \psi(u, t)$, where $\psi(u, t)$ is the expression which Cayley calls the "Schwarzian derivative," and which led Sylvester to the theory of reciprocants. Schwarz's developments on minimum surfaces, his work on hypergeometric series, his inquiries on the existence of solutions to important partial differential equations under prescribed conditions, have secured a prominent place in mathematical literature.

The modern theory of functions of one real variable was first worked out by H. Hankel, Dedekind, G. Cantor, Dini, and Heine, and then carried further, principally, by Weierstrass, Schwarz, Du Bois-Reymond, Thomae, and Darboux. Hankel established the principle of the condensation of singularities; Dedekind and Cantor gave definitions for irrational numbers; definite integrals were studied by Thomae, Du Bois-Reymond, and Darboux along the lines indicated by the definitions of such integrals given by Cauchy, Dirichlet, and Riemann. Dini wrote a text-book on functions of a real variable (1878), which was translated into German, with additions, by J. Lüroth and A. Schepp. Important works on the theory of functions are the *Cours de M. Hermite*, *Tannery's Théorie des Fonctions d'une variable seule*, *A Treatise on the Theory of Functions* by *James Harkness* and *Frank Morley*, and *Theory of Functions of a Complex Variable* by *A. R. Forsyth*.

THEORY OF NUMBERS.

"Mathematics, the queen of the sciences, and arithmetic, the queen of mathematics." Such was the dictum of Gauss, who was destined to revolutionise the theory of numbers. When asked who was the greatest mathematician in Germany, Laplace answered, Pfaff. When the questioner said he should have thought Gauss was, Laplace replied, "Pfaff is by far the greatest mathematician in Germany; but Gauss is the greatest in all Europe."[83] Gauss is one of the three greatest masters of modern analysis, — Lagrange, Laplace, Gauss. Of these three contemporaries he was the youngest. While the first two belong to the period in mathematical history preceding the one now under consideration, Gauss is the one whose writings may truly be said to mark the beginning

of our own epoch. In him that abundant fertility of invention, displayed by mathematicians of the preceding period, is combined with an absolute rigorousness in demonstration which is too often wanting in their writings, and which the ancient Greeks might have envied. Unlike Laplace, Gauss strove in his writings after perfection of form. He rivals Lagrange in elegance, and surpasses this great Frenchman in rigour. Wonderful was his richness of ideas; one thought followed another so quickly that he had hardly time to write down even the most meagre outline. At the age of twenty Gauss had overturned old theories and old methods in all branches of higher mathematics; but little pains did he take to publish his results, and thereby to establish his priority. He was the first to observe rigour in the treatment of infinite series, the first to fully recognise and emphasise the importance, and to make systematic use of determinants and of imaginaries, the first to arrive at the method of least squares, the first to observe the double periodicity of elliptic functions. He invented the heliotrope and, together with Weber, the bifilar magnetometer and the declination instrument. He reconstructed the whole of magnetic science.

Carl Friedrich Gauss[47] (1777–1855), the son of a bricklayer, was born at Brunswick. He used to say, jokingly, that he could reckon before he could talk. The marvellous aptitude for calculation of the young boy attracted the attention of Bartels, afterwards professor of mathematics at Dorpat, who brought him under the notice of Charles William, Duke of Brunswick. The duke undertook to educate the boy, and sent him to the Collegium Carolinum. His progress in languages there was quite equal to that in mathematics. In 1795 he went to Göttingen, as yet undecided whether to pursue philology or mathematics. Abraham Gotthelf Kästner, then professor of mathematics there, and now chiefly remembered for

his *Geschichte der Mathematik* (1796), was not an inspiring teacher. At the age of nineteen Gauss discovered a method of inscribing in a circle a regular polygon of seventeen sides, and this success encouraged him to pursue mathematics. He worked quite independently of his teachers, and while a student at Göttingen made several of his greatest discoveries. Higher arithmetic was his favourite study. Among his small circle of intimate friends was Wolfgang Bolyai. After completing his course he returned to Brunswick. In 1798 and 1799 he repaired to the university at Helmstadt to consult the library, and there made the acquaintance of Pfaff, a mathematician of much power. In 1807 the Emperor of Russia offered Gauss a chair in the Academy at St. Petersburg, but by the advice of the astronomer Olbers, who desired to secure him as director of a proposed new observatory at Göttingen, he declined the offer, and accepted the place at Göttingen. Gauss had a marked objection to a mathematical chair, and preferred the post of astronomer, that he might give all his time to science. He spent his life in Göttingen in the midst of continuous work. In 1828 he went to Berlin to attend a meeting of scientists, but after this he never again left Göttingen, except in 1854, when a railroad was opened between Göttingen and Hanover. He had a strong will, and his character showed a curious mixture of self-conscious dignity and child-like simplicity. He was little communicative, and at times morose.

A new epoch in the theory of numbers dates from the publication of his *Disquisitiones Arithmeticæ*, Leipzig, 1801. The beginning of this work dates back as far as 1795. Some of its results had been previously given by Lagrange and Euler, but were reached independently by Gauss, who had gone deeply into the subject before he became acquainted with the writings of his great predecessors. The *Disquisitiones Arithmeticæ*

was already in print when Legendre's *Théorie des Nombres* appeared. The great law of quadratic reciprocity, given in the fourth section of Gauss' work, a law which involves the whole theory of quadratic residues, was discovered by him by induction before he was eighteen, and was proved by him one year later. Afterwards he learned that Euler had imperfectly enunciated that theorem, and that Legendre had attempted to prove it, but met with apparently insuperable difficulties. In the fifth section Gauss gave a second proof of this "gem" of higher arithmetic. In 1808 followed a third and fourth demonstration; in 1817, a fifth and sixth. No wonder that he felt a personal attachment to this theorem. Proofs were given also by Jacobi, Eisenstein, Liouville, Lebesgue, A. Genocchi, Kummer, M. A. Stern, Chr. Zeller, Kronecker, Bouniakowsky, E. Schering, J. Petersen, Voigt, E. Busche, and Th. Pepin.[48] The solution of the problem of the representation of numbers by binary quadratic forms is one of the great achievements of Gauss. He created a new algorithm by introducing the theory of congruences. The fourth section of the *Disquisitiones Arithmeticæ*, treating of congruences of the second degree, and the fifth section, treating of quadratic forms, were, until the time of Jacobi, passed over with universal neglect, but they have since been the starting-point of a long series of important researches. The seventh or last section, developing the theory of the division of the circle, was received from the start with deserved enthusiasm, and has since been repeatedly elaborated for students. A standard work on *Kreistheilung* was published in 1872 by Paul Bachmann, then of Breslau. Gauss had planned an eighth section, which was omitted to lessen the expense of publication. His papers on the theory of numbers were not all included in his great treatise. Some of them were published for the first time after his death in his collected works (1863–1871). He wrote two memoirs on

the theory of biquadratic residues (1825 and 1831), the second of which contains a theorem of biquadratic reciprocity.

Gauss was led to astronomy by the discovery of the planet Ceres at Palermo in 1801. His determination of the elements of its orbit with sufficient accuracy to enable Olbers to rediscover it, made the name of Gauss generally known. In 1809 he published the *Theoria motus corporum coelestium*, which contains a discussion of the problems arising in the determination of the movements of planets and comets from observations made on them under any circumstances. In it are found four formulæ in spherical trigonometry, now usually called "Gauss' Analogies," but which were published somewhat earlier by Karl Brandon Mollweide of Leipzig (1774–1825), and earlier still by Jean Baptiste Joseph Delambre (1749–1822).[44] Many years of hard work were spent in the astronomical and magnetic observatory. He founded the German Magnetic Union, with the object of securing continuous observations at fixed times. He took part in geodetic observations, and in 1843 and 1846 wrote two memoirs, *Ueber Gegenstände der höheren Geodesie.* He wrote on the attraction of homogeneous ellipsoids, 1813. In a memoir on capillary attraction, 1833, he solves a problem in the calculus of variations involving the variation of a certain double integral, the limits of integration being also variable; it is the earliest example of the solution of such a problem. He discussed the problem of rays of light passing through a system of lenses.

Among Gauss' pupils were Christian Heinrich Schumacher, Christian Gerling, Friedrich Nicolai, August Ferdinand Möbius, Georg Wilhelm Struve, Johann Frantz Encke.

Gauss' researches on the theory of numbers were the starting-point for a school of writers, among the earliest of whom was Jacobi. The latter contributed to *Crelle's Journal* an article on cubic residues, giving theorems without proofs. After the

publication of Gauss' paper on biquadratic residues, giving the law of biquadratic reciprocity, and his treatment of complex numbers, Jacobi found a similar law for cubic residues. By the theory of elliptical functions, he was led to beautiful theorems on the representation of numbers by 2, 4, 6, and 8 squares. Next come the researches of Dirichlet, the expounder of Gauss, and a contributor of rich results of his own.

