I. *Mathematical Investigations concerning the Laws of the Equilibrium of Fluids analagous to the Electric Fluid, with other similar Researches.* By George Green, Esq. *Communicated by Sir Edward Ffrench Bromhead, Bart.* M.A. F.R.S.L. and E.

[Read *Nov.* 12, 1832.]

Amongst the various subjects which have at different times occupied the attention of Mathematicians, there are probably few more interesting in themselves, or which offer greater difficulties in their investigation, than those in which it is required to determine mathematically the laws of the equilibrium or motion of a system composed of an infinite number of free particles all acting upon each other mutually, and according to some given law. When we conceive, moreover, the law of the mutual action of the particles to be such that the forces which emanate from them may become insensible at sensible distances, the researches to which the consideration of these forces lead will be greatly simplified by the limitation thus introduced, and may be regarded as forming a class distinct from the rest. Indeed they then for the most part terminate in the resolution of equations between the values of certain functions at any point taken at will in the interior of the system, and the values of the partial differentials of these functions at the same point. When on the contrary the forces in question continue sensible at every finite distance, the researches dependent upon them become far more complicated, and often require all the resources of the modern analysis for their successful prosecution. It would be easy so to exhibit the theories of the equilibrium and motion of ordinary fluids, as to offer instances of researches appertaining to the former class, whilst the mathematical investigations to which the theories of Electricity and Magnetism have given rise may be considered as interesting examples of such as belong to the latter class.

It is not my chief design in this paper to determine mathematically the density of the electric fluid in bodies under given circumstances, having elsewhere* given some general methods by which this may be effected, and applied these methods to a variety of cases not before submitted to calculation. My present object will be to determine the laws of the equilibrium of an hypothetical fluid analagous to the electric fluid, but of which the law of the repulsion of the particles, instead of being inversely as the square of the distance, shall be inversely as any power n of the distance; and I shall have more particularly in view the determination of the density of this fluid in the interior of conducting spheres when in equilibrium, and acted upon by any exterior bodies whatever, though since the general method by which this is effected will be equally applicable to circular plates and ellipsoids. I shall present a sketch of these applications also.

It is well known that in enquiries of a nature similar to the one about to engage our attention, it is always advantageous to avoid the direct consideration of the various forces acting upon any particle p of the fluid in the system, by introducing a particular function V of the co-ordinates of this particle, from the differentials of which the values of all these forces may be immediately deduced†. We have, therefore, in the present paper endeavoured, in the first place, to find the value of V, where the density of the fluid in the interior of a sphere is given by means of a very simple consideration, which in a great measure obviates the difficulties usually attendant on researches of this kind, have been able to determine the value V, where ρ, the density of the fluid in any element dv of the sphere's volume, is equal to the product of two factors, one of which is a very simple function containing an arbitrary exponent β, and the remaining one f is equal to any rational

* Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.

† This function in the present case will be obtained by taking the sum of all the molecules of a fluid acting upon p, divided by the $(n-1)^{\text{th}}$ power of their respective distances from p; and indeed the function which Laplace has represented by V in the third book of the *Mecanique Celeste,* is only a particular value of our more general one produced by writing 2 in the place of the general exponent n.

and entire function whatever of the rectangular co-ordinates of the element dv, and afterwards by a proper determination of the exponent β, have reduced the resulting quantity V to a rational and entire function of the rectangular co-ordinates of the particle p, of the same degree as the function f. This being done, it is easy to perceive that the resolution of the inverse problem may readily be effected, because the coefficients of the required factor f will then be determined from the given coefficients of the rational and entire function V, by means of linear algebraic equations.

The method alluded to in what precedes, and which is exposed in the two first articles of the following paper, will enable us to assign generally the value of the induced density ρ for any ellipsoid, whatever its axes may be, provided the inducing forces are given explicitly in functions of the co-ordinates of p; but when by supposing these axes equal we reduce the ellipsoid to a sphere, it is natural to expect that as the form of the solid has become more simple, a corresponding degree of simplicity will be introduced into the results; and accordingly, as will be seen in the fourth and fifth articles, the complete solutions both of the direct and inverse problems, considered under their most general point of view, are such that the required quantities are there always expressed by simple and explicit functions of the known ones, independent of the resolution of any equations whatever.

The first five articles of the present paper being entirely analytical, serve to exhibit the relations which exist between the density ρ of our hypothetical fluid, and its dependent function V; but in the following ones our principal object has been to point out some particular applications of these general relations.

In the seventh article, for example, the law of the density of our fluid when in equilibrium in the interior of a conductory sphere, has been investigated, and the analytical value of ρ there found admits of the following simple enunciation.

The density ρ of free fluid at any point p within a conducting sphere A, of which O is the centre, is always proportional to the $(n-4)^{\text{th}}$ power of the radius of the circle formed by the intersection of a plane perpendicular to the ray Op with the surface of the sphere itself, provided

n is greater than 2. When on the contrary n is less than 2, this law requires a certain modification; the nature of which has been fully investigated in the article just named, and the one immediately following.

It has before been remarked, that the generality of our analysis will enable us to assign the density of the free fluid which would be induced in a sphere by the action of exterior forces, supposing these forces are given explicitly in functions of the rectangular co-ordinates of the point of space to which they belong. But, as in the particular case in which our formulæ admit of an application to natural phenomena, the forces in question arise from electric fluid diffused in the inducing bodies, we have in the ninth article considered more especially the case of a conducting sphere acted upon by the fluid contained in any exterior bodies whatever, and have ultimately been able to exhibit the value of the induced density under a very simple form, whatever the given density of the fluid in these bodies may be.

The tenth and last article contains an application of the general method to circular planes, from which results, analagous to those formed for spheres in some of the preceding ones are deduced; and towards the latter part, a very simple formula is given, which serves to express the value of the density of the free fluid in an infinitely thin plate, supposing it acted upon by other fluid, distributed according to any given law in its own plane. Now it is clear, that if to the general exponent n we assign the particular value 2, all our results will become applicable to electrical phenomena. In this way the density of the electric fluid on an infinitely thin circular plate, when under the influence of any electrified bodies whatever, situated in its own plane, will become known. The analytical expression which serves to represent the value of this density, is remarkable for its simplicity; and by suppressing the term due to the exterior bodies, immediately gives the density of the electric fluid on a circular conducting plate, when quite free from all extraneous action. Fortunately, the manner in which the electric fluid distributes itself in the latter case, has long since been determined experimentally by Coulomb. We have thus had the advantage of comparing our theoretical results with those of a very accurate observer, and the differences between them are not greater than may be supposed due to the unavoidable errors of experiment, and to that which would necessarily be produced by employing plates of a finite thickness, whilst the theory supposes this thickness infinitely small. Moreover, the errors are all of the same kind with regard to sign, as would arise from the latter cause.

1. If we conceive a fluid analogous to the electric fluid, but of which the law of the repulsion of the particles instead of being inversely as the square of the distance is inversely as some power n of the distance, and suppose ρ to represent the density of this fluid, so that dv being an element of the volume of a body A through which it is diffused, ρdv may represent the quantity contained in this element, and if afterwards we write g for the distance between dv and any particle p under consideration, and these form the quantity

$$V = \int \frac{\rho dv}{g^{n-1}};$$

the integral extending over the whole volume of A, it is well known that the force with which a particle p of this fluid situate in any point of space is impelled in the direction of any line q and tending to increase this line will always be represented by

$$(1) \ldots\ldots\ldots\ldots \frac{1}{1-n}\left(\frac{dV}{dq}\right);$$

V being regarded as a function of three rectangular co-ordinates of p, one of which co-ordinates coincides with the line q, and $\left(\frac{dV}{dq}\right)$ being the partial differential of V, relative to this last co-ordinate.

In order now to make known the principal artifices on which the success of our general method for determining the function V mainly depends, it will be convenient to begin with a very simple example.

Let us therefore suppose that the body A is a sphere, whose centre is at the origin O of the co-ordinates, the radius being 1; and ρ is such a function of x', y', z', that where we substitute for x', y', z' their values in polar co-ordinates

$$x' = r' \cos\theta', \quad y' = r' \sin\theta' \cos\varpi', \quad z' = r' \sin\theta' \sin\varpi',$$

it shall reduce itself to the form

$$\rho = (1 - r'^2)^\beta . f(r'^2);$$

f being the characteristic of any rational and entire function whatever: which is in fact equivalent to supposing

$$\rho = (1 - x'^2 - y'^2 - z'^2)^\beta . f(x'^2 + y'^2 + z'^2).$$

Now, when as in the present case, ρ can be expanded in a series of the entire powers of the quantities x', y', z', and of the various products of these powers, the function V will always admit of a similar expansion in the entire powers and products of the quantities x, y, z, provided the point p continues within the body A*, and as moreover V evidently depends on the distance $Op = r$ and is independent of θ and ϖ, the two other polar co-ordinates of p, it is easy to see that the quantity V when we substitute for x, y, z these values

$$x = r\cos\theta, \quad y = r\sin\theta\cos\varpi, \quad z = r\sin\theta\sin\varpi$$

will become a function of r, only containing none but the even powers of this variable.

But since we have

$$dv = r'^2 dr' \, d\theta' \, d\varpi' \sin\theta', \quad \text{and} \quad \rho = (1 - r'^2)^\beta . f(r'^2),$$

the value of V becomes

$$V = \int \frac{\rho dv}{g^{n-1}} = \int r'^2 dr' d\theta' d\varpi' \sin\theta' \, (1 - r'^2)^\beta f(r'^2) \cdot g^{1-n};$$

the integrals being taken from $\varpi' = 0$ to $\varpi' = 2\pi$, from $\theta' = 0$ to $\theta' = \pi$, and from $r' = 0$ to $r' = 1$.

* The truth of this assertion will become tolerably clear, if we recollect that V may be regarded as the sum of every element ρdv of the body's mass divided by the $(n-1)^{\text{th}}$ power of the distance of each element from the point p, supposing the density of the body A to be expressed by ρ, a continuous function of x', y', z'. For then the quantity V is represented by a continuous function, so long as p remains within A; but there is in general a violation of the law of continuity whenever the point p passes from the interior to the exterior space. This truth, however, as enunciated in the text, is demonstrable, but since the present paper is a long one, I have suppressed the demonstrations to save room.

Now V may be considered as composed of two parts, one V' due to the sphere B whose centre is at the origin O, and surface passes through the point p, and another V'' due to the shell S exterior to B. In order to obtain the first part, we must expand the quantity g^{1-n} in an ascending series of the powers of $\frac{r'}{r}$. In this way we get

$$g^{1-n} = [r^2 - 2rr' \{\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varpi' - \varpi)\} + r'^2]^{\frac{1-n}{2}}$$

$$= r^{1-n} \cdot \left(Q_0 + Q_1 \frac{r'}{r} + Q_2 \frac{r'^2}{r^2} + Q_3 \frac{r'^3}{r^3} + \&c.\right).$$

If then we substitute this series for g^{1-n} in the value of V', and after having expanded the quantity $(1 - r'^2)^\beta$, we effect the integrations relative to r', θ', and ϖ', we shall have a result of the form

$$V' = r^{4-n} \{A + Br^2 + Cr^4 + \&c.\}$$

seeing that in obtaining the part of V before represented by V', the integral relative to r' ought to be taken from $r' = 0$ to $r' = r$ only.

To obtain the value of V'', we must expand the quantity g^{1-n} in an ascending series of the powers of $\frac{r}{r'}$, and we shall thus have

$$g^{1-n} = (r^2 - 2rr' [\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varpi - \varpi')] + r'^2)^{\frac{1-n}{2}}$$

$$= r'^{1-n} \cdot \{Q_0 + Q_1 \frac{r}{r'} + Q_2 \frac{r^2}{r'^2} + Q_3 \frac{r^3}{r'^3} + \&c.\};$$

the coefficients Q_0, Q_1, Q_2, &c. being the same as before.

The expansion here given being substituted in V'', there will arise a series of the form

$$V'' = T_0 + T_1 + T_2 + T_3 + \&c.$$

of which the general term T_s is

$$T_s = \int d\theta' d\varpi' \sin\theta' \, Q_s \int r'^2 dr' \frac{r^s}{r'^{s+n-1}} (1 - r'^2)^\beta . f(r'^2);$$

the integrals being taken from $r' = r$ to $r' = 1$, from $\theta' = 0$ to $\theta' = \pi$, and from $\varpi' = 0$ to $\varpi' = 2\pi$. This will be evident by recollecting that the triple integral by which the value of V'' is expressed, is the same as the one before given for V, except that the integration relative to r', instead of extending from $r'=0$ to $r'=1$, ought only to extend from $r'=r$ to $r=1$.

But the general term in the function $f(r'^2)$ being represented by $A_t r'^{2t}$, the part of T_s dependent on this term will evidently be

$$(2)\ldots\ldots\ldots\ldots A_t r^s \int d\theta' d\varpi' \sin\theta'.\, Q_s \int r'^{2t+3-s-n} dr' (1-r'^2)^\beta;$$

the limits of the integrals being the same as before.

We thus see that the value of T_s and consequently of V'' would immediately be obtained, provided we had the value of the general integral

$$\int_r^1 r'^b dr' (1-r'^2)^\beta,$$

which being expanded and integrated becomes

$$\left.\begin{aligned} &\frac{1}{b+1} - \frac{\beta}{1}.\frac{1}{b+3} + \frac{\beta(\beta-1)}{1.2}.\frac{1}{b+5} - \&c. \\ -&\frac{r^{b+1}}{b+1} + \frac{\beta}{1}.\frac{r^{b+3}}{b+3} - \frac{\beta(\beta-1)}{1.2}.\frac{r^{b+5}}{b+5} + \&c. \end{aligned}\right\}$$

but since the first line of this expression is the well known expansion of

$$\left(\frac{p}{q}\right) \text{ or } \frac{\Gamma\left(\frac{p}{n}\right)\Gamma\left(\frac{q}{n}\right)}{n\Gamma\left(\frac{p+q}{n}\right)},$$

when $n=2.p=b+1$ and $q=2(\beta+1)$ we have ultimately,

$$(3)\ldots\ldots \int_r^1 r'^b dr'(1-r'^2)^\beta = \frac{\Gamma\left(\frac{b+1}{2}\right)\Gamma(\beta+1)}{2\Gamma\left(\frac{b+3}{2}+\beta\right)} - 1\times\frac{r^{b+1}}{b+1} + \frac{\beta}{1}\times\frac{r^{b+3}}{b+3} - \&c.$$

By means of the result here obtained, we shall readily find the value of the expression (2) which will evidently contain one term multiplied by r^s and an infinite number of others, in all of which the quantity r is affected with the exponent n. But as in the case under consideration, n may represent any number whatever, fractionary or irrational, it is clear that none of the terms last mentioned can enter into V, seeing that it ought to contain the even powers of r only, thence the terms of this kind entering into V'', must necessarily be destroyed by corresponding ones in V'. By rejecting them, therefore, the formula (2) will become

$$(2')\ldots\ldots\ldots \frac{\Gamma\left(t+2-\frac{s+n}{2}\right)\Gamma(\beta+1)}{2\Gamma\left(t+\beta+3-\frac{s+n}{2}\right)} A_t r^s . \int d\theta' d\varpi' \sin\theta' Q_s .$$

But as V ought to contain the even powers of r only, those terms in which the exponent s is an odd number, will vanish of themselves after all the integrations have been effected, and consequently the only terms which can appear in V, are of the form

$$(4)\ldots\ldots\ldots \frac{\Gamma\left(t+2-s'-\frac{n}{2}\right)\Gamma(\beta+1)}{2\Gamma\left(t+\beta+3-s'-\frac{n}{2}\right)} A_t r^{2s'} \int d\theta' d\varpi' \sin\theta' Q_{2s'};$$

where, since s is an even number, we have written $2s'$ in the place of s, and as $Q_{2s'}$ is always a rational and entire function of $\cos\theta'$, $\sin\theta' \cos\varpi'$, and $\sin\theta' \sin\varpi'$, the remaining integrations may immediately be effected.

Having thus the part of $T'_{2s'}$ due to any term $A_t r'^{2t}$ of the function $f(r'^2)$ we have immediately the value of $T'_{2s'}$ and consequently of V'', since

$$V'' = U' + T'_0 + T'_2 + T'_4 + T'_6 + \&c.;$$

U' representing the sum of all the terms in V'' which have been rejected on account of their form, and $T'_0\ T'_1\ T'_2$ the value of $T_0\ T_1\ T_2$, &c. obtained by employing the truncated formula (2′) in the place of the complete one (2).

$$\text{But } V = V' + V'' = V' + U' + T'_0 + T'_2 + T'_4 + T'_6 + \&c.$$

or by transposition,

$$V - T'_0 - T'_2 - T'_4 - T'_6 - \&c. = V' + U',$$

and as in this equation, the function on the left side contains none but the even powers of the indeterminate quantity r, whilst that on the right does not contain any of the even powers of r, it is clear that each of its sides ought to be equated separately to zero. In this way the left side gives

$$(5)\ldots\ldots\ldots V = T_0' + T_2' + T_4' + T_6' + \&c.$$

Hitherto the value of the exponent β has remained quite arbitrary, but the known properties of the function Γ will enable us so to determine β, that the series just given shall contain a finite number of terms only. We shall thus greatly simplify the value of V, and reduce it in fact to a rational and entire function of r^2.

For this purpose, we may remark that

$$\Gamma(0) = \infty, \quad \Gamma(-1) = \infty, \quad \Gamma(-2) = \infty, \textit{ in infinitum.}$$

If therefore we make $-\frac{n}{2} + \beta =$ any whole number positive or negative, the denominator of the function (4) will become infinite, and consequently the function itself will vanish when s' is so great that $-\frac{n}{2} + \beta + t + 3 - s'$ is equal to zero or any negative number, and as the value of t never exceeds a certain number, seeing that $f(r'^2)$ is a rational and entire function, it is clear that the series (4) will terminate of itself, and V become a rational and entire function of r^2.

