ON

from the author

THE THEORY

OF CERTAIN

BANDS SEEN IN THE SPECTRUM.

BY

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XVI. *On the Theory of certain Bands seen in the Spectrum.*

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SOME months ago Professor POWELL communicated to me an account of a new case of interference which he had discovered in the course of some experiments on a fluid prism, requesting at the same time my consideration of the theory. As the phenomenon is fully described in Professor POWELL's memoir, and is briefly noticed in art. 1 of this paper, it is unnecessary here to allude to it. It struck me that the theory of the phenomenon was almost identical with that of the bands seen when a spectrum is viewed by an eye, half the pupil of which is covered by a plate of glass or mica. The latter phenomenon has formed the subject of numerous experiments by Sir DAVID BREWSTER, who has discovered a very remarkable polarity, or apparent polarity, in the bands. The theory of these bands has been considered by the Astronomer Royal in two memoirs "On the Theoretical Explanation of an apparent new Polarity of Light," printed in the Philosophical Transactions for 1840 (Part II.) and 1841 (Part I.). In the latter of these Mr. AIRY has considered the case in which the spectrum is viewed in focus, which is the most interesting case, as being that in which the bands are best seen, and which is likewise far simpler than the case in which the spectrum is viewed out of focus. Indeed, from the mode of approximation adopted, the former memoir can hardly be considered to belong to the bands which formed the subject of Sir DAVID BREWSTER's experiments, although the memoir no doubt contains the theory of a possible system of bands. On going over the theory of the bands seen when the spectrum is viewed in focus, after the receipt of Professor POWELL's letter, I was led to perceive that the intensity of the light could be expressed in finite terms. This saves the trouble of Mr. AIRY's quadratures, and allows the results to be discussed with great facility. The law, too, of the variation of the intensity with the thickness of the plate is very remarkable, on account of its discontinuity. These reasons have induced me to lay my investigation before the Royal Society, even though the remarkable polarity of the bands has been already explained by the Astronomer Royal. The observation of these bands seems likely to become of great importance in the determination of the refractive indices, and more especially the laws of dispersion, of minerals and other substances which cannot be formed into prisms which would exhibit the fixed lines of the spectrum.

Section I.—*Explanation of the formation of the bands on the imperfect theory of Interferences. Mode of calculating the number of bands seen in a given part of the spectrum.*

1. The phenomenon of which it is the principal object of the following paper to investigate the theory, is briefly as follows. Light introduced into a room through a horizontal slit is allowed to pass through a hollow glass prism containing fluid, with its refracting edge horizontal, and the spectrum is viewed through a small telescope with its object-glass close to the prism. On inserting into the fluid a transparent plate with its lower edge horizontal, the spectrum is seen traversed from end to end by very numerous dark bands, which are parallel to the fixed lines. Under favorable circumstances the dark bands are intensely black; but in certain cases, to be considered presently, no bands whatsoever are seen. When the plate is cut from a doubly refracting crystal, there are in general two systems of bands seen together; and when the light is analysed each system disappears in turn at every quarter revolution of the analyser.

2. It is not difficult to see that the theory of these bands must be almost identical with that of the bands described by Sir David Brewster in the Report of the Seventh Meeting of the British Association, and elsewhere, and explained by Mr. Airy in the first Part of the Philosophical Transactions for 1841. To make this apparent, conceive an eye to view a spectrum through a small glass vessel with parallel faces filled with fluid. The vessel would not alter the appearance of the spectrum. Now conceive a transparent plate bounded by parallel surfaces inserted into the fluid, the plane of the plate being perpendicular to the axis of the eye, and its edge parallel to the fixed lines of the spectrum, and opposite to the centre of the pupil. Then we should have bands of the same nature as those described by Sir David Brewster, the only difference being that in the present case the retardation on which the existence of the bands depends is the difference of the retardations due to the plate itself, and to a plate of equal thickness of the fluid, instead of the absolute retardation of the plate, or more strictly, the difference of retardations of the solid plate and of a plate of equal thickness of air, contained between the produced parts of the bounding planes of the solid plate. In Professor Powell's experiment the fluid fills the double office of the fluid in the glass vessel and of the prism producing the spectrum in the imaginary experiment just described.

It might be expected that the remarkable polarity discovered by Sir David Brewster in the bands which he has described, would also be exhibited with Professor Powell's apparatus. This anticipation is confirmed by experiment. With the arrangement of the apparatus already mentioned, it was found that with certain pairs of media, one being the fluid and the other the retarding plate, no bands were visible. These media were made to exhibit bands by using fluid enough to cover the plate to a certain depth, and stopping by a screen the light which would otherwise have passed through the thin end of the prism underneath the plate.

3. Although the explanation of the polarity of the bands depends on diffraction, it may be well to account for their formation on the imperfect theory of interferences, in which it is supposed that light consists of rays which follow the courses assigned to them by geometrical optics. It will thus readily appear that the number of bands formed with a given plate and fluid, and in a given part of the spectrum, has nothing to do with the form or magnitude of the aperture, whatever it be, which limits the pencil that ultimately falls on the retina. Moreover, it seems desirable to exhibit in its simplest shape the mode of calculating the number of bands seen in any given case, more especially as these calculations seem likely to be of importance in the determination of refractive indices.

4. Before the insertion of the plate, the wave of light belonging to a particular colour, and to a particular point of the slit, or at least a certain portion of it limited by the boundaries of the fluid, after being refracted at the two surfaces of the prism, enters the object-glass with an unbroken front. The front is here called unbroken, because the modification which the wave suffers at its edges is not contemplated. According to geometrical optics, the light after entering the object-glass is brought to a point near the principal focus, spherical aberration being neglected ; according to the undulatory theory, it forms a small, but slightly diffused image of the point from which it came. The succession of these images due to the several points of the slit forms the image of the slit for the colour considered, and the succession of coloured images forms the spectrum, the waves for the different colours covering almost exactly the same portion of the object-glass, but differing from one another in direction.