Peter Gustav Lejeune Dirichlet[88] (1805–1859) was born in Düren, attended the gymnasium in Bonn, and then the Jesuit gymnasium in Cologne. In 1822 he was attracted to Paris by the names of Laplace, Legendre, Fourier, Poisson, Cauchy. The facilities for a mathematical education there were far better than in Germany, where Gauss was the only great figure. He read in Paris Gauss' *Disquisitiones Arithmeticæ*, a work which he never ceased to admire and study. Much in it was simplified by Dirichlet, and thereby placed within easier reach of mathematicians. His first memoir on the impossibility of certain indeterminate equations of the fifth degree was presented to the French Academy in 1825. He showed that Fermat's equation, $x^n + y^n = z^n$, cannot exist when $n = 5$. Some parts of the analysis are, however, Legendre's. Euler and Lagrange had proved this when n is 3 and 4, and Lamé proved it when $n = 7$. Dirichlet's acquaintance with Fourier led him to investigate Fourier's series. He became docent in Breslau in 1827. In 1828 he accepted a position in Berlin, and finally succeeded Gauss at Göttingen in 1855. The general principles on which depends the average number of classes of binary quadratic forms of positive and negative determinant (a subject first investigated by Gauss) were given by Dirichlet in a memoir, *Ueber die Bestimmung der mittleren Werthe in der Zahlentheorie*, 1849. More recently F. Mertens of Graz has determined the asymptotic values of several numerical functions. Dirichlet gave some

attention to prime numbers. Gauss and Legendre had given expressions denoting approximately the asymptotic value of the number of primes inferior to a given limit, but it remained for Riemann in his memoir, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, 1859, to give an investigation of the asymptotic frequency of primes which is rigorous. Approaching the problem from a different direction, *Patnutij Tchebycheff*, formerly professor in the University of St. Petersburg (born 1821), established, in a celebrated memoir, *Sur les Nombres Premiers*, 1850, the existence of limits within which the sum of the logarithms of the primes P, inferior to a given number x, must be comprised.[89] This paper depends on very elementary considerations, and, in that respect, contrasts strongly with Riemann's, which involves abstruse theorems of the integral calculus. Poincaré's papers, Sylvester's contraction of Tchebycheff's limits, with reference to the distribution of primes, and researches of J. Hadamard (awarded the *Grand prix* of 1892), are among the latest researches in this line. The enumeration of prime numbers has been undertaken at different times by various mathematicians. In 1877 the British Association began the preparation of factor-tables, under the direction of J. W. L. Glaisher. The printing, by the Association, of tables for the sixth million marked the completion of tables, to the preparation of which Germany, France, and England contributed, and which enable us to resolve into prime factors every composite number less than 9,000,000.

Miscellaneous contributions to the theory of numbers were made by *Cauchy*. He showed, for instance, how to find all the infinite solutions of a homogeneous indeterminate equation of the second degree in three variables when one solution is given. He established the theorem that if two congruences, which have the same modulus, admit of a common solution,

the modulus is a divisor of their resultant. **Joseph Liouville** (1809–1882), professor at the Collége de France, investigated mainly questions on the theory of quadratic forms of two, and of a greater number of variables. Profound researches were instituted by **Ferdinand Gotthold Eisenstein** (1823–1852), of Berlin. Ternary quadratic forms had been studied somewhat by Gauss, but the extension from two to three indeterminates was the work of Eisenstein who, in his memoir, *Neue Theoreme der höheren Arithmetik*, defined the ordinal and generic characters of ternary quadratic forms of uneven determinant; and, in case of definite forms, assigned the weight of any order or genus. But he did not publish demonstrations of his results. In inspecting the theory of binary cubic forms, he was led to the discovery of the first covariant ever considered in analysis. He showed that the series of theorems, relating to the presentation of numbers by sums of squares, ceases when the number of squares surpasses eight. Many of the proofs omitted by Eisenstein were supplied by Henry Smith, who was one of the few Englishmen who devoted themselves to the study of higher arithmetic.

Henry John Stephen Smith[90] (1826–1883) was born in London, and educated at Rugby and at Balliol College, Oxford. Before 1847 he travelled much in Europe for his health, and at one time attended lectures of Arago in Paris, but after that year he was never absent from Oxford for a single term. In 1861 he was elected Savilian professor of geometry. His first paper on the theory of numbers appeared in 1855. The results of ten years' study of everything published on the theory of numbers are contained in his Reports which appeared in the British Association volumes from 1859 to 1865. These reports are a model of clear and precise exposition and perfection of form. They contain much original matter, but the chief results of his own discoveries were

printed in the *Philosophical Transactions* for 1861 and 1867. They treat of linear indeterminate equations and congruences, and of the orders and genera of ternary quadratic forms. He established the principles on which the extension to the general case of n indeterminates of quadratic forms depends. He contributed also two memoirs to the *Proceedings of the Royal Society* of 1864 and 1868, in the second of which he remarks that the theorems of Jacobi, Eisenstein, and Liouville, relating to the representation of numbers by 4, 6, 8 squares, and other simple quadratic forms are deducible by a uniform method from the principles indicated in his paper. Theorems relating to the case of 5 squares were given by Eisenstein, but Smith completed the enunciation of them, and added the corresponding theorems for 7 squares. The solution of the cases of 2, 4, 6 squares may be obtained by elliptic functions, but when the number of squares is odd, it involves processes peculiar to the theory of numbers. This class of theorems is limited to 8 squares, and Smith completed the group. In ignorance of Smith's investigations, the French Academy offered a prize for the demonstration and completion of Eisenstein's theorems for 5 squares. This Smith had accomplished fifteen years earlier. He sent in a dissertation in 1882, and next year, a month after his death, the prize was awarded to him, another prize being also awarded to H. Minkowsky of Bonn. The theory of numbers led Smith to the study of elliptic functions. He wrote also on modern geometry. His successor at Oxford was J. J. Sylvester.

Ernst Eduard Kummer (1810–1893), professor in the University of Berlin, is closely identified with the theory of numbers. Dirichlet's work on complex numbers of the form $a+ib$, introduced by Gauss, was extended by him, by Eisenstein, and Dedekind. Instead of the equation $x^4-1=0$, the roots of which yield Gauss' units, Eisenstein used the equation

$x^3 - 1 = 0$ and complex numbers $a + b\rho$ (ρ being a cube root of unity), the theory of which resembles that of Gauss' numbers. Kummer passed to the general case $x^n - 1 = 0$ and got complex numbers of the form $\alpha = a_1 A_1 + a_2 A_2 + a_3 A_3 + \cdots$, where a_i are whole real numbers, and A_i roots of the above equation.[59] Euclid's theory of the greatest common divisor is not applicable to such complex numbers, and their prime factors cannot be defined in the same way as prime factors of common integers are defined. In the effort to overcome this difficulty, Kummer was led to introduce the conception of "ideal numbers." These ideal numbers have been applied by G. Zolotareff of St. Petersburg to the solution of a problem of the integral calculus, left unfinished by Abel (*Liouville's Journal*, Second Series, 1864, Vol. IX.). **Julius Wilhelm Richard Dedekind** of Braunschweig (born 1831) has given in the second edition of Dirichlet's *Vorlesungen über Zahlentheorie* a new theory of complex numbers, in which he to some extent deviates from the course of Kummer, and avoids the use of ideal numbers. Dedekind has taken the roots of any irreducible equation with integral coefficients as the units for his complex numbers. Attracted by Kummer's investigations, his pupil, **Leopold Kronecker** (1823–1891) made researches which he applied to algebraic equations.

On the other hand, efforts have been made to utilise in the theory of numbers the results of the modern higher algebra. Following up researches of Hermite, **Paul Bachmann** of Münster investigated the arithmetical formula which gives the automorphics of a ternary quadratic form.[89] The problem of the equivalence of two positive or definite ternary quadratic forms was solved by L. Seeber; and that of the arithmetical automorphics of such forms, by Eisenstein. The more difficult problem of the equivalence for indefinite ternary forms has been investigated by Edward Selling of Würzburg. On quadratic

forms of four or more indeterminates little has yet been done. Hermite showed that the number of non-equivalent classes of quadratic forms having integral coefficients and a given discriminant is finite, while Zolotareff and A. N. Korkine, both of St. Petersburg, investigated the minima of positive quadratic forms. In connection with binary quadratic forms, Smith established the theorem that if the joint invariant of two properly primitive forms vanishes, the determinant of either of them is represented primitively by the duplicate of the other.

The interchange of theorems between arithmetic and algebra is displayed in the recent researches of J. W. L. Glaisher of Trinity College (born 1848) and Sylvester. Sylvester gave a Constructive Theory of Partitions, which received additions from his pupils, F. Franklin and G. S. Ely.