(2) The method that has been employed in the preceding article where the function by which the density is expressed is of the particular form

$$\rho = (1 - r'^2)^\beta . f(r'^2)$$

may by means of a very slight modification, be applied to the far more general value

$$\rho = (1 - r'^2)^\beta f(x', y', z') = (1 - x'^2 - y'^2 - z'^2)^\beta f(x', y', z')$$

where f is the characteristic of any rational and entire function whatever: and the same value of β which reduces V to a rational and entire function of r^2 in the first case, reduces it in the second to a similar function of x, y, z and the rectangular co-ordinates of p.

To prove this, we may remark that the corresponding value V will become

$$V = \int r'^2 dr' d\theta' d\varpi' \sin\theta' (1 - r'^2)^\beta f(x', y', z') g^{1-n};$$

the integral being conceived to comprehend the whole volume of the sphere.

Let now the function f be divided into two parts, so that

$$f(x', y', z') = f_1(x', y', z') + f_2(x', y', z');$$

f_1 containing all the terms of the function f, in which the sum of the exponents of x', y', z' is an odd number; and f_2 the remaining terms, or those where the same sum is an even number. In this way we get

$$V = V_1 + V_2;$$

the functions V_1 and V_2 corresponding to f_1 and f_2, being

$$V_1 = \int r'^2 dr' d\theta' d\varpi' \sin\theta' (1 - r'^2)^\beta f_1(x', y', z') g^{1-n},$$

$$V_2 = \int r'^2 dr' d\theta' d\varpi' \sin\theta' (1 - r'^2)^\beta f_2(x', y', z') g^{1-n}.$$

We will in the first place endeavour to determine the value V_1; and for this purpose, by writing for x', y', z' their values before given in r', θ', ϖ', we get

$$f_1(x', y', z') = r'\psi(r'^2);$$

the coefficients of the various powers of r'^2 in $\psi(r'^2)$ being evidently rational and entire functions of $\cos\theta'$, $\sin\theta' \cos\varpi'$, and $\sin\theta' \sin\varpi'$. Thus

$$V_1 = \int r'^2 dr' d\theta' d\varpi' \sin\theta' (1 - r'^2)^\beta r' \psi(r'^2) g^{1-n};$$

this integral, like the foregoing, comprehending the whole volume of the sphere.

Now as the density corresponding to the function V_1 is

$$\rho_1 = (1 - x'^2 - y'^2 - z'^2)^\beta f_1(x', y', z'),$$

it is clear that it may be expanded in an ascending series of the entire powers of x', y', z', and the various products of these powers consequently, as was before remarked (Art. 1.), V_1 admits of an analagous expansion in entire powers and products of x, y, z. Moreover, as the density ρ_1

ψ a rational & entire function

retains the same numerical value, and merely changes its sign when we pass from the element dv to a point diametrically opposite, where the co-ordinates x', y', z' are replaced by $-x', -y', -z'$: it is easy to see that the function V_1, depending upon ρ_1, possesses a similar property, and merely changes its sign when x, y, z, the co-ordinates of p, are changed into $-x, -y, -z$. Hence the nature of the function V_1 is such that it can contain none but the odd powers of r, when we substitute for the rectangular co-ordinates x, y, z, their values in the polar co-ordinates r, θ, ϖ.

Having premised these remarks, let us now suppose V_1 is divided into two parts, one V_1' due to the sphere B which passes through the particle p, and the other V_1'' due to the exterior shell S. Then it is evident by proceeding, as in the case where $\rho = (1-r'^2)^\beta f(r^2)$, that V_1' will be of the form

$$V_1' = r^{5-n}\{A + Br^2 + Cr^4 + \&c.\};$$

the coefficients A, B, C, &c. being quantities independent of the variable r.

In like manner we have also

$$V_1'' = \int r'^2 dr' d\theta' d\varpi' \sin\theta' (1-r'^2)^\beta . r'\psi(r'^2) g^{1-n};$$

the integrals being taken from $r' = r$ to $r = 1$, from $\theta' = 0$ to $\theta' = \pi$, and from $\varpi' = 0$ to $\varpi' = 2\pi$.

By substituting now the second expansion of g^{1-n} before used (Art. 1.), the last expression will become

$$V_1'' = T_0 + T_1 + T_2 + T_3 + \&c.$$

of which series the general term is

$$T_s = \int d\theta' d\varpi' \sin\theta' Q_s \int r'^{4-n} dr' (1-r'^2)^\beta \frac{r^s}{r'^s} \psi(r'^2).$$

Moreover, the general term of the function $\psi(r'^2)$ being represented by $A_t r'^{2t}$, the portion of T_s due to this term, will be

$$(a) \ldots\ldots\ldots r^s \int d\theta' d\varpi' \sin\theta' Q_s A_t \int r'^{4-n+2t-s} dr' (1-r'^2)^\beta;$$

the limits of the integrals being the same as before.

If now we effect the integrations relative to r' by means of the formula (3), Art. 1, and reject as before those powers of the variable r, in which it is affected, with the exponent n, since these ought not enter into the function V_1, the last formula will become

$$(a')\ldots\ldots\ldots\frac{\Gamma\left(\frac{5-n+2t-s}{2}\right)\Gamma(\beta+1)}{2\Gamma\left(\frac{7+2\beta-n+2t-s}{2}\right)}r^s\int d\theta' d\varpi' \sin\theta' Q_s A_t,$$

and as V_1 ought to contain none but the odd powers of r, we may make $s=2s'+1$, and disregard all those terms in which s is an even number, since they will necessarily vanish after all the operations have been effected. Thus the only remaining terms will be of the form

$$\frac{\Gamma\left(\frac{4-n+2t-2s'}{2}\right)\Gamma(\beta+1)}{2.\Gamma\left(\frac{6+2\beta-n+2t-2s'}{2}\right)}r^{2s'+1}\int d\theta' d\varpi' \sin\theta' Q_{2s'+1}A_t;$$

where, as A_t and $Q_{2s'+1}$ are both rational and entire functions of $\cos\theta'$, $\sin\theta'\cos\varpi'$, $\sin\theta'\sin\varpi'$, the remaining integrations from $\theta'=0$ to $\theta'=\pi$, and $\varpi'=0$ to $\varpi'=2\pi$, may easily be effected in the ordinary way.

If now we follow the process employed in the preceding article, and suppose T_0', T_1', T_2', &c. are what T_0, T_1, T_2, &c. become when we use the truncated formula (a') instead of the complete one (a), we shall readily get

$$V_1 = T_1' + T_3' + T_5' + T_7' + \&c.$$

In like manner, from the value of V_2 before given, we get

$$V_2'' = \int r'^2 dr' d\theta' d\varpi' \sin\theta'(1-r'^2)^\beta \phi(r'^2) g^{1-n};$$

the integrals being taken from $r'=r$ to $r=1$, from $\theta'=0$ to $\theta'=\pi$, and from $\varpi=0$ to $\varpi=2\pi$.

Expanding now g^{1-n} as before, we have

$$V_2'' = U_0 + U_1 + U_2 + U_3 + \&c.$$

where

$$U_s = \int d\theta' d\varpi' \sin\varpi' Q_s \int_r^1 r'^{3-n} dr'(1-r'^2)^\beta \frac{r^s}{r'^s}\phi(r'^2),$$

and the part of U_s due to the general term $B_t r'^{2t}$ in $\phi(r'^2)$, will be

$$(b)\ldots\ldots\ldots\ r^s \int d\theta' d\varpi' \sin\theta' Q_s B_t \int_r^1 r'^{3-n+2t-s} dr' (1-r'^2)^\beta;$$

which, by employing the formula (3^r) Art. 1., and rejecting the inadmissible terms, gives for truncated formula

$$(b')\ldots\ldots \frac{\Gamma\left(\frac{4-n+2t-s}{2}\right)\Gamma(\beta+1)}{2\Gamma\left(\frac{6-n+2\beta+2t-s}{2}\right)} r^s \int d\theta' d\varpi' \sin\theta' Q_s B_t.$$

By continuing to follow exactly the same process as was before employed in finding the value of V_1, we shall see that s must always be an even number, say $2s'$; and thus the expression immediately preceding will become

$$\frac{\Gamma\left(\frac{4-n+2t-2s'}{2}\right)\Gamma(\beta+1)}{2\Gamma\left(\frac{6-n+2\beta+2t-2s'}{2}\right)} r^{2s'} \int d\theta' d\varpi' \sin\theta' Q_{2s'} B_t.$$

Moreover, the value of V_2 will be

$$V_2 = U_0' + U_2' + U_4' + U_6' + \&c.;$$

U_0', U_1', U_2', U_3', &c. being what U_0, U_1, U_2, &c. become when we use the formula (b') instead of the complete one (b).

The value of V answering to the density

$$\rho = \rho_1 + \rho_2 = (1-r'^2)^\beta . f(x', y', z'),$$

by adding together the two parts into which it was originally divided, therefore, becomes

$$V = V_1 + V_2 = T_1' + T_3' + T_5' + T_7' + \&c.$$
$$+ U_0' + U_2' + U_4' + U_6' + \&c.$$

When β is taken arbitrarily, the two series entering into V extend in infinitum, but by supposing as before, Art. 1.,

$$\frac{-n}{2} + \beta = \omega;$$

ω representing any whole number, positive or negative, it is clear from the form of the quantities entering into $T_{2s'+1}$ and $U_{2s'}$, and from the known properties of the function Γ, that both these series will terminate of themselves, and the value of V be expressed in a finite form; which, by what has preceded, must necessarily reduce itself to a rational and entire function of the rectangular co-ordinates x, y, z. It seems needless, after what has before been advanced, (Art. 1.) to offer any proof of this: we will, therefore, only remark that if γ represents the degree of the function $f(x', y', z')$, the highest degree to which V can ascend will be

$$\gamma + 2\omega + 4.$$

In what immediately precedes, ω may represent any whole number whatever, positive or negative; but if we make $\omega = -2$, and consequently, $\beta = \frac{n-4}{2}$, the degree of the function V is the same as that of the factor

$$f(x', y', z'),$$

comprised in ρ. This factor then being supposed the most general of its kind, contains as many arbitrary constant quantities as there are terms in the resulting function V. If, therefore, the form of the rational and entire function V be taken at will, the arbitrary quantities contained in $f(x', y', z')$ will in case $\omega = -2$ always enable us to assign the corresponding value of ρ, and the resulting value of $f(x', y', z')$ will be a rational and entire function of the same degree as V. Therefore, in the case now under consideration, we shall not only be able to determine the value of V when ρ is given, but shall also have the means of solving the inverse problem, or of determining ρ when V is given; and this determination will depend upon the resolution of a certain number of algebraical equations, all of the first degree.

3. The object of the preceding sketch has not been to point out the most convenient way of finding the value of the function V, but merely to make known the spirit of the method; and to show on what its success depends. Moreover, when presented in this simple form, it has the advantage of being, with a very slight modification, as applicable to any ellipsoid whatever as to the sphere itself. But when

spheres only are to be considered, the resulting formulæ, as we shall afterwards show, will be much more simple if we expand the density ρ in a series of functions similar to those used by Laplace (*Mec. Cel.* Liv. iii.): it will however be advantageous previously to demonstrate a general property of functions of this kind, which will not only serve to simplify the determination of V, but also admit of various other applications of $d\sigma$.

Suppose, therefore, $Y^{(i)}$ is a function of θ and ϖ, of the form considered by Laplace (*Mec. Cel.* Liv. iii.), r, θ, ϖ being the polar co-ordinates referred to the axes X, Y, Z, fixed in space, so that

$$x = r\cos\theta, \quad y = r\sin\theta\cos\varpi, \quad z = r\sin\theta\sin\varpi;$$

then, if we conceive three other fixed axes X_1, Y_1, Z_1, having the same origin but different directions, $Y^{(i)}$ will become a function of θ_1 and ϖ_1, and may therefore be expanded in a series of the form

$$(6)\ldots\ldots\ldots Y^{(i)} = Y_1^{(0)} + Y_1^{(1)} + Y_1^{(2)} + Y_1^{(3)} + \&c.$$

Suppose now we take any other point p' and mark its various co-ordinates with an accent, in order to distinguish them from those of p; then, if we designate the distance pp' by (p, p'), we shall have

$$\frac{1}{(p, p')} = \{r^2 - 2rr'[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varpi - \varpi')] + r'^2\}^{-\frac{1}{2}}$$
$$= \frac{1}{r}\left(Q^{(0)} + Q^{(1)}\cdot\frac{r'}{r} + Q^{(2)}\frac{r'^2}{r^2} + Q^{(3)}\frac{r'^3}{r^3} + \&c.\right),$$

as has been shewn by Laplace in the third book of the *Mec. Cel.*, where the nature of the different functions here employed is completely explained.

In like manner, if the same quantity is expressed in the polar co-ordinates belonging to the new system of axes X_1, Y_1, Z_1, we have, since the quantities r and r' are evidently the same for both systems,

$$\frac{1}{(p, p')} = \frac{1}{r}\left(Q_1^{(0)} + Q_1^{(1)}\frac{r'}{r} + Q_1^{(2)}\frac{r'^2}{r^2} + Q_1^{(3)}\frac{r'^3}{r^3} + \&c.\right);$$

and it is also evident from the form of the radical quantity of which the series just given are expansions, that whatever number i may represent, $Q_1^{(i)}$ will be immediately deduced from $Q^{(i)}$ by changing θ, ϖ, θ', ϖ' into θ_1, ϖ_1, θ_1', ϖ_1'. But since the quantity $\frac{r'}{r}$ is indeterminate, and may be taken at will, we get, by equating the two values of $\frac{1}{(p, p')}$ and comparing the like powers of the indeterminate quantity $\frac{r'}{r}$,

$$Q^{(i)} = Q_1^{(i)}.$$

If now we multiply the equation (6) by the element of a spherical surface whose radius is unity, and then by $Q^{(h)} = Q_1^{(h)}$, we shall have, by integrating and extending the integration over the whole of this spherical surface,

$$\int d\mu d\varpi Q^{(h)} Y^{(i)} = \int d\mu_1 d\varpi_1 Q_1^{(h)} \{ Y_1^{(0)} + Y_1^{(1)} + Y_1^{(2)} + \&c. \}.$$

Which equation, by the known properties of the functions $Q^{(h)}$ and $Y^{(i)}$, reduces itself to

$$0 = \int d\mu_1 d\varpi_1 Q_1^{(h)} Y_1^{(h)},$$

when h and i represent different whole numbers. But by means of a formula given by Laplace (*Mec. Cel.* Liv. iii. No. 17.) we may immediately effect the integration here indicated, and there will thus result

$$0 = \frac{4\pi}{2h+1} Y_1'^{(h)};$$

$Y_1'^{(h)}$ being what $Y_1^{(h)}$ becomes by changing θ_1, ϖ_1 into θ_1', ϖ_1', and as the values of these last co-ordinates, which belong to p', may be taken arbitrarily like the first, we shall have generally $Y_1^{(h)}$, except when $h = i$. Hence, the expansion (6) reduces itself to a single term, and becomes

$$Y^{(i)} = Y_1^{(i)}.$$

We thus see that the function $Y^{(i)}$ continues of the same form even when referred to any other system of axes X_1, Y_1, Z_1, having the same origin O with the first.

This being established, let us conceive a spherical surface whose center is at the origin O of the co-ordinates and radius r', covered with fluid, of which the density $\rho = Y'^{(i)}$; then, if $d\sigma'$ represent any element of this surface, and we afterwards form the quantity

$$V = \int \rho d\sigma' \psi(g^2) = \int Y'^{(i)} d\sigma' \psi(g^2);$$

the integral extending over the whole spherical surface, g being the distance p, $d\sigma'$ and ψ the characteristic of any function whatever. I say, the resulting value of V will be of the form

$$V = Y^{(i)} R;$$

R being a function of r, the distance Op only and $Y^{(i)}$ what $Y'^{(i)}$ becomes by changing θ', ϖ', the polar co-ordinates, into θ, ϖ, the like co-ordinates of the point p.

To justify this assertion, let there be taken three new axes X_1, Y_1, Z_1, so that the point p may be upon the axis X_1; then, the new polar co-ordinates of $d\sigma'$ may be written r', θ', ϖ', those of p being r, 0, ϖ, and consequently, the distance will become

$$g = \sqrt{(r'^2 - 2rr' \cos\theta_1' + r^2)};$$

and as $d\sigma' = r'^2 d\theta_1' d\varpi_1' \sin\theta_1'$, we immediately obtain

$$V = \int Y'^{(i)} r'^2 d\theta_1 d\varpi_1 \sin\theta_1 \psi(r^2 - 2rr' \cos\theta_1' + r'^2)$$
$$= r'^2 \int_0^\pi d\theta_1' \sin\theta_1' \psi(r^2 - 2rr' \cos\theta_1' + r'^2) \int_0^{2\pi} d\varpi_1' Y'^{(i)}.$$

Let us here consider more particularly the nature of the integral

$$\int_0^{2\pi} d\varpi_1' Y'^{(i)}.$$

In the preceding part of the present article, it has been shown that the value of $Y'^{(i)}$, when expressed in the new co-ordinates, will be of the form $Y_1'^{(i)}$; but all functions of this form (Vide *Mec. Cel.* Liv. iii.) may be expanded in a finite series containing $2i+1$ terms, of which the first is independent of the angle ϖ_1', and each of the others has for a factor a sine or cosine of some entire multiple of this same angle. Hence, the integration relative to ϖ_1' will cause all the last mentioned terms to vanish, and we shall only have to attend to the first here. But this term is known to be of the form

$$k\left(\mu_1'^i - \frac{i.i-1}{2.2i-1}\mu_1'^{i-2} + \frac{i.i-1.i-2.i-3}{2.4.2i-1.2i-3}m_1'^{i-4} - \&c.\right),$$

and consequently, there will result

$$\int_0^{2\pi} d\varpi_1' \, Y'^{(i)} = 2\pi k \left(\mu_1'^{i} - \frac{i.i-1}{2.2i-1} \mu_1'^{i-2} + \frac{i.i-1.i-2.i-3}{2.4.2i-1.2i-3} \mu_1'^{i-4} - \&c. \right);$$

where $\mu_1' = \cos\theta_1'$ and k is a quantity independent of θ_1' and ϖ_1', but which may contain the co-ordinates θ, ϖ, that serve to define the position of the axis X_1 passing through the point p.