Apart from all theory, it is certain that the image of a point or line of homogeneous light seen with a small aperture is diffused. As the aperture is gradually widened the extent of diffusion decreases continuously, and at last becomes insensible. The perfect continuity, however, of the phenomenon shows that the true and complete explanation, whatever it may be, of the narrow image seen with a broad aperture, ought also to explain the diffused image seen with a narrow aperture. The undulatory theory explains perfectly both the one and the other, and even predicts the distribution of the illumination in the image seen with an aperture of given form, which is what no other theory has ever attempted.

As an instance of the effect of diffusion in an image, may be mentioned the observed fact that the definition of a telescope is impaired by contracting the aperture. With a moderate aperture, however, the diffusion is so slight as not to prevent fine objects, such as the fixed lines of the spectrum, from being well seen.

For the present, however, let us suppose the light entering the telescope to consist of rays which are brought accurately to a focus, but which nevertheless interfere. When the plate is inserted into the fluid the front of a wave entering the object-glass will no longer be unbroken, but will present as it were a *fault*, in consequence of the retardation produced by the plate. Let R be this retardation measured by actual length in air, ϱ the retardation measured by phase, M the retardation measured by the number of waves' lengths, so that

$$\varrho=\frac{2\pi}{\lambda}\mathrm{R},\ \mathrm{M}=\frac{1}{\lambda}\mathrm{R};$$

then when M is an odd multiple of $\frac{1}{2}$, the vibrations produced by the two streams, when brought to the same focus, will oppose each other, and there will be a minimum of illumination; but when M is an even multiple of $\frac{1}{2}$ the two streams will combine, and the illumination will be a maximum. Now M changes in passing from one colour to another in consequence of the variations both of R and of λ; and since the different colours occupy different angular positions in the field of view, the spectrum will be seen traversed by dark and bright bands. It is nearly thus that Mr. Talbot has explained the bands seen when a spectrum is viewed through a hole in a card which is half-covered with a plate of glass or mica, with its edge parallel to the fixed lines of the spectrum. Mr. Talbot however does not appear to have noticed the polarity of the bands.

Let h, k be the breadths of the interfering streams; then we may take

$$h\sin\frac{2\pi}{\lambda}vt,\ \ k\sin\left(\frac{2\pi}{\lambda}vt-\varrho\right)$$

to represent the vibrations produced at the focus by the two streams respectively, which gives for the intensity I,

$$\mathrm{I}=(h+k\cos\varrho)^2+(k\sin\varrho)^2=h^2+k^2+2hk\cos\varrho, \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

which varies between the limits $(h-k)^2$ and $(h+k)^2$.

5. Although the preceding explanation is imperfect, for the reason already mentioned, and does not account for the polarity, it is evident that if bands are formed at all in this way, the number seen in a given part of the spectrum will be determined correctly by the imperfect theory; for everything will recur, so far as interference is concerned, when M is decreased or increased by 1, and not before. This points out an easy mode of determining the number of bands seen in a given part of the spectrum. For the sake of avoiding a multiplicity of cases, let an acceleration be reckoned as a negative retardation, and suppose R positive when the stream which passes nearer to the edge of the prism is retarded relatively to the other. From the known refractive indices of the plate and fluid, and from the circumstances of the experiment, calculate the values of R for each of the fixed lines B, C . . . . H of the spectrum, or for any of them that may be selected, and thence the values of M, by dividing by the known values of λ. Set down the results with their proper signs opposite to the letters B, C . . . denoting the rays to which they respectively refer, and then form a table of differences by subtracting the value of M for B from the value for C, the value for C from the value for D, and so on. Let N be the number found in the table of differences corresponding to any interval, as for example from F to G; then the numerical value of N, that is to say, N or $-$N, according as N is positive

or negative, gives the number of bands seen between F and G. For anything that appears from the imperfect theory of the bands given in the preceding article, it would seem that the sign of N was of no consequence. It will presently be seen, however, that the sign is of great importance: it will be found in fact that the sign + indicates that the second arrangement mentioned in art. 2 must be employed; that is to say, the plate must be made to intercept light from the thin end of the prism, while the sign − indicates that the first arrangement is required. It is hardly necessary to remark that, if N should be fractional, we must, instead of the number of bands, speak of the number of band-intervals and the fraction of an interval.

Although the number of bands depends on nothing but the values of N, the values of M are not without physical interest. For M expresses, as we have seen, the number of waves' lengths whereby one of the interfering streams is before or behind the other. Mr. Airy speaks of the formation of rings with the light of a spirit-lamp when the retardation of one of the interfering streams is as much as fifty or sixty waves' lengths. But in some of Professor Powell's experiments, bands were seen which must have been produced by retardations of several hundred waves' lengths. This exalts our ideas of the regularity which must be attributed to the undulations.

6. It appears then that the calculation of the number of bands is reduced to that of the retardation R. As the calculation of R is frequently required in physical optics, it will not be necessary to enter into much detail on this point. The mode of performing the calculation, according to the circumstances of the experiment, will best be explained by a few examples.