The conception of "number" has been much extended in our time. With the Greeks it included only the ordinary positive whole numbers; Diophantus added rational fractions to the domain of numbers. Later negative numbers and imaginaries came gradually to be recognised. Descartes fully grasped the notion of the negative; Gauss, that of the imaginary. With Euclid, a ratio, whether rational or irrational, was not a number. The recognition of ratios and irrationals as numbers took place in the sixteenth century, and found expression with Newton. By the ratio method, the continuity of the real number system has been based on the continuity of space, but in recent time three theories of irrationals have been advanced by Weierstrass, J. W. R. Dedekind, G. Cantor, and Heine, which prove the continuity of numbers without borrowing it from space. They are based on the definition of numbers by regular sequences, the use of series and limits, and some new mathematical conceptions.

APPLIED MATHEMATICS.

Notwithstanding the beautiful developments of celestial mechanics reached by Laplace at the close of the eighteenth century, there was made a discovery on the first day of the present century which presented a problem seemingly beyond the power of that analysis. We refer to the discovery of Ceres by Piazzi in Italy, which became known in Germany just after the philosopher Hegel had published a dissertation proving *a priori* that such a discovery could not be made. From the positions of the planet observed by Piazzi its orbit could not be satisfactorily calculated by the old methods, and it remained for the genius of Gauss to devise a method of calculating elliptic orbits which was free from the assumption of a small eccentricity and inclination. Gauss' method was developed further in his *Theoria Motus*. The new planet was re-discovered with aid of Gauss' data by Olbers, an astronomer who promoted science not only by his own astronomical studies, but also by discerning and directing towards astronomical pursuits the genius of Bessel.

Friedrich Wilhelm Bessel[91] (1784–1846) was a native of Minden in Westphalia. Fondness for figures, and a distaste for Latin grammar led him to the choice of a mercantile career. In his fifteenth year he became an apprenticed clerk in Bremen, and for nearly seven years he devoted his days to mastering the details of his business, and part of his nights to study. Hoping some day to become a supercargo on trading expeditions, he became interested in observations at sea. With a sextant constructed by him and an ordinary clock he determined the latitude of Bremen. His success in this inspired him for astronomical study. One work after another was mastered by him, unaided, during the hours snatched from

sleep. From old observations he calculated the orbit of Halley's comet. Bessel introduced himself to Olbers, and submitted to him the calculation, which Olbers immediately sent for publication. Encouraged by Olbers, Bessel turned his back to the prospect of affluence, chose poverty and the stars, and became assistant in J. H. Schröter's observatory at Lilienthal. Four years later he was chosen to superintend the construction of the new observatory at Königsberg.[92] In the absence of an adequate mathematical teaching force, Bessel was obliged to lecture on mathematics to prepare students for astronomy. He was relieved of this work in 1825 by the arrival of Jacobi. We shall not recount the labours by which Bessel earned the title of founder of modern practical astronomy and geodesy. As an observer he towered far above Gauss, but as a mathematician he reverently bowed before the genius of his great contemporary. Of Bessel's papers, the one of greatest mathematical interest is an "*Untersuchung des Theils der planetarischen Störungen, welcher aus der Bewegung der Sonne ensteht*" (1824), in which he introduces a class of transcendental functions, $J_n(x)$, much used in applied mathematics, and known as "Bessel's functions." He gave their principal properties, and constructed tables for their evaluation. Recently it has been observed that Bessel's functions appear much earlier in mathematical literature.[93] Such functions of the zero order occur in papers of Daniel Bernoulli (1732) and Euler on vibration of heavy strings suspended from one end. All of Bessel's functions of the first kind and of integral orders occur in a paper by Euler (1764) on the vibration of a stretched elastic membrane. In 1878 Lord Rayleigh proved that Bessel's functions are merely particular cases of Laplace's functions. J. W. L. Glaisher illustrates by Bessel's functions his assertion that mathematical branches growing out of physical inquiries as a rule "lack the easy flow

or homogeneity of form which is characteristic of a mathematical theory properly so called." These functions have been studied by C. Th. Anger of Danzig, O. Schlömilch of Dresden, R. Lipschitz of Bonn (born 1832), Carl Neumann of Leipzig (born 1832), Eugen Lommel of Leipzig, I. Todhunter of St. John's College, Cambridge.

Prominent among the successors of Laplace are the following: *Siméon Denis Poisson* (1781–1840), who wrote in 1808 a classic *Mémoire sur les inégalités séculaires des moyens mouvements des planètes.* *Giovanni Antonio Amadeo Plana* (1781–1864) of Turin, a nephew of Lagrange, who published in 1811 a *Memoria sulla teoria dell' attrazione degli sferoidi ellitici*, and contributed to the theory of the moon. **Peter Andreas Hansen** (1795–1874) of Gotha, at one time a clockmaker in Tondern, then Schumacher's assistant at Altona, and finally director of the observatory at Gotha, wrote on various astronomical subjects, but mainly on the lunar theory, which he elaborated in his work *Fundamenta nova investigationes orbitæ veræ quam Luna perlustrat* (1838), and in subsequent investigations embracing extensive lunar tables. **George Biddel Airy** (1801–1892), royal astronomer at Greenwich, published in 1826 his *Mathematical Tracts on the Lunar and Planetary Theories.* These researches have since been greatly extended by him. **August Ferdinand Möbius** (1790–1868) of Leipzig wrote, in 1842, *Elemente der Mechanik des Himmels.* **Urbain Jean Joseph Le Verrier** (1811–1877) of Paris wrote, the *Recherches Astronomiques*, constituting in part a new elaboration of celestial mechanics, and is famous for his theoretical discovery of Neptune. **John Couch Adams** (1819–1892) of Cambridge divided with Le Verrier the honour of the mathematical discovery of Neptune, and pointed out in 1853 that Laplace's explanation of the secular acceleration of the moon's mean motion accounted for only half the observed acceleration.

Charles Eugène Delaunay (born 1816, and drowned off Cherbourg in 1872), professor of mechanics at the Sorbonne in Paris, explained most of the remaining acceleration of the moon, unaccounted for by Laplace's theory as corrected by Adams, by tracing the effect of tidal friction, a theory previously suggested independently by Kant, Robert Mayer, and William Ferrel of Kentucky. **George Howard Darwin** of Cambridge (born 1845) made some very remarkable investigations in 1879 on tidal friction, which trace with great certainty the history of the moon from its origin. He has since studied also the effects of tidal friction upon other bodies in the solar system. Criticisms on some parts of his researches have been made by James Nolan of Victoria. **Simon Newcomb** (born 1835), superintendent of the *Nautical Almanac* at Washington, and professor of mathematics at the Johns Hopkins University, investigated the errors in Hansen's tables of the moon. For the last twelve years the main work of the *U. S. Nautical Almanac* office has been to collect and discuss data for new tables of the planets which will supplant the tables of Le Verrier. *G. W. Hill* of that office has contributed an elegant paper on certain possible abbreviations in the computation of the long-period of the moon's motion due to the direct action of the planets, and has made the most elaborate determination yet undertaken of the inequalities of the moon's motion due to the figure of the earth. He has also computed certain lunar inequalities due to the action of Jupiter.

The mathematical discussion of Saturn's rings was taken up first by Laplace, who demonstrated that a homogeneous solid ring could not be in equilibrium, and in 1851 by B. Peirce, who proved their non-solidity by showing that even an irregular solid ring could not be in equilibrium about Saturn. The mechanism of these rings was investigated by James Clerk Maxwell in an essay to which the Adams prize was awarded.

He concluded that they consisted of an aggregate of unconnected particles.

The problem of three bodies has been treated in various ways since the time of Lagrange, but no decided advance towards a more complete algebraic solution has been made, and the problem stands substantially where it was left by him. He had made a reduction in the differential equations to the seventh order. This was elegantly accomplished in a different way by Jacobi in 1843. *R. Radau* (*Comptes Rendus*, LXVII., 1868, p. 841) and *Allégret* (*Journal de Mathématiques*, 1875, p. 277) showed that the reduction can be performed on the equations in their original form. Noteworthy transformations and discussions of the problem have been given by J. L. F. Bertrand, by Émile Bour (1831–1866) of the Polytechnic School in Paris, by Mathieu, Hesse, J. A. Serret. H. Bruns of Leipzig has shown that no advance in the problem of three or of n bodies may be expected by algebraic integrals, and that we must look to the modern theory of functions for a complete solution (*Acta Math.*, XI., p. 43).[93]

Among valuable text-books on mathematical astronomy rank the following works: *Manual of Spherical and Practical Astronomy* by *Chauvenet* (1863), *Practical and Spherical Astronomy* by *Robert Main* of Cambridge, *Theoretical Astronomy* by *James C. Watson* of Ann Arbor (1868), *Traité élémentaire de Mécanique Céleste* of *H. Resal* of the Polytechnic School in Paris, *Cours d'Astronomie de l'École Polytechnique* by *Faye*, *Traité de Mécanique Céleste* by *Tisserand*, *Lehrbuch der Bahnbestimmung* by *T. Oppolzer*, *Mathematische Theorien der Planetenbewegung* by *O. Dziobek*, translated into English by M. W. Harrington and W. J. Hussey.