It now only remains to find the value of the quantity k, which may be done by making $\theta_1' = 0$, for then the line r coincides with the axis X_1, and $Y^{(i)}$ during the integration remains constantly equal to $Y^{(i)}$, the value of the density at this axis. Thus we have

$$2\pi Y^{(i)} = 2\pi k \left(1 - \frac{i.i-1}{2.2i-1} + \frac{i.i-1.i-2.i-3}{2.4.2i-1.2i-3} - \&c. \right):$$

or, by summing the series within the parenthesis, and supplying the common factor 2π,

$$Y^{(i)} = \frac{1.2.3.........\;i}{1.3.5.........2i-1} k,$$

and, by substituting the value of k, drawn from this equation in the value of the required integral given above, we ultimately obtain

$$\int_0^{2\pi} d\varpi_1' \, Y'^{(i)} = 2\pi Y^{(i)} \frac{1.3.5......2i-1}{1.2.3......\;i} \left(\mu_1'^{i} - \frac{i.i-1}{2.2i-1} \mu_1'^{i-2} + \&c. \right).$$

If now, for abridgement, we make

$$(i) = \mu_1'^{i} - \frac{i.i-1}{2.2i-1} \mu_1'^{i-2} + \frac{i.i-1.i-2.i-3}{2.4.2i-1.2i-3} \mu_1'^{i-4} - \&c.$$

we shall obtain, by substituting the value of the integral just found in that of V before given,

$$V = Y^{(i)} . 2\pi r'^2 \frac{1.3.5......2i-1}{1.2.3......\;i} \int_{-1}^{1} d\mu_1' \, (i) \, \psi (r^2 - 2rr'\mu_1' + r'^2);$$

which proves the truth of our assertion.

From what has been advanced in the preceding article, it is likewise very easy to see that if the density of the fluid within a sphere of any radius be every where represented by

$$\rho = Y'^{(i)} \phi(r);$$

ϕ being the characteristic of any function whatever; and we afterwards form the quantity

$$V = \int \rho dv \,\psi\,(g^2),$$

where dv represents an element of the sphere's volume, and g the distance between dv and any particle p under consideration, the resulting value of V will always be of the form

$$V = Y^{(i)}.R;$$

$Y^{(i)}$ being what $Y'^{(i)}$ becomes by changing θ', ϖ', the polar co-ordinates of the element dv into θ, ϖ, the co-ordinates of the point p; and R being a function of r, the remaining co-ordinate of p, only.

4. Having thus demonstrated a very general property of functions of the form $Y^{(i)}$, let us now endeavour to determine the value of V for a sphere whose radius is unity, and containing fluid of which the density is every where represented by

$$\rho = (1 - x'^2 - y'^2 - z'^2)^\beta f(x', y', z');$$

x', y', z' being the rectangular co-ordinates of dv, an element of the sphere's volume, and f, the characteristic of any rational and entire function whatever.

For this purpose we will substitute in the place of the co-ordinates x', y', z' their values

$$x' = r' \cos\theta': \quad y' = r' \sin\theta' \cos\varpi': \quad z' = r' \sin\theta' \sin\varpi';$$

and afterwards expand the function $f(x', y', z')$ by Laplace's simple method, (*Mec. Cel.* Liv. iii. No. 16.). Thus,

$$(7)\ldots\ldots\ldots f(x', y', z') = f'^{(0)} + f'^{(1)} + f'^{(2)} + \&c.\ldots\ldots + f'^{(s)};$$

s being the degree of the function $f(x', y', z')$.

It is likewise easy to perceive that any term $f'^{(i)}$ of this expansion may be again developed thus,

$$f'^{(i)} = f_0'^{(i)} r'^i + f_1'^{(i)} r'^{i+2} + f_2'^{(i)} r'^{i+4} + \&c.;$$

and as every coefficient of the last developement is of the form $U^{(i)}$, (*Mec. Cel.* Liv. iii.), it is easy to see that the general term $f_t'^{(i)} r'^{i+2t}$ may always be reduced to a rational and entire function of the original co-ordinates x', y', z'.

If now we can obtain the part of V due to the term

$$f_t'^{(i)} \cdot r'^{i+2t},$$

we shall immediately have the value of V by summing all the parts corresponding to the various values of which i and t are susceptible. But from what has before been proved (Art. 3.), the part of V now under consideration must necessarily be of the form $Y^{(i)}$; representing, therefore, this part by $V_t^{(i)}$, we shall readily get

$$V_t^{(i)} = \int_0^1 r'^{i+2t+2} dr' (1 - r'^2)^\beta \int d\varpi' d\theta' \sin\theta' f_t'^{(i)} g^{1-n}.$$

Moreover from what has been shown in the same article, it is easy to see that we have generally

$$\int Y'^{(i)} d\varpi' d\theta' \sin\theta' \psi(g^2) = 2\pi Y^{(i)} \frac{1.3.5....2i-1}{1.2.3....i} \int_{-1}^{1} d\mu_1' (i) \psi(r^2 - 2rr'\mu_1' + r'^2);$$

ψ being the characteristic of any function whatever, and $Y^{(i)}$ what $Y'^{(i)}$ becomes by substituting θ, ϖ the polar co-ordinates of p in the place of θ', ϖ', the analogous co-ordinates of the element dv. If therefore in the expression immediately preceding, we make

$$Y'^{(i)} = f_t'^{(i)} \text{ and } \psi(g^2) = g^{n-1} = (g^2)^{\frac{1-n}{2}},$$

and substitute the value of the integral thus obtained for its equal in $V_t^{(i)}$ there will arise

$$(8)\ V_t^{(i)} = 2\pi f_t^{(i)} \frac{1.3.5...2i-1}{1.2.3...i} \int_0^1 r'^{i+2t+2} dr' (1-r'^2)^\beta \int_{-1}^{1} d\mu'_1 (i) . (r^2 - 2rr'\mu'_1 + r'^2)^{\frac{1-n}{2}};$$

where $f_t^{(i)}$ is deduced from $f_t'^{(i)}$ by changing θ', ϖ' into θ, ϖ, and (i), for abridgement, is written in the place of the function

$$\mu_1'^i - \frac{i.i-1}{2.2i-1} \mu_1'^{i-2} + \frac{i.i-1.i-2.i-3}{2.4.2i-1.2i-3} \mu_1'^{i-4} - \&c.$$

As the integral relative to μ'_1 which enters into the expression on the right side of the equation (8) is a definite one, and depends therefore on the two extreme values of μ'_1 only, it is evident that in the determination of this integral, it is altogether useless to retain the accents by which μ'_1 is affected. But by omitting these superfluous accents, we shall have to calculate the value of the quantity

$$\int_{-1}^{1} d\mu\,(i)\,.\,(r^2 - 2rr'\mu + r'^2)^{\frac{1-n}{2}};$$

where

$$(i) = \mu^i - \frac{i\,.\,i-1}{2\,.\,2i-1}\,\mu^{i-2} + \frac{i\,.\,i-1\,.\,i-2\,.\,i-3}{2\,.\,4\,.\,2i-1\,.\,2i-3}\,\mu^{i-4} - \&c.$$

The method which first presents itself for determining the value of the integral in question, is to expand the quantity $(r^2 - 2rr'\mu + r'^2)^{\frac{1-n}{2}}$ by means of the Binomial Theorem, to replace the various powers of μ by their values in functions similar to (i) and afterwards to effect the integrations by the formulæ contained in the third Book of the *Mec. Cel.* For this purpose we have the general equation

$$(9)\ldots\ldots\ldots\mu^i = (i) + \frac{i\,.\,i-1}{2\,.\,2i-1}\,(i-2) + \frac{i\,.\,i-1\,.\,i-2\,.\,i-3}{2\,.\,4\,.\,2i-3\,.\,2i-5}\,(i-4)$$

$$+ \frac{i\,.\,i-1\,.\,i-2\,.\,i-3\,.\,i-4\,.\,i-5}{2\,.\,4\,.\,6\,.\,2i-5\,.\,2i-7\,.\,2i-9}\,(i-6) + \&c.$$

To remove all doubt of the correctness of this equation, we may multiply each of its sides by μ, and reduce the products on the right by means of the relation

$$\mu\,(i) = (i+1) + \frac{i^2}{2i-1\,.\,2i+1}\,(i-1),$$

which it is very easy to prove exists between functions of the form (i). In this way it will be seen that if the equation (9) holds good for any power μ^i it will do so likewise for the following power μ^{i+1}, and as it is evidently correct when $i = 1$, it is therefore necessarily so, whatever whole number i may represent.

Now by means of the Binomial Theorem, we obtain when $r \angle r'$

$$r'^{n-1}.(r^2 - 2rr'\mu + r'^2)^{\frac{1-n}{2}} = \left(1 - 2\mu\frac{r}{r'} + \frac{r^2}{r'^2}\right)^{\frac{1-n}{2}}$$

$$= \Sigma_0^\infty \frac{n-1\,.\,n+1\,.\,n+3\ldots\ldots n+2s-3}{2\;.\;4\;.\;6\;\ldots\ldots\;2s}\left(2\mu\frac{r}{r'} - \frac{r^2}{r'^2}\right)^s;$$

If now we conceive the quantity $\left(2\mu\frac{r}{r'} - \frac{r^2}{r'^2}\right)^s$ to be expanded by the same theorem, it is easy to perceive that the term having $\left(\frac{r}{r'}\right)^{i+2t'}$ for factor is

$$\frac{n-1.n+1.n+3\ldots\ldots n+2i+4t'-3}{2\;.\;4\;.\;6\;\ldots\ldots\;2i+4t'}\,2^{i+2t'}\mu^{i+2t'}\left(\frac{r}{r'}\right)^{i+2t'}$$

$$-\frac{n-1.n+1\ldots\ldots n+2i+4t'-5}{2\;.\;4\;\ldots\ldots\;2i+4t'-2}\,.\,2^{i+2t'-2}\mu^{i+2t'-2}\left(\frac{r}{r'}\right)^{i+2t-2}\frac{r^2}{r'^2}\,.\,\frac{i+2t-1}{1}$$

$$+\frac{n-1.n+1\ldots\ldots n+2i+4t'-7}{2\;.\;4\;\ldots\ldots\;2i+4t'-4}\,(2\mu)^{i+2t'-4}\left(\frac{r}{r'}\right)^{i+2t'-4}\frac{r^4}{r'^4}\,.\,\frac{i+2t-2\,.\,i+2t'-3}{1\;.\;2}$$

$$-\ \&c.\ldots\ldots\ldots\ldots\ldots\ldots\&c.\ldots\ldots\ldots\ldots\ldots\ldots\&c.\ldots\ldots\ldots$$

and therefore the coefficient of $\left(\frac{r}{r'}\right)^{i+2t'}$ in the expansion of the function

$$\left(1-2\mu\frac{r}{r'}+\frac{r^2}{r'^2}\right)^{\frac{1-n}{2}},$$

will be expressed by

$$\Sigma\,\frac{n-1.n+1\ldots\ldots n+2i+4t'-2s-3}{2\;.\;4\;\ldots\ldots\;2i+4t'-2s}\,(2\mu)^{i+2t-2s}.(-1)^s$$

$$.\,\frac{i+2t'-s\,.\,i+2t'-s-1\ldots\ldots i+2t'-2s+1}{1\;.\;2\;\ldots\ldots\;s}.$$

Hence the portion of this coefficient containing the function (i), when the various powers of μ have been replaced by their values in functions of this kind agreeably to the preceding observation will be found, by means of the equation (9), to be

$$(i)\,\Sigma\,\frac{n-1\,.\,n+1\ldots\ldots n+2i+4t'-2s-3}{2\;.\;4\;\ldots\ldots\;2i+4t'-2s}$$

$$\times\,\frac{i+2t'-2s\,.\,i+2t'-2s-1\ldots\ldots i+1}{2\,.\,4\ldots\ldots 2t'-2s\times 2i+2t'-2s+1\,.\,2i+2t'-2s-1\ldots 2i+3}$$

$$\ldots\ldots 2^{i+2t'-2s}(-1)^s\times\frac{i+2t'-s\,.\,i+2t'-s-1\ldots\ldots\ldots i+2t'-2s+1}{1\;.\;2\;.\;3\;\ldots\ldots\ldots\;s}$$

$$=(i)\,\Sigma\,\frac{n-1\,.\,n+1\,.\,n+3\ldots\ldots n+2i+4t'-2s-3}{2\;.\;4\;.\;6\;\ldots\ldots\;2i+4t'-2s}\,.\,2^{i+2t'-2s}(-1)^s$$

$$\ldots\ldots\times\frac{i+1\,.\,i+2\,.\,i+3\,.\,i+4\ldots\ldots\ldots i+2t'-s}{1\,.\,2\,.\,3\ldots\ldots s\times 2\,.\,4\,.\,6\ldots\ldots 2t'-2s\times 2i+2t'-2s+1\ldots\ldots 2i+3}$$

$$=2^i\,.\,(i)\,.\,\Sigma\,\frac{(-1)^s\,.\,n-1\,.\,n+1\,.\,n+3\ldots\ldots\ldots n+2i+4t'-2s-3}{2\,.\,4\ldots\ldots 2i\times 2\,.\,4\ldots\ldots 2s\times 2\,.\,4\ldots\ldots 2t'-2s\times 2i+2t'-2s+1\ldots 2i+3}$$

$$=\frac{3\,.\,5\,.\,7\ldots\ldots 2i+1}{1\,.\,2\,.\,3\ldots\ldots\;i}\,(i)\times\frac{n-1\,.\,n+1\,.\,n+3\ldots\ldots n+2i+2t'-3}{3\;.\;5\;.\;7\;\ldots\ldots\;2i+2t'+1}$$

$$\ldots\ldots\times\Sigma\,.\,(-1)^s\,\frac{n+2i+2t'-1\ldots\ldots n+2i+4t'-2s-3}{2\,.\,4\,.\,6\;\;\ldots\ldots\;\;2t'-2s}$$

$$\times\frac{2i+2t'-2s+3\ldots\ldots 2i+2t'+1}{2\,.\,4\,.\,6\;\;\ldots\ldots\;\;2s},$$

where all the finite integrals may evidently be extended from $s=0$ to $s=\infty$, and it is clear that the last of these integrals is equal to the coefficient of $x^{t'}$ in the product

$$\left\{1+\frac{n+2i+2t'-1}{2}\,x+\frac{n+2i+2t'-1\,.\,n+2i+2t'+1}{2\,.\,4}\,x^2+\&\text{c. } in\ inf.\right\}$$

$$\times\left\{1-\frac{2i+2t'+1}{2}\,x+\frac{2i+2t'+1\,.\,2i+2t'-1}{2\,.\,4}\,x^2-\&\text{c. } in\ inf.\right\}$$

If now we write in the place of the series their known values, the preceding product will become

$$(1-x)^{-\frac{n+2i+2t'-1}{2}}\times(1-x)^{\frac{2i+2t'+1}{2}}=(1-x)^{\frac{2-n}{2}},$$

and consequently the value of the required coefficient of $x^{t'}$ is

$$\frac{n-2\,.\,n\,.\,n+2\ldots\ldots\ldots n+2t'-4}{2\;.\,4\,.\;6\;\ldots\ldots\ldots\;2t'}.$$

This quantity being substituted in the place of the last of the finite integrals gives for the value of that portion of the coefficient of

$$\left(\frac{r}{r'}\right)^{i+2t'}\text{ in }\left(1-2\mu\frac{r}{r'}+\frac{r^2}{r'^2}\right)^{\frac{1-n}{2}},$$

which contains the function (i) the expression

$$\frac{3\,.\,5\,.\,7\ldots\ldots 2i+1}{1\,.\,2\,.\,3\ldots\ldots\;i}\times\frac{n-1\,.\,n+1\ldots\ldots n+2i+2t'-3}{3\;.\;5\;\ldots\ldots\;2i+2t'+1}\times\frac{n-2\,.\,n\ldots\ldots n+2t'-4}{2\,.\,4\ldots\ldots\;2t'}\,(i).$$

By multiplying the last expression by $\left(\frac{r}{r'}\right)^{i+2t'}$, and taking the sum of all the resulting values which arise when we make successively

$$t'=0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \&c.\ \textit{in infinitum},$$

we shall obtain the value of the term $Y^{(i)}$ contained in the expression

$$\left(1-2\mu\frac{r'}{r'}+\frac{r^2}{r'^2}\right)^{\frac{1-n}{2}}=Y^{(0)}+Y^{(1)}+Y^{(2)}+Y^{(3)}+\&c.$$

Hence,

$$Y^{(i)}=\frac{3.5\ldots\ldots 2i+1}{1.2\ldots\ldots\ i}\,(i)\,\Sigma\,\frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{3\ .\ 5\ \ldots\ldots\ 2i+2t'+1}$$

$$\times\frac{n-2.n\ldots\ldots n+2t'-4}{2.4\ldots\ldots\ 2t'}\left(\frac{r}{r'}\right)^{i+2t'};$$

the finite integral extending from $t'=0$ to $t'=\infty$.

