

A treatise on the rectilinear motion and rotation of bodies; with a description of original experiments. Relative to the subject / By G. Atwood.

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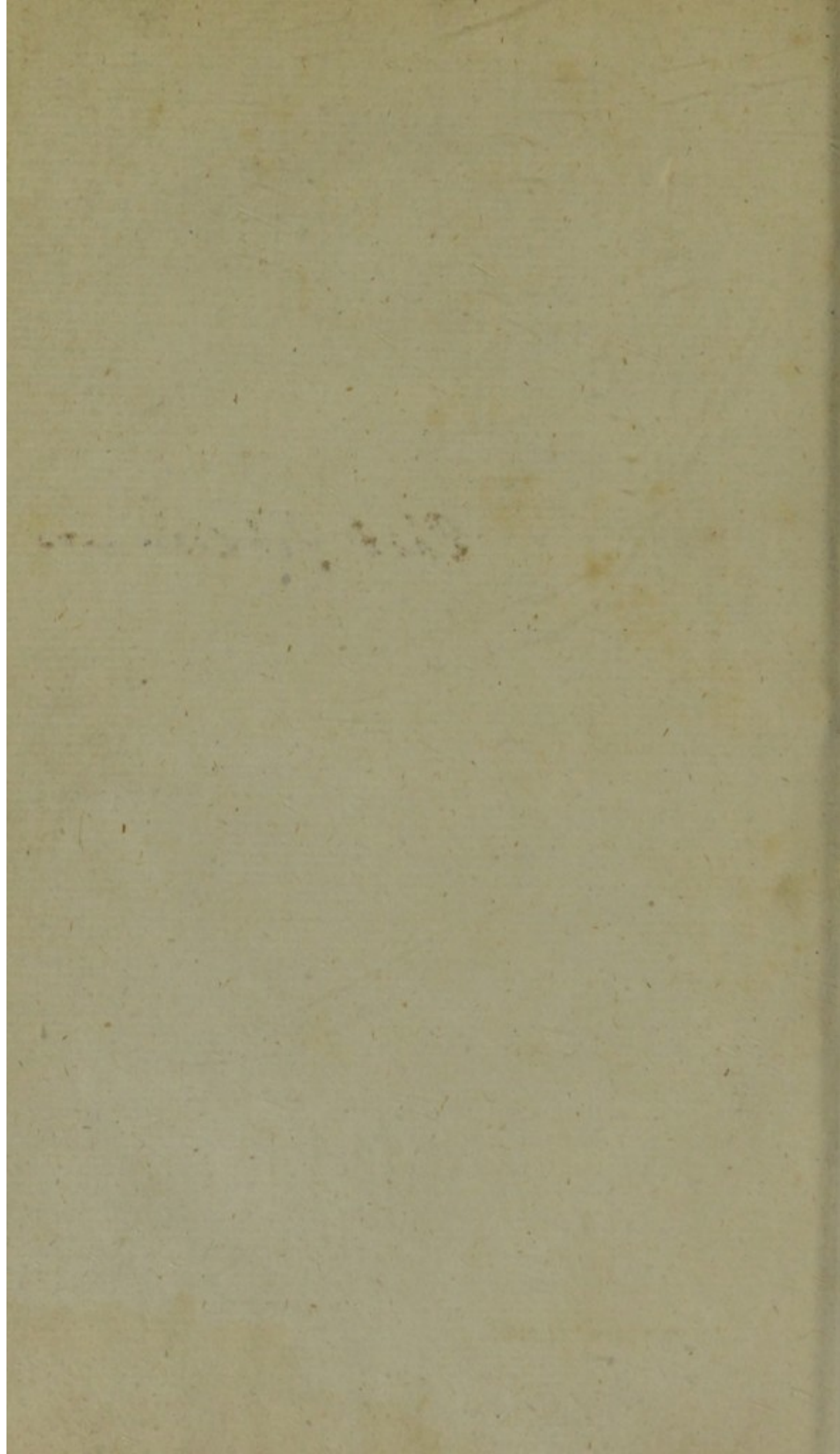
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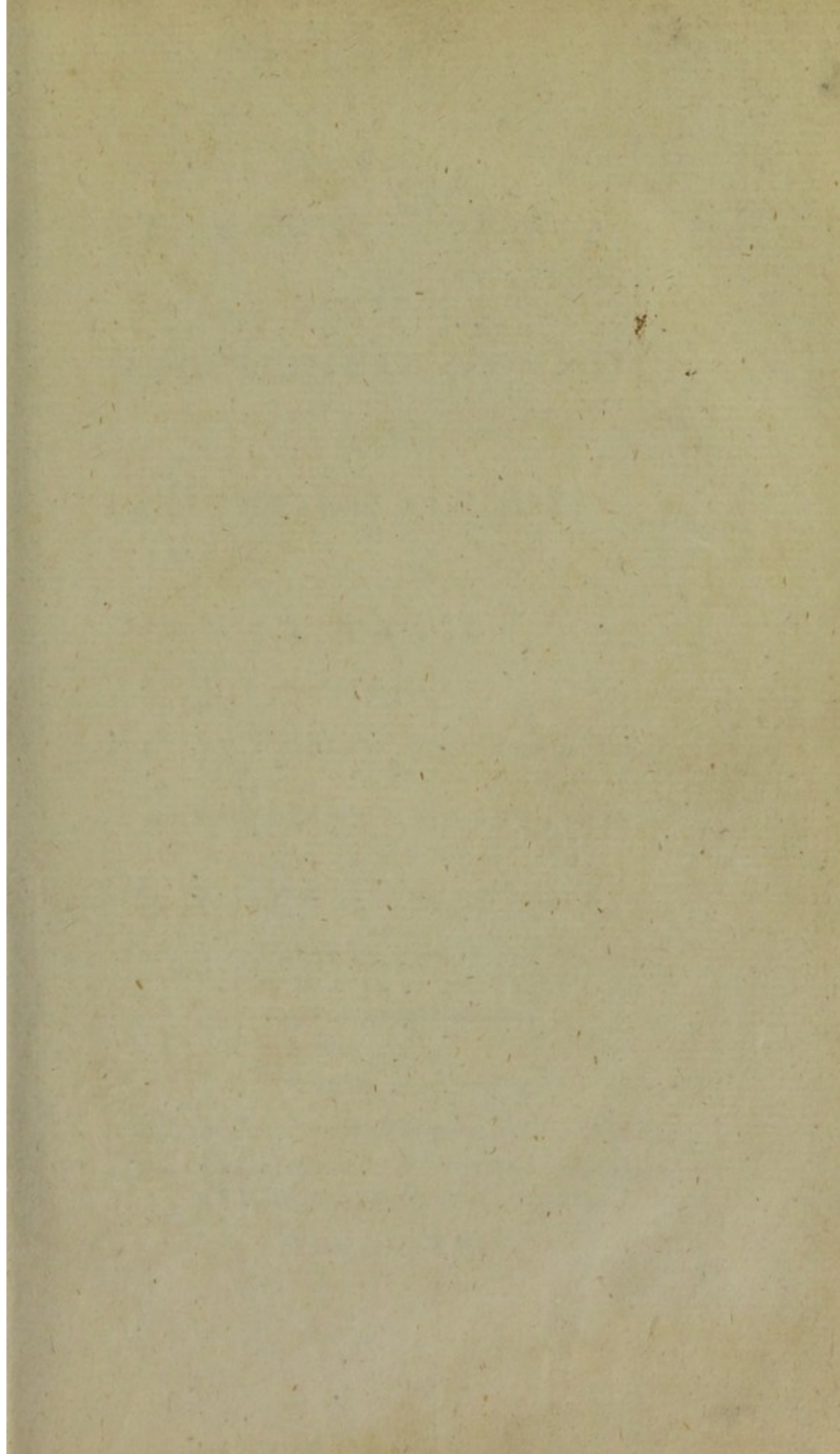
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A
T R E A T I S E
ON THE
RECTILINEAR MOTION
AND
ROTATION OF BODIES;
WITH A
D E S C R I P T I O N
O F
ORIGINAL EXPERIMENTS
RELATIVE TO THE SUBJECT.

By G. ATWOOD, M.A.F.R.S.

LATE FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

C A M B R I D G E,

Printed by J. ARCHDEACON Printer to the UNIVERSITY;
For J. & J. MERRILL, and J. DEIGHTON, in Cambridge;
and T. CADELL, in the Strand, London.

M D C C L X X I V.



TO HIS GRACE
THE DUKE OF RUTLAND,
KNIGHT OF THE MOST NOBLE ORDER OF THE GARTER,
LORD LIEUTENANT OF IRELAND,
&c. &c. &c.

THE FOLLOWING PAGES
ARE WITH GREAT RESPECT

AND GRATITUDE
INSCRIBED

BY HIS GRACE'S
MOST OBEDIENT,

AND MOST OBLIGED,

HUMBLE SERVANT,

GEORGE ATWOOD.

Trinity College, Cambridge,
18 April, 1784.

TO HIS GRACE

THE DUKE OF RUTLAND,

KNIGHT OF THE MOST NOBLE ORDER OF THE GARTER,

LEICESTERIAN OF IRELAND,

NO. 10.

THE FOLLOWING PAGE

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AND GRATITUDE

PRESENTED

TO HIS GRACE

MOST OBEYANT,

AND MOST OBLIGED

SERVANT,

GEORGE ARWOLD.

Printed by J. G. & Co. 17, St. Paul's Churchyard, London.

P R E F A C E.

TH E principles of rectilinear motion and rotation are of considerable extent in the theory of mechanics, and comprize most cases in which this science can be applied to practical use; it is probably from the facility and exactness with which these kinds of motion are produced in bodies, and are communicated from one body to another, that they have been so generally adopted in mechanic constructions of every kind.

A few cursory remarks on the ensuing sections, may be sufficient to explain the plan and design of this treatise.

The first and second sections contain the elementary propositions on which the theory of mechanics is founded. In the third section the rectilinear motion of bodies, impelled or resisted by forces which act uniformly, is considered. The depths to which spherical bodies, impinging with given velocities, sink into
banks

banks of earth, solid blocks of timber, clay, snow, &c. are determined by propositions inserted in the third section, because the forces by which those substances resist the penetration of spherical bodies are, in a physical sense, uniform: in strictness, the forces of resistance are uniform only after the spheres have sunk to a depth equal to their radii; and the greater proportion the space to which any sphere penetrates, before its motion is destroyed, bears to its radius, the less error will be occasioned by assuming the resistance as altogether uniform. If the velocity of impact is produced by the acceleration of gravity, the action of the sphere's weight, after it has sunk to a depth equal to its radius, is too small to have sensible effect in experiments made on the resisting forces of substances, which are in general much greater than the force of gravity.

The notation of ratios explained in the second, and applied in the third section, is somewhat different from that which is commonly used: each ratio is represented by a fraction, the numerator of which is the antecedent, and denominator the consequent of the ratio. Although the meaning and value of any ratio is no
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ways dependent on the manner of expressing it, yet if one method of notation should appear less liable to ambiguity, and better adapted for the purpose of applying and explaining the theory than another, there seems sufficient reason for a preference.

The fourth section contains propositions which determine the motion of bodies, produced by forces varying in some ratio of the distances from a fixed point; by classing propositions according to the particular law in which the accelerating force varies, several phenomena, apparently quite unconnected, admit of solution from the same principle: thus, the vibration of a pendulum in a cycloidal arc, the undulatory motion of fluids in the canal described in page (104)*, the small vibrations of an elastic string, considering the whole mass to be concentrated in the middle point, are all explained from the problem for determining the motion of bodies, which are impelled by forces varying in the direct ratio of the distances from a fixed point.

The propositions just mentioned might
have

* N.B. In the first line of the note instead of *AFIKCM* it should be read *EFIKCD*.

have been inferred from this problem as corollaries, but they have been separately considered on account of the many consequences which follow from each. The imaginary case of the elastic string naturally suggested some further consideration of the subject, on principles more nearly corresponding to the real vibration of an elastic chord.

The theory of resisting forces, which vary in the direct duplicate ratio of the velocities, is considered in the fifth section, and is applied to the explanation of various phenomena: this is the case that really obtains, when bodies move in a fluid, the parts of which are free from cohesion and tenacity, provided the compression of the fluid is such as will cause the space deserted by the moving body during its passage, to be filled up immediately by the surrounding fluid.

Experiments on the resistances of fluids to solid bodies moving through them, have not been very numerous: those which are described in the 2d Vol. of the Principia, are sufficiently decisive to ascertain the agreement between the author's theory and matter of fact; but the repetitions of experiments, by which any im-
portant

portant truths are verified, cannot be thought superfluous: we cannot make too frequent appeals to experience; theories, however perfect, are never so satisfactory, as when they are illustrated by repeated and accurate trials.

The experiments on the descent of spherical bodies in water, inserted in sect. v. were constructed with great attention to exactness, and the results were noted before the computations were made.

In pages (156) and (161) the velocities with which the aerial bubbles or spherules ascend in fluids are estimated, and the theory is applied to the deduction of various consequences: according to Dr. Halley's hypothesis, vapour or steam consists of hollow spherules, filled with an elastic fluid so rare, that the weight of a hollow globule and its contents, shall be lighter than an equal bulk of the air which surrounds it. The ascent of these spherules is similar to that of the air bubbles just referred to, which rise in water and other fluids; and the velocities of ascent in both cases must be estimated from the same principles. A few thoughts which presented themselves on considering the subject in this manner, are contained

tained in page (172,&c.). The hypothesis, which ascribes the production of vapour to chemical solution, may, perhaps, be sufficient to account for the absorption of water in air, but will not so well explain the ascent of steam under all its various appearances. The power by which steam rises from boiling water, or from bodies subjected to degrees of heat in chemical distillation, seems to be different from any that can be attributed to the solution of one fluid in another. Indeed, when metallic or other bodies, are dissolved in their proper menstrua, the small aerial bubbles, which constantly ascend during the solution, carry up particles of the bodies that are dissolved, and in some degree diffuse them over the whole mass of the solvent: but no such bubbles are supposed to exist during the ascent of steam according to the hypothesis which explains that phenomenon by referring it to the effects of chemical solution.

Solid particles of matter of any shape, descend in fluids according to laws corresponding in some degree to those which regulate the descent of spherical particles: for example, large particles of any shape will descend faster than those which are
much

much smaller, if they are of the same density.

According to Dr. Woodward's theory of the deluge, when mineral and other substances were by the violence of the waters, or other causes, dissolved into a mass consisting of heterogeneous particles of solid bodies suspended in water, it is supposed that the densest bodies would subside with the greatest velocities, and consequently, that the strata under the earth's surface, must be disposed according to the order of their specific gravities; but from prop. x. of the fifth section, it appears, that the velocities with which solid particles immersed in a fluid descend by gradually subsiding, depend on their magnitudes as well as on their densities; consequently if the texture of any stratum of earth should be such as disposes it to be divided or broken into larger atoms, than those of which the lighter stratum consists, and their magnitudes are in a due proportion, the lighter particles will subside below those which are specifically heavier.

Strata of fossil and other earths are found, by accurate examination, to be disposed in no regular order in respect of

their specific gravities, but from what has been advanced concerning the manner in which the atoms or particles of matter subside in a fluid, those objections against Dr. Woodward's hypothesis, which are grounded on the promiscuous disposition of the strata, may appear to have less weight.

The principles of rotation are demonstrated in the sixth section. This theory is applied to explain the motion of pendulums which vibrate in circular arcs; to estimate the effects produced by the mechanic powers, or combinations of them, and to the solution of various problems.

In books of mechanics many experiments have been described by which the equilibrium of the mechanic powers, the composition and resolution of forces, and other statical principles are explained and verified; but no account is to be found of methods by which the principles of motion may be subjected to decisive and satisfactory trials. An attempt has been made to supply this deficiency in the seventh and eighth sections of this treatise. The seventh section contains the description of experiments on the rectilinear motion
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of bodies, both accelerated and retarded; and the experiments on the principles of rotation, including those which relate to the vibrations of pendulums, are inserted in the eighth section. Most of the fundamental properties of rectilinear motion and rotation, are illustrated by experiments described in these two sections. The numbers set down were the results of measurements and computations made with great care and attention: in order to insure the results true to one or two decimal places, the quantities are in some instances expressed to a greater number of places than may seem necessary. These experiments and the explanations of them, were part of a course of experimental lectures on the principles of natural philosophy, read in the university of Cambridge.

The hypotheses which ascribe permanent quantities of motion to bodies moving with given velocities, seem to have been adopted for the purpose of avoiding the difficulties which occur in solving most cases in practical mechanics; for if the effects of forces could be truly estimated by a measure, consisting of the quantity of matter moved and any power of the velocities,

cities, there could be no occasion to consider the variation of the forces of acceleration or resistance, since the ultimate effects produced would be known from the due application of the hypotheses without further investigation.

The chief intention of the ninth section is to examine into these hypotheses, and to shew that they ought not be admitted in mechanics as general principles; many instances of error might be produced, which have been occasioned by adopting them; one or two will be sufficient to justify what has been asserted.

In Emerson's Fluxions, p. 177, there is this problem: 'The radii of a wheel and axle are given in the proportion of $b : a$; a weight w acting by means of a line on the circumference of the wheel, elevates a weight y suspended from a line, which goes round the axle; it is required to assign the quantity y , when $y \times$ into its velocity generated in a given time, is the greatest possible.

In the solution the author supposes the momentum of bodies to be as the quantity of matter into the velocity generated; and according to the usual doctrine of momentum, assumes it as an universal truth, that

that if a force acts on any different quantities of matter for a given time, it will always generate the same moment, estimated by the quantity of matter into the velocity. From this reasoning he deduces the weight sought $y = \sqrt{2-1} \times \frac{bw}{a}$, whereas its true value is $y = w \times$

$\sqrt{\frac{b^4}{a^4} + \frac{b^3}{a^3} - \frac{b^2}{a^2}}$ (page 249) agreeing with the former only in the extreme case, when $b = a$, that is, when the radius of the wheel is equal to that of the axle.

Dr. Desaguliers in his course of experimental philosophy, vol. i. p. 173, applies the doctrine of momentum to investigate the pressure sustained by the axis of a wheel and axle, when a weight p acting at the distance b , elevates a weight q applied at the distance a from the axis.

He estimates the momentum of bodies by the quantity of matter into the velocity generated in a given time; and lays it down as a general rule, that "the momentum produced is always equal to the momentum which produces it:" by reasoning from this principle he deduces the pressure
on

on the axis, from the action of the two weights to be $= q + \frac{3pqa - qqa}{pbb + qab}$.

By another solution he finds the pressure on the axis $= \frac{qp \times \overline{b+a}^2}{pb^2 + qa^2}$; wholly different from the former, and yet both the results are consequences from the same data: nor will it be possible from any reasoning the author uses in these solutions, to distinguish which of them is true, or to determine whether both are not erroneous; although from the general principles of motion, independent of any consideration of momentum, it is without difficulty inferred, that the true pressure on the axis is the latter value $\frac{qp \times \overline{b+a}^2}{pb^2 + qa^2}$.

Many other examples of a similar kind might be produced; these two have been mentioned, not with a view of cavilling at writers who have deserved so well of philosophy, but to shew how much prejudice must be occasioned to science by assuming hypotheses of momentum as general principles, when even the most experienced persons have been led into

error by them. In the sections preceding the ninth of this treatise, the reader will find the hypothesis of momentum used in explaining some particular cases in mechanics; but it is in no instance assumed as a principle from which consequences are inferred in the solution of problems.

In the tenth section the principles of rotation in free space are deduced from those which are demonstrated in the sixth section concerning the rotation of bodies round fixed axes.

Analytical demonstrations have been adopted, as being most consistent with the general plan of the work. If this method of arriving at truth appears less eligible in some points of view, in others it certainly has its advantages; particularly in being easily applied to the investigation of problems, and in being comprized in a smaller compass than would in general be necessary for treating the same subject geometrically.

The ensuing treatise, however, is not intended to precede the study of those authors who have written geometrically on the principles of motion, but is rather to be considered as auxiliary and subser-

vient to them. It is generally allowed that no species of mathematical reasoning contributes so much to improve and strengthen the mind, as that which has for its objects the properties of space, whether they are purely abstract, or such as are joined with the explanation of natural phenomena: but whoever wishes to extend the use of the mathematical sciences still further, by applying them to the investigation of new abstract truths, or to the solution of such physical problems as any occasion may present, will soon perceive the necessity of not confining himself to the study of geometry only. It has been often remarked, that those who perfectly understand the geometrical propositions, which explain any branch of philosophy, without having considered the subject under an analytical form, find great difficulty in applying their theory to practice, even in the most ordinary cases.

The following pages are presented to the public as an imperfect sketch, rather than as a work finished according to the wish and satisfaction of the Author, who began this treatise with expectations of
con-

continuing in an academical establishment, which would have afforded him means and opportunities of rendering it more worthy of the public eye. The book, however, such as it is, has cost him some trouble; which he will think amply repaid, if it should appear in any degree to merit the approbation of philosophical and mathematical readers.

The Author takes this opportunity of publicly expressing his thanks and acknowledgements for the liberal assistance he has received toward the expences of printing this volume, from the Vicechancellor and Syndics of the University Press.

Trinity College, Cambridge,
Apr. 18, 1784.

THE

These and innumerable other forces which originate from causes however different, are yet referred to the same general principles of motion, which it is the business of Mechanics to explain.

Among the various laws observed during the acceleration and retardation of bodies, to which every material substance in nature is subject as far as human observation extends, three, called Laws of Motion, are assumed as Physical Axioms; being propositions which although the mind does not assent to on intuition, yet as they are of the most obvious and intelligible kind, suggested constantly by the ordinary motion and quiescence of bodies, and confirmed by every experiment which can be made on the operation of forces, as well as by such * arguments as the nature of the subject will admit of, appear the most proper to be received as principles from which the theory of motion in general may be regularly deduced.

* Vid.
Sect. IX.
p. 358.

II.

AXIOM 1. Every body perseveres in its state of rest or uniform motion in a right line, until a change is effected by the agency of some external force.

2. Any change effected in the quiescence or motion of a body is in the direction of the force impressed, and is proportional to it in quantity.

3. Action and reaction are equal and in contrary directions.

In comparing the changes caused in the motion of bodies, according to the second law of motion, the times of effecting such changes are understood to be equal: Vid. Newt. Princip. Cor. 1. to the axioms in which this second law of motion is referred to.

1. That a body once at rest will continue so until it is acted on by some external force is a truth so obvious that it needs no comment. Also a moving body having no
more

more power to alter the direction or velocity of its motion than to begin at first to move, must necessarily continue to proceed uniformly in a straight line, till a change is effected by the agency of some external force.

2. It is likewise rational to suppose that any change, whether in the direction or velocity of a body's motion, should be proportional to the causes producing such change, that is, to the forces impressed, the time of their action being the same.— Thus imagine a body to be moving from *A* to *B*, Fig. 1. if during the time of its motion, a force be impressed in the direction of *AC* or *BD*, which is parallel to it, this impressed force will deflect the body from its original direction, but will not accelerate or retard the time of its arrival at the line *BE*; it will however cause ~~the~~ distances *BD*, *BE* at which it arrives in that line, to become greater or less, according to the magnitude of the force which acts in the direction *BE*: for the same reason, if during a body's motion in the direction *AC*, a force in the direction *AB* be impressed, the time of the body's arrival at the line *CF* which is parallel to *AB* will not be altered, only the distances *CD*, *CF* in that line, will vary proportionally to the force acting in the direction *CF*. If therefore the two forces which act upon a body at *A* for a given time, be such as when singly applied would cause the body to describe the lines *AC*, *AB* respectively in the same time, these forces being applied together, the body by their joint action must at the end of the time have arrived at both the lines *CF* and *BE*, that is, at their common intersection *D*, in the extremity of the diagonal of the parallelogram *ABCD*, the two sides of which are in the direction of the two forces, and are in quantity equal to the spaces which would be described by the forces acting singly for the same time.

3. When a body describes equal spaces in equal times, the motion is said to be uniform. The degree or celerity of motion is merely relative, and implies the comparison of the spaces described by bodies moving uniformly for the same time: the proportion of the velocities will be the same as that of the spaces. The quantity of motion generated in a body, is the effect whereof the moving force is the cause. Quantities of motion therefore communicated to bodies in the same time are proportional to the moving forces.

Moving bodies also become subsequently the cause of motion in other bodies by impinging on them, but since matter possesses not in itself a power of creating or destroying the least motion, it follows that as much as is gained by

the body struck, will be lost by the other body, so that the quantity of motion existing in bodies, will be the same before and after the impact. From the same principle it follows, that a body any how attracting another body is attracted by it with equal force, and it appears from every experiment which can be made on bodies which approach each other in a right line, by means of their mutual attraction, that they move with velocities which are inversely proportional to the quantities of matter contained in them: It would follow from hence immediately, that the quantities of motion in moving bodies, are proportional to their quantities of matter and velocities jointly; the same conclusion however may be deduced from the laws of motion only, independent of any experiments.

These three physical propositions having been assumed as principles of motion, reduce the science of mechanics to mathematical certainty, arising not only from the strict coherence of innumerable properties of motion deduced from them a priori, but from their agreement with matter of fact, which agreement is ever seen most conspicuous when most severely and minutely examined: It is from these considerations that the laws of motion have been esteemed not only physically but mathematically true.

III.

Any force acting upon a body continually in the same direction, will produce a continual acceleration or retardation of its motion.

* 1st. Law of motion. Any velocity* produced in a body will be continued uniform, after the action of the force ceases; but as long as the force acts, new velocity† must be generated at every instant of the body's motion. Thus a body descending to the earth by the force of gravity, describes unequal spaces in equal times, its motion being continually accelerated. Also the elastic steam of gunpowder acting upon a musket ball, causes a perpetual acceleration of motion during its progress through the barrel; whereas if the ball impinging against a block of wood penetrates into its substance, the retarding force of the block causes a continual diminution of the initial velocity of impact, till the whole motion of the ball is destroyed.

The

The laws of acceleration and retardation will depend on those of the force, which may be infinitely varied. These different forces divide themselves into two sorts, viz. variable and constant, which distinction should be next described.

IV.

If during a body's motion, equal velocities be communicated or destroyed in equal successive portions of time, the force is said to be constant, and the velocity uniformly accelerated or retarded.

By the second law of motion, any change produced in the velocity of a moving body, is in the same direction with the force impressed, and is proportional to it in quantity, the time being given: wherefore by the proposition, equal velocities being produced or destroyed in equal times, it follows that the force acts equably.

Of this sort is the force of gravity, which in the descent or ascent of a body creates or destroys a velocity of $32 \frac{2}{11}$ feet in each second of time; also the force whereby a block of wood resists the progress of a ball entering into it, destroys equal velocities in equal times.

V.

When unequal velocities are generated or destroyed in equal successive portions of time, the force is said to be variable.

By the second law of motion the force is as the velocity generated or destroyed in the same body in a given time, which velocity, by the proposition, is variable: It follows therefore, that the force is variable.

When a body proceeds with accelerated or retarded motion, the velocities in any points of the space described, are estimated by finding what spaces would be described by the velocities acquired in those points, and continued uniform for a given time: the spaces described will give the ratio of the velocities required.

In

In like manner to estimate a variable force acting upon a body in any two points of the space described, it is necessary to find what would be the velocities generated or destroyed in a given time, by the force acting in those points, and continued constant; the ratio of these velocities will give the ratio of the forces required.

It appears therefore that the properties of variable forces must be referred to those of constant forces: these are considered in the ensuing propositions.

Although the properties of acceleration and those of retardation are equally deducible from the laws of motion, and might be demonstrated together, yet it will be more convenient to consider them separately. The laws of acceleration being demonstrated, those of retarded motions will follow from them.

VI.

When a body is acted upon by a constant force, there are four quantities which become the objects of mechanical consideration, viz. the space described, the time of description, the velocity acquired, and the force which produces it; any two of which being given, the others may be ascertained.

It may be remarked, that the force here mentioned, relates to the communication of velocity only, any difference in the quantities of matter moved, not being considered: It is called the accelerating force, being proportional to the velocity generated in a given time.

The moving force relates to the quantity of matter moved, as well as the velocity communicated, and it is proportional to the quantity of motion produced in a given time.

The distinction between these forces will further appear from considering what is the measure of the quantities of motion communicated to bodies, or of the moving forces by which such motion has been generated.

VII.

VII.

The moving forces which communicate the same velocity in a given time, to different bodies, will be as the quantities of matter contained in the bodies moved.

For if one body contain a quantity of matter ten times greater than another, then may the heavier be divided into ten bodies, each equal in quantity of matter to the lighter; and whatever force be required to produce a certain velocity in the lighter body, ten of these forces will be necessary to impel the ten bodies through the same space in the same time, so that the velocities of all the bodies shall be equal at the end of the motion; and it is the same as to the velocity produced, whether the bodies be separated or united, the ten forces still acting upon them.

Thus the moving force exerted by gravity upon bodies descending towards the earth's centre, is proportional to the quantities of matter contained in them; for it appears from experiment, that all bodies, whatever be their weights descend near the earth's surface through equal spaces in equal times, acquiring the same velocity in their descent, the air's resistance being removed, and it is manifest from the proposition, that the heavier bodies will require a greater force to move them than the lighter, according to the quantities of matter contained in them.

VIII.

The moving forces acting upon bodies and the quantities of motion communicated to them in a given time, are proportional to the quantities of matter moved and the velocities communicated jointly.

For when the velocity communicated in a given time is the same, the moving force * is as the quantity of matter * Art VII. moved; and † when the quantity of matter is given, the † 2d. Law. mov- of motion.

moving force is as the velocity communicated in the same time; therefore both quantities of matter and velocities communicated in the same time, being different, the moving forces, and their effects the quantities of motion produced, will be as the quantities of matter and velocities communicated jointly.

It is sometimes asserted, that there are two methods whereby motion may be communicated or destroyed, viz. either by the continual action of a moving force, or by instantaneous impact; but the latter way can obtain only in perfectly hard and inflexible bodies, which exist not in nature; and even in the abstract consideration of these, as well as of other cases in mechanicks, when metaphysical possibilities instead of the natural state of bodies are attended to, difficulties arise hardly explicable by any method of reasoning: but it is certain, that when finite velocity is communicated to any natural body, the time wherein it is communicated must be finite, so that when the body acted upon begins to move from quiescence, it will during the action of the force possess all the intermediate degrees of velocity, between 0 and the velocity ultimately communicated.

To exemplify this further, let us imagine that a soft and flexible substance, such as a ball of clay impinges against another substance of the same sort, in the direction of a line joining the centres of the balls. At the first instant of the impact, the body struck will begin to move, and will proceed with a velocity inferior to that of the impinging body, the velocity of which will continue to decrease, and that of the other body to increase as long as the impact causes a change in the figure of the two bodies, that is, till they shall have possessed a velocity common to both, at which instant all acceleration ceases; provided the bodies be perfectly nonelastic. If the bodies be of such a kind as after having received impression from any impact possess a power of restoring their changed figure with a force equal to that of the impact, it is manifest, that whatever velocity was communicated during the change of figure, an equal velocity will be superadded during the restoration of it. In this case after the acceleration arising from the impact during the change of the bodies figure has ceased, the bodies having then acquired a common velocity, a new acceleration will begin, being caused by the elastic force of the balls, which acting in a direction of the lines joining their centers, tends to separate them, accelerating the ball struck and retarding the other.

From

From these considerations it appears, that in whatever degree the hardness of perfectly elastic bodies may differ, the effects of their impacts on each other will be the same, the weights and velocities before the stroke being given. For the figures of the striking and of the other body must continually change, till they have acquired a common velocity, which depends only on the weights of the bodies and velocity of the impact, and is determined by the rules for the collision of nonelastic bodies. Moreover, the restoration of the changed figures, how small or great soever may have been the change, must cause an addition of velocity in the ball struck equal to that received from the impact.

It follows also, that the effects of the stroke will be the same, whether both bodies be perfectly elastic, or one perfectly elastic, and the other perfectly hard, every thing else being given: for the figure of the elastic body must change till the bodies have obtained a common velocity, which depends on the weights and velocities before the stroke only, and will be the same as if the bodies were nonelastic; the restoration of the figure will in this, as well as in the former case, cause an increase of velocity in the ball struck equal to that before communicated.

Although no substance in nature possesses perfect elasticity, or is entirely destitute of it, yet there are several elastic and nonelastic bodies subject to experimental trials, wherein the laws relating to collision, are found to agree with matter of act, to a considerable degree of exactness.

IX.

The accelerating forces which communicate velocities to bodies, are as the moving forces directly, and the quantities of matter moved inversely.

Since the accelerating force* is as the velocity generated in a given time, and by the last article the moving force is as the quantity of matter and velocity generated in a given time, it follows that the moving force is as the accelerating force, and the quantity of matter moved jointly; that is, the accelerating force is as the moving force directly, and the quantity of matter moved inversely.

To illustrate this, let Q and q denote any two masses of matter: let m be the force or weight by which gravity impels q towards the earth's centre, and let M be any other force

* Art. VI.

force which urges the mass $\mathcal{Q}$: then by the proposition the ratio of the velocities generated in $\mathcal{Q}$ and q in a given time, or which is the same, the ratio of the forces which accelerate $\mathcal{Q}$ and q will be that of $M \times q : m \times \mathcal{Q}$, wherefore if F be put for the force which accelerates $\mathcal{Q}$, and f for the accelerating force of gravity, we shall have F :

$\therefore M \times q : m \times \mathcal{Q}$, or $\frac{F}{f} = \frac{M}{m} \times \frac{q}{\mathcal{Q}}$. Now it is manifest,

that the values of the quantities $\frac{F}{f}$, $\frac{M}{m}$ and $\frac{\mathcal{Q}}{q}$, depend

only on the ratios of the numerators to the denominators; for which reason, the standard or constant quantities f , m and q , with which F , M and $\mathcal{Q}$ are compared, may be *as-

* Vid.
Sect. II.
Prop. VII.

sumed = 1: this will give $F = \frac{M}{\mathcal{Q}}$. Thus, let a mass

of matter = to that contained in 4 ounces of any substance, be impelled by a force = to the weight of 3 ounces, then the force which accelerates the mass of 4 ounces will be $\frac{3}{4}$, when the acceleration of gravity is 1; or, in other words, it will generate in a given time three parts in four of the velocity which gravity generates during any given time in bodies which descend toward the earth's surface.

When any quantity is said to be given, it is meant, that the relation of it to some fixed quantity of the same sort, considered as a standard, is known; in like manner, when any quantity is sought, it is required to find the relation of this unknown quantity to some fixed standard of the same kind. All forces of acceleration are referred to gravity, that is, to the force whereby the earth accelerates adjacent bodies towards its centre. Time is referred to the earth's periodic revolution round its axis, whether it be considered as divided into hours or minutes, &c. These two are definite standards, common to all ages and climes, and it appears from the laws of motion, that standards of space and velocity also are determinable: but many difficulties in practice have hitherto prevented any methods from being carried into execution to fix these; so that different nations continue to make use of different measures to which they refer all other spaces: and even in the same nation these measures from the imperfection of the materials composing them, as well as from various accidents, have been found in a series of ages liable to alteration.

The next section contains a few properties of ratios, which immediately relate to the mechanical propositions ensuing, as well as to the nature of the standard quantities just described.

S E C T.

S E C T. II.

CONTAINING SOME PROPERTIES OF RATIOS.

I.

TWO mathematical quantities of the same kind constitute a ratio.

Thus two lines, two surfaces, &c. constitute a ratio which may always be expressed by numbers, either commensurable or incommensurable with unity.

Physical quantities, which are in themselves not mathematical, are nevertheless capable of mathematical relation, if they are measurable by space or number; thus angles, velocities, times, and forces, and ratios themselves, become capable of mathematical relation.

II.

The ratio of two mathematical or physical quantities may be compared with the ratio of two other quantities, though of a different kind from the former.

Thus when two bodies move uniformly for the same time, the spaces described vary with the velocities, and though space and velocity, being different kinds of quantity, are not comparable, yet the ratio of the spaces is comparable with that of the velocities, these ratios being equal.

III.

If any quantity be divided by a quantity of the same sort, the quotient becomes abstract number.

Thus if U and v represent two velocities, then will $\frac{U}{v}$ be a number which is to 1 as $U : v$.

IV.

The ratio of any mathematical or physical quantities may be expressed by two numbers, if both terms of the ratio be divided by the consequent, or by the antecedent,

Thus if F and f represent two forces, and both be divided by f , the ratio will become $\frac{F}{f} : 1$, which by the last proposition is that of number to number.

V.

Any ratio may be represented by a fraction, the numerator of which is the antecedent, and denominator the consequent of the ratio.

Hence, the addition and subtraction of ratios may be performed by the multiplication and division of fractions. Thus, when two bodies move uniformly, the ratio of the spaces described is equal to the ratio of the velocities added to the ratio of the times of motion, and is expressed thus at length, $S : s :: T \times V : t \times v$; or thus, $\frac{S}{T} = \frac{s}{t} \times \frac{V}{v}$

T

$\frac{T}{t} \times \frac{V}{v}$; which equation implies, that an abstract number $\frac{S}{s}$, is equal to the product of two other abstract numbers $\frac{T}{t}$ and $\frac{V}{v}$: and if the consequents s , t , and v , be made unity, which assumption will not alter the values of the ratios, provided the antecedents S , T , and V vary proportionally, the arithmetic of these ratios will be hereby facilitated, for the last equation will become $\frac{S}{1} = \frac{T}{1} \times \frac{V}{1}$, or $S = T \times V$; S , T and V being abstract numbers.

It must be observed in general, that whenever a mark of equality is interposed between heterogeneous quantities, as $S = T \times V$, no other equality is meant, but that which subsists between the ratios there expressed; that is, the ratio of the spaces, viz. $S : 1$, is equal to the sum of ratios $T : 1$ and $V : 1$; so, since $T = \frac{S}{V}$, we have the ratio of the times, is equal to the ratio of the spaces diminished by the ratio of the velocities. Moreover, when the mark of multiplication is interposed between heterogeneous quantities, it means the addition of two ratios, the antecedents of which are the terms expressed, and consequents are unity.

VI.

Let $\frac{A}{a}$, $\frac{B}{b}$ and $\frac{C}{c}$, be three ratios, consisting of variable terms. If the relation of these quantities be such, that when $\frac{C}{c}$ becomes unity, or the ratio of equality, $\frac{A}{a} = \frac{B}{b}$, also when $\frac{B}{b}$ becomes unity, if $\frac{A}{a} = \frac{C}{c}$, the proposition asserts, that whatever

ever

ever be the magnitudes of $\frac{A}{a}$, $\frac{B}{b}$ and $\frac{C}{c}$,

we always have $\frac{A}{a} = \frac{B}{b} \times \frac{C}{c}$.

For if not, let $\frac{A}{a} = \frac{B^{1+d}}{b^{1+d}} \times \frac{C^{1+e}}{c^{1+e}}$, then when $\frac{B}{b} = 1$, we have $\frac{A}{a} = \frac{C^{1+e}}{c^{1+e}}$, but by the hypothesis, when $\frac{B}{b} = 1$, $\frac{A}{a} = \frac{C}{c}$: wherefore $\frac{C^{1+e}}{c^{1+e}} = \frac{C}{c}$ and $1 + e = 1$. This contradictory conclusion arises from denying the proposition asserted, which is therefore true.

In general let $\frac{A^p}{a^p}$, $\frac{B^m}{b^m}$ and $\frac{C^n}{c^n}$ be three ratios, consisting of the variable terms A a , B b , and C c : if the relation of these quantities be such, that when $\frac{C^n}{c^n} = 1$, $\frac{A^p}{a^p} = \frac{B^m}{b^m}$; also when $\frac{B^m}{b^m} = 1$, $\frac{A^p}{a^p} = \frac{C^n}{c^n}$; then whatever be the magnitudes of $\frac{A^p}{a^p}$, $\frac{B^m}{b^m}$ and $\frac{C^n}{c^n}$, we have $\frac{A^p}{a^p} = \frac{B^m}{b^m} \times \frac{C^n}{c^n}$: for if not, let $\frac{A^p}{a^p} = \frac{B^{m+e}}{b^{m+e}} \times \frac{C^{n+d}}{c^{n+d}}$, then when $\frac{B^m}{b^m} = 1$, we have $\frac{A^p}{a^p} = \frac{C^{n+d}}{c^{n+d}}$; but by the hypothesis, when $\frac{B^m}{b^m} = 1$, $\frac{A^p}{a^p} = \frac{C^n}{c^n}$; wherefore $\frac{C^{n+d}}{c^{n+d}} = \frac{C^n}{c^n}$, and $n = n + d$, the lesser equal to the greater, if d be of finite magnitude; and should d be evanescent, we have $\frac{A^p}{a^p} = \frac{B^{m+e}}{b^{m+e}} \times \frac{C^n}{c^n}$, from which by proceeding in the same manner as before, it would be deduced that $m + e = m$; that is, if e be of any finite magnitude, the lesser is equal to the greater, which is impossible: it follows therefore, the quantities d and e being necessarily evanescent, that $\frac{A^p}{a^p} = \frac{B^m}{b^m} \times \frac{C^n}{c^n}$.

Cor. 1. The weights of bodies depend upon their magnitudes and densities: and if W , w represent the weights of

of two bodies, M, m their magnitudes, and D, d their respective densities, then when $M = m$, or $\frac{M}{m} = 1$, the weights will be as the densities, or $\frac{W}{w} = \frac{D}{d}$; also, if $D = d$, the ratio of the weights $\frac{W}{w} = \frac{M}{m}$ the ratio of the magnitudes: wherefore by the proposition, whatever be the quantities $\frac{W}{w}, \frac{M}{m}$ and $\frac{D}{d}$, we always have $\frac{W}{w} = \frac{M}{m} \times \frac{D}{d}$, or the ratio of the weights = the sum of the ratios of the magnitudes and densities.

Cor. 2. The ratio of equality is in quantity = 0, because, being added to or subtracted from any ratio, it causes neither increase nor diminution of the ratio. Thus, when $\frac{D}{d} = 1$, or when D and d are in the ratio of equality, we have $\frac{W}{w} = \frac{M}{m} \times \frac{1}{1} = \frac{M}{m}$.

Cor. 3. An inverse ratio is that wherein the consequent and antecedent are inverted, or if expressed by a fraction, it is when the numerator becomes the denominator, and the denominator the numerator, thus $\frac{m}{M}$ is the inverted ratio of $\frac{M}{m}$.

Cor. 4. An inverted ratio is equal in quantity to the direct ratio, but is negative in respect of it; for by adding the direct and inverse ratios of two quantities together, the sum becomes = 0, or the ratio of equality: thus the direct ratio $\frac{M}{m}$ added to the inverted ratio $\frac{m}{M}$ the sum becomes $= \frac{mM}{Mm} = \frac{1}{1}$; thus since $\frac{W}{w} = \frac{D}{d} \times \frac{M}{m}$ we have $\frac{M}{m} = \frac{W}{w} \times \frac{d}{D}$, or the ratio of the magnitudes = the ratio of the weights added to the inverted ratio of the densities; or since the inverted ratio of the densities = — the direct ratio of the densities, it follows, that the ratio of the magnitudes will be equal the difference between the ratios of the weights and densities.

Cor. 5. When m, w and d , are each = 1, then will $M,$
 D

D and W be abstract numbers, and we have $M = \frac{W}{D}$ expressing the ratios described in the last Cor.

VII.

In the comparison of the ratios which obtain between mathematical quantities of any sort, the standard to which each of those quantities is referred, may be assumed $= 1$.

This will be best illustrated by an example. Let the weight, magnitude and density of any substance be respectively W , M and D . These relative terms imply the consideration of some standard substance with the weight, magnitude and density of which W , M and D are compared. If the weight, magnitude and density of the standard substance be w , m and d , we have in general $\frac{W}{w} = \frac{M}{m} \times \frac{D}{d}$.

Let a cubic inch of water be assumed as a standard substance, then may w , m and d be each taken $= 1$, the corresponding terms being expressed proportionally to 1, and the equation expressing the relation of weight, magnitude and density of the substance, to those of water, will be $\frac{W}{1} = \frac{M}{1} \times \frac{D}{1}$, or $W = M \times D$; W , M and

D being abstract numbers. This assumption will be most useful in general solutions, but in the application of them to particular cases, since the weight of a cubic inch of water, which was assumed $= 1$, corresponds with none of our common weights, it is more convenient to ascertain by experiment, what is the weight of a cubic inch of water, when referred to any of the standard weights in use among us, viz. to ounces or grains, &c. and it appears upon trial, that the weight sought is .57869 parts of an avoirdupoise ounce: this being therefore substituted in the

preceding equation for w or 1, we have $\frac{W}{.57869} = M \times D$, or $W = .57869 M \times D$, which expresses the relation of the

the weight, magnitude and density of the given substance, compared with the weight, magnitude and density of water, and accommodated to our standard measures of weight and space, that is, the weights being estimated in avoirdupoise ounces, and the magnitudes in cubic inches.

If instead of a cubic inch, m be made $\equiv$ to a cubic foot, this will afford a greater facility in practice: in this case, $m = 1$ cubic foot, and water being still considered as the standard substance, $w =$ the weight of a cubic foot of water $\equiv 1000$ ounces avoirdupoise by experiment, if the specific gravity of water be assumed $\equiv 1000$, we have

$$\frac{W}{1000} = \frac{M}{1} \times \frac{D}{1000}, \text{ or } W = M \times D, \text{ which is the rule}$$

delivered in COTES's Hydrostatics, Lect. 6.

The abbreviations made use of in the following propositions are these; F and f represent any two constant forces, V denotes the velocity generated by the force F in the time T , during which time the space S is described by constant acceleration; v is the velocity generated during the time t , by the action of an accelerating force f , the space described during the time t being s .

F and f are always understood to be accelerating forces proportional to the velocities generated in a given time.

S E C T. III.

CONCERNING THE RECTILINEAR MOTION
OF BODIES IMPELLED BY FORCES
WHICH ACT UNIFORMLY.

I.

THE velocities generated in bodies
by the action of constant forces, are
as those forces and the times in which
they act jointly, or $\frac{V}{v} = \frac{F}{f} \times \frac{T}{t}$.

For when the times are the same, the velocities gene-
rated are as the forces of acceleration,* that is, when $\frac{T}{t} = 1$,

* 2d. Law
of motion.

$\frac{V}{v} = \frac{F}{f}$: and if the forces are the same, the velocities ge-
nerated are as the times wherein the forces act; because
when the force is given, equal velocities † are generated
in equal times, and consequently the whole velocities ac-
quired are as the times wherein the given force acts; that

† Sect. I.
Prop. IV.

is, when $\frac{F}{f} = 1$, $\frac{V}{v} = \frac{T}{t}$; wherefore, both times and ac-

‡ Sect. II.
Prop. VI.

celerating forces being different, ‡ the velocities generated
will be as the forces and the times of their action jointly:

whatever therefore be the magnitude of $\frac{T}{t}$, $\frac{V}{v}$ and $\frac{F}{f}$, it is

proved, that $\frac{V}{v} = \frac{F}{f} \times \frac{T}{t}$.

In this proposition, F represents any constant force, which in the time T , generates a velocity V ; f represents any other constant force, which in the time t , generates a velocity v ; this proposition, therefore, is wholly independent of the absolute magnitudes of the forces, velocities and times, and consequently, no absolute quantity can be inferred from it. The use of the proposition is, however, to compare any undetermined velocity, the constant force which produces it, and the time wherein it is acquired, with a known velocity, the known force by which it is generated, and the definite or standard time wherein the known force acts respectively: the effects of this known force, as to the velocity produced and time of producing it, being determined by actual observation. In this case, therefore, f will represent a standard force, v the velocity generated by it in the time t , all which might be $\dagger$ assumed $\dagger$ Sect. II. equal 1, so as to make $V = F \times T$: and in general solutions Prop. VII. unapplied to any absolute known force, such assumption is allowable, because there is as much reason to denominate any one space (for example $32\frac{2}{3}$ feet) unity, as any other.

But since most physical propositions are either immediately or ultimately applied to the operation of natural forces, the effects of which are estimated by measures of space, which, however arbitrarily fixed at first, still continue in common use, it may be proper to describe such physical standards of force, time and velocity, with which all other forces, times and velocities may be conveniently compared.

Of these three standard quantities, two must be first given or fixed on at pleasure, and the third obtained by actual observation. Now gravity is a force, the general and permanent nature of which renders it the most proper to be assumed as a standard, to which all other forces may be referred: wherefore, this must be one of the given quantities, and may be $\ast$ denoted by 1, every other force, $\ast$ Sect. II. with which it is compared, being represented by a num- Prop. VII. ber which is to 1, as that force is to gravity: on the other hand, bodies may acquire different velocities, descending by gravity in different times. We may therefore chuse whether to assume a certain velocity, and denominating it 1, observe in what time the action of gravity generates this velocity; or we may assume a time $= 1$, and observe the velocity generated during the action of gravity for that time. The latter method, however, from the nature of the case is evidently preferable to the other; which will appear from the following considerations. Time is ne-

cessarily referred to the revolution of the earth round its axis, whether it be considered as divided into minutes or seconds, &c. this periodic revolution being liable to no alteration or variation whatever, and consequently the measures of its parts, consisting of the portions of a given angle, described uniformly will be invariable, and therefore common to all ages, and places upon earth the most distant from each other.

We observe, from hence, that standards of force and time are pointed out by nature herself, and so plainly as to have found universal reception among mankind: whereas, although standards or invariable measures of space and velocity, are also equally fixed by nature, yet the determination of them requiring skill in the theory, as well as the practical parts of mechanics, has among other causes prevented such measures from becoming any where adopted; the reason of which is, that men find it necessary to weigh and to measure, before they become philosophers; and in more improved times, philosophy is of too little consequence to alter what has been established by centuries of continual use.

It is, however, necessary in physical enquiries, to obtain a standard velocity, by actually observing, or inferring from some other actual observations, what velocity is generated by the force of gravity during a given time, for example one second: this velocity being expressed in the measures of space, which are in common use.

The determination of this standard velocity will render the proposition above demonstrated, applicable to the comparison of all other velocities generated in any different times, and by any constant forces, with such velocities, as are generated by the force of gravity; the times and forces themselves being at the same time compared: For suppose it to have been observed, that in a body's descent by the force of gravity for one second, a velocity was generated which would, if uniformly continued, carry the body through a space $= v$ in one second of time: then

* Sect. III. referring to the general * equation $\frac{V}{v} = \frac{T}{t} \times \frac{F}{f}$, we have Prop. I.

† Sect. II. † $t = 1$ second, $f =$ the force of gravity $= 1$, and consequently F and T abstract numbers, wherefore $V = v \times T \times F$. Prop. VII.

It has been hitherto supposed, for the sake of illustration, that v may be obtained by observing the velocity

city acquired in a body's descent for one second of time ; but since the acceleration of gravity causes a continual change in the velocity of the descending body, so as to render an estimation of the velocity at any particular point of time difficult and hitherto unattempted, the value of v , or the velocity acquired by a body which has descended for one second, may be more easily ascertained by the help of the next proposition.

In the mean time, this will be the proper place to remark concerning the theorem already demonstrated, that it is applicable to the motion of bodies acted upon by variable forces also, provided the times wherein they act be taken so small, that the forces may be regarded as constant.

Thus, let f , t and v represent any standard force, time and velocity ; and let F , T and V be other quantities of the same kind, which are compared with the former respectively ; then however the force F may vary, yet if an element of time represented by $\dot{T}$, be taken for the time of its action, it will have the properties of a constant force, as far as regards the particle of time above described. Let

$\dot{V}$ represent the velocity generated by the force F in the time $\dot{T}$, wherefore we have by the theorem $\frac{\dot{V}}{v} = \frac{F}{f} \times$

$\frac{\dot{T}}{t}$; and if v , f and t be assumed each = 1,* the equa-
* Sect. II.
Prop. VII.

tion will be $\dot{V} = F \times \dot{T}$.

II.

If a quiescent body be impelled by any constant force acting upon it for a given time, the space described will be to the space described in the same time, by the body moving uniformly with the last acquired velocity in the ratio of one to two.

For let the given time be divided into equal evanescent instants, the number of which is n ; then the † velocity ge-
† Sect. III.
Prop. I.

nerated being as the time, and continuing uniform during any one instant, we shall have the space described in any one instant proportional to the number of instants comprehended in the time of motion: so that if during the first instant the space described be s , in the next instant the space described will be $2s$, in the third $3s$, and in the three first instants the space described will be $s + 2s + 3s$; so, in the n first instants the space described will be

* Sect. III.
Prop. I. $s + 2s + 3s \dots ns = \frac{n+1 \times ns}{2}$: and since the * velo-

city last acquired is as the time, the force being given, and the space described by any uniform velocity, is as the time and velocity jointly, it follows that the space described by the last acquired velocity continued uniform for the time of the accelerated motion, will be as the square of that time; so that if s be the space uniformly described in the first instant of motion, $n^2 s$ will be the space described in n instants with the velocity last acquired: wherefore the space described by acceleration from quiescence, is to the space described uniformly with the last acquired velocity in the same time, as $n^2 s + ns : 2n^2 s$, or as $n+1 : 2n$; and since gravity acts not by successive impulses, but by unceasing acceleration, the magnitude of each instant must be diminished, and consequently their number increased, fine limite; the last proportion therefore of $n+1, 2n$, will become that of $1 : 2$.

From this proposition, the velocity which a body acquires in its descent by the acceleration of gravity for one
† Sect. III.
Prop. II. second, is easily obtained: for since this velocity † is such as if continued uniform would carry the body in one second through a space twice greater, than that from which the body has descended, it follows, that to find the velocity acquired during any given time, for example one second, it is only necessary, that the space which a body describes in its descent from quiescence in one second should be observed. It is found from actual observation, (but more exactly from other methods,) that the space through which a body descends from rest in one second, is equal 193 English inches, or 16 feet and one inch; and in this

† Sect. III.
Prop. II. descent it appears from the proposition, † that such a velocity is acquired as would carry the body uniformly over $2 \times 16\frac{1}{2}$, or $32\frac{1}{2}$ feet in a second of time; wherefore, if $16\frac{1}{2}$ feet, or 193 inches, be put equal l , then will the standard velocity which was before represented by u
 $= 2l$.

By

By the first proposition † it is found, that in general $\frac{V}{v} = \frac{F}{f} \times \frac{T}{t}$, assuming therefore * f and t each = 1, and $\frac{V}{v} = \frac{F}{f} \times \frac{T}{t}$, $\frac{V}{v} = F \times T$ and $V = v \times F \times T$. † Sect. III, Prop. I.
* Sect. II, Prop. VII.

substituting $2l$ for v , we have $\frac{V}{2l} = F \times T$ and $V = 2l \times F \times T$.

This proposition is applicable to the motion of bodies impelled by variable forces, if the times of motion be assumed evanescent. Since $V = 2l \times F \times T$, by taking the contemporary variations, we have $\dot{V} = 2l \times \dot{F} \times \dot{T}$, because during the least variation of the time $\dot{T}$ the force F is constant.

III.

The spaces which bodies describe from rest by the action of constant forces, are in a compound ratio of the velocities last acquired, and times of motion, or

$$\frac{S}{s} = \frac{V}{v} \times \frac{T}{t}.$$

For the spaces described by the last acquired velocities continued uniform, are as those velocities and the times of motion jointly: and the † spaces described by the accelerating forces acting constantly for equal respective times, are half the former spaces by the last proposition. † Sect. III, Prop. I.

Since in general $\frac{S}{s} = \frac{T}{t} \times \frac{V}{v}$, making $\S t =$ one second, $\frac{S}{s} = \frac{T}{1} \times \frac{V}{v}$, $\frac{S}{s} = T \times \frac{V}{v}$, $S = \frac{V \times T}{2}$, which is an equation expressing the relation § Sect. II, Prop. VII.

$v = 2l$, and consequently $\parallel t = l$, we have $\frac{S}{l} = \frac{T}{1} \times \frac{V}{2l}$, or $S = \frac{V \times T}{2}$, || Sect. III, Prop. II.

of S the space described, and V , a measure of the velocity, being also a space, which would be uniformly described by the last acquired velocity in the time 1: it appears that the space described by the accelerated body, is equal to half of the space V multiplied into an abstract

abstract number T , which expresses the number of seconds contained in the time of motion.

This proposition is also true when applied to the motion of a body which is accelerated by a variable force, provided the time in which it acts be diminished sine limite, so that the force acting for an instant, may become constant.

Since $S = \frac{V \times T}{2}$, we have by taking the cotemporary variations $\dot{S} = \frac{\dot{V}T + V\dot{T}}{2}$; but $V = 2lFT$ and $\dot{V} = 2lF\dot{T}$, or $\dot{V}T = 2lFT\dot{T} = V\dot{T}$, which being substituted in the equation $\dot{S} = \frac{\dot{V}T + V\dot{T}}{2}$, it appears that $\dot{S} = V \times$

† Sect. III.
Prop. II.

$\dot{T}$ and $\dot{T} = \frac{\dot{S}}{V}$ in every scale, whatever be the value of the standard space l ; $\dot{S}$ and V being referred to the same standard space, and $\dot{T}$ and V to the same standard time; for every velocity respects two standard measures, one of space, and the other of time.

It appears therefore that the quantity $\dot{T}$, which is a number referred to any standard 1, will be equal to the quotient of the space described in the time $\dot{T}$, divided by the space which would be uniformly described with the velocity of the body while it is describing $\dot{S}$, in the time 1; whatever be the standard space l , to which S and V are referred.

IV.

Constant forces which accelerate bodies, cause them to describe from rest spaces which are as the forces and squares of the times wherein they act jointly,

that is, $\frac{S}{s} = \frac{T^2}{t^2} \times \frac{F}{f}$.

For

For the spaces described, estimated from quiescence, are as the velocities * last acquired and the times of motion * Señt. III, Prop. III. jointly, or $\frac{S}{s} = \frac{T}{t} \times \frac{V}{v}$, and the velocities last acquired, are

as the forces and the times of their acting jointly, † that is, † Señt. III, Prop. I. $\frac{V}{v} = \frac{F}{f} \times \frac{T}{t}$; wherefore substituting $\frac{F}{f} \times \frac{T}{t}$ for its equal $\frac{V}{v}$, in the former equation, we have $\frac{S}{s} = \frac{T^2}{t^2} \times \frac{F}{f}$.

Cor. 1. If $F = f$, that is, if the forces are the same, it follows, that $\frac{S}{s} = \frac{T^2}{t^2}$, or the spaces are in a duplicate ratio of the times.

Cor. 2. Since ‡ $\frac{V}{v} = \frac{T}{t}$, when the force is given, † Señt. III, Prop. I. substituting $\frac{V^2}{v^2}$ for $\frac{T^2}{t^2}$ in the last Cor. we have $\frac{S}{s} = \frac{V^2}{v^2}$, that is, the spaces are in a duplicate ratio of the velocities last acquired when the force of acceleration is given.

Cor. 3. Since $\frac{S}{s} = \frac{T^2}{t^2} \times \frac{F}{f}$, we have $\frac{F}{f} = \frac{S}{s} \times \frac{t^2}{T^2}$, that is, the ratio of the forces is compounded of the direct ratio of the spaces described, and the inverse duplicate ratio of the times wherein the spaces are described from rest, and consequently when $T = t$, or the times being given, the spaces described are in the same proportion with the forces of acceleration.

Cor. 4. The proposition contained in the last Cor. is applicable to the nascent motion§, produced by a finite force § Newt. Princip. Sect. I. Lemma X. any how variable, and just beginning to impel a body from quiescence: thus a body will in a given time be attracted by the force of gravity through a greater space near the earth's surface, than if the body were situated at the distance of the moon, in the proportion of $60 \times 60 : 1$, which is known from principles not immediately connected with the present subject; wherefore by the proposition, the forces being as the spaces described when the time is given, it follows, that the force of gravity at the earth's surface, is to the force which it exerts at the distance of the moon, as $60 \times 60 : 1$.

Cor. 5. To apply this proposition in order to estimate the force, time and space in reference to the standard measures

* Sect. II.
Prop. VII.

tures above described, since $\frac{S}{s} = \frac{T^2}{t^2} \times \frac{F}{f}$, let $*t = 1$ second, and $f = 1$, the force of gravity, which acting for one second, causes a body to describe in its free descent $16\frac{1}{2}$ feet, or 193 inches, let this equal l , wherefore we have $\frac{S}{l} = T^2 \times F$, or $S = T^2 F l$.

V.

The constant forces which accelerate bodies from rest are in a direct duplicate ratio of the velocities generated, and in an inverse ratio of the spaces described, or

$$\frac{F}{f} = \frac{V^2}{v^2} \times \frac{s}{S}.$$

† Sect. III.
Prop. I.
‡ Sect. III.
Prop. III.

† For $\frac{F}{f} = \frac{V}{v} \times \frac{t}{T}$, and $\dagger \frac{t}{T} = \frac{V}{v} \times \frac{s}{S}$, substituting therefore $\frac{V}{v} \times \frac{s}{S}$, for its equal $\frac{t}{T}$ in the former equation, we have $\frac{F}{f} = \frac{V^2}{v^2} \times \frac{s}{S}$.

Cor. 1. The last acquired velocities are in a subduplicate ratio of the accelerating forces, and a subduplicate ratio of the spaces described jointly, for by the proposition

$$\frac{V^2}{v^2} = \frac{F}{f} \times \frac{S}{s}, \text{ or } \frac{V}{v} = \sqrt{\frac{F}{f}} \times \sqrt{\frac{S}{s}}.$$

§ Newton.
Princip.
Sect. II.
Prop. V.

Cor. 2. Let V and v represent the velocities of a body in different points of the same curve, or of two bodies moving in any two points of different curves, the centripetal forces being F and f , which retain the bodies revolving in their orbits. Let S and s represent the spaces, through which a body must move by the acceleration of the forces F and f continued constant, in order to acquire the velocities in the curves V and v respectively; || then will the centripetal forces be in a direct duplicate ratio of those velocities, and an inverse ratio of the spaces described: for according to the proposition

$$\frac{F}{f} = \frac{V^2}{v^2} \times \frac{s}{S}.$$

Cor.

Cor. 3. If gravity be made the standard * force $f = 1$, and $v = 2l$, then will $s = l$, and the equation will become $\frac{F}{1} = \frac{V^2}{4l^2} \times \frac{l}{S}$, or $V^2 = 4lFS$, and $V = \sqrt{4lFS}$. * Sect. II.
Prop. VII.
Sect. II.
Prop. I.

To exemplify: Suppose it were required to assign the velocity generated in a body, descending from rest 3 feet along an inclined plane, the elevation of which above the horizon equal 30° : here $l = 16\frac{1}{2}$, $F = \frac{1}{2}$, $S = 3$, and the velocity required $= \sqrt{4 \times 16\frac{1}{2} \times \frac{1}{2} \times 3} = 9,82$ feet in a second.

Cor. 4. This proposition is also applicable to the action of forces however variable, provided the spaces, through which bodies are accelerated by them, be so diminished, that the forces may be assumed constant.

Since by the proposition $\frac{V^2}{v^2} = \frac{F}{f} \times \frac{S}{s}$, v , f , s and F being constant quantities, and V and S variable, tak-

ing the least cotemporary variations, $\frac{2V\dot{V}}{v^2} = \frac{F}{f} \times \frac{\dot{S}}{s}$, and making $f = 1$, $v = 2l$, and consequently $s = l$, we have $2V\dot{V} = 4lF\dot{S}$. § Newt.
Princip.
Sect. VII.
Prop.
XXXIX.
† Sect. II.
Prop. VII.

In order to accommodate this theorem to practice, let z be a space through which a body must fall by the force of gravity, so that it may acquire the velocity V , $\dagger$ then will $V = \sqrt{4lz}$, and $V^2 = 4lz$, and $2V\dot{V} = 4l\dot{z}$. † Cor. 3.

Substituting therefore $4l\dot{z}$, for $2V\dot{V}$ in the preceding equation, we have $4l\dot{z} = 4lF\dot{S}$, and $\dot{z} = F \times \dot{S}$, which equation expresses the relation of the homogenous quantities $\dot{z}$ and $\dot{S}$, F being a number which is to 1, as the accelerating force acting on the body while it is describing $\dot{S}$ to the force of gravity.

Let AO be any space which a body describes by the action of a variable accelerating force, the quantity of which during the time wherein the body describes the element of space $\dagger Oo = F$. Fig. II.
† Sect. I.
Prop. V.

Then supposing $AO = x$, $Oo = \dot{x}$, and z equal to a space, which a body falls through by the acceleration of gravity, to acquire the velocity which the body describing AO with the variable force possesses in the point O ;

since $\dot{z} = F \dot{x}$, if F can be expressed in the terms of x , and the fluent of $F \dot{x}$ be obtainable, the velocity of the body, while it is describing the space Oo will be known, for any velocity will be determined, if we can ascertain from what altitude a body must fall by the force of gravity to acquire that velocity. This altitude z is obtained from the equation $\dot{z} = F \dot{x}$, z being equal to the fluent of $F \dot{x}$: the velocity therefore with which the body describes the space $Oo = \sqrt{4l \times \text{fluent of } F \dot{x}}$, and if T represent the time of describing AO , we have the time of describ-

ing $\dagger Oo$, that is, $T = \sqrt{\frac{x}{4l \times \text{fluent of } F \dot{x}}}$, and the time

of describing AO , or $T = \text{the fluent of } \sqrt{\frac{x}{4l \times \text{fluent of } F \dot{x}}}$.

This method of applying the Newtonian theory of acceleration to the solution of physical problems being the most commodious of any, especially as it immediately reduces the solution to any required standard of space and velocity, should be further described, as it will frequently hereafter be referred to. The quantity z which has been denominated the space, through which a body must fall by the force of gravity to acquire the velocity in O , Euler,* who first brought this method into use, defines for the sake of brevity, the space due to the velocity at O , from the acceleration of the constant force 1, that of gravity for example. It being evident, that in the

equation $\dot{z} = F \dot{x}$, it is intirely immaterial to what standard measures z and x are referred, provided these measures be the same; if x , the space described, be estimated in English feet, z will be also expressed in English feet, &c. It will therefore be unnecessary in the solution of problems, to consider the standard quantity l , untill z the fluent of $F \dot{x}$ is ascertained; this being effect-

ed, the $\dagger$ velocity $V = \sqrt{4l \times \text{fluent of } F \dot{x}}$ will be expressed in whatever denomination l is taken.

This measure l is frequently estimated by foreigners in terms of the Rhyndland foot, because since the Rhyndland foot is to the English, as 10000:9715, and $l = 16\frac{1}{2}$ English feet, it follows, that l expressed in Rhyndland feet is equal

15.625,

* Motus
Scient.
Vol. I. Sect.
3.

$\dagger$ Prop. V.
Cor. 3.

35.625, or 15625 thousandth parts of a Rhyndland foot; therefore $4l = 62500$, and the velocity $V = 250$

$\sqrt{}$ fluent of Fx expressed by a thousandth part of a Rhyndland foot in a second: this is here inserted, because some authors have adopted these numbers, without the necessary description of the grounds upon which they are founded. The principal convenience of this method, is to avoid the extraction of the root $\sqrt{4l}$, which in no other measure, except that of the Rhyndland foot, is a whole number. In the succeeding propositions, however, it will be more eligible to make use of the English feet and inches, to which we commonly refer the estimation of space.

In the preceding propositions the quantity of matter moved has not been mentioned, because the accelerating force, proportional $\parallel$ to the velocity generated in a given time, being as the moving force $\dagger$ directly, and the quantity of matter moved inversely, the quantities of matter are therefore implied in the consideration of accelerating forces. But some properties of motion require, that both the moving force and quantity of matter moved should be separately considered.

$\parallel$ Sect. I.
Prop. VI.
 $\dagger$ Sect. I.
Prop. IX.

VI.

If bodies unequal in quantities of matter, be impelled from rest through equal spaces, by the action of moving forces which are constant, these forces are in a duplicate ratio of the last acquired velocities, and the ratio of the quantities of

$$\text{matter jointly, or } \frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q}.$$

Let the moving forces be M and m , the quantities of matter moved Q and q , and the other notation remaining,

we have $\ast \frac{V^2}{v^2} = \frac{F}{f} \times \frac{s}{s}$, and $\dagger \frac{F}{f} = \frac{M}{m} \times \frac{q}{Q}$, or the accelerating forces are in a direct ratio of the moving forces, and an inverse ratio of the quantities of matter moved;

$\ast$ Sect. III.
Prop. V.
 $\dagger$ Sect. I.
Prop. IX.

moved; wherefore substituting $\frac{M}{m} \times \frac{q}{\mathcal{Q}}$ for its equal $\frac{F}{f}$, in the former equation, we have $\frac{V^2}{v^2} = \frac{M}{m} \times \frac{q}{\mathcal{Q}} \times \frac{S}{s}$; or when $S = s$, that is, when the spaces described are equal $\frac{V^2}{v^2} = \frac{M}{m} \times \frac{q}{\mathcal{Q}}$, and $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{\mathcal{Q}}{q}$, that is, the spaces being in the ratio of equality, the moving forces will be in a ratio compounded of the duplicate ratio of the velocities, and of the quantities of matter moved.

Cor. When the spaces S, s are unequal, then $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{\mathcal{Q}}{q} \times \frac{s}{S}$, that is, the moving forces are as the quantities of matter and squares of the velocities directly, and as the spaces described inversely.

This, and all other propositions being deduced from the laws of motion, by regular and systematic reasoning, ought not only to be strictly consistent among themselves, but with matter of fact when examined by the severest trials; since any single instance which could be produced of a disagreement or inconsistency, would invalidate the whole theory of motion, by weakening the foundations on which it rests: but it has not yet appeared, that there is any proposition whatever, which is geometrically deduced from the axioms or received principles of motion, whether of the most complex or uncompounded nature, if it be reducible to accurate and decisive examinations, but what is found coincident with the theory to any degree of precision, to the observation of which the human senses are competent.

Many experiments, however, have been produced, as tending to disprove the Newtonian measure of the quantities of motion communicated to bodies, and to establish another measure instead of it, viz. the square of the velocity and quantity of matter; and it immediately belongs to the present subject, to examine whether the conclusions which have been drawn from these experiments arise from any inconsistency between the Newtonian measures of force and matter of fact, or whether these conclusions are not ill founded, and should be attributed to a partial examination of the subject: but some considerations concerning the principles of retarded motions should premised.

In

In the experiments which have been made on the force of bodies, the loss of motion from resistance has been more attended to, than the communication of it by acceleration, and the reason probably arose from a want of adequate methods of subjecting accelerating forces, velocities acquired, and quantities of matter moved, to experimental trials; whereas the impact of bodies on substances, which they penetrate, by affording convenient opportunity for observing the depths to which bodies sink before all motion is destroyed, regard being had to the velocities of impact, and the weight and form of the impinging body has seemed a more eligible (however imperfect) way of examining the principles of motion. This method is doubtless allowable, in order to estimate the force of moving bodies experimentally, if it be sufficiently accurate; because it is universally agreed, that (under certain restrictions not affecting the present question) the laws of accelerated and retarded motions are mutually deducible from each other; and consequently, what is found true in regard to the communication of motion to bodies accelerated, will be equally true when applied to the loss of motion in bodies retarded, every thing else being the same, and vice versa.

To exemplify: When a body descends by the force of gravity for three seconds, it acquires by constant acceleration a velocity of $3 \times 32\frac{1}{2} = 96\frac{1}{2}$ feet in a second; also if a body be projected perpendicularly upwards, with a velocity of $96\frac{1}{2}$ feet in a second, the whole velocity will be destroyed in three seconds; for the second law of motion regards the change occasioned in the velocities by retardation, as well as by acceleration, consequently, if gravity acting in the direction of a body's motion, generates a velocity of $32\frac{1}{2}$ feet in a second, the same force acting in a direction opposite to that wherein a body is moving, must destroy the same quantity of velocity in the same time, that is, $32\frac{1}{2}$ in each second. In like manner, all the other properties which have been demonstrated, concerning the motion of accelerated bodies, are shewn to belong to those of retarded ones, the following circumstances being attended to: If in any proposition relating to accelerated motion, the force is constant, it follows, that when this is applied to retarded motion, the force of retardation must be constant: moreover, since in accelerated motions the spaces are estimated from quiescence, so in retarded motions the bodies are supposed to move to quiescence, that is, till all motion is destroyed by constant retardation.

In

In order therefore to apply the properties hitherto demonstrated concerning accelerated motions to those of retardation, the whole spaces described may be denominated S, s ; and V, v , which before denoted the velocities last acquired, may now represent the initial velocities wherewith bodies are projected, T and t will represent the times of motion; moreover, F, f , will represent constant forces of retardation, measured by the velocities destroyed in a given time. If the force f , be that whereby gravity retards a body thrown perpendicularly upwards, it will become a standard with which other forces of retardation may be compared; also M , which in accelerated motions denoted the moving force, now represents the resisting force of the substance, and is proportional to the quantity of motion destroyed in a given time: the retarding force will therefore be as the force of resistance directly, and the quantity of matter in the moving body inversely.

* Sect. III.
Prop. IV.
Cor. 2. &
p. 31.

These articles being premised, in order more fully to illustrate the subject, it is to be considered, that if a body projected with different initial velocities, be retarded by any given constant force, the whole spaces which the body describes are in a duplicate ratio of the initial velocities; this follows, from what has already been demonstrated*, and conversely, since when bodies are impelled by an accelerating force through various spaces, if these spaces are always as the squares of the last acquired velocities, it follows, that the force of acceleration is constant; so when a given body is projected with different velocities, and is retarded by a given force, if the whole spaces described be always in a duplicate ratio of the initial velocities, it is concluded that the force of retardation is constant.

It is from this argument inferred, that the force whereby blocks of wood, banks of earth, &c. resist the penetration of bodies impinging on them, is constant: for it is observed, that the depths to which military projectiles of a given magnitude and weight, striking against a body of this kind enter into its substance, are in a duplicate ratio of the initial velocities, which has been sufficiently proved by Mr. Robins, who first ascertained with certainty the velocities of military projectiles, and applied his method, among other useful purposes, to the discovery of the retardation, which bodies suffer by passing through resisting substances.

The forces of resistance, which are opposed to the motion of bodies impinging on substances which they penetrate being granted constant, the propositions concerning

acceleration already demonstrated, may be applied to explain the motion of bodies, which having been projected with given initial velocities, are interrupted by such obstacles as blocks of wood, banks of earth, or others of a similar kind.

For example: It has been demonstrated, that bodies moving from rest by the acceleration of constant forces, * describe spaces which are as the accelerating forces and squares of the times jointly. By applying this proposition to retarded motions, we shall have the whole spaces or depths, to which bodies impinging on the substances, penetrate, as the forces of retardation and squares of the times wherein the bodies move jointly. * Sect. III.
Prop. IV.

Moreover, it has been demonstrated, that if different quantities of matter be impelled from rest through equal spaces, ~~the~~ the † moving forces will be a ratio compounded of the duplicate ratio of the velocities last acquired, and the ratio of the quantities of matter moved. It is from hence inferred, that in retarded motions also, if different quantities of matter be projected against any of the substances above described, with different initial velocities, and the whole depths to which the bodies penetrate be equal, the forces whereby the substances resist the progress of the impinging bodies, will be in a duplicate ratio of the initial velocities of impact, and the quantities of matter jointly: That is, if M, m be put to represent the forces of resistance, $\mathcal{Q}, q$ the quantities of matter moved, V, v the initial velocities, the depths being † Sect. III.
Prop. VI.

equal, we have $\parallel \frac{M}{m} = \frac{V^2}{v^2} \times \frac{\mathcal{Q}}{q}$, and if the depths be unequal, the equation will be $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{\mathcal{Q}}{q} \times \frac{s}{\delta}$. $\parallel$ Sect. III.
Prop. VI.

By this proposition we may examine some of the experiments concerning the force of moving bodies, and the conclusions deduced from them by Bernoulli, Leibnitz, Poleni, &c. against the measure of force delivered by Sir I. Newton, which he described in the following definitions.

‡ The quantity of motion is measured by the quantity of matter in a moving body and its velocity jointly. ‡ Newt.
Princip.
Def.

The moving forces whereby bodies tend toward centres of attraction, are as the quantities of motion generated in a given time.

It follows then from these definitions, that the moving forces acting for a given time, will be proportional to the quantities of matter moved, and velocities generated jointly

jointly ; so that if the ratio of the moving forces be known, and we can find by experiment what velocities are generated in given bodies by the action of them for the same time, the quantities of motion generated in the bodies may be estimated according to Sir I. Newton's definition.

Moreover, since it is allowed, that the effects of a resisting force to destroy, are the same as those of an equal moving force to generate motion in a given time, it follows, that if the ratio of two resisting forces be known, the quantities of matter in bodies which impinge on substances which they penetrate, and the velocities destroyed in a given time, will give the ratio of the quantities of motion destroyed according to Sir I. Newton's definition.

* Vid. infra notes on experiments. In many of the experiments * alluded to, which have been greatly varied and multiplied, the resisting forces were made equal by causing spheres equal in magnitude to impinge on a given substance which they penetrated, and the spheres being of different densities, it was observed in experiments, that whenever the densities or weights of these equal spheres, were in an inverse duplicate ratio of the initial velocities, the depths to which they penetrated would be equal.

The conclusions were these ; the quantities of matter displaced by the moving bodies were equal, the depths to which the equal spheres penetrated being the same ; moreover the whole motions which had been communicated to the bodies were destroyed : These motions therefore wherewith the bodies impinged on the substances were equal ; but by the experiments, the quantities of matter were in an inverse duplicate ratio of the velocities, and consequently the square of the velocity into the quantity of matter, equal in both cases ; wherefore the quantities of motion destroyed, that is, the whole motion of the impinging bodies must have been as the squares of the velocities into the quantities of matter.

But it plainly appears, that the conclusion is not applicable to the Newtonian definition, according to which the moving force which generates motion in bodies, and it follows by what has preceded the resisting force by which the motion of bodies is destroyed, is proportional to the quantities of motion generated or destroyed in a given time respectively ; and consequently to estimate the quantities of motion destroyed, the time wherein the resisting forces act should be equal. If therefore the times wherein the bodies in the experiment describe the equal spaces, can be proved different, this will plainly shew

shew that the quantities of motion destroyed cannot be inferred from the experiment, the different times of the bodies describing the depths to which they sink not being taken into the account: this will be easily proved, since from * proposition 3d, we have universally the spaces de- * Sect. III.
scribed as the velocities last acquired and times jointly, this proposition when applied to retarded motions will also be true; wherefore the spaces being given as in the experiment, the times will be inversely as the initial velocities, which velocities being unequal from the construction of the experiment, it follows that the times are unequal. This being the case, it is manifest that no conclusion can be drawn from these experiments concerning the quantities of motion destroyed, tending to prove any inconsistency between the Newtonian estimation of force and matter of fact. It is next to be shewn, that the experiments are strictly consistent with that measure, and with the theory in general.

From the proposition above demonstrated, we have in accelerated motions $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{s}{S}$, and consequently $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{m}{M}$; that is, the spaces described are in a duplicate ratio of the velocities last acquired and the quantities of matter moved, and an inverse ratio of the moving forces; this proposition being applied to retarded motions, it will be, the whole spaces or depths to which the impinging bodies sink are in a duplicate ratio of the initial velocities, the ratio of the quantities of matter, and an inverse ratio of the resisting forces. Wherefore, since in the experiment from the equality of the spheres' diameters $M = m$, we have $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{Q}{q}$; but by the construction of the experiment $\frac{V^2}{v^2} = \frac{q}{Q}$, substituting therefore $\frac{q}{Q}$ for $\frac{V^2}{v^2}$ in the former equation, it will become $\frac{S}{s} = \frac{Q}{q} \times \frac{q}{Q} = \frac{1}{1}$; wherefore $S = s$, or the depths to which the bodies penetrate, must be equal when spheres equal in diameter are projected against a given substance, the weights being in an inverse duplicate ratio of the initial velocities, which we find entirely correspondent with experiment.

It seems rational to suppose, independent of all theory, that in estimating the quantities of motion generated or

destroyed by given moving or resisting forces, regard must be had to the times wherein such forces act; because moving forces or those of resistance may be equal, and may generate or destroy quantities of motion varying in any assigned degree. For it is manifest, that a small resistance opposed to a moving body for a longer time, may destroy more motion than a greater force acting for a less time, which sufficiently shews that the times wherein the moving and resisting forces act must either be equal, or must be taken into the account in estimating the quantities of motion generated or destroyed.

The next proposition will demonstrate, that when the times wherein motion is generated or destroyed, are equal, the moving and resisting forces, and their effects the quantities of motion generated or destroyed in a given time, will be as the quantities of matter moved, and the velocities acquired or destroyed jointly.

VII.

The moving forces which communicate, and the forces of resistance which destroy the motion of bodies in the same time, will be in a compound ratio of the quantities of matter in the moving bodies, and velocities generated or destroyed, or

$$\frac{M}{m} = \frac{V}{v} \times \frac{Q}{q}.$$

* Sect. III. For $\frac{F}{f} = \frac{V}{v} \times \frac{t}{T}$, that is, the accelerating or retarding forces, are as the velocities generated or destroyed directly, and the times wherein the bodies move inversely;

† Sect. I. Prop. IX. moreover, $\frac{F}{f} = \frac{M}{m} \times \frac{q}{Q}$, that is, the accelerating or retarding forces, are as the moving or resisting forces directly, and the quantities of matter inversely, wherefore

we have $\frac{V}{v} \times \frac{t}{T} = \frac{M}{m} \times \frac{q}{Q}$, or $\frac{M}{m} = \frac{V}{v} \times \frac{Q}{q} \times \frac{t}{T}$, and
when

when the times are the same, or $T = t$, it will be $\frac{M}{m} =$

$\frac{V}{v} \times \frac{\mathcal{Q}}{q}$, that is, the moving or resisting forces will be as the quantities of matter and velocities generated or destroyed in a given time: and the quantities of motion generated or destroyed will be in the same proportion.

This and the preceding propositions, when referred to experiments hereafter described, will appear strictly coincident with them as they now do with each other, and with the theory in general.

The following truths have been derived from repeated experiments, and being deducible from the preceding theory, may be here inserted as corollaries.

When musket balls equal in weight and magnitude, impinge on a block of wood with different velocities, the resisting force being constant, we shall have the whole spaces through which the balls move in the wood until their motion is destroyed, as the squares of the velocities; * for * Sect. III. Prop. VI.

$\frac{S}{s} \times \frac{M}{m} = \frac{V^2}{v^2} \times \frac{\mathcal{Q}}{q}$, when $M = m$ and $\mathcal{Q} = q$, that is, the substances on which equal bodies impinge being given $\frac{S}{s} = \frac{V^2}{v^2}$.

Also, if balls of equal diameters but of different weights, impinge against a block with the same velocity, we have the depths to which they penetrate the block as the weights, for since $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{\mathcal{Q}}{q} \times \frac{s}{S}$, when $\frac{M}{m} = \frac{V^2}{v^2} = 1$, it follows that $\frac{S}{s} = \frac{\mathcal{Q}}{q}$.

If balls of the same kind of substance, that is, of the same density but of different diameters, impinge against a given block of wood, or the same bank of earth with equal velocities, the depths to which they penetrate will be directly as the diameters of the balls: for $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{\mathcal{Q}}{q} \times \frac{s}{S}$, and when $V = v$, that is, when the velocities are the

same $\frac{S}{s} = \frac{\mathcal{Q}}{q} \times \frac{m}{M}$: or the spaces are in a direct ratio of the quantities of matter moved, and an inverse ratio of the

the resisting forces, but $\frac{M}{m} = \frac{D^2}{d^2} \times \frac{R}{r}$, $\frac{R}{r}$ denoting the resistances arising from the density and cohesion of the parts: moreover, $\frac{Q}{q} = \frac{D^3}{d^3}$ the specific gravities being the same: by substituting therefore $\frac{D^2}{d^2} \times \frac{R}{r}$ for $\frac{M}{m}$, and $\frac{D^3}{d^3}$ for $\frac{Q}{q}$ in the equation $\frac{S}{s} = \frac{Q}{q} \times \frac{m}{M}$, we have $\frac{S}{s} = \frac{D^3}{d^3} \times \frac{d^2}{D^2} \times \frac{r}{R}$, that is, since in the same substance $R = r$, $\frac{S}{s} = \frac{D}{d}$, or the ratio of the spaces or depths to which the balls penetrate is the same as that of the diameters.

* In tra
Sect. V.

When the force of resistance is not uniform, the same principle obtains in degree, * although according to various laws; for we observe that greater bodies always suffer less retardation than smaller ones of the same density, moving through the same resisting medium, and projected with a given initial velocity; because though the force of resistance increases with the increase of the body's magnitude, yet the weight increases in a higher proportion, the specific gravity being the same, and therefore the retarding force which is measured by the force of resistance directly, and the quantity of matter inversely, is diminished, as the magnitude of the moving body increases.

It having been shewn, that the retarding and resisting forces, whereby the substances above described oppose the passage of bodies impinging on them is uniform, depending no ways on the velocities of motion; in order to apply these principles, the exact quantity of the retarding force existing in a given substance; or the proportion of it to some standard force, such as that of gravity, should be ascertained.

VIII.

If bodies projected with the same velocity, be retarded by different constant forces, these forces will be in an inverse ratio of the whole spaces described by the
the

the projected bodies until all motion is

destroyed, or $\frac{F}{f} = \frac{s}{S}$.

For * $\frac{V^2}{v^2} = \frac{S}{s} \times \frac{F}{f}$, and when $V = v$, as by the pro- * Sect. III.
Prop. V.

position, it follows that $\frac{F}{f} = \frac{s}{S}$, that is, the forces of retardation are in an inverse ratio of the spaces described.

For example: Let a body be projected on an inclined plane, in a direction contrary to that in which gravity accelerates bodies down the plane, and with a velocity of 144.467 inches in a second; suppose the body thus projected ascending along the plane to describe 216 inches before its motion is destroyed, let it be required to ascertain the retarding force which opposes its ascent, that is, the proportion of it to the force of gravity. If the body were projected perpendicularly upward, with the given velocity of 144.467 inches in a second, it would rise to the altitude of

$\frac{144.467^2}{4 \times 193} \dagger = 27$ inches; and since it ascends along the plane $\dagger$ Sect. III.
Prop. V.
Cor. 3.

216 inches, the retarding force on the plane will be to that of gravity, as 27 : 216, or 1 : 8, which is the proportion of the plane's height to the length also.

From this proposition, having given the depth to which a body impinging against a substance with a given velocity penetrates the proportion of the retarding force to that of gravity may be determined.

For example: Mr. Robins found that a leaden ball of $\frac{3}{4}$ of an inch, or $\frac{1}{16}$ of a foot in diameter, impinging perpendicularly on a block of elm with a velocity of 1700 feet in a second, penetrated into its substance to a depth of five inches, that is $\frac{5}{12}$ of a foot; wherefore since a body projected perpendicularly upwards, with a velocity of 1700 feet in a second, would rise to an altitude of

$\frac{1700^2}{4 \times 16 \frac{1}{12}} = \dagger 44922$ feet, we have the force whereby elm $\dagger$ Sect. III.
Prop. V.
Cor. 3.
retards the ball to the force of gravity, as 44922 : $\frac{5}{12}$, or as 107813 to 1.

Moreover, || since $S = \frac{T \times V}{2}$, and therefore $T = \frac{2S}{V}$, || Sect. III.
Prop. III.
it

it follows, in the present case, since $S = \frac{5}{12}$, and $V = 1700$, that the time of the balls describing this space of five inches $= \frac{2 \times 5}{12 \times 1700} = \frac{1}{2040}$ part of a second.

In order to render this theory more general, it must be observed, that the resistances opposed to spherical bodies which impinge on a block of wood, bank of earth, &c. depend not only on the tenacity or density of the parts, whereof the substances penetrated are composed, but upon the diameters of the impinging spheres; so that although the resisting and retarding forces be determined in any substance for a single case, yet when the diameters and weights of the impinging bodies vary, the forces of resistance and retardation opposed to the impact on the same substance will be different: by the preceding proposition, however, we shall be enabled from a single experiment, made on the retardation opposed by any given substance to a sphere, whose weight and diameter is known, to infer the retardation in any other case, however the weights and diameters may vary. Suppose a given ball, impinging perpendicularly with the same velocity on any two substances, sinks into one to the depth r , and into the

† Sect. III. other substance to the depth R , then will $\frac{R}{r}$ † represent the
Prop. VIII.

ratio of the resisting forces whereby these given substances oppose the progress of the impinging body; these resistances being independent of the magnitudes of the impinging bodies, and varying only with the density and cohesion of the parts whereof the substances consist. $\frac{R}{r}$ may

therefore be properly defined, the ratio of the absolute resisting forces of the two substances.

Then if the diameters of the bodies impinging against the two substances be different, the whole resistances being proportional to the quantities of motion destroyed in a given time, will be as the resistances before determined $R : r$, and the squares of the diameters of the impinging spheres; that is, if D, d be the diameters of the spheres,

the ratio of the whole resistances $\frac{M}{m} = \frac{R}{r} \times \frac{D^2}{d^2}$,

by help of which proposition we may deduce a general theorem which will express the relation of forces, which retard the progress of spheres impinging on any substances
how-

however the weights, diameters and specific gravities may vary, provided the ratio $\frac{R}{r}$ be previously determined by experiment.

IX.

If spheres of different diameters and different specific gravities, impinge perpendicularly on fixed obstacles, the resisting forces of which are constant but of different quantities, the forces which retard the progress of the impinging spheres will be in a direct ratio of the absolute forces of resistance, and the joint inverse ratio of the diameters and specific gravities of the spheres.

For since in general* $\frac{M}{m} = \frac{F}{f} \times \frac{Q}{q}$, and by the pre-
ceding note $\frac{M}{m} = \frac{R}{r} \times \frac{D^2}{d^2}$, it follows that $\frac{F}{f} \times \frac{Q}{q} =$
 $\frac{R}{r} \times \frac{D^2}{d^2}$, and $\frac{F}{f} = \frac{R}{r} \times \frac{D^2}{d^2} \times \frac{q}{Q}$; but if N, n repre-
sent the specific gravities of the spheres, we have $\frac{q}{Q} =$
 $\frac{d^3}{D^3} \times \frac{n}{N}$, and substituting $\frac{d^3}{D^3} \times \frac{n}{N}$, for its equal $\frac{q}{Q}$,
in the equation, $\frac{F}{f} = \frac{R}{r} \times \frac{D^2}{d^2} \times \frac{q}{Q}$, it will become $\frac{F}{f}$
 $= \frac{R}{r} \times \frac{d}{D} \times \frac{n}{N}$; that is, the forces of retardation are in a
direct ratio of the absolute resisting forces, and an inverse
joint ratio of the spheres' diameters and specific gravities.
As this proposition only expresses the equality of ratios,
no absolute conclusion can be inferred from it relating to
matter of fact, unless an experiment be first made on the

* Sect. I.
Prop. IX.

‡ P. 39.
† Sect. II.
Prop. VII.

retarding and resisting force of some substance, considered as a standard, to which the forces of retardation and resistance in other substances may be referred; as no standard is in this case pointed out by nature, any one may be assumed; suppose it were the experiment of Mr. Robins before quoted, wherein the resisting substance is elm, the impinging body a sphere of lead, the specific gravity of which = 11,35, and diameter $\frac{3}{4}$ of an inch, or $\frac{1}{16}$ of a foot, and the force of retardation ‡ = 107813; we have therefore $f = 107813$, $d = \frac{1}{16}$, $n = 11,35$, † and $r = 1$; then R being a number which is to 1, as the absolute resisting force of the substance given is to that of elm, we have

$$\frac{F}{107813} = \frac{R}{r} \times \frac{n}{N} \times \frac{d}{D} = \frac{R}{1} \times \frac{11,35}{N} \times \frac{\frac{1}{16}}{D}, \text{ or } F = \frac{107813 \times 11,35 \times \frac{1}{16} \times R}{D \times N} = \frac{76478R}{DN}, \text{ the force which}$$

retards a sphere impinging on a substance, the absolute resistance of which, in reference to that of elm, is as $R:1$; the diameter of the sphere being $= D$, and its specific gravity N . It is next to be enquired, to what depth spheres impinging on resisting substances will penetrate, when the spheres' diameters, densities, velocities of impact, and the absolute resistance of the substances whereon they impinge are given.

X.

The whole spaces or depths to which spheres impinging on different resisting substances penetrate, are in a ratio compounded of the duplicate ratio of the velocities of impact, the joint ratios of the diameters and specific gravities of the spheres, and an inverse ratio of the absolute forces whereby the substances resist the

the progress of the spheres, or $\frac{S}{s} = \frac{V^2}{v^2} \times$

$$\frac{D}{d} \times \frac{N}{n} \times \frac{r}{R}.$$

† For in general $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{f}{F}$, and by the last proposition $\frac{f}{F} = \frac{r}{R} \times \frac{N}{n} \times \frac{D}{d}$, substituting therefore these ratios for their equal $\frac{f}{F}$ in the equation $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{f}{F}$, it will become $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{D}{d} \times \frac{N}{n} \times \frac{r}{R}$. † Sect. III. Prop. V.

From this proposition and the experiment of Mr. Robins which has been assumed as a standard, we may derive a general expression for the value of S , or the depth to which an impinging sphere will penetrate a given substance, provided the quantity $\frac{R}{r}$, which expresses the ratio of the absolute resistance of the substance to that of elm be first ascertained from experiment.

For since in general $\frac{S}{s} = \frac{V^2}{v^2} \times \frac{D}{d} \times \frac{N}{n} \times \frac{r}{R}$, we have from Mr. Robins' standard experiment $s = \frac{5}{12}$ part of a foot, $v^2 = 1700^2$, $d = \frac{1}{16}$ part of a foot, $n = 11.35$, and $r = 1$; wherefore $S = \frac{5}{12} \times \frac{V^2}{1700^2} \times \frac{D}{\frac{1}{16}} \times \frac{N}{11.35} \times \frac{1}{R}$, that is, $S = \frac{V^2 D N}{4920223 R}$ feet.

Suppose for example, an iron 24 pounder were fired against a bank of earth with an initial velocity of 1300 feet in a second; and that the absolute resistance of the earth to that of elm, were as 1 to 11. Let it be required to assign to what depth the ball will penetrate in the bank of earth.

Since the specific gravity of iron is = 7.53, and the balls weight = 24 pound avoirdupoise, we have the diameter

of the ball = .46 parts of an English foot, wherefore $D = .46$, $N = 7.53$, and $R = \frac{1}{11}$, then if the velocity of the cannon ball be that of 1300 feet in a second, the depth to which it will penetrate into the earth = $\frac{V^2 \times D \times N}{4920223 R} = \frac{1300^2 \times .46 \times 7.53 \times 11}{4920223} = 13$ feet.

It appears, therefore, that an iron 24 pounder fired with its full charge, whereby it issues from the mouth of the cannon, with a velocity of 1300 feet in a second, will penetrate into a bank of earth, upon which it impinges perpendicularly, to the depth of 13 feet, provided the resistance of earth to that of elm be rightly assigned, and the bank be so near to the cannon that the air's resistance shall not have sensibly decreased the initial velocity.

In estimating the proportion of the spaces described, times of description, and velocities lost by bodies, which having been projected with given initial velocities, are resisted by constant forces, the whole spaces have been hitherto necessarily supposed to be described: if it be required to assign the proportion of the spaces described by retarded bodies in any given times, estimating them from the first instant of projection, other rules will be necessary; these might be easily derived from geometrical construction, but may be investigated analytically in a more general manner.

XI.

Fig. III.

Let a body be projected from A in the direction A D, with the velocity which would be acquired by it in descending through the space B C from rest, by acceleration of gravity. Let the constant force F be opposed to the motion of the body, it is required to assign the time of describing any space A O, and the velocity of the body at the point O.