During the present century we have come to recognise the advantages frequently arising from a geometrical treatment of mechanical problems. To Poinsot, Chasles, and Möbius we

owe the most important developments made in geometrical mechanics. **Louis Poinsot** (1777–1859), a graduate of the Polytechnic School in Paris, and for many years member of the superior council of public instruction, published in 1804 his *Éléments de Statique.* This work is remarkable not only as being the earliest introduction to synthetic mechanics, but also as containing for the first time the idea of couples, which was applied by Poinsot in a publication of 1834 to the theory of rotation. A clear conception of the nature of rotary motion was conveyed by Poinsot's elegant geometrical representation by means of an ellipsoid rolling on a certain fixed plane. This construction was extended by Sylvester so as to measure the rate of rotation of the ellipsoid on the plane.

A particular class of dynamical problems has recently been treated geometrically by **Sir Robert Stawell Ball**, formerly astronomer royal of Ireland, now Lowndean Professor of Astronomy and Geometry at Cambridge. His method is given in a work entitled *Theory of Screws,* Dublin, 1876, and in subsequent articles. Modern geometry is here drawn upon, as was done also by Clifford in the related subject of Biquaternions. Arthur Buchheim of Manchester (1859–1888), showed that Grassmann's Ausdehnungslehre supplies all the necessary materials for a simple calculus of screws in elliptic space. Horace Lamb applied the theory of screws to the question of the steady motion of any solid in a fluid.

Advances in theoretical mechanics, bearing on the integration and the alteration in form of dynamical equations, were made since Lagrange by Poisson, William Rowan Hamilton, Jacobi, Madame Kowalevski, and others. Lagrange had established the "Lagrangian form" of the equations of motion. He had given a theory of the variation of the arbitrary constants which, however, turned out to be less fruitful in results than a theory advanced by Poisson.[99] Pois-

son's theory of the variation of the arbitrary constants and the method of integration thereby afforded marked the first onward step since Lagrange. Then came the researches of Sir William Rowan Hamilton. His discovery that the integration of the dynamic differential equations is connected with the integration of a certain partial differential equation of the first order and second degree, grew out of an attempt to deduce, by the undulatory theory, results in geometrical optics previously based on the conceptions of the emission theory. The *Philosophical Transactions* of 1833 and 1834 contain Hamilton's papers, in which appear the first applications to mechanics of the principle of varying action and the characteristic function, established by him some years previously. The object which Hamilton proposed to himself is indicated by the title of his first paper, viz. the discovery of a function by means of which all integral equations can be actually represented. The new form obtained by him for the equation of motion is a result of no less importance than that which was the professed object of the memoir. Hamilton's method of integration was freed by Jacobi of an unnecessary complication, and was then applied by him to the determination of a geodetic line on the general ellipsoid. With aid of elliptic coordinates Jacobi integrated the partial differential equation and expressed the equation of the geodetic in form of a relation between two Abelian integrals. Jacobi applied to differential equations of dynamics the theory of the ultimate multiplier. The differential equations of dynamics are only one of the classes of differential equations considered by Jacobi. Dynamic investigations along the lines of Lagrange, Hamilton, and Jacobi were made by Liouville, A. Desboves, Serret, J. C. F. Sturm, Ostrogradsky, J. Bertrand, Donkin, Brioschi, leading up to the development of the theory of a system of canonical integrals.

An important addition to the theory of the motion of a solid body about a fixed point was made by Madame **Sophie de Kowalevski**[96] (1853–1891), who discovered a new case in which the differential equations of motion can be integrated. By the use of theta-functions of two independent variables she furnished a remarkable example of how the modern theory of functions may become useful in mechanical problems. She was a native of Moscow, studied under Weierstrass, obtained the doctor's degree at Göttingen, and from 1884 until her death was professor of higher mathematics at the University of Stockholm. The research above mentioned received the Bordin prize of the French Academy in 1888, which was doubled on account of the exceptional merit of the paper.

There are in vogue three forms for the expression of the kinetic energy of a dynamical system: the Lagrangian, the Hamiltonian, and a modified form of Lagrange's equations in which certain velocities are omitted. The kinetic energy is expressed in the first form as a homogeneous quadratic function of the velocities, which are the time-variations of the co-ordinates of the system; in the second form, as a homogeneous quadratic function of the momenta of the system; the third form, elaborated recently by Edward John Routh of Cambridge, in connection with his theory of "ignoration of co-ordinates," and by A. B. Basset, is of importance in hydrodynamical problems relating to the motion of perforated solids in a liquid, and in other branches of physics.

In recent time great practical importance has come to be attached to the principle of mechanical similitude. By it one can determine from the performance of a model the action of the machine constructed on a larger scale. The principle was first enunciated by Newton (*Principia*, Bk. II., Sec. VIII., Prop. 32), and was derived by Bertrand from the principle of virtual velocities. A corollary to it, applied in ship-build-

ing, goes by the name of William Froude's law, but was enunciated also by Reech.

The present problems of dynamics differ materially from those of the last century. The explanation of the orbital and axial motions of the heavenly bodies by the law of universal gravitation was the great problem solved by Clairaut, Euler, D'Alembert, Lagrange, and Laplace. It did not involve the consideration of frictional resistances. In the present time the aid of dynamics has been invoked by the physical sciences. The problems there arising are often complicated by the presence of friction. Unlike astronomical problems of a century ago, they refer to phenomena of matter and motion that are usually concealed from direct observation. The great pioneer in such problems is Lord Kelvin. While yet an undergraduate at Cambridge, during holidays spent at the seaside, he entered upon researches of this kind by working out the theory of spinning tops, which previously had been only partially explained by Jellet in his *Treatise on the Theory of Friction* (1872), and by Archibald Smith.

Among standard works on mechanics are **Jacobi's** *Vorlesungen über Dynamik*, edited by Clebsch, 1866; **Kirchhoff's** *Vorlesungen über mathematische Physik*, 1876; **Benjamin Peirce's** *Analytic Mechanics*, 1855; **Somoff's** *Theoretische Mechanik*, 1879; **Tait and Steele's** *Dynamics of a Particle*, 1856; **Minchin's** *Treatise on Statics;* **Routh's** *Dynamics of a System of Rigid Bodies;* **Sturm's** *Cours de Mécanique de l'École Polytechnique.*

The equations which constitute the foundation of the theory of fluid motion were fully laid down at the time of Lagrange, but the solutions actually worked out were few and mainly of the irrotational type. A powerful method of attacking problems in fluid motion is that of images, introduced in 1843 by George Gabriel Stokes of Pembroke College, Cambridge. It received little attention until Sir William Thomson's dis-

covery of electrical images, whereupon the theory was extended by Stokes, Hicks, and Lewis. In 1849, Thomson gave the maximum and minimum theorem peculiar to hydrodynamics, which was afterwards extended to dynamical problems in general.

A new epoch in the progress of hydrodynamics was created, in 1856, by Helmholtz, who worked out remarkable properties of rotational motion in a homogeneous, incompressible fluid, devoid of viscosity. He showed that the vortex filaments in such a medium may possess any number of knottings and twistings, but are either endless or the ends are in the free surface of the medium; they are indivisible. These results suggested to Sir William Thomson the possibility of founding on them a new form of the atomic theory, according to which every atom is a vortex ring in a non-frictional ether, and as such must be absolutely permanent in substance and duration. The vortex-atom theory is discussed by J. J. Thomson of Cambridge (born 1856) in his classical treatise on the *Motion of Vortex Rings*, to which the Adams Prize was awarded in 1882. Papers on vortex motion have been published also by Horace Lamb, Thomas Craig, Henry A. Rowland, and Charles Chree.

The subject of jets was investigated by Helmholtz, Kirchhoff, Plateau, and Rayleigh; the motion of fluids in a fluid by Stokes, Sir W. Thomson, Köpcke, Greenhill, and Lamb; the theory of viscous fluids by Navier, Poisson, Saint-Venant, Stokes, O. E. Meyer, Stefano, Maxwell, Lipschitz, Craig, Helmholtz, and A. B. Basset. Viscous fluids present great difficulties, because the equations of motion have not the same degree of certainty as in perfect fluids, on account of a deficient theory of friction, and of the difficulty of connecting oblique pressures on a small area with the differentials of the velocities.

Waves in liquids have been a favourite subject with Eng-

lish mathematicians. The early inquiries of Poisson and Cauchy were directed to the investigation of waves produced by disturbing causes acting arbitrarily on a small portion of the fluid. The velocity of the long wave was given approximately by Lagrange in 1786 in case of a channel of rectangular cross-section, by Green in 1839 for a channel of triangular section, and by P. Kelland for a channel of any uniform section. Sir George B. Airy, in his treatise on *Tides and Waves*, discarded mere approximations, and gave the exact equation on which the theory of the long wave in a channel of uniform rectangular section depends. But he gave no general solutions. J. McCowan of University College at Dundee discusses this topic more fully, and arrives at exact and complete solutions for certain cases. The most important application of the theory of the long wave is to the explanation of tidal phenomena in rivers and estuaries.