But by the known properties of functions of this kind, we have by substituting for $Y^{(i)}$ its value

$$\int_{-1}^{1}d\mu\,(i)\left(1-2\mu\frac{r}{r'}+\frac{r^2}{r'^2}\right)^{\frac{1-n}{2}}=\int_{-1}^{1}d\mu\,(i)\,.\,Y^{(i)}$$

$$=\frac{3.5.7\ldots\ldots 2i+1}{1.2.3\ldots\ldots\ i}\int d\mu\,(i)^2\times\Sigma\,\frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{3\ .\ 5\ \ldots\ldots\ 2i+2t'+1}$$

$$\times\frac{n-2.n\ldots\ldots n+2t'-4}{2.4\ldots\ldots\ 2t'}\left(\frac{r}{r'}\right)^{i+2t'}$$

$$=2\,\frac{1.2.3\ldots\ldots\ i}{1.3.5\ldots\ldots 2i-1}\,\Sigma\,\frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{3\ .\ 5\ \ldots\ldots\ 2i+2t'+2}$$

$$\times\frac{n-2.n\ldots\ldots n+2t'-4}{2.4\ldots\ldots\ 2t'}\left(\frac{r}{r'}\right)^{i+2t'},$$

since by what Laplace has shown (*Mec. Cel.* Liv. iii. No. 17.)

$$\int_{-1}^{1}d\mu\,(i)^2=\frac{2}{2i+1}\left(\frac{1.2.3\ldots\ldots\ i}{1.3.5\ldots\ldots 2i-1}\right)^2.$$

If now we restore to μ the accents with which it was originally affected, and multiply the resulting quantity by r'^{n-1}, we shall have when $r < r'$

$$(10)\quad \int_{-1}^{1} d\mu'_1 \,(i)\,(r^2 - 2rr'\mu'_1 + r'^2)^{\frac{1-n}{2}} = r'^{n-1} \int_{-1}^{1} d\mu'_1\,(i)\left(1 - 2\mu'_1 \frac{r}{r'} + \frac{r^2}{r'^2}\right)^{\frac{1-n}{2}}$$

$$= 2 \,.\, r'^{1-n} . \frac{1 \,.\, 2 \,.\, 3 \ldots\ldots i}{1 \,.\, 3 \,.\, 5 \ldots\ldots 2i-1} \,\Sigma\, \frac{n-1 \,.\, n+1 \ldots\ldots n+2i+2t'-3}{3 \,.\, 5 \ldots\ldots 2i+2t'+1}$$

$$\times \frac{n-2 \,.\, n \ldots\ldots n+2t'-4}{2 \,.\, 4 \ldots\ldots 2t'} \left(\frac{r}{r'}\right)^{i+2t'},$$

and in order to deduce the value of the same integral when $r' \angle r$, we shall only have to change r into r', and reciprocally, in the formula just given.

We may now readily obtain the value of $V_t^{(i)}$ by means of the formula (8). For the density corresponding thereto being

$$'f_t^{(i)} r^{i+2t} (1 - r'^2)^\beta,$$

it follows from what has been observed in the former part of the present article, that $f_t'^{(i)} r'^{i+2t}$ may always be reduced to a rational and entire function of x', y', z' the rectangular co-ordinates of the element dv, and therefore the density in question will admit of being expanded in a series of the entire powers of x', y', z' and the various products of these powers. Hence (Art. 1.) $V_t^{(i)}$ admits of a similar expansion in entire powers, &c. of x, y, z the rectangular co-ordinates of the point p, and by following the methods before exposed Art. 1 and 2, we readily get

$$V_t''^{(i)} = 4\pi f^{(i)} . \int_r^1 r'^{i+2t'+3-n} dr' \,(1 - r'^2)^\beta . \,\Sigma\, \frac{n-1 \,.\, n+1 \ldots\ldots n+2i+2t'-3}{3 \,.\, 5 \ldots\ldots 2i+2t'+1}$$

$$\times \frac{n-2 \,.\, n \,.\, n+2 \ldots\ldots n+2t'-4}{2 \,.\, 4 \,.\, 6 \ldots\ldots 2t'} \left(\frac{r}{r'}\right)^{i+2t'};$$

and thence we have ultimately,

$$(11)\quad V_t^{(i)} = 2\pi f_t^{(i)} \,\Sigma\, \frac{n-1 \,.\, n+1 \ldots\ldots n+2i+2t'-3}{3 \,.\, 5 \ldots\ldots 2i+2t'+1} \times \frac{n-2 \,.\, n \ldots\ldots n+2t'-4}{2 \,.\, 4 \ldots\ldots 2t'}$$

$$r^{i+2t}.\frac{\Gamma\left(\frac{2t-2t'+4-n}{2}\right)\Gamma(\beta+1)}{\Gamma\left(\frac{2t-2t'+2\beta+6-n}{2}\right)}=2\pi f_t^{(i)}.\frac{\Gamma(\beta+1)\Gamma\left(\frac{4-n}{2}\right)}{\Gamma\left(\frac{6+2\beta-n}{2}\right)}r^i\ldots\ldots$$

$$\ldots\ldots\Sigma r^{2t'}\frac{4-n.6-n\ldots\ldots\ 2t-2t'+2-n}{6+2\beta-n\ldots\ldots 2t-2t'+2\beta+4-n}\times\frac{n-2.n\ldots\ldots n+2t'-4}{2.4\ldots\ldots\ 2t'}$$

$$\times\frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{3\ .\ 5\ \ldots\ldots\ 2i+2t'+1};$$

the finite integrals being taken from $t'=0$ to $t'=\infty$ and Γ being the characteristic of the well known function Gamma, which is introduced when we effect the integrations relative to r' by means of the formula (3), Art. 1.

Having thus $V_t^{(i)}$ or the part of V corresponding to the term $f_t'^{(i)}$, in $f(x', y', z')$ we immediately deduce the complete value of V by giving to i and t the various values of which these numbers are susceptible, and taking the sum of all the parts corresponding to the different terms in the expansion of the function $f(x', y', z')$.

Athough in the present Article we have hitherto supposed f to be the characteristic of a rational and entire function, the same process will evidently be applicable, provided $f(x', y', z')$ can be expanded in an infinite series of the entire powers of x', y', z' and the various products of these powers. In the latter case we have as before, the development

$$f(x', y', z')=f'^{(0)}+f'^{(1)}+f'^{(2)}+f'^{(3)}+\&c.$$

of which any term, as for example $f'^{(i)}$ may be farther expanded as follows,

$$f'^{(i)}=f_0'^{(i)}r'^i+f_1'^{(i)}r'^{i+2}+f_2'^{(i)}r'^{i+4}+\&c.$$

and as we have already determined $V_t^{(i)}$ or the part of V corresponding to $f_t'^{(i)}r'^{i+2t'}$, we immediately deduce as before the required value of V, the only difference is, that the numbers i and t, instead of being as in the former case confined within certain limits, may here become indefinitely great.

In the foregoing expression (11) β may be taken at will, but if we assign to it such a value that $\frac{2\beta-n}{2}$ may be a whole number, the series contained therein will terminate of itself, and consequently the value of $V_t^{(i)}$ will be exhibited in a finite form, capable by what has been shown at the beginning of the present Article of being converted into a rational and entire function of x, y, z, the rectangular co-ordinates of p. It is moreover evident, that the complete value of V being composed of a finite number of terms of the form $V_t^{(i)}$ will possess the same property, provided the function $f(x', y', z')$ is rational and entire, which agrees with what has been already proved in the second Article, by a very different method.

(5) We have before remarked, (Art. 2.) that in the particular case where $\beta = \frac{n-4}{2}$, the arbitrary constants contained in $f(x', y', z')$ are just sufficient in number to enable us to determine this function, so as to make the resulting value of V equal to any given rational and entire function of x, y, z, the rectangular co-ordinates of p, and have proved that the corresponding functions V and f will be of the same degree. But when this degree is considerable, the method there proposed becomes impracticable, seeing that it requires the resolution of a system of

$$\frac{s+1 \,.\, s+2 \,.\, s+3}{1 \,.\, 2 \,.\, 3}$$

linear equations containing as many unknown quantities; s being the degree of the functions in question. But by the aid of what has been shown in the preceding Article, it will be very easy to determine for this particular value of β the function $f(x', y', z')$ and consequently the density ρ when V is given, and we shall thus be able to exhibit the complete solution of the inverse problem by means of very simple formulæ.

For this purpose, let us suppose agreeably to the preceding remarks, that ρ the density of the fluid in the element dv is of the form

$$\rho = (1 - r'^2)^{\frac{n-4}{2}} f(x', y', z');$$

f being the characteristic of a rational and entire function of the same degree as V, and which we will here endeavour so to determine, that the value of V thence resulting, may be equal to any given rational and entire function of x, y, z of the degree s.

Then by Laplace's simple method (*Mec. Cel.* Liv. iii. No. 16.) we may always expand V in a series of the form

$$V = V^{(0)} + V^{(1)} + V^{(2)} + \&c. \ldots\ldots + V^{(s)}.$$

In like manner as has before been remarked, we shall have the analogous expansion

$$f(x', y', z') = f'^{(0)} + f'^{(1)} + f'^{(2)} + f'^{(3)} + \&c. \ldots\ldots + f'^{(s)},$$

of which any term $f'^{(i)}$ for example, may be farther developed as follows,

$$f'^{(i)} = f_0'^{(i)} r'^i + f_1'^{(i)} r'^{i+2} + f_2'^{(i)} r'^{i+4} + \&c. = r'^i \left(f'^{(i)} + f_1'^{(i)} r'^2 + f'^{(i)} r'^4 + \&c.\right)$$

$f_0'^{(i)}$, $f_1'^{(i)}$, $f_2'^{(i)}$, &c. being quantities independent of r' and all of the form $Y'^{(i)}$ (*Mec. Cel.* Liv. iii.) Moreover $V_t^{(i)}$ the part of V due to the general term $f_t'^{(i)} r'^{i+2t}$ of the last series, will be obtained by writing $\frac{n-4}{2}$ for β in the equation (11), and afterwards substituting for

$$\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{4-n}{2}\right) \text{ its value } \frac{\pi}{\sin\left(\frac{n-2}{2}\pi\right)}.$$

In this way we get

$$V_t^{(i)} = \frac{2\pi^2 f_t^{(i)} r^i}{\sin\left(\frac{n-2}{2}\pi\right)} \Sigma r^{2t'} \frac{4-n \,.\, 6-n \ldots\ldots 2t-2t'+2-n}{2 \,.\, 4 \ldots\ldots 2t-2t'}$$

$$\times \frac{n-2 \,.\, n \ldots\ldots n+2t'-4}{2 \,.\, 4 \ldots\ldots 2t'} \times \frac{n-1 \,.\, n+1 \ldots\ldots n+2i+2t'-3}{3 \,.\, 5 \ldots\ldots 2i+2t'+1};$$

$f_t^{(i)}$ being what $f_t'^{(i)}$ becomes by changing θ', ϖ' into θ, ϖ, and the finite integral being taken from $t'=0$ to $t'=\infty$.

Let us now for a moment assume

$$\phi(t') = \frac{n-2 \,.\, n \ldots\ldots n+2t'-4}{2 \,.\, 4 \ldots\ldots 2t'} \times \frac{n-1 \,.\, n+1 \ldots\ldots n+2i+2t'-3}{3 \,.\, 5 \ldots\ldots 2i+2t'+1},$$

then the expression immediately preceding may be written

$$V_t^{(i)} = \frac{2\pi^2 . f_t^{(i)} . r^i}{\sin\left(\frac{n-2}{2}\pi\right)} \Sigma \frac{4-n.6-n......2t-2t'+2-n}{2 \ . \ 4 \ ...... \ 2t-2t'} \phi(t').r^{2t'},$$

and by giving to t the various values 0, 1, 2, 3, &c. of which it is susceptible, and taking the sum of all the resulting values of $V_t^{(i)}$ the quantity thus obtained will be equal to $V^{(i)}$ or that part of V which is of the form $Y^{(i)}$. Thus we get

$$V^{(i)} = \frac{2\pi^2 . r^i}{\sin\left(\frac{n-2}{2}\pi\right)} \times$$

$$\begin{aligned}
&...........\phi(0).f_0^{(i)} \\
&+\frac{4-n}{2}\phi(0)f_1^{(i)} + \phi(1)f_1^{(i)}.r^2 \\
&+\frac{4-n.6-n}{2 \ . \ 4}.\phi(0).f_2^{(i)} + \frac{4-n}{2}\phi(1)f_2^{(i)}.r^2 + \phi(2)f_2^{(i)}.r^4 \\
&+\frac{4-n.6-n.8-n}{2 \ . \ 4 \ . \ 6}\phi(0)f_3^{(i)} + \frac{4-n.6-n}{2 \ . \ 4}\phi(1)f_3^{(i)}.r^2 + \frac{4-n}{2}\phi(2)f_3^{(i)}.r^4 + \phi(3)f_3^{(i)}.r^6 \\
&+ \text{\&c.} \ldots\ldots\ldots\ldots\ldots \text{\&c.} \ldots\ldots\ldots\ldots\ldots \text{\&c.} \ldots\ldots\ldots
\end{aligned}$$

since all the terms in the preceding value of $V_t^{(i)}$ in which $t' > t$ vanish of themselves in consequence of the factor

$$\frac{4-n.6-n......2t-2t'+2-n}{2 \ . \ 4 \ ...... \ 2t-2t'} = \frac{\Gamma\left(\frac{2t-2t'+4-n}{2}\right)}{\Gamma(t-t'+1)\,\Gamma\left(\frac{4-n}{2}\right)} = 0 \text{ (when } t' > t).$$

But $V^{(i)}$ as deduced from the given value of V may be expanded in a series of the form

$$V^{(i)} = r^i . \{V_0^{(i)} + V_1^{(i)} r^2 + V_2^{(i)} . r^4 + V_3^{(i)} r^6 + \text{\&c.}\}$$

and if in order to simplify the remaining operations, we make generally

$$V_t^{(i)} = \frac{2\pi^2}{\sin\left(\frac{n-2}{2}\pi\right)} \times \frac{n-2.n......n+2t-4}{2.4...... \ 2t} \times \frac{n-1.n+1......n+2i+2t-3}{3 \ . \ 5 \ ...... \ 2i+2t+1} U_t^{(i)}$$

$$= \frac{2\pi^2}{\sin\left(\frac{n-2}{2}\pi\right)} \times \phi(t) \,.\, U_t^{(i)},$$

the equation immediately preceding will become

$$V^{(i)} = \frac{2\pi^2 \,.\, r^i}{\sin\left(\frac{n-2}{2}\pi\right)} \times \{\phi(0) \,.\, U_0^{(i)} + \phi(1) \,.\, U_1^{(i)} r^2 + \phi(2)\, U_2^{(i)} \,.\, r^4 + \&c.\}$$

which compared with the foregoing value of $V^{(i)}$, will give by suppressing the factor $\dfrac{2\pi^2 \,.\, r^i}{\sin\left(\frac{n-2}{2}\pi\right)}$, common to both, and equating separately the coefficients of the different powers of the indeterminate quantity r the following system of equations

$$U_0^{(i)} = f_0^{(i)} + \frac{4-n}{2} f_1^{(i)} + \frac{4-n \,.\, 6-n}{2 \,.\, 4} f_2^{(i)} + \frac{4-n \,.\, 6-n \,.\, 8-n}{2 \,.\, 4 \,.\, 6} f_3^{(i)} +$$

$$U_1^{(i)} = f_1^{(i)} + \frac{4-n}{2} f_2^{(i)} + \frac{4-n \,.\, 6-n}{2 \,.\, 4} f_3^{(i)} + \&c.$$

$$U_2^{(i)} = f_2^{(i)} + \frac{4-n}{2} f_3^{(i)} + \frac{4-n \,.\, 6-n}{2 \,.\, 4} f_4^{(i)} + \&c.$$

&c = ...&c.......................&c...........................&c.

for determining the unknown functions $f_0^{(i)}$, $f_1^{(i)}$, $f_2^{(i)}$, &c. by means of the known ones $U_0^{(i)}$, $U_1^{(i)}$, $U_2^{(i)}$, &c. In fact the last equation of the system gives $U_s^{(i)} = f_s^{(i)}$, and then by ascending successively from the bottom to the top equation, we shall get the values of $f_s^{(i)}$, $f_{s-1}^{(i)}$, $f_{s-2}^{(i)}$, &c. with very little trouble. It will however be simpler still to remark, that the general type of all our equations is

$$U_u^{(i)} = (1-\epsilon)^{\frac{n-4}{2}} f_u^{(i)},$$

where the symbols of operation have been separated from those of quantity and ϵ employed in its usual acceptation, so that

$$\epsilon f_u^{(i)} = f_{u+1}^{(i)}, \quad \epsilon^2 f_u^{(i)} = \epsilon f_{u+1}^{(i)} = f_{u+2}^{(i)}, \ \&c.$$

But it is evident we may satisfy the last equation by making

$$f_n^{(i)} = (1-\epsilon)^{\frac{4-n}{2}}\, U_n^{(i)}.$$

Expanding now and replacing $\epsilon U_n^{(i)}$; $\epsilon^2 U_n^{(i)}$, &c. by these values $U_{n+1}^{(i)}$, $U_{n+2}^{(i)}$, &c. we get

$$f_n^{(i)} = U_n^{(i)} + \frac{n-4}{2}\, U_{n+1}^{(i)} + \frac{n-4\,.\,n-2}{2\,.\,4}\, U_{n+2}^{(i)} + \frac{n-4\,.\,n-2\,.\,n}{2\,.\,4\,.\,6}\, U_{n+3}^{(i)} + \text{\&c.},$$

from which we may immediately deduce $f_n'^{(i)}$ and thence successively

$$f'^{(i)} = r'^i\,(f_0'^{(i)} + f_1'^{(i)} r'^2 + f_2'^{(i)} r'^4 + f_3'^{(i)} r'^6 + \text{\&c.})$$

$$f(x', y', z') = f'^{(0)} + f'^{(1)} + f'^{(2)} + \text{\&c.} \ldots\ldots + f'^{(\infty)},$$

$$\text{and } \rho = (1 - x'^2 - y'^2 - z'^2)^{\frac{n-4}{2}}\,.\,f(x', y', z').$$

Application of the general Methods exposed in the preceding Articles to Spherical conducting Bodies.