Let

Let $BC = b$, $AO = x$, V = the velocity in O , and let z be the space through which a body must fall by the acceleration of gravity to acquire the velocity V . Then we have $z = -Fx$, and taking the fluents $z = -Fx$, which consist of variable quantities only; these should vanish together if they constituted the whole fluent, but since when $x = 0$, $z = b$, the constant quantity b must be added to $-Fx$, we have therefore $z = b - Fx$, and the velocity in the point O required

$V = \sqrt{4lz} = \sqrt{4l \times b - Fx}$. Moreover, if the time of describing AO be put equal T , then will $\dot{T}$ denote the time of describing the evanescent space Oo , * therefore $\dot{T} =$

$$\frac{\dot{x}}{\sqrt{4l \times b - Fx}}, \text{ and the time } T = - \frac{\sqrt{b - Fx}}{F \sqrt{l}}$$

being the variable part of the fluent, which requires a constant quantity $\frac{\sqrt{b}}{F \sqrt{l}}$ to be added to it, so that when

$x = 0$, T may likewise equal 0. We have therefore the time sought $T = \frac{\sqrt{b} - \sqrt{b - Fx}}{F \sqrt{l}}$ seconds.

Cor. 1. We obtain also the space described in the time T , for by reducing the equation $x = 2T \times \sqrt{bl} - lFT^2$.

To exemplify this, suppose an elm plank of one inch in thickness being immoveably fixed, were perpendicularly struck by a leaden ball of $\frac{3}{4}$ of an inch diameter, with a velocity of 1700 feet in a second: the ball will pass through the elm, and let it be required to assign what velocity will be destroyed during the ball's passage, or what will be the velocity of the ball immediately on its quitting the elm.

Here referring to the solution, we have $x = \frac{1}{12}$ of a

foot, $l = 16\frac{1}{2}$, $F \dagger = 107813$, and $b \ddagger = \frac{1700^2}{4 \times 16\frac{1}{2}} = 44922$ † Sect. III. Prop. VIII. ‡ Sect. III. Prop. V. Cor. 3.

$$\sqrt{4l \times b - Fx} = \sqrt{4 \times 16\frac{1}{2} \times 44922 - \frac{107813}{12}} =$$

1520 feet in a second. From which it appears, that the velocity lost by the impact is no more than $1700 - 1520 = 180$ feet in a second out of the original velocity of impact 1700.

It appears, moreover, that the time of the body's passing through the plank in the preceding case =

$$\frac{\sqrt{b} - \sqrt{b - Fx}}{\sqrt{l \times F}} = \frac{\sqrt{44922} - \sqrt{44922 - \frac{107813}{12}}}{\sqrt{16\frac{1}{2} \times 107813}} =$$

$$\frac{1}{19328} \text{ part of a second.}$$

This will render it probable without further argument, that a musket ball of certain dimensions, may penetrate through an elm or deal plank set up vertically on its base, without other support than its weight, and notwithstanding the great velocity of the impact that it shall not sensibly move the plank from its vertical position. For even in the preceding case, but a small portion of the ball's motion is communicated to the elm, and if any other substance of less thickness and resisting force be used in the experiment, the motion communicated to the plank set up vertically may be imagined so small, as not to impel the line of direction out of the base. It might easily be computed what would be the exact effect of an impact of this sort, (the necessary data being given) in raising the centre of gravity of the plank, and causing the whole to turn upon one of the edges which terminate the base; but the reason of the phenomenon above described is sufficiently evident from the solution of the preceding problem.

Cor, 2. If the space x , described in the time T , by a body which is retarded by a force F be given, the space s , through which a body must fall by the force F to acquire the velocity of projection will be known. For let y equal the space described by constant acceleration of

† Prop. IV. the force F for T seconds, then will $y = \frac{1}{2} l F T^2$, and $x =$
Cor, 5.

$$2 T \times \sqrt{b l - l F T^2} = 2 T \sqrt{b l - y}, \text{ and } x + y = 2 T \times \sqrt{b l}; \text{ wherefore } \overline{x + y}^2 = 4 T^2 b l = \frac{4 b y}{F} \text{ be-}$$

cause $y = T^2 F l$, wherefore the space through which a body must descend freely from rest by the force of gravity to ac-

$$\text{quire the velocity of projection, that is the space } b = \frac{\overline{x + y}^2}{4 y} \times$$

$\times F$, and the space which must be described * by the * Sect. III.
acceleration of the force F to acquire the same velocity, Prop. V.

that is the space $s = \frac{x + y^2}{4y}$.

If therefore a body be projected on an inclined plane in a direction contrary to that in which gravity accelerates bodies down the plane, and x be the space described in the T first seconds, let y be the space through which a body would descend from rest on the plane during the same time T ; then the body which was observed to describe the space x in T seconds, will in ascending along the plane describe

the whole space $= \frac{x + y^2}{4y}$ before it loses all motion.

For example: Suppose it to have been observed that a body projected along an inclined plane, in a direction contrary to that wherein gravity acts, described 868.5 inches during the two first seconds of its motion, the height of the plane being $\frac{1}{8}$ of its length. We have * the space described on the plane by constant acce- + Sect. III.
Prop. IV.

leration from rest during two seconds, or $y = \frac{193 \times 4}{8}$

$= 96.5$; and since by the question $x = 868.5$, we have

$x + y = 868.5 + 96.5 = 965$, and $\frac{x + y^2}{4y} = \frac{965^2}{4 \times 96.5} =$

2412.5 inches, the whole space along the plane to which the body will ascend, if in the two first seconds it describes 868.5 inches: and the time of describing this whole space $=$

$\sqrt{\frac{2412.5 \times 8}{193}} = 10$ seconds.

By this proposition we may also ascertain the space described in the T last seconds, of the time wherein a body moves from rest by a constant accelerating force F , provided the whole space which the body describes be given, and vice versa, if the space described in the last T seconds, and the force of acceleration be given, the whole space may be ascertained.

Thus suppose it to be observed that a drop of rain had during the last two seconds of its descent to the earth, described 400 feet. Let it be required to ascertain the height from which it fell: referring to the solution we have $x = 400$ the space observed, and the space which would be described by the acceleration of gravity for two seconds

seconds $= 64 \frac{1}{3} = y$ and $x + y = 464.333$, and since $F = 1$ we have the altitude required from which the rain

$$\text{descended} = \frac{x + y^2}{4y} = 837.83 \text{ feet.}$$

It has been already observed, that all motion must be at first produced, and afterward increased or diminished be continual acceleration or retardation. It being not conceivable that any natural body should pass from quiescence into motion, without having possessed all the intermediate degrees of velocity. When any body therefore is struck by another body, its parts give way in a greater or lesser degree to the force of the impact.

Let the striking substance and the body struck be each of them spherical, and let the direction of the stroke be in a line which joins their centres.

Then, when the surfaces of these spheres are just in contact, that is, at the first instant of the impact, the distance of the centres will be half the sum of their diameters; but as the figures of both spheres receive a change by the impact, the centres will thereby become nearer to each other than before, and during the time of their approach, the striking ball by moving with a greater velocity than the ball struck, will continue gradually to accelerate it till the centres are at their least distance; at which instant the centres begin to move with a common velocity, the change of the spherical figures being then the greatest.

This will be true whether the bodies be nonelastic or elastic; in the latter case, indeed, when the centres have become the nearest possible to each other, the instant they are beginning to move with common velocity, the elastic force of the substances by endeavouring to restore the changed figure, causes the centres to recede from each other with a force equal to that which caused the change, provided the bodies be perfectly elastic.

Several considerations arise from hence: The time wherein this acceleration is generated, will depend on the force whereby the textures of the bodies resist the force of the impact; the common velocity communicated, depends only on the weights of the bodies and the velocity of the impact: The forces of resistance, varying with the distances of the centres, depend on the different textures of the substances; in most cases the force must be evidently variable, but if the law of the variation can be ascertained, the time wherein the velocity is communicated,

as well as the approach of the centres, from the first instant of the impact, to the instant at which all acceleration ceases might be investigated.

For example: When the resisting force is constant and of a determined quantity, having given the weights of the bodies and the velocity of the impact, from these data, the common velocity communicated and the time of communicating it, the change of the figure occasioned by the impact, and the space described by the body struck before the two bodies have acquired a common velocity, may be ascertained from the general principles of acceleration.

It is however to be observed, that in the following solution, the change in the figure of the body struck being vastly greater than that of the striking body, the latter is not considered.

XI.

H L G K is a block of wood, or other similar substance, existing in a free space without gravity. Let a body, for example a musket ball, impinge perpendi- Fig. IV. cularly on this block of wood when quiescent in the direction I A, which line passes through the centre of gravity; having given the force wherewith the body resists the progress of the ball, the quantities of matter contained in the ball and the block, and the velocity of the impact, it is required to ascertain the velocity communicated.

Let the velocity of the impact be such as a body acquires in descending by gravity from rest through the space s , and let the quantity of matter in the block be

G

2

2, that in the ball P , the resisting force of the block M ; and suppose, that by the force of the impact the block has been propelled from its first position $GHLK$ to $CEDF$, during which time the ball has penetrated the block to the depth OB , having altogether moved over the space $AO + OB$; then since the block being continually accelerated, and the ball continually retarded during the time wherein the ball sinks into the substance of the block, the velocities of the two bodies will be different during that time: let v be the space through which a body must fall by gravity to acquire the velocity of the ball at B , and w that space which must be fallen through to acquire the velocity of the block at the same instant at which the ball has arrived at B . Moreover, let $AO = x$, $OB = y$, then will the space $x + y$ be described by the ball while the block moves through the space x , and the ball will describe the elements of space $\dot{x} + \dot{y}$ in the same time, wherein the block describes the particle $\dot{x}$, and since the velocity of the ball at $O = \sqrt{v}$, and the cotemporary velocity of the body struck $= \sqrt{w}$, we have $\dot{x} + \dot{y} : \dot{x} :: \sqrt{v} : \sqrt{w}$; because the time of describing the spaces $\dot{x} + \dot{y}$ and $\dot{x}$ is the same; wherefore $\dot{x} + \dot{y} = \sqrt{\frac{v}{w}} \dot{x}$. We have

* Sect. III. therefore from the * principles of acceleration $\dot{x} + \dot{y} \times \frac{M}{P} \dagger$
 Prop. V.
 Cor. 4.
 † Sect. I. $= -\dot{v}$, or since $\dot{x} + \dot{y} = \sqrt{\frac{v}{w}} \dot{x}$, it follows that $\sqrt{\frac{v}{w}} \dot{x}$
 Prop. IX.

$\frac{M}{P} = -\dot{v}$: moreover since M is the resisting force whereby the body struck destroys the ball's motion, $\frac{M}{Q}$ will be the

‡ Sect. I. force by which the block is † accelerated, and it will
 Prop. IX.
 § Sect. III. follow § that $\frac{\dot{x} M}{Q} = \dot{v}$; substituting therefore $\frac{Q \dot{v}}{M}$ for $\dot{x}$ in
 Prop. V.
 Cor. 4.

the former equation, we have $\sqrt{\frac{v}{w}} \times \frac{Q \dot{v}}{M} \times \frac{M}{P} = -\dot{v}$, and

by the reduction $\frac{\dot{v}}{\sqrt{v}} \times Q = -\frac{\dot{v}}{\sqrt{v}} \times P$, and taking

the

the fluents $\sqrt{v} \times Q = -\sqrt{v} \times P$, which should vanish together, but when $\sqrt{v}$, the velocity of the block $= 0$, $\sqrt{v}$ the velocity of the ball $= \sqrt{s}$, the whole fluents therefore will be $\sqrt{v} \times Q = P\sqrt{s} - P\sqrt{v}$, and $P\sqrt{v} + Q\sqrt{v} = P\sqrt{s}$, that is, when both bodies move with a common velocity, or when $\sqrt{v} = \sqrt{v}$, it will be $P + Q \times \sqrt{v} = P\sqrt{s}$, and resolving the equation into an analogy, $P + Q : P :: \sqrt{s} : \sqrt{v}$, or the sum of the bodies is to the striking body, as the velocity of the impact to the velocity communicated.

Cor. 1. Since during the whole acceleration, $P\sqrt{v} + Q\sqrt{v} = P\sqrt{s}$, it follows, that the sum of the moments is at every instant $= P\sqrt{s}$, the sum of the moments before the stroke.

Cor. 2. As the quantity M , or the resisting force of the body struck, enters not into the expression for the common velocity, it follows, that whatever be the resisting force of the bodies, the weights of the bodies and the velocity of impact being the same, the velocity communicated will be given, being always equal to the velocity of impact, into the striking body divided by the sum of the bodies.

XIII.

Every thing remaining as in the last proposition, let it be required to assign the depth OB , to which the ball penetrates into the substance of the block whereon it impinges.

Let the depth required $OB = y$, and $AO = x$, the other notation remaining as in the last proposition, we * have * Sect. III.
 $\frac{x+y}{P} \times \frac{M}{P} = -\dot{v}$, and taking the fluents $x+y \times \frac{M}{P}$ Prop. XII.

$= -v$; and since when $x + y = 0$, $v = s$, it follows, that the whole fluents will be $\overline{x + y} \times \frac{M}{P} = s - v$, or

$x + y = \frac{sP - Pv}{M}$; and since $x = \frac{Qv}{M}$, we have $y = \frac{sP - Pv - Qv}{M}$, that is, when $v = v$ or when the bodies

have obtained a common velocity, $y = \frac{sP - \overline{P + Q} \times v}{M}$,

† Sect. III. Prop. XI, P. 51. † but $\overline{P + Q}^2 : P^2 : s : v$, and $v = \frac{P^2 s}{P + Q^2}$; wherefore

$\overline{P + Q} \times v = \frac{\overline{P + Q} \times P^2 s}{P + Q^2} = \frac{P^2 s}{P + Q}$, and $sP -$

$\overline{P + Q} \times v = sP - \frac{P^2 s}{P + Q} = \frac{QsP}{P + Q}$, and the depth

sought $y = \frac{sP - \overline{P + Q} \times v}{M} = \frac{sPQ}{P + Q \times M}$.

Cor. 1. Let s be the space through which a body must descend by the acceleration of gravity, to acquire the velocity with which a body impinges on a block, the resisting force of which $= M$, and the weight $= Q$, let the weight of the impinging body $= P$, then if Q be infinite, that is, if the block be immoveably fixed, the depth to which

* Sect. III. Prop. VIII. the ball * penetrates $= \frac{sP}{M}$.

Cor. 2. If Q be of finite weight and P impinge on it, the depth to which it penetrates will be to the depth in that case wherein the block is immoveable, as the body struck is to the sum of the bodies, that is, $Q : P + Q$.

Cor. 3. To exemplify this, if a musket ball of $\frac{3}{4}$ of an inch in diameter impinge against a block of elm, with a velocity of 1700 feet in a second, and the block be nine times heavier than the ball, then $P = 1$, $Q = 9$ and

$\frac{Q}{P + Q} = \frac{9}{10}$; moreover since the depth to which the ball

would penetrate if the elm were † immoveable $= 5$ inches, it follows, that in the present case, when it is at liberty to

move in free space, the depth will be $\frac{5 \times Q}{P + Q} = \frac{5 \times 9}{10} =$

4.5 inches.

Cor.

† Sect. III. P. 19. Robins' Gunnery, Vol. I. P. 152.

Cor. 4. Since the common velocity or that which is communicated to the block, is in this case $= \frac{1700 \times 1}{10}$
 $= 170$ feet in a second, it follows, that v = the space through which a body must fall by the acceleration of gravity to acquire the common velocity $= \frac{170^2}{4 \times 16\frac{1}{2}} =$

* 449.22 feet $= v$: moreover, since the retarding force $\frac{M}{P}$ * Sect. III. Prop. V. Cor. 3. † Sect. III. Prop. VIII. p. 39.
 opposed to † the ball $= 107813$, it follows, that the resisting force $M = 107813 \times P$, wherefore because $x = \frac{2v}{M}$ and

$$2 = 9P, \text{ from substitution we have } AO \text{ or } x = \frac{9Pv}{107813P}$$

$$= \frac{9 \times 449.22}{107813} = .0375 \text{ part of a foot or } .45 \text{ parts of an inch.}$$

It appears therefore that the space x or AO through which the elm block will be propelled before all acceleration ceases, that is, before the two bodies have acquired a common velocity, $= .45$ parts of an inch.

Cor. 5. The space or depth to which the ball penetrates into the block $= \frac{PQ}{P + Q \times M}$, and the space de-

scribed by the block in the same time $= \frac{2v}{M}$; this time

will be ascertained from the principles before demonstrated, for in the motion of bodies accelerated from rest, by a constant force, if S be the space described, V the last acquired velocity, T the time wherein this velocity

is generated, we have † always $S = \frac{V \times T}{2}$ and $T = \frac{2S}{V}$; † Sect. III. Prop. III.

wherefore in the present case, the block being accelerated

by the constant force $\frac{M}{Q}$, since the velocity acquired is 170×12 inches in a second, and AO the space described $= .45$ parts of an inch, we have the time of describing this space $= \frac{2 \times .45}{170 \times 12} = \frac{1}{2267}$ part of a second.

S E C T. IV.

CONCERNING THE RECTILINEAR MOTION
OF BODIES ACTED ON BY FORCES
WHICH VARY IN SOME RATIO OF THE
DISTANCES FROM A FIXED POINT.

THE laws observed in the motion of bodies which are uniformly accelerated or retarded, will sufficiently explain all the phenomena which arise from the action of constant forces, or even of variable ones in the very beginning of the motion produced by them. Of this sort is the force of gravity, which must in strictness be accounted a variable force, the law of variation being that of the inverse squares of the distances from the earth's centre. But any spaces through which bodies can actually descend, are so small when compared with the radius of the earth, that the greatest ratio of the squares of those distances from the earth's centre differs little from that of equality. It is for this reason, that the force of gravity,

vity, which accelerates the descent of bodies and retards their ascent, has been regarded according to Galilæo's hypothesis as constant. This assumption, however, would be less allowable than it is, were we not enabled by the laws of variable forces to demonstrate how small a deviation from the truth is occasioned by it.

Gallilæo's discoveries in mechanics related chiefly to the operation of constant forces, but were not sufficient for the estimation of the effects produced by those which are variable; for this purpose, methods are required, unknown till Sir I. Newton applied his invention of fluxions, to determine from the necessary data, the spaces described, times of description, and velocities acquired by bodies acted on by forces varying according to any assigned law.

The properties of variable forces being deducible from those of constant ones, acting in evanescent times, or impelling bodies through evanescent spaces, have been inserted as corollaries to the propositions in the preceding section: from these principles, the effects of variable
forces

forces acting during times which are finite, and impelling bodies through finite spaces, are to be inferred.

The extent of the Newtonian theory of acceleration appears from considering that there are few motions produced either by the operations of nature or of mechanic art, but what are the effects of variable forces. These may either accelerate or retard the motion of bodies, and may observe different laws in regard to the intensity with which they act: for example, these forces may vary according to some power of

The proposition contained in Cor. 4. Sect. iii. which is applied to investigate the effects of variable forces, may be inferred from proposition the vth, otherwise as follows: it appears in the proposition, that

$$\frac{V^2}{v^2} = \frac{F}{f} \times \frac{S}{s}.$$

Let s be the space through which a body must move from rest by the acceleration of the force f , so as to acquire a velocity v equal to V , which is generated by the force F acting on a body moving from rest through the space S : then $V = v$ and $F \times S = f \times s$: here, if f be the force of gravity, and s the space through which a body must descend freely by that force from rest, so as to acquire the velocity v or V , f may be denoted by 1, and F by a proportional number. This will give $FS = s$, and F being constant during an instant, $F\dot{s} = \dot{s}$, the same as in Cor. 4. where $\dot{z}$ represents the same quantity, as is here signified by $\dot{s}$. When S decreases while s increases, their increments will be negative in respect of each other; and the fundamental equation will become $\dot{s} = -F\dot{s}$, or $\dot{z} = -F\dot{x}$.

the distances from a fixed point, or according to some power of the velocities, besides innumerable other laws of variation which obtain in particular cases: The ensuing section contains properties of rectilinear motion, caused by the action of forces, which vary in some ratio of the distances from a fixed point, being such as chiefly relate to natural phenomena.

I.

Let c represent a centre, toward which Fig. V.
bodies are accelerated by forces which are always in a direct proportion of their distances from c . Supposing the force at any given point G to be $= F$, it is required to ascertain the velocity acquired by a body, which has moved from quiescence at A through the space AO .

Let any standard force 1 , the earth's gravity for example, be that to which F is referred, $CA = a$, $CG = r$, $AO = x$, also let V be the velocity required; then since by the problem, the force at O is to that at G as CO to CG , that is, as $a - x : r$, we have the force at $O =$

$$\frac{F \times a - x}{r}.$$

Let z be the space through which a body must move from rest by the acceleration of the standard force 1 , continued constant so as to acquire the velocity at O ; this will give $*z = \frac{F \times ax - x^2}{r}$ and $z = \frac{F \times 2ax - x^2}{r}$, * Sect. III.
Prop. V.
Cor. 4.
 which, since when $x = 0$, $z = 0$, is the entire fluent:
H where-

wherefore the velocity $V = \sqrt{\frac{F \times 2ax - xx}{2r}}$, or $V =$

$\sqrt{\frac{F}{2r}} \times \sqrt{2ax - xx}$: moreover, if T be made to repre-

* Sect. III.
Prop. III.

sent the time of describing AO , we have $\dot{T} = \sqrt{\frac{2r}{F}} \times$

$\frac{\dot{x}}{\sqrt{2ax - xx}}$, and $T = \sqrt{\frac{2r}{F}} \times$ fluent of $\frac{\dot{x}}{\sqrt{2ax - xx}}$,

that is, $T = \sqrt{\frac{2r}{Fa^2}} \times$ into an arc of a circle the versed

sine of which $= x$, and radius $= a$. When $x = a$, that

is, when $AO = AC$, the fluent will become the time of

describing the whole space AC , which time will $= \sqrt{\frac{2r}{Fa^2}}$

$\times$ into the quadrant of a circle the radius of which is $= a$.

Let $p = 3.14159$, &c. the quadrant of the circle will be

$\frac{pa}{2}$, and the time of describing $AC = \sqrt{\frac{2r}{Fa^2}} \times \frac{pa}{2} =$

$\frac{p \times \sqrt{r}}{\sqrt{2F}}$.

If the body begins to descend from any other point B ,

the solution continues the same in every step, and the time

of describing the whole space $BC = \frac{p \times \sqrt{r}}{\sqrt{2F}}$, which,

if the distance r or CG and F the force at G be unalter-

ed, will be equal to that wherein the body described the

space AC .

Cor. 1. If C be a given centre of force, varying as de-

scribed in the problem, and the force at any given dis-

tance CG , continue constantly equal to F , then bodies

describing from rest any spaces BC , GC , AC , DC , will

all arrive at the centre in equal times, the time of the de-

scription being $= \frac{p \times \sqrt{CG}}{\sqrt{2F}}$.

Cor. 2. If the given force F be referred to the earth's

gravity, the latter may be denoted by 1, wherefore F will be

a

a number which is to 1, as the force at G to the force of gravity: then referring the times of descent to 1 second, and the spaces described to feet or inches, let $l = 16\frac{1}{2}$ feet, or 193 inches, and we shall have the time of describing

from rest any spaces BC , GC , or $AC = \frac{p \times \sqrt{CG}}{\sqrt{8Fl}}$. * Sect. III. Prop. V. p. 28.

Cor. 3. Let $ADGCE$ represent a cycloid, and let a body suspended from the line or axis SC vibrate in it, and suppose it were required to assign the time of the pendulum's describing any arc DC , beginning its motion from D , C being the lowest point through which the axis SC passes: let AC or $SC = L$: In order to ascertain the force which acts on the pendulum in any given point G , let the line GE be drawn perpendicular to HC , intersecting the generating circle in K : then to find the force at G , the body's weight being represented in quantity and direction by any constant line, which is perpendicular to the horizon, let the line CH (being $\frac{1}{2}$ the axis) denote the weight of the pendulum, this may be resolved into two CK , HK , whereof CK being parallel to a line touching the cycloid in the point G , accelerates the motion of the pendulum in the curve. The other force HK , acting in a direction perpendicular to the former, tends only to stretch the string and affects not the acceleration. The force, therefore, which accelerates the pendulum will be that part of

its weight which is expressed by the fraction $\frac{CK}{CH}$; the force also which accelerates the pendulum in any other point of its vibration D , will be that part of gravity which is expressed by the fraction $\frac{CO}{CH}$, wherefore the force at G

will be to the force at D , in the proportion of $\frac{CK}{CH}$ to $\frac{CO}{CH}$, or as CK to CO , that is, as CG to CD ; because the cycloidal arcs are double to the corresponding chords of the generating circle. The forces which act at G and D being in the direct proportion of the distances of those points from the point C , the time of describing either GC or DC will be obtained from the preceding proposition, if the length of the pendulum SC be given, and the action of the force at any given distance from C be ascertained.

Suppose CG to be $\frac{1}{2} AC$ or SC , then will CK be $\frac{1}{2} HC$,

H 2

and

and the force at $G = \frac{1}{2}$ of the body's weight or $F = \frac{1}{2}$, and

• Cor. 2. since the time of describing any *space $DC = \frac{p \times \sqrt{GC}}{\sqrt{8Fl}}$,

by substituting $\frac{L}{2}$ for GC , and $\frac{1}{2}$ for F , we have the time

of describing the arc GC or $DC = \frac{p \times \sqrt{L}}{\sqrt{8l}}$, and the

time of one entire vibration $= \frac{p \times \sqrt{L}}{\sqrt{2l}}$.

Cor. 4. If the length of a pendulum, and the force of acceleration at the greatest distance from the lowest point A , continue unaltered and equal to F , the pendulum will perform its vibrations in the longest and shortest arcs in the same

time, namely, in the time $\frac{p \times \sqrt{L}}{\sqrt{2Fl}}$. Pendulums of dif-

ferent lengths will vibrate in times which are in a direct subduplicate ratio of their lengths, and an inverse subduplicate ratio of the forces which accelerate them at any distances from the lowest point, which are in a given proportion to the whole lengths.

Cor. 5. Let F be the force which acts upon the pendulum when at the greatest distance from the lowest

point: Since $T = \frac{p \times \sqrt{L}}{\sqrt{2Fl}}$, T being the time of

† Sect. III. one vibration, and ‡ since $\sqrt{\frac{L}{2Fl}}$ is the time of a body's
Prop. IV.
Cor. 5.

describing the space $\frac{L}{2}$ from rest, by the action of a constant force F , by resolving the equation into an analogy

we have $T : \sqrt{\frac{L}{2Fl}} :: p : 1$, that is, the time of one vibration is to the time of a body's describing from rest half the pendulum's length by the constant acceleration of the force F , as the circumference of a circle is to its diameter.

Cor. 6. If a pendulum L could vibrate in the chords of a circle, the time of one vibration would be equal to that wherein a body descending from rest describes $8L$, that is, the

the time of one vibration $= \sqrt{\frac{8L}{l}}$, F being $= 1$: a pendulum of the same length would vibrate in a cycloidal arc in the time $\frac{p \times \sqrt{L}}{\sqrt{2l}}$, the force F being 1; wherefore we

have the following proportion: the time wherein a pendulum would vibrate in the chords of a circle is to the time of its vibration in the cycloidal arc, as $\sqrt{\frac{8L}{l}}$ to $\frac{p \times \sqrt{L}}{\sqrt{2l}}$, that is, as 4 : p , or as 4 diameters of a circle to the circumference.

Cor. 7. The general value for the time of one vibration being $\frac{p \times \sqrt{L}}{\sqrt{2Fl}}$, if in any given part of the earth, the

force of gravity be denoted by 1 , and the space through which a body descends from rest in one second be equal l , we shall obtain the length of a pendulum which vibrates

seconds in that latitude; for since $T = \frac{p \sqrt{L}}{\sqrt{2Fl}}$, we shall

have, T and F being $= 1$, $L = \frac{2l}{p^2}$. If $l = 193$ English inches in that latitude (for example at London) then $L =$

$\frac{2 \times 193}{3.1459 \times 3.1459} = 39.2$ English inches. If the force of gravity at the equator be to that at London as $F : 1$, then the length of a pendulum which vibrates seconds at the equator must be $F \times 39.2$ English inches.

If therefore it be observed, what is the proportion of the lengths of two pendulums which vibrate seconds at any two latitudes, the proportion of these lengths will give the proportion of the forces of gravity at those places, the effects of the earth's centrifugal force on the times of vibrations not being here considered.

Cor. 8. From the third corollary we may obtain a rule for correcting the pendulum of a clock: for the

time of one vibration $T = \frac{p \sqrt{L}}{\sqrt{2l}}$, and $T^2 = \frac{p^2 L}{2l}$,

of which quantities T and L only are variable: wherefore taking the least cotemporary variations of their hyperbolic loga-

logarithms, we have $\frac{2\dot{T}}{T} = \frac{\dot{L}}{L}$, where $\dot{T}$ represents an error in time caused by the variation in the pendulum's length $\dot{L}$: These variations are in theory less than any assignable quantities of the same sort, but in practice the rule will be sufficiently true when the errors bear a very small proportion to the whole quantities. Suppose for instance, a pendulum vibrating seconds, was observed to lose 10 seconds, in 9 hours or 32400 seconds. Let it be required to assign how much the pendulum must be shortened, so as to keep true time. Here $T = 1$, $\dot{T} = \frac{10}{32400}$, and $L = 39.2$, then from the rule we have the alteration sought

$$\dot{L} = L \times \frac{2\dot{T}}{T} = \frac{39.2 \times 20}{32400} = \frac{1}{41.3} \text{ part of an inch.}$$

† Newt.
Princip.
Prop.
LXXIII.

Cor. 9. † Since a body situated within the earth's surface, is attracted by a force which varies in a direct ratio of the distance from the centre, the time wherein a body would descend freely from the surface to the earth's centre, will be $\frac{1}{2}$ the time in which a pendulum, equal in length to the earth's radius, would perform its least vibrations in a circular arc and vibrations of any length in a cycloid; that is, making r equal to the earth's radius, the time would be $\frac{p}{2} \times \sqrt{\frac{r}{2l}} =$ seconds, or twenty one minutes thirteen seconds.

Cor. 10. The relation of the spaces described, times of description, and velocities acquired, may be geometrically constructed thus.

Fig. VII.

Let AC represent the semicycloidal arc drawn out into a strait line, and the force at A being equal to that of gravity, and C the lowest point of the arc: then suppose the pendulum to begin its vibration from any point D , the time of describing DC will be equal to that of describing the whole arc AC . With the centre C and distance CD , describe the quadrant DB , then supposing DO , DM to be two spaces described during the motion of the body from D to C , the † velocities in the points O and M will be as the fines OF , MN : for

† Newt.
Princip.
Prop.
XXXVIII.

* Vid. supra
p. 58.

the velocity* at $O = \sqrt{\frac{2CD \times DO - DO^2}{2CA}}$, and the velocity at $M = \sqrt{\frac{2CD \times DM - DM^2}{2CA}}$, wherefore the velocity

velocity at O will be to that at M , as $\sqrt{\frac{2CD \times DO - DO^2}{2CA}}$
to $\sqrt{\frac{2CD \times DM - DM^2}{2CA}}$, or multiplying both by
 $\sqrt{2CA}$, as $\sqrt{2CD \times DO - DO^2}$ to $\sqrt{2CD \times DM - DM^2}$,
that is, by the properties of the circle, as OF to MN :
Moreover, the time of describing DO will be to the time
of describing DM , as the arc DF to the arc DN , for the
time of describing $DO = \sqrt{\frac{2CA}{CD^2}} \times$ into the arc of a cir-
cle, the radius of which $= CD$, and versed sine $= DO$, that
is, the time $= \sqrt{\frac{2CA}{CD^2}} \times$ arc DF ; for the same reason,
the time of describing $DM = \sqrt{\frac{2CA}{CD^2}} \times$ arc DN ; where-
fore the proportion of these times is as $\sqrt{\frac{2CA}{CD^2}} \times DF$ to
 $\sqrt{\frac{2CA}{CD^2}} \times DN$, or as DF to DN .

II.

Let c be a centre, toward which bo- Fig. VIII.
dies are attracted with forces which are
inversely proportional to the squares of
their distances from c . Moreover, let
the force, which acts at any given point
 G , be F , referred to a standard force I ,
such as that of the earth's gravity. If a
body begins to descend from a point A , by
the action of the centripetal force, it is
required to ascertain the velocity acquired
at any given point o of its descent.

Let

Let $GC = r$, $AC = a$, $AO = x$, V the velocity at O , then we have the force which accelerates the body at O

$$= F \times \frac{r^2}{a-x^2}, \text{ and if } z \text{ be the space due to the velo-}$$

city at O , from the acceleration of the force 1 ,* then will $z =$

$\frac{F r^2 x}{a-x^2}$, and $z = \frac{F r^2}{a-x}$, which should vanish when x

$= 0$, but when $x = 0$, $\frac{F r^2}{a-x} = \frac{F r^2}{a}$. Wherefore the

entire fluent is $z = \frac{F r^2}{a-x} - \frac{F r^2}{a} = \frac{F r^2 x}{a \times a-x}$, and

the velocity in O , or $V = \sqrt{\frac{F r^2 x}{a \times a-x}}$ in general,

that is, when the standards of space, time, velocity, and force are each assumed equal 1 : if the force F be referred to that which accelerates bodies through a space $= l$,

in the time 1 ,† we have $V = \sqrt{\frac{4 l r^2 F x}{a \times a-x}}$ expressed in the

dimensions of a standard space l , and time 1 .

Fig. IX.

Cor. 1. Let DGB represent the surface, C the centre of the earth, AO any space through which a body can actually descend: then referring to the preceding solution, we have $AC = a$, $GC = r$, $F = 1$, being the force of gravity at the earth's surface: $AO = x$, and the velocity generated

in the point O , will be $\sqrt{\frac{4 l r^2 x}{a \times a-x}}$, but if the force of gravity were uniform at all distances, and $= 1$, the velo-

city generated would ‡ be $\sqrt{4 l x}$, the difference of which

velocities $= \sqrt{4 l x} - \sqrt{\frac{4 l r^2 x}{a \times a-x}}$, $= \sqrt{4 l x} \times$

$1 - \sqrt{\frac{r^2}{a \times a-x}}$. In order to reduce this, let $s = AG$

being the altitude or distance above the earth's surface from which the body begins to fall, then will $a = r + s$ and

$a a - a x = r^2 + 2 r s + s^2 - r x - s x$, that is, since $s^2 - s x$ is incomparably smaller than $2 r s - r x$, we shall have

have $a^2 - ax = r^2 + 2rs - rx$, and $\sqrt{a^2 - ax} = r + \frac{2rs - rx}{2r} = \frac{2r^2 + 2rs - rx}{2r}$; the difference of

the velocities therefore will approximate to $\sqrt{4lx} \times$

$$1 - \frac{2r^2}{2r^2 + 2rs - rx} = \sqrt{4lx} \times \frac{2rs - rx}{2r^2} \text{ or}$$

$$\sqrt{4lx} \times \frac{2s - x}{2r} \text{ as a limit, while } x \text{ and } s, \text{ become conti-}$$

nually smaller: we have therefore the following construction: In the earth's radius CG , take $GI = GA$, then if a body falling from A , describes the space AO , the difference of the velocities, or the error occasioned by assuming Galilæo's Hypothesis of the uniform gravity of the earth at all distances from the centre, will be that part of the whole velocity generated, which is expressed by the frac-

tion $\frac{IO}{2CG}$, when x and s vanish in respect of r .

Cor. 2. If the body descends to the earth's surface, that is, to the point at which the earth's gravity is assumed $= 1$, the error in the velocity above mentioned, will be the whole velocity generated $\times \frac{AG}{2CG}$, that is, the error

cannot be to the whole velocity generated in a greater proportion than that of the altitude fallen from to the earth's diameter.

Since in general the velocity $= \sqrt{\frac{Fr^2x}{a \times a - x}}$, the quantities F, r , and a being given, and x being the variable space described, the velocities in different points of the same descent, will be in a direct subduplicate ratio of the space fallen through, and the inverse subduplicate ratio of the space to be described before the body arrives at the centre.

III.

Let a body descend from rest in a right line toward a centre c , the attractive force of which varies in an inverse dupli-

I cate

Fig. IX.

Fig. VIII.

cate proportion of the distance from c ; also let the force at any given point G be F : it is required to assign the time wherein the body describes any given space AO from rest.

Let the given distance $CG = r$, the force at $G = F$, being referred to any standard force 1: let $AO = x$, $AC = a$, and the force at O will be that part of the force at

G , which is expressed by the fraction $\frac{r^2}{a-x}$, and the

* Sect. IV. velocity* at O will be $\sqrt{\frac{F r^2 x}{a \times a - x}}$. If therefore the time Prop. II.

be denoted by T , we have $\dot{x} = \sqrt{\frac{a}{F r^2}} \times \sqrt{\frac{a-x}{x} \times \dot{x}}$,

and the time of describing AO , that is, $T = \sqrt{\frac{a}{F r^2}} \times$

into the fluent of $\dot{x} \sqrt{\frac{a-x}{x}}$; which may be obtained from the following construction.

Fig. X. Let $CA = a$, upon which describe a semicircle APC , the centre of which is D , and in the line AC assuming any point O , draw the right line OP , then if AO be put

$= x$, the fluent of $\sqrt{\frac{a}{F r^2}} \times \sqrt{\frac{a-x}{x} \times \dot{x}} = \sqrt{\frac{a}{F r^2}}$

× $\overline{AP} + \overline{PO} =$ the time of the body's describing AO from rest, when the standards † of space, time, velocity, and force are each assumed = 1, which may be thus demonstrated, † Sect. III. Prop. V.

Draw the chords AP and CP ; then during the time in which the body describes the element of space Oo , the right line OP will become op , and if Ap , Cp be drawn, since $CPA = CmA$ is a ^{right} angle by the properties of the circle, the cotemporary variation of AP will be mp , and the circular area ACP will be during the same

time increased by the addition of $PCp = \frac{CP \times mp}{2}$, which

which will therefore be the fluxion of the area ACP , and this area will be the fluent whereof mCp is the fluxion.

Let the arc $AP = A$, and the sine $OP = S$, then since $AO = x$, $AC = a$, we have the chord $AP = \sqrt{ax}$, and its fluxion $\frac{\dot{x} \sqrt{a}}{2 \sqrt{x}} = mp$: Moreover, $CP = \sqrt{a^2 - ax}$,

wherefore PCp , or $mCp = \frac{CP \times mp}{2} = \frac{\sqrt{a^2 - ax}}{2}$

$\times \frac{\sqrt{ax}}{2 \sqrt{x}} = \frac{a}{4} \times \sqrt{\frac{a-x}{x}} \dot{x}$, the fluent whereof =

the area CAP = sector DPA + the triangle $CDP = \frac{A \times a}{4}$

+ $\frac{\sqrt{ax} \times \sqrt{a^2 - ax}}{4}$ (because the perpendicular $DQ =$

$\frac{AP}{2} = \frac{\sqrt{ax}}{2}$) wherefore the area $CAP = \frac{A \times a}{4} + \frac{a}{4} \times$

$\sqrt{ax - xx}$, and since $\sqrt{ax - xx} = OP = S$, the

area $CAP = \overline{A + S} \times \frac{a}{4} =$ the fluent of $\frac{a}{4} \times \sqrt{\frac{a-x}{x}}$

$\times \dot{x}$, and the fluent of $\sqrt{\frac{a}{Fr^2}} \times \sqrt{\frac{a-x}{x}} \dot{x} = \sqrt{\frac{a}{Fr^2}}$

$\times \overline{A + S} =$ the time of describing the space AO from

rest, when the standards of space, time, velocity and force are each assumed $\dagger$ equal 1; but if the force be referred to gravity, the time to one second, and the space to l , which a body describes from rest by the acceleration of gravity for 1

second, we shall have *the time required = $\sqrt{\frac{a}{4Fr^2l}} \times \overline{A + S}$, expressed in seconds.

Cor. 1. The time of describing from quiescence any space AO , will be proportional to the $\dagger$ circular area ACP , or to the sum of the sine and arc, whereof the space described is the versed sine, the radius being half the distance of the body when quiescent from the centre of attraction.

Cor. 2. It may be remarked concerning this determination,

$\dagger$ Sect. III.
p. 28.

$\dagger$ Newt.
Princip.
Prop.
XXXII.
Vol. I.

nation, that it is immaterial in what measure the spaces a , r , and l are expressed, viz. whether in feet, inches, &c. of any nation, provided they are expressed in the same measure;

for the fluent $\sqrt{\frac{a}{4Fr^2l}} \times \overline{A+S}$ will always be a number, that is, a fraction, the numerator of which contains $\frac{3}{2}$ dimensions of a line, and the denominator the same dimensions of a line, wherefore whatever measure those spaces be expressed in, the value of the fraction must be unaltered.

Cor. 3. If the body descends from quiescence till it arrives at the centre C , the space $AO = AC$, or $a = x$,

in which case $S = 0$, and $T = \sqrt{\frac{a}{4lr^2F}} \times$ the semicircumference of a circle, the radius of which $= \frac{a}{2}$, this

semicircumference $= \frac{p a}{2}$, p denoting the number 3.14159

&c. that is, $T = \frac{p a^{\frac{3}{2}}}{4r \times \sqrt{Fl}}$ seconds,

Cor. 4. For example: If the earth be considered as an immoveable centre of attraction, and the moon were to descend toward it in a right line by the force of the earth's gravity, the last Cor. will give the time of descent; for here, supposing the earth's radius to be $= 3970$ miles, we

have $l = \frac{16\frac{1}{2} \text{ feet}}{3970 \text{ miles}} = \frac{1}{1303311}$, $r = 1$, $F = 1$, $p = 3.14159$, and $a = 60r$, the time therefore of the descent

would be $= \frac{p a^{\frac{3}{2}}}{4r \times \sqrt{Fl}} = \frac{60^{\frac{3}{2}} \times 3.14159}{4 \times \sqrt{\frac{1}{1303311}}} = 416716$ seconds, or about 4 days and 20 hours.

Cor. 5. Since * the periodic time of any of the celestial bodies which compose the solar system $=$ to $4\sqrt{2} \times$ into the time of its rectilinear descent to the centre from rest, by the action of the centripetal force, it follows, that the periodic time of any primary or secondary planet $P =$

* Newt.
Princip.
Prop. XV.
Vol. I.
and Prop.
XXXII.
Excerpt. c
Newton.
Princip.
p. 87.

$\frac{\sqrt{2 p a^3}}{r \sqrt{Fl}}$, where P denotes the periodic time of the planet, a its distance from the centre of force, F a number which

which is to 1, as the force at any given distance from the centre to the force of the earth's gravity at the same distance from the earth's centre, and $r =$ the distance from the former centre at which the force $= F$. Moreover,

$$P^2 = \frac{2 p^2 a^3}{r^2 F l}, \text{ and } F = \frac{2 p^2 a^3}{r^2 P^2 l}, \text{ by which equation the}$$

absolute quantity of the centripetal force, compared with that of gravity at the same distance, will be known from the observation of the revolving body's periodic time and distance from the centre of its orbit. Thus, if the sun's parallax be $11''$, his distance from the earth will equal 18751 of the

earth's radii; let this $= a$, and $l = \frac{1}{1303311}$, being that

part of the earth's radius which a heavy body descending from rest describes in one second: since P , or the periodic time of the earth $= 365.256 \times 24 \times 3600$

seconds, we have the force of the sun $F = \frac{2 p^2 a^3}{r^2 P^2 l} =$

$$\frac{2 \times 3.14159^2 \times 18751^3}{365.256 \times 24 \times 3600^2 \times \frac{1}{1303311}} = 170313, \text{ a number}$$

expressing the ratio of the sun's absolute force to that of the earth. Moreover, since the sun's diameter would be 87.545 times greater than that of the earth, the sun's parallax being $11''$, we have the ratio of the sun and earth's magnitudes equal to that of 670967 : 1; and the densities being as the quantities of matter or absolute forces directly and magnitudes inversely, the density of the sun will be to

that of the earth as $\frac{170313}{670967}$ to 1, or as 1 to 3.939. The sun's

parallax is assumed $11''$ by Newton,* according to the opinion at his time: later observations have shewn the parallax to be considerably less. Euler in his treatise concerning the Tides, † makes it $10''$; if this were the real parallax it would follow, that the sun's distance from us is equal to 20626 radii of the earth, and by the preceding rule the sun's absolute

* Newt.
Princip.
Vol. III.
Prop. VIII.
† Jesuits'
Newton,
Vol. III.
p. 291.

$$\text{force comes out} = \frac{2 \times 3.14159^2 \times 20626^3}{365.256 \times 24 \times 3600^2 \times \frac{1}{1303311}}$$

$= 22667$. But from the observations which were taken of the transit of Venus, over the sun's disk in the year 1769, in different parts of the earth, it is now concluded that the sun's parallax is $8''.6$. This being the true parallax, the sun's absolute force will appear to be

to

to that of the earth, as 356398 : 1.

Cor. 5. When the descending body describes a space x , which is very small in comparison with the distance

AC , then since the time in general $= \sqrt{\frac{a}{4r^2Fl}} \times \overline{A+S}$, A and S at the very beginning of the descent will become equal to each other, each being $\sqrt{ax - xx}$; wherefore the time in the very beginning of the motion $= \sqrt{ax - xx} \times \sqrt{\frac{a}{r^2Fl}}$, and since the time of describing a space x

* Sect. III. with the constant force $*F = \sqrt{\frac{x}{Fl}}$, we have the difference arising from Galilæo's hypothesis $= \sqrt{\frac{a}{Flr^2}} \times$

$$\sqrt{ax - xx} - \sqrt{\frac{x}{Fl}} = \sqrt{\frac{x}{Fl}} \times \sqrt{\frac{a^2 - ax}{r^2}} - 1,$$

that is, since x is incomparably smaller than a , the difference of the times will approximate to $\sqrt{\frac{x}{Fl}} \times$

$$\frac{a - \frac{x}{2}}{r} - 1, = \sqrt{\frac{x}{Fl}} \times \frac{2a - 2r - x}{2r}. \text{ In order to}$$

Fig. IX.

reduce this let $s = AG$ being the altitude above the earth's surface from which a body begins to descend,

then will $a = r + s$, and $\frac{2a - 2r - x}{2r} = \frac{2s - x}{2r}$, and

the force at G , that is F being $= 1$, the difference in time above described $= \sqrt{\frac{x}{l}} \times \frac{2s - x}{2r}$, which gives the

following construction. In the earth's radius CG , take $GI = GA$: then if a body falling from A describes the space AO , the difference in time or error, occasioned by assuming Galilæo's Hypothesis of the earth's uniform gravity at all distances, is that part of the whole time

which is expressed by the fraction $\frac{IO}{2CG}$.

Cor. 6. If the body descends to the point G , where the

the force of gravity $= 1$, the error in time will approximate to that part of the whole time which is expressed by the fraction $\frac{AG}{2CG}$, that is, the error in time will bear the same proportion to the whole, as that of the height fallen from to the earth's diameter, when the quantity s vanishes in respect of the earth's radius.

IV.

Let a body situated at A, be attracted toward a centre of force c, the quantity of attraction being in the inverse ratio of the distances from c; it is required to ascertain the velocity of a body which has arrived at any point o in its descent from A.

Let $AC = a$, $AO = x$, $CG = r$, the force at the given distance $CG = F$, then the force at O will be denoted by

$F \times \frac{r}{a-x}$, and if $z =$ the space through which a body

must fall by the acceleration of the constant force 1, to acquire the velocity in O, then the principles of acceleration

* give us $\dot{z} = \frac{Fr\dot{x}}{a-x}$, and $z = Fr \times -\log. \frac{a-x}{a}$, * Sect. III, Prop. V, Cor. 4.

which should vanish when $x = 0$; but when $x = 0$, $Fr \times -\log. \frac{a-x}{a}$, will $= Fr \times -\log. a$, the whole fluent,

therefore, will be $z = Fr \times -\log. \frac{a-x}{a} + Fr \times \log. a$, or $z = Fr \times \log. \frac{a}{a-x}$, and the velocity, when

F is referred to gravity $= \sqrt{4lFr \times \log. \frac{a}{a-x}}$.

Let AGH represent a circle described by a body retained in its orbit by a centripetal force C , which force varies in an inverse ratio of the distance from C . It is required to assign how far a body must fall within the circle

circle from rest at A , by the action of the variable force at C , so as to acquire the velocity equal to that with which the body revolves in the circle. Let $AC = a$, AO the distance required $= x$, and let the force at $A = 1$, then will the force at $O = \frac{a}{a-x}$, and z fig-

nifying as before, $\dot{z} = \frac{a \dot{x}}{a-x}$, and $z = a \times \log. \frac{a}{a-x}$; but the velocity in the circle $=$ the velocity acquired by a body which has fallen through $\frac{AC}{2}$, by the action of the con-

stant force 1, and will therefore be $= \sqrt{\frac{a}{2}}$, wherefore

we have by the problem $a \times \log. \frac{a}{a-x} = \frac{a}{2}$, and $\log.$

$\frac{a}{a-x} = \frac{1}{2}$. Let $e = 2.7182818$ being the number

whose hyperbolic $\log. = 1$, therefore $\frac{a}{a-x} = e^{\frac{1}{2}}$, $a =$

$a e^{\frac{1}{2}} - e^{\frac{1}{2}} x$, and $x = \frac{a e^{\frac{1}{2}} - a}{e^{\frac{1}{2}}} = a - \frac{a}{e^{\frac{1}{2}}} = a \times$

.39347.

Fig. XI.

If a body be projected perpendicularly from the circumference with the same velocity with which it revolves in the circle, the height to which it will rise may be also ascertained. For in this case, putting AQ the required space $= x$, and proceeding as before, we have $z = -a \times \log.$

$\frac{a}{a+x}$, which should vanish together, but when $x = 0$, $z = -a \times \log. a$, and by the problem when $x = 0$, $z =$

$\frac{a}{2}$, wherefore the entire fluent will be $z = \frac{a}{2} + a \times \log. a$

$- a \times \log. a + x$, that is, $z = \frac{a}{2} + a \times \log. \frac{a}{a+x}$:

and since by the problem the body is supposed to continue ascending till it loses all velocity, it follows, that

$z = \frac{a}{2} + a \times \log. \frac{a}{a+x} = 0$, we have therefore

$\log. \frac{a}{a+x} = -\frac{1}{2}$; let e be the number whose hyperbo-

lic

lic log. = 1, wherefore $\frac{a}{a+x} = \frac{1}{e^{\frac{x}{a}}}$, and $ae^{\frac{x}{a}} = a+x$,

and $x = a \times e^{\frac{x}{a}} - a = a \times .6487$.

From the same principles it also follows, that whatever be the velocity with which a body be projected directly from a centre of force, varying in the inverse ratio of the distances, it will always return: for let y be the space through which a body must descend from rest by the acceleration of the force 1, so that it may acquire the velocity with which the body is projected from the point A , then we have from the preceding solution $z = -a \times \log. \frac{a}{a+x}$, and the whole fluent $z = y + a \times \log. a - a \times \log. a+x$, or $x = y + a \times \log. \frac{a}{a+x}$; and when the velocity of projection is destroyed or $z = 0$, the equation ~~will be~~ will be $a \times \log. \frac{a}{a+x} = -y$, or $\log. \frac{a+x}{a} = y$, and x being infinite, y will be infinite also.

It appears therefore, that if the projected body returns not, the velocity of projection must have been infinite.

V.

Let a body situated at A , be repelled by Fig. XII.
a force acting from a centre c , always being in an inverse ratio of the distances from c : moreover, let the force at any given point $G = F$, it is required to ascertain the velocity of the body when it has arrived at o .

Let $CG = r$, $AC = a$, $CO = x$, then by the problem the force at $O = F \times \frac{r}{x}$, and if z be the space thro' which a body must fall freely by a standard force 1, for example that of the earth's gravity, to acquire the velocity in O , we
K have

* Sect. III. have $\dot{z} = \frac{Fr\dot{x}}{x}$, and $z = Fr \times \log. x$, when there-
Prop. V.
Cor. 4.

fore $x = a$, $z = Fr \times \log. a$, but by the problem when
 $x = a$, $z = 0$, and the entire fluents will be $z = Fr \times \log.$

† Sect. III. $\frac{x}{a}$, and the velocity† acquired in $O = \sqrt{4Fr l \times \log. \frac{x}{a}}$.
p. 28.

VI.

Every thing remaining as in the last proposition, let it be required to determine the time wherein the body describes any given space AO from rest.

† Sect. IV. Since the velocity† at $O = \sqrt{4rlF \times \log. \frac{x}{a}}$, if T be
Prop. V.

§ Sect. III. the time of describing AO , we have § $\dot{T} = \frac{1}{\sqrt{4rlF}} \times$
Prop. III.

$\frac{\dot{x}}{\sqrt{\log. \frac{x}{a}}}$, and the time required $T = \frac{1}{\sqrt{4rlF}} \times$

fluent of $\frac{\dot{x}}{\sqrt{\log. \frac{x}{a}}}$. In order to obtain the fluent, let

$\log. \frac{x}{a} = v$: if e be the number 2.7182818, the hyperbolic logarithm of which $= 1$, then will $\frac{x}{a} = e^v$, and

$x = ae^v$: also since $\log. \frac{x}{a} = v$, taking the fluxions $\frac{\dot{x}}{x} = \dot{v}$, and $\dot{x} = x \dot{v} = a \dot{v} e^v$; wherefore $\frac{\dot{x}}{\sqrt{\log. \frac{x}{a}}} =$

$\frac{a \dot{v} e^v}{\sqrt{v}}$; but $e^v = 1 + \frac{v}{1} + \frac{v^2}{1.2} + \frac{v^3}{1.2.3} + \frac{v^4}{1.2.3.4} \&c.$ and

$$\frac{a \dot{v} e^v}{\sqrt{v}} = \frac{a v^{\frac{1}{2}} \dot{v}}{1} + \frac{a v^{\frac{3}{2}} \dot{v}}{1.2} + \frac{a v^{\frac{5}{2}} \dot{v}}{1.2.3} + \frac{a v^{\frac{7}{2}} \dot{v}}{1.2.3.4}, \text{ \&c.}$$

wherefore the fluent of $\frac{\dot{x}}{\sqrt{\log. \frac{x}{a}}} =$ the

$$\text{fluent of } \frac{a \dot{v} e^v}{\sqrt{v}} = 2 a v^{\frac{1}{2}} + \frac{2 a v^{\frac{3}{2}}}{3.1} + \frac{2 a v^{\frac{5}{2}}}{5.1.2} + \frac{2 a v^{\frac{7}{2}}}{7.1.2.3},$$

$$\text{\&c. and the time of describing } AO \text{ required} = \sqrt{\frac{1}{4 r l F}} \times$$

$$\text{fluent of } \frac{\dot{x}}{\sqrt{\log. \frac{x}{a}}} = \sqrt{\frac{a^2 v}{r F l}} \times 1 + \frac{v}{3.1} + \frac{v^2}{5.1.2} +$$

$$\frac{v^3}{7.1.2.3}, \text{ \&c.}$$

From the last two propositions we might derive an explanation of the laws according to which motion is communicated to bodies, by the elastic force of the air and other similar substances; but the particular construction of the instruments, by means of which the elastic fluid and vapour possessing similar powers of elasticity are applied to communicate motion, require that these subjects should be separately considered.

The air is a fluid, possessing a property which discriminates it from most other fluids; thus, a quantity of water, mercury, &c. of any given magnitude, can never be reduced in its dimensions so as to occupy a smaller portion of space; whereas the air by compression suffers such diminution of its bulk as is in proportion to the compression, that is, the space occupied by it is always less as the compressive force is greater. Now since action and reaction are equal and in contrary directions, it is plain, that whatever force be necessary to reduce the air into dimensions less than those which it naturally possesses in the atmosphere,* the force with which the air endeavours to restore itself must be equal to that of compression: so that if a quantity of air occupies 1000 times less space than when it is diffused in the atmosphere, it will exert an expansive force 1000 times greater than that with which it reacts against the air surrounding it in a natural state: and this conveys an adequate idea of the air's

elastic force: what the quantity of this force is will be made easily to appear; for the elasticity of the air near the surface of the earth being equal to the force which compresses it, we shall have the quantity of the elastic force exerted on each square inch of surface, equal to the weight of the atmosphere pressing upon the same surface, that is, about 15 pounds avoirdupoise. If therefore 1000 cubic inches of air be included in the magnitude of one inch, this included air will exert a force of elasticity upon each surface of the cube equal to the weight of 15000 pounds avoirdupoise. Let us imagine a perforation to be made in one of the sides of this cube: if a body, for example a leaden ball, be placed close to the aperture, the expansive force of the air would propel it; in this case however the air having room to expand itself in all directions, the whole force of it would not be employed in communicating motion to the ball: but if a cylindrical tube, equal in diameter to the ball, be fitted close to the aperture, and the ball be placed as before, the air now having no other passage or means of the expansion but by the protusion of the ball, will be wholly employed in communicating motion to it.

To find the velocity of this motion communicated in given circumstances, it is to be first considered, that if the force of the air's expansion continued to act uniformly during the whole progress of the ball through the barrel, the velocity communicated would easily follow from the principles of constant acceleration: let the length of the barrel be s , the force of the air's expansion $n = 1000$ times the ball's weight, $l = 16\frac{1}{2}$ feet, v the velocity wherewith the ball would in that case quit the barrel, then we should

• Sect. III. have * $v = \sqrt{4lns}$ feet in a second; but this velocity is
Prop. V. manifestly greater than that which is in reality communi-
Cor. 3. cated; for although the first effort of the air compressed

into $\frac{1}{1000}$ of its natural space, is equivalent to a weight of

15000 pounds upon every square inch of surface pressed on, yet after it has by propelling the ball expanded itself into larger dimensions, the force whereby it accelerates the ball's motion is proportionally diminished; when therefore the cubic inch of compressed air has expanded itself, so as to occupy a space of two cubic inches, the force of expansion will be only one half of what it was before, that is, a force of 7500 upon each square inch of surface, and in proportion for lesser or greater surfaces.

The

The elastic force of the air will therefore continually be diminished as the ball proceeds along the barrel, and of course the increment of velocity generated in a given element of time diminished proportionally: the principles contained in the preceding propositions will be sufficient from having given the quantity of compressed air and the density of it, together with the dimensions of the ball and cylinder to deduce the velocity wherewith the ball quits the cylinder, and the time of motion during its acceleration.

VII.

Let B D represent a cavity which contains compressed air, to this cavity let a cylinder A Q C be fixed, and let a leaden ball be applied to A, opposite to a closed aperture of a diameter equal to that of the ball or cylinder: the aperture being suddenly opened the ball will rush forth from the barrel; it is required from the necessary data to determine the velocity with which the ball issues from c. Fig. XIII.

Suppose the compressed air to occupy $\frac{1}{m}$ parts of its natural bulk in the atmosphere, so that the elastic force of it exerted against any surface, may be m times the weight of the atmosphere pressing against the same surface: moreover, let the cavity contain c cubic inches of this compressed air, and let the diameter of the ball or cylinder = d , its weight = w , the length of the cylinder = $AC = s$, $p = 3.14159$, and then will $\frac{d^2 p}{4}$ be the area of a circle, the plane of which bisects the ball, and cuts the cylinder in a direction perpendicular to the axis: and since the air in its natural state presses on a square

square inch of surface with a force equivalent to the weight of 15 pound or 240 ounces avoirdupoise, it follows, that the elastic force of air, if it be compressed into m times less space than that which it occupies in the atmosphere, acting against the surface $\frac{d^2 p}{4}$ will be $\frac{240 m d^2 p}{4}$ ounces, which will be the absolute or moving force of the compressed air impelling the ball at A , and if the weight of the ball be w , the accelerating force which generates velocity in the ball at A , will be $\frac{240 m d^2 p}{4 w} = \frac{60 m d^2 p}{w}$.

Now, suppose the ball to have been advanced to $\mathcal{Q}$, and let $A\mathcal{Q} = x$, then will the cylindrical space $A\mathcal{Q} = \frac{d^2 p x}{4}$; and since the air before its expansion occupied a space $= c$ cubic inches, when the ball arrives at $\mathcal{Q}$, it will occupy $c + \frac{d^2 p x}{4}$ cubic inches, the elastic force therefore of the air's expansion at $\mathcal{Q}$ will be $\frac{60 m d^2 p}{w} \times \frac{4 c}{4 c + d^2 p x} = \frac{240 m d^2 p c}{4 c w + d^2 p w x}$.

† Sect. III.
Prop. V.
Cor. 4.

Let z be the space through which a body must fall from rest by the acceleration of gravity, to acquire the velocity of the ball at $\mathcal{Q}$, † then we have $\dot{z} = \frac{240 m d^2 p c \dot{x}}{4 c w + d^2 p w x}$, and taking the fluents $z = \frac{240 m c}{w} \times \log. \frac{4 c + d^2 p x}{4 c}$; that is, when $x = s$, or when the ball has arrived at the extremity of the cylinder, $z = \frac{240 m c}{w} \times \log. \frac{4 c + d^2 p s}{4 c}$.

Let A be the capacity of the vessel containing the compressed air; also let B be the capacity of the barrel or cylinder, then will the quantity $\frac{4 c + d^2 p s}{4 c} = \frac{A + B}{A}$, and we shall have $z = \frac{240 m A}{w} \times \log. \frac{A + B}{A}$, and the velocity

velocity*required $V = \sqrt{\frac{4 \times 193 \times 240 A m}{w}} \times \sqrt{\log. \frac{A+B}{A}}$, * Sect. III. p. 28.

that is, $V = \sqrt{\frac{185280 m A}{w}} \times \sqrt{\log. \frac{A+B}{A}}$ inches in a second.

To find the time of the ball's motion through any space $AO = x$, let T be the time required, wherefore $\dot{T} = \dagger$ Prop. III. Prop. III.

$$\sqrt{\frac{w}{185280 m A}} \times \frac{\dot{x}}{\sqrt{\log. \frac{4A + d^2 p x}{4A}}}; \text{ then if } v =$$

$\log. \frac{A+B}{A}$ the fluent of the equation is obtained from the preceding solution, § and we shall have the time of de- § Sect. IV. Prop. VI.

$$\text{scribing the space } AO = \sqrt{\frac{w A v}{2895 d^4 p^2 m}} \times 1 + \frac{v}{3.1} +$$

$$\frac{v^2}{5.1.2} + \frac{v^3}{7.1.2.3}, \text{ \&c. a few terms of which will approxi-}$$

mate to the true time with sufficient exactness, if the capacity of the barrel does not exceed the capacity of the vessel which contains the compressed air, so that the logarithm

of the quantity $\frac{A+B}{A}$ shall not exceed .692; but in the

usual construction of the air gun, the quantity $\frac{A+B}{A}$

rarely exceeds $\frac{11}{10}$, and consequently its logarithm will be

less than .133; in which case three terms of the preceding series will give the time true to the sixth decimal place. The logarithms used in this as well as in the preceding and subsequent solutions are always understood to be hyperbolic, unless the contrary be expressed.

Cor. 1. If the capacity of the barrel and of the vessel containing compressed air remain the same, the velocity of the ball will depend only on its weight and the air's compression, and not in the length of the barrel.

Cor. 2. Every thing else being given, the ball's velocity will be in an inverse subduplicate ratio of its specific gravity: thus, balls of iron and lead discharged from the same air gun by the elastic force of air equally compressed and equal

equal in quantity, will issue forth with velocities which are in the proportion of $\sqrt{11.3}$, $\sqrt{7.3}$, or as 1.2 : 1.

Cor. 3. If the specific gravities of two balls be the same, the velocities wherewith they quit the air gun, will be in a direct subduplicate ratio of the air's compression: thus, if two leaden balls be discharged from the same air gun, and the air being compressed in one case 32 times, and in the other 8 times, the velocities of the ball's egress will be as 2 : 1.

Cor. 4. The theory will be further illustrated by referring to an air gun of the usual dimensions, and from the data belonging to the problem ascertaining the actual velocity communicated to the ball, and the time of its acceleration.

Suppose the cavity containing compressed air to be a sphere of 4 inches in diameter, and consequently equal in capacity to 33.510 cubic inches: suppose the ball to be of lead, and in diameter equal .4 parts of an inch, wherefore the specific gravity of lead being 11.3, the ball's weight = .21913 of an avoirdupoise ounce. Moreover, let the air be compressed into 20 times a smaller space than that which it occupies in the atmosphere, and let the length of the barrel = s , we have therefore for the solution the following data, $A = 33.51$, $m = 20.00$, $d = .40$, $p = 3.14159$, &c. $w = .21913$, $s = 42.00$ inches; wherefore the capacity of the barrel = $\frac{d^2 p s}{4} = 5.2778 = B$: we have also

$\frac{A+B}{A} = \frac{38.7878}{33.51}$, the logarithm of which from the tables = .06352, which being multiplied by 2.30258, will become the hyperbolic logarithm of $\frac{A+B}{A} = .14626$,

and the ball's velocity at $C = \sqrt{\frac{4 \times 193 \times 240 m A}{w}} \times$

$\sqrt{\log. \frac{A+B}{A}} = \sqrt{\frac{4 \times 193 \times 240 \times 20 \times 33.51}{.21913}}$

$\times \sqrt{.14626} = 9103.9$ inches, or 758.6 feet in a second.

In order to obtain the time of the ball's motion through the cylinder, we have $v =$ the log. of

$\frac{A+B}{A} = .14626$, and the other notation remaining

✓

$$\sqrt{\frac{wAv}{2895 d^4 p^2 m}} = \sqrt{\frac{.21913 \times 33.51 \times .14626}{2895 \times .4^4 \times 3.14159^2 \times 20}} =$$

$$.0085683, \text{ and } 1 + \frac{v}{3.1} + \frac{v^2}{5.1.2} = 1.0508, \text{ and the time}$$

$$\text{required} = \sqrt{\frac{wAv}{2895 d^4 p^2 m}} \times 1 + \frac{v}{3.1} + \frac{v^2}{5.1.2}, \text{ \&c.} =$$

$$.0085683 \times 1.0508 = .00900 = \frac{1}{111} \text{ part of a second.}$$

In this problem the air's condensation has been assumed as great as can be used conveniently in physical experiments, but far short of that which exists in some substances both natural and artificial: to mention a single instance only: it is well known that the elastic steam of gunpowder, which remains in a fixed state till the instant of explosion, possesses the properties of air, condensed in about 1000 times a less space than that which it occupies in the atmosphere. Three † parts in ten of the gunpowder's weight consist of this condensed air or vapour, which being set at liberty by the application of fire, will exert a force of pressure equivalent to a thousand times the weight of the atmosphere pressing against the same surface, from which cause arises the great power of this substance in communicating motion to ponderous bodies. This elastic steam like the air expands itself with a force inversely proportional to the space which it occupies, and consequently the preceding solution is applicable to the action of gunpowder as well as to that of air absolutely compressed, and will in both cases be sufficiently accurate, provided the velocity communicated to the ball bears a very small proportion to the velocity with which the air or elastic vapour would expand itself if not impeded: although the impelling force of the elastic steam or compressed air has been assumed equal to the force of its pressure acting against a quiescent ball, yet this is strictly true only at the first instant the ball begins to be moved; afterwards the magnitude of the impulse should be estimated by the difference between the velocity with which the air or steam would expand itself if no ways impeded, and the ball's velocity: this consideration is necessary to render the solution general: if the accelerating force be assumed equal to the air's pressure divided by the ball's weight, as it is in this as well as in † Mr. Robins' solution, † Robins' Gunnery, Vol. I. p. 73.

o

and † Robins' Gunnery, Vol. I. p. 75.

and we diminish the ball's weight sine limite, the accelerating force will be infinite, and consequently the velocity generated greater than any that is assignable; whereas it is impossible that the velocity communicated can in any case exceed that with which the fluid would expand itself if not at all impeded: this circumstance therefore must in some degree affect the truth of the solution, (notwithstanding the great density of the ball,) and should be allowed for accordingly whenever great accuracy is required. In the mean time it may be remarked, that the velocity with which the elastic vapour of gunpowder will expand itself, the obstacles which confine it being removed, is not sufficiently ascertained: Mr. Robins estimates it at about 2000 feet in a second, though some other

† D. Bernouilli.

† authors make it five times greater. It is a subject which scarcely admits of theoretical computation; and great difficulties occur in making experiments relative to it. If the density of the vapour of gunpowder in its compressed state were the same as that of atmospheric air, its elasticity being 1000 times greater, and its progress was not impeded by the gross matter which enters into the composition of gunpowder nor by any other obstacle, the velocity with which the compressed vapour expands itself would be 40900 feet in a second.

From the near agreement between the experiments of Mr. Robins on the actual velocities of musket balls, and the deductions from his theory, it may be concluded that the velocity with which the vapour of gunpowder expands itself, when not impeded, is much greater than that of 2000 feet in a second.

VIII.

Fig. VIII. Let a body A be attracted toward a point or centre c, with forces which vary in that power of the distance from c, which is expressed by the general index

dex n . Also, let the force at a given distance $c g$ be $= F$; it is required to assign the velocity acquired by the body in describing any space $A o$ from rest by the attraction of the variable force.

Let $CG = r$, $AC = a$, $AO = x$, $Oo = \dot{x}$, V the velocity of the body while it is describing Oo : then will the

force acting on the body at $O = \frac{F \times \overline{a-x}^n}{r^n}$, and if z be

the space through which a body must move from rest by the action of a standard force 1,* in order to generate the velocity V , the principles of acceleration give us $\dot{z} = \frac{F \times \overline{a-x}^n}{r^n} \dot{x}$, and taking the fluent, $z = \frac{-F \times \overline{a-x}^{n+1}}{n+1 r^n}$, * Sect. III. Prop. V. Cor. 4.

this being the variable part of the fluent which should

vanish when $x = 0$; but in that case $\frac{-F \times \overline{a-x}^{n+1}}{n+1 r^n}$

$= \frac{-F a^{n+1}}{n+1 r^n}$, the constant quantity therefore $\frac{F a^{n+1}}{n+1 \times r^n}$

being added to $\frac{-F \times \overline{a-x}^{n+1}}{n+1 r^n}$, we have $z =$

$\frac{F \times a^{n+1} - \overline{a-x}^{n+1}}{n+1 r^n} =$ the entire fluent, and the velo-

city required in terms of the general standard 1, will be

$V = \sqrt{\frac{F \times a^{n+1} - \overline{a-x}^{n+1}}{n+1 \times r^n}}$.

Cor. 1. If $n = -1$, the expression for the velocity V fails: in this case the solution is obtained otherwise from Prop. IV.

Cor. 2. When $a = x$, and $n+1$ is positive; the whole space to the centre being described, the velocity $V =$

$\sqrt{\frac{F \times a^{n+1}}{n+1 \times r^n}}$.

Cor. 3. When $n + 1$ is negative, let $n + 1 = -m$, and the general equation for the velocity acquired will become $V =$

$$\sqrt{\frac{F \times a^{-m} - a - x^{-m}}{-m r^{-m-1}}} = \sqrt{\frac{F \times a - x^m - a^m}{-m a^m \times a - x^m \times r^{-1-m}}},$$

and when $x = a$, that is, the whole space to the centre being described, the velocity acquired at C , or

$$\sqrt{\frac{F \times a - x^m - a^m}{-m a^m \times a - x^m \times r^{-1-m}}}$$

will be infinite. If therefore the force varies in any inverse ratio of the distances from the centre equal to or greater than that which is expounded by -1 , the velocity acquired by a body which has described from rest any space AC to the centre will be infinite.

Cor. 4. Let the whole distance $AC = A$, and $OC = P$; moreover, let $n + 1 = m$, then if a body descends from A towards C , the velocity acquired at any point $\dagger O$ will =

$$\sqrt{\frac{A^m - P^m \times F}{m r^{m-1}}}, \text{ and } F \times \frac{1}{m r^{m-1}} \text{ being given, the}$$

* Newt.
Princip.
Vol. I.
Prop. XL.
Cor. 2.

velocity will be proportional to* $\sqrt{A^m - P^m}$.

† Supra p.
83.

Cor. 5. Since $\dagger V = \sqrt{\frac{F \times a^{n+1} - a - x^{n+1}}{n + 1 r^n}}$, if the

time of describing AO be T , we have the time of describing

the space Oo or $\dot{T} = \frac{\sqrt{r^n \times n + 1 \times \dot{x}}}{\sqrt{F \times a^{n+1} - a - x^{n+1}}}$, and the

time $T =$ the fluent of $\frac{\sqrt{r^n \times n + 1 \times \dot{x}}}{\sqrt{F \times a^{n+1} - a - x^{n+1}}}$,

which must be determined for each particular value of n .

IX.

Fig. XIV.

Let a body situated at A , be attracted toward a centre c by a standard force 1; let Ac be bisected in G , and supposing the force

force which tends to c , to vary in any ratio of the distance, which is expounded by the general index n ; it is required to assign through what space AO the body must move from rest at A by the attraction of the variable force, so that the velocity acquired shall be equal to that which would be generated in a body, descending from rest through the space AG by the action of the standard force 1 continued uniform.

Let $AC = a$, and $AG = \frac{a}{2}$; moreover, let $AO = x$, and let z be the space through which a body must move from rest by the acceleration of the constant force 1 , to acquire the velocity in O . Then while the body is describing the element of space $Oo = \dot{x}$, the force acting on it will be $\frac{a - x^n}{a^n}$, and by the principles of acceleration * we have $\dot{z} = \frac{a - x^n}{a^n} \dot{x}$, and taking

the fluents $z = \frac{a - x^{n+1}}{n+1 a^n}$, which is the variable part

of the fluent only, and since when $x = 0$, $\frac{a - x^{n+1}}{n+1 a^n} = \frac{a}{n+1}$, the whole fluent will be $\frac{a}{n+1} - \frac{a - x^{n+1}}{n+1 a^n}$

$=$ to the space through which a body must descend from A by the action of the uniform force 1 , so that the velocity may be equal to that which the body in O acquires by the action of the variable force, while it describes

AO : we have therefore from the problem $\frac{a}{n+1} - \frac{a - x^{n+1}}{n+1 a^n} = \frac{a}{2}$, and $\frac{a - x^{n+1}}{a^n} = a - \frac{na + a}{2} =$

$\frac{a - na}{2}$; wherefore $\overline{a - x}^{n+1} = a^{n+1} \times \frac{1-n}{2}$, and

$$a - x = a \times \sqrt[n+1]{\frac{a-n}{2}}, \text{ and } x = a - a \times \sqrt[n+1]{\frac{1-n}{2}}.$$

† Excerpt. e
Newton.
Princip.
p. 38.

Cor. 1. When a body revolves in a circle, being retained in its orbit by a centripetal force at *C*, the velocity with which the body revolves is equal to that which a body would acquire by falling from rest, † through half the radius, by the action of a constant force equal to the centripetal force acting on the body at the circumference of the circle which it describes. It appears, therefore, from the solution, that when a body revolves in a circle, being retained in the circumference by the action of a centripetal force, the variation of which is expounded by any general power of the distance *n*, the space through which a body must be attracted from the circumference by the variable centripetal force, so that it may acquire the velocity in the circumference

is $AO = a - a \sqrt[n+1]{\frac{1-n}{2}}$, *a* being the radius of the circle.

Cor. 2. When $n = 0$, the space $AO = a - \frac{a}{2} = \frac{a}{2}$.

Cor. 3. If $n = 1$, that is, if the force is as the distance

$\sqrt[n+1]{\frac{1-n}{2}} = 0$, and the space $AO = a - a \sqrt[n+1]{\frac{1-n}{2}} = a$, that is, the body must descend through the whole distance to the centre.

Cor. 4. If $n = -2$, $a \times \sqrt[n+1]{\frac{1-n}{2}} = a \times \sqrt[3]{\frac{3}{2}}$

$= a \times \sqrt[3]{\frac{2}{3}}$; and the space *AO*, through which a body must descend toward the centre to acquire the velocity in the

the circle $= a - a \times \frac{2}{3}$, that is, it must descend through $\frac{1}{3}$ radius.

Cor. 5. When n is $= -1$, $AO = a - a \times \frac{2}{2} \Big)^{\frac{1}{0}}$

which being a * logarithm is not reducible to the present * Sect. IV, Prop. IV. solution.

Cor. 6. Also, if n is positive and greater than 1, the problem is impossible, but with some circumstances which become remarkable; for in this case $1 - n$ must be always negative, and if n is an odd number, $n + 1$ will

be an even number, and the quantity $\frac{1-n}{2} \Big)^{\frac{1}{1+n}}$ will be

impossible, as no root of a negative number can be extracted, when the index is the reciprocal of an even number; but if n be an even number, $n + 1$ will be an odd

number, and the root $\frac{1-n}{2} \Big)^{\frac{1}{n+1}}$ will be obtainable in

rational terms, but the answer will be inconsistent with the conditions of the problem; for in this case the root

$\frac{1-n}{2} \Big)^{\frac{1}{1+n}}$ being necessarily negative, the quantity $-a$

$\times \frac{1-n}{2} \Big)^{\frac{1}{1+n}}$ will be positive, and the space AO inter-

cepted between the circumference and centre will become $a +$ the root before described, a quantity greater than radius, which is impossible. It follows, however, when the force varies in a direct ratio of the distance which is expounded by any index greater than 1, that if a body revolves in a circle by the action of the force, it cannot fall from rest through any part of the radius by the attraction of the variable centripetal force, so as to acquire the velocity with which it revolves in the circle.

X.

Fig. XV.

Let BAD be a circle wherein a body revolves, being retained in its orbit by a centripetal force at c ; let the force at the circumference A be denoted by 1, and supposing the variation of the force at other distances to be in that power of the distances, which is expressed by the general index n : if a body be projected from the circumference at A , and in a direction perpendicular to it with the same velocity wherewith the body revolves; it is required to assign the greatest distance to which it will ascend.

Let $AC = r$, the distance required $AO = x$, and the force at A being $= 1$, the force at any point O will be $\frac{r+x^n}{r^n}$, and if z be a space through which a body must move from rest by the constant acceleration of the force 1,

* Sect. III.
Prop. V.
Cor. 4.

to acquire the velocity in O , we * have $\dot{z} = - \frac{r+x^n}{r^n}$

$\times \dot{x}$, and taking the fluent $z = - \frac{r+x^{n+1}}{r^n \times n+1}$, being

the variable part of the fluent only, and requiring a constant quantity to be added to it.

What this constant quantity is appears from the problem, for when $x = 0$, $z =$ the space due to the velocity of projection $= \frac{r}{2}$; but by the preceding equation, when

$x = 0$, $z = - \frac{r}{n+1}$ the whole fluent therefore will be

$\frac{r}{n+1} + \frac{r}{2} - \frac{r+x}{n+1 \times r^n} = z$: but as the body is supposed by the problem to ascend till its velocity = 0,

it follows, that $z = 0 = \frac{r}{n+1} + \frac{r}{2} - \frac{r+x}{n+1 \times r^n}$

wherefore $\frac{r+x}{r^n} = \frac{n+3 \times r}{2}$, and $r+x^{n+1} =$

$\frac{n+3 \times r^{n+1}}{2}$, $r+x = \frac{n+3}{2} r^{n+1} \times r$, and the distance

AO to which the body will ascend = $x = r \times$

$\frac{n+3}{2} r^{n+1} - r$.

Cor. 1. If $n = 1$, that is, if the force be directly as the distance from C , the whole space AO to which the body will ascend, when projected with a velocity equal to that with which it revolves in the circle, will be $= r \times \sqrt{2} - r = r \times \sqrt{2} - 1$.

Cor. 2. If the force be inversely as the square of the distance, that is, if $n = -2$, then the body projected in the manner above described will rise to an height $= 2r - r = r$, that is, to an height equal to the radius.

Cor. 3. If the force varies in an inverse duplicate ratio of the distance, a body having descended from rest through the space OA equal to the radius, will, when it arrives at the circumference, have acquired a velocity equal to that in the circle, because the same motion will be acquired by acceleration in this case, as was lost by retardation in the last.

Cor. 4. Let y be the space through which a body must move from rest by the acceleration of the force at A or r , so as acquire the velocity of projection: we have from the so-

lution, $\frac{r}{n+1} + y - \frac{r+x}{n+1 \times r^n} = z$, being the space

due to the velocity at O when $AO = x$: if then z be made $= 0$, and x increased sine limite, the ultimate value of y will give the velocity with which a body project-

ed

ed from any point A will never return : we have therefore

$$\text{the equation } \frac{r}{n+1} + y - \frac{\overline{r+x}^{n+1}}{n+1 \times r^n} = 0, \text{ or } y = \frac{\overline{r+x}^{n+1}}{n+1 \times r^n} - \frac{r}{n+1}.$$

It appears from this equation, that if n be any positive number, or any negative proper fraction, the velocity with which a body must be projected so as not to return, is greater than any finite velocity; because $\frac{\overline{r+x}^{n+1}}{n+1 \times r^n}$ in this case is infinite.

Cor. 5. If $n = -1$, this solution fails; but the solution is obtained from Prop. IV.

Cor. 6. If n be any negative number greater than unity, $\frac{\overline{r+x}^{n+1}}{n+1 \times r^n}$ becomes evanescent, and $y = -\frac{r}{n+1}$; thus

if $n = -2$, we have $y = \frac{-r}{n+1} = \frac{-r}{-2+1} = r$, that is,

if the force varies in the inverse duplicate ratio of the distances from the centre, and a body be projected from A in the direction AO , with a velocity which would be acquired by a body descending from rest through a space equal to the radius, by the acceleration of a constant force equal to that which acts on the body at the point A , the body thus projected will never return. Thus, if a body were projected from the earth's surface in a direction perpendicular to the horizon, with a velocity equal to that which a body would acquire in descending from rest through the earth's radius, by the acceleration of a force constantly equal to the earth's gravity at the surface, it would never return; let $r =$ the earth's radius $= 20961600$ feet, $t =$ the space described from rest by bodies which descend for one second by the earth's gravity, this * velocity $= \sqrt{4/r}$ $= \sqrt{4 \times 16\frac{1}{2} \times 20961600} = 36722$ feet, or 6.95 miles, that is, the velocity of projection must be that of near 7 miles in a second.

* Sect. III.
Prop. V.
Cor. 3.

Cor. 7. If $n = -3$, the velocity of projection above described is such as is acquired by a body descending from rest through the space $-\frac{r}{-3+1} = \frac{r}{2}$, or half of the radius

radius by the constant acceleration of the force 1, which is well known from other principles: for if the centripetal force, which retains a body revolving in a circle, varies in an inverse triplicate ratio of the distance, and a body be projected perpendicularly from the circumference with the same velocity as that with which it revolves in the circle, it will never return.

XI.

Let AB represent a line void of gravity, Fig. XVI. one extremity of which is fixed at A ; let this line be suspended over a wheel B , and kept in tension by the weight P . Let w be a small body affixed to c the middle point of the string, and let w be drawn out of its quiescent position through a small space CR , by a force acting in the direction CI , which is perpendicular to AB , as soon as this force ceases to act, the weight w will be accelerated toward c : having given the length of the string AB , the weights P and w ; it is required to determine the time wherein the space RC is described by w .

Let $RC = a$ be the distance from which w begins its motion toward C ; then to find the force whereby the body w is impelled towards C , it is to be observed, that this force is equal to that which acting in a direction CR would exactly counteract the impelling force arising from the tension of P . The three forces which sustain the body w in equilibrio, will be in the direction of the three lines RA , RB and CI , which must be in the proportion of the three sides of a triangle, which are respectively parallel to those lines: the quantity of the force in the direction CI will

therefore be easily determined by drawing BD parallel to AR , and joining AD ; then since two of the forces are BR and RA or BD , the third must be DR , it follows therefore, that the effect of the string's tension to impel the body w in the direction IC , is to the tending force P , as DR to RB , that is, since DR is very small compared with CB , as $2CR$ to CB ; in the same manner, when w has arrived at any point O , the impelling force at O will be to the weight P , as twice the distance OC from the quiescent point C to half the string's length.

Let the string's length $AB = L$, $CR = a$, $RO = x$, then from what has preceded, the impelling force at $O = \frac{4P \times a - x}{L}$, and the force of acceleration at the same

point $= \frac{4P \times a - x}{w \times L}$. Let z be a space through which a body must fall from rest by the action of the earth's gravity to acquire the velocity in O , this will give us $z = \frac{4P}{w} \times \frac{ax - xx}{L}$, and $z = \frac{4P}{w} \times \frac{2ax - xx}{2L}$; and if

* Sect. III. $l = 193$, the velocity at $O = \sqrt{\frac{8Pl}{wL}} \times \sqrt{2ax - xx}$,
p. 28.

let therefore T be put to represent the time wherein the

† Sect. III. body describes PO , we † shall have $\dot{T} = \sqrt{\frac{wL}{8Pl}} \times$
Prop. III.

$\frac{\dot{x}}{\sqrt{2ax - xx}}$, and the time $T = \sqrt{\frac{wL}{8Pl a^2}} \times$ into an arc the versed sine of which $= x$, and radius $= a$, and when $x = a$, or when w arrives at C , the time will become

$\sqrt{\frac{wL}{8Pl a^2}} \times \frac{pa}{2}$, p denoting the number 3.14159: and

time of one entire vibration from R to $D = \sqrt{\frac{wL}{2Pl}} \times \frac{p}{2}$

parts of a second, and the number of vibrations in one se-

cond $= \sqrt{\frac{2Pl}{wL}} \times \frac{2}{p}$.

An elastic string performs its vibrations in a manner not unlike those of the body w , and the time of the string's vibration would be equal to that of w , provided the

the weight of the whole string were collected into the middle point, every thing else being the same; but according to the real manner wherein an elastic string vibrates, it is manifest, that the time of vibration will be less than that assigned in the problem, every thing else being given. For according to the preceding solution, the whole weight moves over a space RD , greater than the space described by any other point of the string in the same time; whereas in the elastic vibration, those parts which are nearer to each extremity by vibrating through less spaces, will describe them in less time than in the other case, the accelerating force being the same.

The difference, however, affecting the vibration of different strings proportionally, the † ratio of the times wherein strings of different weights, tensions, diameters, and lengths vibrate, may be obtained from the preceding solution, as it will appear when these deductions are made from the determination of the real time wherein an elastic string vibrates. But it may be useful to give an example of the time of vibration of a string, deduced from the preceding solution, in which it is supposed, that the whole weight of the string is collected into the middle point, in order that the difference or aberration from the truth arising from this assumption may be more obvious: suppose the weight of an elastic string were 40 grains, the length 30 inches, the tending force 10000 grains: according to the rule we shall have the number of vibrations

† Helsham's
Lectures,
p. 273.

in one second $= \sqrt{\frac{2Pl}{wL}} \times \frac{2}{p} = \sqrt{\frac{2 \times 10000 \times 193}{40 \times 30}}$
 $\times \frac{2}{3.14159} = 36$, and the time of one vibration $= \frac{1}{36}$
 part of a second: the true number of vibrations in a second, deduced from the properties of the harmonic curve
 $= 56.8$ in a second, and the time of one vibration $\frac{1}{56.8}$
 part of a second.

This proposition is here inserted, chiefly because in some solutions, according to the hypothesis above described, the force which impels the body w at any point R , has been made † $= \frac{CR}{CB} \times P$, instead of $\frac{2CR}{CB} \times P$ its true value. || Ibid. p. 271.

If the string be imagined always coincident with two right lines BR , AR , so that each may vibrate backward and

and forward in the manner of a cylindrical pendulum, on axes passing through *A* and *B*: this supposition, it is manifest, will not be extremely different from the truth; but in this case it will give the time too small on account of the elastic string by its curvature extending itself further from the axis toward those parts which are adjacent to the middle vibrating point *C*. The times of vibration of the same string derived from this hypothesis, and from that of the harmonic curve, are in the given ratio of 3.14159 to $2 \times \sqrt{3}$, or as 31 to 34 nearly.

The true time wherein an elastic string vibrates, is derived from the nature of the harmonic curve, a few properties of which being premised, the time of vibration may be deduced from the principles of acceleration, which have been demonstrated: these properties are as follow.

* Smith's
Harmonics,
p. 248.

1. The * force by which any small particle of an elastic string is impelled toward the centre of its curvature, is to the tending force as the length of the particle to the radius of the curvature.

† Ibid.
p. 254.

2. The perpendicular † distance *DH* of a particle from the string in its quiescent position is to the string's length, as the same length is to the radius of curvature multiplied into the number p^2 , p being equal to 3.14159, &c. If therefore any perpendicular distance *RC* be put = a , and the string's length = L , we shall have the radius of curvature at $R = \frac{LL}{ap^2}$.

3. If the whole string be divided into particles *Bb*, *Dd*, &c. each particle will arrive at the axis *ACB* at the same instant of time.

It is also supposed, that the greatest distance from the axis *ACB* to which any particle can vibrate, bears no sensible proportion to the string's length.

XII.

Fig. XVIII.

Having given the length and weight of an elastic string, and the weight which stretches it; let it be required to ascertain how

how many times the string when put into motion, will vibrate in a second, one vibration being the time elapsed between the string's leaving any given point R, or one side of the axis, and the instant of its arrival at the same distance on the other side.

Let L be the length of the string, w its weight, P the weight by which it is stretched; then will the * string during any * Smith's Harmonics, p. 255.
instant of its vibration coincide with the harmonic curve ADC , then since the string's weight $= w$, the weight of the

particle $Dd = \frac{Dd}{L} \times w$, and † because the impelling force † Ibid. p. 248.

which acts on Dd is to the weight P , as Dd to the radius of curvature, that is, as § Dd to $\frac{L^2}{p^2 a}$, we shall have the § Ibid. p. 254. Supra p. 94.

force which impels $Dd = \frac{Dd \times p^2 a}{L^2} \times P$, and the weight

of Dd being $= \frac{Dd}{L} \times w$, the force which || accelerates || Sect. I. Prop. IX.

the particle Dd at the first instant of its motion $= \frac{P p^2 a}{L w}$. Suppose Dd to have described the space DO ,

and let $DO = x$, then will $CO = a - x$, and the force which accelerates the particle Dd when at O , will by the

same method of reasoning be $= \frac{P p^2 a - x}{L w}$: let z be the

space which a body must describe from rest by the acceleration of gravity, so as to acquire the velocity of the par-

ticle Dd in O , this will give $z = \frac{P p^2}{L w} \times \frac{a - x}{x}$, and

$z = \frac{P p^2}{L w} \times \frac{2 a x - x x}{2}$, and the † velocity in $O =$ † Sect. III. p. 28.

$$\sqrt{\frac{2 l P p^2}{L w}} \times \sqrt{2 x a - x x}.$$

If T be put equal to the time of describing † DO , we have † Sect. III. Prop. III.

$$\dot{T} = \sqrt{\frac{L w}{2 l P p^2}} \times \frac{\dot{x}}{\sqrt{2 a x - x x}}, \text{ and taking the fluents}$$

$$T = \sqrt{\frac{L w}{2 l P p^2 a^2}} \times \text{into a circular arc, the versed sine of which} = x, \text{ and radius} = a, \text{ and when } DO = DC,$$

$$\text{that is, when } x = a, \text{ the time} = \sqrt{\frac{L w}{2 l P p^2 a^2}} \times \frac{p a}{2}$$

$$= \sqrt{\frac{L w}{2 l P}} \times \frac{1}{2} \text{ being the time of describing } DC, \text{ and}$$

$$\text{the time of one entire vibration} = \sqrt{\frac{L w}{2 l P}} \text{ parts of a}$$

$$\text{second, and the number of vibrations in one second} =$$

$$\sqrt{\frac{2 l P}{L w}}.$$

Cor. 1. Since the quantity a , or the greatest distance of any part of the string from the axis during one vibration enters not into the expression for the time, it follows, that in whatever ratio the distances from the axis to which the same string vibrates, the times of vibration will be the same, provided those distances be very small.

Cor. 2. Let the number of vibrations in a second $= n =$

$$\sqrt{\frac{2 l P}{L w}}, \text{ then having given the weight of the string} =$$

w , and the tending force $= P$, the length of the string which makes n vibrations in a second will be $L =$

$$\frac{2 l P}{n^2 w}.$$

Cor. 3. If half the length of the string be diminished in the proportion of the tending force to the string's weight, the time of one vibration will be equal to that wherein a body describes from rest this diminished length by the force of gravity.

† Vid. supra Since the velocity † in $O = \sqrt{\frac{2 l P p^2}{L w}} \times \sqrt{2 a x - x x},$

p. 95.

when $x = a$, or when D has arrived at C , the velocity will

$$\text{become} = \sqrt{\frac{2 l P p^2}{L w}} \times a.$$

Cor.

Cor. 4. It follows, therefore, that the greatest velocities generated in a given point of the same string, stretched by a given weight, but impelled to different distances from the axis, are proportional to those distances.

To estimate the velocity in a particular case, let the length of a string = 40 inches, its weight 30 grains, the tending force = 10000 grains, then let any given particle

in the string vibrate $\frac{1}{30}$ of an inch from its quiescent position, and the velocity acquired by the particle when

coincident with the axis = $\sqrt{\frac{2 \times 193 \times 10000}{40 \times 30}} \times$

$$\frac{3.14159}{30} = 5.9392 \text{ inches in a second.}$$

Also the velocity acquired by any particle, when it coincides with the axis, may be estimated by this general rule; multiply the distance of the particle from the axis into the number of vibrations in a second, this product increased in the proportion of the diameter of a circle to its circumference, will give the space which the particle would describe uniformly in a second with the velocity acquired.

Cor. 5. An elastic string, the weight, length, and tension of which are as described in the problem, will vibrate in the same time with a pendulum, the length of which is to the string's * length in a ratio of $w : P \times p^2$, p being = the number 3.14159; for the length of such a

* Smith's Harmonics, p. 259.

pendulum = $\frac{Lw}{p^2 P}$: let this = z , and since the time

in which a pendulum of the length z vibrates in a cy-

cloidal arc is $\dagger p \times \sqrt{\frac{z}{2l}}$, substituting $\frac{Lw}{p^2 P}$ for z , we $\dagger$ Sect. IV, Prop. 1.

have the time wherein the pendulum z , or $\frac{Lw}{p^2 P}$ would

vibrate = $p \times \sqrt{\frac{Lw}{2p^2 Pl}} = \sqrt{\frac{Lw}{2Pl}}$, which is the time of one vibration of the elastic string by the preceding solution.

Cor. 6. Let T represent the time of one vibration, then

$$T = \sqrt{\frac{Lw}{2lP}}, \text{ and } T^2 = \frac{Lw}{2lP}; \text{ let } D = \text{the diameter}$$

N of

of the string, and c a number, which being multiplied into $D^2 L$, will express the weight of the string in the same dimensions with those of P , so that w shall equal $D^2 L c$, then we shall have by substituting $D^2 L c$ for w , $T = \sqrt{\frac{D^2 L^2 c}{2 l P}}$, that is, $T = \frac{D L \sqrt{c}}{\sqrt{2 l P}}$: and since $\frac{\sqrt{c}}{\sqrt{2 l}}$

is a constant quantity, while the specific gravity of the string is constant, it follows, that T will be proportional to $\frac{D L}{\sqrt{P}}$, that is, the time of a single vibration of dif-

ferent strings stretched by different weights will be in a ratio compounded of the joint direct ratio of the diameters and lengths, and an inverse subduplicate ratio of the tending forces.

Cor. 7. The tending forces and diameters being the same, the time of vibration will be as the string's length; but if the same string be stretched by different weights, the tending forces will be in an inverse duplicate ratio of the times wherein the string vibrates.

Cor. 8. Since the times wherein strings vibrate are in a subduplicate ratio of their weights and lengths directly, and an inverse subduplicate ratio of the tending forces, the particular note or tone of a given string stretched with a given weight, may be known a priori; provided a single experiment be previously made, by observing the note which is sounded by a string in given circumstances.

† Smith's — An † experiment of this sort was made by the late
Harmonics, Dr. Smith, master of Trinity College, and is fully de-
p. 202. scribed in his excellent treatise of Harmonics.