The mathematical treatment of solitary waves was first taken up by S. Earnshaw in 1845, then by Stokes; but the first sound approximate theory was given by J. Boussinesq in 1871, who obtained an equation for their form, and a value for the velocity in agreement with experiment. Other methods of approximation were given by Rayleigh and J. McCowan. In connection with deep-water waves, Osborne Reynolds gave in 1877 the dynamical explanation for the fact that a group of such waves advances with only half the rapidity of the individual waves.

The solution of the problem of the general motion of an ellipsoid in a fluid is due to the successive labours of Green (1833), Clebsch (1856), and Bjerknes (1873). The free motion of a solid in a liquid has been investigated by W. Thomson, Kirchhoff, and Horace Lamb. By these labours, the motion of a single solid in a fluid has come to be pretty well understood, but the case of two solids in a fluid is not devel-

oped so fully. The problem has been attacked by W. M. Hicks.

The determination of the period of oscillation of a rotating liquid spheroid has important bearings on the question of the origin of the moon. G. H. Darwin's investigations thereon, viewed in the light of Riemann's and Poincaré's researches, seem to disprove Laplace's hypothesis that the moon separated from the earth as a ring, because the angular velocity was too great for stability ; Darwin finds no instability.

The explanation of the contracted vein has been a point of much controversy, but has been put in a much better light by the application of the principle of momentum, originated by Froude and Rayleigh. Rayleigh considered also the reflection of waves, not at the surface of separation of two uniform media, where the transition is abrupt, but at the confines of two media between which the transition is gradual.

The first serious study of the circulation of winds on the earth's surface was instituted at the beginning of the second quarter of this century by *H. W. Dové*, *William C. Redfield*, and *James P. Espy*, followed by researches of *W. Reid*, *Piddington*, and *Elias Loomis*. But the deepest insight into the wonderful correlations that exist among the varied motions of the atmosphere was obtained by **William Ferrel** (1817–1891). He was born in Fulton County, Pa., and brought up on a farm. Though in unfavourable surroundings, a burning thirst for knowledge spurred the boy to the mastery of one branch after another. He attended Marshall College, Pa., and graduated in 1844 from Bethany College. While teaching school he became interested in meteorology and in the subject of tides. In 1856 he wrote an article on "the winds and currents of the ocean." The following year he became connected with the *Nautical Almanac*. A mathematical paper followed in 1858 on "the motion of fluids and solids relative to the earth's

surface." The subject was extended afterwards so as to embrace the mathematical theory of cyclones, tornadoes, water-spouts, etc. In 1885 appeared his *Recent Advances in Meteorology*. In the opinion of a leading European meteorologist (*Julius Hann* of Vienna), Ferrel has "contributed more to the advance of the physics of the atmosphere than any other living physicist or meteorologist."

Ferrel teaches that the air flows in great spirals toward the poles, both in the upper strata of the atmosphere and on the earth's surface beyond the 30th degree of latitude; while the return current blows at nearly right angles to the above spirals, in the middle strata as well as on the earth's surface, in a zone comprised between the parallels 30° N. and 30° S. The idea of three superposed currents blowing spirals was first advanced by James Thomson, but was published in very meagre abstract.

Ferrel's views have given a strong impulse to theoretical research in America, Austria, and Germany. Several objections raised against his argument have been abandoned, or have been answered by W. M. Davis of Harvard. The mathematical analysis of F. Waldo of Washington, and of others, has further confirmed the accuracy of the theory. The transport of Krakatoa dust and observations made on clouds point toward the existence of an upper east current on the equator, and Pernter has mathematically deduced from Ferrel's theory the existence of such a current.

Another theory of the general circulation of the atmosphere was propounded by Werner Siemens of Berlin, in which an attempt is made to apply thermodynamics to aërial currents. Important new points of view have been introduced recently by Helmholtz, who concludes that when two air currents blow one above the other in different directions, a system of air waves must arise in the same way as waves are formed on the

sea. He and A. Oberbeck showed that when the waves on the sea attain lengths of from 16 to 33 feet, the air waves must attain lengths of from 10 to 20 miles, and proportional depths. Superposed strata would thus mix more thoroughly, and their energy would be partly dissipated. From hydrodynamical equations of rotation Helmholtz established the reason why the observed velocity from equatorial regions is much less in a latitude of, say, 20° or 30°, than it would be were the movements unchecked.

About 1860 acoustics began to be studied with renewed zeal. The mathematical theory of pipes and vibrating strings had been elaborated in the eighteenth century by Daniel Bernoulli, D'Alembert, Euler, and Lagrange. In the first part of the present century Laplace corrected Newton's theory on the velocity of sound in gases, Poisson gave a mathematical discussion of torsional vibrations; Poisson, Sophie Germain, and Wheatstone studied Chladni's figures; Thomas Young and the brothers Weber developed the wave-theory of sound. **Sir J. F. W. Herschel** wrote on the mathematical theory of sound for the *Encyclopædia Metropolitana*, 1845. Epoch-making were Helmholtz's experimental and mathematical researches. In his hands and Rayleigh's, Fourier's series received due attention. Helmholtz gave the mathematical theory of beats, difference tones, and summation tones. **Lord Rayleigh** (John William Strutt) of Cambridge (born 1842) made extensive mathematical researches in acoustics as a part of the theory of vibration in general. Particular mention may be made of his discussion of the disturbance produced by a spherical obstacle on the waves of sound, and of phenomena, such as sensitive flames, connected with the instability of jets of fluid. In 1877 and 1878 he published in two volumes a treatise on *The Theory of Sound*. Other mathematical researches on this subject have been made in England by Donkin and Stokes.

The theory of elasticity[42] belongs to this century. Before 1800 no attempt had been made to form general equations for the motion or equilibrium of an elastic solid. Particular problems had been solved by special hypotheses. Thus, James Bernoulli considered elastic laminæ; Daniel Bernoulli and Euler investigated vibrating rods; Lagrange and Euler, the equilibrium of springs and columns. The earliest investigations of this century, by Thomas Young ("Young's modulus of elasticity") in England, J. Binet in France, and G. A. A. Plana in Italy, were chiefly occupied in extending and correcting the earlier labours. Between 1830 and 1840 the broad outline of the modern theory of elasticity was established. This was accomplished almost exclusively by French writers, — Louis-Marie-Henri Navier (1785–1836), Poisson, Cauchy, Mademoiselle Sophie Germain (1776–1831), Félix Savart (1791–1841).

Siméon Denis Poisson[94] (1781–1840) was born at Pithiviers. The boy was put out to a nurse, and he used to tell that when his father (a common soldier) came to see him one day, the nurse had gone out and left him suspended by a thin cord to a nail in the wall in order to protect him from perishing under the teeth of the carnivorous and unclean animals that roamed on the floor. Poisson used to add that his gymnastic efforts when thus suspended caused him to swing back and forth, and thus to gain an early familiarity with the pendulum, the study of which occupied him much in his maturer life. His father destined him for the medical profession, but so repugnant was this to him that he was permitted to enter the Polytechnic School at the age of seventeen. His talents excited the interest of Lagrange and Laplace. At eighteen he wrote a memoir on finite differences which was printed on the recommendation of Legendre. He soon became a lecturer at the school, and continued through life to hold various government scientific posts and professorships. He prepared some 400 publications,

mainly on applied mathematics. His *Traité de Mécanique*, 2 vols., 1811 and 1833, was long a standard work. He wrote on the mathematical theory of heat, capillary action, probability of judgment, the mathematical theory of electricity and magnetism, physical astronomy, the attraction of ellipsoids, definite integrals, series, and the theory of elasticity. He was considered one of the leading analysts of his time.

His work on elasticity is hardly excelled by that of Cauchy, and second only to that of Saint-Venant. There is hardly a problem in elasticity to which he has not contributed, while many of his inquiries were new. The equilibrium and motion of a circular plate was first successfully treated by him. Instead of the definite integrals of earlier writers, he used preferably finite summations. Poisson's contour conditions for elastic plates were objected to by Gustav Kirchhoff of Berlin, who established new conditions. But Thomson and Tait in their *Treatise on Natural Philosophy* have explained the discrepancy between Poisson's and Kirchhoff's boundary conditions, and established a reconciliation between them.