(6) In order to explain the phenomena which electrified bodies present, Philosophers have found it advantageous either to adopt the hypothesis of two fluids, the vitreous and resinous of Dufay for example, or to suppose with Æpinus and others, that the particles of matter when deprived of their natural quantity of electric fluid, possess a mutual repulsive force. It is easy to perceive that the mathematical laws of equilibrium deducible from these two hypotheses, ought not to differ when the quantity of fluid or fluids (according to the hypothesis we choose to adopt) which bodies in their natural state are supposed to contain, is so great, that a complete decomposition shall never be effected by any forces to which they may be exposed, but that in every part of them a farther decomposition shall always be possible by the application of still greater forces. In fact the mathematical theory of electricity merely consists in determining ρ* the analytical value of

* It may not be improper to remark that ρ is always supposed to represent the density of the free fluid, or that which manifests itself by its repulsive force; and therefore, when the hypothesis of two fluids is employed, the measure of the excess of the quantity of either fluid

the fluid's density, so that the whole of the electrical actions exerted upon any point p, situated at will in the interior of the conducting bodies may exactly destroy each other, and consequently p have no tendency to move in any direction. For the electric fluid itself, the exponent n is equal to 2, and the resulting value of ρ is always such as not to require that a complete decomposition should take place in the body under consideration, but there are certain values of n for which the resulting values of ρ will render $\int \rho dv$ greater than any assignable quantity; for some portions of the body it is therefore evident that how great soever the quantity of the fluid or fluids may be, which in a natural state this body is supposed to possess, it will then become impossible strictly to realize the analytical value of ρ, and therefore some modification at least will be rendered necessary, by the limit fixed to the quantity of fluid or fluids originally contained in the body, and as Dufay's hypothesis appears the more natural of the two, we shall keep this principally in view, when in what follows it may become requisite to introduce either.

7. The foregoing general observations being premised, we will proceed in the present article to determine mathematically the law of the density ρ, when the equilibrium has established itself in the interior of a conducting sphere A, supposing it free from the actions of exterior bodies, and that the particles of fluid contained therein repel each other with forces which vary inversely as the n^{th} power of the distance. For this purpose it may be remarked, that the formula (1), Art. 1, immediately gives the values of the forces acting on any particle p, in virtue of the repulsion exerted by the whole of the fluid contained in A. In this way we get

$$\frac{1}{1-n} \cdot \frac{dV}{dx} = \text{the force directed parallel to the axis } X,$$

$$\frac{1}{1-n} \cdot \frac{dV}{dy} = \text{the force directed parallel to the axis } Y,$$

fluid which we choose to consider as positive over that of the fluid of opposite name in any element dv of the volume of the body is expressed by $\rho\, dv$, whereas on the other hypothesis ρdv serves to measure the excess of the quantity of fluid in the element dv over what it would possess in a natural state.

$$\frac{1}{1-n} \cdot \frac{dV}{dz} = \text{the force directed parallel to the axis } Z.$$

But since, in consequence of the equilibrium, each of these forces is equal to zero, we shall have

$$0 = \frac{dV}{dx}dx + \frac{dV}{dy}dy + \frac{dV}{dz}dz = dV;$$

and therefore, by integration,

$$V = \text{const.}$$

Having thus the value of V at the point p, whose co-ordinates are x, y, z, we immediately deduce, by the method explained in the fifth article,

$$\rho = \frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2} V . (1 - r'^2)^{\frac{n-4}{2}};$$

seeing that in the present case the general expansion of V there given reduces itself to

$$V = V^{(0)}.$$

If moreover Q serve to designate the total quantity of free fluid in the sphere, we shall have, by substituting for

$$\sin\left(\frac{n-2}{2}\pi\right) \text{ its value } \frac{\pi}{\Gamma\left(\frac{n-2}{4}\right)\Gamma\left(\frac{4-n}{2}\right)},$$

$$Q = \int \rho dv = \frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2} V \int_0^1 4\pi r'^2 dr' (1-r'^2)^{\frac{n-4}{2}} = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{4-n}{2}\right)} V.$$

See Legendre. *Exer. de Cal. Int. Quatriéme Partie.*

In the preceding values, as in the article cited, the radius of the sphere is taken for the unit of space; but the same formulæ may readily be adapted to any other unit by writing $\frac{r'}{a}$ in the place of r', and recollecting that the quantities ρ, V, and Q, are of the dimensions 0, $4-n$, and 3 respectively, with regard to space; a being the number which represents the radius of the sphere when we employ the new unit. In this way we obtain

$$\rho = \frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2} V (a^2 - r'^2)^{\frac{n-4}{2}}, \quad \text{and } Q = \frac{\Gamma\left(\frac{3}{2}\right) a^{n-1}}{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{4-n}{2}\right)} . V.$$

Hence, when Q, the quantity of redundant fluid originally introduced into the sphere is given, the values of V and of the density ρ are likewise given. In fact, by writing in the preceding equation for

$$\Gamma\left(\frac{3}{2}\right), \text{ and } \sin\left(\frac{n-2}{2}\pi\right),$$

their values, we thence immediately deduce

$$(12) \ldots\ldots\ldots \rho = \frac{1}{\pi\sqrt{\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} a^{1-n} Q (a^2 - r'^2)^{\frac{n-4}{2}},$$

$$\text{and } V = \frac{2\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{4-n}{2}\right)}{\sqrt{\pi}} a^{1-n} . Q.$$

The foregoing formulæ present no difficulties where $n > 2$, but when $n < 2$, the value of ρ, if extended to the surface of the sphere A, would require an infinite quantity of fluid of one name to have been originally introduced into its interior, and therefore, agreeably to a preceding observation, could not be strictly realized. In order then to determine the modification which in this case ought to be introduced, let us in the first place make $n > 2$, and conceive an inner sphere B, whose radius is $a - \delta a$, in which the density of the fluid is still defined by the first of the equations (12); then, supposing afterwards the rest of the fluid in the exterior shell to be considered on A's surface, the portion so condensed, if we neglect quantities of the order δa, compared with those retained, will be

$$\frac{2^{\frac{n}{2}}}{\sqrt{\pi}} . \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} a^{\frac{2-n}{2}} Q . \delta a^{\frac{n-2}{2}},$$

and since, in the transfer of the fluid to A's surface, its particles move over spaces of the order δa only, the alteration which will thence be produced in V will evidently be of the order

$$\delta a^{\frac{n-2}{2}} \times \delta a = \delta a^{\frac{n}{2}},$$

and consequently the value of V will become

$$V = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{4-n}{2}\right) a^{1-n} Q + k \,.\, \delta a^{\frac{n}{2}};$$

k being a quantity which remains finite when δa vanishes.

In establishing the preceding results, n has been supposed greater than 2, but ρ the density of the fluid within B and the quantity of it condensed on A's surface being still determined by the same formulæ, the foregoing value of V ought to hold good in virtue of the generality of analysis whatever n may be, and therefore when n is a positive quantity and δa is exceedingly small, we shall have without sensible errors

$$V = \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{4-n}{2}\right) a^{1-n} . \, Q.$$

Conceiving now P' to represent the density of the fluid condensed on A's surface, $4\pi a^2 P'$ will be the total quantity thereon contained, which being equated to the value before given, there results

$$4\pi a^2 P' = \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} a^{\frac{2-n}{2}} Q \delta a^{\frac{n-2}{2}},$$

and hence we immediately deduce

$$P' = \frac{2^{\frac{n-4}{2}}}{\pi\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} a^{-\frac{2+n}{2}} . Q \,.\, \delta a^{\frac{n-2}{2}}.$$

Moreover as Q represents the total quantity of redundant fluid in the entire sphere A, the quantity contained in B is

$$Q - \frac{2^{\frac{n}{2}}}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} a^{\frac{2-n}{2}} . Q . \delta a^{\frac{n-2}{2}} .$$

If now when n is supposed less than 2, we adopt an hypothesis similar to Dufay's, and conceive that the quantities of fluid of opposite denominations in the interior of A are exceedingly great when this body is in a natural state, then after having introduced the quantity Q of redundant fluid, we may always by means of the expression just given, determine the value of δa, so that the whole of the fluid of contrary name to Q, may be contained in the inner sphere B, the density in every part of it being determined by the first of the equations (12). If afterwards the whole of the fluid of the same name as Q is condensed upon A's surface, the value of V in the interior of B as before determined will evidently be constant, provided we neglect indefinitely small quantities of the order $\delta a^{\frac{n}{2}}$. Hence all the fluid contained in B will be in equilibrium, and as the shell included between the two concentric spheres, A and B is entirely void of fluid, it follows that the whole system must be in equilibrium.

From what has preceded, we see that the first of the formulæ (12) which served to give the density ρ within the sphere A when n is greater than 2, is still sensibly correct when n represents any positive quantity less than 2, provided we do not extend it to the immediate vicinity of A's surface. But as the foregoing solution is only approximative, and supposes the quantities of the two fluids which originally neutralized each other to be exceedingly great, we shall in the following article endeavour to exhibit a rigorous solution of the problem, in case $n < 2$, which will be totally independent of this supposition.

8. Let us here in the first place conceive a spherical surface whose radius is a, covered with fluid of the uniform density P', and suppose it is required to determine the value of the density ρ in the interior of a concentric conducting sphere, the radius of which is taken for the unit of space, so that the fluid therein contained, may be in equilibrium in virtue of the joint action of that contained in the sphere itself, and on the exterior spherical surface.

If now V' represents the value of V due to the exterior surface, it is clear from what Laplace has shown, (*Mec. Cel.* Liv. ii. No. 12.) that

$$V' = \int \frac{d\sigma P'}{g'^{1-n}} = \frac{2\pi a P'}{(3-n) r} \{(a+r)^{3-n} - (a-r)^{3-n}\};$$

$d\sigma$ being an element of this surface, and g' being the distance of this element from the point p to which V' is supposed to belong.

If afterwards we conceive that the function V is due to the fluid within the sphere itself, it is easy to prove as in the last article, that in consequence of the equilibrium we must have

$$V' + V = \text{const.}$$

But V' and consequently V is of the form $Y^{(0)}$, therefore by employing the method before explained, (Art. 4.) we get

$$f(x', y', z') = f'^{(0)} = f_0^{(0)} + f_1^{(0)}.r'^2 + f_2^{(0)}.r'^4 + \&c. = B_0 + B_1 r'^2 + B_2 r'^4 + \&c.;$$

where, as in the present case, $f_0'^{(0)}$, $f_1'^{(0)}$, $f_2'^{(0)}$, &c. are all constant quantities, they have for the sake of simplicity been replaced by

$$B_0,\ B_1,\ B_2,\ \&c.$$

Hitherto the exponent β has remained quite arbitrary, but by making $\beta = \frac{n-2}{2},$ the formula (11) Art. 4. will become when $i=0$,

$$V_t^{(0)} = 2\pi B_t \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{4-n}{2}\right)}{\Gamma(2)} \Sigma r^{2t'} \frac{4-n.6-n\ldots\ldots 2t-2t'+2-n}{4\ .\ 6\ \ldots\ldots\ 2t-2t'+2}$$
$$\times \frac{n-2.n-1\ldots\ldots n+2t'-3}{2.3.4\ \ldots\ldots\ 2t'+1}$$
$$= \frac{(n-2)\pi^2 B_t}{\sin\left(\frac{n-2}{2}\pi\right)} \Sigma.r^{2t'}.\frac{4-n.6-n\ldots\ldots 2t-2t'+2-n}{4\ .\ 6\ \ldots\ldots\ 2t-2t'+2} \times \frac{n-2.n-1\ldots\ldots n+2t'-3}{2\ .\ 3\ \ldots\ldots\ 2t'+1}.$$

Giving now to t the successive values 0, 1, 2, 3, &c. and taking the sum of the functions thence resulting, there arises

$$V = V^{(0)} = V_0^{(0)} + V_1^{(0)} + V_2^{(0)} + V_3^{(0)} + \&c. = S.V_t^{(0)}$$
$$= \frac{(n-2)\pi^2}{\sin\left(\frac{n-2}{2}\pi\right)} SB_t \Sigma r^{2t'} \frac{4-n.6-n\ldots\ldots 2t-2t'+2-n}{4.6.8\ \ldots\ldots\ 2t-2t'+2} \times \frac{n-2.n-1\ldots\ldots n+2t'-3}{2\ .\ 3\ \ldots\ldots\ 2t'+1},$$

where the sign S is referred to the variable t and Σ to t'.

Again, by substituting for V and V' their values in the equation $V'+V=\text{const.}$ and expanding the function V' we obtain

$$\text{const.} = 4\pi P'a^{3-n}.\,\Sigma\frac{r^{2t'}}{a^{2t'}}.\frac{n-2.n-1.n\ldots\ldots n+2t'-3}{2\;.\;3\;.4\ldots\ldots\;2t'+1}$$

$$+\frac{(n-2)\pi^2}{\sin\left(\frac{n-2}{2}\pi\right)}S\Sigma B_t r^{2t'}\frac{4-n.6-n\ldots\ldots 2t-2t'+2-n}{4\;.\;6\;\ldots\ldots 2t-2t'+2}\times\frac{n-2.n-1\ldots\ldots n+2t'+3}{2.3.4\ldots\ldots 2t'+1},$$

which by equating separately the coefficients of the various powers of the indeterminate quantity r, and reducing, gives generally

$$\frac{2P'a^{3-n-2t'}.\sin\left(\frac{n-2}{2}\pi\right)}{\pi}=S\frac{2-n.4-n\ldots\ldots 2s-n}{2\;.\;4\;\ldots\ldots\;2s}B_{t'+s-1}.$$

Then by assigning to t' its successive values 1, 2, 3, &c. there results for the determination of the quantities B_0, B_1, B_2, &c. the following system of equations,

$$\frac{2P'}{\pi}a^{1-n}.\sin\left(\frac{n-2}{2}\pi\right)=B_0+\frac{2-n}{2}B_1+\frac{2-n.4-n}{2\;.\;4}B_2+\&c.$$

$$\frac{2P'}{\pi}a^{1-n}.\sin\left(\frac{n-2}{2}\pi\right).a^{-2}=B_1+\frac{2-n}{2}B_2+\frac{2-n.4-n}{2\;.\;4}B_3+\&c.$$

$$\frac{2P'}{\pi}a^{1-n}.\sin\left(\frac{n-2}{2}\pi\right).a^{-4}=B_2+\frac{2-n}{2}B_3+\frac{2-n.4-n}{2\;.\;4}B_4+\&c.$$

&c.................&c..................&c..................&c........

But it is evident from the form of these equations, that we may satisfy the whole system by making

$$B_1=B_0.a^{-2},\;B_2=B_1.a^{-2},\;B_3=B_2.a^{-2},\;B_4=B_3.a^{-2},\;\&c.$$

provided we determine B_0 by

$$\frac{2P'}{\pi}a^{1-n}\sin\left(\frac{n-2}{2}\pi\right)=B_0\left(1+\frac{2-n}{2}a^{-2}+\frac{2-n.4-n}{2\;.\;4}a^{-4}+\&c.\right)$$

$$=B_0(1-a^{-2})^{\frac{n-2}{2}}.$$

Hence as in the present case, $\beta=\frac{n-2}{2}$, we immediately deduce the successive values

$$B_0=\frac{2P'}{\pi a}\sin\left(\frac{n-2}{2}\pi\right).(a^2-1)^{\frac{2-n}{2}},$$

$$f(x', y', z') = f'^{(0)} = B_0 + B_1 r'^2 + B_2 r'^4 + \&c. = B_0 \left(1 - \frac{r'^2}{a^2}\right)^{-1},$$

$$\text{and } \rho = (1 - r'^2)^{\frac{n-2}{2}} . f(x', y', z') = \frac{2P'a}{\pi} \sin\left(\frac{n-2}{2}\pi\right) . (a^2 - 1)^{\frac{2-n}{2}} \ldots$$

$$\ldots\ldots (a^2 - r'^2)^{-1} (1 - r'^2)^{\frac{n-2}{2}}.$$

In the value of ρ just exhibited, the radius of the sphere is taken as the unit of space, but the same formula may easily be adapted to any other unit by writing $\frac{a}{b}$ and $\frac{r'}{b}$ in the place of a and r' respectively, and recollecting at the same time that in consequence of the equation

$$\text{const.} = V + V' = \int \frac{dv \,.\, \rho}{g} + \int \frac{d\sigma P'}{g'},$$

before given, $\frac{\rho}{P'}$, is a quantity of the dimension -1 with regard to space: b being the number which represents the radius of the sphere when we employ the new unit. Hence we obtain for a sphere whose radius is bg, acted upon by an exterior concentric spherical surface of which the radius is a,

$$(\beta) \ldots\ldots \rho = \frac{2P'a \,.\, \sin\left(\frac{n-2}{2}\pi\right)}{\pi} \, a^2 - b^2)^{\frac{2-n}{2}} \, (a^2 - r'^2)^{-1} \, (b^2 - r'^2)^{\frac{n-2}{2}};$$

P' being the density of the fluid on the exterior surface.

If now we conceive a conducting sphere A whose radius is a, and determine P' so that all the fluid of one kind, viz. that which is redundant in this sphere, may be condensed on its surface, and afterwards find b the radius of the interior sphere B from the condition that it shall just contain all the fluid of the opposite kind, it is evident that each of the fluids will be in equilibrium within A, and therefore the problem originally proposed is thus accurately solved. The reason for supposing all the fluid of one name to be completely abstracted from B, is that our formulæ may represent the state of *permanent* equilibrium, for the tendency of the forces acting within the void shell included between the surfaces A and B, is to abstract continually the fluid of the same name as that on A's surface from the sphere B.