Having fixed a harpsichord wire to a small cylinder of wood, he suspended it so as to hang vertically at the side of the organ in Trinity Chapel, and by turning round the cylinder, to which the string was affixed, was enabled to regulate its length, so that the tone should precisely coincide with any proposed note of the organ: the lower extremity of the string was defined by a loop to which a weight of 7 pounds avoirdupoise, or 49000 grains, was affixed, and it did not appear that any other terminations were necessary, either in the upper or lower extremity, the vibration of the string not being extended lower than the beginning of this loop. When the string's tone became exactly coincident with that of the lower D of the organ, it was found by mensuration, that the string's length

length was equal to 35.55 inches, and the string being cut at the two terminations, that is, at the beginning of the loop and at the tangent to the wooden cylinder, the vibrating part of it weighed 31 grains. Applying Dr. Smith's or the preceding solution to these data for the determination of the time in which the string vibrated, we have from the experiment, $L = 35.55$, $P = 49000$, $w = 31$, $l = 193$, and the number of vibrations in one second, was

$$\sqrt{\frac{2 \times P l}{L w}} = \sqrt{\frac{2 \times 49000 \times 193}{31 \times 35.55}} = 131: \text{ that is, the}$$

string in its vibratory motion passed the axis AB 131 times in a second. It follows also, that since the note sounded on the organ caused synchronous vibrations in the air, † and because any number of these vibrations impresses on the ear half that number of impulses in the same time, if by any means the air can be put into a tremulous motion, so that 65.5 impulses succeeding each other at equal intervals of time shall be impressed on the ear in a second, the idea of the same note with that described in the experiment will be excited; for the tone of a note depends not on the loudness or softness of the sound, but upon the number of vibrations excited in the air in a given time. Thus a large bell being struck will create the same tone as to gravity or acuteness, as a flute or other similar instrument.

† Newt.
Princip.
Vol. II.
Prop. XLII.

It was inferred from this experiment, that the middle note in the organ denominated D vibrated $4 \times 131 = 524$ times in a second, and consequently impressed on the ear 262 impulses in the same time.

It may perhaps be useful to insert a few experiments on harmonic strings, whereby the number of vibrations in a second, excited in the air by any given note according to the pitch in present use may be determined.

The experiment just described would have been sufficient for this purpose, if it were certainly known what note or tone, according to the present pitch, is the same with the lower D of the organ at Trinity College at the time when Dr. Smith tried his experiment: for at present the pitch of this instrument is considerably different from that which is in common use.

In order to ascertain this point, an instrument was made use of purposely constructed by Mr. Ramsden, in the year 1768: the advantages which it possesses over monochords of the usual construction are many. The string AB hangs vertically, and its length is terminated at A by an hori-

Fig. XIX.

zontal edge; also the other point of termination which in the common monochords, as well as in musical instruments, is a bridge over which the string is stretched, is in this construction effected by two steel edges *DC* vertically placed; these being fixed on a frame can be easily moved in a vertical direction, so as to alter the length of the string in any desired proportion: these edges are separated occasionally by a spring, in order to let the string freely pass through when its length is altered, and are closed again so as to press the string slightly when its length is properly adjusted. By means of this construction the alteration of the tending force by the application of bridges, &c. is wholly avoided: moreover, that this monochord may be universal, that is, may serve to exhibit a series of harmonic tones according to any given temperament, viz. Smith's, Huygens', Ptolemy's, &c. the whole length, by means of a scale placed immediately under the string, is divided into 100 equal parts, and each of these by a micrometer screw *E* subdivided into 1000 equal parts, so that the length of a given portion of the string may be adjusted on the monochord true to the $\frac{1}{100,000}$ part of the whole length.

Two or three experiments made on this instrument may be selected for the present purpose of determining the number of vibrations excited in the air by a given note.

A brass wire was suspended on the monochord, and being stretched by 102.1 ounces weight troy, was suffered to remain in that position for several days, in order that the string might not be subject to any variation during the experiment: the steel edges before described were adjusted until the tone of the string was coincident with that of a steel key or fork commonly used for tuning harpsichords, &c. which corresponded with the lowest *D* in the bass. The length of the string appeared by reading it off from the scale = 38.975 inches, and this length being cut off accurately at the points of termination, the weight of it was found to be 24.35 grains, the tending force was 102.1 ounces, or 49008 grains, we have therefore to obtain the number of vibrations in a second, $P = 49008$, $L = 38.975$, $l = 193$, $w = 24.35$, and the number of vibra-

$$\text{tions in a second} = \sqrt{\frac{2Pl}{wL}} = 141.18.$$

In another experiment the length of the string was 40.154 inches, the tending force was 48024 grains, the string's weight = 22.7 grains; this string when sounded was

was also coincident with the same note as the former, viz. the lower *D* in the bass, and by the rule we have the num-

ber of vibrations in a second = $\sqrt{\frac{2Pl}{wL}} = 142.6$: the

mean of these, and of a great many others which it is needless to insert, is 141.89 nearly, which we may estimate at 142 without sensible error; it follows then, that the tone *D* in the middle of the scale excite $142 \times 4 = 568$ vibrations in a second.

It may be added, that out of eight promiscuous experiments similar to those just mentioned, in which the tending forces, lengths, and weights of the strings were in various proportions, the number of vibrations in a second, excited by the string, when in unison with the same tone, viz. the lower *D*, never varied so much as 1.5 vibrations from the mean.

This determination coincides with that of Dr. Smith; for the pitch of the organ of Trinity College is by estimation about a hemitone lower than that which is in common use: the exact difference is easily to be inferred from the experiments above described: for according to these experiments, the vibrations in a second were * 131 and 142 respectively, and the difference of these tones will be expressed by a string divided in the proportion of 131 : 142, which ratio is equal to the sum of the ratios of 15 : 16, and 98403 : 100000; † that is, since 15 : 16 is a hemitone, and the ratio of 80 : 81, denominated a comma, is something less than the ratio of 98403 : 100000; it appears, that the pitch of the organ above mentioned is a hemitone, and more than a comma lower than the pitch now in use. * Supra p. 99.
† Smith's Harmonics, Sect. II.

This solution will enable us to estimate the number of vibrations excited in the air by any given note, without difficulty: for example, the lowest *G* in the scale will vibrate

once, while the middle *D* † makes $\frac{2}{1} \times \frac{2}{1} \times \frac{3}{2} = 6$ vibrations; therefore, since *D* by the preceding determination causes 568 vibrations in a second, the lower *G* will excite $\frac{568}{6} = 94\frac{2}{3}$ vibrations in a second. † Ibid. Sect. II.

In the same manner

it may be inferred, that the highest *G* in the scale causes $568 \times \frac{2}{1} \times \frac{2}{1} \times \frac{4}{3} = 3029$ vibrations in a second. If

we add an octave to each extremity of the scale, we shall pro-

probably limit the sense of hearing as to the distinct perception of harmonic sounds: this being granted, it will follow, that if the air impresses on the ear impulses however distinct, yet if they succeed each other with a velocity greater than that of $\frac{3029 \times 2}{2} = 3029$ in a second, or

slower than at the rate of $\frac{94\frac{2}{3}}{2 \times 2} = 23\frac{2}{3}$ in a second, no distinct idea of sound, as to gravity or acuteness, will be conveyed to the mind.

† Tartini's
Principles
of Har-
mony:
Stilling-
fleet's
Translation,
Chap. I. &
Chap. II.
p. 18.

The properties of the elastic string inferred from experiment have induced some authors, not conversant in the mathematical part of harmonics, to imagine certain analogies between the principles of harmony and of geometrical figures, and to suppose the properties of each deducible from the other. The celebrated † Tartini fell into some fancies of this sort, which discover a considerable degree of a scientific enthusiasm; he found by taking the versed sines in a given circle, in arithmetical progression, if the lengths of elastic strings (the diameters being the same) were as the chords of the arcs corresponding to the versed sines above described, and the tending forces were as the versed sines themselves respectively, that all the strings would give the same tone, and consequently vibrate in equal times: in consequence of this and a great variety of similar deductions of harmonic properties from those of the circle, he concludes, that there is a natural affinity between them, and seems to think it consistent that the science of harmony, which he supposes to be in itself perfect, should be immediately related to the circle as the most perfect figure.

This analogy, however, obtains only from the known property of the circle, in which the versed sines of any arcs are in a duplicate ratio of the corresponding chords, the diameter being constant, and in the vibration of elastic strings, if the time of vibration and the string's diameter be given, while the length and tending force vary, the tending forces must be in a duplicate ratio of the lengths; and a similar analogy might have been as easily deduced from the relation between the ordinates and abscissas of a parabola, as from that of the versed sines and chords of a circle.

Mr. Holder likewise, a gentleman of Oxford, in a letter to Sir I. Newton * mentions his having deduced all the harmonic ratios from the 47th prop. of the first book of Euclid's Elements, and infers analogies between the
pro-

* Nægæ
Antiquæ.

properties of harmony not unlike those of Tartini. It is however plain, that these ratios may be as well deduced from other figures, as from a right angled triangle; there being few geometrical figures, but what admit of similar deductions of the harmonic ratios. As to the essential difference between musical intervals, called concords and discords, there have been various opinions. The following account has been given by some authors: it is said that two notes constituting a concord, vibrate in times, the ratio of which is expressed by whole numbers, either equal or prime to each other and the nearest to unity. Thus, two strings in unison vibrate in times which are in the ratio of equality, and constitute the principal of concords. Two strings vibrating in times which are as two to one, constitute the octave, which is the concord next to the unison in degree of perfection, and so on to the interval expressed by vibrations which are in the ratio of 2 : 3, 3 : 4, 4 : 5, &c. It is added, that the mind rests most satisfied with the perception of those ideas of ratios which are the most precise and definite; and although the ratio of $\frac{5}{6}$ may

excite in the mind as definite an idea as that of $\frac{1}{2}$, when abstractedly considered, yet two lines or two vibrations, which are in the proportion of 1 : 2, will convey a more exact idea of that ratio, than two other lines or vibrations which are in the ratio of 6 : 5, will convey of the latter proportion.

XIII.

Let $AIKN \dagger$ represent a cylindrical tube or canal, whereof the axes of the two arms IA, KN are vertical, and that of IK horizontal: suppose that a fluid were to occupy a part of this tube, so that the surfaces of the fluid when quiescent should coincide with EF and CD ; let one of these sur-

Fig. XX.
† Newt.
Princip.
Vol. II.
Prop.
XLIV.

surfaces CD be depressed by a weight or other means, so as to coincide with QH ; being depressed through the space DH , the surface EF will consequently be raised through an equal space EA : then the weight which kept CD depressed at QH being suddenly removed, the surface at AB in the other arm will descend by the acceleration of the superior weight in that arm, and by the velocity generated in the whole mass, when it has arrived at the former or quiescent position EF will still descend, and consequently elevate with a retarded motion the surface QH or DC to an altitude MN , equal to that of AB from whence the descent began: it is required to assign the time elapsed between the instant the surface AB began to descend, and that at which the other surface becomes coincident with MN .

Let the axis of the whole tube or canal $AFIKCM$ be denoted by L , and the force whereby gravity would accelerate the fluid were it unconfined $= 1$; then suppose EF to be elevated to AB , and consequently the surface CD depressed through a space $CQ = AE$, and let $EA = a$; then continuing the horizontal line QH to RS , since the fluid contained in the part of the tube $RIKH$ is in exact equilibrium, the force whereby the upper surface BA endeavours to descend and to communicate motion to the whole mass will be the column AR , the altitude of which $= 2a$, and this column will be to the whole weight of the fluid as

$2a : L$, wherefore the force which accelerates the descent of $\dagger AB = \frac{2EA}{L} = \frac{2a}{L}$; in the same manner when the ^{† Sect. I. Prop. IX.} surface AB has by descending arrived at O , the force of acceleration $= \frac{2EO}{L}$, that is, if AO , the space described from quiescence, be denoted by x , we have the force at $O = \frac{2 \times a - x}{L}$, wherefore if z be the space through which a body must fall by gravity from rest, so as to acquire the velocity in O , it will follow, that $z = \frac{2ax - xx}{L}$, and V the velocity at $O = \sqrt{\frac{4L}{L} \times \sqrt{2ax - xx}}$: and if T be put to represent the time of describing AO , we have $\dot{T} = \sqrt{\frac{L}{4l}} \times \frac{\dot{x}}{\sqrt{2ax - xx}}$, and $T = \sqrt{\frac{L}{4l \times a^2}} \times \text{arc of a circle, the versed sine of which} = x, \text{ and radius} = a$; and when $AO = AE$, that is, when $x = a$, $T = \sqrt{\frac{L}{4la^2}} \times \frac{pa}{2}$, p being $= 3.14159$, that is, the time in which AB arrives at $EF = T = \sqrt{\frac{L}{4l}} \times \frac{p}{2}$: and the time elapsed between the beginning of the descent of AB , and end of the ascent of QH to $MN = p \times \sqrt{\frac{L}{4l}}$.

Cor. 1. The time of one entire descent of the fluid from AB to RS is equal to the time in which a pendulum, the length of which is half the sum of the tube's axis, performs one vibration; for the time wherein a pendulum, the length of which $= \frac{L}{2}$ makes one vibration $=$

$$p \times \sqrt{\frac{L}{4l}}.$$

* Sect. IV.
Prop. I.

Cor. 2. The time elapsed between the instant AB begins to descend, and the instant at which it returns to the O the

the same position $= p \times \sqrt{\frac{L}{l}}$, being the time wherein a pendulum, the length of which is equal in length to $2L$, performs one vibration.

Cor. 3. Since the distance $AE = a$, enters not into the expression for the time, it follows, that whatever be the altitude AE above the quiescent position, from which the surface AB begins to descend, the time of the entire descent to R will be the same, viz. $= p \times \sqrt{\frac{L}{4l}}$.

Cor. 4. When the lengths of the tubes or canals are different, the time of one entire descent will be in a direct subduplicate ratio of their lengths.

Cor. 5. During the descent of AB to its original quiescent position EF , it will be continually accelerated, and the velocity will be the greatest at E ; this greatest velocity will be a direct ratio of the distance AE , and an inverse subduplicate ratio of the lengths of the tube's axes, that is, the velocity will be in a subduplicate ratio of $\frac{EA^2}{L}$.

Fig. XXI.

Cor. 6. The arms IA , KN , are in the proposition vertical, and at right angles to an horizontal tube of communication IK , but if instead of this construction, a bent tube be made use of, no alteration will be occasioned in the solution of the problem, or the conclusions derived from it; provided those parts of the tube, through which the surfaces ascend and descend, be parallel to each other and perpendicular to the horizon. For example, let ACD represent a bent tube, the curvature not extending above MN ; if this tube be so far filled with mercury that the surfaces may rest on a level at AB and CD , when the arms MA , ND are vertical, and motion be communicated to the mercury by inclining the tube a little to one side or the other, in a plane coincident with the axes, and then restoring it to its vertical situation, the mercury will be seen to vibrate backward and forward, performing oscillations through the longest and shortest spaces in the same time. If the length of the whole tube, i. e. the length of the straight and curve parts taken together, be L , then the time of one oscillation, that is, the time wherein one surface descends and the other ascends from quiescence

escence to rest again, will be $= p \times \sqrt{\frac{L}{4l}}$: for example, if the length of the tube be $= 78.4$ inches, the time of one oscillation will be $= 3.14159 \times \sqrt{\frac{78.4}{4 \times 193}} = 1$ second.

The time by experiment is something more than that which is deduced from the theory, as may be expected from the friction of the mercury against the sides of the tube which the theory takes not into account.

The length of the whole tube or canal remaining, let the vertical arms IA , KN be diminished, and consequently the horizontal part IK increased equally, as in fig. XXII. then will the time of one vibration continue as before, and in different canals, the heights of which are very small when compared with their lengths, the times of one vibration will be nearly in a direct subduplicate ratio of the breadths, that is, of the horizontal parts of the canal or the distances between the vertical arms.

If the undulatory motion of fluids, the † agitations of which are but small, be analogous to the alternate vibrations of the fluid in the canal above mentioned, and supposing the altitude of the waves to be very small when compared with their lengths, the time of one undulatory motion, that is, the time elapsed between a particle of the fluid's leaving the summit of one wave and after its depression again arriving at the same altitude, will be in a direct subduplicate proportion of the curvilinear distance comprehended between the summits of the two waves; that is, since the heights are very small when compared with the lengths, the time of an undulation will be in a direct subduplicate ratio of the distances between the summits of the two contiguous waves.

Any number of forces may act in such directions and quantities, as to exactly counterbalance each other; a body urged by these forces will continue at rest; in this case it is immaterial what be the quantity of matter acted on, as the equilibrium above described depends only on the quantity and direction of the impelling forces: when one of these forces preponderates over the rest, motion is produced, the quantity and direction of which is likewise to be estimated by mechanics, but here the matter moved as well as the moving forces must be taken into account. In theory we may imagine weights to be col-

lected into points and still to bear any assigned proportion to each other, we may also imagine, that the materials whereby motion is communicated in mechanical operations are void of gravity, inertia, and the particles of them to move over each other without friction: conclusions produced from these data will be mathematically true in theory, but will not be coincident with matter of fact, unless the weights of the materials whereby motion is communicated, considering these weights both as altering the moving forces and the resistance to motion, and the effects of friction be taken into consideration.

To estimate the whole resistance opposed to the impelling force, the weights of the materials used in the operation must be added to the weight moved, together with a weight equivalent to the resistance arising from friction. The weights of materials whereof instruments and mechanic engines are composed being constant in the same instrument or engine, their effects are easily allowed for; but the resistance which arises from friction, depending on the velocity of motion, as well as various other circumstances, is not easily reduced to geometrical mensuration: yet by mechanic methods it may be almost wholly removed whenever it is thought necessary to refer principles of motion to decisive experimental trials; so that in any case we may with sufficient certainty ascertain the resistance which on all accounts opposes the communication of motion by a given impelling force. The moving force itself is also liable to alteration from various causes; when the alteration of the moving force, arising from the weight of the materials which are used in the communication of motion, is constant, it is easily allowed for by adding this difference to or subtracting it from the moving force originally impressed: in other cases, however, it happens that this alteration of the moving force proceeding from the weights of the materials used in communicating motion is variable, and may therefore cause a variable increase or decrease of the acceleration. This is frequently the case when motion is communicated by means of lines going over wheels, pulleys, &c. these lines in philosophical experiments are extremely thin and flexible, and the weight of them is commonly neglected, which may be done generally without error, as far as regards the quantity of matter moved; but it will appear, that the moving force in experiment may become so small that the weights of these lines shall bear a sensible proportion to it.

But in all cases where any physical quantity, whether

ther it be weight, time, velocity, &c. is neglected as inconsiderable, the limits of the error occasioned by this omission should be defined generally for all magnitudes of the variable quantities which are concerned in the problem. If the error of the result occasioned by omitting any quantity, be found to bear so small a proportion to the whole, that it will be either entirely insensible in experiment, or of too small consequence to merit consideration, it may be still omitted. One instance of this kind may be here inserted, as it relates to the construction of experiments hereafter described.

Let BCD represent a fixed pully, over which a line $FEBCDA$ is suspended, two equal weights P, P being affixed to the extremities; then if a weight w , heavier than the string EF , be added to the highest weight, it will set the whole in motion, and the string's weight will manifestly cause a retardation of this motion till the weights on each side are at equal altitudes HH , and afterwards will increase the acceleration: the next proposition is intended to estimate the effects of the string, both in causing a variation in the velocities generated as well as in the times of motion.

XIV.

Let BCD be a fixed pully, the weight Fig. XXII. of which is collected into the circumference and $= q$; let two weights P and P , be suspended at the extremities of a line going over the pully, and exactly balancing each other. If a weight w be added to either side, the weights $P + w$ will preponderate over the other, and will descend by continual acceleration; the moving force being constantly $= w$, and the weight moved $= 2P + q + w$,
pro-

provided the string's weight be too small to have any sensible effect; but if the string operate sensibly, it is manifest, that when the moving force w is higher than the weight P on the opposite side at A , the weight of the string EF will retard the descent; but when w has descended so as to become lower than the ascending weight P , the string's weight will cause an acceleration of the descent: having the data above mentioned, together with the length and weight of the string, and $EF =$ the altitude of $w + P$ above the weight P on the opposite side, at the very beginning of the motion; it is required to determine the velocity acquired by w during its descent through a given space.

Let q be the weight of the wheel or pulley collected into the circumference, and let the length of the string $= L$, its weight $= p$; also let the difference of the altitudes of the weights $P + w$ and P at the first instant of motion, that is, let $EF = b$, AO the space described from rest $= x$, then we

shall have the moving force at $O = w + \frac{2px - bp}{L} = \frac{Lw + 2px - bp}{L}$, wherefore if $2P + q + p + w$ be put

$= Q$, the * accelerating force at $O = \frac{Lw + 2px - bp}{LQ}$,

and if z be the space through which a heavy body must fall from rest by gravity, to acquire the velocity in O ,

† Sect. IV. we shall † have $z = \frac{Lw\dot{x} + 2px\dot{x} - bp\dot{x}}{LQ}$, and $z =$
Prop. V. L

$\frac{Lwx + pxx - bpx}{LQ}$, and the velocity at O or $V =$

$\sqrt{\frac{4l \times Lwx + pxx - bpx}{LQ}}$ inches or feet in a second, ac-

cording to the denomination in which l , and the quanti-
ties x , b and L are taken.

Cor. 1. Since the square of the velocity $= \frac{4lwx}{Q} +$

$\frac{px^2 - bpx \times 4l}{LQ}$, $\frac{4lwx}{Q}$ being the $\dagger$ square of the velo- † Sect. III.
Prop. V.
Cor. 3.

city if the line were without weight, it is manifest, that

the quantity $\frac{px^2 - bpx \times 4l}{LQ}$ is the variation of the

square of the velocity occasioned by the string's weight;

by which it appears, that whenever $b = x$, $\frac{px^2 - bpx}{LQ}$

vanishes; that is, when the descending weight w is as
much lower than the ascending weight P at the end of the
descent, as it was above P at the beginning, the velocity
is neither increased nor diminished by the string's weight.
Moreover, this variation in the square of the velocity ad-
mits of a maximum, if x is less than b , the quantity
 $\frac{px^2 - bpx}{LQ}$ being greatest when $x = \frac{b}{2}$, that is, when the

ascending and descending weights are on a level; at which
positions the variation in the square of the velocity of the
descending weight, occasioned by the string's weight,
will be the greatest, decreasing afterwards until it becomes
 $= 0$ when $x = b$: when x is greater than b , it is plain

that $\frac{px^2 - bpx}{LQ}$ admits not of a maximum, increasing
continually as x increases.

Cor. 2. The variation in the square of the velocity on
account of the string's velocity being $\frac{px^2 - bpx \times 4l}{LQ}$,

it appears, that as long as x is less than b , this variation
will be negative; that is, the string's weight will dimi-
nish the velocity generated, until the descending weight
has described a space equal to the difference in altitude of
the

the string's two extremities at the beginning of motion, after which the variation abovementioned becoming positive, shews that the velocity will afterwards be continually increased by the string's weight.

XIV.

Every thing remaining as in the last proposition, let it be required to assign the time in which the descending body describes the space AO from rest.

Sect. IV. Since the velocity at $O = \sqrt{\frac{Lwx + px^2 - bpx}{LQ}} \times$
 Prop. XIII. $\sqrt{4l}$, if T be the time of describing AO , $T = \sqrt{\frac{LQ}{4l}}$
 ¶ Sect. III. $\times \frac{\dot{x}}{\sqrt{Lwx + px^2 - bpx}}$, and the time required
 Prop. III. $T = \text{fluent of } \sqrt{\frac{LQ}{4l}} \times \frac{\dot{x}}{\sqrt{Lwx + px^2 - bpx}} =$
 $\sqrt{\frac{LQ}{lp}} \times \log. \frac{\sqrt{px} + \sqrt{Lw - bp + px}}{\sqrt{Lw - bp}}$, (being the
 entire fluent because it vanishes when $x = 0$) which will
 be a number expressing the time in seconds, if l be equal
 to the space through which the earth's gravity accelerates
 bodies from rest in one second.

Cor. 1. In order to facilitate the application of the solution, the expression for the time should be reduced to an approximate value, which will be mathematically true when the quantity p vanishes in respect of w or q ; and if it be of finite magnitude, the approximate value will deviate less from the truth, the smaller proportion p bears to w or q . For brevity, let $q = Lw - pb$, which will give the time T

$$T = \sqrt{\frac{LQ}{lp}} \times \log. \frac{\sqrt{px} + \sqrt{q+px}}{\sqrt{q}}, \text{ and since } px$$

is incomparably less than q , we shall have $\sqrt{q+px} = \sqrt{q} + \frac{px}{2\sqrt{q}}$, and $\frac{\sqrt{px} + \sqrt{q+px}}{\sqrt{q}} =$

$$\frac{\sqrt{4pqx + 2q + px}}{2q}, \text{ and } T = \sqrt{\frac{LQ}{lp}} \times$$

$\log. \frac{\sqrt{4pqx + 2q + px}}{2q}$; and because $\sqrt{4pqx + px}$

is incomparably smaller $2q$, we have $T = \sqrt{\frac{LQ}{lp}} \times$

$\frac{\sqrt{4pqx + px}}{2q}$, and since px vanishes in respect of

$$\sqrt{4pqx}, T \text{ will become } = \sqrt{\frac{LQ}{lp}} \times \frac{\sqrt{4pqx}}{2q} = \sqrt{\frac{LQx}{l}}$$

$$\times \frac{1}{\sqrt{Lw - bp}} = \sqrt{\frac{LQx}{l}} \times \frac{2\sqrt{wL}}{2wL - bp}, \text{ (when } bp$$

is very small in respect of Lw), that is, T will approxi-

$$\text{mate to } \sqrt{\frac{Qx}{lw}} \times \frac{2Lw}{2Lw - bp} = \sqrt{\frac{Qx}{lw}} \times 1 + \frac{bp}{2Lw}.$$

Cor. 2. The time wherein the weight would descend, if the string were without gravity, would be $\sqrt{\frac{Qx}{lw}}$ * Sect. III, Prop. IV.

seconds, because $\frac{w}{Q}$ is the constant accelerating force;

wherefore the string will increase the time (b being greater than x) in the proportion of $2Lw - bp$ to $2Lw$.

Cor. 3. In the preceding corollaries it has been supposed that p or the string's weight is very small, when compared with the moving force w , but when $\frac{p}{w}$ becomes considerable in respect of unity, this approximation will deviate from the truth; in this case recourse must be had to the expression for the true time derived from the solution.

S E C T. V.

CONCERNING THE RECTILINEAR MOTION
OF BODIES IN FLUIDS.

IN considering the properties of accelerated motion, no other resistance has hitherto been supposed to act in opposition to the moving force, except that which arises from the inertia of the weights moved; and this inertia always resists the communication of motion with forces proportional to those weights.

In the preceding propositions, therefore, relating to acceleration, bodies have been imagined to move in free and unresisting space, and the general properties of retarded motions have been deduced from those of acceleration; the forces having been assumed in each case either as constant, or varying according to some ratio of the distances from a fixed point.

But there are numberless forces, both imaginable in theory as well as really operating in the production of natural phe-

phenomena, which vary in no proportion of this kind, depending on circumstances altogether different from those above described: such, for instance, are the forces which vary in some ratio of the velocity which may be either direct or inverse, and may either accelerate or retard the motion of bodies; all these cases being physically possible, and consequently admitting of solutions derived from the laws of motion.

In the operations of nature, the forces varying with the velocity are chiefly those which are opposed by fluids to bodies moving in them. Let a solid body formed by the revolution of a plane figure about an axis, be projected or any how impelled in a fluid in the direction of that axis; if there be no other force to interfere with those of impulse and resistance, the body will continue to proceed in a straight line; and the ensuing propositions are intended to investigate from the necessary data the spaces described, times of description, and velocities at the end of the given spaces or times.

The properties of resisting forces which are opposed by fluids to bodies which move in them may probably be more obvious, if the nature of fluid substances as

distinguished from solids be first considered. Supposing all bodies to be formed of elementary hard and solid particles of different forms, it appears evident from various experiments, that the force whereby the parts of bodies cohere, depends upon the quantity of surface wherein the elementary particles touch each other, as a proximate or immediate cause, so that if these particles be spherical, the quantity of surface in contact being incomparably less than if the particles were cubes, prisms, pyramids, or other solid figures terminated by plane surfaces, it follows, that the force whereby the particles cohere is in a physical sense evanescent, which will be the more apparent by the following argument.

Let two spheres of equal diameters and sensible magnitudes touch each other: the contact will be incomparably less than that of any two plane surfaces however small; and it is an easy proposition to demonstrate, that the points wherein equal spheres touch each other, are in a direct ratio of their diameters; wherefore, if two spheres physically evanescent be in contact, the quantity of touching surface must be incomparably less, than that between the two spheres of finite mag-

magnitude which were before mentioned; from all which considerations, it plainly appears, upon the hypothesis of the spherical figure of the particles which compose fluids, how great must be the facility of motion among their parts.

To the preceding arguments add that which follows: if the number of points wherein a given spherical particle was touched, increased in the same proportion as the diameters were diminished, it is evident that the quantity of contact on the surface of each sphere would not be at all diminished from the argument of the spheres' diameters being evanescent; but since when a number of equal spheres are included in a solid space, no single sphere can be touched in more than twelve points, it follows, that while the spheres' diameters are diminished, the number of contacts in the surface of each sphere remains the same, and therefore the quantity of contact existing in the surface of any one sphere is still in proportion to the diameters.

Consequently, if these particles be perfectly hard, round and smooth, and of evanescent magnitude, there will be no resistance to the motion of bodies which im-

* Newt.
Princip.
Vol. II.
Prop. XL.
ad finem.

impinge on or move through them, except that which arises from the inertia of the particles displaced, and this conveys to us the idea of a perfect fluid. Few of these, however, exist in nature, perhaps none whose parts are wholly free from friction, cohesion, and tenacity. Air,* mercury, and water, have been esteemed, as to philosophical purposes, perfect fluids, the cohesion, friction, &c. of their parts, being scarcely if at all sensible in experiment. In other cases the degrees of fluidity are various. Of the perfect fluids above mentioned, water and mercury being mixed with heterogeneous substances, become (with very few exceptions) less fluid than before: Of this kind are solutions of gums and salts, together with oils, balsams, &c. which are, however of a more fluid nature than honey, clays, bitumen, and similar substances, the degrees of tenacity and cohesion between the particles of which, rank them among solid as well as fluid bodies, the properties of both which they seem to participate in an imperfect degree. The resistance which substances of this kind, i. e. imperfect fluids, oppose to bodies impinging on them, or moving through them, de-

depends on the cohesion, tenacity, and friction, as well as the inertia of the matter moved; the geometrical estimation of which circumstances being of no material use in physical enquiries,* the illustrious author of this theory, has chiefly considered the properties of retardation which bodies suffer when moving through the perfect fluids, the cohesion and friction among the parts of which are in a physical sense evanescent.

* Newt.
Princip.
Vol. II.
Prop. XL.
ad finem.

Since therefore the resistance which is opposed to solid bodies, moving in the perfect fluids, proceeds from their inertia only, it is to be next observed, that a body when projected or † impelled in empty space, will continue its motion uniformly in a straight line; but a body being projected or any how impelled in a fluid cannot proceed in the direction of its motion without displacing the fluid, and by communicating ‡ motion to it loses an equal quantity of its own motion: this gives us the idea of a fluid's resistance, the quantity of which will manifestly depend on the form and magnitude of the moving body, and the velocity of its motion: for a greater body, will displace a greater quantity of the fluid

† I. Law of Motion.

‡ III. Law of Motion.

fluid than a smaller one, every thing else being the same; and the greater the velocity wherewith a body moves in a fluid, the more motion will be communicated to the fluid (the exact law not being here considered) and consequently lost to itself.

Moreover, the resistance will depend on the fluid's density, every thing else being the same; for it is manifest, that it will require more force to displace a given quantity of mercury than the same quantity of water, and a quantity of water than an equal quantity of air, if the times be equal wherein these effects are produced, and the times are always by the definition of forces, both of acceleration and retardation, assumed equal in estimating the quantities of motion † generated or destroyed.

† II. Law
of Motion.

Since the parts whereof fluids are composed are hard solid bodies, subject to the same general mechanic laws as other bodies, the ratio of the resistances opposed to bodies which move in fluids of given densities, and with given velocities, may be inferred from the principles of mechanics.

I.

If two plane surfaces move in fluids, and in directions perpendicular to the respective planes, then will the resistances to their motion be in a direct duplicate ratio of their velocities, the joint ratios of the areas of the planes and densities of the fluids : that is, if $\frac{M}{m}$ be the

ratio of the resistances, $\frac{V}{v}$ that of the ve-

locities, and $\frac{A}{a}, \frac{N}{n}$ the ratios of the areas

and specific gravities respectively, the ratio of the resistances will be defined by

this equation $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{A}{a} \times \frac{N}{n} .$

For the motion lost by either plane, for example by A , in a given time, will be the same as the motion which would be communicated by the fluid impinging perpendicularly against it when quiescent with the velocity V ; and the same reasoning may be applied to the other plane a : this motion communicated in a given time will be as the number of particles and force of each. The ratio of the number of particles impinging on the planes = $\frac{A}{a} \times \frac{V}{v}$, and the ratio of the forces of each particle

respectively = $\frac{V}{v} \times \frac{N}{n}$, so that the ratio of the quantities of motion communicated to the quiescent planes in a given time, by the fluids impinging perpendicularly with

Q

with

with the velocities V, v , will be the sum of the ratios $\frac{A}{a} \times \frac{V}{v} \times \frac{V}{v} \times \frac{N}{n}$; that is, the ratio of the quantities of motion lost in a given time by the planes A, a , moving with the velocities V, v respectively, or the resistances opposed to their motion will be defined by the equation $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{A}{a} \times \frac{N}{n}$.

Cor. 1. In this proposition the direction of the plane's motion has been assumed always perpendicular to the planes themselves. Now, let their motion be in a direction any how inclined to the planes, the number of the particles will be diminished in the ratio of the radius to the sine of the inclination above described; also, the force of each particle will be diminished in the same ratio: wherefore on both accounts the resistance determined as before will be diminished in a duplicate ratio of the radius to the sine of inclination; let therefore $R =$ radius, and S, s the sines of the angles in which the planes A, a are inclined to the directions of their motion, and it follows, that to estimate the ratio of the resistances to the two planes, the ratios $\frac{S^2}{R^2} \times \frac{R^2}{s^2}$ or $\frac{S^2}{s^2}$ must be added to the ratio of the resistances before determined, so that on all accounts $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{A}{a} \times \frac{N}{n} \times \frac{S^2}{s^2}$.

Cor. 2. If equal planes move in a given fluid, and in directions equally inclined to themselves, then because in this case $\frac{A}{a} = \frac{N}{n} = \frac{S^2}{s^2} = 1$, we have $\frac{M}{m} = \frac{V^2}{v^2}$; that is, the resistances are in a duplicate ratio of the velocities.

Cor. 3. If the same body of any shape, moving in a given fluid with different velocities, always preserves the same position in regard to the direction of its motion, the resistances will be in a duplicate ratio of the velocities.

The preceding proposition expressing the proportion of the resisting forces only, will not enable us to ascertain the absolute quantity of resistance, which a body actually meets with in passing through a medium, but which must necessarily be known, in order to determine the retardation or decrement of velocity in a given time. As when bodies are accelerated the moving force is estimated by weight *, so the resisting force in retarded motions is also equi-

equivalent to a weight: when a body thrown perpendicularly upwards in empty space is resisted by the force of gravity, this resistance is always equal to the body's weight, so that the retardation of all bodies is the same; but the resistance opposed to bodies moving in fluids, is a weight which varies with the velocity being in the duplicate ratio thereof, and the estimation of this resistance a priori, from the necessary data, is attended with considerable difficulties, involving the consideration of impact relatively to weight: for in this case, it must be determined what weight is equal to the impact of the body moving against the particles of the fluid, or which is the same, the impact of the particles of fluid striking against the solid when quiescent with the same velocity.

Sir I. Newton trusted not wholly to theory either for the determination of the ratio of the resistances, nor the absolute quantity of them; but instituted various experiments, wherein it appeared that the resistances were by some trials in a ratio rather greater †, and by others in a ratio something less than the duplicate ratio of the velocities: these differences being no more than what the unavoidable imperfections in experiments of this kind might occasion, he was at length satisfied, that the resistances to a given body moving in the same fluid were in a duplicate ratio of the velocities, under certain restrictions and conditions, &c. hereafter described ||.

He also found that the resistance ‡ opposed to a globe, moving uniformly in a fluid, was to the constant resisting force or weight whereby the whole motion of the globe would be destroyed, during the time of its describing $2\frac{2}{3}$ diameters uniformly with the given velocity, as the density of the fluid to the density of the sphere, which gives the absolute quantity of resistance opposed to the body's motion, the densities of the sphere and fluid, the diameter of the sphere, and velocity of its motion being known, as will hereafter appear.

The principles of resistance being discovered by Sir I. Newton, and confirmed fully and established by the succeeding age, the subject may be probably more obvious if considered synthetically; which method, though unfit for making discoveries, is esteemed the most proper to exhibit the known principles of science in regular and systematic order. The following proposition is intended to determine the absolute resistance opposed to a body moving in a fluid with a given velocity, that is, to determine a weight equivalent to that resistance.

† Newton.
Princip.
Vol. II.
Prop. XL.
Scholium.

|| P. 135.

‡ Newt.
Princip.
Vol. II.
Prop. XXXVII.

II.

A plane surface moving in a fluid, in a direction perpendicular to the plane, is opposed by a resistance equal to the weight of a column of the fluid, the base of which is the resisted surface, and the altitude equal to that through which a body must fall from rest by the acceleration of gravity to acquire the velocity of the moving plane.

Fig. XXIV. Let the line AB represent the projection of the plane moving in the direction CD , which is perpendicular to it; and suppose that during its retarded motion it has advanced from its position AB to ab , describing the elementary space or parallelepiped $ABab$, with a given velocity V . Then will the plane AB communicate motion to the particles displaced, and will itself lose an equal quantity; and by the principles of bodies' collision, the quantity of motion lost by the resisted plane will be the same, as that which the particles would communicate to it in the same time, were the plane quiescent, and the particles impinged upon it with the same velocity. We are only then to ascertain what weight is equivalent to the impact of a given quantity of fluid striking against the plane with a given velocity, and communicating motion to it while it is describing an evanescent space. Let a cylindrical vessel, the base of which is horizontal, be filled with a fluid equally dense with that which resists the moving plane above described, the altitude of the fluid above the base being equal to that from which a body must fall by the acceleration of gravity, so as to acquire the velocity of the plane's motion: then if an $\dagger$ aperture be made in the base, the fluid will rush forth with a velocity equal to that of the plane's motion.

$\dagger$ Bernoulli
Hydrody-
namica.
sect. IV.

Now, let the plane be applied under the aperture and parallel to it, and the vessel being kept constantly full, let the plane be continually raised so as to approach the aperture: the fluid will strike against it with a continually decreasing velocity, till at last when the plane

is

is just contiguous to the aperture, the motion will be evanescent; and when the motion of the fluid is vanishing, the plane will sustain the weight of the incumbent column, that is, a column of the fluid, the base of which is the plane pressed, and height equal to that of the fluid's surface from the aperture: but the velocity of the fluid when just issuing out of the aperture, is equal to that which a heavy body acquires in falling through the perpendicular distance, between the aperture and fluid's surface; wherefore the nascent impulse of a fluid, acting perpendicularly on the plane, is equal to the weight of a column of the fluid: the base of which equals the aperture, the altitude being the same with that of the fluid's surface above the aperture: but the resistance to the plane from the fluid, is equal to the motion communicated to the plane by the nascent impulse of the fluid striking it when quiescent with a velocity equal to that of the plane's motion; wherefore the resistance to a plane, when moving in a fluid in a direction perpendicular to itself, is equal to the weight of a column of the fluid, the base of which is the resisted surface, and altitude equal to that from which a body must fall from rest by the acceleration of gravity to acquire the velocity of the plane's motion.

Cor. 1. The resistances to the same body moving with different velocities in the same fluid, are in a duplicate ratio of their velocities, because the base of the columns of fluid equal to the resistances being given, their altitudes will, by the proposition, be in a duplicate ratio of the velocities wherewith the plane moves. The other corollaries also which were deduced from the first proposition follow from this by the same methods of inference.

Cor. 2. If a cylinder moves in a fluid, so that the direction of its motion shall always coincide with the axis of the cylinder, the anterior plane surface only will communicate motion to the fluid, because the curved surface being parallel to the direction wherein the whole body moves, neither accelerates nor retards the particles of the fluid; these, as well as the cylinder, being supposed so smooth that no friction can have sensible effect. Let then d = the diameter of the cylinder, $p = 3.14159$, &c. V the velocity of the cylinder's motion, z the altitude from which a heavy body must fall to acquire the velocity V ; moreover, let the specific gravity of the fluid be expounded by

1; then will $\frac{d^2 p}{4} =$ the area of the cylinder's base, and

$\frac{z d^2 p}{4}$ will be the magnitude of a cylinder, whose diameter is d , and altitude $= z$, and the weight of a cylindrical column of the fluid, the base of which equal the base of the cylinder and altitude $= z$, is $\frac{z d^2 p \times 1}{4}$,

† Supra p. 125. which is equal to the resistance† opposed to a cylinder moving in a fluid, the density of which is $= 1$, the cylinder's velocity being $V = \sqrt{z}$, when referred to the general standard 1||, or $= \sqrt{4 l z}$, when referred to the velocity of $2 l$ in a second, which gravity generates in one second of time.

|| Sect. III. Prop. V.

Cor. 3. Let the weight of the cylinder $= w$, and let $F =$ its retardation, measured by the velocity destroyed in a given time in reference to that which is destroyed or generated by gravity in the same time; then since we have

† Sect. I. Prop. IX. in † general $\frac{F}{f} = \frac{M}{m} \times \frac{q}{Q}$, that is, the ratio of the forces

* Sect. II. Prop. VII. of retardation is compounded of the direct ratio of the resisting forces, and the inverse ratio of the weights moved, assuming f, m and q each $= 1$ *, it will be $F = \frac{M}{Q}$, and

since by Cor. 2. $M = \frac{z d^2 p}{4}$, and $Q = w$, we have $F =$

$\frac{z d^2 p}{4 w}$, the force which retards the cylinder in the circumstances above described.

The resistance to a plane moving in a fluid with a given velocity, and in a direction always perpendicular to itself, may be denominated the whole resistance, to distinguish it from the diminished resistances, which are opposed to the plane when moving with the same velocity, but in directions inclined to the plane at various angles; and if the base of a column of the fluid be equal to the moving plane, and the altitude the same as that from which a body must fall by the constant acceleration of gravity to acquire the velocity of the plane's motion, the weight of this column diminished in a duplicate proportion of radius to the sine of the angle at which the plane is inclined to the direction of its motion, will be the resistance to the plane's motion estimated in a direction perpendicular

pendicular to the *plane; but the resistance estimated in the direction of the plane's motion will be different from the former as will appear by the subsequent proposition.

* Sect. V.
Prop. I.
Cor. 1.

III.

Let AB represent a plane, moving in a fluid in the direction AF , with such a velocity as would be acquired in falling from rest by the acceleration of gravity through a space $= AF$. Then will the resistance to the plane, estimated in the direction of its motion AF , be equal to the weight of a column of the fluid, whose base $=$ the plane, and altitude $=$ the space AF diminished in a triplicate ratio of the radius to the sine of the angle at which the plane is inclined to the direction of its motion. Fig. XXV.

For the resistance estimated in the direction perpendicular to the plane is equal to the weight of a column of fluid, the base of which is the plane, and altitude AF , diminished in the duplicate † proportion of the radius to the sine of the angle in which the plane is inclined to the direction of its motion. In the line AF take AC to AF as the sine of inclination, that is, the sine of CAB to radius; through C draw CB perpendicular to AB , then will AF be to BC in a duplicate ratio of the radius to the sine of incidence CAB : the resistance, therefore, in the direction BC will be the weight of a column of the fluid, the base of which is the ‡ plane, and altitude BC .

† Sect. V.
Prop. I.

‡ Sect. V.
Prop. I.

The line BC will therefore represent the force of resistance in that direction, the plane's surface remaining the same: to estimate the resistance in the direction AC resolve BC into two, whereof DC is in the direction CA and BD perpendicular to it; then the force DB will not contribute to retard the progress of the plane in the direction

rection AC , but will impel it in a lateral direction only, and CD will be the only force of resistance acting in the direction opposite to that of the plane's motion: but since $AF : AC$, CB and DC are in a continual proportion, we shall have $AF : DC : AF^3 : AC^3$, or as $AC^3 : BC^3$, that is, the whole or greatest resistance, if it were perpendicular to the plane is to the resistance estimated in the direction CA , in a triplicate ratio of radius to the sine of inclination.

Cor. 1. Let another plane IB be joined to the former at
 Fig. XXVI. an angle $ABI =$ twice the angle of inclination BAD , and then let BI be taken $= BA$, and draw IK parallel to AC : also draw BL perpendicular to IK , and supposing these two planes to move in the direction AC or IL , then will the resistance to the plane IB estimated in the direction of its motion, be equal to the weight of a column of fluid, the base of which is equal to IB , and altitude the space due to the *velocity of the plane, diminished in a triplicate ratio of the radius to the sine of inclination, wherefore the whole resistance to both planes in the direction of their motion will be the weight of a column of the fluid, the base of which is equal to the planes, and altitude the same with that just described. Moreover, the lateral force BD will be counteracted by an equal and opposite force LB ; and consequently the remaining forces of resistance being in the direction KL or BC will only retard the progress of the planes, but will not alter the direction of their motion.

* Sect. V.
 Prop. III.

Cor. 2. Let any solid formed by the revolution of a plane figure round an axis, be projected or any how impelled in a fluid in the direction of that axis, and let the planes of two circles be drawn perpendicular to the axis, and contiguous to each other: then will the circumferences of the circles include an annular space, which will be the fluxion of the surface of the solid. If a tangent be drawn to this surface in the same plane with and meeting the axis, it appears from the preceding corollary, that the resistance to the annulus or elementary space, will be equal to the weight of a column of the fluid, the base of which is the annulus, and the altitude equal to the space due to the velocity with which the solid moves, diminished in a triplicate ratio of the tangent to the ordinate corresponding.

Cor. 3. The weight described in the last Cor. will be the fluxion of the resistance, if this be expressed in terms of the abscissa and constant quantities, the entire fluent will give the whole resistance opposed to the solid, estimated in the direction of its motion.

IV.

If a sphere and cylinder of the same diameter, move with equal velocities in the same fluid, and if the direction of the cylinder's motion, coincides with its axis, the resistance opposed to the motion of the globe, will be to the resistance opposed to the cylinder in the ratio of one to two.

Fig. XXVII

Newt.
Princip.
Vol. II.
Prop.
XXXIV.

The fluid in which the sphere and cylinder move is supposed to be so compressed, that the pressure on every part of the moving bodies shall be the same, as when they are at rest: moreover, it is assumed as true, that the hinder part of the solids contributes nothing to the resistance, which will be the same as if the anterior part only were exposed to the fluid, the weight and every thing else remaining.

Let CT be the line which the sphere's centre C describes during its motion, the velocity of C being equal to that which is acquired by a heavy body which descends from rest by the acceleration of gravity through the space z . Also, let GD be a diameter perpendicular to CT , GED being the plane of a semicircle, which passes through CT . In the semicircle GED draw any radius CA ; draw AB perpendicular to CE , ab parallel and contiguous to AB , and am parallel to CT ; moreover, through the point A draw AT , a tangent to the circle at A .

Supposing the plane CEG to revolve round CE as an axis, the evanescent arc Aa will generate an evanescent surface, which will be the fluxion of the surface of the sphere: if this fluxional area, which we may denote by b , moved in the fluid in a direction perpendicular to its plane with the velocity, which a heavy body would acquire by falling freely through a space z , the resistance to its motion would be the weight of a column of the fluid, of which the magnitude $= bz$; but as this evanescent surface is inclined to the direction of its motion CT in an angle ATC , the whole resistance bz must be diminished in a triplicate ratio of radius to the sine of ATC ;

† Sect. V.
Prop. I.

* Sect. V. *ATC*; that is, in the *ratio of $AT^3 : AB^3$; which will
 Prop. III. give the fluxion of the resistance to the solid in the di-
 Cor. 2. rection of its motion $CT = \frac{bx \times AB^3}{AT^3}$ whatever be the
 curve *GE*; and since *GE* is the quadrant of a circle, and
 $\frac{AB^3}{AT^3} = \frac{CB^3}{CA^3}$ the fluxion of the resistance $= bx \times$
 $\frac{CB^3}{AC^3}$: making $EC = r$, $CB = x$, $Bb = \dot{x}$, $p =$
 314159 , &c. from the properties of the circle, $AB =$
 $\sqrt{r^2 - xx}$, and by similar triangles $Aa : ma :: AC : AB$,
 or $Aa : \dot{x} :: r : \sqrt{r^2 - x^2}$, which gives $Aa = \frac{r\dot{x}}{\sqrt{r^2 - x^2}}$:
 moreover, the circumference of the circle generated by
 the point *A*, while *GEC* revolves round *CE* as an axis,
 will be $2p \times \sqrt{r^2 - x^2}$, which being multiplied into
 Aa or $\frac{r\dot{x}}{\sqrt{r^2 - x^2}}$ will give $2pr\dot{x}$, for the fluxion of the
 surface of the solid; and x being the space due to the ve-
 locity of the sphere's motion, $2prx\dot{x}$ will be the whole
 resistance to the † evanescent resisted surface, supposing
 the direction of motion perpendicular to it; this being
 diminished|| in the ratio of $r^3 : x^3$, or being multiplied into
 $\frac{x^3}{r^3}$ gives $\frac{2prx\dot{x}^3}{r^3}$, for the resistance to the evanescent
 area, estimated in the direction of its motion *CT*, and the
 resistance to the whole surface $=$ the fluent of $\frac{2prx\dot{x}^3}{r^3}$
 $= \frac{pzx^4}{2r^2}$, or when $x = r$, the resistance to the hemisphere
 $= \frac{pzr^2}{2}$: but the resistance § to the cylinder moving with
 § Sect. V. the same velocity in the direction of its axis is $\frac{pd^2z}{4}$ or
 Prop. II. pr^2z , which is to the resistance just found $\frac{pr^2z}{2}$ as 2 : 1.
 Cor. 2. If the diameter of the sphere $= d$, the space due to its
 velocity $= z$: the resistance to the sphere's motion will $=$
 $\frac{pd^2z}{8}$, the specific gravity of the fluid being $= 1$.

V.

The force of resistance which is opposed to a sphere, moving in a fluid with any given velocity, is to the force which would destroy the sphere's whole motion, in the same time in which it describes uniformly $\frac{8}{3}$ parts of its diameter, as the density of the fluid to the density of the sphere.

Newt.
Princip.
Vol. II.
Prop.
XXXVIII.

Let the specific gravity of the sphere be to that of the fluid as $n : 1$, also let M be the resisting or moving force which would destroy or generate the sphere's motion in the time described in the proposition: let the sphere's weight $\frac{d^3 p n}{6} = \mathcal{Q}$, V the velocity of the sphere's mo-

tion, $z = \frac{V^2}{4l}$ the altitude, from which a body must fall * Sect. III. Prop. V.

from rest by the acceleration of gravity to acquire the ve-

locity V : then will a body describe the space $\frac{8d}{3}$ uniformly † Sect. III. Prop. II.

with the velocity V , in the same time wherein it describes $\frac{4d}{3}$, when it is retarded or accelerated by a constant force

$\frac{M}{\mathcal{Q}}$, which destroys or generates its whole velocity V . To

find, therefore, the resisting force M , or retarding force

$\frac{M}{\mathcal{Q}}$, which will destroy the whole motion of the sphere

projected with the velocity V , while it describes a space =

$\frac{4d}{3}$ by an uniformly retarded velocity, we shall have $\frac{1}{2}V =$ † Sect. III. Prop. V.

$\sqrt{\frac{M}{\mathcal{Q}} \times \frac{4d}{3} \times 4l}$, but $\parallel V = \sqrt{4lz}$, wherefore $\frac{M}{\mathcal{Q}} \times \parallel$ Supra.

$\frac{4d}{3} = z$, and $M = \frac{3Qz}{4d}$; or since $Q = \frac{p d^3 n}{6}$, it follows that $M = \frac{z p d^2 n}{8}$, the uniform resisting force which would destroy the sphere's whole motion in the same time it describes the space $\frac{4d}{3}$ with an uniformly retarded velocity, or $\frac{8d}{3}$ with the first velocity of projection continued uniform; which velocity is in both cases equal to that acquired by a heavy body descending from rest through the space $z = \frac{V^2}{4l}$. But the resisting force opposed to

* Sect. V.
Prop. IV. the sphere's * motion $= \frac{z p d^2}{8}$, which is to $\frac{z p d^2 n}{8}$, the weight or resistance which would generate or destroy the sphere's velocity V , the same time in which the sphere describes uniformly the space $\frac{8d}{3}$, with the velocity of projection V , as 1 to n ; that is, as the specific gravity of the fluid to that of the sphere.

VI.

Fig.
XXVIII.

Let a sphere of given diameter be projected in a fluid, the specific gravity of which is to the specific gravity of the sphere as 1 : n : having given the velocity of projection from the point c , let it be required to assign the velocity of the sphere's motion at any given point o .

Let the velocity of projection be that which a heavy body acquires in describing the space b from rest; also let the sphere's diameter $= d$, and $CO = x$, $p = 3.14159$, $l = 193$ inches, z the space through which a body descends by gravity from rest to acquire the sphere's velocity in

in O , then the only force which acts on the sphere while it is describing the evanescent space Oo , is that which retards its motion, and is $= -\frac{3z}{4dn}$. If therefore $Oo = \dot{x}$,

we shall have $\dot{z} = -\frac{3z\dot{x}}{4dn}$, and $\frac{\dot{z}}{z} = -\frac{3\dot{x}}{4nd}$, and tak-

ing the fluents $\log. z = -\frac{3x}{4nd}$, which should vanish to-

gether,; but when $x = 0$, it follows from the problem, that $z = b$, consequently $\log. z = \log. b$; the entire flu-

ents therefore will be $\log. z = \log. b - \frac{3x}{4nd}$, and $\frac{3x}{4nd}$

$= \log. b - \log. z = \log. \frac{b}{z}$: let $e = 2.71828$, being

the number the hyperbolic logarithm of which $= 1$, and

we shall have $\frac{b}{z} = e^{\frac{3x}{4nd}}$, and $z = b \times e^{-\frac{3x}{4nd}}$, and

the sphere's velocity at O or $V = \sqrt{b \times e^{-\frac{3x}{4nd}}}$ when the velocity is referred to the general standard 1, but if it be compared with the velocity $2l$ in a second which gravity†† Sect. III. generates in descending bodies in one second of time, we P. 18.

shall have $V = \sqrt{4lb \times e^{-\frac{3x}{4nd}}} = \frac{\sqrt{4lb}}{e^{\frac{3x}{8nd}}}$.

Cor. 1. The velocity lost by a globe projected in a fluid during the time of its describing the space x , will be equal to that part of the initial velocity which is expressed

by the fraction $\frac{e^{\frac{3x}{8nd}} - 1}{e^{\frac{3x}{8nd}}}$, the letters e, x, n, d signifying

as before.

Cor. 2. Thus, suppose a globe be projected in a fluid of the same density, and to describe a space equal to three of its diameters, then $x = 3d$, and $\frac{3x}{8nd} = \frac{9}{8}$, and the

velocity lost $= \frac{e^{\frac{9}{8}} - 1}{e^{\frac{9}{8}}} = \frac{11.48}{17}$, that is, a globe moving

in

in a resisting medium of the same density with itself through a space equal to 3 of its diameters, loses $\frac{11.48}{17}$ parts of the velocity with which it was projected. It may be here remarked, that the editor of Dr. Helsham's

† Helsham's lectures, from the same data † makes the lost velocity $\frac{9}{17}$ Lectures, p. 356.

instead of $\frac{11.48}{17}$ of the whole velocity of projection; but his expressions for the velocity lost by a body moving through a resisting medium, are by no means deducible from or consistent with the Newtonian propositions which are there referred to.

Cor. 3. From this solution we may obtain the quantity of motion which would be lost in any given time by the earth, if the medium through which it moved were of equal density with the atmosphere at the earth's surface. Suppose the time of motion to be one year, and since the sun's parallax is 8''6, the sun's distance from us = about 24000 of the earth's radii, or 12000 diameters, it follows, that in the circumference of the earth's orbit there are contained $24000 \times 3.14159 = 75398$ diameters, and in one year the earth will have moved through a space equal to 75398 of his diameters, and if the mean density of the

|| Newt. Princip. Vol. III. Prop. X.

earth be to that of air || as $5 : \frac{1}{800}$, or as 4000 : 1, we shall have in the preceding expression for the velocity lost $n = 4000$, $\frac{x}{d} = 75398$, and $\frac{3x}{8nd} = 7.068$, then will the velo-

$$\text{city lost} = \frac{e^{\frac{3x}{8nd}} - 1}{e^{\frac{3x}{8nd}}} = \frac{1173}{1174} \text{ part of the whole.}$$

Cor. 4. When the quantity $\frac{3x}{8nd}$ is very small, the velocity lost will approximate very nearly to $\frac{3x}{8nd}$, because

$$\text{in that case } e^{\frac{3x}{8nd}} = 1 + \frac{3x}{8nd}, \text{ and } \frac{e^{\frac{3x}{8nd}} - 1}{e^{\frac{3x}{8nd}}} = \frac{3x}{8nd}.$$

thus

thus to assume an *example from Sir I. Newton, let a globe of the same density with water be projected through a medium of the same density with that of air near the earth's surface, and let it be required to assign the motion lost during the time the sphere is describing a space equal

* Newt.
Princip.
Vol. II.
Prop. XL.
in Scholio.

half its diameter; here we have $x = \frac{d}{2}$, $n = 860$, $\frac{3x}{8nd}$

$= \frac{3}{13760} = \frac{1}{4586}$, so that the globe will have lost that

part of its whole velocity which is expressed by the fraction

$\frac{1}{4586}$. This shews how nearly this approximation is to

the true value; the error not being sensible in the expression above deduced, which coincides with that of Sir I. Newton to an unit in the denominator.

Cor. 5. It follows also, that if the velocity of projection from C be to that of the sphere when at any point of the space described O, as m to 1, then will the space CO or x

Fig.
XXVIII.

$= \frac{8nd}{3} \times \log. m$. Thus, ‡ suppose a sphere moving in a fluid of the same density with itself loses half its velocity, then will $m = 2$, $n = 1$, and the space described by the

‡ Newt.
Princip.
Vol. II.
Prop.
XXXVIII.
Cor. 4.

sphere $= \frac{8d}{3} \times \log. 2 = \frac{8d \times .6931472}{3} = 1.84839 \times d$;

the sphere therefore loses half its velocity before it has described a space equal to 2 of its diameters.

The theory of resistances, † opposed to bodies moving in the perfect fluids, is demonstrated by Sir I. Newton only under certain conditions and restrictions, which may be here mentioned.

† Ibid.
Vol. II.
Prop. XL.
&c.

1. The particles of fluid wherein the body moves are supposed to be perfectly nonelastic.

2. The fluid is imagined to be infinitely compressed.

The former condition is strictly applicable to the perfect fluids, mercury, water, and even to air, when the resisted body moves slowly, in which case the aerial particles singly considered, may be justly imagined to slide away from bodies moving through them without adhesion, in the same manner as those particles which compose water or mercury. This is only meant to distinguish the nonelastic particles from those which by rebounding from an impact would tend to increase the resistance beyond that which is assigned

ed

ed in the theory. The other condition, i. e. that of infinite compression, obtains not in any fluid whatever, some allowance therefore is due on this account: but in experiments made on bodies moving very slowly, this allowance will be of no material consequence, because from the slowness of the motion, the parts left by the sphere during its progress will be instantly occupied by the surrounding fluid; but when the velocity of projection is considerably increased, the laws of resistance demonstrated by Sir I. Newton, depending on the effects of the inertia of the matter displaced only, must not be expected to correspond with matter of fact. Thus, cannon or musket shot projected with velocities from 400 to 1600 feet in a second, by their great velocities leave behind them during their passage through the air, either partial or entire vacuity; when the velocity is equal to that of 1600 feet in a second, the space described by the ball as it moves along, is an absolute vacuum, for the air by its elasticity rushes into empty space with a velocity of no more than 1296 feet in a second, and will not therefore move with sufficient celerity to fill up the space deserted by the ball in its passage. It follows therefore, that a ball moving with this velocity is resisted by the whole pressure of the atmosphere, exclusive of the effects arising from the air displaced, and this additional resistance must in most cases be very great: for example, the diameter of an iron 24 pounder being 5.457 inches, we shall have the area of a great circle of the ball $= 23.385$ inches, and allowing 15 pounds avoirdupoise for the pressure of the atmosphere upon each square inch, the resistance to the ball's motion arising from the air's pressure $= 23.385 \times 15 = 350.78$, or a weight of 350.78 pounds: to this must be added the resistance which is caused by the inertia of the air displaced: suppose the ball's velocity to be 1600 feet in a second, and $z =$ the space through which a body must fall by gravity to acquire the velocity of 1600 feet in a second, then will this space $z = 39793$ feet; and because the ball's diameter $= 5.457$ inches, the area of a great circle $= .16240$ parts of a foot, and the resistance to the ball's motion being equal to the

weight of a column of air, || the altitude of which $= \frac{39793}{2}$
 feet, and base $= .16240$ parts of a square foot will be equal
 in weight to 3231 cubic feet of air, or 3755 ounces $=$
 234.8 pounds: this being added to 350.7 before found,
 becomes 585.5 pounds for the whole resistance arising from
 inertia

|| Sect. V.
 Prop. II. &
 Prop. IV.

inertia and pressure. * Mr. Robins found by experiment, * Robins' that the resistance was about 540 pounds, which is less than Gunners, that just determined by 44.5 pounds; this difference may be Vol. I. p. easily accounted for from the method adopted by Mr. Ro- 142. bins for determining the resistance, which was by finding

what resistance was opposed to a ball of $\frac{3}{4}$ inch, moving

with a given velocity, and inferring from thence what would be the resistance to a ball of 5.457 inches diameter, if the velocity were nearly the same; by which it is manifest, that a small error in the original experiment, for determining the lesser resistance, will cause a considerable variation in the greater resistance, if it be inferred from the lesser.

There is another circumstance also which tends still to increase the resistance to bodies moving with very great velocities, such for instance as 1800 or 2000 feet in a second. In this case the air being condensed before the ball exerts a force of elasticity against it in proportion to the compression; and it is this repulsion which renders any augmentation of the velocities of military projectiles beyond 1200 or 1300 feet in a second, of little use in increasing either the horizontal range, or the effectual impetus of the bullet.

This elastic force which the air exerts against bodies of small weight, but moving with considerable velocities, may become so great in proportion to the weight, as not only to destroy the motion communicated, but even to repel them; which is observed frequently to happen when very small shot are discharged by a large quantity of powder; in which case the shot return back in a direction contrary to that in which they were projected: and it seems at first sight singular, although strictly coincident with the theory, that a smaller charge of powder by giving the shot less initial velocity will cause it to fly further than a greater charge, if the quantity of the charge should exceed a certain limit.

VII.

Every thing remaining as in the last ^{Fig. XXVIII.} proposition, it is required to determine
s the

the time in which a body describes any given space c o.

* Sect. VI. The velocity* at the point O = $\sqrt{4lb} \times e^{-\frac{3x}{8nd}}$.
Prop. VI.

† Sect. III. If therefore the † time required be T, we shall have $\dot{x} =$
Prop. III.

$$\frac{\dot{x} e^{\frac{3x}{8nd}}}{\sqrt{4lb}}, \text{ and the time required} = \text{fluent of } \frac{\dot{x} e^{\frac{3x}{8nd}}}{\sqrt{4lb}} : \text{ to}$$

obtain the fluent let $e^{\frac{3x}{8nd}} = v$, wherefore $\frac{3x}{8nd} = \log. v$,

and $\dot{x} = \frac{8nd}{3} \times \frac{\dot{v}}{v}$, wherefore $\dot{x} e^{\frac{3x}{8nd}} = \frac{8nd}{3} \times \dot{v}$, the

fluent of which = $\frac{8nd}{3} \times v = \frac{8nd}{3} \times e^{\frac{3x}{8nd}}$, and the

time required = $\frac{8nd}{3 \times \sqrt{4lb}} \times e^{\frac{3x}{8nd}}$; but this quantity

should vanish with x the space described, but when $x = 0$,

$$\frac{8nd}{3 \times \sqrt{4lb}} \times e^{\frac{3x}{8nd}} = \frac{8nd}{3 \times \sqrt{4lb}} : \text{ if therefore}$$

$\frac{8nd}{3 \times \sqrt{4lb}}$ be subtracted from the fluent just found, we

shall have the time $T = \frac{8nd}{3 \sqrt{4lb}} \times e^{\frac{3x}{8nd}} - 1$. In these

cases the body is supposed to be acted on by no other force except those of projection and resistance, and consequently moves with a velocity continually retarded; but if a body be impelled through a fluid by the force of gravity, this force compounded with that of resistance may generate either a retarded or accelerated, or in some extreme cases, as it will hereafter appear, an uniform velocity of motion: and if the body be acted on by no force out of the vertical direction in which gravity acts, that is, if it be either gradually accelerated from quiescence, or be projected in a vertical line, the motion will be rectilinear.

VIII.

Let a spherical body descend in a fluid Fig: XXIX.
from rest; having given the diameter of
the sphere and its specific gravity, re-
latively to that of the fluid, it is required
to assign the velocity of the sphere at any
given point o of the space described A o.

Let the $\dagger$ sphere's diameter = d , the specific gra- $\dagger$ Euler.
vity of the sphere = n , that of the fluid = 1; also let Comment,
on Robins'
Gunnery,
Prop. II.
the space AO which the sphere has described from
and = x : since the sphere's weight = $\frac{d^3 p n}{6}$ and

the weight of an equal bulk of the fluid = $\frac{d^3 p}{6}$, n

being greater than 1, the absolute or moving force
whereby the sphere endeavours to descend will be $\frac{d^3 p n}{6}$

— $\frac{d^3 p}{6}$, which will be always the same, whatever be the

velocity of the sphere's motion (it being supposed always
that the fluid is infinitely compressed): but the resistance
opposed on account of the sphere's motion will depend on
the velocity. Let z be the space through which a body
must fall by its gravity, so that it may acquire the velocity
of the sphere in the point O: then we shall have the resist-

ance opposed to the $\dagger$ sphere equal to the weight $\frac{p z d^2}{8}$; the $\dagger$ Sect. V.
Prop. IV.
Cor.
absolute force therefore, whereby the sphere endeavours to

descend will be upon the whole = $\frac{d^3 p n}{6} - \frac{d^3 p}{6} - \frac{p z d^2}{8}$,

and this weight being $\parallel$ divided by the mass moved $\parallel$ Sect. I.
Prop. IX.
will be the force, which accelerates the sphere while it
is describing O o; and since the weight of the sphere

is = $\frac{p d^3 n}{6}$, we shall have the accelerating force =

$1 - \frac{1}{n} - \frac{3z}{4nd}$: and from the principles of * acceleration * Sect. III.
Prop. V.
and p. 56.

$\dot{z} = 1 - \frac{1}{n} - \frac{3z}{4nd} \times \dot{x}$, and by reduction $\frac{\dot{x}}{4nd} =$
 $\frac{\dot{z}}{4nd - 4d - 3z} = \frac{\dot{z}}{4d \times n - 1 - 3z}$: to obtain the fluent,
 let $4d \times n - 1 - 3z = v$: taking the fluxions, we have
 $3\dot{z} = -\dot{v}$, and $\frac{\dot{z}}{4d \times n - 1 - 3z} = -\frac{\dot{v}}{3v}$, the fluent
 of which $= -\frac{1}{3} \times \log. v = -\frac{1}{3} \times \log. 4d \times n - 1$
 $- 3z$, wherefore $\frac{x}{4nd} = -\frac{1}{3} \times \log. 4d \times n - 1 - 3z$,
 which quantities should vanish together, because by the
 prob. when $z = 0, x = 0$: but when $z = 0, -\frac{1}{3} \times \log.$
 $4d \times n - 1 - 3z$ becomes $-\frac{1}{3} \log. 4d \times n - 1$; the
 entire fluent therefore will be $\frac{x}{4nd} = -\frac{1}{3} \log.$
 $\frac{4d \times n - 1 - 3z}{4d \times n - 1}$. Let $e = 2.71828$, &c. being the num-
 ber the hyperbolic log. of which $= 1$, and we shall have
 $\frac{4d \times n - 1 - 3z}{4d \times n - 1} = e^{-\frac{3x}{4nd}} = 1 - \frac{3z}{4d \times n - 1}$, and
 $\frac{3z}{4d \times n - 1} = 1 - e^{-\frac{3x}{4nd}}$, wherefore $z = \frac{4d}{3} \times$
 $n - 1 \times 1 - e^{-\frac{3x}{4nd}}$: and if the velocity at O , be re-
 ferred to the velocity which the earth's gravity generates
 in bodies each second of the time wherein they fall, l be-
 ing $= 16\frac{1}{2}$ feet or 193 inches, we shall have the velocity

* Sect. III. at $O = \sqrt[3]{16ld \times n - 1} \times \sqrt{1 - e^{-\frac{3x}{4nd}}}$.
 p. 28.

Cor. 1. If the specific gravity of the globe be infinite,
 when compared with that of the fluid wherein it de-
 scends, that is, if n be increased sine limite, the quantity

$-\frac{3x}{4nd}$ will be evanescent, and $e^{-\frac{3x}{4nd}}$ will become $=$

$1 - \frac{3x}{4nd}$, and $1 - e^{-\frac{3x}{4nd}} = \frac{3x}{4nd}$: wherefore the

velocity will $= \sqrt{\frac{16ldn}{3}} \times \frac{3x}{4nd} = \sqrt{4lx}$, which

is the velocity * acquired by a body descending from rest * Sect. III. in vacuo by the acceleration of gravity through the space x . Prop. V.

Cor. 2. If the density of the globe is not increased fine limite, but bears a considerable proportion to that of the fluid, the velocity acquired by the globe descending in the fluid through the space x will approximate to the quan-

tity $\sqrt{4lx} \times 1 - \frac{3x}{16nd}$, because $e^{-\frac{3x}{4nd}} = 1 -$

$\frac{3x}{4nd} + \frac{9x^2}{32n^2d^2}$ nearly, from which the approximation is

easily deduced. For example, if a spherical body of the density of water descends in air from rest through any space

x , let $V = \sqrt{4lx}$, be the † velocity which the body would † Sect. III. acquire by descending through the same space in vacuo, Prop. V. then will the air's resistance destroy that part of the whole Cor. 3.

velocity, and which is expressed by the fraction $\frac{3x}{16nd}$,

that is, since in this case $n =$ about 860, the velocity lost

will be $= V \times \frac{3x}{13780d}$, which will be very near the

truth when the quantity x , or the space fallen through, exceeds not the diameter of the ball in a proportion of above 100 to 1.

Cor. 3. Since $V = \sqrt{\frac{16ld \times n-1}{3}} \times 1 - e^{-\frac{3x}{4nd}}$,

if x be increased fine limite, the quantity $e^{-\frac{3x}{4nd}}$ will approximate to and will ultimately $= 0$, wherefore the velocity acquired by the sphere will continually approximate

to $\sqrt{\frac{16ld \times n-1}{3}}$, as a limit which it can never ex-

ceed, or even attain to in any finite time.

Cor. 4. Although a spherical body descending from rest in a fluid, cannot attain to its mathematical limit, or maxi-

maximum of velocity in any finite time, yet it will, in descending through a few diameters, approach so nearly to it, that the velocity will afterwards become to all observation uniform.

Thus a sphere, the density of which is twice as great as that of the fluid wherein it descends, will in describing 16 diameters only, arrive at its maximum of velocity

within $\frac{1}{800}$ part of the whole; for if we put $x = 16d$,

$n = 2$; then $\frac{3x}{4nd} = 6$, and $e^{-6} = \frac{1}{403}$; and $1 -$

$e^{-6} = \frac{402}{403}$, and $\sqrt{1 - e^{-6}} = \frac{805}{806}$, and the velocity after the sphere has described 16 diameters =

$\sqrt{\frac{16ld \times n - 1}{3}} \times \frac{805}{806}$ being equal to the limit or

greatest velocity $\sqrt{\frac{16ld \times n - 1}{3}}$ within less than $\frac{1}{800}$

part of the whole, which difference, is far less than can be observable by the senses; the sphere therefore, after it has described the space just mentioned, may be considered as moving uniformly.

IX.

Newt.
Princip.
Vol. II.
Prop.
XXXVIII.
Cor. 2.

The greatest velocity which can be acquired by a spherical body descending in a fluid is equal to that which would be acquired by it in descending from rest in vacuo by the constant force of its comparative gravity, through a

space which is to $\frac{4}{3}$ of the diameter, as

the

the density of the sphere to the density of the fluid.

For when the sphere by falling from rest in a fluid medium has acquired its greatest velocity, the increment of velocity is nothing, that is, the force of resistance is equal to the body's weight, and consequently the force of acceleration is nothing. Let therefore the density of the globe be to that of the fluid as $n : 1$, the globe's diameter $= d$, z the space through which a body must fall from rest by gravity to acquire the greatest velocity above described,

then we shall have the † accelerating force $= 1 - \frac{1}{n}$ † Sect. V.
Prop. VIII.

$-\frac{3z}{4nd}$, which by the problem must $= 0$; wherefore

$\frac{3z}{4nd} + \frac{1}{n} = 1$, and $3z = 4d \times n - 1$, and $z =$

$\frac{4d \times n - 1}{3}$, equal to the space due to the greatest velocity acquirable by the sphere; and if $l = 193$ inches, we

have the greatest *velocity $= \sqrt{\frac{16ld \times n - 1}{3}}$: but a • Sect. III.
Prop. V.
Cor. 3.

space which is to $\frac{4d}{3}$ as $n : 1$ becomes $\frac{4dn}{3}$, and if the

weight of the sphere in vacuo be Q , its comparative gravity whereby it endeavours to descend in the fluid

$= Q - \frac{Q}{n}$, and the † force whereby it is accelerated in † Sect. I.
Prop. IX.

its descent $= \frac{n-1}{n}$: and the velocity generated in the

sphere descending from rest in vacuo through a space

$\frac{4nd}{3}$ by the || force $\frac{n-1}{n} = \sqrt{4l \times \frac{4dn}{3} \times \frac{n-1}{n}}$ || Sect. III.
Prop. V.
Cor. 3.

$= \sqrt{\frac{16ldn-1}{3}}$; but this is the greatest velocity which

the sphere can acquire in its descent by the preceding part of the solution.

Or thus: Suppose it were required to assign how far the sphere must fall from rest in vacuo by the action of a constant

stant force, equal to that by which it endeavours to descend in the fluid, so that it may acquire the limit or greatest velocity above described. The notation remaining as before, we have the force which accelerates the sphere's descent =

$$\frac{n-1}{n}, \text{ and } V \text{ the greatest velocity} = \sqrt{\frac{16ld \times n-1}{3}} :$$

then since if S is the space which a body describes from rest by the acceleration of any constant force F , while the velocity V is generated,* we have always $S = \frac{V^2}{4lF}$; ap-

|| Sect. III.
Prop. V.
Cor. 3.

plying this to the present case, $V =$ the greatest velocity

† Sect. V.
Prop. VIII.
Cor. 3.

$$\text{acquirable by the sphere} = \sqrt{\frac{16ld \times n-1}{3}}, F =$$

$$\frac{n-1}{n}, \text{ and } S = \frac{V^2}{4lF} = \frac{16ld \times n-1}{3} \times \frac{n}{4 \times n-1 \times l}$$

$$= \frac{4nd}{3}, \text{ the space required, which is to } \frac{4}{3} \text{ parts of the}$$

diameter as $n : 1$, or as the density of the sphere to the density of the fluid.

X.

Spherical bodies of very small diameters descend in fluids specifically lighter than themselves, with nearly uniform velocities, which are in a direct subduplicate ratio of their diameters, the fluid wherein they descend being the same.

* Sect. V.
Prop. VIII.
Cor. 4.

For the velocity* of any sphere which descends in a fluid of half its specific gravity, through a space equal to 16 of its diameters, will approach to its limit or the greatest ac-

quirable velocity within $\frac{1}{800}$ part of the whole; wherefore,

if the descent were continued for an unlimited space, the velocity generated during this descent would not be more than

than $\frac{1}{800}$ part of that which the body possessed after having described 16 diameters; wherefore the velocity after that time may be regarded, in a physical sense, as absolutely uniform: but while the diameter of a sphere is diminished, the space of 16 diameters is diminished in the same proportion; and when the diameter becomes evanescent, the sphere will have described 16 diameters in the least sensible portion of its descent: the velocity therefore of the descent will be as to observation entirely uniform. The same reasoning may be extended to the descent of spherical bodies of any other specific gravity, their diameters being diminished sine limite. Let d = the diameter be expressed in inches, n = the specific gravity of the descending sphere, that of the fluid being 1; moreover, let $l = 193$ inches, then we have the limit or greatest velocity which the sphere can acquire by descending in the † fluid =

† Sect. V.
Prop. IX.

$\sqrt{\frac{16ld \times n - 1}{3}}$ inches in a second; and by what has

preceded,* this will be the uniform motion with which * Supra, spheres of evanescent diameters descend, and in the same

fluid $\sqrt{\frac{16ln - 1}{3}}$ being constant, the velocities will be

in a subduplicate ratio of the diameters.

Although it would be extremely difficult to reduce the propositions, relating to the velocities acquired by spherical bodies which descend in fluids to experimental trials, when the velocities are considerably accelerated; yet by observing the descent of small spheres, the velocities of which are uniform, the theory admits of easy illustrations from matter of fact. The vessel used for the purpose, was a cylinder of 6 inches in diameter and about 5 feet in height. In the first place, such a vessel being filled with water, if a number of small spherical bodies of different diameters, but of the same specific gravity, such as very small leaden shot, globules of mercury, &c. be at once suffered to descend from rest through the water, the larger bodies will be observed to precede the other in their descent, and that in exact order, no small body being ever seen descending so fast as the larger ones, the velocity of descent also will be to all appearance uniform: these experiments so far correspond with the theory, from which it has been deduced, that the uniform velocities with which small spheres descend in fluids, are in a subduplicate ratio of their dia-

T

meters.

meters. This is an illustration of a general kind, that which follows is more particular.

But it may be first noted that the diameter of a sphere is obtained far more exactly from having given its specific gravity and weight, than from direct mensuration. Let d = the diameter of a sphere in inches, also let n be its specific gravity, that of water being = 1, and w the weight in grains determined by observation, then we

shall always have $d = \frac{w}{n}^{\frac{1}{3}} \times .19612$.

To apply this, the specific gravity of a number of small leaden shot was determined by weighing them in air and water, and found to be 11.1, and the weight of a single shot selected from the rest weighed 1.05 grains; wherefore to find the diameter, we have $n = 11.1$, $w = 1.05$

parts of a grain, and the diameter = $\frac{1.05}{11.1}^{\frac{1}{3}} \times .19612$

= .089 parts of an inch: this sphere was observed to descend in the vessel of water 60 inches from rest in something more than two seconds, as nearly as could be determined. According to the approximated theory it should have described almost 61 inches in the same time, for according to this rule, the † velocity of descent =

† Sect. V.
Prop. IX.

$$\sqrt[3]{\frac{161d \times n - 1}{3}} = \sqrt[3]{\frac{16 \times 193 \times .089 \times 10.1}{3}}$$

= 30.48 inches in a second. It will appear from an ensuing proposition, in which the real time is investigated wherein a sphere describes from rest in a fluid any given space by constant acceleration, how much the experiment, and this approximate value for the uniform velocity deviate from the truth.

In the mean time it may be remarked, that the experiment is sufficient to shew the coincidence of the approximate theory with matter of fact, so as to justify the conclusions which will be deduced from this proposition.

It appears therefore, that the velocities acquired by small spherical bodies which descend in fluids are sufficiently consistent with the theory, when examined by such trials as can be made on spheres, the magnitudes of which however small, are yet subject to accurate and decisive mensuration; we may therefore, without danger of error, extend this reasoning to bodies whose minuteness renders such real mensuration impossible. In all cases it follows from the theory, that homogeneous spheres of the least sensible

able magnitude will † descend in a given fluid with velocities to all observation uniform; which velocities are in a subduplicate ratio of their diameters, and the smaller these diameters are, the more near will the approximation be to the theory. The same laws of descent will also obtain in evanescent particles, the figures of which are nearly similar, although not spherical, but in various degrees; this too is confirmed by experiment, although in a less precise way than the preceding truths. For if a quantity of steel or brass filings, or other minute homogeneous bodies of irregular shapes, be caused to descend from rest in a fluid, you will always observe the largest bodies to precede the others in their descent: the velocities also with which these bodies descend, as in the former case, being uniform.

To apply these principles: It was observed that a sphere of lead, the diameter of which = .089 parts of an inch, descended from rest in water with a velocity of about 30 inches in a second. If therefore the diameter of this leaden sphere be diminished in the proportion of $n : 1$, the velocity

of descent * will be only the $\frac{1}{\sqrt{n}}$ part of the former velocity: * Prop. X. hujus.

thus if the diameter be diminished in the ratio of a million to one, that is, if the diameter = .000000089 parts of an inch, the velocity of descent will be = .03 parts of an inch in a second; and if the diameter be again diminished in the same proportion, that is, of 1000000 : 1, the velocity of the particles' descent will

be only .00003 in a second, or about $\frac{1}{10}$ of an inch

in an hour, allowing no additional retardation for the effects of tenacity or cohesion. To apply this reasoning, we observe that if the parts of some substances, having been separated by dissolving menstrua, are shaken together, so as to be diffused uniformly over the whole, they will continue for some time apparently quiescent, and at length will be observed to descend equably; and from what has preceded, with a greater or less velocity, according to their magnitude and density, in reference to that of medium. Thus should the diameters of the particles be equal to .0890 parts of an inch diminished in a duplicate ratio of 1000000 to 1, and their specific gravity 11.1 times greater than that of the fluid, the velocity of

their uniform descent will be that of about $\frac{1}{10}$ of an inch in an hour, provided the particles be spherical, or not greatly deviating from that figure. But the diminution of the parts of solid bodies dissolved in fluids is not limited to that just described, either by observation or any method of argument hitherto made use of. On the contrary, arguments, such as the nature of the subject admits, will induce us to conclude, that since matter considered as occupying space in a metaphysical sense, is divisible sine limite, so the actual division of material substance which is effected by natural causes is unlimited, as to our power of estimating it by observation; there is certainly no subdivision which can be expressed by numbers, but what we may reasonably suppose to be exceeded in the operations of nature. We may therefore justly imagine the magnitude of the particles of some substances existing in a state of solution, to be far less than that of the particles which descend in water with a velocity of .108 parts of an inch in an hour, and that in any assignable ratio: if the diameters are less than that above mentioned in the ratio of 1000000 : 1, the velocity of this descent will be diminished in the ratio of 1000 : 1, and will be therefore equal to that of $\frac{1}{10}$ part of an inch in 1000 hours, being diminished

in a subduplicate ratio of the particles' diameters; wherefore the particles would be above a year in descending through one inch in the fluid; and it is obvious, that the magnitude of the particles may be still diminished, so that the descent shall not be sensible in any assignable time. This conclusion will be true, without making any allowance for the effects of friction, cohesion, and tenacity, which exist in all fluids in a greater or less degree, although in water and mercury, termed perfect fluids, they are scarcely if at all sensible in experiment: but in other fluids, a resistance to the descent of bodies will be opposed by the tenacity, as well as the inertia of their parts.

It will follow upon the whole, that if the diameter of dissolved particles descending with a slow and uniform motion in fluids void of tenacity and friction be known, and their density with that of the fluid be given, the velocity of their descent may be ascertained a priori, and conversely, if their velocity of descent be observed, and their density with that of the fluid be known, the dia-

meter

meter of the particles may be deduced, provided they are spherical, or of other given figure, not deviating greatly from that of a sphere. Let v be the parts of an inch which these particles describe in their descent in one second, also let the density of the particles be to that of the fluid as $n : 1$, then will the diameter of the spheres =

$$\frac{3 v^2}{16 l \times n - 1} \text{ parts of an inch, } l \text{ being } = 193 \text{ inches.}$$

Thus, suppose particles of copper dissolved in spirit of nitre, were observed to subside one inch in .48 parts

of an hour, that is, $\frac{1}{1728}$ part of an inch in a second:

and supposing the particles to be spherical, let it be required to determine their diameter: here referring to the last

article, we have $v = \frac{1}{1728}$, $l = 193$, and if the specific

gravity of copper be to that of aqua fortis, as 9 : 1.3, we

shall have $n = \frac{9}{1.3} = 6.9231$, and the diameter of the

particles = $\frac{3}{1728^2 \times 16 \times 193 \times 5.9231} =$ less than

$\frac{1}{18205000000}$ part of an inch.

XI.

Let a spherical body descend in a fluid Fig. XXIX. from rest at A, having given the diameter of the sphere and its specific gravity relatively to that of the fluid; it is required to assign the time in which the sphere describes any space AO.

The notation of the eighth proposition remaining,

since the velocity $\dagger$ at any point O = $\sqrt{\frac{16 l d \times n - 1}{3}} \dagger$ VIII. huj.

×

Sec. III.
Prop. III.

× $\sqrt{1 - e^{-\frac{3x}{4nd}}}$, if T be the time of motion,* we shall

have $\dot{T} = \sqrt{\frac{3}{16ld \times n - 1}} \times \frac{\dot{x}}{\sqrt{1 - e^{-\frac{3x}{4nd}}}}$, and

the time of describing $AO = \sqrt{\frac{3}{16 \times ld \times n - 1}} \times$

fluent of $\frac{\dot{x}}{\sqrt{1 - e^{-\frac{3x}{4nd}}}}$: to obtain the fluent, let

$\sqrt{1 - e^{-\frac{3x}{4nd}}} = v$, wherefore $1 - e^{-\frac{3x}{4nd}} = v^2$,

and taking the fluxions $\frac{3\dot{x}}{4nd} \times e^{-\frac{3x}{4nd}} = 2v\dot{v}$, and

since $1 - v^2 = e^{-\frac{3x}{4nd}}$, multiplying one side by

$\frac{4nd}{3e^{-\frac{3x}{4nd}}}$, and the other by its equal $\frac{4nd}{3 \times 1 - v^2}$, we

have $\dot{x} = \frac{8v\dot{v}nd}{3 \times 1 - v^2}$, and $\frac{\dot{x}}{\sqrt{1 - e^{-\frac{3x}{4nd}}}} = \frac{8nd}{3}$

× $\frac{\dot{v}}{1 - v^2}$: the fluent of $\frac{\dot{v}}{1 - v^2}$ is $= \frac{1}{2} \times$ hyperbolic

logarithm of $\frac{1+v}{1-v}$: the fluent therefore of $\frac{\dot{x}}{1 - e^{-\frac{3x}{4nd}}}$

$=$ the fluent of $\frac{8nd}{3} \times \frac{\dot{v}}{1 - v^2} = \frac{8nd}{3 \times 2} \times \log. \frac{1+v}{1-v}$

$= \frac{4nd}{3} \times \log. \frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}}$, and the time re-

quired $= \sqrt{\frac{3}{16ld \times n - 1}} \times$ fluent of $\frac{\dot{x}}{\sqrt{1 - e^{-\frac{3x}{4nd}}}}$

$= \sqrt{\frac{3}{16ld \times n - 1}} \times \frac{4nd}{3} \times \log. \frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}}$

$=$

$$= \sqrt{\frac{n^2 d}{3l \times n - 1}} \times \log. \frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}} \text{ seconds.}$$

Cor. I. When n is increased fine limite the sphere's motion will not be retarded by the medium through which it descends, in this case $e^{-\frac{3x}{4nd}}$ approximates to $1 - \frac{3x}{4nd}$,

$$\text{and } \log. \frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}} = \log. \frac{1 + \sqrt{\frac{3x}{4nd}}}{1 - \sqrt{\frac{3x}{4nd}}} =$$

$2 \times \sqrt{\frac{3x}{4nd}}$: the time of describing any space x there-

fore will be $\sqrt{\frac{n^2 d}{3l \times n - 1}} \times \sqrt{\frac{3x}{nd}} = \sqrt{\frac{x}{l}}$ seconds,

which appears also from Sect. III. Prop. IV.

The times wherein spherical bodies descend from rest in fluids, determined by theory, admit of very exact experimental proof. In order to render trials of this sort capable of variety, a hollow brass sphere was constructed, which by means of an aperture could be so loaded with leaden shot, sand, &c. that its specific gravity might be altered in any assigned ratio; the aperture being closed, the whole figure was perfectly spherical: that the diameter might be ascertained, and the specific gravity be afterwards fixed for the experiment with greater certainty, the sphere was so adjusted by inclosed weights, that it would rest in water perfectly quiescent wherever it was placed: after this adjustment, the weight was found to be = 1093 grains, and since the specific gravity was the same as that of the water wherein it was immersed, referring to the preceding rule,† we shall have † Page 146.

the sphere's diameter = $1093^{\frac{1}{3}} \times .19612 = 2.0202$ inches: a weight of 273 grains being inclosed in the sphere when adjusted in the manner above described, the whole weight must now be 1366, and its specific gravity to that of water, as 1366 to 1093, that is, as 1.25 : 1; upon letting this sphere descend in water from rest, it was observed that the time of describing 60 inches was about 3 seconds.

To compare this experiment with the theory, we have the diameter of the sphere $d = 2.0202$ inches,

✱

$x = 60$, $n = 1.25$, and $\frac{3x}{4nd} = \frac{180}{10.101} = 17.82$, and

$$\frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}} = 219360000, \text{ the logarithm of}$$

which from the tables $= 8.3411574$, which being multiplied by 2.3015850 , will become 19.206 , the hyper-

bolic logarithm of $\frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}}$: we have also

$$\sqrt{\frac{n^2 d}{3l \times n - 1}} = \sqrt{\frac{1.25^2 \times 2.0202}{3 \times 193 \times .25}} = .14767, \text{ and}$$

the time of descent by theory $= \sqrt{\frac{n^2 d}{3l \times n - 1}} \times \log.$

$$\frac{1 + \sqrt{1 - e^{-\frac{3x}{4nd}}}}{1 - \sqrt{1 - e^{-\frac{3x}{4nd}}}} = 19.206 \times .14767 = 2.83 \text{ se-}$$

conds. The sphere descended upon trial in 3 seconds, which deviates from the theory $\frac{17}{100}$ parts of a second.

A few particulars relative to the method of making these experiments accurately may be inserted. The sphere should be very smooth, in order that no diminution of motion may arise from friction.

2. That the sphere may begin its descent at a definite instant of time, it should be suspended by a very fine silk line, so as to be just under the water's surface: if the line be divided with scissars at the beat of a pendulum clock, the beginning of the descent will be estimated without error.

3. The weight of the sphere should be so disposed, that the centre of gravity may be in the same vertical line with the centre of the sphere and the point from which the sphere is suspended, and should be somewhere near the lowest surface of the sphere when suspended.

The want of due attention to these particulars has occasioned the results of some experiments, which have been made on the descent of bodies in fluids, to deviate considerably from the theory, whereas the proper precaution in

con-

constructing the experiment being observed, the theory will be found to agree with matter of fact to a very satisfactory degree of precision.

By altering the quantity of weight inclosed within the sphere, the times of descent may be varied in any ratio. The ensuing table will shew the results of several experiments made by inclosing different weights within the sphere, after it had been adjusted, so as to remain in water quiescent wherever it was placed, in which case it has already been shewn, that the sphere's $\dagger$ weight = 1093 grains, and the $\dagger$ Page 151. diameter = 2.0202 inches.

No. I.

The weight inclosed was 11 grains, and consequently n or the sphere's specific gravity was $\frac{1104}{1093}$, that of water being = 1.

Experiment	Time of describing from rest 60 inches in water by experiment.	Mean time of descent.	Time by theory.	Error of experim.
1	$14 \frac{3}{4}$	14.6	13.94	+ .66
2	$14 \frac{1}{2}$			
3	$14 \frac{3}{4}$			

No. II.

The weight inclosed within the sphere was 3 grains, and consequently n , or the sphere's specific gravity in reference to water = $\frac{1096}{1093}$, that of water being = 1.

Experiment	Time of describing from rest by experiment 60 inches.	Mean of these observations.	Time by theory.	Error of experim.
1	$27 \frac{1}{2}$	27.3	26.66	+ .67
2	27			
3	$27 \frac{1}{2}$			

U

No.

No. III.

The weight inclosed in the sphere was 1 grain, and consequently the sphere's specific gravity = $\frac{1094}{1093}$, that of water being = 1.

Experiment	Time of describing 60 inches from rest by experiment.	Mean of these observations.	Time by theory.	Error of experim.
1	45			
2	$44 \frac{1}{4}$	44.33	46.21	- 1.88
3	$43 \frac{3}{4}$			

These experiments are coincident with the computed times to a sufficient degree of exactness, considering the extreme nicety requisite to construct them so as to be consistent with the conditions required by the theory: for it is almost impossible to prevent some small degree of oscillation in the sphere while it descends; it is also equally difficult to make the figure so perfect that the descent may be precisely in a vertical line. It is also to be noted, that when the inclosed weight, which may properly in this case be termed the moving force, is very small, as in the last set of experiments, it is very difficult to say that it

shall not vary $\frac{1}{40}$ or $\frac{1}{50}$ part of the whole from the truth:

for in this case, the unavoidable errors (however small) in the construction of weights may conspire with those of balancing the sphere in the water previous to the experiments. These causes are certainly sufficient to create some aberration of the time by experiment from that which is deduced from computation, no greater disagreement however is observed between the theory and matter of fact, than what should be rather attributed to those causes than to any imperfection in the theory.

* Cotes's
Hydrostatics, Lect. 4.

When a body is immersed in a fluid, it will either * ascend or descend according as its weight is less or greater than that of an equal bulk of the fluid: in either case the body may be considered as acted on by three forces, one
a whereof

whereof is its weight or natural gravity, which is diminished by the weight of an equal bulk of the fluid; these two forces always act in a direction opposite to each other, and are invariable in quantity, whatever be the velocity with which the sphere moves. To these must be added a third, i. e. that of resistance, which will depend both in direction and quantity on the sphere's motion; if the sphere ascends, the force of resistance conspires with that of gravity, but the sum of both forces is less than the weight of a quantity of fluid equal in magnitude to the solid, for which reason, the sphere will continue to ascend in the direction of the stronger impulse.

In the preceding propositions the spherical bodies have been supposed to descend in the fluid, and from the laws of this descent it has appeared, that bodies of the greatest known specific gravity,† if they be divided into sufficiently minute parts by the menstrua wherein they are dissolved, † Page 148. may remain suspended in them for any length of time. In this manner will aqua regia or even ether hold suspended the parts of that most ponderous of all metals gold: and in the same manner air will retain water suspended in it, when once subdivided into sufficiently minute parts; and various similar instances will naturally suggest themselves. But another difficulty attends these phenomena which should be here taken notice of.

Although the laws of resistance will sufficiently explain the reason why bodies when dissolved remain suspended in fluids specifically lighter than themselves, yet it does not follow from any thing that has preceded, how a body divided into parts however minute, can possibly ascend in a fluid specifically lighter than itself; whereas it is well known that in some solutions, when the solid is to be dissolved is placed at the bottom of a vessel into which the dissolving fluid is poured, the parts of the solid during the solution, without any motion whatever being communicated to the vessel, will be diffused throughout the substance of the dissolving fluid, appearing at first sight to overcome the natural tendency of all bodies toward the earth's centre, and to have some new power of ascent impressed upon it in consequence of the disunion of its parts. How far this is in reality the case may probably appear from the subsequent propositions and arguments deduced from them.

XII.

If a spherical body of a given diameter be immersed in a fluid, and if the sphere's weight bears no sensible proportion to the weight of an equal bulk of the fluid, the velocity of its ascent will be uniform from the first instant of its motion, and will be the same as that which a heavy body acquires in falling from rest by the acceleration of gravity through $\frac{4}{3}$ of its diameter.