Important contributions to the theory of elasticity were made by Cauchy. To him we owe the origin of the theory of stress, and the transition from the consideration of the force upon a molecule exerted by its neighbours to the consideration of the stress upon a small plane at a point. He anticipated Green and Stokes in giving the equations of isotropic elasticity with two constants. The theory of elasticity was presented by Gabrio Piola of Italy according to the principles of Lagrange's *Mécanique Analytique*, but the superiority of this method over that of Poisson and Cauchy is far from evident. The influence of temperature on stress was first investigated experimentally by Wilhelm Weber of Göttingen, and afterwards mathematically by Duhamel, who, assuming Poisson's theory of elasticity, examined the alterations of

form which the formulæ undergo when we allow for changes of temperature. Weber was also the first to experiment on elastic after-strain. Other important experiments were made by different scientists, which disclosed a wider range of phenomena, and demanded a more comprehensive theory. Set was investigated by Gerstner (1756–1832) and Eaton Hodgkinson, while the latter physicist in England and Vicat (1786–1861) in France experimented extensively on absolute strength. Vicat boldly attacked the mathematical theories of flexure because they failed to consider shear and the time-element. As a result, a truer theory of flexure was soon propounded by Saint-Venant. Poncelet advanced the theories of resilience and cohesion.

Gabriel Lamé[94] (1795–1870) was born at Tours, and graduated at the Polytechnic School. He was called to Russia with Clapeyron and others to superintend the construction of bridges and roads. On his return, in 1832, he was elected professor of physics at the Polytechnic School. Subsequently he held various engineering posts and professorships in Paris. As engineer he took an active part in the construction of the first railroads in France. Lamé devoted his fine mathematical talents mainly to mathematical physics. In four works: *Leçons sur les fonctions inverses des transcendantes et les surfaces isothermes; Sur les coordonnées curvilignes et leurs diverses applications; Sur la théorie analytique de la chaleur; Sur la théorie mathématique de l'élasticité des corps solides* (1852), and in various memoirs he displays fine analytical powers; but a certain want of physical touch sometimes reduces the value of his contributions to elasticity and other physical subjects. In considering the temperature in the interior of an ellipsoid under certain conditions, he employed functions analogous to Laplace's functions, and known by the name of "Lamé's functions." A problem in elasticity called by Lamé's name, viz.

to investigate the conditions for equilibrium of a spherical elastic envelope subject to a given distribution of load on the bounding spherical surfaces, and the determination of the resulting shifts is the only completely general problem on elasticity which can be said to be completely solved. He deserves much credit for his derivation and transformation of the general elastic equations, and for his application of them to double refraction. Rectangular and triangular membranes were shown by him to be connected with questions in the theory of numbers. The field of photo-elasticity was entered upon by Lamé, F. E. Neumann, Clerk Maxwell. Stokes, Wertheim, R. Clausius, Jellett, threw new light upon the subject of "rari-constancy" and "multi-constancy," which has long divided elasticians into two opposing factions. The uni-constant isotropy of Navier and Poisson had been questioned by Cauchy, and was now severely criticised by Green and Stokes.

Barré de Saint-Venant (1797–1886), ingénieur des ponts et chaussées, made it his life-work to render the theory of elasticity of practical value. The charge brought by practical engineers, like Vicat, against the theorists led Saint-Venant to place the theory in its true place as a guide to the practical man. Numerous errors committed by his predecessors were removed. He corrected the theory of flexure by the consideration of slide, the theory of elastic rods of double curvature by the introduction of the third moment, and the theory of torsion by the discovery of the distortion of the primitively plane section. His results on torsion abound in beautiful graphic illustrations. In case of a rod, upon the side surfaces of which no forces act, he showed that the problems of flexure and torsion can be solved, if the end-forces are distributed over the end-surfaces by a definite law. Clebsch, in his *Lehrbuch der Elasticität*, 1862, showed that this problem is

reversible to the case of side-forces without end-forces. Clebsch[68] extended the research to very thin rods and to very thin plates. Saint-Venant considered problems arising in the scientific design of built-up artillery, and his solution of them differs considerably from Lamé's solution, which was popularised by Rankine, and much used by gun-designers. In Saint-Venant's translation into French of Clebsch's *Elasticität*, he develops extensively a double-suffix notation for strain and stresses. Though often advantageous, this notation is cumbrous, and has not been generally adopted. *Karl Pearson*, professor in University College, London, has recently examined mathematically the permissible limits of the application of the ordinary theory of flexure of a beam.

The mathematical theory of elasticity is still in an unsettled condition. Not only are scientists still divided into two schools of "rari-constancy" and "multi-constancy," but difference of opinion exists on other vital questions. Among the numerous modern writers on elasticity may be mentioned Émile Mathieu (1835–1891), professor at Besançon, Maurice Levy of Paris, Charles Chree, superintendent of the Kew Observatory, A. B. Basset, Sir William Thomson (Lord Kelvin) of Glasgow, J. Boussinesq of Paris, and others. Sir William Thomson applied the laws of elasticity of solids to the investigation of the earth's elasticity, which is an important element in the theory of ocean-tides. If the earth is a solid, then its elasticity co-operates with gravity in opposing deformation due to the attraction of the sun and moon. Laplace had shown how the earth would behave if it resisted deformation only by gravity. Lamé had investigated how a solid sphere would change if its elasticity only came into play. Sir William Thomson combined the two results, and compared them with the actual deformation. Thomson, and afterwards G. H. Darwin, computed that the resistance of the earth to

tidal deformation is nearly as great as though it were of steel. This conclusion has been confirmed recently by Simon Newcomb, from the study of the observed periodic changes in latitude. For an ideally rigid earth the period would be 360 days, but if as rigid as steel, it would be 441, the observed period being 430 days.

Among text-books on elasticity may be mentioned the works of Lamé, Clebsch, Winkler, Beer, Mathieu, W. J. Ibbetson, and F. Neumann, edited by O. E. Meyer.

Riemann's opinion that a *science* of physics only exists since the invention of differential equations finds corroboration even in this brief and fragmentary outline of the progress of mathematical physics. The undulatory theory of light, first advanced by Huygens, owes much to the power of mathematics: by mathematical analysis its assumptions were worked out to their last consequences. **Thomas Young**[95] (1773–1829) was the first to explain the principle of interference, both of light and sound, and the first to bring forward the idea of transverse vibrations in light waves. Young's explanations, not being verified by him by extensive numerical calculations, attracted little notice, and it was not until **Augustin Fresnel** (1788–1827) applied mathematical analysis to a much greater extent than Young had done, that the undulatory theory began to carry conviction. Some of Fresnel's mathematical assumptions were not satisfactory; hence Laplace, Poisson, and others belonging to the strictly mathematical school, at first disdained to consider the theory. By their opposition Fresnel was spurred to greater exertion. Arago was the first great convert made by Fresnel. When polarisation and double refraction were explained by Young and Fresnel, then Laplace was at last won over. Poisson drew from Fresnel's formulæ the seemingly paradoxical deduction that a small circular disc, illuminated by a luminous point,

must cast a shadow with a bright spot in the centre. But this was found to be in accordance with fact. The theory was taken up by another great mathematician, Hamilton, who from his formulæ predicted conical refraction, verified experimentally by Lloyd. These predictions do not prove, however, that Fresnel's formulæ are correct, for these prophecies might have been made by other forms of the wave-theory. The theory was placed on a sounder dynamical basis by the writings of Cauchy, Biot, Green, C. Neumann, Kirchhoff, McCullagh, Stokes, Saint-Venant, Sarrau, Lorenz, and Sir William Thomson. In the wave-theory, as taught by Green and others, the luminiferous ether was an incompressible elastic solid, for the reason that fluids could not propagate transverse vibrations. But, according to Green, such an elastic solid would transmit a longitudinal disturbance with infinite velocity. Stokes remarked, however, that the ether might act like a fluid in case of finite disturbances, and like an elastic solid in case of the infinitesimal disturbances in light propagation.

Fresnel postulated the density of ether to be different in different media, but the elasticity the same, while C. Neumann and McCullagh assume the density uniform and the elasticity different in all substances. On the latter assumption the direction of vibration lies in the plane of polarisation, and not perpendicular to it, as in the theory of Fresnel.

While the above writers endeavoured to explain all optical properties of a medium on the supposition that they arise entirely from difference in rigidity or density of the ether in the medium, there is another school advancing theories in which the mutual action between the molecules of the body and the ether is considered the main cause of refraction and dispersion.[100] The chief workers in this field are J. Boussinesq, W. Sellmeyer, Helmholtz, E. Lommel, E. Ketteler, W. Voigt, and Sir William Thomson in his lectures delivered at the

Johns Hopkins University in 1884. Neither this nor the first-named school succeeded in explaining all the phenomena. A third school was founded by Maxwell. He proposed the electro-magnetic theory, which has received extensive development recently. It will be mentioned again later. According to Maxwell's theory, the direction of vibration does not lie exclusively in the plane of polarisation, nor in a plane perpendicular to it, but something occurs in both planes—a magnetic vibration in one, and an electric in the other. Fitzgerald and Trouton in Dublin verified this conclusion of Maxwell by experiments on electro-magnetic waves.