Suppose the retarding plate to belong to an ordinary medium, and to be placed so as to intercept light from the thin end of the prism, and to have its plane equally inclined to the faces of the prism. Suppose the prism turned till one of the fixed lines, as F, is seen at a minimum deviation; then the colours about F are incident perpendicularly on the plate; and all the colours may without material error be supposed to be incident perpendicularly, since the directions of the different colours are only separated by the dispersion accompanying the first refraction into the fluid, and near the normal a small change in the angle of incidence produces only a very small change in the retardation. The dispersion accompanying the first refraction into the fluid has been spoken of as if the light were refracted from air directly into the fluid, which is allowable, since the glass sides of the hollow prism, being bounded by parallel surfaces, may be dispensed with in the explanation. Let T be the thickness of the plate, μ the refractive index of the fluid, μ' that of the plate; then

$$R=(\mu'-\mu)T. \qquad (2.)$$

If the plate had been placed so as to intercept light from the thick end of the prism, we should have had $-R=(\mu'-\mu)T$, which would have agreed with (2.) if we had supposed T negative. For the future T will be reckoned positive when the plate intercepts light from the thin end of the prism, and negative when it intercepts light from the thick end, so that the same formulæ will apply to both of the arrangements mentioned in art. 2.

If we put $\mu=1$, the formula (2.) will apply to the experiment in which a plate of glass or mica is held so as to cover half the pupil of the eye when viewing a spectrum formed in any manner, the plate being held perpendicularly to the axis of the eye. The effect of the small obliquity of incidence of some of the colours is supposed to be neglected.

The number of bands which would be determined by means of the formula (2.) would not be absolutely exact, unless we suppose the observation taken by receiving each fixed line in succession at a perpendicular incidence. This may be effected in the following manner. Suppose that we want to count the number of bands between F and G, move the plate by turning it round a horizontal axis till the bands about F are seen stationary; then begin to count from F, and before stopping at G incline the plate a little till the bands about G are seen stationary, estimating the fractions of an interval at F and G, if the bands are not too close. The result will be strictly the number given by the formula (2.). The difference, however, between this result and that which would be obtained by keeping the plate fixed would be barely sensible. If the latter mode of observation should be thought easier or more accurate, the exact formula which would replace (2.) would be easily obtained.

7. Suppose now the nearer face of the retarding plate made to rest on the nearer inner face of the hollow prism, and suppose one of the fixed lines, as F, to be viewed at a minimum deviation. Let ϕ, ϕ' be the angles of incidence and refraction at the first surface of the fluid, i, i' those at the surface of the plate, 2ε the angle of the prism. Since the deviation of F is a minimum, the angle of refraction ϕ'_{F} for F is equal to ε, and the angle of incidence ϕ is given by $\sin\phi=\mu_{\mathrm{F}}\sin\phi'_{\mathrm{F}}$, and ϕ is the angle of incidence for all the colours, the incident light being supposed white. The angle of refraction ϕ' for any fixed line is given by the equation $\sin\phi'=\frac{1}{\mu}\sin\phi=\frac{\mu_{\mathrm{F}}}{\mu}\sin\varepsilon$; then $i=2\varepsilon-\phi'$, and i' is known from the equation

$$\mu'\sin i'=\mu\sin i. \quad \text{(3.)}$$

The retardation is given by either of the formulæ

$$\mathrm{R}=\mu'\mathrm{T}\frac{\sin(i-i')}{\sin i}, \quad \text{(4.)}$$

$$\mathrm{R}=\mathrm{T}(\mu'\cos i'-\mu\cos i). \quad \text{(5.)}$$

These formulæ might be deduced from that given in AIRY's Tract, modified so as to suit the case in which the plate is immersed in a fluid; but either of them may be immediately proved independently by referring everything to the wave's front and not the ray.

By multiplying and dividing the second side of (5.) by $\cos i$, and employing (3.), we get

$$\mathrm{R}=\mathrm{T}\sec i\,.(\mu'-\mu)-\mathrm{T}\mu'\sec i\,\mathrm{versin}\,(i-i'). \quad \text{(6.)}$$

When the refractive indices of the plate and fluid are nearly equal, the last term in this equation may be considered insensible, so that it is not necessary to calculate i' at all.

8. The formulæ (2.), (4.), (5.), (6.) are of course applicable to the ordinary ray of a plate cut from a uniaxal crystal. If the plate be cut in a direction parallel to the axis, and if moreover the lower edge be parallel to the axis, so that the axis is parallel to the refracting edge of the prism, the formulæ will apply to both rays. If μ_o, μ_e be the principal indices of refraction referring to the ordinary and extraordinary rays respectively, μ' in the case last supposed must be replaced by μ_o for the bands polarized in a plane perpendicular to the plane of incidence, and by μ_e for the bands polarized in the plane of incidence. In the case of a plate cut from a biaxal crystal in such a direction that one of the principal axes, or axes of elasticity, is parallel to the refracting edge, the same formulæ will apply to that system of bands which is polarized in the plane of incidence.

If the plate be cut from a biaxal crystal in a direction perpendicular to one of the principal axes, and be held in the vertical position, the formula (2.) will apply to both systems of bands, if the small effect of the obliquity be neglected. The formula would be exact if the observations were taken by receiving each fixed line in succession at a perpendicular incidence.