Fig. XXX. Let the ascent of the sphere begin from quiescence at A , and let $AO = x$: moreover, let x be the space through which a body must fall from rest by the acceleration of gravity, so as to acquire the velocity in O . Then will the re-

† Sect. V. distance † to the sphere's motion at $O =$ the weight $\frac{p d^2 x}{8}$,
Prop. IV.

d being the sphere's diameter, and $p = 3.14159$, &c. the sphere's weight will also resist the ascent, and if n be a very small fraction representing the specific gravity of the sphere, its weight will be expressed by $\frac{d^3 p n}{6}$, wherefore $\frac{p d^3 n}{6} +$

$\frac{p d^2 x}{8}$ will be the whole force of resistance opposed to the

sphere's ascent: but the force whereby the sphere endeavours to ascend is the weight of a quantity of the fluid

equal in bulk to the sphere, which weight $= \frac{p d^3}{6}$, because

the specific gravity of the fluid is $= 1$; wherefore the force which upon the whole impels the sphere in its as-

cent $= \frac{p d^3}{6} - \frac{p d^3 n}{6} - \frac{p d^2 x}{8}$, and this being divided

* Sect. I. by the * sphere's weight $\frac{p d^3 n}{6}$, will give the force where-
Prop. IX.

by

by the sphere is accelerated at O , which will therefore be

$\frac{1}{n} - 1 - \frac{3z}{4nd}$; or since 1 vanishes in respect of $\frac{1}{n} -$

$\frac{3z}{4nd}$, the force of acceleration will be $\frac{1}{n} - \frac{3z}{4nd}$, and

from the principles before || demonstrated, we have $\frac{1}{n} -$ || Sect. III.
p. 28.

$\frac{3z}{4nd} \times \dot{x} = \dot{z}$, or $\frac{4d - 3z}{4nd} \times \dot{x} = \dot{z}$, and $\frac{\dot{x}}{4nd} =$

$\frac{\dot{z}}{4d - 3z}$, wherefore $\frac{x}{4nd} = -\frac{1}{3} \log. 4d - 3z$ be-

ing the variable part of the fluent, to which if the con-

stant part $+\frac{1}{3} \log. 4d$ be added, we shall have $\frac{x}{4nd}$

$= -\frac{1}{3} \times \log. \frac{4d - 3z}{4d}$. Let $e = 2.71828$, being

the number the hyperbolic logarithm of which is equal to

1 ; this will give $\frac{4d - 3z}{4d} = e^{-\frac{3x}{4nd}} = 0$, (because $\frac{3x}{4nd}$

is infinite,) and consequently $4d - 3z = 0$, or $z = \frac{4d}{3}$,

and if $l = 193$ inches, the * velocity at $O = \sqrt{\frac{16ld}{3}}$ * Sect. III.
Prop. V.

inches in a second, which is the same velocity which

would be acquired by a heavy body which falls by the

acceleration of gravity from rest through a space equal to

$\frac{4}{3}$ of the sphere's diameter. Since this reasoning is ap-

plicable without the least alteration to any other point Q ,

in which case the velocity at Q will come out the same as

before $= \sqrt{\frac{16ld}{3}}$ inches in a second, it follows, that

the velocity of the sphere's ascent is entirely uniform.

Cor. 1. A globe without weight ascends in fluids of any

different densities with the same uniform velocity, pro-

vided the diameter of the globe be unaltered.

Cor. 2. Globes without weight of different diameters

ascend in fluids with uniform velocities, which are in a

direct subduplicate ratio of their diameters.

Cor.

Cor. 3. The conclusions deduced from this proposition, follow also from Prop. VIII. where the velocity of the

sphere's descent is shewn to be $V = \sqrt{\frac{16ld \times n - 1}{3}}$

$\times \sqrt{1 - e^{-\frac{3x}{4nd}}}$; n being assumed greater than 1. If n be assumed less than 1, so that the sphere may ascend in the fluid, the investigation will be in every step the same as before, except as far as regards the quantity $n - 1$, which will become $1 - n$; so that the velocity of the sphere's ascent will be defined by the equation $V =$

$\sqrt{\frac{16ld \times 1 - n}{3}} \times \sqrt{1 - e^{-\frac{3x}{4nd}}}$; and if $n = 0$,

as in the proposition just demonstrated, $e^{-\frac{3x}{4nd}} = 0$, and

$V = \sqrt{\frac{16ld}{3}}$ as before.

Cor. 4. If the sphere should not be absolutely without weight when compared with the weight of an equal bulk of the fluid wherein it is immersed, yet the preceding principles will be sufficiently true for any practical application of them to matter of fact, provided the specific gravity of the spheres be not in a greater proportion to that of the fluid, than of about 1 to 100: for in this case, the error of the rules for determining the velocity of ascent

will not exceed $\frac{1}{200}$ part of the true velocity, which would

not be sensible in observation: and in general, if the sphere's specific gravity be a small fraction n , that of the fluid wherein it is immersed being $= 1$, the error of the preceding rule arising from the sphere's weight, will be that part of the true velocity which is expressed by the fraction n , which appears from inspecting the rule in Cor. 3.

These conclusions seem to be related to several natural phenomena, the solution of which from mathematical principles requires some considerations to be premised.

Although it is certain that no material substance is destitute of gravity, yet it is equally true that there are fluids in nature of so rare and subtle a texture, that their weight is insensible in regard to any discoverable effects produced by it in some natural phenomena.

Among

Among these phenomena must be reckoned the minute bubbles or spherical cavities, which ascend in fluids during fermentations, solutions, &c. These small cavities are filled with airs or elastic vapours of different kinds, some of which are specifically heavier than common air, and some far lighter than that fluid. Others again consist of a species of vapour so rare that its weight is scarcely sensible: but whatever be the quality of these vapours, their weight affects not perceptibly the velocity of the ascending spherical cavities containing it, which appears from Cor. 4. *huj.* according to which, if the specific gravity of the included vapour were only 600 times less than that of the fluid wherein the spherules ascend, the variation of the velocity occasioned by the weight, will be no more than

$\frac{1}{1200}$ part of the whole, which is intirely insensible in experiment.

That the contents of all these aerial spherules are elastic is manifest from their cavities admitting of dilatation and compression. For it is evident to the senses, that heat or a diminution of external pressure, will cause them to expand into larger portions of space, whereas the application of cold or an additional pressure will create a diminution of their bulk: and if the temperature of the fluid wherein they ascend and the external pressure remain the same, their diameters will be unaltered.

The laws according to which a spherical body without weight ascends in any fluid, demonstrated in Prop. XII. *huj.* are applicable to the ascent of the minute air bubbles above mentioned, although the figure of a spherical bubble will be changed during its ascent in a fluid if the diameter be considerable; yet when the spheres containing these vapours are very small, no such alteration in their figure takes place, which may be known, not only from their apparent perfectly globular form, but from their equable and vertical ascent: whereas when the diameter is so great that the figure suffers alteration, the sphere's motion is neither equable nor in a vertical line, but in various irregular directions, tending however upon the whole constantly upward. It must also be added, that although these spherules being less compressed as they approach the surface of the fluid, the elasticity of their contents will necessarily cause an increase of their diameters, yet this increase will be so small in most cases, that the velocities as determined by Prop. XII. will not be sensibly affected thereby.

To

Fig. XXX. To illustrate this, let FCE represent the surface of a fluid of the same density with water, B a small aerial sphere ascending in the vertical line BC ; produce BC to A , so that CA may be equal to 34 feet: now it is manifest, that the pressure upon the sphere at B will be as the space BA , and consequently if the depth BC should be one foot, the different pressures at B and C will be as $BA : CA$, or as 34 : 33; and the diameter of the sphere at B will be to its diameter at C in an $\dagger$ inverse subtriplicate ratio of 34 : 33, or as 101 : 102 nearly: which proportion of inequality is so small, that the diameter of the sphere may justly be regarded as constant during the whole of its passage from B to C . This being the case, the velocity of the sphere's ascent may be estimated from having given its diameter, by referring to the preceding $\dagger$ Prop. in which it is shewn that the velocity of the ascending spherule is intirely uniform, and equal to that which a heavy body acquires in descending from rest by the acceleration of gravity through

$\dagger$ Vid. infra
P. 163.

$\dagger$ Sect. V.
Prop. XII.

a space equal to $\frac{4}{3}$ of the sphere's diameter. So that if d = the sphere's diameter, v the velocity of its ascent, l = 193 inches, it has been shewn, that $v = \sqrt{\frac{16 dl}{3}}$,

and consequently $d = \frac{3v^2}{16l}$. To apply this, suppose it were observed that a small air bubble ascended in water at the rate of three inches in a second, and let it be required to determine its diameter; here $v = 3$, and by the rule

$$d = \frac{3v^2}{16l} = \frac{3 \times 9}{16 \times 193} = \frac{1}{114} \text{ part of an inch.}$$

This solution is sufficiently exact for determining the ascent of aerial spherules in water and in fluids of similar density, (as long as the atmosphere presses upon their surface,) through a small space. The proportion of BA to CA is increased either by lessening or removing the pressure of the atmosphere, by increasing the space BC through which the sphere ascends, or by causing the sphere to ascend in a fluid of great density compared with that of water, such as mercury: for in this latter case, the height AC = 34 feet must be diminished in the proportion of 14 : 1, so as to become 2.44 feet: let this = CD , if therefore the spherule ascends through BC = 1 foot, the pressures at B and C would be as 3.44 : 2.44, or as $BD : CD$, and the diameters at B and C in a subtriplicate ratio of 2.44 : 3.44; which

Fig. XXX.

which is too great a proportion of inequality to admit of the diameters being regarded as constant. In these cases recourse must be had to a subsequent proposition in order to determine the velocities and times of ascent.

It is next to be considered in what manner these aerial bubbles act during fermentations, solutions, &c.

XIII.

When a solid body is immersed in a fluid which dissolves it, the parts of the dissolved substance although of greater specific gravity than the menstruum, are yet elevated so as to be diffused in some degree over the whole mass, and this elevation of the particles is caused by the globules of vapour which escape from the solid during its solution.

The discoveries of Dr. Black, Sir J. Pringle, Dr. Macbride, &c. relative to certain airs or vapours existing within the substance of bodies in a fixed or latent state, seem to make it evident that the power whereby the parts of bodies cohere, arises from the action of these vapours, or as they have been denominated, aerial fluids.

On this account fixed † air has been termed the cementing principle, and although other causes may possibly concur in the production of the solidity of bodies' textures, yet it follows indubitably from the admirable experiments of the gentlemen just mentioned, that when the fixed air which naturally exists within the pores of a substance is abstracted from it, the cohesion of its parts is at the same time destroyed; and the principle is still rendered more obvious from restoring the cementing power or fixed air, whereby in many cases a new cohesion of parts takes place, causing the separated particles of the body to reunite into a solid mass. Thus, if animal flesh be im-

† Macbride's
Essays.

mersed under water, after some time small bubbles of air will be observed to rise in the water, a sign that fermentation has begun to dissolve the texture of the flesh: which by this process being deprived of its natural quantity of fixed air, loses its cohesive firmness of texture and becomes putrid: but if a sufficient quantity of fixed air be restored to it, the putrid flesh will recover its solidity of texture, and become as sweet to the smell as before. What more particularly belongs to the subject of the proposition, is an appearance always observed during the fermentative process just mentioned; as soon as the air bubbles begin to arise, small particles of the flesh are likewise detached from the putrescent substance, and ascend to the top of the water; and this ascent is caused by the bubbles of fixed air which ascend during the fermentation. Each of these bubbles would if unimpeded rise with a velocity which a heavy body acquires in descending from rest by the acce-

* Sect. V. *Acceleration of gravity* * through a space equal to $\frac{4}{3}$ of the
Prop. XII.

sphere's diameter: but if any solid substance be attached to it, the sphere will nevertheless ascend, although more slowly, provided the weight of the attached particle exceeds not the weight of a quantity of the fluid equal in bulk to itself and the sphere taken together; and this will be true whatever be the specific gravity of the particle which is elevated by the ascending bubble of air.

In like manner during solutions of saline, metalline, or other bodies, vapours of various kinds are observed to arise in small bubbles, and to escape from the surface of the dissolving fluid; and it is not improbable, that however these vapours may differ in possessing specific properties peculiar to each, yet that of causing or contributing to the coherence of the parts of bodies may be a property common to them all: however this may be, it seems manifest, that the numerous aerial spherules which ascend from any substance immersed in a dissolving fluid, carry up with them the disunited parts of the solid, and in greater or less quantity, according to the magnitude of the parts and their density in reference to that of the dissolving medium. An instance of this kind may be easily observed by immersing a piece of sugar in a glass vessel of water, the air bubbles during their ascent visibly carrying up with them long streams of the dissolved sugar.

In what manner the disunited parts of bodies of the greatest specific gravity when once elevated in menstrua-

ums, are notwithstanding their density there held suspended, has been already described. †

† Page 148.

The diameters of the spherules of air which ascend during fermentations, solutions, &c. have been hitherto considered as invariable, which will be nearly true, provided they continue to suffer invariable compression, and the temperature of the fluid continues the same. Although it has been shewn, that during a sphere's † ascent through † any small space, the compression alters not very considerably, as long as the atmosphere continues to press on the surface of the fluid, yet when that pressure is either removed or much diminished; or if the fluid should be exposed to various degrees of heat, the diameters of the ascending bubbles will vary in so great a degree that the velocity of ascent will no longer be deducible from the rules already demonstrated. In order to investigate the velocity with which a spherule of air or vapour ascends, allowing for the variation of the diameter, the effects of heat are not considered, it being impossible to reduce them to geometrical estimation. The temperature of the fluid therefore in which the aerial spherules arise being the same, the only alteration in their magnitudes and diameters will be occasioned by the variable pressure acting upon them at different distances from the surface of the fluid: this alteration of the diameter will depend on the law, which the spaces occupied by the elastic vapour observe in reference to the respective pressures: and it is * found by some experiments which have been made on the subject (although not yet sufficiently extensive to entirely establish the proposition, but which must at present be allowed as true in order to proceed with the argument) that the elastic forces exerted by a given quantity of this vapour, will be greater in the same proportion as the spaces occupied by it are less: and it is evident that the elastic force must be equal to that of compression, whenever the magnitude of the space occupied remains the same. This being the case as long as an air bubble remains at the same depth, the diameter will not be altered; but as the spherule approaches the surface of the fluid, the pressure will be diminished, and consequently the capacity increased in the same proportion: that is, the sphere's magnitude will be in an inverse proportion of the distance of its centre from the surface of the fluid (the pressure of the atmosphere not being at present considered), and consequently the sphere's diameter will be in an inverse subtriplicate ratio of the same distance. If therefore at any given

* Robins' Gunnery.

X 2

distance

distance from the surface a , the diameter is b , let d be the diameter at any other distance x , then by the rule we have

$$b : d :: x^{\frac{1}{3}} : a^{\frac{1}{3}}, \text{ or } d = \frac{b a^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

It has already appeared, that the increase of the sphere's diameter will cause a continual acceleration of the ascent, and this being the real manner in which aerial spheres ascend in fluids, should next be considered.

XIV.

Fig. XXXI. Let FCE represent the surface of any fluid, B a spherule of air or elastic vapour beginning to ascend in it, in the vertical line BC: produce BC to A, so that CA may be the altitude of a column of the fluid, the weight of which is equal to that of the atmosphere pressing against a given surface; then the velocities of the ascending sphere at any given points O, Q, will be in an inverse ratio of the sixth root of the distances OA, QA, when the sphere's specific gravity is diminished sine limite.

Let the sphere's diameter when at B = b , and suppose the sphere to have ascended to the point O; and let ‡ Supra. AO = x , AB = a , then will the diameter of the ‡ sphere at

$$O = \frac{b \times a^{\frac{1}{3}}}{x^{\frac{1}{3}}}; \text{ also the quantity of contained air or va-}$$

pour being invariable, the sphere's specific gravity will be inversely as its magnitude; wherefore if the specific gravity at B be = m , at O it will become = $m \times \frac{x}{a}$.

Let

Let z be the space through which a body must fall from rest by the acceleration of gravity, so that it may acquire the velocity of the sphere in O ; and since the force which accelerates an ascending sphere, the diameter of which $= d$, and its specific gravity $= n$, is

$\frac{1}{n} - 1 = \frac{3z}{4nd}$, substituting $\parallel \frac{ba^{\frac{1}{3}}}{x^{\frac{1}{3}}}$ for d , and $\frac{mx}{a}$ for n , || Prop. XIII. ad finem.
the $\dagger$ force which accelerates the sphere at O will be † Sect. V. Prop. XII.

$\frac{a}{mx} - 1 = \frac{3za^{\frac{2}{3}}}{4bmx^{\frac{2}{3}}}$; and since the fluxion of the

space $= -\dot{x}$, we have $\ast \frac{a}{xm} - 1 = \frac{3za^{\frac{2}{3}}}{4bmx^{\frac{2}{3}}} \times \dot{x} = -\dot{z}$, * Sect. III. Prop. V. Cor. 4. & p. 56.

and by reduction $\frac{4ba\dot{x}}{x^{\frac{1}{3}}} - 4mbx^{\frac{2}{3}}\dot{x} - 3za^{\frac{2}{3}}\dot{x} = -$

$4mbx^{\frac{2}{3}}\dot{z}$, or because m is supposed to be diminished fine limite by the problem $\frac{4ba}{x^{\frac{1}{3}}} = 3za^{\frac{2}{3}}$, and $z =$

$\frac{4ba^{\frac{1}{3}}}{3x^{\frac{1}{3}}}$; and if $l = 193$ inches, the velocity of the sphere

at $O = \sqrt{\frac{16bla^{\frac{1}{3}}}{3x^{\frac{1}{3}}}} = \frac{1}{x^{\frac{1}{6}}} \times \sqrt{\frac{16bla^{\frac{1}{3}}}{3}}$; and $\sqrt{\frac{16bla^{\frac{1}{3}}}{3}}$

being constant, it follows, that the velocity will be inversely as the sixth root of x , or of the sphere's distance from the point A .

When the specific gravity of the spherule is evanescent, and the fluid wherein it ascends is not pressed by the atmospheres, the velocities will be in an inverse ratio of the sixth root of the distances from the fluid's surface: for in this case, the point A will coincide with the fluid's surface at c . Fig. XXXI.

If it be required to assign the velocity at any given point O , supposing the specific gravity of the sphere too considerable to be neglected, the problem will be more difficult;

difficult: for in this case the relation of z and x must be

|| Supra, assigned from the || equation $\frac{a \dot{x}}{m x} - \dot{x} - \frac{3 z a^{\frac{2}{3}} \dot{x}}{4 m x^{\frac{2}{3}} b} = - \dot{z}$,

or $\dot{x} - \frac{a \dot{x}}{m x} = \dot{z} - \frac{3 z a^{\frac{2}{3}} \dot{x}}{4 m x^{\frac{2}{3}} b}$. In order to separate the

+ Vid.
Waring's
Meditat.
Analyt.
p. 123.

unknown quantities, let y be assumed of such a magnitude that the fluent of $\dot{z} y - \frac{3 z a^{\frac{2}{3}} \dot{x} y}{4 m x^{\frac{2}{3}} b}$ shall be $z y$, and let the

whole equation be multiplied into y , the value of which is to be afterwards determined: this will give $\dot{x} y - \frac{a y \dot{x}}{m x}$

$= \dot{z} y - \frac{3 z y a^{\frac{2}{3}} \dot{x}}{4 m x^{\frac{2}{3}} b} = \dot{z} y + \dot{z} y$ (because the fluent of $\dot{z} y$

$- \frac{3 z a^{\frac{2}{3}} \dot{x} y}{4 m x^{\frac{2}{3}} b}$ is by the supposition $= z y$). Subtracting $\dot{z} y$

from each side, and dividing by z , we have $\dot{y} = - \frac{3 y a^{\frac{2}{3}} \dot{x}}{4 m x^{\frac{2}{3}} b}$ and $\frac{\dot{y}}{y} = - \frac{3 a^{\frac{2}{3}} \dot{x}}{4 m x^{\frac{2}{3}} b}$, or $\log. y = - \frac{9 a^{\frac{2}{3}} x^{\frac{1}{3}}}{4 m b}$;

and if $e = 2.7182, \&c.$ it will follow, that $y = e^{-\frac{9 a^{\frac{2}{3}} x^{\frac{1}{3}}}{4 m b}}$

and consequently $\dot{x} y - \frac{a \dot{x} y}{m x} = \dot{x} e^{-\frac{9 a^{\frac{2}{3}} x^{\frac{1}{3}}}{4 m b}} \times 1 - \frac{a}{m x}$
 $=$ the fluxion of $z y$; it appears therefore that the quan-

tity sought or $z = \frac{\text{the fluent of } \dot{x} e^{-\frac{9 a^{\frac{2}{3}} x^{\frac{1}{3}}}{4 m b}} \times 1 - \frac{a}{m x}}{e^{-\frac{9 a^{\frac{2}{3}} x^{\frac{1}{3}}}{4 m b}}}$

XV.

Every thing remaining as in the last proposition, let it be required to determine the time in which the sphere of air or vapour, beginning its ascent from B, describes any given space B O.

Since the fluxion of the space $= -\dot{x}$, and the sphere's velocity * at O $= \sqrt{\frac{16 l b a^{\frac{1}{3}}}{3 x^{\frac{1}{3}}}}$, it follows, if the time re- * Prop. XIV.

quired be put $= T$, that $\dagger \dot{T} = \frac{-\dot{x} \sqrt{3 x^{\frac{1}{3}}}}{\sqrt{16 l b a^{\frac{1}{3}}}}$, and $T = \dagger$ Sect. III. Prop. III.

$$= -\frac{\sqrt{3} \times 6}{7} \times \frac{x^{\frac{7}{6}}}{\sqrt{16 l b a^{\frac{1}{3}}}} = -\sqrt{\frac{108 x^{\frac{7}{3}}}{49 \times 16 l b a^{\frac{1}{3}}}}$$

$$= -\sqrt{\frac{27 x^{\frac{7}{3}}}{196 l b a^{\frac{1}{3}}}}; \text{ but when } x = a, \text{ by the problem}$$

$T = 0$, the entire fluent will therefore be $T = \sqrt{\frac{27 a^2}{196 l b}}$

$$= \sqrt{\frac{27 x^{\frac{7}{3}}}{196 l b a^{\frac{1}{3}}}} \text{ seconds.}$$

Cor. 1. When $x = 0$, the sphere will have ascended through the whole space to the point A, and in this Fig. XXXI.

case, the whole time of ascent $= \sqrt{\frac{27 a^2}{196 l b}}$. Thus

suppose a vessel of 12 inches in height to be filled with water, and let an air bubble of $\frac{1}{100}$ part of an inch in diameter begin its ascent from the bottom of the vessel. Here

Here (the pressure of the atmosphere upon the water's surface being removed by an air pump,) to find the time of the sphere's ascent, we have $a = 12$, $b = \frac{1}{100}$, $l = 193$,

$$\text{and the time required} = \sqrt{\frac{27 \times 144}{196 \times 193 \times \frac{1}{100}}} = 3.2 \text{ seconds.}$$

Cor. 2. If two equal spherules of air begin to ascend in a fluid of a given specific gravity from rest at different depths, the times of ascent to the surface will be in a direct ratio of those depths or of the spaces described by the spheres.

Cor. 3. If the sphere's diameters at the first instant of motion should be unequal, the times will be in a direct ratio of the depths, and an inverse subduplicate ratio of the diameters.



* Sect. V.
Prop. XII.

It appeared from the preceding propositions, that in fluids of constant temperature, a spherical body of evanescent weight would ascend uniformly; and that the velocity of * ascent would be equal to that which a heavy body acquires in falling from rest by the acceleration of gravity

through $\frac{4}{3}$ of the sphere's diameter, provided the diameter

suffered no sensible alteration either from a variable pressure or temperature; but when a vessel of any liquor is placed under the receiver of an air pump, and the air either partially or almost wholly exhausted, the pressures on the air bubbles will vary considerably. Suppose BC to represent the depth from the surface through which the spherules ascend from rest; and let CA be the height of a column of the fluid equal to the pressure of the atmosphere: if that portion of the air be exhausted from the receiver, which is expressed by the

fraction $\frac{DA}{AC}$; the force of pressure at B will be to the force

at C , as $BD : CD$; which pressures may vary in any ratio of inequality, according to the quantity of air remaining in the receiver: and this appears to be the reason why the air bubbles which arise in water, &c. when placed in vacuo, increase to so great a magnitude before they arrive at the surface; whereas if the pressure of the atmosphere is not removed from the fluid's surface, the diameters of the spherules alter not very much in ascending through a small space. The rapid acceleration which is also observed in aerial spherules which ascend in fluids is the

included in the exhausted receiver of an air pump, is the immediate consequence of their increasing diameters; for it has been shewn in this case, that the velocity of ascent on account of their variable † diameter is in an inverse ratio of the sixth root of the sphere's distance from the surface, and consequently when it arrives at the surface, the velocity would be greater than any that could be assigned, were it not for the remaining part of the air, which can never be wholly exhausted from the receiver, and the change of the spherical figure caused by the increased magnitude of the cavity occupied by the elastic vapour.

† Sect. V.
Prop. XIV.

There is also another cause which prevents the acceleration from being increased beyond a certain limit: it may be mentioned in this place: the propositions relating to the ascent of aerial spherules are demonstrated on a supposition, that the force whereby a body endeavours to ascend in a fluid is equal to the difference between the weight of the solid and of an equal bulk of the fluid, which is true (even in a physical sense) only while the velocity of ascent is inconsiderable: this is the case when air bubbles ascend through small spaces in fluids exposed to the pressure of the atmosphere, but when the velocity is very much increased, by removing the compressure of the atmosphere, so as to bear a considerable proportion to that with which the fluid which impels the ascending sphere would rush into empty space, if no ways impeded, the force of ascent will be no longer equal to that assumed in the demonstrations, which is always greater than the true force. This might be allowed for if necessary to the explication of any phenomena relating to this subject: but the demonstrations which have been given are sufficiently exact for any application to practice, where the nature of the case at all admits of reference to theory.

Since when the atmosphere presses on the surfaces of fluids, the aerial spherules ascend in them with velocities which are in a direct * subduplicate ratio of their diameters, it follows, that these diameters may be so minute that their ascent shall be almost wholly impeded; that is, it shall be so slow, that a given space shall not be described by them in any assignable time however great: but if by any means their diameters are increased, the spherules of air will in consequence of this change not only become visible to the eye, but ascend with velocities increased in a subduplicate proportion of their diameters, if the pressure of the fluid remains constant, but if the diameter be increased during the ascent on account of the continual diminution of pressure,

* Sect. V.
Prop. XII.

sure, the velocity will be accelerated. Thus a vessel of water which shews no appearance of these ascending spherules in open air, if it be included under the receiver of an air pump, and the air exhausted, the spherules will be seen to arise in abundance and with accelerated velocities for a considerable length of time. For the pressure being removed, the spherules will of course be expanded by the elastic force of their contents. This process extracts from the water the air contained in it.

Also if common water in which no spherules apparently ascend, be exposed to heat, the air bubbles will soon begin to arise in it, their diameters being enlarged, and consequently their velocities augmented in a subduplicate proportion: for heat has the property of expanding all substances, especially elastic fluids. When the degree of heat is increased so as to greatly enlarge the magnitude of the ascending bubbles, and consequently the rapid velocity of their ascent, the water assumes the appearance of boiling; having acquired the greatest heat which it is possible to communicate to it. This boiling appearance depends on the magnitude of the ascending cavities of elastic vapour; but it has * been shewn, that a diminution of the pressure on a fluid's surface, will increase the magnitude of the ascending spheres of vapour as well as the application of fire. If therefore a portion of the air's pressure be removed from the surface of a fluid, a less degree of heat will cause the appearance of boiling; it is well known, that water will boil with a less degree of heat when the barometer is low, than when it indicates a greater pressure of the atmosphere, and this decrease of heat corresponding to the diminution of the atmosphere's pressure and the heat of a boiling fluid admits of considerable latitude: thus if water moderately warm be placed under the receiver of an air pump, and the air be sufficiently exhausted, the water will be observed to boil. In the same manner newly fermented liquors which are replete with a species of vapour called fixed air, although when viewed in open air they discover none of the aerial bubbles or spherules of fixed air ascending in them, yet when such liquors are included under the receiver of an air pump and the atmosphere removed, the air contained in the liquors will be observed to rise in large and numerous bubbles, especially if a degree of heat be previously applied; but it should be observed, that the tenacity which this sort of fluids generally possess, prevents the air bubbles which arise from being immediately broken and

* Sect. V.
Prop. XIV.

and mixed with the remaining air, instead of which they are collected into a froth, and sometimes rise to a considerable height above the surface of the fluid.

It may be inferred from these principles, that air when intimately combined with the substance of water and other fluids becomes divided into parts so extremely minute as to escape the observation of the senses, until the aerial particles are expanded into larger portions of space, either by removing the pressure of the atmosphere; or by heat: and it appears that this minuteness of the particles, although of far less specific gravity than the fluid, detains them in a quiescent state till their diameters are increased by one of the causes just mentioned.

It may also be added, that exclusive of the atmospheric air contained in the substance of water, &c. some subtle ethereal elastic medium subsists also, of no sensible weight but capable of intumescence by heat: for after all the atmospheric air has been extracted from water, on the continued application of heat, the ascent of bubbles will still be observed.

|| Hawk-
bee's Physi-
co Mechan,
Exp.

Concerning the application of the preceding propositions to matter of fact, it must be remembered, that one of the conditions on which they are demonstrated is, that the elastic force of the vapour contained in the cavities of the aerial spherules observes the same law in regard to the spaces occupied by it as atmospheric air does; that is, the force wherewith the vapour endeavours to expand itself is supposed to be greater in the same proportion as the space occupied by it is less. But although Mr. Robins* has demonstrated that this law obtains in the fixed air or vapour of gunpowder, it seems probable, that the fixed air which rises from some fermented liquors, when the pressure on their surface is suddenly removed expands itself in a higher ratio than is consistent with the preceding hypothesis. This circumstance therefore must be determined by accurate experiments: in the mean time it must be noted, that whatever law of expansive force be observed in respect of the spaces occupied, the ascent of spherules filled with these aerial fluids will not be affected, being the same as that which is demonstrated in prop. xii. huj. provided the compression of the fluid wherein they ascend be invariably the same in all parts of their ascent, and this compression be always equal to the elastic force of the medium contained in the spherules.

* Robins'
Gunnery,
Prop. III.

XVI.

Vapour or steam consists of hollow spherules of the fluid from which they arise, and their ascent is caused partly by the air's gravitation, and partly by the impulse of some power which acts in a direction contrary to that of the earth's gravity.

It is a known fact, that fluids exposed to certain degrees of heat, varying with the density, tenacity, &c. of the fluids, are converted into a rare and elastic substance, called vapour or steam; one peculiar property of which is, that if no ways impeded it always ascends in the atmosphere in a direction perpendicular to the earth's surface.

Concerning the cause of this and the other phenomena observable in the ascent of vapour philosophers have adopted various opinions; none of which however seem to carry that perfect and satisfactory conviction to the mind, which the solution of natural phenomena from mechanical principles is so remarkable for. The crude and hypothetical notions which prevailed concerning this subject in the earlier ages of philosophy, would naturally be exploded by men who to a diligent observation of nature joined those helps which were to be derived from geometry. The opinion of Dr. Newentyt was, that the particles of fire adhering to those of water, constituted what he calls *moleculæ*, which were specifically lighter than the air, and consequently ascended by the same power which causes cork to rise in water or iron in mercury. This solution was not satisfactory to Dr. Hally, who substituted in its place another more consistent with the received principles of hydrostatics. His idea was, that the aqueous particles were by the power of heat formed into hollow spherules, which he could shew, might be specifically lighter than air in any assigned ratio. These therefore must rise perpendicularly upwards until they arrive at those regions of the atmosphere, the rareness of which is the same with that of the vaporous spherules: in this state they remain until compressed by cold or other cause they become specifically

cifically heavier than the surrounding air, and consequently descend in the form of rain, dews, &c.

Although this account carries with it great appearance of probability, yet it has met with several strenuous opponents, among whom is Mr. Clarke in his treatise on the motion of fluids; Mr. Rowning also uses arguments against it, and after them Dr. Hamilton in a tract writ expressly on the subject, undertakes to shew, that Dr. Hally's hypothesis is insufficient to account for the phenomena of evaporation: his chief argument against Dr. Halley is this; if the ascent of vapour were caused by water being converted by heat into an elastic steam consisting of hollow spherules, it would follow, that the quantity of steam raised from a given surface depends on the quantity of heat: whereas he shews, that evaporation is carried on even while the air is at the temperature of freezing, and that the quantity of vapour raised, depends more on the quantity of fresh air passing over a given surface of the fluid, than the temperature of the atmosphere.

Not to enter into any consideration, whether it be certain that the ascent of vapour from warm liquor be analogous to the evaporation of fluids exposed to the action of current air, or whether each phenomenon be inconsistent with the hypothesis, that the particles which compose vapour are hollow spherules, it may be sufficient for the present purpose, to apply the propositions which have been demonstrated concerning the ascent of small spherical bodies in fluids of greater specific gravity, to the illustration and examination of Dr. Hally's theory.

The density of steam is various according to its different degrees of heat. The steam which is raised from boiling water is 14000 times rarer than water, and since water is at a medium about 850 times rarer than air, it will follow, that aqueous vapour heated to the temperature of boiling water, is about $16\frac{1}{2}$ times lighter than air; but this will not give us the rareness of an individual particle or hollow spherule: in order to obtain the specific gravity of which it must be considered, that if innumerable equal spheres be placed contiguous according to fig. xxxii. the empty space, or interstices between the spheres,

will be something less than $\frac{1}{2}$ of the whole space, the proportion being as 47.6 : 100 nearly. To explain this position of the spheres more fully, as the figure of itself may be insufficient, it is only necessary to suppose any given space to be filled with hollow cubes equal to each other and contiguous: if a sphere, of which the diameter is equal

to the side of one of these cubes, be placed within each cube; as one sphere is to its circumscribing cube, so will all the spheres be to all the cubes; and this proportion is that of 3.14159, &c. to 6, wherefore the hollow cubes being removed, the interstices must be to the whole space as 2.85841 : 6, or as 47.6 : 100 nearly: this proportion being the same whatever be the magnitude of the equal spheres: but if the spheres are so placed that the greatest number possible may be included in a given space; so that a plane passing through the centres *I, K, H, L*, &c. shall be distant from a plane passing through the centres *A, B, C, D* by a space equal to the diameter of one of the spheres multiplied into $\sqrt{\frac{1}{2}}$ according to fig. xxxiii, or

into $\sqrt{\frac{2}{3}}$ according to the figure adjacent; in either case

the proportion of the interstices to the whole space will be something more than that of 1 : 4, being in the ultimate proportion of $\sqrt{18} - p$ to $\sqrt{18}$, when the spheres' diameters are diminished sine limite, *p* being = 3.14159, &c. Now we cannot suppose that the aqueous hollow spherules are exactly disposed according to fig. xxxii. so on the other hand, there is no sufficient reason to imagine them ranged as close as possible, as in fig. xxxiii, or in the fig. adjacent, in which the interstices bear the same proportion to the whole ultimately: here then, if the knowledge of the exact manner in which the hollow spherules are disposed were necessary in the present case, we could not proceed for want of data. But as the issue of this reasoning will not be affected † by any error that can probably be committed in assigning the specific gravity of the ascending spherules, we may assume it, merely to proceed with

† Vide
pag. 175.

the argument, as equal to $\frac{1}{10.3}$ part of the density of the atmosphere, which is about a mean specific gravity between the two extremes.

It may be objected, that a greater rareness is assigned to these spherules than they can possibly possess, if they be supposed to consist of shells of water filled with air rarefied by the heat of the subjacent fluid: because air suffers but a very small rarefaction when heated from the temperature of freezing to that of boiling water, a given bulk of the air when at the * temperature of freezing, being expanded by the heat of boiling water into a space greater than before in a proportion no greater than that of about 1.

* Philos.
Transf. for
1777.

to 1.414; and consequently the combination of air, the specific gravity of which is $\frac{1}{1.414}$ less, and water which is

850 times greater than that of common atmospheric air, can never form a spherule the specific gravity of which shall be that which is just mentioned, i. e. $= \frac{1}{10.3}$. But it is

evident from other experiments, that the cavities of the vaporous spherules are not occupied by air, because when steam is condensed by the application of cold, so as to be reduced to water, no air is produced from these cavities. The fluid which occupies the spherules of vapour must be therefore supposed to consist of some elastic medium incomparably more rare and subtle than air: and consequently a hollow sphere of water filled with this vapour may be lighter than the air in any assigned ratio.

Let their least specific gravity, i. e. when at the temperature of boiling water, be assumed $\frac{1}{10.3}$, then by means

of the different temperatures under that of boiling water to which they are exposed, their specific gravity will vary through all the intermediate magnitudes between $\frac{1}{10.3}$

and 1, according to the preceding determination: now it is to be remarked, that if these spherules were physically without weight, when compared with that of an equal bulk of air, their velocity of ascent would be such as a † heavy body acquires in descending from rest by gravity through † Sect. V. Prop. XII.

$\frac{4}{3}$ of their diameter, in which case their specific gravity =

0; but when their specific gravity = a fraction n , their diameter d , and $l = 193$ inches, their velocity of † ascent † Sect. V. Prop. XII.

$$V = \sqrt{\frac{16 l d}{3}} \times \sqrt{1 - n}, \text{ and } d = \frac{3 V^2}{16 l \times 1 - n}, \text{ } V \text{ here}$$

signifying the number of inches through which the vapour ascends in a second.

It appears therefore that the velocity of ascent will be uniform, if the diameters of the aqueous particles continue the same; but these diameters suffer alteration by being exposed to different temperatures of heat and cold, which by increasing or diminishing the elastic force of the medium occupying the cavities of the spherules, will of course

course expand or contract their dimensions, and consequently alter their diameters in a subtriplicate proportion.

Many of the phænomena observable in the ascent of vapours are extremely consistent with the preceding hypothesis. If the substance of mists and vapours which are seen to arise from lakes or damp grounds, be formed of hollow spherules as above described, they will ascend according to the theory with a motion wholly uniform as long as their diameters are invariable, which is consistent with observation; the diameters of the vaporous particles being unaltered while the air's temperature continues the same: also since it is the property of all elastic fluids to be expanded by heat, we may justly suppose, that the magnitude of these spherules will be enlarged by the same means; an immediate consequence as deduced from the theory is, that their velocity of ascent will be increased in a subduplicate $\dagger$ ratio of their diameters. This also is consistent with observation, for while vapours hang as it were suspended in the atmosphere, their specific gravity being but little different from that of air, the heat of the sun's rays by expanding them into larger portions of space, accelerates the velocity of their ascent, and from these principles it would naturally follow, that the diameter of the particles of steam which rises from boiling water should be larger than those of spherules which constitute vapour when heated to any other temperature, and consequently the uniform velocity of its ascent should be greater than that of steam heated to any inferior degree, which is also consistent with experience.

$\dagger$ Sect. V.
Prop. XII.
Cor. 2.

Thus far theory and matter of fact seem to agree; and if every other appearance were equally deducible from the hypothesis, we should not hesitate to pronounce, that the gravitation of the atmosphere on the spherules specifically lighter was the only cause by which vapour ascended: but a difficulty here occurs from the mathematical principles, which if they are not sufficient for the intire explication of the phænomena, are yet serviceable at least in subjecting the hypothesis to examination.

If the diameters of the aqueous spherules be given, together with their specific gravity in reference to that of the air wherein they ascend, their velocity of ascent may be determined a priori as far as it is effected by the air's gravitation, and conversely, if the velocity wherewith vapour ascends, and the specific gravity of the globules be given, their diameter may be ascertained.

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If therefore the vapour which rises from boiling water consists of hollow spherules, and the velocity with which it ascends in the air be observed, the diameter of the spherules will be determined, and should agree with observation if the theory be perfect: but it is evident, that the particles whereof steam consists are of such minuteness as to render it impossible to observe them separate from each other, the substance of steam appearing, even when examined by glasses, as a rare and perfectly continuous mass by no means distinguishable into parts; whereas the diameter of the particles derived from the theory, upon a supposition that the air's pressure is the only cause of their ascent, is of such a magnitude as would render each spherule obvious to the senses: for if we observe the ascent of steam from the surface of boiling water, it will be found to rise at the rate of 4 inches in a second: and since the

specific gravity of a particle is about $\frac{1}{10.3}$, that of air being 1, we have for the determination of the diameter d from the theory, $n = \frac{1}{10.3}$, $l = 193$ inches, $V = 4$, and

the diameter * required $= \frac{3 V^2}{16 l \times 1 - n} = \frac{1}{58}$ part of an * Sect. V. Prop. XII.

inch.

It is to be observed, that this reasoning is not affected by any error in the determination of the specific gravity of the aqueous spherules which rise from boiling

water; for their specific gravity was assumed $\frac{1}{10.3} = n$, that of the air being 1, let n be increased, the diameter

$d = \frac{3 V^2}{16 l \times 1 - n}$ will be also increased, and if the

spherule's specific gravity be diminished, the diameter of

the spherule will decrease, but can never be less than $\frac{1}{64}$

which is its limit as when n is $= 0$, and the same difficulty

occurs from supposing the diameter $= \frac{1}{64}$ as $\frac{1}{58}$ of an

inch.

It is however certain, that the diameters of the particles of steam which rise from boiling water, are far less

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than

than $\frac{1}{64}$ of an inch: let the diameter, for example, be supposed to $= \frac{1}{120}$ of an inch: to find the velocity of their ascent, we have $n = \frac{1}{120}$, $l = 193$, and V the velocity required $= \sqrt{\frac{16ld}{3}} \times \sqrt{1-n} = 2.78$, or not so much as 3 inches in a second; but the steam rises at the rate of about 4 inches in the same time: it would follow therefore, upon the supposition that the diameter of a particle of steam $= \frac{1}{120}$ part of an inch, that this additional velocity over and above that which is caused by the air's pressure must be attributed to some power which acts in a direction contrary to that of the earth's gravity.

It is rather analogous to the operations of nature, than inconsistent with them, to suppose the existence of some such power acting in a direction contrary to that of gravity. The remarkable phenomenon of the perpendicular growth and position of plants and trees must be attributed to the constant agency of some force external to the plant or tree itself; for if the ground on which a vegetable is planted be inclined to the horizon at any angle whatever, the plant will however soon obtain a direction perpendicular to the horizon, and continue to increase in that direction. Another phenomenon may be here mentioned also: if a bar of metal be fixed in a vertical direction, and heat be by any means applied to the middle of it, this heat will be communicated to the upper part of the bar, when the temperature of the lower part is scarcely altered.

Many other instances might be urged to prove the existence of some power acting in a direction contrary to that of the earth's gravity: neither will this hypothesis be at all inconsistent with that general physical truth, i. e. that the weight of every body is proportional to the quantity of matter contained in it; for this will be the case whether bodies be acted on by gravity alone, or by two powers in contrary directions, the strongest of which is gravity.

It must be however remarked, that this power which seems necessary for the elevation of steam, cannot be proved the only cause of the phenomenon: many facts and arguments tending to shew that the gravitation of the air is

greatly

greatly concerned in producing them, although it be alone insufficient to account for them wholly. Thus, when we see the smoke which is impregnated with the matter from which it arises ascending in the open air, and descending when placed in a rarer medium within the receiver of an air pump, we must conclude that the air's pressure was necessary to the ascent, and that the gross nature of the matter carried up with the vapour, renders its weight superior to the force whereby it is impelled upwards. Also the phenomena of the increased velocity of ascent in vapour from the increase of heat tends greatly to strengthen this opinion, it being very consistent with the theory of the ascent of spherules in air above described.

It may be added to these observations, that during the ascent of vapour into the regions of the clouds, two causes seem to operate in a manner contrary to each other. For while the increasing rareness of the air causes an expansion of the spherules, by removing † the compressure upon their surfaces, and consequently increases their velocity of ascent, † Prop. XIV. the colder temperature of the higher regions by contracting the dimension of the spherical cavities diminishes the velocity, and according to the degrees in which these causes operate, the vapour will arise with a greater or less degree of velocity until it becomes stationary when these causes counterbalance each other. Also it is easy to imagine, that atmospheric cold may render the vapour so dense as to make it preponderate against the force which raised it, so that it shall descend in various forms, i. e. in rains, mists, dews, &c. according to its degree of condensation.

XVII.

Let a spherical body consisting of elastic vapour, the expansive force of which is inversely proportional to the space it occupies, ascend in air of uniform density; and let the sphere be of evanescent weight when compared with that of an equal

bulk of air, if the air's density be diminished in the proportion of $n : 1$, the velocity of the sphere's ascent will be increased in the proportion of $1 : n^{\frac{1}{6}}$.

For if the density of the air be diminished in the ratio of $n : 1$, its elastic force as well as the force of pressure will be diminished in the same proportion: the space occupied by the sphere will therefore be increased in the proportion of $1 : n$, and consequently the sphere's dia-

* Sect. V.
Prop. XII.

meter increased in that of $1 : n^{\frac{1}{3}}$: but spheres of evanescent weight ascend in fluids with velocities, which are in the direct subduplicate * ratio of their diameters; the velocities therefore with which the spheres, whose diameters

are as $1 : n^{\frac{1}{3}}$ ascend, are in the proportion of $1 : n^{\frac{1}{6}}$.

* Page 168.

From this proposition may be deduced the explanation of several phenomena relating to the ascent of vapour, that is, as far as the air's gravitation is concerned in producing them. Whatever be the original cause by which the aqueous spherules are detached from the subjacent fluid, it seems obvious, that the same power which contributes to promote the velocity of their ascent, must also tend to augment the quantity of evaporation. From the theory two causes have been deduced whereby this effect is produced, i. e. heat, or a * diminution of the atmosphere's gravity, both of which however operate by the same means, that is, by increasing the magnitudes of the ascending spherules, and in these particulars theory and matter of fact are intirely consistent: for it is obvious to common observation, that an increase of heat augments the quantity of evaporation under a given pressure of the atmosphere, and experiment demonstrates, that fluids of a given temperature being placed under the receiver of an air pump evaporate the faster the more the air is exhausted; so that when either pure water or any moist substance is placed under a receiver, as the air is exhausting the water becomes converted into vapour, rising in greater abundance according to the rareness of the air within the receiver. This vapour by its elasticity supplies in part the air exhausted, and is a very great impediment to the due operation and effect of the air pump, as well as a cause of

un-

uncertainty in estimating the degrees of exhaustion which that instrument is capable of producing. In order therefore to exhaust a receiver as perfectly as possible, no moisture of any kind nor any solid body which may have imbibed it should communicate with the receiver. This has been fully proved by the author of an ingenious and intelligent series of experiments lately published. †

It is manifest that the density of fluids will not obstruct their evaporation so much as their tenacity: a hollow spherule of mercury will rise equally fast in air with one of water of the same diameter, provided the spherical shell of mercury be about 14 times thinner than that of water, both being supposed to be filled with an elastic fluid in a physical sense void of weight; and the very subtle and divisible nature of material substance in general, and especially of this fluid, renders any degree of tenuity in the shells of the mercurial spherules easily imaginable. Moreover, the parts of this fluid being intirely free from cohesion or tenacity, it will readily be converted into vapour by a moderate degree of heat, and in the ascent of these vapours the same laws will be observed as in the ascent of those which arise from water. Thus taking off the pressure of the atmosphere from the surface of mercury will facilitate and accelerate the ascent of vapour from it; in whatever degree the air be rarefied this ascent would still continue were the † mercurial shells absolutely without weight: and it is possible, that there is no vacuum to be made by art, but what will contain air sufficient to elevate the attenuated spherules: it is certain, that they occupy the space between the surface of the mercury and the upper extremity of a barometer,* the altitude of the mercury being sensibly affected thereby; but whether this ascent of mercurial vapour should be attributed to the very small quantity of air, which notwithstanding the utmost care used in constructing the instrument may remain within the substance of the mercury, and at length emerge from its surface, or to that additional power over and above the air's gravitation which has appeared necessary to account for the phenomena of the ascent of || vapour, future experiments must determine.

To examine this subject a little more particularly: the smoke which arises from ignited bodies, it is evident must ascend by the same causes which elevate vapour from the pure fluids, and this being the case, it might be expected that the laws observed in the ascent of them all would be precisely the same: but although vapour from water and mer-

† Philos.
Transact.
for 1777,
p. 614.

† Prop. XII.
Cor. 1.

* Philos.
Transf. for
1777. p. 671.

|| Sect. V.
Prop. XVI.

mercury rise the more copiously and with the greater velocity, according to the rareness of the air wherein they ascend, smoke from ignited bodies descends in air which is rarefied to any considerable degree: and this phenomenon is by no means inconsistent with the theory; for the globules which form smoke are always of an unctuous nature, and consequently require far greater degree of heat to expand them so as to become capable ascending, than that by which the vapour of mercury or water begins to rise: but when once the globules of smoke have ascended from the ignited body, being deprived of their heat, they soon become hard in cooling, and are incapable of any expansion from the elasticity of the contained medium, notwithstanding the external air should be rarefied in any degree however great. Now it *appeared, that the rarefaction of the air in which the vapour of water or mercury ascended, would cause an acceleration of the vapour's velocity from the increase of the diameter of the globule's which from their extreme fluidity expand themselves into larger portions of space as the external pressure of the air is diminished; but in smoke the tenacious quality of the hollow globules when cool prevents such expansion, and consequently when the air is so rarefied that the weight of a globule exceeds that of an equal bulk of air (together with whatever additional force over and above the air's pressure contributed to its ascent) it will descend.

* Prop.
XVII.

The action of any other power, except that of the air's pressure has not been very particularly considered as it is not here intended to set forth a compleat theory, but to apply the mathematical principles as far as they will allow us to the explication of the phenomena. What has been observed in relation to smoke is applicable in some degree to the vapour which arises from heterogeneous fluids, and from solids impregnated by them. Here the globules of vapour which arise from the same substance will differ in quality as well as their velocity of ascent; and while the purer and most fluid parts continue to ascend with additional velocity as the air is † rendered more rare, the grosser vapour, the tenacity of which admits not of an expansion of the globule's diameters like the former, will either rise with a less degree of velocity, continue stationary, or descend, as the tenacity operates in a greater or less degree while the air is rarefied.

† Prop.
XVII.

S E C T. VI.

CONCERNING THE COMMUNICATION OF
MOTION TO BODIES WHICH REVOLVE
ROUND A FIXED AXIS,

TH E principal properties of rectilinear motion, both accelerated and retarded, may be inferred from the preceding propositions; in demonstrating these, the impelling force, as well as that of resistance, has always been supposed to be impressed in the direction of a straight line passing through the centre of gravity of the moving body, and in this case every particle of the body must partake of the same degree of velocity, being equal to that with which the common centre of gravity moves.

But it frequently happens that a body or system of bodies is so constituted, that when any force is impressed upon it, no motion can be produced, except round a fixed axis; so that the velocity of the particles which compose the system will be greater or less, according as these particles

cles are further from the common axis or nearer to it. These circumstances should be attended to, in order to ascertain the motion of revolving bodies, the preceding principles of acceleration being not wholly of themselves sufficient for that purpose, without taking into consideration the particular nature of rotatory motions.

In order to demonstrate the laws observed by bodies which revolve round a fixed axis, two things are principally to be attended to, i.e. the moving force by which the revolving motion is generated, and the inertia of the parts whereof the system is composed; the moving force exerted on any given particle in the system as well as its inertia depends on its distance from the axis of motion, every thing else being the same, and if both these be ascertained, the absolute acceleration of the particle will be determined, and consequently the absolute velocity generated in it in a given time.

The methods therefore of determining these forces in any given circumstances should next be described.

Let $AFGH$ represent the circumference of a wheel which turns in its own plane round an horizontal axis, passing through

s its centre, and let a weight P , fixed at the extremity of a line AP , communicate ^{Fig. XXXIV.} motion to the wheel. Moreover, let the whole weight of the wheel be Q ; and suppose this weight to be collected uniformly into the circumference $AFGH$, then during the descent of the weight P , each point of the circumference must move with a velocity equal to that with which P descends, and consequently since the moving force is the weight P , and the mass moved $P + Q$, the force which accelerates P in its descent, will be that part of the accelerating force of gravity

which is || expressed by the fraction $\frac{P}{P + Q}$. ^{|| Sect. I. Prop. IX.}

The velocity therefore which is generated in P in any given time is found from the rules before *demonstrated: thus, suppose ^{* Sect. III. Prop. I.}

Q to be equal to P , then will $\frac{P}{P + Q} = \frac{1}{2}$,

and the weight P will be accelerated by a force which is to that of gravity as $1 : 2$. and since gravity generates in bodies which descend near the earth's surface one second of time a velocity of $32\frac{1}{2}$ feet in a second, it follows, that the weight P will in the same time have ‡acquired in its descent a

A a

velo-

† II. Law of Motion, or Sect. II. Prop. I.

velocity of $16\frac{1}{2}$ feet in a second only. This case is mentioned rather particularly, as the more complicated properties of revolving bodies demonstrated in the subsequent propositions will be referred to it.

The parts of the weight Q which are uniformly disposed over the circumference $A F G H$, balance each other round the common centre of gravity s , their weight therefore is of no effect in accelerating or retarding the descent of P ; and this will be the case whenever the axis of motion passes through the common centre of gravity: but in order to render the properties of rotatory motions more obvious, it will be convenient to dispose the parts of the revolving system, so that the axis of motion shall not necessarily pass through the common centre of gravity: thus, referring to the preceding case, instead of supposing the weight Q to be uniformly disposed over the circumference $A F G H$, let it be collected into any point Q . Here it is manifest, that if the mass Q be acted on by gravity, the force which communicates motion to the system round s will be variable, it being the greatest when $s Q$ is horizontal, and gradually diminishing till Q has descended to its lowest point.

But

But as we should begin with the most simple cases, the moving force must be constant. This will be effected by supposing the mass which is collected in Q to be destitute of weight, and to possess inertia only; it follows therefore that during the revolution of Q round s as an axis, the moving force will be constantly equal to P , and the mass moved $= P + Q$, and consequently the force which accelerates the descending weight or any point in the circumference will be that part of gravity

which is expressed by the * fraction $\frac{P}{P + Q}$ * Sect. I.
Prop. IX.
the same as before.

In these cases the force which communicates motion to the system, has been supposed a weight or body acted on by the earth's gravity, and consequently constitutes a part of the mass moved, at the same time it acts as a moving force: but motion may be communicated by a force which shall add nothing to the inertia of the matter moved, and it will be convenient in many demonstrations to assume the moving force of this kind: the inertia of the moving force P therefore in the subsequent propositions

A a 2 will

Fig.
XXXIV.

will not be taken into account, unless expressly mentioned. Thus, if any number of bodies without gravity, being collected into the points F, H, Q , are caused to revolve round the axis s , by a moving force P , the force which accelerates these bodies

† Sect. I.
Prop. IX.

in their ‡ revolution will be $\frac{P}{F + H + Q}$;

provided the bodies F, H, Q be disposed at a distance from the axis of motion equal to the radius of the circle $AFGH$, at the circumference of which the moving force P is applied.

In the preceding example F, H, Q , &c. have been supposed to move with the same velocity: but when bodies revolve at unequal distances from the axis, their velocities of motion being different, other rules will be necessary to determine the force whereby any given point of the system is accelerated; these are contained in the following propositions. In demonstrating the properties of rotatory motion, the revolving system will be supposed to consist of one or more of the bodies A, B, C : the magnitude of these is assumed as evanescent, because were the contrary supposition adopted, the particles in each body would be impelled by dif-

different moving forces, and exert different degrees of inertia in opposition to the communication of motion. But the force which impels each individual particle, and the effects of its inertia in given circumstances must be known before the acceleration of the whole system can be determined.

The bodies A, B, C, which may be termed, according to the ideas just described, material points, are imagined to be connected together by some perfectly rigid substance, so as to always possess the same situation in respect of each other, and consequently no motion can be produced in any of them, except all revolve at the same time round the common axis of motion.

All the points in this imaginary substance by which the parts of the system are connected together, partake of the same angular motion describing circles round the common axis s ; a force P therefore being applied to any point in the plane of its motion, and in the direction of any line in that plane which passes not through the axis, will communicate an equal angular motion to the whole. Thus, let B represent a material ^{Fig. XXXV,} point

point moveable about an axis of motion passing through s , with the centre s , and distance sD , describe a circle DGH : now, if B be connected with every point in the area of this circle, which is an inflexible substance, no force can be applied to move the circle, but what must communicate the same angular motion to B . Let the force be applied at the point D ; it is manifest, that in order render its effects constant, the inclination of its direction to sD must be always the same, and in a given plane: and the most obvious method of effecting this, either in considering the subject theoretically, or in the practical illustration of it, is by applying a thin and flexible line $GHD P$ round the circumference of the circle DGH , and stretching this line by a given moving force P . Here it is plain, that in whatever part of the circumference DGH , the point D is situated, the effects of the force P will be the same, as if it were directly applied to D in the direction of the plane of motion and perpendicular to sD , and the point B will revolve with the same absolute and angular velocity in both cases.

These considerations being premised, in order to prevent unnecessary repetitions,

tions, let A, B, C be a system of bodies of evanescent magnitude and without gravity moveable about an axis of motion which passes through s, it is to be noted, that the imaginary substance by which parts of the system A, B, C are connected, must contribute nothing, either by its weight or inertia, to accelerate or retard the motion of the material points A, B, C, which are caused to revolve by the action of the given and constant force P applied at the distance from the axis s D. Fig. XXXVI.

The absolute force of P to move D, or any point in the circumference, will be P, but the communication of motion to this point D is resisted by the inertia of the bodies A, B, C, which being moved with different velocities and acted on by different moving forces, their inertia will not be estimated by their * quantities of matter only, according to the laws observed in rectilinear motion: the force which accelerates D therefore cannot be obtained by dividing P by $A + B + C$; but if an equivalent mass, or a quantity of matter can be assigned, which being collected into any points of the circumference a, b, c will cause an inertia or resistance to the motion of D, equal to that exerted

* Sect. I,
Prop. IX.

erted by the particles A, B, C , when revolving at their respective distances, the force which accelerates the circumference or any point in it D will be determined. Thus, let the mass Q when collected into a , be such as will be equivalent in its inertia to A when revolving at the distance sA ; also, let R be the mass collected into b , which is equivalent to B when revolving at the distance sB ; and let T be the mass collected in c be equivalent to C when revolving at the distance sC ; then will the mass moved by the force P be $Q + R + T$, and the force which accelerates the cir-

* Sect. I.
Prop. IX. circumference = $\frac{P}{Q + R + T}$ *, being equal

to that by which the circumference or any point in it is accelerated, when the system consists of A, B and C , revolving at the respective distances from the axis of motion sA, sB, sC .

N. B. In the subsequent propositions the moving force P is supposed to be applied to any point D , in the direction of a plane which is perpendicular to the axis of motion, and at right angles to sD .

I.

Fig.
XXXV.

Let a body collected into the point B , revolve round an axis of motion passing through

through s by the action of a force P , applied at the distance sD from the axis; it is required to assign what quantity of matter must be collected in the point D , so that D may be accelerated in the same manner as when B revolved at the distance sB , and consequently that the angular velocity of B and D round s may be the same in both cases.

When any two bodies are put in motion by two constant forces acting for the same time, the quantities of matter moved are in a direct || ratio of the moving forces and an inverse ratio of the velocities generated; that

|| Sect. tit.
Prop. VII;

is, if $\frac{M}{m}$ expresses the ratio of the moving forces, $\frac{Q}{q}$ that

of the quantities of matter, and $\frac{V}{v}$ of the velocities generated,

the relations of these quantities is defined by the

$$\text{equation } \frac{Q}{q} = \frac{M}{m} \times \frac{v}{V}.$$

To apply this it must be observed, that although the absolute force of the weight P acting upon the point D , remains constantly the same, yet its effects upon bodies placed at different distances from the axis of motion, are in the inverse proportion of those distances; therefore the moving forces exerted by P on the points B and D , will be in the proportion of SD to SB : also by the problem the angular motion of D and B are equal, and consequently the velocity of B is to the velocity of D , as $SB : SD$; and since the quantities of matter † in B and D are in the direct proportion of the moving forces, or of $SD : SB$, and an inverse proportion of the velocities generated, or of $SB : SD$, we shall have the quantity of matter in B to that contained in D , as $SD^2 : SB^2$, and consequently the

† Supra.

$$\text{weight sought} = B \times \frac{SB^2}{SD^2}.$$

Or thus, let x = the quantity of matter required to be collected in D , M the moving force which acts on B , m that which acts on D , V the velocity of B , v that of D , then † will $\frac{B}{x} = \frac{M}{m} \times \frac{v}{V}$, but $\frac{M}{m} = \frac{SD}{SB}$ by the property of the lever: moreover $\frac{v}{V} = \frac{SD}{SB}$ by the nature of angular motion, wherefore $\frac{B}{x} = \frac{SD^2}{SB^2}$, and $x = B \times \frac{SB^2}{SD^2}$.

Cor. 1. Whenever therefore a body B revolves round an axis by the action of a constant force P , applied at a given distance SD from the axis; in order to find the force which accelerates D , the mass B may be supposed to be removed, and instead of it an equivalent weight $B \times \frac{SB^2}{SD^2}$, collected in the point D to which the force is applied.

† Sect. I. Prop. IX. Cor. 2. Since the † moving force is P , and the mass moved $\frac{B \times SB^2}{SD^2}$, (the inertia of P not being considered,) the force which accelerates D will be || that part of the

|| Page 185. acceleration of gravity which is expressed by the fraction

$$\frac{P}{B \times \frac{SB^2}{SD^2}} = \frac{P \times SD^2}{B \times SB^2}.$$

Fig. XXXVI. Cor. 3. Let any number of bodies A, B, C , &c. be put in motion round a fixed axis, passing through S by a constant force P applied at D , the point D will be accelerated in the same manner, and consequently the whole system will have the same angular velocity, if instead of A, B, C , &c. placed at the distances SA, SB, SC , we substitute the bodies $\frac{A \times SA^2}{SD^2}$

+ $\frac{B \times SB^2}{SD^2}$ + $\frac{C \times SC^2}{SD^2}$, these being collected into the points a, b and c respectively: the moving force in this case is P ;

* Sect. VI. Prop. I. and p. 185. the mass* moved = $\frac{A \times SA^2}{SD^2} + \frac{B \times SB^2}{SD^2} + \frac{C \times SC^2}{SD^2}$;

and consequently the force which accelerates D is that part of the accelerating force of gravity which is expressed by the

$$\text{the fraction } \frac{P}{A \times \frac{SA^2}{SD^2} + B \times \frac{SB^2}{SD^2} + C \times \frac{SC^2}{SD^2}} = \frac{P}{SD^2 \times P} = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2}{SD^2 \times P}.$$

Cor. 4. The velocity $\parallel$ of the point D is uniformly accelerated, because the force above determined is invariable: || Sect. I.
Prop. IV. it follows also, that the angular velocity of the system is uniformly accelerated, because the absolute velocity of any point at a given distance from the axis of motion is as the angular velocity of that point, and consequently of the whole system.

Cor. 5. It is manifest that it is of no consequence whether the bodies A, B, C , &c. revolve in the same or in different planes, if their distances from the axis SA, SB, SC are the same; these distances being estimated by lines drawn through A, B , and C perpendicular to the common axis of motion; if therefore they should be situated in various planes, they may be referred to any one given plane perpendicular to the axis.

Cor. 6. It is obvious also that changing the position of the bodies A, B, C in the same plane will not affect the force which accelerates the system, provided their respective distances from the axis of motion alter not: thus, with the centre S , and distances SB, SC , let the arcs of circles be described; if B is transferred to b , or C to c , the moving force which acts on these bodies respectively will not be altered, and consequently the masses moved being likewise constant, the accelerating force will be the same. Fig.
XXXVI.

Cor. 7. Also the propositions concerning rotation will be equally true, whatever be the force by which the angular motion is generated, provided it be constant; or if variable should its action be considered for an evanescent particle of time only.

Cor. 8. In these propositions the angular motion is supposed to be communicated by the action of a force applied at a single point only, at an invariable distance from the axis. Thus, suppose a revolving system to consist of the bodies A, B, C of evanescent magnitudes and equal to each other in quantity. Let a weight P equal to A, B , or C be applied by means of a line going round a wheel DH , so as to communicate motion to the system during its descent. Moreover, let $SD = 1, SA = 2, SB = 3$, and $SC = 4$, and let it be required to assign the force which accelerates the descent of the weight P . Fig.
XXXVI.

Here referring to the rule, we have the force required

$$= \frac{SD^2 \times P}{A \times AS^2 + B \times SB^2 + C \times SC^2} = \frac{1}{4 + 9 + 16} =$$

$\frac{1}{29}$, which is the true accelerating force when the moving

force is void of inertia; but in the present case, the weight P acts not only as a moving force, but as a part of the mass moved, and as P moves with the same velocity as the point D , the effects of its inertia will be estimated by supposing a mass of matter $= P$ to be collected into the point D : here then the moving force being P ,

and the mass moved $P \times \frac{SD^2}{SD^2} + A \times \frac{AS^2}{SD^2} + B \times \frac{BS^2}{SD^2}$

* Sect. I.
Prop. IX.

$+ C \times \frac{CS^2}{SD^2}$, the force* which accelerates the descent of

P will be $\frac{SD^2 \times P}{P \times SD^2 + A \times AS^2 + B \times SB^2 + C \times SC^2} =$

$\frac{1}{30}$, the line and parts of the wheel not being taken into

account. It follows therefore, that the weight P will descend from rest by the force of this acceleration through

$\frac{16\frac{1}{2}}{30}$ parts of a foot in a second; and in other different

† Sect. III.
Prop. IV.

times of descent the spaces † will be varied in a direct duplicate ratio of the times, because the accelerating force is constant.

Moreover, the velocity acquired by the descending weight P in a given time, or through a given space, will be determined by the rule demonstrated in sect. III. prop. v. Thus

putting $\frac{P \times SD^2}{P \times SD^2 + A \times AS^2 + B \times BS^2 + C \times CS^2}$

$= F$, $l = 193$ inches, $s =$ the space described by P from

rest, the velocity acquired by P will be $\sqrt{4Fs}$ inches in a second, and in the same manner having given any two of the four quantities, i. e. the accelerating force, the space described from rest, the time of description, and the velocity acquired, the other two may be determined as in sect. III.

For the further illustration of this subject, the following property of uniformly accelerated motion may be here inserted.

II.

If any two bodies be acted on by moving forces which are constant, and which are in an inverse proportion of the spaces described from rest, the quantities of matter contained in the bodies will be in an inverse duplicate ratio of the velocities

generated; that is, if $\frac{Q}{q}$ represents the ra-

tio of any two masses of matter, $\frac{M}{m}$ the

ratio of the moving forces by which they are impelled, $\frac{S}{s}$ that of the spaces describ-

ed from rest, and $\frac{V}{v}$ the ratio of the velo-

cities generated, the proposition asserts, if

$$\frac{M}{m} = \frac{s}{S}, \text{ that } \frac{Q}{q} = \frac{v^2}{V^2}.$$

The duplicate ratio of the velocities is equal to the sum of the ratios of the spaces described and of the || accelerating || Sect. III. Prop. V.

forces, or $\frac{V^2}{v^2} = \frac{F}{f} \times \frac{S}{s}$, but * $\frac{F}{f} = \frac{M}{m} \times \frac{q}{Q}$, we have * Sect. I. Prop. IX.

therefore $\frac{V^2}{v^2} = \frac{M}{m} \times \frac{S}{s} \times \frac{q}{Q}$, and when as in the pro-

blem $\frac{M}{m} = \frac{s}{S}$, or $\frac{M}{m} \times \frac{S}{s} = 1$, the equation which defines

the

the velocities and quantities of matter will be $\frac{V^2}{v^2} = \frac{q}{Q}$, that is, the quantities of matter are in an inverse duplicate ratio of the velocities generated.

This conclusion immediately applies to the preceding proposition, in which it is required to assign the ratio of two quantities of matter under the following conditions, i. e. 1. They are to revolve round a common axis S with the same angular velocity, the absolute velocities therefore must be in the direct proportion of their distances from the axis, or of the spaces described. 2. While a constant force P is applied to any given point D , the effects of these forces to impel bodies at different distances from the axis will be in an inverse ratio of those distances, we have therefore the spaces described in an inverse ratio of the forces which impel the two bodies, and consequently, by the proposition just demonstrated, the quantities of matter in an inverse duplicate proportion of the velocities generated, or of the distances from the axis.

Cor. 1. Let it be required to assign what quantity of matter must be collected into any point E , so that the system in fig. xxxv. shall revolve with the same angular velocity as when B revolved at the distance SB , every thing else remaining, by the rule we have the weight sought to be collected in $E = \frac{B \times SB^2}{SE^2}$.

III.

In a system of bodies which revolve round a fixed axis with an uniformly accelerated motion, the forces which accelerate particles at different distances from the axis will be in a direct ratio of those distances.

By the nature of angular motion, the velocities of the different points in any revolving system will be in a direct ratio of their distances from the axis of motion: but whenever bodies are † uniformly accelerated, the velocities generated

† Sect. III.
Prop. I.

generated in a given time will be directly as the accelerating forces; and consequently the forces which accelerate the different points in a system of revolving bodies, must be in a direct ratio of those distances from the axis.

Cor. 1. It followed from prop. 1. cor. 3. that the force which accelerated the point D was =

$$\frac{P \times SD^2}{A \times SA^2 + B \times SB^2 + C \times SC^2}, \text{ and by } \dagger \text{ this } \dagger \text{ Prop. III.}$$

proposition, the force which accelerates D is to that by which B is accelerated in the proportion of $SD : SB$: it follows therefore, that the force which accelerates B

$$= \frac{P \times SB \times SD}{A \times SA^2 + B \times SB^2 + C \times SC^2}; \text{ in like manner}$$

the force which accelerates any other point C =

$$\frac{P \times SC \times SD}{A \times SA^2 + B \times SB^2 + C \times SC^2}, \text{ and consequently the}$$

the sum of the forces which accelerate all the particles A ,

$$B, C \text{ will be } \frac{AS + SB + SC \times SD \times P}{A \times SA^2 + B \times SB^2 + C \times SC^2}; \text{ and since}$$

the quantities of motion or moments generated in bodies in a given || time, are as the quantities of matter and the || Sect. I. forces which accelerate them jointly, the sum of the Prop. IX. moments generated in a given time will be expressed by

$$\frac{A \times SA + B \times SB + C \times SC \times P \times SD}{A \times AS^2 + B \times SB^2 + C \times SC^2}.$$

It is manifest, that although the bodies A, B and C , and the quantities P and SD remain unaltered in quantity, yet the sum of the moments generated in a given time will depend on the distances SA, SB, SC ; but diminution and augmentation of these may be such that the sum of the moments generated in a given time may still continue the same; there is therefore some intermediate distance at which if all the bodies be collected, the same moment will be produced by the force P acting for a given time, as when the bodies A, B, C are disposed at their respective distances SA, SB, SC .

IV.

In a system of bodies A, B, C moveable round an axis s , to which bodies motion is communicated by a force P acting at a given distance from s ; it is required to assign some distance so , at which if all the bodies are disposed, the sum of the moments generated in them in a given time shall be the same as before, every thing else remaining.

Let A, B and C be collected at the distance SO , then will the sum of the moments generated in a given time

be expressed by $\parallel \frac{A + B + C \times SO \times P \times SD}{SO^2 \times A + B + C} =$
 $\parallel$ Sect. VI.
 Prop. III.
 Cor. 1.

$\frac{A \times AS + B \times BS + C \times CS}{A \times AS^2 + B \times BS^2 + C \times CS^2} \times P \times SD$ by
 the problem, and SO the distance required $=$
 $\frac{A \times AS^2 + B \times BS^2 + C \times CS^2}{A \times AS + B \times BS + C \times CS}$. Now it is manifest,

† Sect. VI.
 Prop. I.
 Cor. 6.

that if the bodies A, B , &c. be transferred to the line SO , † keeping their respective distances from S , the moments generated in a given time will be the same as before.

Let G be the common centre of gravity of A, B, C when collected in the line SO , and since in this case by the property of the centre of gravity $A \times SA + B \times SB + C \times SC$
 $= \frac{A + B + C \times SG}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, it follows, that $SO =$
 $\frac{A + B + C \times SG}{A \times SA^2 + B \times SB^2 + C \times SC^2}$.

Fig.
 XXXVII.

Cor. 1. Thus let FGH represent a circle moveable in its own plane about an axis which passes through its centre S . Supposing the circle material, let the whole mass be collected into any radius SC , the distance of each particle from S remaining the same; then will the density of the matter in the radius increase in the ratio of the distances from S . Let G be the centre of gravity of the weight thus

thus disposed SG is found by the usual rule to be $\frac{2}{3}$ of SC .

Moreover, if the particles thus condensed in the line SC be denominated A, B, C , &c. the sum of all the products $A \times AS^2 + B \times BS^2 + C \times CS^2$, &c. appears to be $\overline{A + B + C} \times \frac{SC^2}{2}$. We have therefore the distance

SO as described in the problem $= \overline{A + B + C} \times \frac{SC^2}{2}$

divided by $\overline{A + B + C} \times \frac{2SC}{3}$, wherefore $SO = \frac{3SC}{4}$.

Consequently if the whole matter of the circle be collected into one or more points at the distance of $\frac{3}{4}$ of the radius

from the axis, the same moment will be generated in a given time in the mass thus disposed, as if it were diffused uniformly over the whole area.

Cor. 2. In a given system the sum of the moments generated in the same time will be in a direct proportion of the distances at which the moving forces are applied from the axis; that is, if these forces be constant and every other circumstance remain the same in both cases.

In these propositions the effects of gravity on the bodies *A, B* and *C* have not been considered, motion being communicated to the system by the action of a force applied at a given distance from the axis, and this force has been assumed of invariable quantity. When gravity acts upon the system as a moving force, each particle contributes to the communication of motion, as well as to the inertia of the mass moved; also the force of acceleration will be variable, unless the time of its action be taken evanescent: but in this case also the velocity generated in a system in a given time, will be obtainable from the preceding rules, because the force exerted by gravity on any body or system of bodies will have precisely the same effects to create motion (the mass moved not being here considered) as if the whole weight were collected into the centre of gravity, and will be constant while the centre of gravity is describing an evanescent arc.

Suppose therefore the bodies *A, B* and *C* to gravitate towards the earth's centre, and let *G* be their common centre of gravity, and draw SGO ; it is manifest that the point *G* will not be quiescent, except when it is placed

C c

coin-

Fig.

XXXVI.

Fig.

XXXVIII.

|| Fig.
XXXIX.

‡ Fig. XL.

coincident with its lowest point, in which case the line SG is perpendicular || to the horizon: let it be moved out of that situation, suppose into the position † SBG : with the centre S and distance SG describe the arc GG and GD equal to it, and draw GB perpendicular to SBG ; then the weight of the system being denoted by the line SCG in quantity, and the direction in which gravity acts, let this be resolved into two, i. e. SB which urges the system directly from S , and BG perpendicular to SB which impels it round the axis of motion: in this situation therefore the point G will be acted on by a force in the direction BG , which is to the weight of the system as BG to SG . This force will decrease continually as SBG approaches SCG , and when it coincides with SCG will be evanescent: the centre of gravity having then acquired its greatest velocity will begin to ascend on the contrary side, being retarded according to the same laws by which it was before accelerated, until it has lost all motion at D ; this motion of the system round an axis through the angle GSD , when effected by gravity, is called a vibration or oscillation, and the vibrating system is called a pendulum.

It is next to be shewn, at what distance from the axis the whole weight of the system must be collected, so that it shall vibrate through a given angle in the same time as before.

V.

Fig. XLI.

In a system of bodies A, B, C , moveable round an horizontal axis which passes through s : let G be their common centre of gravity, when referred to a vertical plane passing through s , and in the line sG produced, let a point o be supposed such, that if $A + B + C$ be collected into o , the system will perform its vibrations through a given angle in the same time, as when the bodies A, B, C are disposed at
the

the distances sA , sB and sC respectively, and in their original positions in respect of each other; it is required to assign the distance sO .

Here two pendulums are to be considered, the one consisting of the bodies $A+B+C$, disposed at their respective given positions and distances from the axis SA , SB , SC , the other being formed by collecting the whole mass $A+B+C$ into some point O ; and it is required by the problem to assign the distance SO , when these two pendulums perform the whole and every corresponding part of their vibration through a given angle in the same time. Let each pendulum begin its vibration from the line $SGNO$, and with the centre S and distances SG , SO describe the arcs GgG , OoO , and through g which is contiguous to G draw the line So : moreover, draw the right fine GM . Then during the time in which any point G at a given distance from the axis in both pendulums describes the evanescent $\dagger$ arc Gg , the force which accelerates that point will be constant in both pendulums, and if these forces are equal to each other at the first instant of motion, the point G in both pendulums $\parallel$ will describe the given arc Gg , and any subsequent $\S$ given increment of the arc GgG in the same elementary particle of time, and consequently that whole arc in the same time. We have therefore only to ascertain at what distance SO , the bodies $A+B+C$ must be collected, so that the point G may be accelerated by the same force as when the pendulum consisted of the bodies $A+B+C$ vibrating at their respective given positions and distances SA , SB , SC .

The moving force which acts upon $\dagger$ the point G (being the centre of gravity of the first mentioned pendulum)

is $\overline{A+B+C} \times \frac{MG}{SG}$, and this force will in a given time

generate the same velocity in G , if A, B, C were removed,

and the equivalent $\dagger$ masses $\frac{A \times SA^2}{SG^2} + \frac{B \times SB^2}{SG^2} + \frac{C \times SC^2}{SG^2}$ were collected in G , the force therefore which

accelerates the point $*G = \frac{\overline{A+B+C} \times MG \times SG}{A \times AS^2 + B \times BS^2 + C \times CS^2}$

C c 2

This

$\dagger$ Sect. III.
Prop. IV.
Cor. 4.

$\parallel$ Page 202.
and
Sect. III.
Prop. IV.

$\S$ Page 21.
and 24.

$\dagger$ Page 201.

$\dagger$ Sect. VI.
Prop. I.

* Sect. VI.
Prop. I.
Cor. 3.

This is the case when the parts of the system A, B and C are disposed at their respective distances and positions SA, SB, SC : now let $A + B + C$ be collected in some point O , and through O draw ON perpendicular to SGO ,

the force which accelerates O will be $= \frac{A + B + C \times ON}{A + B + C \times SO}$

† Sect. VI. $= \frac{GM}{SG}$, and the force which † accelerates $G = \frac{MG}{SO}$.
Prop. II.

If the forces which accelerate the point G in both cases be made equal, the absolute velocity generated in equal particles of time in G will be the same in both pendulums, which
* Sect. III. will therefore describe the given * arc of vibration GgG in
Prop. IV. the same time. Since therefore the forces which accelerate the point G in the two cases described in the problem are to

be equal, we have $\frac{MG}{SO} = \frac{A + B + C \times MG \times SG}{A \times AS^2 + B \times BS^2 + C \times CS^2}$,

and $SO = \frac{A \times AS^2 + B \times BS^2 + C \times CS^2}{A + B + C \times SG}$.

Cor. 1. The point O is called the centre of oscillation, and the distance intercepted between the axis and this point determines the length of the pendulum.

Cor. 2. The whole mass therefore which is contained in a pendulum may be supposed, as far as regards the time of its vibration through a given angle, to be collected into the centre of oscillation.

Cor. 3. The sum of all the products, which are formed by multiplying each particle of a system into the square of its distance from the axis of motion, is equal to the rectangle under the distances of the centres of gravity and oscillation from the axis of motion, into the weight of the whole mass, that is, if $A + B + C, \&c. = W$, $A \times AS^2 + B \times BS^2 + C \times CS^2, \&c. = SO \times SG \times W$.

Cor. 4. In these demonstrations the bodies A, B and C which compose the revolving system have been assumed of evanescent magnitude; and the rules demonstrated are easily applied to the determination of the centres of oscillation in natural bodies, which are made up of innumerable elementary particles, by finding the sum of all the products which can be formed by multiplying each particle into the square of its distance from the axis of motion, and dividing this sum by the bodies whole weight into the distance of the common centre of gravity from the axis.

Cor.

Cor. 5. When the revolving system consists of several bodies, the centres of oscillation and gravity in each of which when revolving round the common axis is known, the centre of oscillation of the whole system is determined

thus. Let O , O , O represent the distances of the centres of oscillation of the separate parts, from the axis, and G , G , and G be the distances of their respective centres of gravity; also let W , W , W be the weights of the separate parts, and SG the distance of their common centre of gravity from the axis; then the distance of the centre of oscillation of the whole system from this axis will be

$$\frac{OGW + OGW + OGW}{SG \times W + W + W}$$

Cor. 6. Suppose the weight of a pendulum to be collected into the centre of oscillation O , and let the pendulum perform its vibration in the arcs of a circle. The force which accelerates the pendulum when at P , will be to the force which accelerates it when at any other point Q as the sines OM , ON . If these sines were the same proportion as the arcs OP , OQ , the arcs would be described in the same time, provided the vibrations commenced from the points Q and P ; but since the sines increase not in so great a proportion as the arcs, it is manifest, that the time in which the pendulum describes the larger arc of its vibration, will be greater than that in which the smaller vibrations are performed. But when the arcs are very small, the ratio of the sines becomes more nearly coincident with that of the arcs themselves, and consequently the forces which accelerate the pendulum are more nearly in that proportion, and accurately so when the arcs are evanescent. The times therefore wherein a pendulum performs its least vibrations in circular arcs will be equal, whatever be the proportion of those arcs, and may be determined by referring to prop. 1. sect. III. Suppose SO to be perpendicular to the horizon, being the quiescent position of the pendulum, and let it be deflected from its vertical situation through an arc PO , then will the force accelerating the point P , be that part of gravity which is expressed by the fraction $\frac{OM}{SO}$, and when the arc PO is

Fig. XLII.

* Sect. III.
Prop. I.

evanescent, the force $= \frac{PO}{SO}$; let $p = 3.14159$, $l = 193$; then will the time wherein the pendulum describes OP

† Sect. III.
Prop. 1.
Cor. 3.
$$= \frac{p \times \sqrt{PO}}{\sqrt{8 \frac{PO}{SO} \times l}} = \frac{p \times \sqrt{SO}}{\sqrt{8l}} \dagger$$
, and the time of one

entire vibration $= \frac{p \times \sqrt{SO}}{\sqrt{2l}}$ parts of a second.

Fig. XLIII. Let ABC be system of bodies without weight, moveable about an axis which passes through S . Let $So = \frac{A \times AS^2 + B \times BS^2 + C \times CS^2}{A \times AS + B \times BS + C \times CS}$, then if the whole matter

|| Sect. VI.
Prop. IV. were collected in o , the same || quantity of motion would be generated in the system in a given time, as when $A + B + C$ revolved at their respective distances SA, SB, SC : if any given moving force W , by which the angular motion is generated, be applied at the same distance from the axis SG in both cases, and in a direction perpendicular to SG ; the sum of the moments generated in a given time will in this case be expressed by the quantity $\frac{SG \times W}{So}$.

Now let the whole mass $A + B + C$ be collected into any other point O , and the force W be applied in a direction perpendicular to SO to communicate motion to the point O , then will the quantity of motion generated in a given time be denoted by W , and the proportion of these quantities of motion generated in a given time will be that of SG to So . This is applicable to the vibratory motion of a pendulum which is effected by gravity. The force to create motion in pendulums is the gravity of each particle, and this will be equivalent in its effect to the entire weight applied at the centre of gravity: therefore, let G be the centre of gravity, and $So = \frac{A \times AS^2 + B \times BS^2 + C \times CS^2}{A \times AS + B \times BS + C \times CS}$,

and let O be the centre of oscillation: the quantity of motion generated in the pendulum by gravity in a given time will be to the quantity of motion, which would be generated in it when the whole mass is collected into the centre of oscillation, in the proportion of $SG : So$.

VI.

In a given system of bodies $A+B+C$ if Fig. XLIV.
the distance of the centre of gravity from
the axis of motion be increased, the di-
stance between the centres of gravity and
oscillation will be decreased in the same
proportion, provided the plane wherein
the pendulum vibrates be the same in re-
spect of the bodies A, B and c .

Let S be the point through which the axis of motion
passes; also let G be the centre of gravity, and O the
centre of oscillation, and draw SGO ; join GA, GB and
 GC , and through A, B and C draw the lines Aa, Bb, Cc
perpendicular to SO : then we have $AS^2 = GS^2 + AG^2 +$
 $2SG \times Ga$, $BS^2 = GS^2 + BG^2 + 2SG \times Gb$, $CS^2 = GS^2$
 $+ CG^2 - 2SG \times Gc$, (because $CS^2 = CG^2 - GS^2 - 2GS$
 $\times Sc = CG^2 + GS^2 - 2GS^2 - 2GS \times Sc = CG^2 + GS^2$
 $- 2SG \times GS + Sc$) and $A \times AS^2 + B \times BS^2 + C \times CS^2$
 $= A \times GS^2 + GA^2 + 2SG \times Ga + B \times GS^2 + GB^2 +$
 $2SG \times Gb + C \times GS^2 + GC^2 - 2SG \times Gc$: but by the
property of the centre of gravity, $A \times Ga + B \times Gb -$
 $C \times Gc = 0$, wherefore $A \times AS^2 + B \times BS^2 + C \times CS^2 =$
 $A + B + C \times GS^2 + A \times AG^2 + B \times BG^2 + C \times CG^2$, and
 $SO = \frac{A + B + C \times GS^2}{A + B + C \times GS} + \frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{A + B + C \times GS}$
 $= GS + GO$, and taking GS from both sides, we
have $GO = \frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{A + B + C \times SG}$, and
 $\frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{A + B + C}$ being constant in the

same system, when the plane of vibration alters not, it fol-
lows, that GO will be inversely proportional to SG .

Cor. 1. With the centre G and distance SG , describe a
circle SRF . Let RQ be the distance of the centre of oscil-
lation

lation from the axis of motion when the pendulum vibrates in the same plane as before, round any point in the circumference SRF , for example R , then will RQ

$$= SO, \text{ for } GQ = \frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{GR \times A + B + C} =$$

GO by the proposition, and consequently $SO = RQ$: the same system will therefore perform its vibrations in a given arc of a circle in the same time, wherever its centre of suspension be situated, provided it be always at the same distance from the centre of gravity, and the plane of its vibration in respect of the system be not altered.

Fig. XLV. Cor. 2. If G be the common centre of gravity of the whole system, and of any two or more parts, for example A and B , then the distance BA remaining, let these bodies revolve round G , in the plane which is perpendicular to the axis of motion; wherever they are fixed during this revolution, the system will vibrate through a given angle round the axis of motion S in the same time as before; for by the revolution of AB , the quantity $A \times AG^2 + B \times BG^2 + C \times CG^2$ is not altered, and SG being likewise the same, $GO = \frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{A + B + C \times SG}$ will be unvaried, and consequently SO the same as before.

Fig. XLIV. Cor. 3. Let A, B, C represent any system revolving round an axis passing through S ; let G be the centre of gravity, O the centre of oscillation, and W the weight of the body; then if the system when referred to any one plane, which is perpendicular to the axis, consists of the particles A, B, C , &c. the sum of the products which are formed by multiplying each particle into the square of its distance from the centre of gravity $= SG \times GO \times W$: that is, when these distances are estimated by lines drawn through each particle perpendicular to a line which passes through the centre of gravity; or if the whole figure be projected into a plane perpendicular to the axis and passing through the centre of gravity, these distances may be estimated by lines intercepted between the particles and the centre of gravity itself.

Fig. XLVI. Cor. 4. Let ABC represent the plane which passes through the centre of gravity of any irregular solid, and let a line be drawn perpendicular to this plane passing through the centre of gravity G . Any where in the plane SAB , let two points S, B found by trial be such, that when they are supported, the solid will remain balanced in any

any position, the centre of gravity must be in a line which joins these points. Let the body vibrate in the plane $SABC$ round an horizontal axis which passes through S . Elevate the point B so that SB may be horizontal, and let the weight q acting vertically by means of a fixed pulley C be such as is just sufficient to support SB horizontal: also let $SB = a$, the weight of the body $= w$, then will $SG = \frac{aq}{w}$. Suppose that the system performs n very small vi-

brations in a circular arc round the given axis which passes through S in t seconds; then the distance of the centre of

oscillation from the axis $SO = \frac{2t^2 l}{n^2 p^2}$ inches*, l being = * Sect. III. Prop. I.

193 inches, and $p = 3.14159$, &c. wherefore $GO =$

$\frac{2t^2 l}{n^2 p^2} - \frac{aq}{w} = \frac{2wt^2 l - n^2 p^2 aq}{n^2 p^2 w}$, and $\dagger w \times GO \times \dagger$ Sect. V. Prop. I.

$SG = A \times AG^2 + B \times BG^2 + C \times CG^2$, &c. = Cor. 2.

$\frac{2waqt^2 l - a^2 q^2 n^2 p^2}{n^2 p^2 w}$. This proposition is of consider-

able use in the practical construction of the balance, when the motion of that instrument as well as its equilibrium is considered, which will be shewn on a future occasion.

Suppose, that during the vibration of a system of bodies round a fixed axis, such an obstacle were opposed to any point O , as to entirely destroy the motion of that point. The point O being quiescent, each particle of the system will endeavour by its inertia to proceed in the direction of its $\dagger$ motion, that is, of the tangent to the circular arc $\dagger$ I. Law of Motion, which it was describing the instant O was stopped. These forces will therefore act on the system to turn it round O , and if the sum of the forces on each side of O should be unequal, the motion of the system will not be destroyed when O is stopped: but since the forces which act on the pendulum between O and S will have an effect to continue the motion of the system contrary to those which are impressed on the other side of O , if the point O be so situated that the sum of the forces to turn the system round O on each side of that point may be exactly equal, the whole motion of the system will be destroyed the instant O is stopped. This point, determined according to the conditions just described, is called the centre of percussion, and if a pendulum vibrating with a given angular motion strikes an obstacle, the effect of the blow will be the greatest when the impact is made in that point: for in

Fig. XLVII.

this case, the obstacle receives the whole revolving motion of the pendulum; whereas if the blow be struck in any other point, a part of the pendulum's motion will be employed in endeavouring to continue its rotation.

VII.

Fig. XLVII. In a given system of bodies $A + B + C$ moveable about an axis of motion passing through s ; it is required to determine the distance of the centre of percussion from the axis of motion.

Let G be the common centre of gravity, draw $SGOL$, and let O be the centre of percussion. Through A , B and C draw Aa , Bb and Cc perpendicular to SA , SB and SC respectively, and the lines AD , BE , CF perpendicular to SO : then the instant O is stopped, the body A will endeavour to proceed in the direction Aa , being perpendicular to SA , and thereby urge forward the point a with a force proportional to $A \times SA$; this may be expounded in quantity and direction by the line Aa , but the direction in which it acts being oblique to SO , in order to find what part of it is employed to turn the system round O , it must be resolved into two Da and DA , whereof Da tends to urge the system from S , and AD to impel it round O : consequently this latter force is that part of the whole force

$A \times SA$, which is expressed by the fraction $\frac{DA}{Aa}$, or because of the similar triangles DAa , SDA , by the fraction $\frac{SD}{SA}$: the force of A therefore applied to turn the system

when O is stopped $= \frac{A \times AS \times SD}{SA} = A \times SD$, and the

effect of this force to generate angular motion round O is $A \times SD \times Oa = A \times SD \times SO - Sa$; and since by similar

triangles $* Sa = \frac{SA^2}{SD}$, it follows, that the force of A to

turn the system round $O = A \times SD \times SO - A \times \frac{SA^2}{SD}$; and

† Euclid.
Elem.
Lib. 6.
Prop. 13.

and by the same reasoning the force of B to turn the system round O in a contrary direction $= B \times SB^2 - B \times SE \times SO$, and that of $C = C \times SC^2 - C \times SF \times SO$: and since these forces are to balance each other round O by the problem, we have $A \times SD \times SO - A \times SA^2 = B \times SB^2 - B \times SE \times SO + C \times SC^2 - C \times SF \times SO$, and $SO = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2}{A \times SD + B \times SE + C \times SF}$; and since by property of the centre of gravity $A \times SD + B \times SE + C \times SF = SG \times A + B + C$, it follows, that $SO = \frac{A \times AS^2 + B \times BS^2 + C \times CS^2}{A + B + C \times SG}$.

Cor. 1. The distance* of the centre of percussion is * Sect. VI. equal to the distance of the centre of oscillation from the axis of motion. Prop. V.

The centre of oscillation has been defined that point wherein if all the matter of a system were collected, the time of vibration through a given angle would be the same as before, and consequently the angular velocity generated in a given time must be the same in both cases: but here a circumstance is to be taken notice of. In determining the centre of oscillation, when the parts of a system A, B, C are disposed at the respective given position and distances SA, SB, SC , the moving force which generates the angular velocity, that is, the natural weight of the body is applied at the common centre of gravity; whereas, when the matter of the system is collected into O , so that the time of vibration in a given angle may be the same as before, the moving force is applied at O , having changed its place from G to O . Fig. XLI.

This is the case when a pendulum is put in motion by gravity, but when any system of bodies A, B, C is caused to revolve by the action of a force applied at any given distance which changes not, if the whole mass $A + B + C$ be collected into the centre of oscillation, the angular velocity generated in a given time will not be the same as before: there exists therefore some other point R , in which if all the matter $A + B + C$ be collected, a given force P applied at an invariable distance from the axis, will generate the same angular velocity in the same time as if the bodies were disposed at their respective distances SA, SB, SC . This point is called the centre of gyration. Fig. XLIII.

N. B. To avoid repetitions, whenever any moving force is applied to turn a system round its axis, it is al-

ways meant to act in the direction of the plane wherein the system revolves, and at right angles to a line, drawn from the point at which it is applied, perpendicular to the axis.

VIII.

Fig. XLIII. In a system of bodies A, B, C revolving round an axis which passes through s: having given the quantities of matter contained in A, B, C, and the distances SA, SB, SC, it is required to determine SR the distance of the centre of gyration from the axis of motion.

* Sect. III.
Prop. I.

|| Sect. VI.
Prop. I.
Cor. 4.

† Sect. VI.
Prop. I.
Cor. 3.

Let any force P be applied to turn the system at any given distance from the axis SD . If the force which accelerates a given point D be the same in any two cases, the * absolute velocity of that point generated in a given time must be the same, and consequently the angular velocity || generated in the system equal in both cases: the force which accelerates the point $D =$

$\frac{P \times SD^2}{A \times AS^2 + B \times BS^2 + C \times CS^2}$: † now, let $A+B+C$ coincide with R , and the force which accelerates D will be

$\frac{P \times SD^2}{A+B+C \times SR^2}$, these forces must be equal by the hypo-

thesis: making therefore $\frac{P \times SD^2}{A \times AS^2 + B \times BS^2 + C \times CS^2}$

$= \frac{P \times SD^2}{A+B+C \times SR^2}$, we shall have $SR^2 =$
 $\frac{A \times AS^2 + B \times BS^2 + C \times CS^2}{A+B+C}$, and the distance of the centre

of gyration required $= \sqrt{\frac{A \times AS^2 + B \times BS^2 + C \times CS^2}{A+B+C}}$.

Cor. 1. Let $A+B+C=W$, and since $A \times AS^2 + B \times BS^2 + C \times CS^2 = SR^2 \times W$, it follows, that $SR^2 =$

$= SO \times SG$: † wherefore the distance of the centre of gyration is a mean proportional between the distances of the centres gravity and oscillation from the axis of motion. † Sect. VI, Prop. VI, Cor. 3.

Cor. 2. Every thing remaining, the angular velocity generated in a system will be the same, whether the mass

W is placed at the distance $\parallel SR = \sqrt{SO \times SG}$, or whe- $\parallel$ Supra.

ther the mass $\frac{W \times SR^2}{SD^2}$ is placed * at the distance SD . * Sect. VI, Prop. I.

Cor. 3. Let the system revolve in the same plane as before, and round an axis which passes through the centre of

gravity; and let $GQ = \sqrt{\frac{A \times GA^2 + B \times GB^2 + C \times GC^2, \&c.}{A + B + C}}$

being the distance of the centre of gyration † from the axis † Sect. VI, of motion: from the proposition it appears, that $GQ^2 = \frac{SG \times GO}{SD^2}$. Prop. I.