To prove the truth of what has just been asserted, we will begin with determining the repulsion exerted by the inner sphere itself, on any point p exterior to it, and situate at the distance r from its centre O. But by what Laplace has shown (*Mec. Cel.* Liv. ii. No. 12.) the repulsion on an exterior point p, arising from a spherical shell of which the radius is r', thickness dr' and center is at O will be measured by

$$\frac{2\pi r' dr' \rho}{1-n\,.\,3-n} \cdot \frac{d}{dr} \cdot \frac{(r+r')^{3-n}-(r-r')^{3-n}}{r},$$

the general term of which when expanded in an ascending series of the powers of $\frac{r'}{r}$ is,

$$+4\pi\,.\,\frac{-2+n\times n\,.\,n+1\,.\,n+2\ldots\ldots n+2s-3\times n+2s-1}{2\;.\;3\;.\;4\;.\;5\;\ldots\ldots\;2s+1}\,r^{n-2s}\,.\,r'^{2s+2}\,.\,\rho dr',$$

and the part of the required repulsion due thereto will, by substituting for ρ its value before found, become

$$+\frac{8P'}{a}\sin\left(\frac{n-2}{2}\pi\right).\,(a^2-b^2)^{\frac{2-n}{2}}\,\frac{-2+n\times n\,.\,n+1\ldots\ldots n+2s-3\times n+1s-1}{2\,.\,3\;.\;4\;\ldots\ldots\;2s+1}\,r^{-n-2s}$$

$$\times\int_0^b\left(1-\frac{r'^2}{a^2}\right)^{-1}(b^2-r'^2)^{\frac{n-2}{2}}\,r'^{2s+2}dr'.$$

It now remains to find the value of the definite integral herein contained. But when $\left(1-\frac{r'^2}{a^2}\right)^{-1}$ is expanded, and the integrations are effected by known formulæ, we obtain

$$(14)\quad \int_0^b\left(1-\frac{r'^2}{a^2}\right)^{-1}(b^2-r'^2)^{\frac{n-2}{2}}\,r'^{2s+2}dr' = \int_0^b \Sigma_0^\infty \frac{r'^{2t}}{a^{2t}}\,(b^2-r'^2)^{\frac{n-2}{2}}\,.\,r'^{2s+2}dr'$$

$$=\tfrac{1}{2}b^{2s+1+n}\,\Sigma_0^\infty\frac{b^{2t}}{a^{2t}}\times\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(s+t+\frac{3}{2}\right)}{\Gamma\left(s+t+\frac{3}{2}+n\right)} = \tfrac{1}{2}b^{2s+1+n}.\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}{\Gamma\left(s+\frac{n}{2}+\frac{3}{2}\right)}$$

$$\left\{1+\frac{2s+3}{2s+3+n}\,\frac{b^2}{a^2}+\frac{2s+3\,.\,2s+5}{2s+3+n\,.\,2s+5+n}\,\frac{b^4}{a^4}+\text{\&c.}\right\}$$

$$=\tfrac{1}{2}b^{2s+1+n}.\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(s+\frac{3}{2}\right)}{\Gamma\left(s+\frac{n}{2}+\frac{3}{2}\right)}\times\frac{(2s+1+n)\,(1-x^2)^{\frac{n-2}{2}}}{x^{2s+1+n}}\int_0\frac{x^{2s+n}dx}{(1-x^2)^{\frac{n}{2}}}$$

$$=\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)}\cdot\frac{1\ .\ 3\ .\ 5\ \ldots\ldots 2s+1}{1+n\,.\,3+n\,.\,5+n\ldots\ldots\ 2s-1+n}\,\frac{b^{2s+1+n}}{}\times\frac{(1-x^2)^{\frac{n-2}{2}}}{x^{2s+1+n}}\int_0\frac{x^{2s+n}dx}{(1-x^2)^{\frac{n}{2}}};$$

where after the integrations have been effected, x ought to be made equal to $\frac{b}{a}$.

The value of the integral last found being substituted in the expression immediately preceding, and the finite integral taken relative to s from $s=0$ to $s=\infty$ gives for the repulsion of the inner sphere,

$$-\frac{4\pi P' b}{a}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}(a^2-b^2)^{\frac{2-n}{2}}$$

$$\times\Sigma_0^\infty\frac{n-2\,.\,n\,.\,n+2\ldots\ldots n+2s-4}{2\,.\,4\,.\,6\ \ldots\ldots\ 2s}\left(\frac{b}{r}\right)^{2s+n}\frac{(1-x^2)^{\frac{n-2}{2}}}{x^{2s+1+n}}\int_0\frac{x^{2s+n}dx}{(1-x^2)^{\frac{n}{2}}}$$

$$=\frac{-4\pi\sqrt{\pi}P'a^2r^{-n}}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}\Sigma_0^\infty\frac{n-2\,.\,n\,.\,n+2\ldots\ldots n+2s-4}{2\,.\,4\,.\,6\ \ldots\ldots\ 2s}\left(\frac{a}{r}\right)^{2s}\int_0 dx\,x^{2s+n}.(1-x^2)^{\frac{-n}{2}};$$

since $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, $\sin\left(\frac{n-2}{2}\pi\right)=\frac{-\pi}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}$,

and as was before observed, $x=\frac{b}{a}$.

But we have evidently by means of the binomial theorem,

$$\left(1-\frac{a^2x^2}{r^2}\right)^{\frac{2-n}{2}}=\Sigma_0^\infty\frac{n-2\,.\,n\,.\,n+2\ldots\ldots n+2s-4}{2\,.\,4\,.\,6\ \ldots\ldots\ 2s}\left(\frac{ax}{r}\right)^{2s};$$

and therefore the preceding quantity becomes

$$(15)\ldots\ldots\ldots\ldots -\frac{4\pi\sqrt{\pi}\,.\,P'a^2.r^{-n}}{\Gamma\left(\frac{1-n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}\int_0^{\frac{b}{a}}dx\,x^n\left(1-\frac{a^2x^2}{r^2}\right)^{\frac{2-n}{2}}(1-x^2)^{\frac{-n}{2}}.$$

If now we make $x = \frac{rx'}{a}$, the same quantity may be written

$$(16)\ldots\ldots\ldots - \frac{4\pi\sqrt{\pi}P'a^{1-n}r}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}\int_0^{\frac{b}{r}} x'^n dx'(1-x'^2)^{\frac{2-n}{2}}\left(1-\frac{r^2x'^2}{a^2}\right)^{\frac{-n}{2}}.$$

Having thus the value of the repulsion due to the inner sphere B on an exterior point p, it remains to determine that due to the fluid on A's surface. But this last is represented by

$$(17)\ldots\ldots\ldots \frac{2\pi aP'}{1-n.3-n}\frac{d.}{dr}\frac{(a+r)^{3-n}-(a-r)^{3-n}}{r}.$$

(*Mec. Cel.* Liv. ii. No. 12.) Now by expanding this function there results

$$4\pi P'a^{1-n}r.\frac{2-n}{3}.\left\{1+\frac{n.n+1}{4.5}2\frac{r^2}{a^2}+\frac{n.n+1.n+2.n+3}{4.5.6.7}.3\frac{r^4}{a^4}+\&c.\right\}$$

$$=4\pi P'a^{1-n}r.\frac{2-n}{3}.\Sigma_0^\infty\frac{n.n+1.n+2\ldots\ldots\ldots n+2s-1}{4\ .\ 5\ .\ 6\ \ldots\ldots\ldots\ 2s+3}(s+1)\frac{r^{2s}}{a^{2s}}.$$

The last of these expressions may readily be exhibited under a finite form, by remarking that

$$\int_0^1 x^n dx(1-x^2)^{\frac{2-n}{2}}\left(1-\frac{r^2x^2}{a^2}\right)^{\frac{-n}{2}}=\int_0^1 x^n dx(1-x^2)^{\frac{2-n}{n}}\Sigma\frac{n.n+2\ldots n+2s-2}{2.4.6\ldots\ 2s}.\frac{r^{2s}x^{2s}}{a^{2s}}$$

$$=\Sigma_0^\infty\frac{n.n+2.n+4\ldots\ldots\ldots n+2s-2}{2.\ 4\ .\ 6\ \ldots\ldots\ldots\ 2s}.\frac{r^{2s}}{a^{2s}}.\frac{\Gamma\left(\frac{2s+n+1}{2}\right)\Gamma\left(\frac{4-n}{2}\right)}{2\Gamma\left(\frac{2s+5}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{2-n}{2}\right)\Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}.\frac{2-n}{3}.\Sigma_0^\infty\frac{n.n+1.n+2\ldots\ldots\ldots n+2s-1}{4.\ 5\ .\ 6\ \ldots\ldots\ldots\ 2s+3}(s+1)\frac{r^{2s}}{a^{2s}}.$$

Hence, since $\Gamma(\frac{1}{2})=\sqrt{\pi}$, the value of the repulsion arising from A's surface becomes

$$\frac{4\pi\sqrt{\pi}.P'a^{1-n}.r}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}\int_0^1 x^n dx(1-x^2)^{\frac{2-n}{2}}\left(1-\frac{r^2x^2}{a^2}\right)^{\frac{-n}{2}}.$$

Now by adding the repulsion due to the inner sphere which is given by the formula (16), we obtain, (since it is evidently indifferent what variable enters into a definite integral, provided each of its limits remain unchanged)

$$\frac{4\pi\sqrt{\pi}P'a^{1-n}r}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}.\left\{\int_0^1 x^n dx(1-x^2)^{\frac{2-n}{2}}\left(1-\frac{r^2x^2}{a^2}\right)^{\frac{-n}{2}}-\int_0^{\frac{b}{r}} x^n dx(1-x^2)^{\frac{2-n}{2}}.\left(1-\frac{r^2x^2}{a^2}\right)^{\frac{-n}{2}}\right\}$$

$$=\frac{4\pi\sqrt{\pi}.P'a^{1-n}r}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}\int_{\frac{b}{r}}^1 x^n dx(1-x^2)^{\frac{2-n}{2}}.\left(1-\frac{r^2x^2}{a^2}\right)^{\frac{-n}{2}},$$

for the value of the total repulsion upon a particle p of positive fluid situate within the sphere A and exterior to B. We thus see that when P' is positive the particle p is always impelled by a force which is equal to zero at B's surface, and which continually increases as p recedes farther from it. Hence, if any particle of positive fluid is separated ever so little from B's surface, it has no tendency to return there, but on the contrary, it is continually impelled therefrom by a regularly increasing force; and consequently, as was before observed, the equilibrium can not be permanent until all the positive fluid has been gradually abstracted from B and carried to the surface of A, where it is retained by the non-conducting medium with which the sphere A is conceived to be surrounded.

Let now q represent the total quantity of fluid in the inner sphere, then the repulsion exerted on p by this will evidently be

$$qr^{-n},$$

when r is supposed infinite. Making therefore r infinite in the expression (15), and equating the value thus obtained to the one just given, there arises

$$q=\frac{-4\pi\sqrt{\pi}.P'a^2}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)}\int_0^{\frac{b}{a}} dx.x^n(1-x^2)^{\frac{-n}{2}}.$$

When the equilibrium has become permanent, $\dot{q}$ is equal to the total quantity of that kind of fluid, which we choose to consider negative, originally introduced into the sphere A; and if now q_1 represent the total quantity of fluid of opposite name contained within A, we shall have, for the determination of the two unknown quantities P' and b, the equations

$$q_1 = 4\pi a^2 . P',$$

$$\text{and } \frac{q}{q_1} = \frac{-\sqrt{\pi}}{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{2-n}{2}\right)} \int_0^{\frac{b}{a}} dx\, x^n (1-x^2)^{\frac{-n}{2}},$$

and hence we are enabled to assign accurately the manner in which the two fluids will distribute themselves in the interior of A; q and q_1, the quantities of the fluids of opposite names originally introduced into A being supposed given.

9. In the two foregoing articles we have determined the manner in which our hypothetical fluids will distribute themselves in the interior of a conducting sphere A when in equilibrium and free from all exterior actions, but the method employed in the former is equally applicable when the sphere is under the influence of any exterior forces. In fact, if we conceive them all resolved into three X, Y, Z, in the direction of the co-ordinates x, y, z of a point p, and then make, as in Art. 1,

$$V = \int \frac{\rho\, dv}{g^{n-1}}.$$

we shall have, in consequence of the equilibrium,

$$0 = \frac{1}{1-n} \frac{dV}{dx} + X, \quad 0 = \frac{1}{1-n} \frac{dV}{dy} + Y, \quad 0 = \frac{1}{1-n} \frac{dV}{dz} + Z,$$

which, multiplied by dx, dy and dz respectively, and integrated, give

$$\text{const.} = \frac{1}{1-n} V + \int (X dx + Y dy + Z dz);$$

where $Xdx + Ydy + Zdz$ is always an exact differential.

We thus see that when X, Y, Z are given rational and entire functions V will be so likewise, and we may thence deduce (Art. 5.)

$$\rho = (1 - x'^2 - y'^2 - z'^2)^{\frac{n-4}{2}} . f(x', y', z'),$$

where f is the characteristic of a rational and entire function of the same degree as V.

The preceding method is directly applicable when the forces X, Y, Z are given explicitly in functions of x, y, z. But instead of these forces, we may conceive the density of the fluid in the exterior bodies as given, and thence determine the state which its action will induce in the conducting sphere A. For example, we may in the first place suppose the radius of A to be taken as the unit of space, and an exterior concentric spherical surface, of which the radius is a, to be covered with fluid of the density $U''^{(i)}$: $U''^{(i)}$ being a function of the two polar co-ordinates θ'' and ϖ'' of any element of the spherical surface of the same kind as those considered by Laplace (*Mec. Cel.* Liv. iii.). Then it is easy to perceive by what has been proved in the article last cited, that the value of the induced density will be of the form

$$\rho = U'^{(i)} r'^i (1 - r'^2)^{\frac{n-4}{2}} . f(r'^2);$$

r', θ', ϖ' being the polar co-ordinates of the element dv, and $U'^{(i)}$ what $U''^{(i)}$ becomes by changing θ'', ϖ'' into θ', ϖ'.

Still continuing to follow the methods before explained, (Art. 4. and 5.) we get in the present case

$$f(x', y', z') = U'^{(i)} r'^i f(r'^2) = f^{(i)},$$

and by expanding $f(r'^2)$, we have

$$f(r'^2) = B_0 + B_1 r'^2 + B_2 r'^4 + B_3 r'^6 + \&c.$$

Hence, $f_t^{(i)} = B_t U'^{(i)}$, and

$$V_t^{(i)} = \frac{2\pi^2 U^{(i)} r^i}{\sin\left(\frac{n-2}{2}\pi\right)} B_t \Sigma_0^\infty r^{2t'} \frac{4-n.6-n.....2t-2t'+2-n}{2.4.6 ..... 2t-2t'} \times \frac{n-2.n.....n+2t'-4}{2.4..... 2t'}$$

$$\times \frac{n-1.n+1.....n+2i+2t'-3}{3 . 5 ..... 2i+2t'+1}.$$

Then, by giving to t all the values 1, 2, 3, &c. of which it is susceptible, and taking the sum of all the resulting quantities, we shall have, since in the present case V reduces itself to the single term $V^{(i)}$,

$$V = \frac{2\pi^2 U^{(i)} r^i}{\sin\left(\frac{n-2}{2}\pi\right)} S B_t \Sigma r^{2t'} . \frac{4-n.6-n.....2t-2t'+2-n}{2 . 4 ..... 2t-2t'} \times \frac{n-2.n.....n+2t'-4}{2.4..... 2t'}$$

$$\times \frac{n-1.n+1.....n+2i+2t'-3}{3 . 5 ..... 2i+2t'+1};$$

the sign S belonging to the unaccented letter t.

If now V' represents the function analagous to V and due to the fluid on the spherical surface, we shall obtain by what has been proved (Art. 3.)

$$V' = U^{(i)}.\,2\pi a^2 \frac{1.3.5.....2i-1}{1.2.3.....\ i} \int_{-1}^{1} d\mu\,(i)\,(r^2 - 2ar\mu + a^2)^{\frac{1-n}{2}};$$

(i) representing the same function as in the article just cited.

Moreover, it is evident from the equation (10) Art. 4, that

$$\int_{-1}^{1} d\mu (i)\,(r^2 - 2ar\mu + a^2)^{\frac{1-n}{2}} = 2a^{1-n} \frac{1.2.3.....\ i}{1.3\ .....2i-1} \Sigma \frac{n-1.n+1.....n+2i+2t'-3}{3\ .\ 5\ .....\ 2i+2t'+1}$$

$$\times \frac{n-2.n.....n+2t'-4}{2.4.....\ 2t'} \left(\frac{r}{a}\right)^{i+2t'};$$

and consequently,

$$(19)......V' = U^{(i)}.\,4\pi a^{3-n}.\,\Sigma \frac{n-1.n+1.....n+2i+2t'-3}{3\ .\ 5\ .....\ 2i+2t'+1}$$

$$\times \frac{n-2.n.....n+2t'-4}{2.4.....\ 2t'} \left(\frac{r}{a}\right)^{i+2t};$$

the finite integrals extending from $t'=0$ to $t'=\infty$.

Substituting now for V and V' their values in the equation of equilibrium,

$$(20)\ ....\ \text{const.} = V' + V,$$

we immediately obtain

$$\text{const.} = U^{(i)}.\,4\pi a^{3-n}.\,\Sigma \frac{n-1.n+1.....n+2i+2t'-3}{3\ .\ 5\ .....\ 2i+2t'+1}$$

$$\times \frac{n-2.n.....n+2t'-4}{2\ .4.....\ 2t'} \left(\frac{r}{a}\right)^{i+2t'}$$

$$+ \frac{2\pi^2}{\sin\left(\frac{n-2}{2}\pi\right)} U^{(i)} S B_t \Sigma r^{i+2t'}.\,\frac{n-1.n+1.....n+2i+2t'-3}{3\ .\ 5\ .....\ 2i+2t'-1}$$

$$\times \frac{n-2.n.....n+2t'-4}{2\ .4.....\ 2t'} \times \frac{4-n.6-n.....2t-2t'+2-n}{2\ .\ 4\ .....2t-2t'},$$

the constant on the left side of this equation being equal to zero, except when $i = 0$.