If the plate be cut from a uniaxal crystal in a direction perpendicular to the axis, and be held obliquely, we have for the extraordinary bands, which are polarized in a plane perpendicular to the plane of incidence,

$$R=T\left(\frac{\mu_o}{\mu_e}\sqrt{\mu_e^2-\mu^2\sin^2 i}-\mu\cos i\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

which is the same as the formula in AIRY's Tract, only modified so as to suit the case in which the plate is immersed in fluid, and expressed in terms of refractive indices instead of velocities. If we take a subsidiary angle j, determined by the equation

$$\sin j=\frac{\mu}{\mu_e}\sin i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

the formula (7.) becomes

$$R=T(\mu_o\cos j-\mu\cos i), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

which is of the same form as (5.), and may be adapted to logarithmic calculation if required by assuming $\frac{\mu_o}{\mu}=\tan\theta$. The preceding formula will apply to the extraordinary bands formed by a plate cut from a biaxal crystal in the manner described in the last paragraph, and held obliquely, the extraordinary bands being understood to mean those which are polarized in a plane perpendicular to the plane of incidence. In this application we must take for μ_e, μ_o those two of the three principal indices of refraction which are symmetrically related to the axis normal to the plate, and to the axis parallel to the plate, and lying in the plane of incidence, respectively; while in applying the formula (4.), (5.) or (6.) to the other system of bands, the third principal index must be substituted for μ'.

It is hardly necessary to consider the formula which would apply to the general case, which would be rather complicated.

9. If a plate cut from a uniaxal crystal in a direction perpendicular to the axis be placed in the fluid in an inclined position, and be then gradually made to approach the vertical position, the breadths of the bands belonging to the two systems will become more and more nearly equal, and the two systems will at last coalesce. This statement indeed is not absolutely exact, because the whole spectrum cannot be viewed at once by light which passes along the axis of the crystal, on account of the dispersion accompanying the first refraction, but it is very nearly exact. With quartz it is true there would be two systems of bands seen even in the vertical position, on account of the peculiar optical properties of that substance; but the breadths of the bands belonging to the two systems would be so nearly equal, that it would require a plate of about one-fifth of an inch thickness to give a difference of one in the number of bands seen in the whole spectrum in the case of the two systems respectively. If the plate should be thick enough to exhibit both systems, the light would of course have to be circularly analysed to show one system by itself.

SECTION II.—*Investigation of the intensity of the light on the complete theory of undulations, including the explanation of the apparent polarity of the bands.*

10. The explanation of the formation of the bands on the imperfect theory of interferences considered in the preceding section is essentially defective in this respect, that it supposes an annihilation of light when two interfering streams are in opposition; whereas it is a most important principle that *light is never lost by interference.* This statement may require a little explanation, without which it might seem to contradict received ideas. It is usual in fact to speak of light as *destroyed* by interference. Although this is true, in the sense intended, the expression is perhaps not very happily chosen. Suppose a portion of light coming from a luminous point, and passing through a moderately small aperture, to be allowed to fall on a screen. We know that there would be no sensible illumination on the screen except almost immediately in front of the aperture. Conceive now the aperture divided into a great number of small elements, and suppose the same quantity of light as before to pass through each element, the only difference being that now the vibrations in the portions passing through the several elements are supposed to have no relation to each other. The light would now be diffused over a comparatively large portion of the screen, so that a point P which was formerly in darkness might now be strongly illuminated. The disturbance at P is in both cases the aggregate of the disturbances due to the several elements of the aperture; but in the first case the aggregate is insensible on account of interference. It is only in this sense that light is *destroyed* by interference, for the total illumination on the screen is the same in the two cases;

the effect of interference has been, not to annihilate any light, but only to alter the "distribution of the illumination," so that the light, instead of being diffused over the screen, is concentrated in front of the aperture.

Now in the case of the bands considered in Section I., if we suppose the plate extremely thin, the bands will be very broad; and the displacement of illumination due to the retardation being small compared with the breadth of a band, it is evident, without calculation, that at most only faint bands can be formed. This particular example is sufficient to show the inadequacy of the imperfect theory, and the necessity of an exact investigation.

11. Suppose first that a point of homogeneous light is viewed through a telescope. Suppose the object-glass limited by a screen in which there is formed a rectangular aperture of length $2l$. Suppose a portion of the incident light retarded, by passing through a plate bounded by parallel surfaces, and having its edge parallel to the length of the aperture. Suppose the unretarded stream to occupy a breadth h of the aperture at one side, the retarded stream to occupy a breadth k at the other, while an interval of breadth $2g$ exists between the streams. In the apparatus mentioned in Section I., the object-glass is not limited by a screen, but the interfering streams of light are limited by the dimensions of the fluid prism, which comes to the same thing. The object of supposing an interval to exist between the interfering streams, is to examine the effect of the gap which exists between the streams when the retarding plate is inclined. In the investigation the effect of diffraction before the light reaches the object-glass of the telescope is neglected.

Let O be the image of the luminous point, as determined by geometrical optics, f the focal length of the object-glass, or rather the distance of O from the object-glass, which will be a little greater than the focal length when the luminous point is not very distant. Let C be a point in the object-glass, situated in the middle of the interval between the two streams, and let the intensity be required at a point M, near O, situated in a plane passing through O and perpendicular to OC. The intensity at any point of this plane will of course be sensibly the same as if the plane were drawn perpendicular to the axis of the telescope instead of being perpendicular to OC. Take OC for the axis of z, the axes of x and y being situated in the plane just mentioned, and that of y being parallel to the length of the aperture. Let p, q be the coordinates of M; x, y, z those of a point P in the front of a wave which has just passed through the object-glass, and which forms part of a sphere with O for its centre. Let c be the coefficient of vibration at the distance of the object-glass; then we may take

$$\frac{c}{\lambda}\cdot\frac{1}{\text{PM}}\sin\frac{2\pi}{\lambda}(vt-\text{PM})\,dxdy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a.)$$

to represent the disturbance at M due to the element $dxdy$ of the aperture at P, P being supposed to be situated in the unretarded stream, which will be supposed to lie at the negative side of the axis of x. In the expression $(a.)$ it is assumed that the proper multiplier of $\frac{c}{\text{PM}}$ is $\frac{1}{\lambda}$. This may be shown to be a necessary consequence of the principle mentioned in the preceding article, that light is never lost by interference; and this principle follows directly from the principle of *vis viva*. In proving that λ^{-1} is the proper multiplier, it is not in the least necessary to enter into the consideration of the law of the variation of intensity in a secondary wave, as the angular distance from the normal to the primary wave varies; the result depends merely on the assumption that in the immediate neighbourhood of the normal the intensity may be regarded as sensibly constant.