Cor. 4. Let $ABCD$ represent any pendulous body movable round an axis of motion HI which passes through S , and let any impact be impressed on the point D , (while the pendulum is at rest,) in an horizontal direction and perpendicular to the vertical plane $ABCD$. To find the angular velocity, which is communicated to the pendulum by the impact, let the weight of the pendulum be W , let G be the centre of gravity, O the centre of oscillation, and R the centre of gyration; then let a weight be assumed which is to W as SR^2 , or $SO \times SG$ to SD^2 , this weight

$= \frac{W \times SO \times SG}{SD^2}$ being collected into the point D , the

same angular velocity will be generated in the system, as when the pendulum is of the prior form, every thing else being the same.

This is the rule delivered in Mr. Robins' Gunnery, without demonstration, page 86. vol. 1. and by the application of it to his experiments, Mr. Robins was the first who ascertained with any tolerable degree of certainty the initial velocity of military projectiles.

Gallilæo had demonstrated that the track of a body projected in any direction not perpendicular to the horizon, was a parabola, the air's resistance being not considered, and this resistance was imagined to be so small, that bodies moving through it would not sensibly deviate from the path which they would describe in vacuo, the velocity and direction of projection being the same. § This opinion however erroneous was adopted by Dr. Hally, as well as several

Fig. XLVIII.

§ Philos. Transact. No. 129.

veral other eminent philosophers. That either the theory of of projectiles was defective, or the resistance of the air such as rendered it inapplicable to practice, was however afterwards discovered. When equal bullets are shot off from the same piece of ordnance, and with equal charges and at the same elevation above the horizon, the horizontal ranges and times of flight must be the same or very nearly so, (provided the figure of the balls be perfectly spherical, and their substance altogether homogeneous,) however great the air's resistance may be. This being the case, if the horizontal range were observed on any one discharge at a given angle of elevation, the initial velocity wherewith the shot quitted the cannon would be deduced from Galilæo's rules, provided the air's resistance were inconsiderable in its effects. But it appeared from experiments made on the same piece of ordnance from which bullets exactly equal were discharged with equal quantities of powder and at different elevations, that the initial velocities deduced by Galilæo's theory from the different horizontal ranges observed were entirely different. Neither did the horizontal ranges follow that law in respect of the elevations, which is demonstrated by Galilæo.

Various hypotheses were formed in order to account for these irregularities. Mr. Robins perceived that their true cause could not be determined, except the initial velocity of the shot in given circumstances were first with certainty known. The great velocities communicated to bullets by gunpowder, rendered it impossible to make any direct observation on them: Mr. Robins therefore used the following indirect method, which seemed fully to answer his intentions.

Fig.
XLVIII.

SABCD represents a pendulum moveable about an horizontal axis *IH*: a heavy block of wood *ABCD* was so affixed to the rod *ES*, that its anterior surface might be vertical when quiescent and parallel to a plane which passed through the axis of motion. When a bullet impinged perpendicularly on the surface *AD*, the pendulum was caused to vibrate on its axis of motion, and the weight of it was so adjusted in respect of the bullet's weight and velocity, that the angle of vibration might be such as admitted of the most exact determination by experiment.

Suppose then that a musket or other bullet were to strike perpendicularly against the point *D*, and the angle through which the pendulum is impelled by the shot observed. Mr. Robins' problem is to infer the velocity with which the bullet impinged against the pendulum from having

having given this angle, together with the dimensions and weight of the pendulum: the solution of this may serve as an illustration of the principles of rotation just demonstrated.

IX.

Let a musket or other ball perpendicu-
larly strike against and penetrate into the
substance of the pendulum $ABCD$ in the
point D ; let the angle through which
the pendulum is impelled by the shot be
 A° : having given the weight of the pen-
dulum and of the ball, and the distance
from the axis of the centres of oscillation
and gravity of the pendulum together
with the distance SD ; it is required to
assign the velocity with which the ball
struck the pendulum.

Fig.
XLVIII.

Let the weight of the pendulum $= W$, and that of the
ball $= p$; also, let O be the centre of oscillation, G the centre
of gravity, R the centre of gyration, and let Q be a weight
which being collected into the point D , a given force ap-
plied to D will generate the same angular velocity as if it
were applied against the pendulum itself in the point D .

This || equivalent weight $Q = \frac{W \times SG \times SO}{SD^2} = \frac{W \times SR^2}{SD^2}$ || Sect. VI.
Prop. VIII.

by the last proposition: also, let V be the velocity com-
municated to the point D , x the velocity of the bullet re-
quired, then the block of wood being nonelastic the laws
of collision give the following proportion $Q + P : P$
 $:: x : V$; wherefore the velocity of the ball $x = V \times$
 $\frac{Q + P}{P}$. Consequently since Q and P are known, it only

Cor. 4. and
Prop. I.

remains

remains to assign the velocity V which has been communicated to the point D , from having given the angle A° , through which the pendulum is impelled by the stroke. Let $SO = s$, $SD = d$, then the velocity acquired by the centre of oscillation in a pendulum, which describing from rest any arc of a circle has arrived at its lowest point, is equal to that acquired by a heavy body which has descended freely from rest by the acceleration of gravity, through a space equal to the versed sine of the arc described by the pendulum: In like manner, if any given velocity be communicated to the centre of oscillation of the pendulum when quiescent, it will rise through an arc whose versed sine is equal to the space through which a body falls freely from rest in order to acquire that velocity. This being the case, since the pendulum in the problem describes an arc A° , let b be the versed sine of the arc A° to radius $= 1$, then will the centre of oscillation or the point O describe an arc during its motion, the versed sine of which $= sb$, and consequently if $l = 16\frac{1}{2}$ feet, the initial velocity* of O will be

* Sect. VI.
Prop. V.
Cor. 3.

$= \sqrt{4lbs}$ feet in a second, and the velocity of the point $D = \sqrt{\frac{4lbd^2}{s}} = V$. This therefore will give us the velocity of the bullet the instant it struck the pendulum or $x = V \times \frac{Q+P}{P} = \sqrt{\frac{4lbd^2}{s}} \times \frac{Q+P}{P}$ feet in a second.

To illustrate this, let an example be assumed from Mr. Robins. The pendulum which he made use of weighed 56 pound 3 oz. or 675 oz. The distance of the impact from the axis $= 66$ inches $= SD$, $SO = 62.66 = s$, $SG = 52$. Mr. Robins † after having described the construction of his instrument and the conditions of this problem, says, “In the compound ratio of 66 to $62\frac{2}{3}$, and 66 to 52, take the quantity of matter in the pendulum to a fourth quantity, which is 42 pounds $\frac{1}{2}$ oz. Now geometers will know, that if the blow be struck in the centre of the piece of wood $ABCD$, the pendulum will resist the stroke in the same manner, as if this quantity of matter only (42 pounds $\frac{1}{2}$ oz.) was concentrated in that point, and the rest of the pendulum was taken away.”

† Robins’
Gunnery,
Vol. I.

† Prop.
VIII, Cor. 3.

This is manifestly the rule just demonstrated for a compound ratio of 66 to $62\frac{2}{3}$, and 66 to 52 is the same as the square

square of the distance of the impact to the rectangle under the distances of the centres of gravity and oscillation from the axis of motion; that is, in a duplicate ratio of the distance of impact to that of the centre of gyration from the axis. The equivalent weight therefore which in the preceding solution was denoted by $\mathcal{Q}$ will $= W \times$

$$\frac{SG \times SO}{SD^2} = \frac{675 \times 62.66\frac{2}{3} \times 52}{66 \times 66} = 504.95 \text{ oz.}$$

In the experiment of Mr. Robins, the chord of the angle through which the pendulum was impelled† by the stroke, was observed to be 17.25 inches to a radius of 71.125 inches. This gives the angle itself $= 13^\circ 55' 48''$; we have moreover for the determination of the bullet's velocity of impact from the proposition, $d = 66$, $s = 62\frac{2}{3}$

and $\mathcal{Q} = 42$ pounds $\frac{95}{100}$ of an oz. or 504.95 oz. $b =$

the versed sine of $13^\circ 55' 48''$ to radius 1 $= .02941057$,

$P = 1$ oz. and the bullet's velocity required $= \sqrt{\frac{4lbd^2}{s}}$

$\times \frac{\mathcal{Q} + P}{P} = 39.727 \times 505.95 = 20100$ inches, or 1675

feet in a second, l being $= 193$ inches: if according to Mr. Robins, l be assumed $= 196.06$, the velocity of the bullet will be $= 1688$. Mr. Robins† makes the velocity from the same data to be 1641 feet in a second: or if the computation had been exact according to his principles, the velocity would have been 1645. This difference is remarked in the excellent comment* on Mr. Robins' Principles of Gunnery: it was occasioned by his having assumed as true, that a given velocity being communicated to any point of the pendulum D when quiescent, will cause that point to describe an arc of which the versed sine is equal to the space through which a body must fall from rest by its gravity, to acquire that velocity: but this is true only when the given velocity is communicated to the centre of oscillation: if therefore a given velocity be communicated to any other point D , in order to find to what perpendicular altitude that point will rise, the velocity at the same time impressed on the centre of oscillation must be first found; then the perpendicular altitude to which the centre of oscillation rises will be equal to the space through which a body must fall freely by gravity to acquire that velocity, and from this the altitude to which the point struck will rise, may be inferred by the rule of

† Robins' Gunnery, page 87.

† Page 88.

* Euler's Comm. on Robins' Gunnery, p. 55. English Edit.

† Fig.
XLVIII.

• Sect. III.
Prop. V.
Cor. 3.

proportion. Thus let V be the velocity communicated to the point D †, then will $V \times \frac{SO}{SD}$ be the velocity communicated to the centre of oscillation, and if $l = 193$ inches, or $16\frac{1}{2}$ feet, the versed sine of the arc through which the centre of oscillation is impelled* $= \frac{V^2 \times SO^2}{4lSD^2}$, wherefore the perpendicular altitude through which the point struck D will ascend $= \frac{V^2 \times SO}{4l \times SD}$; and consequently if this altitude $= a$ by observation, V the velocity communicated will be inferred; for since $\frac{V^2 \times SO}{4l \times SD} = a$, we have $V =$

$$\sqrt{\frac{4l \times SD \times a}{SO}}. \text{ Mr. Robins' rule makes it } \sqrt{4la},$$

which will give the result too small in the proportion of $\sqrt{SO} : \sqrt{SD}$, when SO is less than SD , and greater in the same proportion when SO is greater than SD : thus in the example just referred to $1645 : 1688 : \sqrt{62\frac{2}{3}} : \sqrt{66}$.

In this solution several minute circumstances which may in some degree affect the velocity deduced from the rule are not taken into account, being in general insensible in experiment: such as the resistance of the air, the figure of the ball which has in the solution been supposed a point: also the ball's weight will alter the centres of oscillation and gravity, and consequently the equivalent weight $\mathcal{Q}$. Moreover, the time during which the ball penetrates the block is assumed as an instant, whereas it is of finite quantity, however small: but these circumstances are in general too minute to have any sensible effect: in particular cases corrections may be applied, if they should be necessary.

X.

Fig. XLIX.

In a system of bodies which may vibrate as a pendulum round any axis of motion perpendicular to the plane ABC : having given the centre of gravity G , draw any line $SGOL$ through G in the plane of motion

A

ABC; let it be required to assign in this line a point Q, through which if the axis of motion passes, the pendulum shall vibrate in the least time possible.

|| Let O be the centre of oscillation, Q the centre of suspension, and W the weight of the pendulum: moreover, let the sum of $A \times AG^2 + B \times BG^2 + C \times CG^2$, &c. = S: also, let $GQ = x$, $GO = y$; then $x + y$ will be the length of the pendulum which is to be a minimum by the problem, wherefore

$\dot{x} + \dot{y} = 0$, and $\dot{x} = -\dot{y}$; but *since $xy = \frac{S}{W}$, taking the

fluxions $x\dot{y} = -y\dot{x}$, in which equation substituting $-\dot{y}$

for $\dot{x}$, we have $x = y$, that is, $GQ = GO$, and $xy = \frac{S}{W} =$

$\frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{A + B + C} = GQ^2$, wherefore $GQ =$

$\sqrt{\frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{A + B + C}} =$ to the † distance of

the centre of gyration from the axis of motion, when that axis passes through the centre of gravity.

Let therefore the system revolve round the axis which passes through the centre of gravity; and suppose R to be the centre of gyration when all the matter is collected into any line SGL, each particle keeping its respective distance from G, then with the centre G and distance GR describe a circle, and if the system vibrates in the same plane as before, the time of vibration will be less when the axis of suspension passes through any point in the circumference of the circle just described, than when it is further from the centre of gravity or nearer to it.

Thus, let ABC represent a cylinder, if the mass contained in that body were collected into the line GC, each particle preserving the same distance from the axis as before, the centre of gyration R would be distant from the centre of the circle G by $\frac{1}{\sqrt{2}}$ the radius: if therefore the

radius GC be divided into 100 parts, and GR be taken equal to about 71 of them, and the cylinder vibrates in its own plane round an horizontal axis passing through R, the time of vibration will be less than if R were placed at

a greater or less distance from the centre of gravity. Thus also, if a right line AB or slender cylinder vibrates about a point R , and if $AB = 100$ parts, and AR about 21 of them, the line will vibrate about R in less time than if AR were greater or less, the angle through which the pendulum vibrates being the same in all these cases.

Whenever the centre of gyration of any figure which revolves round an axis passing through the centre of gravity G , is mentioned, it is meant to be the centre of gyration of the body when collected into any given line in the plane of motion passing through the axis, and each particle keeping its respective distance from the axis.

This is here mentioned, because when the axis of motion passes through the centre of gravity, $SG = 0$, and SO is infinite; but $SR^2 = SG \times SO = \frac{A \times GA^2 + B \times BG^2 + C \times GC^2}{A + B + C}$

also in this case, and may be determined either by finding the sum of $A \times GA^2 + B \times BG^2$, &c. and dividing it by $A + B + C$, or by supposing the mass to be collected into any line GD , which passes through the centre of gravity, each particle preserving its respective distance from the axis. Let o be the centre of oscillation of the line GD , when the mass is so collected into it, and g its centre of gravity, then will $SR^2 = Sg \times So$ as before. The quantity SR being ascertained, with the centre G and distance GR , let a circle be described: then the centre of gyration may be any where in the circumference of the circle, the angular velocity * generated in the system in a given time depending on its distance from the axis only.

* Sect. VI.
Prop. I.
Cor. 6.

XI.

Fig. LI.

Let ABC represent a section or plane which passes through the centre of gravity of any irregular body. Suppose this body to vibrate as a pendulum in the plane ABC round an axis which passes through s ; it is required to determine the distance from the centre of gravity of any other

other point Q , through which if the axis passes, the time in which the pendulum vibrates through a given angle shall be the same as before, the plane of vibration not being altered.

Let R be the centre of gyration of the system, when it revolves round an axis which passes through the centre of gravity, and take a third proportional GQ to the lines GS , GR . With the centre G , and distances GS , GQ describe the circles LSV , LQV : if the axis of the pendulum passes through any point in the circumferences of these two circles, the distance of the centre of oscillation from the axis of motion will be always equal to the sum of the radii of the circles, the time of vibration therefore through a given angle will be the same in both cases.

Let the pendulum vibrate round S , and let O be its centre of oscillation: then supposing the particles which compose the system when referred to the plane of vibration to be A , B , C , &c. the sum of all the products $A \times AG^2 + B \times BG^2 + C \times CG^2$, &c. $= GR^2 \times A + B + C = SG \times GO \dagger$ || Sect. VII. Prop. VI. Cor. 3.
 $\times A + B + C$, &c. wherefore $GO = \frac{GR^2}{SG}$, and $SO = GO + SG$ † Sect. VI. Prop. VIII.
 $= \frac{GR^2}{SG} + SG = \frac{GR^2 + SG^2}{SG}$; but by construction $GR^2 = SG$
 $\times QG$, wherefore $SO = \frac{SG \times QG + SG^2}{SG} = QG + SG$
 $=$ to the sum of the radii. Now let the pendulum vibrate in the same plane round Q , and let H be its centre of oscillation, then will $\dagger GH = \frac{GR^2}{GQ}$, and $QH = GQ + \dagger$ † Sect. VI. Prop. VIII. Cor. 3.
 $\frac{GR^2}{GQ}$: but $GR^2 = GS \times GQ$ by construction, wherefore by
substitution $QH = GQ + \frac{GS \times GQ}{GQ} = GQ + GS$, or
the sum of the circle's radii as before.

Cor. 1. It appears that there are two distances of the axis of motion from the centre of gravity of any pendulum, either of which being assumed, the time wherein the pendulum

pendulum vibrates through a given arc will be the same, the plane of vibration not being altered.

* Sect. VI.
Prop. X.

Cor. 2. When the axis of motion passes through the centre of gyration R , there is no other distance of the axis of motion from the centre of gravity, which will cause the pendulum to vibrate in the same time, the two distances before determined now coinciding with GR ; and in this case the * pendulum vibrates in a given angle in the least time possible.

Cor. 3. In any pendulum which vibrates in a given plane if an axis of motion be made to pass through the point which was before the centre of oscillation, the former point through which the axis passed will become the centre of oscillation.

XII.

Fig. LII.

In a system moveable round a fixed axis, and consisting of bodies the weights and figures of which are known: having given the moving force P , and the distance from the axis SD , at which it is constantly applied, let it be required to determine the force which accelerates D .

Let the moving force be a weight P applied so as to act on D by means of a line wound round the circumference DFE : whatever be the figures of the bodies which compose the system, if the component particles be $A+B+C$, &c. $+E+F+G$, &c. the force which accelerates D will be =

$$P \times SD^2$$

$$P \times SD^2 + A \times AS^2 + B \times BS^2, \&c. + E \times ES^2 + F \times FS^2, \&c.$$

|| Sect. VI.
Prop. V.
Cor. 3.

but in any body which revolves round a fixed axis S , if SO be the distance of the centre of oscillation, SG the distance of the centre of gravity from the axis, and W the bodies' weight, the || sum of the products $A \times AS^2 + B \times BS^2 + C \times CS^2$, &c. will = $SO \times SG \times W$: let therefore the weights of the bodies which constitute the system be W, W, W ; let the distances of the centres of oscillation, belong-

belonging to the bodies considered to revolve separately round the axis, be O, O, O , &c. respectively, and the distances of the centres of gravity be G, G, G ; then the force

$$\text{which accelerates } D = \frac{P \times S D^2}{P \times S D^2 + O G W + O G W + O G W}.$$

When this rule is applied to practice, it is manifest, that the axis of motion must pass through the common centre of gravity, (this being the best method of rendering the moving force constant) in which case the centre of oscillation of the whole system is * infinitely distant; but both the centres of gravity and oscillation of the separate parts are determined as if these bodies revolved round the axis of motion separately: thus, let AB represent a cylinder, at the extremity of which two equal spheres are fixed, so that the axis of the cylinder shall pass through the centres of the spheres F and G . Let $SF = s$, $SG = a$, $FH = r$, the radius of the cylinder $= c$, and $SA = b$, and let the weight of either sphere $= W$, and that of the cylindrical arm $SA = W$; also let $SG = \frac{SA}{2}$, G being the

* Sect. VI.
Prop. VI.

Fig. LIII.

centre of gravity of SA , then if the sphere A were to revolve horizontally round the vertical axis LK the distance of the centre of oscillation from the axis of motion $= a + \frac{2r^2}{5a}$, let this $= SO$; also, if the cylinder SA revolves round the axis passing through S , the distance of the centre of oscillation from the axis $= \frac{4b^2 + 3c^2}{6b}$; let

this be represented by SO ; then if a force P be applied to the point D , the force by which P is accelerated will be

$$\frac{P \times S D^2}{P \times S D^2 + 2SO \times SG \times W + 2SO \times SG \times W}, \text{ or if } SD = d,$$

$$\text{the force} = \frac{P \times d^2}{P d^2 + 2W \times \frac{5a^2 + 2r^2}{5} + 2W \times \frac{4b^2 + 3c^2}{12}}.$$

When a body revolving round an axis which passes through its centre of gravity, is contained under any single geometrical figure, the force of acceleration will be at once obtained by finding the sum of the products which arise from multiplying each particle into the square of its distance from

Fig. LIV.

from the axis; thus let a cylinder *DEB* revolve round its axis, passing through the centre *S* by the action of a force *P* applied at *D*, the force which accelerates *D* will be

$$\frac{P \times SD^2}{A \times AS^2 + B \times BS^2 + C \times CS^2, \&c.} : \text{supposing } A, B \text{ and } C$$

to be the particles which compose the cylinder; all these particles may be referred to one plane, i. e. that of *EBD*. To find the sum of $A \times AS^2 + B \times BS^2, \&c.$ let $SA = x$, $SD = d$, the cylinder's weight = W , and let a circle *AFH* be drawn through *A* with the centre *S* and distance *SA*. Moreover, let *a* be a point contiguous to *A*, and with the centre *S* and distance *SA* describe a circle *afb*: the area intercepted between these two circles will be to the whole area, as the weight of an increment of the cylinder *A* is to its whole weight; but if $p = 3.14159$, the circumference of the annulus = $2px$, and its area = $2pxx$, and since the whole area = pd^2 , we have this

$$\text{proportion } 2pxx : pd^2 :: A : W, \text{ or } A = \frac{2pxx \times W}{pd^2},$$

and this particle multiplied into the square of its distance from the axis = $\frac{2W \times x^3 x}{d^2}$, and the sum of all the pro-

$$\text{ducts similarly taken} = \frac{Wx^4}{2d^2}, \text{ or when } x = d, \text{ this sum}$$

$$= \frac{Wd^2}{2}; \text{ the moving force therefore is in this case } P \text{ and}$$

$$\text{the mass moved } \frac{Wd^2}{2d^2} = \frac{W}{2}. \text{ The force therefore which}$$

$$\text{accelerates the point } D = \frac{2P}{W}. \text{ If the force } P \text{ should}$$

consist of a weight possessing inertia, applied to turn the cylinder by means of a line going round it, and terminating in *P*, the inertia of *P* must be added to the mass

$$\text{moved, this will give the accelerating force } \frac{2Pd^2}{2Pd^2 + Wd^2}$$

$$= \frac{2P}{2P + W}. \text{ In the same manner, let a solid be formed}$$

by the revolution of any plane figure round a strait line considered as an axis, if the solid should be put into motion round this axis by the action of any force applied at a given point, the force by which that point is accelerated may be determined.

By

By these propositions the revolving motion of regular bodies may be ascertained from the necessary data, and of irregular bodies also when the axis of motion passes not through the centre of gravity; for if the distances of the centres of gravity and * oscillation from the axis of motion can be found by experiment, the distance of the centre of gyration from the axis will be a mean + proportional between them. But when a system of irregular bodies revolves round an axis which passes through their common centre of gravity, the system must be caused to vibrate as a pendulum round some other axis parallel to the former: and if the distance of the centre of oscillation from the axis of vibration be thence inferred, this with the distance of the centre of gravity from the centre of oscillation will give the distance of the centre of gyration from the axis which passes through the centre of gravity, and round which the system revolves.

* Prop. VI.

Cor. 4.

† Sect. VI.

Prop. VIII.

Cor. 1.

XIII.

Let sik represent the plane in which any irregular body or system of bodies revolves round an axis of motion passing through the centre of gravity G : having given the weight of the system and the point D , to which a given force is applied, it is required to assign the distance of the centre of gyration from the axis of motion.

Fig. LVI

Let G be the common centre of gravity of the system, and suppose the whole to be referred to the plane sik , let the system revolve in this plane round a fixed horizontal axis which passes through G : through G draw any line SGo , and with the centre G and any distance GD draw the circle $SDIK$, and suppose all the matter which is † coincident with the circumference sik just described, to be collected into A : in like manner imagine all the matter contained in other concentric circles to be collected into the corresponding points in the line Go , let g be the

† Sect. VI.

Prop. I.

Cor. 6.

F f

centre

centre of gravity of the matter contained in this line, and let o be the centre of oscillation; then if a weight $=$

* Sect. VI. $\frac{W \times S_o \times S_g}{GD^2}$ * were collected into the circumference SIK ,
Prop. VIII.
Cor. 2.

or any point of it D , the other parts of the system being removed, the point D would be accelerated in the same manner as when the system itself is acted on by the same force applied at the same distance from the axis. In this

† Sect. VI. case the force which accelerates $\dagger D = \frac{SD^2 \times P}{S_g \times S_o \times W}$,
Prop. VIII.

or if the inertia of P is considered, the force $= \frac{SD^2 P}{P \times SD^2 + S_g \times S_o \times W}$: but since the figure of the system

is not geometrical, the quantity $W \times \frac{S_g \times S_o}{SD^2}$, or the equivalent weight cannot be estimated by theory; it may be determined by experiment in the following manner.

Let a weight P be applied to communicate motion to the system by means of a very slender and flexible line going round the wheel SDK , through the centre of which the axis passes: let this weight descend from rest through any convenient space s inches, and let the observed time of its descent be t seconds, then if l be the space through which bodies descend freely by gravity in one second, the equivalent weight sought $= \frac{W \times S_g \times S_o}{SD^2} = \frac{P \times t^2 l}{s}$

— P . Let this weight $= \mathcal{Q}$, and we shall obtain the sum of each particle multiplied into the square of its distance from the axis of motion; and consequently the distance of the centre of gyration from the same axis: for since $\mathcal{Q}:$

† Sect. VI. $W :: Gg \times Go : GD^2$, † we have $A \times GA^2 + B \times GB^2 +$
Prop. X.
ad finem. $C \times GC^2, \&c. = GD^2 \times \mathcal{Q} = Go \times Gg \times W$, and since $Go \times Gg = GR^2$ †, it follows, that the distance of the centre of

† Ibid.

gyration from the axis, or $GR = GD \times \sqrt{\frac{\mathcal{Q}}{W}}$. This hav-

ing been once determined, the force which accelerates any point of the system in other cases, at whatever distance the moving force be applied, will be known. For the angular velocity of the system will always be the same, as if the whole mass were collected into the centre of gyration.

§ Sect. VI.
Prop. VIII.

|| Petrop.
Comm.
Vol. II.
p. 214.

An experimental method is also delivered by Bernoulli, and long ago || transcribed: a demonstration of it may be here inserted as a further illustration of the principles of rotation.

rotation. In any irregular figure referred to the plane Fig. LVI. SIK , let G be the centre of gravity, D the point to which a moving force p is applied to turn the system round an axis which is perpendicular to the plane SIK . With the centre G and distance GD describe a circle SLM ; then taking any point in the circumference S , let the system vibrate in the plane SIK perpendicular to the axis which passes through S , and let O be the centre of oscillation of the system so vibrating: then let $GO = b$, $SG = GD = d$, the weight of the system $= W$, and the moving force

$$= p: \text{the force which accelerates } D = \frac{p \times GS}{p \times GS + w \times GO}$$

$$= \frac{pd}{pd + wb}: \text{the demonstration of this rule follows im-}$$

mediately from the preceding principles; for since the centre of gravity is G , and of oscillation O , it follows,

that $*SG \times GO \times w = A \times AG^2 + B \times BG^2 + C \times CG^2$, * Sect. VI.
Prop. VI.
Cor. 3.

or the sum of all the products which are formed by multiplying each particle in the system into the square of its distances from the axis, when that axis passes through the common centre of gravity: wherefore $SG \times GO = bd =$

$$\frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{w} = \text{the square of the di-}$$

stance of the $\dagger$ centre of gyration from the axis, when the system revolves round G ; let the mass w be collected into $\dagger$ Sect. VI.
Prop. VIII.

the centre of gyration at the distance $\sqrt{bd}$ from the axis; then will the communication of motion be the same as if

the equivalent weight $w \times \frac{bd}{d^2}$ were collected in the point

$D\dagger$, the other parts of the system being removed: here the $\dagger$ Sect. VI.
Prop. VIII.
Cor. 2. force which communicates motion to the point D is p , the

mass moved $p + \frac{bw}{d}$, and the $\parallel$ force which accelerates D $\parallel$ Sect. I.
Prop. IX.

$$= \frac{p}{p + \frac{bw}{d}} = \frac{pd}{pd + bw}.$$

The distance SO is obtained practically thus; suppose that the system performed n least vibrations in t seconds,

if $l = 193$ inches, $p = 3.14159$, &c. $SO = \frac{2lt^2}{n^2p^2}$ the length required.

The objection against the application of this method to practice is the difficulty of ascertaining the distance of the centre of oscillation from the axis S , or the space SO by experiment.

This arises from the friction of the axis, the air's resistance and the inequality of the arcs through which the pendulum vibrates, to all which it is difficult to apply the exact corrections; and it is obvious, that since the lengths of pendulums vary in a duplicate ratio of the times of vibration, a small error in estimating the time causes a considerable variation in the lengths deduced: so small an error as 9 seconds in 10 hours causes * a change of length in

* Sect. IV.
Prop. 1.
Cor. 8.

a pendulum which vibrates seconds $= \frac{1}{41}$ part of an inch, and this error even some pendulum clocks are liable to: but when pendulums are suspended in a manner less accurate, the danger of error in determining the centre of oscillation becomes much greater, the lengths thus deduced having been mistaken by persons well versed in experiments, but who have trusted to experiment only, by $\frac{1}{2}$, or even a greater proportion of the true length.

In the propositions concerning rotation which have been demonstrated, the axis of motion has been supposed fixed; but there are other cases in which the axis itself moves with the common centre of gravity through which it passes. To exemplify this, let G represent the centre of gravity of a sphere placed and sustained quiescent on an inclined plane, by a force which acts in a direction parallel to the plane: it is known, that in this case, the force in the direction of the plane is to the body's weight, as the height of the plane to the length. When therefore the sphere is left to itself, it will endeavour to descend along the plane by a force which is to its natural gravity as the plane's height is to its length.

Fig. LVII.

Part of the sphere's natural weight is lost by its pressure against the plane in a direction perpendicular to it: let the line AB represent in quantity and direction the sphere's natural gravity; this being resolved into two, EB perpendicular to plane, and AE in the direction of it. Then the force which impels the sphere will be that part of its natural weight which is expressed by the fraction $\frac{AE}{AB}$.

If the sphere was at liberty to move freely in the direction

tion of this force, each particle would partake of an equal motion, being the same as that with which the centre of gravity moved: and this would be the sphere's real motion were there no adhesion between the surfaces which prevented the sphere from sliding down the plane; it is upon this principle only, that the laws demonstrated by Galilæo can be applied to the descent of bodies along inclined planes. Indeed the ratios of the spaces described on a given inclined plane, will be the same in given circumstances, whether the sphere rolls or slides; but the absolute quantity of them cannot be from hence assigned: and as the surfaces cannot be made so smooth as to prevent their rotation when they descend along inclined planes, it must not be expected that their motion will correspond with the theory, unless the rotation be taken into account. To consider this a little further: While the impelling force acts on the centre of gravity G , in the direction parallel to the plane HG , the point A , which is contiguous to the plane's surface, will be prevented from moving along with the centre of gravity by the adhesive force existing between the point A and the plane: by which means, while the centre of gravity descends along the plane in a right line, the sphere will revolve on an axis, and all the particles except those in the axis of motion will describe cycloids.

The force which accelerates the centre of gravity of a sphere or cylinder, when rolling down an inclined plane, is next to be considered.

XIV,

The force which accelerates the centre of gravity of a sphere, while it rolls down an inclined plane, is to the force by which it would be accelerated were it to slide, in the ratio of five to seven.

Let the height of the plane be H , its length L , and the sphere's weight $= W$; then will the force by which the
centre

centre of gravity endeavours to descend along the plane be $\frac{W \times H}{L}$. Also, if the sphere were to slide, the weight

moved being $= W$, the accelerating force $= \frac{H}{L}$: but

since the surface revolves with a velocity equal to that with which the centre of gravity moves, the inertia arising from the sphere will consist not only of the sphere's mass of matter, but of the resistance which proceeds from the rotatory motion. Let the sphere G be supposed to descend by sliding, then will the inertia be W ; but let a line passing through the centre G be caused to communicate with the vertical circumference of a great circle of another sphere H , equal in all respects to G , and moveable round an horizontal axis of motion by means of this string: here it is manifest, that the circumference of the sphere must be accelerated exactly in the same manner as the centre of gravity G . Now it is immaterial whether this rotatory motion be produced in the descending sphere G , or in any other equal to it in weight and magnitude, the motion produced in a given time being the same in both cases: it follows, that the force which accelerates the rolling sphere will be the same as that which accelerates the sphere which slides, and at the same time communicates a rotatory motion to an equal sphere H , so that the circumference of the latter shall move equally fast with the centre of the descending sphere.

Let R be the centre of gyration of either sphere, W the weight; then will the inertia of the sphere H which resists the communication of motion to its circumference to which the moving force is applied be the same, as if

* Sect. VI.
Prop. VIII.
Cor. 2. $W \times \frac{HR^2}{HD^2}$ * were collected into the circumference, since

it moves equally fast with the centre G ; when therefore the sphere rolls down the plane, the whole mass moved or resistance arising from inertia will be $W + \frac{W \times HR^2}{HD^2}$, or

since in the sphere $HR^2 = \frac{2HD^2}{5}$, the inertia will be $\frac{7W}{5}$,

and because the moving force is $\frac{W \times H}{L}$, it follows, that the

force which accelerates the centre of gravity $G = \frac{5WH}{7L}$,
which

which is to the force which would accelerate the sphere when sliding down as 5 : 7.

Cor. 1. The absolute force whereby motion is generated in the circumference of the sphere A is $= \frac{2W}{7} \times$

$\frac{H}{L}$. For the equivalent weight at the circumference of the

sphere H is $= \frac{2W}{5}$, and the force which accelerates

the circumference $= \frac{5}{7} \times \frac{H}{L}$, wherefore the moving force

which is always equal to the weight moved multiplied in- † Sect. I.
Prop. IX.
to the number expressing the accelerating force will be $=$

$$\frac{2W}{5} \times \frac{5}{7} \times \frac{H}{L} = \frac{2W}{7} \times \frac{H}{L}.$$

Cor. 2. In the same manner, let a cylinder revolve down an inclined plane the axis being always horizontal, the force which accelerates the axis will be found that

part of gravity which is expressed by the fraction $\frac{2}{3} \times \frac{H}{L}$.

Let ABC represent the section of any irregular figure Fig. LVIII.
whatever passing through its centre of gravity G : with the

centre G and any distance GS describe a circle. Suppose the whole system to roll down an inclined plane, being sustained on the circumference DS which is always in contact with the plane and vertical. If R be the centre of gyration of the figure, the force which accelerates the centre of gravity will be that part of the acceleration of gravity,

which is expressed by the fraction $\frac{GD^2}{GD^2 + GR^2} \times \frac{H}{L}$, and

that part of the moving force which causes the rotation $=$
 $\frac{W \times GR^2}{GD^2 + GR^2} \times \frac{H}{L}$, the demonstration of which follows from the preceding solution.

Cor. 3. Let the figure be suspended on an axis passing through any point S in the circumference; and when it vibrates in its own plane, let O be the centre of oscillation:

the force which accelerates the centre G will also $= \frac{SG}{SO}$

$\times \frac{H}{L}$: for $\parallel GR^2 = \frac{A \times AG^2 + B \times BG^2 + C \times CG^2, \&c.}{A + B + C} = \parallel$ Sect. VI.
Prop. VI.
Cor. 3.
 GO

$GO \times SG$, or $GO \times GD$; substituting therefore $GO \times GD$ for GR^2 in the former expression, we have $\frac{GD^2}{GD^2 + GR^2} = \frac{GD^2}{GD^2 + GO \times GD} = \frac{GD}{GD + GO} = \frac{SG}{SO}$, and the force which accelerates the centre of gravity $= \frac{SG}{SO} \times \frac{H}{L}$, and that part of the moving force by which the rotation of the circumference is caused $= \frac{GO \times W}{SO} \times \frac{H}{L}$.

The force whereby spheres, cylinders, &c. are caused to revolve as they move down an inclined plane instead of sliding, is the adhesion of their surfaces occasioned by the pressure of the rolling body against the plane: this pressure is part of the body's weight; for the weight being resolved into two, one force in the direction of the plane, and the other perpendicular to it, the latter is the force of the pressure, and while the same body rolls down the plane, will be that part of its weight which is expressed by the cosine of the angle of the plane's elevation divided by radius. It appears from hence, that since the cosine of an arc is decreased while the arc increases, the pressure may by elevating the plane to a certain angle above the horizon become so small, that the adhesion shall be less than the force necessary to make the circumference of the sphere revolve so fast as the centre of gravity descends: in this case the sphere descends by sliding partly and partly by rotation.

The same will happen when the surfaces of the revolving body and of the plane are so smooth, that their mutual adhesion is less than the moving force or weight necessary to generate the rotation of the sphere; that is, re-

ferring to the example just || mentioned less than $\frac{W \times GO}{SO}$.

¶ Supra.

But a body may be caused to roll at all elevations of the inclined plane, by winding a line round the cylindrical surface in a manner represented in fig. LVIII. as the cylinder G descends it unwinds itself from the line, which in this case supplies the place of the adhesion between the surfaces; let the rolling body be suspended on any point of the circumference S , let G be its centre of gravity being coincident with the axis of the cylinder, and suppose O to be the centre of oscillation of the figure, when vibrating round S in its own plane, then will the force which

which accelerates the centre of gravity be that part of the accelerating force of gravity which is expressed by $\frac{SG}{SO} \times \frac{H}{L}$, and if the body's weight $= W$, the absolute force or

weight whereby the rotation is || caused $= W \times \frac{GO}{SO} \times \frac{H}{L}$; || Sect. I, Prop. IX.

wherefore let a line KS be drawn coincident with the plane, and a wheel K be placed in a vertical plane in contact with KS ; moreover, let a line wound round the descending cylinder DS be applied over the wheel or pulley, and to the extremity of this line, let a weight $p = W \times \frac{GO}{SO}$

$\times \frac{H}{L}$ be affixed; this weight will remain quiescent while the body ABC rolls down the plane, being just equal to the tension of the string Kp .

It is obvious, that if the plane is elevated so as to be perpendicular to the horizon, the body ABC will descend in a vertical line, by the action of an accelerating force $= \frac{SG}{SO}$, since $H = L$, and the weight p , which is equivalent to the tension of the string Kp , will become $= W \times \frac{GO}{SO}$.

XV.

Let ABC represent a body moveable round its centre of gravity s , through which an horizontal axis of motion passes; let R represent the centre of gyration when the whole mass is collected into the line $sGRo$, each particle keeping its respective distance from the axis: having given the weight of the whole body, and of p which communicates motion to the system

Fig. LIX.

Gg

tem

tem by means of a line DP wound round the circle DEF , it is required to ascertain the angular velocity generated in the system in a given time.

Let the distance of the centre of gyration from the axis $SR = r$, $SD = d$, the weight of the body $ABC = w$, then

* Sect. VI.
Prop. VIII.
Cor. 2.

will the force which * accelerates D be $\frac{p d^2}{r^2 w}$, were p destitute of inertia; but if the inertia of p be considered, the

force of acceleration is $\frac{p d^2}{d^2 p + r^2 w}$: let $c = 3.14159$, $l = 193$ inches, then in the time t the velocity generated in the point D , or descending weight p , will be that of

† Sect. III.
Prop. II.

$\frac{2 l t p d^2}{d^2 p + r^2 w}$ † inches in a second; and since the circumference of the circle $EFD = 2 c d$, the angular velocity generated in the time $t = \frac{360^\circ \times 2 l t p d^2}{2 c d \times d^2 p + r^2 w} = \frac{360^\circ l t p d}{c \times d^2 p + c r^2 w}$

degrees, or $\frac{l t p d}{c d^2 p + c r^2 w}$ revolutions in a second.

degrees, or $\frac{l t p d}{c d^2 p + c r^2 w}$ revolutions in a second.

Cor. 1. The angular velocity generated in the system during the descent of the weight p through any space $s =$

$\sqrt{\frac{l s p}{c^2 d^2 p + c^2 r^2 w}} \times 360^\circ$ degrees, or $\sqrt{\frac{l s p}{c^2 d^2 p + c^2 r^2 w}}$

revolutions in a second. Let F be the force which accelerates p ; then the absolute velocity generated in D during the descent of p through a space s by the accelerating

† Sect. III.
Prop. V.
Cor. 3.

force $F = \sqrt{4 l s F}$ † inches in a second; and any arc $\sqrt{4 l s F}$ is to $2 c d$ the circumference of the circle EFD ,

as A° the angle subtended by $\sqrt{4 l s F}$ is to 360° : wherefore $A^\circ = \sqrt{4 l s F} \times \frac{360}{2 c d}$, being the angular velocity

expressed in the number of degrees which would be described in a second by the system, were the acceleration to cease immediately after the weight p has descended through the space s ; and restoring the value of $F =$

$\frac{p d^2}{d^2 p + r^2 w}$

$\frac{p d^2}{d^2 p + r^2 w}$, the angular velocity $= \sqrt{\frac{l s p}{c^2 d^2 p + c^2 r^2 w}}$
 $\times 360^\circ$ degrees, or $\sqrt{\frac{l s p}{c^2 d^2 p + c^2 r^2 w}}$ revolutions in a
 second.

Cor. 2. The space described by the weight p in its de-
 scent from rest during t * seconds $= \frac{l t^2 p d^2}{d^2 p + r^2 w}$, and con- * Sect. III.
 frequently the time of describing any space s , that is, $t =$ Prop. IV.
 $\sqrt{\frac{s d^2 p + s r^2 w}{l p d^2}}$ seconds. Cor. 5.

Cor. 3. The space described by p from rest while an an-
 gular velocity of n revolutions in a second is generated $=$
 $\frac{n^2 c^2 d^2 p + n^2 c^2 r^2 w}{l p}$.

Cor. 4. The force which † accelerates the centre of gyra- † Sect. VI.
 tion $= \frac{p d r}{d^2 p + r^2 w}$. Prop. II.

Cor. 5. The absolute velocity generated in the weight
 p while it descends from rest through the space $s =$
 $\sqrt{\frac{4 l s p d^2}{d^2 p + r^2 w}}$, and the velocity generated during the
 same descent in the centre of gyration $= \sqrt{\frac{4 l s p r^2}{d^2 p + r^2 w}}$.

Cor. 6. The velocity generated in the centre of gyra-
 tion in the time $t = \frac{2 l t d p r}{d^2 p + r^2 w}$ inches in a second.

While motion is communicated to a system of revolving
 bodies by the action of a descending weight p , the force
 of this weight is employed partly to move the system and
 partly to move itself: but it frequently happens in the
 application of the principles of rotation to practice, that
 the inertia of the weight p is too inconsiderable to have
 sensible effect.

XVI.

In any systems of bodies revolving by the action of weights (considered as without inertia) the absolute forces which impel the centres of gyration, are in a compound ratio of the quantities of matter or weights of the systems and the velocities generated in the centres of gyration, if the times of motion be equal: and the absolute forces or weights which are applied to communicate motion to the systems, are in a compound ratio of the quantities of matter moved or weights of the systems and a duplicate ratio of the velocities generated in the centres of gyration, provided the spaces described by the descending weights be equal.

The notation of the last proposition remaining, let v be the velocity generated in the centre of gyration in t seconds, then † we have $v = \frac{2lt p dr}{d^2 p + r^2 w}$, and when the inertia of $p = 0$, $v = \frac{2lt p d}{wr}$, and $\frac{p d}{r} = \frac{v \times w}{2lt}$; but the absolute force which acts on D is p , and that which acts on the centre of gyration $= \frac{p d}{r}$: wherefore the force which acts on the centre of gyration $\frac{p d}{r}$ is proportional to the quantity $\frac{v \times w}{2lt}$, and in a given time $2lt$ being constant,

† Prop. XV.
Cor. 6.

stant, will be as $v \times w$, that is, as the velocity generated into the quantity of matter, which is the first part of the proposition.

The velocity generated in the centre of gyration, while the descending weight p describes the * space s is $\sqrt{\frac{4 l s p}{w}}$ * Sect. VI. Prop. XV. Cor. 5.
inches in a second: let this $= v$, we have therefore $v^2 = \frac{4 l s p}{w}$ and $p = \frac{v^2 w}{4 l s}$; but p is the absolute force which acts on the system, and this force or weight, if the space s which it descends through be the same, will be as $v^2 \times w$, or in the ratio of the quantities of matter moved and a duplicate ratio of the velocities generated jointly, which is the second part of the proposition to be proved.

The following conclusions are inferred from the two last propositions, on a supposition that the inertia of the moving force is $= 0$.

1. If a force p communicating motion to a given system of revolving bodies, is applied at different distances SD , SI from the axis, the forces which accelerate the points D and I in the two cases, are in a direct duplicate ratio of these distances.

2. The notation remaining the space described by the weight p from rest in t seconds $= \frac{l t^2 d^2 p}{w r^2}$.

3. The absolute velocity generated in the descending weight p in describing a space s from rest $= \sqrt{\frac{4 l s d^2 p}{w r^2}}$.

4. The velocity generated in p during its descent from rest in t seconds $= \frac{2 l t d^2 p}{w r^2}$ inches in a second.

5. The angular velocity generated in the system in t seconds $= \frac{l t p d}{c r^2 w} \times 360$ degrees, or $\frac{l t p d}{c r^2 w}$ revolutions in a second.

6. The angular velocity generated during the descent of the weight p through a space $s = \sqrt{\frac{l s p}{c^2 r^2 w}}$ revolutions in a second.

7. The

7. The velocity generated in the centre of gyration in t seconds $= \frac{2 l t d p}{\omega r}$.

8. The velocity generated in the centre of gyration during the descent of the weight p through a space $s = \sqrt{\frac{4 l s p}{\omega r}}$.

9. The time in which the centre of gyration describes from rest any space $s = \sqrt{\frac{s \omega r}{p l d}}$.

10. The velocity generated in the centre of gyration during the time that point describes any space $S = \sqrt{\frac{4 l S p d}{\omega r}}$.

From the 8th Cor. it follows, that the motion of the centre of gyration generated by the force of a weight descending through a given space S , applied at any distance from the axis whatever, is the same as would be generated in the centre of gravity of the whole system, when at liberty to move freely; if the given weight were applied to accelerate it from rest through a space $= S$.

From these propositions the principle denominated *Conservatio virium vivarum*, as far as it is consistent with truth, is readily deduced. This principle was formerly supposed to constitute a theory not entirely consistent with that which is established upon the Newtonian laws of motion; but the different opinions concerning the measures of force having been long acknowledged a dispute about words, it has become unnecessary to consider the subject in a controversial manner.

According to this theory, Leibnitz and Bernoulli estimate the force of bodies in motion by the sum of the products, which are formed by multiplying each particle into the square of its velocity.

Though this has long ago been demonstrated by various writers to be false, when applied in a general way to the collision of bodies; yet it is certainly true as far as regards the communication of motion by constant acceleration, under certain restrictions and conditions, and is in fact no other than a proposition easily deduced from the Newtonian laws

laws of motion. This has been sufficiently shewn when the motion communicated is * rectilinear; the application * Sect. III. of it to bodies which revolve round a fixed axis will serve Prop. VI. for the further illustration of the theory.

XVII.

Let ABC represent the plane in which a Fig. LIX.
body or system of bodies revolves round
an horizontal axis which passes through
the centre of gravity s . Suppose mo-
tion to be communicated to this system by
the action of a weight p , affixed to a line
wound round the circumference of a cir-
cle, the plane of which is the same with
that of the rotation, and the centre coin-
cident with the axis of motion: then if p
by descending through any space s , ge-
nerates velocity in the revolving system,
the sum of the products of each particle
in the system, multiplied into the square
of the velocity generated in it, will be
equal to the sum of the products of each
particle in the body p into the square
of the velocity which would be generated
in it by gravity, if it descended freely from
rest through the same space s .

The notation of the preceding propositions remain-
ing, the velocity generated in the point D during the

descent of p through the space $s = \sqrt{\frac{4 l s p d^2}{\pi v r^2}}$, and || Sect. VI.
Prop. XVI.
Cor. 3.
con-

consequently the velocity of any particle $A = \sqrt{\frac{4ls p d^2}{w r^2}} \times \frac{SA}{d}$, and the square of the velocity generated in $A = \frac{4ls p d^2}{w r^2} \times \frac{SA^2}{d^2}$: and in like manner the square of the velocity generated in any other particle B during the descent of p through s will be $\frac{4ls p d^2}{w r^2} \times \frac{SB^2}{d^2}$, and the sum of the products of each particle into the square of its velocity $= \frac{4ls p d^2}{w r^2} \times \frac{A \times AS^2 + B \times BS^2 + C \times CS^2, \&c.}{d^2}$: but

† Sect. VI. $A \times AS^2 + B \times BS^2 + C \times CS^2, \&c. = w r^2$, wherefore the
Prop. VIII. sum of the products before mentioned $= 4ls p$. Now let
Cor. 1. the weight p descend from rest through the space s by its

‖ Sect. III. natural gravity, the velocity generated ‖ will be $\sqrt{4ls}$, and
Prop. V. the square of the velocity $= 4ls$, and this multiplied into
Cor. 3. each particle of the body $p = 4ls p$; the same as the sum of the products of each particle in the system into the square of its velocity.

The explanation of these propositions was the subject of a discourse on the force of bodies in motion, delivered at a public course of lectures read at the university of Cambridge, in the years 1776 and 1777; and the truth of them was confirmed by many experiments, as well on bodies which moved in right lines, as on those which revolved round a fixed axis of motion: it was shewn, that the Newtonian measure of the quantities of motion by the joint ratios of the quantities of matter and velocities generated, and that of Leibnitz and Bernoulli, according to whom the forces communicated to bodies are proportional to the quantities of matter moved, and the squares of the velocities generated in them, are consistent with each other, and with the theory of motion in general: and for the following reasons: namely, since if the times of motion be the same, the forces which * generate velocities in bodies, are as the quantities of matter and velocities jointly, according to the Newtonian measure; whereas if the spaces described from rest are the same, the forces to create motion in the bodies moved are as the quantities of matter and the † squares of the velocities jointly, as Bernoulli and others affirm in general, and without the necessary
limi-

* Sect. III.
Prop. VII.

limitation: and it was observed, that the principle upon which the coincidence of the two theories depends, is the following proposition, i. e. that when bodies are uniformly accelerated through equal spaces, the accelerating forces are as the squares of the velocities generated.

XVIII.

Let AB represent a straight lever move-Fig. LX.
able round an horizontal axis of motion
which passes through s: let the arms be
sB, sA: suppose a weight w to be affixed
to the extremity of the shorter arm, and
to be raised by the weight p, applied at
the extremity of the longer arm, when
the lever is horizontal; and let it be re-
quired to assign in what time, w will be
raised through a given altitude; the weight
and inertia of the lever itself not being
considered.

When there is an equilibrium on any mechanic power, the proportion of the weight sustained to the power sustaining it, will in all cases be assigned from having given the dimensions of the mechanic power.

An equilibrium having been once formed, the smallest addition of weight will cause the body to which it is applied on either side to preponderate: in this case a certain degree of motion is generated, and since the uses of the mechanic powers are not only to sustain forces in equilibrium, but to raise weights and overcome resistances, it is a problem of principal consequence, to assign the absolute quantity of motion generated by a known moving force, in given circumstances.

Let $SA = a$, $SB = r$. First to consider the effects of the force p to turn the system at the first instant of motion.

H h

The

The force applied to B is the weight p , but since the weight w being suspended from the extremity of the arm SA will balance a weight $= \frac{wa}{r}$ when placed at B , this force must be subducted from p ; the impelling force therefore which acts on B upon the whole will be $p -$

* Sect. VI. $\frac{wa}{r} = \frac{rp - wa}{r}$. The * inertia or mass moved at B will
Prop. 1.
Cor. 3.

be obtained by supposing the bodies p and w to be removed, and an equivalent mass $p + \frac{wa^2}{r^2}$ to be concentrated in B ; the inertia therefore at B will be that of the mass $\frac{pr^2 + wa^2}{r^2}$, and the force which accelerates B or p at the first instant of motion when the lever is horizontal $=$

|| Sect. VI. $\frac{r^2p - r a w}{pr^2 + wa^2}$. Let this $= F$; suppose the point p to have
Prop. 1.
Cor. 3.

descended through the arc BP , and through P draw PE perpendicular to SB , and let $PE = x$, $SB = r$, then will the force which accelerates the point P be less than when the lever is horizontal, in the proportion of $SB : SE$, or of $r :$

$\sqrt{r^2 - x^2}$, and will therefore $= F \times \frac{\sqrt{r^2 - x^2}}{r}$: let z be

the altitude through which a body must fall freely to acquire the velocity of p , during the time of its describing

the elementary arc Po ; and since the arc $Po = \frac{r \times \dot{x}}{\sqrt{r^2 - x^2}}$,

† Sect. III. the principles of acceleration † give us $\dot{z} = \frac{F \times \sqrt{r^2 - x^2}}{r}$
Prop. V.
Cor. 4.

$\times \frac{r \dot{x}}{\sqrt{r^2 - x^2}} = \dot{x} F$, wherefore $z = x F$, and if $l = 193$

inches, the velocity of p while it describes $oP = \sqrt{4lFx}$ inches in a second.

If $\dot{T}$ be put to represent the time of describing the

Sect. III. arc Po or $\frac{r \dot{x}}{\sqrt{r^2 - x^2}}$, this will give $\dot{T} = \frac{r \dot{x}}{\sqrt{r^2 - x^2}}$
Prop. III.

$\times \frac{1}{\sqrt{4lFx}} = \sqrt{\frac{r^2}{4lF}} \times \frac{\dot{x}}{\sqrt{r^2 x - x^3}}$, the fluent of
which

which will be the time of describing the arc Bp : this may be found by an infinite series. Since $\frac{\dot{x}}{\sqrt{r^2 x - x^3}}$

$$= \frac{\dot{x}}{r \sqrt{x}} + \frac{x^{\frac{3}{2}} \dot{x} \cdot 1}{r^3 \cdot 1 \cdot 2} + \frac{x^{\frac{7}{2}} \dot{x} \cdot 1 \cdot 3}{r^5 \cdot 2 \cdot 4} + \frac{x^{\frac{11}{2}} \dot{x} \cdot 1 \cdot 3 \cdot 5}{r^7 \cdot 2 \cdot 4 \cdot 6},$$

&c. the fluent of $\frac{\dot{x}}{\sqrt{r^2 x - x^3}}$ will $= \frac{2x^{\frac{1}{2}}}{r} \times$

$$1 + \frac{x^2 \times 1}{r^2 \cdot 2 \cdot 5} + \frac{x^4 \cdot 1 \cdot 3}{r^4 \cdot 2 \cdot 4 \cdot 9} + \frac{x^6 \cdot 1 \cdot 3 \cdot 5}{r^6 \cdot 2 \cdot 4 \cdot 6 \cdot 13}, \text{ \&c. and the}$$

time in which the weight p describes the arc $Bp = \sqrt{\frac{x}{lF}}$

$$\times 1 + \frac{x^2}{r^2 \cdot 2 \cdot 5} + \frac{x^4 \cdot 1 \cdot 3}{r^4 \cdot 2 \cdot 4 \cdot 9} + \frac{x^6 \cdot 1 \cdot 3 \cdot 5}{r^6 \cdot 2 \cdot 4 \cdot 6 \cdot 13}, \text{ \&c. ex-}$$

pressed in seconds. As the arms of a lever in raising weights are supposed to move through but small angles, a few terms of the series will be a sufficient approximation: if the angle through which the weight w is raised, be not greater than thirty degrees, two terms will give the time true within about $\frac{1}{380}$ part of the whole.

In this case, the weight and figure of the materials of which the lever is composed, have not been taken into account. The following is a more general solution; let AB be the lever, w the weight moved by the power p , each acting in a direction perpendicular to the horizon. Let G be the common centre of gravity of the whole system, including the weights p and w , and the lever itself, and O the centre of oscillation when AB vibrates round the axis S : the force which accelerates B when the lever

is * horizontal $= \frac{SG \times SB}{SG \times SO} = \frac{SB}{SO}$. If this be put $= F$, * Sect. VI. Prop. V. Prop. VIII. and Prop. II.

the time wherein p descends through a perpendicular space x , and consequently wherein w ascends through

$$x \times \frac{SA}{SB} = \sqrt{\frac{x}{lF}} \times 1 + \frac{x^2}{r^2 \cdot 2 \cdot 5}, \text{ \&c. } = \sqrt{\frac{x}{l}} \times \frac{SO}{SB} \text{ Fig. LXI.}$$

$$\times 1 + \frac{x^2}{r^2 \cdot 2 \cdot 5}, \text{ \&c.}$$

When a wheel and axle is employed to raise any weight q applied to the circumference of the axle, by means of the power p applied to the circumference of the wheel, the axis of the wheel is supposed horizontal, and the moving force being constant, the force which accelerates any given point in the system will be obtained from the principles already demonstrated, and consequently the space described by the elevated weight in a given time will be ascertained.

XIX.

Fig. LXI. Let ABC represent a wheel and axle, its weight w , and let the axis be horizontal: having given a weight q applied to the circumference of the axle, and p applied to the circumference of the wheel in order to raise q , it is required to assign the space described by the elevated weight q from rest in any given time.

The absolute force which impels D is p , and since q acts in a direction contrary to p with a force $= \frac{q \times SA}{SD}$,

this being subtracted from p will give $p - \frac{q \times SA}{SD} =$

$\frac{p \times SD - q \times SA}{SD}$ for the force which upon the whole

impels D . Let the centre of gyration of the wheel be R ; then suppose the mass of matter in the whole system

* Sect. VI.
Prop. VIII.
and
Prop. I.

removed, if the mass $\frac{w \times SR^2 + q \times SA^2 + p \times SD^2}{SD^2}$ be

concentrated in D , the point D will be accelerated in the same manner as when the parts of the system are disposed as they are described in the problem. Since then

the force which impels $D = \frac{p \times SD - q \times SA}{SD}$, and the

inertia

inertia which resists the communication of motion to D is $\frac{w \times SR^2 + q \times SA^2 + p \times SD^2}{D S^2}$, we have the force which ac-

celerates $D = \frac{p \times SD - q \times SA \times SD}{w \times SR^2 + p \times SD^2 + q \times SA^2}$, and consequently the force which accelerates the weight w in its as-

cent* = $\frac{p \times SD - q \times SA \times SA}{w \times SR^2 + p \times SD^2 + q \times SA^2}$: let this = F , and * Sect. VI.
Prop. II.

$l = 193$ inches, then the space described from $\dagger$ rest by q † Sect. III.
Prop. IV.
Cor. 5. in the time t seconds = $t^2 l F$ inches.

Cor. 1. Let SD be to SA as $n : 1$; then if $SA = a$, $SD = na$: let $SR = ra$, ~~the~~ the force which accelerates

the point $D = \frac{n^2 p - n q}{w r^2 + p n^2 + q}$.

Cor. 2. If the inertia of the wheel and axle is evanescent or too small to have sensible effect, the force which

accelerates $D = \frac{n^2 p - n q}{p n^2 + q}$.

Cor. 3. If the inertia of the moving force is also = 0, the force which accelerates the point $D = \frac{n^2 p - n q}{q}$.

Cor. 4. If the mass moved has no weight but possesses inertia only, the force which accelerates $D = \frac{n^2 p}{p n^2 + q}$.

Whenever motion is communicated to a body, a certain resistance must have been overcome by the moving force: this resistance is of various kinds, i. e. 1. The inertia of the mass moved whereby it endeavours to persevere in its state of quiescence, or of uniform motion in a right line. 2. That of a weight or other absolute force opposed to the action of the moving power. 3. Obstacles upon which the moving body impinging is retarded in its progress: such for example is the resistance which arises from the particles of a fluid through which a body moves. The estimation of these resistances, and their effects in retarding the motion of bodies acted on by a given force, are deducible from the laws of motion, and constitute a part of the solution of almost all problems relating to the motion of bodies.

The moving forces also are of various kinds, i. e. the earth's gravity, muscular power, the impact of bodies solid or fluid, &c. It has been shewn, that the effects of these moving forces which are exerted on bodies in order to create motion, exclusive of the resistance opposed to them, depend on the various circumstances of the time in which they act, and on the spaces through which the bodies moved are impelled, &c.

These considerations are urged to shew, that from the great variety of undetermined conditions which may enter into mechanical problems, there must be of course various methods of producing the same mechanical effect; and it is a very material part of the art, considered either in a theoretical or practical view, to proportion the means to the end, and to effect this with all the advantages which the nature of the case is capable of. It is the due observation of these particulars which contributes to render mechanic instruments compleat, and the neglect of them defective in their construction. This proper choice of means to produce mechanical effect is frequently the result of long continued experience independent of all theory; the knowledge of which however when immediately applied to practice would save the artist much time and trouble, as well as would be productive of other advantages which experience alone must be destitute of.

XX.

Fig. LXI. ABC is a wheel and axle moveable round an horizontal axis which passes through s . Suppose a given weight q , which is applied to the circumference of the axle, to be raised by the application of a given moving force p , which is applied to the circumference of the wheel: let it be required to assign the proportion of the radii of the wheel and axle, so that the time in which the weight w ascends

cends through any given space shall be the least possible.

Let the given radius of the axle $SA = a$, the radius of the wheel sought $= x$, let the wheel's weight $= w$, the distance of the centre of gyration from the axis of motion $= r$, then will the force which accelerates the weight p

during its *descent $= \frac{x^2 p - q a x}{w r^2 + p x^2 + q a^2}$, that which ac-

* Sect. VI.
Prop. XIX.

celerates q in its † ascent $= \frac{a x p - q a^2}{w r^2 + p x^2 + q a^2}$.

† Sect. VI.
Prop. II.

The square of ‡time therefore in which any space s is de-

‡ Sect. III.
Prop. IV.
Cor. 5.

scribed by the ascending weight $q = \frac{s}{t} \times \frac{w r^2 + p x^2 + q a^2}{a p x - q a^2}$,

which is to be the least possible by the problem; making therefore its fluxion $= 0$, we have $p^2 a x^2 \dot{x} - 2 p q a^2 x \dot{x} - w r^2 p a \dot{x} - p q a^3 \dot{x} = 0$, which being reduced gives $x =$

$$\frac{a q + \sqrt{q^2 a^2 + p w r^2 + q p a^2}}{p}.$$

Cor. 1. Suppose the inertia of the wheel to be evanescent, then $x = \frac{a q + \sqrt{q^2 a^2 + q p a^2}}{p}$, and if $q = p$,

that is, the weight moved being equal to the moving force,

it will follow, that $x = a + a \times \sqrt{2} = a \times 1 + \sqrt{2}$.

Let ABC represent a cylindrical wheel, the radius of which $= 10$ inches, and its weight 20 oz. let the radius of the axle $SA = 1$ inch, the weight to be raised through any given space be 100 oz. the moving force by which it is raised $= 33$ oz. since the radius of the cylinder $= 10$ inches, the distance of the centre of gyration from the

axis $= \sqrt{50}$ inches ||: then to find a distance SD , at which the moving force p is applied, so that the time in which q ascends through a given space shall be the least possible, we have $q = 100$, $p = 33$, $w = 20$, $r^2 = 50$, $a = 1$, and

|| Sect. VI.
Prop. XII.
P. 224.

the distance sought $= \frac{100 + \sqrt{10000 + 33000 + 3300}}{33}$

$$= \frac{315.17}{33} = 9.55 \text{ inches.}$$

This

This therefore will be the most convenient distance to apply the given moving force, when the chief object is to lessen the time of ascent. If it be required to assign the distance SD , when the moment communicated to w while it ascends through a given space is the greatest possible, the solution will be the same as before, which therefore answers to two conditions; that is, it will render the time in which q ascends through a given space the least, and the moment generated during the same ascent the greatest possible.

Fig. LXII. If the weight q instead of ascending in a vertical direction is drawn along an horizontal plane, and the surfaces be perfectly free from friction, the weight of q will $= 0$. In this case if it be required to assign the distance SD , at which if the given force p be applied, the time of describing a given space shall be the least, and the moment generated in q the greatest possible, we shall have

$$SD = \sqrt{\frac{r^2 w + q a^2}{p}}, \text{ and if the inertia of the wheel should be too small to have sensible effect, } SD = a \times \sqrt{\frac{q}{p}}.$$

Thus, let the quantity of matter to be drawn along the plane be four times greater than that which is contained in the moving force, the radius of the axle SA being given; in order that it may be impelled with the greatest velocity possible and with the greatest moment, the radius of the wheel should be double to that of the axle, when the inertia of the wheel is not considered.

XXI.

Fig. LXI.

ABC is a wheel and axle, the axis of which is horizontal: having given a moving force or weight p acting on the circumference of the wheel in order to raise a weight y which is applied to the circumference of the axle, it is required to assign

assign the quantity y , when the moment generated in it in any given time shall be the greatest possible, the inertia of the wheel and axle not being considered.

Let $SD = d$, $SA = a$, the force which accelerates the ascent* of $y = \frac{a dp - y a^2}{d^2 p + y a^2}$; therefore if $l = 193$ inches, * Sect. VI. Prop. XIX. the velocity generated in y in the $\dagger$ time t will be $2 l t \times \dagger$ Sect. III. Prop. II. $\frac{a dp - y a^2}{d^2 p + y a^2}$, and the moment generated in y in the same time t will be expressed by $2 l t \times \frac{a d p y - a^2 y^2}{d^2 p + y a^2}$; and as this is to be the greatest possible by the problem, its fluxion must be $= 0$, which will give $a d^3 p^2 \dot{y} - 2 a^2 d^2 p y \dot{y} - a^4 y^2 \dot{y} = 0$, and the weight sought $= \frac{\sqrt{d^4 p^2 + d^3 p^2 a - d^2 p}}{a^2}$.

Cor. 1. If SD is to SA as $n : 1$, $y = p \times \sqrt{n^4 + n^3 - n^2 p}$: if the radius of the axle $=$ the radius of the wheel, that is, if $n = 1$, the weight y will be $= p \sqrt{2 - 1}$: the weight moved therefore must be about $\frac{5}{12}$ parts of the moving force.

Cor. 2. If the wheel's weight be taken into account, let its weight $= w$, the distance of the centre of gyration from the axis $= r$, and the moment generated in the ascending weight the time t will be expressed by $2 l t \times \frac{a d p y - a^2 y^2}{w r^2 + d^2 p + y a^2}$, which is the greatest possible when $y = \frac{\sqrt{d^4 p^2 + 2 d^2 p r^2 w + r^4 w^2 + p w d a r^2 + p^2 d^3 a - d^2 p - r^2 w}}{a^2}$.

XXII.

Fig. LXIII. Let $ABCH$ be a system of bodies moveable round a vertical axis which passes through the common centre of gravity of the system. Suppose DEG to be a wheel the axis of which is vertical and coincident with that of the system, let motion be communicated by means of a line going round this wheel, the string DP being stretched by a given weight p ; let it be required to assign the radius of the wheel EGD , so that the angular velocity communicated to the system in any given time may be the greatest possible.

Let the weight of the system $= w$, the distance of the centre of gyration from the axis of motion $= r$, the radius sought $SD = x$: then will the velocity generated in a given time in the * descending weight p be proportional to

* Sect. VI.
Prop. XV.

$$\frac{px^2}{wr^2 + px^2}, \text{ and the angular velocity generated in the same time as } \frac{px}{wr^2 + px^2}, \text{ which is to be a maximum by the conditions of the problem; we have therefore its fluxion}$$

$$\frac{pwr^2\dot{x} + p^2x^2\dot{x} - 2p^2x^2\dot{x}}{(wr^2 + px^2)^2} = 0, \text{ and the distance sought}$$

$$x = \sqrt{\frac{wr}{p}} \times r.$$

Thus, suppose the moving force is $\frac{1}{4}$ of the weight of the system, it should be applied at a distance from the axis equal

equal to twice the distance of the centre of gyration, in order to produce the greatest angular velocity in a given time.

In order to increase the action of a given moving force against a weight to be raised, or resistance to be overcome, a combination of two or more mechanic powers is frequently made use of. Let p be a power applied by means of a line to the vertical wheel C : suppose the circumference of the axle K to be in contact with the circumference of any other vertical wheel B , so that the circumference of the wheel B may always move equally fast with that of the axle which belongs to C : let also the axle of B communicate motion to a vertical wheel A , to the axle of which a weight q is suspended, so as to act in opposition to p : moreover let $lmn:1$ be the sum of the ratios of the radius of each wheel to that of its axle; then if $p l m n = q$, the two weights p and w will sustain each other in equilibrio, but if $l m n p$ is at all greater than q , the equilibrium will be destroyed, and the next problem is intended to assign the motion communicated to p or q in given circumstances.

Fig. LXIV.

XXIII.

In a system of wheels and axles just described, let the radii of the wheel and of the axle A be in the ratio of $1:1$, the radii of the wheel and axle in B as $m:1$, and of c as $n:1$; also let the distance of the centre of gyration from the axis in A be r , the distance of the centre of gyration in $B = s$, in $c = t$: and let the weight of the wheel and axle $A = a$, that of $B = b$, and of $c = c$: suppose the weight p sufficient to communicate motion to q ; and let it be required to assign

Fig. LXIV.

the space described by q in its ascent from rest during a given time.

The absolute force of p to move itself $= p$, and since q acts in opposition to this force, its effects must be subtracted from p in order to obtain the force which impels p on the whole; and since q would balance a weight $= \frac{q}{lmn}$ if applied at p , the force which impels p on the

whole will be $\frac{p l m n - q}{l m n}$. In the next place, the inertia

which resists the communication of motion to p must be ascertained. Motion is communicated to the wheel A from the circumference of the axle B , and the inertia of A and of the weight q which resists the * communication of a force

* Sect. VI.
Prop. VIII.

applied at $I = \frac{a r^2 + q}{l^2}$: in regard therefore to the inertia of A and q , these may be supposed to be removed,

† Sect. VI.
Prop. I.

and the equivalent † mass $\frac{a r^2 + q}{l^2}$ collected into the cir-

cumference of the wheel A , or of the axle B . Since motion is communicated to B by the circumference of the axle C , the inertia of B together with the equivalent mass

$\frac{v r^2 + q}{l^2}$ will be $\frac{s^2 l^2 b + a r^2 + q}{l^2 m^2}$: in like manner

since motion is communicated to C by the weight p acting at D , the inertia which resists the communica-

tion of motion to D or p will be $\frac{p n^2 + c t^2}{n^2} +$

$\frac{s^2 l^2 b + a r^2 + q}{l^2 m^2 n^2} = \frac{l^2 m^2 n^2 p + c t^2 l^2 m^2 + s^2 l^2 b + r^2 a + q}{l^2 m^2 n^2}$,

and the force which accelerates p in its descent from rest $=$

$\frac{p l m n - q \times l m n}{l^2 m^2 n^2 p + c t^2 l^2 m^2 + s^2 l^2 b + r^2 a + q}$, and that which

accelerates $q = \frac{p l m n - q}{l^2 m^2 n^2 p + c t^2 l^2 m^2 + s^2 l^2 b + r^2 a + q}$.

If therefore this force be put $= F$, and l be $= 193$ inches,

the space described by q in its ascent from rest in the time

t seconds ‡ will be $= t^2 l F$ inches.

‡ Sect. III.
Prop. V.
Cor. 5.

XXIV.

Let A, B represent a single moveable Fig. LXV. and a fixed pully, by means of which the power P elevates the weight w : having given P and w , together with the weights of the cylindrical pullies A and B, it is required to assign the space which the descending weight P describes in a given time, the weight of the moveable pully being included in the weight w .

The absolute force which impels P in its descent, is its own weight or P ; but since the weight W acts against this descent with a force $= \frac{W}{2}$, the whole force by which

P endeavours to descend is $P - \frac{W}{2}$. The inertia which

resists the communication of motion to P , depends on the mass contained in P , together with that contained in the two pullies and the weight W . Let the weight of each pully be $\mathcal{Q}$, the inertia of P is its own mass P , and that of

the pully $\dagger A = \frac{\mathcal{Q}}{2}$: the inertia of the pully B , is the same † Page 225.

as if $\frac{\mathcal{Q}}{2}$ were uniformly accumulated into its circumference;

but since the velocity of this circumference is less than the velocity of P in the proportion of 2 : 1, its inertia

will be equivalent to a mass $= \frac{\mathcal{Q}}{2 \times 4}$ descending with || Sect. VI. Prop. II.

P ; for the same reason, since the weight W moves with a velocity only one half of that with which P descends,

its inertia referred to P 's motion will be $\frac{W}{4}$: the inertia

of the whole mass will therefore be $P + \frac{\mathcal{Q}}{2} + \frac{\mathcal{Q}}{8} + \frac{W}{4}$

$\frac{W}{4} = \frac{8P + 5Q + 2W}{8}$; and because the impelling force

is $\frac{2P - W}{2}$, the accelerating force will $= \frac{2P - W}{2} \times$

Sect. VI.
Prop. IV.

$\frac{8}{8P + 5Q + 2W} = \frac{8P - 4W}{8P + 5Q + 2W}$: if therefore l be
 $= 193$ inches, the space described by P from rest in t se-
conds will be $\frac{8P - 4W}{8P + 5Q + 2W} \times t^2 l$.

Cor. If W should be greater than $2P$, W will descend,
and by the same method of reasoning, the force which ac-
celerates the descent of W will be found $= \frac{2W - 4P}{2W + 5Q + 8P}$.

XXV.

In a system of pulleys in which the same
string goes round all the pulleys contained
in two blocks: having given the power
 P and the weight w raised by it, toge-
ther with the number of the pulleys and
the weight and figure of each, it is requir-
ed to assign the force which accelerates
the descent of P , the weight of the lower
block being included in the weight w .

Let the number of pulleys be n , and suppose them as
before to be cylindrical, and that the weight of each $=$
 Q ; then the absolute moving force which acts at $P =$ its
own weight P : but because W acts at P in a contrary di-
rection with a force $= \frac{W}{n}$, the force which upon the whole

impels P in its descent $= \frac{Pn - W}{n}$.

Moreover, the inertia of P is its own mass P , that of
the weight W , the velocity of which $= \frac{1}{n}$ part of P 's ve-

locity † will be $= \frac{W}{n^2}$, and because the circumference of † Sect. VI.
Prop. II.

the slowest pulley moves with the same velocity as the weight W , and its inertia is the same as if $\frac{2}{2}$ were collected into its circumference, the inertia by which it resists the descent of P will be $\frac{2}{2n^2}$, the circumference of

of the next pulley revolving twice as fast as the former, its inertia will be four † times as great, and will therefore be † Sect. VI.
Prop. II.

$= \frac{4 \cdot 2}{2 \times n^2}$, and so on; the sum of the inertia of the pulleys

being $= \frac{2}{2n^2} \times 1 + 4 + 9$, &c. continued to n terms,

which is $= \frac{2}{2n^2} \times \frac{2n^3 + 3n^2 + n}{6}$: the whole inertia

therefore which resists the communication of motion to P will $= P + \frac{W}{n^2} + 2 \times \frac{2n^3 + 3n^2 + n}{12n^2} =$

$\frac{12n^2P + 12W + 2 \times 2n^3 + 3n^2 + n}{12n}$, and the force which acce-

lerates the descent of $P = \frac{12Pn^2 - 12Wn}{12n^2P + 12W + 2 \times 2n^3 + 3n^2 + n}$

XXVI.

Let ABCD represent a system of pul- Fig. LXVI.
lies in which the string that goes round each pulley is fixed to the weight, as represented in fig. LXVI: having given the weight w , and the power which raises it p , together with the number and weights of the cylindrical and equal pulleys A, B, C, D, &c. it is required to assign the force which

which accelerates the descent of the power P , the weights of the pullies B, C, D being included in P .

Let the weight of each pulley be $\mathcal{Q}$, and suppose their number $= n$; then the absolute force which urges P downwards is its own weight P : but since W acts against it with a force $= \frac{W}{2^n - 1}$, the force by which P endeavours to descend upon the whole will be $\frac{P \times 2^n - 1 - W}{2^n - 1}$.

The inertia of W which retards the motion of $P = \frac{W}{2^n - 1}$,

because the velocity with which W ascends is $\frac{1}{2^n - 1}$ part

* Sect. VI.
Prop. II.

of the cotemporary * velocity of P : and since the velocity of the circumference of the pulley A is equal to that with which W ascends, the inertia of the pulley A

inferred to P 's motion will be $\frac{\mathcal{Q}}{2 \times 2^n - 1}$, and because

the angular velocities of the pullies A, B, C, D are as 1, 3, 7, 15, &c. the inertia of B will be $= \frac{9\mathcal{Q}}{2 \times 2^n - 1}$,

that of $C = \frac{49\mathcal{Q}}{2 \times 2^n - 1}$, and the inertia of all the pul-

lies A, B, C and D , &c. which arises from their rotation, $=$

$\frac{\mathcal{Q}}{2 \times 2^n - 1} \times 1^2 + 3^2 + 7^2 + 15^2$, &c. continued to

n terms $= \frac{\mathcal{Q}}{2 \times 2^n - 1} \times \frac{2^{2n+2} - 3 \times 2^{n+2} + 3n + 8}{3}$.

Moreover, the inertia which arises from the motion of the centres of gravity of the pullies B, C, D , &c. as they de-

scend, will be found by the same reasoning to be $\frac{\mathcal{Q}}{2^n - 1}$

$\times \frac{2^{2n} - 3 \times 2^{n+1} + 3n + 5}{3}$, which being added to the

inertia

inertia before found, will give the inertia of the pullies

$$\frac{2}{6 \times 2^n - 1} \times \frac{2^{2n+2} + 2^{2n+1} - 6 \times 2^{n+2} + 9n + 18}{2^n - 1}$$

the whole inertia therefore which resists the communication

$$\text{of motion to } P \text{ during its descent} = P + \frac{W}{2^n - 1} +$$

$$\frac{2}{6 \times 2^n - 1} \times \frac{2^{2n+2} + 2^{2n+1} - 6 \times 2^{n+2} + 9n + 18}{2^n - 1} =$$

$$\frac{6P \times 2^n - 1^2 + 6W + 2 \times 2^{2n+2} + 2^{2n+1} - 6 \times 2^{n+2} + 9n + 18}{6 \times 2^n - 1^2},$$

and because the moving force which acts on P during its

$$\text{descent is } \frac{P \times 2^n - 1 - W}{2^n - 1}, \text{ the force which accelerates } P =$$

$$\frac{6P \times 2^n - 1 - 6W \times 2^n - 1}{6P \times 2^n - 1^2 + 6W + 2 \times 2^{2n+2} + 2^{2n+1} - 6 \times 2^{n+2} + 9n + 18},$$

and the force which accelerates W in its ascent =

$$\frac{6P \times 2^n - 1 - 6W}{6P \times 2^n - 1^2 + 6W + 2 \times 2^{2n+2} + 2^{2n+1} - 6 \times 2^{n+2} + 9n + 18}.$$

In order to render the solutions of these last propositions less complicated, the weights of the moveable pullies have been included either in the weight W or the power P . In the system described in the * proposition just demonstrated, * Prop. XXVI. there is this particular advantage, i. e. the weights of the pullies constitute a part of the moving force and consequently facilitate the elevation of any weight to be raised by it, whereas the weights of the pullies in the other constructions act against the moving force. The weights of the pullies in the system described in the proposition have been supposed to be included in the power P , but as they are not equal in their effect on the raised weight, it may be necessary to mention in what manner their forces when referred to the power P are estimated. It is manifest, that the pulley A being fixed contributes nothing to the moving power: also if the weight of each pulley be 2 , the pulley B acts on W with a force $= 2$, the pulley C with a force $= 3 \times 2$, and D with a force $= 7 \times 2$; if therefore the number of pullies be n , the moving force exerted by them on the weight to be raised $= 2 \times 1 + 3 + 7 + 15$, &c. continued to $n - 1$ terms $= 2 \times 2^n - 1 - n$, and this

this weight would be balanced by $\frac{2 \times 2^n - 1 - n}{2^n - 1}$ acting at P , wherefore if a power G be applied at P , exclusive of the weights of the pullies, the whole moving force which acts on P will be $G + \frac{2 \times 2^n - 1 - n}{2^n - 1} - \frac{W}{2^n - 1}$.

If this value for the moving force be substituted for P in the preceding solution, that is, as far as that body is concerned in contributing to the moving force, we shall have a general expression for the acceleration of the descending weight, every circumstance being taken into account, except those of friction, the line's weight, and the air's resistance; the two latter of which are too small to have sensible effect, and the former may be so diminished, that the real motion of the descending weight shall scarcely deviate from that which is deduced from the theory. Let therefore G be any moving force applied to raise the weight W by means of the system of pullies described in the proposition, every thing else remaining, we shall have the force which accelerates the descending weight G equal to that part of gravity expressed by the fraction

$$\frac{6G \times 2^n - 1 + 6 \times 2^n - 1 - n - 6W \times 2^n - 1}{6G \times 2^n - 1^2 + 6W + 2 \times 2^{2n+2} + 2^{2n+1} - 6 \times 2^{n+2} + 9n + 18}$$

In the same manner the motion generated in any of the mechanic powers may be ascertained from the necessary data, and in other cases, provided the forces by which each particle of the system is moved be constant. Although the following proposition be not immediately connected with the subject of rotatory motion, yet as the solution of it is included in a more general theory hereafter demonstrated, of which the principles of rotation are likewise particular cases, it may be not useless to insert it.

XXVII.

Fig. LXVII. Let ABC be an isosceles wedge, the base of which is AB ; let two equal spheres be applied so as to touch the sides in the points

points *D* and *E* equally distant from *c*: then the base being bisected in *G* let a weight *P* be applied so as to urge the wedge forward in the direction *GCH*, and to communicate motion at the same time to the spheres *D* and *E* in free space, the wedge and spheres being considered without gravity, and their surfaces being perfectly smooth: having given the necessary conditions, it is required to assign the force which accelerates the weight *P* and the spheres *D* and *E*, and the direction in which the spheres move.

Let the mass contained in the wedge = *W*, that in either sphere = *Q*, then will the force which tends to communicate motion to the weight *P* be = *P*; to find the mass moved, since each particle of the wedge moves equally fast with *P*, its inertia will be = *W*: now to find the inertia of the sphere *D*, suppose the wedge to have moved through a space = *CH*, draw *HI* parallel to *AC*, and *HK* to *CB*, then if *DI* be drawn perpendicular to *HI*, and *EK* to *HK*, when the point *C* arrives at *H*, the spheres *D* and *E* will be coincident with *I* and *K* respectively: and if *IA*, *KB* be drawn parallel to *CG*, the point *A* will become coincident with the sphere *D* at *I*, and the point *B* with the sphere *E* at *K*: it follows, that the force which impels the sphere *D* will be equal to that

part of the force *P* which is expressed by the fraction $\frac{AG}{AC}$, and the velocity of *D* will be that part of *P*'s cotemporary velocity which is expressed by the fraction $\frac{DI}{IA}$ or $\frac{AG}{AC}$.

The inertia therefore of the sphere *D* whereby it resists the

communication of motion * to *P* is = $\frac{Q \times AG^2}{CA^2}$, and the

* Sect. VI.
Prop. 11.

inertia of both the spheres = $\frac{2Q \times AG^2}{CA^2}$: wherefore
 since the force of P to move itself = P , and the inertia
 exerted against the communication of motion to it = $P +$
 $\frac{2Q \times AG^2}{CA^2} + W$, the force which accelerates P will be =

† Sect. I.
 Prop. IX. $\frac{P \times CA^2}{P \times CA^2 + W \times CA^2 + 2Q \times AG^2}$ †, and that which accele-

rates the spheres D, E will = $\frac{P \times AG \times CA}{P \times CA^2 + W \times CA^2 + 2Q \times AG^2}$ *

Fig.
 LXVIII.

|| Sect. I.
 p. 8. and
 Sect. III.
 p. 48.

Let a body be moved from rest at A in the direction AB
 by any constant force; then if the force ceases to act at B ,
 the body will proceed with the same uniform velocity which
 it had acquired at B . Suppose when it arrives at C some
 obstacle or resistance is opposed to its progress: if the oppo-
 sition arises from an inert and nonelastic obstacle on which
 it impinges, it will after a || certain time, still proceed uni-
 formly through with a diminished velocity. If instead of
 an inert obstacle opposed to the body's progress at C , it be
 acted upon by an uniformly resisting force, the body will
 be continually retarded until its motion is destroyed. Most
 mechanical operations consist either in communicating
 motion to quiescent and inert bodies or in overcoming
 resistances, in which operations the action of the moving
 force and its mechanical effect are not always cotempo-
 rary; but motion is frequently first generated, and at sub-
 sequent times employed in producing the desired result.
 Thus, in the preceding example, motion is generated
 while the body is describing AB , and it is subsequently
 employed in overcoming the resistance opposed while the
 body describes some space CD . The mechanical effect of
 the motion generated will be greater or less according to
 various circumstances: for a greater mass of matter will
 overcome a greater resistance than a smaller moving with
 the same velocity, every thing else being equal: it is evi-
 dent that the effect will depend on the velocities also, but
 the exact quantity of motion communicated to a body by
 a moving force or destroyed by any resistance, cannot be
 estimated by the quantity of matter contained in the body
 and its velocity, neither by assuming the quantity of mo-
 tion as the mass into the square of the velocity, nor as the
 quantity of matter into the velocity, and for these rea-
 sons; there are five quantities concerned in the genera-
 tion of motion, i. e. the quantity of matter moved, the
 force

force which moves it, the space described, time of description, and velocity generated, and any two of these being given will not determine the rest or any one of them. If

therefore we were to affirm, either that $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{\mathcal{Q}}{q}$,

or that $\frac{M}{m} = \frac{V}{v} \times \frac{\mathcal{Q}}{q}$, either affirmation would be as in-

conclusive, as if estimating the spaces described by bodies moving with uniform velocities, we were to say, that the spaces described are in the same ratio as the velocities of motion; but any three of the five quantities just mentioned

being given determine the rest: wherefore if $\frac{S}{s}$ be the ra-

tio of the spaces described, and that of the times of mo-

tion be $\frac{T}{t}$,* since $\frac{M}{m} = \frac{\mathcal{Q}}{q} \times \frac{V}{v} \times \frac{t}{T}$, if $T = t$, then $\frac{\mathcal{Q}}{q} \times$ * Sect. III. Prop. VII.

$\frac{V}{v} = \frac{M}{m}$; that is, the ratio of the moving forces is equal

to the sum of the ratios of the velocities generated, and of the quantities of matter moved when the times are the

same: moreover, since $\dagger \frac{M}{m} = \frac{V^2}{v^2} \times \frac{\mathcal{Q}}{q} \times \frac{s}{S}$, if $\frac{V^2}{v^2}$ and $\dagger$ Sect. III. Prop. VI.

$\frac{\mathcal{Q}}{q}$ be given, when $S = s$, $\frac{M}{m}$ is determined, and conse-

quently when the spaces described are equal, the ratio of the moving forces $= \frac{V^2}{v^2} \times \frac{\mathcal{Q}}{q}$. Now it is certain, that

we are at liberty to estimate mechanical effects and the forces by the operation of which they are produced, by the velocities destroyed or generated in bodies, during their motion through a given space or in a given time; and according to which of these hypotheses is assumed, the moving forces or resistances will be as the squares of the velocities into the quantities of matter moved, or as the velocities into the quantities of matter. Thus Leibnitz, Bernoulli, and many of the foreign philosophers, estimated the motion of bodies according to the former supposition; Newton, Keil, and most of the British, according to the latter: the same conclusions will of course be produced by each method from given data; in regard to the truth of these conclusions therefore, it is a matter of indifference which method be used in the investigation of them, al-

although particular problems may come out in a more easy and simple manner from one hypothesis than the other.

These considerations being premised, when any mechanical effect is to be produced, in order to proportion duly the means to the end required, it must be considered, that if only one mechanic quantity be given in the problem, there are four others remaining, two of which may be varied sine limite, and combined with all the different constructions which mechanic instruments are capable of. Thus, suppose it were required only to put in motion a given body A : of the force, the space described, the velocity communicated, and the time of motion, any two may be infinitely varied, but the rest are determined when any two with the original quantity are given.

2. Suppose it were required to communicate to a given body A a certain velocity V , then of the force, the space described by A before it has acquired the given velocity as well as the time of acceleration, any one of these may be assumed of any magnitude at pleasure. In the same manner, if it were required to stop or destroy the motion of a body A , the velocity of which is V ; here a great force may stop the body in a small time, or a small force in a great time, the space being determined by either of these quantities; but the same effect, i. e. the destroying the motion of A , is obtained in any case, however the means may vary: if it be required to generate a velocity V in a given body A during the time T , the force to be applied as well as the space described, before A has acquired that velocity cease to be a matter of choice; there being but one force that can be applied to satisfy the conditions of the problem which limits the space described also; and the same method of reasoning is applicable to the destroying the motion of A by the action of an uniformly resisting force.

From observing these circumstances, an useful problem in mechanics may be solved, such as having given the resistance overcome by a given body moving through a given space, to assign the quantity of matter contained in some other body moving with a given velocity which shall overcome the same resistance, or produce a mechanical effect equal to the former, under certain conditions and restrictions required by the nature of the case.

XXVIII.

Having given the quantity of matter contained in a body $= Q$, and its velocity $= v$; then suppose it to be resisted by a force which is to that which gravity would oppose to the same body, when thrown perpendicularly upwards, as $m : Q$, and let the space described by Q before its motion is wholly destroyed by the resisting force m be s , the whole time of its motion being $= T$; let it be required to assign what must be the quantity of matter q moving with any other given velocity u , so that the resistance m shall destroy its whole motion: 1. while it describes the same space as before s ; 2. in the given time T .

Since universally* $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{s}{S}$, when $S = s$ * Sect. III. Prop. VI.

and $M = m$, as in the first part of the problem, $\frac{V^2}{v^2} = \frac{q}{Q}$,

wherefore the weight required $q = \frac{Q \times V^2}{v^2}$, when the same

space is described as in the former case. 2d. † Since in † Sect. III. Prop. VII.

general $\frac{M}{m} = \frac{V^2}{v^2} \times \frac{Q}{q} \times \frac{t}{T}$, when $T = t$ and $M = m$, as

in the second part of the problem, it follows that $\frac{V}{v} = \frac{q}{Q}$,

wherefore the weight sought $q = \frac{V}{v} \times Q$.

Let

† Robins' Gunnery, Vol. I.

Let an example be taken from Mr. Robins†: he found that a leaden ball of $\frac{3}{4}$ of an inch diameter, impinging perpendicularly with a velocity of 1700 feet in a second, on an elm block penetrated into its substance 5 inches: let it be required to assign what quantity of matter must impinge on the same substance with a velocity of one foot in a second, so that its whole motion shall be destroyed while it describes the same space, that is, 5 inches: here referring to the solution we have $V = 1700$, $\mathcal{Q} = 1$, $v = 1$, and the

mass required $q = \frac{V^2}{v^2} \times \mathcal{Q} = 1700^2 \mathcal{Q}$, and $q = 2890000$

times the ball's weight. This is the solution when the ball's motion is destroyed during the time it describes 5 inches in the elm, being the whole depth to which it penetrates. If it be required to assign the mass, when the ball's motion is destroyed in the same time as before, the

quantity will be $\frac{V}{v} \mathcal{Q} = 1700$ times the ball's weight.

Fig. LXIX. It remains to shew in what manner this theory may be practically illustrated. Let I represent the bullet, such as is referred to in the preceding example, and which would if it impinged on a block of elm with a velocity of 1700 feet in a second, penetrate into its substance to the depth of 5 inches. In the preceding problem, when the mass contained in the ball is increased, in order to compensate for the diminished velocity, so that its effects shall be the same, the resistance of the block m is supposed to be neither greater or less than before: it follows from these considerations, that in augmenting the quantity of matter contained in the ball, the magnitude of it must not be altered; for if it were, the † resistance of the elm would not continue the same. The following means may be made use of, whereby the magnitude of the ball will not be altered, and yet the impulse exerted by it when impinging against an elm block with the given velocity of one foot in a second shall be the same as if it were in fact 2890000 times heavier than before. Suppose the ball I affixed to an iron rod, which is fastened into a heavy block of wood or other substance, the weight of which is 2890000 times greater than that of the ball I : and suppose this block so suspended as to revolve about an horizontal axis of motion LM at a convenient height, and which is parallel to the plane of the elm block. Let the rod ID pass through the centre of percussion, the pendulum vibrating in a plane perpendicular to the axis: then let the block be brought to
i such

† Sect. III. P. 38.

such a position, that it shall touch the bullet when the pendulum is quiescent, *ID* being horizontal: also let the pendulum be drawn out of its vertical position by any force

through an arc the versed sine of which is equal to $\frac{1}{64}$ part

of a foot: if it be suffered to descend from rest along this arc to its lowest point, it will there have acquired a velocity of one foot in a second, and if the block be immovably fixed, the whole motion of the impinging body will be destroyed. Moreover, by the problem, the depth to which the bullet *I* will penetrate into the substance of the block will be 5 inches. The same mechanical effect is therefore produced by an inconsiderable, as by a very great velocity, compensation being made for the diminished velocity by increasing the mass of matter moved. And it is plain that the same means may be used to produce by vast and massy bodies moving with small velocities, the effects of cannon balls impinging against obstacles consisting of stone walls, earth, or wood. These two methods have been made use of in different ages for the purpose of demolishing fortifications: the battering rams of the ancients consisted of very large beams of wood, terminated by solid bodies of iron or brass; such a mass being suspended as a pendulum, and driven partly by its gravity and partly by the impulse of men against the walls of a fortification, exerted a force which in some respects exceeded the utmost effects of our battering cannon, though in others it was probably inferior to the modern ordnance. To compare the effects of a battering ram, the metal extremity of which suppose equal in magnitude to a 24 pounder, with that of a cannon ball of 24 pounds weight: in order that the two bodies may have the same effect in cutting a wall, or making a breach in it, the weight of the aries must exceed that of the ball in the proportion of about 1700^2 to the square of the velocity with which the battering ram could be made to impinge against a wall expressed in feet; if this may be estimated at about 10 feet in a second, the proportion of the weights will be that of about 2890000 to 100, or 28900 to 1: the weight of the battering ram must therefore = 346 ton. In this case the battering ram and the cannon ball moving with the velocities of 10 feet and 1700 feet respectively in a second, would have the same effect in penetrating the substance of an opposed obstacle; but it is probable, that the weight of the aries never amounted to so much as is above described, and consequently the effects of the cannon ball to cut down walls by making a breach in them, must exceed those of the

† Sect. VI.
p. 209.

* Sect. VI.
Prop.
XXIV.

ancient battering rams: but the momentum of these, or the impetus whereby they communicated a shock to the whole building was far greater than the utmost force of cannon balls; for if the weight of the battering ram were no more than 170 times greater than that of a cannon ball, each moving with its respective velocity, the moments of both would be equal: but as it is certain, that the weight of these ancient machines was far more than 170 times our heaviest cannon balls, it follows, that their moment or impetus to shake or overturn walls, &c. was far superior to that which is exerted by the modern artillery. And since the strength of fortifications will in general be proportioned to the means which can be used for their demolition, the military walls of the moderns have been constructed with less attention to their solidity and massy weight, than the ancients thought a necessary defence against the aries: that sort of cohesive firmness of texture which resists the penetration of bodies being now more necessary than in ancient times; but it is manifest, that even now, solidity or weight in fortifications also is of material consequence to the effectual construction of a wall or battery. This remark has been urged only to shew such variations in the degrees of solidity and firmness of texture in the fabric of military walls, as have been occasioned by the change which took place in the practice of artillery, when lighter bodies impelled with greater velocities were substituted instead of the ponderous machines of the ancients, to which but inconsiderable velocities could be communicated.

Fig. LXX.

There is also another method of producing great mechanical effects by means of small velocities generated in ponderous bodies by inconsiderable moving forces. Suppose *ABC* were an heavy cylinder of iron or lead, moveable about its axis and in a vertical plane: a small force being applied to turn the cylinder, if long continued, will generate such a force in it as will produce effects in raising weights and overcoming resistances, by no means obtainable by the moving force immediately applied. Suppose, for instance, it were required to assign what must be the weight of the wheel, so that when the bullet referred to in the last example is affixed to any point *D*, and revolves with the cylinder at a given rate, for example, with a † velocity of 1 foot in a second, if it should impinge perpendicularly against an immoveable block of elm, it shall penetrate into that substance to the same depth as when it impinged freely upon the immoveable block with a velocity of 1700 feet in a second.