Of recent mathematical and experimental contributions to optics, mention must be made of H. A. Rowland's theory of concave gratings, and of A. A. Michelson's work on interference, and his application of interference methods to astronomical measurements.

In electricity the mathematical theory and the measurements of **Henry Cavendish** (1731–1810), and in magnetism the measurements of **Charles Augustin Coulomb** (1736–1806), became the foundations for a system of measurement. For electro-magnetism the same thing was done by **Andrè Marie Ampère** (1775–1836). The first complete method of measurement was the system of absolute measurements of terrestrial magnetism introduced by *Gauss* and **Wilhelm Weber** (1804–1891) and afterwards extended by Wilhelm Weber and F. Kohlrausch to electro-magnetism and electro-statics. In 1861 the British Association and the Royal Society appointed a special commission with Sir William Thomson at the head, to consider the unit of electrical resistance. The commission recommended a unit in principle like W. Weber's, but greater than Weber's by a factor of 10^7.[101] The discussions and labours on this subject continued for twenty years, until in 1881 a general agreement was reached at an electrical congress in Paris.

A function of fundamental importance in the mathematical theories of electricity and magnetism is the "potential." It was first used by Lagrange in the determination of gravitational attractions in 1773. Soon after, Laplace gave the celebrated differential equation,

$$\frac{\partial^2 V}{dx^2} + \frac{\partial^2 V}{dy^2} + \frac{\partial^2 V}{dz^2} = 0,$$

which was extended by Poisson by writing $-4\pi k$ in place of zero in the right-hand member of the equation, so that it applies not only to a point external to the attracting mass, but to any point whatever. The first to apply the potential function to other than gravitation problems was **George Green** (1793–1841). He introduced it into the mathematical theory of electricity and magnetism. Green was a self-educated man who started out as a baker, and at his death was fellow of Caius College, Cambridge. In 1828 he published by subscription at Nottingham a paper entitled *Essay on the application of mathematical analysis to the theory of electricity and magnetism*. It escaped the notice even of English mathematicians until 1846, when Sir William Thomson had it reprinted in *Crelle's Journal*, vols. xliv. and xlv. It contained what is now known as "Green's theorem" for the treatment of potential. Meanwhile all of Green's general theorems had been re-discovered by Sir William Thomson, Chasles, Sturm, and Gauss. The term *potential function* is due to Green. Hamilton used the word *force-function*, while Gauss, who about 1840 secured the general adoption of the function, called it simply *potential*.

Large contributions to electricity and magnetism have been made by **William Thomson**. He was born in 1824 at Belfast, Ireland, but is of Scotch descent. He and his brother James studied in Glasgow. From there he entered Cambridge, and was graduated as Second Wrangler in 1845. William Thom-

son, Sylvester, Maxwell, Clifford, and J. J. Thomson are a group of great men who were Second Wranglers at Cambridge. At the age of twenty-two W. Thomson was elected professor of natural philosophy in the University of Glasgow, a position which he has held ever since. For his brilliant mathematical and physical achievements he was knighted, and in 1892 was made Lord Kelvin. His researches on the theory of potential are epoch-making. What is called "Dirichlet's principle" was discovered by him in 1848, somewhat earlier than by Dirichlet. We owe to Sir William Thomson new synthetical methods of great elegance, viz. the theory of electric images and the method of electric inversion founded thereon. By them he determined the distribution of electricity on a bowl, a problem previously considered insolvable. The distribution of static electricity on conductors had been studied before this mainly by Poisson and Plana. In 1845 F. E. Neumann of Königsberg developed from the experimental laws of Lenz the mathematical theory of magneto-electric induction. In 1855 W. Thomson predicted by mathematical analysis that the discharge of a Leyden jar through a linear conductor would in certain cases consist of a series of decaying oscillations. This was first established experimentally by Joseph Henry of Washington. William Thomson worked out the electro-static induction in submarine cables. The subject of the screening effect against induction, due to sheets of different metals, was worked out mathematically by Horace Lamb and also by Charles Niven. W. Weber's chief researches were on electro-dynamics. Helmholtz in 1851 gave the mathematical theory of the course of induced currents in various cases. **Gustav Robert Kirchhoff**[97] (1824–1887) investigated the distribution of a current over a flat conductor, and also the strength of current in each branch of a network of linear conductors.

The entire subject of electro-magnetism was revolutionised

by **James Clerk Maxwell** (1831–1879). He was born near Edinburgh, entered the University of Edinburgh, and became a pupil of Kelland and Forbes. In 1850 he went to Trinity College, Cambridge, and came out Second Wrangler, E. Routh being Senior Wrangler. Maxwell then became lecturer at Cambridge, in 1856 professor at Aberdeen, and in 1860 professor at King's College, London. In 1865 he retired to private life until 1871, when he became professor of physics at Cambridge. Maxwell not only translated into mathematical language the experimental results of Faraday, but established the electro-magnetic theory of light, since verified experimentally by Hertz. His first researches thereon were published in 1864. In 1871 appeared his great *Treatise on Electricity and Magnetism.* He constructed the electro-magnetic theory from general equations, which are established upon purely dynamical principles, and which determine the state of the electric field. It is a mathematical discussion of the stresses and strains in a dielectric medium subjected to electro-magnetic forces. The electro-magnetic theory has received developments from Lord Rayleigh, J. J. Thomson, H. A. Rowland, R. T. Glazebrook, H. Helmholtz, L. Boltzmann, O. Heaviside, J. H. Poynting, and others. **Hermann von Helmholtz** turned his attention to this part of the subject in 1871. He was born in 1821 at Potsdam, studied at the University of Berlin, and published in 1847 his pamphlet *Ueber die Erhaltung der Kraft.* He became teacher of anatomy in the Academy of Art in Berlin. He was elected professor of physiology at Königsberg in 1849, at Bonn in 1855, at Heidelberg in 1858. It was at Heidelberg that he produced his work on *Tonempfindung.* In 1871 he accepted the chair of physics at the University of Berlin. From this time on he has been engaged chiefly on inquiries in electricity and hydrodynamics. Helmholtz aimed to determine in what direction experiments should be made to

decide between the theories of W. Weber, F. E. Neumann, Riemann, and Clausius, who had attempted to explain electro-dynamic phenomena by the assumption of forces acting at a distance between two portions of the hypothetical electrical fluid,—the intensity being dependent not only on the distance, but also on the velocity and acceleration, — and the theory of Faraday and Maxwell, which discarded action at a distance and assumed stresses and strains in the dielectric. His experiments favoured the British theory. He wrote on abnormal dispersion, and created analogies between electro-dynamics and hydrodynamics. Lord Rayleigh compared electro-magnetic problems with their mechanical analogues, gave a dynamical theory of diffraction, and applied Laplace's coefficients to the theory of radiation. Rowland made some emendations on Stokes' paper on diffraction and considered the propagation of an arbitrary electro-magnetic disturbance and spherical waves of light. Electro-magnetic induction has been investigated mathematically by Oliver Heaviside, and he showed that in a cable it is an actual benefit. Heaviside and Poynting have reached remarkable mathematical results in their interpretation and development of Maxwell's theory. Most of Heaviside's papers have been published since 1882; they cover a wide field.

One part of the theory of capillary attraction, left defective by Laplace, namely, the action of a solid upon a liquid, and the mutual action between two liquids, was made dynamically perfect by Gauss. He stated the rule for angles of contact between liquids and solids. A similar rule for liquids was established by Ernst Franz Neumann. Chief among recent workers on the mathematical theory of capillarity are Lord Rayleigh and E. Mathieu.