In the expression (*a*.) we have

$$\text{PM}=\sqrt{\{z^2+(x-p)^2+(y-q)^2\}}=\sqrt{\{f^2+p^2+q^2-2px-2qy\}}=f-\frac{1}{f}(px+qy),\text{nearly,}$$

if we write f for $\sqrt{f^2+p^2+q^2}$. It will be sufficient to replace $\frac{1}{\text{PM}}$ outside the circular function by $\frac{1}{f}$. We may omit the constant f under the circular function, which comes to the same thing as changing the origin of t. We thus get for the disturbance at M due to the unretarded stream,

$$\frac{c}{\lambda f}\int_{-(g+h)}^{-g}\int_{-l}^{l}\sin\frac{2\pi}{\lambda}\left\{vt+\frac{1}{f}(px+qy)\right\}dxdy,$$

or on performing the integrations and reducing,

$$\frac{2chl}{\lambda f}\cdot\frac{\lambda f}{2\pi ql}\sin\frac{2\pi ql}{\lambda f}\cdot\frac{\lambda f}{\pi ph}\sin\frac{\pi ph}{\lambda f}\cdot\sin\frac{2\pi}{\lambda}\left(vt-\frac{pg}{f}-\frac{ph}{2f}\right). \quad . \quad . \quad . \quad . \quad (b.)$$

For the retarded stream, the only difference is that we must subtract R from vt, and that the limits of x are g and $g+k$. We thus get for the disturbance at M due to this stream,

$$\frac{2ckl}{\lambda f}\cdot\frac{\lambda f}{2\pi ql}\sin\frac{2\pi ql}{\lambda f}\cdot\frac{\lambda f}{\pi pk}\sin\frac{\pi pk}{\lambda f}\cdot\sin\frac{2\pi}{\lambda}\left(vt-\text{R}+\frac{pg}{f}+\frac{pk}{2f}\right). \quad . \quad . \quad . \quad . \quad (c.)$$

If we put for shortness τ for the quantity under the last circular function in (*b*.), the expressions (*b*.), (*c*.) may be put under the forms $u\sin\tau$, $v\sin(\tau-\alpha)$, respectively; and if I be the intensity, I will be measured by the sum of the squares of the coefficients of $\sin\tau$ and $\cos\tau$ in the expression

$$u\sin\tau+v\sin(\tau-\alpha),$$

so that

$$\text{I}=u^2+v^2+2uv\cos\alpha,$$

which becomes, on putting for u, v and α, their values, and putting

$$\left\{\frac{\lambda f}{2\pi ql}\sin\frac{2\pi ql}{\lambda f}\right\}^2=\text{Q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

$$\text{I}=\text{Q}.\frac{4c^2l^2}{\pi^2p^2}\left\{\left(\sin\frac{\pi ph}{\lambda f}\right)^2+\left(\sin\frac{\pi pk}{\lambda f}\right)^2+2\sin\frac{\pi ph}{\lambda f}\cdot\sin\frac{\pi pk}{\lambda f}\cos\left[\varrho-\frac{\pi p}{\lambda f}(4g+h+k)\right]\right\}. \quad (11.)$$

12. Suppose now that instead of a point we have a line of homogeneous light, the

line being parallel to the axis of y. The luminous line is supposed to be a narrow slit, through which light enters in all directions, and which is viewed in focus. Consequently each element of the line must be regarded as an independent source of light. Hence the illumination on the object-glass due to a portion of the line which subtends the small angle β at the distance of the object-glass varies as β, and may be represented by $A\beta$. Let the former origin O be referred to a new origin O′ situated in the plane xy, and in the image of the line; and let η, q' be the ordinates of O, M referred to O′, so that $q=q'-\eta$. In order that the luminous point considered in the last article may represent an element of the luminous line considered in the present, we must replace c^2 by $Ad\beta$ or $\frac{A}{f}d\eta$; and in order to get the aggregate illumination due to the whole line, we must integrate from a large negative to a large positive value of η, the largeness being estimated by comparison with $\frac{\lambda f}{l}$. Now the angle $\frac{2\pi ql}{\lambda f}$ changes by π when q changes by $\frac{\lambda f}{2l}$, which is therefore the breadth, in the direction of y, of one of the diffraction bands which would be seen with a luminous point. Since l is supposed not to be extremely small, but on the contrary moderately large, the whole system of diffraction bands would occupy but a very small portion of the field of view in the direction of y, so that we may without sensible error suppose the limits of η to be $-\infty$ and $+\infty$. We have then

$$\int_{-\infty}^{\infty} Q d\eta = \int_{-\infty}^{\infty} \left\{ \frac{\lambda f}{2\pi l(q'-\eta)} \sin \frac{2\pi l(q'-\eta)}{\lambda f} \right\}^2 d\eta = \frac{\lambda f}{2\pi l} \int_{-\infty}^{\infty} \left(\frac{\sin\theta}{\theta} \right)^2 d\theta,$$

by taking the quantity under the circular function in place of η for the independent variable. Now it is known that the value of the last integral is π, as will also presently appear, and therefore we have for the intensity I at any point,

$$I = \frac{2A\lambda l}{\pi^2 p^2} \left\{ \left(\sin \frac{\pi ph}{\lambda f} \right)^2 + \left(\sin \frac{\pi pk}{\lambda f} \right)^2 + 2 \sin \frac{\pi ph}{\lambda f} \cdot \sin \frac{\pi pk}{\lambda f} \cdot \cos \left[\varrho - \frac{\pi p}{\lambda f}(4g+h+k) \right] \right\}, \quad (12.)$$

which is independent of q', as of course it ought to be.