† Page 265.

cond. Here the velocity of the ball = 1 foot in a second, let p = its weight, and x the weight of the cylinder sought; moreover, let R be the centre of gyration of the cylinder, then will the effect of D striking against an obstacle be the same as if the whole weight of the wheel were

removed, and the equivalent *mass $\frac{x \times SR^2}{SD^2}$ collected into ^{* Sect. VI. Prop. VIII.}

the point D . And since p or D 's velocity is that of one foot in a second, the mass at D must be = $p \times 1700^2$, or 2890000 || times the weight of p , putting therefore $p \times$ || ^{Supra.}

$2890000 = \frac{x \times SR^2}{SD^2}$, we have the weight of the cylinder,

which will answer the conditions of the problem, =

$2890000 p \times \frac{SD^2}{SR^2}$; and if the radius of the cylinder =

$10 \times SD$, then will $SR^2 = 50 \times SD^2$ †, and the cylinder's ^{† Sect. VI. p. 224.}

weight = $\frac{2890000 p}{50} = 57800 p$, that is, if $p = 1.3048$ oz.

the weight would become = 4713 pounds.

It remains therefore only to assign for what time or through what space a moving force, which may be assumed of convenient magnitude, must act on the wheel, so that a velocity of 1 foot in a second may be generated in

the point D , the distance of which from the axis is $\frac{1}{10}$

part of the wheel's radius. The solution of this may serve as an example to the principles already demonstrated, as well as for the purpose of illustrating the subject in question: the moving force may be assumed equal to that which the human body can apply without any remarkable exertion.

XXIX.

Let a cylinder of any given radius, for Fig. LXX. example 10 feet, the weight of which = 4713 pounds, revolve round an horizontal axis of motion; it is required to assign how long a moving force of 20 pounds

weight must act on the circumference H , in order to generate a velocity of one foot in a second in the point D , SD being $= 1$ foot.

When $SD = 1$ foot let SR be the distance of the centre of gyration from the axis, and when the velocity in D is that of one foot in a second, the velocity of the circumference will be 10 feet in a second; and since the cylinder's weight will resist the communication of motion to the circumference in the same manner, as if the whole mass were removed and the equivalent mass $\frac{4713 \times SR^2}{SD^2}$

* Sect. VI.
p. 224.

were collected into the circumference, that is,* if $\frac{4713}{2} = 2356.5$ pounds were so collected, we shall have the force which accelerates the circumference $= \frac{20}{20 + 2356.5} = \frac{1}{118.8}$, and the time in which this force will generate a velocity of 10 feet in a second $= \frac{10 \times 118.8}{32^{\frac{1}{2}}} = 37$ se-

conds. A weight therefore of 20 pounds acting for 37 seconds at the circumference of such a cylinder as is described in the problem, will generate a moment which being accumulated in a musket ball fixed at the distance of $\frac{1}{10}$ the cylinder's radius from the axis will produce an effect in penetrating an opposed block of elm immoveably fixed, equal to that which is exerted by the same musket ball fired with its full charge of gunpowder against the same block.

It must be here observed, that the strength of the human body cannot exert constantly so great a quantity of force as that of 20 pounds, if it be applied so far from the axis as 10 feet: the velocity of 10 feet in a second is much greater than that which the velocity of the arm could keep up to, for when the velocity generated is no more than 4 or 5 feet in a second, a person by pulling can scarcely keep the rope stretched with a force equal to 20 pounds: but then

it may be urged, that the force which a man can exert by pulling downward, when the velocity of the mass moved is inconsiderable, is nearly equal to his weight, and therefore may in some measure compensate for the deficiency just described; but at all events the solution may be accommodated to any particular case. For instance, let the wheel be turned by a handle, the distance of which from the axis = 1 foot, then will the force which accelerates the circumference be * diminished in the ratio of 10 : 1, and the time of generating the given angular velocity above described will be † increased in the ratio of 10 : 1, and so will become 370 seconds, or 6 minutes 10 seconds, provided a force equal to 20 pounds be constantly exerted.

* Sect. VI,
Prop. III.

† Prop.
XVI.
Cor. 5.

This accumulation of mechanic force appears extraordinary at first sight, and might suggest wrong notions concerning the subject unless fully considered. It might seem from this great power gained, that by increasing the time in which the moving force acts, the moment communicated to a ponderous wheel might be applied to elevate a weight to an altitude greater than that from which an equal weight descended from rest, in order to generate the motion in the system; and were this really the case, nothing more would be wanting to constitute a machine containing the principle of perpetual motion within itself: but it will appear from the ensuing propositions, that the altitude to which a weight is elevated by a power which communicates motion to a revolving system, and is any how applied to raise it, can never be so great as that from which an equal weight must descend, in order to generate the motion so applied: the times wherein the weights descend and ascend not being of any consequence, the proposition depending on the altitudes only.

XXX.

Let ABC represent a ponderous cylinder moveable in a vertical plane about its axis: let motion be communicated to the whole by a weight p descending through a space s; when the action of p ceases after it has descended through the space s, the

Fig. LXXI.

the system will continue to revolve uniformly; let the moment thus generated be employed to elevate another weight q : the proposition affirms, that the space through which the wheel's motion elevates q , can never be so great as $s \times \frac{p}{q}$,

but will be ultimately equal to $s \times \frac{p}{q}$ when the mass of the revolving body is increased sine limite, the effects of friction not being here considered.

There are two cases contained in this proposition; first, the weight q when quiescent is acted on by the revolving system: 2dly, the weight q always possesses motion equal to that of the circumference to which it is applied. To consider both these; let the radius $SD = d$, $SE = a$, the centre of gyration $= R$, $SR = r$, the weight of the revolving body $= w$, the space through which p descends from rest $= s$. 1st. Let p by describing the space s generate motion in the system before q is applied at the circumference EH : then since the force which accelerates the descent of $p = \frac{p d^2}{p d^2 + w r^2}$, the velocity acquired by p during its de-

* Sect. VI.
Prop. XV.
Cor. 5.

scent through the space $* s = \sqrt{\frac{4 l s p d^2}{p d^2 + w r^2}}$, l being = 193 inches, and the velocity generated in E at the same

|| Sect. VI.
Prop. III.

|| instant $= \sqrt{\frac{4 l s p a^2}{p d^2 + w r^2}}$. Now, suppose p to be removed, at that instant let E be connected with q while quiescent, then will the wheel have the same effect in communicating motion to q as if the whole mass were re-

† Sect. VI.
Prop. VIII.

moved, and the † equivalent mass $\frac{w \times r^2}{a^2}$ were collected into

into E . In order therefore to determine the motion of q , the laws of collision give us this proportion $\frac{wr^2}{a^2} + q$

$\therefore \frac{wr^2}{a^2} \therefore \sqrt{\frac{4lspa^2}{pd^2 + wr^2}}$ to the common velocity with which E and q begin to move, which will therefore $= \sqrt{\frac{4lspa^2}{pd^2 + wr^2}} \times \frac{wr^2}{wr^2 + a^2q}$: and since the constant

force which always retards the ascent of $q = \frac{qa^2}{wr^2 + qa^2}$,

the height S to which q will ascend will $=$ + Sect. III.
Prop. V.
Cor. 3.

$\frac{4lspa^2}{4l \times pd^2 + wr^2} \times \frac{wr^2}{wr^2 + qa^2} \times \frac{wr^2 + qa^2}{qa^2} =$
 $\frac{sp}{q} \times \frac{wr^2}{pd^2 + wr^2 \times wr^2 + qa^2}$, which is always less than

$\frac{sp}{q}$: but approaches $\frac{sp}{q}$ as a limit when wr^2 is increased or $pd^2 + qa^2$ diminished sine limite.

In the preceding case, motion has been supposed to be generated in the system ABC by a weight p descending through a given space s ; and this motion being communicated to a quiescent weight q , applied at any distance whatever from the axis, has been shewn insufficient to elevate q to an altitude which is to s , the space descended through by the weight p , in so great a proportion as that of $p:q$. There is still another case to be considered: suppose the weight q to begin its motion with the revolving system, being elevated by the descent of $q + p$, p being here also the moving force: then let $q + p$ descend through a given space s ; it is to be proved that if $q + p$ be removed the instant the space s has been described, the velocity generated in the system will not be sufficient

to elevate q to an altitude so great as $s \times \frac{p}{q}$. Let the weight q acquire gradually from quiescence a velocity equal to that of the point E : the force which accelerates p in its descent $= \frac{pd^2}{qa^2 + q + p \times d^2 + wr^2}$,

and the velocity generated in p during its descent through a

a space $s = \sqrt{\frac{4 l s p d^2}{w r^2 + q a^2 + q + p \times d^2}}$, and the velocity generated in $E = \sqrt{\frac{4 l s p a^2}{w r^2 + q a^2 + q + p \times d^2}}$.
 Now let $q + p$ be removed; then will the weight q begin to ascend with the velocity $\sqrt{\frac{4 l s p a^2}{w r^2 + q a^2 + q + p \times d^2}}$, and being retarded by the constant force $\frac{q a^2}{w r^2 + q a^2}$, the space S through which q will rise $= \frac{s p a^2}{w r^2 + q a^2 + q + p \times d^2}$
 $\times \frac{w r^2 + q a^2}{q a^2} = \frac{s p}{q} \times \frac{w r^2 + q a^2}{w r^2 + q a^2 + p + q \times d^2}$
 which is always less than $s \times \frac{p}{q}$ while the quantities w, r , &c. are finite: but approaches $s \times \frac{p}{q}$ as a limit while $w r^2$ is increased, or $p + q \times d^2$ is diminished fine limite.

It appears from both cases, that when any motion is generated by the descent of a heavy body, and this motion is afterwards employed to elevate another weight, the altitude of the elevated weight into its quantity of matter, will be always less than the altitude from which the former body descended multiplied into the body itself. If therefore this operation were repeated, and the motion last generated were applied to elevate another weight w by the descent of q , the motion generated by the prior descent would be diminished; this diminution being the consequence of every successive elevation, until the motion is entirely destroyed.

This seems to strengthen those arguments which have been urged to shew the impossibility of a machine possessing the principle of perpetual motion within itself: and it is worthy of remark, that the continual loss of motion above demonstrated, is wholly independent of the effects of friction, and arises from the inertia of the system only; and this appears to be in fact the greatest obstacle which prevents the construction of a self mover: for, notwithstanding what has been urged concerning the effects of friction and the air's resistance, in order to account for the failure of the very many attempts to execute this famous pro-

project, it does not seem that they are such very material impediments to the success of it: for if it were possible from the principles of mechanics, independent of all obstacles, to generate motion in a body by the descent of a weight through any space, so that this motion when applied to raise a weight equal to the former, could elevate it to an altitude in the smallest degree greater than that from which the former weight descended; the reason does not appear why the same means by which this small mechanical advantage was gained, should not be employed in accumulating a still greater accession of mechanic force, such as would be more than sufficient to overcome the effects of friction and other resistances.

The motion generated in ponderous wheels is applied to many practical purposes. Thus, machines to lift weights, to grind corn, &c. as well as many others, having a large wheel of massy substance affixed to some part of them, so that it shall revolve round its axis, have been found to produce far greater mechanical effects than when they are without such additional mass of matter.

Let ABC represent a wheel and axle moveable round an horizontal axis, passing through S by means of an arm or handle E , which is impelled by muscular force: let w be a weight raised by the machine; and let $SD = d$, $SF = a$: part of the moving force applied at E or F to turn the wheel will be employed to balance w . Suppose, for example, the whole force applied at E were p , then will that

Fig. LXXII,

part of it which is expressed by $\frac{Wa}{d}$ be employed in ba-

lancing w , and the other part $\frac{pd - aW}{d}$ will communi-

cate motion to the weight w : this is upon a supposition, that the force impressed on the handle E is constant. But this is not always the case: for in the first place, since the force which acts at E is the pressure exerted by muscular force, it is manifest, that when the velocity generated in E is equal to that with which the muscular power could move if no ways impeded, all pressure must cease: moreover, in the intermediate velocities a given exertion of muscular force will always press with less effect according as the velocity of the handle is greater; for which reason there is some velocity of the handle which will be continued uniform: because if it be greater the moving force diminishes, and the force of the weight w preponderates against it; if it be less than that mean velocity, the

M m

moving

moving force will accelerate it until it comes at the degree of motion above described. This being the case, the use of a ponderous wheel *ABC*, affixed so as to revolve on its axis with the system is manifest: for, suppose a certain uniform velocity generated in it, this will continue for some time to elevate the weight *w*, although the moving force of the arm be discontinued, which must in some degree happen when the arm is ascending; but if there were no motion in the wheel *ABC* to continue the ascent of *w*, the weight *w* would begin to preponderate as soon as the moving force was at all diminished, from which it is manifest how much motion must be continually lost. And these arguments are further strengthened by remarking, that a given muscular force will produce by gradual acceleration as great an uniform velocity when the heavy wheel is applied to the machine, as when the machine is without it, the effects of friction not being considered; the only difference is in the time wherein this velocity is produced, the time being increased by the wheel's weight.

In the preceding propositions the force by which bodies or systems of bodies are caused to revolve, has been supposed constant; and the properties relating to gyration which have been demonstrated are true, when the forces are variable, provided the time of their action be taken evanescent: from which conclusion, and having given the law according to which the force varies, the effects produced by a variable force on a revolving system may be investigated. In the last example, where rotatory motion is supposed to be generated in a ponderous wheel by the action of muscular strength, (of the human arm for instance,) it was observed, that the moving force was not constant, being greatest at the first instant of motion, and absolutely evanescent when the velocity of that part of the wheel, to which the hand was applied, became equal to the greatest velocity with which the hand could move when not impeded.

If the laws according to which muscular force varies in respect of the velocity of the body to which it is applied were known, the effects produced might be computed; but as this kind of power is irregular in its action, being in general greater as the point to which it is applied is descending, geometrical reasoning cannot be used to estimate the motion generated by it, although there would be perhaps little difficulty in forming rules which should be sufficiently near the truth. In the next problem, rotation is generated by the action of a moving force, upon

a principle not entirely dissimilar to that just described ; but the laws according to which the force varies in this case being known, (setting aside the effects of friction) a solution may be obtained and will be a proper illustration of this part of the subject.

XXXI.

Let ABC represent a water wheel which Fig. LXXIII. revolves round an horizontal fixed axis, passing through its centre s . Suppose DEF to be the axle of this wheel, and that a weight w is affixed to a line DW so wound round the axle, that while the wheel is driven round in its own plane by the force of the water impinging at I , the weight w may be raised in a vertical line : having given the area of the boards I, I, I , against which the stream impinges perpendicularly, and the altitude from which the water descends, it is required to assign the greatest velocity with which the wheel can revolve.

When a stream of any fluid impinges perpendicularly against a plane and quiescent surface, the exact quantity of the moving force is equal to the weight of a column of the fluid, the base of which is the area upon which the fluid impinges, and the altitude that from which a body must descend freely from rest by gravity to * acquire that velocity : this will be the moving force which impels * Sect. V. Prop. II. the body when quiescent, or just beginning to move, but after some motion has been communicated to the body upon which the fluid impinges, the impulsive force of the fluid will be diminished ; being the same as if the body were quiescent, and the water impinged upon it with the || difference of the former velocities. Wherefore the altitude || III. Law of Motion. of the column of the fluid, which is equal to its impelling

† Sect. III.
Prop. IV.
Cor. 2.

force, will be always as the difference between the velocity of the impact and that of the body itself; and since the altitudes from † which bodies fall from rest are in a duplicate ratio of the velocities acquired, it follows, that the force of the impact will be in a duplicate ratio of the difference between the velocity of the wheel and that of the impact.

Fig.
LXXIII.

Let the weight of a column of water equal to the force of the impact, when the wheel is quiescent, be A , and in order to facilitate the solution, it must be supposed that the force of the stream acts perpendicularly to the planes on the points I, I, I , at a given distance from the axis, the radius $SI = a$, $SD = b$, V the velocity with which the water impinges on the boards I, I, I , y the velocity of the circumference I, I, I ; then we have this proportion for determining f the force of the water to turn the wheel when its velocity is y , $V - 0^2 : V - y^2 :: A : f = \frac{A \times V - y^2}{V^2}$. Let R be the centre of gyration of the wheel

† Sect. VI.
Prop. VIII.

and the weight W taken together, and let G be the wheel's weight, and $SR = r$; then shall the inertia of the system which resists the communication † of motion to I, I, I , be $\frac{W + G \times r^2}{a^2}$, and the force which accelerates the circum-

ference when its velocity $= y$ is $\frac{A}{W + G} \times \frac{V - y^2}{V^2} \times \frac{a^2}{r^2}$, but from this must be subducted the force of the weight

|| Sect. VI.
Prop. III.
and VIII.

W to retard the circumference, which is || $= \frac{W a b}{W + G \times r^2}$.

The force therefore which accelerates the circumference will be upon the whole $= \frac{A}{W + G} \times \frac{V - y^2}{V^2} \times \frac{a^2}{r^2} -$

$\frac{W a b}{W + G \times r^2}$, and when the wheel's velocity becomes uniform, this force of acceleration must be evanescent, which will give $\frac{A \times V - y^2}{V^2} \times a - W b = 0$, and $y = V - V \times$

$\sqrt{\frac{W}{A} \times \frac{b}{a}}$ the velocity required.

XXXII.

XXXII.

Every thing else remaining as in the last proposition, let the weight w vary, and let it be required to assign the weight w , so that when the wheel has acquired its uniform velocity, the moment of w may be the greatest possible.

Maclaurin's
Fluxions,
Vol. II.
p. 728.

Let the weight sought $= x$, and since the uniform velocity of the * wheel's circumference $= V - V \times \sqrt{\frac{x b}{A a}}$, the uniform velocity of the ascent of x will be $\frac{V b}{a} - V \times \sqrt{\frac{x b^3}{A a^3}}$, and the moment of x will be $\frac{V b x}{a} - V \times \sqrt{\frac{x^3 b^3}{A a^3}}$, the fluxion of which being made $= 0$, we have $\dot{x} - \frac{3 \sqrt{x b} \times \dot{x}}{2 \sqrt{A a}} = 0$, when the moment of x is the greatest possible, and $x = \frac{4 A a}{9 b}$ the weight required.

* Prop. XXXI.

Cor. 1. Let such a weight $\mathcal{Q}$ be suspended from the axle, as will exactly balance the force of the water A acting at the circumference, then will $\mathcal{Q} = \frac{a \times A}{b}$; wherefore when the moment of the ascending weight x is the greatest possible, that weight will be $\frac{4 \mathcal{Q}}{9}$ or $\frac{4}{9}$ parts of the weight which would, if suspended from the axle, balance the force of the stream.

Cor. 2. Since the uniform velocity of the wheel's circumference

* Prop.
XXXI.
‡ Supra.

cumference is $V - V \times \sqrt{\frac{Wb}{Aa}}$, and $\parallel W = \frac{4Aa}{9b}$ when the moment generated in the ascending weight is the greatest, by substituting $\frac{4Aa}{9b}$ for W , we have the velocity $= V - V \times \sqrt{\frac{4}{9}} = \frac{V}{3}$, that is, the velocity of the wheel's circumference will be $\frac{1}{3}$ the velocity of the stream impinging against it, when the moment generated in the weight ascending uniformly is the greatest possible.

Cor. 3. The last Cor. suggests another problem which may be of use: having given a weight W to be raised by the action of the stream of water, the force of which is $= A$, against a quiescent surface; let it be required to assign what must be the proportion between the radius of the wheel and that of the axle, so that the uniform velocity of the ascending weight may be the greatest possible. The value of V remaining, if the radius of the wheel be put $= x$, and $SD = b$, the uniform velocity of the ascending weight $= \frac{bV}{x} - V \times \sqrt{\frac{Wb^3}{Ax^3}}$, of which if the fluxion be made $= 0$, it will give $x = \frac{9Wb}{4A}$.

‡ Supra
p. 277.

Cor. 4. Every thing else remaining, if the velocity with which the water impinges against the boards I, I, I , be doubled, the greatest moment communicated to a weight W ascending uniformly will be increased in the ratio of 8:1: for since the uniform velocity of the ascending weight is $\frac{bV}{3a}$, and the weight $\parallel$ moved $W = \frac{4Aa}{9b}$, the moment generated in the ascending weight will be expressed by $\frac{4AV}{27}$, and $\frac{4}{27}$ being given, the moment will be

‡ Page 276. proportional to $A \times V$, but $\dagger A$ is as V^2 , wherefore the moment communicated to the ascending weight will be as V^3 , or in a triplicate ratio of the velocity wherewith the water impinges on the wheel.

The force which communicates motion to water wheels, and the resistances which are occasioned by friction, tenacity, and various other causes, render the application of the theory

theory of mechanics to practice in these cases extremely difficult. It is probably from this reason, that the art of constructing machines, which are moved by the force of water, &c. has been almost wholly practical; the best improvements having been deduced from continued observation of the results produced in given circumstances, whereby the gradual correction of error, and varied experience of what is most effectual, have supplied the place of a more perfect investigation from the laws of motion.

This seems to be the best method, as far as regards the practical construction of these machines, the nature of the case will admit of; for although there may be two roads leading ultimately to the same truths, i. e. a direct investigation from the laws of motion, and long continued observation, independent of theory; the latter is frequently the most easy and most eligible, although less direct and less scientific; the former being inaccessible to those who possess the elementary parts of mechanics only. It is in vain to attempt the application of the theory of mechanics to the motion of bodies, except every cause which can sensibly influence the moving power and the resistance to motion be taken into account: if any of these be omitted, error and inconsistency in the conclusions deduced must be the consequence. It was at one time supposed from this inadequate application of the theory, that the same laws of motion would not extend to all branches of mechanics, but that different principles were to be accommodated to different kinds of motion. If this were truly the case, the science of mechanics would fall short of that superior excellence and extent, which it is generally allowed to possess. For it is probable, that there is no kind of motion, whether of the most simple or complicated nature, but what may be referred to * three easy and obvious propo-
* Laws of motion.

sitions, the truth of which it is impossible to doubt: and if we are not enabled to investigate the effects from the data in all cases, the deficiency must not be imputed to the science of mechanics, but to the want of methods of applying mathematics to it.

This may be illustrated by an example in order to shew that the motion communicated to water wheels, however complicated the data may be, is equally referable to the laws of motion, with the effects of the most uncompounded force. If a stream of water falls perpendicularly on a plane surface, the moving force arising from the impact only is equal to the weight of a column of water, the base of which is the surface † upon which the water impinges,
† Sect. V. Prop. I.
 and

Fig.
LXXIII.

and altitude that through which a body must fall from rest to acquire the velocity of impact. If the inclination of the stream to the surface should be changed, the force exerted in a direction perpendicular to the plane will be diminished in a duplicate ratio of the radius to the sine of inclination, the surface on which the water impinges remaining. Now when water falls on the boards of a water wheel, the direction of the stream makes different angles with the planes of those boards; for since the particles of water descend in curve lines, they will strike any plane surface in the direction of a tangent to the curve on the point of impact. Moreover, the water will strike the higher boards T, T , with a less velocity and in a direction more inclined to their planes, than when it impinges on the boards I, I : it is also to be considered, that the stream will impinge on the boards at different distances from the axis of motion: all which circumstances must be taken into account to find the force which tends to communicate motion to the wheel when quiescent; and when motion has been communicated, the force of the stream to turn the wheel will be determined as in p. 276. But this is not the only consideration which affects the moving force; the force hitherto considered has been supposed to proceed from the impact of the particles only; in which case each particle, after it has struck the board, is imagined to be of no other effect in communicating motion; but this is not wholly the case: for after the particle has impinged on the board, it will continue some time to operate by its weight, and this time will be longer or shorter according to the different constructions of the wheel. In the overshot wheel, the continuance of the pressure arising from the weight of the water, will be longer than in the undershot, the force which arises from the impact of the water being nearly the same in each case. The whole moving force therefore will consist of the impact determinable as above, and of the weight of the water descending along with the circumference and communicating additional motion to it: this entire moving force being determined either by theory or experiment may be denoted by A . After the moving force which impels the circumference has been determined, the resistance to this force must be found; for on the proportion between the moving force and the resistance, the acceleration of the machine will depend. This resistance is of various kinds: 1. That of inertia: let the centre of gyration of the wheel $= R$, $SR = r$, $SI = a$, the wheel's weight $= G$, then will the inertia of the wheel which resists the communication

munication of motion to the circumference, $\ast = \frac{W \times r^2}{a^2}$: in $\ast$ Sect. VI. Prop. VIII.

the same manner, the inertia of any parts of the system may be obtained from having given the weights, figures, and distances from the axis by prop. XII. 2. If the machine is of that kind which raises weights, such for instance as water, the weight raised, allowing for its mechanical effect on the point the acceleration of which is sought, must be subducted from the moving force before found; and this will be a constant quantity. There are other resistances also homogeneal to weight, i. e. those of friction and tenacity, &c. which are variable in some ratio of the machine's velocity: and in order to proceed with the investigation, the exact quantity of weight which the friction is equal to, when the wheel moves with a given velocity, must be known, and the variation of the resistances in respect of the velocities, which circumstances must be determined by experiment. If the force equivalent to the friction, &c. be subducted from the moving force, the remainder will give the moving power by which the circumference is impelled upon the whole; this being divided by the inertia of the mass moved, will give the force which accelerates the circumference. The notation of the last proposition therefore remaining, if, when the wheel's velocity is V , the friction is equal to the weight $\mathcal{Q}$ applied at the circumference, and varies in the n th power of the velocity, we shall have the force which accelerates the

circumference $\frac{A}{W+G} \times \frac{V-y^2}{V^2} \times \frac{a^2}{r^2} - \frac{Wab}{W+G \times r^2} - \frac{\mathcal{Q}y^n \times a^2}{V^n \times W+G \times r^2}$; wherefore if x be the space which has

been described by the circumference when the velocity is y , and l is the space through which bodies are impelled by the force of gravity in one second, the principles of

acceleration $\dagger$ give us $\frac{y\dot{y}}{2l} = \frac{A}{W+G} \times \frac{V-y^2}{V^2} \times \frac{a^2}{r^2} - \dagger$ Sect. III. Prop. V.

$\frac{Wab}{W+G \times r^2} - \frac{\mathcal{Q}y^n \times a^2}{V^n \times W+G \times r^2} \times \dot{x}$, from which if y be

determined in terms of x and constant quantities, the velocity communicated to the wheel will be known. It is only by taking in all these circumstances into the account, which can produce coincidence between theory and matter of fact.

† Prop.
XXXII.
Cor. 2.

In Mr. Parent's † proposition above demonstrated, the effects of friction are not considered, nor is the time in which the uniform motion is produced taken into account: it is not therefore to be wondered, that if this proposition be referred to experiments made on machines, wherein friction bears a considerable proportion to the moving force, the results shall be different from those demonstrated from different data. The motion of machines impelled by the force of water or wind very soon becomes uniform; the reason of which is, that the friction of the parts vary in some direct ratio of the velocity: for were friction entirely removed, the motion of the machine would be continually accelerated, and would not acquire its uniform velocity in any finite time. If the exact laws according to which the friction of the machine varied in respect of the velocity were known, together with the absolute quantity of friction corresponding to a given velocity, the uniform or greatest velocity acquirable by the machine, and the finite time of acquiring it might be ascertained; but it will appear from the following proposition, that when friction is not taken into account, the time of acquiring the uniform velocity, such as is referred to in Mr. Parent's proposition, is infinite.

XXXIII.

Every thing remaining as in the xxxist prop. let it be required to assign the time during which the wheel A B C is accelerated before it acquires a given velocity, effects of friction, cohesion, &c. not being considered.

The notation remaining, let x be the space which has been described by any point of the wheel's circumference when the velocity generated has become $= y$: the force

‡ Page 276. which † accelerates the wheel's circumference $= \frac{A}{W+G} \times$

$$\frac{V-y}{V^2} \times \frac{a^2}{r^2} - \frac{Wab}{W+G \times r^2}, \text{ for brevity let } \frac{A}{W+G} \times \frac{a^2}{r^2} =$$

$= F$, and $\frac{Wab}{W + G \times r^2} = K$; we have therefore the force

which accelerates the circumference $= \frac{\overline{V-y}^2 \times F}{V^2} - K$:

let z be the space through which a body falls freely from rest to acquire the velocity y , then * will $z = \frac{y^2}{4l}$, and the

* Sect. III.
Prop. V.

principles of acceleration give us this equation $\frac{y \dot{y}}{2l} =$

$$\frac{\overline{V-y}^2 \times F - V^2 K \times \dot{x}}{V^2}, \text{ and } \dot{x} = \frac{V^2 \dot{y} y}{2l \times \overline{V-y}^2 \times F - V^2 K}$$

and the fluxion of the time $t = \frac{\dot{x}}{y} = \frac{V^2 \dot{y}}{2l \times \overline{V-y}^2 \times F - V^2 K}$;

wherefore taking the fluents, the time itself $= \frac{V}{\sqrt{4FKl^2}} \times$

$$\log. \frac{V \sqrt{K} + \overline{V-y} \sqrt{F}}{\sqrt{\overline{V-y}^2 \times F - V^2 K}}: \text{ but this should vanish when}$$

$y = 0$; wherefore the entire fluent or the time required t

$$= \frac{V}{\sqrt{4l^2 FK}} \times \log. \frac{V \sqrt{K} + \overline{V-y} \sqrt{F}}{\sqrt{\overline{V-y}^2 \times F - V^2 K}} \times \frac{\sqrt{F-K}}{\sqrt{F+K}}.$$

In order to reduce this, let $\frac{\sqrt{F-K}}{\sqrt{F+K}} = n$, $V-y = z$,

and e = the number, the hyperbolic log. of which = 1, and

let $\sqrt{\frac{4l^2 FK t^2}{V^2}} = p$; this will give $\frac{nV \sqrt{K} + nz \sqrt{F}}{\sqrt{z^2 F - V^2 K}} = e^p$,

and squaring the whole $\frac{n^2 V^2 K + 2n^2 V z \sqrt{FK} + n^2 z^2 F}{F z^2 - V^2 K}$

$= e^{2p}$, and $F z^2 - V^2 K = e^{-2p} \times n^2 V^2 K + 2n^2 V z \sqrt{FK} e^{-2p} + n^2 z^2 F e^{-2p}$; this being resolved gives

$$z = V-y = \frac{n^2 V \sqrt{FK} e^{-2p}}{F - n^2 F e^{-2p}} + \sqrt{\frac{n^4 V^2 F K e^{-4p}}{F - n^2 F e^{-2p}^2}}$$

$$+ \frac{V^2 K + e^{-2p} \times n^2 V^2 K}{F - n^2 F e^{-2p}}; \text{ which being subtracted}$$

from V , leaves y = the velocity of the circumference generated in the time t from rest.

Cor. As the time t increases, the wheel's velocity increases, and if it has any limit by making t infinite, the ultimate value of y will give that limit: here since $p =$

$$\sqrt{\frac{4l^2 t^2 F K}{V^2}}, \text{ it is manifest, that if } t \text{ be infinite, } p \text{ will}$$

be also infinite, and e^{-2p} evanescent; wherefore all the terms in the preceding expression which are multiplied into e^{-2p} will become $= 0$: this will give the ultimate

$$\text{value of } z = V - y = \text{to } V \sqrt{\frac{K}{F}}, \text{ and } y \text{ the limit of the}$$

$$\text{velocity sought will be } = V - V \sqrt{\frac{K}{F}}; \text{ but } K =$$

$$\frac{Wab}{G + W \times r^2}, \text{ and } F = \frac{A \times a^2}{G + W \times r^2}, \text{ wherefore } \frac{K}{F} =$$

† Prop.
XXXI.

$$\frac{Wb}{Aa}, \text{ and } y \text{ the greatest velocity } \dagger = V - V \sqrt{\frac{Wb}{Aa}}: \text{ this}$$

was otherwise shewn to be the uniform or greatest velocity of motion which could be acquired by the circumference, which it appears from this solution it never can arrive at in any finite time.

The chief proposition upon which the estimation of the velocities generated in a system of bodies revolving round a fixed axis depends, is that by which the inertia of the various parts of the system is determined: for after it has been shewn what mass, when acted on immediately by the moving force, is equivalent in its inertia to that which is exerted by the parts of the system revolving at their respective distances, the acceleration of the point at which the moving force is applied becomes known: it has been shewn, that the inertia exerted by the particles, which re-

† Sect. VI.
Prop. I. & II.

volve at different distances † from the axis of motion, are in a ratio compounded of the direct duplicate ratio of their velocities or distances from the axis, and the direct ratio of the quantities of matter contained in them; from which principle the figures and weights of the various parts which compose the system being known, its revolving motion generated by a given moving force can be estimated: but these propositions cannot be generally applied to the determination of the inertia exerted by bodies against the action of moving forces, because the conditions on which they are demonstrated are particular in belonging to rotatory motion, or others of the same kind; the cotemporary

rary

rary velocities of any given points in the system are in a constant ratio from the first instant of motion ; moreover, the moving forces, as well as the inertia of the various parts are constant, because in revolving bodies the distance from the axis of each particle, as well as the distance from the axis at which the moving force is applied, is invariable : and it follows, that in other cases, where these conditions exist not, the laws which have been demonstrated for the estimation of the acceleration of revolving bodies cannot be applied ; other rules therefore will be necessary for this purpose.

XXXIV.

Let AFC , BGC be two curves, the Fig.
LXXIV. planes of which are vertical, and let a line ACB be stretched over a fixed pulley c by two given weights A and B , of which B preponderates against A , and descends along the curve cGB : it is required to assign the velocity of B when it has descended through a given perpendicular altitude, and A has ascended through a given altitude in the same time.

Let BL be the perpendicular altitude through which B has descended, and let AH be the perpendicular altitude thro' which A has ascended in the same time, and let $BL = y$, $AH = q$. Suppose the weight B to describe in its descent the evanescent arc bB , and during the same time let the other weight A rise through the arc aA . Through b and A draw bo and Al parallel to the horizon : also, through b draw bn perpendicular to CB , and Am perpendicular to Ca , and let $CB = x$, $Bn = \dot{x} = Am$, $Bo = \dot{y}$, $Bb = \dot{r}$, $Aa = \dot{r}$, and $al = \dot{q}$; and let z be the space through which a body must fall freely by the acceleration of gravity from rest so as to acquire the velocity of B , while it is describing the elementary space bB .

The

The force which impels B in the direction bB is $= \frac{B\dot{y}}{s}$, and since A acts in opposition to this force, its effects when referred to the direction bB must be subducted from $\frac{B\dot{y}}{s}$ in order to obtain the force which upon the whole impels B in the direction bB : the force of A in the direction $BC = \frac{A\dot{q}}{x}$, which being resolved into the direction bB becomes $= \frac{A\dot{q}}{s}$; wherefore the whole force by which B is urged in the direction bB will $= \frac{B\dot{y}}{s} - \frac{A\dot{q}}{s}$.

Also, the force with which B impels A in the direction $AC = \frac{B\dot{y}}{x}$, and that of A in the same direction $= \frac{A\dot{q}}{x}$, wherefore the entire force which urges A in the direction $AC = \frac{B\dot{y} - A\dot{q}}{x}$, and this force being resolved into the direction aA becomes $\frac{B\dot{y} - A\dot{q}}{r}$: with this force there-

fore B impels A in the direction aA . It is next to be enquired, what is the mass moved by B , or the inertia which is opposed to the communication of motion to B : the inertia of B opposed by its own mass is B , but as the bodies A and B move with different velocities, the inertia opposed by A will not be equivalent to the mass A , for reasons before assigned. In order to determine an equivalent mass, which, being accumulated in B when A is removed, will resist the communication of motion in the same manner, as the mass A ascending by the force of B , let B be first supposed without inertia, and to possess weight only: then since the force which impels A in the direction aA is $\frac{B\dot{y} - A\dot{q}}{r}$, and the mass moved A , B being supposed without inertia, the force which accelerates A will be denoted by $\frac{B\dot{y} - A\dot{q}}{rA}$, because A is now the only mass of matter moved. But A describes the space i in the same particle of time in which B describes bB or

$\dot{s}$; and since when the fluxion of the time is given, the force which accelerates bodies describing different spaces will be $\frac{A\ddot{q}}{\dot{s}}$ the second fluxions of those spaces, we have the

force which accelerates the body $B = \frac{B\dot{y} - A\dot{q}}{\dot{s}A} \times \frac{\ddot{s}}{\dot{s}}$,

from hence we shall obtain the mass, which if placed at B would exert a resistance or inertia equivalent to A : for since in this case, the moving force and inertia are each applied at the same point, and referred to the same direction, the quantity of matter * moved will be equal to the moving force divided by the accelerating force: and since

* Sect. I.
Prop. IX.

the force which impels B is $\frac{B\dot{y} - A\dot{q}}{\dot{s}}$, and the force

which accelerates $B = \frac{B\dot{y} - A\dot{q}}{A\dot{s}} \times \frac{\ddot{s}}{\dot{s}}$, we have the mass

moved at B or the inertia equivalent to that of $A = \frac{A\dot{s}\ddot{r}}{\dot{s}\ddot{s}}$.

From hence a solution to the problem above described immediately follows; for let z be the space through which a body must fall freely to acquire the velocity in B , then the force which impels B in the direction bB is $\frac{B\dot{y} - A\dot{q}}{\dot{s}}$, and the inertia of B is B , that of A equivalent to a mass $\frac{A\dot{s}\ddot{r}}{\dot{s}\ddot{s}}$ when accumulated in B , A itself being

taken away, the whole mass moved therefore by B will be equivalent to $B + \frac{A\dot{s}\ddot{r}}{\dot{s}\ddot{s}} = \frac{B\dot{s}\ddot{s} + A\dot{s}\ddot{r}}{\dot{s}\ddot{s}}$, and the force

which accelerates $B = \frac{\dot{s} \times B\dot{y} - A\dot{q}}{B\dot{s}\ddot{s} + A\dot{s}\ddot{r}}$: and since $\dot{s}$ is the fluxion of the space described by B , the principles of acceleration give us $\dot{z} = \frac{\dot{s}\ddot{s} \times B\dot{y} - A\dot{q}}{B\dot{s}\ddot{s} + A\dot{s}\ddot{r}}$, and $z =$ the

fluent of $\dot{s}\ddot{s} \times \frac{B\dot{y} - A\dot{q}}{B\dot{s}\ddot{s} + A\dot{s}\ddot{r}}$.

Cor. 1. Let AFC become a straight line perpendicular to the horizon, then will $\dot{r} = \dot{x} = \dot{q}$, it follows therefore,

that $\dot{z} = \frac{\dot{s}\ddot{s} \times B\dot{y} - A\dot{x}}{B\dot{s}\ddot{s} + A\dot{x}\ddot{x}}$.

Cor.

Aug
LXXV

Cor. 2. Let CFA be a straight line perpendicular to the horizon, and let the curve CGB be a logarithmic spiral, the ordinates of which intersect the curve at an angle the cosine of which is to radius in the proportion of c to r ; then the notation remaining, $r\dot{x} = c\dot{y}$ by the property of the curve, and $r\ddot{x} = c\ddot{y}$, wherefore $\frac{\dot{y}\ddot{y}}{B\dot{y}\ddot{y} + A\dot{x}\ddot{x}}$

$$= \frac{r^2}{r^2 B + c^2 A}, \text{ and } z = \frac{r^2 \times By - Ax}{Br^2 + c^2 A}, \text{ and if } l = 193$$

inches, the velocity acquired by $B = \sqrt{\frac{4lr^2 \times By - Ax}{Br^2 + c^2 A}}$.

Cor. 3. The laws observed by bodies which revolve round a fixed axis of motion are particular cases of this general proposition, and may be inferred from it; thus, let LMN represent a wheel and axle moveable round an horizontal axis of motion; and suppose that a weight A is applied to the circumference of the axle to be raised by the weight B applied to the wheel's circumference: and let the radius $SL = r$, and $SM = c$, the force where-

by B endeavours to descend $= B - \frac{Ac}{r} = \frac{Br - Ac}{r}$: let

any space described by B during its descent from rest $= y$, and the space described by A in its ascent during the same time $= x$; then will $rx = cy$, wherefore $r\dot{x} = c\dot{y}$ and $r\ddot{x} = c\ddot{y}$; and since the inertia opposed by B to its own motion is B , and the inertia opposed by A to the motion

of B is by the preceding solution $= \frac{A\dot{x}\ddot{x}}{\dot{y}\ddot{y}} = \frac{Ac^2}{r^2}$, the

whole inertia opposed to the motion of $B = B + \frac{Ac^2}{r^2} =$

$\frac{Br^2 + Ac^2}{r^2}$, and since the force which impels $B =$

* Sect. I. Prop. IX. $\frac{Br - Ac}{r}$, the force which * accelerates the descent of B

will be $= \frac{Br^2 - Ac^2}{Br^2 + Ac^2}$. Let z be the space through

which a body must fall from rest freely, by the acceleration of gravity, to acquire the velocity of B at the instant it has described the space y ; this † will give $z =$

† Sect. III. Prop. V. $\frac{Br^2 y - Acr y}{Br^2 + Ac^2}$, and $z = \frac{Br^2 y - Acr y}{Br^2 + Ac^2}$; or since $y =$

rx

$\frac{rx}{c}, z = \frac{Byr^2 - Ar^2x}{Br^2 + Ac^2}$, which is precisely the same value

as is otherwise obtained † from prop. xix. the wheel's inertia not being considered. In the same manner, the inertia of the whole system, including that of the wheel, might be inferred from the preceding general solution. † Sect. VI.

In the examples which have been given to illustrate this proposition, the ratio of the fluxions $\dot{x} : \dot{y}$ & $\ddot{y} : \ddot{x}$ is constant; in that which follows they are variable. Let two equal weights A, A be fastened to the extremities of a line which goes over the fixed points E and C , which are horizontal: when the line $AECA$ is stretched by the equal weights A, A , let a body B be fixed to the middle point D , it will descend from rest at D in the direction DB , perpendicular to the horizon, at the same time elevating the weights A, A . Suppose it were required to assign the velocity of the descending weight B when it has described any space DB . Let $ED = a$, $EB = y$, and $BD = x$: to determine the velocity of B , while it describes the evanescent space $bB = \dot{x}$. B is impelled downward by

its own weight diminished by $2A \times \frac{x}{y} = 2A \times \frac{x}{\sqrt{a^2 + x^2}}$ acting in the contrary direction BD : the whole force therefore by which B tends downwards $= B - \frac{2Ax}{\sqrt{a^2 + x^2}}$:

the mass moved by B is its own quantity of matter, together with the inertia of the weights A, A referred to the direction Bb ; the inertia of $2A$ estimated in the direction

$Bb = 2A \times \frac{\dot{y}\ddot{y}}{\dot{x}\ddot{x}}$ by the proposition above demonstrated,

and because $\frac{\dot{y}}{\dot{x}} = \frac{x}{\sqrt{a^2 + x^2}}$, and $\frac{\ddot{y}}{\ddot{x}} = \frac{a^2 + 2x^2}{2x \times \sqrt{a^2 + x^2}}$,

the inertia of $2A$ will be $2A \times \frac{\dot{y}\ddot{y}}{\dot{x}\ddot{x}} = A \times \frac{a^2 + 2x^2}{a^2 + x^2}$,

and that of $B + 2A$ will be $\frac{B \times a^2 + x^2 + A \times a^2 + 2x^2}{a^2 + x^2}$:

let z be the space due to the velocity of B while describing Bb , and since the force which impels $B =$

$\frac{B \times \sqrt{a^2 + x^2} - 2Ax}{\sqrt{a^2 + x^2}}$, we shall * have $\dot{z} =$

Ba^2

Fig. LXXXIX.

* Sect. III.
Prop. V.
Cor. 4.

$$\frac{Ba^2\dot{x} + Bx^2\dot{x} - 2Ax\dot{x}\sqrt{a^2+x^2}}{Ba^2 + Bx^2 + Aa^2 + 2Ax^2}, \text{ and taking the fluents } z =$$

$$\frac{BAa \times M^o}{\sqrt{B+A} \times B + 2A^{\frac{3}{2}}} + \frac{Bx + 2Aa - 2A \times \sqrt{a^2+x^2}}{B + 2A} +$$

$$\frac{2A^{\frac{3}{2}}a}{B + 2A^{\frac{3}{2}}} \times \log. \frac{\sqrt{Aa^2} + \sqrt{B + 2A} \times \sqrt{a^2+x^2}}{\sqrt{B+A} \times a^2 + B + 2A \times x^2} \times$$

$$\times \frac{\sqrt{B+A}}{\sqrt{A} + \sqrt{B+2A}} : \text{ in which } M^o \text{ signifies the arc of a}$$

circle, the tangent of which is $\sqrt{\frac{B+2A}{B+A}} \times \frac{x}{a}$ to radius

$= 1$.

Cor. 1. By inspecting the value of z , it appears, that when A is $= 0$, or when B is infinite, $z = x$.

Cor. 2. If $B = 2A$, B will descend continually, but the velocity acquired will be limited; by making x infinite, $A = 1$, and $B = 2$, $p = 3.14159$, &c. the limit of

z will become $\frac{pa}{8\sqrt{3}} - \frac{a \times \log. \sqrt{3}}{4} = a \times .2161$, so that

the velocity of the descending weight B , is always less than that which a heavy body acquires by falling freely from rest through .2161 part of half the string's length.

Cor. 3. If B be in the least greater than $2A$, the value of z , and consequently the velocity, when x is infinite, will increase sine limite.

Cor. 4. When B is less than $2A$, the velocity of B during its descent will first increase and afterwards decrease, until the weights become stationary: B will then begin to ascend.

Cor. 5. By making the fluxion of $z = 0$, or $\dot{z} =$

$$\frac{Ba^2\dot{x} + Bx^2\dot{x} - 2Ax\dot{x}\sqrt{a^2+x^2}}{Ba^2 + Bx^2 + Aa^2 + 2Ax^2} = 0, \text{ and solving the}$$

equation, we shall have the value of x when the velocity of B is the greatest possible in general terms, in which case it appears that $x = \frac{Ba}{\sqrt{4A^2 - B^2}}$.

S E C T. VII.

CONTAINING A DESCRIPTION OF EXPERIMENTS ON THE RECTILINEAR MOTION OF BODIES, WHICH ARE ACTED ON BY CONSTANT FORCES.

MECCHANICAL experiments are of two sorts; the one relating to the quiescence of bodies, and the other to their motion. Among the former are included those which demonstrate, or rather make evident to the senses, the equilibrium of the mechanic powers, and the corresponding proportions of the weights sustained to the forces which sustain them, the properties of the centre of gravity, the composition and resolution of forces, &c.: by the latter kind of experiments are shewn the laws of collision, of acceleration, and the various effects of forces, which communicate motion to bodies.

Concerning mechanical experiments, it is observable, that those wherein an equilibrium is formed, will frequently appear coincident with the theory, although considerable errors are committed in construct-

Fig.
LXXVI.

ing them. To exemplify this, let AB represent a rod moveable round an horizontal axis of motion, which passes through its centre of gravity s . Suppose SA to SC in the proportion of $2 : 1$; and let any weight w be suspended from A ; in order to balance w by a weight acting at C , this weight must be accurately $= 2w$ by the theory: but if, by an error in constructing the experiment, instead of $2w$, another weight greater or less than $2w$ be applied at C , an equilibrium will be still produced, provided the friction of the axis be sufficient to counteract the moving force arising from the erroneous weight. In the same manner, it appears in other cases, that from the effects of friction, tenacity, &c. experiments relating to the equilibrium of forces will derive an appearance of greater exactness, than would be observed in them were friction, &c. wholly removed. The case is different in experiments concerning the motion of bodies, in which, whatever care be taken to render the proportion of the forces and of the weights moved, such as is required by the theory, the interference of friction and other resistances, which contribute to render the experiments on the equilibrium of

of forces apparently more perfect than they really are, causes the motion of bodies, which are the objects of experiments, to differ from the theory, with which it would precisely coincide were those obstacles removed.

According to the greater or less degree of exactness, experiments on the motion of bodies are required to correspond with the theory, more or less attention is necessary in removing the impediments above described, as well as others which will be hereafter considered: if the experiments are designed only to assist the imagination by substituting sensible objects, instead of abstract and ideal quantities, an apparent agreement between the theory and experiment may be sufficient to answer this purpose, although it may be produced from an erroneous construction; but it must be allowed, that experiments of this kind are extremely defective, and entirely insufficient to impress the mind with that satisfactory conviction, which always attends the observation of experiments accurately made.

It may perhaps be required to assign what degree of precision is necessary in experiments on the motion of bodies since
ma-

Fig.
LXXVI.

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It may perhaps be required to assign what degree of precision is necessary in experiments on the motion of bodies since
ma-

periments on bodies, which descend through considerable altitudes, and if the altitudes be diminished, the times of motion will be so small as to render the observation of them very precarious and unsatisfactory.

If to remedy this inconvenience bodies be caused to descend along inclined planes, according to the experiments of the celebrated author* of this theory, by varying the proportion of the planes' heights to their lengths, the force of acceleration may be diminished in any ratio, so that the descending bodies shall move sufficiently slow to allow of the times of motion from rest being accurately observed; and the effects of the air's resistance to bodies moving with these small velocities will be absolutely insensible: the principal difficulty however which here occurs, arises from the rotation of the descending bodies, which cannot be prevented without increasing their friction far beyond what the experiment will allow of.

To consider this a little further: Bodies thus descending along inclined planes by rotation, with an uniformly accelerated motion, may be either spherical or cylindrical. Let A B C represent an inclined plane,
the

the height of which is to its length as 1 : 9.2; then if a body descends from rest by sliding along this plane, and the surfaces of the plane, and of the descending body be perfectly free from friction, since the force

of acceleration is $\frac{1}{9.2}$ part of acceleration of gravity, the space described in one se-

cond from *rest = $\frac{193}{9.2} = 21$ inches: but * Sect. III. Prop. IV.

if a sphere descends by rolling along the same inclined plane, the space described in one second † from rest will be only † Sect. VI. Prop. XIV.

$\frac{21 \times 5}{7} = 15$ inches; and if a cylinder

should roll down the plane instead of a sphere, the space described in one ‡ second ‡ Sect. VI. Prop. XIV. Cor. 2. would be no more than 14 inches, instead

of 21 inches, the space which would be described from rest in one second, by a cylinder or any other body sliding along the plane: it may however be observed, that in each of these descents, the body will be uniformly accelerated, and consequently the spaces described in a duplicate ratio of the times of motion estimated from the body's quiescence.

But in these methods of making experiments on the accelerated motion of bodies,

dies, there are no means separating the mass moved from the moving force; we cannot therefore apply different forces to move the same quantity of matter on a given plane, or the same force to different quantities of matter. Moreover, the accelerating force being constant and inseparable from the body moved, its velocity will be continually accelerated, so as to render the observation of the velocity acquired at any given instant impossible. It may be therefore not altogether useless to describe such an instrument as will subject to experimental examination, the properties of the five mechanical quantities, i. e. the quantity of matter moved, the constant force which moves it, the space described from rest, the time of description, and the velocity acquired.

Of the mass
moved.

1. In order to observe the effects of the moving force, which is the object of any experiment, the interference of all other forces should be prevented: the quantity of matter moved therefore, considering it before any impelling force has been applied, should be without weight. Although it be impossible to abstract the natural gravity or weight from any substance whatever, yet the weight may be so counteracted

teracted as to be of no sensible effect in experiments. Thus in the instrument ^{Fig.} LXXVIII. constructed to illustrate this subject experimentally, A, B represent two equal weights affixed to the extremities of a very fine and flexible silk line: this line is stretched over a wheel or fixed pully a b c d, moveable round an horizontal axis: the two weights A, B being precisely equal and acting against each other, remain in equilibrio; and when the least weight is superadded to either (setting aside the effects of friction) it will preponderate. When A, B are set in motion, by the action of any weight m, the sum $A + B + m$ would constitute the whole mass moved, but for the inertia of the materials which must necessarily be used in the communication of motion: these materials consist of 1. the wheel a b c d over which the line sustaining A and B passes. 2. The four friction wheels on which the axle of the wheel a b c d rests: the use of these wheels is to prevent the loss of motion, which would be occasioned by the friction of the axle if it revolved on an immoveable surface. 3. The line by which the bodies A and B are connected so as when set in motion to

move with equal velocities. The weight and inertia of the line are too small to have sensible effect on the experiments, as will appear in a subsequent note: but the inertia of the other materials just mentioned constitute a considerable proportion of the mass moved, and must be taken into account. Since when A and B are put in motion, they must necessarily move with a velocity equal to that of the circumference of the wheel a b c d, to which the line is applied; it follows, that if the whole mass of the wheels were accumulated in this circumference, its inertia would be truly estimated by the quantity of matter moved: but since the parts of the wheels move with different velocities, their effects in resisting the communication of motion to A and B by their inertia will be different; those parts which are furthest from the axis resisting more than those which revolve nearer, in a duplicate proportion* of those distances. If the figures of the wheels were regular, from knowing their weights and figures, the distances of their centres of gyration from their axes of motion would become known, and consequently an equivalent weight, which being accumulated uniformly in the † circumference

* Sect. VI.
Prop. II.

† Sect. VI.
Prop. VIII.

‡ Prop.
VIII.
and XXIII.

cumference a b c d, would exert an inertia equal to that of the wheels in their constructed form. But as the figures are wholly irregular, recourse must be had to experiment in order to assign what equivalent quantity of matter, which being accumulated uniformly in the circumference of the wheel a b c d, would resist the communication of motion to A in the same manner as the wheels themselves.

In order to ascertain the inertia * of the wheel a b c d, with that of the friction ~~of~~ ~~the~~ wheels, the weights A, B being removed, the following experiment was made.

* Sect. VI.
Prop. XIII.

A weight of 30 grains was affixed to a silk line (the weight of which was not so much as $\frac{1}{4}$ of a grain, and consequently too inconsiderable to have sensible effect in the experiment) this line being wound round the wheel a b c d, the weight 30 grains by descending from rest communicated motion to the wheel, and by many trials was observed to describe a space of about $38\frac{1}{2}$ inches in 3 seconds. From these data the equivalent mass or inertia of the wheels will be known from the rule contained ‡ in prop. † Page 226. XIII. sect. VI. Applying this rule to the present

lent case, we have $p = 30$ grains, $t = 3$ seconds; $l = 193$ inches, $s = 38.5$ inches,

and the inertia fought $= \frac{p \times t^2 l}{s} - p =$

$$\frac{30 \times 9 \times 193}{38.5} - 30 = 1323 \text{ grains, or } 2\frac{3}{4}$$

ounces. This is the inertia equivalent to that of the wheel $a b c d$, and the friction wheels together; for the rule to which the preceding experiment was referred, extends to the estimation of the inertia of the mass contained in all the wheels.

Fig.
LXXVIII.

The resistance to motion therefore arising from the wheel's inertia will be the same as if they were absolutely removed,

and a mass of $2\frac{3}{4}$ ounces were uniformly

accumulated in the circumference of the wheel $a b c d$. This being premised, let the boxes A and B be replaced, being suspended by the silk line over the wheel or pulley $a b c d$, and balancing each other: suppose that any weight m be added to A so that it shall descend; the exact quantity of matter moved, during the descent of the weight A , will be ascertained, for the

whole mass will be $A + B + m + 2\frac{3}{4}$ oz.

To

To proceed in describing the construction of the ensuing experiments. In order to avoid troublesome computations in adjusting the quantities of matter moved and the moving forces, some determinate weight of convenient magnitude may be assumed as a standard, to which all the others are referred. This standard weight in the sub-

sequent experiments is $\frac{1}{4}$ of an ounce, and

is represented by the letter m . The in-

ertia of the wheels being therefore $= 2\frac{3}{4}$

ounces will be denoted by $11m$. A and B

are two boxes constructed so as to contain

different quantities of matter, according as

the experiment may require them to be

varied: the weight of each box, including

the hook to which it is suspended $= 1\frac{1}{2}$

oz. or according to the preceding estimation,

the weight of each box will be de-

noted by $6m$; these boxes contain such

weights as are represented by c , each of

which weighs an ounce, so as to be equi-

valent to $4m$: other weights of $\frac{1}{2}$ an oz.

$= 2m$, $\frac{1}{4} = m$, and aliquot parts of m ,

such

Fig.
LXXIX.

such as $\frac{1}{2}m$, $\frac{1}{4}m$, may be also included in the boxes, according to the conditions of the different experiments hereafter described.

If $4\frac{3}{4}$ oz. or $19m$, be included in either box, this with the weight of the box itself will be $25m$; so that when the weights A and B, each being $25m$, are balanced in the manner above represented, their whole mass will be $50m$, which being added to the inertia of the wheels $11m$, the sum will be $61m$. Moreover, three circular weights, such as that which is represented at fig. LXXX. are constructed; each of which $= \frac{1}{4}$ oz. or m : if one of these be added to A and one to B, the whole mass will now become $63m$, perfectly in equilibrium, and moveable by the least weight added to either (setting aside the effects of friction) in the same manner precisely as if the same weight or force were applied to communicate motion to the mass $63m$, existing in free space and without gravity.

Of the
moving
force.

2. Since the natural weight or gravity of any given substance is constant, and the exact quantity of it easily estimated, it will

will be convenient in the subsequent experiments to apply a weight to the mass A as a moving force: thus, when the system consists of a mass = 63 m, according to the preceding description, the whole being perfectly balanced, let a weight $\frac{1}{4}$ oz.

or m, such as is represented in fig. LXXX. be applied on the mass A, this will communicate motion to the whole system: by adding a quantity of matter m to the former mass 63 m, the whole quantity of matter moved will now become 64 m, and the moving force being = m, this will give the force which accelerates the

* descent of A = $\frac{m}{64 m}$ or $\frac{1}{64}$ part of the * Sect. I.
Prop. IX.

accelerating force by which the bodies descend freely towards the earth's surface.

By the preceding construction the moving force may be altered without altering the mass moved; for suppose the three weights m, two of which are placed on A, and one on B to be removed, then will A balance B. If the weights 3 m be all placed on A, the moving force will now become 3 m, and the mass moved 64 m as before, and the force which acce-

accelerates the descent of A $= \frac{3m}{64m} = \frac{3}{64}$ part of the force by which gravity accelerates bodies in their free descent to the earth's surface.

Suppose it were required to make the moving force 2 m, the mass moved continuing the same. In order to effect this, let the three weights, each of which = m, be removed; A and B will balance each other; and the whole mass will be 61 m:

Fig.
LXXXI.

let $\frac{1}{2}m$ be added to A, and $\frac{1}{2}m$ to B, the equilibrium will still be preserved, and the mass moved will be 62 m; now let 2 m be added to A, the moving force will be 2 m, and the mass moved 64 m as before;

wherefore the force of acceleration $= \frac{1}{32}$

part of the acceleration of gravity. These alterations in the moving force may be made with great ease and convenience in the more obvious and elementary experiments, there being no necessity for altering the contents of the boxes A and B: but the proportion and absolute quantities of the moving force and mass moved may be of any assigned magnitude, according

according to the conditions of the proposition to be illustrated: thus, let any number of the weights m , for example $11m$, be included in A , and $8m$ in B , then will the moving force be $3m$, and the mass moved $14m + 17m + 11m$, (because $11m$ is the inertia of the wheels, and each of the boxes $= 6m$) making altogether $42m$; wherefore the accelerating force in this case is that part of the acceleration of gravity which is expressed by the fraction $\frac{3m}{42m} =$

$$\frac{1}{14}.$$

3. The moving force and the mass moved, and consequently the force which accelerates the bodies, having been constructed; the method of estimating practically the space described from quiescence is next to be considered. The body A descends in a vertical line; and a scale about 64 inches in length graduated into inches and tenths of an inch is adjusted vertical, and so placed that the descending weight A may fall in the middle of a square stage, fixed to receive it at the end of the descent: the beginning of the descent is estimated from 0 on the scale, when the bottom of the box

Of the space described,

Fig. LXXVIII,

A is on a level with o. The descent of A is terminated when the bottom of the box strikes the stage, which may be fixed at different distances from the point o, so that by altering the position of the stage, the space described from quiescence may be of any given magnitude less than 64 inches.

Concerning
the time of
motion.

4. The time of motion is observed by the beats of a pendulum which vibrates seconds: and the experiments, intended to illustrate the elementary propositions, may be easily so constructed that the time of motion shall be a whole number of seconds; the estimation of the time therefore admits of considerable exactness, provided the observer take care to let the bottom of the box A begin its descent precisely at any beat of the pendulum; then the coincidence of the stroke of the box against the stage, and the beat of the pendulum at the end of the time of motion, will shew how nearly the experiment and the theory agree together. There might be various mechanical devices thought of for letting the weight A begin its descent at the instant of a beat of the pendulum; but the following method may perhaps be sufficient: let the bottom of the box A, when at o on the scale, rest on a flat rod held in the hand

hand horizontally, its extremity being coincident with o ; by attending to the beats of the pendulum, and with a little practice the rod which supports the box A may be removed at the instant the pendulum beats, so that the descent of A shall commence at the same instant.

5. It remains only to describe in what manner the velocity acquired by the descending weight A , at any given point of the space through which it has descended, is made evident to the senses. The velocity of A 's descent being continually accelerated will be the same in no two points of the space described: this is occasioned by the constant action of the moving force; and since the velocity of A at any instant, is measured by the space which would be described by it, moving uniformly for a given time with the velocity it had acquired at that instant, this measure cannot be experimentally obtained, except by removing the force by which the descending bodies' acceleration was caused.

In order to shew in what manner this is effected practically, let us suppose that, according to a former example, the boxes A and $B = 25$ m each, so as together to be $= 50$ m; this with the wheel's in-

Of the velocity acquired.

Fig.
LXXX.

inertia $11m$ will make $61m$: now let m be added to A , and an equal weight m to B , these bodies will balance each other, and the whole mass will be $63m$. If a weight m be added to A motion will be communicated, the moving force being m , and the mass moved $64m$. In a former

|| Page 305.

* Fig.
LXXX.

|| example, the circular weight $= *m$ was made use of as a moving force; but for the present purpose of shewing the velocity acquired, it will be convenient to use a flat rod, the weight of which is also $= m$ †. Let the bottom of the box

† Fig.
LXXXII.

A be placed on a level with o on the scale, the whole mass being as described above $= 63m$, perfectly balanced in equilibrio. Now let the rod, the weight of which $= m$, be placed on the upper surface of A ; this body will descend along the scale precisely in the same manner as when the moving force m was applied in the form of a circular weight. Suppose the mass A to have descended by constant acceleration of the force m , for any given time

Fig.
LXXXIII.

or through a given space: let a circular frame be so affixed to the scale, contiguous to which the weight descends, that A may pass centrally through it, and that this circular frame may intercept the rod m by which

which the body A has been accelerated from quiescence. After the moving force m has been intercepted at the end of the given space or time, there will be no force operating on any part of the system, which can either accelerate or retard its motion; this being the case, the weight A, the instant after m has been removed, must proceed uniformly $\ddagger$ with the velocity which it had acquired that instant: in the subsequent part of its descent, the velocity being uniform will be measured by the space described in any convenient number of seconds, according to the examples given in the ensuing experiments.

$\ddagger$ I. Law of motion.

It is needless to describe particularly, but it may not be improper just to mention the further uses of this instrument; such as the experimental estimation of the velocities communicated by the impact of bodies elastic and nonelastic; the quantity of resistance opposed by fluids, as well as for various other purposes; these uses will be shewn on a future occasion: but the properties of retarded motion being a part of the present subject, it may be necessary to shew in what manner the motion of bodies resisted by constant forces are reduced to experiment by means of the

Other uses of the instrument.

in-

instrument above described, with as great ease and precision as the properties of bodies uniformly accelerated. A single instance will be sufficient: thus, suppose the

Fig.
LXXXIII.

mass contained in the weights A and B and the wheels to be 61 m, when perfectly

|| Page 305.

Fig.
LXXX.

in equilibrio, as in a || former example;

Fig.
LXXXII.

let a circular weight m be applied to B, and let two long weights or rods, each =

m, be applied to A, then will A descend by the action of the moving force m, the

mass moved being 64 m: suppose that when it has described any given space by

constant acceleration, the two rods m are intercepted by the circular frame above

described, while A is descending through it; the ‡ velocity acquired by that descent

‡ Sect. III.
Prop. V.

is known, and when the two rods are intercepted, the weight A will begin to move

on with the velocity acquired, being now retarded by the constant force m; and since

the mass moved is 62 m, it follows, that

* Sect. I.
Prop. IX.
p. 10.

the force of *retardation will be $\frac{1}{62}$ part of

that force whereby gravity retards 'bodies thrown perpendicularly upwards. The

weight A will therefore proceed along the graduated scale in its descent with an uni-

formly retarded motion, and the spaces de-

described, times of motion, and velocities destroyed by the resisting force, will be subject to the same measures as in the examples of accelerated motion above described.

The uses of the instrument, in verifying practically the properties of rotatory motion, will appear in the next section.

In the foregoing descriptions, two suppositions have been assumed, neither of which are mathematically true: but it may be easily shewn that they are so in a physical sense; the errors occasioned by them in practice being insensible:

1. The force which communicates motion to the system has been assumed constant, which will be true only on a supposition that the line, at the extremities of which the weights A and B are affixed, is without weight. In order to make it evident, that the line's weight and inertia are of no sensible effect, let a case be referred to, wherein the body A descends through 48 inches from rest by the action of the moving force m, when the mass moved is $64m$: the time * wherein A describes

Fig.
LXXVIII,

* In these experiments the length of the line by which the weights are suspended is about 72 inches, its weight

scribes 48 inches is increased by the effects of the line's weight by no more than

$\frac{312}{10000}$ part of a second; the time of descent

being 3.9896 seconds, when the string's weight is not considered, and the time when the string's weight is taken into account = 4.0208 seconds; the difference between which is wholly insensible by observation.

2. The bodies have also been supposed to move in vacuo, whereas the air's resistance

3 grains: the vertical distance between the weights at the beginning of motion = 60 inches: suppose the space described by the descending weight $A = 48$ inches, the mass moved = $64m + 3$ grains = 7683 grains; wherefore in order to ascertain the time in which the descending weight A describes 48 inches from rest, taking into account the weight and inertia of the line, we have by referring to prop. xiv. sect. iv. $Q = 7683$, $x = 48$ inches, $L = 72$, $b = 60$, $p = 3$ grains, w (or m) = 120, wherefore

$$\frac{\sqrt{px} + \sqrt{Lw + px} - pb}{\sqrt{Lw - bp}} = \frac{\sqrt{144} + \sqrt{8640 + 144} - 180}{\sqrt{8640 - 180}}$$

= $\frac{104.75774}{91.97826} = 1.13891$, the hyperbolic logarithm of which = .13010; wherefore the time wherein the descending

weight describes 48 inches = $\sqrt{\frac{LQ}{lp}} \times .13010 =$

$$\sqrt{\frac{72 \times 7683}{193 \times 3}} \times .13010 = 4.0208 \text{ seconds.}$$

If the string's weight and inertia be not considered, the time

will be = $\sqrt{\frac{48 \times 7680}{120 \times 193}} = 3.9896$, and the difference

between the times = 0.0312 parts of a second.*

* Sect. III.
Prop. IV.
Cor. 5.

distance will have some effect in retarding their motion: but as the greatest velocity communicated in these experiments, cannot much exceed that of about 26 inches in a second, (suppose the limit 26.2845) and the cylindrical boxes being about $1\frac{3}{4}$ inches in diameter, the air's resistance can never increase the time of descent in so great a proportion * as that of 240 : 241; its effects therefore will be insensible in experiment.

The

* If the diameter of a cylinder moving in the direction of its axis be d , $p = 3.14159$, * $\approx$ the altitude through which a body must fall freely by the acceleration of gravity to acquire the cylinder's velocity, the resistance which the air opposes to the motion of the cylinder = the weight of $\frac{\pi p d^2}{4}$ cubic inches of air; that is, since according to the

* Sect. V.
Prop. II.
Cor. 2.

present case, $d = 1.75$, $\dagger \approx = \frac{26.2845^2}{4 \times 193} = .8949$ parts of $\dagger$ Sect. III.
Prop. V.

an inch, and $\frac{2}{7}$ of a grain, being equivalent in weight to each cubic inch of air, the resistance to the two cylinder's

motion = $\frac{.8949 \times 3.14159 \times 1.75^2 \times 2 \times 2}{4 \times 7} = 1.23$ parts

of a grain. This is on a supposition, that the resistance is always equal to that which opposes the weights, when moving at the rate of 26.2845 inches in a second; and since the velocities are in every point of the descents less than that above mentioned, it is wholly needless to estimate the exact increase of time arising from the air's resistance, which might if necessary be obtained from prop. XI, sect. v. Supposing therefore the air's resistance to be al-

R r 2

ways

The effects of friction are almost wholly removed by the friction wheels: for when the surfaces are well polished and free from dust, &c. if the weights A and B be balanced in perfect equilibrio, and the whole mass consists of 63 m, according to || Page 305. the example already || described, a weight of $1\frac{1}{2}$ grains, or at most 2 grains, being added either to A or B, will communicate motion to the whole, which shews that the effects of friction will not be so great as a weight of $1\frac{1}{2}$ or 2 grains. In some cases, however, especially in experiments relating to retarded motion, the effects of friction become sensible; but may be very readily and exactly removed by adding a small weight 1.5 or 2 grains to the descending body, taking care that the weight added is such as is in the least degree smaller than that which is just sufficient to set the whole in motion, when A and B are equal and balance each other, before the moving force is applied.

In

ways so great as 1.23 grains, and the moving force = 120 grains = m , the moving force will be diminished by the air's resistance in the ratio of 120 to $120 - 1.23$, or of 120 to 118.77, and the time will be increased in a subduplicate proportion of 118.77 to 120, which differs not greatly from that of 240 to 241.

In the following description of experiments, at least of the more elementary ones, the spaces described, times of description, &c. are so proportioned as to produce results in whole numbers, and are of such a magnitude as admit of the most easy and convenient observation. The results are set down according to the theory from which the experiments will differ insensibly, if due care be taken in constructing and adjusting the instrument.

N. B. The space which bodies describe by falling freely from rest in 1 second is ≈ 193 inches; but in the ensuing experiments, this space is assumed ≈ 192 inches, (which will be productive of no sensible error) in order to avoid fractions, which would render the use of the instrument less easy and intelligible: in those experiments at the end of the section, which are intended to illustrate propositions of a more complex nature, this assumption is useless; in these cases therefore the space through which bodies descend from rest by their natural gravity in one second is taken ≈ 193 inches, its true value.

I.

If a quantity of matter $\approx 64 m$ be acted on by a force $\approx m$, the space described from rest in one second will be three inches.

If the quantity of matter denoted by $64 m$ be acted on by its natural gravity or weight $64 m$, it will describe from rest 192 inches in one second of time: and if the same quantity of matter $64 m$ be acted on by a force m , being $\parallel$ Sect. III.
64 times smaller than before, it will $\parallel$ describe only a sixty- Prop. IV.
fourth Cor. 3.

Fig.
LXXVIII.

† Page 303.

‖ Page 308.

fourth part of the former space, that is, it will describe only three inches in a second from rest, which may be experimentally illustrated by the instrument above mentioned. Supposing the boxes *A* and *B* to be each $= 6m$ in weight, let $19m$, or $4\frac{3}{4}$ oz. be included in each; then will *A* and *B* together $= 50m$: also, apply two circular weights $2m$ upon *A*, and one of these weights or m upon *B*, then will the mass moved be $50m + 3m + 11m = 64m$, (the mass $11m$ being the inertia of the † wheels); and since $A = 27m$, and $B = 26m$, the moving force will be m . Having affixed the square stage to 3 on the graduated scale, let the ‖ under surface of the weight *A* begin to descend from 0 on the graduated scale at the instant of any beat of the pendulum: it will be heard to strike the stage at the next successive beat of the pendulum, having described three inches in one second from rest.

Another example may be subjoined; every thing else remaining as in the last example, remove the circular weight m from *B* and place it on *A*, which will give $A = 28m$, $B = 25m$, then will the moving force $= 3m$, and the mass moved $25m + 28m + 11m = 64m$, and the space described from rest by *A* in one second $= \frac{3 \times 19^2}{64} = 9$ inches, which is experimentally verified in the same manner as was just described.

II.

* Sect. III.
Prop. IV.
Cor. 1.

Let the quantity of matter $64m$ be put in motion by the action of the constant force m . * If the times of motion be 1 second, 2 seconds, and 3 seconds, the spaces described from rest by the descending weight *A* in those times will be 3 inches, $3 \times 4 = 12$ inches, and $3 \times 9 = 27$ inches respectively; the spaces described from rest being in a duplicate ratio of the times of motion.

Here

Here let $A = 27 m$, and $B = 26 m$: this experiment is made exactly in the same manner as the last by setting the stage to 3, 9, and 27 on the scale, in the different experiments. These practical examples shew, that if the accelerating forces be equal, the spaces described from rest are in a duplicate ratio of the times of motion. Fig. LXXVIII.

The following experiments also will illustrate this truth: let $36\frac{3}{4} m$ be included in A , and $36\frac{1}{4} m$ in B , then the experiments being made in the same manner as before, the spaces described in the times of descent 1 second, 2 seconds, &c. will be as in the subjoined table.

Moving force.	Mass moved.	Accelerating force.	Times of motion in seconds.	Spaces described from rest estimated in inches.
$\frac{1}{2} m$	$96 m$	$\frac{1}{192}$	1	1
			2	4
			3	9
			4	16
			5	25
			6	36
			7	49
			8	64

III.

If the quantity of $\frac{1}{2}$ matter $64 m$ be acted on by different moving forces, viz. m , $2 m$, and $3 m$, for any given time, for example 2 seconds, the spaces described will † Sect. III.
Prop. IV.
Cor. 3.

will be 12 inches, 24 inches, and 36 inches respectively, being in the same proportion with the moving forces.

Fig.
LXXVIII.

1. Let $A = 27m$, $B = 26m$, then will the moving force $= m$, and the mass moved $= 64m$: let the stage be placed at 12 on the graduated scale, and the under surface of A being coincident with 0, let it begin to descend at any beat of the pendulum; the weight will be heard to strike the stage at the next beat but one, having described 12 inches in 2 seconds.

2. Let $A = 27\frac{1}{2}m$, and $B = 25\frac{1}{2}m$, then will the moving force $= 2m$, and the mass moved $= 64m$; let the stage be set to 24 on the graduated scale, and by the same process as was described in the last article, the weight A will be observed to describe 24 inches from rest in 2 seconds.

3. Let $A = 28m$, and $B = 25m$, then will the moving force $= 3m$, and the mass moved $= 64m$, and the space described from rest by A in 2 seconds will be observed to $= 36$ inches.

* Sect. I.
Prop. IX.

It appears from these experiments, that when the times are the same, the spaces described from rest are as the accelerating forces, or the quantities of matter being * given as the moving forces.

It being frequently necessary to vary the accelerating forces in the subsequent experiments, the following problem may be here inserted, whereby the weights to be included in the boxes A and B may be in all cases known, from having given the moving force and accelerating force required: suppose the given moving force were w , and the accelerating force were intended to be that part of the accelerating force of gravity which is expressed by

the fraction $\frac{1}{n}$. Let x be the weight to be included in B ,

then will $x + w$ be the weight included in A : but the weight of the box $A = 6m =$ the weight of the box B , and the inertia of the friction wheels $= 11m$, wherefore the whole mass moved will $= 2x + 23m + w$: and be-

cause the given moving force is w , we have $\frac{w}{2x + 23m + w} = \frac{1}{n}$, and $x = \frac{n w - w - 23m}{2}$.

Thus,

Thus, suppose it were required to assign what weight must be included in the boxes *A* and *B*, so that the force which accelerates the descent of *A* shall be $\frac{1}{96}$ part of the accelerating force of gravity, the given moving force being *m*; here referring to the rule $n = 96$, $w = m$, and $x = \frac{96 - 1 - 23 \times m}{2} = 36m$, or 9 oz. the weight included in *B*, and consequently 37 *m*, the weight to be included in *A*.

IV.

If the quantity of matter 64 *m* be acted on by a moving force $= \frac{1}{2}m$, it will describe* from rest 54 inches in 6 seconds; if the moving force be increased in the ratio of 1 : 4, so as to be $= 2m$, the quantity of matter or 64 *m* not being altered, the time of describing from rest 54 inches, will be diminished in the ratio of 2 : 1, so as to become only 3 seconds.

* Sect. IV;
Prop. IV.
Cor. 5.

Let $20\frac{3}{4}m$ be included in *A*, and $20\frac{1}{4}m$ in *B*, then will the mass moved $= 64m$, and the moving force $\frac{1}{2}m$, so that the force of acceleration $= \frac{1}{128}$, and the space described from rest in 6 seconds $= \frac{192 \times 36}{128} = 54$ inches.

The latter part of the experiment shews, that if the space described remains the same, while the time of description is diminished, the force of acceleration must be increased in a duplicate proportion of the times' diminution.

It has been sufficiently described in what manner to dispose the weights included in the boxes *A* and *B*, so that the accelerating force shall be of any assigned portion of

the accelerating force of gravity, and the methods of making these experiments having been illustrated by several examples, it may be sufficient in the subsequent experiments just to mention the numbers suited to such cases, as may be most conveniently repeated; but these experiments may be varied at pleasure.

V.

Let the quantity of matter $64m$, be acted on by a moving force* m for any given time, for example 2 seconds, the descending weight A will describe 12 inches, and at the end of that descent will have acquired such a velocity as, if uniformly continued, will cause the body to describe 24 inches in the same time, i. e. in 2 seconds.

Fig. LXXXIII. Let $A = 26m$, and $B = 26m$, then will the weights A and B precisely balance each other: in some former experiments, a circular weight m was applied to A as a moving force, and A was observed to describe from rest by constant acceleration † 12 inches in 2 seconds: if instead of a circular weight m , applied as a moving force, a flat rod, the weight of which is also m , be placed on the upper surface of A , the acceleration of A 's descent will be the same as before, i. e. such as causes it to describe 12 inches in 2 seconds from rest. This being premised, let the circular frame, represented in fig. LXXXIII. be so affixed to the graduated scale, that the box A may in descending pass centrally through it, and let its height be such, that the instant the lower surface of the box A arrives at 12 on the graduated scale, the rod m may be intercepted by falling on the horizontal surface of the frame, and thereby be prevented from the further acceleration of the system. Moreover, set the square stage on the scale at 36; that is, 24 inches from the circular frame. The experiment will be as follows: let the weight A begin to descend from 0 on

on the scale, at any beat of the pendulum. At the end of 2 seconds the rod will be heard to strike against the circular frame, having described 12 inches with an uniformly accelerated motion, and at the fourth beat of the pendulum, the box *A* will strike against the square stage at 36, having described 24 inches in 2 seconds with an uniform motion.

Although it be certain, that the weight *A* described the latter 24 inches with a † motion perfectly uniform, this may be further confirmed by setting the square stage at 36, 48, and 60 successively, every thing else remaining. Since the velocity is uniform, and *A* describes 24 inches in 2 seconds, it will of course describe 36 inches in 3 seconds, and 48 inches in 4 seconds: if the stage be fixed at 60 inches, and the experiment be repeated, the rod *m* will be removed at the second beat of the pendulum as before, and the box *A* will strike the stage at 60 at the sixth beat of the pendulum, having described 48 inches uniformly in 4 seconds.

† I. Law of motion. and p. 309.

In like manner it may be shewn experimentally, that during the motion of any body acted on by any constant force, a velocity is acquired, which if continued uniform will cause the body to describe in the same time, a space twice as great as that through which the body was accelerated, in order to acquire that velocity. Thus, let $A = 25\frac{1}{2}m$ and $B = 25\frac{1}{2}m$, then let two rods, each $= m$, be placed on the upper surface of *A*, and place the circular frame at 6, and the square stage at 18; then the weight *A* beginning its descent at the instant of any beat of the pendulum, the two rods will strike the frame, being removed from *A*, at the next successive beat, and at the second beat will strike the stage underneath, having described 6 inches with an uniformly accelerated, and 12 inches with an uniform motion in equal times, i. e. 1 second.

The method of affixing the circular frame, through which the weight *A* in its descent passes, should be here described. Suppose, as in the preceding example, it were required to set the frame at the division 12 on the graduated scale; it is meant to be so affixed, that the instant the rod *m*, descending along with the weight *A*, strikes the circular frame, the lower surface of the box (by which the spaces described are always estimated) shall arrive at the division 12. In order to effect which, when *A* and *B* are in equilibrio, and *A* is lower than the circular frame, as is represented in fig. LXXXIII. add any weight, for example *m* to *B*, then will *B* preponderate, and *A* will ascend: if the rod *m* or other flat substance be held firmly over the circular aperture, it will prevent the weight *A* from ascending

cending further, its upper surface being now on a level with the horizontal surface of the circular frame: in this situation the weight *A* must be kept, until by altering the position of the frame, the under surface of the box *A* shall be on a level with the division 12, or any other division which the experiment may require: but it would be still more eligible to have an index on that part of the frame which is contiguous to the divisions, and at a distance from the frame's horizontal surface, equal to the length of the box *A*; by this means the circular frame may be truly adjusted without further trouble, than setting the index to the proper division on the scale.

VI.

† Sect. III. If a quantity of matter = 64 m be act-
Prop. I. ed on by any constant † force m, during the different times 1 second, 2 seconds, 3 seconds, &c. the velocities generated will be those of 6 inches, 12 inches, and 18 inches in a second respectively, being in the same proportion with the times wherein the given force acts.

|| Exp. V. 1. The construction of the preceding || experiment remaining, let the circular frame be set to 3, and the stage underneath at 9; the weight *A* beginning its descent at any beat of the pendulum, the rod will strike the horizontal circular frame at the next beat, and the rod *m* being at that instant removed, the weight *A* will describe the remaining 6 inches uniformly in 1 second, striking the stage at 9 at the second beat.

Fig.
LXXXIII.

2. Set the circular frame at 12, and the stage at 36, and the weight *A* will descend by constant acceleration through the first 12 inches in 2 seconds, and the remaining 24 inches in 2 seconds with an uniform velocity of 12 inches in a second.

3. Set the circular frame at 27, and the stage at 45, and the weight *A* will describe the 27 first inches from rest

rest by constant acceleration in 3 seconds, and the remaining 18 inches in 1 second, with an uniform velocity.

These experiments illustrate the proposition * wherein * Sect. III. it is demonstrated, that if the force by which bodies are accelerated is the same, the velocities generated are in the same proportion as the times wherein the given force acts. Prop. I.

VII.

If a quantity of matter $= 64 m$ be acted on for any given $\frac{1}{2}$ time, for example 1 † Sect. III.
Prop. I. second, by the different forces m , $2 m$, and $3 m$, the velocities generated will be 6 inches, 12 inches, and 18 inches in a second, being in the same proportion with the forces which accelerate the mass moved.

1. Let $A = 26 m$ and $B = 26 m$, and let the rod $= m$ Fig. be applied to A as a moving force; then the experiment LXXXII. being made as before described, the force m acting for 1 Fig. second, generates in the mass $64 m$ a velocity of 6 inches LXXXIII. in a second.

2. Let $A = 25\frac{1}{2} m$ and $B = 25\frac{1}{2} m$, and let two rods, each $= m$, be applied to A as a moving force; then will the mass moved be $64 m$: set the circular frame at 6, and the stage at 18; and the experiment being made as before, the two rods will be intercepted when the under surface of the weight A arrives at 6, and the remaining 12 inches of the descent will be described uniformly in 1 second, which shews that the force $2 m$ acting on the mass $64 m$ for 1 second, generates in it a velocity of 12 inches in a second.

3. Let $A = 25 m$ and $B = 25 m$, and let three rods, each equal to m , be applied to A as a moving force; then will the mass moved be $64 m$ as before: set the circular frame at 9, then will the three rods be intercepted when the weight A has descended 1 second from rest at 0, and A will then go on uniformly describing 18 inches in the next

next second: this shews that a force $3m$, acting on a mass $64m$ for 1 second, generates in it such a velocity as causes it to describe uniformly 18 inches in a second.

It appears therefore, that if different forces accelerate the same body from quiescence during a given time, the velocities generated will be in the same proportion with those forces. These experiments may be varied at pleasure, to shew that the velocities generated in any different quantities of matter, the time in which the forces act being the same, will be as the accelerating forces.

VIII.

† Sect. III. Prop. I. If bodies be acted on by † accelerating forces, which are in the proportion of 3 : 4, and for times, which are as 1 : 2, the velocities acquired will be in the ratio of 1×3 to 2×4 , or as 3 to 8.

+ Sect. VII. Exp. VI. Art. 1. 1. A † force m acting on a mass $64m$ for 1 second, generates a velocity of 6 inches in a second; here the accelerating force $= \frac{1}{64}$.

2. Now let $A = 19m$ and $B = 18m$, then will the moving force $= m$, and the mass moved $19m + 18m + 11m = 48m$, wherefore the accelerating force is $\frac{1}{48}$.

Fig. LXXXIII.

Set the circular frame at 16, and the stage at 48: the weight A in descending by the force of the rod m will describe 16 inches in 2 seconds with an uniformly accelerated motion, and the remaining 32 inches until it strikes the stage with an uniform motion of 16 inches in a second. The velocities generated therefore in these two examples, are as 6 : 16, or as 3 : 8. In these experiments the accelerating forces are $\frac{1}{64}$ and $\frac{1}{48}$, or as 3 : 4. The times during which the bodies are accelerated from rest 1 : 2: add these ratios the sum will be the ratio of the velocities generated, or that of 3 : 8.

IX.

IX.

If a quantity of matter $64\ m$ be accelerated through the same $\dagger$ space, i. e. 12 inches by the forces m and $4\ m$, the velocities generated will be in a subduplicate ratio of these forces, or in the ratio of 1 to 2 . $\dagger$ Sect. III.
Prop. V.

1. Let $A = 26\ m$ and $B = 26\ m$, and let a rod m be applied to A as a moving force, the mass moved will be $64\ m$; then if the circular frame be set to 12 , as in a former experiment, and the stage at 36 ; the weight A will descend through 12 inches in 2 seconds, and the rod m being then removed will proceed with an uniform velocity of 12 inches in a second, to the stage underneath at 36 .

2. Let $A = 24\frac{1}{2}\ m$ and $B = 24\frac{1}{2}\ m$, and let 4 rods, each equal to m , be applied as a moving force; then the quantity of matter moved will be denoted by $64\ m$; wherefore

the accelerating force $= \frac{4\ m}{64\ m} = \frac{1}{16}$, being increased in

the ratio of 1 to 4 : every thing else remaining as in the last article, the weight A will descend through the first 12 inches from 0 with an uniformly accelerated motion in a second, and will describe the remaining 24 inches with an uniform velocity in 1 second; the velocities generated in these two examples, are 12 inches and 24 inches in a second, being in the ratio of $1:2$; whereas the ratio of the accelerating forces is that of $1:4$, the space through which the bodies are accelerated being the same in both cases.

X.

If different quantities of $\parallel$ matter $64\ m$ and $48\ m$ be impelled through the same space, $\parallel$ Sect. III.
Prop. VI.

space, i. e. 12 inches, and acquire the same velocity, the moving forces must be in the same ratio as the quantities of matter moved, that is, as 4 : 3.

Experiments to illustrate this truth may be comprized in the subjoined table.

Moving forces.	Quantities of matter moved.	Accelerating forces.	Spaces described in inches.	Velocities acquired in inches in a second.
m	$64 m$	$\frac{1}{64}$	12	12
$\frac{3 m}{2}$	$96 m$	$\frac{1}{64}$	12	12
$\frac{3 m}{4}$	$48 m$	$\frac{1}{64}$	12	12

It is collected from the two last sets of experiments, that the moving forces, which impel bodies through the same space, are in the joint ratios of the quantities of matter moved and squares of the velocities generated.

XI.

* Sect. III.
Prop. IV.
Cor. 2.

If a quantity of *matter $64 m$ be impelled through different spaces from rest, 3 inches and 27 inches, by the action of the same force m , the velocities generated will be in a subduplicate ratio of the spaces described, or as 1 : 3.

If

1. If the mass moved is $64m$ and the moving force m , the velocity generated in descending through 3 inches, has been shewn in a former experiment to be 6 inches in a second.

2. Now let A descend by the same force through 27 inches, and set the circular frame to 27, and the stage at 45, the weight A will describe the first 27 inches with an uniformly accelerated motion in 3 seconds, and the rod m being there removed will strike the stage at 45 at the fourth beat of the pendulum, having described the last 18 inches uniformly in a second. In these experiments the spaces described during the body's acceleration are 3 and 27, or as 1 : 9, the velocities acquired as 6 : 18, or as 1 : 3, being in a subduplicate ratio of the spaces.

It appears therefore that when the forces of acceleration are the same, the velocities acquired are in a subduplicate ratio of the spaces; and from a former set of * experiments it is collected, that if the spaces described are the same, the velocities generated are in a subduplicate ratio of the forces of acceleration: it may be concluded therefore, that when both spaces described and accelerating forces vary, the velocities † generated will be in a subduplicate ratio of the accelerating forces and of spaces described jointly. * Exp. IX. † Sect. III. Prop. V.

The laws observed during the motion of uniformly accelerated bodies are made evident to the senses, by the experiments which have been already described: those which follow are designed to illustrate the properties of uniformly retarded motion. When a body is thrown perpendicularly upward from the earth's surface, it is continually resisted by a force which is equal to the body's weight; and the weights of all substances being proportional to the quantities of matter contained in them, it follows, that the force which retards the perpendicular ascent of any body being † measured by its weight divided by the ascending mass is the same, being such as destroys a velocity of $32\frac{1}{6}$ feet in a second, in each second of the body's motion. † Sect. I. Prop. IX. But in order to illustrate by experiment the general laws according to which bodies are retarded by the action of constant forces, such methods should be made use of as will enable us to apply different resisting forces to the same mass of matter and the same resisting force to different quantities of matter; both of which conditions will be satisfied by the instrument which has been already described, and will be again referred to in the subsequent experiments. In order to render these more obvious, a

few considerations concerning resistances may be inserted previous to the description of the experiments. A resisting force is to be understood to convey precisely the same idea as the term moving force, except as far as regards the directions in which these forces act in respect of the bodies' motion, these directions being contrary to each other. To exemplify this, let a musket ball of $\frac{3}{4}$ of an inch in diameter, and 1.3 oz. weight, descend toward the earth's surface by its own gravity: the moving force is in this case equal a weight of 1.3 oz. Let the same ball be thrown perpendicularly upward; the resisting force is likewise a weight of 1.3 oz. but let the same ball impinge perpendicularly against an immoveable block of elm, so as to penetrate its substance, the ball's motion will be resisted by a force equal to $* 107000 \times 1.3$ oz. if therefore the earth's gravity were 107000 times greater than it really is, and the leaden ball before referred to were projected perpendicularly upward with any given velocity, the resistance opposed to the ball's motion would be exactly the same, as when the ball is projected against an immoveable block of elm with the same velocity, and consequently the ball so projected would rise to a perpendicular altitude equal to the depth to which it would penetrate into the substance of the elm, if it impinged against the elm with the same velocity of projection.

* p. 39.

These principles admit of very exact illustration from experiments, which also confirm those mechanical propositions, which † Leitnitz and Bernouilli, &c. deduced from observing the depths to which spherical bodies penetrated the substance of clays and other soft bodies. These ensuing experiments will also be the more satisfactory from their admitting of an exact estimation of the moving force and the mass moved, in which the experiments before mentioned are defective; neither is the force of resistance constant in the experiments on the depths to which spheres penetrate into the substance of clays, on account of their spherical figure, and the law according to which the forces vary, it would be difficult to assign.

† p. 333.

In the subsequent experiments, the force of resistance is a weight, and therefore of invariable quantity, during the whole time it is opposed to a body's motion, and the exact quantity of it is moreover easily assigned.

XII.

Let a given mass 61 m be projected with a velocity of 18 inches in a second: then if it be resisted by a constant force m , the space described by the resisted mass, before its motion is destroyed, will be 25.6; if it be resisted by a force $= 2m$, the space described will be $12.8 = \frac{25.6}{2}$, and if the resistance be $3m$, the space will $8.533 = \frac{25.6}{3}$ inches; these spaces being in the inverse ratio of the resisting forces.

Let $A = 24\frac{1}{2}m$ and $B = 25\frac{1}{2}m$, and apply to the upper surface of A two rods, each of which $= m$, then will the weight A preponderate and descend by the action of a moving force $= m$, the whole mass moved being $= 63m$. Set the circular frame to 26.44; then the weight A by describing from rest the space 26.44 inches will acquire a

$$\text{velocity} = \sqrt{\frac{4 \times 193 \times 26.44}{63}} = 18^* \text{ inches in a se-} \quad \begin{array}{l} * \text{ Sect. III.} \\ \text{Prop. V.} \\ \text{and p. 32.} \end{array}$$

cond: and at that instant the two rods, each of which $= m$, being intercepted, the weight A will continue to descend with an uniformly retarded motion; which will be precisely the same as if a mass of 61 m , without gravity, were projected with a velocity of 18 inches in a second in free space, and a force of resistance $= m$ were opposed to its motion; wherefore A (with the other parts of the system) will lose its motion gradually, and will describe a space $=$

$$\frac{18^2 \times 61}{4 \times 193} = 25.6 \text{ inches before its motion is entirely de-}$$

stroyed: A will therefore be observed in the experiment

to descend as low as 52 inches, before it begins to ascend by the superior weight of B .

2. Let $A = 24m$ and $B = 26m$, and let three rods, each equal to m , be applied at the upper surface of A , and set the circular frame at 26.86; then will A descend by a moving force m , the whole mass moved being $64m$, and consequently while A descends through 26.86 inches, it will acquire a velocity of 18 inches in a second: and the three rods being intercepted by the circular frame, A will go on with an uniformly retarded motion, being resisted by a force $= 2m$, and the mass moved $= 61m$, wherefore the force of retardation being $= \frac{2}{61}$, the space de-

† Sect. III.
Prop. V.
and p. 32.

scribed by A before its motion is † destroyed $= \frac{18^2 \times 61}{4 \times 193 \times 2} = 12.8$ inches; and will therefore be observed from the experiment to descend as low as the division on the scale which corresponds to 39.66 inches.

3. Let $A = 23\frac{1}{2}m$ and $B = 26\frac{1}{2}m$, and let four rods, each of which $= m$, be applied on the upper surface of A , and set the circular frame to 27.28; then will A descend by the force m , the whole mass moved being $65m$, and the four rods will be intercepted by the circular frame at 27.28, and A having acquired at that instant the velocity of 18 inches in a second, will begin to be resisted by a force of $3m$, the mass moved being $61m$, and A will describe a space $= \frac{18^2 \times 61}{4 \times 193 \times 3} = 8.533$ inches, before its motion is destroyed: A will therefore be observed in the experiment to descend to the division on the scale which answers to 35.81.

In the same manner, other principles relating to retarded motions may be verified experimentally. It is well known, that if spheres of the same magnitude, but of different specific gravities fall on an horizontal surface of clay or any similar substance, having descended from altitudes which are in an inverse proportion of their specific gravities, that the depth to which they penetrate will be equal. To analyze this is not difficult; for assuming the force which resists the progress of the balls as constant, it will be of the same quantity when opposed to the different balls, because their magnitudes † are equal; and since the quantities of matter contained in the balls are directly as their specific gravities, the force which re-

† p. 40.

tards

tards the balls will be inverfely as their fpecific gravities:

let therefore $\frac{F}{f}$ represent the direct ratio of the forces which retard any two of the fpheres, or the inverfe ratio of their fpecific gravities: $\frac{A}{a}$ the ratio of the altitudes from

which they fall, $\frac{V}{v}$ the ratio of the velocities with which

they impinge on the furface of the fubftance which they penetrate to the depths of S, s refpectively; then from the

principles of uniformly retarded motion, we have $\frac{F}{f} \propto \frac{S}{s}$ * Sect. III, Prop. V.