By equating separately the coefficients of the various powers of the indeterminate quantity r, we get the following system of equations:

$$-\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{3-n-1} = B_0 + B_1\frac{4-n}{2} + B_2\frac{4-n.6-n}{2\;.\;4} + \&c.$$

$$-\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{3-n-i-2} = B_1 + B_2\frac{4-n}{2} + B_3\frac{4-n.6-n}{2\;.\;4} + \&c.$$

$$-\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{3-n-i-4} = B_2 + B_3\frac{4-n}{2} + B_4\frac{4-n.6-n}{2\;.\;4} + \&c.$$

&c.......................&c..................&c.......................

But it is evident from the form of these equations, that if we make generally $B_{t+1} = a^{-2}B_t$, they will all be satisfied provided the first is, and as by this means the first equation becomes

$$-\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{3-n-i} = B_0\left(1 + \frac{4-n}{2}a^{-2} + \frac{4-n.6-n}{2\;.\;4}a^{-4} + \&c.\right)$$

$$= B_0(1-a^{-2})^{\frac{n-4}{2}} = B_0a^{4-n}(a^2-1)^{\frac{n-4}{2}},$$

there arises

$$B_0 = -\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{-i-1}(a^2-1)^{\frac{4-n}{2}},\quad B_1 = B_0.a^{-2},\quad B_2 = B_0.a^{-4},\ \&c.$$

Hence

$$f(r'^2) = B_0 + B_1r'^2 + B_2r'^4 + \&c. = B_0\left(1 + \frac{r'^2}{a^2} + \frac{r'^4}{a^4} + \&c.\right)$$

$$= B_0\left(1-\frac{r'^2}{a^2}\right)^{-1} = B_0a^2(a^2-r'^2)^{-1} = -\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}a^{i+1}(a^2-1)^{\frac{4-n}{2}}(a^2-r'^2)^{-1},$$

and the required value of ρ becomes

$$(21)\ldots\ldots \rho = U'^{(i)}r'^i(1-r'^2)^{\frac{n-4}{2}}f(r'^2)$$

$$= -\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi}(a^2-1)^{\frac{4-n}{2}}aU'^{(i)}\left(\frac{r'}{a}\right)^i(a^2-r'^2)^{-1}(1-r'^2)^{\frac{n-4}{2}}.$$

But whatever the density P on the inducing spherical surface may be, we can always expand it in a series of the form

$$P = U''^{(0)} + U''^{(1)} + U''^{(2)} + U''^{(3)} + \&c. \textit{ in inf.}$$

and the corresponding value of ρ by what precedes will be

$$\rho = -\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi} a(a^2-1)^{\frac{4-n}{2}} . (a^2-r'^2)^{-1}(1-r'^2)^{\frac{n-4}{2}} \ldots\ldots$$

$$\ldots\ldots \times \left\{U'^{(0)} + U'^{(1)}\frac{r'}{a} + U'^{(2)}\frac{r'^2}{a^2} + U'^{(3)}\frac{r'^3}{a^3} + \&c. \textit{ in inf.}\right\};$$

$U'^{(0)}$, $U'^{(1)}$, $U'^{(2)}$, &c. being what $U''^{(0)}$, $U''^{(1)}$, $U''^{(2)}$, &c. become by changing θ'', ϖ'' into θ', ϖ', the polar co-ordinates of the element dv. But, since we have generally

$$\int d\theta'' d\varpi'' \sin\theta'' P Q^{(i)} = \int d\theta'' d\varpi'' \sin\theta'' U''^{(i)} Q^{(i)} = \frac{4\pi}{2i+1} U'^{(i)},$$

(*Mec. Cel.* Liv. iii.) the preceding expression becomes

$$\rho = \frac{-\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2} a(a^2-1)^{\frac{4-n}{2}} (a^2-r'^2)^{-1}(1-r'^2)^{\frac{n-4}{2}} \int d\theta'' d\varpi'' \sin\theta'' \ldots\ldots$$

$$\ldots\ldots \Sigma_0^\infty (2i+1) P Q^{(i)} \frac{r'^i}{a^i};$$

the integrals being taken from $\theta''=0$ to $\theta''=\pi$, and from ϖ'' to $\varpi''=2\pi$.

In order to find the value of the finite integral entering into the preceding formula, let R represent the distance between the two elements $d\sigma$, dv; then by expanding $\frac{a}{R}$ in an ascending series of the powers of $\frac{r'}{a}$ we shall obtain

$$\frac{a}{R} = \frac{a}{\sqrt{a^2 - 2ar'[\cos\theta'\cos\theta'' + \sin\theta'\sin\theta''\cos(\varpi'-\varpi'') + r'^2}} = \Sigma_0^\infty Q^{(i)} . \frac{r'^i}{a^i},$$

Mec. Cel. Liv. iii.). Hence we immediately deduce

$$\frac{a\sqrt{r'}}{R} = \Sigma_0^\infty Q^{(i)} \frac{r'^{i+\frac{1}{2}}}{a^i}, \text{ and } 2\sqrt{r'}\frac{d.}{dr'}\frac{a\sqrt{r'}}{R} = \Sigma_0^\infty (2i+1) Q^{(i)} \frac{r'^i}{a^i}.$$

If now we substitute this in the value of ρ before given, and afterwards write $\frac{ds}{a^2}$ and $\frac{a^2-r'^2}{2R^3}$ in the place of their equivalents,

$$d\theta'' d\varpi'' \sin\theta'', \text{ and } \sqrt{r'}\frac{d.}{dr'}\frac{\sqrt{r'}}{R},$$

we shall obtain

$$\rho = -\frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2}(a^2-1)^{\frac{4-n}{2}}(1-r'^2)^{\frac{n-4}{2}}\int\frac{d\sigma P}{R^3};$$

the integral relative to $d\sigma$ being extended over the whole spherical surface.

Lastly, if ρ_1 represents the density of the reducing fluid disseminated over the space exterior to A, it is clear that we shall get the corresponding value of ρ by changing P into $\rho_1 da$ in the preceding expression, and then integrating the whole relative to a. Thus,

$$\rho = -\frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2}(1-r'^2)^{\frac{n-4}{2}}\int(1-a^2)^{\frac{4-n}{2}}\int\frac{d\sigma da\rho_1}{R^3}.$$

But $d\sigma da = dv_1$; dv_1 being an element of the volume of the exterior space, and therefore we ultimately get

$$(22)\ldots\ldots\rho = -\frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2}(1-r'^2)^{\frac{n-4}{2}}.\int\rho_1 dv_1\frac{(a^2-1)^{\frac{4-n}{2}}}{R^3};$$

where the last integral is supposed to extend over all the space exterior to the sphere and R, to represent the distance between the two elements dv and dv_1.

It is easy to perceive from what has before been shown (Art. 7.), that we may add to any of the preceding values of ρ, a term of the form

$$h(1-r'^2)^{\frac{n-4}{2}};$$

h being an arbitrary constant quantity: for it is clear from the article just cited, that the only alteration which such an addition could produce would be to change the value of the constant on the left side of the general equation of equilibrium; and as this constant is arbitrary, it is evident that the equilibrium will not be at all affected by the change in question. Moreover, it may be observed, that in general the additive term is necessary to enable us to assign the proper value of ρ, when Q, the quantity of redundant fluid originally introduced into the sphere, is given.

In the foregoing expressions the radius of the sphere has been taken as the unit of space, but it is very easy thence to deduce formulæ adapted to any other unit, by recollecting that $\frac{\rho}{\rho_1}$, $\frac{\rho}{P}$, $\frac{\rho}{U'^{(i)}}$ and $\frac{V_1}{U^{(i)}}$, are quantities of the dimensions 0, -1, -1 and $3-n$ respectively with regard to space: for if b represents the sphere's radius, when we employ any other unit we shall only have to write, $\frac{r}{b}$, $\frac{r'}{b}$, $\frac{R}{b}$, $\frac{dv_1}{b^3}$ and $\frac{a}{b}$ in the place of r, r', R, dv_1 and a, and afterwards to multiply the resulting expressions by such powers of b, as will reduce each of them to their proper dimensions.

If we here take the formula (22) of the present article as an example, there will result,

$$(23)\ldots\ldots\ldots \rho = -\frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2}(b^2-r'^2)^{\frac{n-4}{2}}\int \rho_1 dv_1 \frac{(a^2-b^2)^{\frac{4-n}{2}}}{R^3},$$

for the value of the density which would be induced in a sphere A, whose radius is b, by the action of any exterior bodies whatever.

When $n > 2$, the value of ρ or of the density of the free fluid here given offers no difficulties, but if $n < 2$, we shall not be able strictly to realize it, for reasons before assigned (Art. 6. and 7.) If however n is positive, and we adopt the hypothesis of two fluids, supposing that the quantities of each contained by bodies in a natural state are exceedingly great, we shall easily perceive by proceeding as in the last of the articles here cited, that the density given by the formula (23) will be sensibly correct except in the immediate vicinity of A's surface, provided we extend it to the surface of a sphere whose radius is $b-\delta b$ only, and afterwards conceive the exterior shell entirely deprived of fluid: the surface of the conducting sphere itself having such a quantity condensed upon it, that its density may every where be represented by

$$P' = -\frac{\sin\left(\frac{n-2}{2}\pi\right)}{2\pi^2} \times \frac{b^{\frac{n-4}{2}}(2\delta b)^{\frac{n-2}{2}}}{n-2} \int \rho_1 dv_1 \frac{(a^2-b^2)^{\frac{4-n}{2}}}{R^3}.$$

Application of the general Methods to circular conducting Planes, &c.

10. Methods in every way similar to those which have been used for a sphere, are equally applicable to a circular plane as we shall immediately proceed to show, by endeavouring in the first place to determine the value of V when the density of the fluid on such a plane is of the form

$$\rho = (1-r'^2)^\beta . f(x', y'):$$

f being the characteristic of a rational and entire function of the degree s; x', y' the rectangular co-ordinates of any element $d\sigma$ of the plane's surface, and r', θ' the corresponding polar co-ordinates.

Then we shall readily obtain the formula

$$V = \int \frac{\rho d\sigma}{g^{n-1}} = \iint \frac{r dr' d\theta' (1-r'^2)^\beta . f(x', y')}{(r^2 - 2rr' \cos(\theta - \theta') + r'^2)^{\frac{n-1}{2}}};$$

where r, θ are the polar co-ordinates of p, and the integrals are to be taken from $\theta'=0$ to $\theta'=2\pi$, and from $r'=0$ to $r'=1$; the radius of the circular plane being for greater simplicity considered as the unit of distance.

Since the function $f(x', y')$ is rational and entire of the degree s, we may always reduce it to the form

$$(24) \ldots\ldots\ldots f(x', y') = A^{(0)} + A^{(1)} \cos\theta' + A^{(2)} \cos 2\theta' + A^{(3)} \cos 3\theta' + \\ + B^{(1)} \sin\theta' + B^{(2)} \sin 2\theta' + B^{(3)} \sin 3\theta' +$$

the coefficients $A^{(0)}$, $A^{(1)}$, $A^{(2)}$, &c. $B^{(1)}$, $B^{(2)}$, $B^{(3)}$, &c. being functions of r' only of a degree not exceeding s, and such that

$$A^{(0)} = a_0^{(0)} + a_1^{(0)} r'^2 + a_2^{(0)} r'^4 + \&c.; \qquad A^{(1)} = (a_0^{(1)} + a_1^{(1)} r^2 + a_2^{(1)} r'^4 +) r';$$

$$B^{(1)} = (b_0^{(1)} + b_1^{(1)} r'^2 + b_2^{(1)} r'^4 + \&c.) r'; \quad B^{(2)} = (b_0^{(2)} + b_1^{(2)} r'^2 + \&c.) r'^2.$$

We will now consider more particularly the part of V due to any of the terms in f as $A^{(i)}\cos i\theta'$ for example. The value of this part will evidently be

$$\iint \frac{r'dr'd\theta'(1-r'^2)^\beta A^{(i)}\cos i\theta'}{(r^2-2rr'\cos(\theta-\theta')+r'^2)^{\frac{n-1}{2}}};$$

the limits of the integrals being the same as before. But if we make $\theta'=\theta+\phi$, there will result $d\theta'=d\phi$, and $\cos i\theta'=\cos i\theta\cos i\phi-\sin i\theta\sin i\phi$, and hence the double integral here given by observing that the term multiplied $\sin i\phi$ vanishes when the integration relative to ϕ is effected, becomes

$$\cos i\theta \int_0^1 A^{(i)}r'dr'(1-r'^2)^\beta \int_0^{2\pi}\frac{d\phi\cos i\phi}{(r^2-2rr'\cos\phi+r'^2)^{\frac{n-1}{2}}};$$

If now we write $V_t^{(i)}$ for that portion of V which is due to the term $a_t^{(i)}.r'^{i+2t}$ in the coefficient $A^{(i)}$ we shall have

$$V_t^{(i)}=a_t^{(i)}.\cos i\theta\int_0^1 r'^{i+2t+1}dr'(1-r'^2)^\beta\int_0^{2\pi}\frac{d\phi\cos i\phi}{(r^2-2rr'\cos\phi+r'^2)^{\frac{n-1}{2}}}.$$

But by well known methods we readily get

$$\int_0^{2\pi}\frac{d\phi\cos i\phi}{(r^2-2rr'\cos\phi+r'^2)^{\frac{n-1}{2}}}$$

$$=2\pi r^i.r'^{1-n-i}\Sigma_0^\infty\frac{r^{2t'}}{r'^{2t'}}.\frac{n-1.n+1\ldots\ldots n+2t'-3}{2\ .\ 4\ \ldots\ldots\ 2t'}\times\frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{2\ .\ 4\ \ldots\ldots\ 2i+2t'},$$

when $r'>r$, and when $r'<r$, the same expression will still be correct, provided we change r into r' and reciprocally.

This value being substituted in that of $V_t^{(i)}$ we shall readily have by following the processes before explained, (Art. 1. and 2.)

$$V_t^{(i)}=2\pi a_t^{(i)}r^i\cos i\theta\,\Sigma_0^\infty r^{2t'}\frac{n-1.n+1\ldots\ldots n+2t'-3}{2\ .\ 4\ \ldots\ldots\ 2t'}$$

$$\times\frac{n-1.n+1\ldots\ldots n+2i+2t'-3}{2\ .\ 4\ \ldots\ldots\ 2i+2t'}\times\frac{\Gamma(\beta+1)\Gamma\left(\frac{3+2t-2t'-n}{2}\right)}{2\Gamma\left(\frac{2\beta+5+2t-2t'}{2}\right)}$$

$$= \pi a_t^{(i)} r^i \cos i\theta \, . \frac{\Gamma(\beta+1)\,\Gamma\left(\frac{3-n}{2}\right)}{\Gamma\left(\frac{2\beta+5-n}{2}\right)} \, .$$

$$\Sigma_0^{\infty} r^{2t'} \frac{n-1 \, . \, n+1 \ldots\ldots n+2t'-3}{2 \, . \, 4 \ldots\ldots 2t'} \times \frac{n-1 \, . \, n+1 \ldots\ldots n+2i+2t'-3}{2 \, . \, 4 \ldots\ldots 2i+2t'}$$

$$\times \frac{3-n \, . \, 5-n \ldots\ldots 1+2t-2t'-n}{2\beta+5-n \ldots\ldots 2\beta+3+2t+2t'-n} \, ;$$

the sign of integration Σ belonging to the variable t'.

Having thus the part of V due to the term $a_t^{(i)} \cos i\theta'$ in the expansion of $f(x', y')$ it is clear that we may thence deduce the part due to the analogous term $b_t^{(i)} \sin i\theta'$ by simply changing $a_t^{(i)} \cos i\theta$ into $b_t^{(i)} \sin i\theta$, and consequently we shall have the total value of V itself, by taking the sum of the various parts due to all the different terms which enter into the complete expansion of $f(x', y')$.