13. Suppose now that instead of a line of homogeneous light we have a line of white light, the component parts of which have been separated, whether by refraction or by diffraction is immaterial, so that the different colours occupy different angular positions in the field of view. Let $B\beta\psi$ be the illumination on the object-glass due to a length of the line which subtends the small angle β, and to a portion of the spectrum which subtends the small angle ψ at the centre of the object-glass. In the axis of x take a new origin O″, and let ξ, p' be the abscissæ of O′, M reckoned from O″, so that $p=p'-\xi$. In order that (12.) may express the intensity at M due to an elementary portion of the spectrum, we must replace A by $Bd\psi$, or $\frac{B}{f}d\xi$; and in order to find the aggregate illumination at M, we must integrate so as to include all values of ξ which are sufficiently near to p' to contribute sensibly to the illumination

at M. It would not have been correct to integrate using the displacement instead of the intensity, because the different colours cannot interfere. Suppose the angular extent, in the direction of x, of the system of diffraction bands which would be seen with homogeneous light, or at least the angular extent of the brighter part of the system, to be small compared with that of the spectrum. Then we may neglect the variations of B and of λ in the integration, considering only those of ξ and ϱ, and we may suppose the changes of ϱ proportional to those of ξ; and we may moreover suppose the limits of ξ to be $-\infty$ and $+\infty$. Let ϱ' be the value of ϱ, and $-\varpi$ that of $\frac{d\varrho}{d\xi}$, when $\xi=p'$, so that we may put $\varrho=\varrho'+\varpi(p'-\xi)$; and take p instead of ξ for the independent variable. Then putting for shortness

$$\frac{\pi h}{\lambda f}=h_{\prime},\qquad \frac{\pi k}{\lambda f}=k_{\prime},\qquad \varpi-\frac{\pi}{\lambda f}(4g+h+k)=g_{\prime}, \quad . \quad . \quad . \quad . \quad . \quad (13.)$$

we have for the intensity,

$$\mathrm{I}=\frac{2\mathrm{B}\lambda l}{\pi^2 f}\int_{-\infty}^{\infty}\{\sin^2 h_{\prime}p+\sin^2 k_{\prime}p+2\sin h_{\prime}p\,.\sin k_{\prime}p\,.\cos(\varrho'-g_{\prime}p)\}\frac{dp}{p^2}.$$

Now $\int_{-\infty}^{\infty}\sin^2 h_{\prime}p\frac{dp}{p^2}=h_{\prime}\int_{-\infty}^{\infty}\sin^2\theta\frac{d\theta}{\theta^2}=\pi h_{\prime}$. Similarly, $\int_{-\infty}^{\infty}\sin^2 k_{\prime}p\,.\frac{dp}{p^2}=\pi k_{\prime}$.

Moreover, if we replace

$$\cos(\varrho'-g_{\prime}p)\text{ by }\cos\varrho'\,.\cos g_{\prime}p+\sin\varrho'\,.\sin g_{\prime}p,$$

the integral containing $\sin\varrho'$ will disappear, because the positive and negative elements will destroy each other, and we have only to find w, where

$$w=\int_{-\infty}^{\infty}\sin h_{\prime}p\,.\sin k_{\prime}p\,.\cos g_{\prime}p\,.\frac{dp}{p^2}.$$

Now we get by differentiating under the integral sign,

$$\frac{dw}{dg_{\prime}}=-\int_{-\infty}^{\infty}\sin h_{\prime}p\,.\sin k_{\prime}p\,.\sin g_{\prime}p\,.\frac{dp}{p}$$

$$=\frac{1}{4}\int_{-\infty}^{\infty}\Big\{\sin(g_{\prime}+h_{\prime}+k_{\prime})p+\sin(g_{\prime}-h_{\prime}-k_{\prime})p-\sin(g_{\prime}+h_{\prime}-k_{\prime})p-\sin(g_{\prime}+k_{\prime}-h_{\prime})p\Big\}\frac{dp}{p}.$$

But it is well known that

$$\int_{-\infty}^{\infty}\frac{\sin sp}{p}\,dp=\pi,\text{ or }=-\pi,$$

according as s is positive or negative. If then we use $\mathrm{F}(s)$ to denote a discontinuous function of s which is equal to $+1$ or -1 according as s is positive or negative, we get

$$\frac{dw}{dg_{\prime}}=\frac{\pi}{4}\Big\{\mathrm{F}(g_{\prime}+h_{\prime}+k_{\prime})+\mathrm{F}(g_{\prime}-h_{\prime}-k_{\prime})-\mathrm{F}(g_{\prime}+h_{\prime}-k_{\prime})-\mathrm{F}(g_{\prime}+k_{\prime}-h_{\prime})\Big\}.$$