$= \frac{V^2}{v^2} \dagger = \frac{A}{a}$, which is $= \frac{F}{f}$ by the conftruction of the ex- † Sect. III, Prop. IV, Cor. 2.

periment; wherefore $\frac{S}{s} = 1$, or the ratio of the depths to

which they penetrate the fubftance on which they impinge will be that of equality. Whenever therefore the retarding forces are in a direct duplicate ratio of the velocities where-with bodies begin to move, the fpaces defcribed by them with an uniformly retarded motion will be equal. Thus, let a mafs $= 64m$ be projected with a velocity of 12 inches in a fecond, and let it be refifted by a force m ; moreover, let a mafs $48m$ be projected with a velocity of 24 inches in a fecond, and fuppofe it to be refifted by a force $= 3m$, then will the force which retards the motion of the bodie

be as $\frac{1}{64}$ to $\frac{3}{48}$, or as $1 : 4$: and becaufe the velocities of projection are as $1 : 2$, the fpaces defcribed by the uniformly retarded bodies will be equal, being in each cafe 11.94 inches.

Thefe experiments are eafily made, as in the preceding cafes, and the adjustment of the weights neceffary may ferve as an example to render the ufe of the inftrument familiar.

In thefe instances the motion of the retarded body is continued, until it is gradually deftroyed by the refifting force: it may be proper to fubjoin one practical example concerning retarded motion, wherein the body A , moving with a known velocity, is retarded by a given force, and the time of defcribing any given fpace eftimated from the beginning of its retarded motion is obferved experimentally.

XIII.

Let a mass $= 63\frac{1}{2}m$ move with a velocity of 11.87 inches in a second; if it be retarded by a resisting force $\frac{1}{2}m$, it will describe 21.95 inches in 3 seconds, the beginning of motion being estimated from the first instant of its retardation.

Let $A = 26m$ and $B = 26\frac{1}{2}m$, and let a flat rod, similar to those which have been before referred to (but the weight must in this instance be $1\frac{1}{2}m$) be applied on the upper surface of A , and set the circular frame to 11.877 on the scale; then will A descend, the moving force being m and the mass moved $= 65m$, and when it has described 11.877 inches, it will have acquired a velocity of 11.877 inches in a second; and the rod $1\frac{1}{2}m$ being at that instant intercepted, the mass A will begin to descend with an uniformly retarded motion, the resisting force being $\frac{1}{2}m$, and the mass moved $63\frac{1}{2}m$, and consequently the force of retardation $\frac{.5}{63.5} = \frac{1}{127}$; and since if the retarding force

$= F$, and the initial velocity be equal to that which a body acquires in falling from rest freely by its natural gravity through the space b , and $l = 193$ inches, the space described in

• See A. III. T seconds $= 2T \times \sqrt{bl} - lFT^2$: this * rule being applied to the present case, gives $T = 3$, $b = .1827$ part of an inch, being the space through which a body descends freely from rest, by the acceleration of gravity, to acquire the velocity of 11.877 inches in a second, $F = \frac{1}{127}$, and the space

required $= 6 \times \sqrt{.1827 \times 193} - \frac{9 \times 193}{127} = 21.95$ inches.

If

If therefore the square stage be affixed to 33.83, and the weight *A* begins to descend from 0, at any beat of the pendulum, the rod $1\frac{1}{2}m$ will strike the circular frame at the end of two seconds, having described 11.877 with an uniformly accelerated motion, and the rod being intercepted, *A* will go on with an uniformly retarded motion, and will strike the stage at 33.83 at the fifth beat, having described the last 21.95 inches in 3 seconds.

The following articles may tend to facilitate the construction and adjustment of the instrument represented in Fig. 78 and 83 and referred to throughout this section.

1. The pendulum of the clock which is affixed to the pillar of the instruments vibrates seconds; it has only one wheel which immediately acts on the pendulum: the small weight which continues the pendulum's motion, after it has been wound up, is half an hour in descending to the ground. The clock will be sufficiently exact if it keeps time with a common well regulated clock for this half hour.

2. The feet of the frame on which the friction wheels rest, should not be fixed immoveably to the frame underneath, because in order to adjust the axis of the wheel, *a b c d* horizontal, it will be necessary to alter the height of the two anterior or posterior feet, by inserting small plates of wood or brass under them, or by means of screws acting immediately on the feet. — A hanging level applied to the wheel's axle, will shew when the axis is horizontal.

3. When the axis of the wheel *a b c d* has been adjusted Fig. LXXVIII, horizontal, let two equal weights *A* and *B* be suspended from the extremities of a silk line of proper length, the thickness of which is no greater than is just sufficient to sustain the weights. When these weights are perfectly quiescent, a small impulse being applied to either, in a vertical direction, will set the whole in motion; which will be continued uniform till one of the boxes arrives at the extremity of the scale. When the box *A* is at the bottom of the scale and quiescent, it must be observed whether the middle line on the scale is every where exactly opposite to the line sustaining *A*; or in other words, whether the line in the middle of the scale is in the same vertical plane with the line which sustains *A*. If it is not, the lower extremity of the scale must be moved along the arm of the base until the adjustment is correct. 2. It is also to be observed whether the line is every where at equal perpendicular distances from the

Fig.
LXXXIII.

the middle line on the scale: if it is not, the upper extremity of the scale, must be moved further from or nearer to the silk line until the distances are every where equal. The middle line on the scale will now be vertical, and the circular frame described in page 310, must be so constructed, that the box *A* may pass centrally through it when these adjustments are correct.

4. The wheel *a b c d* previous to its adjustment, should weigh something more than six ounces, provided the figure of the wheels be not very different from those represented in the Fig. 78, 83. In the circumference of the wheel *a b c d*, a small groove should be cut and well polished, to admit the line which goes over the wheel and sustains the weights *A* and *B*.

5. The weight of the wheel *a b c d* must be so adjusted that the resistance to motion of its inertia, with that of the friction wheels shall be equivalent to $2\frac{3}{4}$ ounces. This may be effected according to the method described in page 301, that is, by winding a very slender line round the wheel, and applying to the extremity a weight of 30 grains: let the square stage be fixed to the scale at $38\frac{1}{2}$ inches, and let the weight 30 grains, begin to descend at any beat of the pendulum from 0 on the scale. If it describes the $38\frac{1}{2}$ inches in less than three seconds, the wheel is too light to admit of the experiments being made according to the numbers used in this section. Suppose then the weight to be more than 3 seconds in describing the $38\frac{1}{2}$ inches from rest: the wheel's weight must be gradually diminished until the time of descent is exactly three seconds. It is however to be noted that in first making this experiment for the adjustment of the wheel's weight, the line was wound round the cylindrical surface of the wheel's circumference, no groove having been at first found necessary. When a groove is cut the line must no longer be wound round the wheel for reasons too obvious to need mentioning: in this case therefore the adjustment of the wheel's weight is to be obtained by another method. Instead of the weights *A* and *B*, let the weights 48 grains and 24 grains be suspended from the extremities of the line which goes over the wheel *a b c d*, and set the square stage at 30; then letting the weight 48 grains, begin to descend from 0 on the scale at any beat of the pendulum, observe the time of its describing the 30 inches as measured on the scale: if the time of descent should be less than three seconds the wheel is too light; suppose therefore the time to be greater than three seconds; the wheel's weight

weight must be gradually diminished until the time of descent is precisely three seconds. When this adjustment is correct, the resistance from the wheel's inertia including that of the friction wheels, will be exactly the same as if those wheels were removed and the mass $2\frac{3}{4}$ oz, was uniformly accumulated in the circumference of the wheel to which the line is applied.

6. Great attention should be paid to the adjustment of the weights used in these experiments as a very small error such as is scarcely perceivable in each, will tend greatly to affect the exactness of the experiments in which many weights are used.

7. In letting the box *A* begin to descend at any beat of the pendulum according to the method described in page 308, the observer must not wait until he hears the beat, at which he intends *A*'s descent shall begin; for in this case *A*'s descent will always commence too late: the proper method is to attend to the beats of the pendulum until an exact idea of their succession is obtained; then the extremity of the rod being withdrawn from the bottom of the box *A* directly downward at the instant of any beat, the descent will commence at the same instant.

This instrument was executed with great mechanical skill, partly by Mr. *L. Martin*, and partly Mr. *G. Adams*, mathematical instrument makers in London.

S E C T. VIII.

CONTAINING A DESCRIPTION OF EXPERIMENTS ON THE MOTION OF BODIES WHICH REVOLVE ROUND FIXED AXES, AND ROUND SUCH AS MOVE WITH THE SAME VELOCITIES AS THE CENTRES OF GRAVITY OF THE REVOLVING BODIES.

ALTHOUGH experiments may be made on bodies or systems of bodies, which revolve round fixed axes inclined to the horizon at any different angles, it is however most convenient that the axes should be either horizontal or vertical.

Which of these positions should be preferred to the other, will depend on the circumstances of the different experiments; chiefly on the weights and figures of the parts which compose the revolving system. When the axle is horizontal it is absolutely necessary that it should be supported on friction wheels, which greatly diminish, or altogether prevent the loss of motion which would be caused by the friction of the axle, if it revolved on an immoveable surface: If therefore the weight of the system be such as can be supported easily on friction wheels,
and

and the parts of it be of proper shape and dimensions, the position of the axle may be horizontal. In Fig. 75. $P L Q$ represents a wheel and axle which in the experiments rest on the friction wheels*. Suppose that two weights ^{Fig.} B and A act on the wheel and axle respectively, and that B preponderates against A , descending in a vertical line; the graduated scale must be adjusted so that the space described by B in its descent may be measured on it, or if A preponderates against B , the scale must be placed so as to become a measure of A 's descent: by this means the spaces described, times of description, real velocities of the descending weight as well as the angular velocities generated in the system, will be experimentally obtained. In other cases the system will revolve on a vertical axis as in Fig. 84. where $A B$ represents a cylindrical axle; $C D$ a cylindrical rod, the axis of which cuts the vertical axis at right angles; $C O$, $D Q$ are two equal spheres, through the centres of which the axis of the cylinder passes; these spheres are fastened to the cylinder by screws, at equal distances from the vertical axis: motion is communicated to the system by the line

LXXVIII.

Fig.
LXXVIII.

EF, stretched by a weight p ; the line EF having been wound round the vertical axle passes horizontally and goes over a wheel or pulley FG,* which revolves on the friction wheels so often referred to in the last section. It is easy to see that the frame UXYZ must be fastened to the continuation of the stage on which the friction wheels are placed, i. e. to that part of the stage which projects beyond the pillar.

Fig.
LXXVIII.

The graduated† scale is to be so adjusted as to measure the space described by p , as it descends by communicating motion to the revolving system. The representation of the frame UXYZ, and the system CDAB is omitted in Fig. 78, and 81. for want of room.

In the ensuing experiments, the moving force is applied at the circumference of a wheel or axle, the axis of which coincides with the axis of the system; it is therefore necessary that the distance of this circumference from the axis, or the radius of the axle, should be very precisely known; because the quantity of the moving force, exerted by a given weight depends entirely on it. In direct mensuration, errors must be unavoidably committed which bear a considerable proportion to the whole, espe-

especially when the radius to be measured is small; in order to obviate this difficulty, the following indirect method was used, by which the radii of the cylinders used in the experiments were obtained to very great exactness.

Having fixed to the extremity of a very fine and flexible line a weight sufficient to keep the line stretched, fasten the other extremity to the axle of which the radius is required; the line being stretched by the weight mentioned above, measure by a scale of even parts, any convenient length, 6 inches for example, and mark the extremities of the length so measured; then holding the axle horizontal, let the measured part of the line be wound round it in the form of an helix, the circumferences being every where contiguous. Count the number of revolutions, and suppose them $= n$, also measure the length of the cylinder occupied by the helix; let this $= a$, and the length of the helix or line first mentioned $= l$, $p = 3.14159$, &c. then, the

$$\text{radius of the cylinder} = \frac{\sqrt{l^2 - a^2}}{2 p n}.$$

The exactness of this method may be known by observing, that if the cylinder be truly made,

made, and the process carefully repeated with different values of l and a , the radius deduced will however always come out the same to the 4th or even the 5th decimal place.

I.

Fig.
LXXV.

In the wheel and axle $P L Q$, the radius of the wheel $s L$ is to the radius of the axle $s O$, as 10 to 1: the distance of the centre of gyration from the axis $s R = 6.7345$ parts, the radius of the axle being estimated as one of them; the wheels' weight $= 6.6635$ oz. then if two weights A and B act against each other on the axle and wheel respectively, whereof $A = 10$ oz. and $B = 1$ oz. these weights will exactly balance each other. Let $\frac{1}{10}$ oz. be added to B , so that B shall become $= 1.1$ oz. it will descend, and will describe from rest 19.54 inches in two seconds.

Fig.
LXXVIII.

The wheel and axle $P L Q$ is exactly equal in weight and dimensions to the wheel and axle $a b c d$, except only in having no groove cut in the circumference, which in this case is a plane cylindrical surface, terminated on each side by an edge, to prevent the line from falling off: this wheel and axle is to be placed on the friction wheels, and the weights being applied as described above, adjust the graduated scale so that it shall be opposite to the descending weight B , and set the square stage at 19.54: then let B begin to descend from 0 on the

the scale at any beat of the pendulum, it will strike the stage at the second beat, having described 19.54 in two seconds.

In the description of this experiment it is said, that the distance of the centre of gyration from the axis is 6.7345 parts, the radius of the axle being one of them. This is obtained from having the equivalent weight, which being uniformly accumulated, in the circumference resists the communication of motion by its inertia, equally with the mass contained in the wheels: * This equivalent weight ^{P. 302.} was found to be 2.75 oz. Suppose then the mass of the wheels to be removed, and the 2.75 oz. to be accumulated uniformly in the circumference, at the distance from the axis SL , and let $2.75 = P$, and the wheels' weight W ; then according to the rule demonstrated in Prop. VIII.

$$\text{Sect. VI. } SR = \sqrt{\frac{P \times SL^2}{W}} = \sqrt{\frac{2.75 \times 100}{6.0635}} = 6.7345.$$

To compare this experiment with the theory we must refer to Sect. VI. Prop. XIX. Cor. 1. which being applied to the present case, gives $n = 10$, $p = 1.1$, $q = 10$, $w = 6.0635$, $r = 6.7345$, and the force which accelerates the descent of $B = \frac{n^2 p - n q}{w r^2 + p n^2 + q} = \frac{110 - 100}{275 + 110 + 10} = \frac{1}{39.5}$. Wherefore the space described from rest by

$$\text{the descending weight } B \text{ in two } \dagger \text{seconds} = \frac{193 \times 4}{39.5} = \dagger \text{ Sect. III. Prop. IV. Cor. 5.}$$

19.54.

This experiment may be varied by making B descend for 1 second, 2 seconds, and 3 seconds: the spaces described from rest, will be 4.88, 4.88×4 and 4.88×9 , whereby it is practically shewn, that the spaces described are in a duplicate ratio of the times of motion; and consequently that the rotation of the system is equably accelerated.

II.

On the same wheel and axle PQL , ^{Fig. LXXV.} let a weight $A = 10$ oz. be suspended from the axle by means of a line wound round it, and $B = 1$ oz. from a line going round the

the circumference of the wheel so as to act against the former, these will balance each other. Now let $\frac{3}{4}$ oz. be added to A, so that it shall become $= 10\frac{3}{4}$ oz. then will A descend and will describe 3.37 inches in 3 seconds, from rest.

If the weight which descends from the axle be q , the weight acting against it on the circumference of the wheel $= p$, the radius of the wheel $= n$, that of the axle $= 1$, the distance of the centre of gyration from the axis $= r$, and the wheels weight $= w$; the force which accelerates the descent of $q = \frac{q - p n}{w r^2 + p n^2 + q}$ which is demonstrated exactly in the same manner as Prop. XIX. Sect. VI. Applying this to the present case, we have $A = q = 10.75$, $B = p = 1$, $SL = n = 10$, $w = 6.0635$, $r = 6.7345$, and the force of acceleration $= \frac{10.75 - 10}{275 + 100 + 10.75} = \frac{.75}{385.75}$; wherefore the space described by the descending weight A in one second $= \frac{.75 \times 193}{385.75} = .3752$ parts of an inch, and in 3 seconds the space will be $= .3752 \times 9 = 3.37$ inches.

In all the preceding experiments, the weights have been estimated according to the troy measure, the ounce consisting of 480 grains; in the ensuing experiments the avordupoise ounce happening to have been used when they were originally constructed, it was afterwards thought needless to alter the numbers; this is here mentioned, not because it is of any consequence to what standard the weights in any experiments are referred as far as regards the weights themselves, but because according to the known specific gravities of the substances used, the dimensions and weights of the parts, of which the revolving systems consist would not be consistent with the numbers set down in the experiments, according to the troy measure; the standard weight therefore referred to in the remaining experiments will be the avordupoise oz. consisting of 437.5 grains.

III.

$A B O Q$ is the system described in page 239, of which the dimensions, &c. are as follows: Radius of the steel axle $A B = .20133$ parts of an inch, its weight = 2.4713 oz. the radius of the steel rod $O Q = .0671$, length of the rod $O Q = 12.55$; its weight = .8034 oz: radius of either brass sphere $C O, D Q = .74$, the weight of each sphere = 7.985 oz. a line is wound round the axle $A B$, and passing horizontally over the wheel $F G$, is stretched by a weight $p = 10$ oz. which describes from rest during its descent 15.5 inches in ten seconds, or 5.59 inches in six seconds.

Fig.
LXXXIV.

The wheel $F G$ as in the last experiment revolves on the friction wheels, and the spaces described by p in its descent are estimated by means of the graduated scale as before. To compare this experiment with the theory, we may have recourse to Prop. I. Cor. 2. Sect. VI. where it is shewn that in any revolving system, the force which accelerates the point to which the moving force is applied, is that part of the acceleration of gravity which is expressed by a fraction, of which the numerator is the square of the distance at which the force is applied from the axis, multiplied into the moving force, and the denominator the sum of all the products formed by multiplying each particle of the system into the square of its distance from the axis of motion.

Fig.
LXXVIII.

In the present case therefore it is necessary to obtain the sum of all the products which can be formed by multiplying each particle of the system $A B O Q$ into the square of

X x

its

its distance from the axis of motion, according to the dimensions given.

In order to estimate the sum of these products in the two spheres, some few considerations are necessary before the usual rules can be applied. It is to be observed that in each sphere there is a cylindrical cavity terminated at each end by portions of the sphere; but these portions are far too small to have sensible effect in the computation, even if it were carried beyond the usual number of decimals. The cavities therefore being cylinders of which the length is equal to the sphere's diameter or $= 1.48$, and their radius $= 0.0671$, and the specific gravity of brass being 8 when that of water is 1; it appears that if these cylinders were of brass, the weight of each would be $= 0.0969$ part of an oz. which being added to 7.985 the weight of each sphere when the cavity is not filled up, will give the weight of the entire sphere of brass $= 8.0819$ oz.

When any sphere revolves round an axis of motion, from which the distance of the sphere's centre is d , the radius of the sphere being r and its weight w , the sum of the products which can be formed by multiplying each particle into the square of its distance from the axis of

† Page 224.

† motion $= \frac{5 d^2 w + 2 r^2 w}{5}$. Applying this to the present case, we have $d = SE = 5.535$, $r = .74$, $w = 8.0819$, and the sum of the products required $= \frac{5 \times 5.535^2 \times 8.0819 + 2 \times .74^2 \times 8.0819}{5} = 249.37$;

but from this sum of products just found must be subtracted the sum of those which are formed by multiplying each particle of the brass cylinder, imagined to be inserted in the sphere, into the square of its distance from the axis.

† Page 223.

Now when any cylinder revolves round † an axis of motion, the axis of the cylinder being perpendicular to that axis, the sum of the products mentioned above may be obtained by the following rule. Let the radius of the cylinder $= r$, its weight w , the distance of its centre of gravity from the axis of motion $= d$; the distance of the nearest extremity from the axis of motion $= a$, that of the farthest $= b$: then will the sum of all the products under each particle into the square of its respective distance from the

axis of motion $= \frac{8 dbw + 4 a^2 w + 3 w r^2}{12}$: by apply-

ing

ing this to the brass cylinder imagined to be inserted on the sphere we have $w = 0.0969$, $d = 5.535$, $b = 6.275$, $a = 4.795$, $r = 0.0671$, and the sum of the products =

$$\frac{8 \times 5.535 \times 6.275 + 4 \times 4.795^2 + 3 \times 0.0671^2 \times .0969}{12}$$

= 2.98, which being subtracted from the sum of the products 249.37, there will remain 246.39, for the sum of the products of each particle in the sphere with the cylindrical cavity multiplied into the square of its distance from the axis of motion; and the sum of these products in both spheres = 492.78.

To find the sum of the products in the rod SQ we have by referring to the rule above, $a = 0$, $d = 3.1375$, $b = 6.275$, $r = 0.0671$, $w = .4017$ oz. and the sum of

$$\text{the products required} = \frac{8 \times 3.1375 \times 6.275}{12} +$$

$$\frac{3 \times .0671^2}{12} \times .4017 = 5.272, \text{ and the sum for the whole}$$

rod = 10.544.

The sum of the products similarly taken in the axis is obtained from the rule contained in page 224, where it is shewn to be half the cylinder's weight multiplied into the square of its radius: this will give the sum of the products in question =

$$\frac{2.4713 \times 0.20133^2}{2} = .050: \text{ the sum}$$

of the products therefore in the whole system = 492.78 + 10.544 + .05 = 503.37, and because the weight of the system is 18.244 ounces, the distance of the centre of gyration † from the axis will be =

$$\sqrt{\frac{503.37}{18.244}} = 5.2527. \text{ Let this} = r, \text{ and } w = 18.244,$$

$p = 10$ ounces = the descending weight: $q =$ a weight equivalent to the inertia of the wheel FG and the friction wheels when accumulated into the circumference of the wheel FG , this has been || found = 2.75 troy ounces, which are equal to 3.017 avoirdupoize ounces used in this experiment. Moreover, let d be the radius of the axle $AB = .20133$, and $t = 10$ seconds: then the space described † by the descending weight p in ten seconds =

$$\frac{1t^2 d^2 p}{p + q \times d^2 + wr^2} = 15.52 \text{ inches.}$$

X x 2

In

† Sect. VI.
Prop. VIII.

|| Page 302.

† Sect. VI.
Prop. XV.
Cor. 2.

† Sect. VI. In six seconds the space described from rest by the descending ‡ weight is observed = 5.59, which is to 15.52, as the square of 6 to the square of 10; thus it appears by experiment, that the descent of p is uniformly accelerated, and that the spaces described from rest are in a duplicate ratio of the times of motion.

IV.

Fig.
LXXXIV.

In the system A B O Q; let the length of that part of the line which is wound round the axle = 15.5 inches, and let the extremity of it be so applied to the axle by a loop and pin, that when $p = 10$ oz. has during its descent described 15.5 inches, the string may fall off: At this instant the force which accelerated the motion of the system being removed, the system will continue to revolve uniformly, and will perform 24.5 revolutions in 10 seconds.

In Prop. XV. Sect. VI. Cor. 1. it is shewn that if p be the moving force generating angular velocity in any system, d the radius of the axle to which the moving force is applied, r the distance of the centre of gyration from the axis of motion, w the weight of the system, $l = 193$ inches, and $c = 3.14159$, &c. the weight p in descending through any space s , will generate such an angular velocity in the system, as will cause it to describe uni-

formly $\sqrt{\frac{l s p}{c^2 d^2 p + c^2 r^2 w}}$ revolutions in a second;

provided after p has described the space s , the system suffers neither acceleration nor retardation. Applying this rule to the present case, we have $s = 15.5$, $p = 10$, $d = .20133$, $w = 18.244$, $r = 5.2527$, and the number of revolutions

tions in one second according to the rule will be =

$$\sqrt{\frac{193 \times 15.5 \times 10}{3.14159^2 \times .20133^2 \times 10 + 3.14159^2 \times 18.244 \times 5.2527^2}} \\ = 2.453, \text{ and consequently in 10 seconds the number of} \\ \text{revolutions will be } 24.53.$$

V.

D C E is a brass cylindrical plate of which the weight = 188.5 ounces; this cylinder turns horizontally round a fixed vertical axle A B: the axle is sustained in the frame U X Y Z. Motion is communicated to the cylindrical plate by means of a line I G wound round its circumference and from thence going over the wheel F G which rests on the friction wheels; or by the line I G which is wound round the axle and goes over the same wheel F G: the radius of the wheel is to the radius of the axle as 12.6. 1. If the line I G be stretched by a weight $p = 7$ oz. p will describe during its descent 51.84 inches in two seconds. Fig. LXXXV.

The sum of the $\dagger$ products which are formed by multiplying each particle of the cylinder into the square of its distance from the axis of motion is equal to half the cylinder's weight multiplied into the square of its radius, that is in the present case, the sum of the products just mentioned = $\frac{12.6^2 \times 188.5}{2} = 14963$; and the sum of the products similarly taken for the axle being .52, the whole sum will be 14963.52, to this must be added the pro- $\dagger$ Page 224

product of the equivalent weight of the wheel FG , and the descending weight, multiplied into the square of the wheel's radius, that is, $3 + 7 \times 12.6^2 = 1587$; which will make the sum of the products for the entire system $= 16550.52$; wherefore the force which accelerates the descent

* Sect. VI. of $p = \frac{12.6^2 \times 7}{16550.52} = \frac{1}{14.89}$ and the space described
Prop. I.
Cor. 3.

from rest in one second $= \frac{193}{14.892} = 12.96$ inches, and in two seconds the space will be $12.96 \times 4 = 51.84$, as appear by experiment.

VI.

Every thing else remaining as in the last experiment: let the line be wound round the axle, and let a moving force $= 10$ oz. be applied to stretch the line which goes over the wheel LG ; the space described by the descending weight in 10 seconds will be 12.8 inches.

† Exp. V. The sum of the products formed by multiplying each particle into the square of its distance from the axis in the whole system being 14963.52; to this add the products formed by multiplying† the equivalent weight of the wheel FG , and the descending weight into the square of the distance at which they act on the system from the axis, and since that distance is 1, the product required will be 13; this being added to the sum before found, the whole will be 14976.52, and since the moving force p is 10 oz. the force which accelerates the descent of $p = \frac{10}{14976.52} =$

$\frac{1}{1497.652}$. The space therefore described by p during

† Sect. VI. one second from rest $= \frac{193}{1497.652} = .1288$ † inches and
Prop. I.
Cor. 4. in ten seconds the space described $= 12.88$ inches.

Cor.

Cor. The distance of the centre of gyration from the axis = $\sqrt{\frac{14963.52}{188.5}} = 8.913$, the radius of the axle being = 1.

N. B. The aperture in the cylinder which admits the axle is so small that whether it is taken into account or not, no alteration is occasioned in the number set down.

VII.

Let A B C represent an inclined plane of which the height is to the length as 1 : 16, and let a sphere descend by rolling down this plane: the centre of the ball will describe 34 inches in 2 seconds.

Se&t. VI.
Prop. XIV.
Fig.
LXXVII.

If the * sphere's surface and the inclined plane were perfectly smooth, so that the sphere might slide down the plane, the force which accelerated the descent would be to the force of gravity as the plane's height to the

* Page 233.

length, that is, it would be $\frac{1}{16}$ part of the acceleration of gravity; but as the sphere rolls down the plane, the force of acceleration is diminished in the proportion of 7 : 5, so as to become $\frac{5}{7}$ of $\frac{1}{16} = \frac{5}{112}$; wherefore the sphere's centre

will describe by the action of this † force $\frac{193 \times 5}{112}$ or 8.616 inches in one second, and $4 \times 8.616 = 34.4$ in two seconds.

† Se&t. III.
Prop. IV.
Cor. 5.

Cor. As all spheres, whatever be their magnitudes descend by rolling down a given plane with equal absolute velocities, the angular velocities acquired in a given time will be inversely as their diameters: so that by decreasing the diameters, this angular velocity might be increased sine limite, unless the effects of friction encreased the same time, which obstruct the motion of small spheres down inclined planes far more than larger ones.

2. If

2. If a cylinder be accelerated along the same inclined plane by rolling; in this case the force of acceleration
 ¶ Page 231. being $= \frac{1}{16} \times \frac{2}{3} = \frac{1}{24}$ ||, the space described from rest in
 one second $= \frac{193}{24} = 8.041$, and in two seconds the space
 $= 32.164$.

Hence it is manifest how much experiment will disagree with the theory of the descent of bodies along planes, unless their rotation be taken into account; for if the cylinder were to slide without rotation along the plane, during two seconds as in the last experiment, the space described would be 48.25 inches, instead of 32.165, the difference of which is no less than 16.08 inches.

VIII.

Let the system represented in fig. 84. be removed from the frame; and let two lines be wound round the axle in the manner represented in fig. 86. let these lines be fastened to the points L and I at a considerable height, L and I being in an horizontal line, and make $LT = IH$, that the axle may be horizontal. When the system is unsustained, it will gradually descend by unwinding itself from the line, and the centre of the axle will describe 28.3 inches in ten seconds.

† Sect. VI. Through the † centre of gravity G draw G R O perpen-
 Prop. XIV. dicular to the horizon when the rod OQ is vertical; and
 Cyr. 3. let R be the centre of gyration of the system when it revolves round the axis AB: assume TH in the surface of the axle parallel to the axis AB: suppose the system to vibrate as a pendulum on this line, and let O be the centre

tre of oscillation; then if the system descends by unrolling itself from the lines TL , IH , the centre G will be accelerated by a force which is that part of the acceleration of gravity, which is expressed by the fraction $\frac{SG}{SO}$. In

order to find the value of this fraction by theory, we have $SG = .20133$, $GR = 5.2527$, and because $SG : GR ::$

$GR : GO$ †, it follows, that $GO = \frac{5.2527^2}{.20133} = 137$, ^{† Sect. VI. Prop. VI. Cor. 3.}

wherefore $GO + GS = SO = 137 + .20133 = 137.201$, ^{and Prop. VIII. Cor. 3.}

and $\frac{SG}{SO} = \frac{.20133}{137.201} = \frac{1}{681.4}$, being the force which ac-

celerates the descent of G : which therefore in ten seconds will describe $\frac{100 \times 100}{681.48} = 28.32$ inches.

If the axle AB were to be placed between two inclined planes of the same heights, lengths and positions, and were to descend by rolling along them, and the cohesion subsisting between the surfaces of the plane and the axle were sufficient to supply the place of the line, by the tension of which the rotatory motion was before generated; in this case the accelerating force would be equal the force in the last example multiplied into the plane's height and divided by its length.

IX.

Every thing else remaining as in the last experiment, let TL , HI (no longer fixed to L and I) go round two fixed pulleys F and M , and to the extremities let two weights p , p , each of which is equal to 9.108 oz. be suspended: these weights will exactly counterbalance the system as it is descending, and will remain quiescent.

That moving force which causes the rotation of the system, or that which stretches the strings LT , HI is
Y y
that

† Sect. VI.
Prop. XIV.
Cor. 3.

that part † of the system's weight which is expressed by the fraction $\frac{GO}{SO}$, but in the present case $GO = 137$, and $SO = 137.20133$, and the weight $= 18.244$; wherefore the force which stretches the string in exp. viii. $= \frac{18.244 \times 137}{137.2033} = 18.217$, the half of which is 9.108 , if therefore each of the weights p be each 9.108 , they will together be equivalent to the string's tension as the system is descending, and consequently will remain quiescent.

Let a small piece of steel be fitted to the rod GC , so as to admit of being fixed to it at any given distance from the centre G , passing through the axis of the rod CD : let two edges proceed from the opposite sides of this piece of steel in directum in regard to each other, and parallel to the horizontal axis AB : so that when the system rests on these steel edges, the axis of the cylinder DC shall be vertical, as represented in fig. 86. when motion is communicated to the system, it will vibrate as a pendulum on these steel edges as an axis of motion.

X.

Let the steel edges (vid. preceding note) be fixed at $.717$ parts of an inch from the centre G : on setting the pendulum in motion it will be observed to vibrate once in each second, provided the arc of vibration be small.

In this case $SG = .717$, and by what has preceded $GR = 5.2527$, if therefore O be the centre of oscillation. we

† Sect. VI.
Prop. VI. have † $GO = \frac{5.2527^2}{.717} = 38.48$, which being added to $SG = .717$, gives $SO = 39.197$, or nearly 39.2 for the pendulum's length; and a pendulum of this length is known to perform its vibrations in seconds.

XI.

XI.

Let the steel edges be fixed at different distances from the centre G : it will be observed that the time of the pendulum's vibration will be the least of all when the axis or edges pass through R the centre of gyration.

This has been demonstrated in prop. x. sect. vi. but may be more easily shewn thus. Let S be the point through which the axis of suspension passes, O the centre of oscillation, R the centre of gyration, then in general $SG \dagger : GR :: GR : GO$: now when four quantities are in $\dagger$ Sect. VI. geometrical $\dagger$ proportion, the sum of the two extremes is Prop. VI. always greater than the sum of the two middle terms, Cor. 3. except when the four terms are all equal: therefore SG and Prop. VIII. $+ GO$ is always greater than $2GR$, except when $SG =$ Cor. 3. $GR = GO$; $SG + GO$ therefore will be the least possible $\dagger$ Eucild. when $SG = GR$, that is, when the point S through El. Lib. V. which the axis of suspension before coincides with R the Prop. ult. centre of gyration.

Cor. When a given body vibrates in the least time possible, the plane of vibration being the same, the distance between the axis of suspension and centre of gravity is equal to the distance between the centre of gravity and the centre of oscillation, for by what has preceded $SG = GR$, and $GO = \frac{GR^2}{SG} = GR = GS$.

It must be observed concerning the experiments made on bodies which revolve round axes, that the friction is diminished, or indeed almost wholly removed as to any sensible effects three ways: 1. by making the axle revolve horizontally on friction wheels: 2dly, by placing the axle vertical, the lower point of the axis being sustained in a very small and well polished cavity, as in fig. 84. here the smallness of the surfaces in contact has an effect in removing tenacity, and we may say friction too, notwithstanding some experiments to the contrary equal to that produced by the friction wheels: 3dly, when the system vibrates; by suspending it on sharp edges as in the latter experiment.

S E C T. IX.

ON THE MOMENTUM † OF BODIES IN
MOTION.

• Page 1.

FORCE has been * defined that which causes a change in the state of motion or quiescence of substances: The action of animal muscles, the impact of bodies either solid or fluid, attractions and repulsions of various sorts have been enumerated as instances of this motive power. The Idea of force is originally acquired from the sense of feeling, and through the medium of that sense by the eye also; but this idea extends only to the degree of force, not to the quantity or any measure of it. Thus when muscular force is exerted in order to overcome resistances of any kind, there is no difficulty in determining when the force is greater or lesser, but as it is not in itself a mathematical quantity, the proportion between the greater and lesser, and the exact increase or diminution of its intensity cannot be known except by applying either ‡ space or number as measures of it.

† Sect. II.
Prop. I.

Application
of mathe-
matics.

This application is effected by the axioms or laws of motion, which may be term-

† The expressions, quantity of motion, momentum, and moment, are used synonymously throughout this section.

termed propositions intermediate between Geometry and Philosophy; through these, mechanics become a mathematical branch of physics, and its conclusions possessed of such coherence and consistency among themselves and with matter of fact, as are rarely to be found in other^a branches, which admit not of so intimate an union with the science of quantity.

Since the Newtonian laws apply mathematical relations to the motion of bodies, it may be expected that the truth of them should rest on mathematical proof: but it is to be remembered, that these laws are assumed in mechanics as axioms are in geometry; and an axiom is understood to be a proposition which either from being intuitive requires not, or from the nature of the objects to which it is applied is incapable of, demonstrative evidence.

Laws of
motion not
demon-
strated.

This is the case with the laws of motion: They cannot be deduced from any reasoning *à priori*, for we know nothing concerning matter but from experience; and though experiment confirms the truth of them in a variety of instances, and to a certain degree of exactness, yet be-

^a Chemistry and Electricity are of this sort: the principles of these are arranged systematically, and are referable to a few general laws; but they are not universally and without exception consistent among themselves and with experience, as the principles of mechanics are.

between that degree and the mathematical relation expressed in the laws, there may be infinite varieties of different relations which experiment cannot discover.

Laws of
motion col-
lected from
various ar-
guments.

^b If then it be enquired, from what evidence our assent to these axioms is derived, it may be replied, the evidences are of various kinds. 1. ^c From the constant observation of our senses, which tend to suggest the truth of them in the ordinary motion of bodies, as far as the experience of mankind extends. 2. From ^d experiments, properly so called: In these it always appears, that when the most effectual means are used in order to subject these axioms or laws of motion to the severest and most minute examinations, by removing all impediments which may tend to occasion any deviation from the truth, the result is a more near agreement between the experiments and the axioms just mentioned, the more perfectly the intentions of the experiments are accomplished. This will lead to

a

^b The laws of motion not admitting of demonstration, the truth of them must be collected from the best evidence the nature of the subject will admit.

^c This argument depends not on the accuracy of these common observations, but from their extent; nothing contrary to the axioms being observed in the great variety of motions which are the daily objects of the senses.

^d In experiments every power is removed as far ^{as} may be, except those which are the objects of examination.

a belief, that if it were possible to remove every imperfection from the construction of experiments, and to increase the acuteness of the senses, of which they are the objects, in an infinite degree, a mathematical coincidence between such experiments and the laws of motion would be observed.

3. From arguments *à posteriori*: — Let a proposition be assumed as true even without evidence of any kind; if by strict and logical reasoning, various conclusions are deduced, which upon examination are found consistent among themselves, and with experience, this will be a presumptive proof in favour of the principle assumed; and our assent to it will be the more strongly enforced in proportion as the conclusions inferred and the comparison of them with experience have been more extensive. From the Newtonian axioms assumed as true, a system has been deduced and compared with phenomena in numberless cases: it has been applied to the motion of the planets and comets, to that of bodies on the earth's surface; even to the motion of those minute particles which compose both solid and fluid substances: A perfect agreement between these consequences of the axioms and matter of fact has been the result, no one instance excepted. These and other similar
ar-

arguments, upon the whole amounting to evidence scarcely inferior to mathematical demonstration, are the grounds on which the laws of motion, although incapable of direct proof, are received as axioms, from which the various theorems concerning the effects of forces are systematically demonstrated.

In treating of mechanical as well as other subjects, it is certainly allowable to describe terms by any form of words that may seem best adapted to the purpose of conveying a distinct and adequate idea of the object described. * The measure of the quantities of motion impressed on bodies, or destroyed in them, may be defined by the joint ratios of their quantities of matter and velocities, or of the quantities of matter and squares of their velocities: but as definitions affirm nothing, being in themselves incapable of expressing truth or falsity, if either of these measures were said to be right exclusively of the other, this would be only misapplying the definition by converting it

* The definition does not imply the description of any properties of motion, but only of the measure of its quantity. By the ratio of the quantities of motion in any two moving bodies, is defined to be meant, the joint ratios of the velocities and quantities of matter contained in the bodies: it is therefore impossible to mistake what is meant by the expression, quantity of motion, according to this description; and this is the only object of the definition: — the same may be applied to any other definition of the moments of bodies.

it into an affirmation or proposition. The same objection may be urged against many arguments which have been brought both from theory and experiment in the way of proof, in order to establish one of these measures, or controvert the other. The following are of this kind, and when cleared from the multitude of words which controversy has heaped upon them may be described thus. Let the measure

expressed ^f by $\frac{V^2}{v^2} \times \frac{Q}{q}$ be assumed as that

by which the quantity of motion is defined: because the total effects of bodies in motion estimated by the resistances which they overcome, and the whole spaces described during their constant retardation jointly, are observed in experiments

^f Whenever any measure of the quantity of motion is mentioned, it implies the consideration of two bodies, the moments of which are compared: let the two bodies be Q and q , and their respective velocities V and v : then the joint ratios of their quantities of matter and velocities will be $V \times Q$ to $v \times q$, or according to the notation used

in sect. ii. $\frac{V}{v} \times \frac{Q}{q}$ to 1. In the following pages when

$\frac{V}{v} \times \frac{Q}{q}$ is mentioned as expressing the quantities of bodies' moments, it means that the quantity in the body Q is to that in q as $\frac{V}{v} \times \frac{Q}{q}$ to 1, or as $V \times Q$ to $v \times q$: a similar construction is to be understood concerning the other measure $\frac{V^2}{v^2} \times \frac{Q}{q}$.

ments to be always proportional to $\frac{V^2}{v^2} \times$

$\frac{Q}{q}$, or in other expressions, because

‡ Page 35.
261.
Sect. VII.
Exp. XII.
page 333.

$$\frac{V^2}{v^2} \times \frac{Q}{q} = \frac{M}{m} \times \frac{S}{s}^\dagger, \text{ either } \frac{V^2}{v^2} \times \frac{Q}{q} \text{ or } \frac{M}{m}$$

$\times \frac{S}{s}$ was concluded^h to be theⁱ true mea-

sure

§ The notation used in prop. v. and vi. sect. iii. is adopted in this section, and the signification of the quantities is the same as described in those propositions.

h The quantity of rectilinear motion any how estimated is referred to the same direction: two equal quantities of motion acting in opposite directions exactly counteract each other, and vice versa, if two moments acting in contrary directions counteract each other, they are inferred to be equal. It is needless to mention cases of oblique forces, since nothing is gained or lost in a given direction by the resolution or composition of forces.

i A practical instance or two in which both measures of the quantities of motion are applied, may be here inserted. 1. When piles are driven into the earth by the percussion of heavy bodies impinging on them, the spaces they describe at each impact of the weight are estimated from the theory of Leibnitz, &c. thus. Let the force with which the earth resists the entrance of the piles, be to the weight of the impinging body as $m : M$; and let the space through which the pile is driven, be to the space through which the heavy body falls as $S : s$; then, as the whole motion of the impinging weight is destroyed by resistance in the opposite direction, the moments of both are equal and contrary: in the equation $\frac{V^2}{v^2} \times \frac{Q}{q} = \frac{M}{m} \times \frac{S}{s}$, since $V^2 Q = v^2 q$ by the present case, it follows that $M \times S = m \times s$: that is, the impinging weight multiplied into the height from

sure of the quantities of motion destroyed. This is only one out of a many instances of similar experiments made on bodies both accelerated and retarded, which have been brought to establish this measure of momentum. It is however evident, that nothing is proved by these experiments, except the truth of the mechanical proposition expressed in the equation

$$\frac{V^2}{v^2} \times \frac{Q}{q} = \frac{M}{m} \times \frac{S}{s}, \S \text{ which, without any } \begin{matrix} \S \text{ Sect. III.} \\ \text{Prop. VI.} \end{matrix}$$

reference to experiments, is deduced from the laws of motion.

Exactly in the same manner experiments shew, that $\frac{V}{v} \times \frac{Q}{q} = \frac{M}{m} \times \frac{T}{t}$, from $\begin{matrix} \S \text{ Sect. VII.} \\ \text{Exp. VI. \& VII. \& VIII.} \end{matrix}$ which

from which it falls, is equal to the resistance exerted against the pile's entrance, multiplied into the space to which it is driven. This, it is evident, holds true, whatever be the resistance which is opposed to the pile, because while it moves through a small space the resistance is constant, and its proportion to the impinging weight must be always expressed in the rule by the ratio $m : M$.

2. Let two boats of different weights be connected together by a line, and let the line be stretched by pulling the larger vessel toward the smaller, or the smaller toward the larger; applying the equation $\frac{V}{v} \times \frac{Q}{q} = \frac{M}{m} \times \frac{T}{t}$, $M = m$, because the tension of the rope acts equally on each body: and $T = t$, because they begin and continue to move together; wherefore, because $MT = mt$, it follows, that $V \times Q = v \times q$, or the velocities of the two boats, at any given instant, will be in the inverse proportion of their quantities of matter, the resistance of the water not being considered.

which it may with equal reason be concluded, that either $\frac{V}{v} \times \frac{Q}{q}$ or $\frac{M}{m} \times \frac{T}{t}$ is the true measure of the quantities of motion in bodies.

These arguments, however inconclusive in themselves^k, being extended and diversified, gave rise to others less ambiguous: whatever difference of opinion might subsist concerning the measures in question, it was agreed universally, that the quantity of motion, if measurable at all, should be proportional to the causes by which it was generated, or to the effects produced by it, estimated according to the same measure; and that a constant equality between the quantity of motion, before and after its communication, should be preserved in bodies of every kind and any how moved.

This principle applied to the ¹measure
which

^k The experiments first made on the effects of percussion on soft bodies, &c. in order to prove the truth of the measure of momentum expressed by $\frac{V^2}{v^2} \times \frac{Q}{q}$, were prior in time to Bernoulli's theory denominated *Conservatio virium vivarum*.

¹ Bernoulli termed this principle *virium vivarum conservatio* when applied to the measure $\frac{V^2}{v^2} \times \frac{Q}{q}$ only; but *motus* being the word used to imply quantity of motion in general, *conservatio motus* will express the application of the same principle, to the motion of bodies estimated as to quantity by either of the measures.

which estimates the motion of bodies by their quantities of matter into the squares of their velocities, Bernoulli denominat-
ed *virium vivarum conservatio*, imagining any given moment thus measured to be universally permanent and immutable in quantity: † “Hinc patet vim vivam esse
“aliquid reale et substantiale quod per
“se subsistit, &c. — unde concludimus
“quamlibet vim vivam habere suam de-
“terminatam quantitatem de quâ nihil
“perire potest quod non in effectu edito
“reperiatur.”

† Bernoulli,
Vol. III.
De Vera
Notione
Virium vi-
varum.

Perceiving his principle to obtain in several instances, he too hastily concluded it to be universal; and, as is well known, by reasoning from it ^m systematically, was led into mistakes and inconsistencies, which he chose to misemploy his ingenuity in defending, rather than forego a preconceived theory. If indeed the nature of motion were such, that a given quantity of it, any how estimated remained in all cases permanent, suffering neither augmentation nor diminution in its communication, this immutable quality would in itself constitute the reality and truth of the measure thus
ap-

^m In some *cases he refers the effects of forces to direc-
tions different from those in which the forces were im-
pressed, and thence deduces conclusions favourable to his
theory, but as is well known contrary to the laws of mo-
tion.

* Discours
sur le
Mouve-
ment,
Vol. III.

applied; but not exclusively of other measures with which (if possible) the general principle “*conservatio motus*” were consistent.

To consider this more particularly, let either of the measures, that of Bernoulli for example, be assumed. It is plain, that unless the permanency in a given quantity of motion thus expressed, were demonstrated in all kinds of bodies any how put in motion, or unless the truth of this principle were collected by induction sufficiently extensive, and contradicted in no one instance, it should rather be accounted an individual property of motion, than a general law: but to give the argument its due force, let it be supposed that the “*conservatio motus*” estimated by

$\frac{V^2}{v^2} \times \frac{Q}{q}$ ■ obtained in general, without

limitation or exception; still if the same universality could be proved to belong to the permanency of motion as expressed

by $\frac{V}{v} \times \frac{Q}{q}$, there seems no reason for as-

suming one of them as the true measure of the moments of bodies in motion rather than the other. But the truth is, the principle obtains not according to either of the measures, except in particular cases, which may be demonstrated as the other proper-

ties

ties of forces are from the general laws or axioms.

In the rectilinear motion of bodies, accelerated from quiescence or retarded § ^{Note h} p. 362. until they are at rest, the permanency of any given quantity of † motion is demonstrated from the axioms, whether that † Sect. III. Prop. VI. and VII. motion be estimated by one measure or by the other.

In bodies which revolve round fixed axes, the principle obtains without exception when the moment is measured by the quantity of || matter into the square of the velocity, but fails when measured by the quantity of ⁿ matter into the velocity; a given quantity of motion thus estimated being alterable in any assigned ratio.

In

ⁿ This may be deduced from * Cor. 7. Prop. 16. If it were true that a given quantity of motion, estimated by * Sect. VI.

the measure $\frac{V}{v} \times \frac{Q}{q}$, is permanent, it would follow that

any force p acting for a given time t , must produce such a velocity in a revolving system, that the sum of the products formed by multiplying each particle into its velocity, shall be the same as the body p multiplied into the velocity which it would acquire in falling freely by its gravity during the time t : but the contrary is easily shewn. Let the whole mass in the system ABC be collected into R : then the notation of † prop. xvi. remaining, the velocity

generated in R in the time $t = \frac{2lt dp}{wr}$, which being mul-

tiplied into the mass w will give $\frac{2lt dp}{r}$ for the product

under the quantity of matter and into its velocity: but the velocity generated in p when it falls freely for t seconds is $= 2lt$, which being multiplied into p gives $2ltp$ for † Sect. VI. Prop. II. the product of the mass into the velocity acquired, which it is plain may differ from the former in any ratio.

Fig. LIX.
† Sect. VI.
Cor. 7.

In the communication of motion to bodies by collision, when the direction of the stroke passes through the centres of gravity, the principle in question holds

universally according to the measure $\frac{V}{v} \times \frac{Q}{q}$, but fails when the moments are esti-

mated by $\frac{V^2}{v^2} \times \frac{Q}{q}$ in every case, except when both bodies are perfectly elastic, or one perfectly elastic and the other perfectly hard.

Lastly°, when motion is communicated to bodies by impact, the direction of which

° This is an argument against the permanency of a given quantity of motion according to any measure: if a body *A* impinge directly with the velocity *V* on a quiescent body or system of bodies *B*, in the direction of a line which joins their centres of gravity, and the bodies should be nonelastic, the whole mass will go on after the

impact with the velocity $\frac{AV}{A+B}$, or if the inertia of *A* be incomparably smaller than that of *B*, the common velocity will be $\frac{AV}{B}$. If the impact be still perpendicular to

the plane which it strikes, although the direction of it passes not through the centres of gravity, the velocity of the

centre of gravity will be the same as before, i. e. $\frac{AV}{B}$, and

at the same time the system will revolve round that centre with different degrees of angular velocity, according as the direction of the impact passed nearer to the centre of gravity, or farther from it, as is demonstrated in sect. x. of this book: it is also deduced from different principles in

which passes not through the centres of gravity, the quantity of motion communicated, whether estimated by one measure or the other, preserves neither equality nor any constant proportion to the quantity of motion impressed: these considerations make it evident, that from the fundamental properties of forces as expressed in the general laws or axioms, which are controverted by no one, any given quantity of a body's motion, as measured by the sum of the products formed by multiplying each particle into the velocity, or into the square of the velocity,

a very ingenious * paper on the subject of rotation by the Rev. Mr. Vince of the university of Cambridge, to whom the prize medal of Sir G. Copley, for the year 1780, was adjudged by the president and council of the Royal Society.

* Philof.
Transf. for
1780, p. 550.

When bodies revolve round a progressive axis of motion, passing through the center of gravity, the sum of the moments, if they were estimated in a given direction, as far as regards the rotation, would be $= 0$. This would also be true when the axis is fixed; by which method of reasoning, the moment of every body revolving round a fixed axis, which passes through the centre of gravity would be nothing; whereas according to any supposed measure, the quantities of motion impressed on revolving bodies, are not only finite, but are in assignable proportions to each other: this being granted, when any quantity of matter revolves in free space, by the impulse of a body impinging on it with a given velocity, the quantity of motion in the impinging body remaining the same, any different quantities of motion may be produced in the body struck, according as the impact is made nearer to the centre of gravity, or further from it; a conclusion which is contrary to the hypothesis of moment according to any measure.

locity, is not of that permanent and immutable nature, which if really existing, would be the only ground on which any measure of such quantity could be established. This being the case, all controversy concerning which of the expressions, i. e. that of $\frac{V^2}{v^2} \times \frac{Q}{q}$, or $\frac{V}{v} \times \frac{Q}{q}$ is the true measure of the quantities of motion in bodies, must be regarded as verbal, i. e. a dispute about a term or definition, except only as far as relates to the assumed principle, denominated * “conservatio motûs,” which implies the quality of the sums of products formed by multiplying each particle of a body into some power of its velocity: this being a distinct proposition admits of such examination as is necessary to confirm or disprove it.

* Page 364.
note 1.

It follows from what has preceded, that the permanency of motion, during its communication, should not be applied in the demonstration of mechanical propositions, because in systematic reasoning, no proposition is to be assumed as true, except the axioms or fundamental laws, and such as have been regularly inferred from them. Thus in demonstrating the laws of collision, the equality of moments before and after the stroke, as estimated by any measure, should

should not be assumed as a step in the demonstration, because nothing is mentioned concerning the equality of these moments in the laws of motion: the equality of moments according to the measure

$\frac{V}{v} \times \frac{Q}{q}$ can indeed be deduced from the

axioms, which deduction is the same as the demonstration of the laws of collision.

Let A and B be the centres of gravity Fig. LXXVII.
of two bodies moving in free space, and let A be supposed to act on B in the direction BD with any kind of force variable in any manner; then will B react on A in the direction AC with equal force; and if the time in which the effects of these forces are produced, be assumed evanescent, the force will for that instant be constant. In the corollary to prop. 1. sect. III. it appears, that in any given particle of time, the variation of velocity generated in that time will be always proportional to the accelerating force, that is, to the moving force directly, and quantity of matter inversely; or, in the present case, since the force with which A impels B is equal to that with which B reacts on A , the velocities generated in these bodies during the same time, will be inversely as the quantities of matter contained in them: any velocity

III. Law of motion.

generated in A in the direction AC being considered as a decrement of its velocity referred to the direction AD .

When a sphere A impinges directly on any other sphere B , moving in the same direction, the acceleration of B and retardation of A will be effected gradually during the time in which the figures of the two bodies are changed by the impact; while therefore the centre of A moves with a velocity greater than that of B , so long will B continue to gain and A to lose motion, until the change of their figures is the greatest, the centres being then nearest to each other; at this instant they will begin to go on with the same velocity, in bodies of every kind of texture. If the bodies be such as possess a power of restoring the figures which have been thus changed, this power also will act equally on each body in contrary directions, for the same time, and will create the greater augmentation of B 's motion, and the greater diminution of A 's, according to the proportion of the restitutive power to that whereby the bodies resisted the force which effected the change of their figures: this restitutive power in bodies perfectly void of elasticity being nothing. Suppose the bodies A and B to be of this sort, and to move before the impact with the velocities a and b :
let

III, Law of
motion.

let the time in which the change of figure is effected be divided into innumerable instants. Suppose during the first of these instants, that is, at the first contact between A and B , A loses any small part of its velocity represented by p , and during the same instant that B gains an increment of velocity represented by u ; in the next instant let q be the loss of A 's velocity, and w the gain of B 's, and so on during innumerable instants, let $p, q, r, s, \&c.$ be the successive values of the loss of A 's velocity, and $u, w, t, z, \&c.$ the contemporary increments of the velocity of B : because each of these instants is evanescent, the force which urges the two bodies in contrary directions will be constant in any given instant, and acting for the same time will generate increments or decrements of velocity which are inversely proportional to the *quantities of matter * Page 371. contained in A and B ; for this reason in the first instant we shall have $p : u :: B : A$; moreover in the second instant $q : w :: B : A$, and so on: † wherefore † Euclid. El. Lib. V. Prop. XII. $p + q + r + s, \&c. : u + w + t + z, \&c.$ as $B : A$; and if $x = p + q + r + s, \&c.$ represents the whole loss of A 's velocity, and $y = u + w + t + z, \&c.$ the total increase in that of B , the proportion will be as $x : y :: B : A$: but when $x =$ the whole sum of $p + q + r + s, \&c.$ and $y =$ the other

other corresponding sum, the bodies begin to move with a common velocity, and since a and b were the respective velocities before the impact, it will follow that after the impact $a - x = b + y$, or $x + y$

Page 373. $= a - b$: that is, since $x = \frac{By}{A}$, $\frac{By + Ay}{A}$

$= a - b$, and $y = \frac{a - b \times A}{A + B}$, which gives

this analogy, as $A + B : A :: a - b : y$, $=$ the velocity gained by B : if therefore

we add to $\frac{a - b \times A}{A + B}$, the original velocity

of B which was b , the sum will be the

velocity of B after the stroke $= \frac{Aa - Ab}{A + B}$

$+ b = \frac{Aa + Bb}{A + B}$, or the common velocity,

with which the two bodies begin to move when their change of figure is the greatest, whatever be their texture.

Cor. Multiply $\frac{Aa + Bb}{A + B}$ the common velocity by the sum of the bodies: the product will be $= Aa + Bb$, which is the same as the sum of the products of each body into its respective velocity before the stroke.

When the bodies are possessed of a power by which they endeavour to restore the

the change of figure effected by the impact, this power may act with different degrees of intensity, and for different times according to the textures of the bodies: but the time of its action, whatever be the quantity of it, may be divided into innumerable instants, corresponding to those into which the time of effecting the greatest change was divided. Now suppose the restitutive power such as will generate in the body struck, during any instant, a velocity which is to the velocity before generated, while the figures were changing in the corresponding instant, as m to 1: m being assumed as a measure of the elastic force: then during the successive instants in which the changed figures are restored, the diminution of A 's velocity will be $m \times p + q + r + s, \&c. = mx$, and $m \times u + w + t + z, \&c. = my$ the cotemporary increase of B 's velocity: and since before the elastic power took place A 's velocity was $a - x$, and B 's $= b + y$, after that power has ceased to act A 's velocity must be $= a - x - mx$, and B 's velocity $= b + y + my$, or if $m + 1$ be put $= n$ the velocity of A after the impact will be $a -$

nx , and that of $B = b + ny$; but $x = \frac{By}{A} + \frac{1}{2}$ Page 373.

in general; this will give the velocity of A after the stroke $= \frac{Aa - Bny}{A}$ and the velocity of $B = b + ny$.

Cor.

Cor. Multiply each body after the stroke into its respective velocity, the products will be $Aa - Bny + Bb + Bny$, the sum of which is $Aa + Bb$, the same as the sum of the products formed by multiplying each body into its velocity before the stroke: if therefore the quantity of motion be defined by the sum of these products, it will follow, that in the collision of all kinds of bodies, the quantity of motion is not altered by the effects of the impact.

If the elastic power be equal to that which resists the change of figure, m will be $= 1$, and $m + 1 = n = 2$; in this case therefore A 's velocity after the impact will be $a - 2x$, that of $B = b + 2y$; and † because $x + y = a - b$, by substitution, the difference of the velocities after the stroke will be $= a - b - 2a + 2b = b - a$, the same as the difference of velocities before the stroke, only negative in respect of it. From hence follows the proposition so much insisted on by Bernouilli; that in the impact of perfectly elastic bodies, the sum of the products formed by multiplying each body into the square of its velocity is not altered by the impact.

† Supra.
* Discours
sur le
Mouvement,
Vol. III.

His demonstration is this: The other notation remaining, let p be the velocity of A and q that of B after the stroke; then since by what has † preceded $Aa + Bb = Ap + Bq$, or * $Aa - Ap = Bq - Bb$,

Bb , * and $a - b = q - p$, or $a + p =$ * *Supra*,
 $q + b$: by multiplying $Aa - Ap$ into $a +$
 p , and $Bq - Bb$ into $q + b$, we have Aa^2
 $- Aap + Aap - Ap^2 = Bq^2 - Bqb$
 $+ Bqb - Bb^2$, or $Aa^2 + Bb^2 = Ap^2$
 $+ Bq^2$, which is the equality affirmed: a
 geometrical demonstration is given of the
 same proposition by Maclaurin †.

† View of
 Sir J. Newt.,
 Phil.
 Book II.
 Chap. IV.

From the ‡ laws of collision it appears,
 that the quantity of motion as measured
 by the sum of the products expressed by
 $V \times Q$, is not altered by the collision of
 bodies of any kind: but the equality of
 the quantities of motion, before and after
 the impact, according to the measure ex-
 pressed by $V^2 \times Q$, obtains only in those
 bodies which are perfectly elastic.

† Page 376.

It is extraordinary that Bernoulli
 should insist upon the demonstration of
 this particular property of perfectly ela-
 stic bodies, in order to confirm an hypo-
 thesis which he assumed as a general law
 of motion, and consequently ought to
 extend to bodies of every kind; had his
 process been inverted, more general con-
 clusions would have been the result. In-
 stead of demonstrating the equality of the
 sums of the products formed by multi-
 plying each particle into the square of
 its velocity before and after the impact,
 on a supposition that the elastic force
 is equal to that which resists the change

of figure, it may be more expedient to inquire, having given the sums of the products expressed by the mass multiplied into any power of the velocity as V^n , what must be the elastic force, in order that the sums of the products, before and after the stroke, shall be equal. Let the notation used above be resumed; then if u be put to denote the restitutive power when r is that which resists the change of figure, and let $r + u = s$, we shall have the velocity of A after the stroke $= a -$

‡ Page 373. δx , or since $\dagger x = \frac{By}{A}$, the velocity of $A =$

$a - \frac{Bsy}{A}$, and that of $B = b + sy$; and the sum of the products formed by multiplying each mass into the n^{th} power of its velocity after the stroke will be $\frac{Aa - Bsy^n}{A^{n-1}}$

$+ B \times \overline{b + sy}^n$, which by the problem is equal to $Aa^n + Bb^n$: by solving the

equation $\frac{Aa - Bsy^n}{A^{n-1}} + B \times \overline{b + sy}^n = Aa^n$

$+ Bb^n$, the value of s will give that of u , or the restitutive power sought. Suppose for example, it were required to assign u when the sums of the products formed by multiplying the mass into the square of the velocity are equal before and after the impact:

paſt: here $n=2$, which being ſubſtituted, the equation will become $A^2 a^2 - 2AB a s y + B^2 s^2 y^2 + AB b^2 + 2B A b s y + AB s^2 y^2 = A^2 a^2 + B A b^2$, or $s^2 \times AB y^2 + B^2 y^2 - s \times 2AB y \times a - b = 0$, or ſince $y = \frac{a - b \times A}{A + B}$, the equation will be $s \times \frac{A + B \times a - b \times A}{A + B} = 2A \times a - b$, and

$s = 2$; wherefore $u = s - 1 = 1$; that is, the force of elasticity muſt be equal to that which reſiſts the change of figure. Suppoſe $n = 3$, or, let it be required to aſſign the elastic force, ſo that the ſums of the products under the maſs and cube of the velocity ſhall be equal before and after the impact: in the general equation

$$\frac{Aa - Bsy}{A^{n-1}} + B \times b + sy^n = Aa^n + Bb^n,$$

ſubſtitute 3 for n , and $\frac{A \times a - b}{A + B}$ for y ; the

equation will become $s^2 + s \times \frac{3Ba + 3Ab}{A - B \times a - b}$

$$= \frac{3A + 3B \times a + b}{A - B \times a - b}, \text{ from which } s \text{ is eaſi-}$$

ly obtained. It is manifeſt here, that the elastic power to ſaſisfy the conditions of the problem, will depend on the quantities of matter of the bodies, and on the rela-

tive velocities before the stroke: if $b = 0$, that is, if the body struck should be quiescent before the impact, s will be found =

$$\sqrt{\frac{9B^2}{4 \times A - B^2}} + \frac{3A + 3B}{A - B} - \frac{3B}{2 \times A - B},$$

in which case it appears, that the quantity of elasticity necessary to satisfy the conditions of the problem, will now not at all depend on the velocity of the impinging body, but on the mass in the bodies only: if $A = 2$ and $B = 1$, s will be = 1.85410, and the restitutive force $u = s - 1 = .85410$; according therefore to the conditions mentioned above, if the force whereby the spheres A and B restore the change of figure be to the force which resisted that change in the proportion of .85410 to 1; the sums of the products of each particle into the cube of its velocity, before and after the impact, will be equal: so that if the quantities of motion in bodies were defined to be measured by the mass into the cube of the velocity, the sum of the moments before and after the impact would be equal in this particular case.

It is not probable that the theory of motion, however incontestable its principles may be, can afford much assistance to the practical mechanic; and there appears as little reason to imagine that

that any errors or misconceptions which may have been propagated concerning the effects of forces considered in a theoretical view, have at all impeded the due construction of useful machines, such as are impelled by the force of wind or water, by springs, or any other kind of motive power. Machines of this sort, owe their origin and improvement to other sources: it is from long experience of repeated trials, errors, deliberations, corrections, continued throughout the lives of individuals, and by successive generations of them, that the sciences strictly called practical, derive their gradual advancement from feeble and awkward beginnings, to their most perfect state of excellence.

Those however, who are the most conversant in the construction of machines, and to whom the manner of their operation is most familiar, will naturally be inclined to inquire into the reasons or causes of motions thus observed, and to compare them with the rules which are deduced from the general and undisputed principles of mechanics; but it is to be apprehended, that the obstacles which are occasioned by friction, tenacity, the irregular action of the impelling force, and of the resistances, which scarcely admit of precise estimation, must greatly obstruct,
if

if not wholly prevent, a satisfactory and accurate comparison of the theory with practice, in any ordinary way of considering the subject. These difficulties will appear to be greater, from considering, that the motion of machines which are impelled by wind or by water is uniform (setting aside the irregularities of the moving power, &c. which affect not the present argument) from whence it is known that in these machines, when there is no weight which acts against the moving force, the resistance of friction must be equal to the whole moving power, whereas in the theory of mechanics, friction is considered as having no effect.

Mr. Smeaton in his paper on mechanic power (published in the Philosophical Transactions for the year 1776) allows, that the theory usually given, will not correspond with matter of fact, when compared with the motion of machines, and seems to attribute this disagreement rather to some deficiency in the theory, than to the obstacles which have prevented the application of it, to the complicated motion of engines, &c. In order to satisfy himself, concerning the reasons of this disagreement, he constructed a set of experiments, which, from the known abilities and ingenuity of the author, certainly deserve great consideration and
atten-

attention from every one, who is interested in these inquiries : and if any doubt could be entertained concerning the truth of the theory, as deduced from the Newtonian laws of motion, these experiments, among others, might be assumed as standards, to which its examination might be referred : but as it must be confessed, that the evidence upon which the certainty of this theory rests, is scarcely less than mathematical, it will be more eligible to refer mechanical experiments of every kind to the theory, as a means of discovering how far, the unavoidable imperfections of construction and observation, the resistance of friction, &c. may have caused them to deviate from the truth. The ingenious author himself, has afforded us sufficient means for this purpose, by giving the exact weights and dimensions of the parts of his instrument ; the outline of which is represented in figure 88 : this system is caused to turn round a vertical axis, by the action of a weight, placed in the scale *S*, which is suspended from a line that goes ‡ over the pulley *R*, and is wound round the barrel *M* or *N*, the radii of which are as two to one ; when the string is wound round the || barrel *N*, during the descent of the scale 25.25 inches, the system makes 10 revolutions, by which it appears, that the circumference of the barrel

† *Philos.*
Trans. for
 1776.
 p. 466.

|| Page 461.

barrel $N = 25.25$ inches, and consequently the radius $= .401866$, and the radius of the barrel $^P M = .803732$ parts of an inch: let the radius of the barrel M (or of the other when it is used) $= d$: the weight of the whole system is 103.53 oz. avoirdupoise, let this $= w$: the distance of the centre of gyration a from the axis of motion

P If Mr. Smeaton had mentioned the lengths of the barrel M and N , occupied by the line which is wound round them, estimated in a direction parallel to their axis, the exact circumferences and radii of the barrels might have been ascertained from the rule in page 341. But this will not sensibly affect the theoretical conclusions with which the experiments are compared: the length of the barrel M occupied by 5 helical circumferences, if the line $= 25.25$ inches, will not probably exceed $.4$ parts of an inch, from which the true circumference of the barrel M appears to be $= 5.0493$, and its radius $= .803631$, instead of $.803732$, being the radius deduced from assuming 25.25 inches $=$ to 5 circular circumferences of the barrel: the times of descent in the experiments are not af-

fected by this difference in the radius by so much as $\frac{1}{100}$ part of a second, and in the other experiments, which are made on the angular velocities, the error in the times of describing uniformly 20 revolutions being (in a physical sense) evanescent, it is unnecessary to apply any correction on that account.

a The $\dagger$ weights and dimensions of the several parts of the instrument, necessary for the estimation of the centre of gyration, are as follow, the weights being expressed in avoirdupoise ounces, and the distances in inches. K and L are two equal cylinders of lead, perforated by cylindrical cavities, in order to admit the wooden arms to which they are affixed. The weight of each is $= 48$: the external diameter $= 2.57$, the diameter of the internal cylindrical cavity $= .72$, the distance of the centres of the leaden cylinders $= 8.25$ from the axis: the length of the fir arm $TV = 10$; and its diameter being $= .72$, since the specific

$\dagger$ Philos.
Transf. for
1776. p. 461.

tion = 8.03383, which put = r . According to Mr. Smeaton's first † experiment, a weight

† Philof.
Transf. for
1776, p. 463.

fic gravity of fir is about .54, when that of water is 1, the weight of the arm TV will appear to be = 1.272 oz.

The fir axle is an irregular figure, but by taking the parts of it proportionally to the rest, its length will be = about 13, and its diameter at a medium = 1.25; this will give the weight of the axle = 4.986 oz; the weights of the separate parts being added, the weight of the whole system will appear to be = 103.53.

The distance between the centre of gyration, and the axis, is found by multiplying each particle of the system into the square of its distance from the axis, and dividing the sum by the † weight of the whole system: the square root of the quotient, will be the distance required: the sum of these products, must be found separately, for each of the parts. To obtain this sum for one of the leaden cylinders, it will be requisite to know the weight of a cylinder of the same kind of lead, which would exactly fill the cylindrical cavity occupied by the wooden arm: let w represent this weight, R the radius of the external cylinder, r the radius of the internal cavity, W the weight of of the hollow cylinder; then will the weight required

† Sect. VI.
Prop. VIII.

$w = \frac{Wr^2}{R^2 - r^2}$ by the properties of the cylinder, and making

$W = 48$, $r = .36$, $R = 1.285$, it appears that $w = 4.08826$ = the weight of the internal cylinder of lead; if therefore the hollow cylinder, were exactly filled with lead, of equal specific gravity, its weight would be = 52.08826 = $W + w$.

The notation just used remaining, making also the length of the leaden cylinder = l , and the distance between the axis and the centre of gravity of the cylinder = g ; from the rule investigated in p. 223, it is easily inferred that the sum of the products formed by multiplying each particle of the leaden cylinder, weighing 48 oz. into the square

of its distance = $\frac{W \times 12g^2 + l^2 + 3R^2 \times W + w - 3r^2w}{12}$;

in the present case $W = 48$, $w = 4.08826$, $l = 1.56$, $g = 8.25$, which being substituted, the sum of the products required will be = 3298.103, and in both the cylinders, the sum will be 6596.206.

* Sect. VI.
Prop. XV.
Cor. 2.

weight of 8 oz. stretching the string by being placed in the scale, gradually communicates motion to the system, and describes 25.25 inches in $14\frac{1}{4}$ seconds of time. To compare this with the theory: if w be the weight of the system, r the distance of the centre of gyration from the axis, p = the moving force, d the distance from the axis at which it is applied, $l = 193$ inches: the time in which the descending weight describes* the space s from rest

will be $\sqrt{\frac{s d^2 p + s r^2 w}{l p d^2}}$ seconds: now

in this experiment the larger barrel is used, of which the radius $= .803732 = d$, $p = 8$, and $d^2 p = 5.1679$, $r^2 w = 6682.080$, and $s = 25.25$, which quantities being substituted for their respective values, the time

sought $= \sqrt{\frac{5.1679 + 6682.080}{193 \times 5.1679}} \times \sqrt{25.25}$
 $= 13.01$ seconds; by experiment the time was

From the same rule, making $r = 0$, $w = 0$, $W = 1.272$ being the weight of the fir arm TV , $l = 10$ being the length of the arm TV , $R = .36$, and $g = 5$, we shall obtain the sum of the products of each particle in the fir arm TV , multiplied into the square of its distance, from the axis $= 42.45$, and for both arms, the sum will be $= 84.90$. Moreover, for the vertical axis, the sum required $=$ half the weight multiplied into the square of the radius $= .974$, and the sum of the products for the whole system $= 6682.080$; and because the whole weight of the system $= 103.53$, the distance between the centre of

gyration, and axis of motion $= \sqrt{\frac{6682.080}{103.53}} = 8.03383$.

was 14.25, which appears to be too great by 1.24 seconds.

It was also observed in the experiment, that during the descent of the weight 8 oz. through * 25.25 inches, such an angular velocity was generated, as caused the system to perform uniformly 20 revolutions in 29 seconds: by the theory making $c = 3.14159$, the angular velocity generated in the circumstances of the experiment is such as would cause the system to describe † uni-

* Philos.
Transf. for
1776. p. 463.

† Sect. VI.
Prop. XV.
Cor. 1.

formly $\sqrt{\frac{lsp}{c^2 d^2 p + c^2 r^2 w}} = \frac{1}{3.14159} \times$

$$\sqrt{\frac{193 \times 25.25 \times 8}{5.1679 + 6682.080}} = .76858 \text{ parts of}$$

a revolution in a second, and consequently 20 revolutions in 26.02 seconds: in the experiment, the time of describing 20 revolutions was 29 seconds, which is greater than appears from the theory by 2.98 seconds and parts.

By proceeding in the same manner, the time in which the moving forces 8 oz. or 32 oz. describe the different spaces, and the angular velocities acquired in the subsequent experiments, are obtained from the principles of rotation: the annexed table contains the results both by theory and experiment.

Table of Mr. Smeaton's Experiments, compared with the theory.

No.	Ounces avoirdupoise in the scale.	Barrel used <i>M</i> the larger, <i>N</i> the smaller.	Spaces described by the descending weight from rest.	Time of the descent of the weight in the scale by theory.	Time of descent by experiment.	Time too great by experiment.	Time of making 20 revolutions by theory.	Time of making 20 revolutions by experiment.	Time too great by experiment.
1	8	<i>M</i>	25.25	13.01	14.25	+ 1.24	26.02	29	2.98
2	8	<i>N</i>	25.25	26.01	28.25	+ 2.24	26.01	29.25	3.24
3	8	<i>N</i>	6.31	13.00	14.25	+ 1.25	52.02	58.5	6.5
4	32	<i>M</i>	25.25	6.51	7	+ .49	13.03	14	.97
5	32	<i>N</i>	25.25	13.01	14	+ .99	13.01	14.75	1.74
6	32	<i>N</i>	6.31	6.50	7	+ .50	26.02	28.75	2.73
7	8	<i>M</i>	25.25	6.42	7	+ .58	12.84	14.75	1.91
8	8	<i>N</i>	25.25	12.83	14	+ 1.17	12.83	15	2.18
9	8	<i>N</i>	6.31	6.41	7	+ .59	25.66	30.25	4.59

The difference between these results evidently indicates a loss of motion in all the experiments, and some account should be given of the causes of this disagreement. The weight of the scale *S*, which is not taken into the theoretical estimation, nor mentioned by Mr. Smeaton, would manifestly tend in the experiments to lessen the differences, by causing an addition to the moving force; but then it must be considered, that three causes operate in a contrary way, which all together are more than sufficient to counteract its effects: 1. the inertia of the pulley *R*; 2. the friction of the pulley; 3. the friction of the machine. The inertia of the pulley is too small to make any sensible alteration in the experiments, being only $=$ to $\frac{1}{2}$ its weight multiplied into the square of the radius, which must bear a very small proportion to $w r^2 + p d^2 = 6682.080$, to which it should be added: the friction of the machine is not probably very great: when the centre[†] of oscillation is brought nearer to the axis in the proportion of 2 : 1 by means of counting the number of vibrations performed in 60 seconds, if the friction of the axis operated considerably, it is not probable that it should act equally on the pendulum, moving with the different velocities, the weight not being altered, so that

[†] Philof.
Trans.
p. 461.

that if it were not very small, the true distances of the centres of oscillation could not be so nearly in the proportion of 2 : 1, as they appear to be from the dimensions, &c. the greater distance being 8.3052, and the lesser, when the leaden cylinders are brought nearer, being = 4.1750. If it had been mentioned at what angle the axis of motion was inclined to the horizon, when the system performed 92 vibrations in 60 seconds, the effects of friction might have been in some degree estimated; for the length of such a pendulum is 16.66 inches, and the length of Mr. Smeaton's pendulum when the axis is horizontal = 8.3052. Suppose then that the inclination of ^{the} axis to the horizon was 60°; this would have increased the true length in the proportion of 1 : 2, so as to have rendered it = 16.6104 : as it appeared to be nearly of this length by experiment, when, in the author's words, the machine was converted into a kind of pendulum, it is plain, that there could have been little or no loss of motion from friction, had the inclination just mentioned been 60°: if the length had come out greater by experiment, an approximation to the quantity of friction might have been obtained from the data.

It seems probable, however, that the
angle,

angle, at which the axis was inclined to the horizon, when it performed 92 vibrations in 60 seconds, was not much less than 60° ; the friction of the pulleys' axle, and of the line unwinding itself from the barrel, might have been the principal causes, which diminished the effect of the moving force; but as the different results are affected nearly in proportion to their magnitudes, their ratios are scarcely altered from the truth.

When the leaden cylinders were brought nearer to the axis of motion, it was evidently meant, that the whole mass should be virtually at half its former distance, or in other words, if the whole were concentrated into two points successively, so that the rotation should not be altered from that which takes place in the constructed forms, the nearest of these points should be at a distance from the axis half as great as the other: but to effect this, the leaden cylinders should have been moved to such a distance from the axis, that the centre of gyration, and not the centre of oscillation, might be at half its former distance; this will make some little difference in the estimation. When the centres of the leaden cylinders are moved to the distance of 3.92 inches from the axis, the distance of the centre of gyration is $= 3.9596$; but the distance should

should have been $= 4.0169$, being the half of 8.0338 , the original distance.

Although this difference will tend to lessen the effects of friction in exp. 7, 8, and 9, yet the centre of gyration being brought much nearer to the axis, will at the same time operate in the contrary manner; for it is always observed in revolving systems, that the more near the parts are brought to the axis, the greater diminution of motion is occasioned by friction, every thing else being the same.

† *Philos.*
Transf. for
1776. p. 466,
&c.

The inferences deduced by Mr. Smeaton from these † experiments, are a very good illustration of the theory denominated *Conservatio virium vivarum*, which, as far as regards the rotation of bodies on fixed axes, when the moving forces are constant, as well as other particular kinds of motion, has been demonstrated from the Newtonian laws or axioms. As these inferences or principles are deduced by the author from his experiments only, he must certainly be considered to have invented them, although they had been before thought of by others: in order to compare these inferences with the theory, we must follow Mr. Smeaton's example in neglecting the inertia of the weights in the scales as inconsiderable: this being premised, let d denote the radius of the barrel, which is used in the experiment, r the di-

distance of the centre of gyration from the axis, v the velocity generated in that centre, or it may be expressed in the common phrase, the velocity generated in the two bodies K and L ; let $l = 193$ inches, w the weight of the system, p the weight in the scale. In order to prevent misapplication of terms, the definitions premised by Mr. Smeaton, may be here inserted and expressed according to the notation used in the theory.

“By $\ddagger$ impulse, impulsion, or impelling
 “force, is meant, the uniform endeavour
 “that one body exerts upon another, in
 “order to make it move:” By this moving
 force in the experiments, we must understand, the force of the descending weight p by which motion in the system is generated: but this moving power exerts a greater force to communicate motion to the cylinders K and L (considered as concentrated in the centre of gyration) according as the distance at which it acts from the axis, or the radius of the barrel round which the line is wound, is greater, and as the distance of the bodies K and L from the axis becomes smaller; wherefore on the whole, the effect of this power to generate motion in the bodies K and L will be ex-

pressed by $\frac{d}{r}$, which is called by Mr. Smeaton, the intensity with which the power

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for 1776.

$\ddagger$ Page 464.

p acts, so that the impelling power compounded with its intensity will be $\frac{dp}{r}$:

moreover, he defines the quantity of mechanic power by the product under the impelling force p and the space through which it descends or by ps . By cor. 8. prop. XVI.

sect. VI. $v = \sqrt{\frac{4lsp}{w}}$, or as $4l$ is a quantity constantly the same, and the ratios only of the different quantities are inquired into,

$v = \sqrt{\frac{sp}{w}}$: it follows, that $sp = v^2 w$,

3rd & 2^d
Observation,
p. 465.

that is, the mechanic power is as the square of the velocity into the quantity of matter moved: if therefore the quantity of mechanic power sp is unaltered, and w the weight of the system remains the same, the square of the velocity generated will not be altered, and the velocity of course the same, in whatever manner the times of generating this velocity may vary; as appears in Mr. Smeaton's first and second

† Sect. VI.
Prop. XVI.
Cor. 7.

observations. Moreover†, since $v = \frac{2lt p}{w}$

$\times \frac{d}{r}$, or omitting $2l$ as the relative values only are concerned, it appears that $t = \frac{vw}{p} \times \frac{r}{d}$; when v , w and p are the same,

as

as in the first and second experiments and observations||; it follows, that the time of generating the given velocity v will be

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Page 465.

proportional to the quantity $\frac{r}{d}$: but $\frac{d}{r}$ is the direct ratio of the intensity of the given power p , wherefore the time must be in these circumstances in the “ simple “ inverse ratio of the intensity of the im- “ pulsive power;” this is affirmed in the deduction from the second observation †.

† Page 466.

Because $*sp = v^2w$, if sp the mechanic power be decreased in the proportion of 4 : 1, the square of the velocity generated in the heavy bodies must be diminished in the same proportion, w remaining constant, and consequently the velocity v must be diminished in the proportion of 2 : 1; this is the case in the third experiment, as explained in the third observation. It appears also from the numbers 2 and 3 in the sixth column, that the velocity produced is as the time that a given impelling power of the same intensity continues to act upon it; this is expressed by

3d Obser-
vation.

* Sect. VI.
Prop. XVI.
Cor. 8.

the § equation $t = \frac{vw}{p} \times \frac{r}{d}$: in the numbers

§ Sect. VI.
Prop. XVI.
Cor. 7.

2 and 3 in the sixth † column, $\frac{w}{p}$ is the

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for 1776.

† Table of
Experim.
p. 463. or
p. 388. huj.

same, and the intensity $\frac{r}{d}$ being also the

same, the time must be as the velocity acquired.

4th Obser-
vation, and
4th & 5th
Experim.
Sect. VI.
Prop. XVI.
Cor. 8.

Since in general $v = \sqrt{\frac{4 l s p}{w}}$, if the impelling power p be quadrupled, it appears, that v will be doubled when $\frac{s}{w}$ is

given: in the experiment referred to in this observation, and in the 5th experiment, the velocity of the bodies is increased in that proportion, when compared with the velocity generated in the first experiment.

It is said in this observation, that the velocity acquired is simply as the impelling power compounded with the time of its action: "for a quadruple impulsion acting for 7", instead of 14", generates a "double velocity:" but we must here understand the impelling power to mean that power compounded with its intensity, which (the ratio of the time in which it acts being added,) is proportional to the velocity generated. In experiment 1

and 4, the intensity is the same, $\frac{d}{r}$ being given in both cases; therefore in these instances, the velocity generated in the heavy bodies will be simply as the impelling power. But in the 1st and 7th experiments, the impelling power being
the

the same, the intensities measured by $\frac{d}{r}$

are as 1 : 2, and the times of action being as 2 : 1, these proportions compounded give the ratio of equality in the * velocities generated, as they appear in the experiments: if the impelling powers or weights in the scale had been different, the proportion of the two weights must have been compounded with the others.

* P. 395. &
Sect. VI.
Prop. XVI.
Cor. 7.

Since $v = \frac{p}{w} \times \frac{t d}{r}$, $\frac{d}{r}$ being the inten-

Phil. Transf.
for 1776.
5th & 6th
Observa-
tion, and
Experi-
ment, p. 467.

sity of the force p , by doubling $\frac{d}{r}$ and di-

minishing t one half, $\frac{t d}{r}$ is not altered,

and consequently $\frac{p}{w}$ being constant, the

velocity $= \frac{p t d}{w r}$ must be the same as be-

fore. This is what is expressed in the sixth observation †; where it is said, that † Page 468.

“an impulsive power of double the intensity acting for half the time, produces the same effect in generating motion, as an impulsive power of half the intensity acting for the whole time.”

In the 7th observation*, Mr. Smeaton * Page 469.
infers it to be “an universal law of nature, that the mechanic powers to be
“ ex-

“ expended” (measured by the weights and the spaces through which they descend jointly) “ are as the squares of the “ velocities to be generated;” and “ that “ the velocities generated are as the impelling power compounded with or “ multiplied by the time of its action “ and vice versâ.” Here also by the impelling power must be understood, the moving force or weight compounded with its intensity; which quantity (the ratio of the time in which it acts being added) has appeared in the experiments, as well as from the theory, to be, as Mr. Smeaton affirms, proportional to the velocity generated in the revolving bodies.

The axle of Mr. Smeaton's instrument makes no sensible alteration in the time wherein the descending weight describes a given space, or in the angular velocity generated; so that, whether it be taken into account or neglected, will be wholly immaterial; nor will there be a perceptible difference in the result, whether the axle be made of the lightest or heaviest kind of wood. To exemplify this, let the first experiment of Mr. Smeaton's be assumed as an example; and let the axis be neglected: the time of describing* 25.25 inches will be 13.0103 seconds; if the axis be made of fir wood, the time will be 13.0113: lastly, if the axis should be of mahogany, the specific gravity of which is about 1.1, when that of water is = 1, the time will appear to be = 13.0122.

* Table of
Experiments.
p. 388.

Here we observe that it makes a difference of no more than a five hundredth part of a second, whether the axle be wholly neglected, or be taken into account on a supposition that its specific gravity is the same as that of mahogany. Mr. Smeaton does not mention of what substance his axle was formed; he describes the arms to have been fir; in the computation† the axle was supposed to be

of

of that substance, but it appears from this note, that the specific gravity of the axle is wholly immaterial, and that the numbers in the table p. 388. will not at all be altered, whatever be the specific gravity of the wood of which the axle was formed.

The numbers inserted in pages 391 and 392, relating to the distances of the centre of gyration, will be altered by the different specific gravities of the wooden axle, but they will be altered proportionally, so as to remain in a physical sense in the same ratio as is there expressed: the reasoning therefore to which these numbers are applied, will not be effected by any alteration in the specific gravity of the axle.

SECT.

S E C T. X.

R
CONCERNING THE ROTATION OF BODIES
IN FREE SPACE, AND ON THE CENTRE
OF SPONTANEOUS ROTATION.

THE motion of bodies in free space, that is, of bodies at liberty to move freely by the action of any force impressed, being unopposed by a fixed axis or other obstacle, is immediately deduced from the principles of rotation about fixed axes, which have been demonstrated in Sect. VI.

The fixed axis round which a body revolves, is pressed by the impelling force while it generates rotatory motion; but the axis being (by the hypothesis) immoveable, reacts equally against that pressure, and *when it passes through the centre of gravity, the force of pressure urging the axis to motion is such, as, if
un-

* In the propositions throughout this section relating to the motion of bodies in free space, the plane of rotation is supposed to be unaltered in respect of the revolving body, consequently in the examples of bodies revolving on fixed axes, assumed to illustrate the subject, the pressure against the axis must be considered as equal in every part, so that if the axis be unopposed, it may move by the action of the impelling force always parallel to itself.

unopposed, would cause each particle of the system to move with the same velocity, and in the direction in which the force acts. If then the force which presses against a fixed axis, on which a body revolves in given circumstances be ascertained, the motion of the body in free space, when the axis is removed, will be known. For the motion of the body in free space, will consist of the rotatory motion round the axis, passing through the centre of gravity considered as fixed, which is determined by the propositions in Sect. VI. compounded with the motion of the centre of gravity caused by the force now free to impel that centre, the fixed axis which passed through it being removed.

By the centre of gravity in any body or system of bodies, is usually understood to be meant, a point in which the gravitation of each individual particle may be supposed concentrated; because if the gravity of each particle in reality acted on that point, the same motion would be produced in the system, as when the gravity of each particle acts at its respective distance from the centre of gravity, the direction of these forces being always parallel to each other. But the point which answers this description, being determined in any system by the known

geometrical rules, many properties relating to the quiescence as well as the motion of bodies, are demonstrated to belong to it wholly independent of gravity, and which would be equally efficient if gravity existed not. Thus, if each particle of a body without gravity, moves in free space with an uniform velocity, and an immoveable obstacle is opposed to that centre, the whole motion of the system will be destroyed; whereas if the obstacle be opposed to any other point, the system will continue to move although each particle will suffer alteration in the velocity and direction of its motion.

Moreover, if a system revolves in free space round any axis which passes through the centre of gravity, that centre will not be affected by the action of the parts of the system on each other; if therefore the centre of gravity be quiescent when the rotation commences, it will continue quiescent during the rotation of the system round it, nor will it change its place until some external force is impressed upon it.

If by any force acting for a portion of time, either small or great, motion should be communicated to this centre, after the force ceases to impel it, the centre of gravity will move on uniformly in a right line with the velocity acquired,

acquired, having power to alter neither the velocity nor determination of its motion*.

These properties are evidently independent of gravity and other similar powers, which shews that the point usually denominated the centre of gravity, might with equal propriety be called the centre of inertia, when applied to the motion of bodies in free space, especially since gravity or other motive power may be caused to act partially on a system, and it is often necessary to consider a power so acting in theory in order to demonstrate some properties of motion, in which case the centres of gravity and inertia do not coincide, this coincidence hap-

* The principles of centripetal and centrifugal forces, Fig. XC. are demonstrated on a supposition, that the centre of gravity proceeds in a right line until it is compelled to change its direction by the agency of some external force. Let G be the centre of gravity of a system impelled in the direction Gg ; then will G continue to move in the direction Gg , until some external force shall act on it: if any force of this kind always tends towards a centre C , and is sufficient to retain the centre of gravity G in the circular arc GB , the quantity of this centripetal force will be measured by the line Bg or the versed sine of the arc GB , this arc being in its evanescent state: and the same line Bg will measure the force by which G endeavours to recede from the centre, and as soon as the central force ceases to act, will proceed in the direction of a tangent to the arc which it is describing when the central force ceases. conversely from the properties of centripetal and centrifugal forces assumed as true, the rectilinear motion of the centre of gravity might be inferred.

happening when each particle in a system is impelled by gravity or other similar power, which acts in right lines parallel to each other, and which is proportional to the quantity of matter moved, and in other particular cases.

Fig. XCI.

To illustrate these remarks by an example, let A, B, C, D, E, F be small bodies or material points without gravity, situated in the right line AF , and let them be united by some perfectly rigid substance without weight or gravity. If AG

$$\text{be made} = \frac{A \times o + B \times BA + C \times CA, \&c.}{A + B + C, \&c.},$$

then will G be the point denominated the centre of gravity of the bodies $A, B, C, \&c.$ or according to the reasoning just alledged, since these bodies are by the supposition void of gravity, G may be termed the centre of inertia. Moreover, let forces of any kind, represented in quantity by the letters a, b, c, d, e, f , act on the system at the points $a, b, c, \&c.$ respectively in any given direction, for example in a direction perpendicular to AF : if Ag be

$$\text{made} = \frac{a \times Aa + b \times Ab + c \times Ac, \&c.}{a + b + c, \&c.},$$

then will g be the centre of gravity of the system, properly so called, and the forces $a, b, c, \&c.$ will have the same effect to communicate motion to the system, as if the

the

the sum of the forces $a + b + c$, &c. acted altogether in the point g .

Here we observe, that the points G and g , which have been denominated the centres of inertia and gravity, may be distant from each other by any space within the limits of the system, but will coincide when the forces by which each particle of a body is impelled, act in parallel lines, and are proportional to the quantities of matter contained in the particles moved.

Suppose the sum of the forces $a + b + c$, &c. to act at g , and to be represented in quantity by g : concerning the application of these principles, it is also to be remarked, that if the force g , &c. be of the same kind as gravity, the elastic force or pressure of a fluid without inertia, the place of the point G will not be altered by the application of the force g . But if g be a force which possesses inertia, G will begin to move from its place the instant g is applied to act on the system, and will approach nearer to g , the greater proportion the inertia of g bears to that of $A + B + C$, &c. But as taking into consideration the motion of the point G during the action of the force applied at g , would render the ensuing propositions too complex: it will be expedient to consider the inertia of the force applied to

to move a system in free space, as evanescent, as far as regards any motion of the point G which may be occasioned by it.

A distinction between the centres of inertia and gravity was inserted only for the purpose of explaining the subject in question; but there is no necessity to introduce the use of a new term, as the centre of gravity of a system such as $A + B + C$, &c. without weight, will be always understood as determining the position of that point in the system, and not as implying the existence of gravity.

Fig. XCII.

Let SIK represent a plane, into which the matter contained in a system is projected: let G be the centre of gravity through which a fixed axis of motion passes perpendicular to the plane SIK ; let motion be communicated to the system by means of a line $DMEDp$, wound round the circle EDM , of which the centre is G , and stretched by a weight or force of any kind p ; then the axis sustains the weight of the system SIK added to the tension of the string Dp , or if the system be considered as without weight, the pressure against the axis will be the tension of the string Dp only. Now suppose the axis to be at liberty to move in free space (always however being parallel

rallel to its first position) then the weight p , acting by stretching the string Dp , will urge the centre of gravity G by a force equal to the string's tension, while the system revolves round G by the action of the same force. In order therefore to ascertain the motion of the system in free space, it will be necessary to determine the tension of the string Dp while the system revolves round the fixed axis which passes through G .

I.

Let SIK represent a system which is moveable round a fixed axis passing through its centre of gravity G : with the centre G and any distance GD , let a circle DEM be described, and let motion be communicated by a weight or force p , stretching a line which is wound round the circumference of the circle MED ; it is required to assign the tension of the string Dp , while the system is accelerated in its angular motion round G . Fig. XCH.

Let R be the centre of gyration of the system, and let $GR = r$, $GD = d$, the weight or inertia of the system $= w$; then the acceleration of the system will be the same as if the whole mass being removed, that part of it which is

expressed* by the fraction $\frac{wr^2}{d^2}$ were uniformly accumulat-

ed in the circumference MED : moreover, since the force which accelerates the circumference is that part of the acceleration of gravity which is expressed by the fraction

a

$p d^2$

* Sect. VI.
Prop. VIII.
Cor. 2.

† Sect. VI. Prop. XV. $\frac{p d^2}{p d^2 + w r^2}$, † multiplying this into $\frac{w r^2}{d^2}$, we have the

† Sect. I. Prop. IX. tension of the † string $Dp = \frac{p w r^2}{p d^2 + w r^2}$.

Cor. 1. The quantities p and w are in this proposition understood, as usual, to represent both the weight and inertia of the bodies respectively. If it should be preferred to represent the inertia and weight by different quantities, the solution will in no respect be altered: thus, let w be the absolute weight or gravity of the system, q its inertia, p the absolute force which acts on the point D , i its inertia, then will the force which accelerates the circumference of the circle

DEM be $\frac{p d^2}{i d^2 + q r^2}$, which being multiplied into the equivalent mass $\frac{q r^2}{d^2}$ will give the string's tension $= \frac{p q r^2}{i d^2 + q r^2}$.

Cor. 2. It appears that in whatever manner the weight of the system w may vary, the tension of the string will not be altered.

Cor. 3. If the inertia of the force which stretches the string, that is, if i be evanescent, the tension of the string will become $= p$; and this will be the case whatever be the magnitude of the force p .

Cor. 4. When r is infinite, although p , q , i and d remain finite, the tension in this case also becomes $= p$.

Cor. 5. When i the inertia of the moving force is evanescent, the pressure on the axis, which is equal to the string's tension, will be $= p$. The system therefore being without weight, and the axis passing through G being sustained by a line GA * which goes over a fixed pulley, if the line GA be stretched by a weight p , it will exactly counterbalance the pressure on the axis occasioned by the tension of the string Dp as p descends, so that G will remain quiescent.

Fig. XCIII.

II.

Fig. XCIV.

Every thing else remaining, let the axis which passes through the centre of gravity G , be moveable in free space; it is required to determine the point round which

which the solid begins to revolve in free space at the first instant of motion.

Let G be the centre of gravity of the system, being coincident with the centre of the circle DME , round which the line $DMEDp$ is wound; a force or weight p by stretching the line Dp , acts on the system in the direction of a tangent Dp to the circle at the point D , which is perpendicular to GD .