If now we make $\beta = \frac{n-3}{2}$ and recollect that

$$\Gamma\left(\frac{n-1}{2}\right) \Gamma\left(\frac{3-n}{2}\right) = \frac{\pi}{\sin\left(\frac{n-1}{2}\pi\right)},$$

the foregoing expression will undergo simplifications analogous to those before noticed (Art. 5.) Thus we shall obtain

$$V_t^{(i)} = \frac{\pi^2 a_t^{(i)}}{\sin\left(\frac{n-1}{2}\pi\right)} r^i \cos i\theta \, . \, \Sigma r^{2t'} . \frac{n-1 \, . \, n+1 \ldots\ldots n+2t'-3}{2 \, . \, 4 \ldots\ldots 2t'}$$

$$\times \frac{n-1 \, . \, n+1 \ldots\ldots n+2i+2t'-3}{2 \, . \, 4 \ldots\ldots 2i+2t'} \times \frac{3-n \, . \, 5-n \ldots\ldots 1+2t-2t'-n}{2 \, . \, 4 \ldots\ldots 2t-2t'} \, ,$$

or by writing for abridgment

$$\phi(i, t') = \frac{n-1 \, . \, n+1 \ldots\ldots n+2t'-3}{2 \, . \, 4 \ldots\ldots 2t'} \times \frac{n-1 \, . \, n+1 \ldots\ldots n+2i+2t'-3}{2 \, . \, 4 \ldots\ldots 2i+2t'} \, ,$$

there will result this particular value of β

$$V_t^{(i)} = \frac{\pi^2 a_t^{(i)}}{\sin\left(\frac{n-1}{2}\pi\right)} r^i \cos i\theta \, . \, \Sigma r^{2t'} . \frac{3-n \, . \, 5-n \ldots\ldots 1+2t-2t'-n}{2 \, . \, 4 \, . \, 6 \ldots\ldots 2t-2t'} . \phi(i \, ; \, t'),$$

and afterwards by making

$$V^{(i)} = V_0^{(i)} + V_1^{(i)} + V_2^{(i)} + V_3^{(i)} + V_4^{(i)} + \&c.$$

we shall have

$$V^{(i)} = \frac{\pi^2}{\sin\left(\frac{n-1}{2}\pi\right)} r^i \cos i\theta \text{ into } \times \ldots\ldots$$

$$a_0^{(i)} . 1 . \phi(i;\,0)$$

$$+ a_1^{(i)} . \frac{3-n}{2} . \phi(i;\,0) + a_1^{(i)} . 1 . \phi(i;\,1) . r^2$$

$$+ a_2^{(i)} . \frac{3-n \,.\, 5-n}{2 \,.\, 4} . \phi(i;\,0) + a_2^{(i)} . \frac{3-n}{2} . \phi(i;\,1) . r^2 + a_2^{(i)} \phi(i;\,2) . r^4$$

$$+ a_3^{(i)} . \frac{3-n . 5-n . 7-n}{2 \,.\, 4 \,.\, 6} . \phi(i;\,0) + a_3^{(i)} \frac{3-n . 5-n}{2 \,.\, 4} . \phi(i;\,1) . r^2$$

$$+ a_3^{(i)} . \frac{3-n}{2} . \phi(i;\,2) . r^4 + a_3^{(i)} . 1 . \phi(i;\,3) . r^6$$

$$+ \&c. \ldots\ldots\ldots\ldots + \&c. \ldots\ldots\ldots\ldots + \&c. \ldots\ldots\ldots\ldots + \&c. \ldots\ldots\ldots\ldots$$

Conceiving in the next place that V is a given rational and entire function of x, y, the rectangular co-ordinates of p, we shall have since $x = r\cos\theta$, $y = r\sin\theta$.

$$(25) \ldots\ldots V = C^{(0)} + C^{(1)}\cos\theta + C^{(2)}\cos 2\theta + C^{(3)}\cos 3\theta + \&c.$$
$$+ E^{(1)}\sin\theta + E^{(2)}\sin 2\theta + E^{(3)}\sin 3\theta + \&c.$$

of which expansion any coefficient as $C^{(i)}$ for example, may be still farther developed in the form

$$C^{(i)} = \frac{\pi^2 . r^i}{\sin\left(\frac{n-1}{2}\pi\right)} \{c_0^{(i)} . \phi(i;\,0) + c_1^{(i)} . \phi(i;\,1) . r^2 + c_2^{(i)} . \phi(i;\,2) . r^4 + \&c.\}.$$

Now it is clear that the term $C^{(i)}\cos i\theta$ in the developement (25) corresponds to that part of V which we have designated by $V^{(i)}$, and hence by equating these two forms of the same quantity, we get

$$V^{(i)} = C^{(i)}\cos i\theta,$$

which by substituting for $V^{(i)}$ and $C^{(i)}$ their values before exhibited, and comparing like powers of the indeterminate quantity r gives

$$c_0^{(i)} = 1 \,.\, a_0^{(i)} + \frac{3-n}{2} a_1^{(i)} + \frac{3-n \,.\, 5-n}{2 \,.\, 4} a_2^{(i)} + \frac{3-n \,.\, 5-n \,.\, 7-n}{2 \,.\, 4 \,.\, 6} a_3^{(i)} + \&\text{c.}$$

$$c_1^{(i)} = 1 \,.\, a_1^{(i)} + \frac{3-n}{2} a_2^{(i)} + \frac{3-n \,.\, 5-n}{2 \,.\, 4} a_3^{(i)} + \&\text{c.}$$

$$c_2^{(i)} = 1 \,.\, a_2^{(i)} + \frac{3-n}{2} a_3^{(i)} + \&\text{c.}$$

&c. = ...&c..........&c.......

of which system the general type is

$$c_u^{(i)} = (1-\epsilon)^{\frac{n-3}{2}} \,.\, a_u^{(i)};$$

the symbols of operation being here separated from those of quantity, and ϵ being used in its ordinary acceptation with reference to the lower index u, so that we shall have generally

$$\epsilon^m \,.\, a_u^{(i)} = a_{u+m}^{(i)}.$$

The general equation between $a_u^{(i)}$ and $c_u^{(i)}$ being resolved, evidently gives by expanding the binomial and writing in the place of $\epsilon c_u^{(i)}$, $\epsilon^2 c_u^{(i)}$, $\epsilon^3 c_u^{(i)}$, &c. their values $c_{u+1}^{(i)}$, $c_{u+2}^{(i)}$, $c_{u+3}^{(i)}$, &c.

$$(26)\ldots\ldots\ldots\ldots a_u^{(i)} = (1-\epsilon)^{\frac{3-n}{2}} c_u^{(i)} = c_u^{(i)} + \frac{n-3}{2} c_{u+1}^{(i)} + \frac{n-3 \,.\, n-1}{2 \,.\, 4} c_{u+2}^{(i)} + \frac{n-3 \,.\, n-1 \,.\, n+1}{2 \,.\, 4 \,.\, 6} c_{u+3}^{(i)} + \&\text{c.}$$

Having thus the value of $a_u^{(i)}$ we thence immediately deduce the value of $A^{(i)}$ and this quantity being known, the first line of the expansion (25) evidently becomes known.

In like manner when we suppose that the quantity $E^{(i)}$ is expanded in a series of the form

$$E^{(i)} = \frac{\pi^2 r^4}{\sin\left(\frac{n-1}{2}\pi\right)} \{e_0^{(i)} \,.\, \phi(i;\,0) + e_1^{(i)} \phi(i;\,1) \,.\, r^2 + e_2^{(i)} \phi(i;\,2) \,.\, r^4 + \&\text{c.}\}$$

we shall readily deduce

$$b_u^{(i)} = (1-\epsilon)^{\frac{3-n}{2}} e_u^{(i)} = e_u^{(i)} + \frac{n-3}{2} e_{u+1}^{(i)} + \frac{n-3 \,.\, n-1}{2 \;.\; 4} e_{u+2}^{(i)} + \&c.,$$

and $b_u^{(i)}$ being thus given, $B^{(i)}$ and consequently the second line of the expansion (25) are also given.

From what has preceded, it is clear that when V is given equal to any rational and entire function whatever of x and y, the value of $f(x', y')$ entering into the expression

$$\rho = (1 - r'^2)^{\frac{n-3}{2}} . f(x', y'),$$

will immediately be determined by means of the most simple formulæ.

The preceding results being quite independent of the degree s of the function $f(x', y')$ will be equally applicable when s is infinite, or wherever this function can be expanded in a series of the entire powers of x', y', and the various products of these powers.

We will now endeavour to determine the manner in which one fluid will distribute itself on the circular conducting plane A when acted upon by fluid distributed in any way in its own plane.

For this purpose, let us in the first place conceive a quantity q of fluid concentrated in a point P, where $r=a$ and $\theta=0$, to act upon a conducting plate whose radius is unity. Then the value of V due to this fluid will evidently be

$$\frac{q}{(a^2 - 2ar \cos\theta + r^2)^{\frac{n-1}{2}}} = V',$$

and consequently the equation of equilibrium analogous to the one marked (20) Art. 10., will be

$$(27) \ldots\ldots\ldots\ldots\ldots \text{const.} = \frac{q}{(a^2 - 2ar \cos\theta + r^2)^{\frac{n-1}{2}}} + V;$$

V being due to the fluid on the conducting plate only.

If now we expand the value of V deduced from this equation, and then compare it with the formulæ (25) of the present article, we shall have generally $E^{(i)}=0$, and

$$C^{(i)}=-2qa^{-n}\frac{r^i}{a^i}\cdot\{\phi(i;\,0)+\phi(i;\,1)\frac{r^2}{a^2}+\phi(i;\,2)\frac{r^4}{a^4}+\phi(i;\,3)\frac{r^6}{a^6}+\&c.\},$$

except when $i=0$, in which case we must take only half the quantity furnished by this expression in order to have the correct value of $C^{(0)}$. Hence whatever u may be,

$$e_u^{(i)}=0,\quad \text{and } c_u^{(i)}=-\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}qa^{1-n-i-2u};$$

the particular value $i=0$ being excepted, for in this case we have agreeably to the preceding remark

$$c_u^{(0)}=-\frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}q\cdot a^{1-n-2u},$$

and then the only remaining exception is that due to the constant quantity on the left side of the equation (27). But it will be more simple to avoid considering this last exception here, and to afterwards add to the final result the term which arises from the constant quantity thus neglected.

The equation (26) of the present article gives by substituting for $c_u^{(i)}$ its value just found.

$$a_u^{(i)}=-\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}qa^{1-n-i-2u}\cdot\{1+\frac{n-3}{2}\cdot a^{-2}$$

$$+\frac{n-3\,.\,n-1}{2\,.\,4}\cdot a^{-4}+\frac{n-3\,.\,n-1\,.\,n-1}{2\,.\,4\,.\,6}\cdot a^{-6}+\&c.\}$$

$$=-\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}qa^{1-n-i-2u}(1-a^{-2})^{\frac{3-n}{2}}$$

$$=-\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}qa^{-2-i-2u}\cdot(a^2-1)^{\frac{3-n}{2}},$$

and consequently,

$$A^{(i)} = \{a_0^{(i)} + a_1^{(i)} r'^2 + a_2^{(i)} r'^4 + \&c.\} \, r'^i$$

$$= -\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q a^{-2-i} (a^2-1)^{\frac{3-n}{2}} r'^i . \{1 + \frac{r'^2}{a^2} + \frac{r'^4}{a^2} + \&c.\}$$

$$= -\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q a^{-2-i} (a^2-1)^{\frac{3-n}{2}} r'^i \left(1 - \frac{r'^2}{a^2}\right)^{-1}$$

$$= -\frac{2\sin\left(\frac{n-2}{2}\pi\right)}{\pi^2} q (a^2-1)^{\frac{3-n}{2}} (a^2 - r'^2)^{-1} . \frac{r'^i}{a^i};$$

the particular value $A^{(0)}$ being one half only of what would result from making $i=0$ in this general formulæ.

But $e_s^{(i)} = 0$ evidently gives $E^{(i)} = 0$, and therefore the expansion of $f(x', y')$ before given becomes

$$f(x', y') = A^{(0)} + A^{(1)} \cos\theta' + A^{(2)} \cos 2\theta' + A^{(3)} \cos 3\theta' + \&c.$$

$$= -\frac{2\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q (a^2-1)^{\frac{3-n}{2}} (a^2 - r'^2)^{-1} . \{\tfrac{1}{2} + \frac{r'}{a}\cos\theta' + \frac{r'^2}{a^2}\cos 2\theta' + \&c.\}$$

or by summing the series included between the braces,

$$f(x', y') = -\frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q \frac{(a^2-1)^{\frac{3-n}{2}}}{a^2 - 2ar'\cos\theta' + r'^2}$$

$$= -\frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q \frac{(a^2-1)^{\frac{3-n}{2}}}{R^2};$$

R being the distance between P, the point in which the quantity of fluid q is concentrated, and that to which the density ρ is supposed to belong.

Having thus the value of $f(x', y')$ we thence deduce

$$\rho = (1 - r'^2)^{\frac{n-3}{2}} f(x', y') = -\frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} (1 - r'^2)^{\frac{n-3}{2}} q \frac{(a^2-1)^{\frac{3-n}{2}}}{R^2}.$$

The value of ρ here given being expressed in quantities perfectly independent of the situation of the axis from which the angle θ' is measured, is evidently applicable when the point P is not situated upon this axis, and in order to have the complete value of ρ, it will now only be requisite to add the term due to the arbitrary constant quantity on the left side of the equation (26), and as it is clear from what has preceded, that the term in question is of the form

$$\text{const.} \times (1 - r'^2)^{\frac{n-3}{2}},$$

we shall therefore have generally, wherever P may be placed,

$$\rho = (1 - r'^2)^{\frac{n-3}{2}} \cdot \left\{ \text{const.} - \frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} q \cdot \frac{(a^2 - 1)^{\frac{3-n}{2}}}{R^2} \right\}.$$

The transition from this particular case to the more general one, originally proposed is almost immediate: for if ρ represents the density of the inducing fluid on any element $d\sigma_1$ of the plane coinciding with that of the plate, $\rho_1 d\sigma_1$ will be the quantity of fluid contained in this element, and the density induced thereby will be had from the last formula, by changing q into $\rho_1 d\sigma_1$. If then we integrate the expression thus obtained, and extend the integral over all the fluid acting on the plate, we shall have for the required value of ρ

$$\rho = (1 - r'^2)^{\frac{n-3}{2}} \cdot \left\{ \text{const.} - \frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2} \int \rho_1 d\sigma_1 \frac{(a^2 - 1)^{\frac{3-n}{2}}}{R^2} \right\};$$

R being the distance of the element $d\sigma_1$ from the point to which ρ belongs, and a the distance between $d\sigma_1$ and the center of the conducting plate.

Hitherto the radius of the circular plate has been taken as the unit of distance, but if we employ any other unit, and suppose that b is the measure of the same radius, in this case we shall only have to write $\frac{a}{b}$, $\frac{r'}{b}$, $\frac{d\sigma_1}{b^2}$ and $\frac{R}{b}$ in the place of a, r', $d\sigma_1$ and R respectively, recollecting that $\frac{\rho}{\rho_1}$ is a quantity of the dimension 0 with regard to space, by so doing the resulting value of ρ is

$$(28)\ldots\ldots\ldots\rho=(b^2-r'^2)^{\frac{n-3}{2}}.\left\{\text{const.}-\frac{\sin\left(\frac{n-1}{2}\pi\right)}{\pi^2}\int\rho_1 d\sigma_1\frac{(a^2-b^2)^{\frac{3-n}{2}}}{R^2}\right\}.$$

By supposing $n=2$, the preceding investigation will be applicable to the electric fluid, and the value of the density induced upon an infinitely thin conducting plate by the action of a quantity of this fluid, distributed in any way at will in the plane of the plate itself will be immediately given. In fact, when $n=2$, the foregoing value of ρ becomes

$$\rho=\frac{1}{\sqrt{b^2-r'^2}}\left\{\text{const.}-\frac{1}{\pi^2}\int\rho_1 d\sigma_1\frac{\sqrt{a^2-b^2}}{R^2}\right\}.$$

If we suppose the plate free from all extraneous action, we shall simply have to make $\rho_1=0$ in the preceding formula; and thus

$$(29)\ldots\ldots\ldots\rho=\frac{\text{const.}}{\sqrt{b^2-r'^2}}.$$

Biot (*Traité de Physique*, Tom. II. p. 277.), has related the results of some experiments made by Coulomb on the distribution of the electric fluid when in equilibrium upon a plate of copper 10 inches in diameter, but of which the thickness is not specified. If we conceive this thickness to be very small compared with the diameter of the plate, which was undoubtedly the case, the formula just found ought to be applicable to it, provided we except those parts of the plate which are in the immediate vicinity of its exterior edge. As the comparison of any results mathematically deduced from the received theory of electricity with those of the experiments of so accurate an observer as Coulomb must always be interesting, we will here give a table of the values of the density at different points on the surface of the plate, calculated by means of the formula (29), together with the corresponding values found from experiment.

Distances from the Plate's edge.	Observed densities.	Calculated densities.
5 in.......	1,	1,
4	1,001	1,020
3	1,005	1,090
2	1,17	1,250
1	1,52	1,667
,5........	2,07	2,294
0	2,90	infinite.

We thus see that the differences between the calculated and observed densities are trifling; and moreover, that the observed are all something smaller than the calculated ones, which it is evident ought to be the case, since the latter have been determined by considering the thickness of the plate as infinitely small, and consequently they will be somewhat greater than when this thickness is a finite quantity, as it necessarily was in Coulomb's experiments.

It has already been remarked that the method given in the second article is applicable to any ellipsoid whatever, whose axes are a, b, c. In fact, if we suppose that x, y, z are the co-ordinates of a point p within it, and x', y', z' those of any element dv of its volume, and afterwards make

$$x = a.\cos\theta, \quad y = b.\sin\theta\ \cos\varpi, \quad z = c.\sin\theta\ \sin\varpi,$$

$$x' = a.\cos\theta', \quad y' = b.\sin\theta'\ \cos\varpi', \quad z' = c.\sin\theta'\ \sin\varpi',$$

we shall readily obtain by substitution,

$$V = abc\int\rho.r'^2 dr' d\theta' d\varpi' \sin\theta'.(\lambda r^2 - 2\mu rr' + \nu r'^2)^{\frac{1-n}{2}};$$

the limits of the integrals being the same as before (Art. 2.), and

$$\lambda = a^2\cos\theta^2 + b^2\sin\theta^2\cos\varpi^2 + c^2\sin\theta^2\sin\varpi^2,$$

$$\mu = a^2\cos\theta\cos\theta' + b^2\sin\theta\sin\theta'\cos\varpi\cos\varpi' + c^2\sin\theta\sin\theta'\sin\varpi\sin\varpi',$$

$$\nu = a^2\cos\theta'^2 + b^2\sin\theta'^2\cos\varpi'^2 + c^2\sin\theta'^2\sin\varpi'^2.$$

Under the present form it is clear the determination of V can offer no difficulties after what has been shown (Art. 2.). I shall not therefore insist upon it here more particularly, as it is my intention in a future paper to give a general and purely analytical method of finding the value of V, whether p is situated within the ellipsoid or not. I shall therefore only observe, that for the particular value

$$(30)\ldots\ldots\ldots \rho = k\left(1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2} - \frac{z'^2}{c^2}\right)^{\frac{n-4}{2}} = k(1-r'^2)^{\frac{n-4}{2}},$$

the series $U_0' + U_2' + U_4' + \&c.$ (Art. 2.) will reduce itself to the single term U_0', and we shall ultimately get

$$V = \frac{\pi k a b c}{2\sin\left(\frac{n-2}{2}\pi\right)} \int_0^\pi d\theta' \sin\theta' \int_0^{2\pi} d\varpi' (a^2\cos\theta'^2 + b^2\sin\theta'^2\cos\varpi'^2 + c^2\sin\theta'^2\sin\varpi'^2)^{\frac{1-n}{2}},$$

which is evidently a constant quantity. Hence it follows that the expression (30) gives the value of ρ when the fluid is in equilibrium within the ellipsoid, and free from all extraneous action. Moreover, this value is subject, when $n < 2$, to modifications similar to those of the analagous value for the sphere (Art. 7.).

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