The great principle of the conservation of energy was established by **Robert Mayer** (1814–1878), a physician in Heilbronn, and again independently by Colding of Copen-

hagen, Joule, and Helmholtz. **James Prescott Joule** (1818–1889) determined experimentally the mechanical equivalent of heat. Helmholtz in 1847 applied the conceptions of the transformation and conservation of energy to the various branches of physics, and thereby linked together many well-known phenomena. These labours led to the abandonment of the corpuscular theory of heat. The mathematical treatment of thermic problems was demanded by practical considerations. Thermodynamics grew out of the attempt to determine mathematically how much work can be gotten out of a steam engine. **Sadi-Carnot,** an adherent of the corpuscular theory, gave the first impulse to this. The principle known by his name was published in 1824. Though the importance of his work was emphasised by *B. P. E. Clapeyron*, it did not meet with general recognition until it was brought forward by William Thomson. The latter pointed out the necessity of modifying Carnot's reasoning so as to bring it into accord with the new theory of heat. William Thomson showed in 1848 that Carnot's principle led to the conception of an absolute scale of temperature. In 1849 he published "an account of Carnot's theory of the motive power of heat, with numerical results deduced from Regnault's experiments." In February, 1850, **Rudolph Clausius** (1822–1888), then in Zürich (afterwards professor in Bonn), communicated to the Berlin Academy a paper on the same subject which contains the Protean second law of thermodynamics. In the same month **William John M. Rankine** (1820–1872), professor of engineering and mechanics at Glasgow, read before the Royal Society of Edinburgh a paper in which he declares the nature of heat to consist in the rotational motion of molecules, and arrives at some of the results reached previously by Clausius. He does not mention the second law of thermodynamics, but in a subsequent paper he declares that it could be derived

from equations contained in his first paper. His proof of the second law is not free from objections. In March, 1851, appeared a paper of William Thomson which contained a perfectly rigorous proof of the second law. He obtained it before he had seen the researches of Clausius. The statement of this law, as given by Clausius, has been much criticised, particularly by Rankine, Theodor Wand, P. G. Tait, and Tolver Preston. Repeated efforts to deduce it from general mechanical principles have remained fruitless. The science of thermodynamics was developed with great success by Thomson, Clausius, and Rankine. As early as 1852 Thomson discovered the law of the dissipation of energy, deduced at a later period also by Clausius. The latter designated the non-transformable energy by the name *entropy*, and then stated that the entropy of the universe tends toward a maximum. For entropy Rankine used the term *thermodynamic function*. Thermodynamic investigations have been carried on also by G. Ad. Hirn of Colmar, and Helmholtz (monocyclic and polycyclic systems). Valuable graphic methods for the study of thermodynamic relations were devised in 1873–1878 by **J. Willard Gibbs** of Yale College. Gibbs first gives an account of the advantages of using various pairs of the five fundamental thermodynamic quantities for graphical representation, then discusses the entropy-temperature and entropy-volume diagrams, and the volume-energy-entropy surface (described in Maxwell's *Theory of Heat*). Gibbs formulated the energy-entropy criterion of equilibrium and stability, and expressed it in a form applicable to complicated problems of dissociation. Important works on thermodynamics have been prepared by Clausius in 1875, by R. Rühlmann in 1875, and by Poincaré in 1892.

In the study of the law of dissipation of energy and the principle of least action, mathematics and metaphysics met on

common ground. The doctrine of least action was first propounded by Maupertius in 1744. Two years later he proclaimed it to be a universal law of nature, and the first scientific proof of the existence of God. It was weakly supported by him, violently attacked by König of Leipzig, and keenly defended by Euler. Lagrange's conception of the principle of least action became the mother of analytic mechanics, but his statement of it was inaccurate, as has been remarked by Josef Bertrand in the third edition of the *Mécanique Analytique*. The form of the principle of least action, as it now exists, was given by Hamilton, and was extended to electrodynamics by F. E. Neumann, Clausius, Maxwell, and Helmholtz. To subordinate the principle to all reversible processes, Helmholtz introduced into it the conception of the "kinetic potential." In this form the principle has universal validity.

An offshoot of the mechanical theory of heat is the modern kinetic theory of gases, developed mathematically by *Clausius*, *Maxwell*, *Ludwig Boltzmann* of Munich, and others. The first suggestions of a kinetic theory of matter go back as far as the time of the Greeks. The earliest work to be mentioned here is that of Daniel Bernoulli, 1738. He attributed to gas-molecules great velocity, explained the pressure of a gas by molecular bombardment, and deduced Boyle's law as a consequence of his assumptions. Over a century later his ideas were taken up by Joule (in 1846), A. K. Krönig (in 1856), and Clausius (in 1857). Joule dropped his speculations on this subject when he began his experimental work on heat. Krönig explained by the kinetic theory the fact determined experimentally by Joule that the internal energy of a gas is not altered by expansion when no external work is done. Clausius took an important step in supposing that molecules may have rotary motion, and that atoms in a molecule may move relatively to each other. He assumed that the force acting

between molecules is a function of their distances, that temperature depends solely upon the kinetic energy of molecular motions, and that the number of molecules which at any moment are so near to each other that they perceptibly influence each other is comparatively so small that it may be neglected. He calculated the average velocities of molecules, and explained evaporation. Objections to his theory, raised by Buy's-Ballot and by Jochmann, were satisfactorily answered by Clausius and Maxwell, except in one case where an additional hypothesis had to be made. Maxwell proposed to himself the problem to determine the average number of molecules, the velocities of which lie between given limits. His expression therefor constitutes the important law of distribution of velocities named after him. By this law the distribution of molecules according to their velocities is determined by the same formula (given in the theory of probability) as the distribution of empirical observations according to the magnitude of their errors. The average molecular velocity as deduced by Maxwell differs from that of Clausius by a constant factor. Maxwell's first deduction of this average from his law of distribution was not rigorous. A sound derivation was given by O. E. Meyer in 1866. Maxwell predicted that so long as Boyle's law is true, the coefficient of viscosity and the coefficient of thermal conductivity remain independent of the pressure. His deduction that the coefficient of viscosity should be proportional to the square root of the absolute temperature appeared to be at variance with results obtained from pendulum experiments. This induced him to alter the very foundation of his kinetic theory of gases by assuming between the molecules a repelling force varying inversely as the fifth power of their distances. The founders of the kinetic theory had assumed the molecules of a gas to be hard elastic spheres; but Maxwell, in his second presentation of the theory in 1866,

went on the assumption that the molecules behave like centres of forces. He demonstrated anew the law of distribution of velocities; but the proof had a flaw in argument, pointed out by Boltzmann, and recognised by Maxwell, who adopted a somewhat different form of the distributive function in a paper of 1879, intended to explain mathematically the effects observed in Crookes' radiometer. Boltzmann gave a rigorous general proof of Maxwell's law of the distribution of velocities.

None of the fundamental assumptions in the kinetic theory of gases leads by the laws of probability to results in very close agreement with observation. Boltzmann tried to establish kinetic theories of gases by assuming the forces between molecules to act according to different laws from those previously assumed. Clausius, Maxwell, and their predecessors took the mutual action of molecules in collision as repulsive, but Boltzmann assumed that they may be attractive. Experiment of Joule and Lord Kelvin seem to support the latter assumption.

Among the latest researches on the kinetic theory is Lord Kelvin's disproof of a general theorem of Maxwell and Boltzmann, asserting that the average kinetic energy of two given portions of a system must be in the ratio of the number of degrees of freedom of those portions.

ADDENDA.

PAGE 14. The new *Akhmim papyrus*, written in Greek, is probably the copy of an older papyrus, antedating Heron's works, and is the oldest extant text-book on practical Greek arithmetic. It contains, besides arithmetical examples, a table for finding "unit-fractions," identical in scope with that of Ahmes, and, like Ahmes's, without a clue as to its mode of construction. See BIBLIOTH. MATH., 1893, p. 79–89. The papyrus is edited by J. Baillet (*Mémoires publiés par les membres de la mission archéologique française au Caire*, T. IX., 1r *fascicule*, *Paris*, 1892, p. 1–88).

PAGE 39. Chasles's or Simson's definition of a Porism is preferable to Proclus's, given in the text. See Gow, p. 217–221.

PAGE 114. Nasir Eddin for the first time elaborated trigonometry independently of astronomy and to such great perfection that, had his work been known, Europeans of the 15th century might have spared their labours. See BIBLIOTH. MATH., 1893, p. 6.

PAGE 116. This law of sines was probably known before Gabir ben Aflah to Tabit ben Korra and others. See BIBLIOTH. MATH., 1893, p. 7.

PAGE 125. Athelard was probably not the first to translate Euclid's *Elements* from the Arabic. See M. Cantor's VORLESUNGEN, Vol. II., p. 91, 92.

PAGE 240. G. Eneström argues that Taylor and not Nicole is the real inventor of finite differences. See BIBLIOTH. MATH., 1893, p. 91.

PAGE 250. An earlier publication in which 3.14159 ··· is designated by π, is W. Jones's *Synopsis palmariorum matheseos*, London, 1706, p. 243, 263 *et seq*. See BIBLIOTH. MATH., 1894, p. 106.

PAGE 335. Before Gauss a theorem on convergence, usually attributed to Cauchy, was given by Maclaurin (*Fluxions*, § 350). A rule of convergence was deduced also by Stirling. See *Bull. N. Y. Math. Soc.*, Vol. III., p. 186.

PAGE 358. The surface of a solid with p holes was considered before Clifford by Tonelli, and was probably used by Riemann himself. See MATH. ANNALEN, Vol. 45, p. 142.

PAGE 361. As early as 1835, Lobachevsky showed in a memoir the necessity of distinguishing between continuity and differentiability. See G. B. Halsted's transl. of A. Vasiliev's *Address* on Lobachevsky, p. 23.

Recent deaths. Johann Rudolf Wolf, Dec. 6, 1893; Heinrich Hertz, Jan. 1, 1894; Eugène Catalan, Feb. 14, 1894; Hermann von Helmholtz, Sept. 8, 1894; Arthur Cayley, Jan. 26, 1895.

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