Produce DG indefinitely: The points G , D , &c. in the line OGD must begin to move in the direction of the impelling force; that is, in a direction which is perpendicular to OD : and because the determination of motion once impressed on the centre of gravity, is not altered, except by the impulse of external force acting in some other direction, it follows, that the direction in which the point G proceeds from the very beginning of its motion, will be perpendicular to the line OD . + Page 403.

Draw Gg perpendicular to OD , being equal to the space through which the tension of the string applied to act on the centre of gravity would impel it in any particular of time t : through g draw gd parallel to OD ; make dge equal to the angle described about G , considered as fixed during the time t ; through e and g draw eg intersecting OGD in the point O ; about this point the system will begin to revolve in free space. * Sect. X.
Prop. 1.
Cor. 3.

Make $go = GO$ and join Oo : then since by construction Oo is equal to Gg , Oo is the space through which the force impressed on the centre of gravity carries the point O , in the direction Oo parallel to Gg ; and oO is the space through which the rotation of the system carries the point O in a contrary direction in the same time t ; and as this is applicable every instant during the motion of the point G through the evanescent space Gg , it follows, that while the other points of the system are changing their places, the point O will be quiescent, or in other words, the system will begin to move round the point O , as determined by this construction.

III.

The construction remaining, if gravity be supposed to act on the system, and a fixed horizontal axis be caused to Fig. XCIV.

pass through the point O perpendicular to the plane SIK , D will be the centre of oscillation.

¶ Sect. VI. When the system is impelled by the force p in free
Prop. VIII. space, let t be a very small portion of time in which
motion is produced in the system: let $\mathcal{Q}$ be the principal
centre of gyration, that is, the centre of gyration when
the system revolves in the plane SIK , round an axis which
passes through the ||centre of gravity; and let $G\mathcal{Q} = r$, w
= the weight of the system, $l = 193$ inches, $GD = d$:
then supposing a fixed axis to pass through G , the space
through* which the weight p descends in the time t or the
* Sect. VI. evanescent arc de will = $\frac{p l t^2 d^2}{d^2 p + w r^2}$, or p being very small,
Prop. XV. that is, $p d^2$ being inconsiderable when compared with $w r^2$,
Cor. 2. $de = \frac{p l t^2 d^2}{w r^2}$: Secondly, When the centre of gravity G is

at liberty to move freely in the direction of the impelling
force p , being unopposed by a fixed axis, the space Gg
= Dd described in the † time t will be = $\frac{p t^2 l}{w}$; where-
† Sect. III.
Prop. IV.
Cor. 5.

fore $De = \frac{r^2 + d^2 \times p t^2 l}{w r^2}$: and because of the similar tri-
angles DOe , GgO , we have De , $Gg :: DO : GO$, or as DO
: $GO :: \frac{r^2 + d^2 \times p t^2 l}{w r^2} : \frac{p l t^2}{w} :: r^2 + d^2 : r^2$; wherefore
 $DO - GO : GO :: d^2 : r^2$, that is, since $d = GD = DO -$
 GO , and $r = G\mathcal{Q}$, we have $DG^2 : G\mathcal{Q}^2 :: DG : GO$, and
 $GO = \frac{G\mathcal{Q}^2}{DG}$; but $G\mathcal{Q}$ being the distance of the principal

† Sect. VI. centre of gyration from the † centre of gravity, whenever
Prop. VIII. $GD \times GO = G\mathcal{Q}^2$, if D be made the centre of suspension,
Cor. 3. O will be the centre of oscillation, and vice versa: from
which it follows in the present case, that O is the centre
of oscillation of the system, when it vibrates freely round
an horizontal axis passing through D perpendicular to the
plane SIK .

The demonstration here given, might be applied to
the rotation of bodies in free space, which is effected by
percussion, since it has been shewn, that motion commu-
nicated

nicated in this manner, is the effect of † continual acceleration: but for the sake of explaining the proposition more distinctly, the following demonstration may be subjoined, a few considerations relating to the subject being premised.

Let $ABCD$, &c. be a system of bodies consisting of particles or material inert points without gravity, being connected together by some perfectly rigid substance without weight or inertia; and suppose these points to be † projected into the plane $AODBK$ by right lines perpendicular to it: let the system be moveable round a fixed axis passing through O , and perpendicular to the plane $AODBK$. Through O and the centre of gravity G draw the line OGE , and let a body impinge on the system in the direction LE perpendicular to OGE , and in the plane $AODBK$. This impulse will urge the system round the axis which passes through O , each body beginning to describe a circular arc of which the centre is the point O , and the radius the distance of the body from that point respectively. But each body by its inertia will resist the communication of motion in a direction contrary to that in which it is impelled; thus the body A beginning to move in the direction Aa , which is perpendicular to OA , will resist the communication of motion in the direction aA : in the same manner the point B will resist the communication of motion in the direction bB perpendicular to BO , and so on of the rest. It is plain, that the effect of the impact at E on the point O will depend on the resistance of inertia which is opposed by the particles of the system to the communication of motion. If, for example, the greater part of this resistance acts between E and a , then the impulse at E will be applied between the point O and resistance of inertia, which will serve as a fulcrum for the lever to act against O ; the point O therefore will be propelled by the impact in the direction of the stroke: but if the resistance of inertia acts chiefly between O and E , the impulses at E and O will be on opposite sides of the resistance of inertia, which serving as a fulcrum for the lever to act against, will cause the point O to be impelled by the impact in a direction contrary to that of the stroke, and consequently if the inertia is disposed so as to act on the system to impel the point O equally in opposite directions, the axis passing through O will be not be urged either in one direction or the other, while the system begins to revolve round it. In this case therefore the fixed axis will not at all affect the motion of the system.

† Page 8. &
Sect. IX.
P. 377.

† Sect. VI.
Prop. I.
Cor. 5.

tem, which when struck in the point E will begin to revolve spontaneously about the point O in free space, when the fixed axis passing through it is removed. The point O answering to this description, is called the centre of spontaneous rotation*.

† Bernoulli,
Vol. IV.
p. 265.

IV.

Fig. XCV. Let E be the point in the system $ABCD$, which is struck by a body impinging in the direction LE perpendicular to OGE , and let O be the centre of spontaneous rotation; then if O is the point round which the system revolves on a fixed axis perpendicular to the plane $ADGB$, E will be the centre of percussion.

Through the points A, B, C and D draw Aa, Bb, Cc and Dd perpendicular to AO, BO, CO and DO respectively; and AK, BH, CI and DF perpendicular to OGE , and let O be any point round which the system is moveable by the force of the impact at E : at the instant of the impact, the body A will be impelled in the direction Aa , and consequently will oppose a resistance of inertia in the opposite direction aA . The force of the impact at E to impel the point O will depend on the inertia of each particle, that of A for example, acting on it by the lever, of which the arms are Ea, EO , the force of the impact at E being the fulcrum: and although the fulcrum at E be not immovable, and will oppose different degrees of force, according to the different velocities and quantities of matter in the impinging body, yet in a single impact, the force at the fulcrum E will be the same in respect of all the particles A, B, C and D ; and therefore the relative values of the forces by which the inertia of the different particles A, B, C and D , &c. impels the point O will be obtained from considering E as an immovable fulcrum.

This being considered, the inertia of the particle A which it opposes to the communication of motion at E is

$A \times$

$\frac{A \times AO^2}{OE^2}$; and consequently equal to the force of inertia * Sect. VI. Prop. II.

$\frac{A \times OA}{OE}$ acting at A : this inertia acting on the line Oa

in the point a , and in the direction aA may be represented by that line Aa in quantity and direction; but this force acting obliquely must be resolved into two KA , perpendicular to Oa and Ka coincident with it, of which KA only is efficient to oppose the angular motion of the system: this will be to the whole inertia as AK to Aa , or because the triangles AKa and AOK are similar, as OK to OA ; we have therefore the inertia opposed at a to the communication of angular motion by the particle $A =$

$\frac{A \times OA}{OE} \times \frac{OK}{OA} = \frac{A \times OK}{OE}$, acting at a in a direction

perpendicular to Oa . Now the impact being made at E , the inertia opposed by A at the point a , in a direction perpendicular to Oa , acting against E as a fulcrum, will impel the point O in a direction contrary to that of the stroke; but the effect of any force applied at a to urge

O round the point E will be that force $\times \frac{Ea}{OE}$, that is, the

force of A 's inertia which impels O will be $\frac{A \times OK \times Ea}{OE^2}$:

in the same manner, the force to impel O arising from the inertia of C will be $\frac{A \times OI \times Ec}{OE^2}$ acting in the same

direction; the forces which arise from the inertia of B and D will impel O in a contrary direction, and will be expressed

by $\frac{B \times OH \times Eb}{OE^2}$ and $\frac{D \times OF \times Ed}{OE^2}$ respectively: but be-

cause O is the centre of spontaneous rotation by the proposition, it is not affected by the impulses communicated to the system, being quiescent while the other parts of the system begin to move with velocities which are proportional to their distances from it. The forces therefore which act on O in contrary directions must be equal,

which will give $\frac{A \times OK \times Ea + C \times OI \times Ec}{OE^2} =$

$\frac{B \times OH \times Eb + D \times OF \times Ed}{OE^2}$: or because $Ea = Oa -$

OE

$$\begin{aligned}
 OE &= \frac{OA^2}{OK} - OE, \quad Ec = Oc - OE = \frac{OC^2}{OI} - OE, \quad Eb \\
 &= OE - Ob = OE - \frac{OB^2}{OH}, \quad \text{and } Ed = OE - Od = OE \\
 &- \frac{OD^2}{OF}, \text{ by substituting these values and multiplying by } OE^2, \text{ we have } \\
 A \times OA^2 - A \times OK \times OE + C \times OC^2 - C \times OI \times OE &= B \times OH \times OE - B \times BO^2 + D \times OF \times OE - D \times OD^2, \text{ from which we obtain } OE = \\
 \frac{A \times OA^2 + B \times OB^2 + C \times OC^2 + D \times OD^2}{A \times OK + B \times OH + C \times OI + D \times OF}, \text{ or } OE &= \\
 = \frac{A \times OA^2 + B \times OB^2 + C \times OC^2 + D \times OD^2}{A + B + C + D \times OG}, \text{ which}
 \end{aligned}$$

is the distance of the centre of percussion from the axis of motion, *when that axis passes through O.

* Sect. VI.
Prop. VII.

Cor. 1. The same demonstration precisely is applied to any indefinite number of material points which constitute natural bodies.

Cor. 2. If W be the mass contained in the system, and R be the centre of gyration when it revolves round O ; a given body impinging perpendicularly against E , the system existing in free space, will generate the same motion in that point, as if the mass $\frac{W \times OR^2}{OE^2}$ were concentrated

in E , the matter of the system being removed: for the same reason, if the entire system moves in the direction EL , which is always perpendicular to OE , and an immoveable obstacle at E be struck by it, the force of the impact will be the same as if the obstacle were struck by a

† Sect. VI.
Prop. VIII.

body = $\frac{W \times OR^2}{OE^2} = \frac{W \times OG}{OE}$ † concentrated into the point which strikes on the obstacle, with a velocity equal to that with which the system moved.

Fig. XCVI.

Cor. 3. Let SIK be a system existing in free space, G being the centre of gravity, E the point on which a body impinges in the direction LE perpendicular to GE , and let O be the centre of spontaneous rotation; if the impact be made on the point F or any other point in the line FE instead of on E , the centre of spontaneous rotation O will not be altered if the direction of the impact remains the same; because it is indifferent as to the particles which compose the system, whether the impulse is communicated

communicated to them from one point of the given line EF , or from any other: moreover, when the system revolves about an axis which passes through O , any point in the line EF will be a centre of percussion.

Cor. 4. Since when an impulse of any magnitude is impressed on E , the centre of spontaneous rotation O is not affected by it, it follows, that if during the motion of the system in the direction HI parallel to EF , the point E should impinge against an immoveable obstacle, the motion of the point O will not be affected by it: the point O will in this case proceed after the impact with a velocity equal to that with which it before moved.

Cor. 5. If any point F in the line EF impinges in the direction EL against an immoveable obstacle, the motion of the point O will not be affected by it, provided the surface of the impinging body at F be perpendicular to the direction in which the point F strikes the obstacle, and both be so smooth and hard that there shall be no adhesion or friction between them: for the point O endeavouring to proceed with a velocity equal to that which it had before the impact, when the point E is stopped will cause the system to revolve round E for a small particle of time, and consequently the surface of the body at F must slide over the fixed obstacle for a small space: it is only on this supposition, that the force of the impulse on the obstacle at F is the same as that which it would receive if an equivalent mass equal to that of the system

multiplied into the fraction $\frac{OG}{OE}$ impinged on it with the same velocity. If the surfaces at F adhere, it is manifest that the point O instead of proceeding in a direction parallel to EF , will be deflected into a direction perpendicular to OF , and will consequently lose part of its motion estimated in the direction EF or HI .

Cor. 6. Since when a system is struck in free space, the point denominated the centre of spontaneous rotation is quiescent while the other parts of the system begin to revolve, it is plain that the absolute velocity of the point struck, that of the centre of gravity, and the angular velocity generated will be the same as if a fixed axis passed through the centre of spontaneous rotation, we may therefore refer to what has been before demonstrated in sect. vi. in order to determine these quantities.

Postulate for the ensuing propositions. It is required that in the bodies or system of bodies, which revolve round an axis perpendicular to any constant plane of motion,

tion, the sum of the products which are formed by multiplying each particle into the square of its distance from the axis passing through the centre of gravity may be determined. If the figure be regular, this may be effected by geometrical rules, if irregular, it may be obtained from experiment by the rules given in page 227.

V.

Fig. XCVI. Let SIK represent a body existing in free space, and let a body impinge on it in any given point F , and in any direction LF : having the mass contained in the body struck, the velocity of the impact, the perpendicular distance of its direction GE from the centre of gravity, and the quantity of matter in the striking body; it is required to assign the velocity of the centre of gravity, and the angular velocity generated by the impact, supposing the system to be moveable in the plane SIK only.

Let G be the centre of gravity of the system; produce LF indefinitely in the direction LFM , and through G draw OGK perpendicular to LM . Suppose the system to revolve in the plane SIK , on a fixed axis which passes through E , and let O be the centre of percussion; this point O will also be the centre of spontaneous rotation, about which the system begins to revolve in free space by the force * of the impact at F . The point O being therefore immovable at the first instant of the system's revolution, the angular velocity generated and the initial velocity of the centre of gravity will be the same as when a fixed axis passes through O . Let W be the quantity of matter in the system, A that of the striking body, and V the velocity of impact: also let R be the centre of gyration, when the system revolves round O ; then the velocity generated in E by a given impact at E , will be

* Sect. X.
Prop. IV.
Cor. 3.

be the same as if the whole mass of the system were re-

moved, and the sequivalent mass $\frac{W \times OR^2}{OE^2}$ were concen- † Sect. VI.
Prop. VIII.
Cor. 2. & 4.

trated in E , and the bodies being nonelastic, the laws of collision give us the following proportion, as $A + \frac{W \times OR^2}{OE^2} : A :: V$ to the velocity communicated, if A

impinged on the point E ; but here as F is at a greater distance from the centre of motion O than E , the inertia of A will be greater in a duplicate † proportion, and will † Sect. VI.
Prop. II.

therefore be $= \frac{A \times FO^2}{OE^2}$, which mass being concentrated with the rest at E , will be equivalent to the inertia of A ; this being the case, the laws of collision will give the proportion: as $\frac{A \times FO^2 + W \times OR^2}{OE^2} : A :: V$ to the

velocity communicated to E , which will be $= \frac{AV \times OE^2}{W \times OR^2 + A \times OF^2}$: and consequently this being diminished in the proportion of OE to OG will become

$\frac{AV \times OE \times OG}{W \times OR^2 + A \times OF^2} =$ the initial velocity of the centre

of gravity. Moreover, for the angular velocity, making $3.14159 = C$, we shall have the circumference of the circle which would be uniformly described by the point E round O , if O were fixed, $= 2C \times OE$; which will

give this proportion: as $\frac{VA \times OE^2}{W \times OR^2 + A \times OF^2} : 2C \times OE$, so is 1 second to the time of one revolution in seconds $= \frac{2C \times W \times OR^2 + A \times OF^2}{VA \times OE}$, (1 second being the standard

time to which the velocities are referred;) and consequently the number of revolutions or parts of a revolution is a second, or the angular velocity round the point O required $=$

$\frac{AV \times OE}{2C \times W \times OR^2 + A \times OF^2}$.

Cor. 1. If the inertia of the striking body be evanescent, the velocity communicated to the point E will

* Sect. VI.
Prop. VIII. become $= \frac{AV \times OE^2}{W \times OR^2}$, or because * $OR^2 = OE \times OG$, the
Cor. 1.

initial velocity of E will $= \frac{VA \times OE}{W \times OG}$.

Cor. 2. For the same reason the velocity of the centre of gravity will become $= \frac{VA \times OE \times OG}{W \times OR^2} = \frac{VA \times OE \times OG}{W \times OE \times OG} = \frac{VA}{W}$, being the same as would be generated in the centre of gravity G if the body A impinged directly on it with the velocity V .

Cor. 3. The inertia of P being evanescent as in the last corollaries, we have also the angular velocity generated in the system $= \frac{AV}{2CW \times OG}$.

Cor. 4. Let Q be the centre of gyration of the system when it revolves round the centre of gravity; then $GQ^2 = GO \times GE$ †; wherefore the angular velocity generated is equal to $\frac{VA \times GE}{2CW \times GQ^2}$. From this and the last cor.

it follows, that every thing else being the same, the angular velocity generated will be directly as the distance of the impact from the centre of gravity GE , and inversely as OG , GQ being a constant quantity in a given system, revolving in the same plane.

VI.

The conditions of the last proposition remaining, let it be required to assign the motion of the centre of gravity in free space, and the angular velocity of the system round it.

Fig. XCVII. O being the centre of spontaneous rotation, each particle in the system after the impulse begins to move in a direction perpendicular to the line which joins the particle and the point O respectively. If a fixed axis passed through that point, each particle would continue to describe

scribe uniformly with the velocity acquired a circular arc, of which the radius is the distance between the moving particle and the axis of motion respectively. But the point G not being confined by the action of a fixed axis, will go on in the direction of its first $\dagger$ impulse perpendicular to OE , having described the right line Gg with the

velocity $\frac{AV \times OE \times OG}{W \times OR^2 + A \times OF^2}$ *, and if no rotatory motion ^{* Sect. X. Prop. V.}

were communicated to the system, the line OGE would have moved into the position ogE parallel to OGE ; but the angular motion described about O during the same time is $=$ the angle $EOG = Ogo$ the cotemporary angular velocity round the centre of gravity. The motion therefore of the system in free space will be compounded of the uniform rectilinear velocity generated in the centre of gravity, in the direction Gg perpendicular to OE , and the angular motion generated round the centre of gravity, which is equal to that round the point O , or $=$

$$\frac{VA \times OE}{2C \times W \times OR^2 + A \times OF^2}.$$

Cor. 1. When the inertia of A is very small, the angular velocity round the centre of gravity will become

$$\frac{VA \times OE}{2C \times W \times OR^2}, \text{ or because } \dagger OR^2 = OG \times OE, \text{ the an-} \begin{matrix} \dagger \text{ Sect. VI.} \\ \text{Prop. VIII.} \end{matrix}$$

$$\text{gular velocity} = \frac{VA \times OE}{2C \times W \times OG \times OE} = \frac{VA}{2C \times OG \times W}$$

$$= \frac{VA \times GE}{2CW \times GQ^2}, \text{ the same as in prop. v. cor. 3. \& 4.}$$

Cor. 2. If a fixed axis passes through the centre of gravity, and a body A impinges on the system in the point F in a direction which is perpendicular to GE , Q being the principal centre of gyration, the resistance

of inertia opposed at E || will be the same as if $\frac{W \times GQ^2}{GE^2}$ || ^{Sect. VI. Prop. VIII. & Sect. X. Prop. IV. Cor. 3.} were accumulated in the point E , the other matter being removed: this will give the following proportion, as

$$\frac{A \times GF^2 + W \times GQ^2}{GE^2} : A :: V : \frac{VA \times GE^2}{W \times GQ^2 + A \times GF^2} \text{ the } \S \text{ Page 417.}$$

velocity of the point E after the impact; when A is

very small, the velocity will become $= \frac{VA \times GE^2}{W \times GQ^2}$, and

the angular velocity round the centre of gravity $= \frac{VA}{3G2}$

† Sect. X.
Prop. V.
Cor. 4.

$\frac{VA \times GE}{2CW \times GQ^2}$, † the same as when the system moved in free space unconfined by any axis, from which we have this general conclusion, that when a body is struck in the manner described in prop. v. by a body of very small inertia, the angular velocity in free space will be the same as that which is generated round a fixed axis which passes through the centre of gravity; and the motion of the centre of gravity in free space is the same as would be generated in that centre if the striking body impinged directly on it, every thing else being the same.

Cor. 3. The velocity with which the point *O* revolves round the centre of gravity considered as fixed, is equal to the velocity of the centre of gravity in free space: for the angular velocity of the system revolving round the

* Cor. 2. centre of gravity* is $\frac{VA}{2CW \times OG}$, which being multiplied into $2C \times OG$ will be $\frac{VA}{W} =$ the velocity of the

† Sect. X.
Prop. V.
Cor. 2.

point *O*, being the † same as the velocity of the centre of gravity moving in free space.

Cor. 4. Since the centre *G* moves with a velocity equal to that of the point *O* round *G* considered as a fixed centre, it is plain that the line described by the point *O*, during the motion of the system, must be the common cycloid.

VII.

Fig. XCVI.

Let *G* represent the centre of gravity of a body *SIK* moving in the direction *HGI* with the velocity *V*, each particle also moving with the same velocity *V*; and let a fixed obstacle be opposed to any point *E* of the line *GK*, which is perpendicular to *HG*; it is required to assign the motion of the system after the impact.

The system *SIK* is considered to be moveable in the plane *SGIK* only: this being premised, if *E* be made

made the centre of suspension, O will be the centre of percussion corresponding, O will also be the centre of spontaneous * rotation: when therefore the system impinges against the fixed obstacle at E , that point of the system E will be quiescent, and the point O not being affected by the impact, will go on with the velocity V , and consequently the centre of gravity will proceed with the velocity $\frac{V \times EG}{OE}$; and if $C = 3.14159$, &c. the angular velocity of the system round E , at the instant of the impact will be $= \frac{V}{2C \times OE}$.

If any other point of the line EF parallel to GI impinges against the immovable obstacle, the effect will be the same, provided the surface of the body at the point of impact be perpendicular to the direction in which it strikes the obstacle, and there be no friction or adhesion between the obstacle and striking body.

If the direction of the impact against the obstacle be oblique to the surface of the impinging body at the point of impact, it must be resolved into two, of which one is perpendicular and the other parallel to the surface; the latter of these will have no effect in altering the motion of the body struck, as the surfaces are by the supposition perfectly smooth and hard; the perpendicular force will act precisely as in the case of the direct impact.

When an irregular body LMN impinges on an irregular body SIK , several circumstances relating to the impact are to be attended to in order to ascertain the motion of each body after the impact: let the direction of the impact be QFS ; and let the velocity of the impact be represented by QF ; resolve this into two, FV perpendicular to a tangent to the curve at F , and VQ parallel to that tangent. 1. At the instant of the impact, the body LMN may adhere firmly to the body SKI , so that the whole must move altogether as one substance: this is the case when a bullet of any kind penetrates the substance of an obstacle, and after the impact becomes so united with it as to entirely partake of its motion. 2dly. The bodies may not be so united by the impact as to become one substance, but the surface of the striking body may nevertheless so adhere to the body struck, that not only the perpendicular force FV shall be efficient to propel it, but the lateral force also VQ shall contribute to generate rotatory motion round the centre of gravity, and rectilinear

* Sect. X.
Prop. IV.
Cor. 4.

Fig.
XCVIII.

near

near motion in that centre. 3dly. When the surfaces are hard and smooth the lateral force QV will have no effect on the motion of the body struck, which will in this case be impelled by the perpendicular force VF only.

VIII.

Fig.
XCVIII.

Let a body LMN impinge on the body MIH at rest, the point of impact being at F , and the direction of the impact QS : it is required from the necessary conditions to ascertain the motion of the two bodies after the impact.

Let G be the mass of the striking body, I the mass of the body struck: let QF represent the velocity of the impact in quantity and direction: this being oblique to the surfaces at F must be resolved into two FV , QV , the former perpendicular and the latter parallel to the curves at F : the surfaces of the bodies being perfectly hard and smooth, the force VF only will be effectual in impelling the body struck. Produce VF indefinitely: and through the centres of gravity G and I draw OGE , KIH perpendicular to VFH : and suppose the body G to vibrate round an axis which passes through E , let O be the centre of percussion; and when the body I revolves round H , let K be the centre of percussion. It is here to be observed, that because the surfaces at F are not supposed to adhere, the motion of the body I will not be impeded by the inertia of the striking body: the force VF impelling the body I in the direction FH , will cause it to begin its motion round the centre of spontaneous rotation K , while F describes an evanescent arc of which the radius is KF ; but the striking body by its inertia will endeavour to proceed in the direction FH until the bodies have a common velocity in that direction; this will manifestly cause the surfaces to slide over each other near F through a small space, but the time in which the impact is performed being in a physical sense an instant, the force will during the whole of the time be applied to move the point H : this being premised, let the perpendicular velocity of impact VF be represented by V : through G and I draw CGL , MIX parallel to VFH ; then will the point I describe the line
IX

IX after the impact, and G will describe the line GL , (the lateral motion QV being considered afterwards:) the resistance of inertia opposed by the body I , is the same as if the whole mass were removed, and the equivalent

mass $\frac{I \times KI}{KH}$ were *concentrated in H ; and the system G

which impinges on I will strike that body with the same force as if the whole mass in G were removed and the

equivalent† mass $\frac{G \times OG}{OE}$ were concentrated in E : where-

fore the laws of collision will give this proportion, as $\frac{I \times KI}{HK}$

+ $\frac{G \times OG}{EO} : \frac{G \times GO}{OE} :: V$: to the velocity communicat-

ed to H , which being diminished in the proportion of HK to KI will be the velocity of the centre of gravity in the direction IX : moreover, the velocity of the centre of gravity and its distance from the centre of spontaneous rotation will give the angular velocity of the system about K or I . And because O is the centre of spontaneous rotation of the body G , that centre †will not be at all affected by the impact, and the velocity of the point F in the striking body estimated in the direction of the perpendicular FV will be the same with that which is communicated to H : let this latter velocity be expounded by the line EU when OT represents the original perpendicular velocity of impact, and join TUW , this will give the perpendicular velocity of the centre of gravity $= Gg$, and the angle described by the body G in its rotation during the same time $= TWO$ being the measure of the angular velocity of the striking body after the impact. The velocity of the centre of gravity Gg in the direction GL , was by construction to the entire perpendicular velocity of impact as Gg to TO ; we have therefore the perpendicular velocity of G after the impact in the direction $GL = V \times \frac{Gg}{TO}$, which being compounded with the lateral motion of G denoted by QV , will give the true velocity and direction of the centre G after the impact.

¶ Sect. X.
Prop. VI.

* Sect. VI.
Prop. VIII.
& Sect. X.
Prop. IV.
Cor. 6.

† Sect. X.
Prop. IV.
Cor. 2.

† Sect. X.
Prop. IV.

IX.

Fig. XCVI. In any system SIK to which when quiescent, motion has been communicated by the impulse of a force without inertia; that is, rectilinear motion to the centre of gravity G measured by the space D , which the point G would describe uniformly in any given time, and rotatory or angular motion measured by the number M revolutions, which it would describe uniformly round G in the same given time: it is required to ascertain at what perpendicular distance GE from the centre of gravity, the direction of the impelling force LE must have passed, so as to have generated the rectilinear and rotatory motions described in the proposition.

Let LE be the direction of the impulse, and through G the centre of gravity of the system, draw OGE perpendicular to LE , and let O be the centre of spontaneous rotation of the system when impelled at E or F . Let $\mathcal{Q}$ be the principal centre of gyration of the system when it revolves round the centre of gravity. Moreover, let A be a body which impinges on F with a velocity V , W being the weight of the system: then the velocity communicat-

† Sect. X. Prop. V. Cor. 2. ed to the centre of gravity = $\frac{VA}{W}$, and the angular velocity generated in the system round G is = $\frac{VA}{2CW \times OG}$,

† Sect. X. Prop. III. Cor. 3. or $\frac{VA \times GE}{2CW \times G\mathcal{Q}^2}$ †, C being = 3.14159, &c. but by the problem, the velocity communicated to the centre of gravity is D , and the angular motion is M ; that is, the num-

number of revolutions or parts of a revolution described while the centre of gravity describes the space $D = M$: we have therefore from the conditions, this equation: $\frac{D \times GE}{2C \times GQ^2} = M$: and the distance sought or $GE = \frac{2CM \times GQ^2}{D}$.

The earth revolves about an axis passing through its centre of gravity, while that centre is carried on in free space describing an orbit nearly circular. This double motion has been discovered in several of the planets, and by analogy may be supposed to exist in the others, although their distances from us, unvaried surfaces, or different causes have prevented any satisfactory observation of their revolutions round their axes.

Most of the other phenomena in the planets' motions are acknowledged to be intimately related, and dependant on each other, and to be regulated by certain definite laws, from which they deviate not in the smallest degree; yet the rotation of the planets round their axes, as far as they have been observed †, seem to follow no rule or order whatever, either in respect of their distances from the sun, their periodic times, quantities of matter, or any other circumstance of appearance or motion. This is no more, however, than saying that no law has been hitherto discovered to which these phenomena can be referred; but to argue from hence, that no law exists in nature by which the rotations of the planets round their axes were primarily adjusted, would be to acknowledge, that this part of the solar system is exempt from that harmony and order, which are so conspicuous in the other appearances relating to the planets' motions. The proposition above demonstrated immediately applies to the subject in question, at first sight suggesting the probability of discovering some proportion or relation between the rotation of the planets and their other motions, their distances from the sun, &c. and although little at present is collected from the examination, yet when the revolutions of the planets round their axes are more generally and more exactly known, and other phenomena combined with them, future experience may perceive, that the times of rotation could have been no other than what they are, according to the known laws of motion.

To apply the proposition above demonstrated to the motion of the earth. Let $SIKG$ represent a section of the earth passing through its centre of gravity G , which
 3 H considering

† J. Bernoulli,
Vol. IV.
p. 280.

Fig. XCIX.

† Sect. X.
Prop. VI.

* Sect. X.
Prop. V.
Cor. 2 & 3.

† Sect. II.
Prop. VII.

considering the earth as spherical and homogeneous will be the centre of the sphere. Suppose the earth to have been originally quiescent, and that an impulse was impressed on it in the direction CGA ; if the direction of this impulse passed through the centre of gravity G , no rotation round an axis would have been the consequence; if the impulse passed in the same direction, but not through the centre of gravity, a motion of rotation † round the centre G would be generated while the centre of gravity proceeded in the right line GA parallel to the direction of the impelling force. It is manifest, that the attraction of the sun which causes the centre G to deviate from the right line GA , and to describe an arc GH (that may be here assumed as circular without error) will alter neither the velocity of the centre G nor the motion of rotation. Through G draw $OGEB$ perpendicular to LE . Having given the magnitude of an impulse relatively to the earth's mass, impressed in the direction LE , and the distance GE on the earth's radius from the centre of gravity at which it was applied, the absolute velocity of the centre of gravity G , and the angular motion round G would be inferred*; and conversely according to the conditions of the proposition above demonstrated, if the angular motion and the absolute velocity of the centre of gravity be given, the distance GE at which the impulse was applied from the earth's centre will be known: to apply this, let GQ be the distance of the principal centre of gyration from the earth's centre: let the earth's radius be $= 1$, being assumed as a standard of space; and let the earth's periodic time round its axis, or $23^h. 56'. 4''$. be $= 1$, a standard † of time, being a sidereal day; in order to express the velocity of the earth's centre of gravity, we must determine the space described by it in one sidereal day; and since the sun's parallax is about $8\frac{1}{2}''$, the circumference of the earth's orbit will contain 152470 of the earth's radii, and because this space is described by the earth's centre in 365.24 mean days or 366.24 sidereal days, the space described by the earth's centre in one sidereal day will be $= \frac{152470}{366.24} = 416.31$ of the earth's radii $= D$ by the proposition, which will be the earth's velocity in its orbit referred to the time of one sidereal day. We have therefore in the equation $GE = \frac{2CM \times GQ^2}{D}$, $C = 3.14159$, $M = 1$ being the angular velocity of the earth round its axis,

axis, or the number of revolutions described while the centre G passes through the space D : moreover GQ^2 being the square of the distance of the principal centre of gyration from the centre of gravity $= \frac{2}{5} \times \text{square of the radius}$, * Sect. VI, Prop. XII.

radius, or since the radius $= 1$, $GQ^2 = \frac{2}{5}$, and $D = \frac{2}{5}$, p. 225.

416.31, which being substituted, we shall have $GE = \frac{2 \times 3.14159 \times \frac{2}{5}}{416.31} = \frac{1}{165.6}$. From hence it appears, that

if an impulse be impressed on a quiescent sphere, and the direction of the force should be at a perpendicular distance

from the centre of gravity of $\frac{1}{165.6}$ part of the radius:

the angular motion of the sphere, and the absolute motion of the centre, will be proportional to those of the earth. From this determination, the earth's centre of spontaneous rotation is obtained: for making O that centre, we

have $\dagger GO = \frac{GQ^2}{GE} = \frac{2}{5} \times 165.6 = 66.25$, which is some- † Sect. VI, Prop. VIII.

what more than the moon's greatest distance from the earth's centre.

Bernoulli makes $\dagger$ the distance $GE = \frac{1}{150}$ and $GO = 60$, † J. Bernoulli, Vol. IV. p. 283.
from assuming the sun's parallax $9''.375$, which is now known not to exceed $8\frac{1}{2}''$.

In the same manner the distances GE and GO may be ascertained for the other planets, their times of revolution round their axes and the absolute velocities of their centres of gravity being known.

X.

GIF represents a system existing in free Fig. C.
space: suppose that a body A impinges on the point I in the direction HI with the velocity V , and that at the same instant another body B impinges on the point F in any other direction LF with
3 H 2 the

the velocity U ; let it be required to determine the rectilinear motion of the centre of gravity G after the impact, and the angular velocity generated in the system, supposing that the inertia of the striking bodies is inconsiderable.

Through the centre of gravity G draw GC parallel to HI , and GD parallel to LF ; make GC to GD in the proportion compounded of the ratio of $A : B$, and of $V : U$; compleat the parallelogram, and draw the diagonal GK : then will GK be the direction in which the centre of gravity G moves after the impact, its velocity being such as will carry it uniformly over the space GK in the same time that GC would be described by it, if the body A only impinged on the point I in the direction HI with the velocity V . Produce HI and LF indefinitely, and through G draw GP perpendicular to HIP , and GE perpendicular to LFE . Let $\mathcal{Q}$ be the principal centre of gyration, W the mass of the body: then referring the velocities to any standard of time 1, the angular velocity generated by the impact of the body A , will be that which

* Sect. X.
Prop. V.
Cor. 4.

would cause the system to describe $\frac{VA \times GP}{2CW \times G\mathcal{Q}^2}$ revolutions in the time 1, C being the number 3.14159, &c. And the impact of the body B if applied alone would generate an angular velocity of $\frac{UB \times GE}{2CW \times G\mathcal{Q}^2}$ revolutions in the same standard time 1; and if the two impacts at F and I act to turn the system round G contrary ways, the angular velocity after the impact will on the whole be $\frac{VA \times GP - UB \times GE}{2CW \times G\mathcal{Q}^2}$ or $\frac{UB \times GE - VA \times GP}{2CW \times G\mathcal{Q}^2}$ according as $VA \times GP$ is greater or less than $UB \times GE$.

XI.

Fig. CI.

G represents the centre of gravity of a body moving in free space with the velocity

city V ; and at the same time revolving round G with an angular velocity of M revolutions in the standard time to which V is referred; the mass of the system is W , and Q is the principal centre of gyration: it is required to assign the velocity with which a body A must impinge on any given point K , so that the point K may be quiescent the instant after the impact, it being supposed that the direction of the impact NK is perpendicular to KG passing through the centre of gravity, and that the inertia of A is inconsiderable in comparison of W .

It is manifest from the construction in prop. ii. that if a line KG be drawn through the centre of gravity perpendicular to the direction of impact, or which is the same thing, perpendicular to the direction in which the centre of gravity moves, the centre of spontaneous rotation will be somewhere in this perpendicular line KG produced.

Let O be the centre of spontaneous rotation, then because V is the velocity of the centre of gravity, and M is

angular velocity*, $GO = \frac{V}{2C \times M}$, C being $= 3.14159$, * Sect. X.
Prop. V.
Cor. 2 & 3.

and $GE = \frac{GQ^2 \times 2CM}{V}$, Moreover since V is the velocity of the point G , $\frac{V \times OK}{QG}$ will be the velocity of the point K .

Let T be the centre of spontaneous rotation corresponding to the point K : the force of the stroke is the same as if an equivalent mass, $=$ the whole mass diminished in the proportion of $KT:GT$ †, being concentrated into a point, impinged directly on it with the same velocity. † Sect. X.
Prop. IV.
Cor. 2.

To solve the problem therefore it will be sufficient to assign the velocity of the body A , when $A \times$ into that velocity shall be equal to the equivalent mass into the velocity of its motion. Let z be the velocity of A required;

quired: the velocity of K is $\frac{V \times KO}{GO}$, and because $\mathcal{T}$ is the centre of spontaneous rotation corresponding to K , the equivalent mass to be concentrated into K , so as to cause a resistance of inertia equal to that of the whole system is $\frac{W \times GT}{KT}$, from which the following equation is

|| Sect. VI. derived: $Az = \frac{V \times KO \times W \times GT}{GO \times KT}$, but || $GT = \frac{GQ^2}{GK}$,
Prop. VIII.

$$GO = \frac{GQ^2}{GE}, = \frac{V}{2CM} \text{ and } KO = GK + \frac{GQ^2}{GE} = GK$$

+ Sect. X. $+ \frac{V}{2CM}$, and $KT = GK + \frac{GQ^2}{GK}$, wherefore $Az =$
Prop. V.

Cor. 2 & 3. $\frac{V \times W \times 2CM \times GK + V \times GT \times GK}{2CM \times GO \times GK^2 + GQ^2}$; but $2CM \times GO = V$,

and $GT \times GK = GQ^2$, we have therefore $Az =$
 $\frac{W \times GQ^2 \times V + 2CM \times GK}{GK^2 + GQ^2}$, and the velocity acquired, or

$$z = \frac{W \times GQ^2 \times V + 2CM \times GK}{A \times GK^2 + GQ^2}.$$

Cor. 1. If the centre of gravity moves in the direction GI with the velocity V , the angular velocity being M , and any point K impinges on an opposed obstacle when the line GK is perpendicular to GI , the force of the stroke will be

$$Az = \frac{W \times GQ^2 \times V + 2CM \times GK}{GK^2 + GQ^2}.$$

Cor. 2. While the system is moving in free space in the manner described in the proposition, if a body impinges on any point F , not in the line GK , which is perpendicular to the impact, it is manifest, that the velocity cannot be so adjusted so that this point shall be motionless immediately after the impact, for if the point F were quiescent, H the centre of spontaneous rotation corresponding to it, would move in a direction perpendicular to HF ; but from what has been demonstrated, it must proceed in the direction perpendicular to OK , in which it was moving at the time of the impact.

XII.

Let $WTEF$, $QBGF$ be two systems Fig. CII.
 revolving in free space about their centres of gravity G and W ; let the angular velocity of the system $QBGF$ be such as causes it to make M revolutions or parts of a revolution in any standard time 1, and in the same time let the system $WTEF$ perform N revolutions or parts of a revolution in the same plane, but in a direction contrary to that in which the former system revolves: moreover, let the centre of gravity G move in the right line XB with the velocity V , and let the centre W move in the opposite direction WD parallel to BX : suppose the system $QBGF$ to impinge on the body $WTEF$ in the point F and in the direction LF parallel to HD or BX and perpendicular to the surface which it strikes; it is required to assign the motion of each body after the impact.

Through G and W draw YL and OE perpendicular to the direction in which the centres of gravity move, and let the principal centre of gyration of the system $QBGF$ be Q , and that of the system $WTEF$ be T . Moreover, let the mass of the system $QBGF$ be G , and that of the other system W : the point F in the system $WTEF$ will be impelled by the impact with the same force as if the mass

$\frac{G \times GQ^2 \times V + 2CM \times GL}{L \times GQ^2 + GL^2}$ • impinged against F being con- * Prop. XI,
 centrated into a point with the same velocity, on Cor. 1.

a supposition that the inertia of the striking body at F is

F is evanescent or inconsiderable. But because the angular velocity of the striking body is M , and the velocity of its centre of gravity $= V$, the velocity of L is $= V + 2MC \times GL$: the equivalent mass will therefore be $\frac{G \times GQ^2}{GQ^2 + GL^2}$ impinging with the velocity $V + 2MC \times GL$. This impact will generate in the centre of

* Sect. X.
Prop. V.
Cor. 2.

gravity W a velocity $= \frac{G \times GQ^2 \times V + 2MC \times GL}{W \times GQ^2 + GL^2}$ in the

direction WH : wherefore after the impact the velocity of the centre of gravity W in the same direction will be

$V - \frac{G \times GQ^2 \times V + 2MC \times GL}{W \times GQ^2 + GL^2}$: moreover, the same force of

impact will generate an angular velocity in the system

* Cor. 3.

$WTEF = \frac{G \times WE \times GQ^2 \times V + 2MC \times GL}{2C \times W \times WT^2 \times GQ^2 + GL^2}$, which being

added to the angular velocity before the impact N or subtracted from it, according as the impact tends to augment or counteract the rotation of the body struck, the angular velocity of the system $WTEF$ will be obtained.

* Ad finem.

Let A represent the angular velocity of the system $WTEF$ immediately after the impact which has been just determined, then if C be $= 3.14152$, &c. $2CA \times WE$ will be the velocity of the point E , or of the point F in the striking body: for according to the reasoning contained in page 422*, when the surfaces at F are perfectly smooth the point F of the striking body will proceed at the instant after the impact in the direction FE , until the two bodies have a common velocity in the direction FE : in the line LE take LI equal to the space which would be described by the point E with the velocity $2CA \times WE$ in any standard particle of time.

Let Y be the centre of spontaneous rotation in the system $QBGF$: the motion of this point will not be affected by the impact; its velocity and direction therefore will be the same after the impact as it was before the system $QDBGF$ impinged on the other body: because the angular velocity of the striking body is M , in the direction from L to F , and the velocity of the centre of gravity V in the direction GB , the velocity of the point Y before the impact

will be $V - 2CM \times GY$ or $V - \frac{2CM \times GQ^2}{GL}$: make YZ

perpen-

perpendicular to LY and equal to the space through which point Y is carried uniformly by the velocity $V - \frac{2CM \times GQ^2}{GL}$ in the standard particle of time; through I draw IZP intersecting GB in g and LG produced in P ; then Gg will be the space which the centre of gravity G will describe uniformly after the impact in the particle of time 1, and will therefore represent the velocity of the centre of gravity G after the impact: and IPL will be the angle described in the same time, and will therefore be a measure of the angular velocity of the striking body after the impact.

Cor. 1. If $\frac{G \times WE \times GQ^2 \times V + 2MC \times GL}{2C \times W \times WT^2 \times GQ^2 + GL^2}$ be less than

N , the rotation of the system $WTEF$ will be in the same direction after as before the stroke. If the two quantities just mentioned are equal, the body struck will have no angular velocity after the stroke, and if the former should exceed the latter, the direction in which the system revolved will be changed by the impact.

Cor. 2. If $LI = zY$, the striking body will not revolve after the impact, its angular velocity being destroyed. If LI exceeds zY , the rotation will be continued in the same direction after the impact as before it: if zy exceeds LI the direction of rotation will be changed by the impact.

A P P E N D I X.

CONTAINING A DESCRIPTION OF THE ADJUSTMENTS AND PRACTICAL USE OF THE MONOCHORD REPRESENTED IN FIG. XIX. Vid. p. 99.

IN this construction the string is not terminated by a bridge, but between metallic edges, which not only define the string's length with great precision, but prevent any irregularity in the tending force, which is the unavoidable consequence of bridges: the two steel edges at *C* are separated from each other by the screw *F*, in order to give room for the string, while the frame *TQRS* is moved along the scale. When the screw *N* is loosened, the frame *TQRS* moves freely along the scale, carrying with it the steel edges, &c. By turning the screw *N*, the frame *TQRS* is fastened to the scale, while the other parts *OGDC* are moveable only by means of the screw *EW P*, which serves the purpose of adjustment as well as for other uses hereafter mentioned. *KL* is a part of the brass scale *AB* represented in the left hand figure, the whole length is about 40 inches, and the breadth about $\frac{1}{2}$ an inch. In this scale 100 equal parts are set off, and each part is bisected by a shorter division. In the square aperture *M*, an index is engraved the distance of it from the steel edges, estimated in the direction of the scale, is such, that when this index is set to any division, the length of the string intercepted between the steel edges *C*, and the upper termination of the string, may be exactly equal to the number of parts expressed by the division. Suppose the frame *TQRS* to be fastened, the screw *EW P* being turned twenty revolutions, carries the steel edges, or the index at *M*, through a space equal to one whole division, and consequently

sequently one revolution of the screw carries the steel edges through one twentieth part of a division.

The circumference of the screw is divided into 50 equal parts, each of which must therefore answer to one thousandth part of a division, that is, if the whole length of a string be made $= 100$ parts, one of the divisions on the screw's circumference will answer to the hundred thousandth part of the whole length. By this means, it will be easy to adjust the length of string intercepted between the steel edges *C* and the upper extremity, of any magnitude less than 100000: but because any musical interval implies the vibrations of two strings, of which the lengths are in the direct proportion of the times of vibration, a base note is always assumed as a standard to which the others are referred. In order therefore to exemplify the use of this instrument, the length of the base note must be first adjusted. The length of the string which constitutes this base note, is most conveniently assumed $= 10, 100, 1000$, or some number in that progression. In this instrument, the number corresponding to the base note is 100000: after having separated the steel edges at *D* by the screw *F*, bring the index in the square aperture to the division on the scale marked 100, as near as can be judged by the naked eye, and fix the frame *TQRS* by turning the screw marked *N*. Then to adjust the coincidence of index precise, apply the microscope *G* immediately over the index, and move the screw *EPW* until the index coincides with the division 100: the two edges being now suffered to compress the string, the note sounded by it will be the base or fundamental tone. Now, let it be required to adjust the length of the string $= 66667$ parts, the whole length being 100000. The operation will be as follows; loosen the screw *N* that the frame *TQRS* may move freely: bring the index on the square aperture to 66 on the scale, and adjust by the glass as before; bring the moveable index *x* to the point *o* on the circumference *WEP*: then, since the third figure in the number proposed is 6, answering to 6 thousandths of the whole length, this will be set off by turning the screw *EPW* twelve revolutions: by which the three first figures 666 will be expressed in the length of the string. Moreover, as there are 67 hundred thousandth parts remaining to be added to the length, and one of the division, on the screw's circumference answers to one hundred thousandth part, it follows, that the screw must be turned through 67 of these divisions; that is, because the circumference of the screw is divided into 50 equal

equal parts, the screw must be turned one revolution and 17 divisions. Very little time is taken in making these adjustments, the use of which is to define the string to any given length and true to the hundred thousandth part of the whole length; by which the different systems of musical sounds, which have been before invented, such as Huygens's, Ptolemy's, Smith's, &c. may be brought into comparison, as well as others which may be suggested either from theory or trials. The monochords of the old construction, were divided according to one or two given scales only, which rendered them wholly useless in making experiments on harmonic temperaments in general, for which purpose the instrument above described was constructed.

The method of terminating the string between metallic edges, in preference to that of supporting it by bridges, was copied from the monochord of a very ingenious and worthy friend the Rev. Mr. L. Huddleston, of the University of Oxford.



E R R A T A.

PAGE 3. line 14. *for its distances, read the distances;*
p. 7. l. 19. *after descend, add from rest;* p. 8. 3d l.
from the bottom, *f. in a direction, read in the direction;*
p. 14. l. 14. *f.* $\frac{B}{b} = 1$, *r.* $\frac{B^m}{b^m}$; *ibid.* last l. *f.* and if *W*, *r.*
if *W*; p. 18. l. 2. *after impelled, insert or resisted;* p. 29. l. 3.
f. a thousandth part, *r.* in thousandth parts; p. 50. l. 19. *f.*
O, *r.* *B*; p. 52. in the references, *f.* prop. XI. *r.* prop. XII;
p. 62. l. 26. *f.* and the force, *r.* the force; p. 68. l. 11.
f. *F*, *r.* *T*; *ibid.* l. 14. *f.* a^3 , *r.* $p a^3$; p. 73. l. 4. *f.* the
velocity, *r.* the finite velocity; *ibid.* l. 13. *dele* of it; *ibid.*
l. 14. *f.* $\text{Log} \frac{a+x}{a}$, *r.* $a \times \text{Log} \frac{a+x}{a}$; p. 74. l. 16. *f.*
 $\dot{v} = x \dot{w}$, *r.* $\dot{x} = x \dot{v}$; p. 79. l. 25. *f.* in the length, *r.* on
the length; p. 83. l. 15. *f.* $-F a^{r+1}$, *r.* $-F a^n + 1$;
p. 86. l. 20. *f.* $\frac{1}{1-n}$, *r.* $\frac{1}{1+n}$; p. 87. l. 4. *f.* a logarithm,
r. a logarithmitic expression; p. 92. l. 20. *f.* *PO*, *r.* *RO*;
p. 95. l. 5. *f.* or one side, *r.* on one side; p. 96. l. 13.
after string vibrates, insert should vary; p. 97. l. 14. *f.* the
distance, *r.* the greatest distance; p. 102. l. 40. *f.* Holder,
r. Harrington; p. 109. l. 18. *f.* *HH*, *r.* *HO*; *ibid.* *f.* Fig.
XXII, *r.* Fig. XXIII; p. 118. l. 8. *f.* opposed, *r.* oppose;
p. 126. l. 3. *f.* equal, *r.* equals; p. 135. l. 9. *f.* how nearly,
r. how near; p. 141. l. 18. *f.* 13780, *r.* 13760; p. 143. l. 6.
f. body's weight, *r.* body's weight in the fluid; p. 148. l. 7.
f. on the contrary arguments, *r.* on the contrary, argu-
ments; p. 152. l. 4. *f.* 2301585, *r.* 2302585; p. 155. l. 32.
del e is; p. 156. l. 6. *f.* from, *r.* after; p. 160. *f.* XXX, *r.* XXXI;
ibid. l. 4. *f.* 34, *r.* 33; p. 173. l. 5. *f.* Clarke, *r.* Clare; *ibid.*
l. 33. *f.* rarer, *r.* denser; p. 182. l. 7. *f.* degree, *r.* degrees;
p. 193. ult. *f.* weight, *r.* mass; p. 199. l. 22. *f.* Diminution,
r. The diminution; p. 214. l. 22. *f.* by Gallileo, *r.* from
Gallileo's principles; p. 268. l. 11. *f.* SD^2 , *r.* SH^2 ; p. 269.
l. 11. *f.* 10 : 1, *r.* 1 : 10; p. 281. l. 1. *f.* $\frac{Wr^2}{a^2}$, *r.* $\frac{Gr^2}{a^2}$;
p. 283. l. 4. *f.* *x*, *r.* *z*; *ibid.* l. 6. *f.* $V^2 \dot{y}$, *r.* $V^2 y \dot{y}$; p. 286.
l. 11. *f.* same, *r.* contrary; p. 290. l. 15. *after string's length,*
insert estimated from C to D; p. 301. l. 5. *dele* which; *ibid.*
dele of the; p. 331. l. 9. *after will, insert be;* p. 335. l. 13.
f. instruments, *r.* instrument; p. 337. l. 5. *f.* $3\frac{3}{4}$ oz. *r.* $2\frac{3}{4}$
oz. p. 345. l. 2. *f.* 239, *r.* 339; p. 355. l. 5. *f.* centre, *r.*
principal centre; p. 356. l. 5. *f.* measures, *r.* a measure;
p. 363. l. 2. *dele* a; p. 362. l. 25. *f.* $S : s$, *r.* $s : S$; p. 366. l.
19. *dele* is; p. 367. l. 2. *f.* accelerated, *r.* uniformly acce-
lerated; p. 384. l. 1. *f.* 25.25, *r.* 2.525; p. 404. l. 12. *f.*
weight, *r.* inertia; p. 1415. l. 14. *dele* immoveable.

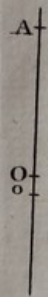
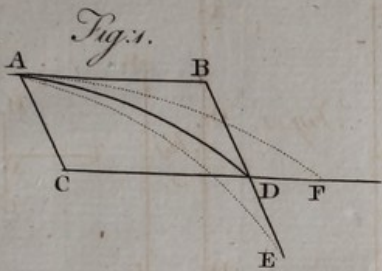


Fig. 2.

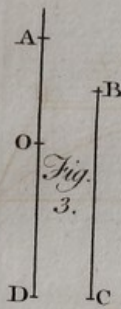


Fig. 3.

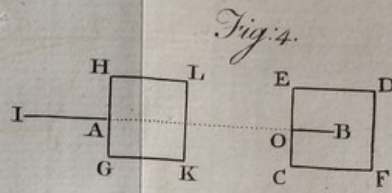


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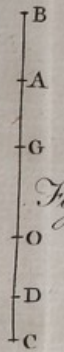


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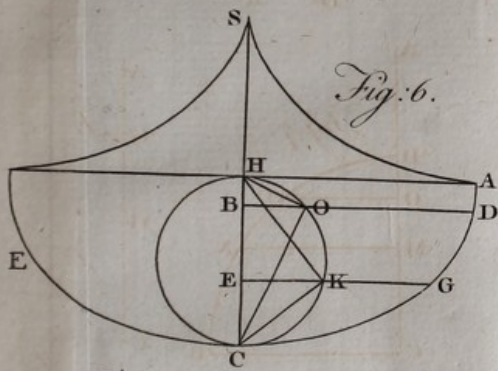


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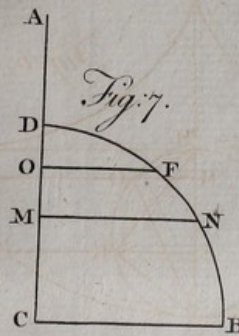


Fig. 7.

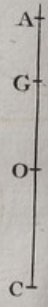


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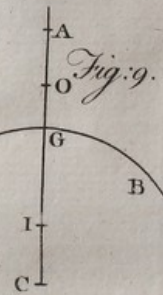


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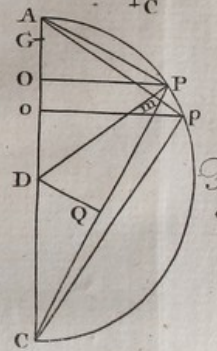


Fig. 10.

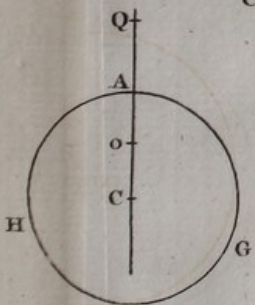


Fig. 11.

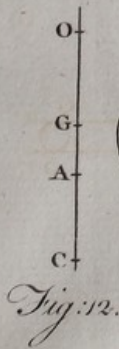


Fig. 12.

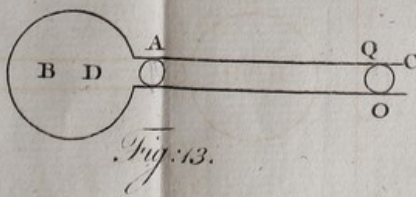


Fig. 13.

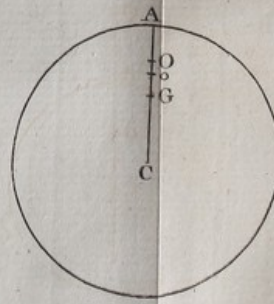


Fig. 14.

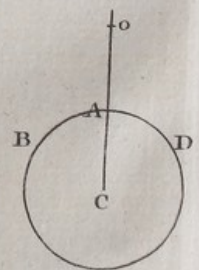
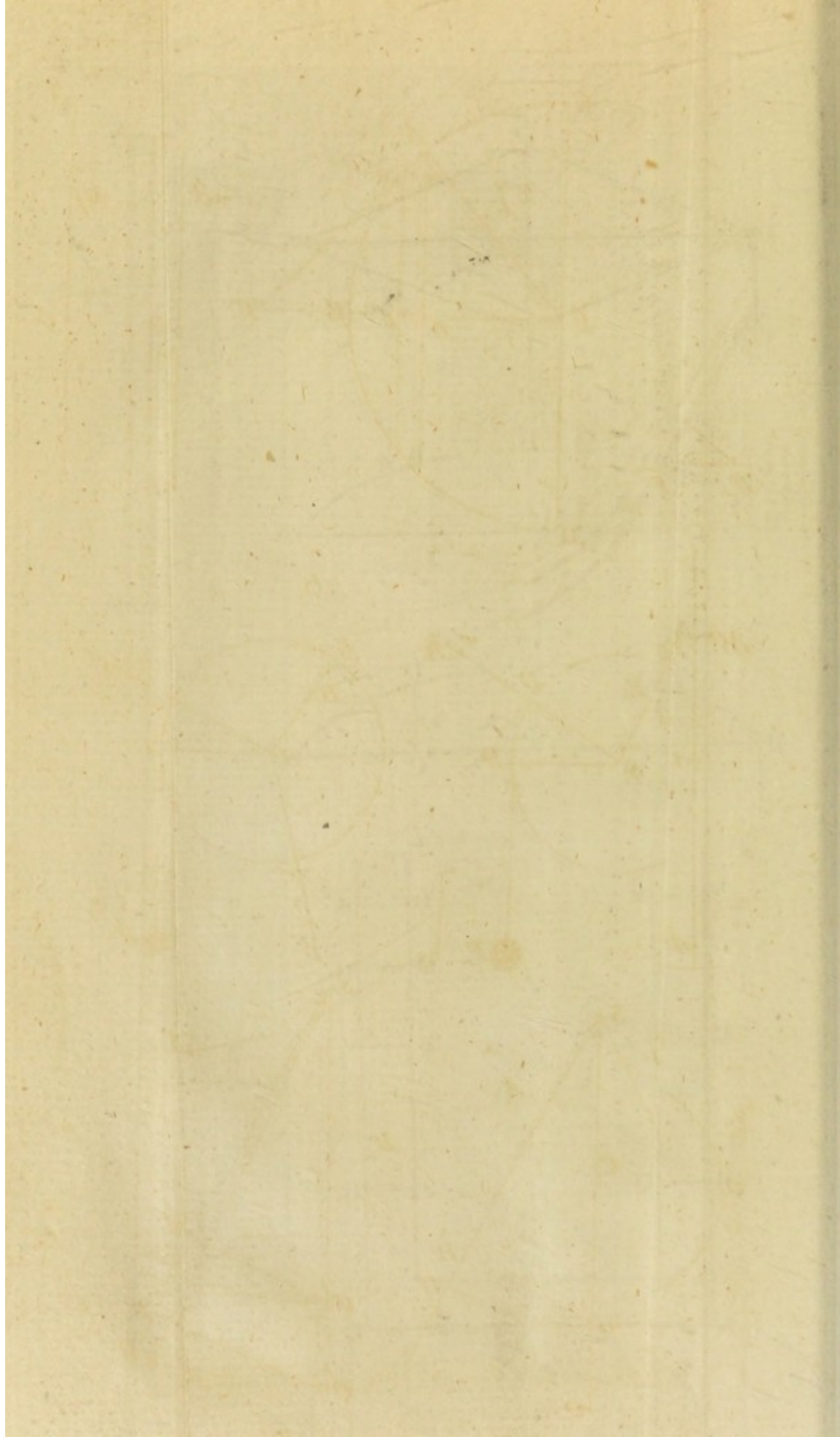
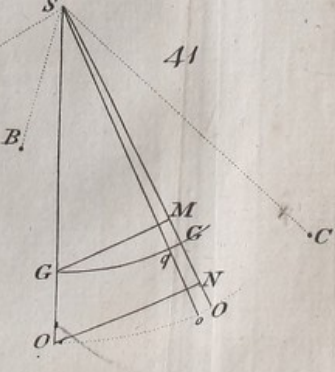
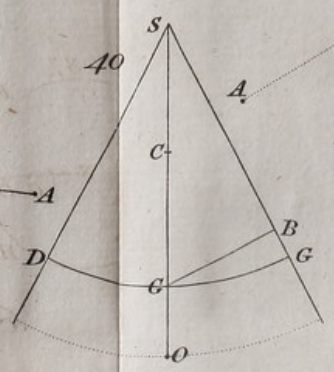
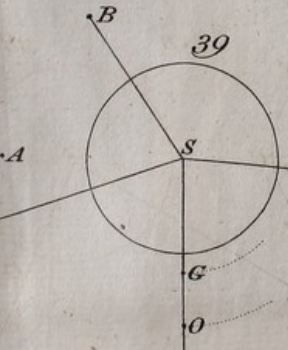
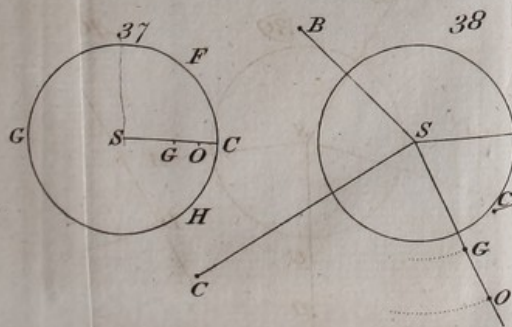
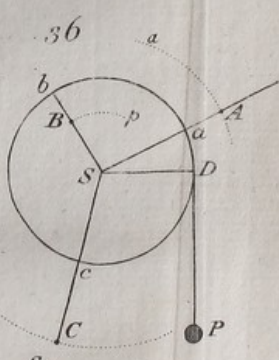
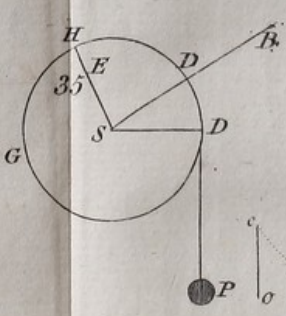
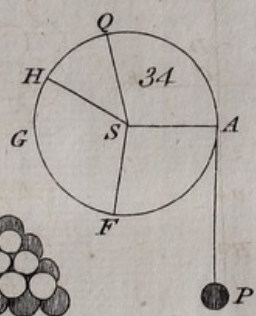
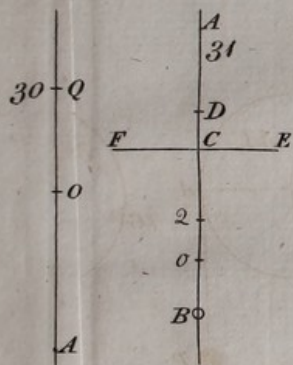
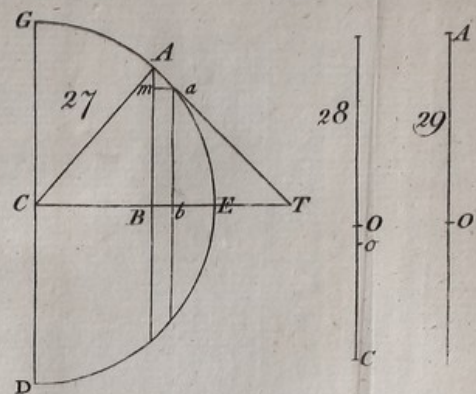
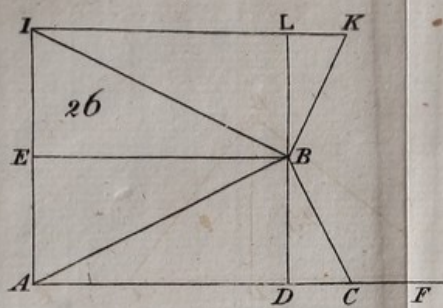
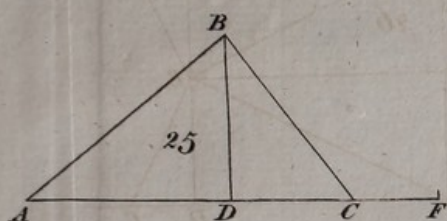
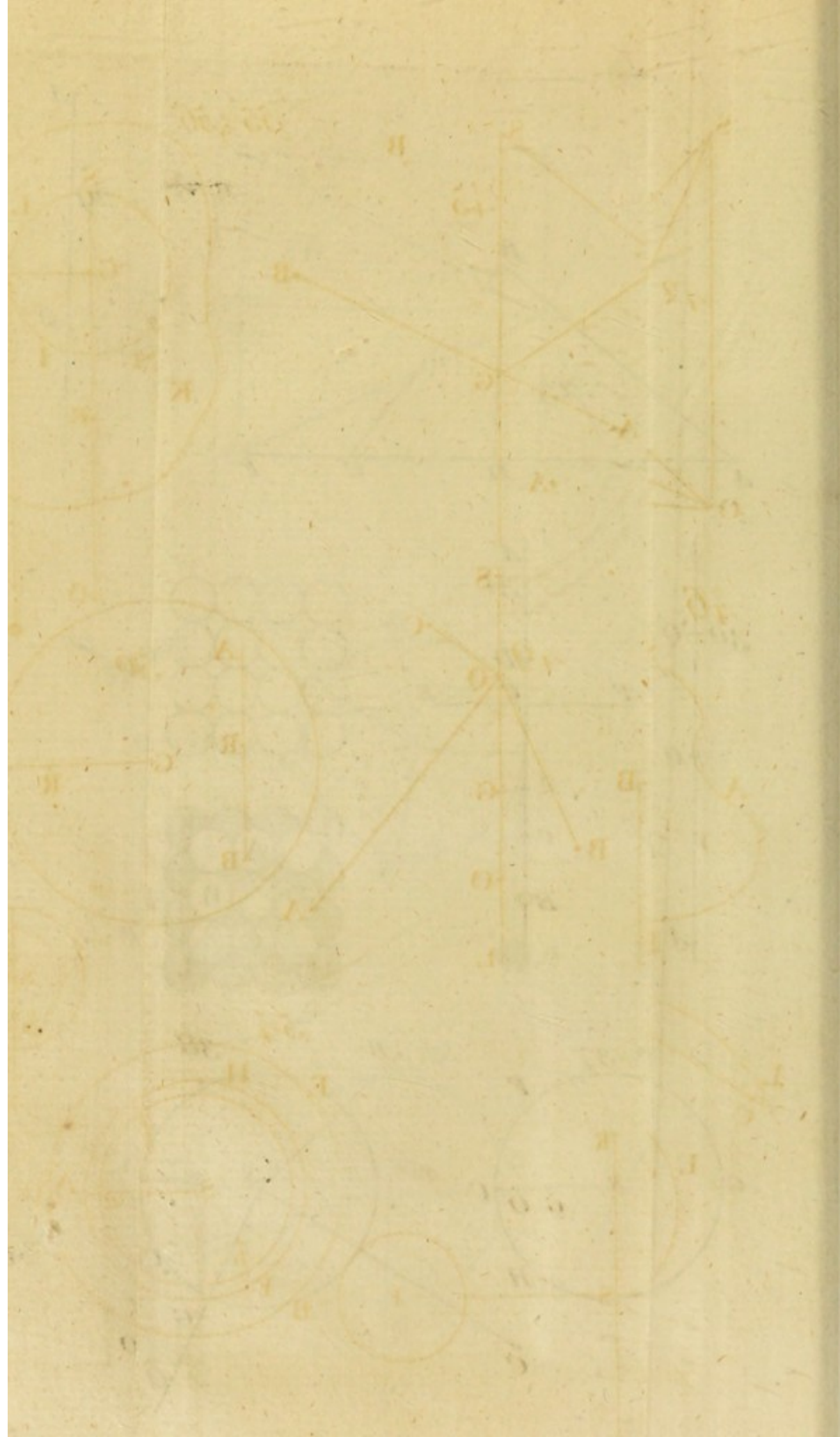
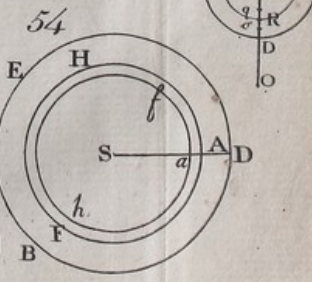
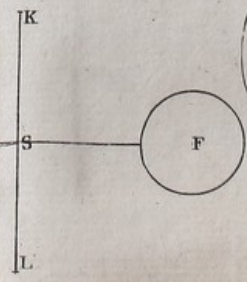
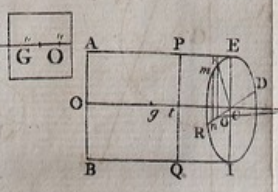
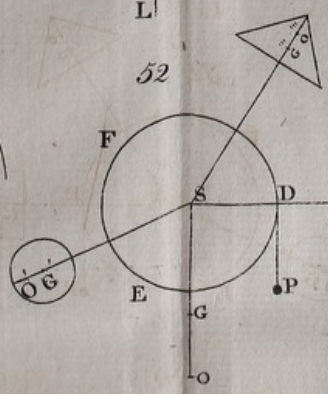
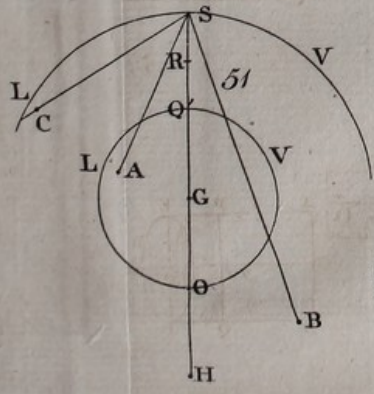
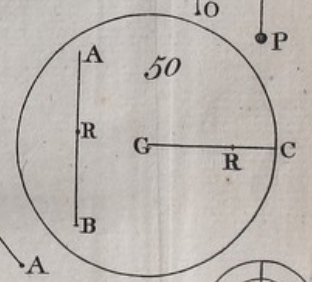
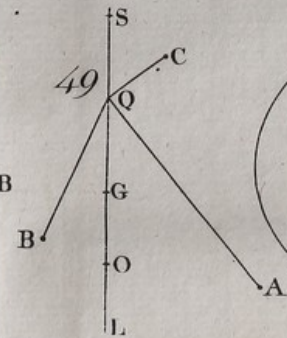
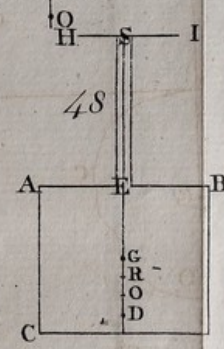
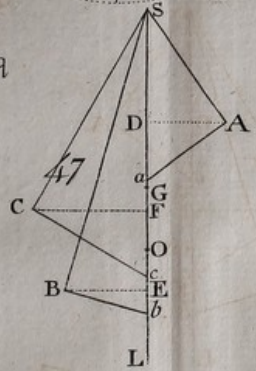
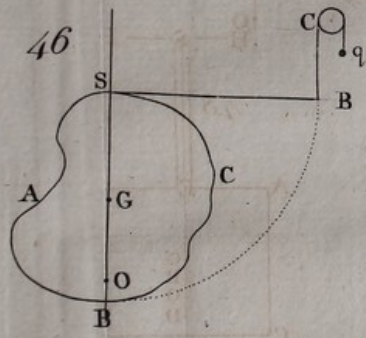
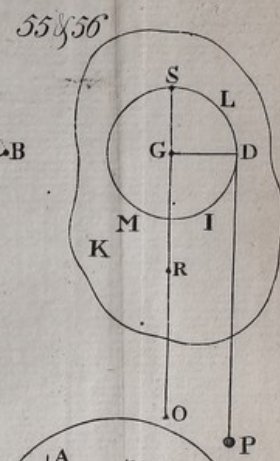
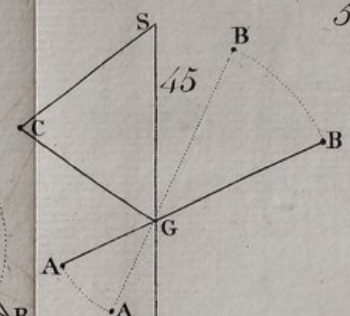
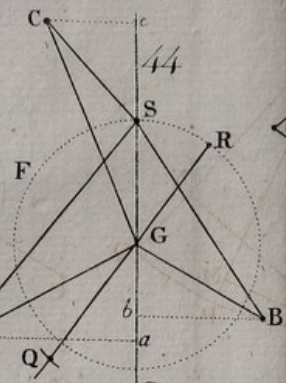
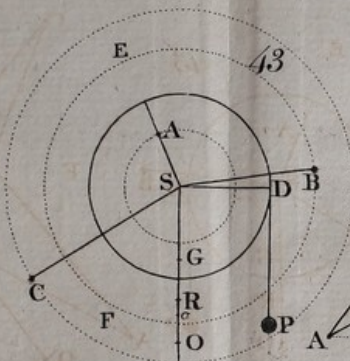
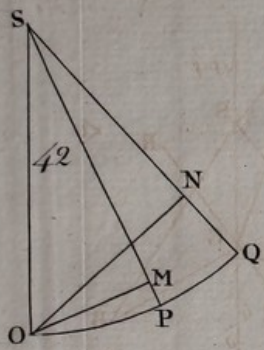


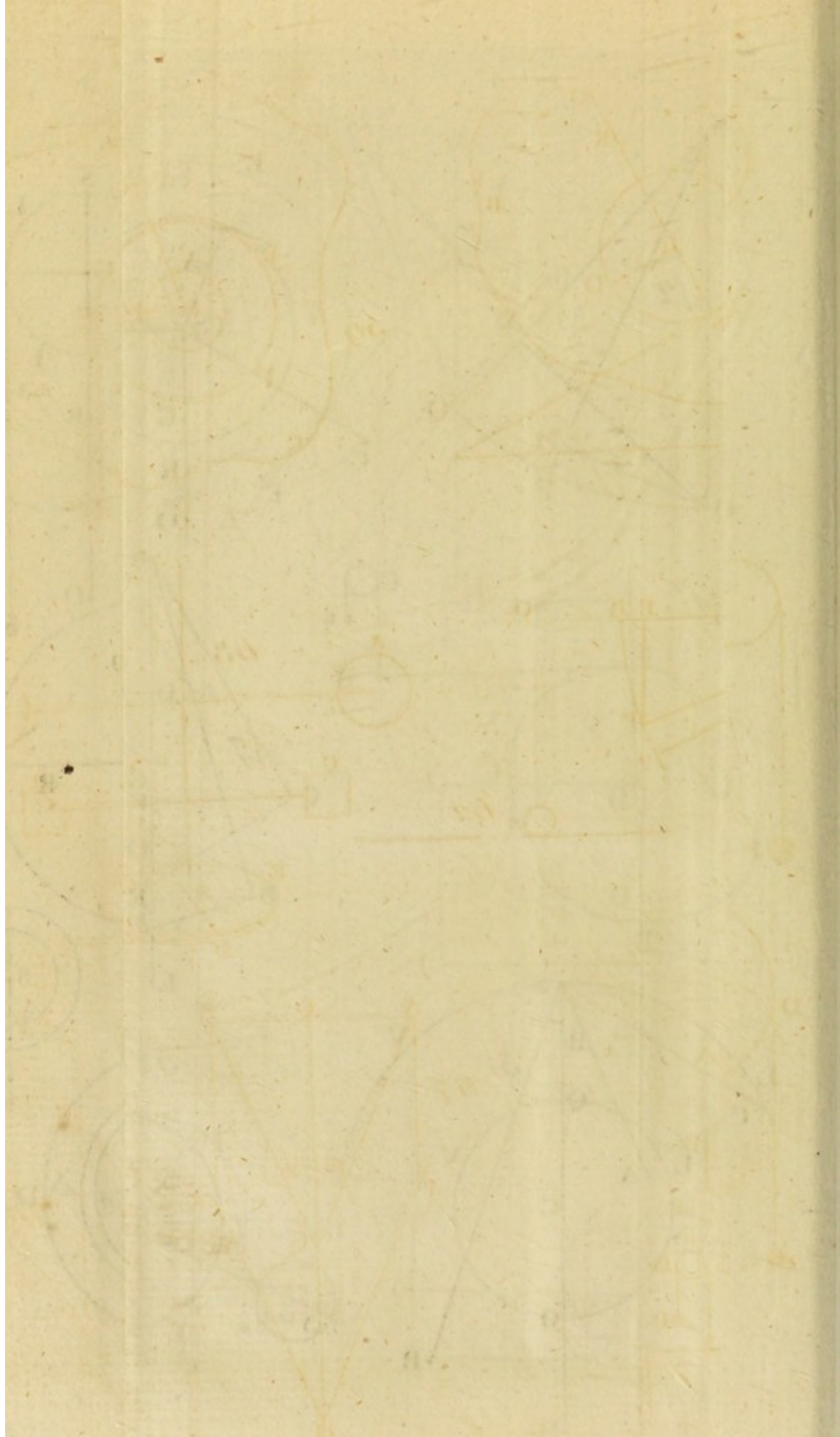
Fig. 15.

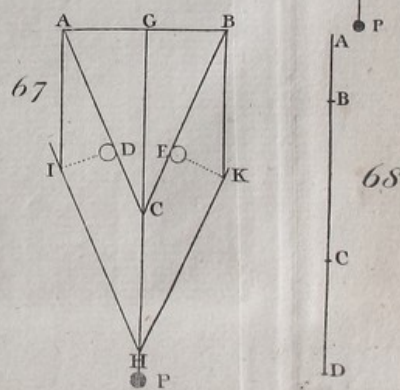
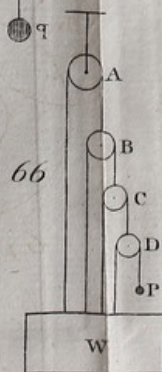
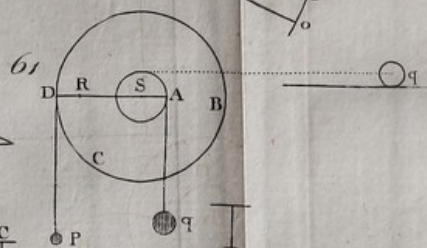
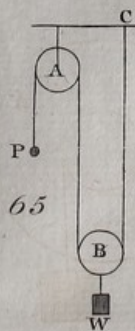
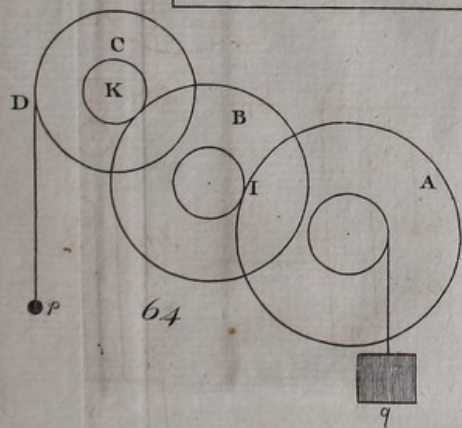
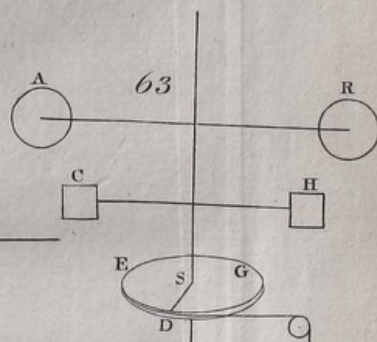
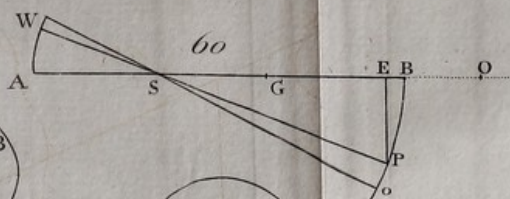
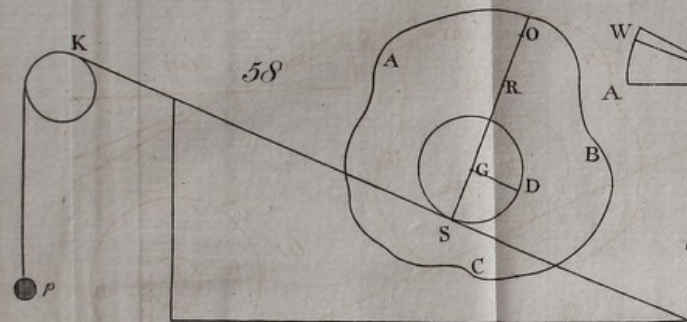
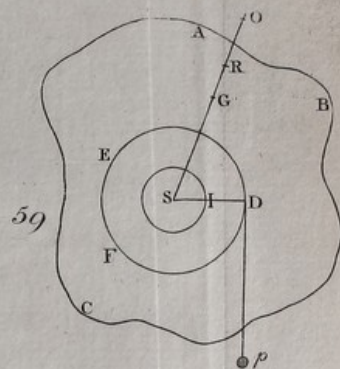
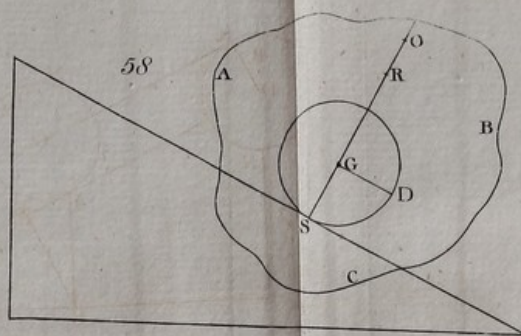


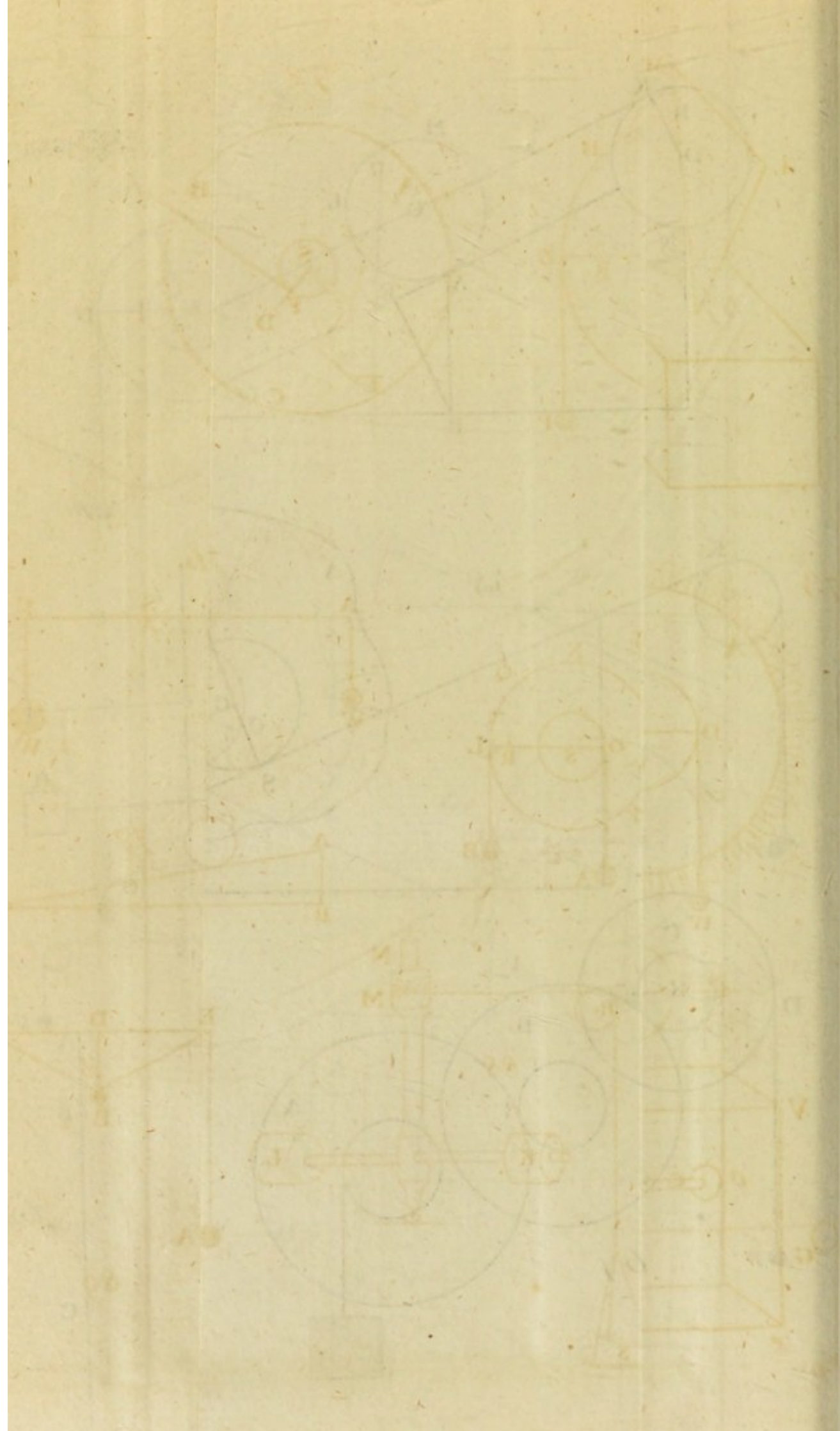


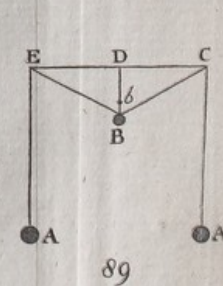
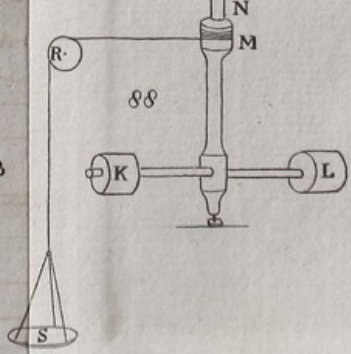
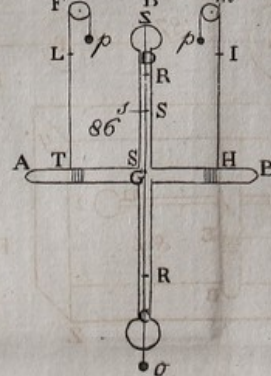
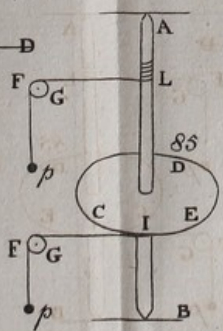
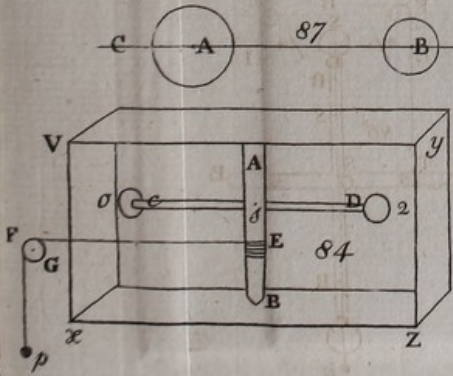
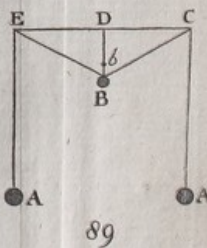
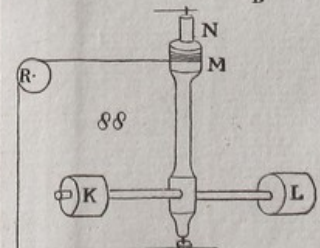
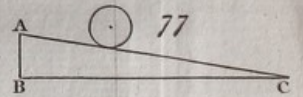
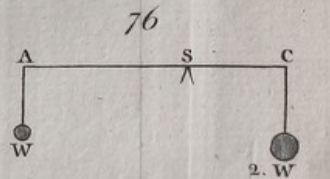
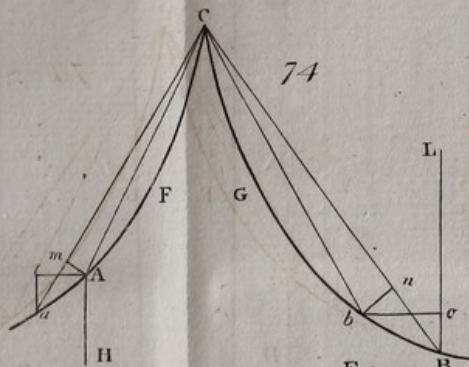
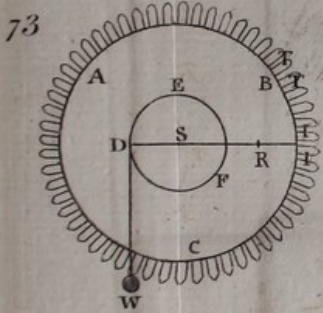
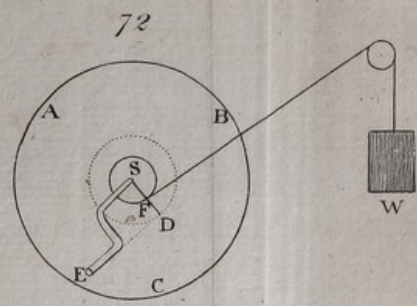
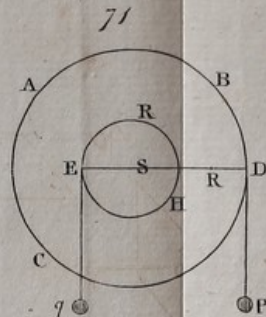
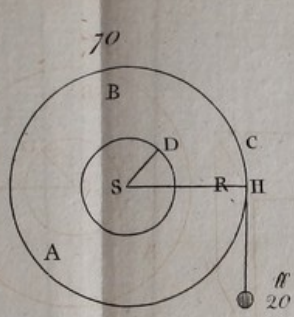
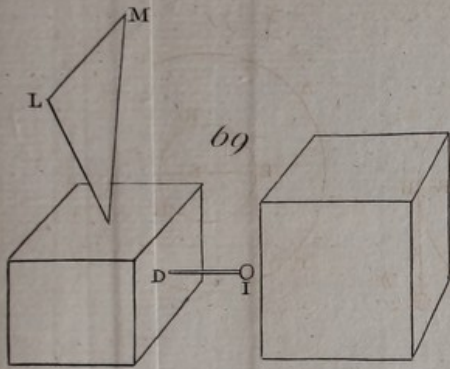








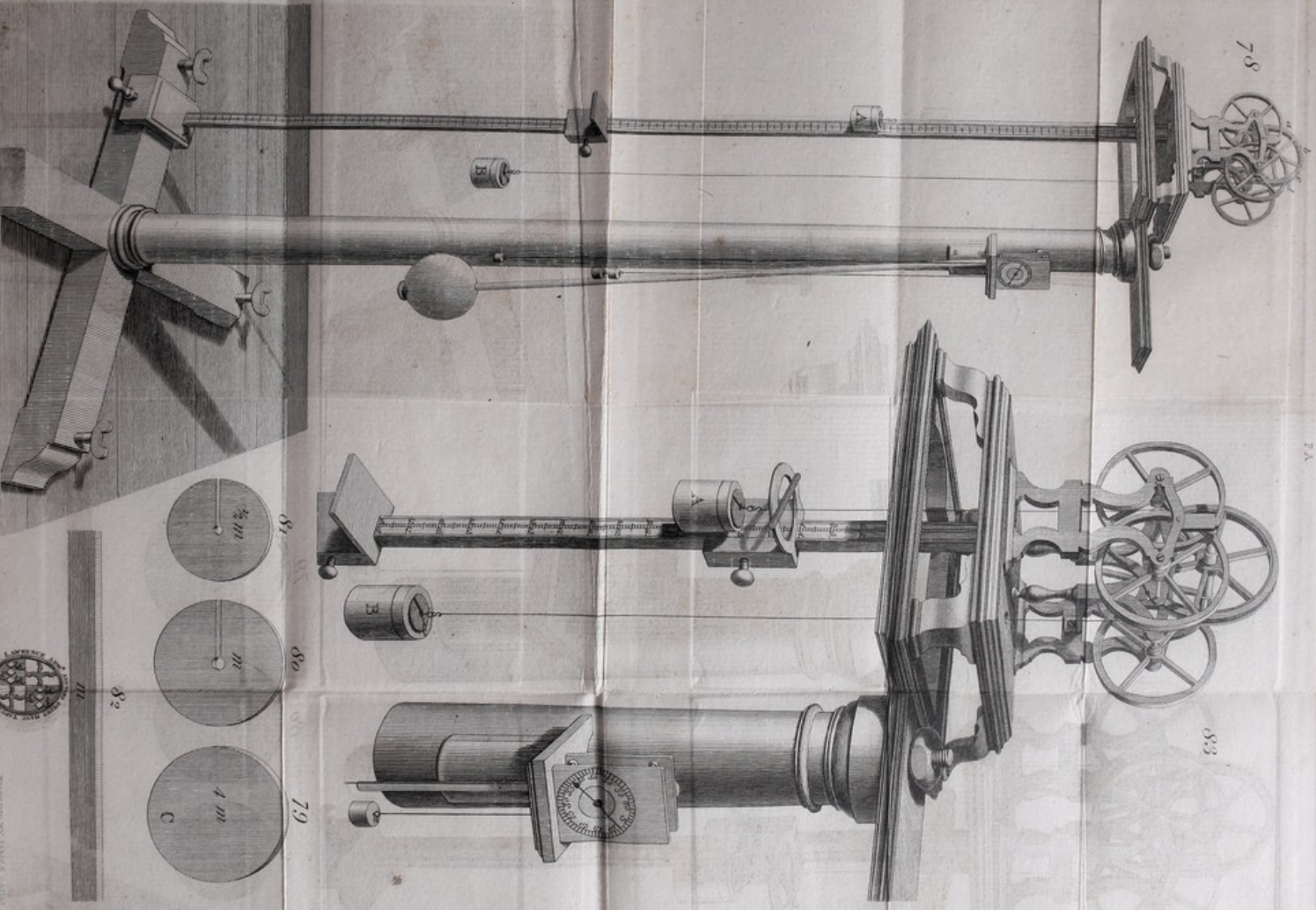






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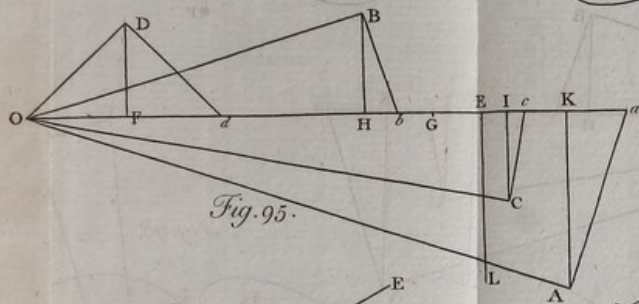
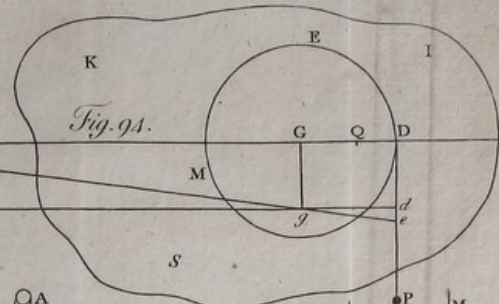
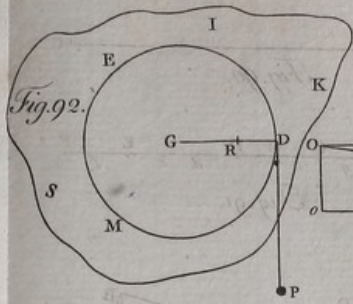
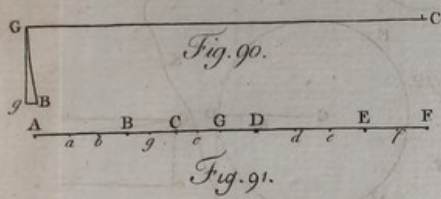


Fig. 98.