This equation gives

$$\frac{dw}{dg_{\prime}}=0,\text{ from }g_{\prime}=-\infty\text{ to }g_{\prime}=-(h_{\prime}+k_{\prime})$$

$$=\tfrac{\pi}{2}, \text{ from } g_{,}=-(h_{,}+k_{,}) \text{ to } g_{,}=-(h_{,}\backsim k_{,})$$

$$=0, \text{ from } g_{,}=-(h_{,}\backsim k_{,}) \text{ to } g_{,}=+(h_{,}\backsim k_{,})$$

$$=-\tfrac{\pi}{2}, \text{ from } g_{,}=h_{,}\backsim k_{,} \text{ to } g_{,}=h_{,}+k_{,}$$

$$=0, \text{ from } g_{,}=h_{,}+k_{,} \text{ to } g_{,}=\infty .$$

Now w vanishes when $g_{,}$ is infinite, on account of the fluctuation of the factor $\cos g_{,}p$ under the integral sign, whence we get by integrating the value of $\frac{dw}{dg_{,}}$ given above, and correcting the integral so as to vanish for $g_{,}=-\infty$,

$w=0$, from $g_{,}=-\infty$ to $g_{,}=-(h_{,}+k_{,})$;

$w=\frac{\pi}{2}(h_{,}+k_{,}+g_{,})$, from $g_{,}=-(h_{,}+k_{,})$ to $g_{,}=-(h_{,}\backsim k_{,})$;

$w=\pi k_{,}$ or $=\pi h_{,}$, (according as $h_{,}>k_{,}$ or $h_{,}<k_{,}$,) from $g=-(h_{,}\backsim k_{,})$ to $g_{,}=+(h_{,}\backsim k_{,})$;

$w=\frac{\pi}{2}(h_{,}+k_{,}-g_{,})$, from $g_{,}=h'\backsim k_{,}$ to $g_{,}=h_{,}+k_{,}$;

$w=0$, from $g_{,}=h_{,}+k_{,}$ to $g_{,}=\infty$.

Substituting in the expression for the intensity, and putting $g_{,}=\frac{\pi g'}{\lambda f}$, so that

$$g'=\frac{\varpi\lambda f}{\pi}-4g-h-k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14.)$$

we get

$$\mathrm{I}=\frac{2\mathrm{B}l}{f^2}(h+k), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15.)$$

when the numerical value of g' exceeds $h+k$;

$$\mathrm{I}=\frac{2\mathrm{B}l}{f^2}\left\{h+k+(h+k-\sqrt{g'^2})\cos g'\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16.)$$

when the numerical value of g' lies between $h+k$ and $h\backsim k$;

$$\mathrm{I}=\frac{2\mathrm{B}l}{f^2}(h+k+2h\cos g'), \text{ or } =\frac{2\mathrm{B}l}{f^2}(h+k+2k\cos g'), \quad . \quad . \quad . \quad . \quad (17.)$$

according as h or k is the smaller of the two, when the numerical value of g' is less than $h\backsim k$.

The discontinuity of the law of intensity is very remarkable.

By supposing $g_{,}=0$, $k_{,}=h_{,}$ in the expression for w, and observing that these suppositions reduce w to $\int_{-\infty}^{\infty}\sin^2 h_{,}p\,.\frac{dp}{p^2}$ we get

$$\int_{-\infty}^{\infty}\left(\frac{\sin h_{,}p}{p}\right)^2 dp=\pi h_{,},$$

a result already employed. This result would of course have been obtained more readily by differentiating with respect to $h_{,}$.

14. The preceding investigation will apply, with a very trifling modification, to Sir David Brewster's experiment, in which the retarding plate, instead of being placed in front of the object-glass of a telescope, is held close to the eye. In this case the eye itself takes the place of the telescope; and if we suppose the whole refraction to take place at the surface of the cornea, which will not be far from the truth, we must replace f by the diameter of the eye, and ψ by the angular extent of the portion of the spectrum considered, diminished in the ratio of m to 1, m being the refractive index of the cornea. When a telescope is used in this experiment, the retarding plate being still held close to the eye, it is still the naked eye, and not the telescope, which must be assimilated to the telescope considered in the investigation; the only difference is that ψ must be taken to refer to the magnified, and not the unmagnified spectrum.

Let the axis of x be always reckoned positive in the direction in which the blue end of the spectrum is seen, so that in the image formed at the focus of the object-glass or on the retina, according as the retarding plate is placed in front of the object-glass or in front of the eye, the blue is to the negative side of the red. Although the plate has been supposed at the positive side, there will thus be no loss of generality, for should the plate be at the negative side it will only be requisite to change the sign of ϱ.

First, suppose ϱ to decrease algebraically in passing from the red to the blue. This will be the case in Sir David Brewster's experiment when the retarding plate is held at the side on which the red is seen. It will be the case in Professor Powell's experiment when the first of the arrangements mentioned in art. 2 is employed, and the value of N in the table of differences mentioned in art. 5 is positive, or when the second arrangement is employed and N is negative. In this case ϖ is negative, and therefore $g' < -(h+k)$, and therefore (15.) is the expression for the intensity. This expression indicates a uniform intensity, so that there are no bands at all.

Secondly, suppose ϱ to increase algebraically in passing from the red to the blue. This will be the case in Sir David Brewster's experiment when the retarding plate is held at the side on which the blue is seen. It will be the case in Professor Powell's experiment when the first arrangement is employed and N is negative, or when the second arrangement is employed and N is positive. In this case ϖ is positive; and since ϖ varies as the thickness of the plate, g' may be made to assume any value from $-(4g+h+k)$ to $+\infty$ by altering the thickness of the plate. Hence, provided the thickness lie within certain limits, the expression for the intensity will be (16.) or (17.). Since these expressions have the same form as (1.), the magnitude only of the coefficient of $\cos \varrho'$, as compared with the constant term, being different, it is evident that the number of bands and the places of the minima are given correctly by the imperfect theory considered in Section I.

15. The plate being placed as in the preceding paragraph, suppose first that the breadths h, k of the interfering streams are equal, and that the streams are contiguous,

so that $g=0$. Then the expression (17.) may be dispensed with, since it only holds good when $g'=0$, in which case it agrees with (16.). Let T_0 be the value of the thickness T for which $g'=0$. Then $T=0$ corresponds to $g'=-(h+k)$, $T=T_0$ to $g'=0$, and $T=2T_0$ to $g'=h+k$; and for values of T equidistant from T_0, the values of g' are equal in magnitude but of opposite signs. Hence, provided T be less than $2T_0$, there are dark and bright bands formed, the vividness of the bands being so much the greater as T is more nearly equal to T_0, for which particular value the minima are absolutely black.

Secondly, suppose the breadths h, k of the two streams to be equal as before, but suppose the streams separated by an interval $2g$; then the only difference is that $g'=-(h+k)$ corresponds to a positive value, T_2 suppose, of T. If T be less than T_2, or greater than $2T_0-T_2$, there are no bands; but if T lie between T_2 and $2T_0-T_2$ bands are formed, which are most vivid when $T=T_0$, in which case the minima are perfectly black.

Thirdly, suppose the breadths h, k of the interfering streams unequal, and suppose, as before, that the streams are separated by an interval $2g$; then $g'=-(h+k)$ corresponds to a positive value, T_2 suppose, of T: $g'=-(h \backsim k)$ corresponds to another positive value, T_1 suppose, of T, T_1 lying between T_2 and T_0, T_0 being, as before, the value of T which gives $g'=0$. As T increases from T_0, g' becomes positive and increases from 0, and becomes equal to $h \backsim k$ when $T=2T_0-T_1$, and to $h+k$ when $T=2T_0-T_2$. When $T<T_2$ there are no bands. As T increases to T_1 bands become visible, and increase in vividness till $T=T_1$, when the ratio of the minimum intensity to the maximum becomes that of $h-k$ to $h+3k$, or of $k-h$ to $k+3h$, according as h or k is the greater of the two, h, k. As T increases to $2T_0-T_1$, the vividness of the bands remains unchanged; and as T increases from $2T_0-T_1$ to $2T_0-T_2$, the vividness decreases by the same steps as it before increased. When $T=2T_0-T_2$, the bands cease to exist, and no bands are formed for a greater value of T.

Although in discussing the intensity of the bands the aperture has been supposed to remain fixed, and the thickness of the plate to alter, it is evident that we might have supposed the thickness of the plate to remain the same and the aperture to alter. Since $\varpi \propto T$, the vividness of the bands, as measured by the ratio of the maximum to the minimum intensity, will remain the same when T varies as the aperture. This consideration, combined with the previous discussion, renders unnecessary the discussion of the effect of altering the aperture. It will be observed, that, as a general rule, fine bands require a comparatively broad aperture in order that they may be well-formed, while broad bands require a narrow aperture.

16. The particular thickness T_0 may be conveniently called the *best thickness*. This term is to a certain extent conventional, since when h and k are unequal the thickness may range from T_1 to $2T_0-T_1$ without any change being produced in the vividness of the bands. The best thickness is determined by the equation

$$\varpi=-\frac{d\varrho}{d\xi}=\frac{\pi}{\lambda f}(4g+h+k).$$

Now in passing from one band to its consecutive, ϱ changes by 2π, and ξ by e, if e be the linear breadth of a band; and for this small change of ξ we may suppose the changes of ϱ and ξ proportional, or but $-\frac{d\varrho}{d\xi}=\frac{2\pi}{e}\cdot$ Hence the best aperture for a given thickness is that for which

$$4g+h+k=\frac{2\lambda f}{e}\cdot$$

If $g=0$ and $k=h$, this equation becomes $h=\frac{\lambda f}{e}\cdot$

The difference of distances of a point in the plane xy whose coordinates are ξ, 0 from the centres of the portions of the object-glass which are covered by the interfering streams, is nearly

$$\frac{\xi}{f}\left(g+\frac{1}{2}h\right)-\frac{\xi}{f}\left(-g-\frac{1}{2}k\right),\text{ or }\frac{\xi}{2f}(4g+h+k);$$

and if δ be the change of ξ when this difference changes by λ,

$$4g+h+k=\frac{2\lambda f}{\delta}\cdot$$

Hence, when the thickness of the plate is equal to the best thickness, $e=\delta$, or the interval between the bands seen in the spectrum is equal to the interval between the bands formed by the interference of two streams of light, of the colour considered, coming from a luminous line seen in focus, and entering the object-glass through two very narrow slits parallel to the axis of y, and situated in the middle of the two interfering streams respectively. This affords a ready mode of remembering and calculating the best thickness of plate for a given aperture, or the best aperture for a given thickness of plate.

17. According to the preceding explanation, no bands would be formed in Sir David Brewster's experiment when the plate was held on the side of the spectrum on which the red was seen. Mr. Airy has endeavoured to explain the existence of bands under such circumstances*. Mr. Airy appears to speak doubtfully of his explanation, and in fact to offer it as little more than a conjecture to account for an observed phenomenon. In the experiments of Mr. Talbot and Mr. Airy, bands appear to have been seen when the retarding plate was held at the red side of the spectrum; whereas Sir David Brewster has stated that he has repeatedly looked for the bands under these circumstances and has never been able to find the least trace of them; and he considers the bands seen by Mr. Talbot and Mr. Airy in this case to be of the nature of Newton's rings. While so much uncertainty exists as to the experimental circumstances under which the bands are seen when the retarding plate is held at the red side of the spectrum, if indeed they are seen at all, it does not seem to be desirable to enter into speculations as to the cause of their existence.

* Philosophical Transactions for 1841, Part I. p. 6.